On the Asymptotic Behavior of Modular Forms and Related Objects

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Gewidmet meiner Familie.
Danke für eure Unterstützung und Liebe.

Abstract

This thesis consists of research articles on the asymptotic behavior of modular forms and various related objects.

First we determine the bivariate asymptotic behavior of Fourier coefficients for a wide class of eta-theta quotients with simple poles in the upper half plane by employing a variant of Wright's Circle Method. These kind of quotients show up in many different areas not only in mathematics. For example they show up in investigations into Vafa–Witten invariants or the counting of so-called BPS-states via wall-crossing, but also in Watson's quintuple product formula which has many applications in number theory and combinatorics.

Further, we offer a general framework to prove asymptotic equidistribution, convexity, and log-concavity of coefficients of generating functions in arithmetic progressions. We do this by using a variant of Wright's Circle Method and give a selection of different examples of such results for various (modular typed) objects.

We end the thesis by employing the Circle Method to prove exact formulae for Fourier coefficients of an infinite family of weight zero mixed false modular forms showing up as characters of modules of rational vertex operator algebras.

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Chapter I

Introduction and statement of objectives

This thesis mainly consists of the three research articles [Ces23, CCM21, CM21] that study the asymptotic behavior of various objects. In this chapter we restate parts of their introductions and collect their main theorems to recall their scientific context.

I.1 Definitions and previous results

This preliminary section is intended to classify the following chapters in the technical context and at the same time create a solid basis for understanding them. Therefore we summerize some basic definitions and previous results on modular forms and a few of their generalisations as well as on partition theory and the Circle Method.

I.1.1 Modular forms

Modular forms and their generalisations play a very important role in many fields of number theory and other areas. Particularly interesting, especially for this thesis, are their Fourier coefficients, which often encode valuable arithmetic, geometric or combinatorial information.

To start, we would like to collect some background on this theory by explaining the basic notations. More details can for example be found in [DS00, KK07, Miy06].

As usual we denote the *complex upper half plane* by $\mathbb{H} := \{\tau = u + iv \in \mathbb{C} : v > 0\}$. Throughout we always use $z =: x + iy \in \mathbb{C}$ and $\tau =: u + iv \in \mathbb{H}$, unless we say otherwise.

A standard group in the theory of modular forms is the *special linear group*

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \ a, b, c, d \in \mathbb{Z},\ \det(M) = 1 \right\},$$

which acts on $\mathbb{H}$ via *Möbius transformations*,

$$M\tau := \frac{a\tau + b}{c\tau + d}, \qquad \text{for } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

since we have that
$$\mathrm{Im}(M\tau) = \frac{v(ad-bc)}{|c\tau+d|^2} = \frac{v}{|c\tau+d|^2} > 0.$$
Another one is the *Hecke congruence subgroup of level* N,
$$\Gamma_0(N) := \{M \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod N\}.$$
Of course there are a lot more congruence subgroups, but we stick to this one to keep the introduction here as simple as possible.

Definition I.1.1.
(a) Let $k \in \mathbb{Z}$. We call a function $f : \mathbb{H} \to \mathbb{C}$ *modular form of weight* k *and level* N, with multiplier $\chi : \Gamma_0(N) \to \{z \in \mathbb{C} : |z| = 1\}$, if the following conditions hold
(1) the function is modular of weight k, i.e.,
$$f\left(\frac{a\tau+b}{c\tau+d}\right) = \chi(M)(c\tau+d)^k f(\tau) \tag{I.1.1}$$
for all $\tau \in \mathbb{H}$ and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$,
(2) the function f is holomorphic,
(3) the function f has at most polynomial growth at all cusps $\Gamma_0(N)\backslash(\mathbb{Q} \cup \{i\infty\})$ of $\Gamma_0(N)$.
(b) We call f a *cusp form* if it vanishes at all cusps.
(c) Functions that instead of satisfying (2) are allowed to have isolated poles in $\mathbb{H} \cup \{i\infty\}$ are called *meromorphic modular forms*, and those that satisfy (I.1.1) but are allowed to have poles at cusps are called *weakly holomorphic modular forms.*

Note that for $4 \mid N$ we can extend this definition to $k \in \frac{1}{2} + \mathbb{Z}$ to obtain *modular forms of half-integral weight*, by replacing (I.1.1) with
$$f\left(\frac{a\tau+b}{c\tau+d}\right) = \varepsilon_d^{-2k} \left(\frac{c}{d}\right) \chi(M)(c\tau+d)^k f(\tau),$$
where
$$\varepsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4, \\ i & \text{if } d \equiv 3 \pmod 4, \end{cases}$$
and where $\left(\frac{\cdot}{\cdot}\right)$ is the Kronecker symbol. This goes back to Shimura [Shi73], who introduced half-integral weight modularity in the context of powers of the theta function, which we will discuss in the next subsection.

I.1.2 Jacobi forms

Jacobi forms are ubiquitous throughout number theory and beyond. For example, they appear in string theory, the theory of black holes, and the combinatorics of partition statistics (see e.g., [BD16, DMZ12, Mal20, RT96]). Note that the original motivation for looking at Jacobi forms comes from their important role in the proof of the Saito–Kurokawa Conjecture proven by Eichler and Zagier [EZ85, Zag79]. We follow [EZ85].

Definition I.1.2. Let $k, m \in \mathbb{N}$. We call a (holomorphic) function $\phi : \mathbb{C} \times \mathbb{H} \to \mathbb{C}$ *(holomorphic) Jacobi form of weight k and index m* on $\Gamma_0(N)$, if it satisfies the two transformation equations, for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and $\lambda, \mu \in \mathbb{Z}$,

$$\begin{aligned} \phi\left(\frac{z}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right) =& (c\tau+d)^k e^{\frac{2\pi i m c z^2}{c\tau+d}} \phi(z;\tau), \\ \phi(z+\lambda\tau+\mu;\tau) =& e^{-2\pi i m\left(\lambda^2\tau+2\lambda z\right)} \phi(z;\tau), \end{aligned}$$

(i.e., ϕ transforms modular in τ and elliptic in z) and has a Fourier expansion of the form

$$\phi(z;\tau) = \sum_{n\geq 0} \sum_{\substack{r\in\mathbb{Z} \\ r^2\leq 4nm}} c(n,r) q^n \zeta^r,$$

where $q := e^{2\pi i\tau}$ and $\zeta := e^{2\pi i z}$. Additionally, we call ϕ a *Jacobi cusp form* if we have that $c(n,r) = 0$, whenever $4nm = r^2$.

Remarks.
(1) One can extend this definition to Jacobi forms of negative index (see e.g., [BFOR17, Section 11.3]).
(2) As for modular forms we are additionally able to extend the definition to Jacobi forms of half-integal weight and index whenever $4 \mid N$ by replacing the modular transformation in τ by

$$\phi\left(\frac{z}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right) = \varepsilon_d^{-2k} \left(\frac{c}{d}\right) (c\tau+d)^k e^{\frac{2\pi i m c z^2}{c\tau+d}} \phi(z;\tau).$$

We immediately note that $\phi(0;\tau)$ is a modular form of weight k, which easily demonstrates that Jacobi forms are two-variable genaralisations of the aforementioned modular forms.

Further we have the theta decomposition (see [EZ85, pages 57–58])

$$\phi(z;\tau) = \sum_{\mu \pmod{2m}} h_\mu(\tau)\vartheta_{m,\mu}(z;\tau), \tag{I.1.2}$$

with

$$h_\mu(\tau) := \sum_{N\geq 0} c\left(\frac{N+r^2}{4m}, r\right) q^{\frac{N}{4m}}, \qquad \vartheta_{m,\mu}(z;\tau) := \sum_{\substack{r\in\mathbb{Z} \\ r\equiv\mu \pmod{2m}}} q^{\frac{r^2}{4m}} \zeta^r.$$

By definition we have that ϕ is a Jacobi form of weight k and index m. Since the theta functions $\vartheta_{m,\mu}(z;\tau)$ have weight $\frac{1}{2}$ and index m, (I.1.2) provides that $h_\mu(\tau)$ has to be a modular form of weight $k-\frac{1}{2}$, which gives another connection between Jacobi forms and modular forms of half-integral weight. To be more precise, this theta decomposition gives an isomorphism between the Jacobi forms of weight k and index m and the space of vector-valued modular forms $(h_\mu)_{\mu \pmod{2m}}$ on $\mathrm{SL}_2(\mathbb{Z})$ satisfying certain transformation laws and that are bounded as $\mathrm{Im}(\tau)\to\infty$ (see [EZ85, Theorem 5.1]).

In this thesis, we are particularly interested in the *Jacobi theta function* defined by[1]

$$\vartheta(z;\tau) := iq^{\frac{1}{8}}\zeta^{\frac{1}{2}} \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{n^2+n}{2}} \zeta^n. \tag{I.1.3}$$

I.1.3 False theta functions as characters of vertex operator algebras

Characters of modules of rational vertex operator algebras are often of the form

$$\frac{f(\tau)}{\eta(\tau)^k},$$

where η is the *Dedekind η-function*, which is a weight $\frac{1}{2}$ modular form for $\mathrm{SL}_2(\mathbb{Z})$ defined by

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n\geq 1} (1-q^n).$$

In [CM14] the authors observed that some numerators of atypical characters of the so-called $(1,p)$-singlet algebra are false theta functions of Rogers (see [AB09]). In particular, the numerators of characters of the atypical irreducible modules of the $(1,p)$-singlet vertex operator algebra $M_{1,s}$, for $1\leq s\leq p-1$ and $p\in\mathbb{N}_{\geq 2}$, that have been studied in [BM15, CM14, CMW17], are essentially the false theta functions, for $j\in\mathbb{Z}$ and $N\in\mathbb{N}_{>1}$,

$$F_{j,N}(\tau) := \sum_{n\in\mathbb{Z}} \mathrm{sgn}\left(n+\frac{j}{2N}\right) q^{N\left(n+\frac{j}{2N}\right)^2}, \tag{I.1.4}$$

[1] Note that we have the connection

$$\vartheta_{m,\mu}(z;\tau) = -q^{\frac{\mu^2}{4m}+\frac{m}{4}-\frac{\mu}{2}} \zeta^{\mu-m} \vartheta\left((\mu-m)\tau + 2mz + \frac{1}{2}; 2m\tau\right).$$

with $\operatorname{sgn}(n) := \frac{n}{|n|}$ for $n \neq 0$, $\operatorname{sgn}(0) := 0$. Note that this sgn-factor prevents modularity, which is why we call these functions “false”. Removing the sgn-factor would give classical theta functions, which are modular forms of weight $\frac{1}{2}$. Recently Bringmann and Nazaroglu [BN19] constructed a certain modular completion $\widehat{F_{j,N}}$, resolving this obstuction.

We call a function a *mixed false modular form* if it is a linear combination of false theta functions multiplied by modular forms. One could make this definition precise by adapting [BFOR17, Definition 13.1].

I.1.4 Partitions, their generalizations, and some statistics

An (*integer*) *partition* λ of a non-negative integer n is a list of non-increasing positive integrers, say $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, that satisfies $|\lambda| := \lambda_1 + \cdots + \lambda_m = n$. We denote the number of partitions of n by $p(n)$ and set $p(0) = 1$, as usual. For example the partitions of 4 are given by

$$(4), \quad (3,1), \quad (2,2), \quad (2,1,1), \quad (1,1,1,1).$$

We refer the readers to [And98] for an excellent survey on partitions.

One of the most famous results in partition theory is due to Ramanujan, who proved in [Ram21] that for $n \geq 0$ the following congruences hold

$$p(5n+4) \equiv 0 \pmod{5}, \qquad p(7n+5) \equiv 0 \pmod{7}, \qquad p(11n+6) \equiv 0 \pmod{11}.$$

This inspired further research in the field of partition congruences see for example Atkin–O'Brian [AO67] and Ono [Ono00].

Euler proved that one may write the generating function of integer partitions as the following infinite product

$$P(q) := \sum_{n\geq 0} p(n)q^n = \prod_{n\geq 1} \frac{1}{1-q^n} = \frac{q^{\frac{1}{24}}}{\eta(\tau)},$$

which provides an archetypal example of the close connection between partitions and modular forms.

One of the most popular statistics of integer partitions is the so-called *rank* of λ, which is given by the largest part minus the number of parts, and was introduced by Dyson [Dys44] to motivate the Ramanujan congruences combinatorially. As conjectured by Dyson [Dys44] and later proved by Atkin and Swinnerton-Dyer [ASD54] the rank in fact gives a combinatorial explanation for the first and second congruence. Dyson additionally conjectured the existence of another statistic, which he called the *crank* and which should explain all Ramanujan congruences. It was later found by Andrews and Garvan [AG88, Gar88] and is given by

$$\operatorname{crank}(\lambda) := \begin{cases} \lambda_1 & \text{if } \lambda \text{ contains no ones,} \\ \mu(\lambda) - \omega(\lambda) & \text{if } \lambda \text{ contains ones,} \end{cases}$$

where $\omega(\lambda)$ denotes the number of ones in λ, and $\mu(\lambda)$ denotes the number of parts greater than $\omega(\lambda)$.

Since that time, a large area of research has developed around this topic. For example it turns out that the two-parameter generating functions of the rank and crank function are closely related to (mock) modular forms [AG88, ASD54]. Essentially they turn out to be a mock Jacobi form and a Jacobi form of weight and index $-\frac{1}{2}$. For a nice overview on more properties of these functions we refer the reader to [BFOR17, Subsection 14.3] or the introductions of the Chapters II and III for some further results.

As we will see in Chapter III, not only are integer partitions a very popular area of research in a wide variety of fields, but their generalizations are also receiving increasing attention. Here, we want to briefly mention two examples that will be studied later on.

An *overpartition* is a partition where the first occurrence of each distinct number may be overlined. For example there are fourteen overpartitions of 4, given by

$$(4),\ \left(\overline{4}\right),\ (3,1),\ \left(\overline{3},1\right),\ \left(3,\overline{1}\right),\ \left(\overline{3},\overline{1}\right),\ (2,2),\ \left(\overline{2},2\right),$$
$$(2,1,1),\ \left(\overline{2},1,1\right),\ \left(2,\overline{1},1\right),\ \left(\overline{2},\overline{1},1\right),\ (1,1,1,1),\ \left(\overline{1},1,1,1\right).$$

The generating function for overpartitions is given by [CL04]

$$\prod_{n\geq 1} \frac{1+q^n}{1-q^n},$$

since the non-overlined parts form an integer partition, while the overlined parts form a partition into distinct parts (their generating function is given by $\prod_{n\geq 1}(1+q^n)$). The *first residual crank* of an overpartition, which was introduced by Bringmann, Lovejoy, and Osburn in [BLO09], is given by the crank of the subpartition consisting of the non-overlined parts.

A *plane partition* of n (see e.g., [And98]) is a two-dimensional array $\pi_{j,k}$ of non-negative integers $j, k \geq 1$, that is non-increasing in both variables, i.e., $\pi_{j,k} \geq \pi_{j+1,k}$, $\pi_{j,k} \geq \pi_{j,k+1}$ for all j and k, and fulfills $|\Lambda| := \sum_{j,k} \pi_{j,k} = n$. For example there are six plane partitions of 3, given in Figure III.2. We let $\operatorname{pp}(n)$ denote the number of plane partitions of n. One of the more famous statistics associated to plane partitions $\Lambda = \{\pi_{j,k}\}_{j,k\geq 1}$ is its *trace* $t(\Lambda)$, which is defined by

$$t(\Lambda) = \sum_{j\geq 1} \pi_{j,j}.$$

Certain asymptotic properties of the trace have been studied by Kamenov and Mutafchiev [KM07] and Mutafchiev [Mut18], where the limiting distribution and expected value of $t(\Lambda)$ were considered. In Chapter III we study the distribution of the trace in residue classes.

I.1.5 The Circle Method

Investigating the asymptotic behavior of Fourier coefficients of modular forms is not only interesting within number theory, but has applications in various mathematical fields. One of the most famous asymptotic formulae in partition theory was found in 1918 by Hardy and Ramanujan [HR18] who proved that

$$p(n) \sim \frac{1}{4\sqrt{3}n} \cdot e^{\pi\sqrt{\frac{2n}{3}}}$$

as $n \to \infty$. Their work marked the birth of the so-called (Hardy–Ramanujan) Circle Method. Our exposition follows [BFOR17, Sketch of Proof of Theorem 14.3]. Suppose that one is interested in the asymptotic behavior of a sequence $\{a(n)\}$ of "moderate growth" as $n \to \infty$. One builds a generating function for that sequence $A(q) := \sum_{n\geq 0} a(n)q^n$, suppose that it has radius of convergence equal to 1. Using Cauchy's Theorem one is thus able to extract the coefficients

$$a(n) = \frac{1}{2\pi i} \int_C \frac{A(q)}{q^{n+1}}\, dq,$$

where C is an arbitrary path inside the unit disk, that loops around zero in the counter-clockwise direction exacly once. For many interesting sequences, e.g., $\{a(n)\} = \{p(n)\}$, the singularities of the generating function A show up as roots of unity on the unit disk. When the generating function is modular, the behavior near the singularities is well-approximable. This further means that one can often find nice approximations for A near these singularities, which provide the main terms of the approximation. The leftover terms then contribute to a much smaller error term.

Half a century after Hardy and Ramanujan proved their famous asymptotic result, Wright [Wri68, Wri71] developed a modified version of the Circle Method, referred to as Wright's Circle Method, which provides a general method for studying the Fourier coefficients of functions with known asymptotic behavior near cusps. The essence of Wright's method is to use Cauchy's theorem to recover the coefficients as seen before. One then splits the integral into two arcs, the "major arc" and "minor arc", where the generating function has large growth (towards the dominant pole(s)[2]) and small relative

[2]The poles, where the most growth appears.

growth (away from these pole(s)), respectively (see Figure I.1). Even though this version of the Circle Method gives weaker bounds than the original techniques of Hardy and Ramanujan, it is more flexible when working with non-modular generating functions. It has been used extensively in the literature, see e.g., [BM14, KKS15, Mal20, Mal21a, Mao18] for several examples closely related to Chapters II and III.

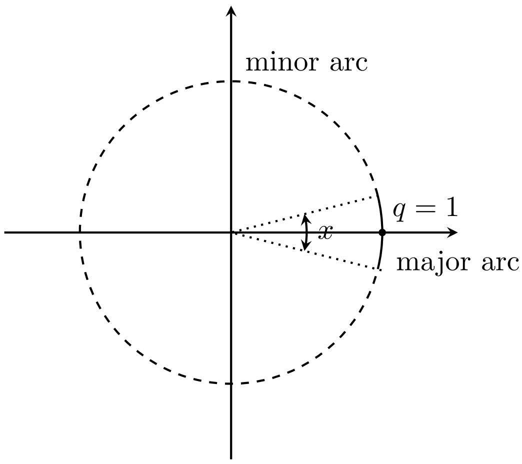

Figure I.1: Idea behind Wright's Circle Method

I.1.6 *I*-Bessel functions and Kloosterman sums

For $x > 0$ the *I-Bessel function of order* ℓ may be defined as (see e.g., [Arf85, BD16])

$$I_\ell(x) := \frac{1}{2\pi i}\int_\Gamma t^{-\ell-1} e^{\frac{x}{2}\left(t+\frac{1}{t}\right)} dt,$$

where Γ is a contour which starts in the lower half plane at $-\infty$, surrounds the origin counterclockwise and returns to $-\infty$ in the upper half plane. We are particularly interested in the asymptotic behavior of I_ℓ as $x \to \infty$, given for fixed ℓ by (see e.g., [AAR99, equation (4.12.7)])

$$I_\ell(x) = \frac{e^x}{\sqrt{2\pi x}} + O\left(\frac{e^x}{x^{\frac{3}{2}}}\right).$$

In Chapters II and IV we will see that these *I*-Bessel functions play an important part in the calculation of the main term arising from the Circle Method. For example they show up in the Fourier expansion of the weakly holomorphic Poincaré series of exponential type as e.g., stated in [BFOR17, Theorem 6.9].

Moreover Kloosterman sums play a very important role in the calculations of the exact formulae in Chapter IV, since they show up in the Fourier expansion of modular-type objects. Here we give some general background. We follow [Est29, Sal32].

Let $k \geq 1$ be a positive integer and h an integer such that $0 \leq h < k$. Further let $\gcd(h,k) = 1$. Then there exists an unique integer h' such that $hh' \equiv 1 \pmod{k}$ with $0 \leq h' < k$. For $n, m \in \mathbb{Z}$ we thus define a (standard) *Kloosterman sum* by

$$S(n,m;k) := \sum_{\substack{0 \leq h < k \\ \gcd(h,k)=1}} \exp\left(2\pi i \frac{nh + mh'}{k}\right).$$

A famous estimate on these sums goes back to Weil and states that

$$|S(n,m;k)| \leq \tau(k) \gcd(n,m,k)^{\frac{1}{2}} k^{\frac{1}{2}},$$

where $\tau(k) := \sum_{d \geq 1, d|k} 1$.

I.2 Statement of objectives

I.2.1 Bivariate asymptotics for eta-theta quotients with simple poles

In the first project of this thesis, see Chapter II, we give an example of how the theory of modular forms can be used in areas outside of number theory, namely string theory, by determining the asymptotic profile of a family of eta-theta quotients with multiple simple poles.

We consider the weight $\sum_{j=1}^{N} \frac{\alpha_j}{2}$ and index $c - b^2$ meromorphic Jacobi form

$$f(z;\tau) := \frac{\vartheta(z;\tau)}{\vartheta(bz;c\tau)} \prod_{j=1}^{N} \eta(a_j \tau)^{\alpha_j},$$

where $a_j, b, c \in \mathbb{N}$, $N \in \mathbb{N}_{>1}$, and $\alpha_j \in \mathbb{Z}$. Such eta-theta quotients appear in numerous places, for example theta quotients [GSZ19] and investigations into the counting of so-called BPS-states [Wot13], to just name a few.

Defining

$$f(z;\tau) =: \sum_{\substack{n \geq 0 \\ m \in \mathbb{Z}}} c(m,n) \zeta^m q^n,$$

for z in a small neighborhood of 0 that is pole-free as a function in z, we show how to obtain a bivariant asymptotic behavior for those coefficients $c(m,n)$. We prove the following theorem.

Theorem I.2.1 (Theorem II.1.2). *Define* $\beta = \beta(n) := \pi\sqrt{\frac{2}{n}}$ *and* $w := \frac{1}{2}\sum_{j=1}^{N} \alpha_j \in \frac{1}{2}\mathbb{Z}$, *which is the weight of the eta quotient part of our function* f, *along with*

$$\Lambda_1 := (-1)^{2w+1} (2\pi)^w \frac{c^{\frac{3}{2}}}{4\pi^2 (2b^2 - b - c)} \prod_{j=1}^{N} a_j^{-\frac{\alpha_j}{2}},$$

and

$$\Lambda_2 := \frac{b^2}{c} - \frac{b}{c} + \frac{1}{4c} - \frac{1}{4} - \sum_{j=1}^{N} \frac{\alpha_j}{12a_j}.$$

Assume that $0 < 1 - \sum_{j=1}^{N} \frac{\alpha_j}{12a_j} < \sqrt{\Lambda_2}$, $\sum_{j=1}^{N} \frac{\alpha_j}{a_j} < 0$, b *even with* $b \neq c$, $b^2 > c$, *and* $m = m(n)$ *with* $|m| \leq \frac{1}{6\beta} n^{-\delta} \log(n)$ *for some* $0 < \delta < \frac{1}{2}$ *such that* $m \to \infty$ *as* $n \to \infty$. *Then*

$$c(m,n) = \frac{1}{2\pi i} \Lambda_1 \beta^{2-w} \sqrt{\Lambda_2}^{-w} \frac{e^{2\pi\sqrt{2\Lambda_2 n}}}{2\pi (2\Lambda_2 n)^{\frac{1}{4}}} + O\left(\beta^{3-w} \frac{e^{2\pi\sqrt{2\Lambda_2 n}}}{2\pi (2\Lambda_2 n)^{\frac{1}{4}}}\right)$$

as $n \to \infty$.

To prove this theorem we follow the work of Males [Mal20, Mal21a], who proved the bivariant asymptotic behavior of the certain eta-theta quotient

$$\frac{\vartheta(z;\tau)^4}{\eta(\tau)^9 \vartheta(2z;\tau)}.$$

As in his work we use an extension of Wright's Circle Method that was pioneered by Bringmann and Dousse [BD16], respectively Dousse and Mertens [DM15], to study the bivariant asymptotic behavior of the Fourier coefficients of the partition crank, respectively partition rank, function. However we need to modify their arguments a little, since our family of functions has multiple simple poles.

I.2.2 Asymptotic equidistribution for partition statistics and topological invariants

In the second project of this thesis, see Chapter III, we provide a general framework for proving asymptotic equidistribution, convexity, and log-concavity of coefficients of generating functions on arithmetic progressions.

Throughout many areas in pure mathematics, the equidistribution properties of certain objects are a central theme studied by many authors, including areas of algebraic and arithmetic geometry [CM15, GT12, Kat15] and number theory [OS18, Xi20].

The primary aim of this project is for proving large families of so-called Dirichlet-type equidistribution theorems. Suppose $c(n)$ is an arithmetic function which counts something of interest. Let $q = e^{-z}$, where $z = x + iy \in \mathbb{C}$ with $x > 0$ and $|y| < \pi$. Furthermore let $\zeta = \zeta_b^a := e^{\frac{2\pi i a}{b}}$ be a b-th root of unity for some natural number $b \geq 2$ and $0 \leq a < b$. Assume that we have a generating function on arithmetic progressions $a \pmod b$ given by

$$H(a, b; q) = \sum_{n \geq 0} c(a, b; n) q^n,$$

for some coefficients $c(a, b; n)$ such that

$$H(a, b; q) = \frac{1}{b} \sum_{j=0}^{b-1} \zeta_b^{-aj} H\left(\zeta_b^j; q\right),$$

for some generating functions $H(\zeta; q)$, with $H(q) := H(1; q) = \sum_{n \geq 0} c(n) q^n$. To say that equidistribution of $c(a, b; n)$ holds is to say that $c(a, b; n) \sim \frac{1}{b} c(n)$ as $n \to \infty$. Our framework may be summarized as follows.

Result (see Theorem III.3.1 for a precise statement)**.** *Assume that on both the major and minor arcs $H(q)$ dominates $H(\zeta; q)$, and $H(q)$ is dominant on the major arc as $q \to 1$. Then $c(a, b; n)$ are equidistributed as $n \to \infty$.*

We use this framework to offer a selection of different examples of such results, proving asymptotic equidistribution for various partition statistics and topological invariants (see Theorems III.1.1 to III.1.6). For example, let $M(a, b; n)$ be the number of partitions of n with crank congruent to $a \pmod b$. We prove the following theorem.

Theorem I.2.2 (see Theorem III.1.2)**.** *Let $0 \leq a < b$ and $b \geq 2$. Then as $n \to \infty$ we have that*

$$M(a, b; n) = \frac{1}{b} p(n) \left(1 + O\left(n^{-\frac{1}{2}}\right)\right).$$

Additionally we use our framework to immediately conclude asymptotic convexity and log-concavity for a large class of functions (see Corollaries III.3.2 and III.3.3).

A central ingredient employed in the proof of our framework is a variant of Wright's Circle Method, which was recently developed by Bringmann, Craig, Males, and Ono [BCMO22, Proposition 4.4], following work of Ngo and Rhoades [NR17].

I.2.3 Fourier coefficients of weight zero mixed false modular forms

In the third and last project of this thesis, see Chapter IV, we give, to the best of the author's knowledge, a first example of exact formulae for Fourier coefficients of an infinite family of weight zero mixed false modular forms.

In 1937 Rademacher [Rad37] proved the following exact formula for the partition function

$$p(n) = \frac{2\pi}{(24n-1)^{\frac{3}{4}}} \sum_{k\geq 1} \frac{A_k(n)}{k} I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right),$$

with a Kloosterman sum $A_k(n)$ and an I-Bessel function of half-integral order (defined in (IV.1.2) and (IV.1.3)), from which one can deduce the famous asymptotic behavior of the partition function, mentioned above.

We find Rademacher-type exact formulae for Fourier coefficients of an infinite family of weight zero mixed false modular forms

$$\mathcal{A}_{j,N}(\tau) := \frac{F_{j,N}(\tau)}{\eta(\tau)},$$

where, for $\tau \in \mathbb{H}$, $j \in \mathbb{Z}$ and $N \in \mathbb{N}_{>1}$, $F_{j,N}(\tau)$ defined in (I.1.4) is a false theta function at rank one. Defining their coefficients by

$$\mathcal{A}_{j,N}(\tau) =: q^{\frac{j^2}{4N}-\frac{1}{24}}\left(a_{j,N}(0) + \sum_{n\geq 1} a_{j,N}(n) q^n\right)$$

we prove the following theorem.

Theorem I.2.3 (Theorem IV.1.1). *For all $n \geq 1$ and $\sqrt{\frac{N}{6}} \notin \mathbb{Z}$ we have*

$$a_{j,N}(n) = -\frac{2\pi i}{\sqrt{n+\frac{j^2}{4N}-\frac{1}{24}}} \sum_{k\geq 1}\sum_{r=1}^{N-1}\sum_{\kappa=0}^{k-1} \frac{K_{k,j,N}(n,r,\kappa)}{k^2}$$
$$\times \mathrm{P.V.}\int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}} \sqrt{\frac{1}{24}-Nx^2}\cot\left(\pi\left(-\frac{x}{k}+\frac{\kappa}{k}+\frac{r}{2Nk}\right)\right) I_1\left(\frac{4\pi\sqrt{n+\frac{j^2}{4N}-\frac{1}{24}}}{k}\sqrt{\frac{1}{24}-Nx^2}\right) dx,$$

where I_α denotes the I-Bessel function of order α and $K_{k,j,N}(n,r,\kappa)$ is a Kloosterman sum defined as

$$K_{k,j,N}(n,r,\kappa) := \sum_{\substack{0\leq h<k \\ \gcd(h,k)=1}} \chi_{j,r}(N, M_{h,k}) \zeta_{24k}^{\left(24N\left(\kappa+\frac{r}{2N}\right)^2-1\right)h' - 24\left(n+\frac{j^2}{4N}-\frac{1}{24}\right)h},$$

with h' *a solution of* $hh' \equiv -1 \pmod{k}$*,* $M_{h,k} = \begin{pmatrix} h' & -\frac{hh'+1}{k} \\ k & -h \end{pmatrix}$*,* $\chi_{j,r}(N, M)$ *the multiplier defined in* (IV.2.6)*, and* $\zeta_\ell := e^{\frac{2\pi i}{\ell}}$ *with* $\ell \in \mathbb{N}$ *an* ℓ*-th root of unity.*

Additionally we show that our Kloosterman sum satisfies the following bound by rewriting it into a sort of Salié sum and using a bound of Malishev (see Lemma IV.1.3).

Theorem I.2.4 (Theorem IV.1.2)**.** *For* $\varepsilon > 0$ *we have that*

$$K_{k,j,N}(n, r, \kappa) = O_N\left(nk^{\frac{1}{2}+\varepsilon}\right)$$

as $k \to \infty$*.*

The proof of these results requires considerably more effort in comparison to negative weight functions, or weight zero modular forms (see e.g., [BN19, Rad38]). Compared to negative weight functions, for example, we have to take special care of the bound of the Kloosterman sum occuring to ensure that the error term in the Circle Method vanishes. Additionally our transformation behavior is not as simple as the one of a modular form, which results in more difficulties. To prove our exact formulae we investigate the “false” modular transformation behavior of our family of functions, following [BN19], and use the Circle Method along with ideas of Rademacher and Zuckerman [Rad38, Rad37, RZ38].

Chapter II

Bivariate asymptotics for eta-theta quotients with simple poles

This chapter is based on a preprint of the same title recommended for publication in *The Ramanujan Journal* and is joint work with Dr. Joshua Males [CM21].

II.1 Introduction and statement of results

Jacobi forms (see e.g., [EZ85]) are ubiquitous throughout number theory and beyond. For example, they appear in string theory [Mal20, RT96], the theory of black holes [DMZ12], and the combinatorics of partition statistics [BD16]. The Fourier coefficients of Jacobi forms often encode valuable arithmetic information. To describe a motivating example, let λ be a *partition* of a positive integer n, i.e., a list of non-increasing positive integers λ_j with $1 \leq j \leq s$ that sum to n. We denote the number of partitions of n by $p(n)$, as usual. One of the most famous results in partition theory is due to Ramanujan, who proved in [Ram21] that for $n \geq 0$ the following congruences hold

$$p(5n+4) \equiv 0 \pmod{5}, \qquad p(7n+5) \equiv 0 \pmod{7}, \qquad p(11n+6) \equiv 0 \pmod{11}.$$

The *rank* [Dys44] of λ is given by the largest part minus the number of parts. It offers a combinatorial explanation for the first and second congruence as conjectured by Dyson [Dys44] and later proved by Atkin and Swinnerton-Dyer [ASD54], since the partitions of $5n+4$ (respectively $7n+5$) form 5 (respectively 7) equal-sized groups when sorted by their ranks modulo 5 (respectively 7). Dyson additionally conjectured the existence of another statistic, which he called the crank and which should explain all Ramanujan congruences. The *crank* of λ was later found by Andrews and Garvan [AG88, Gar88] and is given by

$$\begin{cases} \lambda_1 & \text{if } \lambda \text{ contains no ones,} \\ \mu(\lambda) - \omega(\lambda) & \text{if } \lambda \text{ contains ones.} \end{cases}$$

Here, $\omega(\lambda)$ denotes the number of ones in λ, and $\mu(\lambda)$ denotes the number of parts greater than $\omega(\lambda)$. We denote by $M(m,n)$, respectively $N(m,n)$, the number of partitions of n

with crank m, respectively rank m. Throughout the rest of this chapter we let $\zeta := e^{2\pi iz}$ for $z \in \mathbb{C}$, and $q := e^{2\pi i\tau}$ with $\tau \in \mathbb{H}$, the upper half plane. It is well-known that the generating function of M is given by (see [BD16, equation (2.1)])

$$\sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}} M(m,n)\zeta^m q^n = \frac{i\left(\zeta^{\frac{1}{2}}-\zeta^{-\frac{1}{2}}\right)q^{\frac{1}{24}}\eta^2(\tau)}{\vartheta(z;\tau)},$$

which is a weak Jacobi form (up to rational powers of ζ and q). Here, the *Dedekind η-function* is given by

$$\eta(\tau) := q^{\frac{1}{24}}\prod_{n\geq 1}\left(1-q^n\right),$$

and the *Jacobi theta function* is defined by

$$\vartheta(z;\tau) := iq^{\frac{1}{8}}\zeta^{\frac{1}{2}}\sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{n^2+n}{2}}\zeta^n. \tag{II.1.1}$$

Note that a similar formula can be found for the generating function of N as a mock Jacobi form involving an eta-theta quotient. In general Jacobi forms have a Fourier expansion of the form

$$\sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}} a(m,n)\zeta^m q^n.$$

Many interesting examples of Jacobi forms arise as quotients of η- and ϑ-functions. As an illuminating example, for $a_k, b_j \in \mathbb{N}$ and $n \in \mathbb{Z}$, consider the study of theta quotients [GSZ19, equation (13)],

$$\frac{\vartheta(a_1z;\tau)\vartheta(a_2z;\tau)\cdots\vartheta(a_kz;\tau)}{\vartheta(b_1z;\tau)\vartheta(b_2z;\tau)\cdots\vartheta(b_jz;\tau)}\eta(\tau)^n,$$

which provide new constructions of (not necessarily holomorphic) Jacobi and Siegel modular forms. As highlighted by Gritsenko, Skoruppa, and Zagier, theta quotients also have deep applications to areas such as Fourier analysis over infinite-dimensional Lie algebras and the moduli spaces in algebraic geometry. In this chapter, we obtain the bivariate asymptotic behavior of the coefficients of a prototypical family of such theta quotients, while the steps presented here also offer a pathway to obtain similar results for more general families. Our framework covers theta quotients for $k = j = 1$, $a_1 = 1, b_1 \in \mathbb{N}$, and $n \in \mathbb{Z}$.

In [BD16] Bringmann and Dousse pioneered the use of new techniques in the study of the bivariate asymptotic behavior of the Fourier coefficients and applied them to the partition crank function. In [DM15] Dousse and Mertens used these techniques to study the rank function. In particular, each of these papers used an extension of Wright's Circle Method [Wri34, Wri71] to obtain bivariate asymptotics of $N(m,n)$ and $M(m,n)$, with m in a certain range depending on n.

Recently, Males extended these techniques to an example appearing in the partition function for entanglement entropy in string theory. In particular, [Mal20, Mal21a] considered the eta-theta quotient

$$\frac{\vartheta(z;\tau)^4}{\eta(\tau)^9\vartheta(2z;\tau)} =: \sum_{\substack{m\in\mathbb{Z}\\ n\geq 0}} b(m,n)\zeta^m q^m$$

with a simple pole at $z=\frac{1}{2}$. The bivariate asymptotic behavior of the coefficients $b(m,n)$ is given by [Mal21a, Theorem 1.1].

Theorem II.1.1. *For $\beta := \pi\sqrt{\frac{2}{n}}$ and $|m| \leq \frac{1}{6\beta}\log(n)$ we have that*

$$b(m,n) = (-1)^{m+\delta+1}\frac{i\beta^6 m}{8\pi^5(2n)^{\frac{1}{4}}}e^{2\pi\sqrt{2n}} + O\left(mn^{-\frac{15}{4}}e^{2\pi\sqrt{2n}}\right)$$

as $n\to\infty$. Here, $\delta := 1$ if $m<0$ and $\delta = 0$ otherwise.

This chapter serves to extend these results to a large family of eta-theta quotients with multiple simple poles[1]. Such eta-theta quotients appear in numerous places. For example, investigations into Vafa–Witten invariants [Ale21, equation (2.5)] involve the functions

$$\frac{i}{\eta(\tau)^{N-1}\vartheta(2z;\tau)},$$

which also appear in investigations into the counting of so-called BPS-states via wall-crossing [Wot13, equation (5.114)]. The asymptotics of this family of functions were studied in [BM13]. Other examples of similar shapes also arise as natural pieces of functions in investigations into BPS-states, see e.g., [Wot13, Section 5.6.2]. Similar functions also appear prominently in Watson's well-known quintuple product formula

$$\vartheta^*(z;\tau) := \sum_{r\in\mathbb{Z}}\left(\frac{12}{r}\right)q^{\frac{r^2}{24}}\zeta^{\frac{r}{2}} = \frac{\eta(\tau)\vartheta(2z;\tau)}{\vartheta(z;\tau)},$$

[1] A similar framework exists for those without poles by simply extending the results of [BD16, DM15].

which has a plethora of applications in number theory and combinatorics, and our main theorem gives a bivariate asymptotic for the coefficients of $\vartheta^*(z;\tau)^{-1}$. Such asymptotics for inverse theta functions are a topic currently in vogue in the literature, see e.g., [BM13, LZ22] and the references contained therein.

Throughout, we consider an eta-theta quotient of the form

$$f(z;\tau) \coloneqq \frac{\vartheta(z;\tau)}{\vartheta(bz;c\tau)} \prod_{j=1}^{N} \eta(a_j\tau)^{\alpha_j},$$

where $a_j, b, c \in \mathbb{N}$, $N \in \mathbb{N}_{>1}$, and $\alpha_j \in \mathbb{Z}$. Since we require asymptotic growth, we assume that $\sum_{j=1}^{N} \frac{\alpha_j}{a_j} < 0$. We omit the dependency on these parameters for notational ease. We assume that b is even, $b \neq c$, and $b^2 > c$, and indicate the differences that would occur if b were odd. In the language of [GSZ19], this is a family of theta quotients.

Remarks.

(1) Note that by the conditions from above we assume that we have exponential growth towards the cusp 0 and therefore ensure that the Circle Method works by choosing the major arc around $q = 1$.

(2) The exposition presented here may be easily generalized to include products of theta functions in both the numerator and denominator of f, although this becomes lengthy to write out for the general case.

(3) We include a theta function in the numerator to allow us to assume that there are no poles of f at the lattice points 0 or 1. However, using the techniques presented here and shifting integrals to not have endpoints at 0 or 1, a similar method holds for functions without a theta function in the numerator.

We define the coefficients $c(m,n)$ by

$$f(z;\tau) \eqqcolon \sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}} c(m,n)\zeta^m q^n,$$

for some z in a small neighborhood of 0 that is pole-free, and investigate their bivariate asymptotic behavior. To this end, we employ and extend the techniques of [BD16], which also appear in [DM15, Mal20, Mal21a], using Wright's Circle Method to arrive at the following theorem.

Theorem II.1.2. *Define $\beta = \beta(n) \coloneqq \pi\sqrt{\frac{2}{n}}$ and $w \coloneqq \frac{1}{2}\sum_{j=1}^{N} \alpha_j \in \frac{1}{2}\mathbb{Z}$, which is the weight of the eta quotient part of our function f, along with*

$$\Lambda_1 \coloneqq (-1)^{2w+1} (2\pi)^w \frac{c^{\frac{3}{2}}}{4\pi^2 \left(2b^2 - b - c\right)} \prod_{j=1}^{N} a_j^{-\frac{\alpha_j}{2}},$$

and

$$\Lambda_2 := \frac{b^2}{c} - \frac{b}{c} + \frac{1}{4c} - \frac{1}{4} - \sum_{j=1}^{N} \frac{\alpha_j}{12a_j}.$$

Assume that $0 < 1 - \sum_{j=1}^{N} \frac{\alpha_j}{12a_j} < \sqrt{\Lambda_2}$, $\sum_{j=1}^{N} \frac{\alpha_j}{a_j} < 0$, b *even with* $b \neq c$, $b^2 > c$, *and* $m = m(n)$ *with* $|m| \leq \frac{1}{6\beta} n^{-\delta} \log(n)$ *for some* $0 < \delta < \frac{1}{2}$ *such that* $m \to \infty$ *as* $n \to \infty$. *Then*

$$c(m,n) = \frac{1}{2\pi i} \Lambda_1 \beta^{2-w} \sqrt{\Lambda_2}^{-w} \frac{e^{2\pi\sqrt{2\Lambda_2 n}}}{2\pi \left(2\Lambda_2 n\right)^{\frac{1}{4}}} + O\left(\beta^{3-w} \frac{e^{2\pi\sqrt{2\Lambda_2 n}}}{2\pi \left(2\Lambda_2 n\right)^{\frac{1}{4}}}\right)$$

as $n \to \infty$.

Remark. Note that the restriction on Λ_2 still leaves infinitely many choices.

Since we assumed that $b^2 > c$ we only have functions of negative index. Therefore one might be able to use [BRZ16] to obtain our results.

This chapter is structured as follows. We begin in Section II.2 by recalling results that are relevant to the rest of this chapter. Section II.3 deals with defining the Fourier coefficients of ζ^m of f. In Section II.4 we investigate the behavior of f toward the dominant pole $q = 1$. We follow this in Section II.5 by bounding the contribution away from the pole at $q = 1$. In Section II.6 we obtain the asymptotic behavior of $c(m,n)$ and hence prove Theorem II.1.2.

II.2 Preliminaries

Here we recall relevant definitions and results which will be used throughout the rest of this chapter.

II.2.1 Properties of ϑ and η

When determining the asymptotic behavior of f we require the modular properties of both ϑ and η. We from now on define the square root using the principal branch, which means that we exclude the negative reals and impose positive square roots for positive real numbers.

It is well-known that ϑ is a Jacobi form (see e.g., [Mum07]).

Lemma II.2.1. *The function* ϑ *satisfies*

$$\vartheta(z;\tau) = -\vartheta(-z;\tau), \quad \vartheta(z;\tau) = -\vartheta(z+1;\tau), \quad \vartheta(z;\tau) = \frac{i}{\sqrt{-i\tau}} e^{\frac{-\pi i z^2}{\tau}} \vartheta\left(\frac{z}{\tau}; -\frac{1}{\tau}\right).$$

We also have the well-known triple product formula (see e.g., [Zwe02, Proposition 1.3] for this explicit formulation), that yields

$$\vartheta(z;\tau) = i\zeta^{\frac{1}{2}} q^{\frac{1}{8}} \prod_{n\geq 1} (1-q^n)(1-\zeta q^n)\left(1-\zeta^{-1}q^{n-1}\right). \tag{II.2.1}$$

Furthermore, we have the following modular transformation formula of η (see e.g., [KK07]).

Lemma II.2.2. *We have that*

$$\eta(\tau) = \sqrt{\frac{i}{\tau}}\eta\left(-\frac{1}{\tau}\right).$$

II.2.2 Integrals over segments of circles

Let $U_r(z_0) := \{z : |z-z_0| < r\}$ be the open disk around $z_0 \in \mathbb{C}$ with radius $r > 0$. Then we have the following result [Cur78, page 263].

Lemma II.2.3. *Let $g : U_r(z_0)\backslash\{z_0\} \to \mathbb{C}$ be analytic and have a simple pole at z_0. Let $\gamma(\delta)$ be a circular arc with parametric equation $z = z_0 + \delta e^{i\theta}$, for $-\pi < \theta_1 \leq \theta \leq \theta_2 \leq \pi$ and $0 < \delta < r$. Then*

$$\lim_{\delta\to 0}\int_{\gamma(\delta)} g(z)dz = i(\theta_2-\theta_1)\operatorname{Res}_{z_0}(g),$$

where $\operatorname{Res}_{z_0}(g)$ denotes the residuum of g at z_0.

See Figure II.1 for a pictorial explanation of this result.

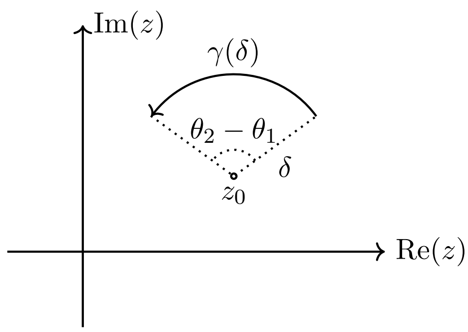

Figure II.1: Segment of a circle with radius δ around a simple pole z_0.

II.2.3 A particular bound

We require a bound on the size of

$$P(q) := \frac{q^{\frac{1}{24}}}{\eta(\tau)},$$

away from the pole at $q = 1$. For this we use [BD16, Lemma 3.5].

Lemma II.2.4. *Let $M > 0$ be a fixed constant. Let $\tau = u + iv \in \mathbb{H}$ with $Mv \leq u \leq \frac{1}{2}$ for $u > 0$ and $v \to 0$. Then*

$$|P(q)| \ll \sqrt{v} \exp\left[\frac{1}{v}\left(\frac{\pi}{12} - \frac{1}{2\pi}\left(1 - \frac{1}{\sqrt{1+M^2}}\right)\right)\right].$$

In particular, with $v = \frac{\beta}{2\pi}$, $u = \frac{\beta m^{-\frac{1}{3}} x}{2\pi}$ and $M = m^{-\frac{1}{3}}$ this gives for $1 \leq x \leq \frac{\pi m^{\frac{1}{3}}}{\beta}$ the bound

$$|P(q)| \ll n^{-\frac{1}{4}} \exp\left[\frac{2\pi}{\beta}\left(\frac{\pi}{12} - \frac{1}{2\pi}\left(1 - \frac{1}{\sqrt{1+m^{-\frac{2}{3}}}}\right)\right)\right]. \tag{II.2.2}$$

II.2.4 I-Bessel functions

Here we recall relevant results on the *I-Bessel function* which for $x > 0$ may be written as (see e.g., [Arf85, BD16])

$$I_\ell(x) := \frac{1}{2\pi i}\int_\Gamma t^{-\ell-1} e^{\frac{x}{2}\left(t+\frac{1}{t}\right)} dt,$$

where Γ is a contour which starts in the lower half plane at $-\infty$, surrounds the origin counterclockwise and returns to $-\infty$ in the upper half plane. We are particularly interested in the asymptotic behavior of I_ℓ, given in the following lemma (see e.g., [AAR99, equation (4.12.7)]).

Lemma II.2.5. *For fixed ℓ we have*

$$I_\ell(x) = \frac{e^x}{\sqrt{2\pi x}} + O\left(\frac{e^x}{x^{\frac{3}{2}}}\right)$$

as $x \to \infty$.

II.3 Fourier Coefficients of f

Note that $f(-z;\tau) = f(z;\tau)$ by Lemma II.2.1, and so $c(-m,n) = c(m,n)$. For the case $m = 0$ one can use classical results (see e.g., [BFOR17, Theorem 15.10]) to calculate the Fourier coefficients. We therefore restrict our attention to the case $m > 0$.

We first define the Fourier coefficients of ζ^m of f. Since shifts under $z \mapsto z+1$ of ϑ are understood we focus only on the case $z \in [0,1]$, we let $h_1, ..., h_s \in \mathbb{Q}$ denote the poles of f in this range, each of the form $\frac{d}{b}$, with $1 \leq d \leq b-1$ and $d \in \mathbb{N}$. Note that the distribution of the poles is symmetric on the interval in question.

Define the path of integration $\Gamma_{\ell,r}$ by

$$\Gamma_{\ell,r} := \begin{cases} 0 \text{ to } h_1 - r & \text{if } \ell = 0, \\ h_\ell + r \text{ to } h_{\ell+1} - r & \text{if } 1 \leq \ell \leq s-1, \\ h_s + r \text{ to } 1 & \text{if } \ell = s, \end{cases}$$

for some $r > 0$ sufficiently small. Note that in our setting we have $s = b-1$. Following the framework of [DMZ12, Mal20, Mal21a], we define

$$f_m^{\pm}(\tau) := \sum_{\ell=0}^{s} \int_{\Gamma_{\ell,r}} f(z;\tau) e^{-2\pi i m z} dz + \sum_{\ell=1}^{s} G_{\ell,r}^{\pm}, \qquad \text{where} \quad G_{\ell,r}^{\pm} := \int_{\gamma_{\ell,r}^{\pm}} f(z;\tau) e^{-2\pi i m z} dz$$

for a fixed pole h_ℓ ($1 \leq \ell \leq s$). Here, $\gamma_{\ell,r}^{+}$ is the semi-circular path of radius r passing above the pole h_ℓ and $\gamma_{\ell,r}^{-}$ is the semi-circular path passing below the pole h_ℓ, see Figures II.2 and II.3.

Figure II.2: The path of integration taking $\gamma_{\ell,r}^{+}$ at each pole.

Following [DMZ12] the Fourier coefficient of ζ^m of f, for fixed m, is given by

$$\begin{aligned} f_m(\tau) &:= \lim_{r \to 0^+} \frac{f_m^+(\tau) + f_m^-(\tau)}{2} \\ &= \lim_{r \to 0^+} \frac{1}{2} \left(2 \sum_{\ell=0}^{s} \int_{\Gamma_{\ell,r}} f(z;\tau) e^{-2\pi i m z} dz + \sum_{\ell=1}^{s} G_{\ell,r}^{+} + G_{\ell,r}^{-} \right). \end{aligned} \tag{II.3.1}$$

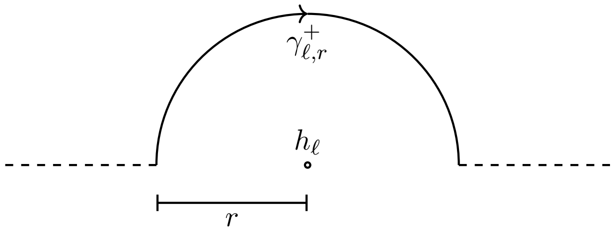

Figure II.3: The contour $\gamma^+_{\ell,r}$ for a fixed ℓ.

For fixed ℓ we use Lemma II.2.3 to see that

$$\lim_{r\to 0^+}\left(G^+_{\ell,r}+G^-_{\ell,r}\right)=0,$$

since we only have simple poles.

The substitution $z\mapsto 1-z$ gives us

$$\sum_{\ell=0}^{s}\int_{\Gamma_{\ell,r}} f(z;\tau)e^{-2\pi imz}dz=-\sum_{\ell=0}^{s}\int_{\Gamma_{\ell,r}} f(z;\tau)e^{2\pi imz}dz,$$

since b is even and using that $f(1-z;\tau)=(-1)^{b+1}f(z;\tau)$ by Lemma II.2.1. Thus, (II.3.1) simplifies to

$$f_m(\tau)=-i\lim_{r\to 0^+}\sum_{\ell=0}^{s}\int_{\Gamma_{\ell,r}} f(z;\tau)\sin(2\pi mz)dz. \tag{II.3.2}$$

Remark. For odd b one would obtain a similar formula with the integrand replaced by $f(z;\tau)\cos(2\pi mz)$.

In the following two sections we determine the asymptotic behavior of f towards and away from the dominant pole at $q=1$, ectively. From now on we let $\tau=\frac{i\varepsilon}{2\pi}$, $\varepsilon:=\beta(1+ixm^{-\frac{1}{3}})$, $\beta=\pi\sqrt{\frac{2}{n}}$, and $|m|\le\frac{1}{6\beta}n^{-\delta}\log(n)$ for some $0<\delta<\frac{1}{2}$ such that $m\to\infty$ as $n\to\infty$.

II.4 Bounds toward the dominant pole

In this section we consider the behavior of f_m toward the dominant pole at $q=1$. Remember that we have $w\in\frac{1}{2}\mathbb{Z}$ by definition (see Theorem II.1.2).

Lemma II.4.1. *Let $\tau = \frac{i\varepsilon}{2\pi}$, with $0 < \mathrm{Re}(\varepsilon) \ll 1$, let z be away from the poles, let $\mathcal{M}(z)$ be the function defined in* (II.4.2) *which is positive for all $z \in (0,1)$, and let*

$$\mathcal{C}(z;\tau) := (-1)^{2w} \left(\frac{2\pi}{\varepsilon}\right)^w c^{\frac{1}{2}} \left(\prod_{j=1}^N a_j^{-\frac{\alpha_j}{2}}\right) \frac{\sinh\left(\frac{2\pi^2 z}{\varepsilon}\right)}{\sinh\left(\frac{2\pi^2 bz}{c\varepsilon}\right)} e^{\frac{2\pi^2}{\varepsilon}\left(\frac{4b^2z^2+1}{4c} - \frac{4z^2+1}{4} - \sum_{j=1}^N \frac{\alpha_j}{12 a_j}\right)}.$$

Then we have that

$$f\left(z;\frac{i\varepsilon}{2\pi}\right) = \mathcal{C}\left(z;\frac{i\varepsilon}{2\pi}\right)\left(1 + O\left(e^{-\frac{4\pi^2}{\varepsilon}\mathcal{M}(z)}\right)\right)$$

as $n \to \infty$.

Proof. Using Lemmata II.2.1 and II.2.2 we see that $f = f_1 f_2 f_3$, where

$$\begin{aligned} f_1(z;\tau) &:= i^{-3w}\tau^{-w} c^{\frac{1}{2}} \prod_{j=1}^N a_j^{-\frac{\alpha_j}{2}}, \\ f_2(z;\tau) &:= e^{\frac{\pi i z^2}{\tau}\left(\frac{b^2}{c}-1\right)}, \\ f_3(z;\tau) &:= \frac{\vartheta\left(\frac{z}{\tau};-\frac{1}{\tau}\right)\prod_{j=1}^N \eta\left(-\frac{1}{a_j\tau}\right)^{\alpha_j}}{\vartheta\left(\frac{bz}{c\tau};-\frac{1}{c\tau}\right)}. \end{aligned}$$

Using the definition of η, (II.2.1), and setting $q_t := e^{-\frac{2\pi i}{t\tau}} = e^{-\frac{4\pi^2}{t\varepsilon}}$ for $t \in \mathbb{N}$ one may easily show that $f_3(z;\tau)$ becomes

$$\frac{i\zeta^{\frac{1}{2\tau}} q_1^{\frac{1}{8}} \prod_{\kappa\ge1}\left(1-q_1^\kappa\right)\left(1-\zeta^{\frac{1}{\tau}}q_1^\kappa\right)\left(1-\zeta^{-\frac{1}{\tau}}q_1^{\kappa-1}\right) \prod_{j=1}^N q_{a_j}^{\frac{\alpha_j}{24}} \prod_{\kappa\ge1}\left(1-q_{a_j}^\kappa\right)^{\alpha_j}}{i\zeta^{\frac{b}{2c\tau}} q_c^{\frac{1}{8}} \prod_{\kappa\ge1}\left(1-q_c^\kappa\right)\left(1-\zeta^{\frac{b}{c\tau}}q_c^\kappa\right)\left(1-\zeta^{-\frac{b}{c\tau}}q_c^{\kappa-1}\right)}$$

$$= e^{-\sum_{j=1}^N \frac{\pi i \alpha_j}{12 a_j \tau} - \frac{\pi i}{4\tau} + \frac{\pi i}{4c\tau}} \frac{\sinh\left(\frac{\pi i z}{\tau}\right)}{\sinh\left(\frac{\pi i b z}{c\tau}\right)} \prod_{\kappa\ge1} \frac{\left(\prod_{j=1}^N \left(1-q_{a_j}^\kappa\right)^{\alpha_j}\right)\left(1-q_1^\kappa\right)\left(1-\zeta^{\frac{1}{\tau}}q_1^\kappa\right)\left(1-\zeta^{-\frac{1}{\tau}}q_1^\kappa\right)}{\left(1-q_c^\kappa\right)\left(1-\zeta^{\frac{b}{c\tau}}q_c^\kappa\right)\left(1-\zeta^{-\frac{b}{c\tau}}q_c^\kappa\right)},$$

by using the trick

$$\prod_{\kappa\ge1}\left(1-\zeta^{-x}q^{\kappa-1}\right) = \left(1-\zeta^{-x}\right)\prod_{\kappa\ge2}\left(1-\zeta^{-x}q^{\kappa-1}\right) = \left(1-\zeta^{-x}\right)\prod_{\kappa\ge1}\left(1-\zeta^{-x}q^\kappa\right)$$

and the fact that $\zeta^{\frac{x}{2}}(1-\zeta^{-x}) = (\zeta^{\frac{x}{2}} - \zeta^{-\frac{x}{2}}) = 2\sinh(\pi i x z)$. Putting those results together and setting $\tau = \frac{i\varepsilon}{2\pi}$ yields

$$f\left(z;\frac{i\varepsilon}{2\pi}\right) = \mathcal{C}\left(z;\frac{i\varepsilon}{2\pi}\right) \times \prod_{\kappa\geq 1} \frac{\left(1-e^{-\frac{4\pi^2\kappa}{\varepsilon}}\right)\left(1-e^{\frac{4\pi^2}{\varepsilon}(z-\kappa)}\right)\left(1-e^{\frac{4\pi^2}{\varepsilon}(-z-\kappa)}\right)\prod_{j=1}^{N}\left(1-e^{-\frac{4\pi^2\kappa}{a_j\varepsilon}}\right)^{\alpha_j}}{\left(1-e^{-\frac{4\pi^2\kappa}{c\varepsilon}}\right)\left(1-e^{\frac{4\pi^2}{c\varepsilon}(bz-\kappa)}\right)\left(1-e^{\frac{4\pi^2}{c\varepsilon}(-bz-\kappa)}\right)}.$$

In order to find a bound we inspect the asymptotic behavior of the product over κ. Splitting α_j into positive and negative powers, labeled by $\gamma_j, \delta_j \in \mathbb{N}$, and a_j, into x_j and y_j, respectively, we first rewrite this as

$$\prod_{\kappa\geq 1} \frac{\left(1-e^{-\frac{4\pi^2\kappa}{\varepsilon}}\right)\left(1-e^{-\frac{4\pi^2}{\varepsilon}(\kappa-z)}\right)\left(1-e^{-\frac{4\pi^2}{\varepsilon}(\kappa+z)}\right)\prod_{j=1}^{N_1}\left(1-e^{-\frac{4\pi^2\kappa}{x_j\varepsilon}}\right)^{\gamma_j}\prod_{k=1}^{N_2}\left(\sum_{\mu\geq 0} e^{-\frac{4\pi^2\mu\kappa}{y_k\varepsilon}}\right)^{\delta_k}}{\left(1-e^{-\frac{4\pi^2\kappa}{c\varepsilon}}\right)\left(1-e^{-\frac{4\pi^2}{c\varepsilon}(\kappa-bz)}\right)\left(1-e^{-\frac{4\pi^2}{c\varepsilon}(\kappa+bz)}\right)}, \tag{II.4.1}$$

since $|e^{-\frac{4\pi^2\kappa}{y_k\varepsilon}}| < 1$ for all $\kappa \geq 1$. We also have that $|e^{-\frac{4\pi^2\kappa}{c\varepsilon}}| < 1$ and $|e^{-\frac{4\pi^2}{c\varepsilon}(\kappa+bz)}| < 1$ for all $\kappa \geq 1$ since $b, c \in \mathbb{N}$. Therefore, we have that

$$\frac{1}{\left(1-e^{-\frac{4\pi^2\kappa}{c\varepsilon}}\right)\left(1-e^{-\frac{4\pi^2}{c\varepsilon}(\kappa+bz)}\right)} = \sum_{\lambda\geq 0} e^{-\frac{4\pi^2\lambda\kappa}{c\varepsilon}} \sum_{\xi\geq 0} e^{-\frac{4\pi^2\xi}{c\varepsilon}(\kappa+bz)}.$$

Up to this point our calculations are independent of the size of z. The remaining term is

$$\frac{1}{1-e^{-\frac{4\pi^2}{c\varepsilon}(\kappa-bz)}}.$$

Let κ_0 be the smallest $\kappa \geq 1$ such that $(\kappa - bz) \geq 0$. We may rewrite

$$\prod_{\kappa\geq 1}\frac{1}{\left(1-e^{-\frac{4\pi^2}{c\varepsilon}(\kappa-bz)}\right)} = \prod_{\kappa=1}^{\kappa_0-1}\frac{1}{\left(1-e^{-\frac{4\pi^2}{c\varepsilon}(\kappa-bz)}\right)}\prod_{\kappa\geq\kappa_0}\sum_{\mu\geq 0} e^{-\frac{4\pi^2\mu}{c\varepsilon}(\kappa-bz)}.$$

The first product is

$$\prod_{\kappa=1}^{\kappa_0-1}\frac{1}{\left(1-e^{-\frac{4\pi^2}{c\varepsilon}(\kappa-bz)}\right)} = \prod_{\kappa=1}^{\kappa_0-1}\left(-e^{\frac{4\pi^2}{c\varepsilon}(\kappa-bz)}\right)\sum_{\nu\geq 0} e^{\frac{4\pi^2\nu}{c\varepsilon}(\kappa-bz)}.$$

Additionally we have $e^{-\frac{4\pi^2}{\varepsilon}(1+z)} < e^{-\frac{4\pi^2}{\varepsilon}} < e^{-\frac{4\pi^2}{\varepsilon}(1-z)}$.

Let

$$\mathcal{M}(z) := \begin{cases} \min\left(1-z, \frac{1}{x_j}, \frac{1}{y_k}, \frac{1}{c}, \frac{\kappa_0 - bz}{c}, \frac{bz+1-\kappa_0}{c}\right) & \text{if } \kappa_0 \neq 1, \\ \min\left(1-z, \frac{1}{x_j}, \frac{1}{y_k}, \frac{1}{c}, \frac{\kappa_0 - bz}{c}\right) & \text{if } \kappa_0 = 1, \end{cases} \tag{II.4.2}$$

running over all $1 \leq j \leq N_1$ and $1 \leq k \leq N_2$. Note that for $0 < \mathrm{Re}(\varepsilon) \ll 1$, and $z \in (0,1)$ we have $\mathcal{M}(z) > 0$, so the product in (II.4.1) is of order

$$1 + O\left(e^{-\frac{4\pi^2}{\varepsilon}\mathcal{M}(z)}\right),$$

which finishes the proof. □

Remark. By separating into cases, one is able to obtain more precise asymptotics. However, this is not required for what follows and we leave the details for the interested reader.

Theorem II.4.2. *Let Λ_1 and Λ_2 be defined as in Theorem II.1.2. For $|x| \leq 1$ we have that*

$$f_m\left(\frac{i\varepsilon}{2\pi}\right) = \Lambda_1 \varepsilon^{1-w} e^{\frac{2\pi^2}{\varepsilon}\Lambda_2} + O\left(\beta^{2-w} e^{\frac{2\pi^2}{\varepsilon}\Lambda_2}\right)$$

as $n \to \infty$.

Proof. Plugging Lemma II.4.1 into (II.3.2) yields

$$f_m\left(\frac{i\varepsilon}{2\pi}\right) = -i\sum_{\ell=0}^{s} \lim_{r\to 0^+} \int_{\Gamma_{\ell,r}} \mathcal{C}\left(z; \frac{i\varepsilon}{2\pi}\right)\left(1 + O\left(e^{-\frac{4\pi^2}{\varepsilon}\mathcal{M}(z)}\right)\right) \sin(2\pi m z) dz. \tag{II.4.3}$$

We have that

$$\frac{\sinh\left(\frac{2\pi^2 z}{\varepsilon}\right)}{\sinh\left(\frac{2\pi^2 bz}{c\varepsilon}\right)} = \frac{e^{\frac{2\pi^2 z}{\varepsilon}}\left(1 - e^{-\frac{4\pi^2 z}{\varepsilon}}\right)}{e^{\frac{2\pi^2 bz}{c\varepsilon}}\left(1 - e^{-\frac{4\pi^2 bz}{c\varepsilon}}\right)} = e^{\frac{2\pi^2}{\varepsilon}z\left(1-\frac{b}{c}\right)}\left(1 - e^{-\frac{4\pi^2 z}{\varepsilon}}\right)\sum_{\lambda\geq 0} e^{-\frac{4\pi^2 \lambda bz}{c\varepsilon}}$$
$$= e^{\frac{2\pi^2}{\varepsilon}z\left(1-\frac{b}{c}\right)}\left(1 + O\left(e^{-\frac{4\pi^2 z}{\varepsilon}}\right)\right),$$

using $|e^{-\frac{4\pi^2 bz}{c\varepsilon}}| < 1$. Additionally we see that

$$e^{\frac{2\pi^2}{\varepsilon}\left(\frac{4b^2z^2+1}{4c} - \frac{4z^2+1}{4} - \sum_{j=1}^{N}\frac{\alpha_j}{12a_j}\right)} = e^{\frac{2\pi^2}{\varepsilon}\left(\frac{b^2}{c}-1\right)z^2} e^{\frac{2\pi^2}{\varepsilon}\left(\frac{1}{4c} - \frac{1}{4} - \sum_{j=1}^{N}\frac{\alpha_j}{12a_j}\right)}.$$

Defining

$$\Omega(m,n) := (-1)^{2w}\left(\frac{2\pi}{\varepsilon}\right)^{w} c^{\frac{1}{2}}\left(\prod_{j=1}^{N} a_j^{-\frac{\alpha_j}{2}}\right) e^{\frac{2\pi^2}{\varepsilon}\left(\frac{1}{4c}-\frac{1}{4}-\sum_{j=1}^{N}\frac{\alpha_j}{12a_j}\right)}$$

and $\mathcal{N}(z) := \min(z,\mathcal{M}(z))$, we can rewrite (II.4.3) as

$$-i\Omega(m,n)\sum_{\ell=0}^{s}\lim_{r\to 0^+}\int_{\Gamma_{\ell,r}} e^{\frac{2\pi^2}{\varepsilon}\left(\frac{b^2}{c}-1\right)z^2} e^{\frac{2\pi^2}{\varepsilon}\left(1-\frac{b}{c}\right)z}\left(1+O\left(e^{-\frac{4\pi^2}{\varepsilon}\mathcal{N}(z)}\right)\right)\sin(2\pi mz)dz.$$

We immediately see that this splits up into two integrals

$$\sum_{\ell=0}^{s}\lim_{r\to 0^+}\int_{\Gamma_{\ell,r}} e^{\frac{2\pi^2}{\varepsilon}\left(\frac{b^2}{c}-1\right)z^2} e^{\frac{2\pi^2}{\varepsilon}\left(1-\frac{b}{c}\right)z}\sin(2\pi mz)dz \tag{II.4.4}$$

and

$$\sum_{\ell=0}^{s}\lim_{r\to 0^+}\int_{\Gamma_{\ell,r}} e^{\frac{2\pi^2}{\varepsilon}\left(\frac{b^2}{c}-1\right)z^2} e^{\frac{2\pi^2}{\varepsilon}\left(1-\frac{b}{c}\right)z} O\left(e^{-\frac{4\pi^2}{\varepsilon}\mathcal{N}(z)}\right)\sin(2\pi mz)dz. \tag{II.4.5}$$

Let $\operatorname{erf}(z) := \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt$ denote the *error function* and note that $\frac{d}{dz}\operatorname{erf}(z) = \frac{2e^{-z^2}}{\sqrt{\pi}}$. For arbitrary $\mathcal{H}_1,\mathcal{H}_2\in\mathbb{C}$, with $\mathcal{H}_2\neq 0$ a straightforward calculation, using the identity $\frac{1}{2i}(e^{2\pi imz}-e^{-2\pi imz}) = \sin(2\pi mz)$, gives us that

$$\begin{aligned}\frac{d}{dz}\Bigg[\frac{\sqrt{\pi}}{4\sqrt{\mathcal{H}_2}}\Bigg(& e^{-\frac{1}{4}\frac{(\mathcal{H}_1+2\pi im)^2}{\mathcal{H}_2}}\operatorname{erf}\left(\frac{1}{2}\frac{\mathcal{H}_1+2\pi im+2\mathcal{H}_2 z}{\sqrt{-\mathcal{H}_2}}\right)\\ &+e^{-\frac{1}{4}\frac{(-\mathcal{H}_1+2\pi im)^2}{\mathcal{H}_2}}\operatorname{erf}\left(\frac{1}{2}\frac{-\mathcal{H}_1+2\pi im-2\mathcal{H}_2 z}{\sqrt{-\mathcal{H}_2}}\right)\Bigg)\Bigg] = e^{\mathcal{H}_1 z}e^{\mathcal{H}_2 z^2}\sin(2\pi mz).\end{aligned}$$

Therefore the following formula holds

$$\begin{aligned}\int_t^u e^{\mathcal{H}_1 z}e^{\mathcal{H}_2 z^2}\sin(2\pi mz)dz =& \frac{\sqrt{\pi}}{4\sqrt{\mathcal{H}_2}}\Bigg(e^{-\frac{1}{4}\frac{(\mathcal{H}_1+2\pi im)^2}{\mathcal{H}_2}}\operatorname{erf}\left(\frac{1}{2}\frac{2\mathcal{H}_2 t+\mathcal{H}_1+2\pi im}{\sqrt{-\mathcal{H}_2}}\right)\\ &+e^{-\frac{1}{4}\frac{(-\mathcal{H}_1+2\pi im)^2}{\mathcal{H}_2}}\operatorname{erf}\left(\frac{1}{2}\frac{-2\mathcal{H}_2 t-\mathcal{H}_1+2\pi im}{\sqrt{-\mathcal{H}_2}}\right)\\ &-e^{-\frac{1}{4}\frac{(\mathcal{H}_1+2\pi im)^2}{\mathcal{H}_2}}\operatorname{erf}\left(\frac{1}{2}\frac{2\mathcal{H}_2 u+\mathcal{H}_1+2\pi im}{\sqrt{-\mathcal{H}_2}}\right)\\ &-e^{-\frac{1}{4}\frac{(-\mathcal{H}_1+2\pi im)^2}{\mathcal{H}_2}}\operatorname{erf}\left(\frac{1}{2}\frac{-2\mathcal{H}_2 u-\mathcal{H}_1+2\pi im}{\sqrt{-\mathcal{H}_2}}\right)\Bigg).\end{aligned}$$

For arbitrary $\mathcal{H}_1, \mathcal{H}_2 \in \mathbb{C}$, with $\mathcal{H}_2 \neq 0$ we thus obtain

$$\begin{aligned}
&\sum_{\ell=0}^{s} \lim_{r\to 0^+} \int_{\Gamma_{\ell,r}} e^{\mathcal{H}_2 z^2} e^{\mathcal{H}_1 z} \sin(2\pi m z) dz \qquad \text{(II.4.6)}\\
=&\frac{\sqrt{\pi}}{4\sqrt{\mathcal{H}_2}} \left(e^{-\frac{1}{4}\frac{(\mathcal{H}_1+2\pi i m)^2}{\mathcal{H}_2}} \operatorname{erf}\left(\frac{1}{2}\frac{\mathcal{H}_1+2\pi i m}{\sqrt{-\mathcal{H}_2}}\right) + e^{-\frac{1}{4}\frac{(-\mathcal{H}_1+2\pi i m)^2}{\mathcal{H}_2}} \operatorname{erf}\left(\frac{1}{2}\frac{-\mathcal{H}_1+2\pi i m}{\sqrt{-\mathcal{H}_2}}\right)\right.\\
&\left. -e^{-\frac{1}{4}\frac{(\mathcal{H}_1+2\pi i m)^2}{\mathcal{H}_2}} \operatorname{erf}\left(\frac{1}{2}\frac{2\mathcal{H}_2+\mathcal{H}_1+2\pi i m}{\sqrt{-\mathcal{H}_2}}\right) - e^{-\frac{1}{4}\frac{(-\mathcal{H}_1+2\pi i m)^2}{\mathcal{H}_2}} \operatorname{erf}\left(\frac{1}{2}\frac{-2\mathcal{H}_2-\mathcal{H}_1+2\pi i m}{\sqrt{-\mathcal{H}_2}}\right)\right),
\end{aligned}$$

since all the other terms cancel. If $|\operatorname{Arg}(\pm z)| < \frac{\pi}{4}$, we have that (see e.g., [BN19, page 10])

$$\operatorname{erf}(iz) = \frac{ie^{z^2}}{\sqrt{\pi}z}\left(1 + O\left(|z|^{-2}\right)\right) = \frac{ie^{z^2}}{\sqrt{\pi}z} + O\left(e^{z^2}|z|^{-3}\right), \qquad \text{(II.4.7)}$$

as $|z| \to \infty$. Note that taking the limit $|z| \to \infty$ is equivalent to taking the limit $n \to \infty$ in our setting.

Consider the integral (II.4.4), so set $\mathcal{H}_1 = \frac{2\pi^2}{\varepsilon}(1 - \frac{b}{c})$ and $\mathcal{H}_2 = \frac{2\pi^2}{\varepsilon}(\frac{b^2}{c} - 1)$. In this case, since $\frac{b^2}{c} > 1$, we obtain

$$\frac{1}{2}\frac{\pm\mathcal{H}_1 + 2\pi i m}{\sqrt{-\mathcal{H}_2}} = \frac{\pm\left(\frac{\pi}{\varepsilon}\left(1-\frac{b}{c}\right)\right) + im}{i\sqrt{\frac{2}{\varepsilon}\left(\frac{b^2}{c}-1\right)}} =: iz_1,$$

respectively

$$\frac{1}{2}\frac{\pm 2\mathcal{H}_2 \pm \mathcal{H}_1 + 2\pi i m}{\sqrt{-\mathcal{H}_2}} = \frac{\pm 2\left(\frac{\pi}{\varepsilon}\left(\frac{b^2}{c}-1\right)\right) \pm \left(\frac{\pi}{\varepsilon}\left(1-\frac{b}{c}\right)\right) + im}{i\sqrt{\frac{2}{\varepsilon}\left(\frac{b^2}{c}-1\right)}} =: iz_2.$$

Using $\varepsilon = \beta(1 + ixm^{-\frac{1}{3}})$ and $\sqrt{z} = \sqrt{|z|}\cos(\frac{1}{2}\operatorname{Arg}(z)) + i\sqrt{|z|}\sin(\frac{1}{2}\operatorname{Arg}(z))$ a straightforward calculation shows that

$$\begin{aligned}
z_1 =& 2\left(\frac{b^2}{c}-1\right)\sqrt{\frac{2\left(\frac{b^2}{c}-1\right)}{\beta\sqrt{1+x^2m^{-\frac{2}{3}}}}}\\
&\times\left[\left(\left(\mp\pi\left(1-\frac{b}{c}\right)+\beta x m^{\frac{2}{3}}\right)\cos\left(\frac{1}{2}\operatorname{Arg}\left(1-ixm^{-\frac{1}{3}}\right)\right) + \beta m \sin\left(\frac{1}{2}\operatorname{Arg}\left(1-ixm^{-\frac{1}{3}}\right)\right)\right)\right.\\
&\left. +i\left(-\beta m\cos\left(\frac{1}{2}\operatorname{Arg}\left(1-ixm^{-\frac{1}{3}}\right)\right) + \left(\mp\pi\left(1-\frac{b}{c}\right)+\beta x m^{\frac{2}{3}}\right)\sin\left(\frac{1}{2}\operatorname{Arg}\left(1-ixm^{-\frac{1}{3}}\right)\right)\right)\right],
\end{aligned}$$

respectively

$$
\begin{aligned}
z_2 =& 2\left(\frac{b^2}{c}-1\right)\sqrt{\frac{2\left(\frac{b^2}{c}-1\right)}{\beta\sqrt{1+x^2m^{-\frac{2}{3}}}}} \\
&\times\Bigg[\left(\mp\left(\pi\left(1-\frac{b}{c}\right)+2\pi\left(\frac{b^2}{c}-1\right)\right)+\beta x m^{\frac{2}{3}}\right)\cos\left(\frac{1}{2}\operatorname{Arg}\left(1-ixm^{-\frac{1}{3}}\right)\right)+\beta m\sin\left(\frac{1}{2}\operatorname{Arg}\left(1-ixm^{-\frac{1}{3}}\right)\right) \\
&+i\left(-\beta m\cos\left(\frac{1}{2}\operatorname{Arg}\left(1-ixm^{-\frac{1}{3}}\right)\right)+\left(\mp\left(\pi\left(1-\frac{b}{c}\right)+2\pi\left(\frac{b^2}{c}-1\right)\right)+\beta x m^{\frac{2}{3}}\right)\sin\left(\frac{1}{2}\operatorname{Arg}\left(1-ixm^{-\frac{1}{3}}\right)\right)\right)\Bigg].
\end{aligned}
$$

Since $|x|<1$ we see that $|\operatorname{Arg}(1-ixm^{-\frac{1}{3}})|<\frac{\pi}{4}$ and thus we have

$$
\left|\cos\left(\frac{1}{2}\operatorname{Arg}\left(1-ixm^{-\frac{1}{3}}\right)\right)\right|>\left|\sin\left(\frac{1}{2}\operatorname{Arg}\left(1-ixm^{-\frac{1}{3}}\right)\right)\right|. \tag{II.4.8}
$$

From the assumption $|m|\le\frac{1}{6\beta}n^{-\delta}\log(n)$ for some $0<\delta<\frac{1}{2}$ we not only ensure that $m\to\infty$ as $n\to\infty$ but additionally that $\beta m\to 0$ and $\beta m^{\frac{2}{3}}\to 0$ as $n\to\infty$. Thus, together with (II.4.8), we see that $|\operatorname{Re}(z_1)|>|\operatorname{Im}(z_1)|$, respectively $|\operatorname{Re}(z_2)|>|\operatorname{Im}(z_2)|$, for n sufficiently large.

Therefore the arguments of the error functions in (II.4.6) satisfy the condition of (II.4.7). Plugging in yields

$$
\begin{aligned}
&\sum_{\ell=0}^{s}\lim_{r\to0^+}\int_{\Gamma_{\ell,r}}e^{\mathcal{H}_2z^2}\,e^{\mathcal{H}_1z}\sin(2\pi mz)dz \\
=&\frac{\sqrt{\pi}}{4\sqrt{\mathcal{H}_2}}\left(\frac{i}{\sqrt{\pi}\left(-i\frac{1}{2}\frac{\mathcal{H}_1+2\pi im}{\sqrt{-\mathcal{H}_2}}\right)}+O\left(\left|\left(-i\frac{1}{2}\frac{\mathcal{H}_1+2\pi im}{\sqrt{-\mathcal{H}_2}}\right)\right|^{-3}\right)\right. \\
&+\frac{i}{\sqrt{\pi}\left(-i\frac{1}{2}\frac{-\mathcal{H}_1+2\pi im}{\sqrt{-\mathcal{H}_2}}\right)}+O\left(\left|\left(-i\frac{1}{2}\frac{-\mathcal{H}_1+2\pi im}{\sqrt{-\mathcal{H}_2}}\right)\right|^{-3}\right) \\
&-\frac{ie^{\frac{1}{4}\frac{4\mathcal{H}_2^2+4\mathcal{H}_2(\mathcal{H}_1+2\pi im)}{\mathcal{H}_2}}}{\sqrt{\pi}\left(-i\frac{1}{2}\frac{2\mathcal{H}_2+\mathcal{H}_1+2\pi im}{\sqrt{-\mathcal{H}_2}}\right)}+O\left(e^{\frac{1}{4}\frac{4\mathcal{H}_2^2+4\mathcal{H}_2(\mathcal{H}_1+2\pi im)}{\mathcal{H}_2}}\left|\left(-i\frac{1}{2}\frac{2\mathcal{H}_2+\mathcal{H}_1+2\pi im}{\sqrt{-\mathcal{H}_2}}\right)\right|^{-3}\right) \\
&\left.-\frac{ie^{\frac{1}{4}\frac{4\mathcal{H}_2^2-4\mathcal{H}_2(-\mathcal{H}_1+2\pi im)}{\mathcal{H}_2}}}{\sqrt{\pi}\left(-i\frac{1}{2}\frac{-2\mathcal{H}_2-\mathcal{H}_1+2\pi im}{\sqrt{-\mathcal{H}_2}}\right)}+O\left(e^{\frac{1}{4}\frac{4\mathcal{H}_2^2-4\mathcal{H}_2(-\mathcal{H}_1+2\pi im)}{\mathcal{H}_2}}\left|\left(-i\frac{1}{2}\frac{-2\mathcal{H}_2-\mathcal{H}_1+2\pi im}{\sqrt{-\mathcal{H}_2}}\right)\right|^{-3}\right)\right) \\
=&\frac{\sqrt{\pi}}{4\sqrt{\mathcal{H}_2}}\left(\frac{i}{\sqrt{\pi}\left(-i\frac{1}{2}\frac{\mathcal{H}_1+2\pi im}{\sqrt{-\mathcal{H}_2}}\right)}+O\left(\left|\left(-i\frac{1}{2}\frac{\mathcal{H}_1+2\pi im}{\sqrt{-\mathcal{H}_2}}\right)\right|^{-3}\right)\right. \\
&+\frac{i}{\sqrt{\pi}\left(-i\frac{1}{2}\frac{-\mathcal{H}_1+2\pi im}{\sqrt{-\mathcal{H}_2}}\right)}+O\left(\left|\left(-i\frac{1}{2}\frac{-\mathcal{H}_1+2\pi im}{\sqrt{-\mathcal{H}_2}}\right)\right|^{-3}\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{ie^{\mathcal{H}_2+\mathcal{H}_1}}{\sqrt{\pi}\left(-i\frac{1}{2}\frac{2\mathcal{H}_2+\mathcal{H}_1+2\pi im}{\sqrt{-\mathcal{H}_2}}\right)}+O\left(e^{\mathcal{H}_2+\mathcal{H}_1}\left|\left(-i\frac{1}{2}\frac{2\mathcal{H}_2+\mathcal{H}_1+2\pi im}{\sqrt{-\mathcal{H}_2}}\right)\right|^{-3}\right)\\
&-\frac{ie^{\mathcal{H}_2+\mathcal{H}_1}}{\sqrt{\pi}\left(-i\frac{1}{2}\frac{-2\mathcal{H}_2-\mathcal{H}_1+2\pi im}{\sqrt{-\mathcal{H}_2}}\right)}+O\left(e^{\mathcal{H}_2+\mathcal{H}_1}\left|\left(-i\frac{1}{2}\frac{-2\mathcal{H}_2-\mathcal{H}_1+2\pi im}{\sqrt{-\mathcal{H}_2}}\right)\right|^{-3}\right)\Bigg)\\
=&\frac{\sqrt{\pi}}{4\sqrt{\mathcal{H}_2}}\left(-\frac{ie^{\mathcal{H}_2+\mathcal{H}_1}}{\sqrt{\pi}\left(-i\frac{1}{2}\frac{2\mathcal{H}_2+\mathcal{H}_1+2\pi im}{\sqrt{-\mathcal{H}_2}}\right)}+O\left(e^{\mathcal{H}_2+\mathcal{H}_1}\left|i\frac{1}{2}\frac{2\mathcal{H}_2+\mathcal{H}_1+2\pi im}{\sqrt{-\mathcal{H}_2}}\right|^{-3}\right)\right)\\
=&\frac{e^{\mathcal{H}_2+\mathcal{H}_1}}{4i\left(\frac{1}{2}(2\mathcal{H}_2+\mathcal{H}_1+2\pi im)\right)}+O\left(\frac{\sqrt{\pi}}{4\sqrt{\mathcal{H}_2}}e^{\mathcal{H}_2+\mathcal{H}_1}\left|i\frac{1}{2}\frac{2\mathcal{H}_2+\mathcal{H}_1+2\pi im}{\sqrt{-\mathcal{H}_2}}\right|^{-3}\right)\\
=&\frac{e^{\mathcal{H}_2+\mathcal{H}_1}}{4i\mathcal{H}_2+2i\mathcal{H}_1-4\pi m}+O\left(\frac{\sqrt{\pi}e^{\mathcal{H}_2+\mathcal{H}_1}}{4\sqrt{\mathcal{H}_2}}\left|\frac{1}{2}\frac{2\mathcal{H}_2+\mathcal{H}_1+2\pi im}{\sqrt{-\mathcal{H}_2}}\right|^{-3}\right).
\end{aligned}
$$

We thus obtain that (II.4.4) equals

$$
\frac{e^{\frac{2\pi^2}{\varepsilon}\left(\frac{b^2}{c}-\frac{b}{c}\right)}}{\frac{4i\pi^2}{\varepsilon}\left(\frac{2b^2}{c}-1-\frac{b}{c}\right)-4\pi m}+O\left(\frac{\sqrt{\pi}}{4}\frac{e^{\frac{2\pi^2}{\varepsilon}\left(\frac{b^2}{c}-\frac{b}{c}\right)}}{\sqrt{\frac{2\pi^2}{\varepsilon}\left(\frac{b^2}{c}-1\right)}}\left|\frac{\frac{\pi^2}{\varepsilon}\left(\frac{2b^2}{c}-1-\frac{b}{c}\right)+\pi im}{\sqrt{\frac{2\pi^2}{\varepsilon}\left(\frac{b^2}{c}-1\right)}}\right|^{-3}\right).
$$

Combining this along with the fact that $\mathcal{N}(z)>0$ and recycling the same arguments for (II.4.5), yields

$$
\begin{aligned}
f_m\left(\frac{i\varepsilon}{2\pi}\right)=&-i\Omega(m,n)\left(\frac{e^{\frac{2\pi^2}{\varepsilon}\left(\frac{b^2}{c}-\frac{b}{c}\right)}}{\frac{4i\pi^2}{\varepsilon}\left(\frac{2b^2}{c}-1-\frac{b}{c}\right)-4\pi m}\right.\\
&\left.+O\left(\frac{\sqrt{\pi}}{4}\frac{e^{\frac{2\pi^2}{\varepsilon}\left(\frac{b^2}{c}-\frac{b}{c}\right)}}{\sqrt{\frac{2\pi^2}{\varepsilon}\left(\frac{b^2}{c}-1\right)}}\left|\frac{\frac{\pi^2}{\varepsilon}\left(\frac{2b^2}{c}-1-\frac{b}{c}\right)+\pi im}{\sqrt{\frac{2\pi^2}{\varepsilon}\left(\frac{b^2}{c}-1\right)}}\right|^{-3}\right)\right).
\end{aligned}
$$

Plugging in $\Omega(m,n)$ yields the claim. The main term here simplifies to

$$
\Lambda_1\varepsilon^{1-w}e^{\frac{2\pi^2}{\varepsilon}\Lambda_2}
$$

since $4\pi m\varepsilon\to 0$ as $n\to\infty$, which holds since we have

$$
|m\varepsilon|\le|m\beta|+\left|\beta ixm^{\frac{2}{3}}\right|\le\frac{1}{6}n^{-\delta}\log(n)+\beta^{\frac{1}{3}}x\left(\frac{1}{6}n^{-\delta}\log(n)\right)^{\frac{2}{3}}\to 0,
$$

as $n\to\infty$, and for $0<\delta<\frac{1}{2}$. □

II.5 Bounds away from the dominant pole

In this section we investigate the contribution of f_m away from the dominant pole at $q = 1$, and show that it forms part of the error term. Recall that from (II.3.2) we have

$$f_m(\tau) = -i \lim_{r \to 0^+} \left(\sum_{\ell=0}^{s} \int_{\Gamma_{\ell,r}} f(z;\tau) \sin(2\pi m z) dz \right).$$

One immediately sees that

$$\left| \sum_{\ell=0}^{s} \int_{\Gamma_{\ell,r}} f(z;\tau) \sin(2\pi m z) dz \right| \ll \sum_{\ell=0}^{s} \int_{\Gamma_{\ell,r}} |f(z;\tau) \sin(2\pi m z)| \, dz.$$

Consider

$$|f(z;\tau)\sin(2\pi m z)| = \left| \prod_{j=1}^{N} \eta(a_j\tau)^{\alpha_j} \frac{\vartheta(z;\tau)}{\vartheta(bz;c\tau)} \right| |\sin(2\pi m z)| \ll \left| \prod_{j=1}^{N} \eta(a_j\tau)^{\alpha_j} \right| \left| \frac{\vartheta(z;\tau)}{\vartheta(bz;c\tau)} \right|$$

away from the dominant pole. We begin with the term $\prod_{j=1}^{N} \eta(a_j\tau)^{\alpha_j}$. As in [Mal20] we write

$$\prod_{j=1}^{N} \eta(a_j\tau)^{\alpha_j} = \prod_{j=1}^{N_1} \eta(x_j\tau)^{\gamma_j} \prod_{k=1}^{N_2} q^{-\frac{y_k\delta_k}{24}} P(q^{y_k})^{\delta_k}.$$

Using Lemma II.2.2 we see that

$$\eta\left(\frac{ix_j\varepsilon}{2\pi}\right)^{\gamma_j} \ll \left(\frac{2\pi}{x_j\beta}\right)^{\frac{\gamma_j}{2}} e^{-\frac{\pi^2\gamma_j}{6x_j\beta}}.$$

By (II.2.2) we also obtain that

$$|P(q^{y_k})| \ll n^{-\frac{1}{4}} \exp\left[\frac{2\pi}{y_k\beta}\left(\frac{\pi}{12} - \frac{1}{2\pi}\left(1 - \frac{1}{\sqrt{1+m^{-\frac{2}{3}}}}\right)\right)\right].$$

Therefore we find

$$\prod_{j=1}^{N} \eta\left(\frac{ia_j\varepsilon}{2\pi}\right)^{\alpha_j} \ll \left(\prod_{j=1}^{N_1} \frac{2\pi}{x_j\beta}\right)^{\frac{\gamma_j}{2}} e^{-\sum\limits_{j=1}^{N_1} \frac{\pi^2\gamma_j}{6x_j\beta}}$$
$$\times \prod_{k=1}^{N_2} n^{-\frac{\delta_k}{4}} \exp\left[\frac{2\pi\delta_k}{y_k\beta}\left(\frac{\pi}{12} - \frac{1}{2\pi}\left(1 - \frac{1}{\sqrt{1+m^{-\frac{2}{3}}}}\right)\right)\right],$$

and thus we obtain

$$\left|\prod_{j=1}^{N} \eta(a_j\tau)^{\alpha_j}\right| \left|\frac{\vartheta(z;\tau)}{\vartheta(bz;c\tau)}\right| \ll \left(\prod_{j=1}^{N_1} \frac{2\pi}{x_j\beta}\right)^{\frac{\gamma_j}{2}} e^{-\sum\limits_{j=1}^{N_1} \frac{\pi^2\gamma_j}{6x_j\beta}} \prod_{k=1}^{N_2} n^{-\frac{\delta_k}{4}} \tag{II.5.1}$$
$$\times \exp\left[\frac{2\pi\delta_k}{y_k\beta}\left(\frac{\pi}{12} - \frac{1}{2\pi}\left(1 - \left(1+m^{-\frac{2}{3}}\right)^{-\frac{1}{2}}\right)\right)\right] \left|\left|\frac{\vartheta(z;\tau)}{\vartheta(bz;c\tau)}\right|\right|.$$

Plugging in (II.1.1), using Lemma II.2.1, and rearranging leads to

$$\left|\frac{\vartheta(z;\tau)}{\vartheta(bz;c\tau)}\right| = \left|q^{-\frac{c}{8}}\right| \frac{|\vartheta(z;\tau)|}{\left|\sum\limits_{\kappa\in\mathbb{Z}} (-1)^\kappa q^{c\frac{\kappa^2+\kappa}{2}} \zeta^{b\kappa}\right|}$$
$$\ll \sqrt{\frac{2\pi}{\beta}} e^{-\frac{2\pi^2}{\beta} \min\limits_{z\in\Gamma_{\ell,r}} \left|z-\frac{1}{2}\right|^2} \left|\sum_{\kappa\in\mathbb{Z}} (-1)^\kappa e^{-\frac{2\pi^2}{\varepsilon}\left(\kappa^2+(1-2z)\kappa\right)}\right|$$
$$\ll \beta^{-\frac{1}{2}} e^{-\frac{2\pi^2}{\beta} \min\limits_{z\in\Gamma_{\ell,r}} \left|z-\frac{1}{2}\right|^2} \ll n^{\frac{1}{4}} e^{-\frac{2\pi^2}{\beta} \min\limits_{z\in\Gamma_{\ell,r}} \left|z-\frac{1}{2}\right|^2} \ll n^{\frac{1}{4}}. \tag{II.5.2}$$

Define

$$\mathcal{B}(m,n) := n^{\frac{1}{4} - \sum\limits_{k=1}^{N_2} \frac{\delta_k}{4}} \prod_{j=1}^{N_1} \left(\frac{2\pi}{x_j\beta}\right)^{\frac{\gamma_j}{2}}.$$

Then equations (II.5.1) and (II.5.2) imply that for $r \to 0^+$

$$\left|\sum_{\ell=0}^{s} \int_{\Gamma_{\ell,r}} f(z;\tau)\sin(2\pi mz)dz\right|$$
$$\ll \sum_{\ell=0}^{s} \mathcal{B}(m,n) \exp\left[\sum_{k=1}^{N_2} \frac{2\pi\delta_k}{y_k\beta}\left(\frac{\pi}{12} - \frac{1}{2\pi}\left(1 - \frac{1}{\sqrt{1+m^{-\frac{2}{3}}}}\right)\right) - \sum_{j=1}^{N_1} \frac{\pi^2\gamma_j}{6x_j\beta}\right].$$

Hence, away from the dominant pole at $q = 1$ we have shown the following proposition.

Proposition II.5.1. *For* $1 \leq x \leq \frac{\pi m^{\frac{1}{3}}}{\beta}$ *we have that*

$$\left| f_m\left(\frac{i\varepsilon}{2\pi}\right) \right| \ll (s+1)\mathcal{B}(m,n) \times \exp\left[\sum_{k=1}^{N_2} \frac{2\pi\delta_k}{y_k\beta}\left(\frac{\pi}{12} - \frac{1}{2\pi}\left(1 - \frac{1}{\sqrt{1+m^{-\frac{2}{3}}}}\right)\right) - \sum_{j=1}^{N_1} \frac{\pi^2\gamma_j}{6x_j\beta}\right]$$

as $n \to \infty$.

II.6 The Circle Method

In this section we use Wright's variant of the Circle Method to complete the proof of Theorem II.1.2. Cauchy's Theorem implies that

$$c(m,n) = \frac{1}{2\pi i}\int_C \frac{f_m(\tau)}{q^{n+1}}dq,$$

where $C \coloneqq \{q \in \mathbb{C} : |q| = e^{-\beta}\}$ is a circle centered at the origin of radius less than 1 with the path taken in the counter-clockwise direction, traversing the circle exactly once. Making a change of variables, reversing the direction of the path of integration, and recalling that $\varepsilon = \beta(1 + ixm^{-\frac{1}{3}})$ we have

$$c(m,n) = \frac{\beta}{2\pi m^{\frac{1}{3}}}\int_{|x| \leq \frac{\pi m^{\frac{1}{3}}}{\beta}} f_m\left(\frac{i\varepsilon}{2\pi}\right)e^{\varepsilon n}dx.$$

Splitting this integral into two pieces, we have $c(m,n) = M + E$, where

$$M \coloneqq \frac{\beta}{2\pi m^{\frac{1}{3}}}\int_{|x|\leq 1} f_m\left(\frac{i\varepsilon}{2\pi}\right)e^{\varepsilon n}dx,$$

and

$$E \coloneqq \frac{\beta}{2\pi m^{\frac{1}{3}}}\int_{1\leq|x| \leq \frac{\pi m^{\frac{1}{3}}}{\beta}} f_m\left(\frac{i\varepsilon}{2\pi}\right)e^{\varepsilon n}dx.$$

Next we determine the contributions of each of the integrals M and E, and see that M contributes to the main asymptotic term, while E is part of the error term.

II.6.1 The major arc

Considering the contribution of M, we obtain the following proposition.

Proposition II.6.1. *We have that*

$$M = \frac{1}{2\pi i}\Lambda_1\beta^{2-w}\sqrt{\Lambda_2}^{-w}\frac{e^{2\pi\sqrt{2\Lambda_2 n}}}{2\pi\left(2\Lambda_2 n\right)^{\frac{1}{4}}} + O\left(\beta^{3-w}\frac{e^{2\pi\sqrt{2\Lambda_2 n}}}{2\pi\left(2\Lambda_2 n\right)^{\frac{1}{4}}}\right)$$

as $n \to \infty$.

Proof. By definition we have that

$$M = \frac{\beta}{2\pi m^{\frac{1}{3}}}\Lambda_1\int_{|x|\leq 1}\varepsilon^{1-w}e^{\frac{2\pi^2}{\varepsilon}\Lambda_2}e^{\varepsilon n}dx + \frac{\beta}{2\pi m^{\frac{1}{3}}}\int_{|x|\leq 1}O\left(\beta^{2-w}e^{\frac{2\pi^2}{\varepsilon}\Lambda_2}\right)e^{\varepsilon n}dx. \quad \text{(II.6.1)}$$

Making the change of variables $v = 1 + ixm^{-\frac{1}{3}}$ and then $v \mapsto \sqrt{\Lambda_2}v$ we obtain that the first term equals

$$\frac{1}{2\pi i}\Lambda_1\beta^{2-w}\Lambda_2^{-\frac{w}{2}}P_{1-w,12\Lambda_2}, \quad \text{(II.6.2)}$$

where

$$P_{s,k} := \int_{\frac{1-im^{-\frac{1}{3}}}{\sqrt{\Lambda_2}}}^{\frac{1+im^{-\frac{1}{3}}}{\sqrt{\Lambda_2}}} v^s e^{\pi\sqrt{\frac{kn}{6}}\left(v+\frac{1}{v}\right)}dv.$$

One may relate $P_{s,k}$ to I-Bessel functions in exactly the same way as in [BD16, Lemma 4.2], making the adjustment for $\sqrt{\Lambda_2}$ where necessary, to obtain that

$$P_{s,k} = I_{-s-1}\left(\pi\sqrt{\frac{2kn}{3}}\right) + O\left(\exp\left(\pi\sqrt{\frac{kn}{6}}\left(1+\frac{1}{1+m^{-\frac{2}{3}}}\right)\right)\right).$$

Using the asymptotic behavior of the I-Bessel function given in Lemma II.2.5 we obtain

$$P_{1-w,12\Lambda_2} = \frac{e^{2\pi\sqrt{2\Lambda_2 n}}}{2\pi\left(2\Lambda_2 n\right)^{\frac{1}{4}}} + O\left(\frac{e^{2\pi\sqrt{2\Lambda_2 n}}}{\left(8\pi^2\Lambda_2 n\right)^{\frac{3}{4}}}\right) + O\left(e^{\pi\sqrt{2\Lambda_2 n}\left(1+\frac{1}{1+m^{-\frac{2}{3}}}\right)}\right),$$

and therefore (II.6.2) becomes

$$\frac{1}{2\pi i}\Lambda_1\beta^{2-w}\sqrt{\Lambda_2}^{-w}\frac{e^{2\pi\sqrt{2\Lambda_2 n}}}{2\pi\left(2\Lambda_2 n\right)^{\frac{1}{4}}}+O\left(\beta^{2-w}\frac{e^{2\pi\sqrt{2\Lambda_2 n}}}{\left(8\pi^2\Lambda_2 n\right)^{\frac{3}{4}}}\right)+O\left(\beta^{2-w}e^{\pi\sqrt{2\Lambda_2 n}\left(1+\frac{1}{1+m^{-\frac{2}{3}}}\right)}\right).$$

Analogously the second term of (II.6.1) is

$$\frac{1}{2\pi i}\beta^{3-w}\sqrt{\Lambda_2}^{-1}P_{0,12\Lambda_2}=O\left(\beta^{3-w}\frac{e^{2\pi\sqrt{2\Lambda_2 n}}}{2\pi\left(2\Lambda_2 n\right)^{\frac{1}{4}}}\right).$$

This yields

$$\begin{aligned}M=&\frac{1}{2\pi i}\Lambda_1\beta^{2-w}\sqrt{\Lambda_2}^{-w}\frac{e^{2\pi\sqrt{2\Lambda_2 n}}}{2\pi\left(2\Lambda_2 n\right)^{\frac{1}{4}}}+O\left(\beta^{2-w}\frac{e^{2\pi\sqrt{2\Lambda_2 n}}}{\left(8\pi^2\Lambda_2 n\right)^{\frac{3}{4}}}\right)\\&+O\left(\beta^{2-w}e^{\pi\sqrt{2\Lambda_2 n}\left(1+\frac{1}{1+m^{-\frac{2}{3}}}\right)}\right)+O\left(\beta^{3-w}\frac{e^{2\pi\sqrt{2\Lambda_2 n}}}{2\pi\left(2\Lambda_2 n\right)^{\frac{1}{4}}}\right)\end{aligned}$$

and finishes the proof. □

II.6.2 The error arc

Finally, we bound E as follows.

Proposition II.6.2. *We have $E \ll M$ as $n \to \infty$.*

Proof. By Proposition II.5.1 we have

$$\begin{aligned}E\ll&\frac{\beta}{2\pi m^{\frac{1}{3}}}(s+1)\mathcal{B}(m,n)\exp\left[\sum_{k=1}^{N_2}\frac{2\pi\delta_k}{y_k\beta}\left(\frac{\pi}{12}-\frac{1}{2\pi}\left(1-\frac{1}{\sqrt{1+m^{-\frac{2}{3}}}}\right)\right)-\sum_{j=1}^{N_1}\frac{\pi^2\gamma_j}{6x_j\beta}\right]\\&\times e^{\beta n}\int_{1\leq|x|\leq\frac{\pi m^{\frac{1}{3}}}{\beta}}e^{\beta nixm^{-\frac{1}{3}}}dx\\\ll&\frac{s+1}{\pi}\mathcal{B}(m,n)\exp\left[\pi\sqrt{2n}\left(1-\sum_{j=1}^{N}\frac{\alpha_j}{12a_j}\right)-\sum_{k=1}^{N_2}\frac{\delta_k}{y_k\beta}\left(1-\frac{1}{\sqrt{1+m^{-\frac{2}{3}}}}\right)\right],\end{aligned}$$

where we trivially estimate the final integral. Using $1-\sum_{j=1}^{N}\frac{\alpha_j}{12a_j}<2\sqrt{\Lambda_2}$ the result follows immediately by comparing to M and therefore also finishes the proof of Theorem II.1.2. □

II.7 Further questions

We end by briefly commenting on some related questions that could be the subject of further research.

(1) Here we only discussed the case of eta-theta quotients with simple poles. A natural question to ask is: Does a similar story hold for functions with higher order poles? The situation is of course expected to be more complicated, in particular finding Fourier coefficients with the method presented here seems to be much more difficult. One could attempt to build a framework by following the definitions of Fourier coefficients given in [DMZ12, Section 8].
For example, in [MZR15] Manschot and Zapata Rolón studied a Jacobi form with a double pole related to χ_y-genera of Hilbert schemes on K3. They obtain bivariate asymptotic behavior in a similar flavor to those here. Can one extend this family?

(2) Although the functions considered in this chapter provide a wide family of results, it should be possible to extend the method to other related families of functions. In particular, it would be instructive to consider similar approaches for prototypical examples of mock Jacobi forms.

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Chapter III

Asymptotic equidistribution for partition statistics and topological invariants

This chapter is based on a prerpint of the same title submitted for publication and is joint work with Dr. William Craig and Dr. Joshua Males [CCM21].

III.1 Introduction and statement of results

A *partition* λ of a non-negative integer n is a list of non-increasing positive integrers, say $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, that satisfies $|\lambda| := \lambda_1 + \cdots + \lambda_m = n$, and we let $p(n)$ denote the number of such partitions. In 1918 Hardy and Ramanujan [HR18] proved

$$p(n) \sim \frac{1}{4\sqrt{3}n} \cdot e^{\pi\sqrt{\frac{2n}{3}}}$$

as $n \to \infty$, one of the most famous asymptotic formulae in partition theory. Their work marked the birth of the so-called Circle Method.

Half a century later Wright [Wri68, Wri71] developed a modified version of the Circle Method which provides a general method for studying the Fourier coefficients of functions with known asymptotic behavior near cusps. The essence of Wright's method is to use Cauchy's theorem to recover the coefficients as the integral over a circle of the generating function. One then splits the integral into two arcs, the major arc and minor arc, where the generating function has large growth and small relative growth, respectively. Even though this version of the Circle Method gives weaker bounds than the original techniques of Hardy and Ramanujan, it is more flexible when working with non-modular generating functions. It has been used extensively in the literature, see e.g., [BM14, KKS15, Mao18] for several examples closely related to this chapter.

Throughout mathematics, the equidistribution properties of certain objects are a central theme studied by many authors, including in areas of algebraic and arithmetic geometry [CM15, GT12, Kat15] and number theory [OS18, Xi20]. Recently, there has been

$$\begin{array}{llllllll} \bullet & \bullet & \bullet & \dots & \bullet & \leftarrow & \lambda_1 \text{ many nodes} \\ \bullet & \bullet & \dots & \bullet & & \leftarrow & \lambda_2 \text{ many nodes} \\ \vdots & \vdots & \vdots & & & & \\ \bullet & \dots & \bullet & & & \leftarrow & \lambda_m \text{ many nodes} \end{array}$$

Figure III.1: The Ferres-Young diagram of a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$.

a body of work in analogy with Dirichlet's theorem on the asymptotic equidistribution (or non-equidistribution) on arithmetic progressions of various objects. For example, Males showed the asymptotic equidistribution of the partition ranks in [Mal21b], Ciolan proved asymptotic equidistribution results for the number of partitions of n into k-th powers in [Cio20], Gillman, Gonzalez, Ono, Rolen, and Schoenbauer proved asymptotic equidistribution for Hodge numbers and Betti numbers of certain Hilbert schemes of surfaces [GGORS20], and Zhou proved asymptotic equidistribution of a wide class of partition objects in [Zho21].

Another example is one of Craig and Pun [CP21], wherefore we need to define hook lengths. For each partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ we can draw a *Ferres-Young diagram* as shown in Figure III.1. The node in row k and column j has *hook length*

$$h(k, j) := (\lambda_k - k) + (\lambda'_j - j) + 1,$$

where λ'_j denotes the number of nodes in column j. We let $\mathcal{H}_t(\lambda)$ denote the multiset of *t-hooks*, those hook lengths which are multiples of a fixed positive integer t, of a partition λ. Craig and Pun investigated the t-hook partition functions

$$\begin{aligned} p_t^e(n) &:= \#\{\lambda \text{ a partition of } n : \#\mathcal{H}_t(\lambda) \text{ is even}\}, \\ p_t^o(n) &:= \#\{\lambda \text{ a partition of } n : \#\mathcal{H}_t(\lambda) \text{ is odd}\}, \end{aligned}$$

which divide the partitions of n into two subsets, those with an even (respectively odd) number of t-hooks. For even t, they proved that the partitions of n are asymptotically equidistributed between these two subsets, while for odd t they found the surprising phenomenon that they are not. Following this example, Bringmann, Craig, Males, and Ono [BCMO22] showed that on arithmetic progressions modulo odd primes t-hooks are not asymptotically equidistributed, while the Betti numbers of two specific Hilbert schemes are. Their results centrally used a variant of Wright's Circle Method (see Proposition III.2.3).

The primary aim of this chapter is for proving large families of Dirichlet-type equidistribution theorems. We begin by making more precise the meaning of a Dirichlet-type theorem. Suppose $c(n)$ is an arithmetic function which counts something of interest. Let $q = e^{-z}$, where $z = x + iy \in \mathbb{C}$ with $x > 0$ and $|y| < \pi$. Furthermore let $\zeta = \zeta_b^a := e^{\frac{2\pi i a}{b}}$ be a b-th root of unity for some natural number[1] $b \geq 2$ and $0 \leq a < b$. Assume that we have a generating function on arithmetic progressions $a \pmod b$ given by

$$H(a, b; q) = \sum_{n \geq 0} c(a, b; n) q^n, \tag{III.1.1}$$

for some coefficients $c(a, b; n)$ such that

$$H(a, b; q) = \frac{1}{b} \sum_{j=0}^{b-1} \zeta_b^{-aj} H\left(\zeta_b^j; q\right), \tag{III.1.2}$$

for some generating functions $H(\zeta; q)$, with $H(q) := H(1; q) = \sum_{n \geq 0} c(n) q^n$. To say that equidistribution holds is to say that $c(a, b; n) \sim \frac{1}{b} c(n)$ as $n \to \infty$. We are concerned with relating analytic properties of the functions $H(\zeta; q)$ to equidistribution results for $c(a, b; n)$. We provide a general framework for answering this question for a large class of generating functions by applying the spirit of Wright's Circle Method along with ideas of [BCMO22] (see Theorem III.3.1 for a precise statement). Since our aim is to unify differing approaches to asymptotic equidistribution, we also collect many known or partially-known results and prove them using our framework, which may be summarized as follows.

Result. *Assume that on both the major and minor arcs $H(q)$ dominates $H(\zeta; q)$, and $H(q)$ is dominant on the major arc as $q \to 1$. Then $c(a, b; n)$ are eqidistributed as $n \to \infty$.*

Theorem III.1.1 is already known, Theorem III.1.2 is partially known, while (to the best of the author's knowledge) Theorems III.1.3, III.1.4, III.1.5, and III.1.6 are new.

Because this method also naturally produces asymptotic formulae for the coefficients $c(a, b; n)$, we may also derive other interesting results, namely results about convexity and log-concavity. Convexity-type results of partition theoretic objects have been studied in recent years, for example in [BO16] Bessenrodt and Ono showed that if $n_1, n_2 \geq 1$ and $n_1 + n_2 \geq 9$, then

$$p(n_1)p(n_2) > p(n_1 + n_2).$$

A similar phenomenon for partition ranks congruent to $a \pmod b$, denoted by $N(a, b; n)$, was investigated by Hou and Jagadeesan [HJ18], who gave an explicit lower bound on n

[1] The case $b = 1$ is clearly trivial for coefficients that are integral.

for convexity of $N(a,3;n)$. Confirming a conjecture of [HJ18], Males showed in [Mal21b] that for large enough n_1, n_2 we have

$$N(a,b;n_1)N(a,b;n_2) > N(a,b;n_1+n_2).$$

A direct corollary to Proposition III.2.3 shows that $c(a,b;n)$ arising from functions that satisfy the conditions of Proposition III.2.3 also satisfy the convexity result

$$c(a,b;n_1)c(a,b;n_2) > c(a,b;n_1+n_2) \tag{III.1.3}$$

for large enough n_1, n_2. A further corollary yields that the coefficients are *asymptotically log-concave*, i.e., for large enough n_1, n_2,

$$c(a,b;n)^2 \geq c(a,b;n-1)c(a,b;n+1). \tag{III.1.4}$$

Such log-concavity results have been obtained for various arithmetic coefficients in the literature, including [BJSMR19, LDM19, DP15] among many others. In particular, all of the coefficients discussed in the following sections asymptotically satisfy (III.1.3) and (III.1.4). To the best of the author's knowledge, this gives new results for the first residual crank, traces of plane partitions, Betti numbers of the two- and three-flag Hilbert schemes we consider, as well as the cells of the scheme $V_{n,k}$ of Göttsche, each defined in the following subsections.

III.1.1 Partition statistics

We next consider various statistics on partitions, beginning with the asymptotic equidistribution properties of two of the most famous partition statistics: the rank and the crank.

In [Ram21] Ramanujan proved that for $n \geq 0$

$$p(5n+4) \equiv 0 \pmod 5, \qquad p(7n+5) \equiv 0 \pmod 7, \qquad p(11n+6) \equiv 0 \pmod{11}.$$

The *rank* [Dys44] of a partition λ is given by the largest part minus the number of parts. Dyson [Dys44] conjectured, and Atkin and Swinnerton-Dyer [ASD54] later proved, that the partitions of $5n+4$ (respectively $7n+5$) form 5 (respectively 7) groups of equal size when sorted by their ranks modulo 5 (respectively 7), thereby combinatorially explaining two of Ramanujan's congruences. Moreover, Dyson posited the existence of another statistic which should explain all Ramanujan congruences, which he called the *crank*. The crank was later found by Andrews and Garvan [AG88, Gar88], and is given by

$$\begin{cases} \lambda_1 & \text{if } \lambda \text{ contains no ones,} \\ \mu(\lambda) - \omega(\lambda) & \text{if } \lambda \text{ contains ones,} \end{cases}$$

where $\omega(\lambda)$ denotes the number of ones in λ and $\mu(\lambda)$ denotes the number of parts greater than $\omega(\lambda)$.

The function $N(a, b; n)$, which is the number of partitions of n with rank congruent to $a \pmod b$, was shown to be asymptotically equidistributed by Males in [Mal21b], making use of Ingham's Tauberian theorem and monotonicity properties[2], and may also be concluded from [Bri08]. We reprove this result.

Theorem III.1.1. *Let $0 \leq a < b$ and $b \geq 2$. Then as $n \to \infty$ we have that*

$$N(a, b; n) = \frac{1}{b} p(n) \left(1 + O\left(n^{-\frac{1}{2}}\right)\right).$$

In a similar vein, it is natural to consider the asymptotic behavior of the crank on arithmetic progressions. For odd b, the asymptotic equidistribution is known by Hamakiotes, Kriegman, and Tsai [HKT21], who used results on the asymptotic of cranks given by Zapata Rolón in [Zap15]. With our framework we are able to extend this result to all b. Note that our method is simpler than the full Circle Method, allowing us to easily extend to include the case of b even. However, the asymptotic formulae obtained in [HKT21] are far more precise than ours. Let $M(a, b; n)$ be the number of partitions of n with crank congruent to $a \pmod b$.

Theorem III.1.2. *Let $0 \leq a < b$ and $b \geq 2$. Then as $n \to \infty$ we have that*

$$M(a, b; n) = \frac{1}{b} p(n) \left(1 + O\left(n^{-\frac{1}{2}}\right)\right).$$

In [BLO09], Bringmann, Lovejoy, and Osburn introduced two so-called residual cranks on overpartitions. Recall that an *overpartition* is a partition where the first occurrence of each distinct number may be overlined. The *first residual crank* of an overpartition is given by the crank of the subpartition consisting of the non-overlined parts. Let $\overline{M}(a, b; n)$ denote the number of overpartitions of n whose first residual crank is congruent to $a \pmod b$.

Theorem III.1.3. *Let $0 \leq a < b$ and $b \geq 2$. Then as $n \to \infty$ we have that*

$$\overline{M}(a, b; n) = \frac{1}{8bn} e^{\pi\sqrt{n}} \left(1 + O\left(n^{-\frac{1}{2}}\right)\right).$$

Remark. One could obtain a similar result for the second residual crank of [BLO09], which we omit here for succinctness.

[2]Since the proof in [Mal21b] used Ingham's Tauberian theorem, there was no error term.

1	1	1	1	1	1	2	1	2	3
			1		1			1	
					1				

Figure III.2: The plane partitions of 3.

Our framework applies to a larger realm than just the classical theory of partitions. In fact, we now demonstrate an example where we can prove equidistribution in congruence classes for a plane partition statistic. A *plane partition* of n (see e.g., [And98]) is a two-dimensional array $\pi_{j,k}$ of non-negative integers $j, k \geq 1$, that is non-increasing in both variables, i.e., $\pi_{j,k} \geq \pi_{j+1,k}$, $\pi_{j,k} \geq \pi_{j,k+1}$ for all j and k, and fulfills $|\Lambda| := \sum_{j,k} \pi_{j,k} = n$. For example there are six plane partitions of 3, which we list in Figure III.2 using the standard visual representation of plane partitions. We let $\mathrm{pp}(n)$ denote the number of plane partitions of n, so $\mathrm{pp}(3) = 6$. Plane partitions were famously studied by MacMahon [Mac04], who established the generating function

$$\mathrm{PP}(q) := \sum_{n\geq 0} \mathrm{pp}(n) q^n = \prod_{n\geq 1} \frac{1}{(1-q^n)^n} = 1 + q + 3q^2 + 6q^3 + 13q^4 + 24q^5 + \cdots .$$

As with regular partitions, many authors have studied asymptotic properties of families of plane partitions and their statistics. For instance, in 1931 Wright [Wri31] established the asymptotic formula

$$\mathrm{pp}(n) \sim \frac{\zeta(3)^{\frac{7}{36}}}{\sqrt{12\pi}} \left(\frac{n}{2}\right)^{-\frac{25}{36}} \exp\left(3\zeta(3)^{\frac{1}{3}} \left(\frac{n}{2}\right)^{\frac{2}{3}} + \zeta'(-1)\right) \tag{III.1.5}$$

as $n \to \infty$, where $\zeta(s) := \sum_{k\geq 1} \frac{1}{k^s}$ with $\mathrm{Re}(s) > 1$ is the *Riemann zeta function*. One of the more famous statistics associated to plane partitions $\Lambda = \{\pi_{j,k}\}_{j,k\geq 1}$ is its *trace* $t(\Lambda)$, which is defined by

$$t(\Lambda) = \sum_{j\geq 1} \pi_{j,j}.$$

In [Sta73], Stanley generalized MacMahon's generating function to a two-variable function which keeps track of the values of $t(\Lambda)$, proving

$$\sum_{\Lambda} \zeta^{t(\Lambda)} q^{|\Lambda|} = \prod_{n\geq 1} \frac{1}{(1-\zeta q^n)^n}.$$

Certain asymptotic properties of the trace have been studied by Kamenov and Mutafchiev [KM07] and Mutafchiev [Mut18], where the limiting distribution and expected value of

$t(\Lambda)$ were considered. Here, we study the distribution of the trace in residue classes. In particular, for integers $0 \leq a < b$ we define the function $\mathrm{pp}(a, b; n)$ as the number of plane partitions of n whose trace is congruent to $a \pmod b$, that is,

$$\mathrm{pp}(a, b; n) := \#\{\Lambda : |\Lambda| = n, t(\Lambda) \equiv a \pmod b\}.$$

For example, from the plane partitions of 3 given above we can see that $\mathrm{pp}(0, 2; 3) = 2$ and $\mathrm{pp}(1, 2; 3) = 4$.

Theorem III.1.4. *Let $0 \leq a < b$ and $b \geq 2$. Then as $n \to \infty$ we have that*

$$\mathrm{pp}(a, b; n) \sim \frac{1}{b}\,\mathrm{pp}(n) \sim \frac{1}{b}\frac{\zeta(3)^{\frac{7}{36}}}{\sqrt{12\pi}}\left(\frac{n}{2}\right)^{-\frac{25}{36}} \exp\left(3\zeta(3)^{\frac{1}{3}}\left(\frac{n}{2}\right)^{\frac{2}{3}} + \zeta'(-1)\right).$$

There are a plethora of other partition statistics in the literature for which one could obtain similar theorems using our framework. For example, such results could be proved for more residual crank-like statistics [Jen15], ranks for overpartition pairs [BL08], or the full rank of k-marked Durfee symbols [BGM09].

III.1.2 Betti numbers of Hilbert schemes

In topology a fundamental goal is to determine whether two spaces have the same topological, differential, or complex analytic structure. Topological invariants are important tools for determining when spaces have different structure. A prominent example are Betti numbers, which count the dimension of certain vector spaces of differential forms of a manifold. Often, the generating function of the Betti numbers are related to modular forms. Two prominent examples were investigated by Bringmann, Craig, Males, and Ono in [BCMO22], where it was shown that the Betti numbers of the Hilbert scheme of n points on $\mathbb{C}^2$ as well as its quasihomogenous counterpart are each (essentially) asymptotically equidistributed[3] as $n \to \infty$. Here we provide further examples of this phenomenon.

For a Hilbert scheme X, let $b_j(X) := \dim(H_j(X, \mathbb{Q}))$ be the *Betti numbers*. Here, $H_j(X, \mathbb{Q})$ denotes the j-th homology group of X with rational coefficients. Then the generating function in a formal variable T for the Betti numbers is known as the *Poincaré polynomial*, defined by[4]

$$P(X; T) := \sum_j b_j(X)T^j = \sum_j \dim(H_j(X, \mathbb{Q}))T^j.$$

[3]Here we mean equidistributed up to a trivial modification which comes from the fact that certain Betti numbers in this setting are identically zero. See the definition of $d(a, b)$ as below.

[4]The reader should be aware that often the Poincaré polynomial is written in the formal variable $T^{\frac{1}{2}}$, which explains some apparent mismatches between the referenced sources for generating functions in Section III.4 and those quoted in this chapter.

We consider the modular sums of Betti numbers on congruence classes $a \pmod b$, and define

$$B(a,b;X) := \sum_{j\equiv a \pmod b} b_j(X).$$

Define the three-step flag Hilbert scheme by

$$\begin{aligned} X_1 :=& \mathrm{Hilb}^{n,n+1,n+2}(0) \\ =& \left\{\mathbb{C}[[x,y]] \supset I_n \supset I_{n+1} \supset I_{n+2} \colon I_k \text{ ideals with } \dim_{\mathbb{C}}^{\mathbb{C}[[x,y]]}/I_k = k\right\}, \end{aligned}$$

and the two-step flag scheme

$$X_2 := \mathrm{Hilb}^{n,n+2}(0) = \left\{\mathbb{C}[[x,y]] \supset I_n \supset I_{n+2} \colon I_k \text{ ideals with } \dim_{\mathbb{C}}^{\mathbb{C}[[x,y]]}/I_k = k\right\}.$$

Furthermore, let (J,I) be a point in

$$\mathrm{Hilb}^{n,n+2}\left(\mathbb{C}^2\right) := \left\{I_n \in \mathrm{Hilb}^n\left(\mathbb{C}^2\right), I_{n+2} \in \mathrm{Hilb}^{n+2}(\mathbb{C}^2) \colon I_n \supset I_{n+2}\right\},$$

where $\mathrm{Hilb}^n(\mathbb{C}^2)$ denotes the usual Hilbert scheme of n points over $\mathbb{C}^2$. Then J, I are said to be *trivially related* if $J/I \cong \mathbb{C}^2$ as trivial $\mathbb{C}[x,y]$ modules (see [Boc16, Definition 4.2.1]). We also consider

$$X_3 := \mathrm{Hilb}^{n,n+2}\left(\mathbb{C}^2\right)_{\mathrm{tr}},$$

which is the subspace of $\mathrm{Hilb}^{n,n+2}(\mathbb{C}^2)$ of trivially related points (see also [NY11a]). For $m \in \mathbb{N}$, we also regard the certain perverse coherent sheaves (defined explicitly in [NY11b]), called $X_4 := \widehat{M}^m(c_N)$ where c_N is some prescribed homological data.

Let

$$d(a,b) := \begin{cases} \frac{1}{b} & \text{if } b \text{ is odd,} \\ \frac{2}{b} & \text{if } a \text{ and } b \text{ are even,} \\ 0 & \text{if } a \text{ is odd and } b \text{ is even.} \end{cases}$$

We prove the following result, which shows that the Betti numbers of these schemes are (essentially) asymptotically equidistributed.

Theorem III.1.5. *Let $0 \leq a < b$ and $b \geq 2$. Then as $n \to \infty$ we have that*

$$\frac{1}{2}B(a,b;X_1) \sim B(a,b;X_2) \sim B(a,b;X_3) = \frac{d(a,b)\sqrt{3}}{4\pi^2}e^{\pi\sqrt{\frac{2n}{3}}}\left(1 + O\left(n^{-\frac{1}{2}}\right)\right)$$

and

$$B(a,b;X_4) = \frac{d(a,b)n^{\frac{m-2}{2}}}{6^{\frac{1-m}{2}}2\sqrt{2}c_m\pi^m} e^{\pi\sqrt{\frac{2n}{3}}}\left(1+O\left(n^{-\frac{1}{2}}\right)\right)$$

where $\prod_{j=1}^{m}\frac{1}{1-e^{-jz}} =: \frac{1}{c_m z^m} + O(z^{-m+1})$.

Remark. It is possible to obtain further terms in the asymptotic expansion directly from the application of Theorem III.3.1, which highlights the difference in lower-order terms of $B(a,b;X_j)$. Moreover, for a odd and b even, one may easily show that $B(a,b;X_j)$ identically vanish.

Since many generating functions for topological invariants arise as infinite q-products, one may conclude similar results for many other functions. For example, in [MZR15] Manschot and Zapata Rolón investigated the asymptotics of the χ_y-genera of Hilbert schemes of n points on $K3$ surfaces, centrally using Wright's Circle Method. Since their generating function is a quotient of infinite q-products (see [MZR15, page 2]), it is likely that one may conclude similar equidistribution properties for these genera.

III.1.3 A particular scheme of Göttsche

Let $\mathrm{Hilb}_n(S)$ denote the Hilbert scheme which parametrises finite subschemes of length n on a smooth projective surface S. We follow Fulton [Ful84] and Ellingsrud–Strømme [ES87], and say that a scheme X has a *cellular decomposition* if there is a filtration $X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset X_{-1} = \varnothing$ by closed subschemes with each $X_j - X_{j-1}$ a disjoint union of schemes $U_{\ell j}$ isomorphic to certain affine spaces. Then the $U_{\ell j}$ are known as the *cells* of the decomposition.

Let k be an algebraically closed field. Let $\boldsymbol{m}$ be the maximal ideal in $k[[x,y]]$, and define

$$V_{n,k} := \mathrm{Hilb}_n\left(\mathrm{spec}\left(k[[x,y]]/\boldsymbol{m}^n\right)\right).$$

The scheme $V_{n,k}$ was a central tool of Göttsche in obtaining the famous formula for the Betti numbers of any Hilbert scheme of points on a smooth projective variety [Goe90], via the Weil conjectures. Let $v(a,b;n)$ count the number of cells of $V_{n,k}$ whose dimension is congruent to $a \pmod{b}$.

Theorem III.1.6. *Let* $0 \le a < b$ *and* $b \ge 2$*. As* $n \to \infty$ *we have that*

$$v(a,b;n) = \frac{1}{b}p(n)\left(1+O\left(n^{-\frac{1}{2}}\right)\right).$$

This chapter is structured as follows. In Section III.2 we recall relevant results from previous works in the literature. In Section III.3 we then state our central theorem on the asymptotic equidistribution of coefficients of certain generating functions and show how convexity and log-concavity immediately follow from the asymptotics produced by Wright's Circle Method. Finally we prove the remaining theorems in Section III.4.

III.2 Preliminaries

III.2.1 Asymptotics of infinite q-products

Here we recall the asymptotic behavior of various infinite q-products. One helpful tool is the modularity of the partition generating function

$$P(q) \coloneqq \sum_{n\geq 0} p(n)q^n = \frac{1}{(q;q)_\infty} = \frac{q^{\frac{1}{24}}}{\eta(\tau)},$$

where we set $(a)_j = (a;q)_j \coloneqq \prod_{\ell=0}^{j-1}(1-aq^\ell)$ for $j \in \mathbb{N}_0 \cup \{\infty\}$, $q = e^{2\pi i\tau}$ and the Dedekind η-function

$$\eta(\tau) \coloneqq q^{\frac{1}{24}} \prod_{n\geq 1}(1-q^n),$$

which is a modular form of weight $\frac{1}{2}$. We also have [BD16, Lemma 3.5].

Lemma III.2.1. *Let $M > 0$ be a fixed constant. Assume that $\tau = u + iv \in \mathbb{H}$, with $Mv \leq |u| \leq \frac{1}{2}$ for $u > 0$ and $v \to 0$. We have that*

$$|P(q)| \ll \sqrt{v} \exp\left[\frac{1}{v}\left(\frac{\pi}{12} - \frac{1}{2\pi}\left(1 - \frac{1}{\sqrt{1+M^2}}\right)\right)\right].$$

This gives us the asymptotic behavior of $P(q)$ on the so-called minor arc.

Using the transformation property of η we obtain the following classical transformation behavior (see e.g., [BCMO22, equation (2.7)] with $k = 1, h = 0$ and shifting $z \mapsto \frac{z}{2\pi}$)

$$\left(e^{-z};e^{-z}\right)_\infty = \left(\frac{2\pi}{z}\right)^{\frac{1}{2}} e^{\frac{\pi}{12}\left(\frac{z}{2\pi} - \frac{2\pi}{z}\right)} \left(e^{-\frac{4\pi^2}{z}}, e^{-\frac{4\pi^2}{z}}\right)_\infty, \tag{III.2.1}$$

for $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$.

Keeping the naming convention of [BCMO22], we let

$$F_1(\zeta;q) \coloneqq \prod_{n\geq 1}\left(1 - \zeta q^n\right), \qquad F_3(\zeta;q) \coloneqq \prod_{n\geq 1}\left(1 - \zeta^{-1}(\zeta q)^n\right).$$

Recall *Lerch's transcendent*

$$\Phi(z,s,a) \coloneqq \sum_{n\geq 0} \frac{z^n}{(n+a)^s},$$

and for $0 \leq \theta < \frac{\pi}{2}$ define the domain $D_\theta \coloneqq \left\{z = re^{i\alpha} \colon r \geq 0 \text{ and } |\alpha| \leq \theta\right\}$. Throughout, the *Gamma function* is defined by $\Gamma(z) \coloneqq \int_0^\infty t^{z-1}e^{-t}dt$, for $\mathrm{Re}(z) > 0$. Then we have [BCMO22, Theorem 2.1] (see also [BFG22] for the first case) which enables us to determine the asymptotics of F_1 and F_3 on major arcs.

Theorem III.2.2. *For $b \geq 2$, let ζ be a primitive b-th root of unity. Then the following are true.*
(1) *As $z \to 0$ in D_θ, we have*

$$F_1\left(\zeta; e^{-z}\right) = \frac{1}{\sqrt{1-\zeta}}\, e^{-\frac{\zeta\Phi(\zeta,2,1)}{z}} \left(1 + O\left(|z|\right)\right).$$

(2) *As $z \to 0$ in D_θ, we have*

$$F_3\left(\zeta; e^{-z}\right) = \frac{\sqrt{2\pi}\left(b^2 z\right)^{\frac{1}{2}-\frac{1}{b}}}{\Gamma\left(\frac{1}{b}\right)} \prod_{j=1}^{b-1} \frac{1}{(1-\zeta^j)^{\frac{j}{b}}}\, e^{-\frac{\pi^2}{6b^2 z}} \left(1 + O\left(|z|\right)\right).$$

Remark. Note that the proof of (1) does not require ζ to be primitive but only $\zeta \neq 1$.

III.2.2 Wright's Circle Method

We require the following variant of Wright's Circle Method, which was proved by Bringmann, Craig, Males, and Ono [BCMO22, Proposition 4.4], following work of Wright [Wri71], see also Ngo and Rhoades [NR17].

Proposition III.2.3. *Suppose that $F(q)$ is analytic for $q = e^{-z}$ where $z = x + iy \in \mathbb{C}$ satisfies $x > 0$ and $|y| < \pi$, and suppose that $F(q)$ has an expansion $F(q) = \sum_{n\geq 0} c(n)q^n$ near 1. Let $N, M > 0$ be fixed constants. Consider the following hypotheses:*
(1) *As $z \to 0$ in the bounded cone $|y| \leq Mx$ (major arc), we have*

$$F(e^{-z}) = z^B e^{\frac{A}{z}} \left(\sum_{j=0}^{N-1} \alpha_j z^j + O_M\left(|z|^N\right)\right),$$

where $\alpha_j \in \mathbb{C}$, $A \in \mathbb{R}^+$, and $B \in \mathbb{R}$.

(2) *As $z \to 0$ in the bounded cone $Mx \le |y| < \pi$ (minor arc), we have*

$$|F(e^{-z})| \ll_M e^{\frac{1}{\mathrm{Re}(z)}(A-\kappa)},$$

for some $\kappa \in \mathbb{R}^+$.
If (1) and (2) hold, then as $n \to \infty$ we have for any $N \in \mathbb{R}^+$

$$c(n) = n^{\frac{1}{4}(-2B-3)} e^{2\sqrt{An}} \left(\sum_{r=0}^{N-1} p_r n^{-\frac{r}{2}} + O\left(n^{-\frac{N}{2}}\right)\right),$$

where $p_r := \sum_{j=0}^{r} \alpha_j c_{j,r-j}$ *and* $c_{j,r} := \dfrac{(-\frac{1}{4\sqrt{A}})^r \sqrt{A}^{j+B+\frac{1}{2}}}{2\sqrt{\pi}} \dfrac{\Gamma(j+B+\frac{3}{2}+r)}{r!\Gamma(j+B+\frac{3}{2}-r)}.$

III.3 The central theorem

Recall that we have the functions $H(a,b;q)$, $H(q)$, and $H(\zeta;q)$ as in (III.1.1) and (III.1.2), respectively. We now prove a theorem regarding asymptotic equidistribution of the coefficients $c(a,b;n)$.

Theorem III.3.1. *Let $H(a,b;q)$ and $H(\zeta;q)$ be analytic on $|q|<1$, $|\zeta|=1$ such that*

$$H(a,b;q) = \frac{1}{b} \sum_{j=0}^{b-1} \zeta_b^{-aj} H\left(\zeta_b^j; q\right).$$

Suppose $c(a,b;n)$ and $c(n)$ are the Fourier coefficients of $H(a,b;q)$ and $H(1;q)$, respectively. Let $C = C_n$ be a sequence of circles centered at the origin inside the unit disk with radii $r_n \to 1$ as $n \to \infty$ that loops around zero exactly once. For $0 < \theta$, let $\widetilde{C} := C \cap D_\theta$ and $C \backslash \widetilde{C}$ be arcs such that the following hypotheses hold.
(1) *As $z \to 0$ outside of D_θ, we have*

$$\sum_{j=1}^{b-1} \zeta_b^{-aj} H\left(\zeta_b^j; e^{-z}\right) = O\left(H(1;e^{-z})\right).$$

(2) *As $z \to 0$ in D_θ, we have for each $1 \le j \le b-1$ that*

$$H\left(\zeta_b^j; e^{-z}\right) = o\left(H(1;e^{-z})\right).$$

(3) *As $n \to \infty$, we have*

$$c(n) \sim \frac{1}{2\pi i} \int_{\widetilde{C}} \frac{H(1;q)}{q^{n+1}} dq.$$

Then as $n \to \infty$, we have

$$c(a,b;n) \sim \frac{1}{b} c(n).$$

In particular, if $H(1;q)$ and $H(\zeta;q)$ satisfy the conditions of Proposition III.2.3 we have that

$$c(a,b;n) \sim \frac{1}{b} c(n) \sim \frac{1}{b} n^{\frac{1}{4}(-2B-3)} e^{2\sqrt{An}} \left(\sum_{r=0}^{N-1} p_r n^{-\frac{r}{2}} + O\left(n^{-\frac{N}{2}}\right) \right)$$

as $n \to \infty$.

Proof. By Cauchy's theorem and the decomposition of $H(a,b;q)$ we have

$$c(a,b;n) = \frac{1}{2\pi i} \int_C \frac{H(a,b;q)}{q^{n+1}} dq = \frac{1}{b} \left[\frac{1}{2\pi i} \int_C \frac{\sum_{j=0}^{b-1} \zeta_b^{-aj} H(\zeta_b^j;q)}{q^{n+1}} dq \right].$$

We now break down the integral over C into the components $\widetilde{C}$ and $C \backslash \widetilde{C}$. Along $C \backslash \widetilde{C}$, we have by (1) that

$$\frac{1}{2\pi i} \int_{C \backslash \widetilde{C}} \frac{\sum_{j=0}^{b-1} \zeta_b^{-aj} H(\zeta_b^j;q)}{q^{n+1}} dq = O\left(\frac{1}{2\pi i} \int_{C \backslash \widetilde{C}} \frac{H(1;q)}{q^{n+1}} dq \right).$$

From (3) along with Cauchy's integral formula for $c(n)$ it follows that

$$\frac{1}{2\pi i} \int_{C \backslash \widetilde{C}} \frac{H(1;q)}{q^{n+1}} dq = o\left(\frac{1}{2\pi i} \int_{\widetilde{C}} \frac{H(1;q)}{q^{n+1}} dq \right)$$

as $n \to \infty$, and therefore

$$\frac{1}{2\pi i} \int_{C \backslash \widetilde{C}} \frac{\sum_{j=0}^{b-1} \zeta_b^{-aj} H(\zeta_b^j;q)}{q^{n+1}} dq = o\left(\frac{1}{2\pi i} \int_{\widetilde{C}} \frac{H(1;q)}{q^{n+1}} dq \right).$$

On $\widetilde{C}$ we have by (2) that $\sum_{j=0}^{b-1} \zeta_b^{-aj} H(\zeta_b^j;q) = H(1;q) + o\left(H(1;q)\right)$, from which it follows that

$$\frac{1}{2\pi i} \int_{\widetilde{C}} \frac{\sum_{j=0}^{b-1} \zeta_b^{-aj} H(\zeta_b^j;q)}{q^{n+1}} dq \sim \frac{1}{2\pi i} \int_{\widetilde{C}} \frac{H(1;q)}{q^{n+1}} dq$$

as $n \to \infty$. Therefore, combining the estimates along $\widetilde{C}$ and $C \backslash \widetilde{C}$ we have by (3) that

$$c(a,b;n) \sim \frac{1}{b} \left[\frac{1}{2\pi i} \int_{\widetilde{C}} \frac{H(1;q)}{q^{n+1}} dq \right] \sim \frac{1}{b} c(n)$$

as $n \to \infty$. This proves the first claim. If we now assume $H(1;q)$ and $H(\zeta_b^j;q)$ satisfy the hypotheses of Proposition III.2.3, then it is clear that each of (1) – (3) are satisfied and the result follows by the asymptotic for $c(n)$ in Proposition III.2.3. □

Using this result, we may immediately conclude asymptotic convexity for a large class of functions.

Corollary III.3.2. *Let $0 \leq a < b$ and $b \geq 2$. Assume that $H(1;q)$ and $H(\zeta;q)$ satisfy the conditions of Proposition III.2.3. Then for large enough n_1, n_2 we have that*

$$c(a,b;n_1)c(a,b;n_2) > c(a,b;n_1+n_2).$$

Remark. The proof also works for the plane partition functions $\mathrm{pp}(a,b;n)$ by Wright's asymptotic formula (III.1.5). Higher order Turán inequalities for plane partitions have recently been studied by Ono, Pujahari, and Rolen [OPR22].

Proof of Corollary III.3.2. We use the description of the asymptotics of $c(a,b;n)$ from the proof of Theorem III.3.1 for $N=1$. Then

$$\begin{aligned} &c(a,b;n_1)c(a,b;n_2) \\ &\quad = \frac{p_0^2}{b^2}(n_1n_2)^{\frac{1}{4}(-2B-3)}e^{2\sqrt{An_1}+2\sqrt{An_2}}\left(1+O\left(\max\left(n_1^{-\frac{1}{2}}, n_2^{-\frac{1}{2}}, (n_1n_2)^{-\frac{1}{2}}\right)\right)\right) \end{aligned}$$

and

$$c(a,b;n_1+n_2) = \frac{p_0}{b}(n_1+n_2)^{\frac{1}{4}(-2B-3)}e^{2\sqrt{A(n_1+n_2)}}\left(1+O\left((n_1+n_2)^{-\frac{1}{2}}\right)\right).$$

Comparing the exponential growth of the main terms immediately yields the conclusion. □

A very similar calculation gives the following log-concavity result.

Corollary III.3.3. *Let $0 \leq a < b$ and $b \geq 2$. Assume that $H(1;q)$ and $H(\zeta;q)$ satisfy the conditions of Proposition III.2.3. For large enough n, we have*

$$c(a,b;n)^2 \geq c(a,b;n-1)c(a,b;n+1).$$

We consider the case of partition statistics in slightly more detail. Let $s(\lambda)$ be a partition statistic, i.e., s is a map from the set of all partitions to $\mathbb{Z}$, and let

$$H_s(\zeta;q) = \sum_{\lambda} \zeta^{s(\lambda)} q^{|\lambda|}.$$

Note that $H_s(1;q) = \sum_\lambda q^{|\lambda|}$ is the generating function of $p(n)$. Then by orthogonality of roots of unity we have that (see e.g., [And98])

$$H_s(a,b;q) := \frac{1}{b}\sum_{j=0}^{b-1} \zeta_b^{-aj} H_s\left(\zeta_b^j;q\right) = \sum_{\substack{\lambda \\ s(\lambda)\equiv a \pmod{b}}} q^{|\lambda|}. \tag{III.3.1}$$

A direct corollary of Theorem III.3.1 is the following.

Corollary III.3.4. *Assume that $H_s(1;q)$ and $H_s(\zeta;q)$ satisfy the conditions of Theorem III.3.1, and let $s(a,b;n)$ count the number of partitions of n with statistic s congruent to $a \pmod b$. Then as $n \to \infty$ we have that $s(a,b;n) \sim \frac{1}{b}p(n)$. If furthermore the conditions of Proposition III.2.3 are satisfied, we have the error term which yields*

$$s(a,b;n) = \frac{1}{b}p(n)\left(1 + O\left(n^{-\frac{1}{2}}\right)\right).$$

III.4 Proofs of Theorems III.1.1 to III.1.6

In this section we prove each of the theorems from the introduction in turn. Each proof relies on the asymptotic equidistribution result concluded in Theorem III.3.1.

III.4.1 Proof of Theorem III.1.1

In accordance with (III.3.1), we have

$$\sum_{n\geq 0} N(a,b;n)q^n = \frac{1}{b}\sum_{n\geq 0} p(n)q^n + \frac{1}{b}\sum_{j=1}^{b-1}\zeta_b^{-aj}R\left(\zeta_b^j;q\right),$$

where

$$R(\zeta;q) \coloneqq \sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}} N(m,n)\zeta^m q^n.$$

To conclude the asymptotic equidistribution in the framework presented here, one needs only check that the conditions of Theorem III.3.1 apply. Since the asymptotics of $(q;q)_\infty^{-1}$ follow from (III.2.1) and satisfy the required properties on both the major and minor arcs, one simply needs to show that

$$R\left(\zeta_b^j;q\right) = o\left((q;q)_\infty^{-1}\right), \qquad \left|R\left(\zeta_b^j;q\right)\right| < \left|(q;q)_\infty^{-1}\right|,$$

on the major arc and minor arcs respectively. In fact, in [Mal21b] it was shown that as $z \to 0$ with positive real part we have $R(\zeta_b^j;q) \to 0$. Thus clearly each inequality is satisfied, the assumptions of Theorem III.3.1 (and Corollary III.3.4) apply, and we conclude the result.

III.4.2 Proof of Theorem III.1.2

Let $M(m,n)$ denote the number of partitions of n with crank m. In accordance with (III.3.1), we have (see e.g., [Mah05, equation (3.2)])

$$\sum_{n\geq 0} M(a,b;n)q^n = \frac{1}{b}\sum_{n\geq 0} p(n)q^n + \frac{1}{b}\sum_{j=1}^{b-1}\zeta_b^{-aj} C\left(\zeta_b^j;q\right),$$

where

$$C\left(\zeta;q\right) \coloneqq \sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}} M(m,n)\zeta^m q^n = \frac{(q;q)_\infty}{F_1\left(\zeta;q\right)F_1\left(\zeta^{-1};q\right)}.$$

We have from Theorem III.2.2, as $z\to 0$ in D_θ (so on the major arc), for $q=e^{-z}$ and ζ a b-th root of unity not equal to 1, that

$$F_1\left(\zeta;e^{-z}\right) = \frac{1}{\sqrt{1-\zeta}}\, e^{-\frac{\zeta\Phi(\zeta,2,1)}{z}}\left(1+O\left(|z|\right)\right).$$

Equation (III.2.1) implies that on the major arc we have

$$\left(e^{-z};e^{-z}\right)_\infty^{-1} = \sqrt{\frac{z}{2\pi}}e^{\frac{\pi^2}{6z}}(1+O(|z|)),$$

while Lemma III.2.1 gives us

$$\left|\left(e^{-z};e^{-z}\right)_\infty^{-1}\right| \leq \sqrt{x}e^{\frac{\pi^2}{6x}-\frac{\mathcal{C}}{x}},$$

for some $\mathcal{C}>0$ on the minor arc.

Moreover, one may conclude in a similar way to [BCMO22, Proof of Theorem 1.4 (2)] that

$$\left|C\left(\zeta_b^j;q\right)\right| < \left|(q;q)_\infty^{-1}\right|$$

on the minor arcs. For the major arcs we obtain that

$$C\left(\zeta_b^j;q\right) \ll_{j,b} e^{-\frac{\pi^2}{6}\operatorname{Re}\left(\frac{1}{z}\right)+\operatorname{Re}\left(\frac{\zeta\Phi(\zeta,2,1)}{z}\right)+\operatorname{Re}\left(\frac{\zeta^{-1}\Phi\left(\zeta^{-1},2,1\right)}{z}\right)},$$

using the asymptotics of F_1 and the Pochhammer symbol. This gives us that

$$C\left(\zeta_b^j;q\right) = o\left((q;q)_\infty^{-1}\right)$$

holds if and only if

$$\left(\frac{\pi^2}{3} - \varepsilon - \phi_1 - \phi_1^*\right) \frac{x}{|z|^2} > (\phi_2 + \phi_2^*) \frac{y}{|z|^2},$$

for $\phi_1 + i\phi_2 \coloneqq \zeta_b^j \Phi(\zeta_b^j, 2, 1)$ and $\phi_1^* + i\phi_2^* \coloneqq \zeta_b^{-j}\Phi(\zeta_b^{-j}, 2, 1)$. A straightforward calculation shows that

$$\zeta_b^{\pm j} \Phi\left(\zeta_b^{\pm j}, 2, 1\right) = \sum_{n\geq 1} \frac{\cos\left(\frac{2\pi nj}{b}\right)}{n^2} \pm i \sum_{n\geq 1} \frac{\sin\left(\frac{2\pi nj}{b}\right)}{n^2}.$$

Using that $\sum_{n\geq 1} \frac{\cos(n\theta)}{n^2} = \frac{\pi^2}{6} - \frac{\theta(2\pi-\theta)}{4}$ for some $0 \leq \theta \leq 2\pi$ (see [Zag88, page 238]) then gives that $\phi_1 = \frac{\pi^2}{6} - \frac{\pi^2 j}{b}(1 - \frac{j}{b}) = \phi_1^*$ and $\phi_2 = -\phi_2^*$. Therefore, our assumption reduces to

$$\left(\frac{2\pi^2 j}{b}\left(1 - \frac{j}{b}\right) - \varepsilon\right) \frac{x}{|z|^2} > 0,$$

which holds, since we have $b > 0$, $1 \leq j \leq b-1$ and $x = \mathrm{Re}(z) > 0$. Thus all the assumptions of Theorem III.3.1 apply, and we conclude the result.

III.4.3 Proof of Theorem III.1.3

In [BLO09, equation (2.1)] it was shown that the generating function of the number of overpartitions of n with residual crank m, denoted by $\overline{M}(m,n)$, is

$$C(\zeta; q) \coloneqq \sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}} \overline{M}(m,n)\zeta^m q^n = \frac{\left(q^2;q^2\right)_\infty}{F_1(\zeta;q)F_1(\zeta^{-1};q)}.$$

We thus have

$$C(a,b;q) \coloneqq \sum_{n\geq 0} \overline{M}(a,b;n)q^n = \frac{1}{b}\sum_{j=0}^{b-1} \zeta_b^{-aj} C\left(\zeta_b^j; q\right).$$

By a similar argument as before, the asymptotic behavior toward $z = 0$ is dominated by the $j = 0$ term on both the major and minor arcs. If $j = 0$, we have

$$\frac{(q^2;q^2)_\infty}{(q;q)_\infty^2}.$$

Using (III.2.1) and standard arguments, this is seen to satisfy

$$\frac{(q^2;q^2)_\infty}{(q;q)_\infty^2} = \frac{\sqrt{z}}{2\sqrt{\pi}} e^{\frac{\pi^2}{4z}} \left(1 + O\left(|z|\right)\right)$$

on the major arc and, for some $\mathcal{C}' > 0$,

$$\left|\frac{(q^2;q^2)_\infty}{(q;q)_\infty^2}\right| \ll \sqrt{x}e^{\frac{\pi^2}{4z}-\frac{\mathcal{C}'}{x}}$$

on the minor arc. This means that the conditions of Proposition III.2.3 are satisfied here with $B = \frac{1}{2}$, $A = \frac{\pi^2}{4}$, and $\alpha_0 = \frac{1}{2\sqrt{\pi}}$. Thus applying Theorem III.3.1 yields the claimed result.

III.4.4 Proof of Theorem III.1.4

Let $\mathrm{pp}(m,n)$ be the number of plane partitions of n with trace m. We have MacMahon's classical generating function [Mac04]

$$\sum_{n\geq 0} \mathrm{pp}(n)q^n = \prod_{n\geq 1} \frac{1}{(1-q^n)^n}$$

and the trace generating function [Sta73]

$$\mathrm{PP}(\zeta;q) := \sum_{n,m\geq 0} \mathrm{pp}(m,n)\zeta^m q^n = \prod_{n\geq 1} \frac{1}{(1-\zeta q^n)^n}.$$

Following the strategy of [BCMO22] we have, for $q = e^{-z}$ and ζ a b-th root of unity, that

$$\begin{aligned}\mathrm{Log}(\mathrm{PP}(\zeta;q)) &= -\sum_{n\geq 1} n \,\mathrm{Log}\,(1-\zeta q^n) = \sum_{n\geq 1} n \sum_{m\geq 1} \frac{\zeta^m q^{nm}}{m} \\ &= \sum_{m\geq 1} \zeta^m \,\frac{q^m}{m(1-q^m)^2} = z\sum_{m\geq 1} \zeta^m \,\frac{q^m}{mz(1-q^m)^2}.\end{aligned}$$

Recall that the generating function for the Bernoulli numbers B_n is given by (see e.g., [BFOR17])

$$\sum_{n\geq 0} \frac{B_n}{n!} z^n = \frac{z}{e^z-1} = \frac{ze^{-z}}{1-e^{-z}}.$$

Defining $B(z) := \frac{1}{z}\sum_{n\geq 0} \frac{B_n}{n!} z^n = \frac{e^{-z}}{1-e^{-z}}$, differentiating gives us that $B'(z) = -\frac{e^{-z}}{(1-e^{-z})^2}$ and yields the identity

$$\frac{-B'(mz)}{mz} = \frac{e^{-mz}}{mz\left(1-e^{-mz}\right)^2}.$$

Therefore, if we set $F(z) := \frac{-B'(z)}{z}$, we obtain

$$\mathrm{Log}\left(\mathrm{PP}\left(\zeta; e^{-z}\right)\right) = z \sum_{m \geq 1} \zeta^m F(mz).$$

For $\zeta = \zeta_b^a := e^{\frac{2\pi i a}{b}}$ a b-th root of unity not equal to 1 and by substituting $m \mapsto bm + j$ for $m \in \mathbb{N}_0$, $1 \leq j \leq b$ this yields

$$\mathrm{Log}\left(\mathrm{PP}\left(\zeta_b^a; e^{-z}\right)\right) = z \sum_{j=1}^{b} \zeta_b^{aj} \sum_{m \geq 0} F\left(\left(m + \frac{j}{b}\right) bz\right). \tag{III.4.1}$$

We turn to evaluating the inner sum. We note that $F(z)$ has the Laurent expansion

$$F(z) = \frac{-B'(z)}{z} = -\sum_{n \geq -3} \frac{(n+2)B_{n+3}}{(n+3)!} z^n.$$

By Euler–Maclaurin summation we have for $c_n := \frac{-(n+2)B_{n+3}}{(n+3)!}$ the identity (see e.g., [BCMO22, Lemma 2.2])

$$\sum_{m \geq 0} F\left(\left(m + \frac{j}{b}\right) bz\right) \tag{III.4.2}$$
$$\sim \frac{\zeta\left(3, \frac{j}{b}\right)}{b^3 z^3} + \frac{I_{F,1}^*}{bz} + \frac{1}{12bz}\left[\mathrm{Log}(bz) + \psi\left(\frac{j}{b}\right) + \gamma\right] - \sum_{n \geq 0} c_n \frac{B_{n+1}\left(\frac{j}{b}\right)}{n+1} b^n z^n$$

as $z \to 0$ in D_θ. Here $\zeta(s, z) := \sum_{n \geq 0} \frac{1}{(n+z)^s}$ is the *Hurwitz zeta function*, $\psi(x) := \frac{\Gamma'(x)}{\Gamma(x)}$ is the *digamma function*, γ is the *Euler–Mascheroni constant*, $B_n(x)$ denotes the *n-th Bernoulli polynomial* defined via its generating function $\frac{te^{xt}}{e^t - 1} = \sum_{n \geq 0} B_n(x) \frac{t^n}{n!}$, and for some $A \in \mathbb{R}^+$ we define

$$I_{F,A}^* := \int_0^\infty \left(F(u) - \sum_{n=n_0}^{-2} c_n u^n - \frac{c_{-1} e^{-Au}}{u}\right) du.$$

Here and throughout we say that

$$f(z) \sim \sum_{n \geq 0} a_n z^n,$$

if $f(z) = \sum_{n=0}^{N} a_n z^n + O(|z|^{N+1})$, for any $N \in \mathbb{N}_0$. Applying (III.4.2) to (III.4.1) and using that $\sum_{j=1}^{b} \zeta_b^{aj} = 0$, we obtain

$$\begin{aligned}
&\operatorname{Log}\left(\mathrm{PP}\left(\zeta_b^a; e^{-z}\right)\right)\\
&\quad \sim z\sum_{j=1}^{b}\zeta_b^{aj}\left[\frac{\zeta\left(3,\frac{j}{b}\right)}{b^3z^3}+\frac{I_{F,1}^*}{bz}+\frac{1}{12bz}\left[\operatorname{Log}(bz)+\psi\left(\frac{j}{b}\right)+\gamma\right]-\sum_{n\geq 0}c_n\frac{B_{n+1}\left(\frac{j}{b}\right)}{n+1}b^nz^n\right]\\
&\quad = \frac{1}{b^3z^2}\sum_{j=1}^{b}\zeta_b^{aj}\zeta\left(3,\frac{j}{b}\right)+\frac{1}{12b}\sum_{j=1}^{b}\zeta_b^{aj}\psi\left(\frac{j}{b}\right)+O(|z|).
\end{aligned}$$

We have the well-known identity (see e.g., [BCMO22, equation (2.6)])

$$\sum_{j=1}^{b}\zeta_b^{aj}\psi\left(\frac{j}{b}\right)=b\operatorname{Log}\left(1-\zeta_b^a\right)$$

and by elementary manipulations we furthermore obtain

$$\sum_{j=1}^{b-1}\zeta_b^{aj}\zeta\left(3,\frac{j}{b}\right)=\sum_{j=1}^{b-1}\zeta_b^{aj}\sum_{n\geq 0}\frac{b^3}{(bn+j)^3}=b^3\operatorname{Li}_3\left(\zeta_b^a\right),$$

where $\operatorname{Li}_3(z)=\sum_{k\geq 1}\frac{z^k}{k^3}$ is the *third polylogarithm function*. Therefore on the major arc, we conclude by exponentiating that, for $\zeta_b^a\neq 1$, we have

$$\mathrm{PP}\left(\zeta_b^a; e^{-z}\right)=\left(1-\zeta_b^a\right)^{\frac{1}{12}}e^{\frac{\operatorname{Li}_3\left(\zeta_b^a\right)}{z^2}}\left(1+O(|z|)\right)$$

and otherwise by [Wri31]

$$\mathrm{PP}\left(1; e^{-z}\right)=z^{\frac{1}{12}}e^{\frac{\zeta(3)}{z^2}-\kappa}\left(1+O(|z|)\right),$$

where $\kappa=\zeta'(-1)<0$. An analogous argument to the one of (III.3.1) yields that $\mathrm{PP}(a,b;q)$ and $\mathrm{PP}(\zeta;q)$ are analytic such that

$$\mathrm{PP}(a,b;q)=\frac{1}{b}\sum_{j=0}^{b-1}\zeta_b^{-aj}\,\mathrm{PP}\left(\zeta_b^j;q\right).$$

Comparing exponents, we see that $\mathrm{PP}(\zeta_b^a;e^{-z})=o(\mathrm{PP}(1;e^{-z}))$, and therefore the second hypothesis of Theorem III.3.1 is true for $\mathrm{PP}(\zeta;q)$.

We now consider $\mathrm{PP}(\zeta;q)$ on the minor arc. By definition, we have

$$\frac{\mathrm{PP}\left(\zeta_b^a;e^{-z}\right)}{\mathrm{PP}\left(1;e^{-z}\right)}=\prod_{n\geq 1}\left(\frac{1-e^{-nz}}{1-\zeta_b^ae^{-nz}}\right)^n.$$

As $z \to 0$ with $\mathrm{Re}(z) > 0$, we see that $1 - e^{-nz} \to 0$ while $1 - \zeta_b^a e^{-nz} \not\to 0$. Thus, for all z on the minor arc with $|z|$ sufficiently small, we see that $|\frac{\mathrm{PP}(\zeta_b^a;e^{-z})}{\mathrm{PP}(1;e^{-z})}| < 1$. This proves that the first hypothesis of Theorem III.3.1 holds for $\mathrm{PP}(\zeta;q)$. The third condition of Theorem III.3.1 follows by noting that the integral of $\mathrm{PP}(1;q)$ along the major arc gives Wright's asymptotic (III.1.5), and so equidistribution follows by Theorem III.3.1.

III.4.5 Proof of Theorem III.1.5

For X a Hilbert scheme, letting

$$G_X(T;q) \coloneqq \sum_{n\geq 0} P(X;T)q^n,$$

a standard argument with orthogonality of roots of unity yields

$$\sum_{n\geq 0} B(a,b;X)q^n = \frac{1}{b}\sum_{r=0}^{b-1} \zeta_b^{-ar} G_X\left(\zeta_b^r;q\right). \tag{III.4.3}$$

The main result of Boccalini's thesis [Boc16, equation (4.1)] states that

$$G_{X_1}(\zeta;q) = \sum_{n\geq 0} P\left(\mathrm{Hilb}^{n,n+1,n+2}(0);\zeta\right)q^n = \frac{1+\zeta^2}{(1-\zeta^2 q)(1-\zeta^4 q^2)} F_3\left(\zeta^2;q\right)^{-1}.$$

By (III.4.3) we have that

$$\begin{aligned} H_{X_1}(a,b;q) \coloneqq& \sum_{n\geq 0} B(a,b;X_1)q^n \\ =& \frac{1}{b}\left(1+(-1)^a\delta_{2|b}\right)G_{X_1}(1;q) + \frac{1}{b}\sum_{\substack{0<r\leq b-1\\ r\neq \frac{b}{2}}} \zeta_b^{-ar} G_{X_1}\left(\zeta_b^r;q\right). \end{aligned}$$

Since

$$\begin{aligned} G_{X_1}(1;e^{-z}) =& \frac{2}{(1-e^{-z})(1-e^{-2z})}(e^{-z};e^{-z})_\infty^{-1} \\ =& (e^{-z};e^{-z})_\infty^{-1}\left(\frac{1}{z^2} + \frac{3}{2z} + \frac{11}{12} + O(z)\right), \end{aligned}$$

the asymptotic behavior is essentially controlled by the Pochhammer symbol. It is then enough to show that on the major and minor arcs, $G_{X_1}(\zeta_b^r;q) = o(G_{X_1}(1;q))$ for $\zeta_b^r \neq 1$. This follows directly from the asymptotics of F_3 given in Theorem III.2.2 in a similar fashion to [BCMO22, Theorem 1.4 (1)] for the major arc[5], and a similar calculation to the arguments of [BCMO22] for the minor arc. Thus toward $z = 0$ on the major arc we have

$$H_{X_1}(a, b; e^{-z}) = \frac{d(a, b)}{\sqrt{2\pi} z^{\frac{3}{2}}} e^{\frac{\pi^2}{6z}} (1 + O(|z|)).$$

We are left to apply Proposition III.2.3 with $A = \frac{\pi^2}{6}, B = -\frac{3}{2}$, and $\alpha_0 = \frac{d(a,b)}{\sqrt{2\pi}}$ which yields that

$$B(a, b; X_1) = \frac{\sqrt{3} d(a, b)}{2\pi^2} e^{\pi\sqrt{\frac{2n}{3}}} \left(1 + O\left(n^{-\frac{1}{2}}\right)\right),$$

from which one may also conclude asymptotic equidistribution.

Similarly, it was shown in [Boc16, equation (4.2)] that we have

$$G_{X_2}(\zeta; q) \coloneqq \sum_{n\geq 0} P\left(\mathrm{Hilb}^{n,n+2}(0); \zeta\right) q^n = \frac{1 + \zeta^2 - \zeta^2 q}{(1 - \zeta^2 q)(1 - \zeta^4 q^2)} F_3\left(\zeta^2; q\right)^{-1}.$$

An analogous argument and application of Proposition III.2.3 to the case of X_1 holds. Using the generating functions [Boc16, equation (4.15)] (which in turn cites [NY11a, Corollary 5.4]) and [NY11b, Corollary 5.4]

$$G_{X_3}(\zeta; q) \coloneqq \frac{1}{(1 - \zeta^2 q)(1 - \zeta^4 q^2)} F_3\left(\zeta^2; q\right)^{-1},$$
$$G_{X_4}(\zeta; q) \coloneqq F_3\left(\zeta^2; q\right)^{-1} \prod_{j=1}^{m} \frac{1}{1 - \zeta^{2j} q^j},$$

the cases for X_3 and X_4 follow in the same way.

III.4.6 Proof of Theorem III.1.6

The results [ES87, Proposition 4.2] and [Goe90, Proposition 2.8] show that $V_{n,k}$ has a cell decomposition and that, letting $v(m, n) \coloneqq \#\{m\text{-dimensional cells of } V_{n,k}\}$, we have

$$V(\zeta; q) \coloneqq \sum_{m,n\geq 0} v(m, n)\zeta^m q^n = \prod_{n\geq 1} \frac{1}{1 - \zeta^{n-1} q^n} = F_3(\zeta; q)^{-1}.$$

[5] Note that for $\gcd(r, b) > 1$ we could simply generalize the result of Theorem III.2.2 (2) by replacing b by $\frac{b}{\gcd(b,r)}$.

Then by orthogonality of roots of unity, we have

$$\sum_{n\geq 0} v(a,b;n)q^n = \frac{1}{b}\sum_{j=0}^{b-1} \zeta_b^{-aj} V\left(\zeta_b^j; q\right).$$

Note that the $j=0$ term corresponds to $\frac{1}{b}(q;q)_\infty^{-1}$. Combining this with Theorem III.2.2 (2), one can show in the same way as [BCMO22, Theorem 1.4 (1)] that on both the major and minor arcs, the asymptotic behavior of the $j=0$ term dominates as $z \to 0$. An application of Corollary III.3.4 immediately yields the claimed asymptotic.

III.5 References

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Chapter IV

Fourier coefficients of weight zero mixed false modular forms

This chapter is based on a prerpint of the same title submitted for publication [Ces23].

IV.1 Introduction and statement of results

In [BN19] Bringmann and Nazaroglu embedded *false theta functions*, functions that resemble theta functions but do not have modular transformation properties, into a modular framework. An example is given by

$$\psi(z;\tau) := i \sum_{n\in\mathbb{Z}} \operatorname{sgn}\left(n+\frac{1}{2}\right)(-1)^n q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} \zeta^{n+\frac{1}{2}},$$

where here and throughout $\zeta := e^{2\pi i z}$ for $z \in \mathbb{C}$, $q := e^{2\pi i\tau}$, with $\tau \in \mathbb{H}$, and

$$\operatorname{sgn}(n) := \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{if } n > 0, \\ -1 & \text{if } n < 0, \end{cases}$$

as usual. They found the modular completion[1] of those false theta functions, with $w \in \mathbb{H}$, given by (see [BN19, equation (1.2)])

$$\widehat{\psi}(z;\tau,w) := i \sum_{n\in\mathbb{Z}} \operatorname{erf}\left(-i\sqrt{\pi i(w-\tau)}\left(n+\frac{1}{2}+\frac{\operatorname{Im}(z)}{\operatorname{Im}(\tau)}\right)\right)(-1)^n q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} \zeta^{n+\frac{1}{2}},$$

and repaired the modular invariance, where $\operatorname{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$ denotes the *error function* and where $\widehat{\psi}$ satisfies (see [BN19, equation (1.3)])

$$\lim_{t\to\infty} \widehat{\psi}(z;\tau,\tau+it+\varepsilon) = \psi(z;\tau) \tag{IV.1.1}$$

[1] These are modular objects from which the original function can be easily recovered, here for example by taking the limit (see (IV.1.1)).

if $-\frac{1}{2} < \frac{\mathrm{Im}(z)}{\mathrm{Im}(\tau)} < \frac{1}{2}$ and $\varepsilon > 0$ arbitrary. Note that here and in the following we define the square root on a cut-plane excluding the negative reals and imposing positive square roots for positive real numbers.

As an application from this framework they considered the false theta functions at rank one (see [BN19, equation (1.6)])

$$F_{j,N}(\tau) \coloneqq \sum_{\substack{n\in\mathbb{Z} \\ n\equiv j \pmod{2N}}} \operatorname{sgn}(n)\, q^{\frac{n^2}{4N}},$$

with $j \in \mathbb{Z}$ and $N \in \mathbb{N}_{>1}$ and showed how the quantum modularity[2] of these functions follows from the construction of their completions.

The motiviation for looking at this functions comes from W-algebraic characters, see for example [BM15, CM14, CMW17, Mil14]. Characters of modules of rational vertex operator algebras are often of the form

$$\frac{f(\tau)}{\eta(\tau)^k},$$

where $\eta(\tau) \coloneqq q^{\frac{1}{24}} \prod_{n\geq 1}(1-q^n)$ is *Dedekind's eta function.* In [CM14] the authors observed that some numerators of atypical characters of the so-called $(1,p)$-singlet algebra are false theta functions of Rogers (see [AB09]). In particular, the functions

$$\mathcal{A}_{j,N}(\tau) \coloneqq \frac{F_{j,N}(\tau)}{\eta(\tau)}$$

show up as characters of the atypical irreducible modules of the $(1,p)$-singlet vertex operator algebra $M_{1,s}$, for $1 \leq s \leq p-1$ and $p \in \mathbb{N}_{\geq 2}$, that have been studied in [BM15, CM14, CMW17].

In 1937 Rademacher [Rad37] proved the following exact formula for the partition function

$$p(n) = \frac{2\pi}{(24n-1)^{\frac{3}{4}}} \sum_{k\geq 1} \frac{A_k(n)}{k} I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right),$$

where $A_k(n)$ is a Kloosterman sum given by

$$A_k(n) \coloneqq \sum_{\substack{h \pmod{k} \\ \gcd(h,k)=1}} e^{\pi i s(h,k)} e^{-2\pi i n \frac{h}{k}}, \tag{IV.1.2}$$

[2]For a so-called quantum set $\mathcal{Q} \subset \mathbb{Q}$ we call a function $f : \mathcal{Q} \to \mathbb{C}$ *quantum modular form of weight* k, if its obstruction to modularity, namely $f(\tau) - (c\tau + d)^{-k} f(M\tau)$, for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \subset \mathrm{SL}_2(\mathbb{Z})$, behaves "nice" in some analytical sense. See e.g., [Zag10] for more background on quantum modular forms.

with $s(h,k)$ the Dedekind sum defined in (IV.2.4) and where I_α denotes the I-Bessel function of order α, which in the special case of order $\frac{3}{2}$ can be written as

$$I_{\frac{3}{2}}(z) = \sqrt{\frac{2z}{\pi}}\frac{d}{dz}\left(\frac{\sinh(z)}{z}\right). \tag{IV.1.3}$$

Our goal is to find Rademacher-type exact formulae for the Fourier coefficients of the infinite family of weight zero *mixed false modular forms*[3] $\mathcal{A}_{j,N}(\tau)$. Note that it requires considerably more work to obtain an exact formula for a weight zero function than for a function of negative weight. In contrast to negative weight functions, as for example in the work of Bringmann and Nazaroglu [BN19], we have to take special care of the bound of the Kloosterman sum occuring to ensure that the error term in the Circle Method vanishes. In comparison to [Rad38] for example, where Rademacher studied the coefficients of the modular invariant $j(\tau)$ of weight zero, we have the additional problems that the Kloosterman sum showing up in our work is much more complicated and can not be immediately bounded by the famous Weil bound and that the transformation behavior of our family of functions is not as simple as the one of a modular form.

In this chapter we let

$$\mathcal{A}_{j,N}(\tau) =: q^{\frac{j^2}{4N}-\frac{1}{24}}\left(a_{j,N}(0) + \sum_{n\geq 1} a_{j,N}(n)q^n\right). \tag{IV.1.4}$$

Extending the techniques presented in [BN19, Section 3] and [Rad38] we prove the following theorem, which, to the best of the author's knowledge, is the first example of an exact formula of a weight zero mixed false modular form.

Theorem IV.1.1. *For all $n \geq 1$ and $\sqrt{\frac{N}{6}} \notin \mathbb{Z}$ we have*

$$\begin{aligned} a_{j,N}(n) = &-\frac{2\pi i}{\sqrt{n+\frac{j^2}{4N}-\frac{1}{24}}}\sum_{k\geq 1}\sum_{r=1}^{N-1}\sum_{\kappa=0}^{k-1}\frac{K_{k,j,N}(n,r,\kappa)}{k^2} \\ &\times \mathrm{P.V.}\int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}}\sqrt{\frac{1}{24}-Nx^2}\cot\left(\pi\left(-\frac{x}{k}+\frac{\kappa}{k}+\frac{r}{2Nk}\right)\right)I_1\left(\frac{4\pi\sqrt{n+\frac{j^2}{4N}-\frac{1}{24}}}{k}\sqrt{\frac{1}{24}-Nx^2}\right)dx, \end{aligned} \tag{IV.1.5}$$

where $K_{k,j,N}(n,r,\kappa)$ is a Kloosterman sum defined as

$$K_{k,j,N}(n,r,\kappa) := \sum_{\substack{0\leq h<k \\ \gcd(h,k)=1}} \chi_{j,r}(N,M_{h,k})\zeta_{24k}^{\left(24N\left(\kappa+\frac{r}{2N}\right)^2-1\right)h'-24\left(n+\frac{j^2}{4N}-\frac{1}{24}\right)h}, \tag{IV.1.6}$$

[3]These are in general linear combinations of false theta functions multiplied by modular forms (see [Bri21, Section 4]).

with h' *a solution of* $hh' \equiv -1 \pmod{k}$, $M_{h,k} = \begin{pmatrix} h' & -\frac{hh'+1}{k} \\ k & -h \end{pmatrix}$, $\chi_{j,r}(N,M)$ *the multiplier defined in* (IV.2.6), *and* $\zeta_\ell := e^{\frac{2\pi i}{\ell}}$ *with* $\ell \in \mathbb{N}$ *an* ℓ*-th root of unity.*

Remarks.
(1) Although this representation as a convergent series does not hold for $n = 0$ we obtain that $a_{j,N}(0) = 1$, independent of j and N.
(2) Note that we are able to split the principal value integral in (IV.1.5), which gives us a more explicit but also more complicated version of our main result, as can be seen in (IV.4.26).

As a second result, which will be extremely helpful in the proof of Theorem IV.1.1, we are able to give a bound on the Kloosterman sum defined in (IV.1.6). In particular, we show the following theorem.

Theorem IV.1.2. *For* $\varepsilon > 0$ *we have that*

$$K_{k,j,N}(n,r,\kappa) = O_N\left(nk^{\frac{1}{2}+\varepsilon}\right) \tag{IV.1.7}$$

as $k \to \infty$*.*

As the main tool to prove this theorem we use the following result by Malishev.

Lemma IV.1.3. (see [KS64, page 482]) *Let*

$$K_\rho(\mu_*,\nu_*;G) := \sum_{\substack{h \pmod{G} \\ \gcd(h,G)=1}} \left(\frac{h}{\rho}\right) \exp\left(\frac{2\pi i}{G}(\mu_* h + \nu_* h')\right),$$

where μ_* *and* ν_* *are integers,* G *is a positive integer, and* ρ *is an odd positive integer all of whose prime factors devide* G*. Furthermore* h' *is any integral solution of the congruence* $hh' \equiv 1 \pmod{G}$ *and* $\left(\frac{h}{\rho}\right)$ *is the Jacobi symbol. Then*

$$|K_\rho(\mu_*,\nu_*;G)| \leq A(\varepsilon) G^{\frac{1}{2}+\varepsilon} \min\left(\gcd(\mu_*,G)^{\frac{1}{2}}, \gcd(\nu_*,G)^{\frac{1}{2}}\right)$$

for each $\varepsilon > 0$*, where* $A(\varepsilon) > 0$ *depends only on* ε*.*

The chapter is structured as follows. In Section IV.2 we use the modular completion of $F_{j,N}$ and its modular transformation behavior to determine the "false" modular behavior of $\mathcal{A}_{j,N}$. With this we rewrite the obstruction to modularity term, respectively the error of modularity plus the holomorphic part of our function, as a Mordell-type integral. In Section IV.3 we go on by proving Theorem IV.1.2 and use the Circle Method, to prove Theorem IV.1.1 in Section IV.4. We end the chapter with some numerical results in Section IV.5.

IV.2 Modular Transformations with Mordell-type Integrals

IV.2.1 Modular transformations

We first note that a simple straight-forward calculation shows that

$$F_{0,N}(\tau) = 0$$

for every $\tau \in \mathbb{H}$. Thus we from now on assume that $j \neq 0$. Furthermore we can restrict to $1 \leq j \leq N-1$, since $F_{j,N}(\tau) = -F_{-j,N}(\tau)$ and $F_{j\pm 2N,N}(\tau) = F_{j,N}(\tau)$.

According to [BN19, Section 4] a modular completion of $F_{j,N}$ can be written as (for $\tau, w \in \mathbb{H}$)

$$\widehat{F}_{j,N}(\tau, w) \coloneqq \sum_{\substack{n\in\mathbb{Z} \\ n\equiv j \pmod{2N}}} \operatorname{erf}\left(-i\sqrt{\pi i(w-\tau)}\frac{n}{\sqrt{2N}}\right) q^{\frac{n^2}{4N}}.$$

This modular completion can conveniently be rewritten as (see [BN19, equation (4.2)])

$$\widehat{F}_{j,N}(\tau, w) = \pm F_{j,N}(\tau) - \sqrt{2N}\int_w^{\tau+i\infty\pm\varepsilon} \frac{f_{j,N}(\mathfrak{z})}{\sqrt{i(\mathfrak{z}-\tau)}}d\mathfrak{z}, \qquad \text{(IV.2.1)}$$

where $\varepsilon > 0$ and $f_{j,N}$ are the vector-valued cusp forms of weight $\frac{3}{2}$

$$f_{j,N}(\tau) \coloneqq \frac{1}{2N}\sum_{\substack{n\in\mathbb{Z} \\ n\equiv j \pmod{2N}}} n\, q^{\frac{n^2}{4N}} = \sum_{n\in\mathbb{Z}}\left(n + \frac{j}{2N}\right) q^{N\left(n+\frac{j}{2N}\right)^2}.$$

Equation (IV.2.1) can also be understood from the writing of $F_{j,N}(\tau)$ as a holomorphic Eichler integral[4].

The modular transformations of $F_{j,N}$ can be deduced from [BN19, equation (4.5)]

$$\widehat{F}_{j,N}\left(\frac{a\tau+b}{c\tau+d}, \frac{aw+b}{cw+d}\right) = \chi_{\tau,w}(M)\,(c\tau+d)^{\frac{1}{2}}\sum_{r=1}^{N-1}\psi_{j,r}(N,M)\,\widehat{F}_{r,N}(\tau,w), \qquad \text{(IV.2.2)}$$

where $M \coloneqq \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$,

$$\chi_{\tau,w}(M) \coloneqq \sqrt{\frac{i(w-\tau)}{(c\tau+d)(cw+d)}}\frac{\sqrt{c\tau+d}\sqrt{cw+d}}{\sqrt{i(w-\tau)}},$$

[4]For a cusp form f of weight $k \in 2\mathbb{N}$, Eichler introduced in [Eic57] the integral

$$\int_{\tau_0}^{\tau} (\tau-\mathfrak{z})^{k-2} f(\mathfrak{z})d\mathfrak{z},$$

which is independent of the path of integration. Integrals of this shape are now called Eichler integrals.

and, for $j, r \in \{1, \dots, N-1\}$,

$$\psi_{j,r}(N,M) := \begin{cases} e^{2\pi i ab \frac{j^2}{4N}} e^{-\frac{\pi i}{4}(1-\operatorname{sgn}(d))} \delta_{j,r} & \text{if } c = 0, \\ e^{-\frac{3\pi i}{4}\operatorname{sgn}(c)} \sqrt{\frac{2}{N|c|}} \sum_{\ell=0}^{|c|-1} e^{\frac{\pi i}{2Nc}\left(a(2N\ell+j)^2+dr^2\right)} \sin\left(\frac{\pi r(2N\ell+j)}{N|c|}\right) & \text{if } c \neq 0. \end{cases}$$

For reference let us also note the modular transformation of the eta function given by

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \nu_\eta(M)\,(c\tau+d)^{\frac{1}{2}}\,\eta(\tau), \tag{IV.2.3}$$

where, for $c > 0$, we have

$$\nu_\eta(M) := \exp\left(\pi i\left(\frac{a+d}{12c} - \frac{1}{4} + s(-d,c)\right)\right),$$

with the *Dedekind sum* given by

$$s(h,k) := \sum_{r=1}^{k-1} \frac{r}{k}\left(\frac{hr}{k} - \left\lfloor \frac{hr}{k} \right\rfloor - \frac{1}{2}\right). \tag{IV.2.4}$$

Remark. For $M := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ an alternate representation of the eta-multiplier is given by (see e.g., [BJSM18, Lemma 2.1])

$$\nu_\eta(M) = \begin{cases} \left(\frac{d}{|c|}\right) e^{\frac{\pi i}{12}\left((a+d)c - bd\left(c^2-1\right) - 3c\right)} & \text{if } c \text{ is odd}, \\ \left(\frac{c}{d}\right) e^{\frac{\pi i}{12}\left(ac\left(1-d^2\right) + d(b-c+3) - 3\right)} & \text{if } c \text{ is even}, \end{cases}$$

where $\left(\frac{\cdot}{\cdot}\right)$ is the extended Legendre symbol, also known as Kronecker symbol.

By combining (IV.2.1) and (IV.2.2) we can write the modular transformation of $F_{j,N}$ as

$$\begin{aligned} &F_{j,N}(\tau) - \chi_{\tau,w}(M)(c\tau+d)^{-\frac{1}{2}} \sum_{r=1}^{N-1} \psi_{j,r}\left(N, M^{-1}\right) F_{r,N}\left(\frac{a\tau+b}{c\tau+d}\right) \\ =&\sqrt{2N} \int_w^{\tau+i\infty+\varepsilon} \frac{f_{j,N}(\mathfrak{z})}{\sqrt{i(\mathfrak{z}-\tau)}} d\mathfrak{z} \\ &- \sqrt{2N}\chi_{\tau,w}(M)(c\tau+d)^{-\frac{1}{2}} \sum_{r=1}^{N-1} \psi_{j,r}\left(N, M^{-1}\right) \int_{\frac{aw+b}{cw+d}}^{\frac{a\tau+b}{c\tau+d}+i\infty+\varepsilon} \frac{f_{r,N}(\mathfrak{z})}{\sqrt{i\left(\mathfrak{z} - \frac{a\tau+b}{c\tau+d}\right)}} d\mathfrak{z}. \end{aligned}$$

Assuming $c>0$ and taking $w\to\tau+i\infty+\varepsilon$ we get $\chi_{\tau,w}\to 1$ and hence

$$\begin{aligned}F_{j,N}(\tau)-(c\tau+d)^{-\frac{1}{2}}\sum_{r=1}^{N-1}\psi_{j,r}\left(N,M^{-1}\right)F_{r,N}\left(\frac{a\tau+b}{c\tau+d}\right)\\ =-\sqrt{2N}(c\tau+d)^{-\frac{1}{2}}\sum_{r=1}^{N-1}\psi_{j,r}\left(N,M^{-1}\right)\int_{\frac{a}{c}}^{\frac{a\tau+b}{c\tau+d}+i\infty+\varepsilon}\frac{f_{r,N}(\mathfrak{z})}{\sqrt{i(\mathfrak{z}-\frac{a\tau+b}{c\tau+d})}}d\mathfrak{z}.\end{aligned}\qquad\text{(IV.2.5)}$$

Now define

$$\chi_{j,r}(N,M):=\nu_\eta(M)\,\psi_{j,r}\left(N,M^{-1}\right).\qquad\text{(IV.2.6)}$$

Also for $\varrho\in\mathbb{Q}$ we define

$$\mathcal{E}_{j,N,\varrho}(\tau):=\sqrt{2N}\int_{\varrho}^{\tau+i\infty+\varepsilon}\frac{f_{j,N}(\mathfrak{z})}{\sqrt{i(\mathfrak{z}-\tau)}}d\mathfrak{z}.$$

Using this together with (IV.2.3) and (IV.2.5) immediately gives the modular transformation equation for $\mathcal{A}_{j,N}$.

Lemma IV.2.1. *For* $M:=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$ *with* $c>0$ *we have*

$$\mathcal{A}_{j,N}(\tau)=\sum_{r=1}^{N-1}\chi_{j,r}(N,M)\left(\mathcal{A}_{r,N}\left(\frac{a\tau+b}{c\tau+d}\right)-\eta\left(\frac{a\tau+b}{c\tau+d}\right)^{-1}\mathcal{E}_{r,N,\frac{a}{c}}\left(\frac{a\tau+b}{c\tau+d}\right)\right).\qquad\text{(IV.2.7)}$$

IV.2.2 Mordell-type integrals

Next we want to rewrite the obstruction to modularity term as a Mordell-type integral. If $\varrho\in\mathbb{Q}$ and $V\in\mathbb{C}$ with $\mathrm{Re}(V)>0$, then we have,

$$\begin{aligned}\mathcal{E}_{j,N,\varrho}(\varrho+iV)&=\sqrt{2N}\int_{\varrho}^{\varrho+iV+i\infty+\varepsilon}\frac{f_{j,N}(\mathfrak{z})}{\sqrt{i(\mathfrak{z}-(\varrho+iV))}}d\mathfrak{z}\\&=i\sqrt{2N}\int_{-V}^{\infty-i\varepsilon}\frac{f_{j,N}(\varrho+i(\mathfrak{z}+V))}{\sqrt{-\mathfrak{z}}}d\mathfrak{z}\\&=i\sqrt{2N}\int_{-V}^{\infty-i\varepsilon}\sum_{n\in\mathbb{Z}}\left(n+\frac{j}{2N}\right)e^{-2\pi N\left(n+\frac{j}{2N}\right)^2(\mathfrak{z}+V)+2\pi iN\left(n+\frac{j}{2N}\right)^2\varrho}\frac{d\mathfrak{z}}{\sqrt{-\mathfrak{z}}}.\end{aligned}$$

First note that the integral is absolutely convergent, since the integrand is a cusp form and therefore exponentially decaying as $\mathfrak{z}\to -V$ and as $\mathfrak{z}\to\infty$. Also note that each summand is exponentially decaying as $\mathfrak{z} \to \infty$ but we lose this condition as $\mathfrak{z} \to -V$. Since we want to be able to interchange the sum and the integral, we rewrite

$$\mathcal{E}_{j,N,\varrho}\left(\varrho + iV\right) = i\sqrt{2N} \lim_{\delta \to 0^+} \int_{-V+\delta}^{\infty - i\varepsilon} \sum_{n \in \mathbb{Z}} \left(n + \frac{j}{2N}\right) e^{-2\pi N \left(n + \frac{j}{2N}\right)^2 (\mathfrak{z} + V) + 2\pi i N \left(n + \frac{j}{2N}\right)^2 \varrho} \frac{d\mathfrak{z}}{\sqrt{-\mathfrak{z}}}.$$

We can exchange the sum and the integral now and get

$$\begin{aligned}
&\mathcal{E}_{j,N,\varrho}\left(\varrho + iV\right) \\
&\quad = i\sqrt{2N} \lim_{\delta \to 0^+} \sum_{n \in \mathbb{Z}} \left(n + \frac{j}{2N}\right) e^{-2\pi N \left(n + \frac{j}{2N}\right)^2 V + 2\pi i N \left(n + \frac{j}{2N}\right)^2 \varrho} \int_{-V+\delta}^{\infty - i\varepsilon} e^{-2\pi N \left(n + \frac{j}{2N}\right)^2 \mathfrak{z}} \frac{d\mathfrak{z}}{\sqrt{-\mathfrak{z}}}.
\end{aligned}$$

Using the identity (see Lemma V.1.1)

$$\begin{aligned}
&\int_{-V+\delta}^{\infty - i\varepsilon} \frac{e^{-2\pi N (n + \frac{j}{2N})^2 \mathfrak{z}}}{\sqrt{-\mathfrak{z}}} d\mathfrak{z} \\
&\qquad = -\frac{i}{\sqrt{2N}(n + \frac{j}{2N})} \left(\operatorname{sgn}\left(n + \frac{j}{2N}\right) + \operatorname{erf}\left(i\left(n + \frac{j}{2N}\right)\sqrt{2\pi N (V - \delta)}\right)\right),
\end{aligned}$$

yields

$$\begin{aligned}
\mathcal{E}_{j,N,\varrho}\left(\varrho + iV\right) = &\lim_{\delta \to 0^+} \sum_{n \in \mathbb{Z}} e^{-2\pi N \left(n + \frac{j}{2N}\right)^2 V + 2\pi i N \left(n + \frac{j}{2N}\right)^2 \varrho} \\
&\times \left(\operatorname{sgn}\left(n + \frac{j}{2N}\right) + \operatorname{erf}\left(i\left(n + \frac{j}{2N}\right)\sqrt{2\pi N \left(V - \delta\right)}\right)\right). \quad \text{(IV.2.8)}
\end{aligned}$$

To check the convergence as $\delta \to 0^+$ we start analogously to [BN19, page 10]. First we notice that the definition of the error function yields the asymptotic behavior

$$\operatorname{erf}\left(iz\right) = \frac{ie^{z^2}}{\sqrt{\pi} z}\left(1 + O\left(|z|^{-2}\right)\right),$$

if $|\operatorname{Arg}(\pm z)| < \frac{\pi}{4}$ as $|z| \to \infty$. Because of this we note that (IV.2.8) does not converge absolutely at $\delta = 0$ and we have to be careful by taking the limit $\delta \to 0^+$. Seperating this main term of the error function as

$$\begin{aligned}
&\left(\operatorname{erf}\left(i\left(n + \frac{j}{2N}\right)\sqrt{2\pi N \left(V - \delta\right)}\right) - \frac{ie^{2\pi N \left(n + \frac{j}{2N}\right)^2 (V - \delta)}}{\pi \left(n + \frac{j}{2N}\right)\sqrt{2N\left(V - \delta\right)}}\right) \\
&\qquad + \frac{ie^{2\pi N \left(n + \frac{j}{2N}\right)^2 (V - \delta)}}{\pi \left(n + \frac{j}{2N}\right)\sqrt{2N\left(V - \delta\right)}}, \qquad \text{(IV.2.9)}
\end{aligned}$$

we find that the term in the brackets is absolutely and uniformly convergent on compact subsets $\mathrm{Re}(V) > 0$ and $0 \leq \delta \leq \delta_0$ for sufficently small δ_0, so we can plug in $\delta = 0$ for these terms to take the limit.

We go on by focussing on the last term of (IV.2.9) whose contibution to $\mathcal{E}_{j,N,\varrho}(\varrho + iV)$ is given by

$$\lim_{\delta\to 0^+} \frac{i}{\pi\sqrt{2N(V-\delta)}} \sum_{n\in\mathbb{Z}} \frac{e^{2\pi i N\left(n+\frac{j}{2N}\right)^2 \varrho}}{n+\frac{j}{2N}} e^{-2\pi N\left(n+\frac{j}{2N}\right)^2 \delta}.$$

Since $|e^{-2\pi N\left(n+\frac{j}{2N}\right)^2\delta}| < 1$ for all $\delta > 0$ this series is absolutely convergent for any $\delta > 0$. If the corresponding series is also convergent for $\delta = 0$ the limit as $\delta \to 0^+$ is simply the value at $\delta = 0$, by Abel's Theorem (viewing it as a power series in $e^{-2\pi N\delta}$).

To prove convergence at $\delta = 0$, recall that we assume $j \neq 0$. Let $\varrho = \frac{h}{k}$ with $\gcd(h,k) = 1$ and $k > 0$ and consider for $\nu \in \mathbb{N}$ the following sum

$$\sum_{-\nu\leq n\leq\nu} \frac{e^{2\pi i N\left(n+\frac{j}{2N}\right)^2 \frac{h}{k}}}{n+\frac{j}{2N}}. \tag{IV.2.10}$$

We immediately see that

$$e^{2\pi i N\left(n+k+\frac{j}{2N}\right)^2 \frac{h}{k}} = e^{2\pi i N\left(n+\frac{j}{2N}\right)^2 \frac{h}{k}},$$

which means that the phase is periodic in n with period k. Denoting the average as

$$\mathfrak{a} := \frac{1}{k} \sum_{n \pmod{k}} e^{2\pi i N\left(n+\frac{j}{2N}\right)^2 \frac{h}{k}},$$

which is convergent by definition, we can rewrite (IV.2.10) as

$$\sum_{-\nu\leq n\leq\nu} \frac{e^{2\pi i N\left(n+\frac{j}{2N}\right)^2 \frac{h}{k}} - \mathfrak{a}}{n+\frac{j}{2N}} + \sum_{-\nu\leq n\leq\nu} \frac{\mathfrak{a}}{n+\frac{j}{2N}}. \tag{IV.2.11}$$

We first look at the second sum in (IV.2.11). We have that

$$\sum_{-\nu\leq n\leq\nu} \frac{\mathfrak{a}}{n+\frac{j}{2N}} = \frac{\mathfrak{a}}{\frac{j}{2N}} + \mathfrak{a} \sum_{1\leq n\leq\nu} \left(\frac{1}{n+\frac{j}{2N}} + \frac{1}{-n+\frac{j}{2N}} \right) = \frac{\mathfrak{a}}{\frac{j}{2N}} + \mathfrak{a} \sum_{1\leq n\leq\nu} \frac{\frac{j}{N}}{\left(\frac{j}{2N}\right)^2 - n^2},$$

where the summand is $O(n^{-2})$, which gives us that the sum converges absolutely. Looking at the first sum in (IV.2.11) and writing $n = km + r$ we obtain

$$\begin{aligned}\sum_{-\nu\leq n\leq\nu} \frac{e^{2\pi iN\left(n+\frac{j}{2N}\right)^2\frac{h}{k}} - \mathfrak{a}}{n+\frac{j}{2N}} &= \sum_{-\frac{\nu}{k}\leq m\leq\frac{\nu}{k}} \sum_{r=0}^{k-1} \frac{e^{2\pi iN\left(km+r+\frac{j}{2N}\right)^2\frac{h}{k}} - \mathfrak{a}}{km+r+\frac{j}{2N}} + O\left(\frac{k}{\nu}\right) \\ &= \sum_{-\frac{\nu}{k}\leq m\leq\frac{\nu}{k}} \sum_{r=0}^{k-1} \frac{e^{2\pi iN\left(r+\frac{j}{2N}\right)^2\frac{h}{k}} - \mathfrak{a}}{km+r+\frac{j}{2N}} + O\left(\frac{k}{\nu}\right), \qquad \text{(IV.2.12)}\end{aligned}$$

using the periodicity of the exponential. For simplicity we denote $d_r \coloneqq e^{2\pi iN\left(r+\frac{j}{2N}\right)^2\frac{h}{k}} - \mathfrak{a}$. Since we have that

$$\sum_{r=0}^{k-1} d_r = -k\mathfrak{a} + \sum_{r=0}^{k-1} e^{2\pi iN\left(r+\frac{j}{2N}\right)^2\frac{h}{k}} = 0$$

by definition of $\mathfrak{a}$, we can write $d_{k-1} = -d_0 - d_1 - \cdots - d_{k-2}$. With this we can rewrite (IV.2.12) as

$$\begin{aligned}\sum_{-\frac{\nu}{k}\leq m\leq\frac{\nu}{k}} \Bigg(d_0 \left(\frac{1}{km+\frac{j}{2N}} - \frac{1}{km+k-1+\frac{j}{2N}}\right) + d_1 \left(\frac{1}{km+1+\frac{j}{2N}} - \frac{1}{km+k-1+\frac{j}{2N}}\right) \\ + \cdots + d_{k-2}\left(\frac{1}{km+k-2+\frac{j}{2N}} - \frac{1}{km+k-1+\frac{j}{2N}}\right)\Bigg) + O\left(\frac{k}{\nu}\right),\end{aligned}$$

where each term in the brackets is $O(m^{-2})$, which gives us that (IV.2.12) and thus (IV.2.10) is absolutely convergent, by taking the limit $\nu \to \infty$.

Therefore the last term of (IV.2.9) is convergent for $\delta = 0$ and with this we see that (IV.2.8) is convergent. Thus we are allowed to set $\delta = 0$ in (IV.2.8) to obtain

$$\begin{aligned}&\mathcal{E}_{j,N,\varrho}(\varrho + iV) \qquad \text{(IV.2.13)} \\ &= \sum_{n\in\mathbb{Z}} \left(\operatorname{sgn}\left(n+\frac{j}{2N}\right) + \operatorname{erf}\left(i\left(n+\frac{j}{2N}\right)\sqrt{2\pi NV}\right)\right) e^{2\pi iN\left(n+\frac{j}{2N}\right)^2(\varrho+iV)}.\end{aligned}$$

Hence, using (IV.2.7) and the definition of $\mathcal{A}_{j,N}$, we get

$$\begin{aligned}\mathcal{A}_{j,N}(\tau) &= \sum_{r=1}^{N-1} \chi_{j,r}(N,M) \left(\mathcal{A}_{r,N}(\varrho+iV) - \eta(\varrho+iV)^{-1}\,\mathcal{E}_{r,N,\varrho}(\varrho+iV)\right) \\ &= \sum_{r=1}^{N-1} \chi_{j,r}(N,M) \Bigg(\mathcal{A}_{r,N}(\varrho+iV) - \eta(\varrho+iV)^{-1} \sum_{n\in\mathbb{Z}} \operatorname{sgn}\left(n+\frac{r}{2N}\right) e^{2\pi iN\left(n+\frac{r}{2N}\right)^2(\varrho+iV)} \\ &\quad -\eta(\varrho+iV)^{-1} \sum_{n\in\mathbb{Z}} \operatorname{erf}\left(i\left(n+\frac{r}{2N}\right)\sqrt{2\pi NV}\right) e^{2\pi iN\left(n+\frac{r}{2N}\right)^2(\varrho+iV)}\Bigg)\end{aligned}$$

$$= \sum_{r=1}^{N-1} \chi_{j,r}(N,M)\, \eta(\varrho+iV)^{-1} \left(-\sum_{n\in\mathbb{Z}} \operatorname{erf}\left(i\left(n+\frac{r}{2N}\right)\sqrt{2\pi NV}\right) e^{2\pi i N\left(n+\frac{r}{2N}\right)^2(\varrho+iV)} \right). \quad \text{(IV.2.14)}$$

We see that the first term of (IV.2.13) cancels against the contribution of $\mathcal{A}_{r,N}(\varrho+iV)$, so we focus on the second term of (IV.2.13) and define

$$\mathcal{I}_{j,N,\varrho}(\varrho+iV) := -\sum_{n\in\mathbb{Z}} \operatorname{erf}\left(i\left(n+\frac{j}{2N}\right)\sqrt{2\pi NV}\right) e^{2\pi i N\left(n+\frac{j}{2N}\right)^2(\varrho+iV)},$$

which is basically our error of modularity plus the holomorphic part of our function. Using the identity, for $s \in \mathbb{R}\setminus\{0\}$ and $\mathrm{Re}(V) > 0$, (see Lemma V.1.2)

$$e^{-\pi s^2 V} \operatorname{erf}\left(is\sqrt{\pi V}\right) = -\frac{i}{\pi}\,\mathrm{P.V.}\int_{-\infty}^{\infty} \frac{e^{-\pi V x^2}}{x-s}\,dx := -\frac{i}{\pi}\lim_{\varepsilon\to 0^+}\left(\int_{-\infty}^{s-\varepsilon} \frac{e^{-\pi V x^2}}{x-s}\,dx + \int_{s+\varepsilon}^{\infty} \frac{e^{-\pi V x^2}}{x-s}\,dx\right),$$

we obtain

$$\mathcal{I}_{j,N,\varrho}(\varrho+iV) = \frac{i}{\pi}\sum_{n\in\mathbb{Z}} e^{2\pi i N\left(n+\frac{j}{2N}\right)^2\varrho}\,\mathrm{P.V.}\int_{-\infty}^{\infty} \frac{e^{-2\pi NVx^2}}{x-\left(n+\frac{j}{2N}\right)}\,dx. \quad \text{(IV.2.15)}$$

IV.2.3 Splitting of the Mordell-type integral

Let $\varrho = \frac{h'}{k}$ with $h',k\in\mathbb{Z}$, $\gcd(h',k)=1$, and $k>0$. For a real number d with $0 \le d < N$ such that $2\sqrt{dN} \notin \mathbb{Z}\setminus\{0\}$ we split $\mathcal{I}_{j,N,\frac{h'}{k}}$ as follows

$$e^{2\pi dV}\mathcal{I}_{j,N,\frac{h'}{k}}\left(\frac{h'}{k}+iV\right) = \mathcal{I}^*_{j,N,\frac{h'}{k},d}\left(\frac{h'}{k}+iV\right) + \mathcal{I}^e_{j,N,\frac{h'}{k},d}\left(\frac{h'}{k}+iV\right),$$

where

$$\mathcal{I}^*_{j,N,\frac{h'}{k},d}\left(\frac{h'}{k}+iV\right) = \frac{i}{\pi}e^{2\pi dV}\sum_{n\in\mathbb{Z}} e^{2\pi i N\left(n+\frac{j}{2N}\right)^2\frac{h'}{k}}\,\mathrm{P.V.}\int_{-\sqrt{\frac{d}{N}}}^{\sqrt{\frac{d}{N}}} \frac{e^{-2\pi NVx^2}}{x-\left(n+\frac{j}{2N}\right)}\,dx, \quad \text{(IV.2.16)}$$

$$\mathcal{I}^e_{j,N,\frac{h'}{k},d}\left(\frac{h'}{k}+iV\right) = \frac{i}{\pi}e^{2\pi dV}\sum_{n\in\mathbb{Z}} e^{2\pi i N\left(n+\frac{j}{2N}\right)^2\frac{h'}{k}}\,\mathrm{P.V.}\int_{|x|\ge\sqrt{\frac{d}{N}}} \frac{e^{-2\pi NVx^2}}{x-\left(n+\frac{j}{2N}\right)}\,dx. \quad \text{(IV.2.17)}$$

Note that the assumption $2\sqrt{dN} \notin \mathbb{Z}\setminus\{0\}$ ensures the well-definedness of the principal value integral, since we avoid having poles on the boundary.

IV.3 Proof of Theorem IV.1.2

In this section we prove Theorem IV.1.2 by using a bound of Malishev, which we stated as Lemma IV.1.3.

We note that $K_{k,j,N}(n,r,\kappa)$ from (IV.1.6) is well-defined and a Kloosterman sum of modulus k, which follows from a lengthy but straightforward calculation using the Chinese Remainder Theorem, quadratic reciprocity, and some formulae on the Kronecker symbol (see Lemmata V.2.1 and V.2.2). Thus we can rewrite it as

$$K_{k,j,N}(n,r,\kappa) = \sum_{\substack{h \pmod k \\ \gcd(h,k)=1}} \chi_{j,r}(N,M_{h,k}) \exp\left(-\frac{2\pi i}{k}\left(\left(n+\frac{j^2}{4N}-\frac{1}{24}\right)h - \left(N\left(\kappa+\frac{r}{2N}\right)^2-\frac{1}{24}\right)h'\right)\right).$$

Note that for even k we have

$$\chi_{j,r}(N,M_{h,k}) = \left(\frac{k}{-h}\right)\exp\left(\frac{\pi i}{12}\left(h'k\left(1-(-h)^2\right)+(-h)\left(-\frac{hh'+1}{k}-k+3\right)-3\right)\right)$$
$$\times \exp\left(\frac{3\pi i}{4}\right)\sqrt{\frac{2}{Nk}}\sum_{s=0}^{k-1}\exp\left(-\frac{\pi i}{2Nk}\left(-h(2Ns+j)^2+h'r^2\right)\right)\sin\left(\frac{\pi r(2Ns+j)}{Nk}\right),$$

while for odd k we have

$$\chi_{j,r}(N,M_{h,k}) = \left(\frac{-h}{k}\right)\exp\left(\frac{\pi i}{12}\left((h'-h)k-\frac{hh'+1}{k}h\left(k^2-1\right)-3k\right)\right)$$
$$\times \exp\left(\frac{3\pi i}{4}\right)\sqrt{\frac{2}{Nk}}\sum_{s=0}^{k-1}\exp\left(-\frac{\pi i}{2Nk}\left(-h(2Ns+j)^2+h'r^2\right)\right)\sin\left(\frac{\pi r(2Ns+j)}{Nk}\right).$$

The strategy of the proof is to rewrite our Kloosterman sum into a sort of Salié sum

$$K_{k,j,N}(n,r,\kappa) = \epsilon(k,j,N,r)\sum_{\substack{h \pmod{Gk} \\ \gcd(h,Gk)=1}}\left(\frac{h}{\rho}\right)\exp\left(\frac{2\pi i}{Gk}\left(\mu_* h - \nu_*[h]'_{Gk}\right)\right),$$

where $\mu_*,\nu_*\in\mathbb{Z}$, $G\in\mathbb{N}$, $\rho\in\mathbb{N}$ odd such that all his prime divisors divide Gk, $[h]'_{Gk}$ the negative modular inverse of h modulo Gk, and some $\epsilon(k,j,N,r)=O_N(1)$. Then we bound it using [KS64, equation (12)]. Note that we use the $[\cdot]_\cdot'$ notation from now on to denote the negative modular inverse of given modulus.

We write

$$\sin\left(\frac{\pi r(2Ns+j)}{Nk}\right) = \frac{1}{2i}\left(\exp\left(\frac{\pi i r(2Ns+j)}{Nk}\right)-\exp\left(-\frac{\pi i r(2Ns+j)}{Nk}\right)\right),$$

which yields that

$$\sum_{s=0}^{k-1}\exp\left(-\frac{\pi i}{2Nk}\left(-h(2Ns+j)^2+h'r^2\right)\right)\sin\left(\frac{\pi r(2Ns+j)}{Nk}\right)$$

$$
\begin{aligned}
=&\frac{1}{2i}\sum_{s=0}^{k-1}\left(\exp\left(\frac{2\pi i}{k}\left(hNs^2+(hj+r)\,s\right)\right)\exp\left(\frac{2\pi i}{4Nk}\left(hj^2-h'r^2+2rj\right)\right)\right.\\
&\left.\qquad-\exp\left(\frac{2\pi i}{k}\left(hNs^2+(hj-r)\,s\right)\right)\exp\left(\frac{2\pi i}{4Nk}\left(hj^2-h'r^2-2rj\right)\right)\right).
\end{aligned}
$$

We additionally see that this equals

$$
\begin{aligned}
&\frac{1}{2i}\left(\exp\left(\frac{2\pi i}{4Nk}\left(hj^2-h'r^2+2rj\right)\right)G\left(hN,hj+r,k\right)\right.\\
&\qquad\left.-\exp\left(\frac{2\pi i}{4Nk}\left(hj^2-h'r^2-2rj\right)\right)G\left(hN,hj-r,k\right)\right), \qquad \text{(IV.3.1)}
\end{aligned}
$$

where

$$
G\left(a,b,c\right):=\sum_{s=0}^{c-1}\exp\left(2\pi i\frac{as^2+bs}{c}\right)
$$

denotes the *generalized quadratic Gauss sum*[5]. From this point on we have to look at odd, respectively even, k seperately.

IV.3.1 Odd k

We have that

$$
\begin{aligned}
\chi_{j,r}(N,M_{h,k})=&\left(\frac{-h}{k}\right)\sqrt{\frac{2}{Nk}}\exp\left(2\pi i\left(\frac{1}{24}\left((h'-h)k-\frac{hh'+1}{k}h\left(k^2-1\right)-3k\right)+\frac{3}{8}\right)\right)\\
&\times\sum_{s=0}^{k-1}\exp\left(-\frac{\pi i}{2Nk}\left(-h(2Ns+j)^2+h'r^2\right)\right)\sin\left(\frac{\pi r(2Ns+j)}{Nk}\right).
\end{aligned}
$$

Using (IV.3.1) we can thus rewrite this as

$$
\begin{aligned}
&\chi_{j,r}(N,M_{h,k})\\
&=-i\left(\frac{-h}{k}\right)\sqrt{\frac{1}{2Nk}}\exp\left(2\pi i\left(\frac{1}{24}\left((h'-h)k-\frac{hh'+1}{k}h\left(k^2-1\right)-3k\right)+\frac{3}{8}+\frac{1}{4Nk}\left(hj^2-h'r^2+2rj\right)\right)\right)\\
&\quad\times G\left(hN,hj+r,k\right)\\
&\quad+i\left(\frac{-h}{k}\right)\sqrt{\frac{1}{2Nk}}\exp\left(2\pi i\left(\frac{1}{24}\left((h'-h)k-\frac{hh'+1}{k}h\left(k^2-1\right)-3k\right)+\frac{3}{8}+\frac{1}{4Nk}\left(hj^2-h'r^2-2rj\right)\right)\right)\\
&\quad\times G\left(hN,hj-r,k\right). \qquad \text{(IV.3.2)}
\end{aligned}
$$

Note that $\gcd(Nh,k)=\gcd(N,k)$, since $\gcd(h,k)=1$. Set

$$
\varepsilon_m:=\begin{cases}1 & \text{if } m\equiv 1\pmod 4,\\ i & \text{if } m\equiv 3\pmod 4,\end{cases} \qquad \text{(IV.3.3)}
$$

[5] Note that this sum is well-defined for any $a,c\in\mathbb{N}$ and $b\pmod c$.

for every odd integer m. For odd k we obtain that

$$G(hN, hj \pm r, k) = \begin{cases} 0 & \text{if } \gcd(N,k) > 1, \text{ and } \gcd(N,k) \nmid (hj \pm r), \\ \gcd(N,k) G\left(\frac{Nh}{\gcd(N,k)}, \frac{hj\pm r}{\gcd(N,k)}, \frac{k}{\gcd(N,k)}\right) & \text{if } \gcd(N,k) > 1, \text{ and } \gcd(N,k) \mid (hj \pm r), \\ \varepsilon_k \sqrt{k} \left(\frac{Nh}{k}\right) \exp\left(-2\pi i \frac{\psi(Nh)(hj\pm r)^2}{k}\right) & \text{if } \gcd(N,k) = 1, \end{cases}$$
$$= \begin{cases} 0 & \text{if } \gcd(N,k) \nmid (hj \pm r), \\ \gcd(N,k) \varepsilon_{\frac{k}{\gcd(N,k)}} \sqrt{\frac{k}{\gcd(N,k)}} \left(\frac{\frac{Nh}{\gcd(N,k)}}{\frac{k}{\gcd(N,k)}}\right) \exp\left(-2\pi i \frac{\psi^*\left(\frac{Nh}{\gcd(N,k)}\right)\left(\frac{hj\pm r}{\gcd(N,k)}\right)^2}{\frac{k}{\gcd(N,k)}}\right) & \text{otherwise,} \end{cases} \tag{IV.3.4}$$

where $\psi(a)$ and $\psi^*(a)$ are some numbers satisfying[6]

$$4\psi(a)a \equiv 1 \pmod{k} \qquad \text{and} \qquad 4\psi^*(a)a \equiv 1 \left(\mod \frac{k}{\gcd(N,k)}\right).$$

We can thus rewrite (IV.3.2) as

$$\begin{aligned} &\chi_{j,r}(N, M_{h,k}) \\ &= -i\left(\frac{-h}{k}\right)\sqrt{\frac{1}{2Nk}} \exp\left(2\pi i\left(\frac{1}{24}\left((h'-h)k - \frac{hh'+1}{k}h\left(k^2-1\right) - 3k\right) + \frac{3}{8} + \frac{1}{4Nk}\left(hj^2 - h'r^2 + 2rj\right)\right)\right) \\ &\quad \times \gcd(N,k)\varepsilon_{\frac{k}{\gcd(N,k)}} \sqrt{\frac{k}{\gcd(N,k)}} \left(\frac{\frac{Nh}{\gcd(N,k)}}{\frac{k}{\gcd(N,k)}}\right) \exp\left(-2\pi i \frac{\psi^*\left(\frac{Nh}{\gcd(N,k)}\right)\left(\frac{hj+r}{\gcd(N,k)}\right)^2}{\frac{k}{\gcd(N,k)}}\right) \delta_{\gcd(N,k)\mid(hj+r)} \\ &\quad + i\left(\frac{-h}{k}\right)\sqrt{\frac{1}{2Nk}} \exp\left(2\pi i\left(\frac{1}{24}\left((h'-h)k - \frac{hh'+1}{k}h\left(k^2-1\right) - 3k\right) + \frac{3}{8} + \frac{1}{4Nk}\left(hj^2 - h'r^2 - 2rj\right)\right)\right) \\ &\quad \times \gcd(N,k)\varepsilon_{\frac{k}{\gcd(N,k)}} \sqrt{\frac{k}{\gcd(N,k)}} \left(\frac{\frac{Nh}{\gcd(N,k)}}{\frac{k}{\gcd(N,k)}}\right) \exp\left(-2\pi i \frac{\psi^*\left(\frac{Nh}{\gcd(N,k)}\right)\left(\frac{hj-r}{\gcd(N,k)}\right)^2}{\frac{k}{\gcd(N,k)}}\right) \delta_{\gcd(N,k)\mid(hj-r)}, \end{aligned} \tag{IV.3.5}$$

using

$$\delta_{\text{condition}} := \begin{cases} 1 & \text{if this condition is true,} \\ 0 & \text{otherwise,} \end{cases}$$

here and throughout the rest of the chapter. By definition we have that

$$-4a\psi^*(a) \equiv -1 \left(\mod \frac{k}{\gcd(N,k)}\right),$$

which gives us that

$$\psi^*(a) = [-4a]'_{\frac{k}{\gcd(N,k)}}.$$

[6]Note that $\psi(a)$ and $\psi^*(a)$ exist, since we assumed that k, and thus $\frac{k}{\gcd(N,k)}$, are odd and that $\gcd(Nh,k) = 1$ by assumption of the first case and $\gcd(Nh, \frac{k}{\gcd(N,k)}) = 1$.

Using that $[ab]'_x = -[a]'_x[b]'_x$, for any modulus $x \in \mathbb{N}$ and arbitrary $a, b \in \mathbb{N}$, we obtain that

$$\psi^*(a) = [4]'_{\frac{k}{\gcd(N,k)}} [a]'_{\frac{k}{\gcd(N,k)}}$$

and thus

$$\psi^*\left(\frac{Nh}{\gcd(N,k)}\right) = [4]'_{\frac{k}{\gcd(N,k)}} \left[\frac{Nh}{\gcd(N,k)}\right]'_{\frac{k}{\gcd(N,k)}} = -[4]'_{\frac{k}{\gcd(N,k)}} \left[\frac{N}{\gcd(N,k)}\right]'_{\frac{k}{\gcd(N,k)}} [h]'_{\frac{k}{\gcd(N,k)}}.$$

Note that (IV.3.5) is well-defined for $[a]'_{\frac{k}{\gcd(N,k)}}$ the negative modular inverse of a, i.e., a solution of $a\,[a]'_{\frac{k}{\gcd(N,k)}} \equiv -1 \pmod{\frac{k}{\gcd(N,k)}}$, since we have that

$$\exp\left(-2\pi i \frac{\psi^*\left(\frac{Nh}{\gcd(N,k)}\right)\left(\frac{hj\pm r}{\gcd(N,k)}\right)^2}{\frac{k}{\gcd(N,k)}}\right)$$
$$= \exp\left(-2\pi i \frac{-[4]'_{\frac{k}{\gcd(N,k)}} \left[\frac{N}{\gcd(N,k)}\right]'_{\frac{k}{\gcd(N,k)}} [h]'_{\frac{k}{\gcd(N,k)}} \left(\frac{hj\pm r}{\gcd(N,k)}\right)^2}{\frac{k}{\gcd(N,k)}}\right)$$

is invariant under any shifts by $\frac{k}{\gcd(N,k)}$, since $\frac{hj\pm r}{\gcd(N,k)} \in \mathbb{Z}$.

For simplicity we stick to the notation $[h]'_k = h'$. We obtain

$$\begin{aligned}
&\chi_{j,r}(N, M_{h,k}) \\
&= -i\varepsilon_{\frac{k}{\gcd(N,k)}} \left(\frac{-h}{k}\right)\left(\frac{\frac{Nh}{\gcd(N,k)}}{\frac{k}{\gcd(N,k)}}\right)\sqrt{\frac{\gcd(N,k)}{2N}} \exp\left(-2\pi i \frac{-[4]'_{\frac{k}{\gcd(N,k)}} \left[\frac{N}{\gcd(N,k)}\right]'_{\frac{k}{\gcd(N,k)}} [h]'_{\frac{k}{\gcd(N,k)}} \left(\frac{hj+r}{\gcd(N,k)}\right)^2}{\frac{k}{\gcd(N,k)}}\right) \\
&\quad \times \exp\left(2\pi i\left(\frac{1}{24}\left((h'-h)k - \frac{hh'+1}{k}h\left(k^2-1\right) - 3k\right) + \frac{3}{8} + \frac{1}{4Nk}\left(hj^2 - h'r^2 + 2rj\right)\right)\right)\delta_{\gcd(N,k)|(hj+r)} \\
&\quad + i\varepsilon_{\frac{k}{\gcd(N,k)}} \left(\frac{-h}{k}\right)\left(\frac{\frac{Nh}{\gcd(N,k)}}{\frac{k}{\gcd(N,k)}}\right)\sqrt{\frac{\gcd(N,k)}{2N}} \exp\left(-2\pi i \frac{-[4]'_{\frac{k}{\gcd(N,k)}} \left[\frac{N}{\gcd(N,k)}\right]'_{\frac{k}{\gcd(N,k)}} [h]'_{\frac{k}{\gcd(N,k)}} \left(\frac{hj-r}{\gcd(N,k)}\right)^2}{\frac{k}{\gcd(N,k)}}\right) \\
&\quad \times \exp\left(2\pi i\left(\frac{1}{24}\left((h'-h)k - \frac{hh'+1}{k}h\left(k^2-1\right) - 3k\right) + \frac{3}{8} + \frac{1}{4Nk}\left(hj^2 - h'r^2 - 2rj\right)\right)\right)\delta_{\gcd(N,k)|(hj-r)}
\end{aligned}$$

and see that

$$\left(\frac{-h}{k}\right)\left(\frac{\frac{Nh}{\gcd(N,k)}}{\frac{k}{\gcd(N,k)}}\right) = \left(\frac{-h}{k}\right)\left(\frac{-h}{\frac{k}{\gcd(N,k)}}\right)\left(\frac{\frac{-N}{\gcd(N,k)}}{\frac{k}{\gcd(N,k)}}\right) = \left(\frac{-h}{\gcd(N,k)}\right)\left(\frac{\frac{-N}{\gcd(N,k)}}{\frac{k}{\gcd(N,k)}}\right).$$

Therefore our Kloosterman sum equals

$$\begin{aligned}
K_{k,j,N}(n,r,\kappa) =& i\varepsilon_{\frac{k}{\gcd(N,k)}} \left(\frac{\frac{-N}{\gcd(N,k)}}{\frac{k}{\gcd(N,k)}}\right) \sqrt{\frac{\gcd(N,k)}{2N}} \exp\left(\frac{2\pi i}{24k}\left(-3k^2+9k\right)\right) \sum_{\substack{h \pmod k\\ \gcd(h,k)=1}} \left(\frac{-h}{\gcd(N,k)}\right)\\
&\times \exp\left(\frac{2\pi i}{24k}\left(\left(-24n+2-2k^2\right)h - \left(-24N\kappa^2-24\kappa r+1-k^2\right)h'\right)\right)\\
&\times \left(\delta_{\gcd(N,k)|(hj-r)} \exp\left(2\pi i \frac{[4]'_{\frac{k}{\gcd(N,k)}} \left[\frac{N}{\gcd(N,k)}\right]'_{\frac{k}{\gcd(N,k)}} [h]'_{\frac{k}{\gcd(N,k)}} \left(\frac{hj-r}{\gcd(N,k)}\right)^2}{\frac{k}{\gcd(N,k)}}\right)\right.\\
&\qquad \times \exp\left(\frac{2\pi i}{24k}\left(-h^2h'k^2+h^2h'-\frac{12rj}{N}\right)\right)\\
&\qquad -\delta_{\gcd(N,k)|(hj+r)} \exp\left(2\pi i \frac{[4]'_{\frac{k}{\gcd(N,k)}} \left[\frac{N}{\gcd(N,k)}\right]'_{\frac{k}{\gcd(N,k)}} [h]'_{\frac{k}{\gcd(N,k)}} \left(\frac{hj+r}{\gcd(N,k)}\right)^2}{\frac{k}{\gcd(N,k)}}\right)\\
&\qquad \left.\times \exp\left(\frac{2\pi i}{24k}\left(-h^2h'k^2+h^2h'+\frac{12rj}{N}\right)\right)\right).
\end{aligned}$$

We already saw that the following is well-defined and now observe that

$$\begin{aligned}
&\exp\left(2\pi i \frac{[4]'_{\frac{k}{\gcd(N,k)}} \left[\frac{N}{\gcd(N,k)}\right]'_{\frac{k}{\gcd(N,k)}} [h]'_{\frac{k}{\gcd(N,k)}} \left(\frac{hj\pm r}{\gcd(N,k)}\right)^2}{\frac{k}{\gcd(N,k)}}\right)\\
=&\exp\left(\frac{2\pi i}{k\gcd(N,k)}\left([4]'_{\frac{k}{\gcd(N,k)}} \left[\frac{N}{\gcd(N,k)}\right]'_{\frac{k}{\gcd(N,k)}} [h]'_{\frac{k}{\gcd(N,k)}} \left(h^2j^2 \pm 2hjr + r^2\right)\right)\right).
\end{aligned}$$

Choose $[h]'_{\frac{k}{\gcd(N,k)}} = h'$ from now on[7]. Let $x \in \mathbb{N}$ such that $\gcd(x,h) = 1$ (note that this condition is necessary to make sure that the negative modular inverse is well-defined) and $[h]'_{xk}$ the negative modular inverse of h modulo xk, i.e.,

$$h[h]'_{xk} \equiv -1 \pmod{xk}.$$

Then we see that we also have $h[h]'_{xk} \equiv -1 \pmod k$, since $k \mid xk$. This yields that

$$h' \equiv [h]'_{xk} \pmod k.$$

[7]Note that $hh' \equiv -1 \pmod k$ implies that $hh' \equiv -1 \pmod{\frac{k}{\gcd(N,k)}}$, since $\frac{k}{\gcd(N,k)} \mid k$. Thus h' is a possible choice for $[h]'_{\frac{k}{\gcd(N,k)}}$.

Thus we can choose h' such that $hh' \equiv -1 \pmod{xk}$. Taking $x = \gcd(N,k)$ we obtain

$$\begin{aligned}
&K_{k,j,N}(n,r,\kappa)\\
&= i\varepsilon_{\frac{k}{\gcd(N,k)}} \left(\frac{\frac{-N}{\gcd(N,k)}}{\frac{k}{\gcd(N,k)}}\right)\left(\frac{-1}{\gcd(N,k)}\right)\sqrt{\frac{\gcd(N,k)}{2N}}\exp\left(\frac{2\pi i}{24k}\left(-3k^2+9k\right)\right)\sum_{\substack{h \pmod k\\ \gcd(h,k)=1}}\left(\frac{h}{\gcd(N,k)}\right)\\
&\quad\times\exp\left(\frac{2\pi i}{24\gcd(N,k)k}\left(\left(\left(-24n+2-2k^2\right)\gcd(N,k)-24j^2[4]'_{\frac{k}{\gcd(N,k)}}\left[\frac{N}{\gcd(N,k)}\right]'_{\frac{k}{\gcd(N,k)}}\right)h\right.\right.\\
&\qquad\left.\left.-\left(\left(-24N\kappa^2-24\kappa r+1-k^2\right)\gcd(N,k)-24r^2[4]'_{\frac{k}{\gcd(N,k)}}\left[\frac{N}{\gcd(N,k)}\right]'_{\frac{k}{\gcd(N,k)}}\right)[h]'_{k\gcd(N,k)}\right)\right)\\
&\quad\times\left(\delta_{\gcd(N,k)|(hj-r)}\exp\left(\frac{2\pi i}{k\gcd(N,k)}\left([4]'_{\frac{k}{\gcd(N,k)}}\left[\frac{N}{\gcd(N,k)}\right]'_{\frac{k}{\gcd(N,k)}}2jr\right)\right)\right.\\
&\qquad\times\exp\left(\frac{2\pi i}{24k}\left(h^2[h]'_{k\gcd(N,k)}\left(1-k^2\right)-\frac{12rj}{N}\right)\right)\\
&\qquad-\delta_{\gcd(N,k)|(hj+r)}\exp\left(\frac{2\pi i}{k\gcd(N,k)}\left(-[4]'_{\frac{k}{\gcd(N,k)}}\left[\frac{N}{\gcd(N,k)}\right]'_{\frac{k}{\gcd(N,k)}}2jr\right)\right)\\
&\qquad\left.\times\exp\left(\frac{2\pi i}{24k}\left(h^2[h]'_{k\gcd(N,k)}\left(1-k^2\right)+\frac{12rj}{N}\right)\right)\right).
\end{aligned}$$

We now need to split into two cases, $3 \nmid k$ and $3 \mid k$. In the first case we have $1-k^2 \equiv 0 \pmod{24}$. Thus we obtain[8]

$$K_{k,j,N}(n,r,\kappa) = K_{k,j,N,+}(n,r,\kappa) + K_{k,j,N,-}(n,r,\kappa),$$

with

$$\begin{aligned}
&K_{k,j,N,\pm}(n,r,\kappa)\\
&:= \mp i\varepsilon_{\frac{k}{\gcd(N,k)}}\left(\frac{\frac{-N}{\gcd(N,k)}}{\frac{k}{\gcd(N,k)}}\right)\left(\frac{-1}{\gcd(N,k)}\right)\sqrt{\frac{\gcd(N,k)}{2N}}\exp\left(\frac{2\pi i}{24k}\left(-3k^2+9k\right)\right)\\
&\quad\times\exp\left(\frac{2\pi i}{24k\gcd(N,k)}\left(\mp 48jr[4]'_{\frac{k}{\gcd(N,k)}}\left[\frac{N}{\gcd(N,k)}\right]'_{\frac{k}{\gcd(N,k)}}\pm\frac{12rj}{N}\gcd(N,k)\right)\right)\\
&\quad\times\sum_{\substack{h \pmod k\\ \gcd(h,k)=1}}\left(\frac{h}{\gcd(N,k)}\right)\delta_{\gcd(N,k)|(hj\pm r)}\\
&\quad\times\exp\left(\frac{2\pi i}{24\gcd(N,k)k}\left(\left(\left(-24n+1-k^2\right)\gcd(N,k)-24j^2[4]'_{\frac{k}{\gcd(N,k)}}\left[\frac{N}{\gcd(N,k)}\right]'_{\frac{k}{\gcd(N,k)}}\right)h\right.\right.\\
&\qquad\left.\left.-\left(\left(-24N\kappa^2-24\kappa r+1-k^2\right)\gcd(N,k)-24r^2[4]'_{\frac{k}{\gcd(N,k)}}\left[\frac{N}{\gcd(N,k)}\right]'_{\frac{k}{\gcd(N,k)}}\right)[h]'_{k\gcd(N,k)}\right)\right).
\end{aligned}$$

[8]Using that $h[h]'_{k\gcd(N,k)} \equiv -1 \pmod k$ since $k \mid (k\gcd(N,k))$.

We set

$$
\begin{aligned}
&K_{k,j,N,\pm}(n,r,\kappa)\\
=:&\epsilon_{o,\pm}(k,j,N,r)\frac{1}{\gcd(N,k)}\sum_{\substack{h \pmod{\gcd(N,k)k}\\ \gcd(h,\gcd(N,k)k)=1}}\left(\frac{h}{\gcd(N,k)}\right)\delta_{\gcd(N,k)|(hj\pm r)}\\
&\times\exp\left(\frac{2\pi i}{\gcd(N,k)k}\left(\left(\left(-n+\frac{1-k^2}{24}\right)\gcd(N,k)-j^2[4]'_{\frac{k}{\gcd(N,k)}}\left[\frac{N}{\gcd(N,k)}\right]'_{\frac{k}{\gcd(N,k)}}\right)h\right.\right.\\
&\qquad\left.\left.-\left(\left(-N\kappa^2-\kappa r+\frac{1-k^2}{24}\right)\gcd(N,k)-r^2[4]'_{\frac{k}{\gcd(N,k)}}\left[\frac{N}{\gcd(N,k)}\right]'_{\frac{k}{\gcd(N,k)}}\right)[h]'_{k\gcd(N,k)}\right)\right)\\
=:&\epsilon_{o,\pm}(k,j,N,r)\frac{1}{\gcd(N,k)}\sum_{\substack{h \pmod{\gcd(N,k)k}\\ \gcd(h,\gcd(N,k)k)=1}}\left(\frac{h}{\gcd(N,k)}\right)\delta_{\gcd(N,k)|(hj\pm r)}\\
&\times\exp\left(\frac{2\pi i}{\gcd(N,k)k}\left(\mu_1 h-\nu_1[h]'_{k\gcd(N,k)}\right)\right)
\end{aligned}
$$

and note that, by orthogonality of roots of unity, we have

$$
\delta_{\gcd(N,k)|(hj\pm r)}=\frac{1}{\gcd(N,k)}\sum_{s=0}^{\gcd(N,k)-1}\exp\left(2\pi i\frac{(hj\pm r)s}{\gcd(N,k)}\right),
$$

which finally gives us that

$$
\begin{aligned}
K_{k,j,N,\pm}(n,r,\kappa)=&\epsilon_{o,\pm}(k,j,N,r)\frac{1}{\gcd(N,k)^2}\sum_{s=0}^{\gcd(N,k)-1}\exp\left(\pm 2\pi i\frac{rs}{\gcd(N,k)}\right)\\
&\times\sum_{\substack{h \pmod{\gcd(N,k)k}\\ \gcd(h,\gcd(N,k)k)=1}}\left(\frac{h}{\gcd(N,k)}\right)\exp\left(\frac{2\pi i}{\gcd(N,k)k}\left(\left(\mu_1+jsk\right)h-\nu_1[h]'_{k\gcd(N,k)}\right)\right).
\end{aligned}
\tag{IV.3.6}
$$

In the second case, $3\mid k$, we have $1-k^2\equiv 0 \pmod 8$ and $3\nmid h$. Thus, choosing $[h]'_{k\gcd(N,k)}$ such that $h[h]'_{k\gcd(N,k)}\equiv -1 \pmod{3k\gcd(N,k)}$ analogously to above, we obtain[9]

$$
\begin{aligned}
&K_{k,j,N}(n,r,\kappa)\\
=&i\varepsilon_{\frac{k}{\gcd(N,k)}}\left(\frac{\frac{-N}{\gcd(N,k)}}{\frac{k}{\gcd(N,k)}}\right)\left(\frac{-1}{\gcd(N,k)}\right)\sqrt{\frac{\gcd(N,k)}{2N}}\exp\left(\frac{2\pi i}{24k}\left(-3k^2+9k\right)\right)\sum_{\substack{h \pmod k\\ \gcd(h,k)=1}}\left(\frac{h}{\gcd(N,k)}\right)\\
&\times\left(\delta_{\gcd(N,k)|(hj-r)}\exp\left(\frac{2\pi i}{24k\gcd(N,k)}\left(48jr[4]'_{\frac{k}{\gcd(N,k)}}\left[\frac{N}{\gcd(N,k)}\right]'_{\frac{k}{\gcd(N,k)}}-\frac{12rj}{N}\gcd(N,k)\right)\right)\right.\\
&\qquad\left.-\delta_{\gcd(N,k)|(hj+r)}\exp\left(\frac{2\pi i}{24k\gcd(N,k)}\left(-48jr[4]'_{\frac{k}{\gcd(N,k)}}\left[\frac{N}{\gcd(N,k)}\right]'_{\frac{k}{\gcd(N,k)}}+\frac{12rj}{N}\gcd(N,k)\right)\right)\right)
\end{aligned}
$$

[9]Using that $h[h]'_{3k\gcd(N,k)}\equiv -1 \pmod{3k}$ since $(3k)\mid(3k\gcd(N,k))$.

$$\times \exp\left(\frac{2\pi i}{24\gcd(N,k)k}\left(\left(\left(-24n+1-k^2\right)\gcd(N,k) - 24j^2[4]'_{\frac{k}{\gcd(N,k)}}\left[\frac{N}{\gcd(N,k)}\right]'_{\frac{k}{\gcd(N,k)}}\right)h\right.\right.$$
$$\left.\left. - \left(\left(-24N\kappa^2 - 24\kappa r + 1 - k^2\right)\gcd(N,k) - 24r^2[4]'_{\frac{k}{\gcd(N,k)}}\left[\frac{N}{\gcd(N,k)}\right]'_{\frac{k}{\gcd(N,k)}}\right)[h]'_{3k\gcd(N,k)}\right)\right)$$
$$=: K^*_{k,j,N,-}(n,r,\kappa) + K^*_{k,j,N,+}(n,r,\kappa).$$

Here we set

$$K^*_{k,j,N,\pm}(n,r,\kappa)$$
$$= \epsilon_{o,\pm}(k,j,N,r)\frac{1}{3\gcd(N,k)}\sum_{\substack{h \pmod{3\gcd(N,k)k}\\ \gcd(h,3\gcd(N,k)k)=1}}\left(\frac{h}{\gcd(N,k)}\right)\delta_{\gcd(N,k)|(hj+r)}$$
$$\times \exp\left(\frac{2\pi i}{3\gcd(N,k)k}\left(\left(\left(-3n+\frac{1-k^2}{8}\right)\gcd(N,k) - 3j^2[4]'_{\frac{k}{\gcd(N,k)}}\left[\frac{N}{\gcd(N,k)}\right]'_{\frac{k}{\gcd(N,k)}}\right)h\right.\right.$$
$$\left.\left. - \left(\left(-3N\kappa^2 - 3\kappa r + \frac{1-k^2}{8}\right)\gcd(N,k) - 3r^2[4]'_{\frac{k}{\gcd(N,k)}}\left[\frac{N}{\gcd(N,k)}\right]'_{\frac{k}{\gcd(N,k)}}\right)[h]'_{3k\gcd(N,k)}\right)\right)$$
$$=: \epsilon_{o,\pm}(k,j,N,r)\frac{1}{3\gcd(N,k)}$$
$$\times \sum_{\substack{h \pmod{3\gcd(N,k)k}\\ \gcd(h,3\gcd(N,k)k)=1}}\left(\frac{h}{\gcd(N,k)}\right)\delta_{\gcd(N,k)|(hj+r)}\exp\left(\frac{2\pi i}{3\gcd(N,k)k}\left(\mu_2 h - \nu_2[h]'_{3k\gcd(N,k)}\right)\right)$$

and, by orthogonality of roots of unity, we finally have

$$K^*_{k,j,N,\pm}(n,r,\kappa) = \epsilon_{o,\pm}(k,j,N,r)\frac{1}{3\gcd(N,k)^2}\sum_{s=0}^{\gcd(N,k)-1}\exp\left(\pm 2\pi i\frac{rs}{\gcd(N,k)}\right) \tag{IV.3.7}$$
$$\times \sum_{\substack{h \pmod{3\gcd(N,k)k}\\ \gcd(h,3\gcd(N,k)k)=1}}\left(\frac{h}{\gcd(N,k)}\right)\exp\left(\frac{2\pi i}{3\gcd(N,k)k}\left(\left(\mu_2+3jsk\right)h - \nu_2[h]'_{3k\gcd(N,k)}\right)\right).$$

Since in (IV.3.6) and (IV.3.7) both sums over h are of the required shape we can bound them using Malishev's result (see Lemma IV.1.3) and obtain that they are

$$\begin{cases} O\left((\gcd(N,k)k)^{\frac{1}{2}+\varepsilon}\min\left(\gcd\left(\mu_1+jsk,\gcd(N,k)k\right)^{\frac{1}{2}},\gcd\left(\nu_1,\gcd(N,k)k\right)^{\frac{1}{2}}\right)\right) & \text{if } 3\nmid k,\\ O\left((3\gcd(N,k)k)^{\frac{1}{2}+\varepsilon}\min\left(\gcd\left(\mu_2+3jsk,3\gcd(N,k)k\right)^{\frac{1}{2}},\gcd\left(\nu_2,3\gcd(N,k)k\right)^{\frac{1}{2}}\right)\right) & \text{if } 3\mid k,\end{cases}$$

for $\varepsilon > 0$. We see that $\gcd(N,k) \le N = O_N(1)$, and, by Lemma V.2.4,

$$\min\left(\gcd\left(\mu_1+jsk,\gcd(N,k)k\right)^{\frac{1}{2}},\gcd\left(\nu_1,\gcd(N,k)k\right)^{\frac{1}{2}}\right) = O_N\left(n^{\frac{1}{2}}\right),$$

and

$$\min\left(\gcd\left(\mu_2+3jsk,3\gcd(N,k)k\right)^{\frac{1}{2}},\gcd\left(\nu_2,3\gcd(N,k)k\right)^{\frac{1}{2}}\right) = O_N\left(n^{\frac{1}{2}}\right).$$

Thus we showed that

$$\begin{aligned}K_{k,j,N,\pm}(n,r,\kappa) =&O_N\left(\left|\epsilon_{o,\pm}(k,j,N,r)\frac{1}{\operatorname{gcd}(N,k)^2}\right|\sum_{s=0}^{\operatorname{gcd}(N,k)-1}\left|\exp\left(\pm 2\pi i\frac{rs}{\operatorname{gcd}(N,k)}\right)\right|n^{\frac{1}{2}}k^{\frac{1}{2}+\varepsilon}\right)\\ =&O_N\left(n^{\frac{1}{2}}k^{\frac{1}{2}+\varepsilon}\right)\end{aligned}$$

and analogously $K^*_{k,j,N,\pm}(n,r,\kappa) = O_N(n^{\frac{1}{2}}k^{\frac{1}{2}+\varepsilon})$, which yields

$$K_{k,j,N}(n,r,\kappa) =O_N\left(n^{\frac{1}{2}}k^{\frac{1}{2}+\varepsilon}\right)$$

and finishes the proof for odd k.

IV.3.2 Even k

We go on with the case of even k and have that

$$\begin{aligned}&\chi_{j,r}(N,M_{h,k})\\ &=-i\left(\frac{k}{-h}\right)\sqrt{\frac{1}{2Nk}}\exp\left(2\pi i\left(\frac{1}{24}\left(h'k\left(1-(-h)^2\right)+(-h)\left(-\frac{hh'+1}{k}-k+3\right)-3\right)+\frac{3}{8}\right)\right)\\ &\qquad\times\exp\left(\frac{2\pi i}{4Nk}\left(hj^2-h'r^2+2rj\right)\right)G\left(hN,hj+r,k\right)\\ &\quad+i\left(\frac{k}{-h}\right)\sqrt{\frac{1}{2Nk}}\exp\left(2\pi i\left(\frac{1}{24}\left(h'k\left(1-(-h)^2\right)+(-h)\left(-\frac{hh'+1}{k}-k+3\right)-3\right)+\frac{3}{8}\right)\right)\\ &\qquad\times\exp\left(\frac{2\pi i}{4Nk}\left(hj^2-h'r^2-2rj\right)\right)G\left(hN,hj-r,k\right),\end{aligned}$$

using (IV.3.1). For even k we can write $k = 2^\nu\mu$ with $\nu \geq 1$ and μ odd. Using the multiplicativity of the generalized quadratic Gauss sum[10], we thus have that

$$G(hN,hj\pm r,k) = G(hN,hj\pm r,2^\nu\mu) = G(hN2^\nu,hj\pm r,\mu)G(hN\mu,hj\pm r,2^\nu).$$

Defining $\alpha \coloneqq \max(x : 2^x \mid (hN\mu)) = \max(x : 2^x \mid N)$ we obtain

$$\begin{aligned}&G(hN\mu,hj\pm r,2^\nu)\\ &=\begin{cases}2^\nu & \text{if } \nu-\alpha=1 \text{ and } hj\pm r\not\equiv 0 \pmod{2},\\ 2^{\frac{\nu+\alpha}{2}}(i+1)\left(\frac{-2^{\nu+\alpha}}{\frac{hN\mu}{2^\alpha}}\right)\varepsilon_{\frac{hN\mu}{2^\alpha}}\exp\left(-2\pi i\frac{\left[\frac{hN\mu}{2^\alpha}\right]'_{2^{\nu+\alpha+2}}\frac{(hj\pm r)^2}{4}}{2^{\nu+\alpha}}\right) & \text{if } \nu-\alpha>1 \text{ and } hj\pm r\equiv 0 \pmod{2^{\alpha+1}},\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

[10]For given $a,c,d\in\mathbb{N}$, $b \pmod{c}$ and $\operatorname{gcd}(c,d)=1$ we have that $G(a,b,cd)=G(ac,b,d)G(ad,b,c)$.

Noting that $\gcd(hN2^\nu, \mu) = \gcd(hN, \mu) = \gcd(N, \mu)$, $hj \pm r \equiv hj + r \pmod 2$, and combining this with (IV.3.4) yields

$$G(hN, hj \pm r, 2^\nu \mu) = \begin{cases} 2^\nu A_\pm & \text{if } \nu - \alpha = 1,\ hj + r \not\equiv 0 \pmod 2, \\ & \text{and } \gcd(N,\mu) \mid (hj \pm r), \\ 2^{\frac{\nu+\alpha}{2}} A_\pm (i+1) \left(\frac{-2^{\nu+\alpha}}{\frac{hN\mu}{2^\alpha}}\right) \varepsilon_{\frac{hN\mu}{2^\alpha}} \exp\left(-2\pi i \frac{\left[\frac{hN\mu}{2^\alpha}\right]'_{2^{\nu+\alpha+2}} \frac{(hj\pm r)^2}{4}}{2^{\nu+\alpha}}\right) & \text{if } \nu - \alpha > 1,\ hj \pm r \equiv 0 \pmod{2^{\alpha+1}}, \\ & \text{and } \gcd(N,\mu) \mid (hj \pm r), \\ 0 & \text{otherwise,} \end{cases}$$

with

$$A_\pm := \gcd(N,\mu)\varepsilon_{\frac{\mu}{\gcd(N,\mu)}} \sqrt{\frac{\mu}{\gcd(N,\mu)}} \left(\frac{\frac{Nh2^\nu}{\gcd(N,\mu)}}{\frac{\mu}{\gcd(N,\mu)}}\right) \exp\left(-2\pi i \frac{\widetilde{\psi}\left(\frac{Nh2^\nu}{\gcd(N,\mu)}\right)\left(\frac{hj\pm r}{\gcd(N,\mu)}\right)^2}{\frac{\mu}{\gcd(N,\mu)}}\right),$$

$\varepsilon_{\frac{hN\mu}{2^\alpha}}$ and $\varepsilon_{\frac{\mu}{\gcd(N,\mu)}}$ as in (IV.3.3), and where $\widetilde{\psi}(a)$ is some number satisfying

$$4\widetilde{\psi}(a)a \equiv 1 \left(\operatorname{mod} \frac{\mu}{\gcd(N,\mu)}\right).$$

Note that in the first case we have that $\nu = \alpha + 1$ which gives us that $2^\nu = 2^{\alpha+1} \leq 2N$, and allows us to say that $2^\nu = O_N(1)$.

Using that for even k the h we are summing over have to be odd we split our Kloosterman sum as follows

$$\begin{aligned} &K_{k,j,N}(n,r,\kappa) \\ &= \Bigg(\delta_{\substack{\nu-\alpha=1 \\ j\not\equiv r \pmod 2}} \Bigg(\sum_{\substack{0\le h<k \\ \gcd(h,k)=1 \\ \gcd(N,\mu)\mid(hj+r)}} + \sum_{\substack{0\le h<k \\ \gcd(h,k)=1 \\ \gcd(N,\mu)\mid(hj-r)}}\Bigg) + \delta_{\nu-\alpha>1}\Bigg(\sum_{\substack{0\le h<k \\ \gcd(h,k)=1 \\ (\gcd(N,\mu)2^{\alpha+1})\mid(hj+r)}} + \sum_{\substack{0\le h<k \\ \gcd(h,k)=1 \\ (\gcd(N,\mu)2^{\alpha+1})\mid(hj-r)}}\Bigg)\Bigg) \\ &\quad \times \chi_{j,r}(N, M_{h,k}) \zeta_{24k}^{\left(24N\left(\kappa+\frac{r}{2N}\right)^2-1\right)h' - 24\left(n+\frac{j^2}{4N}-\frac{1}{24}\right)h} \\ &=: K_{k,j,N,1,+}(n,r,\kappa) + K_{k,j,N,1,-}(n,r,\kappa) + K_{k,j,N,2,+}(n,r,\kappa) + K_{k,j,N,2,-}(n,r,\kappa) \\ &=: K_{k,j,N,1}(n,r,\kappa) + K_{k,j,N,2}(n,r,\kappa). \end{aligned}$$

For $K_{k,j,N,1}(n,r,\kappa)$ we can run a similar calculation as in the odd k case. By definition we have that $-4a\widetilde{\psi}(a) \equiv -1 \ (\operatorname{mod} \frac{\mu}{\gcd(N,\mu)})$, which gives us that

$$\widetilde{\psi}(a) = [-4a]'_{\frac{\mu}{\gcd(N,\mu)}} = [4]'_{\frac{\mu}{\gcd(N,\mu)}}[a]'_{\frac{\mu}{\gcd(N,\mu)}}$$

and thus

$$\widetilde{\psi}\left(\frac{Nh2^\nu}{\gcd(N,\mu)}\right) = [4]'_{\frac{\mu}{\gcd(N,\mu)}} \left[\frac{Nh2^\nu}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}} = -[4]'_{\frac{\mu}{\gcd(N,\mu)}} \left[\frac{N2^\nu}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}} [h]'_{\frac{\mu}{\gcd(N,\mu)}}.$$

Note that

$$\exp\left(-2\pi i \frac{\widetilde{\psi}\left(\frac{Nh2^\nu}{\gcd(N,\mu)}\right)\left(\frac{hj\pm r}{\gcd(N,\mu)}\right)^2}{\frac{\mu}{\gcd(N,\mu)}}\right) = \exp\left(-2\pi i \frac{-[4]'_{\frac{\mu}{\gcd(N,\mu)}} \left[\frac{N2^\nu}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}} [h]'_{\frac{\mu}{\gcd(N,\mu)}} \left(\frac{hj\pm r}{\gcd(N,\mu)}\right)^2}{\frac{\mu}{\gcd(N,\mu)}}\right)$$

is well-defined for $[a]'_{\frac{\mu}{\gcd(N,\mu)}}$ a solution of $a\,[a]'_{\frac{\mu}{\gcd(N,\mu)}} \equiv -1 \pmod{\frac{\mu}{\gcd(N,\mu)}}$, since it is invariant under any shifts by $\frac{\mu}{\gcd(N,\mu)}$, because $\frac{hj\pm r}{\gcd(N,\mu)} \in \mathbb{Z}$ by assumption.

For simplicity we stick to the notation $[h]'_k = h'$. For $K_{k,j,N,1}(n,r,\kappa)$ we obtain that

$$\begin{aligned}
&\chi_{j,r}(N, M_{h,k}) \\
&= -i\varepsilon_{\frac{\mu}{\gcd(N,\mu)}} \left(\frac{k}{-h}\right) \left(\frac{\frac{Nh2^\nu}{\gcd(N,\mu)}}{\frac{\mu}{\gcd(N,\mu)}}\right) \sqrt{\frac{\gcd(N,\mu)2^\nu}{2N}} \delta_{\gcd(N,\mu)|(hj+r)} \\
&\quad \times \exp\left(-2\pi i \frac{-[4]'_{\frac{\mu}{\gcd(N,\mu)}} \left[\frac{N2^\nu}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}} [h]'_{\frac{\mu}{\gcd(N,\mu)}} \left(\frac{hj+r}{\gcd(N,\mu)}\right)^2}{\frac{\mu}{\gcd(N,\mu)}}\right) \\
&\quad \times \exp\left(2\pi i \left(\frac{1}{24}\left(h'k\left(1-h^2\right) - h\left(-\frac{hh'+1}{k} - k + 3\right) - 3\right) + \frac{3}{8} + \frac{hj^2 - h'r^2 + 2rj}{4Nk}\right)\right) \\
&\quad + i\varepsilon_{\frac{\mu}{\gcd(N,\mu)}} \left(\frac{k}{-h}\right) \left(\frac{\frac{Nh2^\nu}{\gcd(N,\mu)}}{\frac{\mu}{\gcd(N,\mu)}}\right) \sqrt{\frac{\gcd(N,\mu)2^\nu}{2N}} \delta_{\gcd(N,\mu)|(hj-r)} \\
&\quad \times \exp\left(-2\pi i \frac{-[4]'_{\frac{\mu}{\gcd(N,\mu)}} \left[\frac{N2^\nu}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}} [h]'_{\frac{\mu}{\gcd(N,\mu)}} \left(\frac{hj-r}{\gcd(N,\mu)}\right)^2}{\frac{\mu}{\gcd(N,\mu)}}\right) \\
&\quad \times \exp\left(2\pi i \left(\frac{1}{24}\left(h'k\left(1-h^2\right) - h\left(-\frac{hh'+1}{k} - k + 3\right) - 3\right) + \frac{3}{8} + \frac{hj^2 - h'r^2 - 2rj}{4Nk}\right)\right).
\end{aligned}$$

Using quadratic reciprocity together with $\left(\frac{k}{-h}\right) = \operatorname{sgn}(k)\left(\frac{k}{h}\right) = \left(\frac{k}{h}\right)$ we have

$$\begin{aligned}
\left(\frac{k}{-h}\right)\left(\frac{\frac{Nh2^\nu}{\gcd(N,\mu)}}{\frac{\mu}{\gcd(N,\mu)}}\right) &= \left(\frac{2}{h}\right)^\nu \left(\frac{\mu}{h}\right) \left(\frac{h}{\frac{\mu}{\gcd(N,\mu)}}\right) \left(\frac{\frac{N2^\nu}{\gcd(N,\mu)}}{\frac{\mu}{\gcd(N,\mu)}}\right) \\
&= \left((-1)^{\frac{h^2-1}{8}}\right)^\nu (-1)^{\frac{(\mu-1)(h-1)}{4}} \left(\frac{h}{\gcd(N,\mu)}\right) \left(\frac{\frac{N2^\nu}{\gcd(N,\mu)}}{\frac{\mu}{\gcd(N,\mu)}}\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
K_{k,j,N,1}(n,r,\kappa) =& i\varepsilon_{\frac{\mu}{\gcd(N,\mu)}} \left(\frac{\frac{N2^\nu}{\gcd(N,\mu)}}{\frac{\mu}{\gcd(N,\mu)}}\right) \sqrt{\frac{\gcd(N,\mu)2^\nu}{2N}} \exp\left(\frac{2\pi i}{4}\right) \delta_{\substack{\nu-\alpha=1\\ j\not\equiv r \pmod 2}} \\
&\times \sum_{\substack{h \pmod k\\ \gcd(h,k)=1}} \left((-1)^{\frac{h^2-1}{8}}\right)^\nu (-1)^{\frac{(\mu-1)(h-1)}{4}} \left(\frac{h}{\gcd(N,\mu)}\right) \\
&\times \exp\left(\frac{2\pi i}{24k}\left(\left(-24n+2+k^2-3k\right)h - \left(-24N\kappa^2 - 24\kappa r + 1 - k^2\right)h'\right)\right) \\
&\times \left(-\delta_{\gcd(N,\mu)|(hj+r)} \exp\left(2\pi i \frac{[4]'_{\frac{\mu}{\gcd(N,\mu)}} \left[\frac{N2^\nu}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}} [h]'_{\frac{\mu}{\gcd(N,\mu)}} \left(\frac{hj+r}{\gcd(N,\mu)}\right)^2}{\frac{\mu}{\gcd(N,\mu)}}\right)\right. \\
&\qquad \times \exp\left(\frac{2\pi i}{24k}\left(-h^2h'k^2 + h^2h' + \frac{12rj}{N}\right)\right) \\
&\qquad +\delta_{\gcd(N,\mu)|(hj-r)} \exp\left(2\pi i \frac{[4]'_{\frac{\mu}{\gcd(N,\mu)}} \left[\frac{N2^\nu}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}} [h]'_{\frac{\mu}{\gcd(N,\mu)}} \left(\frac{hj-r}{\gcd(N,\mu)}\right)^2}{\frac{\mu}{\gcd(N,\mu)}}\right) \\
&\qquad \left.\times \exp\left(\frac{2\pi i}{24k}\left(-h^2h'k^2 + h^2h' - \frac{12rj}{N}\right)\right)\right).
\end{aligned}$$

We already saw that the following is well-defined and now observe that

$$\begin{aligned}
&\exp\left(2\pi i \frac{[4]'_{\frac{\mu}{\gcd(N,\mu)}} \left[\frac{N2^\nu}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}} [h]'_{\frac{\mu}{\gcd(N,\mu)}} \left(\frac{hj\pm r}{\gcd(N,\mu)}\right)^2}{\frac{\mu}{\gcd(N,\mu)}}\right) \\
=& \exp\left(\frac{2\pi i}{k\gcd(N,\mu)}\left(2^\nu [4]'_{\frac{\mu}{\gcd(N,\mu)}} \left[\frac{N2^\nu}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}} [h]'_{\frac{\mu}{\gcd(N,\mu)}} \left(h^2j^2 \pm 2hjr + r^2\right)\right)\right).
\end{aligned}$$

Choose $[h]'_{\frac{\mu}{\gcd(N,\mu)}} = h'$ here[11]. Analogously to above we can choose h' such that $hh' \equiv -1 \pmod{xk}$ for some $x \in \mathbb{N}$ such that $\gcd(x,h) = 1$. Taking $x = \gcd(N,\mu)$ we obtain

$$\begin{aligned}
&K_{k,j,N,1}(n,r,\kappa) \\
&= i\varepsilon_{\frac{\mu}{\gcd(N,\mu)}} \left(\frac{\frac{N2^\nu}{\gcd(N,\mu)}}{\frac{\mu}{\gcd(N,\mu)}}\right) \sqrt{\frac{\gcd(N,\mu)2^\nu}{2N}} \exp\left(\frac{2\pi i}{4}\right) \delta_{\substack{\nu-\alpha=1\\ j\not\equiv r \pmod 2}}
\end{aligned}$$

[11]Note that $hh' \equiv -1 \pmod k$ implies that $hh' \equiv -1 \pmod{\frac{\mu}{\gcd(N,\mu)}}$, since $\frac{\mu}{\gcd(N,\mu)} \mid k$. Thus h' is a possible choice for $[h]'_{\frac{\mu}{\gcd(N,\mu)}}$.

$$
\begin{aligned}
&\times \sum_{\substack{h \pmod{k}\\ \gcd(h,k)=1}} \left((-1)^{\frac{h^2-1}{8}}\right)^{\nu} (-1)^{\frac{(\mu-1)(h-1)}{4}} \left(\frac{h}{\gcd(N,\mu)}\right)\\
&\times \exp\left(\frac{2\pi i}{24k\gcd(N,\mu)}\left(\left(\left(-24n+2+k^2-3k\right)\gcd(N,\mu) - 24j^2 2^{\nu}[4]'_{\frac{\mu}{\gcd(N,\mu)}} \left[\frac{N2^{\nu}}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}}\right) h\right.\right.\\
&\qquad \left.\left. - \left(\left(-24N\kappa^2 - 24\kappa r + 1 - k^2\right)\gcd(N,\mu) - 24r^2 2^{\nu}[4]'_{\frac{\mu}{\gcd(N,\mu)}} \left[\frac{N2^{\nu}}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}}\right)[h]'_{k\gcd(N,\mu)}\right)\right)\\
&\times \left(-\delta_{\gcd(N,\mu)|(hj+r)} \exp\left(\frac{2\pi i}{k\gcd(N,\mu)}\left(-2^{\nu+1}jr[4]'_{\frac{\mu}{\gcd(N,\mu)}}\left[\frac{N2^{\nu}}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}}\right)\right)\right.\\
&\qquad \times \exp\left(\frac{2\pi i}{24k}\left(h^2[h]'_{k\gcd(N,\mu)}(1-k^2) + \frac{12rj}{N}\right)\right)\\
&\qquad + \delta_{\gcd(N,\mu)|(hj-r)} \exp\left(\frac{2\pi i}{k\gcd(N,\mu)}\left(2^{\nu+1}jr[4]'_{\frac{\mu}{\gcd(N,\mu)}}\left[\frac{N2^{\nu}}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}}\right)\right)\\
&\qquad \left.\times \exp\left(\frac{2\pi i}{24k}\left(h^2[h]'_{k\gcd(N,\mu)}(1-k^2) - \frac{12rj}{N}\right)\right)\right).
\end{aligned}
$$

We now need to split into two cases, namely $3 \nmid k$ and $3 \mid k$. In the first case we obtain that $3 \mid (k^2-1)$. Choosing $[h]'_{k\gcd(N,\mu)}$ such that $h[h]'_{k\gcd(N,\mu)} \equiv -1 \pmod{8k\gcd(N,\mu)}$, analogously to above[12], yields[13]

$$
\begin{aligned}
&K_{k,j,N,1}(n,r,\kappa)\\
=& i\varepsilon_{\frac{\mu}{\gcd(N,\mu)}} \left(\frac{\frac{N2^{\nu}}{\gcd(N,\mu)}}{\frac{\mu}{\gcd(N,\mu)}}\right) \sqrt{\frac{\gcd(N,\mu)2^{\nu}}{2N}} \exp\left(\frac{2\pi i}{4}\right) \delta_{\substack{\nu-\alpha=1\\ j\not\equiv r \pmod{2}}}\\
&\times \sum_{\substack{h \pmod{k}\\ \gcd(h,k)=1}} \left((-1)^{\frac{h^2-1}{8}}\right)^{\nu} (-1)^{\frac{(\mu-1)(h-1)}{4}} \left(\frac{h}{\gcd(N,\mu)}\right)\\
&\times \left(-\delta_{\gcd(N,\mu)|(hj+r)} \exp\left(\frac{2\pi i}{24k\gcd(N,\mu)}\left(-48\cdot 2^{\nu}jr[4]'_{\frac{\mu}{\gcd(N,\mu)}}\left[\frac{N2^{\nu}}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}} + \frac{12rj}{N}\gcd(N,\mu)\right)\right)\right.\\
&\qquad \left.+ \delta_{\gcd(N,\mu)|(hj-r)} \exp\left(\frac{2\pi i}{24k\gcd(N,\mu)}\left(48\cdot 2^{\nu}jr[4]'_{\frac{\mu}{\gcd(N,\mu)}}\left[\frac{N2^{\nu}}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}} - \frac{12rj}{N}\gcd(N,\mu)\right)\right)\right)\\
&\times \exp\left(\frac{2\pi i}{24k\gcd(N,\mu)}\left(\left(\left(-24n+1+2k^2-3k\right)\gcd(N,\mu) - 24j^2 2^{\nu}[4]'_{\frac{\mu}{\gcd(N,\mu)}}\left[\frac{N2^{\nu}}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}}\right) h\right.\right.\\
&\qquad \left.\left. - \left(\left(-24N\kappa^2 - 24\kappa r + 1 - k^2\right)\gcd(N,\mu) - 24r^2 2^{\nu}[4]'_{\frac{\mu}{\gcd(N,\mu)}}\left[\frac{N2^{\nu}}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}}\right)[h]'_{8k\gcd(N,\mu)}\right)\right)\\
=:& K_{k,j,N,1,+}(n,r,\kappa) + K_{k,j,N,1,-}(n,r,\kappa),
\end{aligned}
$$

[12]We are allowed to do this since $\gcd(8,h)=1$, this is because we know that h is odd.

[13]Using that $h[h]'_{8k\gcd(N,\mu)} \equiv -1 \pmod{8k}$ since $(8k) \mid (8k\gcd(N,\mu))$.

where

$$
\begin{aligned}
K_{k,j,N,1,\pm}(n,r,\kappa) =& \epsilon^*_{e,\pm}(k,j,N,r)\frac{1}{8\gcd(N,\mu)^2}\sum_{s=0}^{\gcd(N,\mu)-1}\exp\left(\pm 2\pi i\frac{rs}{\gcd(N,\mu)}\right)\\
&\times \sum_{\substack{h \pmod{8\gcd(N,\mu)k}\\ \gcd(h,8\gcd(N,\mu)k)=1}} (-1)^{\frac{h^2-1}{8}\nu+\frac{(\mu-1)(h-1)}{4}}\left(\frac{h}{\gcd(N,\mu)}\right)\\
&\times \exp\left(\frac{2\pi i}{8k\gcd(N,\mu)}\left((\mu_3+8jsk)\,h-\nu_3[h]'_{8k\gcd(N,\mu)}\right)\right),
\end{aligned}
$$

with

$$
\begin{aligned}
\epsilon^*_{e,\pm}(k,j,N,r) :=& \mp i\varepsilon_{\frac{\mu}{\gcd(N,\mu)}}\left(\frac{\frac{N2^\nu}{\gcd(N,\mu)}}{\frac{\mu}{\gcd(N,\mu)}}\right)\sqrt{\frac{\gcd(N,\mu)2^\nu}{2N}}\exp\left(\frac{2\pi i}{4}\right)\delta_{\substack{\nu-\alpha=1\\ j\not\equiv r \pmod 2}}\\
&\times \exp\left(\frac{2\pi i}{24k\gcd(N,\mu)}\left(\mp 48\cdot 2^\nu jr[4]'_{\frac{\mu}{\gcd(N,\mu)}}\left[\frac{N2^\nu}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}}\pm\frac{12rj}{N}\gcd(N,\mu)\right)\right),\\
\mu_3 :=& \left(-8n+\frac{1+2k^2}{3}-k\right)\gcd(N,\mu)-8j^2 2^\nu[4]'_{\frac{\mu}{\gcd(N,\mu)}}\left[\frac{N2^\nu}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}},\\
\nu_3 :=& \left(-8N\kappa^2-8\kappa r+\frac{1-k^2}{3}\right)\gcd(N,\mu)-8r^2 2^\nu[4]'_{\frac{\mu}{\gcd(N,\mu)}}\left[\frac{N2^\nu}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}}.
\end{aligned}
$$

Note that $\mu_3,\nu_3\in\mathbb{Z}$, since $k^2-1\equiv 0 \pmod 3$ is equivalent to $2k^2+1\equiv 0 \pmod 3$.

Lastly we use a small trick to rewrite our Kloosterman sum into the shape that we want. First we note that $16 \mid (8k\gcd(N,\mu))$ and that $(-1)^{\frac{h^2-1}{8}\nu+\frac{(\mu-1)(h-1)}{4}}$ only depends on h modulo 16. Thus we obtain

$$
\begin{aligned}
&K_{k,j,N,1,\pm}(n,r,\kappa) \qquad\qquad (\text{IV.3.8})\\
=& \epsilon^*_{e,\pm}(k,j,N,r)\frac{1}{8\gcd(N,\mu)^2}\sum_{s=0}^{\gcd(N,\mu)-1}\exp\left(\pm 2\pi i\frac{rs}{\gcd(N,\mu)}\right)\frac{1}{16}\sum_{j \pmod{16}}(-1)^{\frac{j^2-1}{8}\nu+\frac{(\mu-1)(j-1)}{4}}\sum_{\ell \pmod{16}}e^{\frac{-2\pi i j\ell}{16}}\\
&\times \sum_{\substack{h \pmod{8\gcd(N,\mu)k}\\ \gcd(h,8\gcd(N,\mu)k)=1}}\left(\frac{h}{\gcd(N,\mu)}\right)\exp\left(\frac{2\pi i}{8k\gcd(N,\mu)}\left(\left(\mu_3+8jsk+\frac{8\ell\gcd(N,\mu)k}{16}\right)h-\nu_3[h]'_{8k\gcd(N,\mu)}\right)\right),
\end{aligned}
$$

using the orthogonality of roots of unity

$$
\frac{1}{16}\sum_{\ell \pmod{16}} e^{\frac{2\pi i a\ell}{16}} = \begin{cases} 1 & \text{if } 16 \mid a,\\ 0 & \text{otherwise.}\end{cases}
$$

In the second case, $3 \mid k$, we have that $3 \nmid h$ and thus $\gcd(24,h)=1$. Choosing $[h]'_{k\gcd(N,\mu)}$ such that $h[h]'_{k\gcd(N,\mu)} \equiv -1 \pmod{24k\gcd(N,\mu)}$, analogously to above, yields[14]

$$K_{k,j,N,1}(n,r,\kappa) =: K^*_{k,j,N,1,+}(n,r,\kappa) + K^*_{k,j,N,1,-}(n,r,\kappa),$$

where analogously to the first case

$$\begin{aligned} &K^*_{k,j,N,1,\pm}(n,r,\kappa) \\ &= \epsilon^*_{e,\pm}(k,j,N,r)\frac{1}{24\gcd(N,\mu)^2}\sum_{s=0}^{\gcd(N,\mu)-1}\exp\left(\pm 2\pi i\frac{rs}{\gcd(N,\mu)}\right)\frac{1}{16}\sum_{j \pmod{16}}(-1)^{\frac{j^2-1}{8}\nu+\frac{(\mu-1)(j-1)}{4}}\sum_{\ell \pmod{16}} e^{\frac{-2\pi i j\ell}{16}} \\ &\times \sum_{\substack{h \pmod{24\gcd(N,\mu)k} \\ \gcd(h,24\gcd(N,\mu)k)=1}}\left(\frac{h}{\gcd(N,\mu)}\right)\exp\left(\frac{2\pi i}{24k\gcd(N,\mu)}\left(\left(\mu_4+24jsk+\frac{24\ell\gcd(N,\mu)k}{16}\right)h-\nu_4[h]'_{24k\gcd(N,\mu)}\right)\right), \end{aligned} \tag{IV.3.9}$$

with

$$\mu_4 := \left(-24n+1+2k^2-3k\right)\gcd(N,\mu) - 24j^2 2^\nu [4]'_{\frac{\mu}{\gcd(N,\mu)}}\left[\frac{N2^\nu}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}},$$

$$\nu_4 := \left(-24N\kappa^2-24\kappa r+1-k^2\right)\gcd(N,\mu) - 24r^2 2^\nu [4]'_{\frac{\mu}{\gcd(N,\mu)}}\left[\frac{N2^\nu}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}}.$$

We now note that we can bound (IV.3.8), respectively (IV.3.9), by

$$\begin{aligned} &\left|K_{k,j,N,1,\pm}(n,r,\kappa)\right| \\ &\le \left|\epsilon^*_{e,\pm}(k,j,N,r)\frac{1}{8\gcd(N,\mu)^2}\frac{1}{16}\right|\sum_{s=0}^{\gcd(N,\mu)-1}\sum_{j \pmod{16}}\sum_{\ell \pmod{16}} \\ &\times \left|\sum_{\substack{h \pmod{8\gcd(N,\mu)k} \\ \gcd(h,8\gcd(N,\mu)k)=1}}\left(\frac{h}{\gcd(N,\mu)}\right)\exp\left(\frac{2\pi i}{8k\gcd(N,\mu)}\left(\left(\mu_3+8jsk+\frac{8\ell\gcd(N,\mu)k}{16}\right)h-\nu_3[h]'_{8k\gcd(N,\mu)}\right)\right)\right|, \end{aligned}$$

respectively

$$\begin{aligned} &\left|K^*_{k,j,N,1,\pm}(n,r,\kappa)\right| \\ &\le \left|\epsilon^*_{e,\pm}(k,j,N,r)\frac{1}{24\gcd(N,\mu)^2}\frac{1}{16}\right|\sum_{s=0}^{\gcd(N,\mu)-1}\sum_{j \pmod{16}}\sum_{\ell \pmod{16}} \\ &\times \left|\sum_{\substack{h \pmod{24\gcd(N,\mu)k} \\ \gcd(h,24\gcd(N,\mu)k)=1}}\left(\frac{h}{\gcd(N,\mu)}\right)\exp\left(\frac{2\pi i}{24k\gcd(N,\mu)}\left(\left(\mu_4+24jsk+\frac{24\ell\gcd(N,\mu)k}{16}\right)h-\nu_4[h]'_{24k\gcd(N,\mu)}\right)\right)\right|. \end{aligned}$$

Both sums over h are of the required shape, so we can bound them using Malishev's result (see Lemma IV.1.3) and obtain that they are

$$O\left((8\gcd(N,\mu)k)^{\frac{1}{2}+\varepsilon}\min\left(\gcd\left(\mu_3+8jsk+\frac{8\ell\gcd(N,\mu)k}{16}, 8\gcd(N,\mu)k\right)^{\frac{1}{2}}, \gcd\left(\nu_3, 8\gcd(N,\mu)k\right)^{\frac{1}{2}}\right)\right),$$

[14]Using that $h[h]'_{24k\gcd(N,\mu)} \equiv -1 \pmod{24k}$ since $(24k) \mid (24k\gcd(N,\mu))$.

respectively

$$O\left((24\gcd(N,\mu)k)^{\frac{1}{2}+\varepsilon}\min\left(\gcd\left(\mu_4+24jsk+\frac{24\ell\gcd(N,\mu)k}{16},24\gcd(N,\mu)k\right)^{\frac{1}{2}},\gcd\left(\nu_4,24\gcd(N,\mu)k\right)^{\frac{1}{2}}\right)\right),$$

for $\varepsilon > 0$.

We see that $8\gcd(N,\mu) \leq 24\gcd(N,\mu) \leq 24N = O_N(1)$ and, analogously to Lemma V.2.4,

$$\min\left(\gcd\left(\mu_3+8jsk+\frac{8\ell\gcd(N,\mu)k}{16},8\gcd(N,\mu)k\right)^{\frac{1}{2}},\gcd\left(\nu_3,8\gcd(N,\mu)k\right)^{\frac{1}{2}}\right)=O_N\left(n^{\frac{1}{2}}\right),$$

and

$$\min\left(\gcd\left(\mu_4+24jsk+\frac{24\ell\gcd(N,\mu)k}{16},24\gcd(N,\mu)k\right)^{\frac{1}{2}},\gcd\left(\nu_4,24\gcd(N,\mu)k\right)^{\frac{1}{2}}\right)=O_N\left(n^{\frac{1}{2}}\right).$$

This yields

$$\begin{aligned}K_{k,j,N,1,\pm}(n,r,\kappa)=&O_N\left(\left|\epsilon^*_{e,\pm}(k,j,N,r)\frac{1}{8\gcd(N,\mu)^2}\frac{1}{16}\right|\sum_{s=0}^{\gcd(N,\mu)-1}\sum_{j \pmod{16}}\sum_{\ell \pmod{16}}k^{\frac{1}{2}+\varepsilon}n^{\frac{1}{2}}\right)\\=&O_N\left(\left|\epsilon^*_{e,\pm}(k,j,N,r)\frac{1}{8\gcd(N,\mu)^2}\frac{1}{16}\right|16^2\gcd(N,\mu)k^{\frac{1}{2}+\varepsilon}n^{\frac{1}{2}}\right)\\=&O_N\left(n^{\frac{1}{2}}k^{\frac{1}{2}+\varepsilon}\right),\end{aligned}$$

and analogously $K^*_{k,j,N,1,\pm}(n,r,\kappa) = O_N(n^{\frac{1}{2}}k^{\frac{1}{2}+\varepsilon})$, since $\epsilon^*_{e,\pm}(k,j,N,r) = O_N(1)$. We thus showed that

$$K_{k,j,N,1}(n,r,\kappa)=O_N\left(n^{\frac{1}{2}}k^{\frac{1}{2}+\varepsilon}\right).$$

The only thing left to do now is to look at $K_{k,j,N,2}(n,r,\kappa)$, where

$$G(hN,hj\pm r,2^\nu\mu)=2^{\frac{\nu+\alpha}{2}}A_\pm(i+1)\left(\frac{-2^{\nu+\alpha}}{\frac{hN\mu}{2^\alpha}}\right)\varepsilon_{\frac{hN\mu}{2^\alpha}}\exp\left(-2\pi i\frac{\left[\frac{hN\mu}{2^\alpha}\right]'_{2^{\nu+\alpha+2}}\frac{(hj\pm r)^2}{4}}{2^{\nu+\alpha}}\right).$$

Analogously to the calculations of $K_{k,j,N,1}$ we obtain that

$$\begin{aligned}&\chi_{j,r}(N,M_{h,k})\\&=\delta_{(\gcd(N,\mu)2^{\alpha+1})|(hj+r)}(1-i)\varepsilon_{\frac{hN\mu}{2^\alpha}}\varepsilon_{\frac{\mu}{\gcd(N,\mu)}}\left(\frac{k}{-h}\right)\left(\frac{-2^{\nu+\alpha}}{\frac{hN\mu}{2^\alpha}}\right)\left(\frac{\frac{Nh2^\nu}{\gcd(N,\mu)}}{\frac{\mu}{\gcd(N,\mu)}}\right)\sqrt{\frac{2^{\alpha-1}\gcd(N,\mu)}{N}}\end{aligned}$$

$$\times \exp\left(2\pi i\left(\frac{1}{24}\left(h'k\left(1-(-h)^2\right)+(-h)\left(-\frac{hh'+1}{k}-k+3\right)-3\right)+\frac{3}{8}\right)\right)$$
$$\times \exp\left(\frac{2\pi i}{4Nk}\left(hj^2-h'r^2+2rj\right)\right)\exp\left(-2\pi i\frac{\left[\frac{hN\mu}{2^\alpha}\right]'_{2^{\nu+\alpha+2}}\frac{(hj+r)^2}{4}}{2^{\nu+\alpha}}\right)$$
$$\times \exp\left(-2\pi i\frac{-[4]'_{\frac{\mu}{\gcd(N,\mu)}}\left[\frac{N2^\nu}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}}[h]'_{\frac{\mu}{\gcd(N,\mu)}}\left(\frac{hj+r}{\gcd(N,\mu)}\right)^2}{\frac{\mu}{\gcd(N,\mu)}}\right)$$
$$+\,\delta_{(\gcd(N,\mu)2^{\alpha+1})|(hj-r)}(i-1)\varepsilon_{\frac{hN\mu}{2^\alpha}}\varepsilon_{\frac{\mu}{\gcd(N,\mu)}}\left(\frac{k}{-h}\right)\left(\frac{-2^{\nu+\alpha}}{\frac{hN\mu}{2^\alpha}}\right)\left(\frac{\frac{Nh2^\nu}{\gcd(N,\mu)}}{\frac{\mu}{\gcd(N,\mu)}}\right)\sqrt{\frac{2^{\alpha-1}\gcd(N,\mu)}{N}}$$
$$\times \exp\left(2\pi i\left(\frac{1}{24}\left(h'k\left(1-(-h)^2\right)+(-h)\left(-\frac{hh'+1}{k}-k+3\right)-3\right)+\frac{3}{8}\right)\right)$$
$$\times \exp\left(\frac{2\pi i}{4Nk}\left(hj^2-h'r^2-2rj\right)\right)\exp\left(-2\pi i\frac{\left[\frac{hN\mu}{2^\alpha}\right]'_{2^{\nu+\alpha+2}}\frac{(hj-r)^2}{4}}{2^{\nu+\alpha}}\right)$$
$$\times \exp\left(-2\pi i\frac{-[4]'_{\frac{\mu}{\gcd(N,\mu)}}\left[\frac{N2^\nu}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}}[h]'_{\frac{\mu}{\gcd(N,\mu)}}\left(\frac{hj-r}{\gcd(N,\mu)}\right)^2}{\frac{\mu}{\gcd(N,\mu)}}\right).$$

Using quadratic reciprocity we have

$$\left(\frac{k}{-h}\right)\left(\frac{-2^{\nu+\alpha}}{\frac{hN\mu}{2^\alpha}}\right)\left(\frac{\frac{Nh2^\nu}{\gcd(N,\mu)}}{\frac{\mu}{\gcd(N,\mu)}}\right)$$
$$=\left(\frac{2}{h}\right)^\nu\left(\frac{\mu}{h}\right)\left(\frac{-1}{h}\right)\left(\frac{2}{h}\right)^{\nu+\alpha}\left(\frac{-2^{\nu+\alpha}}{\frac{N\mu}{2^\alpha}}\right)\left(\frac{h}{\frac{\mu}{\gcd(N,\mu)}}\right)\left(\frac{\frac{N2^\nu}{\gcd(N,\mu)}}{\frac{\mu}{\gcd(N,\mu)}}\right)$$
$$=\left((-1)^{\frac{h^2-1}{8}}\right)^\nu(-1)^{\frac{(\mu-1)(h-1)}{4}}(-1)^{\frac{h-1}{2}+\frac{h^2-1}{8}(\nu+\alpha)}\left(\frac{h}{\gcd(N,\mu)}\right)\left(\frac{-2^{\nu+\alpha}}{\frac{N\mu}{2^\alpha}}\right)\left(\frac{\frac{N2^\nu}{\gcd(N,\mu)}}{\frac{\mu}{\gcd(N,\mu)}}\right).$$

Therefore our Kloosterman sum equals

$$K_{k,j,N,2}(n,r,\kappa)$$
$$=(1-i)\varepsilon_{\frac{\mu}{\gcd(N,\mu)}}\sqrt{\frac{2^{\alpha-1}\gcd(N,\mu)}{N}}\left(\frac{-2^{\nu+\alpha}}{\frac{N\mu}{2^\alpha}}\right)\left(\frac{\frac{N2^\nu}{\gcd(N,\mu)}}{\frac{\mu}{\gcd(N,\mu)}}\right)\exp\left(\frac{2\pi i}{4}\right)\delta_{\nu-\alpha>1}$$
$$\times\sum_{\substack{h \pmod k\\ \gcd(h,k)=1}}(-1)^{\frac{h^2-1}{8}\nu+\frac{(\mu-1)(h-1)}{4}+\frac{h-1}{2}+\frac{h^2-1}{8}(\nu+\alpha)}\varepsilon_{\frac{hN\mu}{2^\alpha}}\left(\frac{h}{\gcd(N,\mu)}\right)$$

$$\times \exp\left(\frac{2\pi i}{24k}\left(\left(-24n+2+k^2-3k\right)h-\left(-24N\kappa^2-24\kappa r+1-k^2\right)h'\right)\right)$$

$$\times\left(\delta_{\left(\gcd(N,\mu)2^{\alpha+1}\right)|(hj+r)}\exp\left(\frac{2\pi i}{24k}\left(-h^2h'k^2+h^2h'+\frac{12rj}{N}\right)\right)\exp\left(-2\pi i\frac{\left[\frac{hN\mu}{2^\alpha}\right]'_{2^{\nu+\alpha+2}}\frac{(hj+r)^2}{4}}{2^{\nu+\alpha}}\right)\right.$$

$$\times\exp\left(2\pi i\frac{[4]'_{\frac{\mu}{\gcd(N,\mu)}}\left[\frac{N2^\nu}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}}[h]'_{\frac{\mu}{\gcd(N,\mu)}}\left(\frac{hj+r}{\gcd(N,\mu)}\right)^2}{\frac{\mu}{\gcd(N,\mu)}}\right)$$

$$-\delta_{\left(\gcd(N,\mu)2^{\alpha+1}\right)|(hj-r)}\exp\left(\frac{2\pi i}{24k}\left(-h^2h'k^2+h^2h'-\frac{12rj}{N}\right)\right)\exp\left(-2\pi i\frac{\left[\frac{hN\mu}{2^\alpha}\right]'_{2^{\nu+\alpha+2}}\frac{(hj-r)^2}{4}}{2^{\nu+\alpha}}\right)$$

$$\left.\times\exp\left(2\pi i\frac{[4]'_{\frac{\mu}{\gcd(N,\mu)}}\left[\frac{N2^\nu}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}}[h]'_{\frac{\mu}{\gcd(N,\mu)}}\left(\frac{hj-r}{\gcd(N,\mu)}\right)^2}{\frac{\mu}{\gcd(N,\mu)}}\right)\right).$$

We observe that

$$\exp\left(-2\pi i\frac{\left[\frac{hN\mu}{2^\alpha}\right]'_{2^{\nu+\alpha+2}}\frac{(hj\pm r)^2}{4}}{2^{\nu+\alpha}}\right)=\exp\left(\frac{2\pi i}{2^{\nu+\alpha+2}}\left(\left[\frac{N\mu}{2^\alpha}\right]'_{2^{\nu+\alpha+2}}\left(-hj^2\mp 2jr+[h]'_{2^{\nu+\alpha+2}}r^2\right)\right)\right).$$

Choose $[h]'_{\frac{\mu}{\gcd(N,\mu)}}=h'$ from now on[15]. Analogously to the odd k case or the calculations of $K_{k,j,N,1}$ we are able to choose h' such that $hh'\equiv -1 \pmod{2^{\alpha+2}\gcd(N,\mu)k}$. Choosing $[h]'_{2^{\nu+\alpha+2}}=[h]'_{2^{\alpha+2}k\gcd(N,\mu)}$ in addition[16], we obtain that

$$K_{k,j,N,2}(n,r,\kappa)$$
$$=(1-i)\varepsilon_{\frac{\mu}{\gcd(N,\mu)}}\sqrt{\frac{2^{\alpha-1}\gcd(N,\mu)}{N}}\left(\frac{-2^{\nu+\alpha}}{\frac{N\mu}{2^\alpha}}\right)\left(\frac{\frac{N2^\nu}{\gcd(N,\mu)}}{\frac{\mu}{\gcd(N,\mu)}}\right)\exp\left(\frac{2\pi i}{4}\right)\delta_{\nu-\alpha>1}$$
$$\times\sum_{\substack{h\pmod k\\ \gcd(h,k)=1}}(-1)^{\frac{h^2-1}{8}\nu+\frac{(\mu-1)(h-1)}{4}+\frac{h-1}{2}+\frac{h^2-1}{8}(\nu+\alpha)}\varepsilon_{\frac{hN\mu}{2^\alpha}}\left(\frac{h}{\gcd(N,\mu)}\right)$$
$$\times\exp\left(\frac{2\pi i}{24\cdot 2^{\alpha+2}k\gcd(N,\mu)}\left(\left(\left(-24n+2+k^2-3k\right)2^{\alpha+2}\gcd(N,\mu)-24j^2 2^{\nu+\alpha+2}[4]'_{\frac{\mu}{\gcd(N,\mu)}}\left[\frac{N2^\nu}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}}\right.\right.\right.$$
$$\left.-24\mu j^2\left[\frac{N\mu}{2^\alpha}\right]'_{2^{\nu+\alpha+2}}\gcd(N,\mu)\right)h$$
$$-\left(\left(-24N\kappa^2-24\kappa r+1-k^2\right)2^{\alpha+2}\gcd(N,\mu)-24r^2 2^{\nu+\alpha+2}[4]'_{\frac{\mu}{\gcd(N,\mu)}}\left[\frac{N2^\nu}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}}\right.$$

[15] Note that $hh'\equiv -1 \pmod k$ implies that $hh'\equiv -1 \pmod{\frac{\mu}{\gcd(N,\mu)}}$, since $\frac{\mu}{\gcd(N,\mu)}\mid k$. Thus h' is a possible choice for $[h]'_{\frac{\mu}{\gcd(N,\mu)}}$.

[16] Note that the equivalence $h[h]'_{2^{\alpha+2}k\gcd(N,\mu)}\equiv -1 \pmod{2^{\alpha+2}k\gcd(N,\mu)}$ implies that we additionally have $h[h]'_{2^{\alpha+2}k\gcd(N,\mu)}\equiv -1 \pmod{2^{\nu+\alpha+2}}$, since $2^{\nu+\alpha+2}\mid\left(2^{\alpha+2}k\gcd(N,\mu)\right)$. Thus $[h]'_{2^{\alpha+2}k\gcd(N,\mu)}$ is a possible choice for $[h]'_{2^{\nu+\alpha+2}}$.

$$-24\mu r^2\left[\frac{N\mu}{2^\alpha}\right]'_{2^{\nu+\alpha+2}}\gcd(N,\mu)\Big)[h]'_{2^{\alpha+2}k\gcd(N,\mu)}\Big)\Big)$$

$$\times\Bigg(\delta_{\left(\gcd(N,\mu)2^{\alpha+1}\right)|(hj+r)}\exp\left(\frac{2\pi i}{24k}\left(h^2[h]'_{2^{\alpha+2}k\gcd(N,\mu)}(1-k^2)+\frac{12rj}{N}\right)\right)$$

$$\times\exp\left(\frac{2\pi i}{2^{\nu+\alpha+2}}\left(-2jr\left[\frac{N\mu}{2^\alpha}\right]'_{2^{\nu+\alpha+2}}\right)\right)\exp\left(\frac{2\pi i}{k\gcd(N,\mu)}\left(-2^{\nu+1}jr[4]'_{\frac{\mu}{\gcd(N,\mu)}}\left[\frac{N2^\nu}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}}\right)\right)$$

$$-\delta_{\left(\gcd(N,\mu)2^{\alpha+1}\right)|(hj-r)}\exp\left(\frac{2\pi i}{24k}\left(h^2[h]'_{2^{\alpha+2}k\gcd(N,\mu)}(1-k^2)-\frac{12rj}{N}\right)\right)$$

$$\times\exp\left(\frac{2\pi i}{2^{\nu+\alpha+2}}\left(2jr\left[\frac{N\mu}{2^\alpha}\right]'_{2^{\nu+\alpha+2}}\right)\right)\exp\left(\frac{2\pi i}{k\gcd(N,\mu)}\left(2^{\nu+1}jr[4]'_{\frac{\mu}{\gcd(N,\mu)}}\left[\frac{N2^\nu}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}}\right)\right)\Bigg).$$

We now need to split into two cases, $3\nmid k$ and $3\mid k$. In the first case we obtain $3\mid(k^2-1)$. Choosing $[h]'_{2^{\alpha+2}k\gcd(N,\mu)}$ such that

$$h[h]'_{2^{\alpha+2}k\gcd(N,\mu)}\equiv -1\pmod{2^{\alpha+5}k\gcd(N,\mu)},$$

analogously to above[17], yields[18]

$$K_{k,j,N,2}(n,r,\kappa)$$
$$=(1-i)\varepsilon_{\frac{\mu}{\gcd(N,\mu)}}\sqrt{\frac{2^{\alpha-1}\gcd(N,\mu)}{N}}\left(\frac{-2^{\nu+\alpha}}{\frac{N\mu}{2^\alpha}}\right)\left(\frac{\frac{N2^\nu}{\gcd(N,\mu)}}{\frac{\mu}{\gcd(N,\mu)}}\right)\exp\left(\frac{2\pi i}{4}\right)\delta_{\nu-\alpha>1}$$
$$\times\sum_{\substack{h\pmod k\\ \gcd(h,k)=1}}(-1)^{\frac{h^2-1}{8}\nu+\frac{(\mu-1)(h-1)}{4}+\frac{h-1}{2}+\frac{h^2-1}{8}(\nu+\alpha)}\varepsilon_{\frac{hN\mu}{2^\alpha}}\left(\frac{h}{\gcd(N,\mu)}\right)$$

$$\times\Bigg(\delta_{\left(\gcd(N,\mu)2^{\alpha+1}\right)|(hj+r)}\exp\left(\frac{2\pi i}{8k}\left(\frac{12rj}{3N}\right)\right)$$

$$\times\exp\left(\frac{2\pi i}{2^{\nu+\alpha+2}}\left(-2jr\left[\frac{N\mu}{2^\alpha}\right]'_{2^{\nu+\alpha+2}}\right)\right)\exp\left(\frac{2\pi i}{k\gcd(N,\mu)}\left(-2^{\nu+1}jr[4]'_{\frac{\mu}{\gcd(N,\mu)}}\left[\frac{N2^\nu}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}}\right)\right)$$

$$-\delta_{\left(\gcd(N,\mu)2^{\alpha+1}\right)|(hj-r)}\exp\left(\frac{2\pi i}{8k}\left(-\frac{12rj}{3N}\right)\right)$$

$$\times\exp\left(\frac{2\pi i}{2^{\nu+\alpha+2}}\left(2jr\left[\frac{N\mu}{2^\alpha}\right]'_{2^{\nu+\alpha+2}}\right)\right)\exp\left(\frac{2\pi i}{k\gcd(N,\mu)}\left(2^{\nu+1}jr[4]'_{\frac{\mu}{\gcd(N,\mu)}}\left[\frac{N2^\nu}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}}\right)\right)\Bigg)$$

$$\times\exp\Bigg(\frac{2\pi i}{24\cdot 2^{\alpha+2}k\gcd(N,\mu)}\Bigg(\Bigg(\left(-24n+1+2k^2-3k\right)2^{\alpha+2}\gcd(N,\mu)-24j^22^{\nu+\alpha+2}[4]'_{\frac{\mu}{\gcd(N,\mu)}}\left[\frac{N2^\nu}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}}$$
$$-24\mu j^2\left[\frac{N\mu}{2^\alpha}\right]'_{2^{\nu+\alpha+2}}\gcd(N,\mu)\Bigg)h$$

[17] We are allowed to do this since $\gcd(8,h)=1$, this is because we know that h is odd.

[18] Using that $h[h]'_{2^{\alpha+5}k\gcd(N,\mu)}\equiv -1\pmod{8k}$ since $(8k)\mid(2^{\alpha+5}k\gcd(N,\mu))$.

$$- \left(\left(-24N\kappa^2 - 24\kappa r + 1 - k^2\right) 2^{\alpha+2} \gcd(N,\mu) - 24r^2 2^{\nu+\alpha+2}[4]'_{\frac{\mu}{\gcd(N,\mu)}} \left[\frac{N2^\nu}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}} \right.$$
$$\left. -24\mu r^2 \left[\frac{N\mu}{2^\alpha}\right]'_{2^{\nu+\alpha+2}} \gcd(N,\mu)\right) [h]'_{2^{\alpha+5}k\gcd(N,\mu)}\Bigg)\Bigg)$$
$$=: K_{k,j,N,2,+}(n,r,\kappa) + K_{k,j,N,2,-}(n,r,\kappa),$$

where

$$K_{k,j,N,2,\pm}(n,r,\kappa) = \epsilon_{e,\pm}(k,j,N,r) \frac{1}{2^{2\alpha+6}\gcd(N,\mu)^2} \sum_{s=0}^{\gcd(N,\mu)2^{\alpha+1}-1} \exp\left(\pm 2\pi i \frac{rs}{\gcd(N,\mu)2^{\alpha+1}}\right)$$
$$\times \sum_{\substack{h \pmod{2^{\alpha+5}k\gcd(N,\mu)} \\ \gcd(h,2^{\alpha+5}k\gcd(N,\mu))=1}} (-1)^{\frac{h^2-1}{8}\nu + \frac{(\mu-1)(h-1)}{4} + \frac{h-1}{2} + \frac{h^2-1}{8}(\nu+\alpha)} \varepsilon_{\frac{hN\mu}{2^\alpha}} \left(\frac{h}{\gcd(N,\mu)}\right)$$
$$\times \exp\left(\frac{2\pi i}{2^{\alpha+5}k\gcd(N,\mu)}\left((\mu_5 + 16jks)\,h - \nu_5[h]'_{2^{\alpha+5}k\gcd(N,\mu)}\right)\right),$$

with

$$\epsilon_{e,\pm}(k,j,N,r) := \pm(1-i)\varepsilon_{\frac{\mu}{\gcd(N,\mu)}} \sqrt{\frac{2^{\alpha-1}\gcd(N,\mu)}{N}} \left(\frac{-2^{\nu+\alpha}}{\frac{N\mu}{2^\alpha}}\right) \left(\frac{\frac{N2^\nu}{\gcd(N,\mu)}}{\frac{\mu}{\gcd(N,\mu)}}\right) \delta_{\nu-\alpha>1}$$
$$\times \exp\left(\frac{2\pi i}{4}\right) \exp\left(\frac{2\pi i}{8k}\left(\pm\frac{12rj}{3N}\right)\right) \exp\left(\frac{2\pi i}{2^{\nu+\alpha+2}}\left(\mp 2jr\left[\frac{N\mu}{2^\alpha}\right]'_{2^{\nu+\alpha+2}}\right)\right)$$
$$\times \exp\left(\frac{2\pi i}{k\gcd(N,\mu)}\left(\mp 2^{\nu+1}jr[4]'_{\frac{\mu}{\gcd(N,\mu)}}\left[\frac{N2^\nu}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}}\right)\right),$$
$$\mu_5 := \left(-8n + \frac{1+2k^2}{3} - k\right) 2^{\alpha+2}\gcd(N,\mu) - 8j^2 2^{\nu+\alpha+2}[4]'_{\frac{\mu}{\gcd(N,\mu)}} \left[\frac{N2^\nu}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}}$$
$$- 8\mu j^2 \left[\frac{N\mu}{2^\alpha}\right]'_{2^{\nu+\alpha+2}} \gcd(N,\mu),$$
$$\nu_5 := \left(-8N\kappa^2 - 8\kappa r + \frac{1-k^2}{3}\right) 2^{\alpha+2}\gcd(N,\mu) - 8r^2 2^{\nu+\alpha+2}[4]'_{\frac{\mu}{\gcd(N,\mu)}} \left[\frac{N2^\nu}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}}$$
$$- 8\mu r^2 \left[\frac{N\mu}{2^\alpha}\right]'_{2^{\nu+\alpha+2}} \gcd(N,\mu).$$

Note that $\mu_5, \nu_5 \in \mathbb{Z}$, since $3 \mid (k^2-1)$ is equivalent to $3 \mid (2k^2+1)$.

Analogously to above we note that $(-1)^{\frac{h^2-1}{8}\nu + \frac{(\mu-1)(h-1)}{4} + \frac{h-1}{2} + \frac{h^2-1}{8}(\nu+\alpha)}$ only depends on h modulo 16, $16 \mid (2^{\alpha+5}k\gcd(N,\mu))$, and $\varepsilon_{\frac{hN\mu}{2^\alpha}}$ only depends on h modulo 4, means we can also look at it modulo 16, since $4 \mid 16$. Thus we obtain

$$K_{k,j,N,2,\pm}(n,r,\kappa) = \epsilon_{e,\pm}(k,j,N,r) \frac{1}{2^{2\alpha+6}\gcd(N,\mu)^2} \frac{1}{16} \sum_{s=0}^{\gcd(N,\mu)2^{\alpha+1}-1} \exp\left(\pm 2\pi i \frac{rs}{\gcd(N,\mu)2^{\alpha+1}}\right)$$

$$\times \sum_{j \pmod{16}} (-1)^{\frac{j^2-1}{8}\nu+\frac{(\mu-1)(j-1)}{4}+\frac{j-1}{2}+\frac{j^2-1}{8}(\nu+\alpha)}\varepsilon_{\frac{jN\mu}{2^\alpha}} \sum_{\ell \pmod{16}} e^{\frac{-2\pi ij\ell}{16}}$$
$$\times \sum_{\substack{h \pmod{2^{\alpha+5}k\gcd(N,\mu)} \\ \gcd(h,2^{\alpha+5}k\gcd(N,\mu))=1}} \left(\frac{h}{\gcd(N,\mu)}\right) \tag{IV.3.10}$$
$$\times \exp\left(\frac{2\pi i}{2^{\alpha+5}k\gcd(N,\mu)}\left(\left(\mu_5+16jks+\frac{2^{\alpha+5}\ell k\gcd(N,\mu)}{16}\right)h-\nu_5[h]'_{2^{\alpha+5}k\gcd(N,\mu)}\right)\right).$$

In the second case, $3 \mid k$, we have that $3 \nmid h$ and thus $\gcd(24, h) = 1$. Choosing $[h]'_{2^{\alpha+2}k\gcd(N,\mu)}$ such that $h[h]'_{2^{\alpha+2}k\gcd(N,\mu)} \equiv -1 \pmod{24 \cdot 2^{\alpha+2}k\gcd(N,\mu)}$, analogously to above, yields[19]

$$K_{k,j,N,2}(n,r,\kappa) =: K^*_{k,j,N,2,+}(n,r,\kappa) + K^*_{k,j,N,2,-}(n,r,\kappa),$$

where analogously to the first case

$$K_{k,j,N,2,\pm}(n,r,\kappa)$$
$$=\epsilon_{e,\pm}(k,j,N,r)\frac{1}{3\cdot 2^{2\alpha+6}\gcd(N,\mu)^2}\frac{1}{16}\sum_{s=0}^{\gcd(N,\mu)2^{\alpha+1}-1}\exp\left(\pm 2\pi i\frac{rs}{\gcd(N,\mu)2^{\alpha+1}}\right)$$
$$\times \sum_{j \pmod{16}} (-1)^{\frac{j^2-1}{8}\nu+\frac{(\mu-1)(j-1)}{4}+\frac{j-1}{2}+\frac{j^2-1}{8}(\nu+\alpha)}\varepsilon_{\frac{jN\mu}{2^\alpha}} \sum_{\ell \pmod{16}} e^{\frac{-2\pi ij\ell}{16}}$$
$$\times \sum_{\substack{h \pmod{3\cdot 2^{\alpha+5}k\gcd(N,\mu)} \\ \gcd(h,3\cdot 2^{\alpha+5}k\gcd(N,\mu))=1}} \left(\frac{h}{\gcd(N,\mu)}\right) \tag{IV.3.11}$$
$$\times \exp\left(\frac{2\pi i}{3\cdot 2^{\alpha+5}k\gcd(N,\mu)}\left(\left(\mu_6+48jks+\frac{3\cdot 2^{\alpha+5}\ell k\gcd(N,\mu)}{16}\right)h-\nu_6[h]'_{3\cdot 2^{\alpha+5}k\gcd(N,\mu)}\right)\right),$$

with

$$\mu_6 := \left(-24n+1+2k^2-3k\right)2^{\alpha+2}\gcd(N,\mu) - 24j^2 2^{\nu+\alpha+2}[4]'_{\frac{\mu}{\gcd(N,\mu)}}\left[\frac{N2^\nu}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}}$$
$$- 24\mu j^2\left[\frac{N\mu}{2^\alpha}\right]'_{2^{\nu+\alpha+2}}\gcd(N,\mu),$$
$$\nu_6 := \left(-24N\kappa^2-24\kappa r+1-k^2\right)2^{\alpha+2}\gcd(N,\mu) - 24r^2 2^{\nu+\alpha+2}[4]'_{\frac{\mu}{\gcd(N,\mu)}}\left[\frac{N2^\nu}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}}$$
$$- 24\mu r^2\left[\frac{N\mu}{2^\alpha}\right]'_{2^{\nu+\alpha+2}}\gcd(N,\mu).$$

We now note that we can bound (IV.3.10) by

[19]Using that $h[h]'_{24\cdot 2^{\alpha+2}k\gcd(N,\mu)} \equiv -1 \pmod{24k}$ since $(24k) \mid (24\cdot 2^{\alpha+2}k\gcd(N,\mu))$.

$$\begin{aligned}
&\left|K_{k,j,N,2,\pm}(n,r,\kappa)\right| \\
&\leq \left|\epsilon_{e,\pm}(k,j,N,r)\frac{1}{2^{2\alpha+6}\gcd(N,\mu)^2}\frac{1}{16}\right| \sum_{s=0}^{\gcd(N,\mu)2^{\alpha+1}-1} \sum_{j \pmod{16}} \sum_{\ell \pmod{16}} \\
&\times \left| \sum_{\substack{h \pmod{2^{\alpha+5}k\gcd(N,\mu)} \\ \gcd(h,2^{\alpha+5}k\gcd(N,\mu))=1}} \left(\frac{h}{\gcd(N,\mu)}\right) \right. \\
&\left. \times \exp\left(\frac{2\pi i}{2^{\alpha+5}k\gcd(N,\mu)}\left(\left(\mu_5+16jsk+\frac{2^{\alpha+5}\ell k\gcd(N,\mu)}{16}\right)h-\nu_5[h]'_{2^{\alpha+5}k\gcd(N,\mu)}\right)\right)\right|.
\end{aligned}$$

Moreover we obtain that (IV.3.11) is bounded by

$$\begin{aligned}
&|K_{k,j,N,2,\pm}(n,r,\kappa)| \\
&\leq \left|\epsilon_{e,\pm}(k,j,N,r)\frac{1}{3\cdot 2^{2\alpha+6}\gcd(N,\mu)^2}\frac{1}{16}\right| \sum_{s=0}^{\gcd(N,\mu)2^{\alpha+1}-1} \sum_{j \pmod{16}} \sum_{\ell \pmod{16}} \\
&\times \left| \sum_{\substack{h \pmod{3\cdot 2^{\alpha+5}k\gcd(N,\mu)} \\ \gcd(h,3\cdot 2^{\alpha+5}k\gcd(N,\mu))=1}} \left(\frac{h}{\gcd(N,\mu)}\right) \right. \\
&\left. \times \exp\left(\frac{2\pi i}{3\cdot 2^{\alpha+5}k\gcd(N,\mu)}\left(\left(\mu_6+48jsk+\frac{3\cdot 2^{\alpha+5}\ell k\gcd(N,\mu)}{16}\right)h-\nu_6[h]'_{3\cdot 2^{\alpha+5}k\gcd(N,\mu)}\right)\right)\right|.
\end{aligned}$$

Both last sums over h are of the required shape, so we can bound them using Malishev's result (see Lemma IV.1.3) and obtain that they are

$$O\left((2^{\alpha+5}k\gcd(N,\mu))^{\frac{1}{2}+\varepsilon}\min\left(\gcd\left(\mu_5+16jks+\frac{2^{\alpha+5}\ell k\gcd(N,\mu)}{16},2^{\alpha+5}k\gcd(N,\mu)\right)^{\frac{1}{2}},\gcd\left(\nu_5,2^{\alpha+5}k\gcd(N,\mu)\right)^{\frac{1}{2}}\right)\right),$$

respectively

$$\begin{aligned}
&O\Big((3\cdot 2^{\alpha+5}k\gcd(N,\mu))^{\frac{1}{2}+\varepsilon} \\
&\times \min\left(\gcd\left(\mu_6+48jks+\frac{3\cdot 2^{\alpha+5}\ell k\gcd(N,\mu)}{16},3\cdot 2^{\alpha+5}k\gcd(N,\mu)\right)^{\frac{1}{2}},\gcd\left(\nu_6,3\cdot 2^{\alpha+5}k\gcd(N,\mu)\right)^{\frac{1}{2}}\right)\Big),
\end{aligned}$$

for $\varepsilon > 0$.

We see that $2^{\alpha+5}\gcd(N,\mu) \leq 3\cdot 2^{\alpha+5}\gcd(N,\mu) \leq 3\cdot 2^5 N^2 = O_N(1)$ and, by Lemma V.2.5,

$$\min\left(\gcd\left(\mu_5+16jks+\frac{2^{\alpha+5}\ell k\gcd(N,\mu)}{16},2^{\alpha+5}k\gcd(N,\mu)\right)^{\frac{1}{2}},\gcd\left(\nu_5,2^{\alpha+5}k\gcd(N,\mu)\right)^{\frac{1}{2}}\right) = O_N(n),$$

and

$$\min\left(\gcd\left(\mu_6+48jks+\frac{3\cdot 2^{\alpha+5}\ell k\gcd(N,\mu)}{16},3\cdot 2^{\alpha+5}k\gcd(N,\mu)\right)^{\frac{1}{2}},\gcd\left(\nu_6,3\cdot 2^{\alpha+5}k\gcd(N,\mu)\right)^{\frac{1}{2}}\right)=O_N\left(n\right).$$

This yields

$$\begin{aligned}K_{k,j,N,2,\pm}(n,r,\kappa)=&O_N\left(\left|\epsilon_{e,\pm}(k,j,N,r)\frac{1}{2^{2\alpha+6}\gcd(N,\mu)^2}\frac{1}{16}\right|\sum_{s=0}^{\gcd(N,\mu)2^{\alpha+1}-1}\sum_{j\pmod{16}}\sum_{\ell\pmod{16}}k^{\frac{1}{2}+\varepsilon}n\right)\\=&O_N\left(\left|\epsilon_{e,\pm}(k,j,N,r)\frac{1}{2^{2\alpha+6}\gcd(N,\mu)^2}\frac{1}{16}\right|16^2\gcd(N,\mu)2^{\alpha+1}k^{\frac{1}{2}+\varepsilon}n\right)\\=&O_N\left(nk^{\frac{1}{2}+\varepsilon}\right),\end{aligned}$$

and analogously $K^*_{k,j,N,2,\pm}(n,r,\kappa)=O_N(nk^{\frac{1}{2}+\varepsilon})$, since $\epsilon_{e,\pm}(k,j,N,r)=O_N(1)$. We thus showed that

$$K_{k,j,N,2}(n,r,\kappa)=O_N\left(nk^{\frac{1}{2}+\varepsilon}\right),$$

which finally gives

$$K_{k,j,N}(n,r,\kappa)=O_N\left(nk^{\frac{1}{2}+\varepsilon}\right)$$

and finishes the proof for k even and therefore the proof of Theorem IV.1.2.

IV.4 Applying the Circle Method

In this section we use the Circle Method and ideas of Rademacher and Zuckerman [Rad38, Rad37, RZ38] to finally prove Theorem IV.1.1. As we already mentioned in the introduction of this chapter, the Kloosterman sum and transformation behavior of our family of functions is a little more complicated here than it is in [Rad38], for example. Even though we now have a nice bound for our Kloosterman sum this will cause extra work in bounding the error parts.

Let $0\leq h<k\leq J$ with $\gcd(h,k)=1$ and a parameter $J\in\mathbb{N}$ that later tends to infinity. Furthermore let $\frac{h_1}{k_1}<\frac{h}{k}<\frac{h_2}{k_2}$ be consecutive fractions in the *Farey sequence* of order J (a series of fractions $\frac{p_j}{q_j}$ with $p_j\leq q_j\leq J$, $\gcd(p_j,q_j)=1$ and $\frac{p_j}{q_j}<\frac{p_\ell}{q_\ell}$ for all $j<\ell$). We denote the *Farey arc* $\xi_{h,k}$ to be the image of $(\frac{h_1+h}{k_1+k},\frac{h_2+h}{k_2+k})$ under the map (see e.g., [Rad37])

$$\phi\mapsto e^{-2\pi J^{-2}+2\pi i\phi}$$

and $\xi_{0,1}$ to be the image of $(-\frac{1}{J+1},\frac{1}{J+1})$ (see e.g., [And98, equation (5.2.9)]).

Note that for the fraction $\frac{h}{k}$ and its neighbors we have (see [Rad38, page 503])

$$hk_1 - h_1k = 1 \quad \text{and} \quad h_2k - hk_2 = 1,$$

which is equivalent to

$$hk_1 \equiv 1 \pmod{k} \quad \text{and} \quad hk_2 \equiv -1 \pmod{k},$$

or, using that $hh' \equiv -1 \pmod{k}$,

$$k_1 \equiv -h' \pmod{k} \quad \text{and} \quad k_2 \equiv h' \pmod{k}. \tag{IV.4.1}$$

Since $\frac{h_1+h}{k_1+k}$ and $\frac{h_2+h}{k_2+k}$ do not belong to the Farey sequence of order J we have $k_1 + k > J$ and $k_2 + k > J$, which, together with $k_1, k_2 \leq J$, enclose k_1 and k_2 to the intervals

$$J - k < k_1 \leq J, \qquad J - k < k_2 \leq J. \tag{IV.4.2}$$

The formulae (IV.4.1) and (IV.4.2) thus determine k_1 and k_2 uniquely as functions of h and k.

Using Cauchy's formula and (IV.1.4) we write (see e.g., [BFOR17, equation (14.4)])

$$a_{j,N}(n) = \frac{1}{2\pi i} \int_{C_J} \frac{q^{\frac{1}{24} - \frac{j^2}{4N}} \mathcal{A}_{j,N}(\tau)}{q^{n+1}} dq, \tag{IV.4.3}$$

where C_J is an arbitrary path inside the unit disk that loops around zero in the counter-clockwise direction exactly once. Here we choose C_J to be the circle of radius $e^{-2\pi J^{-2}} < 1$ and note that we can split this circle into disjoint Farey arcs as done in Rademacher's original works [Rad38, Rad37] by

$$\bigcup_{\substack{0 \leq h < k \leq J \\ \gcd(h,k)=1}} \xi_{h,k} = C_J,$$

which will allow us to focus on the most important cusps. Using this we are able to rewrite (IV.4.3) as

$$a_{j,N}(n) = \sum_{\substack{0 \leq h < k \leq J \\ \gcd(h,k)=1}} \frac{1}{2\pi i} \int_{\xi_{h,k}} \frac{\mathcal{A}_{j,N}(\tau)}{q^{g_{j,N}(n)+1}} dq,$$

where we denoted $g_{j,N}(n) := n + \frac{j^2}{4N} - \frac{1}{24}$ for simplicity. Defining (see e.g., [Rad38, equation (3.5)])

$$\vartheta'_{h,k} := \frac{1}{k(k_1 + k)}, \qquad \vartheta''_{h,k} := \frac{1}{k(k_2 + k)}$$

and substituting $\tau = \frac{h}{k} + i(J^{-2} - i\phi)$ (arc length centered at $e^{2\pi i \frac{h}{k}}$) thus leads to

$$a_{j,N}(n) = \sum_{\substack{0\leq h<k\leq J \\ \gcd(h,k)=1}} e^{-\frac{2\pi i h}{k} g_{j,N}(n)} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{2\pi g_{j,N}(n)\left(J^{-2}-i\phi\right)} \mathcal{A}_{j,N}\left(\frac{h}{k} + i\left(J^{-2} - i\phi\right)\right) d\phi. \tag{IV.4.4}$$

Let $\omega \coloneqq J^{-2} - i\phi$. To better control the integrand's behavior near rational numbers we use the modular transformation $M_{h,k}$ from Theorem IV.1.1. In addition equation (IV.2.14) gives us

$$\mathcal{A}_{j,N}\left(\frac{h}{k} + i\omega\right) = \sum_{r=1}^{N-1} \chi_{j,r}(N, M_{h,k})\, \eta\left(\frac{h'}{k} + \frac{i}{k^2\omega}\right)^{-1} \mathcal{I}_{r,N,\frac{h'}{k}}\left(\frac{h'}{k} + \frac{i}{k^2\omega}\right), \tag{IV.4.5}$$

where we used that

$$M_{h,k}\left(\frac{h}{k} + i\omega\right) = \frac{h'\left(\frac{h}{k} + i\omega\right) - \frac{hh'+1}{k}}{k\left(\frac{h}{k} + i\omega\right) - h} = \frac{ih'\omega - \frac{1}{k}}{ik\omega} = \frac{h'}{k} + \frac{i}{k^2\omega}.$$

Taking a closer look at (IV.4.5) we obtain

$$\begin{aligned}\mathcal{A}_{j,N}\left(\frac{h}{k} + i\omega\right) = \sum_{r=1}^{N-1} \chi_{j,r}(N, M_{h,k}) \Bigg(&\left(\eta\left(\frac{h'}{k} + \frac{i}{k^2\omega}\right)^{-1} - e^{-\frac{\pi i}{12}\left(\frac{h'}{k}+\frac{i}{k^2\omega}\right)}\right) \mathcal{I}_{r,N,\frac{h'}{k}}\left(\frac{h'}{k} + \frac{i}{k^2\omega}\right) \\ &+ \zeta_{24k}^{-h'}\left(\mathcal{I}^{e}_{r,N,\frac{h'}{k},\frac{1}{24}}\left(\frac{h'}{k} + \frac{i}{k^2\omega}\right) + \mathcal{I}^{*}_{r,N,\frac{h'}{k},\frac{1}{24}}\left(\frac{h'}{k} + \frac{i}{k^2\omega}\right)\right)\Bigg).\end{aligned}$$

Note that in this calculation we set $d = \frac{1}{24}$ in the splitting of our Mordell-type integral such that our assumption from above simplifies to $\sqrt{\frac{N}{6}} \notin \mathbb{Z}$.

Plugging this into (IV.4.4) gives us

$$a_{j,N}(n) = a_{\mathcal{I},j,N}(n) + a_{\mathcal{I}^e,j,N}(n) + a_{\mathcal{I}^*,j,N}(n), \tag{IV.4.6}$$

with

$$\begin{aligned} a_{\mathcal{I},j,N}(n) \coloneqq \sum_{\substack{0\leq h<k\leq J \\ \gcd(h,k)=1}} e^{-\frac{2\pi i h}{k} g_{j,N}(n)} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{2\pi g_{j,N}(n)\omega} \sum_{r=1}^{N-1} \chi_{j,r}(N, M_{h,k}) \\ \times \left(\eta\left(\frac{h'}{k} + \frac{i}{k^2\omega}\right)^{-1} - e^{-\frac{\pi i}{12}\left(\frac{h'}{k}+\frac{i}{k^2\omega}\right)}\right) \mathcal{I}_{r,N,\frac{h'}{k}}\left(\frac{h'}{k} + \frac{i}{k^2\omega}\right) d\phi, \end{aligned}$$

$$a_{\mathcal{I}^e,j,N}(n) \coloneqq \sum_{\substack{0\leq h<k\leq J \\ \gcd(h,k)=1}} e^{-\frac{2\pi i h}{k} g_{j,N}(n)} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{2\pi g_{j,N}(n)\omega} \sum_{r=1}^{N-1} \chi_{j,r}(N, M_{h,k}) \zeta_{24k}^{-h'} \mathcal{I}^{e}_{r,N,\frac{h'}{k},\frac{1}{24}}\left(\frac{h'}{k} + \frac{i}{k^2\omega}\right) d\phi,$$

$$a_{\mathcal{I}^*,j,N}(n) \coloneqq \sum_{\substack{0\leq h<k\leq J \\ \gcd(h,k)=1}} e^{-\frac{2\pi i h}{k} g_{j,N}(n)} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{2\pi g_{j,N}(n)\omega} \sum_{r=1}^{N-1} \chi_{j,r}(N, M_{h,k}) \zeta_{24k}^{-h'} \mathcal{I}^{*}_{r,N,\frac{h'}{k},\frac{1}{24}}\left(\frac{h'}{k} + \frac{i}{k^2\omega}\right) d\phi.$$

IV.4.1 Principal part

We now look at each of the terms in $a_{j,N}(n)$ seperately. We start with the part that contains the principal part, namely $a_{\mathcal{I}^*,j,N}(n)$. Analogously to [Rad38] we split our integral as

$$\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} = \int_{-\frac{1}{k(J+k)}}^{\frac{1}{k(J+k)}} + \int_{-\vartheta'_{h,k}}^{-\frac{1}{k(J+k)}} + \int_{\frac{1}{k(J+k)}}^{\vartheta''_{h,k}},$$

since we have

$$-\vartheta'_{h,k} = -\frac{1}{k(k_1+k)} \leq -\frac{1}{k(J+k)} \leq \frac{1}{k(J+k)} \leq \frac{1}{k(k_2+k)} = \vartheta''_{h,k}.$$

Defining

$$a_{\mathcal{I}^*,j,N,0}(n) := \sum_{\substack{0\leq h<k\leq J\\ \gcd(h,k)=1}} e^{-\frac{2\pi ih}{k}g_{j,N}(n)} \int_{-\frac{1}{k(J+k)}}^{\frac{1}{k(J+k)}} e^{2\pi g_{j,N}(n)\omega} \sum_{r=1}^{N-1} \chi_{j,r}(N,M_{h,k})\zeta_{24k}^{-h'} \mathcal{I}^*_{r,N,\frac{h'}{k},\frac{1}{24}}\left(\frac{h'}{k}+\frac{i}{k^2\omega}\right) d\phi,$$

$$a_{\mathcal{I}^*,j,N,1}(n) := \sum_{\substack{0\leq h<k\leq J\\ \gcd(h,k)=1}} e^{-\frac{2\pi ih}{k}g_{j,N}(n)} \int_{-\vartheta'_{h,k}}^{-\frac{1}{k(J+k)}} e^{2\pi g_{j,N}(n)\omega} \sum_{r=1}^{N-1} \chi_{j,r}(N,M_{h,k})\zeta_{24k}^{-h'} \mathcal{I}^*_{r,N,\frac{h'}{k},\frac{1}{24}}\left(\frac{h'}{k}+\frac{i}{k^2\omega}\right) d\phi,$$

$$a_{\mathcal{I}^*,j,N,2}(n) := \sum_{\substack{0\leq h<k\leq J\\ \gcd(h,k)=1}} e^{-\frac{2\pi ih}{k}g_{j,N}(n)} \int_{\frac{1}{k(J+k)}}^{\vartheta''_{h,k}} e^{2\pi g_{j,N}(n)\omega} \sum_{r=1}^{N-1} \chi_{j,r}(N,M_{h,k})\zeta_{24k}^{-h'} \mathcal{I}^*_{r,N,\frac{h'}{k},\frac{1}{24}}\left(\frac{h'}{k}+\frac{i}{k^2\omega}\right) d\phi,$$

we thus obtain

$$a_{\mathcal{I}^*,j,N}(n) = a_{\mathcal{I}^*,j,N,0}(n) + a_{\mathcal{I}^*,j,N,1}(n) + a_{\mathcal{I}^*,j,N,2}(n). \tag{IV.4.7}$$

We go on by estimating $a_{\mathcal{I}^*,j,N,0}(n)$. Using (IV.2.16) we see that

$$\begin{aligned} a_{\mathcal{I}^*,j,N,0}(n) =& \frac{i}{\pi}\sum_{k=1}^{J}\int_{-\frac{1}{k(J+k)}}^{\frac{1}{k(J+k)}} e^{2\pi g_{j,N}(n)\omega} e^{\frac{2\pi}{24k^2\omega}} \sum_{r=1}^{N-1}\sum_{\kappa\in\mathbb{Z}} \text{P.V.}\int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}} \frac{e^{-2\pi N\frac{1}{k^2\omega}x^2}}{x-\left(\kappa+\frac{r}{2N}\right)}\,dx \\ &\times \sum_{\substack{0\leq h<k\\ \gcd(h,k)=1}} e^{-\frac{2\pi ih}{k}g_{j,N}(n)}\chi_{j,r}(N,M_{h,k})\zeta_{24k}^{-h'} e^{2\pi iN\left(\kappa+\frac{r}{2N}\right)^2\frac{h'}{k}}\,d\phi. \end{aligned}$$

Plugging in the definition of $K_{k,j,N}(n,r,\kappa)$ from (IV.1.6), which is well-defined and a Kloosterman sum of modulus k, and taking the finite sum over r out of the integral gives us

$$a_{\mathcal{I}^*,j,N,0}(n) = \frac{i}{\pi}\sum_{k=1}^{J}\sum_{r=1}^{N-1}\int_{-\frac{1}{k(J+k)}}^{\frac{1}{k(J+k)}} e^{2\pi\left(g_{j,N}(n)\omega+\frac{1}{24k^2\omega}\right)} \sum_{\kappa\in\mathbb{Z}} K_{k,j,N}(n,r,\kappa)\,\text{P.V.}\int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}} \frac{e^{-2\pi N\frac{1}{k^2\omega}x^2}}{x-\left(\kappa+\frac{r}{2N}\right)}\,dx\,d\phi.$$

Note that, for arbitrary $\ell \in \mathbb{Z}$, we have

$$\zeta_{24k}^{\left(24N\left((\kappa+\ell k)+\frac{r}{2N}\right)^2-1\right)h'}=\zeta_{24k}^{\left(24N\left(\kappa+\frac{r}{2N}\right)^2-1\right)h'}$$

and therefore $K_{k,j,N}(n,r,\kappa+\ell k)=K_{k,j,N}(n,r,\kappa)$. Shifting $\kappa\mapsto\kappa+\ell k$ for $\ell\in\mathbb{Z}$ we thus obtain

$$\begin{aligned}a_{\mathcal{I}^*,j,N,0}(n)=&\frac{i}{\pi}\sum_{k=1}^{J}\sum_{r=1}^{N-1}\sum_{\kappa=0}^{k-1}K_{k,j,N}(n,r,\kappa)\\&\times\int_{-\frac{1}{k(J+k)}}^{\frac{1}{k(J+k)}}e^{2\pi\left(g_{j,N}(n)\omega+\frac{1}{24k^2\omega}\right)}\lim_{L\to\infty}\sum_{\ell=-L}^{L}\text{P.V.}\int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}}\frac{e^{-2\pi N\frac{1}{k^2\omega}x^2}}{x-\left(\kappa+\ell k+\frac{r}{2N}\right)}\,dx\,d\phi.\end{aligned}$$

Note that the convergence is uniform in our finite range so we are allowed to switch the order of the integral and the sum over ℓ. Using the equality (see [BN19, equation (3.10)])

$$\pi\cot(\pi x)=\lim_{L\to\infty}\sum_{\ell=-L}^{L}\frac{1}{x+\ell},\tag{IV.4.8}$$

which holds for all $x\in\mathbb{C}\backslash\mathbb{Z}$, we thus obtain

$$\begin{aligned}a_{\mathcal{I}^*,j,N,0}(n)=&-i\sum_{k=1}^{J}\sum_{r=1}^{N-1}\sum_{\kappa=0}^{k-1}\frac{K_{k,j,N}(n,r,\kappa)}{k}\int_{-\frac{1}{k(J+k)}}^{\frac{1}{k(J+k)}}e^{2\pi\left(g_{j,N}(n)\omega+\frac{1}{24k^2\omega}\right)}\\&\times\text{P.V.}\int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}}e^{-2\pi N\frac{1}{k^2\omega}x^2}\cot\left(\pi\left(-\frac{x}{k}+\frac{\kappa}{k}+\frac{r}{2Nk}\right)\right)dx\,d\phi.\end{aligned}$$

Here the possible poles in $\mathbb{Z}$ have already been excluded by the principal value integral.

Note that we only have a simple pole in $x=\kappa+\ell k+\frac{r}{2N}$ if and only if $\kappa=\ell=0$ and $r<\sqrt{\frac{N}{6}}$. Since one can show that there exists a constant $C_{\varepsilon,k}$ ($C_{\varepsilon,k}\to0$ as $\varepsilon\to0$ and k fix) such that

$$\left|\left(\lim_{\varepsilon\to0}\int_{\substack{\left|x-\frac{r}{2N}\right|\geq\varepsilon\\|x|\leq\sqrt{\frac{1}{24N}}}}e^{-2\pi N\frac{1}{k^2\omega}x^2}\cot\left(\pi\left(-\frac{x}{k}+\frac{r}{2Nk}\right)\right)dx\right)-\int_{\substack{\left|x-\frac{r}{2N}\right|\geq\varepsilon\\|x|\leq\sqrt{\frac{1}{24N}}}}e^{-2\pi N\frac{1}{k^2\omega}x^2}\cot\left(\pi\left(-\frac{x}{k}+\frac{r}{2Nk}\right)\right)dx\right|\leq C_{\varepsilon,k}$$

uniformly in ϕ, using the Taylor expansion of the exponential together with (IV.4.8), we see that we only have integrals over compact subsets with continuous integrands and can additionally switch the integrals over x and ϕ to get

$$a_{\mathcal{I}^*,j,N,0}(n) = -\,i\sum_{k=1}^{J}\sum_{r=1}^{N-1}\sum_{\kappa=0}^{k-1}\frac{K_{k,j,N}(n,r,\kappa)}{k}\,\text{P.V.}\int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}}\cot\left(\pi\left(-\frac{x}{k}+\frac{\kappa}{k}+\frac{r}{2Nk}\right)\right)$$
$$\times\int_{-\frac{1}{k(J+k)}}^{\frac{1}{k(J+k)}} e^{2\pi\left(g_{j,N}(n)\omega+\frac{1}{24k^2\omega}-\frac{Nx^2}{k^2\omega}\right)}\,d\phi\,dx.$$

To evaluate the integral over ϕ we substitute $\omega = J^{-2} - i\phi$ to obtain

$$\int_{-\frac{1}{k(J+k)}}^{\frac{1}{k(J+k)}} e^{2\pi\left(g_{j,N}(n)\omega+\frac{1}{24k^2\omega}-\frac{Nx^2}{k^2\omega}\right)}\,d\phi = -i\int_{J^{-2}-\frac{i}{k(J+k)}}^{J^{-2}+\frac{i}{k(J+k)}} e^{2\pi\left(g_{j,N}(n)\omega+\frac{1}{k^2}\left(\frac{1}{24}-Nx^2\right)\frac{1}{\omega}\right)}\,d\omega.$$

Then we view it as an integral over the right vertical of a rectangle in the complex ω-plane and denote the integrals over the other sides, γ_1, γ_2, and γ_3, by $R_1(x), R_2(x)$, and $R_3(x)$, respectively, where we dropped the dependence on the other parameters for simplicity (see Figure IV.1).

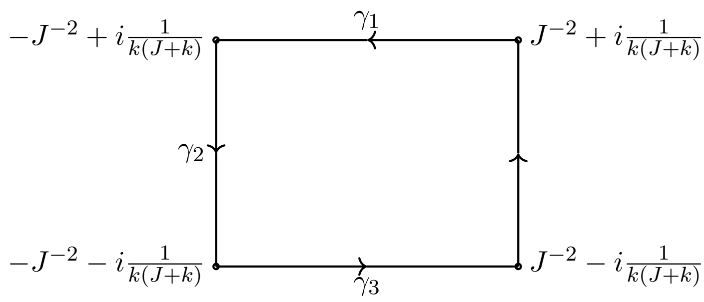

Figure IV.1: Rectangle in the complex ω-plane.

Let $R_{k,j,J,N}(n,x)$ denote the integral over the whole rectangle and let R denote the rectangle itself with counterclockwise orientation such that

$$a_{\mathcal{I}^*,j,N,0}(n) = -\,i\sum_{k=1}^{J}\sum_{r=1}^{N-1}\sum_{\kappa=0}^{k-1}\frac{K_{k,j,N}(n,r,\kappa)}{k}\text{P.V.}\int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}}\cot\left(\pi\left(-\frac{x}{k}+\frac{\kappa}{k}+\frac{r}{2Nk}\right)\right) \quad \text{(IV.4.9)}$$
$$\times\,(-i)\left(R_{k,j,J,N}(n,x)-R_1(x)-R_2(x)-R_3(x)\right)dx.$$

We now have

$$\frac{1}{2\pi i}R_{k,j,J,N}(n,x) = \frac{1}{2\pi i}\int_R e^{2\pi\left(g_{j,N}(n)\omega+\frac{1}{k^2}\left(\frac{1}{24}-Nx^2\right)\frac{1}{\omega}\right)}\,d\omega$$
$$= \frac{1}{2\pi i}\int_R\sum_{\mu\geq 0}\frac{(2\pi g_{j,N}(n)\omega)^\mu}{\mu!}\sum_{\nu\geq 0}\frac{\left(\frac{2\pi}{k^2\omega}\left(\frac{1}{24}-Nx^2\right)\right)^\nu}{\nu!}\,d\omega,$$

using the Taylor expansion of the exponential function. According to the residue theorem this integral equals zero unless there is a simple pole in $\omega = 0$, which requires $\nu = \mu + 1$. Thus we obtain

$$\frac{1}{2\pi i} R_{k,j,J,N}(n,x) = \frac{\sqrt{\frac{1}{24} - Nx^2}}{k\sqrt{g_{j,N}(n)}} \sum_{\mu \geq 0} \frac{\left(\frac{2\pi\sqrt{g_{j,N}(n)}}{k} \sqrt{\frac{1}{24} - Nx^2} \right)^{2\mu+1}}{\mu!(\mu+1)!}.$$

Using the representation (see e.g., [NIST, equation 10.25.2])

$$I_\alpha(z) = \sum_{m \geq 0} \frac{1}{m!\Gamma(m+\alpha+1)} \left(\frac{z}{2}\right)^{2m+\alpha}$$

of the I-Bessel function of first kind and order α we furthermore see that

$$\frac{1}{2\pi i} R_{k,j,J,N}(n,x) = \frac{\sqrt{\frac{1}{24} - Nx^2}}{k\sqrt{g_{j,N}(n)}} I_1\left(\frac{4\pi\sqrt{g_{j,N}(n)}}{k} \sqrt{\frac{1}{24} - Nx^2} \right).$$

Plugging this into (IV.4.9) we obtain

$$\begin{aligned} &a_{\mathcal{I}^*,j,N,0}(n) \\ &= -\sum_{k=1}^{J} \sum_{r=1}^{N-1} \sum_{\kappa=0}^{k-1} \frac{K_{k,j,N}(n,r,\kappa)}{k} \mathrm{P.V.} \int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}} \cot\left(\pi\left(-\frac{x}{k} + \frac{\kappa}{k} + \frac{r}{2Nk}\right)\right) \\ &\qquad \times \frac{2\pi i\sqrt{\frac{1}{24} - Nx^2}}{k\sqrt{g_{j,N}(n)}} I_1\left(\frac{4\pi\sqrt{g_{j,N}(n)}}{k} \sqrt{\frac{1}{24} - Nx^2} \right) dx \\ &\quad + \sum_{k=1}^{J} \sum_{r=1}^{N-1} \sum_{\kappa=0}^{k-1} \frac{K_{k,j,N}(n,r,\kappa)}{k} \mathrm{P.V.} \int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}} \cot\left(\pi\left(-\frac{x}{k} + \frac{\kappa}{k} + \frac{r}{2Nk}\right)\right) (R_1(x) + R_2(x) + R_3(x))\, dx \\ &=: M + E. \end{aligned}$$

We are now left with estimating E. We can rewrite it as

$$\begin{aligned} &E_1 + E_2 + E_3 \\ &:= \sum_{k=1}^{J} \sum_{r=1}^{\left\lceil \sqrt{\frac{N}{6}} - 1 \right\rceil} \frac{K_{k,j,N}(n,r,0)}{k} \mathrm{P.V.} \int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}} \cot\left(\pi\left(-\frac{x}{k} + \frac{r}{2Nk}\right)\right) (R_1(x) + R_2(x) + R_3(x))\, dx \\ &\quad + \sum_{k=1}^{J} \sum_{r=\left\lceil \sqrt{\frac{N}{6}} \right\rceil}^{N-1} \frac{K_{k,j,N}(n,r,0)}{k} \int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}} \cot\left(\pi\left(-\frac{x}{k} + \frac{r}{2Nk}\right)\right) (R_1(x) + R_2(x) + R_3(x))\, dx \\ &\quad + \sum_{k=1}^{J} \sum_{r=1}^{N-1} \sum_{\kappa=1}^{k-1} \frac{K_{k,j,N}(n,r,\kappa)}{k} \int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}} \cot\left(\pi\left(-\frac{x}{k} + \frac{\kappa}{k} + \frac{r}{2Nk}\right)\right) (R_1(x) + R_2(x) + R_3(x))\, dx, \end{aligned}$$

since the integral over x only has a simple pole in $x = \frac{r}{2N}$ for $\kappa = 0$ and $r < \sqrt{\frac{N}{6}}$.

To bound the remaining sides of the rectangle we start with $R_1(x)$ and $R_3(x)$. On this paths of integration we have $\omega = u \pm i\frac{1}{k(J+k)}$, where $-J^{-2} \leq u \leq J^{-2}$, and

$$\mathrm{Re}\left(\frac{1}{\omega}\right) = \frac{u}{u^2 + \frac{1}{k^2(J+k)^2}} = \frac{uk^2(J+k)^2}{u^2k^2(J+k)^2+1} < J^{-2}k^2(J+k)^2 \leq 4k^2.$$

Thus we see that the integrand is less than $e^{2\pi g_{j,N}(n)J^{-2}+8\pi\left(\frac{1}{24}-Nx^2\right)}$ and obtain that

$$|R_1(x)| \text{ and } |R_3(x)| < 2J^{-2}e^{2\pi g_{j,N}(n)J^{-2}+8\pi\left(\frac{1}{24}-Nx^2\right)}. \tag{IV.4.10}$$

For $R_2(x)$ the path of integration is given by $\omega = -J^{-2} - iv$, where we have $-\frac{1}{k(J+k)} \leq v \leq \frac{1}{k(J+k)}$. Note that $g_{j,N}(n)$, $\frac{1}{24} - Nx^2 \geq 0$. Since the real part of ω is always $-J^{-2} < 0$ and $\mathrm{Re}(\frac{1}{\omega}) = \frac{-J^{-2}}{J^{-4}+v^2} < 0$ we conclude that the integrand is $O(1)$ and therefore

$$|R_2(x)| < \frac{2}{k(J+k)} < 2k^{-1}J^{-1}. \tag{IV.4.11}$$

From (IV.4.10) and (IV.4.11) we conclude that

$$R_1(x) + R_2(x) + R_3(x) = O\left(k^{-1}J^{-1}e^{2\pi g_{j,N}(n)J^{-2}+8\pi\left(\frac{1}{24}-Nx^2\right)}\right). \tag{IV.4.12}$$

We start by evaluating E_3, which, using the calculations before together with (IV.1.7), equals

$$O_N\left(\frac{n}{J}e^{2\pi g_{j,N}(n)J^{-2}}\sum_{k=1}^{J}\sum_{r=1}^{N-1}\sum_{\kappa=1}^{k-1}k^{-\frac{3}{2}+\varepsilon}\int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}}\left|\cot\left(\pi\left(-\frac{x}{k}+\frac{\kappa}{k}+\frac{r}{2Nk}\right)\right)\right| dx\right).$$

We note that

$$\left|\cot\left(\pi\left(-\frac{x}{k}+\frac{\kappa}{k}+\frac{r}{2Nk}\right)\right)\right| = \frac{\left|\cos\left(\pi\left(-\frac{x}{k}+\frac{\kappa}{k}+\frac{r}{2Nk}\right)\right)\right|}{\left|\sin\left(\pi\left(-\frac{x}{k}+\frac{\kappa}{k}+\frac{r}{2Nk}\right)\right)\right|} \leq \frac{1}{\left|\sin\left(\pi\left(-\frac{x}{k}+\frac{\kappa}{k}+\frac{r}{2Nk}\right)\right)\right|}.$$

Similar to [Bri09, page 11] we furthermore have

$$\begin{aligned}\left|\sin\left(\pi\left(-\frac{x}{k}+\frac{\kappa}{k}+\frac{r}{2Nk}\right)\right)\right| &= \left|\sin\left(\pi\left|-\frac{x}{k}+\frac{\kappa}{k}+\frac{r}{2Nk}\right|\right)\right| \\ &\gg \min\left(\left\{\left|-\frac{x}{k}+\frac{\kappa}{k}+\frac{r}{2Nk}\right|\right\}, 1-\left\{\left|-\frac{x}{k}+\frac{\kappa}{k}+\frac{r}{2Nk}\right|\right\}\right) \\ &= \begin{cases}\left\{\left|-\frac{x}{k}+\frac{\kappa}{k}+\frac{r}{2Nk}\right|\right\} & \text{if } 0 \leq \left\{\left|-\frac{x}{k}+\frac{\kappa}{k}+\frac{r}{2Nk}\right|\right\} \leq \frac{1}{2}, \\ 1-\left\{\left|-\frac{x}{k}+\frac{\kappa}{k}+\frac{r}{2Nk}\right|\right\} & \text{else,}\end{cases}\end{aligned}$$

where $\{x\} \coloneqq x - \lfloor x \rfloor$ is the *fractional part* of a real number x.

Taking a closer look at $-\frac{x}{k} + \frac{\kappa}{k} + \frac{r}{2Nk}$ we observe that

$$-\frac{x}{k} + \frac{\kappa}{k} + \frac{r}{2Nk} \leq 1 + \frac{1}{k\sqrt{24N}} - \frac{1}{2k} - \frac{1}{2Nk} = 1 - \frac{\sqrt{6N} + \sqrt{\frac{6}{N}} - 1}{k\sqrt{24N}} < 1$$

and

$$-\frac{x}{k} + \frac{\kappa}{k} + \frac{r}{2Nk} \geq \frac{1}{k}\left(\frac{1}{2N} + \kappa - \frac{1}{\sqrt{24N}}\right) \geq \frac{47}{48k},$$

since we have $\kappa \geq 1$. In particular this gives us that

$$\begin{aligned}
\min&\left(\left\{\left|-\frac{x}{k} + \frac{\kappa}{k} + \frac{r}{2Nk}\right|\right\}, 1 - \left\{\left|-\frac{x}{k} + \frac{\kappa}{k} + \frac{r}{2Nk}\right|\right\}\right) \\
&= \min\left(\frac{1}{k}\left(-x + \kappa + \frac{r}{2N}\right), 1 - \frac{1}{k}\left(-x + \kappa + \frac{r}{2N}\right)\right) \\
&= \begin{cases} \frac{1}{k}\left(-x + \kappa + \frac{r}{2N}\right) & \text{if } 0 \leq \frac{1}{k}\left(-x + \kappa + \frac{r}{2N}\right) \leq \frac{1}{2}, \\ 1 - \frac{1}{k}\left(-x + \kappa + \frac{r}{2N}\right) & \text{if } \frac{1}{2} < \frac{1}{k}\left(-x + \kappa + \frac{r}{2N}\right) < 1, \end{cases} \\
&= \begin{cases} \frac{1}{k}\left(-x + \kappa + \frac{r}{2N}\right) & \text{if } x \geq \frac{r}{2N} + \kappa - \frac{k}{2}, \\ 1 - \frac{1}{k}\left(-x + \kappa + \frac{r}{2N}\right) & \text{if } x < \frac{r}{2N} + \kappa - \frac{k}{2}. \end{cases}
\end{aligned}$$

Using this our O-term contributes to

$$\begin{aligned}
&O_N\left(\frac{n}{J}e^{2\pi g_{j,N}(n)J^{-2}} \sum_{k=1}^{J} k^{-\frac{3}{2}+\varepsilon} \sum_{r=1}^{N-1}\sum_{\kappa=1}^{k-1} \int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}} \frac{1}{\min\left(\frac{1}{k}\left(-x+\kappa+\frac{r}{2N}\right), 1-\frac{1}{k}\left(-x+\kappa+\frac{r}{2N}\right)\right)}dx\right) \\
=&O_N\left(\frac{n}{J}e^{2\pi g_{j,N}(n)J^{-2}} \sum_{k=1}^{J} k^{-\frac{3}{2}+\varepsilon} \sum_{r=1}^{N-1}\sum_{\kappa=1}^{k-1} \left(\int\limits_{\substack{|x|\leq\sqrt{\frac{1}{24N}} \\ x\geq\frac{r}{2N}+\kappa-\frac{k}{2}}} \frac{k}{-x+\kappa+\frac{r}{2N}}dx + \int\limits_{\substack{|x|\leq\sqrt{\frac{1}{24N}} \\ x<\frac{r}{2N}+\kappa-\frac{k}{2}}} \frac{1}{1-\frac{1}{k}\left(-x+\kappa+\frac{r}{2N}\right)}dx\right)\right) \\
=&O_N\left(\frac{n}{J}e^{2\pi g_{j,N}(n)J^{-2}} \sum_{k=1}^{J} k^{-\frac{1}{2}+\varepsilon} \sum_{r=1}^{N-1}\sum_{\kappa=1}^{k-1} \left(\delta_{\max\left(-\sqrt{\frac{1}{24N}}, \frac{r}{2N}+\kappa-\frac{k}{2}\right)<\sqrt{\frac{1}{24N}}} \int_{\max\left(-\sqrt{\frac{1}{24N}}, \frac{r}{2N}+\kappa-\frac{k}{2}\right)}^{\sqrt{\frac{1}{24N}}} \frac{1}{-x+\kappa+\frac{r}{2N}}dx\right.\right. \\
&\left.\left.+\delta_{\min\left(\sqrt{\frac{1}{24N}}, \frac{r}{2N}+\kappa-\frac{k}{2}\right)>-\sqrt{\frac{1}{24N}}} \int_{-\sqrt{\frac{1}{24N}}}^{\min\left(\sqrt{\frac{1}{24N}}, \frac{r}{2N}+\kappa-\frac{k}{2}\right)} \frac{1}{k+x-\kappa-\frac{r}{2N}}dx\right)\right) \\
=&O_N\left(\frac{n}{J}e^{2\pi g_{j,N}(n)J^{-2}} \sum_{k=1}^{J} k^{-\frac{1}{2}+\varepsilon} \sum_{r=1}^{N-1} \left(\sum_{\kappa=1}^{k-1}\delta_{\max\left(-\sqrt{\frac{1}{24N}}, \frac{r}{2N}+\kappa-\frac{k}{2}\right)<\sqrt{\frac{1}{24N}}} \int_{\max\left(-\sqrt{\frac{1}{24N}}, \frac{r}{2N}+\kappa-\frac{k}{2}\right)}^{\sqrt{\frac{1}{24N}}} \frac{1}{-x+\kappa+\frac{r}{2N}}dx\right.\right. \\
&\left.\left.+\sum_{\kappa'=1}^{k-1}\delta_{\min\left(\sqrt{\frac{1}{24N}}, \frac{r}{2N}+\frac{k}{2}-\kappa'\right)>-\sqrt{\frac{1}{24N}}} \int_{-\sqrt{\frac{1}{24N}}}^{\min\left(\sqrt{\frac{1}{24N}}, \frac{r}{2N}+\frac{k}{2}-\kappa'\right)} \frac{1}{\kappa'+x-\frac{r}{2N}}dx\right)\right),
\end{aligned} \tag{IV.4.13}$$

by substituting $\kappa' = k - \kappa$.

We note that in the first integral we have $\frac{r}{2N} - x > -\sqrt{\frac{1}{48}} =: C_1$, while in the second integral we have $x - \frac{r}{2N} > -\frac{25}{48} =: C_2$. Thus we obtain that (IV.4.13) equals

$$O_N\left(\frac{n}{J}e^{2\pi g_{j,N}(n)J^{-2}}\sum_{k=1}^{J}k^{-\frac{1}{2}+\varepsilon}\sum_{r=1}^{N-1}\right.$$
$$\times\left(\sum_{\kappa=1}^{k-1}\frac{1}{\kappa+C_1}\delta_{\max\left(-\sqrt{\frac{1}{24N}},\frac{r}{2N}+\kappa-\frac{k}{2}\right)<\sqrt{\frac{1}{24N}}}\int_{\max\left(-\sqrt{\frac{1}{24N}},\frac{r}{2N}+\kappa-\frac{k}{2}\right)}^{\sqrt{\frac{1}{24N}}}dx\right.$$
$$\left.\left.+\sum_{\kappa'=1}^{k-1}\frac{1}{\kappa'+C_2}\delta_{\min\left(\sqrt{\frac{1}{24N}},\frac{r}{2N}+\frac{k}{2}-\kappa'\right)>-\sqrt{\frac{1}{24N}}}\int_{-\sqrt{\frac{1}{24N}}}^{\min\left(\sqrt{\frac{1}{24N}},\frac{r}{2N}+\frac{k}{2}-\kappa'\right)}dx\right)\right).$$

Using that (combining [NIST, equations 5.7.6 and 5.11.2])

$$\sum_{\kappa=1}^{k-1}\frac{1}{\kappa+C} = O\left(\log(k)\right),$$

as $k\to\infty$ and for a $C>-1$, this yields[20]

$$\begin{aligned}E_3 =&O_N\left(\frac{n}{J}e^{2\pi g_{j,N}(n)J^{-2}}\sum_{k=1}^{J}k^{-\frac{1}{2}+\varepsilon}\sum_{r=1}^{N-1}N^{-\frac{1}{2}}\log(k)\right)\\ =&O_N\left(ne^{2\pi g_{j,N}(n)J^{-2}}J^{-\frac{1}{2}+\varepsilon}\log(J)\right),\end{aligned} \tag{IV.4.14}$$

which tends to 0 as $J\to\infty$.

We go on by evaluating E_2. Using the fact that $\cot(z) = O\left(\frac{1}{z}\right)$ as $z\to 0$ we see that

$$\cot\left(\pi\left(-\frac{x}{k}+\frac{r}{2Nk}\right)\right) = O\left(\frac{k}{\pi\left(-x+\frac{r}{2N}\right)}\right) = O_N(k).$$

Together with (IV.4.12) and (IV.1.7) this gives us that

$$E_2 = O_N\left(ne^{2\pi g_{j,N}(n)J^{-2}}J^{-\frac{1}{2}+\varepsilon}\right). \tag{IV.4.15}$$

[20]We used that if we had a function $f(x)$ on $[1,\infty)$ with $f(x) = O(\log(x))$ as $x\to\infty$ we know that we have $f(x)\le C_1\log(x+1)$ for all $x\ge x_0$ and that f is bounded by $\frac{f(x)}{\log(x+1)}\le C_2$ on $[1,x_0)$. In total this would give us $\frac{f(x)}{\log(x+1)}\le C := C_1+C_2$ everywhere.

Lastly we evaluate the part with poles, namely E_1. Extracting the pole in $\frac{r}{2N}$ yields

$$
\begin{aligned}
E_1 =& \sum_{k=1}^{J} \sum_{r=1}^{\left\lceil \sqrt{\frac{N}{6}}-1 \right\rceil} \frac{K_{k,j,N}(n,r,0)}{k} \int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}} \cot\left(\pi\left(-\frac{x}{k}+\frac{r}{2Nk}\right)\right)\left(R_1(x)-R_1\left(\frac{r}{2N}\right)\right) dx \\
&+ \sum_{k=1}^{J} \sum_{r=1}^{\left\lceil \sqrt{\frac{N}{6}}-1 \right\rceil} \frac{K_{k,j,N}(n,r,0)}{k} \int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}} \cot\left(\pi\left(-\frac{x}{k}+\frac{r}{2Nk}\right)\right)\left(R_2(x)-R_2\left(\frac{r}{2N}\right)\right) dx \\
&+ \sum_{k=1}^{J} \sum_{r=1}^{\left\lceil \sqrt{\frac{N}{6}}-1 \right\rceil} \frac{K_{k,j,N}(n,r,0)}{k} \int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}} \cot\left(\pi\left(-\frac{x}{k}+\frac{r}{2Nk}\right)\right)\left(R_3(x)-R_3\left(\frac{r}{2N}\right)\right) dx \\
&+ \sum_{k=1}^{J} \sum_{r=1}^{\left\lceil \sqrt{\frac{N}{6}}-1 \right\rceil} \frac{K_{k,j,N}(n,r,0)}{k} \left(R_1\left(\frac{r}{2N}\right)+R_2\left(\frac{r}{2N}\right)+R_3\left(\frac{r}{2N}\right)\right) \mathrm{P.V.} \int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}} \cot\left(\pi\left(-\frac{x}{k}+\frac{r}{2Nk}\right)\right) dx.
\end{aligned}
$$

We first concentrate on the parts

$$
\sum_{k=1}^{J} \sum_{r=1}^{\left\lceil \sqrt{\frac{N}{6}}-1 \right\rceil} \frac{K_{k,j,N}(n,r,0)}{k} \int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}} \cot\left(\pi\left(-\frac{x}{k}+\frac{r}{2Nk}\right)\right)\left(R_m(x)-R_m\left(\frac{r}{2N}\right)\right) dx,
$$

with $m \in \{1,2,3\}$. Defining $f_m(x) := R_m(x) - R_m(\frac{r}{2N})$ and using the Taylor expansion we obtain that

$$
f_m(x) = f_m\left(\frac{r}{2N}\right) + f_m'\left(\frac{r}{2N}\right)\left(x-\frac{r}{2N}\right) + \cdots + \frac{f_m^{(\ell)}\left(\frac{r}{2N}\right)}{\ell!}\left(x-\frac{r}{2N}\right)^{\ell} + \widetilde{R}_\ell(x),
$$

where $\widetilde{R}_\ell(x)$ is the *remainder term* defined as

$$
\widetilde{R}_\ell(x) := \frac{f_m^{(\ell+1)}(\xi_x)}{(\ell+1)!}\left(x-\frac{r}{2N}\right)^{\ell+1},
$$

for some real ξ_x betwen $\frac{r}{2N}$ and x. Choosing $\ell = 0$ we obtain

$$
f_m(x) = f_m\left(\frac{r}{2N}\right) + f_m'(\xi_x)\left(x-\frac{r}{2N}\right) = f_m'(\xi_x)\left(x-\frac{r}{2N}\right).
$$

Next we focus on $m \in \{1,3\}$ and thus have

$$
\sum_{k=1}^{J} \sum_{r=1}^{\left\lceil \sqrt{\frac{N}{6}}-1 \right\rceil} \frac{K_{k,j,N}(n,r,0)}{k} \int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}} \cot\left(\pi\left(-\frac{x}{k}+\frac{r}{2Nk}\right)\right)\left(R_m(x)-R_m\left(\frac{r}{2N}\right)\right) dx
$$

$$=\sum_{k=1}^{J}\sum_{r=1}^{\left\lceil\sqrt{\frac{N}{6}}-1\right\rceil}\frac{K_{k,j,N}(n,r,0)}{k}\int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}}\left(x-\frac{r}{2N}\right)\cot\left(\pi\left(-\frac{x}{k}+\frac{r}{2Nk}\right)\right)f_m'\left(\xi_x\right)\,dx. \quad \text{(IV.4.16)}$$

Now we want to bound $|(x-\frac{r}{2N})\cot(\pi(-\frac{x}{k}+\frac{r}{2Nk}))|$ and $|f_m'(\xi_x)|$ seperately. For the first one we use the Taylor series expansion around $\frac{r}{2N}$ and see that

$$\left(x-\frac{r}{2N}\right)\cot\left(\pi\left(-\frac{x}{k}+\frac{r}{2Nk}\right)\right)=-\frac{k}{\pi}+\frac{1}{3}\frac{\pi}{k}\left(\frac{r}{2N}-x\right)^2+O\left(\frac{\left(\frac{r}{2N}-x\right)^4}{k^3}\right)=O_N(k)$$

as $k\to\infty$. For the second one we see that

$$\left|f_m'\left(\xi_x\right)\right|=\left|\left(R_m(x)-R_m\left(\frac{r}{2N}\right)\right)'\Big|_{x=\xi_x}\right|=\left|R_m'\left(\xi_x\right)\right|,$$

since $R_m\left(\frac{r}{2N}\right)$ is independent of x. We note that

$$R_m(x)=\int_{\gamma_m}e^{2\pi\left(g_{j,N}(n)\omega+\frac{1}{k^2}\left(\frac{1}{24}-Nx^2\right)\frac{1}{\omega}\right)}\,d\omega=\int_{\gamma_m}e^{2\pi g_{j,N}(n)\omega}e^{\frac{\pi}{12k^2\omega}}e^{-\frac{2\pi Nx^2}{k^2\omega}}\,d\omega,$$

where we have an integral over a compact set and continuously differentiable integrand and thus are allowed to switch the integral with a derivative. This yields

$$\left|R_m'\left(\xi_x\right)\right|=\left|\int_{\gamma_m}e^{2\pi g_{j,N}(n)\omega}e^{\frac{\pi}{12k^2\omega}}\left(-\frac{4\pi N\xi_x}{k^2\omega}\right)e^{-\frac{2\pi N\xi_x^2}{k^2\omega}}\,d\omega\right|.$$

Remember that we have $\omega=u\pm i\frac{1}{k(J+k)}$ with $-J^{-2}\leq u\leq J^{-2}$ and $\mathrm{Re}(\frac{1}{\omega})\leq 4k^2$. Additionally we see that

$$\left|\frac{1}{\omega}\right|=\left(\frac{u^2+\left(\mp\frac{1}{k(J+k)}\right)^2}{\left(u^2+\frac{1}{k^2(J+k)^2}\right)^2}\right)^{\frac{1}{2}}=\left(\frac{1}{u^2+\frac{1}{k^2(J+k)^2}}\right)^{\frac{1}{2}}=k(J+k)\left(\frac{1}{k^2(J+k)^2u^2+1}\right)^{\frac{1}{2}}<k(J+k)$$

and therefore $|\frac{1}{k^2\omega}|<\frac{J+k}{k}$. Thus our integrand in this cases is less than

$$\frac{4\pi N|\xi_x|(J+k)}{k}e^{2\pi g_{j,N}(n)J^{-2}}e^{\frac{\pi}{3}}e^{-8\pi N\xi_x^2},$$

which finally yields that

$$\begin{aligned}|R_1'(\xi_x)|\text{ and }|R_3'(\xi_x)|<&8\pi N|\xi_x|\frac{J+k}{J^2k}e^{2\pi g_{j,N}(n)J^{-2}}e^{\frac{\pi}{3}}e^{-8\pi N\xi_x^2}\\ \leq&\frac{16\pi N|\xi_x|}{Jk}e^{2\pi g_{j,N}(n)J^{-2}}e^{\frac{\pi}{3}}e^{-8\pi N\xi_x^2}.\end{aligned}$$

This gives us that (IV.4.16) equals

$$O\left(\sum_{k=1}^{J}\sum_{r=1}^{\left\lceil\sqrt{\frac{N}{6}}-1\right\rceil}\frac{nk^{\frac{1}{2}+\varepsilon}}{k}\int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}}k\frac{16\pi N|\xi_x|}{Jk}e^{2\pi g_{j,N}(n)J^{-2}}\,dx\right)=O_N\left(ne^{2\pi g_{j,N}(n)J^{-2}}J^{-\frac{1}{2}+\varepsilon}\right).$$

For $m=2$ we define $\delta:=\min(kJ^{-\frac{3}{4}},|\sqrt{\frac{1}{24N}}-\frac{r}{2N}|)$ and split our integral over x as follows

$$\begin{aligned}&\sum_{k=1}^{J}\sum_{r=1}^{\left\lceil\sqrt{\frac{N}{6}}-1\right\rceil}\frac{K_{k,j,N}(n,r,0)}{k}\int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}}\cot\left(\pi\left(-\frac{x}{k}+\frac{r}{2Nk}\right)\right)\left(R_2(x)-R_2\left(\frac{r}{2N}\right)\right)dx\\&=\sum_{k=1}^{J}\sum_{r=1}^{\left\lceil\sqrt{\frac{N}{6}}-1\right\rceil}\frac{K_{k,j,N}(n,r,0)}{k}\\&\qquad\times\left(\left(\int_{-\sqrt{\frac{1}{24N}}}^{\frac{r}{2N}-\delta}+\int_{\frac{r}{2N}+\delta}^{\sqrt{\frac{1}{24N}}}+\int_{\frac{r}{2N}-\delta}^{\frac{r}{2N}+\delta}\right)\cot\left(\pi\left(-\frac{x}{k}+\frac{r}{2Nk}\right)\right)\left(R_2(x)-R_2\left(\frac{r}{2N}\right)\right)dx\right).\end{aligned}$$

Using (IV.1.7) we see that

$$\begin{aligned}&\sum_{k=1}^{J}\sum_{r=1}^{\left\lceil\sqrt{\frac{N}{6}}-1\right\rceil}\frac{K_{k,j,N}(n,r,0)}{k}\left(\left(\int_{-\sqrt{\frac{1}{24N}}}^{\frac{r}{2N}-\delta}+\int_{\frac{r}{2N}+\delta}^{\sqrt{\frac{1}{24N}}}\right)\cot\left(\pi\left(-\frac{x}{k}+\frac{r}{2Nk}\right)\right)\left(R_2(x)-R_2\left(\frac{r}{2N}\right)\right)dx\right)\\&=O_N\left(\sum_{k=1}^{J}\sum_{r=1}^{\left\lceil\sqrt{\frac{N}{6}}-1\right\rceil}\frac{k^{\frac{1}{2}+\varepsilon}n}{k}\left(\left(\int_{-\sqrt{\frac{1}{24N}}}^{\frac{r}{2N}-\delta}+\int_{\frac{r}{2N}+\delta}^{\sqrt{\frac{1}{24N}}}\right)\left|\cot\left(\pi\left(-\frac{x}{k}+\frac{r}{2Nk}\right)\right)\right|\left|R_2(x)-R_2\left(\frac{r}{2N}\right)\right|dx\right)\right).\end{aligned}$$

Since we are away from $x=\frac{r}{2N}$ we can bound

$$\left|\cot\left(\pi\left(-\frac{x}{k}+\frac{r}{2Nk}\right)\right)\right|=O\left(\frac{k}{\pi\left(-x+\frac{r}{2N}\right)}\right)=O\left(\frac{k}{\delta}\right)$$

and

$$\left|R_2(x)-R_2\left(\frac{r}{2N}\right)\right|=O\left(|R_2(x)|\right)=O\left(k^{-1}J^{-1}\right),$$

using (IV.4.11). Thus we can simplify our O-term to

$$O_N\left(\sum_{k=1}^{J}\sum_{r=1}^{\left\lceil\sqrt{\frac{N}{6}}-1\right\rceil}\frac{k^{\frac{1}{2}+\varepsilon}n}{k}\frac{k}{\delta}k^{-1}J^{-1}\left(\left(\int_{-\sqrt{\frac{1}{24N}}}^{\frac{r}{2N}-\delta}+\int_{\frac{r}{2N}+\delta}^{\sqrt{\frac{1}{24N}}}\right)dx\right)\right)=O_N\left(nJ^{-1}\sum_{k=1}^{J}k^{-\frac{1}{2}+\varepsilon}\delta^{-1}\right).$$

For $\delta = kJ^{-\frac{3}{4}}$ this equals

$$O_N\left(nJ^{-\frac{1}{4}}\sum_{k=1}^{J}k^{-\frac{3}{2}+\varepsilon}\right) = O_N\left(nJ^{-\frac{1}{4}}\right),$$

while for $\delta = |\sqrt{\frac{1}{24N}} - \frac{r}{2N}|$ it equals

$$O_N\left(nJ^{-1}\sum_{k=1}^{J}k^{-\frac{1}{2}+\varepsilon}\right) = O_N\left(nJ^{-\frac{1}{2}+\varepsilon}\right).$$

For the last integral, the one close to $\frac{r}{2N}$, we use the Taylor expansion as seen in the cases $m \in \{1, 3\}$ and have

$$\begin{aligned}
&\sum_{k=1}^{J}\sum_{r=1}^{\left\lceil\sqrt{\frac{N}{6}}-1\right\rceil}\frac{K_{k,j,N}(n,r,0)}{k}\int_{\frac{r}{2N}-\delta}^{\frac{r}{2N}+\delta}\cot\left(\pi\left(-\frac{x}{k}+\frac{r}{2Nk}\right)\right)\left(R_2(x)-R_2\left(\frac{r}{2N}\right)\right)dx\\
=&\sum_{k=1}^{J}\sum_{r=1}^{\left\lceil\sqrt{\frac{N}{6}}-1\right\rceil}\frac{K_{k,j,N}(n,r,0)}{k}\int_{\frac{r}{2N}-\delta}^{\frac{r}{2N}+\delta}\cot\left(\pi\left(-\frac{x}{k}+\frac{r}{2Nk}\right)\right)\left(x-\frac{r}{2N}\right)R_2'(\xi_x)\,dx. \qquad \text{(IV.4.17)}
\end{aligned}$$

Since $|\cot(\pi(-\frac{x}{k}+\frac{r}{2Nk}))(x-\frac{r}{2N})| = O_N(k)$, as seen before, we only need to look at $|R_2'(\xi_x)|$. Recall that on γ_2 we had $\omega = -J^{-2} - iv$ with $-\frac{1}{k(J+k)} \leq v \leq \frac{1}{k(J+k)}$ and $\mathrm{Re}(\omega), \mathrm{Re}\left(\frac{1}{\omega}\right) < 0$. Using that

$$\left|\frac{1}{\omega}\right| = \left(\frac{J^{-4}+v^2}{(J^{-4}+v^2)^2}\right)^{\frac{1}{2}} = \left(\frac{1}{J^{-4}+v^2}\right)^{\frac{1}{2}} = \left(\frac{J^4}{1+J^4v^2}\right)^{\frac{1}{2}},$$

we obatin

$$\begin{aligned}
|R_2'(\xi_x)| =& \left|-i\int_{-\frac{1}{k(J+k)}}^{\frac{1}{k(J+k)}} e^{2\pi g_{j,N}(n)\left(-J^{-2}-iv\right)}e^{\frac{2\pi}{k^2(-J^{-2}-iv)}\left(\frac{1}{24}-N\xi_x^2\right)}\left(-\frac{4\pi N\xi_x}{k^2\left(-J^{-2}-iv\right)}\right)dv\right|\\
\leq&\frac{4\pi N|\xi_x|}{k^2}\int_{-\frac{1}{k(J+k)}}^{\frac{1}{k(J+k)}}\frac{1}{|-J^{-2}-iv|}\,dv = \frac{4\pi N|\xi_x|}{k^2}\int_{-\frac{1}{k(J+k)}}^{\frac{1}{k(J+k)}}\left(\frac{J^4}{1+J^4v^2}\right)^{\frac{1}{2}}dv\\
\leq&\frac{4\pi N|\xi_x|}{k^2}\int_{-\frac{1}{k(J+k)}}^{\frac{1}{k(J+k)}}\frac{J}{\sqrt{2v}}\,dv = \frac{4\pi N|\xi_x|J}{\sqrt{2}k^2}\int_{-\frac{1}{k(J+k)}}^{\frac{1}{k(J+k)}}v^{-\frac{1}{2}}\,dv = O_N\left(\frac{J}{k^2\left(k(J+k)\right)^{\frac{1}{2}}}\right)\\
=&O_N\left(\frac{1}{k^2}\frac{J^{\frac{1}{2}}}{k^{\frac{1}{2}}}\right) = O_N\left(\frac{J^{\frac{1}{2}}}{k^{\frac{5}{2}}}\right).
\end{aligned}$$

Thus we can simplify (IV.4.17) to

$$O_N\left(\sum_{k=1}^{J}\sum_{r=1}^{\left\lceil\sqrt{\frac{N}{6}}-1\right\rceil}\frac{nk^{\frac{1}{2}+\varepsilon}}{k}\int_{\frac{r}{2N}-\delta}^{\frac{r}{2N}+\delta}k\frac{J^{\frac{1}{2}}}{k^{\frac{5}{2}}}\,dx\right)=O_N\left(nJ^{\frac{1}{2}}\sum_{k=1}^{J}k^{-2+\varepsilon}\delta\right)=O_N\left(nJ^{-\frac{1}{4}+\varepsilon}\right),$$

where we used that $\delta \leq kJ^{-\frac{3}{4}}$.

Thus we overall see that

$$\begin{aligned}E_1=&\sum_{k=1}^{J}\sum_{r=1}^{\left\lceil\sqrt{\frac{N}{6}}-1\right\rceil}\frac{K_{k,j,N}(n,r,0)}{k}\left(R_1\left(\frac{r}{2N}\right)+R_2\left(\frac{r}{2N}\right)+R_3\left(\frac{r}{2N}\right)\right)\\&\times \text{P.V.}\int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}}\cot\left(\pi\left(-\frac{x}{k}+\frac{r}{2Nk}\right)\right)\,dx+O_N\left(ne^{2\pi g_{j,N}(n)J^{-2}}J^{-\frac{1}{4}+\varepsilon}\right).\end{aligned}$$

Substituting $y=\frac{r}{2N}-x$ yields that the principal value integral equals

$$\begin{aligned}\text{P.V.}\int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}}\cot\left(\frac{\pi}{k}\left(-x+\frac{r}{2N}\right)\right)\,dx=&\,\text{P.V.}\int_{\frac{r}{2N}-\sqrt{\frac{1}{24N}}}^{\frac{r}{2N}+\sqrt{\frac{1}{24N}}}\cot\left(\frac{\pi}{k}y\right)\,dy\\=&\,\text{P.V.}\int_{\frac{r}{2N}-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}-\frac{r}{2N}}\cot\left(\frac{\pi}{k}y\right)\,dy+\int_{\sqrt{\frac{1}{24N}}-\frac{r}{2N}}^{\frac{r}{2N}+\sqrt{\frac{1}{24N}}}\cot\left(\frac{\pi}{k}y\right)\,dy.\end{aligned}$$

We obtain that the leftover principal value integral equals zero, since we have an odd function and a symmetric interval, and by using the Taylor expansion of $\cot(z)$ we see that

$$\int_{\sqrt{\frac{1}{24N}}-\frac{r}{2N}}^{\sqrt{\frac{1}{24N}}+\frac{r}{2N}}\cot\left(\frac{\pi}{k}y\right)\,dy=O_N\left(\int_{\sqrt{\frac{1}{24N}}-\frac{r}{2N}}^{\sqrt{\frac{1}{24N}}+\frac{r}{2N}}k\,dy\right)=O_N\left(k\right).$$

Additionally we obtain, using (IV.4.12), that

$$\begin{aligned}R_1\left(\frac{r}{2N}\right)+R_2\left(\frac{r}{2N}\right)+R_3\left(\frac{r}{2N}\right)=&\,O\left(k^{-1}J^{-1}e^{2\pi g_{j,N}(n)J^{-2}+8\pi\left(\frac{1}{24}-\frac{r^2}{4N}\right)}\right)\\=&\,O_N\left(k^{-1}J^{-1}e^{2\pi g_{j,N}(n)J^{-2}}\right),\end{aligned}$$

which yields

$$\sum_{k=1}^{J}\sum_{r=1}^{\left\lceil\sqrt{\frac{N}{6}}-1\right\rceil}\frac{K_{k,j,N}(n,r,0)}{k}\left(R_1\left(\frac{r}{2N}\right)+R_2\left(\frac{r}{2N}\right)+R_3\left(\frac{r}{2N}\right)\right)\text{P.V.}\int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}}\cot\left(\pi\left(-\frac{x}{k}+\frac{r}{2Nk}\right)\right)dx$$

$$=O_N\left(\sum_{k=1}^{J}\sum_{r=1}^{\left\lceil\sqrt{\frac{N}{6}}-1\right\rceil}\frac{nk^{\frac{1}{2}+\varepsilon}}{k}k^{-1}J^{-1}e^{2\pi g_{j,N}(n)J^{-2}}k\right)=O_N\left(ne^{2\pi g_{j,N}(n)J^{-2}}J^{-\frac{1}{2}+\varepsilon}\right)$$

using (IV.1.7). Overall we therefore showed that

$$E_1=O_N\left(ne^{2\pi g_{j,N}(n)J^{-2}}J^{-\frac{1}{4}+\varepsilon}\right). \tag{IV.4.18}$$

Combining (IV.4.14), (IV.4.15), and (IV.4.18) finally gives

$$\begin{aligned}a_{\mathcal{I}^*,j,N,0}(n)=&-\frac{2\pi i}{\sqrt{g_{j,N}(n)}}\sum_{k=1}^{J}\sum_{r=1}^{N-1}\sum_{\kappa=0}^{k-1}\frac{K_{k,j,N}(n,r,\kappa)}{k^2}\\&\times\text{P.V.}\int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}}\sqrt{\frac{1}{24}-Nx^2}\cot\left(\pi\left(-\frac{x}{k}+\frac{\kappa}{k}+\frac{r}{2Nk}\right)\right)I_1\left(\frac{4\pi\sqrt{g_{j,N}(n)}}{k}\sqrt{\frac{1}{24}-Nx^2}\right)dx\\&+O_N\left(ne^{2\pi g_{j,N}(n)J^{-2}}\max\left(J^{-\frac{1}{2}+\varepsilon}\log(J),J^{-\frac{1}{4}+\varepsilon}\right)\right).\end{aligned} \tag{IV.4.19}$$

IV.4.2 Error part

It is left to show that all the other parts of $a_{j,N}(n)$ are relatively small compared to $a_{\mathcal{I}^*,j,N,0}(n)$. Therefore we go on by analyzing $a_{\mathcal{I}^*,j,N,1}(n)$ and $a_{\mathcal{I}^*,j,N,2}(n)$, where we only discuss $a_{\mathcal{I}^*,j,N,1}(n)$ in detail, since $a_{\mathcal{I}^*,j,N,2}(n)$ can be treated accordingly.

Recalling the definitions from above we have that

$$\begin{aligned}a_{\mathcal{I}^*,j,N,1}(n)=&\sum_{k=1}^{J}\sum_{\substack{0\le h<k\\ \gcd(h,k)=1}}e^{-\frac{2\pi ih}{k}g_{j,N}(n)}\int_{-\frac{1}{k(k_1+k)}}^{-\frac{1}{k(J+k)}}e^{2\pi g_{j,N}(n)\omega}\sum_{r=1}^{N-1}\chi_{j,r}(N,M_{h,k})\zeta_{24k}^{-h'}\frac{i}{\pi}e^{\frac{2\pi}{24k^2\omega}}\\&\times\sum_{\kappa\in\mathbb{Z}}e^{2\pi iN\left(\kappa+\frac{r}{2N}\right)^2\frac{h'}{k}}\text{P.V.}\int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}}\frac{e^{-2\pi N\frac{1}{k^2\omega}x^2}}{x-\left(\kappa+\frac{r}{2N}\right)}dx\,d\phi.\end{aligned}$$

Following [Rad38], using (IV.4.2), and splitting the integral over ϕ into integrals running over segments $[-\frac{1}{k\ell},-\frac{1}{k(\ell+1)}]$, for $k_1+k\le\ell\le J+k-1$, it follows that

$$\begin{aligned}a_{\mathcal{I}^*,j,N,1}(n)=&\sum_{k=1}^{J}\sum_{\substack{0\le h<k\\ \gcd(h,k)=1}}e^{-\frac{2\pi ih}{k}g_{j,N}(n)}\sum_{\ell=k_1+k}^{J+k-1}\int_{-\frac{1}{k\ell}}^{-\frac{1}{k(\ell+1)}}e^{2\pi g_{j,N}(n)\omega}\sum_{r=1}^{N-1}\chi_{j,r}(N,M_{h,k})\zeta_{24k}^{-h'}\frac{i}{\pi}e^{\frac{2\pi}{24k^2\omega}}\\&\times\sum_{\kappa\in\mathbb{Z}}e^{2\pi iN\left(\kappa+\frac{r}{2N}\right)^2\frac{h'}{k}}\text{P.V.}\int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}}\frac{e^{-2\pi N\frac{1}{k^2\omega}x^2}}{x-\left(\kappa+\frac{r}{2N}\right)}dx\,d\phi\end{aligned}$$

$$=\frac{i}{\pi}\sum_{k=1}^{J}\sum_{\ell=J+1}^{J+k-1}\int_{-\frac{1}{k\ell}}^{-\frac{1}{k(\ell+1)}} e^{2\pi g_{j,N}(n)\omega}e^{\frac{2\pi}{24k^2\omega}}\sum_{r=1}^{N-1}\sum_{\kappa\in\mathbb{Z}}\text{P.V.}\int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}}\frac{e^{-2\pi N\frac{1}{k^2\omega}x^2}}{x-\left(\kappa+\frac{r}{2N}\right)}dx$$
$$\times\sum_{\substack{0\le h<k\\ \gcd(h,k)=1\\ J<k_1+k\le\ell}} e^{-\frac{2\pi ih}{k}g_{j,N}(n)}\chi_{j,r}(N,M_{h,k})\zeta_{24k}^{-h'}e^{2\pi iN\left(\kappa+\frac{r}{2N}\right)^2\frac{h'}{k}}\,d\phi.$$

The sum

$$K^*_{k,j,N}(n,r,\kappa,\ell):=\sum_{\substack{0\le h<k\\ \gcd(h,k)=1\\ J<k_1+k\le\ell}}\chi_{j,r}(N,M_{h,k})\zeta_{24k}^{\left(24N\left(\kappa+\frac{r}{2N}\right)^2-1\right)h'-24g_{j,N}(n)h} \tag{IV.4.20}$$

is again well-defined for $hh'\equiv -1 \pmod k$ and of modulus k. For $\mu\in\mathbb{Z}$ we observe that $K^*_{k,j,N}(n,r,\kappa+\mu k,\ell)=K^*_{k,j,N}(n,r,\kappa,\ell)$, which, shifting $\kappa\to\kappa+\mu k$, gives us

$$a_{\mathcal{I}^*,j,N,1}(n)=\frac{i}{\pi}\sum_{k=1}^{J}\sum_{\ell=J+1}^{J+k-1}\sum_{r=1}^{N-1}\sum_{\kappa=0}^{k-1}K^*_{k,j,N}(n,r,\kappa,\ell)$$
$$\times\int_{-\frac{1}{k\ell}}^{-\frac{1}{k(\ell+1)}} e^{2\pi g_{j,N}(n)\omega}e^{\frac{2\pi}{24k^2\omega}}\lim_{L\to\infty}\sum_{\mu=-L}^{L}\text{P.V.}\int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}}\frac{e^{-2\pi N\frac{1}{k^2\omega}x^2}}{x-\left(\kappa+\mu k+\frac{r}{2N}\right)}dx\,d\phi.$$

Completely analogously to the calculations of $a_{\mathcal{I}^*,j,N,0}(n)$ we obtain

$$a_{\mathcal{I}^*,j,N,1}(n)=-i\sum_{k=1}^{J}\sum_{\ell=J+1}^{J+k-1}\sum_{r=1}^{N-1}\sum_{\kappa=0}^{k-1}\frac{K^*_{k,j,N}(n,r,\kappa,\ell)}{k}\text{P.V.}\int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}}\cot\left(\pi\left(-\frac{x}{k}+\frac{\kappa}{k}+\frac{r}{2Nk}\right)\right)$$
$$\times\int_{-\frac{1}{k\ell}}^{-\frac{1}{k(\ell+1)}} e^{2\pi\left(g_{j,N}(n)\omega+\frac{1}{24k^2\omega}-\frac{Nx^2}{k^2\omega}\right)}d\phi\,dx.$$

Since $\omega=J^{-2}-i\phi$ we obtain

$$\int_{-\frac{1}{k\ell}}^{-\frac{1}{k(\ell+1)}} e^{2\pi\left(g_{j,N}(n)\omega+\frac{1}{24k^2\omega}-\frac{Nx^2}{k^2\omega}\right)}d\phi=-i\int_{J^{-2}+\frac{i}{k(\ell+1)}}^{J^{-2}+\frac{i}{k\ell}} e^{2\pi\left(g_{j,N}(n)\omega+\left(\frac{1}{24}-Nx^2\right)\frac{1}{k^2\omega}\right)}d\omega$$

and note that for $v:=-\phi$

$$\text{Re}\left(2\pi\left(g_{j,N}(n)\omega+\left(\frac{1}{24}-Nx^2\right)\frac{1}{k^2\omega}\right)\right)=2\pi\left(g_{j,N}(n)J^{-2}+\left(\frac{1}{24}-Nx^2\right)\frac{J^{-2}}{k^2\left(J^{-4}+v^2\right)}\right).$$

Summing over all ℓ we see that $\frac{1}{k(J+k)}\le v\le\frac{1}{k(J+1)}$ and thus

$$\frac{J^{-2}}{k^2\left(J^{-4}+v^2\right)}\le\frac{J^{-2}}{k^2\left(J^{-4}+\left(\frac{1}{k(J+k)}\right)^2\right)}=\frac{(J+k)^2J^{-2}}{(J+k)^2k^2J^{-4}+1}\le(J+k)^2J^{-2}=\left(\frac{J+k}{J}\right)^2<4.$$

This gives that

$$\mathrm{Re}\left(2\pi\left(g_{j,N}(n)\omega+\left(\frac{1}{24}-Nx^2\right)\frac{1}{k^2\omega}\right)\right)\le 2\pi g_{j,N}(n)J^{-2}+8\pi\left(\frac{1}{24}-Nx^2\right)$$

and therefore

$$\begin{aligned}H_\ell(x):=&\int_{J^{-2}+\frac{i}{k(\ell+1)}}^{J^{-2}+\frac{i}{k\ell}} e^{2\pi\left(g_{j,N}(n)\omega+\left(\frac{1}{24}-Nx^2\right)\frac{1}{k^2\omega}\right)}\,d\omega\\ =&O_N\left(\left(\frac{1}{k\ell}-\frac{1}{k(\ell+1)}\right)e^{2\pi g_{j,N}(n)J^{-2}}\right).\end{aligned}\tag{IV.4.21}$$

Splitting $a_{\mathcal{I}^*,j,N,1}(n)$ analogously to E in the case $a_{\mathcal{I}^*,j,N,0}(n)$ gives

$$\begin{aligned}a_{\mathcal{I}^*,j,N,1}(n)=&E_1^*+E_2^*+E_3^*\\ :=&-\sum_{k=1}^{J}\sum_{\ell=J+1}^{J+k-1}\sum_{r=1}^{\left\lceil\sqrt{\frac{N}{6}}-1\right\rceil}\frac{K^*_{k,j,N}(n,r,0,\ell)}{k}\,\mathrm{P.V.}\int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}}\cot\left(\pi\left(-\frac{x}{k}+\frac{r}{2Nk}\right)\right)H_\ell(x)\,dx\\ &-\sum_{k=1}^{J}\sum_{\ell=J+1}^{J+k-1}\sum_{r=\left\lceil\sqrt{\frac{N}{6}}\right\rceil}^{N-1}\frac{K^*_{k,j,N}(n,r,0,\ell)}{k}\int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}}\cot\left(\pi\left(-\frac{x}{k}+\frac{r}{2Nk}\right)\right)H_\ell(x)\,dx\\ &-\sum_{k=1}^{J}\sum_{\ell=J+1}^{J+k-1}\sum_{r=1}^{N-1}\sum_{\kappa=1}^{k-1}\frac{K^*_{k,j,N}(n,r,\kappa,\ell)}{k}\int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}}\cot\left(\pi\left(-\frac{x}{k}+\frac{\kappa}{k}+\frac{r}{2Nk}\right)\right)H_\ell(x)\,dx.\end{aligned}$$

Following [Est29, page 94] we set $h_0:=\ell-J$, which gives that $1\le h_0<k$. Additionally we define

$$g_1(m):=\begin{cases}1 & \text{if } 0<m\le h_0,\\ 0 & \text{if } h_0<m\le k,\end{cases}$$

and

$$g_1(m+k)=g_1(m)$$

for all integers m. Using this setting together with (IV.4.1) we obtain that

$$\delta_{J<k_1+k\le\ell}=\delta_{0<k_1+k-J\le h_0}=g_1(k_1+k-J)=g_1(k_1-J)=g_1(-h'-J)=:\delta_{\sigma_1\le h'<\sigma_2},$$

for some $0\le\sigma_1<\sigma_2\le k$. This yields that the extra restriction on k_1 in the sum $K^*_{k,j,N}(n,r,\kappa,\ell)$ constrains the choice of h' to an interval mod k. Therefore $K^*_{k,j,N}(n,r,\kappa,\ell)$ is an incomplete Kloosterman sum and can be bounded by (IV.1.7) following the techniques by Lehner [Leh41, Section 10] (see Lemma V.3.1). We thus obtain that

$$\begin{aligned}E_3^* =&O_N\left(ne^{2\pi g_{j,N}(n)J^{-2}}\sum_{k=1}^{J}\sum_{\ell=J+1}^{J+k-1}\sum_{r=1}^{N-1}\sum_{\kappa=1}^{k-1}k^{-\frac{1}{2}+\varepsilon}\left(\frac{1}{k\ell}-\frac{1}{k(\ell+1)}\right)\int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}}\left|\cot\left(\pi\left(-\frac{x}{k}+\frac{\kappa}{k}+\frac{r}{2Nk}\right)\right)\right|dx\right)\\
=&O_N\left(ne^{2\pi g_{j,N}(n)J^{-2}}\sum_{k=1}^{J}\sum_{r=1}^{N-1}\sum_{\kappa=1}^{k-1}k^{-\frac{1}{2}+\varepsilon}\left(\frac{1}{k(J+1)}-\frac{1}{k(J+k)}\right)\int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}}\left|\cot\left(\pi\left(-\frac{x}{k}+\frac{\kappa}{k}+\frac{r}{2Nk}\right)\right)\right|dx\right)\\
=&O_N\left(\frac{n}{J}e^{2\pi g_{j,N}(n)J^{-2}}\sum_{k=1}^{J}\sum_{r=1}^{N-1}\sum_{\kappa=1}^{k-1}k^{-\frac{3}{2}+\varepsilon}\int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}}\left|\cot\left(\pi\left(-\frac{x}{k}+\frac{\kappa}{k}+\frac{r}{2Nk}\right)\right)\right|dx\right)\\
=&O_N\left(ne^{2\pi g_{j,N}(n)J^{-2}}J^{-\frac{1}{2}+\varepsilon}\log(J)\right),\end{aligned}$$

as seen before. Similar to that we redo the calculations for bounding E_2 to prove that

$$E_2^* = O_N\left(ne^{2\pi g_{j,N}(n)J^{-2}}J^{-\frac{1}{2}+\varepsilon}\right).$$

Lastly we take care of E_1^*. We rewrite

$$\begin{aligned}E_1^* =&-\sum_{k=1}^{J}\sum_{\ell=J+1}^{J+k-1}\sum_{r=1}^{\left\lceil\sqrt{\frac{N}{6}}-1\right\rceil}\frac{K_{k,j,N}^*(n,r,0,\ell)}{k}\int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}}\cot\left(\pi\left(-\frac{x}{k}+\frac{r}{2Nk}\right)\right)\left(H_\ell(x)-H_\ell\left(\frac{r}{2N}\right)\right)dx\\
&-\sum_{k=1}^{J}\sum_{\ell=J+1}^{J+k-1}\sum_{r=1}^{\left\lceil\sqrt{\frac{N}{6}}-1\right\rceil}\frac{K_{k,j,N}^*(n,r,0,\ell)}{k}H_\ell\left(\frac{r}{2N}\right)\text{ P.V.}\int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}}\cot\left(\pi\left(-\frac{x}{k}+\frac{r}{2Nk}\right)\right)dx.\end{aligned}$$

Using $\delta=\min(kJ^{-\frac{3}{4}},|\sqrt{\frac{1}{24N}}-\frac{r}{2N}|)$ as before and splitting our integral yields

$$\begin{aligned}E_1^* =&-\sum_{k=1}^{J}\sum_{\ell=J+1}^{J+k-1}\sum_{r=1}^{\left\lceil\sqrt{\frac{N}{6}}-1\right\rceil}\frac{K_{k,j,N}^*(n,r,0,\ell)}{k}\\
&\times\left(\int_{-\sqrt{\frac{1}{24N}}}^{\frac{r}{2N}-\delta}+\int_{\frac{r}{2N}+\delta}^{\sqrt{\frac{1}{24N}}}+\int_{\frac{r}{2N}-\delta}^{\frac{r}{2N}+\delta}\right)\cot\left(\pi\left(-\frac{x}{k}+\frac{r}{2Nk}\right)\right)\left(H_\ell(x)-H_\ell\left(\frac{r}{2N}\right)\right)dx\\
&-\sum_{k=1}^{J}\sum_{\ell=J+1}^{J+k-1}\sum_{r=1}^{\left\lceil\sqrt{\frac{N}{6}}-1\right\rceil}\frac{K_{k,j,N}^*(n,r,0,\ell)}{k}H_\ell\left(\frac{r}{2N}\right)\text{ P.V.}\int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}}\cot\left(\pi\left(-\frac{x}{k}+\frac{r}{2Nk}\right)\right)dx.\end{aligned}$$

Using (IV.1.7) and (IV.4.21) we see that

$$-\sum_{k=1}^{J}\sum_{\ell=J+1}^{J+k-1}\sum_{r=1}^{\left\lceil\sqrt{\frac{N}{6}}-1\right\rceil}\frac{K_{k,j,N}^*(n,r,0,\ell)}{k}\left(\int_{-\sqrt{\frac{1}{24N}}}^{\frac{r}{2N}-\delta}+\int_{\frac{r}{2N}+\delta}^{\sqrt{\frac{1}{24N}}}\right)\cot\left(\pi\left(-\frac{x}{k}+\frac{r}{2Nk}\right)\right)\left(H_\ell(x)-H_\ell\left(\frac{r}{2N}\right)\right)dx$$

$$=O_N\left(\sum_{k=1}^{J}\sum_{\ell=J+1}^{J+k-1}\sum_{r=1}^{\left\lceil\sqrt{\frac{N}{6}}-1\right\rceil}\frac{nk^{\frac{1}{2}+\varepsilon}}{k}\left(\int_{-\sqrt{\frac{1}{24N}}}^{\frac{r}{2N}-\delta}+\int_{\frac{r}{2N}+\delta}^{\sqrt{\frac{1}{24N}}}\right)\frac{k}{\delta}\left(\frac{1}{k\ell}-\frac{1}{k(\ell+1)}\right)e^{2\pi g_{j,N}(n)J^{-2}}\,dx\right)$$

$$=O_N\left(\frac{n}{J}e^{2\pi g_{j,N}(n)J^{-2}}\sum_{k=1}^{J}\sum_{r=1}^{\left\lceil\sqrt{\frac{N}{6}}-1\right\rceil}k^{-\frac{1}{2}+\varepsilon}\delta^{-1}\left(\int_{-\sqrt{\frac{1}{24N}}}^{\frac{r}{2N}-\delta}+\int_{\frac{r}{2N}+\delta}^{\sqrt{\frac{1}{24N}}}\right)dx\right)=O_N\left(ne^{2\pi g_{j,N}(n)J^{-2}}J^{-\frac{1}{4}}\right),$$

as before.

Defining $h_\ell(x) := H_\ell(x)-H_\ell(\frac{r}{2N})$ and using the Taylor expansion we obtain that

$$h_\ell(x)=h_\ell'(\xi_x)\left(x-\frac{r}{2N}\right),$$

for some ξ_x between $\frac{r}{2N}$ and x and see that $|h_\ell'(\xi_x)|=|H_\ell'(\xi_x)|$ as before. For the integral close to $\frac{r}{2N}$ we first note that

$$\sum_{\ell=J+1}^{J+k-1}|H_\ell'(\xi_x)|=O_N\left(\frac{J^{\frac{1}{2}}}{k^{\frac{5}{2}}}e^{2\pi g_{j,N}(n)J^{-2}}\right),$$

using the techniques from before. Therefore we obtain

$$-\sum_{k=1}^{J}\sum_{\ell=J+1}^{J+k-1}\sum_{r=1}^{\left\lceil\sqrt{\frac{N}{6}}-1\right\rceil}\frac{K^*_{k,j,N}(n,r,0,\ell)}{k}\int_{\frac{r}{2N}-\delta}^{\frac{r}{2N}+\delta}\cot\left(\pi\left(-\frac{x}{k}+\frac{r}{2Nk}\right)\right)\left(H_\ell(x)-H_\ell\left(\frac{r}{2N}\right)\right)dx$$
$$=O_N\left(\sum_{k=1}^{J}\sum_{\ell=J+1}^{J+k-1}\sum_{r=1}^{\left\lceil\sqrt{\frac{N}{6}}-1\right\rceil}nk^{\frac{1}{2}+\varepsilon}\int_{\frac{r}{2N}-\delta}^{\frac{r}{2N}+\delta}|H_\ell'(\xi_x)|\,dx\right)=O_N\left(ne^{2\pi g_{j,N}(n)J^{-2}}J^{-\frac{1}{4}+\varepsilon}\right),$$

which yields

$$E_1^*=-\sum_{k=1}^{J}\sum_{\ell=J+1}^{J+k-1}\sum_{r=1}^{\left\lceil\sqrt{\frac{N}{6}}-1\right\rceil}\frac{K^*_{k,j,N}(n,r,0,\ell)}{k}H_\ell\left(\frac{r}{2N}\right)\text{ P.V.}\int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}}\cot\left(\pi\left(-\frac{x}{k}+\frac{r}{2Nk}\right)\right)dx$$
$$+O_N\left(ne^{2\pi g_{j,N}(n)J^{-2}}J^{-\frac{1}{4}+\varepsilon}\right).$$

As seen before we have

$$\text{P.V.}\int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}}\cot\left(\pi\left(-\frac{x}{k}+\frac{r}{2Nk}\right)\right)dx=O_N(k).$$

Additionally we obtain

$$\sum_{\ell=J+1}^{J+k-1} H_\ell\left(\frac{r}{2N}\right) = O_N\left(k^{-1}J^{-1}e^{2\pi g_{j,N}(n)J^{-2}}\right),$$

using (IV.4.21). Analogously to above this yields

$$-\sum_{k=1}^{J}\sum_{\ell=J+1}^{J+k-1}\sum_{r=1}^{\left\lceil\sqrt{\frac{N}{6}}-1\right\rceil}\frac{K^*_{k,j,N}(n,r,0,\ell)}{k}H_\ell\left(\frac{r}{2N}\right)\text{P.V.}\int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}}\cot\left(\pi\left(-\frac{x}{k}+\frac{r}{2Nk}\right)\right)dx$$
$$= O_N\left(ne^{2\pi g_{j,N}(n)J^{-2}}J^{-\frac{1}{2}+\varepsilon}\right)$$

and therefore overall

$$E_1^* = O_N\left(ne^{2\pi g_{j,N}(n)J^{-2}}J^{-\frac{1}{4}+\varepsilon}\right).$$

This finally gives

$$a_{\mathcal{I}^*,j,N,1}(n) = O_N\left(ne^{2\pi g_{j,N}(n)J^{-2}}\max\left(J^{-\frac{1}{2}+\varepsilon}\log(J), J^{-\frac{1}{4}+\varepsilon}\right)\right)$$

and the analog result for $a_{\mathcal{I}^*,j,N,2}(n)$.

Plugging (IV.4.19) and this results into (IV.4.7) yields that

$$\begin{aligned}a_{\mathcal{I}^*,j,N}(n) = &-\frac{2\pi i}{\sqrt{g_{j,N}(n)}}\sum_{k=1}^{J}\sum_{r=1}^{N-1}\sum_{\kappa=0}^{k-1}\frac{K_{k,j,N}(n,r,\kappa)}{k^2}\\ &\times \text{P.V.}\int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}}\sqrt{\frac{1}{24}-Nx^2}\cot\left(\pi\left(-\frac{x}{k}+\frac{\kappa}{k}+\frac{r}{2Nk}\right)\right)I_1\left(\frac{4\pi\sqrt{g_{j,N}(n)}}{k}\sqrt{\frac{1}{24}-Nx^2}\right)dx\\ &+O_N\left(ne^{2\pi g_{j,N}(n)J^{-2}}\max\left(J^{-\frac{1}{2}+\varepsilon}\log(J), J^{-\frac{1}{4}+\varepsilon}\right)\right).\end{aligned} \tag{IV.4.22}$$

Lastly we have to take care of $a_{\mathcal{I},j,N}(n)$ and $a_{\mathcal{I}^e,j,N}(n)$. From the definition and (IV.2.17) we have

$$\begin{aligned}a_{\mathcal{I}^e,j,N}(n) = &\sum_{k=1}^{J}\sum_{\substack{0\le h<k\\ \gcd(h,k)=1}} e^{-\frac{2\pi ih}{k}g_{j,N}(n)}\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{2\pi g_{j,N}(n)\omega}\sum_{r=1}^{N-1}\chi_{j,r}(N,M_{h,k})\zeta_{24k}^{-h'}\frac{i}{\pi}e^{\frac{2\pi}{24k^2\omega}}\\ &\times\sum_{\kappa\in\mathbb{Z}}e^{2\pi iN\left(\kappa+\frac{r}{2N}\right)^2\frac{h'}{k}}\text{P.V.}\int_{|x|\ge\sqrt{\frac{1}{24N}}}\frac{e^{-2\pi N\frac{1}{k^2\omega}x^2}}{x-\left(\kappa+\frac{r}{2N}\right)}dx\,d\phi.\end{aligned}$$

By decomposing the Farey segment $-\vartheta'_{h,k} \le \phi \le -\vartheta''_{h,k}$ as seen before we obtain

$$\begin{aligned}
a_{\mathcal{I}^e,j,N}(n) =& \sum_{k=1}^{J} \sum_{\substack{0\le h<k \\ \gcd(h,k)=1}} e^{-\frac{2\pi i h}{k} g_{j,N}(n)} \int_{-\frac{1}{k(J+k)}}^{\frac{1}{k(J+k)}} e^{2\pi g_{j,N}(n)\omega} \sum_{r=1}^{N-1} \chi_{j,r}(N, M_{h,k}) \zeta_{24k}^{-h'} \frac{i}{\pi} e^{\frac{2\pi}{24k^2\omega}} \\
&\times \sum_{\kappa\in\mathbb{Z}} e^{2\pi i N\left(\kappa+\frac{r}{2N}\right)^2 \frac{h'}{k}} \, \mathrm{P.V.} \int_{|x|\ge\sqrt{\frac{1}{24N}}} \frac{e^{-2\pi N \frac{1}{k^2\omega} x^2}}{x-\left(\kappa+\frac{r}{2N}\right)} \, dx \, d\phi \\
&+ \sum_{k=1}^{J} \sum_{\substack{0\le h<k \\ \gcd(h,k)=1}} e^{-\frac{2\pi i h}{k} g_{j,N}(n)} \sum_{\ell=k_1+k}^{J+k-1} \int_{-\frac{1}{k\ell}}^{-\frac{1}{k(\ell+1)}} e^{2\pi g_{j,N}(n)\omega} \sum_{r=1}^{N-1} \chi_{j,r}(N, M_{h,k}) \zeta_{24k}^{-h'} \frac{i}{\pi} e^{\frac{2\pi}{24k^2\omega}} \\
&\times \sum_{\kappa\in\mathbb{Z}} e^{2\pi i N\left(\kappa+\frac{r}{2N}\right)^2 \frac{h'}{k}} \, \mathrm{P.V.} \int_{|x|\ge\sqrt{\frac{1}{24N}}} \frac{e^{-2\pi N \frac{1}{k^2\omega} x^2}}{x-\left(\kappa+\frac{r}{2N}\right)} \, dx \, d\phi \\
&+ \sum_{k=1}^{J} \sum_{\substack{0\le h<k \\ \gcd(h,k)=1}} e^{-\frac{2\pi i h}{k} g_{j,N}(n)} \sum_{\ell=k_2+k}^{J+k-1} \int_{\frac{1}{k(\ell+1)}}^{\frac{1}{k\ell}} e^{2\pi g_{j,N}(n)\omega} \sum_{r=1}^{N-1} \chi_{j,r}(N, M_{h,k}) \zeta_{24k}^{-h'} \frac{i}{\pi} e^{\frac{2\pi}{24k^2\omega}} \\
&\times \sum_{\kappa\in\mathbb{Z}} e^{2\pi i N\left(\kappa+\frac{r}{2N}\right)^2 \frac{h'}{k}} \, \mathrm{P.V.} \int_{|x|\ge\sqrt{\frac{1}{24N}}} \frac{e^{-2\pi N \frac{1}{k^2\omega} x^2}}{x-\left(\kappa+\frac{r}{2N}\right)} \, dx \, d\phi \\
=:& a_{\mathcal{I}^e,j,N,1}(n) + a_{\mathcal{I}^e,j,N,2}(n) + a_{\mathcal{I}^e,j,N,3}(n).
\end{aligned}$$

We first note that

$$\begin{aligned}
a_{\mathcal{I}^e,j,N,1}(n) =& \frac{i}{\pi} \sum_{k=1}^{J} \sum_{r=1}^{N-1} \sum_{\kappa=0}^{k-1} K_{k,j,N}(n,r,\kappa) \\
&\times \int_{-\frac{1}{k(J+k)}}^{\frac{1}{k(J+k)}} e^{2\pi g_{j,N}(n)\omega} \lim_{L\to\infty} \sum_{\mu=-L}^{L} \mathrm{P.V.} \int_{|x|\ge\sqrt{\frac{1}{24N}}} \frac{e^{-2\pi\left(Nx^2-\frac{1}{24}\right)\frac{1}{k^2\omega}}}{x-\left(\kappa+\mu k+\frac{r}{2N}\right)} \, dx \, d\phi
\end{aligned}$$

completely analogously to the calculations of $a_{\mathcal{I}^*,j,N,0}(n)$, while

$$\begin{aligned}
a_{\mathcal{I}^e,j,N,2}(n) =& \frac{i}{\pi} \sum_{k=1}^{J} \sum_{\ell=J+1}^{J+k-1} \sum_{r=1}^{N-1} \sum_{\kappa=0}^{k-1} K^*_{k,j,N}(n,r,\kappa,\ell) \\
&\times \int_{-\frac{1}{k\ell}}^{-\frac{1}{k(\ell+1)}} e^{2\pi g_{j,N}(n)\omega} \lim_{L\to\infty} \sum_{\mu=-L}^{L} \mathrm{P.V.} \int_{|x|\ge\sqrt{\frac{1}{24N}}} \frac{e^{-2\pi\left(Nx^2-\frac{1}{24}\right)\frac{1}{k^2\omega}}}{x-\left(\kappa+\mu k+\frac{r}{2N}\right)} \, dx \, d\phi, \\
a_{\mathcal{I}^e,j,N,3}(n) =& \frac{i}{\pi} \sum_{k=1}^{J} \sum_{\ell=J+1}^{J+k-1} \sum_{r=1}^{N-1} \sum_{\kappa=0}^{k-1} \widetilde{K}_{k,j,N}(n,r,\kappa,\ell) \\
&\times \int_{\frac{1}{k(\ell+1)}}^{\frac{1}{k\ell}} e^{2\pi g_{j,N}(n)\omega} \lim_{L\to\infty} \sum_{\mu=-L}^{L} \mathrm{P.V.} \int_{|x|\ge\sqrt{\frac{1}{24N}}} \frac{e^{-2\pi\left(Nx^2-\frac{1}{24}\right)\frac{1}{k^2\omega}}}{x-\left(\kappa+\mu k+\frac{r}{2N}\right)} \, dx \, d\phi
\end{aligned}$$

analogously to the calculation of $a_{\mathcal{I}^*,j,N,1}(n)$, where

$$\widetilde{K}_{k,j,N}(n,r,\kappa,\ell) := \sum_{\substack{0\le h<k\\ \gcd(h,k)=1\\ J<k_2+k\le\ell}} \chi_{j,r}(N,M_{h,k})\zeta_{24k}^{\left(24N\left(\kappa+\frac{r}{2N}\right)^2-1\right)h'-24g_{j,N}(n)h}.$$

For $a_{\mathcal{I},j,N}(n)$ we have

$$\begin{aligned} a_{\mathcal{I},j,N}(n) = &\sum_{k=1}^{J} \sum_{\substack{0\le h<k\\ \gcd(h,k)=1}} e^{-\frac{2\pi i h}{k}g_{j,N}(n)} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{2\pi g_{j,N}(n)\omega} \sum_{r=1}^{N-1}\chi_{j,r}(N,M_{h,k}) \\ &\times\left(\eta\left(\frac{h'}{k}+\frac{i}{k^2\omega}\right)^{-1} - e^{-\frac{\pi i}{12}\left(\frac{h'}{k}+\frac{i}{k^2\omega}\right)}\right)\frac{i}{\pi}\sum_{\kappa\in\mathbb{Z}} e^{2\pi i N\left(\kappa+\frac{r}{2N}\right)^2\frac{h'}{k}}\,\mathrm{P.V.}\int_{-\infty}^{\infty}\frac{e^{-2\pi\frac{N}{k^2\omega}x^2}}{x-\left(\kappa+\frac{r}{2N}\right)}\,dx\,d\phi \end{aligned}$$

from the definition and (IV.2.15). Additionally we observe that

$$\eta\left(\frac{h'}{k}+\frac{i}{k^2\omega}\right)^{-1} - e^{-\frac{\pi i}{12}\left(\frac{h'}{k}+\frac{i}{k^2\omega}\right)} = \sum_{m\ge1} p(m) e^{-\frac{2\pi m}{k^2\omega}}\zeta_{24k}^{(24m-1)h'} e^{\frac{2\pi}{24k^2\omega}},$$

where $p(m)$ is the partition function.

By decomposing the Farey segment $-\vartheta'_{h,k}\le\phi\le-\vartheta''_{h,k}$ as seen before we obtain

$$\begin{aligned} a_{\mathcal{I},j,N}(n) = &\sum_{k=1}^{J} \sum_{\substack{0\le h<k\\ \gcd(h,k)=1}} e^{-\frac{2\pi i h}{k}g_{j,N}(n)} \int_{-\frac{1}{k(J+k)}}^{\frac{1}{k(J+k)}} \sum_{m\ge1} p(m)e^{-\frac{2\pi m}{k^2\omega}} e^{2\pi g_{j,N}(n)\omega}\sum_{r=1}^{N-1}\chi_{j,r}(N,M_{h,k}) \\ &\times \zeta_{24k}^{(24m-1)h'}\frac{i}{\pi}e^{\frac{2\pi}{24k^2\omega}}\sum_{\kappa\in\mathbb{Z}} e^{2\pi i N\left(\kappa+\frac{r}{2N}\right)^2\frac{h'}{k}}\,\mathrm{P.V.}\int_{-\infty}^{\infty}\frac{e^{-2\pi\frac{N}{k^2\omega}x^2}}{x-\left(\kappa+\frac{r}{2N}\right)}\,dx\,d\phi \\ &+\sum_{k=1}^{J} \sum_{\substack{0\le h<k\\ \gcd(h,k)=1}} e^{-\frac{2\pi i h}{k}g_{j,N}(n)} \sum_{\ell=k_1+k}^{J+k-1}\int_{-\frac{1}{k\ell}}^{-\frac{1}{k(\ell+1)}} \sum_{m\ge1} p(m)e^{-\frac{2\pi m}{k^2\omega}} e^{2\pi g_{j,N}(n)\omega}\sum_{r=1}^{N-1}\chi_{j,r}(N,M_{h,k}) \\ &\times \zeta_{24k}^{(24m-1)h'}\frac{i}{\pi}e^{\frac{2\pi}{24k^2\omega}}\sum_{\kappa\in\mathbb{Z}} e^{2\pi i N\left(\kappa+\frac{r}{2N}\right)^2\frac{h'}{k}}\,\mathrm{P.V.}\int_{-\infty}^{\infty}\frac{e^{-2\pi\frac{N}{k^2\omega}x^2}}{x-\left(\kappa+\frac{r}{2N}\right)}\,dx\,d\phi \\ &+\sum_{k=1}^{J} \sum_{\substack{0\le h<k\\ \gcd(h,k)=1}} e^{-\frac{2\pi i h}{k}g_{j,N}(n)} \sum_{\ell=k_2+k}^{J+k-1}\int_{\frac{1}{k(\ell+1)}}^{\frac{1}{k\ell}} \sum_{m\ge1} p(m)e^{-\frac{2\pi m}{k^2\omega}} e^{2\pi g_{j,N}(n)\omega}\sum_{r=1}^{N-1}\chi_{j,r}(N,M_{h,k}) \\ &\times \zeta_{24k}^{(24m-1)h'}\frac{i}{\pi}e^{\frac{2\pi}{24k^2\omega}}\sum_{\kappa\in\mathbb{Z}} e^{2\pi i N\left(\kappa+\frac{r}{2N}\right)^2\frac{h'}{k}}\,\mathrm{P.V.}\int_{-\infty}^{\infty}\frac{e^{-2\pi\frac{N}{k^2\omega}x^2}}{x-\left(\kappa+\frac{r}{2N}\right)}\,dx\,d\phi \\ =:&a_{\mathcal{I},j,N,1}(n)+a_{\mathcal{I},j,N,2}(n)+a_{\mathcal{I},j,N,3}(n). \end{aligned}$$

Define

$$K_{k,j,N}(n,m,r,\kappa) := \sum_{\substack{0\le h<k\\ \gcd(h,k)=1}} \chi_{j,r}(N,M_{h,k})\zeta_{24k}^{\left(24N\left(\kappa+\frac{r}{2N}\right)^2+24m-1\right)h'-24g_{j,N}(n)h},$$

$$K^*_{k,j,N}(n,m,r,\kappa,\ell) := \sum_{\substack{0\le h<k\\ \gcd(h,k)=1\\ J<k_1+k\le\ell}} \chi_{j,r}(N,M_{h,k})\zeta_{24k}^{\left(24N\left(\kappa+\frac{r}{2N}\right)^2+24m-1\right)h'-24g_{j,N}(n)h},$$

$$\widetilde{K}_{k,j,N}(n,m,r,\kappa,\ell) := \sum_{\substack{0\le h<k\\ \gcd(h,k)=1\\ J<k_2+k\le\ell}} \chi_{j,r}(N,M_{h,k})\zeta_{24k}^{\left(24N\left(\kappa+\frac{r}{2N}\right)^2+24m-1\right)h'-24g_{j,N}(n)h},$$

which are all well-defined Kloosterman sums of modulus k. We thus note that

$$a_{\mathcal{I},j,N,1}(n) = \frac{i}{\pi}\sum_{k=1}^{J}\sum_{r=1}^{N-1}\sum_{\kappa=0}^{k-1}\int_{-\frac{1}{k(J+k)}}^{\frac{1}{k(J+k)}}\sum_{m\ge1}p(m)e^{-\frac{2\pi\left(m-\frac{1}{24}\right)}{k^2\omega}}K_{k,j,N}(n,m,r,\kappa)e^{2\pi g_{j,N}(n)\omega}$$
$$\times\lim_{L\to\infty}\sum_{\mu=-L}^{L}\mathrm{P.V.}\int_{-\infty}^{\infty}\frac{e^{-2\pi Nx^2\frac{1}{k^2\omega}}}{x-\left(\kappa+\mu k+\frac{r}{2N}\right)}\,dx\,d\phi$$

completely analogously to the calculations of $a_{\mathcal{I}^*,j,N,0}(n)$, while

$$a_{\mathcal{I},j,N,2}(n) = \frac{i}{\pi}\sum_{k=1}^{J}\sum_{\ell=J+1}^{J+k-1}\sum_{r=1}^{N-1}\sum_{\kappa=0}^{k-1}\int_{-\frac{1}{k\ell}}^{-\frac{1}{k(\ell+1)}}\sum_{m\ge1}p(m)e^{-\frac{2\pi\left(m-\frac{1}{24}\right)}{k^2\omega}}K^*_{k,j,N}(n,m,r,\kappa,\ell)e^{2\pi g_{j,N}(n)\omega}$$
$$\times\lim_{L\to\infty}\sum_{\mu=-L}^{L}\mathrm{P.V.}\int_{-\infty}^{\infty}\frac{e^{-2\pi Nx^2\frac{1}{k^2\omega}}}{x-\left(\kappa+\mu k+\frac{r}{2N}\right)}\,dx\,d\phi$$

and

$$a_{\mathcal{I},j,N,3}(n) = \frac{i}{\pi}\sum_{k=1}^{J}\sum_{\ell=J+1}^{J+k-1}\sum_{r=1}^{N-1}\sum_{\kappa=0}^{k-1}\int_{\frac{1}{k(\ell+1)}}^{\frac{1}{k\ell}}\sum_{m\ge1}p(m)e^{-\frac{2\pi\left(m-\frac{1}{24}\right)}{k^2\omega}}\widetilde{K}_{k,j,N}(n,m,r,\kappa,\ell)e^{2\pi g_{j,N}(n)\omega}$$
$$\times\lim_{L\to\infty}\sum_{\mu=-L}^{L}\mathrm{P.V.}\int_{-\infty}^{\infty}\frac{e^{-2\pi Nx^2\frac{1}{k^2\omega}}}{x-\left(\kappa+\mu k+\frac{r}{2N}\right)}\,dx\,d\phi$$

analogously to the calculation of $a_{\mathcal{I}^*,j,N,1}(n)$.

To be able to bound all parts of $a_{\mathcal{I}^e,j,N}(n)$, respectively $a_{\mathcal{I},j,N}(n)$, we need the following lemma.

Lemma IV.4.1. *For $0 \leq d < N$, N, k, ω, κ, and r as above, and $2\sqrt{Nd} \notin \mathbb{Z}\backslash\{0\}$ we have*

$$\lim_{L\to\infty} \sum_{\mu=-L}^{L} \text{P.V.} \int_{|x|\geq\sqrt{\frac{d}{N}}} \frac{e^{-2\pi\left(Nx^2-d\right)\frac{1}{k^2\omega}}}{x-\left(\kappa+\mu k+\frac{r}{2N}\right)}\, dx = O_N\left(\frac{1}{\min\left(\kappa+\frac{r}{2N}, k-\kappa-\frac{r}{2N}\right)}\right),$$

as $k \to \infty$.

Proof. We follow the ideas of [BN19, Proof of Lemma 3.3]. Combining the integral over negative and positive reals gives us that

$$\begin{aligned}
\mathcal{P}_{d,N}(k,\omega,\kappa,r) :=& \text{P.V.} \int_{|x|\geq\sqrt{\frac{d}{N}}} \frac{e^{-2\pi\left(Nx^2-d\right)\frac{1}{k^2\omega}}}{x-\left(\kappa+\mu k+\frac{r}{2N}\right)}\, dx \\
=& \text{P.V.} \int_{\sqrt{\frac{d}{N}}}^{\infty} e^{-2\pi\left(Nx^2-d\right)\frac{1}{k^2\omega}} \left(\frac{1}{x-\left(\kappa+\mu k+\frac{r}{2N}\right)} - \frac{1}{x+\left(\kappa+\mu k+\frac{r}{2N}\right)}\right) dx \\
=& 2\left(\kappa+\mu k+\frac{r}{2N}\right) \text{P.V.} \int_{\sqrt{\frac{d}{N}}}^{\infty} \frac{e^{-2\pi N\left(x^2-\frac{d}{N}\right)\frac{1}{k^2\omega}}}{x^2-\left(\kappa+\mu k+\frac{r}{2N}\right)^2}\, dx \\
=& \left(\kappa+\mu k+\frac{r}{2N}\right) \text{P.V.} \int_{0}^{\infty} \frac{e^{-2\pi Nu\frac{1}{k^2\omega}}}{\sqrt{u+\frac{d}{N}}\left(u+\frac{d}{N}-\left(\kappa+\mu k+\frac{r}{2N}\right)^2\right)}\, du,
\end{aligned}$$

where we substituted $u = x^2 - \frac{d}{N}$ in the last step. We go on by writing

$$\begin{aligned}
\frac{1}{u+\frac{d}{N}-\left(\kappa+\mu k+\frac{r}{2N}\right)^2} =& \left(\frac{1}{u+\frac{d}{N}-\left(\kappa+\mu k+\frac{r}{2N}\right)^2} + \frac{1}{\left(\kappa+\mu k+\frac{r}{2N}\right)^2}\right) - \frac{1}{\left(\kappa+\mu k+\frac{r}{2N}\right)^2} \\
=& \frac{u+\frac{d}{N}}{\left(\kappa+\mu k+\frac{r}{2N}\right)^2\left(u+\frac{d}{N}-\left(\kappa+\mu k+\frac{r}{2N}\right)^2\right)} - \frac{1}{\left(\kappa+\mu k+\frac{r}{2N}\right)^2}
\end{aligned}$$

and consider the contribution of each term seperately, where we denote them by $\mathcal{P}_{d,N,1}(k,\omega,\kappa,r)$, respectively $\mathcal{P}_{d,N,2}(k,\omega,\kappa,r)$. We start by looking at

$$\mathcal{P}_{d,N,2}(k,\omega,\kappa,r) = -\frac{1}{\kappa+\mu k+\frac{r}{2N}} \int_{0}^{\infty} \frac{e^{-2\pi Nu\frac{1}{k^2\omega}}}{\sqrt{u+\frac{d}{N}}}\, du.$$

Using that $\text{Re}(\frac{2N}{k^2\omega}) \geq N$ together with $u \geq 0$ we see that

$$\left|\int_{0}^{\infty} \frac{e^{-2\pi Nu\frac{1}{k^2\omega}}}{\sqrt{u+\frac{d}{N}}}\, du\right| \leq \int_{0}^{\infty} \frac{e^{-\pi Nu}}{\sqrt{u+\frac{d}{N}}}\, du = O_N(1).$$

Using (IV.4.8) we additionally see that

$$-\lim_{L\to\infty}\sum_{\mu=-L}^{L}\frac{1}{\kappa+\mu k+\frac{r}{2N}}=-\frac{1}{k}\lim_{L\to\infty}\sum_{\mu=-L}^{L}\frac{1}{\mu+\frac{\kappa}{k}+\frac{r}{2Nk}}=-\frac{\pi}{k}\cot\left(\pi\left(\frac{\kappa}{k}+\frac{r}{2Nk}\right)\right).$$

Since $0<\frac{\kappa}{k}+\frac{r}{2Nk}<1$ we obtain that (see [BN19, page 13])

$$\left|\cot\left(\pi\left(\frac{\kappa}{k}+\frac{r}{2Nk}\right)\right)\right|\ll\frac{1}{\frac{\kappa}{k}+\frac{r}{2Nk}}+\frac{1}{1-\frac{\kappa}{k}-\frac{r}{2Nk}},$$

which yields that

$$\begin{aligned}\lim_{L\to\infty}\sum_{\mu=-L}^{L}\mathcal{P}_{d,N,2}(k,\omega,\kappa,r)=&O_N\left(\frac{1}{k}\left(\frac{1}{\frac{\kappa}{k}+\frac{r}{2Nk}}+\frac{1}{1-\frac{\kappa}{k}-\frac{r}{2Nk}}\right)\right)\\=&O_N\left(\frac{1}{\min\left(\kappa+\frac{r}{2N},k-\kappa-\frac{r}{2N}\right)}\right).\end{aligned}$$

Next we look at $\mathcal{P}_{d,N,1}(k,\omega,\kappa,r)$. We start by writing

$$\mathcal{P}_{d,N,1}(k,\omega,\kappa,r)=\frac{1}{\kappa+\mu k+\frac{r}{2N}}\,\text{P.V.}\int_0^\infty\frac{e^{-2\pi Nu\frac{1}{k^2\omega}}\sqrt{u+\frac{d}{N}}}{u+\frac{d}{N}-\left(\kappa+\mu k+\frac{r}{2N}\right)^2}\,du.$$

Our pole thus lies in $u=(\kappa+\mu k+\frac{r}{2N})^2-\frac{d}{N}\in\mathbb{R}$. We further investigate that since $d<N$ we only have a pole in 0 if $\kappa=\mu=0$ and $r=2\sqrt{Nd}$, which cannot happen since $r\geq 1$ and we assumed that $2\sqrt{Nd}\notin\mathbb{Z}\backslash\{0\}$.

We rewrite our principal value integral as the average of the paths $\gamma_{\varepsilon,+}$ and $\gamma_{\varepsilon,-}$, where $\gamma_{\varepsilon,+}$, respectively $\gamma_{\varepsilon,-}$, is the path of integration along the positive real axis taking a semicircular path of radius ε above, respectively below, the pole (see Figure IV.2).

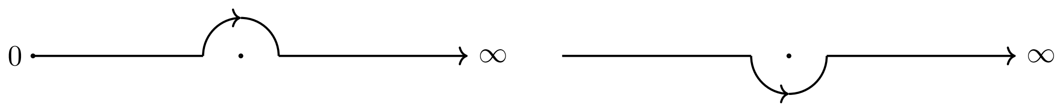

Figure IV.2: The paths of integration $\gamma_{\varepsilon,+}$, respectively $\gamma_{\varepsilon,-}$.

We obtain that

$$\mathcal{P}_{d,N,1}(k,\omega,\kappa,r)=\frac{1}{\kappa+\mu k+\frac{r}{2N}}\lim_{\varepsilon\to0}\left(\frac{1}{2}\left(\int_{\gamma_{\varepsilon,+}}+\int_{\gamma_{\varepsilon,-}}\right)\frac{e^{-2\pi Nu\frac{1}{k^2\omega}}\sqrt{u+\frac{d}{N}}}{u+\frac{d}{N}-\left(\kappa+\mu k+\frac{r}{2N}\right)^2}\,du\right).$$

We note that

$$\mathrm{Re}\left(\frac{2N}{k^2\omega}e^{\pm\frac{\pi i}{4}}\right) = \frac{\mathrm{Re}\left(\frac{2N}{k^2\omega}\right)}{\sqrt{2}} \mp \frac{\mathrm{Im}\left(\frac{2N}{k^2\omega}\right)}{\sqrt{2}},$$

since $e^{\pm\frac{\pi i}{4}} = \frac{1}{\sqrt{2}}(1 \pm i)$. Choosing the $\pm$ to be $-\operatorname{sgn}(\mathrm{Im}(\frac{2N}{k^2\omega}))$ thus gives us that

$$\mathrm{Re}\left(\frac{2N}{k^2\omega}e^{\pm\frac{\pi i}{4}}\right) = \frac{\mathrm{Re}\left(\frac{2N}{k^2\omega}\right)}{\sqrt{2}} + \frac{\left|\mathrm{Im}\left(\frac{2N}{k^2\omega}\right)\right|}{\sqrt{2}} \geq \frac{\mathrm{Re}\left(\frac{2N}{k^2\omega}\right)}{\sqrt{2}} \geq \frac{N}{\sqrt{2}},$$

since $\mathrm{Re}(\frac{2N}{k^2\omega}) \geq N$. This means we either have $\mathrm{Re}(\frac{2N}{k^2\omega}e^{\frac{\pi i}{4}}) \geq \frac{N}{\sqrt{2}}$ or $\mathrm{Re}(\frac{2N}{k^2\omega}e^{-\frac{\pi i}{4}}) \geq \frac{N}{\sqrt{2}}$. Using Cauchy's Theorem we now want to rotate our paths of integration either to $e^{\frac{\pi i}{4}}\mathbb{R}^+$ if we have $\mathrm{Re}(\frac{2N}{k^2\omega}e^{\frac{\pi i}{4}}) \geq \frac{N}{\sqrt{2}}$ or to $e^{-\frac{\pi i}{4}}\mathbb{R}^+$ if $\mathrm{Re}(\frac{2N}{k^2\omega}e^{-\frac{\pi i}{4}}) \geq \frac{N}{\sqrt{2}}$ picking up the residues from the poles that lie on the real line. Since both rotations follow the same argument we from now on assume that we have $\mathrm{Re}(\frac{2N}{k^2\omega}e^{\frac{\pi i}{4}}) \geq \frac{N}{\sqrt{2}}$. Note that for this rotation we only pick up poles by performing the rotation on $\gamma_{\varepsilon,-}$ as can by easily seen in Figure IV.3.

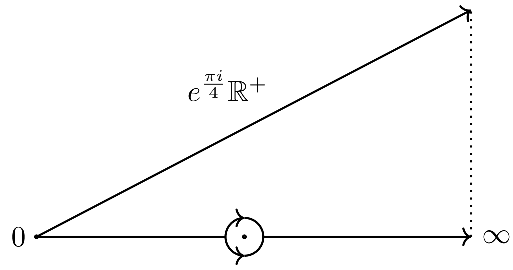

Figure IV.3: Rotation of the paths of integration.

We compute

$$\begin{aligned}
&\mathrm{Res}_{u=\left(\kappa+\mu k+\frac{r}{2N}\right)^2-\frac{d}{N}} \frac{e^{-2\pi N u\frac{1}{k^2\omega}}\sqrt{u+\frac{d}{N}}}{u+\frac{d}{N}-\left(\kappa+\mu k+\frac{r}{2N}\right)^2} \\
=& \lim_{u\to\left(\kappa+\mu k+\frac{r}{2N}\right)^2-\frac{d}{N}} \left(u-\left(\left(\kappa+\mu k+\frac{r}{2N}\right)^2-\frac{d}{N}\right)\right)\frac{e^{-2\pi N u\frac{1}{k^2\omega}}\sqrt{u+\frac{d}{N}}}{u+\frac{d}{N}-\left(\kappa+\mu k+\frac{r}{2N}\right)^2} \\
=& e^{-2\pi N\left(\left(\kappa+\mu k+\frac{r}{2N}\right)^2-\frac{d}{N}\right)\frac{1}{k^2\omega}}\sqrt{\left(\kappa+\mu k+\frac{r}{2N}\right)^2} = e^{-2\pi N\left(\left(\kappa+\mu k+\frac{r}{2N}\right)^2-\frac{d}{N}\right)\frac{1}{k^2\omega}}\left|\kappa+\mu k+\frac{r}{2N}\right|,
\end{aligned}$$

which gives us that the contribution of this poles on the positive real line and with respect to $\lim_{L\to\infty}\sum_{\mu=-L}^{L}\mathcal{P}_{d,N,1}(k,\omega,\kappa,r)$ sums up to

$$\pi i \lim_{L\to\infty}\sum_{\mu=-L}^{L}\delta_{\left(\kappa+\mu k+\frac{r}{2N}\right)^2-\frac{d}{N}\geq 0}e^{-2\pi N\left(\left(\kappa+\mu k+\frac{r}{2N}\right)^2-\frac{d}{N}\right)\frac{1}{k^2\omega}}\operatorname{sgn}\left(\kappa+\mu k+\frac{r}{2N}\right).$$

Next we define the *Jacobi theta function*[21] as

$$\vartheta_3(z;\tau):=\sum_{n\in\mathbb{Z}}e^{\pi i n^2\tau+2\pi i n z}.$$

Noting that it transforms modular by (see e.g., [Mum83, page 32])

$$\vartheta_3(z;\tau)=\frac{1}{\sqrt{-i\tau}}e^{-\frac{\pi i z^2}{\tau}}\vartheta_3\left(\frac{z}{\tau};-\frac{1}{\tau}\right)$$

we can bound the absolute value of this contribution against

$$\pi e^{\pi d}\sum_{\mu\in\mathbb{Z}}e^{-\pi N k^2\left(\mu+\frac{\kappa}{k}+\frac{r}{2Nk}\right)^2}=\pi e^{\pi d}\frac{1}{\sqrt{Nk^2}}\vartheta_3\left(\frac{\kappa}{k}+\frac{r}{2Nk};-\frac{1}{iNk^2}\right)=O_N\left(\frac{1}{k}\right),$$

where we used [NIST, Figure 20.3.4] in the last step[22].

We are left with bounding the integrals on the rotated paths. We see that they sum up to

$$\begin{aligned}\mathcal{Q}_{d,N}(k,\omega,\kappa,r):=&\frac{1}{\kappa+\mu k+\frac{r}{2N}}\int_{e^{\frac{\pi i}{4}}\mathbb{R}^+}\frac{e^{-2\pi N u\frac{1}{k^2\omega}}\sqrt{u+\frac{d}{N}}}{u+\frac{d}{N}-\left(\kappa+\mu k+\frac{r}{2N}\right)^2}du\\ =&\frac{1}{\kappa+\mu k+\frac{r}{2N}}e^{\frac{\pi i}{4}}\int_0^\infty\frac{e^{-2\pi N\frac{1+i}{\sqrt{2}}u\frac{1}{k^2\omega}}\sqrt{\frac{1+i}{\sqrt{2}}u+\frac{d}{N}}}{\frac{1+i}{\sqrt{2}}u+\frac{d}{N}-\left(\kappa+\mu k+\frac{r}{2N}\right)^2}du,\end{aligned}$$

by changing variables $u\mapsto e^{\frac{\pi i}{4}}u$. Since $\operatorname{Re}(\frac{2N}{k^2\omega}e^{\frac{\pi i}{4}})\geq\frac{N}{\sqrt{2}}$ and $u\geq 0$ we note that

$$\left|e^{-2\pi N\frac{1+i}{\sqrt{2}}u\frac{1}{k^2\omega}}\right|=e^{-\pi u\operatorname{Re}\left(\frac{2N}{k^2\omega}e^{\frac{\pi i}{4}}\right)}\leq e^{-\frac{\pi u N}{\sqrt{2}}}.$$

Furthermore we have

$$\frac{1}{\left|\frac{1+i}{\sqrt{2}}u+\frac{d}{N}-\left(\kappa+\mu k+\frac{r}{2N}\right)^2\right|}\leq\frac{\sqrt{2}}{\left|\left(\kappa+\mu k+\frac{r}{2N}\right)^2-\frac{d}{N}\right|},$$

[21] Note that $\vartheta_3(z;\tau)=-iq^{-\frac{1}{8}}e^{-\pi i\left(z-\frac{1}{2}-\frac{\tau}{2}\right)}\vartheta\left(z-\frac{1}{2}-\frac{\tau}{2};\tau\right)$, where $\vartheta(z;\tau)$ is the Jacobi theta function defined in (I.1.3).

[22] Note that $\vartheta_3(z;\tau)=\theta_3(\pi z;e^{\pi i\tau})$ from [NIST, Figure 20.3.4].

which, together with the other bound, yields

$$|\mathcal{Q}_{d,N}(k,\omega,\kappa,r)| \leq \frac{\sqrt{2}}{\left|\kappa+\mu k+\frac{r}{2N}\right| \left|\left(\kappa+\mu k+\frac{r}{2N}\right)^2-\frac{d}{N}\right|} \int_0^\infty e^{-\frac{\pi u N}{\sqrt{2}}} \left|\frac{1+i}{\sqrt{2}}u+\frac{d}{N}\right|^{\frac{1}{2}} du.$$

Noting that $e^{-\frac{\pi u N}{\sqrt{2}}}|\frac{1+i}{\sqrt{2}}u+\frac{d}{N}|^{\frac{1}{2}}$ is integrable and only depending on d and N gives us that the integral occuring above is $O_N(1)$. Therefore we are left with showing

$$\lim_{L\to\infty} \sum_{\mu=-L}^{L} \frac{\sqrt{2}}{\left|\kappa+\mu k+\frac{r}{2N}\right| \left|\left(\kappa+\mu k+\frac{r}{2N}\right)^2-\frac{d}{N}\right|} = O_N\left(\frac{1}{\min\left(\kappa+\frac{r}{2N}, k-\kappa-\frac{r}{2N}\right)}\right).$$

Using elementary estimates and the fact that $d<N$ we see that

$$\begin{aligned}
&\frac{\sqrt{2}}{k^3}\sum_{\mu\in\mathbb{Z}} \frac{1}{\left|\mu+\frac{\kappa}{k}+\frac{r}{2Nk}\right| \left|\left(\mu+\frac{\kappa}{k}+\frac{r}{2Nk}\right)^2-\frac{d}{Nk^2}\right|}\\
&<\frac{\sqrt{2}}{k^3}\left[\frac{k^3}{\left(\kappa+\frac{r}{2N}\right)\left|\left(\kappa+\frac{r}{2N}\right)^2-\frac{d}{N}\right|}+\frac{k^3}{\left(k-\kappa-\frac{r}{2N}\right)\left|\left(k-\kappa-\frac{r}{2N}\right)^2-\frac{d}{N}\right|}\right.\\
&\qquad+\frac{k^3}{\left(k+\kappa+\frac{r}{2N}\right)\left(\left(k+\kappa+\frac{r}{2N}\right)^2-1\right)}+\frac{k^3}{\left(2k-\kappa-\frac{r}{2N}\right)\left(\left(2k-\kappa-\frac{r}{2N}\right)^2-1\right)}\\
&\qquad\left.+\sum_{\mu\geq 2}\frac{1}{\mu\left(\mu^2-1\right)}+\sum_{\mu\geq 3}\frac{1}{(\mu-1)\left((\mu-1)^2-1\right)}\right]\\
&=O_N\left(\frac{1}{\min\left(\left(\kappa+\frac{r}{2N}\right)\left|\left(\kappa+\frac{r}{2N}\right)^2-\frac{d}{N}\right|, \left(k-\kappa-\frac{r}{2N}\right)\left|\left(k-\kappa-\frac{r}{2N}\right)^2-\frac{d}{N}\right|\right)}\right).
\end{aligned} \tag{IV.4.23}$$

For $\kappa\in\{0,1,k-2,k-1\}$ the required bound holds, so we can restrict to the case $2\leq\kappa\leq k-3$. Here we have

$$\left(\kappa+\frac{r}{2N}\right)^2-\frac{d}{N}>1 \qquad \text{and} \qquad \left(k-\kappa-\frac{r}{2N}\right)^2-\frac{d}{N}>1,$$

which yields that we can also bound (IV.4.23) against

$$\frac{1}{\min\left(\kappa+\frac{r}{2N}, k-\kappa-\frac{r}{2N}\right)}$$

and finishes the proof. ∎

Using Lemma IV.4.1 with $d = \frac{1}{24}$, (IV.1.7) for all the Kloosterman sums, and noting that

$$\sum_{\kappa=0}^{k-1} \frac{1}{\min\left(\kappa + \frac{r}{2N}, k - \kappa - \frac{r}{2N}\right)} = O(\log(k))$$

yields

$$\begin{aligned}
a_{\mathcal{I}^e,j,N,1}(n) =& O_N\left(\sum_{k=1}^{J}\sum_{r=1}^{N-1} nk^{\frac{1}{2}+\varepsilon}\log(k)\int_{-\frac{1}{k(J+k)}}^{\frac{1}{k(J+k)}} \left|e^{2\pi g_{j,N}(n)\omega}\right| d\phi\right)\\
=& O_N\left(ne^{2\pi g_{j,N}(n)J^{-2}}J^{-1}\sum_{k=1}^{J}k^{-\frac{1}{2}+\varepsilon}\log(k)\right) = O_N\left(ne^{2\pi g_{j,N}(n)J^{-2}}J^{-\frac{1}{2}+\varepsilon}\log(J)\right),\\
a_{\mathcal{I}^e,j,N,2}(n) =& O_N\left(\sum_{k=1}^{J}\sum_{\ell=J+1}^{J+k-1}\sum_{r=1}^{N-1} nk^{\frac{1}{2}+\varepsilon}\log(k)\int_{-\frac{1}{k\ell}}^{-\frac{1}{k(\ell+1)}} \left|e^{2\pi g_{j,N}(n)\omega}\right| d\phi\right)\\
=& O_N\left(ne^{2\pi g_{j,N}(n)J^{-2}}J^{-1}\sum_{k=1}^{J}k^{-\frac{1}{2}+\varepsilon}\log(k)\right) = O_N\left(ne^{2\pi g_{j,N}(n)J^{-2}}J^{-\frac{1}{2}+\varepsilon}\log(J)\right),
\end{aligned}$$

and analogously $a_{\mathcal{I}^e,j,N,3}(n) = O_N(ne^{2\pi g_{j,N}(n)J^{-2}}J^{-\frac{1}{2}+\varepsilon}\log(J))$. Overall we thus obtain

$$a_{\mathcal{I}^e,j,N}(n) = O_N\left(ne^{2\pi g_{j,N}(n)J^{-2}}J^{-\frac{1}{2}+\varepsilon}\log(J)\right). \tag{IV.4.24}$$

Using the same bounds as used above for Lemma IV.4.1 with $d = 0$, $\mathrm{Re}(\frac{2}{k^2\omega}) \geq 1$, and noting that

$$\sum_{m\geq 1} p(m)e^{-\pi\left(m-\frac{1}{24}\right)} = \eta\left(\frac{i}{2}\right)^{-1} - e^{\frac{\pi}{24}} = O(1)$$

yields

$$\begin{aligned}
a_{\mathcal{I},j,N,1}(n) =& O_N\left(\sum_{k=1}^{J}\sum_{r=1}^{N-1}\int_{-\frac{1}{k(J+k)}}^{\frac{1}{k(J+k)}}\sum_{m\geq 1}p(m)\left|e^{-\frac{2\pi\left(m-\frac{1}{24}\right)}{k^2\omega}}\right| nk^{\frac{1}{2}+\varepsilon}\log(k)\left|e^{2\pi g_{j,N}(n)\omega}\right| d\phi\right)\\
=& O_N\left(ne^{2\pi g_{j,N}(n)J^{-2}}\sum_{k=1}^{J}\sum_{r=1}^{N-1}k^{\frac{1}{2}+\varepsilon}\log(k)\int_{-\frac{1}{k(J+k)}}^{\frac{1}{k(J+k)}} d\phi\right)\\
=& O_N\left(ne^{2\pi g_{j,N}(n)J^{-2}}J^{-\frac{1}{2}+\varepsilon}\log(J)\right),
\end{aligned}$$

and analogously to above

$$a_{\mathcal{I},j,N,2}(n) = O_N\left(\sum_{k=1}^{J}\sum_{\ell=J+1}^{J+k-1}\sum_{r=1}^{N-1}\int_{-\frac{1}{k\ell}}^{-\frac{1}{k(\ell+1)}}\sum_{m\geq 1}p(m)\left|e^{-\frac{2\pi\left(m-\frac{1}{24}\right)}{k^2\omega}}\right| nk^{\frac{1}{2}+\varepsilon}\log(k)\left|e^{2\pi g_{j,N}(n)\omega}\right| d\phi\right)$$

$$=O_N\left(ne^{2\pi g_{j,N}(n)J^{-2}}J^{-\frac{1}{2}+\varepsilon}\log(J)\right),$$

and $a_{\mathcal{I},j,N,3}(n)=O_N(ne^{2\pi g_{j,N}(n)J^{-2}}J^{-\frac{1}{2}+\varepsilon}\log(J))$. Overall we thus obtain

$$a_{\mathcal{I},j,N}(n)=O_N\left(ne^{2\pi g_{j,N}(n)J^{-2}}J^{-\frac{1}{2}+\varepsilon}\log(J)\right). \tag{IV.4.25}$$

IV.4.3 Combining the results

Plugging (IV.4.22), (IV.4.24), and (IV.4.25) into (IV.4.6), using the definition of $g_{j,N}(n)$, and taking $J\to\infty$ gives

$$a_{j,N}(n)=-\frac{2\pi i}{\sqrt{n+\frac{j^2}{4N}-\frac{1}{24}}}\sum_{k\geq 1}\sum_{r=1}^{N-1}\sum_{\kappa=0}^{k-1}\frac{K_{k,j,N}(n,r,\kappa)}{k^2}$$
$$\times\,\mathrm{P.V.}\int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}}\sqrt{\frac{1}{24}-Nx^2}\cot\left(\pi\left(-\frac{x}{k}+\frac{\kappa}{k}+\frac{r}{2Nk}\right)\right)I_1\left(\frac{4\pi\sqrt{n+\frac{j^2}{4N}-\frac{1}{24}}}{k}\sqrt{\frac{1}{24}-Nx^2}\right)dx.$$

This finishes the proof of Theorem IV.1.1.

Again by noting that the integral over x only has a simple pole in $x=\frac{r}{2N}$ for $\kappa=0$ and $r<\sqrt{\frac{N}{6}}$ we additionally obtain

$$a_{j,N}(n)=-\frac{2\pi i}{\sqrt{n+\frac{j^2}{4N}-\frac{1}{24}}}\sum_{k\geq 1}\sum_{r=1}^{\left\lceil\sqrt{\frac{N}{6}}-1\right\rceil}\frac{K_{k,j,N}(n,r,0)}{k^2} \tag{IV.4.26}$$
$$\times\lim_{\varepsilon\to 0}\left(\int_{-\sqrt{\frac{1}{24N}}}^{\frac{r}{2N}-\varepsilon}\sqrt{\frac{1}{24}-Nx^2}\cot\left(\pi\left(-\frac{x}{k}+\frac{r}{2Nk}\right)\right)I_1\left(\frac{4\pi\sqrt{n+\frac{j^2}{4N}-\frac{1}{24}}}{k}\sqrt{\frac{1}{24}-Nx^2}\right)dx\right.$$
$$\left.+\int_{\frac{r}{2N}+\varepsilon}^{\sqrt{\frac{1}{24N}}}\sqrt{\frac{1}{24}-Nx^2}\cot\left(\pi\left(-\frac{x}{k}+\frac{r}{2Nk}\right)\right)I_1\left(\frac{4\pi\sqrt{n+\frac{j^2}{4N}-\frac{1}{24}}}{k}\sqrt{\frac{1}{24}-Nx^2}\right)dx\right)$$
$$-\frac{2\pi i}{\sqrt{n+\frac{j^2}{4N}-\frac{1}{24}}}\sum_{k\geq 1}\sum_{r=\left\lceil\sqrt{\frac{N}{6}}\right\rceil}^{N-1}\frac{K_{k,j,N}(n,r,0)}{k^2}$$
$$\times\int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}}\sqrt{\frac{1}{24}-Nx^2}\cot\left(\pi\left(-\frac{x}{k}+\frac{r}{2Nk}\right)\right)I_1\left(\frac{4\pi\sqrt{n+\frac{j^2}{4N}-\frac{1}{24}}}{k}\sqrt{\frac{1}{24}-Nx^2}\right)dx$$
$$-\frac{2\pi i}{\sqrt{n+\frac{j^2}{4N}-\frac{1}{24}}}\sum_{k\geq 1}\sum_{r=1}^{N-1}\sum_{\kappa=1}^{k-1}\frac{K_{k,j,N}(n,r,\kappa)}{k^2}$$
$$\times\int_{-\sqrt{\frac{1}{24N}}}^{\sqrt{\frac{1}{24N}}}\sqrt{\frac{1}{24}-Nx^2}\cot\left(\pi\left(-\frac{x}{k}+\frac{\kappa}{k}+\frac{r}{2Nk}\right)\right)I_1\left(\frac{4\pi\sqrt{n+\frac{j^2}{4N}-\frac{1}{24}}}{k}\sqrt{\frac{1}{24}-Nx^2}\right)dx.$$

Remark. Note that we had to exclude $n = 0$ in our calculation. This is not caused by the fact that we have $\sqrt{g_{j,N}(n)}$ in the denominator of (IV.4.19) (which equals 0 if and only if $n = 0$ and $j = \sqrt{\frac{N}{6}}$), but because the estimates of our Kloosterman sums would break down for this special case (see [Rad38, Section 8]).

IV.5 Numerical Results

In this section we offer some numerical results and compare the value of $a_{j,N}(n)$ for a number of cases to the results from Theorem IV.1.1, where we numerically perform the sum over k from 1 to J. We offer the code to obtain those values in the Appendix.

	J=1	J=3	J=20	J=25	J=50
$a_{1,3}(3) = 2$	$2.3181\ldots$	$2.2886\ldots$	$2.0990\ldots$	$2.0875\ldots$	$2.0527\ldots$
$a_{1,3}(10) = 30$	$29.8989\ldots$	$30.2442\ldots$	$30.0866\ldots$	$30.0789\ldots$	$30.0418\ldots$
$a_{1,3}(18) = 272$	$271.3098\ldots$	$272.2656\ldots$	$272.0720\ldots$	$272.0651\ldots$	$272.0408\ldots$
$a_{5,8}(3) = 2$	$2.5197\ldots$	$2.2200\ldots$	$1.9993\ldots$	$1.9830\ldots$	$1.9892\ldots$
$a_{5,8}(10) = 27$	$26.2697\ldots$	$26.9853\ldots$	$26.9856\ldots$	$26.9997\ldots$	$26.9991\ldots$
$a_{5,8}(18) = 216$	$214.4979\ldots$	$216.0557\ldots$	$215.9830\ldots$	$215,9893\ldots$	$216.0044\ldots$
$a_{3,10}(3) = 3$	$3.1624\ldots$	$3.0544\ldots$	$3.0307\ldots$	$3.0222\ldots$	$2.9985\ldots$
$a_{3,10}(10) = 39$	$38.5337\ldots$	$38.9965\ldots$	$39.0080\ldots$	$39.0001\ldots$	$38.9982\ldots$
$a_{3,10}(18) = 336$	$334.3940\ldots$	$336.0237\ldots$	$336.0058\ldots$	$336.0254\ldots$	$336.0111\ldots$

Table IV.1: Numerical results for Fourier coefficients of $q^{\frac{1}{24}-\frac{j^2}{4N}}\mathcal{A}_{j,N}(\tau)$.

IV.6 Further Questions

To end this chapter we want to briefly mention some related questions that could be the topic for possible follow up projects.

(1) By splitting the Mordell-type integral in Section IV.2.3 we avoided the special case $2\sqrt{Nd} \in \mathbb{Z}\backslash\{0\}$ to verify the well-definedness of the principal value integral. Is there a possibility to get rid of the extra condition? What happens if we have a pole right at the edge of our integration path?

(2) If one is interested in figuring out what happens if we let N tend to ∞ one could be more precise about the dependence of N in the error terms while running the Circle Method as well as in the bound of the Kloosterman sum.

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Chapter V

Additional Details for Chapter IV

In this chapter we give some additional proofs for Chapter IV. Note that we therefore keep the notation of the sections we add details on.

V.1 Additional proofs for Section IV.2

In this section we prove two identities. The first one is stated in the following lemma.

Lemma V.1.1. *We have*

$$\int_{-V+\delta}^{\infty-i\varepsilon} \frac{e^{-2\pi N(n+\frac{j}{2N})^2\mathfrak{z}}}{\sqrt{-\mathfrak{z}}} d\mathfrak{z} = -\frac{i}{\sqrt{2N}(n+\frac{j}{2N})} \left(\operatorname{sgn}\left(n+\frac{j}{2N}\right) + \operatorname{erf}\left(i \left(n+\frac{j}{2N}\right) \sqrt{2\pi N\left(V-\delta\right)} \right) \right).$$

Proof. Similar to [BN19, page 12] we split the integral as

$$\int_{-V+\delta}^{\infty-i\varepsilon} \frac{e^{-2\pi N(n+\frac{j}{2N})^2\mathfrak{z}}}{\sqrt{-\mathfrak{z}}} d\mathfrak{z} = \int_{0}^{\infty-i\varepsilon} \frac{e^{-2\pi N(n+\frac{j}{2N})^2\mathfrak{z}}}{\sqrt{-\mathfrak{z}}} d\mathfrak{z} + \int_{-V+\delta}^{0} \frac{e^{-2\pi N(n+\frac{j}{2N})^2\mathfrak{z}}}{\sqrt{-\mathfrak{z}}} d\mathfrak{z}.$$

To simplify our calculations we look at each integral seperately.

First we change variables as $\sqrt{2\pi N}|n+\frac{j}{2N}|\sqrt{\mathfrak{z}} = x$ in the first integral and obtain

$$\begin{aligned}\int_{0}^{\infty-i\varepsilon} \frac{e^{-2\pi N(n+\frac{j}{2N})^2\mathfrak{z}}}{\sqrt{-\mathfrak{z}}} d\mathfrak{z} = &\int_{0}^{\infty-i\varepsilon} \frac{2e^{-x^2}}{i\sqrt{2\pi N}\left|n+\frac{j}{2N}\right|} dx \\ =&\frac{-i}{\sqrt{2N}\left(n+\frac{j}{2N}\right)} \operatorname{sgn}\left(n+\frac{j}{2N}\right) \frac{2}{\sqrt{\pi}} \int_{0}^{\infty-i\varepsilon} e^{-x^2} dx \\ =&\frac{-i}{\sqrt{2N}\left(n+\frac{j}{2N}\right)} \operatorname{sgn}\left(n+\frac{j}{2N}\right). \qquad \text{(V.1.1)}\end{aligned}$$

Next we also change variables in the second integral as $\sqrt{2\pi N}i|n+\frac{j}{2N}|\sqrt{-\mathfrak{z}} = x$ and get

$$\int_{-V+\delta}^{0} \frac{e^{-2\pi N(n+\frac{j}{2N})^2\mathfrak{z}}}{\sqrt{-\mathfrak{z}}} d\mathfrak{z} = -\int_{i\sqrt{2\pi N(V-\delta)}|n+\frac{j}{2N}|}^{0} \frac{2e^{-x^2}}{i\sqrt{2\pi N}\left|n+\frac{j}{2N}\right|} dx$$

$$
\begin{aligned}
&= \int_0^{i\sqrt{2\pi N(V-\delta)}\left|n+\frac{j}{2N}\right|} \frac{2e^{-x^2}}{i\sqrt{2\pi N}\left|n+\frac{j}{2N}\right|} dx \\
&= -\frac{i}{\sqrt{2N}} \frac{1}{\left|n+\frac{j}{2N}\right|} \frac{2}{\sqrt{\pi}} \int_0^{i\sqrt{2\pi N(V-\delta)}\left|n+\frac{j}{2N}\right|} e^{-x^2} dx \\
&= -\frac{i}{\sqrt{2N}} \frac{1}{\left|n+\frac{j}{2N}\right|} \operatorname{erf}\left(i\sqrt{2\pi N(V-\delta)}\left|n+\frac{j}{2N}\right|\right) \\
&= -\frac{i}{\sqrt{2N}} \frac{\operatorname{erf}\left(i\sqrt{2\pi N(V-\delta)}\left|n+\frac{j}{2N}\right|\right)}{i\left|n+\frac{j}{2N}\right|\sqrt{2\pi N(V-\delta)}} i\sqrt{2\pi N(V-\delta)} \\
&= \frac{-i}{\sqrt{2N}\left(n+\frac{j}{2N}\right)} \operatorname{erf}\left(i\sqrt{2\pi N(V-\delta)}\left(n+\frac{j}{2N}\right)\right), \qquad \text{(V.1.2)}
\end{aligned}
$$

using that

$$
\frac{\operatorname{erf}\left(i\sqrt{2\pi N(V-\delta)}\left|n+\frac{j}{2N}\right|\right)}{i\left|n+\frac{j}{2N}\right|\sqrt{2\pi N(V-\delta)}} = \frac{\operatorname{erf}\left(i\sqrt{2\pi N(V-\delta)}\left(n+\frac{j}{2N}\right)\right)}{i\left(n+\frac{j}{2N}\right)\sqrt{2\pi N(V-\delta)}},
$$

with $\frac{\operatorname{erf}(|x|)}{|x|} = \frac{\operatorname{erf}(x)}{x}$ and $\operatorname{erf}(-x) = -\operatorname{erf}(x)$ in the last step. Combining (V.1.1) and (V.1.2) yields the claim. □

We go on by proving the second identity, which is stated in the following lemma.

Lemma V.1.2. *For* $s \in \mathbb{R} \setminus \{0\}$ *and* $\operatorname{Re}(V) > 0$ *we have*

$$
\begin{aligned}
e^{-\pi s^2 V} \operatorname{erf}\left(is\sqrt{\pi V}\right) &= -\frac{i}{\pi} \text{P.V.} \int_{-\infty}^{\infty} \frac{e^{-\pi V x^2}}{x-s} dx \qquad \text{(V.1.3)} \\
&:= -\frac{i}{\pi} \lim_{\varepsilon \to 0^+} \left(\int_{-\infty}^{s-\varepsilon} \frac{e^{-\pi V x^2}}{x-s} dx + \int_{s+\varepsilon}^{\infty} \frac{e^{-\pi V x^2}}{x-s} dx \right).
\end{aligned}
$$

Proof. Since both sides of (V.1.3) are odd in s we can assume that $s > 0$. Additionally we first assume $V > 0$ and get that the left hand side of (V.1.3) equals

$$
e^{-\left(s\sqrt{\pi V}\right)^2}\left(1+\operatorname{erf}\left(is\sqrt{\pi V}\right)\right) - e^{-\left(s\sqrt{\pi V}\right)^2} = \omega\left(s\sqrt{\pi V}\right) - e^{-\left(s\sqrt{\pi V}\right)^2},
$$

where ω is the *Faddeeva function* defined as (see [NIST, equations 7.2.2 and 7.2.3])

$$
\omega(z) := e^{-z^2}\left(1+\operatorname{erf}(iz)\right).
$$

Using the identity (see [NIST, equation 7.7.2])

$$\omega(\tau) = -\frac{i}{\pi}\int_{-\infty}^{\infty} \frac{e^{-t^2}}{t-\tau}\,dt, \qquad \text{for } \operatorname{Im}(\tau) > 0$$

we obtain, for $V > 0$ and some $\delta > 0$,

$$\omega\left(s\sqrt{\pi V}(1+i\delta)\right) = -\frac{i}{\pi}\int_{-\infty}^{\infty} \frac{e^{-t^2}}{t-\sqrt{\pi V}s\,(1+i\delta)}\,dt.$$

Since both sides are holomorphic for $\operatorname{Re}(V) > 0$, the identity holds for such complex values as well by analytic continuation. Using this and substituting $t = \sqrt{\pi V}x$ we see that the left hand side of (V.1.3) becomes

$$\begin{aligned}
&\lim_{\delta\to 0^+}\left(-\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{e^{-t^2}}{t-\sqrt{\pi V}s\,(1+i\delta)}\,dt\right) - e^{-\left(s\sqrt{\pi V}\right)^2}\\
&= \lim_{\delta\to 0^+}\left(-\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{e^{-\pi V x^2}}{\sqrt{\pi V}x-\sqrt{\pi V}s\,(1+i\delta)}\sqrt{\pi V}\,dx\right) - e^{-\pi V s^2}\\
&= -\frac{i}{\pi}\lim_{\delta\to 0^+}\left(\int_{-\infty}^{\infty}\frac{e^{-\pi V x^2}}{x-s(1+i\delta)}\,dx\right) - e^{-\pi V s^2}\\
&= -\frac{i}{\pi}\lim_{\delta\to 0^+}\left(\int_{-\infty}^{s-\varepsilon}\frac{e^{-\pi V x^2}}{x-s(1+i\delta)}\,dx + \int_{s-\varepsilon}^{s+\varepsilon}\frac{e^{-\pi V x^2}}{x-s(1+i\delta)}\,dx + \int_{s+\epsilon}^{\infty}\frac{e^{-\pi V x^2}}{x-s(1+i\delta)}\,dx\right) - e^{-\pi V s^2}\\
&= -\frac{i}{\pi}\left(\int_{-\infty}^{s-\varepsilon}\frac{e^{-\pi V x^2}}{x-s}\,dx + \int_{s+\epsilon}^{\infty}\frac{e^{-\pi V x^2}}{x-s}\,dx\right) - \lim_{\delta\to 0^+}\frac{i}{\pi}\int_{s-\varepsilon}^{s+\varepsilon}\frac{e^{-\pi V x^2}}{x-s(1+i\delta)}\,dx - e^{-\pi V s^2} \qquad \text{(V.1.4)}
\end{aligned}$$

for every $\varepsilon > 0$.

By shifting the path of integration to the lower half plane we are allowed to also take the limit in δ in the last integral and get

$$-\frac{i}{\pi}\int_{\gamma_{s,\varepsilon}}\frac{e^{-\pi V x^2}}{x-s}\,dx,$$

where $\gamma_{s,\varepsilon}$ is the semi-circular path of radius ε passing below s (see Figure V.1).

Plugging in the Taylor series expansion of the exponential arround s gives us

$$\begin{aligned}
-\frac{i}{\pi}\int_{\gamma_{s,\varepsilon}}\frac{e^{-\pi V x^2}}{x-s}\,dx &= -\frac{i}{\pi}\int_{\gamma_{s,\varepsilon}}\frac{1}{x-s}\left(e^{-\pi V s^2} + \sum_{n\geq 1}\frac{(x-s)^n\left(\frac{\partial^n}{\partial x^n}e^{-\pi V x^2}\right)\Big|_{x=s}}{n!}\right)dx\\
&= -\frac{i}{\pi}e^{-\pi V s^2}\int_{\gamma_{s,\varepsilon}}\frac{1}{x-s}\,dx - \frac{i}{\pi}\int_{\gamma_{s,\varepsilon}}\frac{1}{x-s}\sum_{n\geq 1}\frac{(x-s)^n\left(\frac{\partial^n}{\partial x^n}e^{-\pi V x^2}\right)\Big|_{x=s}}{n!}\,dx
\end{aligned}$$

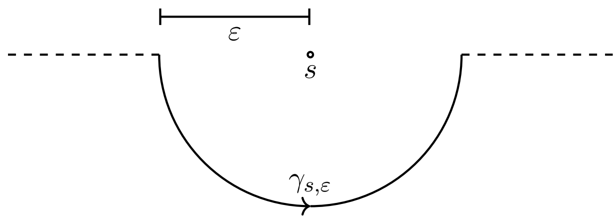

Figure V.1: The contour $\gamma_{s,\varepsilon}$.

$$= -\frac{i}{\pi} e^{-\pi V s^2} \int_{\gamma_{s,\varepsilon}} \frac{1}{x-s}\, dx - \frac{i}{\pi} \int_{\gamma_{s,\varepsilon}} \sum_{n\geq 1} \frac{(x-s)^{n-1} \left(\frac{\partial^n}{\partial x^n} e^{-\pi V x^2} \right)\Big|_{x=s}}{n!}\, dx.$$

Now we see that the second integral does not have any poles on the path of integration and therefore it vanishes by taking $\varepsilon \to 0^+$ since the path of integration vanishes. For the first integral we notice, that we have a simple pole and therefore

$$\lim_{\varepsilon\to 0^+} \int_{\gamma_{s,\varepsilon}} \frac{1}{x-s} = \frac{1}{2}(2\pi i) \operatorname{Res}_{x=s} \left(\frac{1}{x-s} \right) = \pi i \lim_{x\to s} (x-s) \frac{1}{x-s} = \pi i.$$

Taking the limit $\varepsilon \to 0^+$ in (V.1.4) gives the claim. □

V.2 Additional proofs for Section IV.3

In this section we first prove the well-definedness of the Kloosterman sum defined in (IV.1.6).

Lemma V.2.1. *We have that* $K_{k,j,N}(n,r,\kappa)$*, from* (IV.1.6)*, is well-defined.*

Proof. For $M_{h,k} = \begin{pmatrix} h' & -\frac{hh'+1}{k} \\ k & -h \end{pmatrix}$ and $hh' \equiv -1 \pmod{k}$ we define

$$K_{k,j,N}(n,r,\kappa) = \sum_{\substack{0\leq h<k \\ \gcd(h,k)=1}} \chi_{j,r}(N, M_{h,k}) \zeta_{24k}^{\left(24N\left(\kappa+\frac{r}{2N}\right)^2-1\right)h'-24g_{j,N}(n)h} =: \sum_{\substack{0\leq h<k \\ \gcd(h,k)=1}} a(h').$$

To make sure that the sum is well-defined, we need to prove that $a(h'+\alpha k) = a(h')$ for every choice of $\alpha \in \mathbb{Z}$, which ensures that the summand is independet of the choice of h'.

For odd k we first note that we have

$$\begin{aligned}
&\chi_{j,r}(N, M_{h,k}) \\
&:=\nu_\eta(M)\cdot\psi_{j,r}\left(N, M^{-1}\right) \\
&=\left(\frac{-h}{|k|}\right)\exp\left(\frac{\pi i}{12}\left((h'-h)k-\frac{hh'+1}{k}h\left(k^2-1\right)-3k\right)\right) \\
&\quad\times\exp\left(-\frac{3\pi i}{4}\operatorname{sgn}(-k)\right)\sqrt{\frac{2}{N|-k|}}\sum_{s=0}^{|-k|-1}\exp\left(-\frac{\pi i}{2Nk}\left(-h(2Ns+j)^2+h'r^2\right)\right)\sin\left(\frac{\pi r(2Ns+j)}{N|-k|}\right).
\end{aligned}$$

We therefore obtain

$$\begin{aligned}
a(h') &= \left(\frac{-h}{|k|}\right)\exp\left(\frac{\pi i}{12}\left((h'-h)k-\frac{hh'+1}{k}h\left(k^2-1\right)-3k\right)\right) \\
&\quad\times\exp\left(-\frac{3\pi i}{4}\operatorname{sgn}(-k)\right)\sqrt{\frac{2}{N|-k|}}\sum_{s=0}^{|-k|-1}\exp\left(-\frac{\pi i}{2Nk}\left(-h(2Ns+j)^2+h'r^2\right)\right)\sin\left(\frac{\pi r(2Ns+j)}{N|-k|}\right) \\
&\quad\times\exp\left(\frac{2\pi i}{24k}\left(\left(24N\left(\kappa+\frac{r}{2N}\right)^2-1\right)h'-24g_{j,N}(n)h\right)\right) \\
&= \left(\frac{-h}{k}\right)\exp\left(\frac{\pi i}{12}\left((h'-h)k-\frac{hh'+1}{k}h\left(k^2-1\right)-3k\right)\right) \\
&\quad\times\exp\left(\frac{3\pi i}{4}\right)\sqrt{\frac{2}{Nk}}\sum_{s=0}^{k-1}\exp\left(-\frac{\pi i}{2Nk}\left(-h(2Ns+j)^2+h'r^2\right)\right)\sin\left(\frac{\pi r(2Ns+j)}{Nk}\right) \\
&\quad\times\exp\left(\frac{2\pi i}{24k}\left(\left(24N\left(\kappa+\frac{r}{2N}\right)^2-1\right)h'-24g_{j,N}(n)h\right)\right).
\end{aligned}$$

Let α be an arbitrary integer, then we see that

$$\begin{aligned}
&a(h'+\alpha k) \\
&=\left(\frac{-h}{k}\right)\exp\left(\frac{\pi i}{12}\left((h'+\alpha k-h)k-\frac{h(h'+\alpha k)+1}{k}h\left(k^2-1\right)-3k\right)\right) \\
&\quad\times\exp\left(\frac{3\pi i}{4}\right)\sqrt{\frac{2}{Nk}}\sum_{s=0}^{k-1}\exp\left(-\frac{\pi i}{2Nk}\left(-h(2Ns+j)^2+(h'+\alpha k)r^2\right)\right)\sin\left(\frac{\pi r(2Ns+j)}{Nk}\right) \\
&\quad\times\exp\left(\frac{2\pi i}{24k}\left(\left(24N\left(\kappa+\frac{r}{2N}\right)^2-1\right)(h'+\alpha k)-24g_{j,N}(n)h\right)\right) \\
&=\left(\frac{-h}{k}\right)\exp\left(\frac{\pi i}{12}\left((h'-h)k-\frac{hh'+1}{k}h\left(k^2-1\right)-3k\right)\right)\exp\left(\frac{\pi i}{12}\left(\alpha k^2-\frac{h\alpha k}{k}h(k^2-1)\right)\right) \\
&\quad\times\exp\left(\frac{3\pi i}{4}\right)\sqrt{\frac{2}{Nk}}\sum_{s=0}^{k-1}\exp\left(-\frac{\pi i\alpha k r^2}{2Nk}\right)\exp\left(-\frac{\pi i}{2Nk}\left(-h(2Ns+j)^2+h'r^2\right)\right) \\
&\quad\times\sin\left(\frac{\pi r(2Ns+j)}{Nk}\right)\exp\left(\frac{2\pi i}{24k}\left(\left(24N\left(\kappa+\frac{r}{2N}\right)^2-1\right)h'-24g_{j,N}(n)h\right)\right) \\
&\quad\times\exp\left(\frac{2\pi i}{24k}\left(\left(24N\left(\kappa+\frac{r}{2N}\right)^2-1\right)\alpha k\right)\right) \\
&=\left(\frac{-h}{k}\right)\exp\left(\frac{\pi i}{12}\left((h'-h)k-\frac{hh'+1}{k}h\left(k^2-1\right)-3k\right)\right) \\
&\quad\times\exp\left(\frac{3\pi i}{4}\right)\sqrt{\frac{2}{Nk}}\sum_{s=0}^{k-1}\exp\left(-\frac{\pi i}{2Nk}\left(-h(2Ns+j)^2+h'r^2\right)\right)
\end{aligned}$$

$$\times \sin\left(\frac{\pi r(2Ns+j)}{Nk}\right) \exp\left(\frac{2\pi i}{24k}\left(\left(24N\left(\kappa+\frac{r}{2N}\right)^2-1\right)h'-24g_{j,N}(n)h\right)\right)$$
$$\times \exp\left(\frac{\pi i}{12}\alpha\left(k^2-h^2(k^2-1)\right)\right)\exp\left(-\frac{\pi i\alpha r^2}{2N}\right)\exp\left(\frac{2\pi i}{24k}\left(\left(24N\left(\kappa^2+\frac{\kappa r}{N}+\frac{r^2}{4N^2}\right)-1\right)\alpha k\right)\right).$$

We have

$$\exp\left(\frac{2\pi i}{24k}\left(\left(24N\left(\kappa^2+\frac{\kappa r}{N}+\frac{r^2}{4N^2}\right)-1\right)\alpha k\right)\right)$$
$$= \exp\left(2\pi i N\alpha\kappa^2\right)\exp\left(2\pi i\alpha\kappa r\right)\exp\left(\frac{\pi i\alpha r^2}{2N}\right)\exp\left(-\frac{\pi i\alpha}{12}\right)$$
$$= \exp\left(\frac{\pi i\alpha r^2}{2N}\right)\exp\left(-\frac{\pi i\alpha}{12}\right),$$

since $\alpha \in \mathbb{Z},\ N \in \mathbb{N}_{>1},\ \kappa \in \mathbb{Z},\ r \in \mathbb{N}$ and thus get that

$$\exp\left(\frac{\pi i}{12}\alpha\left(k^2-h^2(k^2-1)\right)\right)\exp\left(-\frac{\pi i\alpha r^2}{2N}\right)\exp\left(\frac{2\pi i}{24k}\left(\left(24N\left(\kappa^2+\frac{\kappa r}{N}+\frac{r^2}{4N^2}\right)-1\right)\alpha k\right)\right)$$
$$= \exp\left(\frac{\pi i}{12}\alpha\left(k^2-h^2(k^2-1)-1\right)\right) = \exp\left(2\pi i\alpha\frac{k^2-h^2(k^2-1)-1}{24}\right).$$

If we have that $k^2 - h^2(k^2-1) \equiv 1 \pmod{24}$ this exponential is simply 1 and we proved that $a(h') = a(h'+\alpha k)$ for every choice of $\alpha \in \mathbb{Z}$. To see this remember that we have $k > 0$ and odd. The Chinese Remainder Theorem says that for some $m, n \in \mathbb{Z}$ with $\gcd(m,n) = 1$ we have $f(x) \equiv 1 \pmod{mn}$ if and only if

$$f(x) \equiv 1 \pmod{m} \quad \text{and} \quad f(x) \equiv 1 \pmod{n}.$$

We want to have $k^2 - h^2(k^2-1) \equiv 1 \pmod{24}$, so we prove $k^2 - h^2(k^2-1) \equiv 1 \pmod 8$ and $k^2 - h^2(k^2-1) \equiv 1 \pmod 3$. Since k is odd we know that $k^2 \equiv 1 \pmod 8$. Therefore we obtain

$$k^2 - h^2\left(k^2-1\right) \equiv 1 - h^2(1-1) \pmod 8 \equiv 1 \pmod 8.$$

To prove $k^2 - h^2(k^2-1) \equiv 1 \pmod 3$ we seperate the two cases $3 \nmid k$ and $3 \mid k$. For $3 \nmid k$ we either have $k \equiv 1 \pmod 3$ or $k \equiv 2 \pmod 3$ but in both cases we have $k^2 \equiv 1 \pmod 3$ so we again obtain

$$k^2 - h^2\left(k^2-1\right) \equiv 1 - h^2(1-1) \equiv 1 \pmod 3.$$

For $3 \mid k$ we obatin that $3 \nmid h$, since we have $\gcd(h,k) = 1$ so we, analogously to the first case, obtain $h^2 \equiv 1 \pmod 3$ and thus

$$k^2 - h^2\left(k^2-1\right) \equiv k^2 - 1\left(k^2-1\right) \equiv k^2 - k^2 + 1 \equiv 1 \pmod 3.$$

So we conclude, for odd k, $a(h'+\alpha k)=a(h')$ for every choice of $\alpha\in\mathbb{Z}$.

Next we do the same calculations for even k. We have that

$$\begin{aligned}\chi_{j,r}(N,M_{h,k}) :=&\nu_\eta(M_{h,k})\cdot\psi_{j,r}\left(N,M_{h,k}^{-1}\right)\\ =&\left(\frac{k}{-h}\right)e^{\frac{\pi i}{12}\left(h'k\left(1-(-h)^2\right)+(-h)\left(-\frac{hh'+1}{k}-k+3\right)-3\right)}\\ &\times\exp\left(\frac{3\pi i}{4}\right)\sqrt{\frac{2}{Nk}}\sum_{s=0}^{k-1}\exp\left(-\frac{\pi i}{2Nk}\left(-h(2Ns+j)^2+h'r^2\right)\right)\sin\left(\frac{\pi r(2Ns+j)}{Nk}\right).\end{aligned}$$

Therefore we obtain

$$\begin{aligned}a(h')=&\left(\frac{k}{-h}\right)e^{\frac{\pi i}{12}\left(h'k\left(1-(-h)^2\right)+(-h)\left(-\frac{hh'+1}{k}-k+3\right)-3\right)}\\ &\times\exp\left(\frac{3\pi i}{4}\right)\sqrt{\frac{2}{Nk}}\sum_{s=0}^{k-1}\exp\left(-\frac{\pi i}{2Nk}\left(-h(2Ns+j)^2+h'r^2\right)\right)\sin\left(\frac{\pi r(2Ns+j)}{Nk}\right)\\ &\times\exp\left(\frac{2\pi i}{24k}\left(\left(24N\left(\kappa+\frac{r}{2N}\right)^2-1\right)h'-24g_{j,N}(n)h\right)\right).\end{aligned}$$

Similar to above we get

$$\begin{aligned}a(h'+\alpha k)=&\left(\frac{k}{-h}\right)e^{\frac{\pi i}{12}\left((h'+\alpha k)k\left(1-h^2\right)+h\left(\frac{h(h'+\alpha k)+1}{k}+k-3\right)-3\right)}\exp\left(-\frac{\pi i\alpha}{12}\right)\\ &\times\exp\left(\frac{3\pi i}{4}\right)\sqrt{\frac{2}{Nk}}\sum_{s=0}^{k-1}\exp\left(-\frac{\pi i}{2Nk}\left(-h(2Ns+j)^2+h'r^2\right)\right)\sin\left(\frac{\pi r(2Ns+j)}{Nk}\right)\\ &\times\exp\left(\frac{2\pi i}{24k}\left(\left(24N\left(\kappa+\frac{r}{2N}\right)^2-1\right)h'-24g_{j,N}(n)h\right)\right)\\ =&\exp\left(-\frac{\pi i\alpha}{12}\right)\exp\left(\frac{\pi i}{12}\left(\alpha k^2\left(1-h^2\right)+\alpha h^2\right)\right)a(h')\\ =&\exp\left(2\pi i\alpha\frac{k^2\left(1-h^2\right)+h^2-1}{24}\right)a(h').\end{aligned}$$

If we have $k^2\left(1-h^2\right)+h^2\equiv 1 \pmod{24}$ the additional exponential is simply 1 and we proved that $a(h'+\alpha k)=a(h')$ for even k and every choice of $\alpha\in\mathbb{Z}$. Since k is even, we know that h has to be odd, because $\gcd(h,k)=1$. Analogously to above (changing the roles of h and k) we have

$$k^2\left(1-h^2\right)+h^2=k^2-k^2h^2+h^2=h^2-k^2\left(h^2-1\right)\equiv 1 \pmod{24}.$$

So we conclude, for even k, $a(h'+\alpha k)=a(h')$ for every choice of $\alpha\in\mathbb{Z}$. □

Next we determine the modulus of the Kloosterman sum defined in (IV.1.6).

Lemma V.2.2. *We have that $K_{k,j,N}(n,r,\kappa)$, from* (IV.1.6)*, is a Kloosterman sum of modulus k.*

Proof. For $M_{h,k} = \begin{pmatrix} h' & -\frac{hh'+1}{k} \\ k & -h \end{pmatrix}$ and

$$K_{k,j,N}(n,r,\kappa) = \sum_{\substack{0\le h<k \\ \gcd(h,k)=1}} \chi_{j,r}(N, M_{h,k}) \zeta_{24k}^{\left(24N\left(\kappa+\frac{r}{2N}\right)^2-1\right)h'-24g_{j,N}(n)h} =: \sum_{\substack{0\le h<k \\ \gcd(h,k)=1}} a(h)$$

we, analogously to above and for odd k, obtain that

$$\begin{aligned} a(h) &:= \left(\frac{-h}{k}\right) \exp\left(\frac{\pi i}{12}\left((h'-h)k - \frac{hh'+1}{k}h\left(k^2-1\right) - 3k\right)\right) \\ &\quad \times \exp\left(\frac{3\pi i}{4}\right)\sqrt{\frac{2}{Nk}}\sum_{s=0}^{k-1}\exp\left(-\frac{\pi i}{2Nk}\left(-h(2Ns+j)^2+h'r^2\right)\right)\sin\left(\frac{\pi r(2Ns+j)}{Nk}\right) \\ &\quad \times \exp\left(\frac{2\pi i}{24k}\left(\left(24N\left(\kappa+\frac{r}{2N}\right)^2-1\right)h' - 24g_{j,N}(n)h\right)\right). \end{aligned}$$

Therefore

$$\begin{aligned} &a(h+\alpha k) \\ &= \left(\frac{-(h+\alpha k)}{k}\right)\exp\left(\frac{\pi i}{12}\left((h'-(h+\alpha k))k - \frac{(h+\alpha k)h'+1}{k}(h+\alpha k)\left(k^2-1\right) - 3k\right)\right) \\ &\quad \times \exp\left(\frac{3\pi i}{4}\right)\sqrt{\frac{2}{Nk}}\sum_{s=0}^{k-1}\exp\left(-\frac{\pi i}{2Nk}\left(-(h+\alpha k)(2Ns+j)^2+h'r^2\right)\right)\sin\left(\frac{\pi r(2Ns+j)}{Nk}\right) \\ &\quad \times \exp\left(\frac{2\pi i}{24k}\left(\left(24N\left(\kappa+\frac{r}{2N}\right)^2-1\right)h' - 24g_{j,N}(n)(h+\alpha k)\right)\right) \\ &= \left(\frac{-h}{k}\right)\exp\left(\frac{\pi i}{12}\left((h'-h)k - \frac{(h+\alpha k)h'+1}{k}(h+\alpha k)\left(k^2-1\right) - 3k\right)\right)\exp\left(\frac{\pi i}{12}\left(-\alpha k^2\right)\right) \\ &\quad \times \exp\left(\frac{3\pi i}{4}\right)\sqrt{\frac{2}{Nk}}\sum_{s=0}^{k-1}\exp\left(-\frac{\pi i}{2Nk}\left(-h(2Ns+j)^2-\alpha k(2Ns+j)^2+h'r^2\right)\right) \\ &\quad \times \sin\left(\frac{\pi r(2Ns+j)}{Nk}\right)\exp\left(\frac{2\pi i}{24k}\left(\left(24N\left(\kappa+\frac{r}{2N}\right)^2-1\right)h' - 24g_{j,N}(n)h - 24g_{j,N}(n)\alpha k\right)\right) \\ &= \left(\frac{-h}{k}\right)\exp\left(\frac{\pi i}{12}\left((h'-h)k - \frac{hh'+1}{k}h\left(k^2-1\right) - 3k\right)\right) \\ &\quad \times \exp\left(\frac{\pi i}{12}\left(-\alpha k^2 - \left(\left(hh'+1\right)\alpha + \alpha hh' + \alpha^2 h'k\right)\left(k^2-1\right)\right)\right) \\ &\quad \times \exp\left(\frac{3\pi i}{4}\right)\sqrt{\frac{2}{Nk}}\sum_{s=0}^{k-1}\exp\left(-\frac{\pi i}{2Nk}\left(-h(2Ns+j)^2+h'r^2\right)\right)\exp\left(\frac{\alpha\pi i}{2N}(4N^2s^2+4Nsj+j^2)\right) \\ &\quad \times \sin\left(\frac{\pi r(2Ns+j)}{Nk}\right)\exp\left(\frac{2\pi i}{24k}\left(\left(24N\left(\kappa+\frac{r}{2N}\right)^2-1\right)h' - 24g_{j,N}(n)h\right)\right)\exp\left(-2\pi i g_{j,N}(n)\alpha\right) \\ &= \left(\frac{-h}{k}\right)\exp\left(\frac{\pi i}{12}\left((h'-h)k - \frac{hh'+1}{k}h\left(k^2-1\right) - 3k\right)\right) \\ &\quad \times \exp\left(\frac{\pi i}{12}\alpha\left(-k^2 - \left(hh'+1+hh'+\alpha h'k\right)\left(k^2-1\right)\right)\right) \end{aligned}$$

$$
\begin{aligned}
&\times \exp\left(\frac{3\pi i}{4}\right)\sqrt{\frac{2}{Nk}}\sum_{s=0}^{k-1}\exp\left(-\frac{\pi i}{2Nk}\left(-h(2Ns+j)^2+h'r^2\right)\right)\exp\left(2\alpha\pi iNs^2+2\alpha\pi isj+\frac{\alpha\pi ij^2}{2N}\right)\\
&\times \sin\left(\frac{\pi r(2Ns+j)}{Nk}\right)\exp\left(\frac{2\pi i}{24k}\left(\left(24N\left(\kappa+\frac{r}{2N}\right)^2-1\right)h'-24g_{j,N}(n)h\right)\right)\exp\left(-2\pi ig_{j,N}(n)\alpha\right)\\
=&\exp\left(\frac{\alpha\pi ij^2}{2N}\right)\exp\left(\frac{\pi i}{12}\alpha\left(-k^2-\left(2hh'+1+\alpha h'k\right)\left(k^2-1\right)\right)\right)\\
&\times\left(\frac{-h}{k}\right)\exp\left(\frac{\pi i}{12}\left((h'-h)k-\frac{hh'+1}{k}h\left(k^2-1\right)-3k\right)\right)\\
&\times \exp\left(\frac{3\pi i}{4}\right)\sqrt{\frac{2}{Nk}}\sum_{s=0}^{k-1}\exp\left(-\frac{\pi i}{2Nk}\left(-h(2Ns+j)^2+h'r^2\right)\right)\\
&\times \sin\left(\frac{\pi r(2Ns+j)}{Nk}\right)\exp\left(\frac{2\pi i}{24k}\left(\left(24N\left(\kappa+\frac{r}{2N}\right)^2-1\right)h'-24g_{j,N}(n)h\right)\right)\exp\left(-2\pi ig_{j,N}(n)\alpha\right)\\
=&\exp\left(\frac{\pi i}{12}\alpha\left(\frac{12j^2}{2N}\right)\right)\exp\left(\frac{\pi i}{12}\alpha\left(-k^2-\left(2hh'+1+\alpha h'k\right)\left(k^2-1\right)\right)\right)\exp\left(-2\pi ig_{j,N}(n)\alpha\right)a(h)\\
=&\exp\left(\frac{\pi i}{12}\alpha\left(\frac{12j^2}{2N}-k^2-\left(2hh'+1+\alpha h'k\right)\left(k^2-1\right)\right)-2\pi ig_{j,N}(n)\alpha\right)a(h)\\
=&\exp\left(\frac{\pi i}{12}\alpha\left(\frac{6j^2}{N}-k^2-\left(2hh'k^2+k^2+\alpha h'k^3-2hh'-1-\alpha h'k\right)\right)-2\pi ig_{j,N}(n)\alpha\right)a(h)\\
=&\exp\left(\frac{\pi i}{12}\alpha\left(\frac{6j^2}{N}-2hh'k^2-2k^2-\alpha h'k^3+2hh'+1+\alpha h'k\right)-2\pi ig_{j,N}(n)\alpha\right)a(h)\\
=&\exp\left(2\pi i\alpha\,\frac{\frac{6j^2}{N}-2hh'k^2-2k^2-\alpha h'k^3+2hh'+1+\alpha h'k-24g_{j,N}(n)}{24}\right)a(h).
\end{aligned}
$$

If we have that $\frac{6j^2}{N}-2hh'k^2-2k^2-\alpha h'k^3+2hh'+2+\alpha h'k-24g_{j,N}(n)\equiv 1 \pmod{24}$ this additional exponential is simply 1 and we proved that $a(h)=a(h+\alpha k)$ for every choice of $\alpha\in\mathbb{Z}$ and odd k.

We go on as before and want to prove that it is 1 modulo 3 and 8. Remember that since k is odd we know that $k^2\equiv 1 \pmod 8$. We see that, by definition of $g_{j,N}(n)$,

$$
\begin{aligned}
&\frac{6j^2}{N}-2hh'k^2-2k^2-\alpha h'k^3+2hh'+2+\alpha h'k-24g_{j,N}(n)\\
&\qquad=\frac{6j^2}{N}+(1-k^2)(2hh'+2+\alpha h'k)-24n-\frac{6j^2}{N}+1\equiv 1 \pmod 8.
\end{aligned}
$$

To prove $\frac{6j^2}{N}-2hh'k^2-2k^2-\alpha h'k^3+2hh'+2+\alpha h'k-24g_{j,N}(n)\equiv 1 \pmod 3$ we seperate the two cases $3\nmid k$ and $3\mid k$. For $3\nmid k$ we have $k^2\equiv 1 \pmod 3$ analogously to above and obtain

$$
\begin{aligned}
&\frac{6j^2}{N}-2hh'k^2-2k^2-\alpha h'k^3+2hh'+2+\alpha h'k-24g_{j,N}(n)\\
&\qquad=\frac{6j^2}{N}+(1-k^2)(2hh'+2+\alpha h'k)-24n-\frac{6j^2}{N}+1\equiv 1 \pmod 3.
\end{aligned}
$$

For $3 \mid k$ we have that $hh' \equiv -1 \pmod{k}$ is equivalent to $k \mid (hh'+1)$, which implies $3 \mid (hh'+1)$ and therefore gives $hh' \equiv -1 \pmod{3}$. We thus obtain that

$$\begin{aligned}
&\frac{6j^2}{N} - 2hh'k^2 - 2k^2 - \alpha h'k^3 + 2hh' + 2 + \alpha h'k - 24g_{j,N}(n) \\
&\quad = \frac{6j^2}{N} - 2hh'k^2 - 2k^2 - \alpha h'k^3 + 2hh' + 2 + \alpha h'k - 24n - \frac{6j^2}{N} + 1 \\
&\quad = -2hh'k^2 - 2k^2 - \alpha h'k^3 + 2hh' + 3 + \alpha h'k - 24n \\
&\quad \equiv 2hh' \equiv 2 \cdot (-1) \equiv -2 \equiv 1 \pmod{3}.
\end{aligned}$$

Next we do the same calculations for even k. From now on we write $k = 2^\nu \mu$ for some $\nu \geq 1$ and some odd $\mu \in \mathbb{N}$ and $x = 2\widetilde{x} + 1$ with $\widetilde{x} \in \mathbb{N}$ for every odd element x. Analogously to above we have

$$\begin{aligned}
a(h) = &\left(\frac{k}{-h}\right) e^{\frac{\pi i}{12}\left(h'k\left(1-h^2\right)+h\left(\frac{hh'+1}{k}+k-3\right)-3\right)} \\
&\times \exp\left(\frac{3\pi i}{4}\right)\sqrt{\frac{2}{Nk}}\sum_{s=0}^{k-1}\exp\left(-\frac{\pi i}{2Nk}\left(-h(2Ns+j)^2+h'r^2\right)\right)\sin\left(\frac{\pi r(2Ns+j)}{Nk}\right) \\
&\times \exp\left(\frac{2\pi i}{24k}\left(\left(24N\left(\kappa+\frac{r}{2N}\right)^2-1\right)h' - 24g_{j,N}(n)h\right)\right).
\end{aligned}$$

We look at the cases $\nu \geq 2$ and $\nu = 1$ seperately.

For $\nu \geq 2$ and using Lemma V.2.3 we obtain

$$\begin{aligned}
&a(h+\alpha k) \\
&= \left(\frac{k}{-(h+\alpha k)}\right) e^{\frac{\pi i}{12}\left(h'k\left(1-(h+\alpha k)^2\right)+(h+\alpha k)\left(\frac{(h+\alpha k)h'+1}{k}+k-3\right)-3\right)} \\
&\quad \times \exp\left(\frac{3\pi i}{4}\right)\sqrt{\frac{2}{Nk}}\sum_{s=0}^{k-1}\exp\left(-\frac{\pi i}{2Nk}\left(-(h+\alpha k)(2Ns+j)^2+h'r^2\right)\right)\sin\left(\frac{\pi r(2Ns+j)}{Nk}\right) \\
&\quad \times \exp\left(\frac{2\pi i}{24k}\left(\left(24N\left(\kappa+\frac{r}{2N}\right)^2-1\right)h' - 24g_{j,N}(n)(h+\alpha k)\right)\right) \\
&= \exp\left(\frac{\pi i\alpha j^2}{2N}\right)\exp\left(\frac{\pi i}{12}\left(h'k(-2h\alpha k-\alpha^2k^2)+\alpha k\left(\frac{hh'+1}{k}+k-3\right)+(h+\alpha k)\alpha h'\right)\right) \\
&\quad \times \exp\left(-2\pi i g_{j,N}(n)\alpha\right) a(h) \\
&= \exp\left(\frac{\pi i}{12}\alpha\left(\frac{12j^2}{2N}+h'k(-2hk-\alpha k^2)+k\left(\frac{hh'+1}{k}+k-3\right)+(h+\alpha k)h'\right)-2\pi i g_{j,N}(n)\alpha\right)a(h) \\
&= \exp\left(\frac{\pi i}{12}\alpha\left(\frac{6j^2}{N}-2hh'k^2-h'\alpha k^3+2hh'+1+k^2-3k+\alpha kh'\right)-2\pi i g_{j,N}(n)\alpha\right)a(h) \\
&= \exp\left(2\pi i\alpha\left(\frac{\frac{6j^2}{N}-2hh'k^2-h'\alpha k^3+2hh'+1+k^2-3k+\alpha kh'-24g_{j,N}(n)}{24}\right)\right)a(h).
\end{aligned}$$

If we have that

$$\alpha\left(\frac{6j^2}{N} - 2hh'k^2 - h'\alpha k^3 + 2hh' + 1 + k^2 - 3k + \alpha kh' - 24g_{j,N}(n)\right) \equiv 0 \pmod{24}$$

this additional exponential is simply 1 and we proved that $a(h) = a(h + \alpha k)$ for every choice of $\alpha \in \mathbb{Z}$, even k, and $\nu \geq 2$.

To show this we seperate the following cases. For even α we need to show that

$$\frac{6j^2}{N} - 2hh'k^2 - h'\alpha k^3 + 2hh' + 2 + k^2 - 3k + \alpha kh' - 24g_{j,N}(n) \equiv 1 \pmod{12},$$

i.e., that this expression is congruent to 1 modulo 4 and 3, by the Chinese Remainder Theorem.
For odd α we need to show that

$$\frac{6j^2}{N} - 2hh'k^2 - h'\alpha k^3 + 2hh' + 2 + k^2 - 3k + \alpha kh' - 24g_{j,N}(n) \equiv 1 \pmod{24},$$

i.e., that this expression is congruent to 1 modulo 8 and 3.

To prove the congruences modulo 4 and 8 we additionally seperate the cases $\nu \geq 3$ and $\nu = 2$.

For $\nu \geq 3$ we know that $k \equiv 0 \pmod 8$ and thus $hh' \equiv -1 \pmod 8$ analogously to above. We see that

$$\begin{aligned}
&\frac{6j^2}{N} - 2hh'k^2 - h'\alpha k^3 + 2hh' + 2 + k^2 - 3k + \alpha kh' - 24g_{j,N}(n) \\
&\qquad = \frac{6j^2}{N} - 2hh'k^2 - h'\alpha k^3 + 2hh' + 2 + k^2 - 3k + \alpha kh' - 24n - \frac{6j^2}{N} + 1 \\
&\qquad = -2hh'k^2 - h'\alpha k^3 + 2hh' + 3 + k^2 - 3k + \alpha kh' - 24n \\
&\qquad \equiv -2 + 3 \equiv 1 \pmod 8.
\end{aligned}$$

Since $k \equiv 0 \pmod 8$ implies $k \equiv 0 \pmod 4$ the same calculation holds modulo 4 (independent of α).

For $\nu = 2$, so $k = 4\mu = 4(2\widetilde{\mu} + 1)$, we have $hh' \equiv -1 \pmod 4$ and therefore $2hh' \equiv -2 \pmod 8$. For odd α and noting that h' has to be odd[1] we thus obtain

$$\begin{aligned}
&\frac{6j^2}{N} - 2hh'k^2 - h'\alpha k^3 + 2hh' + 2 + k^2 - 3k + \alpha kh' - 24g_{j,N}(n) \\
&\qquad = -2hh'k^2 - h'\alpha k^3 + 2hh' + 3 + k^2 - 3k + \alpha kh' - 24n
\end{aligned}$$

[1] Note that h is odd since k is even. Assume that h' would be even, then hh' would be even and $hh' + 1$ would be odd, which would be a contradiction to $hh' \equiv -1 \pmod 4$.

$$\begin{aligned}
&\equiv -2+3-3k+\alpha k h' \\
&\equiv 1-3(4(2\widetilde{\mu}+1))+4(2\widetilde{\alpha}+1)(2\widetilde{\mu}+1)(2\widetilde{h}'+1) \\
&\equiv 1-4+4 \equiv 1 \pmod 8 .
\end{aligned}$$

For even α we get

$$\begin{aligned}
&\frac{6j^2}{N} - 2hh'k^2 - h'\alpha k^3 + 2hh' + 2 + k^2 - 3k + \alpha k h' - 24 g_{j,N}(n) \\
&\quad = -2hh'k^2 - h'\alpha k^3 + 2hh' + 3 + k^2 - 3k + \alpha k h' - 24n \\
&\quad \equiv 2hh' + 3 \equiv -2 + 3 \equiv 1 \pmod 4 .
\end{aligned}$$

To prove $\frac{6j^2}{N} - 2hh'k^2 - h'\alpha k^3 + 2hh' + 2 + k^2 - 3k + \alpha k h' - 24 g_{j,N}(n) \equiv 1 \pmod 3$ we seperate the two cases $3 \nmid k$ and $3 \mid k$, but are independent of the choice of α and ν. For $3 \nmid k$ we have $k^2 \equiv 1 \pmod 3$ analogously to above and obtain

$$\begin{aligned}
&\frac{6j^2}{N} - 2hh'k^2 - h'\alpha k^3 + 2hh' + 2 + k^2 - 3k + \alpha k h' - 24 g_{j,N}(n) \\
&\quad = -2hh'k^2 - h'\alpha k^3 + 2hh' + 3 + k^2 - 3k + \alpha k h' - 24n \\
&\quad \equiv (1-k^2)(2hh' + h'\alpha k + 2) + 1 \equiv 1 \pmod 3 .
\end{aligned}$$

For $3 \mid k$ we obtain

$$\begin{aligned}
&\frac{6j^2}{N} - 2hh'k^2 - h'\alpha k^3 + 2hh' + 2 + k^2 - 3k + \alpha k h' - 24 g_{j,N}(n) \\
&\quad = -2hh'k^2 - h'\alpha k^3 + 2hh' + 3 + k^2 - 3k + \alpha k h' - 24n \\
&\quad \equiv 2hh' \equiv 1 \pmod 3 ,
\end{aligned}$$

analogously to above.

Lastly we look at what happens for $\nu = 1$. Using Lemma V.2.3 we obtain that

$$\begin{aligned}
& a(h+\alpha k) \\
&= \left(\frac{k}{-(h+\alpha k)}\right) e^{\frac{\pi i}{12}\left(h'k\left(1-(h+\alpha k)^2\right)+(h+\alpha k)\left(\frac{(h+\alpha k)h'+1}{k}+k-3\right)-3\right)} \\
&\quad \times \exp\left(\frac{3\pi i}{4}\right)\sqrt{\frac{2}{Nk}}\sum_{s=0}^{k-1}\exp\left(-\frac{\pi i}{2Nk}\left(-(h+\alpha k)(2Ns+j)^2+h'r^2\right)\right)\sin\left(\frac{\pi r(2Ns+j)}{Nk}\right) \\
&\quad \times \exp\left(\frac{2\pi i}{24k}\left(\left(24N\left(\kappa+\frac{r}{2N}\right)^2-1\right)h' - 24 g_{j,N}(n)(h+\alpha k)\right)\right) \\
&= \exp\left(2\pi i \alpha\left(\frac{3hk+\frac{3}{2}\alpha k^2 - 3(\mu-1)k}{24}\right)\right) \\
&\quad \times \exp\left(2\pi i\alpha\left(\frac{\frac{6j^2}{N} - 2hh'k^2 - h'\alpha k^3 + 2hh' + 1 + k^2 - 3k + \alpha k h' - 24 g_{j,N}(n)}{24}\right)\right) a(h)
\end{aligned}$$

$$= \exp\left(2\pi i\alpha\left(\frac{\frac{6j^2}{N} - 2hh'k^2 - h'\alpha k^3 + 2hh' + 1 + k^2 - 3k + \alpha kh' - 24g_{j,N}(n) + 3hk + \frac{3}{2}\alpha k^2 - 3(\mu-1)k}{24}\right)\right)$$
$$\times a(h).$$

Again, if we have that

$$\frac{6j^2}{N} - 2hh'k^2 - h'\alpha k^3 + 2hh' + 2 + k^2 - 3k + \alpha kh' - 24g_{j,N}(n) + 3hk + \frac{3}{2}\alpha k^2 - 3(\mu-1)k \equiv 1 \pmod{24}$$

this additional exponential is simply 1 and we proved that $a(h) = a(h + \alpha k)$ for every choice of $\alpha \in \mathbb{Z}$ and even k.

We go on as before and want to prove that it is 1 modulo 3 and 1 modulo 8, respectively 4, for odd, respectively even, α.

We start with the congruences modulo 8 and 4. We know that

$$k^2 = 4\mu^2 = 4\left(2\widetilde{\mu}+1\right)^2 = 4\left(4\widetilde{\mu}^2 + 4\widetilde{\mu} + 1\right) \equiv 4 \pmod 8$$

and $k^3 = 8\mu^3 \equiv 0 \pmod 8$. Since h is odd we have $h^2 \equiv 1 \pmod 8$ therefore we have $h(-h) = -h^2 \equiv -1 \pmod 8$, so we can choose $h' \equiv -h \pmod 8$. For odd α we see that

$$\begin{aligned}
&\frac{6j^2}{N} - 2hh'k^2 - h'\alpha k^3 + 2hh' + 2 + k^2 - 3k + \alpha kh' - 24g_{j,N}(n) + 3hk + \frac{3}{2}\alpha k^2 - 3(\mu-1)k \\
&\quad = - 2hh'k^2 - h'\alpha k^3 + 2hh' + 3 + k^2 - 3(1 + (\mu-1))k + \alpha kh' - 24n + 3hk + \frac{3}{2}\alpha k^2 \\
&\quad \equiv 2hh' + 3 + 4 - 3\mu k + \alpha kh' + 3hk + 6\alpha \equiv -2 + 7 + 3(h-\mu)k + \alpha kh' + 6\alpha \\
&\quad \equiv 5 + 3(h-\mu)k - \alpha kh + 6\alpha \equiv 5 + 4(\widetilde{h} - \widetilde{\mu})\mu - 2(2\widetilde{\alpha}+1)\mu(2\widetilde{h}+1) + 6(2\widetilde{\alpha}+1) \\
&\quad \equiv 5 + 4\widetilde{h}\mu - 4\widetilde{\mu}\mu - (4\widetilde{\alpha}\mu + 2\mu)(2\widetilde{h}+1) + 4\widetilde{\alpha} + 6 \\
&\quad \equiv 3 + 4\widetilde{h}\mu - 4\widetilde{\mu}\mu - (4\widetilde{\alpha}\mu + 4\mu\widetilde{h} + 2\mu) + 4\widetilde{\alpha} \equiv 3 - 4\widetilde{\mu}\mu - 4\widetilde{\alpha}\mu - 2\mu + 4\widetilde{\alpha} \\
&\quad \equiv 3 - 4\widetilde{\mu} - 4\widetilde{\alpha} - 4\widetilde{\mu} - 2 + 4\widetilde{\alpha} \equiv 1 \pmod 8 .
\end{aligned}$$

Note that since $2 \mid k$ we have $2hh' \equiv -2 \pmod 4$. For even α we thus get

$$\begin{aligned}
&\frac{6j^2}{N} - 2hh'k^2 - h'\alpha k^3 + 2hh' + 2 + k^2 - 3k + \alpha kh' - 24g_{j,N}(n) + 3hk + \frac{3}{2}\alpha k^2 - 3(\mu-1)k \\
&\quad = - 2hh'k^2 - h'\alpha k^3 + 2hh' + 3 + k^2 - 3\mu k + \alpha kh' - 24n + 3hk + \frac{3}{2}\alpha k^2 \\
&\quad \equiv - 2 + 3 - 3\mu k + 3hk \equiv 1 + 3(h-\mu)k \equiv 1 \pmod 4 ,
\end{aligned}$$

since $h - \mu$ is even.

Lastly we show the congruence modulo 3, namely

$$\frac{6j^2}{N} - 2hh'k^2 - h'\alpha k^3 + 2hh' + 2 + k^2 - 3k + \alpha kh' - 24g_{j,N}(n) + 3hk + \frac{3}{2}\alpha k^2 - 3(\mu-1)k \equiv 1 \pmod 3 .$$

To do so we seperate the two cases $3 \nmid k$ and $3 \mid k$, but are independent of the choices of α. For $3 \nmid k$ we have $k^2 \equiv 1 \pmod 3$, so we obtain

$$\begin{aligned}\frac{6j^2}{N} &- 2hh'k^2 - h'\alpha k^3 + 2hh' + 2 + k^2 - 3k + \alpha kh' - 24g_{j,N}(n) + 3hk + \frac{3}{2}\alpha k^2 - 3(\mu-1)k \\ &= - 2hh'k^2 - h'\alpha k^3 + 2hh' + 3 + k^2 - 3k + \alpha kh' - 24n + 3hk + 6\alpha\mu^2 - 3(\mu-1)k \\ &\equiv (1-k^2)(2hh' + h'\alpha k + 2) + 1 \equiv 1 \pmod 3 .\end{aligned}$$

For $3 \mid k$ we obatin

$$\begin{aligned}\frac{6j^2}{N} &- 2hh'k^2 - h'\alpha k^3 + 2hh' + 2 + k^2 - 3k + \alpha kh' - 24g_{j,N}(n) + 3hk + \frac{3}{2}\alpha k^2 - 3(\mu-1)k \\ &= - 2hh'k^2 - h'\alpha k^3 + 2hh' + 3 + k^2 - 3k + \alpha kh' - 24n + 3hk + 6\alpha\mu^2 - 3(\mu-1)k \\ &\equiv 2hh' \equiv 1 \pmod 3 ,\end{aligned}$$

analogously to above. □

In the previous proof we used a result on the Kronecker symbol, which we want to summarize in the following lemma.

Lemma V.2.3. *For $k = 2^\nu \mu$ with $\nu \geq 1$ and some odd $\mu \in \mathbb{N}$ we have*

$$\left(\frac{k}{-(h+\alpha k)}\right) = \begin{cases} \left(\frac{k}{-h}\right) & \text{if } \nu \geq 2, \\ \exp\left(\pi i \left(\frac{2\alpha hk + \alpha^2 k^2 - 2(\mu-1)\alpha k}{8}\right)\right)\left(\frac{k}{-h}\right) & \text{if } \nu = 1. \end{cases}$$

Proof. We have $\alpha \in \mathbb{Z}$ and split the Kronecker symbol as follows

$$\left(\frac{k}{-(h+\alpha k)}\right) = \left(\frac{2}{-(h+\alpha k)}\right)^\nu \left(\frac{\mu}{-(h+\alpha k)}\right).$$

By quadratic reciprocity we see that

$$\begin{aligned}\left(\frac{\mu}{-(h+\alpha k)}\right) &= (-1)^{\frac{\mu-1}{2}\frac{-(h+\alpha k)-1}{2}} \left(\frac{-(h+\alpha k)}{\mu}\right) = (-1)^{-\frac{(\mu-1)\alpha k}{4}} (-1)^{\frac{\mu-1}{2}\frac{-h-1}{2}} \left(\frac{-h}{\mu}\right) \\ &= (-1)^{-\frac{(\mu-1)\alpha k}{4}} \left(\frac{\mu}{-h}\right).\end{aligned}$$

Note that $\mu - 1$ is even, since we have that μ is odd, and that $\gcd(-(h+\alpha k), 2) = 1$. If ν is even we obtain that

$$\left(\frac{2}{-(h+\alpha k)}\right)^\nu = 1 = \left(\frac{2}{-h}\right)^\nu, \qquad \left(\frac{\mu}{-(h+\alpha k)}\right) = (-1)^{-\frac{(\mu-1)\alpha k}{4}} \left(\frac{\mu}{-h}\right) = \left(\frac{\mu}{-h}\right),$$

and thus

$$\left(\frac{k}{-(h+\alpha k)}\right) = \left(\frac{2}{-(h+\alpha k)}\right)^{\nu}\left(\frac{\mu}{-(h+\alpha k)}\right) = \left(\frac{k}{-h}\right),$$

using that $-\frac{(\mu-1)\alpha k}{4} \equiv 0 \pmod 2$ since $4 \mid k$ and the fact that $\left(\frac{a}{-b}\right) = \operatorname{sgn}(a)\left(\frac{a}{b}\right)$.

Lets assume ν is odd. We obtain, by quadratic reciprocity that

$$\begin{aligned}\left(\frac{2}{-(h+\alpha k)}\right)\left(\frac{2}{-h}\right) =& \left(\frac{2}{-1}\right)\left(\frac{2}{h+\alpha k}\right)\left(\frac{2}{-1}\right)\left(\frac{2}{h}\right)\\ =&(-1)^{\frac{(h+\alpha k)^2-1}{8}-\frac{h^2-1}{8}} = (-1)^{\frac{2h\alpha k+\alpha^2k^2}{8}}.\end{aligned}$$

Overall we therefore have

$$\begin{aligned}\left(\frac{k}{-(h+\alpha k)}\right) =& \left(\frac{2}{-(h+\alpha k)}\right)^{\nu}\left(\frac{\mu}{-(h+\alpha k)}\right) = (-1)^{\frac{2h\alpha k\nu+\alpha^2k^2\nu}{8}-\frac{(\mu-1)\alpha k}{4}}\left(\frac{2}{-h}\right)^{\nu}\left(\frac{\mu}{-h}\right)\\ =&(-1)^{\frac{2\alpha hk\nu+\alpha^2k^2\nu-2(\mu-1)\alpha k}{8}}\left(\frac{k}{-h}\right).\end{aligned}$$

For $\nu \geq 3$ we have $8 \mid k$ and thus

$$2\alpha hk\nu + \alpha^2k^2\nu - 2(\mu-1)\alpha k \equiv 0 \pmod{16},$$

such that we again obtain $\left(\frac{k}{-(h+\alpha k)}\right) = \left(\frac{k}{-h}\right)$.

For $\nu = 1$ we have

$$\left(\frac{k}{-(h+\alpha k)}\right) =(-1)^{\frac{2\alpha hk+\alpha^2k^2-2(\mu-1)\alpha k}{8}}\left(\frac{k}{-h}\right) = \exp\left(\pi i\left(\frac{2\alpha hk+\alpha^2k^2-2(\mu-1)\alpha k}{8}\right)\right)\left(\frac{k}{-h}\right).$$

This finishes the proof. □

The next lemma gives a bound on the minumum coming from Malishev's result in the odd k case.

Lemma V.2.4. *We have*

$$\min\left(\gcd\left(\mu_1+jsk,\gcd(N,k)k\right)^{\frac{1}{2}},\gcd\left(\nu_1,\gcd(N,k)k\right)^{\frac{1}{2}}\right) = O_N\left(n^{\frac{1}{2}}\right)$$

and

$$\min\left(\gcd\left(\mu_2+3jsk,3\gcd(N,k)k\right)^{\frac{1}{2}},\gcd\left(\nu_2,3\gcd(N,k)k\right)^{\frac{1}{2}}\right) = O_N\left(n^{\frac{1}{2}}\right).$$

Proof. First we note that

$$\min\left(\gcd\left(\mu_1+jsk,\gcd(N,k)k\right)^{\frac{1}{2}},\gcd\left(\nu_1,\gcd(N,k)k\right)^{\frac{1}{2}}\right) \leq \gcd\left(\mu_1+jsk,\gcd(N,k)k\right)^{\frac{1}{2}}.$$

Since $\gcd(a,bc) \leq \gcd(a,b)\gcd(a,c)$, for $a,b,c \in \mathbb{Z}$, we see that

$$\begin{aligned}\gcd\left(\mu_1 + jsk, \gcd(N,k)k\right) =& \gcd\left(\mu_1 + jsk, \frac{k}{\gcd(N,k)}\gcd(N,k)^2\right)\\ \leq& \gcd\left(\mu_1 + jsk, \frac{k}{\gcd(N,k)}\right)\gcd\left(\mu_1 + jsk, \gcd(N,k)^2\right),\end{aligned}$$

where we have that $\gcd\left(\mu_1 + jsk, \gcd(N,k)^2\right) \leq \gcd(N,k)^2 \leq N^2 = O_N(1)$. We note that $\gcd(a,b) \leq \gcd(xa,b)$ for any $x \in \mathbb{Z}\backslash\{0\}$, which implies

$$\gcd\left(\mu_1 + jsk, \frac{k}{\gcd(N,k)}\right) \leq \gcd\left(24(\mu_1 + jsk), \frac{k}{\gcd(N,k)}\right).$$

Since we further have that $\gcd(a,b) = \gcd(a \pmod{b}, b)$ for $b \neq 0$ we can reduce $24(\mu_1 + jsk)$ modulo $\frac{k}{\gcd(N,k)}$ and see that

$$\begin{aligned}&24(\mu_1 + jsk)\\ &= \left(-24n + 1 - k^2\right)\gcd(N,k) - 24j^2[4]'_{\frac{k}{\gcd(N,k)}}\left[\frac{N}{\gcd(N,k)}\right]'_{\frac{k}{\gcd(N,k)}} + 24jsk\\ &\equiv (-24n+1)\gcd(N,k) + 6j^2\left[\frac{N}{\gcd(N,k)}\right]'_{\frac{k}{\gcd(N,k)}}\\ &\equiv \left[\frac{N}{\gcd(N,k)}\right]'_{\frac{k}{\gcd(N,k)}}\left(N\,(24n-1) + 6j^2\right) \pmod{\frac{k}{\gcd(N,k)}}.\end{aligned}$$

We now notice that by definition we have $\gcd(\frac{N}{\gcd(N,k)}, \frac{k}{\gcd(N,k)}) = 1$. By the definition of the negative inverse we thus notice that we additionally have

$$1 = \gcd\left(\frac{N}{\gcd(N,k)}\left[\frac{N}{\gcd(N,k)}\right]'_{\frac{k}{\gcd(N,k)}}, \frac{k}{\gcd(N,k)}\right) = \gcd\left(\left[\frac{N}{\gcd(N,k)}\right]'_{\frac{k}{\gcd(N,k)}}, \frac{k}{\gcd(N,k)}\right),$$

using that $\gcd(ab,c) = \gcd(a,c)$ if $\gcd(b,c) = 1$. This gives us that

$$\gcd\left(24(\mu_1 + jsk), \frac{k}{\gcd(N,k)}\right) = \gcd\left(N\,(24n-1) + 6j^2, \frac{k}{\gcd(N,k)}\right).$$

Therefore we obtain

$$\begin{aligned}\gcd\left(\mu_1 + jsk, \gcd(N,k)k\right)^{\frac{1}{2}} =& O_N\left(\gcd\left(N\,(24n-1) + 6j^2, \frac{k}{\gcd(N,k)}\right)^{\frac{1}{2}}\right)\\ =& O_N\left(\left(N\,(24n-1) + 6j^2\right)^{\frac{1}{2}}\right) = O_N\left(n^{\frac{1}{2}}\right).\end{aligned}$$

The second result follows analogously, with 8 instead of 24. □

A similar result holds in the even k case.

Lemma V.2.5. *We have*

$$\min\left(\gcd\left(\mu_5+16jks+\frac{2^{\alpha+5}\ell k\gcd(N,\mu)}{16},2^{\alpha+5}k\gcd(N,\mu)\right)^{\frac{1}{2}},\gcd\left(\nu_5,2^{\alpha+5}k\gcd(N,\mu)\right)^{\frac{1}{2}}\right)=O_N\left(n\right),$$

and

$$\min\left(\gcd\left(\mu_6+48jks+\frac{3\cdot 2^{\alpha+5}\ell k\gcd(N,\mu)}{16},3\cdot 2^{\alpha+5}k\gcd(N,\mu)\right)^{\frac{1}{2}},\gcd\left(\nu_6,3\cdot 2^{\alpha+5}k\gcd(N,\mu)\right)^{\frac{1}{2}}\right)=O_N\left(n\right).$$

Proof. First we note that

$$\min\left(\gcd\left(\mu_5+16jks+\frac{2^{\alpha+5}\ell k\gcd(N,\mu)}{16},2^{\alpha+5}k\gcd(N,\mu)\right)^{\frac{1}{2}},\gcd\left(\nu_5,2^{\alpha+5}k\gcd(N,\mu)\right)^{\frac{1}{2}}\right)$$
$$\leq\gcd\left(\mu_5+16jks+\frac{2^{\alpha+5}\ell k\gcd(N,\mu)}{16},2^{\alpha+5}k\gcd(N,\mu)\right)^{\frac{1}{2}}.$$

Since $\gcd(a,bc)\leq\gcd(a,b)\gcd(a,c)$ we see that

$$\gcd\left(\mu_5+16jks+\frac{2^{\alpha+5}\ell k\gcd(N,\mu)}{16},2^{\alpha+5}k\gcd(N,\mu)\right)$$
$$\leq\gcd\left(\mu_5+16jks+\frac{2^{\alpha+5}\ell k\gcd(N,\mu)}{16},\frac{\mu}{\gcd(N,\mu)}\right)\gcd\left(\mu_5+16jks+\frac{2^{\alpha+5}\ell k\gcd(N,\mu)}{16},2^{\nu+\alpha+2}\right)$$
$$\times\gcd\left(\mu_5+16jks+\frac{2^{\alpha+5}\ell k\gcd(N,\mu)}{16},2^3\gcd(N,\mu)^2\right),$$

where $\gcd(\mu_5+16jks+\frac{2^{\alpha+5}\ell k\gcd(N,\mu)}{16},2^3\gcd(N,\mu)^2)\leq 2^3\gcd(N,\mu)^2\leq 8N^2=O_N(1)$. Thus we are left with bounding the first and second gcd.

We note that $\gcd(a,b)\leq\gcd(xa,b)$ for any $x\in\mathbb{Z}\backslash\{0\}$, which implies

$$\gcd\left(\mu_5+16jks+\frac{2^{\alpha+5}\ell k\gcd(N,\mu)}{16},\frac{\mu}{\gcd(N,\mu)}\right)\leq\gcd\left(3\left(\mu_5+16jks+2^{\alpha+1}\ell k\gcd(N,\mu)\right),\frac{\mu}{\gcd(N,\mu)}\right)$$

and

$$\gcd\left(\mu_5+16jks+\frac{2^{\alpha+5}\ell k\gcd(N,\mu)}{16},2^{\nu+\alpha+2}\right)\leq\gcd\left(2^{\alpha}6\left(\mu_5+16jks+\frac{2^{\alpha+5}\ell k\gcd(N,\mu)}{16}\right),2^{\nu+\alpha+2}\right).$$

Since we have that $\gcd(a,b)=\gcd(a\pmod b,b)$ for $b\neq 0$ we can reduce the value $3\left(\mu_5+16jks+2^{\alpha+1}\ell k\gcd(N,\mu)\right)$ modulo $\frac{\mu}{\gcd(N,\mu)}$ and see that

$$\begin{aligned}
&3\left(\mu_5+16jks+2^{\alpha+1}\ell k\gcd(N,\mu)\right)\\
&=\left(-24n+1+2k^2-3k\right)2^{\alpha+2}\gcd(N,\mu)-24j^2 2^{\nu+\alpha+2}[4]'_{\frac{\mu}{\gcd(N,\mu)}}\left[\frac{N2^\nu}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}}\\
&\quad-24\mu j^2\left[\frac{N\mu}{2^\alpha}\right]'_{2^{\nu+\alpha+2}}\gcd(N,\mu)+3\cdot 2^{\alpha+1}\ell k\gcd(N,\mu)+48jks\\
&\equiv(-24n+1)\,2^{\alpha+2}\gcd(N,\mu)+6j^2 2^{\alpha+2}\left[\frac{N}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}}\\
&\equiv\left[\frac{N}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}}2^{\alpha+2}\left(N\left(24n-1\right)+6j^2\right)\quad\left(\operatorname{mod}\frac{\mu}{\gcd(N,\mu)}\right).
\end{aligned}$$

Analogously we can reduce $2^\alpha 6\left(\mu_5+16jks+2^{\alpha+1}\ell k\gcd(N,\mu)\right)$ modulo $2^{\nu+\alpha+2}$ and see that

$$\begin{aligned}
&2^\alpha 6\left(\mu_5+16jks+2^{\alpha+1}\ell k\gcd(N,\mu)\right)\\
&\quad=\left(-48n+2+4k^2-6k\right)2^{2\alpha+2}\gcd(N,\mu)-48j^2 2^{\nu+2\alpha+2}[4]'_{\frac{\mu}{\gcd(N,\mu)}}\left[\frac{N2^\nu}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}}\\
&\quad\quad-2^\alpha 48\mu j^2\left[\frac{N\mu}{2^\alpha}\right]'_{2^{\nu+\alpha+2}}\gcd(N,\mu)+3\cdot 2^{2\alpha+2}\ell k\gcd(N,\mu)+2^{\alpha+2}24jks\\
&\quad\equiv(-48n+2)\,2^{2\alpha+2}\gcd(N,\mu)+2^\alpha 48\mu\,[\mu]'_{2^{\nu+\alpha+2}}\,j^2\left[\frac{N}{2^\alpha}\right]'_{2^{\nu+\alpha+2}}\gcd(N,\mu)\\
&\quad\equiv\left[\frac{N}{2^\alpha}\right]'_{2^{\nu+\alpha+2}}\gcd(N,\mu)\left(2^{\alpha+2}N\left(48n-2\right)-2^\alpha 48j^2\right)\quad\left(\operatorname{mod}2^{\nu+\alpha+2}\right).
\end{aligned}$$

We now notice that by definition we have $\gcd(\frac{N}{\gcd(N,\mu)},\frac{\mu}{\gcd(N,\mu)})=1$ as well as $\gcd(\frac{N}{2^\alpha},2^{\nu+\alpha+2})=1$. By the definition of the negative modular inverse we thus notice that we also have

$$1=\gcd\left(\frac{N}{\gcd(N,\mu)}\left[\frac{N}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}},\frac{\mu}{\gcd(N,\mu)}\right)=\gcd\left(\left[\frac{N}{\gcd(N,\mu)}\right]'_{\frac{\mu}{\gcd(N,\mu)}},\frac{\mu}{\gcd(N,\mu)}\right),$$

and

$$1=\gcd\left(\frac{N}{2^\alpha}\left[\frac{N}{2^\alpha}\right]'_{2^{\nu+\alpha+2}},2^{\nu+\alpha+2}\right)=\gcd\left(\left[\frac{N}{2^\alpha}\right]'_{2^{\nu+\alpha+2}},2^{\nu+\alpha+2}\right),$$

using that $\gcd(ab,c)=\gcd(a,c)$ if $\gcd(b,c)=1$. This gives us that

$$\gcd\left(3\left(\mu_5+16jks+2^{\alpha+1}\ell k\gcd(N,\mu)\right),\frac{\mu}{\gcd(N,\mu)}\right)$$

$$= \gcd\left(2^{\alpha+2}\left(N\left(24n-1\right)+6j^2\right), \frac{\mu}{\gcd(N,\mu)}\right)$$

and

$$\gcd\left(2^{\alpha}6\left(\mu_5+16jks+2^{\alpha+1}\ell k\gcd(N,\mu)\right), 2^{\nu+\alpha+2}\right)$$
$$= \gcd\left(\gcd(N,\mu)2^{\alpha}\left(4N\left(48n-2\right)-48j^2\right), 2^{\nu+\alpha+2}\right).$$

Overall we therefore obtain

$$\gcd\left(\mu_5+16jks+\frac{2^{\alpha+5}\ell k\gcd(N,\mu)}{16}, 2^{\alpha+5}k\gcd(N,\mu)\right)^{\frac{1}{2}}$$
$$=O_N\left(\gcd\left(2^{\alpha+2}\left(N\left(24n-1\right)+6j^2\right), \frac{\mu}{\gcd(N,\mu)}\right)^{\frac{1}{2}}\gcd\left(\gcd(N,\mu)2^{\alpha}\left(4N\left(48n-2\right)-48j^2\right), 2^{\nu+\alpha+2}\right)^{\frac{1}{2}}\right)$$
$$=O_N\left(\left(2^{\alpha+2}\left(N\left(24n-1\right)+6j^2\right)\right)^{\frac{1}{2}}\left(\gcd(N,\mu)2^{\alpha}\left(4N\left(48n-2\right)-48j^2\right)\right)^{\frac{1}{2}}\right)=O_N\left(n^{\frac{1}{2}}n^{\frac{1}{2}}\right)=O_N\left(n\right).$$

The second result follows analogously. □

V.3 Additional proofs for Section IV.4

We prove the following lemma, giving a bound on the sum defined in (IV.4.20).

Lemma V.3.1. *The sum $K^*_{k,j,N}(n,r,\kappa,\ell)$ defined in* (IV.4.20) *is an incomplete Kloosterman sum and can be bounded by* (IV.1.7) *following the techniques by Lehner* [Leh41, Section 10].

Proof. In the proof of Theorem IV.1.2 we saw that we can rewrite

$$K_{k,j,N}(n,r,\kappa)=\epsilon(k,j,N,r)\sum_{\substack{h \pmod{Gk}\\ \gcd(h,Gk)=1}}\left(\frac{h}{\rho}\right)\exp\left(\frac{2\pi i}{Gk}\left(\mu_*h-\nu_*[h]'_{Gk}\right)\right),$$

where $\mu_*,\nu_*\in\mathbb{Z}$, $G\in\mathbb{N}$, $\rho\in\mathbb{N}$ odd such that all his prime divisors divide Gk, $[h]'_{Gk}$ the negative modular inverse of h modulo Gk, and some $\epsilon(k,j,N,r)=O_N(1)$. Analogously to [Leh41, equation (3.3)] we denote by $\{a,b\}$ the unique real number defined by

$$\{a,b\}\equiv a \pmod{b}, \qquad 0<\{a,b\}\leq b.$$

Since we noted that the extra restriction on k_1 in the sum $K^*_{k,j,N}(n,r,\kappa,\ell)$ constrains the choice of h' to an interval mod k, this gives us that we can rewrite $K^*_{k,j,N}(n,r,\kappa,\ell)$ as

$$K^*_{k,j,N}(n,r,\kappa,\ell)=\epsilon(k,j,N,r)\sum_{\substack{h \pmod{Gk}\\ \gcd(h,Gk)=1\\ 0\leq[h]'_{Gk}<Gk\\ \sigma_1\leq\{[h]'_{Gk},k\}<\sigma_2}}\left(\frac{h}{\rho}\right)\exp\left(\frac{2\pi i}{Gk}\left(\mu_*h-\nu_*[h]'_{Gk}\right)\right),$$

where the last two conditions in the sum mean that $[h]'_{Gk}$ is restricted to G (or possibly $G+1$) intervals whose endpoints are congruent to σ_1, σ_2 modulo k, with $0 \leq \sigma_1 < \sigma_2 \leq k$.

Analouge to [Leh41, page 650] we define $m(s)$ in the interval $(0, k)$ by

$$m(s) := \begin{cases} 1 & \text{for } \sigma_1 \leq s < \sigma_2, \\ 0 & \text{elsewhere in the interval } 0 \leq s < k, \end{cases}$$

and outside of the interval by periodicity. We have

$$m(s) = \sum_{\ell=0}^{k-1} \alpha_\ell \exp\left(2\pi i \frac{s\ell}{k}\right),$$

where

$$\alpha_j = \frac{1}{k} \sum_{s=0}^{k-1} m(s) \exp\left(-2\pi i \frac{sj}{k}\right) = \frac{1}{k} \sum_{s=\sigma_1}^{\sigma_2 - 1} \exp\left(-2\pi i \frac{sj}{k}\right), \qquad \alpha_0 = \frac{\sigma_2 - \sigma_1}{k}.$$

We see that $|\alpha_0| \leq 1$, while for $j \neq 0$

$$|\alpha_j| \leq \frac{2}{k} \left| 1 - \exp\left(-\frac{2\pi i j}{k}\right) \right|^{-1} = \frac{1}{k} \csc\left(\frac{\pi j}{k}\right).$$

Therefore we obtain

$$\sum_{j=0}^{k-1} |\alpha_j| \ll 1 + \sum_{j=1}^{\frac{k}{2}} \frac{1}{j} = O\left(\log(k)\right). \tag{V.3.1}$$

This gives us

$$\begin{aligned} K^*_{k,j,N}(n, r, \kappa, \ell) =& O\left(\sum_{\substack{h \pmod{Gk} \\ \gcd(h, Gk)=1}} m([h]'_{Gk}) \left(\frac{h}{\rho}\right) \exp\left(\frac{2\pi i}{Gk} \left(\mu_* h - \nu_* [h]'_{Gk}\right)\right) \right) \\ =& O\left(\sum_{\ell=0}^{k-1} \alpha_\ell \sum_{\substack{h \pmod{Gk} \\ \gcd(h, Gk)=1}} \left(\frac{h}{\rho}\right) \exp\left(\frac{2\pi i}{Gk} \left(\mu_* h - (\nu_* - \ell)[h]'_{Gk}\right)\right) \right). \end{aligned}$$

In Section IV.3 we saw that the inner sum is bounded by (IV.1.7), which, using (V.3.1), yields

$$K^*_{k,j,N}(n, r, \kappa, \ell) = O\left(n k^{\frac{1}{2}+\varepsilon} \log(k)\right) = O\left(n k^{\frac{1}{2}+\widetilde{\varepsilon}}\right)$$

for $\varepsilon, \widetilde{\varepsilon} > 0$. ◻

V.4 References

[BN19] K. Bringmann and C. Nazaroglu, *A framework for modular properties of false theta functions*, Res. Math. Sci. **6** (2019), no. 3, Paper No. 30, 23 pp.

[Leh41] J. Lehner, *A partition function connected with the modulus five*, Duke Math. J. 8 (1941), 631–655.

[NIST] *NIST Digital Library of Mathematical Functions*. http://dlmf.nist.gov/, Release 1.0.26 of 2020-03-15. F. Olver, A. Olde Daalhuis, D. Lozier, B. Schneider, R. Boisvert, C. Clark, B. Miller, B. Saunders, H. Cohl, and M. McClain, eds.

Appendix

Here, we offer some numerical data related to Table IV.1.

In Chapter IV we defined the coefficients of $\mathcal{A}_{j,N}$ by

$$\mathcal{A}_{j,N}(\tau) = q^{\frac{j^2}{4N}-\frac{1}{24}}\left(a_{j,N}(0) + \sum_{n\geq 1} a_{j,N}(n)q^n\right),$$

which together with

$$\begin{aligned}\mathcal{A}_{j,N}(\tau) =& q^{\frac{j^2}{4N}-\frac{1}{24}}\left(\sum_{n\geq 0} p(n)q^n\right)\left(\sum_{m\geq 0} q^{(Nm+j)m} - \sum_{m\geq 1} q^{(Nm-j)m}\right)\\ =& q^{\frac{j^2}{4N}-\frac{1}{24}}\left(\sum_{n,m\geq 0} p(n)q^{n+(Nm+j)m} - \sum_{n\geq 0}\sum_{m\geq 1} p(n)q^{n+(Nm-j)m}\right)\\ =:& q^{\frac{j^2}{4N}-\frac{1}{24}}\mathcal{B}_{j,N}(\tau)\end{aligned}$$

gives that

$$\mathcal{B}_{j,N}(\tau) = a_{j,N}(0) + \sum_{n\geq 1} a_{j,N}(n)q^n.$$

We compute $\mathcal{B}_{1,3}(\tau)$, $\mathcal{B}_{5,8}(\tau)$ and $\mathcal{B}_{3,10}(\tau)$ up to q^{20} to obtain the values of the first column of Table IV.1 by the following code implemented in Mathematica [Wol17]. Note that throughout the code we use the letter M instead of N. We could not use N, since this letter is reserved in Mathematica.

```
In[1]:= B[q_, j_, M_] := Sum[Sum[PartitionsP[n]
*q^(n + (M*m + j)*m),
{m, 0, 1000}], {n, 0, 1000}] - Sum[Sum[PartitionsP[n]*
q^(n + (M*m - j)*m), {m, 1, 1000}], {n, 0, 1000}]
In[2]:= Series[B[q, 1, 3], {q, 0, 20}]
Out[2]= 1 + q + q^2 + 2 q^3 + 4 q^4 + 5 q^5 + 8 q^6 + 11 q^7
+ 16 q^8 + 22 q^9 + 30 q^10 + 40 q^11 + 55 q^12 + 72 q^13
+96 q^14 + 125 q^15 + 164 q^16 + 210 q^17 + 272 q^18 + 346 q^19
+ 442 q^20 + O[q]^21
In[3]:=Series[B[q, 5, 8], {q, 0, 20}]
```

```
Out[3]= 1 + q + 2 q^2 + 2 q^3 + 4 q^4 + 5 q^5 + 8 q^6 + 10 q^7
+ 15 q^8 + 19 q^9 + 27 q^10 + 34 q^11 + 47 q^12 + 60 q^13
+ 80 q^14 + 101 q^15 + 133 q^16 + 167 q^17 + 216 q^18
+ 270 q^19 + 345 q^20  + O[q]^21
In[4]:= Series[B[q, 3, 10], {q, 0, 20}]
Out[4]= 1 + q + 2 q^2 + 3 q^3 + 5 q^4 + 7 q^5 + 11 q^6 + 14 q^7
+ 21 q^8 + 28 q^9 + 39 q^10 + 51 q^11 + 70 q^12 + 91 q^13
+121 q^14 + 156 q^15 + 204 q^16 + 260 q^17 + 336 q^18
+ 424 q^19 + 541 q^20 + + O[q]^21
```

Next we implement the coefficients in the way presented in (IV.4.26), where the sum over k runs up to J. Therefore we first implement the multiplier defined in (IV.2.6), a function $u(n)$ that constructs a list of coprime elements to a given integer n, a function $H(k,j)$ that gives the j-th coprime element to k (from the list $u(k)$), and the Kloosterman sum as defined in (IV.1.6). Note that here we use the letter t instead of κ.

```
In[5]:= X[j_, r_, M_, h_, k_] := X[j, r, M, h, k] =
If[Mod[k, 2] == 1, KroneckerSymbol[-h, k]*
Exp[((Pi*I)/(12))*((-ModularInverse[h, k] - h)*
k - ((h*(-ModularInverse[h, k]) + 1)/k)*h*(k^2 - 1) - 3*k)],
KroneckerSymbol[k, -h]*
Exp[((Pi*I)/(12))*(-ModularInverse[h, k]*k*(1 - h^2) -
h*(-((h*(-ModularInverse[h, k]) + 1)/k) - k + 3) - 3)]]*
Exp[(3*Pi*I)/4]*Sqrt[2/(M*k)]*Sum[Exp[((Pi*I)/(2*
M*(-k)))*(-h*(2*M*l + j)^2 + (-ModularInverse[h, k])*r^2)]*
Sin[(Pi*r*(2*M*l + j))/(M*k)], {l, 0, k - 1}]
In[6]:= u[n_Integer] := u[n] = With[{l = Range[n]},
Pick[l, CoprimeQ[l, n]]]
In[7]:= H[k_, j_] := H[k, j] = Extract[u[k], j]
In[8]:= K[k_, j_, M_, n_, r_, t_] := K[k, j, M, n, r, t] =
If[k == 1, X[j, r, M, 0, 1]*Exp[(1/(24))*(2*Pi*
I*((24*M*(t + (r/(2*M)))^2 - 1)*(-ModularInverse[0, 1]) -
24*(n + (j^2/(4*M)) - 1/(24))*0))], Sum[X[j, r, M, H[k, b], k]*
Exp[(1/(24*k))*(2*Pi*I*((24*M*(t + (r/(2*M)))^2 -
1)*(-ModularInverse[H[k, b], k]) -
24*(n + (j^2/(4*M)) - 1/(24))*H[k, b]))], {b, 1, EulerPhi[k]}]]
In[9]:= a[n_, j_, M_, J_] := a[n, j, M, J]
= -((2*Pi*I)/(Sqrt[n + (j^2/(4*M)) - (1/(24))]))
*Sum[Sum[(K[k, j, M, n, r, 0]/(k^2))
*NIntegrate[Sqrt[(1/(24)) - M*x^2]
```

```
*Cot[Pi*(-(x/k) + (r/(2*M*k)))]
*BesselI[1, ((4*Pi*Sqrt[n + (j^2/(4*M)) - (1/(24))])/k)*
Sqrt[(1/(24)) - M*x^2]], {x, -Sqrt[1/(24*M)], r/(2*M),
Sqrt[1/(24*M)]}, Method -> "PrincipalValue",
WorkingPrecision -> 120], {r, 1, Ceiling[Sqrt[M/6] - 1]}], {k,
1, J}] - ((2*Pi*I)/(Sqrt[n + (j^2/(4*M)) - (1/(24))]))*
Sum[Sum[(K[k, j, M, n, r, 0]/(k^2))*NIntegrate[
Sqrt[(1/(24)) - M*x^2]*Cot[Pi*(-(x/k) + (r/(2*M*k)))]*
BesselI[1, ((4*Pi*Sqrt[n + (j^2/(4*M)) - (1/(24))])/k)*
Sqrt[(1/(24)) - M*x^2]], {x, -Sqrt[1/(24*M)],
Sqrt[1/(24*M)]}, WorkingPrecision -> 120], {r,
Ceiling[Sqrt[M/6]], M- 1}], {k, 1,
J}] - ((2*Pi*I)/(Sqrt[n + (j^2/(4*M)) - (1/(24))]))*
Sum[Sum[Sum[(K[k, j, M, n, r, t]/(k^2))
*NIntegrate[Sqrt[(1/(24)) - M*x^2]*
Cot[Pi*(-(x/k) + (t/k) + (r/(2*M*k)))]*
BesselI[1, ((4*Pi*Sqrt[n + (j^2/(4*M)) - (1/(24))])/k)*
Sqrt[(1/(24)) - M*x^2]], {x, -Sqrt[1/(24*M)],
Sqrt[1/(24*M)]}, WorkingPrecision -> 120], {t, 1,
k - 1}], {r, 1, M- 1}], {k, 1, J}]
```

We obtain the following results for the coefficients $a_{1,3}(n)$, $a_{5,8}(n)$, and $a_{3,10}(n)$ for $n \in \{3, 10, 18\}$ and $J \in \{1, 3, 20, 25, 50\}$, which fill the rest of Table IV.1.

```
In[10]:= N[a[3, 1, 3, 1], 20]
Out[10]= 2.3181245751167808453
In[11]:= N[a[3, 1, 3, 3], 10]
Out[11]= 2.288642013 + 0.*10^-10 I
In[12]:= N[a[3, 1, 3, 20], 10]
Out[12]= 2.099000692 + 0.*10^-10 I
In[13]:= N[a[3, 1, 3, 25], 10]
Out[13]= 2.087549722 + 0.*10^-10 I
In[14]:= N[a[3, 1, 3, 50], 10]
Out[14]= 2.052693607 + 0.*10^-10 I
In[15]:= N[a[10, 1, 3, 1], 10]
Out[15]= 29.89888333
In[16]:= N[a[10, 1, 3, 3], 10]
Out[16]= 30.24415241 + 0.*10^-9 I
In[17]:= N[a[10, 1, 3, 20], 10]
Out[17]= 30.08657561 + 0.*10^-9 I
```

```
In[18]:= N[a[10, 1, 3, 25], 10]
Out[18]= 30.07892325 + 0.*10^-9 I
In[19]:= N[a[10, 1, 3, 50], 10]
Out[19]= 30.04183108 + 0.*10^-9 I
In[20]:= N[a[18, 1, 3, 1], 10]
Out[20]= 271.3098369
In[21]:= N[a[18, 1, 3, 3], 10]
Out[21]= 272.2656084 + 0.*10^-8 I
In[22]:= N[a[18, 1, 3, 20], 10]
Out[22]= 272.0719934 + 0.*10^-8 I
In[23]:= N[a[18, 1, 3, 25], 10]
Out[23]= 272.0650969 + 0.*10^-8 I
In[24]:= N[a[18, 1, 3, 50], 10]
Out[24]= 272.0407998 + 0.*10^-8 I

In[25]:= N[a[3, 5, 8, 1], 10]
Out[25]= 2.519680370
In[26]:= N[a[3, 5, 8, 3], 10]
Out[26]= 2.220002095 + 0.*10^-10 I
In[27]:= N[a[3, 5, 8, 20], 10]
Out[27]= 1.999336893 + 0.*10^-10 I
In[28]:= N[a[3, 5, 8, 25], 10]
Out[28]= 1.982974730 + 0.*10^-10 I
In[29]:= N[a[3, 5, 8, 50], 10]
Out[29]= 1.989195022 + 0.*10^-10 I
In[30]:= N[a[10, 5, 8, 1], 10]
Out[30]= 26.26967573
In[31]:= N[a[10, 5, 8, 3], 10]
Out[31]= 26.98533328 + 0.*10^-9 I
In[32]:= N[a[10, 5, 8, 20], 10]
Out[32]= 26.98561400 + 0.*10^-9 I
In[33]:= N[a[10, 5, 8, 25], 10]
Out[33]= 26.99967174 + 0.*10^-9 I
In[34]:= N[a[10, 5, 8, 50], 10]
Out[34]= 26.99908412 + 0.*10^-9 I
In[35]:= N[a[18, 5, 8, 1], 10]
Out[35]= 214.4979032
In[36]:= N[a[18, 5, 8, 3], 10]
Out[36]= 216.0556573 + 0.*10^-8 I
```

```
In[37]:= N[a[18, 5, 8, 20], 10]
Out[37]= 215.9830229 + 0.*10^-8 I
In[38]:= N[a[18, 5, 8, 25], 10]
Out[38]= 215.9893492 + 0.*10^-8 I
In[39]:= N[a[18, 5, 8, 50], 10]
Out[39]= 216.0044062 + 0.*10^-8 I

In[40]:= N[a[3, 3, 10, 1], 10]
Out[40]= 3.162360090
In[41]:= N[a[3, 3, 10, 3], 10]
Out[41]= 3.054430238 + 0.*10^-10 I
In[42]:= N[a[3, 3, 10, 20], 10]
Out[42]= 3.030683577 + 0.*10^-10 I
In[43]:= N[a[3, 3, 10, 25], 10]
Out[43]= 3.022204192 + 0.*10^-10 I
In[44]:= N[a[3, 3, 10, 50], 10]
Out[44]= 2.998494259 + 0.*10^-10 I
In[45]:= N[a[10, 3, 10, 1], 10]
Out[45]= 38.53373501
In[46]:= N[a[10, 3, 10, 3], 10]
Out[46]= 38.99653266 + 0.*10^-9 I
In[47]:= N[a[10, 3, 10, 20], 10]
Out[47]= 39.00798893 + 0.*10^-9 I
In[48]:= N[a[10, 3, 10, 25], 10]
Out[48]= 39.00013349 + 0.*10^-9 I
In[49]:= N[a[10, 3, 10, 50], 10]
Out[49]= 38.99815238 + 0.*10^-9 I
In[50]:= N[a[18, 3, 10, 1], 10]
Out[50]= 334.3940087
In[51]:= N[a[18, 3, 10, 3], 10]
Out[51]= 336.0237112 + 0.*10^-8 I
In[52]:= N[a[18, 3, 10, 20], 10]
Out[52]= 336.0058347 + 0.*10^-8 I
In[53]:= N[a[18, 3, 10, 25], 10]
Out[53]= 336.0254115 + 0.*10^-8 I
In[54]:= N[a[18, 3, 10, 50], 10]
Out[54]= 336.0111158 + 0.*10^-8 I
```

Note that the small imaginary parts occuring in these results are some errors caused by Mathematica [Wol17], since the coefficients $a_{j,N}(n)$ are real for any j, N, and n.

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Declaration

I hereby declare that the article *Bivariate asymptotics for eta-theta quotients with simple poles* [CM21] was jointly written with Dr. Joshua Males and my share of the work amounted to 50%. The article *Asymptotic equidistribution for partition statistics and topological invariants* [CCM21] was jointly written with Dr. William Craig and Dr. Joshua Males and my share of the work amounted to 33%. The article *Fourier Coefficients of Weight Zero Mixed False Modular Forms* [Ces23] was written by myself as sole author and is 100% my own work.

Köln, den 21. April 2023

G. Cesana

Giulia Cesana

Erklärung

(Gemäß §7 Absatz (8) Satz 1 der Promotionsordnung vom 12. März 2020)

Hiermit versichere ich an Eides statt, dass ich die vorliegende Dissertation selbstständig und ohne die Benutzung anderer als der angegebenen Hilfsmittel und Literatur angefertigt habe. Alle Stellen, die wörtlich oder sinngemäß aus veröffentlichten und nicht veröffentlichten Werken dem Wortlaut oder dem Sinn nach entnommen wurden, sind als solche kenntlich gemacht. Ich versichere an Eides statt, dass diese Dissertation noch keiner anderen Fakultät oder Universität zur Prüfung vorgelegen hat; dass sie - abgesehen von unten angegebenen Teilpublikationen und eingebundenen Artikeln und Manuskripten - noch nicht veröffentlicht worden ist sowie, dass ich eine Veröffentlichung der Dissertation vor Abschluss der Promotion nicht ohne Genehmigung des Promotionsausschusses vornehmen werde. Die Bestimmungen dieser Ordnung sind mir bekannt. Darüber hinaus erkläre ich hiermit, dass ich die Ordnung zur Sicherung guter wissenschaftlicher Praxis und zum Umgang mit wissenschaftlichem Fehlverhalten der Universität zu Köln gelesen und sie bei der Durchführung der Dissertation zugrundeliegenden Arbeiten und der schriftlich verfassten Dissertation beachtet habe und verpflichte mich hiermit, die dort genannten Vorgaben bei allen wissenschaftlichen Tätigkeiten zu beachten und umzusetzen. Ich versichere, dass die eingereichte elektronische Fassung der eingereichten Druckfassung vollständig entspricht.

Teilpublikationen:

1. G. Cesana and J. Males, *Bivariate asymptotics for eta-theta quotients with simple poles*, arXiv preprint, (2021), arXiv:2101.10046.
2. G. Cesana, W. Craig, and J. Males, *Asymptotic equidistribution for partition statistics and topological invariants*, arXiv preprint, (2021), arXiv:2111.13766.
3. G. Cesana, *Fourier Coefficients of Weight Zero Mixed False Modular Forms*, arXiv preprint, (2023), arXiv:2303.14025.

Köln, den 21. April 2023

G. Cesana

Giulia Cesana

Curriculum Vitae

Personal Data

Name	Giulia Cesana
Date of Birth	July 29, 1993
Place of Birth	Cologne
Nationality	German

Contact Information

Email	gcesana@math.uni-koeln.de
Homepage	https://sites.google.com/view/giuliacesana

Education

2019 – 2023	Ph.D. candidate at University of Cologne Title: *On the Asymptotic Behavior of Modular Forms and Related Objects* Advisor: Prof. Dr. Kathrin Bringmann
2016 – 2019	Master of Science Title: *Das regularisierte Petersson-Skalarprodukt und die Verbindung zur Riemannschen Zeta-Funktion* Advisor: Prof. Dr. Kathrin Bringmann at the University of Cologne
2012 – 2016	Bachelor of Science Title: *Das Monge-Kantorovich Transportproblem* Advisor: Prof. Dr. Juan J. L. Velázquez at the Rheinische Friedrich-Wilhelms-Universität Bonn
2012	Abitur (Allgemeine Hochschulreife), Grade 1.7 Deutzer Gymnasium Schaurtestraße, Cologne

Employment

2019 – 2023	University of Cologne Wissenschaftliche Mitarbeiterin (Scientific assistant) Advisor: Prof. Dr. Kathrin Bringmann

Research Interests

Number theory
Modular (and Jacobi) forms, partitions and their statistics and applications, generating functions and asymptotics of Fourier coefficients

Publications

submitted	G. Cesana, *Fourier Coefficients of Weight Zero Mixed False Modular Forms*
submitted	K. Benli, G. Cesana, C. Dartyge, C. Dombrowsky, and L. Thompson, *Sums of proper divisors with missing digits*
submitted	G. Cesana, W. Craig, and J. Males, *Asymptotic equidistribution for partition statistics and topological invariants*
submitted	G. Cesana and J. Males, *Bivariate asymptotics for eta-theta quotients with simple poles*

Co-Advising

Summer 2020	Bachelorthesis of David Wambach Title: *Ramanujan's Definition der Mock-Thetafunktionen*

Teaching

Summer 2023	Seminar on modular forms (Prof. Dr. Kathrin Bringmann) Undergraduate/graduate course
	Analytic number theory (Dr. Johann Franke) Graduate course
Winter 2022	Algebra (Prof. Dr. Sander Zwegers) Undergraduate/graduate course
Oct. – Nov. 2022	Mathematics 1 for teaching (PD Dr. Michael Mertens) Undergraduate course
Summer 2022	Elementary number theory (Prof. Dr. Kathrin Bringmann) Undergraduate/graduate course
Winter 2021	Algebra (Prof. Dr. Kathrin Bringmann) Undergraduate/graduate course
Summer 2021	Linear algebra 2 (PD Dr. Stephan Ehlen) Undergraduate course

Winter 2020	Linear algebra 1 (PD Dr. Stephan Ehlen) Undergraduate course
Summer 2020	Seminar on modular forms (Prof. Dr. Kathrin Bringmann) Undergraduate/graduate course
	Linear algebra 2 (Prof. Dr. Peter Littelmann) Undergraduate course
Winter 2019	Linear algebra 1 (Prof. Dr. Peter Littelmann) Undergraduate course

Conferences, Workshops, and Research Visits

Dec. 2022	Luxembourg Number Theory Day 2022[1], Luxembourg
Sep. 2022	ABKLS, Cologne
Aug. 2022	Women in Numbers Europe - 4, Utrecht
Jul. 2022	Ramanujan and Euler: partitions, mock theta functions, and q-series[1]
Jun. 2022	ABKLS, Bonn
Mar. 2022	Young Scholars in the Analytic Theory of Numbers and Automorphic Forms, Bonn
Mar. 2022	Gehaltsverhandlungen beim Jobeinstieg für Frauen[1] (Salary negotiations when starting a job for women), Cologne
Sep. 2021	Women in automorphic forms, Bielefeld
Jun. 2021	Grundlagen der Kommunikation – Freundlich und durchsetzungsstark in jeder Lebenslage[1] (Basics of communication – friendly and assertive in every situation), Cologne
May 2021	Weniger Stress durch mehr Selbstmanagement – Gesunder Umgang mit Stress, Ärger und Hürden des Arbeitsalltags in Studium und Promotion[1] (Less stress through more self-management – healthy handling of stress, anger and hurdles of everyday work in studies and doctorates), Cologne
May 2021	Online Hausdorff school on "The Circle Method: Entering its Second Century"[1], Bonn
Feb. 2020	ABKLS, Aachen

[1]Denotes an online conference/seminar.

Sep. 2019	ABKLS, Cologne
Oct. 2017	ABKLS, Cologne

Professional Activities

Oct. 2022	Co.-Organizer ENTR-Workshop (Early Number Theory Researchers Workshop) jointly with Lars Kleinemeier (University of Bielefeld), Ingmar Metzler (TU Darmstadt), Dr. Andreas Mono (University of Cologne), and Dr. Riccardo Zuffetti (TU Darmstadt) in Darmstadt
Mar. 2022	Lecturer Cologne Children's University Workshop on kryptographie jointly with Dr. Christina Röhrig for 20 schoolchildren
Summer 2022	Co.-Organizer ENTR Seminar[1] (Early Number Theory Researchers Seminar) jointly with Ingmar Metzler (TU Darmstadt) and Dr. Andreas Mono (University of Cologne)
since 2021	Referee Work Research in Number Theory

Invited Talks

Feb. 2023	International Seminar on Automorphic Forms[2] Darmstadt, Germany
Apr. 2022	NAAPing Class Group[2] Working group on Number theory, Arithmetic and Algebraic geometry, and Physics Tokyo, Japan
Mar. 2022	Young Scholars in the Analytic Theory of Numbers and Automorphic Forms Bonn, Germany

Contributed Talks

Sep. 2021	Women in automorphic forms Speed Talk, Bielefeld, Germany

[2]Denotes an online talk.

Jun. 2021	CDE seminar[2] Speed Talk, Early career online seminar on automorphic forms joining Cologne, Darmstadt, and ETH Zürich

Other Skills

Programming	Basic C++
Languages	Native German, English (B2/C1), Italian (A1)
Math. Software	LaTeX, Basic Mathematica
Others	Basic Microsoft Office

Cologne, April 21, 2023