On some percolation problems in correlated systems

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Abstract

In this thesis we explore the framework of the percolation theory and we analyse two models. We investigate the level set of the Gaussian free field on a supercritical Galton–Watson tree conditioned on non-extinction with random conductances, showing that the critical parameter h_* is deterministic and strictly positive, that the level set contains almost surely a transient component for some $h > 0$ and it is stable under perturbation via small quenched noise.

Then we study an infection model with recovery on fractal graphs as the Sierpiński gaskets and carpets and show the survival of the infection for small recovery parameter. To prove the result, we generalize the concept of Lsipschitz surface for the lattice to fractal graphs, and we show the existence and certain connectivity properties of what we call a *Lipschitz cutset*.

Contents

Chapter I

Introduction

In this introduction, we intend to present the framework of the widely studied *percolation theory* and state the first results in the field, which will allow us to consider more complex models. In the first and easiest formulations, there is no correlation involved between the variables, contrary to the two models which we will consider in the following chapters. Let us start presenting the well-known Bernoulli percolation.

I.1 Background

In the field of percolation theory, one usually considers a locally finite graph. We can start by assuming the graph to be deterministic, and in the first case we will restrict to the integer lattice $\mathbb{Z}^d$. The graph $(\mathbb{Z}^d, E)$ is defined as the set of vertices in $\mathbb{Z}^d$ with two points x, y sharing an edge if $|x-y|_1 = 1$, and we say that x and y are neighbors or equivalently $x \sim y$ if they share an edge. Successively, in the classical *site percolation* the vertices in a random subset $\mathcal{O} \subseteq \mathbb{Z}^d$ are declared *open*; in *bond percolation* the edges in a random subset $\mathcal{O}' \subseteq E$ are declared open. One may try to understand various properties of those subsets $\mathcal{O}$ or $\mathcal{O}'$, first of all whether they contain an unbounded and connected component.

The first model introduced in [BH57] deal with *bond percolation* on $(\mathbb{Z}^d, E)$. One defines on some probability space a family of Bernoulli random variables $(B_e)_{e\in E}$ which are independent and identically distributed with some parameter $p \in [0,1]$ under some probability measure $\mathbb{P}_p$, and declares an edge e *open* if the Bernoulli variable on the bond satisfies $\{B_e = 1\}$. One may wonder whether it is possible to find an unbounded cluster inside $\mathcal{O} := \{e \in E \colon B_e = 1\}$, as the parameter p of the model varies. It is easy to verify that the probability of finding such unbounded cluster is non-decreasing in p: for example one could couple the Bernoulli variables with a family of uniform random variables in $[0,1]$ so that the set $\mathcal{O}$ is itself non-decreasing in p. We can hence define

$$p_* := \inf\big\{p \colon \mathbb{P}_p(\mathcal{O} \text{ contains an infinite cluster}) > 0\big\}. \tag{I.1.1}$$

In the case $p < p_*$ there exists no unbounded cluster a.s. and this is usually called the *subcritical phase*, while the *supercritical phase* corresponds to $p > p_*$. The critical parameter in various models is an object of high interest and one tries to gather information about its value or the behavior of the system in the two phases.

While it is trivial to see that in dimension one the problem is not interesting as one immediately gets $p_* = 1$, the problem is harder in higher dimensions. For bond percolation in dimension $d = 2$, first Harris in [Har60] showed the inequality $p_* \geqslant \frac{1}{2}$, and later Kesten in [Kes80] obtained the equality $p_* = \frac{1}{2}$. The latter article uses a property of the lattice valid in two dimensions, namely the self duality. Given a planar graph, one can define the dual graph where the vertices corresponds to the "faces" of the original graph and the edges between two vertices are drawn if the corresponding faces share an edge. The lattice $\mathbb{Z}^2$ has the key property of being its own dual graph, providing one of the few cases where the critical parameter is known exactly.

In higher dimensions, or different graphs, or even the site percolation, one is interested in proving the weaker result of existence of the phase transition, which in the case of independent Bernoulli percolation translates to $p_* \in (0,1)$. A classical result for various models achieved through 0-1 laws or ergodic theory is

$$\mathbb{P}_p(\mathcal{O} \text{ contains an infinite cluster}) \in \{0,1\}. \tag{I.1.2}$$

The previous equation in the case in which $p_* < \frac{1}{2}$ gives rise to phenomenon of *coexistence* for $p_* < p < 1 - p_*$, meaning that is possible to find two unbounded clusters, one in $\mathcal{O}$ and one in its complement $\mathcal{O}^c$.

Other natural questions deal with the uniqueness of the unbounded cluster in the supercritical phase $p > p_*$, or the tails of the distribution of the size of the connected component containing the origin, $\mathbb{P}(|\mathcal{C}(0)| > k)$, where $\mathcal{C}(0)$ denote the connected component of $\mathcal{O}$ containing 0. Other quantities of interest are the *critical exponent* near the critical regime, i.e. the exponent of $|p - p_*|$ in the asymptotic behavior of $\mathbb{E}[|\mathcal{C}(0)|]$ as $p \uparrow p_*$ and of $\mathbb{P}(|\mathcal{C}(0)| = \infty)$ as $p \downarrow p_*$. For more details we refer to the monograph [Gri99].

Independent percolation on lattices was the first model explored, but numerous variations have been studied. Is possible to consider different graphs: for instance graphs with conductances to obtain weighted graphs or random graphs, or one can consider the more challenging problem of considering various type of correlations between the sites. In this thesis we will consider weighted graphs, the supercritical Galton–Watson tree (cf. Subsection II.2.1) and subdiffusive graphs (cf. Subsection I.1.4), and involve correlations through the Gaussian free field (cf. next subsection and Subsection II.2.3) and Poisson random walks (cf. Subsection III.2.4).

I.1.1 Gaussian free field

The first model we consider is a Gaussian field with long range correlations, the Gaussian free field. The first result for the percolation of the associated level set was proven in [BLM87], and from then it has caught a lot of attention. In particular a lot of results have been obtained thanks to the relation first shown in [Szn12a] with an other object called random interlacements. The major obstacle when treating the Gaussian free field resides in the presence of correlations of the variables, so that various techniques that work for independent systems need to be improved or substituted.

We consider a graph (G, E) and maintain the notation $x \sim y$ for adjacent vertices, i.e. $(x, y) \in E$, and we consider the weights $\lambda := (\lambda_{x,y})$, $x, y \in G$ with the assumptions $\lambda_{x,y} = \lambda_{y,x}$ and $\lambda_{x,y} > 0$ if $x \sim y$ and 0 otherwise, we define for $x \in G$ $\lambda_x := \sum_{y \sim x} \lambda_{x,y}$ and we call (G, λ) a weighted graph. Let $(X_k)_{k \in \mathbb{N}}$ be a random walk on the graph starting in x_0 under some measure P_{x_0} with transition rates from $x \in G$ to $y \sim x$ given by $\frac{\lambda_{x,y}}{\lambda_x}$ and we assume (G, λ) to be transient. We can hence define the Green function as

$$g(x,y) := \frac{1}{\lambda_y} E_x\Big[\sum_{k=0}^{\infty} \mathbf{1}_{\{X_k=y\}}\Big] = \frac{1}{\lambda_y}\sum_{k=0}^{\infty} P_x(X_k = y), \tag{I.1.3}$$

i.e. the expected number of visits in the point y starting from x. The assumption of transience assures the finiteness of $g(x,y)$, which is proven to be symmetric and positive definite. Hence we can define the Gaussian free field $(\varphi_x)_{x\in G}$ as the Gaussian field under the measure $\mathbb{P}^{\mathrm{G}}$ with

$$\mathbb{E}^{\mathrm{G}}[\varphi_x] = 0$$
$$\mathbb{E}^{\mathrm{G}}[\varphi_x\varphi_y] = g(x,y)$$

for all $x,y \in G$. We mentioned the percolation of the Gaussian free field referring to the percolation of its *level set*: consider for $h \in \mathbb{R}$

$$E^{\geqslant h} := \{x \in G \colon \varphi_x \geqslant h\}.$$

It is natural to see $E^{\geqslant h}$ as the set $\mathcal{O}$ of the previous subsection, that means declaring a site x “open” when the Gaussian free field in x is bigger than h. A often used metaphor to visualize consist in the analogy between a realization of the field and a landscape, seeing the random set $E^{\geqslant h}$ as the “land above see level h”. As for the case of Bernoulli independent percolation one defines the critical parameter

$$h_* := \sup\big\{h \in \mathbb{R} \colon \mathbb{P}^{\mathrm{G}}(E^{\geqslant h} \text{ contains an unbounded cluster}) > 0\big\}.$$

The analogy with the landscape allows immediately to see that, contrary to the Bernoulli case, $h < h_*$ corresponds to the supercritical phase, while $h > h_*$ to the subcritical phase. Results about the critical parameter on $\mathbb{Z}^d$ — with $d \geqslant 3$ to assure transience — include $h_* \geqslant 0$ from [BLM87], finiteness and asymptotics for large d in [RS13b] which in particular gives the occurrence of a phase transition. On the lattice $\mathbb{Z}^d$ it holds $h_* > 0$ from [DPR18b], and the strict positivity was also shown for $d-$regular trees in [Szn16] and for Galton–Watson trees whose mean offspring size satisfies $m > 2$ in [AS18].

Other critical parameters have been introduced. The value $\bar{h}$ determines the *strong supercritical* phase. It was introduced in [DRS14b] and proven to be positive in [DPR18a] — giving a fortiori $h_* > 0$ — for a large class of graphs which satisfy certain conditions on the volume growth, the random walk dimension and isoperimetric condition (we refer to Subsection I.1.4 for precise definitions of those assumptions). An other critical value, h_{**} determines the strong subcritical phase, and was shown in [DC+20] that the three critical parameters actually coincide. This equality implies for example, that as soon as we are in the subcritical phase $h > h_{**} = h_*$, clusters are exponentially small, and that in the supercritical phase $h < \bar{h} = h_*$ clusters are locally connected, which on a heuristic levels means that two non-small clusters in some ball belongs to the same cluster in a larger ball (we refer to [DPR18a] for a precise definition of the strong supercritical phase).

A lot of the proofs of those results made strong use of an other object which shares a relation with the Gaussian free field, specifically an isomorphism theorem. Let us then introduce it.

I.1.2 Random interlacements

The process of *random interlacements* was introduced by Sznitman in [Szn10] and later generalized to transient graphs in [Tei09]. We give a short definition here of the process, and refer to Section II.2.4 and the monograph [DRS14a] for more details.

Given a transient weighted graph (G, λ), one can define the space of doubly infinite nearest-neighbor trajectories

$$\overleftrightarrow{Z} := \{\overleftrightarrow{w} : \mathbb{Z} \to G \,\big|\, \overleftrightarrow{w}(k) \sim \overleftrightarrow{w}(k-1) \text{ and } |k \colon \overleftrightarrow{w}(k) = x| < \infty,\ \forall x \in G\}.$$

This set is usually indicated with W, but we stick to the notation of subsection II.2.4, where the letter Z was chosen in place of W as the latter will be used for another object introduced later called "watershed". One can consider the relation $\sim$ in $\overleftrightarrow{Z}$, for which two trajectories are equivalent if one is the time-shifted version of the other, and define the quotient space $Z^* := Z/\sim$. It was proven that there exists a measure ν on Z^* such that, for each $K \subseteq G$ and $x \in K$ the measure ν of trajectories hitting K in x modulo time shift is proportional to the probability that the trajectory is at x at time 0 and never returns to K before time 0, cf. (II.2.14). The random interlacements process ω is defined as the Poisson point process on $Z^* \times \mathbb{R}^+$ with intensity measure $\nu \otimes \lambda$, where λ is the 1-dimensional Lebesgue measure, and the random interlacements set $\mathcal{I}^u$ is the set of vertices in G visited by the trajectories in ω which have label in $(0, u)$.

From the point of view of percolation, the set $\mathcal{I}^u$ is not interesting, since on any transient graph it contains a infinite connected component for every value of u since random walk trajectories are connected. It is interesting however to observe the complement $\mathcal{V}^u$, the so-called *vacant set*. Defining the critical parameter u^* as the smallest value for which every component of $\mathcal{V}^u$ is bounded, it was first shown in [SS09] that $u^* \in (0, \infty)$ on the lattice $\mathbb{Z}^d$, $d \geqslant 3$, and a later a shorter proof was presented in [Rát15]. The case of Galton–Watson trees has been treated by [Tas10], which showed that the critical parameter u^* is deterministic, non-trivial and an explicit formula was there provided.

We mentioned already a connection between the Gaussian free field and random interlacements: it consists of a Ray–Knight type isomorphism theorem, first shown in [Szn12a].

It states that for all $x \in G$ and $u \in \mathbb{R}^+$

$$\frac{1}{2}\varphi_x^2 + \ell_{x,u} \text{ has the same law as } \frac{1}{2}\big(\varphi_x - \sqrt{2u}\big)^2$$

where $\ell_{x,u}$ is the occupation time of the random interlacements at level u and φ_x is a Gaussian free field independent of the random interlacements process. A strong improvement of the theorem was the generalization to a *continuous* structure built around the graph, the so-called *cable system*, started by [Lup16] and improved in [Szn16]. Although we will not use explicitly the cable system in Chapter II, it is actually necessary and a key ingredient for the use of the isomorphism theorem.

The cable system is defined as follows. Given a weighted graph (G, λ), to each edge $\{x, y\}$ corresponds a compact interval $I_{x,y} \subseteq \mathbb{R}$ of length $\frac{1}{2\lambda_{x,y}}$, where the endpoints of the interval are identified and glued to x and y. The obtained continuous metric structure is denoted with $\widetilde{G}$ and it is possible to define the Gaussian free field $(\widetilde{\varphi}_z)_{z\in\widetilde{G}}$ and the random interlacements $\widetilde{\omega}$ on the cable system. For a precise construction we refer to [Lup16], but we provide a quick intuition on the method. One can define a diffusion $\widetilde{X}$ on $\widetilde{G}$, whose restriction to G behaves like a continuous time random walk and inside the cables $I_{x,y}$ like a Brownian motion. The field $(\widetilde{\varphi}_z)_{z\in\widetilde{G}}$ is then defined as the Gaussian field with covariance $\widetilde{g}(z, w)$, which is the Green function associated to $\widetilde{X}$; similarly the random interlacements process $\widetilde{\omega}$ is the Poisson point process of trajectories distributed as $\widetilde{X}$ modulo time shifts; the restriction of $(\widetilde{\varphi}_z)_{z\in\widetilde{G}}$ to G has the law of $(\varphi_x)_{x\in G}$. Alternatively one can construct $(\widetilde{\varphi}_z)_{z\in\widetilde{G}}$ from $(\varphi_x)_{x\in G}$ adding to edges $\{x,y\}$ independent Brownian bridges between φ_x and φ_y. Finally it has been shown that the isomorphism holds for the cable system: there exists a coupling between $\widetilde{\varphi}_z$ and $\widetilde{\omega}$ such that for all $z\in\widetilde{G}$ and $u>0$

$$\frac{1}{2}\widetilde{\varphi}_z^2+\widetilde{\ell}_{z,u}=\frac{1}{2}\big(\widetilde{\varphi}_z-\sqrt{2u}\big)^2. \tag{I.1.4}$$

Using equation (I.1.4) we can easily infer the result $h_*\geqslant 0$: for each $u>0$ the set $\{z\colon \widetilde{\ell}_{z,u}>0\}$ contains an unbounded connected component and thus there exists an unbounded connected component of $\{z\in\widetilde{G}\colon \widetilde{\varphi}_z-\sqrt{2u}\neq 0\}$. By continuity in z of $\widetilde{\varphi}_z$ and since (φ_x) is the restriction of $\widetilde{\varphi}_z$ to G, we can find for each $u>0$ an unbounded cluster either in $\{x\in G\colon \varphi_x>\sqrt{2u}\}$ or in $\{x\in G\colon \varphi_x<\sqrt{2u}\}$, and by symmetry both imply $h_*\geqslant 0$.

One can define the critical parameter $\widetilde{h}_*$ for the percolation of the level set of the Gaussian free field $(\widetilde{\varphi}_z)$ on the cable system in the same way of h_* for (φ_x). By restriction, it is immediate to see that $\widetilde{h}_*\leqslant h_*$, and the same argument actually shows $\widetilde{h}_*\geqslant 0$. It was shown in [Lup16] that on $\mathbb{Z}^d$ it holds $\widetilde{h}_*=0$ (in contrast with $h_*>0$ as already mentioned). In [DPR22] a rather weak condition named (Cap) is provided for $\widetilde{h}_*\leqslant 0$ to hold. In [Pré23], Prévost shows that (Cap) is not necessary for $\widetilde{h}_*\leqslant 0$ and gives an example of tree with exponentially small conductances where actually $\widetilde{h}_*=\infty$.

Assuming that $\widetilde{h}_*=0$, so that every component of $\{z\in\widetilde{G}\colon \widetilde{\varphi}_z>\sqrt{2u}\}$ is bounded for any $u>0$, then the same argument below (I.1.4) gives that $\widetilde{\mathcal{I}}^u\subseteq\{z\in\widetilde{G}\colon \widetilde{\varphi}_z<\sqrt{2u}\}$ and, by taking complements and restricting to the graph, $\{x\in G\colon \varphi_x>\sqrt{2u}\}\subseteq\mathcal{V}^u$, which gives

$$h_*\leqslant\sqrt{2u_*}.$$

This inequality is actually strict in the context of trees, as proven in [Szn16] and [AS18].

We mentioned that the first result about u_*, in particular its non-triviality for $\mathbb{Z}^d$, $d\geqslant 3$ was proven in [SS09] using a renormalization scheme. Those arguments, also known as *multi-scale* have been proven useful in a variety of situations.

I.1.3 Multi-scale arguments

In various works a *multi-scale* strategy has been adopted. The various proofs of course differ from each other in the details, but we can sketch a general overview of the similarities between very different models. In the case of the lattice $\mathbb{Z}^d$, a *coarse-grained* (or equivalently a *renormalization*) approach consist, generally speaking, in the partitioning of the graph in various "boxes" of some fixed side length, and requiring some properties of the boxes: this allows to classify boxes as either "good" or "bad". If the probability of being "good" is large enough — where enough depends on the model and the geometry— one should be able to recover a macro-structure of the "good" boxes which allows to conclude.

Sometimes a single partition is not enough and it is necessary to define different scales. At each scale, the graph is partitioned in boxes whose side length is given by the scale and each box is then further partitioned in "boxes" of smaller scale. The classification into "good" or "bad" boxes remains, but it is not excluded that the notion or the probability of goodness depend on the scale.

A (non exhaustive) list of multi-scale arguments applied to different models includes the following works: the already mentioned percolation of the vacant set in [SS09], upper bound on the speed and shape theorems for spread of infection of Poisson random walks [KS05; KS06], positive speed of multi-particle diffusion limited aggregation [SS19], existence of phase transition for activated Random Walk [ST17] and classical Bernoulli percolation [CT18].

Multi-scale arguments can become quite involved and technical, and often needs to be recreated tailor-made for the treated model. A new tool, called Lipschitz surface for independent Bernoulli percolation was introduced in [Dir+10] and further deepened in [GH12], and Gracar and Stauffer in [GS19a] extended the Lipschitz surface in the space-time graph $\mathbb{Z}^d \times \mathbb{Z}$ for non-independent percolation of time-dependent processes. The surface is constructed with a multi-scale argument, but is quite robust and can be applied to various frameworks, such as the spread of an infections, choosing a suitable local, increasing and translation invariant event E accordingly to the model A similar structure was constructed by the authors for the torus $\mathbb{T}^d$ in [GS18], however most works in the field concentrate on the lattice graph $\mathbb{Z}^d$.

I.1.4 Subdiffusive graphs

The lattice graph $\mathbb{Z}^d$ has been widely studied and is well understood from the point of view of random walks. The goal of this subsection is to present a class of graphs which differ substantially from the lattice. To this aim, let us recall some properties which might be considered natural or even trivial.

Let $(X_n)_{n\geqslant 0}$ be a discrete time simple random walk on $\mathbb{Z}^d$ starting in $x_0 \in \mathbb{Z}^d$. Some familiar facts about random walks include that

$$\mathbb{E}[\|X_n\|_2^2] = n \text{ for all } d \geqslant 1;$$

denoting with $d(\ ,\ )$ the graph distance, $B_r(x)$ the open ball of radius $r > 0$ and center $x \in \mathbb{Z}^d$, and $H_A := \min\{n \in \mathbb{N}\colon X_n \in A\}$ the hitting time of a set $A \subseteq \mathbb{Z}^d$ (with the usual convention of $\min \varnothing = \infty$), that

$$\mathbb{E}_{x_0}[H_{B_r(x_0)^c}] \asymp r^2 \text{ for all } d \geqslant 1, \tag{I.1.5}$$

where $\asymp$ means that the ratio between the two sides is bounded from above and below by some positive constant independent of the other variables. Equation (I.1.5) can be easily shown stopping the martingale $M_n := (X_n)^2 - n$, $n \geqslant 0$ at $\tau_r := \min\{n\colon d(x_0, X_n) = r\}$.

Furthermore letting $p_n(x,y) := \mathbb{P}_x(X_n = y)$, for $x, y \in \mathbb{Z}^d, n \geqslant 0$ be the transition density, the following Gaussian estimates for $d(x,y) \leqslant n$ are well known (see for example [Woe00, Corollary 13.11])

$$p_n(x,y) \asymp n^{-\frac{d}{2}} \exp\Big(-\frac{d(x,y)^2}{c_1 n}\Big), \tag{I.1.6}$$

where c_1 is some other constant which might differ for the upper and lower bound.

Similar estimates hold true for continuous time random walks and the Brownian motion on $\mathbb{R}^d$. For a metric measure space let $V_r(x)$ the measure of the open ball of radius r and center x. Li and Yau [LY86] showed that on a complete manifold with non-negative Ricci curvature which satisfies

$$V_r(x) \simeq r^\alpha, \tag{I.1.7}$$

and in particular $\mathbb{R}^d$ with $\alpha = d$, the heat kernel $p_t(x,y)$ satisfies

$$p_t(x,y) \simeq t^{-\frac{\alpha}{2}} \exp\Big(-\frac{d(x,y)^2}{ct}\Big), \tag{I.1.8}$$

where $p_t(x,y) := f(t, x-y)$ is there defined in terms of the fundamental solution f of the heat equation $\frac{\partial f}{\partial t} = \Delta f$.

Equations (I.1.5), (I.1.6), (I.1.8) seem to suggest that on the lattice and on the Euclidean space, informally

$$\textit{time} \;\approx\; \textit{space}^2.$$

One may wonder whether this "ratio" is valid on any graph or there are counterexamples where, for example, Gaussian estimates as in (I.1.6) do not hold.

We are now going to consider more general graphs: let (G,λ) be a weighted graph as defined in Subsection I.1.1.

Some results in this direction were first obtained looking at fractals set such as the Sierpiński gasket K in $\mathbb{R}^2$ and the associated fractal graph, the Sierpiński graph $\mathbb{G}^2$ (we refer to Section III.2 for a precise definition). In [Kus87; Gol87; BP88] a diffusion process was constructed via finer and finer approximation of random walks on a fractal lattice. In particular, the transition density $p_t(x,y)$ of the constructed Brownian motion satisfies sub-Gaussian estimates

$$p_t(x,y) \simeq t^{-\frac{d_f}{d_w}} \exp\Big(-\Big(\frac{|x-y|^{d_w}}{ct}\Big)^{1/(d_w-1)}\Big), \tag{I.1.9}$$

for all $x,y \in K, t > 0$ and where $|x-y|$ is the Euclidean distance in K, $d_f = \log_2(3)$ is the Hausdorff dimension of the fractal K and $d_w = \log_2(5)$ is the walk dimension.

Similar result have been obtained for (generalized) Sierpiński carpets $\mathbb{SC}^d(l_F, m_F)$ — we refer to Section III.8 for precise definitions. Those include the construction of Brownian motion in [BB89] on $\mathbb{SC}^2(3,8)$ and sub-Gaussian estimates similar to (I.1.9) in [BB92; BB99a] with appropriate values of the fractal dimension d_v and walk dimension d_w. When the walk dimension satisfies $\mathrm{d}_w > 2$ those estimates exhibit a *sub*-Gaussian behavior; this is the reason why those fractal graphs are often referred to as *sub*diffusive.

Successive works aimed at obtaining transition density estimates for random walks as in (I.1.6) for graphs with a "fractal-line" structure, with the same self similarities properties in a macro level. The following estimates for the heat kernel $p_n(x,y) := \frac{1}{\lambda_y} P_n(X_n = y)$ were obtained for the Sierpiński gasket by [Jon96] with $d_v = \log_2(3), d_w = \log_2(5)$ for all $x,y \in \mathbb{G}^2$ and $n \geqslant d(x,y)$

$$p_n(x,y) \asymp n^{-\frac{d_v}{d_w}} \exp\Big(-\Big(\frac{d(x,y)^{d_w}}{c_2 n}\Big)^{1/(d_w-1)}\Big). \tag{HKB(d_v,d_w)}$$

In [BB99b], equation (HKB(d_v, d_w)) is proven for generalized Sierpiński carpets for all $x,y \in \mathbb{SC}^d(l_F, m_F)$, $n \geqslant d(x,y)$ if n and $d(x,y)$ both odd or both even, for some value d_v and d_w. More general estimates were obtained on fractal graphs in [HK04], including (HKB(d_v, d_w)) for recurrent nested fractal graphs.

Let us define similarly to (I.1.7) the volume dimension d_v for a general graph G as the value — if exists — such that

$$V_r(x) \asymp r^{d_v}, \text{ for all } x \in G, r > 0 \tag{V(d_v)}$$

where $V_r(x)$ is the measure of a ball of radius r and center $x \in G$ with respect to the graph distance, i.e. $V_r(x) = \lambda(B_r(x)) := \sum_{y \in B_r(x)} \lambda_y$. The walk dimension d_w, similarly to (I.1.5), is the value —if exists — such that the mean exit time from a ball satisfies

$$\mathbb{E}[H_{B_r(x)^c}] \asymp r^{d_w}, \text{ for all } x \in G, r > 0. \tag{E(d_w)}$$

While showing the existence of a value d_v as in (V(d_v)) is easily done not much differently from evaluating the Hausdorff dimension of the fractal (see Section III.B for a proof in the Sierpiński graph $\mathbb{G}^d$), the existence of a value d_w is proven through the construction of the Brownian motion and follows from (I.1.9). In [GY18] they instead evaluated the walk dimension d_w of $\mathbb{G}^d$ with an alternative method without using the diffusion. Hence, in view of (I.1.9), (HKB(d_v, d_w)), (E(d_w)), we can observe at an informal level

$$\textit{time} \;\approx\; \textit{space}^{d_w}.$$

Equations (HKB(d_v, d_w)), (V(d_v)) and (E(d_w)) are clearly related and with the use of the same notation d_v and d_w we silently suggested their correspondence. Before explicitly stating this equivalence and some other results, we introduce further conditions on the graph G.

The first assumption for a weighted graph (G, λ) we consider is the following: there exists $p_0 > 0$

$$p(x, y) := \frac{\lambda_{xy}}{\lambda_x} \geqslant p_0, \quad \text{for all } x \sim y. \tag{p_0}$$

We say that (G, λ) satisfies a *volume doubling* condition if there exists C_1 such that for all $x \in \mathbb{G}$ and $R > 0$

$$V_{2R}(x) \leqslant C_1 V_R(x). \tag{VD}$$

Recalling the definition of the Green function in (I.1.3) consider the condition

$$g(x, y) \asymp (d(x, y))^{-d_v + d_w}, \;\; x \neq y \in \mathbb{G}. \tag{G($d_v - d_w$)}$$

Recalling that a function $h \colon \mathbb{G} \to \mathbb{R}$ is *harmonic* on $A \subseteq \mathbb{G}$ if $\Delta h(x) = 0$ for all $x \in A$, where

$$\Delta h(x) := \frac{1}{\lambda_x} \sum_{y \sim x} \lambda_{xy}(h(y) - h(x)),$$

we say that the graph (G, λ) satisfies an *elliptic Harnack Inequality* if there exists $C_2 > 0$ such that for all $x \in \mathbb{G}$, $R \geqslant 1$ and non-negative $h \colon G \to \mathbb{R}$ harmonic in $B_{2R}(x)$

$$\sup_{B_R(x)} h \leqslant C_2 \inf_{B_{2R}(x)} h. \tag{EHI}$$

The graph satisfies a *parabolic Harnack inequality* with parameter d_w if there exists $C_3 > 0$ such that for all $x \in G$, $R \geqslant 1$ and non-negative $h \colon G \times \mathbb{R} \to \mathbb{R}$ solving the heat equation $\frac{\partial h(x,t)}{\partial t} = \Delta h(x, t)$ in $B_{2R}(x) \times (0, 4R^2)$ it holds that

$$\sup_{B_R(z) \times [R^{d_w}, 2R^{d_w}]} h(x, t) \leqslant C_3 \inf_{B_R(z) \times [3R^{d_w}, 4R^{d_w}]} h(x, t). \tag{PH(d_w)}$$

Previous conditions are related in the following way. The authors in [GT01] showed that for $d_v > d_w$, any infinite connected weighted graph (G, λ) satisfying (p_0)

$$\text{(V(d_v))} \;+\; \text{(G($d_v - d_w$))} \Longleftrightarrow \text{(HKB(d_v, d_w))},$$

and it is known that for $d_v > d_w$

$$\text{(G($d_v - d_w$))} \Longrightarrow \text{(E(d_w))}.$$

The inequality $d_v > d_w$ clearly does not cover all ranges of possible values of d_v and d_w : for example in $\mathbb{Z}^d$ this holds only for $d > 2$.

However, not every choice $(d_v, d_w) \in (\mathbb{R}_+)^2$ is possible: it is known that if an infinite connected weighted graph satisfies (p_0), $(\mathrm{V}(d_v))$, $(\mathrm{E}(d_w))$ then $d_v \geqslant 1$ and

$$2 \leqslant d_w \leqslant 1 + d_v,$$

and a proof can be found in [Bar04]; there is also proven that for any $d_v \geqslant 1$, $2 \leqslant d_w \leqslant 1+d_v$, there exists an infinite connected locally finite graph which satisfies $(\mathrm{V}(d_v))$, $(\mathrm{E}(d_w))$ and (EHI). Furthermore, if $d_w \geqslant d_v$ then the graph is recurrent [Bar04, Proposition 3].

So, including wider ranges of the parameters, in [GT02, Theorem 3.1] the following equivalences are proven for any $d_w \geqslant 2$

$$(\mathrm{HKB}(d_v, d_w)) \;+\; (\mathrm{V}(d_v)) \Longleftrightarrow (\mathrm{PH}(d_w)) \Longleftrightarrow (\mathrm{VD}) + (\mathrm{EHI}) + (\mathrm{E}(d_w)).$$

As those graphs differ substantially from the diffusive lattice $\mathbb{Z}^d$, arises the natural question of whether percolation results for dependent or independent fields can be obtained.

I.2 Percolation in correlated systems

In this Section we introduce the results which we will prove in Chapters II and III which correspond respectively to [DGP22] and [DGG23]. We gave an overview of known results for the percolation of the level set $E^{\geqslant h}$ of the Gaussian free field, and in particular we mentioned that the critical parameter h_* is positive in the case of Galton–Watson trees conditioned on survival with mean offspring size $m > 2$, as proven in [AS18].

The Galton–Watson tree is the primary example of branching process. We can briefly define it here and refer for more detail to subsection II.2.1: consider a probability distribution ν on $\mathbb{N}$, and starting from a vertex $\varnothing$ called *root*, generate $Z_\varnothing$-many vertices, where $Z_\varnothing$ is a random variable distributed according to ν and connect them with an edge to $\varnothing$. For each of those vertices, generate offspring independently again according to ν, and iterate the procedure. One then obtain a graph $\mathcal{T}$ with the structure of a tree, i.e. for each vertex $x \in \mathcal{T}$, there is only a unique shortest path of vertices connecting x to the root $\varnothing$ and a natural orientation, i.e. x^- is the parent of the vertex x, the vertex closer to the root. It is well-known (see [LP16]) that the process has a positive probability to generate an infinite graph if the mean $m := \mathbb{E}^{\mathrm{GW}}[Z_\varnothing] > 1$, and otherwise is finite a.s.. If the tree is infinite, one easily obtains transience, but note that $(\mathrm{V}(d_v))$ does not hold for any d_v.

Since $m = 1$ is the critical value for Galton–Watson tree, one may wonder if the result $h_* > 0$ from [AS18], valid only for $m \in (2, \infty)$, holds as well in the whole supercritical phase, i.e. for all $m \in (1, \infty)$. We will answer positively to the question in Chapter II through a new construction of the graph via *watersheds*: consider under some probability $\mathbb{P}^{\mathrm{GW}}$ a Galton–Watson tree $\mathcal{T}$ with mean $m > 1$ conditioned on survival, and equip the edges $\{x, y\}$ with conductances $\lambda_{x,y}$ such that the family $\{\lambda_{x,y} \colon y^- = x,\ y \in \mathcal{T}\}_{x \in \mathcal{T}}$ is i.i.d. and $\mathbb{E}[\sum_{y \sim x} \lambda_{x,y}] < \infty$. We show that $h_*(\mathcal{T})$ is $\mathbb{P}^{\mathrm{GW}}$-a.s. constant and

$$h_*(\mathcal{T}) > 0. \tag{I.2.1}$$

The result is somehow surprising if compared to the independent case. Let $\mathcal{T}$ be a Galton–Watson tree conditioned on survival, for simplicity with conductances identically 1, and let $(Y_x)_{x\in\mathcal{T}}$ be a centered Gaussian field with variance 1 and covariance 0, which in particular means it is an independent field. Independent Bernoulli percolation is well understood, and the critical parameter p_* for Galton–Watson trees is known explicitly and equals $\frac{1}{m}$, which is usually proved observing that the subtree $\{y \in \mathcal{T} : B_y = 1\}$ constitutes a Galton–Watson tree and hence is supercritical when the mean of its offspring distribution is larger than 1. So for $h > 0$ we can consider

$$E\Big[\big|\{y \sim x : Y_y \geqslant h\}\big|\Big] = mF(-h),$$

with F being the probability distribution function of a standard normal variable. Since we assumed $h > 0$ $F(-h) > \frac{1}{2}$ and it approaches $\frac{1}{2}$ as $h \to 0$. In particular if and only if we assume $m > 2$ we can find a value $h_1 > 0$ depending on m such that $mF(-h_1) > 1$, so that with positive probability there exists an unbounded cluster with $Y > h$, for any $h \leqslant h_1$. This means that $h_*(Y)$ is positive only for $m > 2$.

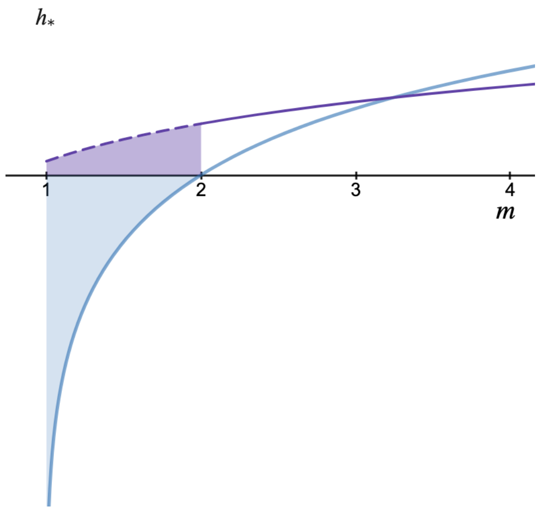


Figure I.1: A visual representation of the values of h_* for the independent field Y (in blue) and the Gaussian free field φ (in violet) as the mean offspring size m varies. While it is easy to obtain that $h_*(Y) = -F^{-1}(\frac{1}{m})$ for the independent field, for the critical value h_* it is only known to be positive and less than u_*, so the graph for $h_*(\varphi)$ is not accurate, and in particular it is unclear if an intersection is absent (as conjectured) or not. Our result $h_* > 0$ for all m highlights the violet area, showing a strict inequality $h_*(Y) < h_*(\varphi)$ for $m \in (1, 2)$ which is new in particular for $m = 2$. The value $h = 0$ is then now supercritical for the Gaussian free field.

The statement in (I.2.1) for all $m > 1$, implies that the correlated field percolates more easily than the independent one, and that in particular $h = 0$ is a value belonging to the supercritical phase for the percolation of the level set of φ but subcritical for level set of Y, showing one of the first example in which the mantra "positive correlation helps percolation" was actually seen to hold (see Figure I.1).

The proof of (I.2.1) is reasonably easier when $m \in (2, \infty)$. In that range, even if φ is a strong correlated field, one uses the previous argument where a spatial Markov property and a clever construction allows to work around the dependencies, as we explain in Section II.3. Our proof instead works for any $m > 1$ and for any distribution of the conductances with $\mathbb{E}[\sum_{y\sim x} \lambda_{x,y}] < \infty$, and uses the isomorphism with random interlacements. However, in order to deal with the random environment and random interlacements on it, we had to provide a new method for generating the tree and random interlacements simultaneously.

As a byproduct of this construction we are able to prove 2 other results. The random interlacements set $\mathcal{I}^u$, which contains trivially an unbounded cluster can be perturbed with some Bernoulli noise, i.e. a i.i.d. family $(\mathcal{B}_x)_{x\in\mathcal{T}}$ of Bernoulli variables of parameter p. We show that for each level $u > 0$ we can find a high intensity p such that $\mathcal{I}^u \cap \{\mathcal{B}_x = 1\}$ still contains an unbounded component, extending the result known for $\mathbb{Z}^d$ from [RS13a].

For the second result we required the conductances to be elliptic, i.e. there exists $\overline{c}_\lambda$, $\overline{C}_\Lambda$ such that for all $x \sim y$

$$\overline{c}_\lambda < \lambda_{x,y} < \overline{C}_\Lambda. \tag{I.2.2}$$

Then, we can find $u > 0, h > 0, p \in (0,1)$ such that the graphs $\mathcal{I}^u \cap \{\mathcal{B}_x = 1\}$ and $E^{\geqslant h} \cap \{\mathcal{B}_x = 1\}$ are almost surely transient, again generalizing the result for $\mathbb{Z}^d$ of [RS13a].

In Chapter II we concentrated on trees; a popular and often more challenging choice are graphs with polynomial growth. We already mentioned that in [DPR18a] the inequalities $\bar{h} > 0$ and $u_* > 0$ are proven for a large class of graphs: precisely, they assume (p_0), the volume growth (V(d_v)) (where d_v is there called α), the Green function decay (G($d_v - d_w$)) (with d_w called β) and weak sectional isoperimetric condition.

As discussed in the previous section, this class includes in particular fractal-like graphs such as Sierpiński gaskets and carpets. Therefore, we asked ourselves if it was possible to extend the concept of Lipschitz surface as in [GS19a] to those graphs.

We present in Chapter III our results about existence and properties of the Lipschitz cutset, the analogous of the Lipschitz surface for $\mathbb{G}^d$ and $\mathbb{SC}^d(l_F, m_F)$. One defines a coarse-graining of the space-time graph $\mathbb{G}^d \times \mathbb{Z}$, subdividing it into space -time cell $R_1(\iota, \tau)$ indexed by some (ι, τ). According to the model in consideration, one define a suitable event $E(\iota, \tau)$ which needs to be increasing and “restricted to a cell $R_1(\iota, \tau)$” (cf. Definition III.2.6 for a proper definition of “restricted”). If the probability of $E(\iota, \tau)$ is high enough for all (ι, τ) then there exist a set F of cells where the event E happens for all cells in F. Unlike its analogous in the lattice, the Lipschitz cutset cannot hope to have such a strong connectivity property due to “holes in the fractal”, but it will still satisfy a Lipschitz-like condition in the time component. However it still behaves as a cutset, meaning that any sequence of adjacent cells with distance from the origin going to infinity intersects the Lipschitz cutset F. Furthermore we show in Theorem (III.2.13), that F surrounds the origin at distance t with exponentially high probability. Those properties together allow to prove various facts about the models in consideration. As an example of a possible application, we present in Section III.9 the survival of an infection with recovery for small intensity of the recovery parameter.

Chapter II

Generating Galton–Watson trees using random walks and percolation for the Gaussian free field

II.1 Introduction

The main subject of this article is the study of level set percolation for the Gaussian free field on supercritical Galton–Watson trees. Due to the strong correlations inherent to the model, the problem of level set percolation induced by the Gaussian free field is quite intricate and significantly harder to understand than that of Bernoulli percolation. In the setting of fairly general transient graphs, the model has received increased attention in the last decade, as it is an important showcase for percolation problems with long-range correlations. A fundamental question in this context is to show the positivity of the associated critical parameter h_* – see (II.1.4) below for its definition – which entails a coexistence phase for $h > 0$ close to zero. It has been investigated on $\mathbb{Z}^d$, $d \geqslant 3$, in [BLM87; RS13b; DPR18b], and on more general graphs with polynomial growth in [DPR18a]. Of particular relevance for us is the setting of the Gaussian free field on trees, which has been studied in [Szn16; AS18; AČ20a]. More precisely, in [AS18, Section 5], Abächerli and Sznitman consider the particular case of the Gaussian free field on supercritical Galton–Watson trees with mean offspring distribution $m \in (1, \infty)$, and prove that $h_* \in [0, \infty)$ for all $m \in (1, \infty)$, as well as the strict inequality $h_* > 0$ when $m > 2$.

The main goal of the current article is to extend this result $h_* > 0$ to all supercritical Galton–Watson trees, i.e. with offspring mean $m \in (1, \infty)$, which along the way solves an open question of [AS18, Remark 5.6]. Moreover, we additionally allow the edges of the tree to be equipped with random conductances with finite mean, and show that the associated critical parameter h_* is still deterministic and strictly positive.

It is intriguing to compare our main result with Bernoulli site percolation on supercritical Galton–Watson trees $\mathcal{T}$, for which – conditioned on survival – the associated critical parameter is known to almost surely equal the inverse of the offspring mean, i.e., $p_c(\mathcal{T}) = 1/m$; see [Lyo90] or [LP16, Proposition 5.9]. Contrasting this well-known result with the inequality $h_*(\mathcal{T}) > 0$ is particularly interesting in the newly investigated range $m \in (1, 2]$ in our article. Indeed, in this range we have that the density of Bernoulli percolation at the critical parameter is given by $p_c(\mathcal{T}) = 1/m \geqslant 1/2$, whereas the density of percolation for the Gaussian free field level sets at the critical parameter is strictly smaller than $1/2$, since $h_*(\mathcal{T}) > 0$. Therefore, when $m \in (1, 2]$ the positive correlations of the Gaussian free field make percolation easier. This is a behavior expected for many percolation models, see in particular [Pra+92] as well as [ML06] for numerical reasonings concerning the setting of percolation with long-range correlations. To the best of our knowledge, the only other class of transient graphs where an inequality between densities at criticality of Gaussian free field and independent percolation has been rigorously proven are d-regular trees, see [Szn16, Corollary 4.5], but it is conjectured to hold for a large class of transient graphs.

A key tool in our proof is based on a construction of the Galton–Watson tree and random walks on it at the same time, see Section II.4. Each random walk will explore a portion of the tree below its starting point, and we call such a subset of the tree a "watershed". The specific exploration via watersheds will prevent the random walks from "predicting the future of the tree" during its construction; that is, we construct each watershed on a part of the Galton–Watson tree while preserving the independence of the rest of the tree. The main feature of the explored tree is its stability to perturbation by small quenched noise. The desired positivity of h_* will then be obtained by means of a Dynkin-type isomorphism theorem between the

Gaussian free field and random walks, see [Eis+00], or more precisely with random interlacements, a random soup of random walks, see [Szn12a; Lup16]. Moreover, we expect that our exploration procedure of the Galton–Watson tree via watersheds can also be used to obtain other interesting results. A first manifestation of this is already provided by the results on noise-stability and transience for the interlacements set as well as for the level sets of the Gaussian free field above small positive levels, see Theorem II.1.2 and II.1.3 below.

II.1.1 Main results

Let us now explain our setting and results in more detail. We consider a

$$\begin{gathered}\text{Galton–Watson random tree } \mathcal{T} \text{ with mean offspring}\\ \text{distribution } m > 1\text{, conditioned on survival,}\end{gathered} \tag{II.1.1}$$

and denote the underlying probability measure by $\mathbb{P}^{\mathrm{GW}}$. We endow the natural graph structure induced by $\mathcal{T}$ with positive random conductances $\lambda_{x,y}$, $x \sim y$, such that, conditionally on $\mathcal{T}$, and denoting by y^- the parent of $y \in \mathcal{T}$, with y different from the root $\varnothing$,

$$\begin{aligned}&\text{the family } \{\lambda_{x,y} : y \in \mathcal{T} \text{ and } y^- = x\}_{x\in\mathcal{T}}\text{, is i.i.d. and}\\ &\mathbb{E}^{\mathrm{GW}}[\lambda_{x,+}] < \infty \quad \forall x \in \mathcal{T},\ \text{where } \lambda_{x,+} := \sum_{y:\, y^-=x} \lambda_{x,y};\end{aligned} \tag{II.1.2}$$

note that this setting is slightly more general than endowing the edges of the Galton–Watson tree with independent conductances. In particular, when the conductances $\lambda_{x,y}$, $x \sim y$, are constant equal to 1, we recover the usual Galton–Watson tree, and in this case condition (II.1.2) simply boils down to the mean offspring distribution m being finite. In a slight abuse of notation, we also denote by $\mathcal{T}$ the weighted graph with the conductances λ, and will explicitly mention when we consider the tree $\mathcal{T}$ to be weightless as in (II.1.1) to avoid confusion. We refer to Section II.2.1 for precise notation and definitions.

It is known that the random tree $\mathcal{T}$ is almost surely transient, cf. Proposition II.2.1, and conditionally on its realization, we denote by $g^{\mathcal{T}}$ the Green function associated to the random walk on $\mathcal{T}$, see below (II.2.10).

Conditionally on the realization of $\mathcal{T}$, we then define the Gaussian free field $(\varphi_x)_{x\in\mathcal{T}}$ under some probability measure $\mathbb{P}^{\mathrm{G}}_{\mathcal{T}}$ as the centered Gaussian field with covariance function $g^{\mathcal{T}}$, see Section II.2.3 for further details. Note that this is a Gaussian free field in a random environment, that is we first generate the Galton–Watson tree $\mathcal{T}$ with random conductances and then – conditionally on the surviving Galton–Watson tree $\mathcal{T}$ – we generate a Gaussian free field on $\mathcal{T}$.

We will study the percolative properties of the *level sets* or *excursion sets* of the Gaussian free field on $\mathcal{T}$, i.e., of the random set

$$E^{\geqslant h} := E^{\geqslant h}(\mathcal{T}) = \{x \in \mathcal{T} \colon \varphi_x \geqslant h\}, \quad h \in \mathbb{R}. \tag{II.1.3}$$

We observe that the level set is clearly decreasing in h, and we define the critical parameter

$$h_* := h_*(\mathcal{T}) := \inf\{h \in \mathbb{R} : \mathbb{P}^{\mathrm{G}}_{\mathcal{T}}\text{-a.s. all connected components of } E^{\geqslant h}(\mathcal{T}) \text{ are bounded}\} \tag{II.1.4}$$

for the corresponding percolation problem.

A priori, it is not known if h_* is deterministic, nor whether the phase transition is nontrivial, i.e., whether $h_* \in \mathbb{R}$. For unitary conductances, the former is proved in [AS18, Lemma 5.1], and the latter – more precisely the inequality $0 \leqslant h_* < \infty$ – is proved in [AS18, Proposition 5.2], taking advantage of [Tas10]. The result $h_* > 0$ is shown to hold in [AS18] for constant conductances under the additional assumption $m \in (2, \infty)$; however, it seems that the assumption of finite mean is not essential to their proof. Let us also note in passing that even for Galton–Watson trees with random i.i.d. conductances, $h_*(\mathcal{T})$ is still deterministic, see Appendix II.A.2. We now state our main result.

Theorem II.1.1. *Under* (II.1.1) *and* (II.1.2)*, there exists* $h > 0$ *such that* $E^{\geqslant h}$ *contains* $\mathbb{E}^{\mathrm{GW}}[\mathbb{P}^{\mathrm{G}}_{\mathcal{T}}(\cdot)]$*-almost surely an unbounded connected component, and hence* $h_*(\mathcal{T}) > 0$.

Note that Theorem II.1.1 does not yet imply that the phase transition is nontrivial, that is $h_*(\mathcal{T}) < \infty$. Indeed, this finiteness property does hold true for i.i.d. weights, but it may fail without this condition – we refer to the discussion below (II.1.6) for details.

In the case $m > 2$, the assumption $\mathbb{E}^{\mathrm{GW}}[\lambda_{x,+}] < \infty$ from (II.1.2) is not necessary to prove the inequality $h_* > 0$ as explained at the end of Section II.3 (for unitary conductances this also follows from [AS18, Theorem 5.5]). In view of Theorem II.1.1, a natural question then is whether $h_* > 0$ under the broader assumptions $\mathbb{E}^{\mathrm{GW}}[\lambda_{x,+}] = \infty$ and $m \in (1, 2]$.

We will now put our result into the context of previous literature on percolation for the Gaussian free field. The study of this percolation problem for unitary conductances had been initiated by Bricmont, Lebowitz and Maes in [BLM87] on the Euclidean lattice $\mathbb{Z}^d$ in transient dimensions $d \geqslant 3$. Using a soft but quite robust contour approach, they proved that $h_*(\mathbb{Z}^d) \geqslant 0$ for all $d \geqslant 3$, as well as $h_*(\mathbb{Z}^3) < \infty$. More recently, on $\mathbb{Z}^d$, it has been established in [RS13b] that $h_*(\mathbb{Z}^d) < \infty$ for all $d \geqslant 3$, as well as $h_*(\mathbb{Z}^d) > 0$ for all sufficiently large d; in [DPR18b] it has then subsequently been shown that $h_*(\mathbb{Z}^d) > 0$ for all $d \geqslant 3$. For trees with unitary conductances, the parameter $h_* \in (0, \infty)$ was first characterized in [Szn16] on d-regular trees, $d \geqslant 3$, and subsequently in [AS18] for a larger class of transient trees, including supercritical Galton–Watson trees with mean $m > 2$.

In [AČ20a], further percolative properties for d-regular trees have then been studied in the super- and sub-critical regime. In [DPR18a], $h_* > 0$, and in fact local uniqueness of the infinite cluster at a positive level, has been shown for a larger class of graphs with polynomial growth. This class of graphs actually include $\mathbb{Z}^d$, $d \geqslant 3$, with bounded conductances as a special case, which was further studied in [CN21]. We also refer to [Szn15; AČ20b; DC+20; GRS22; Con21; Čer21] for further recent progress in this area.

Our proof crucially relies on another important object: the random interlacements set $\mathcal{I}^u$, $u > 0$, which has been introduced in $\mathbb{Z}^d$, $d \geqslant 3$, by [Szn10]. Later on, it has been generalized to transient weighted graphs in [Tei09]. It is related to the Gaussian free field via Ray-Knight type isomorphism theorems, first obtained in [Szn12a], and later on extended in a series of works [Lup16; Szn16; DPR22]. From a heuristic point of view, random interlacements is a random soup of doubly infinite transient random walks, and the union $\mathcal{I}^u$ of their traces thus trivially has an unbounded connected component (and hence percolates). On $\mathbb{Z}^d$, $d \geqslant 3$, it was proved in [RS13a] that $\mathcal{I}^u$ still percolates when perturbed by a small quenched noise, and this property was essential in the proof of $h_* > 0$ from [DPR18b]. Although our approach to proving $h_* > 0$ on Galton–Watson trees is quite different from that of [DPR18b], the stability of $\mathcal{I}^u$ to perturbation via small quenched noise will still play an essential role in our proof of Theorem II.1.1. Note that in the context of random Galton–Watson trees, we will see $\mathcal{I}^u$ as a quenched random interlacements on the realization of the tree $\mathcal{T}$; see Section II.2.4 for details.

We now describe this stability property – which is of independent interest, see its implications in Theorem II.1.3 below – in more detail. Again conditionally on the realization of the tree $\mathcal{T}$, for some $p \in (0,1)$, denote by $\mathcal{B}_x$, $x \in \mathcal{T}$, an independent family of i.i.d. Bernoulli random variables with parameter p and let

$$B_p := \{x \in \mathcal{T} : \mathcal{B}_x = 1\}. \tag{II.1.5}$$

Theorem II.1.2. *Under* (II.1.1) *and* (II.1.2)*, for all $u > 0$, there exists $p \in (0,1)$ such that $\mathcal{I}^u \cap B_p$ contains almost surely an infinite connected component. Moreover, there exist $h > 0$ and $p \in (0,1)$ such that $E^{\geqslant h} \cap B_p$ contains almost surely an infinite connected component.*

In [RS13a], the question of stability of the vacant set $\mathcal{V}^u := (\mathcal{I}^u)^c$ to perturbation by small quenched noise on $\mathbb{Z}^d$ has also been studied. In a similar vein, on Galton–Watson trees one can also easily prove that $\mathcal{V}^u \cap B_p$ percolates for p large enough, see Remark II.2.3. In [RS13a], the proof of stability of $\mathcal{I}^u$ to perturbation by small quenched noise involves some local connectivity result for random interlacements, which can also be used to prove transience of the interlacements set [RS11], or of $\mathcal{I}^u \cap B_p$, see [RS13a]. It turns out that, although our proof of Theorem II.1.2 is entirely different from that of [RS13a], it can also be employed to show transience of $\mathcal{I}^u \cap B_p$, or of $E^{\geqslant h} \cap B_p$ at small, but positive, levels, under some additional assumptions on the conductances.

Theorem II.1.3. *Assume* (II.1.1), (II.1.2) *and that, conditionally on the non-weighted graph $\mathcal{T}$, $(\lambda_{x,y})_{x\sim y\in\mathcal{T}}$ are i.i.d. conductances with compact support in $(0,\infty)$. Then for all $u > 0$, there exists $p \in (0,1)$ such that $\mathcal{I}^u \cap B_p$ contains almost surely a transient connected component. Moreover, there exist $h > 0$ and $p \in (0,1)$ such that $E^{\geqslant h} \cap B_p$ contains almost surely a transient connected component.*

For the reader's convenience we refer to the discussion above (II.6.1) for the precise definition of what means in our context that, conditionally on the non-weighted graph $\mathcal{T}$, $(\lambda_{x,y})_{x\sim y\in\mathcal{T}}$ are i.i.d. conductances with compact support in $(0,\infty)$ – which, in fact, is arguably the "natural" way of endowing a tree with i.i.d. random conductances, but less general when compared to (II.1.2).

Let us finish this subsection with some comments on percolation for the vacant set of random interlacements, and the finiteness of h_*. The random interlacements set $\mathcal{I}^u$ always percolates since the trace of a transient random walk is an unbounded connected set; one may, however, wonder if the same holds true for its complement the *vacant set* $\mathcal{V}^u$ when the intensity parameter varies.

Denoting by u_* the critical parameter associated to the percolation of $\mathcal{V}^u$, $u > 0$, the isomorphism between random interlacements and the Gaussian free field, see Proposition II.2.5 below (which can be used in our context in view of Proposition II.5.8), implies similarly as in [Lup16, Theorem 3] that

$$h_* \leqslant \sqrt{2u_*}. \tag{II.1.6}$$

The inequality (II.1.6) combined with Theorem II.1.1 implies $u_* > 0$, but note that the inequality $u_* > 0$ could be proved via easier means, see Remark II.2.3. Let us note here that in the special case of unitary conductances, an explicit formula for u_* has been derived in [Tas10]. The proof of [Tas10, Theorem 1] can be adapted to random conductances as long as $(\lambda_{x,y})_{x\sim y\in\mathcal{T}}$ are i.i.d. conductances conditionally on the non-weighted graph $\mathcal{T}$. In particular, $u_* < \infty$ under the same conditions, and thus $h_* < \infty$ as well by (II.1.6). However, if we allow the weights $(\lambda_{x,y})_{x\sim y\in\mathcal{T}}$ to not be i.i.d. conditionally on the non-weighted graph $\mathcal{T}$ – but still satisfying the usual setup of (II.1.2) – one can find Galton–Watson trees where $h_* = \infty$, see (II.3.4).

The weak inequality (II.1.6) can actually be improved to $h_* < \sqrt{2u_*}$ on d-regular trees, $d \geqslant 3$, see [Szn16]. In [AS18], the authors provide general enough conditions to obtain $h_* < \sqrt{2u_*}$ on transient trees, and in particular for Galton–Watson trees with unitary conductances this strict inequality holds under additional hypotheses on exponential moments of the offspring distribution, see [AS18, Theorem 5.4]. They also provide an example, namely the tree where each vertex has an offspring size equal to its distance to the root, where actually $0 = h_* = \sqrt{2u_*}$. Note that this entails that Theorem II.1.1 does not hold when removing the assumption $\mathbb{E}^{\mathrm{GW}}[\lambda_{x,+}] < \infty$ from (II.1.2), as well as the assumption that the distribution of the number of children does not depend on the generation.

II.1.2 Outline of the proof

We now comment on the proofs of Theorems II.1.1, II.1.2 and II.1.3 in more detail. Let us first elaborate on the fact that Theorem II.1.2 is useful to obtain Theorem II.1.1. The isomorphism between random interlacements and the Gaussian free field, see Proposition II.2.5, implies that for each $u > 0$, random interlacements and the Gaussian free field on $\mathcal{T}$ can be coupled in such a way that

$$\text{almost surely,} \quad \mathcal{I}^u \subset E^{\geqslant -\sqrt{2u}}. \tag{II.1.7}$$

This implies in particular that $E^{\geqslant -\sqrt{2u}}$ percolates for all $u > 0$, and taking $u \downarrow 0$ we infer that $h_* \geqslant 0$. Note that the validity of the inclusion (II.1.7) requires some condition on the tree to be fulfilled – see (II.2.20) – but we will actually show in Proposition II.5.8 that this condition is always satisfied in our context. In [DPR18b; DPR18a], an extension of the inclusion (II.1.7) to a continuous metric structure associated with the discrete graph, the so-called cable system, was used to lift the inclusion (II.1.7) – when the field was taking not too high values – to level sets of the Gaussian free field at positive levels, which then yielded the desired strict inequality $h_* > 0$. Here, we follow a simpler approach, that is we use an extension of the inclusion (II.1.7), see Proposition II.2.5 below, which includes information about the exact values of the free field, as well as the local times of random interlacements. Proposition II.2.5 is proven using the cable system, cf. [Lup16] for further details. The proposition readily implies that there exists a coupling such that for each $u > 0$,

$$\text{almost surely,} \quad \mathcal{I}^u \cap A_u \subset \widehat{E}^{\geqslant \sqrt{2u}}, \tag{II.1.8}$$

where $\widehat{E}^{\geqslant \sqrt{2u}}$ has the same law as $E^{\geqslant \sqrt{2u}}$, see (II.1.3), and

$$A_u := \left\{ x \in \mathcal{T} \colon \mathcal{E}_x > 4u\lambda_x \text{ or } |\varphi_x| > 2\sqrt{2u} \right\}, \tag{II.1.9}$$

for some i.i.d. exponential random variables $(\mathcal{E}_x)_{x\in\mathcal{T}}$ with parameter one, independent of the Gaussian free field φ and the interlacements set $\mathcal{I}^u$. Note that A_u increases a.s. to $\mathcal{T}$ as $u \to 0$, and one can thus interpret the intersection with A_u as applying a small quenched noise. Theorem II.1.2 then suggests that $\mathcal{I}^u \cap A_u$ might percolate for u small enough, which again would imply Theorem II.1.1 by (II.1.8).

However, one cannot directly use Theorem II.1.2 for proving Theorem II.1.1 for two reasons: first, the variables $\{x \in A_u\}$, $x \in \mathcal{T}$, are not independent, and second, the probability that $x \in A_u$ depends on the parameter u of the interlacements set, and thus, contrary to p in Theorem II.1.2, it cannot be taken arbitrarily close to one for a fixed u. The first problem will be essentially solved by lower bounding the probability that $x \in A_u$ conditionally on $\{y \in A_u\}$, $y \neq x$, using the Markov property of the free field, see (II.5.25). To solve the second problem, we will make the dependency of p on u in Theorem II.1.2 explicit, that is, we find a function $p(u)$, with $p(u) \uparrow 1$ as $u \to 0$, such that $\mathcal{I}^u \cap B_{p(u)}$ percolates for all $u > 0$, and we show that the probability that $x \in A_u$ is larger than $p(u)$ for u small enough, see the proof of Proposition II.5.7.

Therefore, in order to obtain Theorem II.1.1, it is essentially enough to show that $\mathcal{I}^u \cap B_{p(u)}$ percolates, where $p(u)$ is smaller than the probability that $x \in A_u$ for u small enough. The main difficulty is that, when u is small, there are two competing effects at play in this percolation problem. On the one hand, in the $u > 0$ small regime, the interlacements set $\mathcal{I}^u$ consists of few trajectories, and hence is less well-connected; i.e., intersecting $\mathcal{I}^u$ with B_p might break its infinite connected components into finite pieces. This is particularly problematic when m is close to one, since the tree tends to contain long stretches which locally look like $\mathbb{Z}$, and hence the connectivity of such components turns out to be sensitive to an independent noise. On the other hand, as $u \to 0$, for each $x \in \mathcal{T}$, the probability that x is in A_u tends to one, and it thus becomes less likely to break a fixed connected component of $\mathcal{I}^u$ into finite pieces when intersecting with $B_{p(u)}$. The proof of Theorem II.1.1 therefore requires a subtle comparison of the influences of these two opposite effects as $u \to 0$. We now provide a short explanation of how this is done.

The probability that a vertex x is contained in A_u^c can be easily upper bounded by $u^{3/2}\lambda_x^{3/2}$, see (II.5.25) below, and we can thus take $p(u) = 1 - u^{3/2}\lambda_x^{3/2}$ for u small enough. To prove percolation of $\mathcal{I}^u \cap B_{p(u)}$, we use a description of the trajectories in $\mathcal{I}^u$ via their highest (i.e., minimal distance to the root) visited vertex, Theorem II.2.2, which can be seen as a generalization of [Tei09, Theorem 5.1]. This description entails that $\mathcal{I}^u$ can be generated by starting, for each vertex $x \in \mathcal{T}$, an independent Poissonian number Γ_x of random walks starting at x going down the tree. Here, the Poisson distribution underlying Γ_x has parameter $u\check{e}_{\mathcal{T}}(x)$, where $\check{e}_{\mathcal{T}}(x)$ – see (II.2.16) – is a parameter depending on the subtree rooted at x, which bears some similarity with the square of the conductance from x to infinity.

Now in the simpler case where each vertex in the tree $\mathcal{T}$ always had at least two children and the conductances were bounded, one could finish the proof by first conditioning on $\mathcal{T}$ and by then proceeding as follows. One can under these conditions easily show that $\check{e}_{\mathcal{T}}(x)$ is of constant order, uniformly in $x \in \mathcal{T}$. Thus, when $\Gamma_x \geqslant 1$, with high probability, starting a random walk at x going down the tree up to the first time it has visited C/u vertices, for a large constant C, there are at least two vertices y with $\Gamma_y \geqslant 1$ which are not visited by the walk, but children of vertices visited by the walk (the existence of such vertices is guaranteed by the fact that each vertex visited by the walk has at least two children). We will say that such a point y corresponds to a free point, see (II.4.12). Moreover, again with high probability as $u \to 0$, all the vertices visited by this walk are contained in $B_{p(u)}$, with $p(u) = 1 - u^{3/2}\lambda_x^{3/2}$, and in particular there is a path between x and y in

$\mathcal{I}^u \cap B_{p(u)}$. One can now iterate this procedure starting a new trajectory at each y corresponding to a new free point, and show that the tree of free points contains a d-ary tree, see Proposition II.5.5. In particular it percolates, which directly implies the percolation of $\mathcal{I}^u \cap B_{p(u)}$ also.

In this approach, we thus first generate $\mathcal{T}$, and then construct an infinite cluster in $\mathcal{I}^u \cap B_{p(u)}$ on the now fixed tree $\mathcal{T}$. However, when the mean offspring number m is close to one, or the conductances are not bounded, then the tree $\mathcal{T}$ will contain some connected components of vertices, each with exactly one child, with size more than C/u, on which the above approach is bound to fail. Note, however, that as $u \to 0$, condition (II.1.2) in combination with the Marcinkiewicz-Zygmund law of large numbers implies that these bad sequences in $\mathcal{T}$ become rarer when the tree is generated, see (II.5.7). In order to benefit from this information, we are going to generate the interlacements set $\mathcal{I}^u$ and the Galton–Watson tree $\mathcal{T}$ simultaneously. Generating the two processes at the same time is of considerable importance as it allows us to operate with the interlacements process without being forced to generate the whole tree beforehand.

To generate these two processes at the same time, we will explore the Galton–Watson tree using random walks, in the form of an object that we will call *watershed*, as is explained in Section II.4 in more detail. The previously mentioned description of random interlacement trajectories via their highest visited vertex then implies that for each vertex x, if a Poisson random variable with parameter u takes the value at least one, one can start a watershed at x, that is a walk starting at x and exploring the tree below x, which is included in random interlacements at level $u/e_{\{x\},\mathcal{T}_x}(x)$, see Proposition II.4.2; here, $e_{\{x\},\mathcal{T}_x}$ is the equilibrium measure of the set $\{x\}$ for the subtree $\mathcal{T}_x$ of $\mathcal{T}$ rooted in x, see (II.2.12). Now, for each vertex x, we will first generate a portion of the tree to make sure that $e_{\{x\},\mathcal{T}_x}(x) \geqslant c_e$ for some constant c_e, see (II.5.19), and then start a watershed at x if a Poisson random variable with parameter u is at least one, which will thus be included in random interlacements at level u/c_e, see Proposition II.5.6. We can now use the additional randomness of the tree – which in particular entails that with high probability there are no large components of vertices each with exactly one child – to show that, for $u > 0$ small enough, the intersection of all the watersheds and $B_{p(u/c_e)}$ percolates for each $m > 1$, and thus $E^{\geqslant h}$ percolates for h small enough as well; see Section II.5 for details.

Finally, in order to prove Theorem II.1.3, we note that, for uniformly bounded weights, the trace of a random walk on the watersheds is essentially a coarse-grained random walk on the tree of free points with a drift, see (II.6.4). Using an argument from [Col06], we deduce that such a random walk is transient, which finishes the proof using the isomorphism (II.1.8) again.

The structure of the article is as follows: in Section II.2 we will define the main objects and set up notation. In Section II.3 we provide a short and simple proof of Theorem II.1.1 under the additional assumption $m > 2$ – this will turn out instructive for the proof of the general result also. Furthermore, we provide examples of Galton–Watson trees with $h_* = \infty$. In Section II.4 we will introduce the exploration of the Galton–Watson tree through random walks, which is used in Section II.5 to prove Theorems II.1.1 and II.1.2. In Section II.6, we use similar methods to prove Theorem II.1.3. Finally, we prove in Appendix II.A.2 that h_* is deterministic in our setting.

II.2 Notation and definitions

In Sections II.2.1 and II.2.2 we introduce the Galton–Watson trees which we will be considering. Subsequently, Sections II.2.3 and II.2.4 are then devoted to random walks, the Gaussian free field, as well as random interlacements on trees. In Section II.2.5 we introduce the isomorphism theorem between random interlacements and the Gaussian free field.

II.2.1 Galton–Watson trees

We will investigate trees using the Ulam-Harris labeling. For this purpose, consider the space

$$\mathcal{X} := \bigcup_{i=0}^{\infty} \mathbb{N}^i, \tag{II.2.1}$$

where $\mathbb{N}$ is the set of positive integers, $\mathbb{N}_0$ the set of non-negative integers and $\mathbb{N}^0$ is defined as $\{\varnothing\}$. For $i, j \in \mathbb{N}$ as well as $x, y \in \mathcal{X}$ such that $x = (x_1, \dots, x_i) \in \mathbb{N}^i$ and $y = (y_1, \dots, y_j) \in \mathbb{N}^j$, we define the concatenation of x and y as $xy = (x_1, \dots, x_i, y_1, \dots, y_j) \in \mathbb{N}^{i+j} \subseteq \mathcal{X}$. Moreover, for $A \subseteq \mathcal{X}$ and $x \in \mathcal{X}$ we introduce $x \cdot A := \{xy : y \in A\}$; note that in contrast to pointwise concatenation we put an additional dot for aesthetic reasons. For all $x = (x_1, \dots, x_i) \in \mathcal{X}$, $i \in \mathbb{N}$, we define $x^- := (x_1, \dots, x_{i-1})$, the parent of x, with the convention $() = \varnothing$. For a set $A \subseteq \mathcal{X}$ we define its (interior) boundary as $\partial A := \{x \in A : \nexists\, y \in A,\, y^- = x\}$. Note that this is not exactly the natural topological boundary, but this slightly modified definition will turn out useful for our purposes. We moreover introduce, for $A \subseteq \mathcal{X}$ and $x \in A$, the set of children of x in A as

$$G_x^A := \{y \in A \mid y^- = x\}. \tag{II.2.2}$$

We call $T \subset \mathcal{X}$ a tree if for each $x \in T \backslash \{\varnothing\}$, we have $x^- \in T$ and $|G_x^T| < \infty$. We then say that $x \in T \backslash \{\varnothing\}$ is a child of $y \in T$ if $x^- = y$. If the tree T under consideration is clear from the context, for all $x, y \in T$, we write $x \sim y$ if either $x = y^-$ or $y = x^-$. One can also view a tree T as a graph with edges between x and y if and only if $x \sim y$. On this graph, we denote by $d_T(x, y)$ the usual graph distance. We say that T is a weighted tree if each edge between x and y is endowed with a symmetric conductance $\lambda_{x,y} = \lambda_{y,x} \in (0, \infty)$. For $x \in T$ we also define $\lambda_{x,+}$ as in (II.1.2). Since weights are not encoded in $\mathcal{X}$, a weighted tree is not a subset of $\mathcal{X}$. However, to simplify notation, we will often implicitly identify a weighted tree with its set of vertices, a subset of $\mathcal{X}$. Note that most of the previous notation depends on the choice of the tree T, which will always be clear from the context. For $x \in T$, we write T_x for the subtree of T consisting of x and all descendants of x, endowed with the same conductances as in the underlying tree T. In this article, we think of trees as growing from top to bottom, so we sometimes refer to the points in the subtree T_x as the points below x. A priori, T_x may consist of finitely many nodes only, but with a standard pruning procedure, we will actually soon reduce ourselves to the case of infinite Galton–Watson trees, see Section II.2.2.

We now explain how to define a Galton–Watson tree with random weights as a random weighted tree $\mathcal{T}$. We consider a probability measure ν on $[0, \infty)^{\mathbb{N}}$, which will form a canonical probability space, in order to describe the offspring distribution as well as the associated conductances. More precisely, we consider ν such that if the sequence $(\lambda_i)_{i \in \mathbb{N}}$ on $[0, \infty)^{\mathbb{N}}$ has law ν, then there exists $d \in \mathbb{N}$ such that ν-a.s.,

$\lambda_i > 0$ for all $i < d$ and $\lambda_i = 0$ for all $i \geqslant d$. We will soon use ν to assign weights to the edges of the tree by means of a vector $(\lambda_{x,xi})_{i=1}^{\infty}$, distributed according to ν for each vertex x. Throughout this article, except in Section II.3, we moreover assume that the law of the conductances satisfies

$$\mathbb{E}^{\nu}\big[\lambda_+\big] < \infty, \text{ where } \lambda_+ = \sum_i \lambda_i; \tag{II.2.3}$$

essentially, this is just a reformulation of the second condition in (II.1.2). Note that we do not assume the conductances to be bounded away from zero or infinity, nor that the conductances λ_i, $i \in \mathbb{N}$, are independent under ν. Defining the function $\pi\colon [0,\infty)^{\mathbb{N}} \to \mathbb{N}_0$ via $(\lambda_i)_{i\in\mathbb{N}} \mapsto |\{i \in \mathbb{N} : \lambda_i > 0\}|$, we introduce the pushforward probability measure

$$\mu := \nu \circ \pi^{-1} \tag{II.2.4}$$

on $\mathbb{N}_0$. As it corresponds to the law of the number of edges with conductances different from 0, it will play the role of the offspring distribution. We will assume from now on that the mean of the offspring distribution satisfies

$$m := \sum_{i=0}^{\infty} i\mu(i) > 1, \tag{II.2.5}$$

which will correspond to the case of supercritical Galton–Watson trees.

On some rich enough probability space we define the Galton–Watson tree $\mathcal{T}$ by constructing $\mathcal{T} \cap \mathbb{N}^k (\subset \mathcal{X})$, endowed with conductances on the (undirected) edges with the vertices in $\mathcal{T} \cap \mathbb{N}^{k-1}$, recursively in k. For $k = 0$, we simply start with the vertex $\varnothing \in \mathbb{N}^0 \subseteq \mathcal{X}$ called the root. For $k \geqslant 0$, once the tree $\mathcal{T}$ has been generated up to generation k, for each vertex $x \in \mathbb{N}^k \cap \mathcal{T}$ we generate independently a random vector $(\lambda_{x,xi})_{i\in\mathbb{N}}$ with law ν. The vertex x has $\pi((\lambda_{x,xi})_{i\in\mathbb{N}})$ children, and we endow the edge from x to its child xi, $1 \leqslant i \leqslant \pi((\lambda_{x,xi})_{i\in\mathbb{N}})$, with the conductance $\lambda_{x,xi} \in (0,\infty)$. This defines $\mathcal{T} \cap \mathbb{N}^{k+1}$ and its conductances with vertices in $\mathcal{T} \cap \mathbb{N}^k$. The union over $k \in \mathbb{N}_0$ of these sets, endowed with the respective conductances, is denoted by $\mathcal{T}$, the weighted Galton–Watson tree. Note that the structure of the tree is completely determined by the weights λ, and that an edge between two vertices is present if and only if the conductance between them is non-zero. Under our standing assumption (II.2.5), the tree becomes extinct with probability $q < 1$ (cf. for instance the discussion below [LP16, Proposition 5.4]). Hence, it has a positive probability to survive indefinitely, and in order to avoid trivial situations, we will always condition the Galton–Watson tree on this event of survival in what follows. We denote by $\mathbb{P}^{\mathrm{GW}}$ the probability measure underlying the Galton–Watson tree constructed above, conditioned on survival.

Let us also define here already the canonical σ-algebras that we consider throughout the article, and which only become relevant at later points in this article. The set $\mathcal{X}$ is endowed with the σ-algebra $\sigma(\{x\}, x \in \mathcal{X})$, and the space of subsets of $\mathcal{X}$ is endowed with the σ-algebra generated by the coordinate functions $A \mapsto \mathbf{1}_{\{x\in A\}}$, $x \in \mathcal{X}$. If $T \subset \mathcal{X}$, we will often regard $(\lambda_{x,y})_{x\sim y\in T} \in (0,\infty)^{\{x,y\in T:\, x\sim y\}}$ as an element of $[0,\infty)^{\mathcal{X}\times\mathcal{X}}$, endowed with the product of the Borel-σ-algebras, by taking $\lambda_{x,y} = 0$ if either $x \notin T$ or $y \notin T$, or else if x and y are not neighbors in T.

II.2.2 Pruning of the tree

In this subsection we describe a useful pruning procedure for the tree conditioned on survival, which corresponds to chopping all finite branches of the tree – the remaining subtree is known as the reduced subtree in the literature, see e.g. [LP16]. In order to simplify our investigations, we will then observe that the conditioned chopped Galton–Watson tree can also be constructed as a Galton–Watson tree with modified offspring distribution and which then survives almost surely, see (II.2.6). For this purpose, we define the reduced subtree $\mathcal{T}^\infty$ of $\mathcal{T}$ as consisting of those vertices of $\mathcal{T}$ which have an infinite line of descendants:

$$\mathcal{T}^\infty := \{x \in \mathcal{T} : \mathcal{T}_x \text{ is infinite}\},$$

where we recall that the notation $\mathcal{T}_x$ has been introduced in the paragraph below (II.2.2).

Then [LP16, Proposition 5.28 (i)] entails that $\mathcal{T}^\infty$, which can be seen as a tree in $\mathcal{X}$, has – possibly after relabeling and conditionally on survival – the same law as a Galton–Watson tree $\mathcal{T}^*$ with offspring distribution μ^*. The latter is characterized by its probability generating function

$$f^*(s) = \frac{f(q + s(1-q)) - q}{1-q}, \qquad \begin{array}{l}\text{where } q \text{ is the probability that } \mathcal{T} \text{ is finite, and}\\ f \text{ is the probability generating function of } \mu.\end{array} \tag{II.2.6}$$

Note that $f^*(0) = 0$, hence $\mu^*(0) = 0$, i.e. points in $\mathcal{T}^*$ have zero probability of generating no children, and that μ^* has the same mean m as the law μ associated to $\mathcal{T}$.

The behavior of the law of the conductances under pruning is slightly more involved. Indeed, conditionally on $\mathcal{T}$ and for each $x \in \mathcal{T}$, conditionally on its number of children $|G_x^{\mathcal{T}}|$, the weights $(\lambda_{x,y})_{y \sim x}$ are independent of the event $\{x \in \mathcal{T}^\infty\}$. Therefore, one can find a probability measure ν^* on $[0,\infty)^{\mathbb{N}}$ with $\nu^* \circ \pi^{-1} = \mu^*$ such that the weighted tree $\mathcal{T}^\infty$ has – after relabeling – the same law conditionally on survival as a weighted Galton–Watson tree $\mathcal{T}$ obtained from the probability ν^*. The law of ν^* is the same as the law of ν restricted to P positive coordinates chosen uniformly at random among the $K + P$ positive coordinates of ν, where P has law μ^* and K has the law of the number of children of the root which do not survive, given that the root has P surviving children (its probability generating function is described in [LP16, Proposition 5.28 (iv)]).

Note that even under ν^* it holds true that $\mathbb{E}^{\nu^*}[\sum_{i\in\mathbb{N}} \lambda_i] < \infty$. Indeed, we first condition on survival which is an event of positive probability, and then we delete those points not belonging to $\mathcal{T}^\infty$, which can only decrease the respective expected conductance.

We already remark at this point that the above pruning procedure does not change the critical parameter h_* we are interested in, as the Gaussian free field restricted to $\mathcal{T}^\infty$ has the same law on the pruned tree, and similarly for random interlacements. In particular, Theorems II.1.1, II.1.2 and II.1.3 can be proven equivalently on the initial tree or on the pruned tree, and we refer to Remark II.2.4 for further details.

Therefore, without loss of generality, from now on we always work under the standing assumption that

$$\begin{array}{c}\nu \text{ is a probability measure such that } \pi\big((\lambda_i)_{i\in\mathbb{N}}\big) \geqslant 1 \ \nu\text{-a.s.;}\\ \text{i.e., under } \mathbb{P}^{\mathrm{GW}} \text{ all } x \in \mathcal{T} \text{ have a.s. an infinite line of descendants.}\end{array} \tag{SA}$$

In particular, under (SA), $\mathbb{P}^{\mathrm{GW}}$ is the law of a Galton–Watson tree without conditioning on survival, since survival occurs with probability one.

II.2.3 Gaussian free field

Let us now define one of our main objects of interest, the Gaussian free field. We start with some general definitions related to random walks. Let T be a weighted tree with positive weights $(\lambda_{x,y})_{x\sim y\in T}$. For $x_0 \in T$ we define a random walk $(X_n)_{n\in\mathbb{N}_0}$ on T under $P^T_{x_0}$ as the Markov chain on its canonical space $\mathbb{N}_0$ starting in x_0 with transition probabilities

$$P^T_{x_0}(X_{n+1} = y \,|\, X_n = x) = \frac{\lambda_{x,y}}{\lambda_x} \text{ for all } x \sim y \in T, \tag{II.2.7}$$

where the total weight λ_x at x is defined as

$$\lambda_x = \sum_{y\sim x} \lambda_{xy}; \tag{II.2.8}$$

note that the total weight, unlike $\lambda_{x,+}$ in (II.1.2), sums over the conductance λ_{x,x^-} also. For a set $U \subseteq T$, the hitting and return times of X, respectively, are denoted by

$$H_U(X) := H_U := \inf\{n \geqslant 0 \,:\, X_n \in U\} \text{ and } \widetilde{H}_U(X) := \widetilde{H}_U := \inf\{n \geqslant 1 \,:\, X_n \in U\}, \tag{II.2.9}$$

respectively, with the convention $\inf \varnothing = \infty$. In the case of a single point $U := \{x\}$, we will write H_x and $\widetilde{H}_x$ in place of $H_{\{x\}}$ and $\widetilde{H}_{\{x\}}$.

In this section, we assume that the random walk X on T is transient, an assumption which will in particular be satisfied for supercritical Galton–Watson trees conditioned on survival, see Proposition II.2.1. For $U \subset T$, the Green function associated to X, killed upon exiting U under $P^T_\cdot$, is given by

$$g^T_U(x,y) := \frac{1}{\lambda_y} E^T_x\Big[\sum_{k=0}^{H_{T\setminus U}-1} \mathbf{1}_{\{X_k=y\}}\Big] \text{ for all } x,y \in T. \tag{II.2.10}$$

In particular, we note that $g^T_U(x,y) = 0$ if either $x \notin U$ or $y \notin U$. In addition, we write $g^T(x,y) := \frac{1}{\lambda_y} E^T_x[\sum_{k=0}^{\infty} \mathbf{1}_{\{X_k=y\}}]$, where $x,y \in T$, for the Green function associated to X on T.

Then g^T is symmetric positive definite, and we can hence consider a probability measure $\mathbb{P}^{\mathrm{G}}_T$ on $\mathbb{R}^T$ endowed with the canonical σ-algebra generated by the coordinate maps $(\varphi_x)_{x\in T}$ such that

$$\begin{aligned}&(\varphi_x)_{x\in T} \text{ is a centered Gaussian field}\\ &\text{with covariance given by } \mathbb{E}^{\mathrm{G}}_T[\varphi_x\varphi_y] = g^T(x,y),\ x,y \in T.\end{aligned}$$

We call φ the Gaussian free field on the tree T. Let us now recall the Markov property for φ, see for instance [Szn12b, Proposition 2.3]. For a finite set $K \subseteq T$ and $U := T\backslash K$, define for all $z \in T$,

$$\beta^U_z := E^T_z\big[\varphi_{X_{H_K}} \mathbf{1}_{\{H_K<\infty\}}\big] \quad \text{and} \quad \psi^U_z := \varphi_z - \beta^U_z. \tag{II.2.11}$$

Then

$$(\psi^U_z)_{z\in T} \text{ is a centered Gaussian field with covariance function } \mathbb{E}^{\mathrm{G}}_T\big[\psi^U_z\psi^U_w\big] = g^T_U(z,w),$$

which vanishes in K and is independent of $\sigma(\varphi_z, z \in K)$. Note moreover that β^U is $\sigma(\varphi_z, z \in K)$-measurable, and thus independent of ψ^U.

Putting the previous general considerations in our context of interest, we note that for almost all realizations of a weighted Galton–Watson tree $\mathcal{T}$, under $\mathbb{P}^{\mathrm{GW}}$ the Green function $g^{\mathcal{T}}$ is finite since the random walk is transient: the proof in [Gan+12, Proposition 2.1] can be straightforwardly adapted to our case, i.e. the case where for each $x \in \mathcal{X}$, the family $(\lambda_{x,y})_{y\sim x}$, is not necessarily independent. This yields the following result.

Proposition II.2.1 ([Gan+12]). *$\mathbb{P}^{\mathrm{GW}}$-almost surely, the random walk on the tree $\mathcal{T}$ with conductances $(\lambda_{x,y})_{x,y\in\mathcal{T},x\sim y}$ is transient.*

Hence, for almost all realizations of the Galton–Watson tree $\mathcal{T}$, we can define the Gaussian free field on $\mathcal{T}$ as the field φ under $\mathbb{P}^{\mathrm{G}}_{\mathcal{T}}$.

II.2.4 Random interlacements

The random interlacements process has been introduced by Sznitman [Szn10] for $\mathbb{Z}^d$ (see [DRS14a] and [ČT12] for introductory texts) and it has subsequently been generalized to transient weighted graphs in [Tei09]. For a transient weighted tree T with conductances $(\lambda_{x,y})_{x\sim y\in T}$, we define the equilibrium measure and capacity of a finite set $K \subseteq T$ as

$$e_{K,T}(x) := \mathbf{1}_{\{x\in K\}}\lambda_x P_x^T(\widetilde{H}_K = \infty) \text{ and } \mathrm{cap}_T(K) := \sum_{x\in K} e_{K,T}(x). \tag{II.2.12}$$

We also define the capacity of an infinite set $F \subseteq T$ as the limit of the capacity of F_n as $n \to \infty$, where $(F_n)_{n\in\mathbb{N}}$ is a sequence of finite sets increasing to F; we refer for instance to the end of [DPR22, Section 2.2] for as to why this limit exists and does not depend on the choice of the exhausting sequence $(F_n)_{n\in\mathbb{N}}$. We further introduce the set

$$\overrightarrow{Z}_T := \{\overrightarrow{w}: \mathbb{N}_0 \to T \,|\, \overrightarrow{w}_n \sim \overrightarrow{w}_{n+1} \text{ for all } n \geqslant 0 \text{ and } d_T(\varnothing, \overrightarrow{w}_n) \to \infty \text{ as } n \to \infty\}$$

of transient nearest neighbor trajectories on T as well as the set

$$\overleftrightarrow{Z}_T := \big\{\overleftrightarrow{w}: \mathbb{Z} \to T \big| \overleftrightarrow{w}_n \sim \overleftrightarrow{w}_{n+1} \text{ for all } n \in \mathbb{Z} \text{ and } d_T(\varnothing, \overleftrightarrow{w}_n) \to \infty \text{ as } n \to \pm\infty\big\} \tag{II.2.13}$$

of doubly infinite transient nearest neighbor trajectories. In the literature, the set $\overleftrightarrow{Z}_T$ in (II.2.13) is usually denoted by W; in this article, however, in a self-suggestive manner, we reserve W for the notion of watersheds, a key object which will be defined in Section II.4. Denote by $\overleftrightarrow{X}$ the identity map on $\overleftrightarrow{Z}_T$, and we indicate with $\overrightarrow{X}$ and $\overleftarrow{X}$ the forward and backward trajectories

$$(\overrightarrow{X}_n)_{n\in\mathbb{N}_0} := (\overleftrightarrow{X}_n)_{n\in\mathbb{N}_0} \quad \text{and} \quad (\overleftarrow{X}_n)_{n\in\mathbb{N}_0} := (\overleftrightarrow{X}_{-n})_{n\in\mathbb{N}_0}.$$

Let $\overrightarrow{\mathcal{Z}}_T$ and $\overleftrightarrow{\mathcal{Z}}_T$ be the associated σ-algebras on $\overrightarrow{Z}_T$ and $\overleftrightarrow{Z}_T$ generated by the coordinate functions. On $(\overleftrightarrow{Z}_T, \overleftrightarrow{\mathcal{Z}}_T)$ we consider the family of measures Q_K^T, $K \subseteq T$ finite, which is characterized by the identities

$$\begin{aligned} Q_K^T\big((\overleftarrow{X}_n)_{n\in\mathbb{N}} \in A,\, X_0 = x,\, (\overrightarrow{X}_n)_{n\in\mathbb{N}} \in B\big) &= P_x^T\big(A, \widetilde{H}_K = \infty\big)\lambda_x P_x^T(B)\mathbf{1}_{\{x\in K\}} \\ &\overset{\text{(II.2.12)}}{=} P_x^T\big(A \,\big|\, \widetilde{H}_K = \infty\big) e_{K,T}(x) P_x^T(B) \end{aligned} \tag{II.2.14}$$

for all $A, B \in \overrightarrow{\mathcal{Z}}_T, x \in T$; here, $\widetilde{H}_K$ is the return time to K defined in (II.2.9).

Following [Tei09], one can then show that there exists a unique measure μ_T on the quotient space Z_T^* of trajectories in $\overleftrightarrow{Z}_T$ modulo time shift, whose restriction to the trajectories hitting K is the pushforward of the measures Q_K^T by projection onto Z_T^*. Under some probability measure $\mathbb{P}_T^{\mathrm{RI}}$, the random interlacements process on T is then defined as the Poisson point process

$$\sum_{i\in\mathbb{N}} \delta_{(w_i^*, u_i)} \text{ on } Z_T^* \times [0,\infty) \text{ with intensity measure } \mu_T \otimes \lambda, \tag{II.2.15}$$

where λ is the one-dimensional Lebesgue measure restricted to $[0,\infty)$. For $u \in (0,\infty)$ we define the random interlacements process ω_u at level u as the sum of $\delta_{w_i^*}$ over all $i \in \mathbb{N}$ with $u_i \in [0,u]$, and the random interlacements set $\mathcal{I}^u$ at level u as the subset of T visited by the (equivalence classes of) random walks w_i^* in the support of ω_u.

We now present an alternative construction of the random interlacements process on trees, which will turn out useful for our purposes. It consists of partitioning the space $\overleftrightarrow{Z}_T$ into subsets according to the highest visited vertex of the contained trajectories. For this purpose, for $x \in T$ define the quantity

$$\breve{e}_T(x) := P_x^T\big(\widetilde{H}_x = \infty, H_{x^-} = \infty\big)\lambda_x P_x^T(H_{x^-} = \infty), \tag{II.2.16}$$

where we recall that H_x and $\widetilde{H}_x$ are the hitting and return times, respectively, of x, defined in (II.2.9). If $x = \varnothing$, we take the convention that $H_{x^-} = \infty$ occurs almost surely. We also define the law of a doubly infinite random walk with the point x at smallest distance from the root $\varnothing$, and which is reached for the first time at time 0, by

$$\overline{Q}_x^T\big((\overleftarrow{X}_n)_{n\in\mathbb{N}} \in A,\, (\overrightarrow{X}_n)_{n\in\mathbb{N}} \in B\big) := P_x^T(A \mid \widetilde{H}_x = \infty, H_{x^-} = \infty) P_x^T(B \mid H_{x^-} = \infty), \tag{II.2.17}$$

for all $A, B \in \overrightarrow{\mathcal{Z}}_T$. Here, we use the convention $H_{x^-} = \infty$ a.s. if $x = \varnothing$. Note that $\breve{e}_T(\varnothing)\overline{Q}_\varnothing^T = Q_\varnothing^T$. We now show that this alternative construction provides us with a random interlacements process as desired.

Theorem II.2.2. *Denote by T a transient weighted tree with conductances $(\lambda_{x,y})_{x\sim y\in T}$. Let $u > 0$, and independently for each $x \in T$, let Γ_x be a $\mathrm{Poi}(u\breve{e}_T(x))$-distributed random variable. Furthermore, let $X_{x,i}$, $i \in \mathbb{N}$, be an independent i.i.d. family of doubly infinite random walks on T with common law $\overline{Q}_x^T$. Denote by $X_{x,i}^*$ the trajectory $X_{x,i}$ modulo time-shift. Then*

$$\sum_{x\in T}\sum_{i=1}^{\Gamma_x} \delta_{X_{x,i}^*} \text{ has the same law as } \omega_u \text{ under } \mathbb{P}_T^{\mathrm{RI}}.$$

Proof. For $x \in T$ we denote by $\overleftrightarrow{Z}_{x,T}$ the subset of $\overleftrightarrow{Z}_T$, see (II.2.13), which contains only those doubly infinite trajectories with highest point equal to x, reached for the first time at time 0, i.e.,

$$\overleftrightarrow{Z}_{x,T} := \Big\{X \in \overleftrightarrow{Z}_T : X_0 = x, H_{x^-}(\overrightarrow{X}) = H_{x^-}(\overleftarrow{X}) = \widetilde{H}_x(\overleftarrow{X}) = \infty\Big\}.$$

Write $Z_{x,T}^*$ for the quotient space of $\overleftrightarrow{Z}_{x,T}$ modulo time shift. Since trajectories on a tree have a unique highest point, the family of sets $Z_{x,T}^*$, $x \in T$, forms a partition of Z_T^*.

For any measure M and measurable set A, write $M\big|_A$ for the restriction $M(A\cap\cdot)$ to A. Recalling the definitions of Q_K^T, $\check{e}_T$ and $\overline{Q}_x^T$ in (II.2.14), (II.2.16) and (II.2.17), we have for all events $A, B \in \overrightarrow{\mathcal{Z}}$ that

$$\begin{aligned}
&Q_{\{x\}}^T\Big|_{\overleftrightarrow{Z}_{x,T}}\big((\overleftarrow{X}_n)_{n\in\mathbb{N}} \in A, (\overrightarrow{X}_n)_{n\in\mathbb{N}} \in B\big)\\
&= P_x^T\big(A, H_{x^-} = \infty, \widetilde{H}_x = \infty\big)\lambda_x P_x^T(B, H_{x^-} = \infty)\\
&= \check{e}_T(x)P_x^T\big(A \,\big|\, H_{x^-} = \infty, \widetilde{H}_x = \infty\big)P_x^T(B \,|\, H_{x^-} = \infty)\\
&= \check{e}_T(x)\overline{Q}_x^T\big((\overleftarrow{X}_n)_{n\in\mathbb{N}} \in A,\ (\overrightarrow{X}_n)_{n\in\mathbb{N}} \in B\big).
\end{aligned}$$

Next, write $(\overline{Q}_x^T)^*$ for the pushforward of $\overline{Q}_x^T$ into the quotient space. If a trajectory $X_x \in \overleftrightarrow{Z}_T$ is such that $X_x^* \in Z_{x,T}^*$, then $Q_{\{x\}}^T$-a.s. we have $X_x \in \overleftrightarrow{Z}_{x,T}$, so we see that $\frac{1}{\check{e}_T(x)}\mu_T\big|_{Z_{x,T}^*} = (\overline{Q}_x^T)^*$. Hence, since Γ_x is a Poisson random variable with parameter $u\check{e}_T(x)$ we deduce that

$$\sum_{i=1}^{\Gamma_x}\delta_{X_{x,i}^*} \text{ is a Poisson point process on } Z_T^* \text{ with intensity measure } u\mu_T\big|_{Z_{x,T}^*}. \tag{II.2.18}$$

Using the restriction property and the mapping theorem for Poisson point processes in order to first remove the trajectories with label bigger than u and then the labels themselves, we see that the interlacements process ω_u as defined below (II.2.15) has the law of a Poisson point process with intensity measure $u\mu_T$.

Furthermore, since the subsets $Z_{x,T}^*$, $x \in T$, form a partition of Z_T^*, due to the superposition theorem for Poisson point processes, taking the sum of (II.2.18) over $x \in T$ yields the law of a Poisson point process with intensity $u\mu_T$, i.e. of ω_u, and the proof is complete. $\square$

The representation of random interlacements via the highest vertex visited by its trajectories, Theorem II.2.2, will be the base of our construction of the Galton–Watson tree via random interlacements, cf. Proposition II.4.2.

Remark II.2.3. Theorem II.2.2 can be seen as a generalization of [Tei09, Theorem 5.1]. Indeed, if $x \in T$ is such that either $x^- \in \mathcal{V}^u := (\mathcal{I}^u)^c$ or $x = \varnothing$, then $x \in \mathcal{V}^u$ if and only if there are no trajectories in $\overleftrightarrow{Z}_{x,T}$ in the support of ω_u. By Theorem II.2.2, this happens independently for each $x \in T$ with probability $\mathbb{P}\Gamma_x = 0 = \exp(-u\check{e}_T(x))$. In other words, the cluster of $\varnothing$ in $\mathcal{V}^u$ has the same law as the cluster of $\varnothing$ when opening each vertex x of T independently with probability $\exp(-u\check{e}_T(x))$. Moreover, $\check{e}_T(x)$ is equal to the function $f_\varnothing(x)$ from [Tei09, (5.1)], and [Tei09, Theorem 5.1] follows readily after rerooting.

Similarly to [Tei09], this can be used to prove the $\mathbb{P}^{\mathrm{GW}}$-a.s. inequality $u_*(\mathcal{T}) > 0$, where $u_*(\mathcal{T})$ is the critical parameter associated to the percolation of $\mathcal{V}^u$ under $\mathbb{P}_{\mathcal{T}}^{\mathrm{RI}}$. Indeed, this follows from the following facts:

- the inequality $\check{e}_T(x) \leqslant \lambda_x \leqslant \lambda_{x,+} + \lambda_{x^-,+}\mathbf{1}_{\{x\neq\varnothing\}}$, and
- the fact that the cluster of $\varnothing$ for Bernoulli percolation on $\mathcal{T}$ with parameter $e^{-2uC}\mathbf{1}_{\{\lambda_{x,+}\leqslant C\}}$, $x \in \mathcal{T}$, is a Galton–Watson tree since $\lambda_{x,+}$, $x \in \mathcal{T}$, are i.i.d. random variables, which is supercritical for first choosing C large enough and then $u > 0$ small enough.

Note that the inequality $u_*(\mathcal{T}) > 0$ can also be seen as a consequence of Theorem II.1.1 as noted below (II.1.6). One can furthermore also similarly prove that $\mathcal{V}^u \cap B_p$ – see (II.1.5) for notation – percolates for $u > 0$ small enough and $p \in (0,1)$ large enough, since it is minorized by Bernoulli percolation on $\mathcal{T}$ with parameter $pe^{-2uC}\mathbf{1}_{\{\lambda_{x,+} \leqslant C\}}$, $x \in \mathcal{T}$.

Remark II.2.4. Note that the trace random walk on $\mathcal{T}^\infty$ of the random walk on $\mathcal{T}$ is a random walk on $\mathcal{T}^\infty$, as follows from instance from [Szn12b, Proposition 1.11]. Therefore, as in [AS18, (1.30), (1.31)], the restriction of φ to $\mathcal{T}^\infty$ has the same law as the Gaussian free field on $\mathcal{T}^\infty$, and so the critical parameters for level set percolation of the Gaussian free field on $\mathcal{T}$ and $\mathcal{T}^\infty$ coincide – note that this remains true in the case of weighted trees. In particular, one can substitute ν by ν^* when proving Theorem II.1.1. Moreover, one can easily prove that $\mathcal{I}^u \cap \mathcal{T}^\infty$ – where $\mathcal{I}^u$ is the random interlacements set on $\mathcal{T}$ – has the same law as the random interlacements set on the graph $\mathcal{T}^\infty$ (note to this effect that $\lambda_x P_x^T(A, \widetilde{H}_K = \infty)$ is equal to $\sum_{y \in \mathcal{T}^\infty} \lambda_{x,y} P_y^T(A, H_K = \infty)$ for each $x \in K$ in (II.2.14)), and thus one can also substitute ν by ν^* when proving Theorems II.1.2 and II.1.3.

II.2.5 An isomorphism theorem

A key tool in our investigations is provided by certain Ray-Knight isomorphism theorems relating the Gaussian free field to random interlacements. Such results have a long history, dating back to Dynkin's isomorphism theorem and, less explicitly, even earlier work by Symanzik [Sym66] as well as Brydges, Fröhlich and Spencer [BFS82]. The exact isomorphism that we are going to use here has been developed in [Szn12a], [Lup16], [Szn16], and then [DPR22].

As before, we still assume some transient weighted tree T to be given. Recalling the definition below (II.2.15) of the random interlacements process ω_u at level u, for $x \in T$ and $u > 0$ let us denote by

$$\begin{aligned}&N_x(u) \text{ the sum over all equivalence classes of trajectories } w^*\\ &\qquad \text{in } \omega_u \text{ of the total number of times } w^* \text{ visits } x.\end{aligned}$$

On some possibly extended probability space, let $\mathcal{E}_x^{(k)}$, $x \in T$ and $k \in \mathbb{N}$, be an i.i.d. family of exponential random variables with parameter one, independent of the random interlacements. The local time $(\ell_{x,u})_{x \in T}$, of random interlacements at level u can then be defined as

$$\ell_{x,u} := \frac{1}{\lambda_x} \sum_{k=1}^{N_x(u)} \mathcal{E}_x^{(k)} \quad \text{for all } x \in T. \tag{II.2.19}$$

We can now state the isomorphism theorem; note that here and below, we use the convention that $H_{\varnothing^-} = \infty$ holds P_x^T-almost surely for any tree T and $x \in T$.

Proposition II.2.5. *Assume that T is a transient tree verifying that for all $x \in T$,*

$$\mathrm{cap}_T(\{X_i, i \in \mathbb{N}\}) = \infty \quad P_x^T(\cdot \,|\, H_{x^-} = \infty)\text{-}a.s. \tag{II.2.20}$$

Then for each $u > 0$, there exists a coupling $\mathbb{Q}_T^u$ of two Gaussian free fields φ and γ on T, a random interlacements process ω_u on T at level u, and i.i.d. exponential *random variables* $\mathcal{E}_x^{(k)}$, $x \in T$ *and* $k \in \mathbb{N}$, *with parameter one such that* φ, $\mathcal{E}_\cdot^{(\cdot)}$ *and* ω_u *are independent, and* $\mathbb{Q}_T^u$*-a.s.,*

$$\gamma_x = -\sqrt{2u} + \sqrt{2\ell_{x,u} + \varphi_x^2} \qquad \text{for all } x \in \mathcal{I}^u, \tag{II.2.21}$$

where $\ell_{x,u}$ *is defined as in* (II.2.19) *and* $\mathcal{I}^u$ *as below* (II.2.15)*.*

Proof. The isomorphism theorem on the so-called cable system, see [Lup16, Proposition 6.3] or [Szn12a, (0.4)] on general graphs, states that

$$|\widetilde{\gamma}_x + \sqrt{2u}| = \sqrt{2\widetilde{\ell}_{x,u} + \widetilde{\varphi}_x^2} \quad \text{for all } x \in \widetilde{T}. \tag{II.2.22}$$

Here, $\widetilde{T}$ denotes the cable system associated to T, and $\widetilde{\gamma}, \widetilde{\varphi}$ and $\widetilde{\ell}_{\cdot,u}$ correspond to Gaussian free fields and local times of random interlacements on $\widetilde{T}$. We restrain from introducing the cable system $\widetilde{T}$ in this article, as this metric structure will be only used in this proof; see [Lup16] for references. We only note that $T \subset \widetilde{T}$, and that the restrictions γ, φ and $\ell_{\cdot,u}$ of $\widetilde{\gamma}, \widetilde{\varphi}$ and $\widetilde{\ell}_{\cdot,u}$ to T have the same laws as the corresponding fields from Proposition II.2.5. In order to deduce (II.2.21) from (II.2.22), we note that

$$\begin{gathered}\text{each trajectory } w^* \text{ of } \omega_u \text{ is either included in a connected component of} \\ \{x \in \widetilde{T} : \widetilde{\gamma}_x > -\sqrt{2u}\} \text{ or of } \{x \in \widetilde{T} : \widetilde{\gamma}_x < -\sqrt{2u}\},\end{gathered} \tag{II.2.23}$$

which is a simple consequence of [DPR22, (3.19)]. Moreover, by [DPR22, Theorem 1.1, (1)] and symmetry it holds that

$$\text{all the connected components of } \big\{x \in \widetilde{T} : \widetilde{\gamma}_x < -\sqrt{2u}\big\} \text{ have finite capacity.} \tag{II.2.24}$$

Under hypothesis (II.2.20), for each trajectory w^* of ω_u, it follows from Theorem II.2.2 that the capacity of w^* is $\mathbb{P}^{\mathrm{RI}}$-a.s. infinite, and thus by (II.2.23) and (II.2.24), w^* must be included in $\{x \in T : \gamma_x > -\sqrt{2u}\}$. The identity (II.2.21) then follows readily from (II.2.22). □

Actually Proposition II.2.5 remains true on any locally finite graph, but we will only need it on trees in this paper. We will prove that the hypothesis (II.2.20) holds when $T = \mathcal{T}$ is the Galton–Watson tree introduced in Section II.2.1, see Proposition II.5.8. Therefore, in our context, Proposition II.2.5 will readily imply the inclusion (II.1.8) (defining $\widehat{E}^{\geqslant\sqrt{2u}}$ therein as the level sets of the field γ), which is the first step in the proof of Theorem II.1.1 as explained in Section II.1.2.

Remark II.2.6. Following the proof of [AS18, Proposition 5.2], one can easily show that a version of the isomorphism (II.2.21) holds on Galton–Watson trees with unitary conductances and finite mean offspring distribution m. They prove this isomorphism using conditions different from (II.2.20), namely that the sign clusters of the Gaussian free field on the cable system are bounded and a certain boundedness condition of the Green function; in view of [DPR22, Theorem 1.1, (2)], the boundedness of the sign clusters is actually sufficient. It turns out that in the context of random conductances (and in particular, if the mean offspring distribution m is infinite or if $(\lambda_{x,y})_{x\sim y\in\mathcal{T}}$ are not i.i.d. conductances conditionally on the non-weighted graph $\mathcal{T}$), it will be easier to deduce the isomorphism (II.2.21) from condition (II.2.20) instead. Indeed, we will prove that condition (II.2.20) holds in Proposition II.5.8 using tools very similar to the proof Theorem II.1.2.

II.3 Warm up: a first proof in an easier setting

In this section we give a simple proof of the inequality $h_*(\mathcal{T}) > 0$ under the stronger assumption that $m > 2$. Note that this is also proved via different means in the setting of Galton–Watson trees with unit weights in [AS18]. The proof in [AS18] could be adapted to the setting of random weights, but it is currently not clear to us how to adapt it to the setting $m \in (1, 2]$. Moreover, we believe that our proof in this section for $m > 2$ is simpler, and at the same time it exhibits the difficulties that are showing up when proving Theorem II.1.1 for the case $m \in (1, 2]$. What is more, our proof will also provide us with an example of a weighted Galton–Watson tree where $h_* = \infty$, see (II.3.4), showing that the phase transition is not always non-trivial in our context.

In order to introduce our setup, we consider the weighted Galton–Watson tree $\mathcal{T} \subseteq \mathcal{X}$ from Subsection II.2.1. Recall that the law of the weights below each vertex is a probability measure ν on $[0, \infty)^{\mathbb{N}}$, and these weights are chosen independently for different vertices, and that the function $\pi((\lambda_i)_{i\in\mathbb{N}})$ denotes the number of offspring, with mean m, see (II.2.4) and (II.2.5). Contrary to the rest of this article, in this section we do not make the usual assumption (II.2.3) on the weights λ, but keep the assumption $m > 1$. In the following, by F we denote the cumulative distribution function of a standard normal variable.

Proposition II.3.1. *For all $h \geqslant 0$ such that there exists $M > 0$ with*

$$\mathbb{E}^{\nu}\big[\pi((\lambda_i)_{i\in\mathbb{N}})\mathbf{1}_{\{\sum_{i\in\mathbb{N}}\lambda_i \leqslant M\}}\big]F(-h\sqrt{2M}) > 1, \tag{II.3.1}$$

we have $h_* \geqslant h$.

Proof. In this proof, we use the construction of the Gaussian free field as in [AČ20a, Section 2.1] through independent standard normal variables, extended to our case of non-regular trees. Let $(Z_x)_{x\in\mathcal{X}}$ be a family of independent standard normal variables under P. Then, conditionally on the realization of the tree $\mathcal{T}$, define $\varphi_\varnothing := \sqrt{g^{\mathcal{T}}(\varnothing,\varnothing)}Z_\varnothing$ and, recursively in the distance from the root, we set

$$\varphi_x := P_x^{\mathcal{T}}(H_{x^-} < \infty)\varphi_{x^-} + \sqrt{g^{\mathcal{T}}_{\mathcal{T}_x}(x,x)}Z_x.$$

Using the Markov property (II.2.11) with $U = \mathcal{T}_x$, one can check that the field $(\varphi_x)_{x\in\mathcal{T}}$ defined this way has the law of a Gaussian free field on $\mathcal{T}$. Moreover, using the bound $g^{\mathcal{T}}_{\mathcal{T}_x}(x,x) \geqslant \frac{1}{\lambda_x}$, conditioned on the realization of the weighted tree $\mathcal{T}$, the previous display then entails the implication

$$\{Z_x > h\sqrt{\lambda_x}, \varphi_{x^-} > h\} \Rightarrow \{\varphi_x > h\}, \tag{II.3.2}$$

with the convention $\varphi_{x^-} > h$ a.s. if $x = \varnothing$.

We define now the random set $S(h, M) \subseteq \mathcal{T}$ as

$$S(h, M) := \{\varnothing\} \cup \{x \in \mathcal{T}\backslash\{\varnothing\} \colon Z_{x^-} > h\sqrt{2M}, \lambda_{x^-,+} \leqslant M\}.$$

Note that on the event $x \in \mathcal{T}$, the mean number of children the vertex x has in $S(h, M)$ satisfies

$$\begin{aligned}\mathbb{E}^{\mathrm{GW}} \otimes E\big[|G_x^{S(h,M)}| \,\big|\, x \in \mathcal{T}\big] &= \mathbb{E}^{\mathrm{GW}}\Big[\pi((\lambda_{x,xi})_{i\in\mathbb{N}})\mathbf{1}_{\{\lambda_{x,+}\leqslant M\}}P\big(Z_x > h\sqrt{2M}\big) \,\big|\, x \in \mathcal{T}\Big]\\ &= \mathbb{E}^{\nu}\big[\pi((\lambda_i)_{i\in\mathbb{N}})\mathbf{1}_{\{\sum_{i\in\mathbb{N}}\lambda_i\leqslant M\}}\big]F(-h\sqrt{2M}).\end{aligned} \tag{II.3.3}$$

Moreover, for each $x \in \mathcal{T}$, the number of children of x in $S(h, M)$ only depends on $(\lambda_{x,xi})_{i\in\mathbb{N}}$ and Z_x, which are independent in x. Therefore, the connected component of $\varnothing$ in $S(h, M)$ has the law of a Galton–Watson tree with mean given by (II.3.3). Due to assumption (II.3.1), this mean is strictly larger than one and thus this Galton–Watson tree has a positive probability to be infinite. Finally, it follows easily from (II.3.2) and the inequality $\lambda_x \leqslant \lambda_{x,+} + \lambda_{x^-,+}$ that $\varphi_{x^-} \geqslant h$ for each $x \neq \varnothing$ in the connected component of $\varnothing$ in $S(h, M)$, and we can conclude. □

Let us now present two interesting assumptions on the mean offspring m and on the distribution of the weights $(\lambda_i)_{i\in\mathbb{N}}$, under which (II.3.3) is satisfied.

- Assume $m > 2$. We can find some $M > 0$ such that $\mathbb{E}_\nu\big[\pi((\lambda_i)_{i\in\mathbb{N}})\mathbf{1}_{\{\sum_{i\in\mathbb{N}}\lambda_i\leqslant M\}}\big] > 2$ since the left hand side converges to m as $M \to \infty$, and then a positive level h such that $F(-h\sqrt{2M})$ is close enough to $\frac{1}{2}$, so that (II.3.3) is bigger than 1, providing us with $h_* > 0$.
- Let N be a random variable taking values in $\mathbb{N}$ with infinite mean under ν. Define $(\lambda_i)_{i\in\mathbb{N}}$ via $\lambda_i = 1/N$ for all $i \leqslant N$ and $\lambda_i = 0$ for all $i > N$. Then $\sum_{i\in\mathbb{N}}\lambda_i = 1$ and $m = \infty$. Hence for each $h > 0$ since $F(-h\sqrt{2}) > 0$ we have that the left-hand side of (II.3.1) is infinite for $M = 1$, that is
$$h_* = \infty. \tag{II.3.4}$$

Note that we have not taken advantage of the assumption (II.1.2) in this section; as a consequence, the inequality $h_* > 0$ from Theorem II.1.1 holds when $m > 2$ even without this assumption. It is not clear whether this assumption is necessary when $m \in (1, 2]$.

II.4 A simultaneous exploration of the tree via random interlacements

In this section we introduce an explorative construction procedure for supercritical Galton–Watson trees via random interlacements, which is tailor-made for our purposes. To the best of our knowledge, previous approaches to problems related to random interlacements on random graphs generated the random interlacements process only after having complete information on the realization of the graph. In our setting, however – in order to gain a better control on both, the Gaussian free field and the local times of random interlacements – we generate the underlying graph $\mathcal{T}$ and the random interlacements process simultaneously. In some sense, this construction provides us with independence properties that will turn out useful in creating coarse-grained "good" parts of the interlacements set and the level sets of an independent Gaussian free field.

In particular, in Subsection II.4.1 we will first construct a "single small piece" of the tree. This piece will consist of the trace of a finite random walk trajectory exploring the Galton–Watson tree at each vertex visited by the walk. We will call a piece of the tree constructed in this way a *watershed*. Repeating this procedure iteratively for boundary vertices of previously constructed watersheds, in Subsection II.4.2 we will then patch together all watersheds constructed in this way, as well as some remaining ends; the resulting object will be denoted by $\mathcal{T}^{\mathbf{W}}$. It turns out that $\mathcal{T}^{\mathbf{W}}$ will be a tree with the following properties: it is a weighted Galton–Watson tree, and the random walk trajectories used to construct its watersheds can be interpreted as part of a random interlacements process on $\mathcal{T}^{\mathbf{W}}$. This last property will be shown in Subsection II.4.3 with the help of Theorem II.2.2.

II.4.1 Watersheds

We now introduce the notion of a watershed starting at a vertex $x \in \mathcal{X} \backslash \{\varnothing\}$, with parameters $L \in \mathbb{N}, L \geqslant 2$, and $\kappa \in [0, \infty)$, on which all the objects constructed in this subsection will depend implicitly (the case $x = \varnothing$ is excluded for technical reasons). A watershed will form a finite subtree of a Galton–Watson tree, and it will be constructed as the trace of a random walk that is visiting vertices starting at the root x of a subtree of $\mathcal{X}$, until – if successful – at least L vertices of the subtree are explored in a suitable way. The parameter κ will represent the conductance of the edge between x and x^-, which is thus fixed. In order to facilitate readability, we will denote objects pertaining to watersheds by boldface letters throughout.

The watershed will be defined by means of a sequence of triplets

$$(\mathbf{T}_k, (\boldsymbol{\lambda}_{y,z})_{y \sim z, y,z \in \mathbf{T}_k}, \mathbf{X}_k)_{k \in \mathbb{N}_0},$$

such that, for each $k \in \mathbb{N}_0$, we have that

- $\mathbf{T}_k \subset \mathcal{X}$ is connected,
- the $\boldsymbol{\lambda}_{y,z} \in (0, \infty)$ are (symmetric) weights on the edges $\{y, z\}$ of $\mathbf{T}_k$, and
- $\mathbf{X}_k$ is a random variable with $\mathbf{X}_k \in \mathbf{T}_k$.

In order to construct this sequence, we first fix

$$(\boldsymbol{\lambda}_i^{(k)})_{i \in \mathbb{N}}, k \in \mathbb{N}_0, \text{ an i.i.d. family of random variables with common law } \nu, \tag{II.4.1}$$

and proceed by induction. We start with $\mathbf{T}_0$ as being characterized uniquely by the specification of its vertex set $\{x^-, x\}$ (mind that x^- is well-defined as we assumed $x \neq \varnothing$), as well as the conductance $\boldsymbol{\lambda}_{x^-,x} := \kappa$ and the almost sure equality $\mathbf{X}_0 := x$.

We first define the the triplet $(\mathbf{T}_k, (\boldsymbol{\lambda}_{y,z})_{y \sim z, y,z \in \mathbf{T}_k}, \mathbf{X}_k)$ until some stopping time $\widetilde{V}_L(\mathbf{X})$, that we will define in (II.4.3), and thus assume that this triplet is given for some non-negative integer $k < \widetilde{V}_L(\mathbf{X})$. Recalling the definition below (II.2.1) of the boundary $\partial \mathcal{T}$ for a tree $\mathcal{T}$, we then define $(\mathbf{T}_{k+1}, (\boldsymbol{\lambda}_{y,z})_{y \sim z, y,z \in \mathbf{T}_{k+1}}, \mathbf{X}_{k+1})$ as follows:

- if $\mathbf{X}_k \in \partial \mathbf{T}_k$, we proceed as follows. Let $\mathbf{N}_k := |\{\mathbf{X}_0, \ldots, \mathbf{X}_k\}|$, and construct the offspring of $\mathbf{X}_k$ via $\boldsymbol{\lambda}^{(\mathbf{N}_k)}$. More precisely, in Ulam-Harris notation, define $\mathbf{T}_{k+1}$ as the union of $\mathbf{T}_k$ with the set of offspring of $\mathbf{X}_k$, that is with $\{\mathbf{X}_k i, 1 \leqslant i \leqslant \pi((\boldsymbol{\lambda}_i^{(\mathbf{N}_k)})_{i \in \mathbb{N}})\}$, so $\mathbf{T}_{k+1}$ again is a tree. By definition, the number of offspring of $\mathbf{X}_k$ in $\mathbf{T}_{k+1}$ has distribution μ. Furthermore, the weights $\boldsymbol{\lambda}$ on $\mathbf{T}_{k+1}$ are the same as on $\mathbf{T}_k$, where in addition we now attribute weights $\boldsymbol{\lambda}_{\mathbf{X}_k, \mathbf{X}_k i} := \boldsymbol{\lambda}_i^{(\mathbf{N}_k)}$ for $1 \leqslant i \leqslant \pi((\boldsymbol{\lambda}_i^{(\mathbf{N}_k)})_{i \in \mathbb{N}})$ to the edges which are contained in $\mathbf{T}_{k+1}$ but not in $\mathbf{T}_k$.
- if $\mathbf{X}_k \notin \partial \mathbf{T}_k$, then we set $\mathbf{T}_{k+1} := \mathbf{T}_k$, and the weights $\boldsymbol{\lambda}$ on $\mathbf{T}_{k+1}$ are the same as on $\mathbf{T}_k$.

In both of the above cases, in order to construct $\mathbf{X}_{k+1}$, we consider a random walk transition of $\mathbf{X}_k$ on $\mathbf{T}_{k+1}$; hence, independently of everything else, we define the random variable $\mathbf{X}_{k+1}$ as a neighbor of $\mathbf{X}_k$ in $\mathbf{T}_{k+1}$, which is equal to $y \sim \mathbf{X}_k$, $y \in \mathbf{T}_{k+1}$, with probability $\boldsymbol{\lambda}_{\mathbf{X}_k,y}/\boldsymbol{\lambda}_{\mathbf{X}_k}$, where $\boldsymbol{\lambda}_{\mathbf{X}_k}$ is a normalizing constant defined similarly to (II.2.8). Note that, as long as x^- is not reached by $\mathbf{X}$, the event $\{\mathbf{X}_k \in \partial\mathbf{T}_k\}$ above corresponds to the event $\{\mathbf{X}_k \notin \{\mathbf{X}_0, \ldots, \mathbf{X}_{k-1}\}\}$.

We iterate the above procedure in k until reaching the stopping time $\widetilde{V}_L(\mathbf{X})$ that we are about to define. For this purpose, set $H_{x^-}(\mathbf{X})$ to be the first hitting time of x^- by $\mathbf{X}$, defined similarly as in (II.2.9), and

$$V_L := V_L(\mathbf{X}) := \inf\{k \geqslant 0 : |\{\mathbf{X}_0, \ldots, \mathbf{X}_k\}| \geqslant L\} \wedge H_{x^-}(\mathbf{X}) \tag{II.4.2}$$

the first time at which the random walk $\mathbf{X}$ has visited L different vertices, or x^- is hit. Then let

$$\widetilde{V}_L := \widetilde{V}_L(\mathbf{X}) := \begin{cases} \inf\left\{n \geqslant V_L \colon \mathbf{X}_n = \mathbf{X}^-_{V_L}\right\} & \text{if } V_L(\mathbf{X}) < H_{x^-}(\mathbf{X}), \\ H_{x^-}(\mathbf{X}) & \text{if } V_L(\mathbf{X}) = H_{x^-}(\mathbf{X}), \end{cases} \tag{II.4.3}$$

where we always use the convention $\inf \varnothing = \infty$. In words, $\widetilde{V}_L(\mathbf{X})$ is the first time the parent of $\mathbf{X}_{V_L}$ is visited if $H_{x^-} > V_L$, and otherwise it equals H_{x^-}. That is, we stop our recursive construction the first time either x^- is visited by $\mathbf{X}$, or $\mathbf{X}$ has visited L vertices at time V_L, and then $\mathbf{X}^-_{V_L}$ is hit. Note that it is possible that neither x^-, nor $\mathbf{X}^-_{V_L}$ after time V_L, are visited, and in this case $\widetilde{V}_L = \infty$, i.e., we continue our recursive construction indefinitely. Otherwise, we stop the recursion at time $\widetilde{V}_L$, and for each $k \geqslant \widetilde{V}_L$ we define $(\mathbf{T}_k, (\boldsymbol{\lambda}_{y,z})_{y\sim z, y,z\in\mathbf{T}_k}, \mathbf{X}_k) := (\mathbf{T}_{\widetilde{V}_L}, (\boldsymbol{\lambda}_{y,z})_{y\sim z,y,z\in\mathbf{T}_{\widetilde{V}_L}}, \mathbf{X}_{\widetilde{V}_L})$. We also abbreviate $(\mathbf{T}, \boldsymbol{\lambda}, \mathbf{X}) := (\mathbf{T}_k, (\boldsymbol{\lambda}_{y,z})_{y\sim z,y,z\in\mathbf{T}_k}, \mathbf{X}_k)_{k\in\mathbb{N}_0}$. This concludes the recursive construction of this triplet.

The process $(\mathbf{T}, \boldsymbol{\lambda}, \mathbf{X})$ is called *watershed process*, and we denote by

$$\mathbf{Q}_x^{\kappa,L} \text{ the law of the watershed process } (\mathbf{T}, \boldsymbol{\lambda}, \mathbf{X}) \tag{II.4.4}$$

starting at $x \in \mathcal{X}\backslash\{\varnothing\}$, with parameters $L \in \mathbb{N}$ and $\kappa > 0$. Similarly to the above, if we replace the evolving state space of $\mathbf{X}$ by a fixed tree T, under the law P_x^T of the simple random walk X from (II.2.7), we define $\widetilde{V}_L = \widetilde{V}_L(X)$ similarly as in (II.4.3). In the following proposition, we explain how the process $(\mathbf{T}, \boldsymbol{\lambda}, \mathbf{X})$ can be considered a random walk exploration of the initial Galton–Watson tree $\mathcal{T}$ from Section II.2.1.

Proposition II.4.1. *For all $x \in \mathcal{X}\backslash\{\varnothing\}$, $\kappa > 0$, and $L \in \mathbb{N}$, the process $(\mathbf{T}, \boldsymbol{\lambda}, \mathbf{X})$ under $\mathbf{Q}_x^{\kappa,L}$ has the same law as $(\mathcal{T}^X_{k\wedge\widetilde{V}_L}, (\lambda_{y,z})_{y,z\in\mathcal{T}^X_{k\wedge\widetilde{V}_L}}, X_{k\wedge\widetilde{V}_L})_{k\in\mathbb{N}_0}$ under $\mathbb{E}^{\mathrm{GW}}[P_x^{\mathcal{T}}(\cdot) \,|\, \lambda_{x,x^-} = \kappa, x \in \mathcal{T}]$, where:*

- *conditionally on $(\mathcal{T}, (\lambda_{y,z})_{y,z\in\mathcal{T}})$, the process (X_n) is the random walk on $\mathcal{T}$ defined in Subsection II.2.3.*
- *for $k \in \mathbb{N}$, the set $\mathcal{T}_k^X := \{z \in \mathcal{T} : z \sim X_i \text{ for some } i \leqslant k-1\}$ is the subset of $\mathcal{T}$ adjacent to the trace of $\{X_1, \ldots, X_{k-1}\}$.*

Proof. At time k, for $1 \leqslant k \leqslant \widetilde{V}_L$, we sample the offspring of $\mathbf{X}_{k-1}$ independently of everything else via their conductances according to ν if it is the first time $\mathbf{X}_{k-1}$ was visited by $\mathbf{X}$; therefore, $\mathbf{T}_k$ is a Galton–Watson tree restricted to the offspring of the vertices explored by $\mathbf{X}$ before time $k-1$, union with the edge $\mathbf{T}_0 = \{x^-, x\}$. After time $\widetilde{V}_L$ (if it is finite), $\mathbf{T}_k$ stays constant equal to $\mathbf{T}_{\widetilde{V}_L}$, and $\mathbf{X}_k$ constant equal to $\mathbf{X}_{\widetilde{V}_L}$.

Similarly, when $\mathbf{X}$ at time $1 \leqslant k \leqslant \widetilde{V}_L$ performs a jump, the offspring of the point $\mathbf{X}_{k-1}$ has already been generated according to ν, either at step k or in a preceding step, and then $\mathbf{X}_{k-1}$ jumps to $\mathbf{X}_k$ with the probability

$$\frac{\boldsymbol{\lambda}_{\mathbf{X}_{k-1},\mathbf{X}_k}}{\boldsymbol{\lambda}_{\mathbf{X}_{k-1}}},$$

which is analogous to (II.2.7). Hence both $\mathbf{X}$ and X behave like a random walk on their respective trees until time $\widetilde{V}_L$, and $\widetilde{V}_L$ corresponds for both walks to the first time either x^- is hit, or L different vertices have been visited by the walk, and then, denoting by y the last of these L vertices, y^- has been hit. One can easily conclude. □

Let us finish this section with an observation which will be essential in the proof of Lemma II.5.3 below. For this purpose, first define under $\mathbf{Q}_x^{\kappa,L}$ the watershed $\mathbf{W}$ as the path of $\mathbf{X}$ until $V_L - 1$, that is

$$\mathbf{W} := \{\mathbf{X}_0, \ldots, \mathbf{X}_{V_L-1}\}. \tag{II.4.5}$$

Using the convention $\boldsymbol{\lambda}_{y,yi} = 0$ if $yi \notin \mathbf{T}$, by (II.4.2), (II.4.3) and the construction of the weights $\boldsymbol{\lambda}_{y,z}$, $y \sim z \in \mathbf{T}_k$, we have under $\mathbf{Q}_x^{\kappa,L}$ that

$$\begin{aligned} &(\boldsymbol{\lambda}_{x,xi})_{i\in\mathbb{N}} = (\boldsymbol{\lambda}_i^{(1)})_{i\in\mathbb{N}}, \text{ and if } V_L(\mathbf{X}) < H_{x^-}(\mathbf{X}), \text{ then} \\ &\big\{(\boldsymbol{\lambda}_{y,yi})_{i\in\mathbb{N}} : y \in \mathbf{W}\backslash\{x\}\big\} = \big\{(\boldsymbol{\lambda}_i^{(k)})_{i\in\mathbb{N}} : k \in \{2,\ldots,L-1\}\big\}, \end{aligned} \tag{II.4.6}$$

which follows simply from the fact that the conductances $(\boldsymbol{\lambda}_{y,yi})_{i\in\mathbb{N}}$ are equal to $(\boldsymbol{\lambda}_i^{(k)})_{i\in\mathbb{N}}$ if y is the k-th vertex visited by $\mathbf{X}$.

II.4.2 Patching together watersheds

In the previous subsection we explained how to construct a watershed process $(\mathbf{T}, \boldsymbol{\lambda}, \mathbf{X})$ starting at an arbitrary vertex. We will now iteratively patch together watersheds at the endpoints of previously generated watersheds. The union $\mathcal{T}_-^{\mathbf{W}}$ of such watersheds will already constitute a transient subset of the random interlacements set on the Galton–Watson tree. Embellishing $\mathcal{T}_-^{\mathbf{W}}$ with some further "ends" will yield a tree $\mathcal{T}^{\mathbf{W}}$ which has the law of the weighted Galton–Watson tree we are interested in.

We will now give an informal description of this procedure and provide mathematical details below. To patch the watersheds together, we will introduce another tree F, the *tree of free points*. This tree encodes the points at which watersheds will be patched together in the construction outlined above, i.e. F is a tree in $\mathcal{X}$ and, at the same time, to each free point $a \in F$ we associate another point $\widehat{a} \in \mathcal{X}$ – which will turn out to also be an element of the tree $\mathcal{T}^{\mathbf{W}}$ to be constructed – at which we will start a new watershed. Patching up the watersheds through their vertices corresponding to free points, we will then be able to construct inductively the tree $\mathcal{T}_-^{\mathbf{W}}$. We refer to Figure II.1 for an illustration.

We will define the weighted tree F with weights denoted by $\lambda^F_{a,a'}$, $a \sim a' \in F$, through a recursively defined sequence (F_k) of weighted trees, such that to each $a \in F_{k-1}$ we associate a watershed $(\mathbf{T}^a, \boldsymbol{\lambda}^a, \mathbf{X}^a)$ starting in $\widehat{a}$ as defined in the last subsection, and to each vertex $a \in F_k$ we associate another vertex $\widehat{a} \in \mathcal{X}$.

As explained above, this construction of F as well as the corresponding watersheds, will depend on a parameter $L \in \mathbb{N}$, that we fix for the rest of this section. We

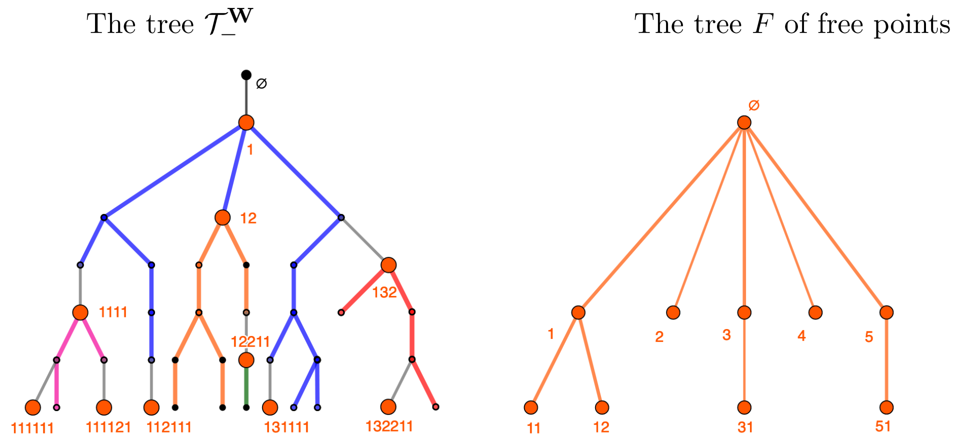


Figure II.1: (A finite subset of) the tree $\mathcal{T}^{\mathbf{W}}$, on the left, has some highlighted vertices, denoted by a coding $\hat{a}$, at which a new watershed is generated. Those points correspond to points in (a finite subset of) the tree of free points F on the right, where they have a different coding a. For instance $\widehat{72} = 132211$. We highlighted with different colors each $a \in F$ on the right and on the left the corresponding point $\hat{a}$ and the path on $\mathcal{T}^{\mathbf{W}}$ visited by the random walk $\mathbf{X}^a$, which generates the watershed below $\hat{a}$. On the right, the points 5 and 6 are part of the tree of free points, but the corresponding vertices $\hat{5}$ and $\hat{6}$ do not appear yet on the left since they are below the 6th generation.

denote by $\mathbf{P}_L^W$ the probability measure under which these objects are constructed. For technical reasons, we will start the first watershed in the point 1 instead of $\varnothing$.

First set $F_{-1} := \varnothing$, $F_0 := \{\varnothing\}$ take $\widehat{\varnothing} = 1$, and generate some weights $(\lambda^{\mathbf{W}}_{\varnothing,i})_{i\in\mathbb{N}}$ with law ν. Now assume F_{k-1} and F_k are given for some $k \in \mathbb{N}_0$, and that each point $a \in F_k$ is associated to a point $\hat{a} \in \mathcal{X}$. We define F_{k+1} as follows. For each $a \in F_k \backslash F_{k-1}$, we generate

$$\text{an independent watershed } (\mathbf{T}^a, \boldsymbol{\lambda}^a, \mathbf{X}^a) \text{ with law } \mathbf{Q}_{\hat{a}}^{\lambda^F_{a^-,a},L}, \tag{II.4.7}$$

as defined in (II.4.4). Note that $\varnothing^-$ is not well-defined, but for $a = \varnothing$ we will take the convention

$$\lambda^F_{a^-,a} := \lambda^{\mathbf{W}}_{\varnothing,1}. \tag{II.4.8}$$

The watershed $(\mathbf{T}^a, \boldsymbol{\lambda}^a, \mathbf{X}^a)$ will be used to encode the set of *free points* via the following set

$$\mathfrak{F}_a := (\partial \mathbf{T}^a_{V_L}) \backslash \{\mathbf{X}^a_{V_L}\}; \tag{II.4.9}$$

in other words, apart from $\mathbf{X}^a_{V_L}$, the set $\mathfrak{F}_a$ corresponds to the vertices on the boundary of the tree $\mathbf{T}^a$ once the walk has either visited L vertices or hit $\hat{a}^-$. The vertex $\mathbf{X}^a_{V_L}$ is excluded from this set since, by definition of V_L, the first generation of the tree below $\mathbf{X}^a_{V_L}$ has already been explored by $\mathbf{T}^a$. Equivalently, the points in $\mathfrak{F}_a$ are vertices not visited by the random walk $\mathbf{X}^a_k$, $1 \leqslant k \leqslant V_L$, but adjacent to its trace, and which have thus already been generated during the construction of the watershed. We will then generate new watersheds from the vertices in $\mathfrak{F}_a$. We can now define the next generation of the tree of free points

$$F_{k+1} := F_k \cup \bigcup_{a \in F_k \setminus F_{k-1}} \bigcup_{i=2}^{|\mathfrak{F}_a|} \{ai\}. \tag{II.4.10}$$

In other words, the sets of points $\mathfrak{F}_a$, $a \in F_k \setminus F_{k-1}$, are used to build the $(k+1)$-st level of the tree of free points, and we define $\widehat{ai}$ as the i-th element (in lexicographic order) of $\mathfrak{F}_a$ for each $1 \leqslant i \leqslant |\mathfrak{F}_a|$. Note that the union over i starts at 2 for technical reasons, cf. property ii) in Definition II.5.1, and the explanation in the second paragraph thereafter. In particular, $\widehat{a1}$ is well-defined but not part of the tree F, for instance $\widehat{1} = 1111$ in Figure II.1.

We moreover define the conductance of the edge above the vertex ai for F_{k+1} as

$$\lambda^F_{a,ai} := \boldsymbol{\lambda}^a_{(\widehat{ai})^-, \widehat{ai}}, \tag{II.4.11}$$

whereas the conductances on $F_k \subset F_{k+1}$ stay the same as before. This concludes the inductive definition of the sequence (F_k), and the tree of free points is simply defined via

$$F := \bigcup_{k \in \mathbb{N}_0} F_k, \tag{II.4.12}$$

endowed with the same conductances as the F_k, $k \in \mathbb{N}_0$.

Let us now explain how to construct a Galton–Watson tree by gluing together the watersheds $(\mathbf{T}^a, \boldsymbol{\lambda}^a, \mathbf{X}^a)$, $a \in F$. We first set

$$\mathcal{T}^{\mathbf{W}}_{-} := \Big\{2, \dots, \pi\big((\lambda^{\mathbf{W}}_{\varnothing,j})_{j \in \mathbb{N}}\big)\Big\} \cup \bigcup_{a \in F} \mathbf{T}^a; \tag{II.4.13}$$

in other words, $\mathcal{T}^{\mathbf{W}}_{-}$ consists of a first generation with weights $(\lambda^{\mathbf{W}}_{\varnothing,j})_{j \in \mathbb{N}}$, and the union of the watersheds $\mathbf{T}^a$, $a \in F$; note that the root $\varnothing$ belongs to $\mathbf{T}^{\varnothing}$ by (II.4.7) and the convention $\widehat{\varnothing} = 1$, cf. (II.4.8) also, and in particular $\varnothing \in \mathcal{T}^{\mathbf{W}}_{-}$. One can view $\mathcal{T}^{\mathbf{W}}_{-}$ as a tree in $\mathcal{X}$, and we endow each of its edges $\{x, y\}$ such that $x, y \in \mathbf{T}^a$ for some $a \in F$ with the conductance $\boldsymbol{\lambda}^a_{x,y}$. Note that each edge $\{x, y\}$ of $\mathcal{T}^{\mathbf{W}}_{-}$ is also an edge of $\mathbf{T}^a$ for some $a \in F$, and in fact, for each $a \in F$, $\mathbf{T}^a$ and $\mathbf{T}^{a^-}$ have exactly one edge in common: $\{\widehat{a}^-, \widehat{a}\}$. Moreover, in view of (II.4.7) and (II.4.11), $\boldsymbol{\lambda}^a_{\widehat{a}^-,\widehat{a}} = \lambda^F_{a^-,a} = \boldsymbol{\lambda}^{a^-}_{\widehat{a}^-,\widehat{a}}$, hence the conductances of the tree $\mathcal{T}^{\mathbf{W}}_{-}$ are uniquely defined.

Observe that the tree $\mathcal{T}^{\mathbf{W}}_{-}$ is not yet a Galton–Watson tree with the desired offspring distribution since for some vertices $x \in \mathcal{T}^{\mathbf{W}}_{-}$ we did not construct their descendants: this is the case if $x = \widehat{a1}$ for some $a \in F$ (see (II.4.10)), or if x is in the boundary of $\mathbf{T}^a_{\tilde{V}_L} \setminus \mathbf{T}^a_{V_L}$ (since no vertices correspond to free points in this part of the watershed). Therefore, we now add some ends to those points in order to complete the construction of the Galton–Watson tree. More precisely, define independently of everything else

$$\begin{gathered}\text{an independent family of Galton–Watson trees } (\mathcal{T}^x)_{x \in \mathcal{X}},\\ \text{each } \mathcal{T}^x \text{ with the same law as } x \cdot \mathcal{T} \text{ under } \mathbb{P}^{\mathrm{GW}}.\end{gathered} \tag{II.4.14}$$

In other words, $\mathcal{T}^x$ is a Galton–Watson tree rooted at x. We now define $\mathcal{T}^{\mathbf{W}}$ as the weighted tree obtained from the union of $\mathcal{T}^{\mathbf{W}}_{-}$ with the $\mathcal{T}^x$, $x \in \partial\mathcal{T}^{\mathbf{W}}_{-}$, endowed with their respective conductances, and we denote by $\lambda^{\mathbf{W}}$ the conductances on $\mathcal{T}^{\mathbf{W}}$. We then have that for all $L \in \mathbb{N}$,

$$\mathcal{T}^{\mathbf{W}} \text{ has the same law under } \mathbf{P}_L^W \text{ as the Galton–Watson tree } \mathcal{T} \text{ under } \mathbb{P}^{\mathrm{GW}}; \tag{II.4.15}$$

indeed, it follows from Proposition II.4.1 and (II.4.7) that, conditionally on $\mathbf{T}^{a'}$, $a' \in F_{k-1}$, a single watershed $\mathbf{T}^a$, $a \in F_k \backslash F_{k-1}$, has the same law as a Galton–Watson tree restricted to this watershed, conditionally on $\boldsymbol{\lambda}^a_{\widehat{a}^-,\widehat{a}} = \lambda^F_{a^-,a}$. Since $\lambda^F_{a^-,a} = \boldsymbol{\lambda}^{a^-}_{\widehat{a}^-,\widehat{a}} (= \boldsymbol{\lambda}^a_{\widehat{a}^-,\widehat{a}})$ by (II.4.7) and (II.4.11) we obtain that the conductances between each vertex $x \in \mathcal{T}^{\mathbf{W}}_- \backslash \partial \mathcal{T}^{\mathbf{W}}_-$ and its offspring are distributed independently according to ν. Note that, for each $x \in \partial \mathcal{T}^{\mathbf{W}}_-$, the subtree $\mathcal{T}^{\mathbf{W}}_x := (\mathcal{T}^{\mathbf{W}})_x$ equals $\mathcal{T}^x$ with the desired offspring distribution by definition in (II.4.14) and below, and we conclude that (II.4.15) holds true.

II.4.3 Watersheds and random interlacements

In the previous subsections, we generated simultaneously the Galton–Watson tree and random walks on it through the structure of watersheds. The next goal now is to interpret these random walks as a part of a random interlacements process, which will essentially follow from Theorem II.2.2 and some additional conditions as in (II.4.18). Under some probability measure $\mathbb{P}^{\Gamma}_{\widetilde{u}}$, $\widetilde{u} > 0$, let

$$(\Gamma_x)_{x \in \mathcal{X}} \text{ be an i.i.d. family of } \mathrm{Poi}(\widetilde{u}) \text{ random variables.} \tag{II.4.16}$$

We denote by $\mathbf{P}^W_{L,\widetilde{u}}$ the product measure $\mathbf{P}^W_L \otimes \mathbb{P}^{\Gamma}_{\widetilde{u}}$, under which the tree $\mathcal{T}^{\mathbf{W}}$ and the Poisson random variables $(\Gamma_x)_{x \in \mathcal{X}}$ are independent. Furthermore, for $a \in F$ let

$$\mathbf{W}^a := \big\{ \mathbf{X}^a_k \,:\, k \in \{0, \dots, V_L(\mathbf{X}^a) - 1\} \big\}. \tag{II.4.17}$$

Recall the definition of $e_{K,T}$ from (II.2.12).

Proposition II.4.2. *Let $\widetilde{u}, u > 0$ and $L \in \mathbb{N}$. On some extension of the probability space corresponding to $\mathbf{P}^W_{L,\widetilde{u}}$, one can couple $\mathcal{T}^{\mathbf{W}}$ defined in (II.4.15) and a set $\mathcal{I}^u$ in such a way that conditionally on $\mathcal{T}^{\mathbf{W}}$, the set $\mathcal{I}^u$ is an interlacements set at level u on $\mathcal{T}^{\mathbf{W}}$, and for all $a \in F$, if*

$$\Gamma_{\widehat{a}} \geqslant 1, \quad \widetilde{V}_L(\mathbf{X}^a) = \infty, \quad \text{and} \quad u \geqslant \frac{\widetilde{u}}{e_{\{\widehat{a}\}, \mathcal{T}^{\mathbf{W}}_{\widehat{a}}}(\widehat{a})}, \tag{II.4.18}$$

where $\mathcal{T}^{\mathbf{W}}_{\widehat{a}}$ is the subtree of $\mathcal{T}^{\mathbf{W}}$ below $\widehat{a}$, then

$$\mathbf{W}^a \subset \mathcal{I}^u.$$

Proof. Conditionally on $\mathcal{T}^{\mathbf{W}}$, for each $a \in F$, define $\overline{\mathbf{X}}^a$ as a process on $\mathcal{T}^{\mathbf{W}}$ such that $\overline{\mathbf{X}}^a_k = \mathbf{X}^a_k$ for $0 \leqslant k \leqslant \widetilde{V}_L(\mathbf{X}^a)$, and such that, if $\widetilde{V}_L(\mathbf{X}^a) < \infty$, the process $\overline{\mathbf{X}}^a_k$, $k \geqslant \widetilde{V}_L(\mathbf{X}^a)$, is a random walk on $\mathcal{T}^{\mathbf{W}}$ starting in $\mathbf{X}^a_{\widetilde{V}_L(\mathbf{X}^a)}$. On some extension of the probability space corresponding to $\mathbf{P}^W_{L,\widetilde{u}}$, conditionally on $\mathcal{T}^{\mathbf{W}}$, start independently from each $x \in \mathcal{T}^{\mathbf{W}}$ i.i.d. random walks $\mathbf{X}^{x,i}$, $i \geqslant 2$, each with law $P^{\mathcal{T}^{\mathbf{W}}}_x(\,\cdot \mid H_{x^-} = \infty)$, with the convention $H_{\varnothing^-} = \infty$. Moreover, take $\mathbf{X}^{x,1} = \overline{\mathbf{X}}^a$ if $x = \widehat{a}$ for some $a \in F$ and $H_{\widehat{a}^-}(\overline{\mathbf{X}}^a) = \infty$, and otherwise let $\mathbf{X}^{x,1}$ be some other independent walk with law $P^{\mathcal{T}^{\mathbf{W}}}_x(\,\cdot \mid H_{x^-} = \infty)$. Taking advantage of the thinning property for Poisson random variables and Proposition II.4.1, one can easily prove that, conditionally on $\mathcal{T}^{\mathbf{W}}$ and for each $a \in F$, the probability $\mathbf{P}^{W}_{L,\tilde{u}}(\Gamma_{\widehat{a}} \geqslant 1, H_{\widehat{a}^-}(\overline{\mathbf{X}}^a) = \infty)$ is smaller than or equal to the probability that a $\mathrm{Poi}(\tilde{u}P_{\widehat{a}}^{\mathcal{T}^{\mathbf{W}}}(H_{\widehat{a}^-} = \infty))$-distributed random variable is larger or equal to one. Noting that $\widetilde{V}_L(\mathbf{X}^a) = \infty$ implies $H_{\widehat{a}^-}(\overline{\mathbf{X}}^a) = \infty$, and taking advantage of the equality

$$
\begin{aligned}
&e_{\{\widehat{a}\},\mathcal{T}^{\mathbf{W}}_{\widehat{a}}}(\widehat{a}) \overset{(\text{II.2.12})}{=} \lambda^{\mathbf{W}}_{\widehat{a},+} P_{\widehat{a}}^{\mathcal{T}^{\mathbf{W}}_{\widehat{a}}}(\widetilde{H}_{\widehat{a}} = \infty) \\
=&\lambda^{\mathbf{W}}_{\widehat{a}} P_{\widehat{a}}^{\mathcal{T}^{\mathbf{W}}}(\widetilde{H}_{\widehat{a}} = \infty, H_{\widehat{a}^-} = \infty) \overset{(\text{II.2.16})}{=} \frac{\check{e}_{\mathcal{T}^{\mathbf{W}}}(\widehat{a})}{P_{\widehat{a}}^{\mathcal{T}^{\mathbf{W}}}(H_{\widehat{a}^-} = \infty)},
\end{aligned}
$$

one can construct conditionally on $\mathcal{T}^{\mathbf{W}}$ for each $x \in \mathcal{T}^{\mathbf{W}}$ a Poisson random variable Γ'_x with parameter $u\check{e}_{\mathcal{T}^{\mathbf{W}}}(x)$ such that for each $a \in F$, the properties in (II.4.18) already entail that $\Gamma'_{\widehat{a}} \geqslant 1$.

Moreover, conditionally on $\mathcal{T}^{\mathbf{W}}$, introduce $\overleftrightarrow{\mathbf{X}}^{x,i}$, $i \geqslant 1$, as doubly infinite random walk trajectories on $\mathcal{T}^{\mathbf{W}}$, whose forward part is defined to be $\mathbf{X}^{x,i}$, and whose backward part is an independent random walk with law $P_x^{\mathcal{T}^{\mathbf{W}}}(\,\cdot \mid H_{x^-} = \infty, \widetilde{H}_x = \infty)$ for each $x \in \mathcal{T}^{\mathbf{W}}$. By Proposition II.4.1, conditionally on $\mathcal{T}^{\mathbf{W}}$, the process $\overleftrightarrow{\mathbf{X}}^{x,i}$ has law $\overline{Q}_x^{\mathcal{T}^{\mathbf{W}}}$ for each $i \geqslant 1$, see (II.2.17). We can now define $\mathcal{I}^u$ as the set of vertices visited by any of the trajectories $\overleftrightarrow{\mathbf{X}}^{x,i}$, $i \in \{1, \dots, \Gamma'_x\}$ and $x \in \mathcal{T}^{\mathbf{W}}$, which has the same law conditionally on $\mathcal{T}^{\mathbf{W}}$ as under $\mathbb{P}^{\mathrm{RI}}_{\mathcal{T}^{\mathbf{W}}}$ by Theorem II.2.2. Since (II.4.18) implies $\Gamma'_{\widehat{a}} \geqslant 1$ and $\mathbf{X}_k^{\widehat{a},1} = \mathbf{X}_k^a$ for each $k \in \mathbb{N}_0$, we can easily conclude by the definition (II.4.17) of $\mathbf{W}^a$.

□

II.5 Percolation of the level set

In this section we prove Theorems II.1.1 and II.1.2. We first define a set of "good" properties, see Definition II.5.1 below, which can be satisfied by a vertex a in the tree of free points F, as defined in Section II.4.2. We will show in Lemma II.5.3 that a is good with not too small probability. Our notion of goodness is chosen so that on the one hand, the watershed associated to each good free point is included in the interlacements set $\mathcal{I}^u$ from Proposition II.4.2, see Proposition II.5.5, and also included in the set A_u from (II.1.9) with high probability, see Proposition II.5.7; on the other hand, it also ensures that the tree of good free points survives, see Proposition II.5.5. We refer to the discussion below Definition II.5.1 for more details. This readily yields the percolation of the set $A_u \cap \mathcal{I}^u$, and an application of the inclusion (II.1.8), which follows from Proposition II.2.5 and Proposition II.5.8 below, completes the proof of Theorems II.1.1 and II.1.2.

Let us now define the properties which make a free point good. For this purpose, recall the watershed $(\mathbf{T}^a, \boldsymbol{\lambda}^a, \mathbf{X}^a)$ from (II.4.7), where $a \in F$, with F the tree of free points defined in (II.4.12). We recall that in this watershed, $\mathbf{X}^a$ is a random walk stopped at time $\widetilde{V}_L(\mathbf{X}^a)$, see (II.4.3), and for $K \subset \mathbf{T}^a$ we denote by $H_K(\mathbf{X}^a)$ the hitting time of K for this stopped random walk similarly to (II.2.9). Recall also the definition of the set $\mathbf{W}^a$ from (II.4.17) and of the Poisson random variable $\Gamma_{\widehat{a}}$ from (II.4.16). Also recall that when $x \in \partial\mathcal{T}_-^{\mathbf{W}}$, the tree $\mathcal{T}^x$, see (II.4.14), is equal to the Galton–Watson tree below x in $\mathcal{T}^{\mathbf{W}}$. Finally, recall that for a set $A \subset \mathcal{X}$, by G_x^A we denote the set of children of x in A, see (II.2.2), and for a transient tree T, by g^T we denote the Green function on T, see below (II.2.10).

Definition II.5.1. Let $\widetilde{u}$, B, c_λ, C_Λ, C_g be positive real numbers, $L \in \mathbb{N}$ and $c_f \in (0,1]$. Under $\mathbf{P}^W_{L,\widetilde{u}}$, we say that $a \in F$ is $(L, B, c_\lambda, C_\Lambda, C_g, c_f)$-*good* if the corresponding watershed $(\mathbf{T}^a, \boldsymbol{\lambda}^a, \mathbf{X}^a)$, the weighted tree $\mathcal{T}^{\widehat{a}1}$ and the Poisson random variable $\Gamma_{\widehat{a}}$ satisfy the following properties:

i) The Poisson variable $\Gamma_{\widehat{a}}$ satisfies $\Gamma_{\widehat{a}} \geqslant 1$.

ii) The watershed satisfies

$$\left|G^{\mathbf{T}^a}_{\widehat{a}}\right| \geqslant 2,\ \boldsymbol{\lambda}^a_{\widehat{a},\widehat{a}1} > c_\lambda \text{ and } (\boldsymbol{\lambda}^a)_{\widehat{a},+} \leqslant C_\Lambda, \tag{II.5.1}$$

and the weighted tree $\mathcal{T}^{\widehat{a}1}$ satisfies

$$g^{\mathcal{T}^{\widehat{a}1}}(\widehat{a}1, \widehat{a}1) \leqslant C_g. \tag{II.5.2}$$

iii) The trajectory $\mathbf{X}^a$ satisfies

$$H_{\{\widehat{a}^-,\widehat{a}1\}}(\mathbf{X}^a) = \widetilde{V}_L(\mathbf{X}^a) = \infty.$$

iv) The set of children of the vertex a in the tree of free points F satisfies

$$\left|\left\{a' \in G^F_a : \lambda^F_{a,a'} \leqslant C_\Lambda\right\}\right| \geqslant c_f L.$$

v) The conductances $\boldsymbol{\lambda}^a$ on $\mathbf{W}^a$ satisfy

$$\frac{1}{L^{\frac{3}{2}}} \sum_{y \in \mathbf{W}^a} (\boldsymbol{\lambda}^a_y)^{\frac{3}{2}} < B. \tag{II.5.3}$$

We now explain how the good properties defined above can be combined in order to deduce the percolation of $A_u \cap \mathcal{I}^u$, see (II.1.9). The first three properties imply that the conditions in (II.4.18) are verified, see the proof of Proposition II.5.6, and so, in view of Proposition II.4.2, the set $\mathbf{W}^a$ of the watershed associated to a good free point $a \in F$ is included in the coupled interlacements set $\mathcal{I}^u$. More precisely, property i) implies the first condition in (II.4.18); property ii) will imply a lower bound on $e_{\{\widehat{a}\},\mathcal{T}^{\mathbf{W}}_{\widehat{a}}}(\widehat{a})$, and thus that the third assumption in (II.4.18) is satisfied for u of the same order as $\widetilde{u}$, see (II.5.19); and property iii) implies that the second condition in (II.4.18) is satisfied. Property iv) ensures the creation of many new free points with bounded conductances to their parent, which will imply – using Lemma II.5.4 below – that the tree of good free points contains a d-ary tree for arbitrarily large d, see Proposition II.5.5. Finally, using (II.5.25), property v) will provide us with a good bound on the probability that $\mathbf{W}^a \subset A_u$. Combining these five properties we will thus obtain percolation of the free points $a \in F$ such that $\mathbf{W}^a \subset A_u \cap \mathcal{I}^u$, and thus percolation of $A_u \cap \mathcal{I}^u$, see Proposition II.5.7.

One of the main difficulties in the previous steps is to understand how property ii) in our notion of goodness is used to bound the equilibrium measure $e_{\{\widehat{a}\},\mathcal{T}^{\mathbf{W}}_{\widehat{a}}}(\widehat{a})$ from below, which implies that we can find $\widetilde{u}$ and u of the same order verifying the third assumption (II.4.18), and, consequently, that there is a random interlacements trajectory starting in $\widehat{a}$ when a is good. When $\widehat{a}1$ is not visited by $\mathbf{X}^a$, which is the case when a is good by property iii), then $\widehat{(a1)} = \widehat{a}1$, so no new watershed is generated starting from $\widehat{a}1$ in view of (II.4.10), and thus $\widehat{a}1 \in \partial\mathcal{T}^{\mathbf{W}}_-$. Therefore, by the construction of the tree $\mathcal{T}^{\mathbf{W}}$ above (II.4.15), we obtain that if a is good, then

$\mathcal{T}^{\widehat{a}1}$ is the tree below $\widehat{a}1$ in $\mathcal{T}^{\mathbf{W}}$. The bound on the Green function on $\mathcal{T}^{\widehat{a}1}$ combined with (II.5.1) in property ii) will then imply the desired lower bound on $e_{\{\widehat{a}\},\mathcal{T}^{\mathbf{W}}_{\widehat{a}}}(\widehat{a})$, see (II.5.22) for details. In other words, the reason we excluded $a1$ from the tree of free points in (II.4.10) is to make sure that $\mathcal{T}^{\widehat{a}1}$ is the tree below $\widehat{a}1$ in $\mathcal{T}^{\mathbf{W}}$, and thus that we can use the independent tree $\mathcal{T}^{\widehat{a}1}$ to bound $e_{\{\widehat{a}\},\mathcal{T}^{\mathbf{W}}_{\widehat{a}}}(\widehat{a})$ without using any information on the other watersheds in $\mathcal{T}^{\mathbf{W}}$.

We now provide lower bounds on the probabilities of the previous properties in the following lemma. Note that in items ii) to v) below we do not consider exactly the same kind of events as in Definition II.5.1; they do, however, present the advantage of having more independence and we will show in Lemma II.5.3 (see for instance (II.5.9)) that the probabilities of the events from Definition II.5.1 are larger than those of the events from Lemma II.5.2. Recall that $(\Gamma_x)_{x\in\mathcal{X}}$ are Poisson random variables with parameter $\widetilde{u}$ under of $\mathbb{P}^{\Gamma}_{\widetilde{u}}$, see (II.4.16), that $(\lambda_i)_{i\geqslant 0}$ under ν represents the law of the weights below any vertex, and that $\mathbf{Q}^{\kappa,L}_x$ denotes the law of the watershed introduced in Section II.4.1, see (II.4.4). Recall also the definition of the (interior) boundary ∂A of a set $A\subset\mathcal{X}$ from the paragraph below (II.2.1), and to simplify notation for $B\subset A$ we will write $\partial A\backslash B$ for $(\partial A)\cap B^c$.

Lemma II.5.2. *There exist positive constants $c_\lambda, C_\Lambda, C_g, c_V, c_f \in (0,\infty)$ such that for each $\varepsilon\in(0,1)$ and $B>0$, there exists $L_0 = L_0(B,\varepsilon)\in\mathbb{N}$ such that for all $x\in\mathcal{X}\backslash\{\varnothing\}$, $L\geqslant L_0$, $\kappa\leqslant C_\Lambda$ and $\widetilde{u}>0$, the following properties hold true:*

i) $\mathbb{P}^{\Gamma}_{\widetilde{u}}(\Gamma_x\geqslant 1) = 1-\exp(-\widetilde{u})$,

ii) $\nu\Big(\pi((\lambda_i)_{i\in\mathbb{N}})\geqslant 2, \lambda_1>c_\lambda, \lambda_2>c_\lambda, \lambda_+\leqslant C_\Lambda\Big)\geqslant\frac{1}{2}(1-\mu(1))$,
$\mathbb{P}^{\mathrm{GW}}\big(g^{x1\cdot\mathcal{T}}(x1,x1)\leqslant C_g\big)\geqslant\frac{1}{2}$,

iii) $\mathbb{E}^{\mathrm{GW}}\Big[\dfrac{c_\lambda\lambda_{x2,x21}}{2C_\Lambda(2C_\Lambda+\lambda_{x2,+})}P^{\mathcal{T}}_{x21}(\widetilde{V}_{L-2}=H_{x2}=\infty)\,\Big|\,x\in\mathcal{T},\,\pi((\lambda_{x,xi})_{i\in\mathbb{N}})\geqslant 2\Big] = c_V$,

iv) $\mathbf{Q}^{\kappa,L}_x\Big(\big|\{y\in\partial\mathbf{T}_{V_L}\backslash\{x1,\mathbf{X}_{V_L}\}:\boldsymbol{\lambda}_{y,y^-}\leqslant C_\Lambda\}\big|<c_fL,\widetilde{V}_L(\mathbf{X})=\infty\Big)\leqslant\varepsilon$,

v) $\mathbf{Q}^{\kappa,L}_x\Bigg(\dfrac{1}{L^{\frac{3}{2}}}\displaystyle\sum_{y\in\mathbf{W}}(\boldsymbol{\lambda}_y)^{\frac{3}{2}}\geqslant B,\widetilde{V}_L(\mathbf{X})=\infty\Bigg)\leqslant\varepsilon$.

Proof. i) This is immediate from the definition in (II.4.16).

ii) First note that $\nu\big(\pi((\lambda_i)_{i\in\mathbb{N}})\geqslant 2\big) = 1-\mu(1)$ by definition (II.2.4) of μ in combination with our assumption (SA) in Subsection II.2.2. Moreover, $\mathcal{T}$ is $\mathbb{P}^{\mathrm{GW}}$-a.s. transient due to Proposition II.2.1. Therefore, the Green function $g^{x1\cdot\mathcal{T}}(x1,x1)$ associated to the tree $\mathcal{T}$ rooted at $x1$ is $\mathbb{P}^{\mathrm{GW}}$-a.s. finite, and its law does not depend on the choice of x. Since probability measures are continuous from below, by definition of the conductances in (II.1.2) and above, one can find a small enough positive constant c_λ as well as large enough finite constants C_Λ and C_g, independent of x, such that ii) holds uniformly in $x\in\mathcal{X}$.

iii) Note that for each $y\in\mathcal{T}\backslash\{\varnothing\}$, since the subtree $\mathcal{T}_{y^-}$ is a.s. transient, for almost all realizations of $\mathcal{T}$, the probability $P^{\mathcal{T}}_y(H_{y^-}=\infty)$ is strictly positive. Therefore, using the strong Markov property at time V_{L-2} – which is finite and larger than H_{x2} with positive probability under $P^{\mathcal{T}}_{x21}$, see its definition in

(II.4.2) – and using the previous with $y = X_{V_{L-2}}$, it follows from the definition of $\widetilde{V}_{L-2}$ in (II.4.3) that the variable appearing in the $\mathbb{P}^{\mathrm{GW}}$-expectation of iii) is a.s. positive, and we can conclude.

iv) We will use twice the weak law of large numbers for the i.i.d. sequence of weights $(\boldsymbol{\lambda}_i^{(k)})_{i\in\mathbb{N}}$, $k \geqslant 2$, from (II.4.1). For this purpose, from the proof of ii) we recall that $\nu(\pi((\lambda_i)_{i\in\mathbb{N}}) \geqslant 2) = 1-\mu(1) > 0$. As a consequence, the sequence of random variables $|\{k \in \{2,\dots,L\} : \pi((\boldsymbol{\lambda}_i^{(k)})_{i\in\mathbb{N}}) \geqslant 2\}|/L$, $L \in \mathbb{N}$, converges to $1-\mu(1)$ in probability as $L \to \infty$ by (II.4.1). Fixing $c_f \in (0, (1-\mu(1))/2)$, we obtain for L large enough that

$$\mathbf{Q}_x^{\kappa,L}\Big(\big|\big\{k \in \{2,\dots,L-1\} : \pi((\boldsymbol{\lambda}_i^{(k)})_{i\in\mathbb{N}}) \geqslant 2\big\}\big| < 2Lc_f\Big) \leqslant \frac{\varepsilon}{2}. \tag{II.5.4}$$

Similarly, fixing C_Λ large enough so that

$$\nu\Big(\sum_i \lambda_i \leqslant C_\Lambda\Big) > 1 - c_f,$$

we have by (II.4.1) that for L large enough

$$\mathbf{Q}_x^{\kappa,L}\Big(\big|\big\{k \in \{2,\dots,L-1\} : \sum_{i\in\mathbb{N}} \boldsymbol{\lambda}_i^{(k)} \leqslant C_\Lambda\big\}\big| < (1-c_f)L\Big) \leqslant \frac{\varepsilon}{2}. \tag{II.5.5}$$

Recalling the notation $\mathbf{W}$ from (II.4.5), and that $\boldsymbol{\lambda}_{y,+} = \sum_{i\in\mathbb{N}} \boldsymbol{\lambda}_{y,yi}$, see (II.1.2), our goal is now to prove that, under $\mathbf{Q}_x^{\kappa,L}$,

$$\begin{aligned} &\text{if } \big|\big\{y \in \mathbf{W}\backslash\{x\} : \boldsymbol{\lambda}_{y,+} \leqslant C_\Lambda\big\}\big| \geqslant (1-c_f)L \\ &\text{and } \big|\big\{y \in \mathbf{W}\backslash\{x\} : \big|G_y^{\mathbf{T}_{V_L}}\big| \geqslant 2\big\}\big| \geqslant 2Lc_f, \\ &\text{then } \big|\big\{y \in \partial\mathbf{T}_{V_L}\backslash\{x1, \mathbf{X}_{V_L}\} : \boldsymbol{\lambda}_{y,y^-} \leqslant C_\Lambda\big\}\big| \geqslant c_f L; \end{aligned} \tag{II.5.6}$$

indeed, in view of (II.4.6), on the event $\widetilde{V}_L(\mathbf{X}) = \infty$, which implies $V_L(\mathbf{X}) < H_{x^-}(\mathbf{X})$, we can take advantage of (II.5.6) in order to use (II.5.4) and (II.5.5) to upper bound the probability of the event appearing in iv) of Lemma II.5.2, and we can conclude.

To prove (II.5.6), let us define $A := \{y \in \mathbf{W}\backslash\{x\} : |G_y^{\mathbf{T}_{V_L}}| \geqslant 2\}$ the set of vertices in $\mathbf{W}\backslash\{x\}$ with at least two children in $\mathbf{T}_{V_L}$. Observe that $|\partial\mathbf{T}_{V_L}\backslash G_x^{\mathbf{T}_{V_L}}| \geqslant |A|+1$, which can easily be proved recursively on $|\mathbf{W}|$ starting at $|\mathbf{W}| = 2$. In addition, for each $y \in \partial\mathbf{T}_{V_L}\backslash G_x^{\mathbf{T}_{V_L}}$ we have $y^- \in \mathbf{W}\backslash\{x\}$ and $\boldsymbol{\lambda}_{y,y^-} \leqslant \boldsymbol{\lambda}_{y^-,+}$, and so $\boldsymbol{\lambda}_{y,y^-} \geqslant C_\Lambda$ for at most $c_f L$ different $y \in \partial\mathbf{T}_{V_L}\backslash G_x^{\mathbf{T}_{V_L}}$ on the first event of the first line of (II.5.6). Therefore, since the second event in the first line of (II.5.6) implies $|A| \geqslant 2Lc_f$, we have at least $c_f L + 1$ many vertices $y \in \partial\mathbf{T}_{V_L}\backslash G_x^{\mathbf{T}_{V_L}}$ with $\boldsymbol{\lambda}_{y,y^-} \leqslant C_\Lambda$, which finishes the proof of (II.5.6).

v) Here we can use the Marcinkiewicz-Zygmund law of large numbers, which states that, if $(Y_k)_{k\in\mathbb{N}}$ is a sequence of i.i.d. random variables with $\mathbb{E}[|Y_1|^r] < \infty$ for some $0 < r < 1$, then

$$\frac{1}{n^{1/r}} \sum_{k=1}^n Y_k \xrightarrow[n\to\infty]{a.s.} 0.$$

A proof of this classical result can be found in [Loè77, Section 17.4, p.254]. We can take $Y_k := (\sum_i \boldsymbol{\lambda}_i^{(k)})^{\frac{3}{2}}$ and $r = \frac{2}{3}$ since the expectation of $Y_k^{\frac{2}{3}}$ under $\mathbf{Q}_x^{\kappa,L}$ is then equal to $\mathbb{E}^{\nu}[\sum_i \lambda_i]$, which is finite by our assumption (II.1.2) (see also (II.2.3)). By (II.4.6), this then entails that $L^{-3/2}\sum_{y\in\mathbf{W}\setminus\{x\}} Y_k$ converges a.s. to 0 as $L\to\infty$, and hence for all $\varepsilon\in(0,1)$ and $B>0$ there exists $L_0 = L_0(B,\varepsilon)$ so that for all $L \geqslant L_0$,

$$\mathbf{Q}_x^{\kappa,L}\left(\frac{1}{L^{\frac{3}{2}}}\sum_{k=1}^{L-1}\big(\sum_{i\in\mathbb{N}}\boldsymbol{\lambda}_i^{(k)}\big)^{\frac{3}{2}} \geqslant \frac{B}{6}\right) \leqslant \varepsilon. \tag{II.5.7}$$

Our goal is now to prove that for $L \geqslant L_0(B,\varepsilon)$,

$$\text{if } \frac{1}{L^{\frac{3}{2}}}\sum_{y\in\mathbf{W}}(\boldsymbol{\lambda}_{y,+})^{\frac{3}{2}} < \frac{B}{6}, \text{ then } \frac{1}{L^{\frac{3}{2}}}\sum_{y\in\mathbf{W}}(\boldsymbol{\lambda}_y)^{\frac{3}{2}} < B; \tag{II.5.8}$$

indeed, in view of (II.4.6), on the event $\widetilde{V}_L(\mathbf{X}) = \infty$, we can use (II.5.8) and then (II.5.7) to upper bound the probability of the event appearing in v) of Lemma II.5.3, so that we can conclude. To prove (II.5.8), we use the bounds $(\boldsymbol{\lambda}_y)^{\frac{3}{2}} \leqslant \sqrt{8}((\boldsymbol{\lambda}_{y,+})^{\frac{3}{2}} + (\boldsymbol{\lambda}_{y,y^-})^{\frac{3}{2}})$ for all $y\in\mathbf{W}$, the bound $\boldsymbol{\lambda}_{y,y^-} \leqslant \boldsymbol{\lambda}_{y^-,+}$ for all $y\in\mathbf{W}\setminus\{x\}$, the inequality $\boldsymbol{\lambda}_{x,x^-} = \kappa \leqslant C_\Lambda$, the fact that $\{y^- : y\in\mathbf{W}\setminus\{x\}\}\subset\mathbf{W}$, and take $L_0(B,\varepsilon)$ much larger than $C_\Lambda/B^{2/3}$.

□

Let us now show that the bounds obtained in Lemma II.5.2 can be combined to lower bound the probability that a vertex $a\in F$ is good, see Definition II.5.1. Recall that $\mathbf{P}^W_{L,\widetilde{u}}$ is the probability measure underlying our tree of free points constructed in Section II.4.2, see also below (II.4.16).

Lemma II.5.3. *Let c_λ, C_Λ, C_g and c_f be as in Lemma II.5.2. There exists $c_p>0$ such that for all $B>0$, there exists $L_0(B)\in\mathbb{N}$ such that for all $a\in\mathcal{X}$, $L\geqslant L_0(B)$ and $\widetilde{u}>0$, on the event $\{\lambda^F_{a,a^-}\leqslant C_\Lambda\}$ we have*

$$\mathbf{P}^W_{L,\widetilde{u}}\Big(a \textit{ is } (L,B,c_\lambda,C_\Lambda,C_g,c_f)\textit{-good}\,\big|\,\lambda^F_{a,a^-}, a\in F\Big) \geqslant c_p(1-e^{-\widetilde{u}}).$$

Proof. We will check the properties of Definition II.5.1. In the first part of the proof, we show that the event appearing in Lemma II.5.2 iii) implies that Definition II.5.1 iii) is fulfilled with positive conditional probabilities under the appropriate conditions. More precisely, we have for all $a\in F$ that

$$\begin{gathered}\text{if } \boldsymbol{\lambda}^a_{\widehat{a},+}\leqslant C_\Lambda, \lambda^F_{a,a^-}\leqslant C_\Lambda \text{ and } \boldsymbol{\lambda}^a_{\widehat{a},\widehat{a}2} > c_\lambda\\ \text{then } P^{\mathcal{T}^{\mathbf{W}}}_{\widehat{a}}\big(H_{\{\widehat{a}^-,\widehat{a}1\}} = \widetilde{V}_L = \infty\big) \geqslant \frac{c_\lambda\boldsymbol{\lambda}^a_{\widehat{a}2,\widehat{a}21}}{2C_\Lambda(2C_\Lambda+\boldsymbol{\lambda}^a_{\widehat{a}2,+})}P^{\mathcal{T}^{\mathbf{W}}}_{\widehat{a}21}(\widetilde{V}_{L-2} = H_{\widehat{a}2} = \infty);\end{gathered} \tag{II.5.9}$$

indeed, under the conditions from (II.5.9), noting that $\boldsymbol{\lambda}^a_{\widehat{a},\widehat{a}^-} = \lambda^F_{a,a^-}$ by (II.4.11), and thus $\boldsymbol{\lambda}^a_{\widehat{a}}\leqslant 2C_\Lambda$, we have that

$$P^{\mathcal{T}^{\mathbf{W}}}_{\widehat{a}}(X_2 = \widehat{a}21) = \frac{\boldsymbol{\lambda}^a_{\widehat{a},\widehat{a}2}\boldsymbol{\lambda}^a_{\widehat{a}2,\widehat{a}21}}{\boldsymbol{\lambda}^a_{\widehat{a}}(\boldsymbol{\lambda}^a_{\widehat{a},\widehat{a}2}+\boldsymbol{\lambda}^a_{\widehat{a}2,+})} \geqslant \frac{c_\lambda\boldsymbol{\lambda}^a_{\widehat{a}2,\widehat{a}21}}{2C_\Lambda(2C_\Lambda+\boldsymbol{\lambda}^a_{\widehat{a}2,+})}.$$

Therefore, (II.5.9) follows easily by using the Markov property at time 2, noting that, under $P^{\mathcal{T}^{\mathbf{W}}}_{\widehat{a}}$ and on the event $\{X_2 = \widehat{a}21\}$, in view of (II.4.2) and (II.4.3), we have $\widetilde{V}_{L-2}((X_{k+2})_{k\geqslant 0})=\widetilde{V}_L((X_k)_{k\geqslant 0})$. Furthermore, if $\widehat{a}2$ is never visited after time 2, then $\widehat{a}1$ and $\widehat{a}^-$ are never visited by X. Moreover, note that the random variable on the right-hand side of the inequality of the second line of (II.5.9) is independent of $\mathcal{T}^{\widehat{a}1}$, $\Gamma_{\widehat{a}}$, $(\boldsymbol{\lambda}^a_{\widehat{a},\widehat{a}i})_{i\in\mathbb{N}}$ and λ^F_{a,a^-}. Combining Proposition II.4.1, (II.4.7), Lemma II.5.2 iii) and (II.5.9), we thus have on the intersection of the events $\{\boldsymbol{\lambda}^a_{\widehat{a},\widehat{a}2}>c_\Lambda\}$, $\{\boldsymbol{\lambda}^a_{\widehat{a},+}\leqslant C_\Lambda\}$ and $\{\lambda^F_{a,a^-}\leqslant C_\Lambda\}$, that

$$\mathbf{P}^{\mathbf{W}}_{L,\tilde{u}}\Big(H_{\{\widehat{a}^-,\widehat{a}1\}}(\mathbf{X}^a)=\widetilde{V}_L(\mathbf{X}^a)=\infty\,\Big|\,\Gamma_{\widehat{a}},\,(\boldsymbol{\lambda}^a_{\widehat{a},\widehat{a}i})_{i\in\mathbb{N}},\,\mathcal{T}^{\widehat{a}1},\,\lambda^F_{a,a^-},\,a\in F\Big)\geqslant c_V. \tag{II.5.10}$$

In this second part of the proof, we aim at combining the estimates from Lemma II.5.2 in order to infer the general lower bound $c_p(1-e^{-\tilde{u}})$ on the probability for a to be good. Obtaining a lower bound on the intersection of the events i), ii) and iii) in Definition II.5.1 is easy by independence, Lemma II.5.2 and (II.5.10). More care is required for the other properties though.

It is not difficult to combine Lemma II.5.2 iv) and v), since the complements of the events there happen with high probability, as we now explain. On the event $\{\lambda^F_{a,a^-}\leqslant C_\Lambda\}$, using the estimates from Lemma II.5.2 iv), v) for $\varepsilon=\frac{1}{3}\frac{c_V(1-\mu(1))}{2}$, and writing them in the form of Definition II.5.1 – see (II.4.7), (II.4.9), (II.4.11) and the definition of the tree of free points from (II.4.10) and below – we thus have for all $L\geqslant L_0(B)$, with $L_0(B)=L_0(B,\varepsilon)$ from Lemma II.5.2 for this choice of ε that

$$\begin{aligned}\mathbf{P}^{\mathbf{W}}_{L,\tilde{u}}&\left(\begin{matrix}\Big\{\big|\{a'\in G^F_a:\,\lambda^F_{a,a'}\leqslant C_\Lambda\}\big|\geqslant c_f L,\\ L^{-\frac{3}{2}}\displaystyle\sum_{y\in\mathbf{W}^a}(\boldsymbol{\lambda}^a_y)^{\frac{3}{2}}<B\Big\}^c,\,H_{\{\widehat{a}^-,\widehat{a}1\}}(\mathbf{X}^a)=\widetilde{V}_L(\mathbf{X}^a)=\infty\end{matrix}\,\middle|\,\Gamma_{\widehat{a}},\,\mathcal{T}^{\widehat{a}1},\,\lambda^F_{a,a^-},\,a\in F\right)\\ &\leqslant\frac{2}{3}\frac{c_V(1-\mu(1))}{2}.\end{aligned} \tag{II.5.11}$$

Here, we used that both, the event $H_{\{\widehat{a}^-,\widehat{a}1\}}(\mathbf{X}^a)=\widetilde{V}_L(\mathbf{X}^a)=\infty$ and the events in Definition II.5.1 iv) and v), are $(\mathbf{T}^a,\boldsymbol{\lambda}^a,\mathbf{X}^a)$-measurable, and thus independent of $\Gamma_{\widehat{a}}$ and $\mathcal{T}^{\widehat{a}1}$, and that $\{\widehat{a}:\,a\in G^F_a\}=\partial\mathbf{T}^a_{V_L(\mathbf{X}^a)}\backslash\{\widehat{a}1,\mathbf{X}^a_{V_L(\mathbf{X}^a)}\}$ when $H_{\{\widehat{a}^-,\widehat{a}1\}}(\mathbf{X}^a)=\infty$ in view of (II.4.9), (II.4.10).

Now we can further combine (II.5.10) with the equation in the first line of ii) of Lemma II.5.2 (recall that the number of children $\big|G^{\mathbf{T}^a_1}_{\widehat{a}}\big|$ of $\widehat{a}$ in $\mathbf{T}^a_1$ is equal to $\pi((\boldsymbol{\lambda}^a_{\widehat{a},\widehat{a}i})_{i\in\mathbb{N}})$. One can combine this with (II.5.11) thanks to the dependence of the bound (II.5.11) on $c_V(1-\mu(1))/2$, noting also that the event in the first line of Definition II.5.1 ii) is independent of $\Gamma_{\widehat{a}}$ and $\mathcal{T}^{\widehat{a}1}$, to obtain that on the event $\{\lambda^F_{a,a^-}\leqslant C_\Lambda\}$, for all $L\geqslant L_0(B)$ we have

$$\begin{aligned}\mathbf{P}^{\mathbf{W}}_{L,\tilde{u}}&\left(\begin{matrix}\big|\{a'\in G^F_a:\,\lambda^F_{a,a'}\leqslant C_\Lambda\}\big|\geqslant c_f L,\\ L^{-\frac{3}{2}}\displaystyle\sum_{y\in\mathbf{W}^a}(\boldsymbol{\lambda}^a_y)^{\frac{3}{2}}<B,\,H_{\{\widehat{a}^-,\widehat{a}1\}}(\mathbf{X}^a)=\widetilde{V}_L(\mathbf{X}^a)=\infty,\\ \big|G^{\mathbf{T}^a_1}_{\widehat{a}}\big|\geqslant 2,\,\boldsymbol{\lambda}^a_{\widehat{a},\widehat{a}1}>c_\lambda,\,\boldsymbol{\lambda}^a_{\widehat{a},+}\leqslant C_\Lambda\end{matrix}\,\middle|\,\Gamma_{\widehat{a}},\,\mathcal{T}^{\widehat{a}1},\,\lambda^F_{a,a^-},\,a\in F\right)\\ &\geqslant\frac{1}{3}\frac{c_V(1-\mu(1))}{2}.\end{aligned} \tag{II.5.12}$$

Finally, for the good events in i) and the second line of ii) in Definition II.5.1, conditionally on $a\in F$ and λ^F_{a,a^-}, the random variables $\Gamma_{\widehat{a}}$ and $\mathcal{T}^{\widehat{a}1}$ have respective laws $\mathbb{P}^{\Gamma}_{\widetilde{u}}(\Gamma_{\widehat{a}} \in \cdot)$ and $\mathbb{P}^{\mathrm{GW}}(\widehat{a}1 \cdot \mathcal{T} \in \cdot)$, (see, respectively, below (II.4.16) and (II.4.14)), and are independent. Therefore, the two estimates provided by Lemma II.5.2 i) and the second line of ii), yield that for all $\widetilde{u} > 0$ one has

$$\mathbf{P}^{\mathbf{W}}_{L,\widetilde{u}}\left(\Gamma_{\widehat{a}} \geqslant 1, g^{\mathcal{T}^{\widehat{a}1}}(\widehat{a}1, \widehat{a}1) \leqslant C_g \,\middle|\, \lambda^F_{a,a^-}, a \in F\right) \geqslant \frac{1}{2}(1 - \exp(-\widetilde{u})). \tag{II.5.13}$$

Combining (II.5.12) and (II.5.13), we can readily conclude by taking $c_p = c_V(1 - \mu(1))/12$. □

We now want to show that the set of good free points introduced in Definition II.5.1 percolates with the help of Lemma II.5.3. This set can be interpreted as a random subset in $\mathcal{X}$, endowed with the σ-algebra introduced at the end of Section II.2.1. Recall the definition G_x^A of the number of children of x in $A \subset \mathcal{X}$ from (II.2.2). In the following technical lemma, we say that a tree is d-ary if it contains $\varnothing$ and every vertex has exactly d children. While it seems like a standard result, we were not able to locate it in the literature and therefore provide a proof here.

Lemma II.5.4. *There exists a function $d : [0, \infty) \to \mathbb{N}_0$ such that $d(t) \to \infty$ as $t \to \infty$ and the following holds. Under some probability measure $\mathbb{P}$, let $S \subset \mathcal{X}$ be a random set containing $\varnothing$ almost surely, such that for some $N \in \mathbb{N}$ and $p \in [0, 1]$, for all $x \in \mathcal{X}$*

$$\mathbb{P}\Big(|G_x^S| \geqslant N \,\Big|\, \mathcal{F}_x\Big) \geqslant p \text{ on the event } \{x \in S\}; \tag{II.5.14}$$

here, $\mathcal{F}_x = \sigma(\mathbf{1}_{\{y \in S\}}, y \in \mathcal{X} \backslash (x \cdot (\mathcal{X} \backslash \{\varnothing\})))$ is the σ-algebra generated by the restriction of S to vertices which are not descendants of x. Then, S contains with positive probability, depending only on p and N, a $d(Np)$-ary tree.

Proof. In this proof, we say that a random subset of $\mathcal{X}$ is a weightless Galton–Watson tree with offspring distribution $p\delta_N + (1-p)\delta_0$ if, after possible reordering of the labels, this set has the same law as the tree $\mathcal{T}$ seen as a subset of $\mathcal{X}$ (that is removing the weights), introduced in Section II.2.1 when the offspring distribution μ from (II.2.4) is $p\delta_N + (1-p)\delta_0$. Note that since we discard the weights here, the law of this tree is entirely determined by its offspring distribution.

Let us first show that we can couple S and a weightless Galton–Watson tree with offspring distribution $p\delta_N + (1-p)\delta_0$, such that S is included in this tree. For this purpose, fix a sequence $x_0, x_1, \dots$ exhausting $\mathcal{X}$ and such that $\{x_0, \dots, x_{k-1}\} \subset (x_k \cdot \mathcal{X})^c$ for each $k \in \mathbb{N}_0$. The result will follow once we have that, under some probability measure $\widetilde{\mathbb{P}}$, there exist an i.i.d. family of Bernoulli random variables ζ_{x_k}, $k \in \mathbb{N}_0$ with parameter p, and random sets $\widetilde{S}_k$, $k \in \mathbb{N}_0$, with the following properties: $\widetilde{S}_k$ is an increasing sequence of sets, each with the same law as $S_k := \{x \in S : x \sim x_i \text{ for some } i \leqslant k\}$ under $\mathbb{P}$, and if $\zeta_{x_k} = 1$ and $x_k \in \widetilde{S}_k$, then $|G_{x_k}^{\widetilde{S}_k}| \geqslant N$ (in order to facilitate reading, the construction of these random variables will take place in the last paragraph of the proof). Indeed, defining $\widetilde{S}$ as the union of $\widetilde{S}_k$, $k \in \mathbb{N}_0$, one obtains that $\widetilde{S}$ has the same law as S under $\mathbb{P}$. Furthermore, the tree T obtained recursively by keeping exactly N children in $\widetilde{S}$ of $x \in \widetilde{S}$ each time $\zeta_x = 1$, and keeping zero children otherwise, is then a Galton–Watson tree with offspring distribution $p\delta_N + (1-p)\delta_0$, which is contained in $\widetilde{S}$.

In order to conclude, we still need to show that for each $\widetilde{d} \in \mathbb{N}_0$, there exists $t = t(\widetilde{d}) \in (0, \infty)$ such that for each $p \in [0, 1]$ and $N \in \mathbb{N}$ with $pN \geqslant t$, a weightless Galton–Watson tree with offspring distribution $p\delta_N + (1-p)\delta_0$ contains with positive probability a $\widetilde{d}$-ary tree, and then take $d(s) := \sup\{\widetilde{d} \in \mathbb{N}_0 : t(\widetilde{d}) \leqslant s\}$ for all $s > 0$, with the convention $\sup \varnothing = 0$. This can be easily proven by noting that, if $G_{\widetilde{d}}$ is the function from [LP16, Theorem 5.29], then $G_{\widetilde{d}}(0) > 0$ and $G_{\widetilde{d}}(1-p/2) < 1-p/2$ if $pN \geqslant t$ for some t large enough. We leave the details to the reader.

It therefore remains to construct construct the sequences $\widetilde{S}_k$ and ζ_{x_k}, $k \in \mathbb{N}_0$. We have $x_0 = \varnothing$, and (II.5.14) applied to $x = \varnothing$ implies that one can indeed define a Bernoulli random variable $\zeta_\varnothing$ with parameter p and $\widetilde{S}_0$ such that $\widetilde{S}_0$ has the same law as $\{x \in S : x \sim \varnothing\}$, and $\zeta_\varnothing = 1$ implies $|G_\varnothing^{\widetilde{S}_0}| \geqslant N$. Assume now that ζ_{x_i}, $i \leqslant k-1$, and $\widetilde{S}_{k-1}$ are constructed. Let $\widetilde{S}_k$ be the union of $\widetilde{S}_{k-1}$ and some children of x_k, constructed so that, conditionally on $(\zeta_{x_i})_{i \leqslant k-1}$ and $\widetilde{S}_{k-1}$, the law of $\widetilde{S}_k$ is the same as law of S_k conditionally on $S_{k-1} = \widetilde{S}_{k-1}$. Then (II.5.14) implies that, conditionally on $(\zeta_{x_i})_{i \leqslant k-1}$ and $\widetilde{S}_{k-1}$, $\mathbf{1}\big\{|G_{x_k}^{\widetilde{S}_k}| \geqslant N\big\}$ stochastically dominates a Bernoulli random variable with parameter p on the event $\{x_k \in \widetilde{S}_{k-1}\}$. Hence, up to extending the probability space $\widetilde{\mathbb{P}}$, we can define a Bernoulli random variable ζ_{x_k} with parameter p, independent of ζ_{x_i}, $i \leqslant k-1$, and $\widetilde{S}_{k-1}$, and such that if $\zeta_{x_k} = 1$ and $x_k \in \widetilde{S}_{k-1}$ then $|G_{x_k}^{\widetilde{S}_k}| \geqslant N$. This concludes the induction, and the proof that $\widetilde{S}$ contains a.s. a weightless Galton–Watson tree with offspring distribution $p\delta_N + (1-p)\delta_0$. □

We now prove that with positive probability, the tree of $(L, B, c_\lambda, C_\Lambda, C_g, c_f)$-good free points contains a d-ary tree for suitable choices of the parameters. To do so, observe that on the one hand, the probability for a free point to be good is bounded from below due to Lemma II.5.3. On the other hand, property iv) of Definition II.5.1 will let us tune the parameter L in such a way that a good free point has many children. We will then be able to use Lemma II.5.4 in order to conclude.

Proposition II.5.5. *Let c_λ, C_Λ, C_g and c_f be as in Lemma II.5.2, c_p as in Lemma II.5.3, and the function d as in Lemma II.5.4. For all $B > 0$, there exists $L_0(B) \in \mathbb{N}$ such that for all $L \geqslant L_0(B)$ and $\widetilde{u} > 0$, the set*

$$F^g := \{\varnothing\} \cup \big\{a \in F \backslash \{\varnothing\} \,\big|\, a^- \textit{ is } (L, B, c_\lambda, C_\Lambda, C_g, c_f)\textit{-good and } \lambda^F_{a,a^-} \leqslant C_\Lambda\big\} \tag{II.5.15}$$

contains with positive $\mathbf{P}^{\mathbf{W}}_{L,\widetilde{u}}$ probability a $d(Lq(\widetilde{u}))$-ary tree, where $q(\widetilde{u}) = c_f c_p(1 - e^{-\widetilde{u}})$.

Proof. Let $B > 0$. Fix c_λ, C_Λ, C_g, c_f, and $L_0(B)$ as in Lemma II.5.3, and fix $L \geqslant L_0(B)$ and $\widetilde{u} > 0$. Throughout the proof we write "good" instead of "$(L, B, c_\lambda, C_\Lambda, C_g, c_f)$-good" to simplify notation, keeping the implicit dependence on the parameters in mind. Let us first extend the definition of the weights λ^F from $\{\{a, a^-\} : a \in F\backslash\{\varnothing\}\}$ to $\{\{a, a^-\} : a \in \mathcal{X}\backslash\{\varnothing\}\}$ by letting $\lambda^F_{a,a^-} = 0$ if $a \in \mathcal{X}\backslash F$. For each $a \in \mathcal{X}\backslash F$, we also fix arbitrarily some $\widehat{a} \in \mathcal{X}$, so that $\widehat{a} \neq \widehat{a'}$ for all $a \neq a' \in \mathcal{X}$. This way, we can also define $(\mathbf{T}^a, \boldsymbol{\lambda}^a, \mathbf{X}^a)$, $a \in \mathcal{X}\backslash F$, as a family of independent watersheds with law $\mathbf{Q}_{\widehat{a}}^{\lambda^F_{a^-,a},L}$, see (II.4.7). Note that for $a \notin F$ we never actually use the additional watershed $(\mathbf{T}^a, \boldsymbol{\lambda}^a, \mathbf{X}^a)$ nor the notation $\widehat{a}$, they are however necessary to define the following σ-algebra

$$\mathcal{W}^a := \sigma\big(\Gamma_{\widehat{a}}, \mathbf{X}^a, (\boldsymbol{\lambda}^a_{x,y})_{x \sim y \in \mathbf{T}^a}, (\lambda^{\widehat{a}1}_{x,y})_{x \sim y \in \mathcal{T}^{\widehat{a}1}}\big) \text{ for all } a \in \mathcal{X},$$

where $\lambda^{\widehat{a}1}$ are the weights of the tree $\mathcal{T}^{\widehat{a}1}$ which was defined in (II.4.14); also recall that $\mathbf{X}^a$, $\boldsymbol{\lambda}^a$ and $\lambda^{\widehat{a}1}$ are random variables whose canonical σ-algebras on their respective state spaces have been defined at the end of Section II.2.1. By construction,

$(\mathbf{T}^{a^-},\boldsymbol{\lambda}^{a^-},\mathbf{X}^{a^-})$, $\mathcal{T}^{\mathbf{W}}_{\widehat{(a^-)1}}$, the weight $\lambda^F_{a^-,a}=\boldsymbol{\lambda}^{a^-}_{\hat{a},\hat{a}^-}$, see (II.4.11), as well as the event $\{a\in F\}=\{\lambda^F_{a^-,a}>0\}$ are $\mathcal{W}^{a^-}$-measurable. Therefore, in view of Definition II.5.1

$$\{a\in F^g\}\in\mathcal{W}^{a^-}\text{ for all }a\in\mathcal{X},\tag{II.5.16}$$

where we recall F^g from (II.5.15), and with the convention $\mathcal{W}^{\varnothing^-}:=\sigma(\{\varnothing\})$ is the trivial σ-algebra. By (II.4.7), a watershed depends on the previous watersheds only through the weights λ^F_{a,a^-}, that is $\mathcal{W}^a$ and $\mathcal{W}^{a'}$, $a'\notin a\cdot\mathcal{X}$, are independent conditionally on λ^F_{a,a^-} for all $a\in F\backslash\{\varnothing\}$. Therefore, defining for each $a\in\mathcal{X}$ the σ-algebra

$$\mathcal{F}^g_a:=\sigma\big(\mathcal{W}^{(a')^-},a'\notin a\cdot(\mathcal{X}\backslash\{\varnothing\})\big)=\sigma(\mathcal{W}^{a'},a'\notin a\cdot\mathcal{X}),\tag{II.5.17}$$

we have that for all $a\in F$,

$$\mathbf{P}^{\mathbf{W}}_{L,\tilde{u}}(a\text{ is good}\,|\,\mathcal{F}^g_a)=\mathbf{P}^{\mathbf{W}}_{L,\tilde{u}}(a\text{ is good}\,|\,\lambda^F_{a,a^-},a\in F),\tag{II.5.18}$$

with the convention $\lambda^F_{\varnothing,\varnothing^-}=0$. Note that, in view of (II.5.16), the σ-algebra $\mathcal{F}^g_a$ contains the σ-algebra $\mathcal{F}_a$ from Lemma II.5.4 when $S=F^g$. By property iv) of Definition II.5.1, we moreover have $|G^{F^g}_a|=|\{a'\in G^F_a:\lambda^F_{a,a'}\leqslant C_\Lambda\}|\geqslant c_fL$ if $a\in F$ is good. Thus since $\{\lambda^F_{a,a^-}\leqslant C_\Lambda\}\subset\{a\in F^g\}\in\mathcal{F}^g_a$ by (II.5.16) and (II.5.17), we have that on the event $\{a\in F^g\}$,

$$\mathbf{P}^{\mathbf{W}}_{L,\tilde{u}}(|G^{F^g}_a|\geqslant c_fL\,|\,\mathcal{F}^g_a)\geqslant\mathbf{P}^{\mathbf{W}}_{L,\tilde{u}}(a\text{ is good}\,|\,\mathcal{F}^g_a)\geqslant c_p(1-e^{-\tilde{u}}),$$

where we used Lemma II.5.3 and (II.5.18) in the last inequality. Using (II.5.17) and Lemma II.5.4 for $S=F^g$, we can conclude.

□

With the help of Proposition II.4.2, we now show that for a suitable choice of the parameters $u,\tilde{u}>0$, under $\mathbf{P}^W_{L,\tilde{u}}$, for each $(L,B,c_\lambda,C_\Lambda,C_g,c_f)$-good free point $a\in F$, one can include the watershed $\mathbf{W}^a$ in the random interlacements set $\mathcal{I}^u$ from Proposition II.4.2. For this purpose, we need to verify that all the assumptions of (II.4.18) are verified for good free points.

Proposition II.5.6. *Let $u,B,c_\lambda,c_\Lambda,C_g,c_f>0$, $L\in\mathbb{N}$, $a\in F$ and*

$$\tilde{u}=uc_e,\ \textit{where } c_e:=\frac{c_\lambda}{c_\lambda C_g+1}.\tag{II.5.19}$$

Then, under the extension of the probability space $\mathbf{P}^W_{L,\tilde{u}}$ from Proposition II.4.2,

$$\mathbf{W}^a\subset\mathcal{I}^u\textit{ for all }(L,B,c_\lambda,c_\Lambda,C_g,c_f)\textit{-good vertices }a\in F.\tag{II.5.20}$$

Proof. Fix some $(L,B,c_\lambda,c_\Lambda,C_g,c_f)$-good vertex $a\in F$. First note that by properties i) and iii) of Definition II.5.1, the first and second condition in (II.4.18) are satisfied, and thus by Proposition II.4.2,

$$\mathbf{W}^a\subset\mathcal{I}^u\quad\text{once we show}\quad u\geqslant\frac{\tilde{u}}{e_{\{\hat{a}\},\mathcal{T}^{\mathbf{W}}_{\hat{a}}}(\hat{a})}.\tag{II.5.21}$$

To bound the parameter $e_{\{\hat{a}\},\mathcal{T}^{\mathbf{W}}_{\hat{a}}}(\hat{a})$ from below we will use property ii) of Definition II.5.1. We use the analogy to electrical circuits, and note that by Rayleigh's

Monotonicity Principle [LP16, (2.5) and Sections 2.3 and 2.4], we have that $g^{\mathcal{T}^{\mathbf{W}}_{\widehat{a}}}(\widehat{a},\widehat{a}) \leqslant g^{\mathcal{T}^{\mathbf{W}}_{\widehat{a},1}}(\widehat{a},\widehat{a})$, where $\mathcal{T}^{\mathbf{W}}_{\widehat{a},1}$ denotes the subtree of $\mathcal{T}^{\mathbf{W}}_{\widehat{a}}$ consisting only of $\widehat{a}$ and $\mathcal{T}^{\mathbf{W}}_{\widehat{a}1}$. Moreover, using a series transformation [LP16, Subsection 2.3.I], equations (II.5.1) and (II.5.2) imply that $g^{\mathcal{T}^{\mathbf{W}}_{\widehat{a},1}}(\widehat{a},\widehat{a}) \leqslant C_g + \frac{1}{c_\lambda}$ since, on the event $H_{\widehat{a}1}(\mathbf{X}^{\widehat{a}}) = \infty$ which is implied by property iii) of Definition II.5.1, $\mathcal{T}^{\widehat{a}1}$ is the subtree $\mathcal{T}^{\mathbf{W}}_{\widehat{a}1}$ of $\mathcal{T}^{\mathbf{W}}$ below $\widehat{a}1$ as explained in the second paragraph below Definition II.5.1. Thus, the equilibrium measure at $\widehat{a}$ for $\mathcal{T}^{\mathbf{W}}_{\widehat{a}}$ is bounded from below by

$$e_{\{\widehat{a}\},\mathcal{T}^{\mathbf{W}}_{\widehat{a}}}(\widehat{a}) = \frac{1}{g^{\mathcal{T}^{\mathbf{W}}_{\widehat{a}}}(\widehat{a},\widehat{a})} \geqslant \frac{c_\lambda}{c_\lambda C_g + 1} =: c_e. \tag{II.5.22}$$

We can conclude by combining (II.5.19), (II.5.21) and (II.5.22).

□

If $q(\widetilde{u})L$ is large enough, combining Propositions II.5.5 and II.5.6 provides us with an infinite tree of good free points a satisfying $\mathbf{W}^a \subset \mathcal{I}^u$. Taking advantage of property v) from Definition II.5.1, we are now ready to prove percolation for the set on the left-hand side of (II.1.8). For each $p \in (0,1)$, under some probability $\mathbb{P}^{\mathrm{E}}_p$, let $(\mathcal{E}_x)_{x\in\mathcal{X}}$ be an independent family of exponential random variables with parameter one, and $(\mathcal{B}_x)_{x\in\mathcal{X}}$ the independent family of Bernoulli random variables defined above (II.1.5). Recall that φ is a Gaussian free field on T under $\mathbb{P}^{\mathrm{G}}_T$, see Section II.2.3, that $\mathcal{I}^u$ is a random interlacements set on T under $\mathbb{P}^{\mathrm{RI}}_T$, see Section II.2.4, that $\mathcal{T}$ is a Galton–Watson tree under $\mathbb{P}^{\mathrm{GW}}$, see Section II.2.1, and let B_p be as in (II.1.5) and A_u as in (II.1.9).

Proposition II.5.7. *There exists $u_0 > 0$ such that for each $u \in (0,u_0]$, there exists $p \in (0,1)$ so that the set $A_u \cap B_p \cap \mathcal{I}^u$ contains $\mathbb{E}^{\mathrm{GW}}[\mathbb{P}^{\mathrm{RI}}_{\mathcal{T}} \otimes \mathbb{P}^{\mathrm{G}}_{\mathcal{T}} \otimes \mathbb{P}^{\mathrm{E}}_p(\cdot)]$-a.s. an unbounded cluster.*

Proof. Under $\mathbf{E}^{\mathbf{W}}_{L,\widetilde{u}}[\mathbb{P}^{\mathrm{G}}_{\mathcal{T}^{\mathbf{W}}} \otimes \mathbb{P}^{\mathrm{E}}_p(\cdot)]$, for some $L \in \mathbb{N}$ and $\widetilde{u} > 0$, consider the event

$$A^{\mathbf{W}}_u := \Big\{x \in \mathcal{T}^{\mathbf{W}} : \mathcal{E}_x > 4u\lambda^{\mathbf{W}}_x \text{ or } |\varphi_x| > 2\sqrt{2u}\Big\} \cap \big\{x \in \mathcal{T}^{\mathbf{W}} : \mathcal{B}_x = 1\big\}. \tag{II.5.23}$$

For $a \in F$, we now evaluate the probability, conditioned on the value of $\varphi_{\widehat{a}-}$, that $\mathbf{W}^a \subset A^{\mathbf{W}}_u$ (recall (II.4.17)). For $\mathcal{E}$ and $\mathcal{B}$, simple estimates for exponential and Bernoulli variables will be sufficient, while for the Gaussian free field we take advantage of the Markov property (II.2.11) applied to the set $U_a := \mathcal{T}^{\mathbf{W}}_{\widehat{a}}$. For each $y \in U_a$, one can decompose the field as $\varphi_y = \psi^{U_a}_y + \beta^{U_a}_y$; here, $\psi^{U_a}_y$ is a centered Gaussian field, independent of $\beta^{U_a}_y$ and $\varphi_{\widehat{a}-}$, and with variance $g^{\mathcal{T}^{\mathbf{W}}}_{U_a}(y,y)$, which by (II.2.10) satisfies

$$g^{\mathcal{T}^{\mathbf{W}}}_{U_a}(y,y) \geqslant \frac{1}{\lambda^{\mathbf{W}}_y} \text{ for all } y \in U_a.$$

Thus, for all $y \in U_a$ we have – using the symmetry and unimodality of the distribution of $\psi^{U_a}_y$ to obtain the first inequality – that

$$\begin{aligned}\mathbb{P}^{\mathrm{G}}_{\mathcal{T}^{\mathbf{W}}}\big(|\varphi_y| \leqslant 2\sqrt{2u} \,\big|\, \varphi_{\widehat{a}-}\big) &= \mathbb{P}^{\mathrm{G}}_{\mathcal{T}^{\mathbf{W}}}\big(\big|\psi^{U_a}_y + \beta^{U_a}_y\big| \leqslant 2\sqrt{2u} \,\big|\, \varphi_{\widehat{a}-}\big)\\ &\leqslant \mathbb{P}^{\mathrm{G}}_{\mathcal{T}^{\mathbf{W}}}\big(\big|\psi^{U_a}_y\big| \leqslant 2\sqrt{2u}\big) \leqslant \frac{4\sqrt{2u}}{\sqrt{2\pi/\lambda^{\mathbf{W}}_y}}.\end{aligned} \tag{II.5.24}$$

Therefore, for all $a \in F$,

$$
\begin{aligned}
&\mathbb{P}^{\mathrm{G}}_{\mathcal{T}^{\mathbf{W}}} \otimes \mathbb{P}^{\mathrm{E}}_{p}\big(\mathbf{W}^a \subset A_u^{\mathbf{W}} \,\big|\, \varphi_{\widehat{a}^-}\big) \\
&\overset{\text{(II.5.23)}}{=} \prod_{y \in \mathbf{W}^a} \mathbb{P}^{\mathrm{E}}_{p}(\mathcal{B}_y = 1)\Big(1 - \mathbb{P}^{\mathrm{G}}_{\mathcal{T}^{\mathbf{W}}} \otimes \mathbb{P}^{\mathrm{E}}_{p}\Big(\bigcup_{y \in \mathbf{W}^a} \{|\varphi_y| \leqslant 2\sqrt{2u}\} \cap \{\mathcal{E}_y \leqslant 4u\lambda_y^{\mathbf{W}}\} \,\Big|\, \varphi_{\widehat{a}^-}\Big)\Big) \\
&\geqslant p^L \Big(1 - \sum_{y \in \mathbf{W}^a} \mathbb{P}^{\mathrm{G}}_{\mathcal{T}^{\mathbf{W}}}\big(|\varphi_y| \leqslant 2\sqrt{2u} \,\big|\, \varphi_{\widehat{a}^-}\big)\mathbb{P}^{\mathrm{E}}_{p}\big(\mathcal{E}_y \leqslant 4u\lambda_y^{\mathbf{W}}\big)\Big) \\
&\overset{\text{(II.5.24)}}{\geqslant} p^L \Big(1 - \sum_{y \in \mathbf{W}^a} \frac{4\sqrt{2u\lambda_y^{\mathbf{W}}}}{\sqrt{2\pi}}\Big(1 - e^{-4u\lambda_y^{\mathbf{W}}}\Big)\Big) \\
&\geqslant p^L \Big(1 - \frac{16}{\sqrt{\pi}} u^{\frac{3}{2}} \sum_{y \in \mathbf{W}^a} (\lambda_y^{\mathbf{W}})^{\frac{3}{2}}\Big),
\end{aligned}
\tag{II.5.25}
$$

taking advantage of the inequality $1 - e^{-x} \leqslant x$ for $x > 0$ in order to obtain the last inequality.

We now fix the parameters and start with choosing $c_\lambda, C_\Lambda, C_g, c_f, c_p > 0$ as well as $L_0(B)$, with B to be fixed later on, as the parameters from Proposition II.5.5, and c_e as the parameter from (II.5.19). Finally, for $u > 0$ define

$$
\widetilde{u}(u) := uc_e,\ L(u,B) := \left\lceil \frac{c_e}{3(1 - e^{-uc_e})}\Big(\frac{\sqrt{\pi}}{32B}\Big)^{\frac{2}{3}} \right\rceil \vee L_0(B) \text{ and } p(u,B) = 2^{-\frac{1}{L(u,B)}}.
\tag{II.5.26}
$$

Using the bound $1 - e^{-x} \geqslant x/2$ for $x > 0$ small enough, we can now find $u_0 = u_0(c_e, B) > 0$ such that

$$
L(u,B) \leqslant \frac{1}{u}\Big(\frac{\sqrt{\pi}}{32B}\Big)^{\frac{2}{3}} \text{ for all } u \in (0, u_0].
\tag{II.5.27}
$$

Then for all $u \in (0, u_0)$, under $\mathbf{P}^W_{L(u,B),\widetilde{u}(u)}$, for each $(L(u,B), B, c_\lambda, C_\Lambda, C_g, c_f)$-good vertex $a \in F$, we can continue the chain of inequalities in (II.5.25) to obtain

$$
\begin{aligned}
\mathbb{P}^{\mathrm{G}}_{\mathcal{T}^{\mathbf{W}}} \otimes \mathbb{P}^{\mathrm{E}}_{p(u,B)}\big(\mathbf{W}^a \subset A_u^{\mathbf{W}} \,\big|\, \varphi_{\widehat{a}^-}\big) &\overset{\text{(II.5.25)}}{\geqslant} p(u,B)^{L(u,B)}\Big(1 - \frac{16}{\sqrt{\pi}} u^{\frac{3}{2}} \sum_{y \in \mathbf{W}^a} (\lambda_y^{\mathbf{W}})^{\frac{3}{2}}\Big) \\
&\overset{\text{(II.5.3)}}{\geqslant} p(u,B)^{L(u,B)}\Big(1 - \frac{16}{\sqrt{\pi}} B(uL(u,B))^{\frac{3}{2}}\Big) \\
&\overset{\text{(II.5.26),(II.5.27)}}{\geqslant} \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.
\end{aligned}
\tag{II.5.28}
$$

With our choice of parameters, see in particular (II.5.26), we can use Proposition II.5.5 to show that the set F^g from (II.5.15) contains with positive probability a $d(c_d B^{-2/3})$-ary tree that we denote by F^{g0}, where $d(c_d B^{-2/3})$ will be large (cf. (II.5.31)), and $c_d := c_e c_p c_f (\sqrt{\pi}/32)^{2/3}/3$. Conditionally on the realization of the Galton–Watson tree $\mathcal{T}^{\mathbf{W}}$, and on the event that F^{g0} exists, we write

$$
\begin{aligned}
F^{g1} &:= \{\varnothing\} \cup \Big\{a \in F^{g0} \backslash \{\varnothing\} : \mathbf{W}^{a^-} \subset A_u^{\mathbf{W}}\Big\} \text{ and} \\
\mathcal{F}^{g1}_a &:= \sigma\big(\mathbf{1}_{\{\mathbf{W}^{(a')^-} \subset A_u^{\mathbf{W}}\}}, a' \in (F \backslash F_a) \cup \{a\}\big)
\end{aligned}
\tag{II.5.29}
$$

for all $a \in F$, where F_a, the subtree below a, was defined in the paragraph below (II.2.2), and where we use the convention $\mathbf{W}^{\varnothing^-} = \varnothing$. Taking advantage of the Markov property, see (II.2.11) and below, under $\mathbb{P}^{\mathrm{G}}_{\mathcal{T}^{\mathbf{W}}}$ and conditionally on $\varphi_{\hat{a}^-}$, the field $\varphi_{|\mathbf{W}^a}$ is independent of $\varphi_{\varnothing}$ and $\varphi_{|\mathbf{W}^{(a')^-}}$ for all $a' \in (F \backslash F_a) \cup \{a\}$. Thus, for all $u \in (0, u_0)$ and $a \in \mathcal{X}$, on the event that F^{g0} exists and $a \in F^{g1}$ (which implies in particular that a is good), we have that

$$
\begin{aligned}
&\mathbb{P}^{\mathrm{G}}_{\mathcal{T}^{\mathbf{W}}} \otimes \mathbb{P}^{\mathrm{E}}_{p(u,B)}\Big(\big|G_a^{F^{g1}}\big| \geqslant d(c_d B^{-2/3}) \,\big|\, \mathcal{F}_a^{g1}, \varphi_{\varnothing}\Big) \\
&= \mathbb{P}^{\mathrm{G}}_{\mathcal{T}^{\mathbf{W}}} \otimes \mathbb{P}^{\mathrm{E}}_{p(u,B)}(\mathbf{W}^a \subset A_u^{\mathbf{W}} \,|\, \varphi_{\hat{a}^-}) \overset{\text{(II.5.28)}}{\geqslant} \frac{1}{4}.
\end{aligned}
\tag{II.5.30}
$$

Therefore, conditionally on the realization of the Galton–Watson tree $\mathcal{T}^{\mathbf{W}}$ and on the event that F^{g0} exists, by Lemma II.5.4, the set F^{g1} contains with positive $\mathbb{P}^{\mathrm{G}}_{\mathcal{T}^{\mathbf{W}}} \otimes \mathbb{P}^{\mathrm{E}}_{p(u,B)}(\,\cdot\,|\,\varphi_{\varnothing})$-probability (not depending on $\varphi_{\varnothing}$) a $d\big(d(c_d B^{-2/3})/4\big)$-ary tree. Moreover, since

$$
d\big(d(c_d B^{-2/3})/4\big) \to \infty \text{ as } B \to 0, \tag{II.5.31}
$$

taking B small enough we get that, under $\mathbf{E}^{W}_{L(u,B),\tilde{u}(u)}[\mathbb{P}^{\mathrm{G}}_{\mathcal{T}^{\mathbf{W}}} \otimes \mathbb{P}^{\mathrm{E}}_{p(u,B)}(\,\cdot\,|\,\varphi_{\varnothing})]$, the set F^{g1} contains an infinite subtree with positive probability that we denote by δ, and which does not depend on $\varphi_{\varnothing}$.

Write $p(u) = p(u,B)$ and $L(u) = L(u,B)$ for this choice of B. For each $a \in F^{g1}$, we have $\mathbf{W}^{a^-} \subset A_u^{\mathbf{W}} \cap \mathcal{I}^u$ by (II.5.15), (II.5.20) and (II.5.29). Since $\hat{a} \in \mathbf{W}^a$ and $\hat{a}^- \in \mathbf{W}^{a^-}$ by construction, and so $\mathbf{W}^a$ and $\mathbf{W}^{a^-}$ are adjacent in $\mathcal{T}^{\mathbf{W}}$ (i.e. $\min_{x \in \mathbf{W}^{a^-}, y \in \mathbf{W}^a} d_{\mathcal{T}^{\mathbf{W}}}(x,y) = 1$) the infinite connected tree in F^{g1} yields an infinite connected subset $\bigcup_{a \in F^{g1}} \mathbf{W}^a$ in $\mathcal{T}^{\mathbf{W}}$ which is included in $A_u^{\mathbf{W}} \cap \mathcal{I}^u$. Since $(\mathcal{T}^{\mathbf{W}}, A_u^{\mathbf{W}}, \mathcal{I}^u)$ under $\mathbf{E}^{\mathbf{W}}_{L(u),\tilde{u}(u)}[\mathbb{P}^{\mathrm{G}}_{\mathcal{T}^{\mathbf{W}}} \otimes \mathbb{P}^{\mathrm{E}}_{p(u)}(\cdot)]$ has the same law as $(\mathcal{T}, A_u \cap B_{p(u)}, \mathcal{I}^u)$ under $\mathbb{E}^{\mathrm{GW}}[\mathbb{P}^{\mathrm{RI}}_{\mathcal{T}} \otimes \mathbb{P}^{\mathrm{G}}_{\mathcal{T}} \otimes \mathbb{P}^{\mathrm{E}}_{p(u)}(\cdot)]$ by (II.1.9), (II.4.15) and (II.5.23), we proved that the root is included in an unbounded connected component of $A_u \cap B_p \cap \mathcal{I}^u$ with positive probability.

In order to conclude, we still need to prove that percolation occurs almost surely. The strategy will be to construct a Galton–Watson tree $\mathcal{T}^Z$ such that there are conditionally independent copies of the tree F^{g1} from (II.5.29) whose associated watersheds can all be embedded into $\mathcal{T}^Z$. Since each of these copies of F^{g1} is infinite with probability at least δ, at least one of them will be infinite a.s., and we can conclude. We now explain how to do this construction in detail. Under some probability measure $\mathbb{P}^Z_u$, let $(Z_k)_{k \in \mathbb{N}}$ be an i.i.d. sequence of subtrees in $\mathcal{X}$, with the same law as the subtree

$$
\mathcal{T}^{\mathbf{W}}_- \cup \bigcup_{a \in F:\, \widetilde{V}_L(\mathbf{X}^a) = H_{\hat{a}1}(\mathbf{X}^a) = \infty} \mathcal{T}^{\hat{a}1}
$$

of $\mathcal{T}^{\mathbf{W}}$ under $\mathbf{P}^{W}_{L(u),\tilde{u}(u)}$, where $\mathcal{T}^{\mathbf{W}}_-$ is defined in (II.4.13) and $\mathcal{T}^x$ in (II.4.14). Since $\mathcal{T}^{\mathbf{W}}_-$ is constructed by the use of watersheds, in a slight abuse of language we will also call watersheds the respective subsets of Z_k corresponding to watersheds in $\mathcal{T}^{\mathbf{W}}_-$, if no confusion is to arise from this. Let us now define recursively a sequence of trees $\mathcal{T}^Z_k$, $k \in \mathbb{N}$, with $\partial \mathcal{T}^Z_k \neq \varnothing$, a.s. as follows: first take $\mathcal{T}^Z_1 = Z_1$. Note that $\partial Z_1 \neq \varnothing$ a.s. since either $\widetilde{V}_L(\mathbf{X}^{\varnothing}) = \infty$, and then ∂Z_1 contains any point of $\partial(\mathbf{T}^{\varnothing} \backslash \mathbf{T}^{\varnothing}_{V_L(X^{\varnothing})})$, which is a.s. non-empty; or otherwise if $\widetilde{V}_L(\mathbf{X}^{\varnothing}) < \infty$ then $\widehat{\varnothing 1} \in \partial Z_1$ (which does not always corresponds to $\widehat{\varnothing}1$) since we did not add the tree $\mathcal{T}^{\widehat{\varnothing}1}$ in the definition of Z_1 and $\widehat{\varnothing 1} \in \partial\mathcal{T}_-^{\mathbf{W}}$ by (II.4.10).

To define $\mathcal{T}_k^Z$ recursively, assume that $\mathcal{T}_{k-1}^Z$ is defined with $\partial\mathcal{T}_{k-1}^Z \neq \varnothing$. Let x_k be the first vertex in $\partial\mathcal{T}_{k-1}^Z$ (in lexicographic order in Ulam-Harris notation). We then define $\mathcal{T}_k^Z$ as the union of $\mathcal{T}_{k-1}^Z$ and $x_k \cdot Z_k$, which also verifies $\partial\mathcal{T}_k^Z \neq \varnothing$.

Let $\mathcal{T}_-^Z$ be the union of $\mathcal{T}_k^Z$, $k \in \mathbb{N}$, and $\mathcal{T}^Z$ be the union of $\mathcal{T}_-^Z$ and some additional independent Galton–Watson trees below each $x \in \partial\mathcal{T}_-^Z$, each with the same law as $x \cdot \mathcal{T}$ under $\mathbb{P}^{\mathrm{GW}}$. Then, by construction, $\mathcal{T}^Z$ has the same law as the usual Galton–Watson tree $\mathcal{T}$ under $\mathbb{P}^{\mathrm{GW}}$. Define F_k^{g0} and $\mathbf{W}_k^a$, $a \in F_k^{g0}$, similarly as above (II.5.29) and in (II.4.17), but corresponding to Z_k, which are i.i.d. copies of F^{g0} and $\mathbf{W}^a$, $a \in F^{g0}$, in $k \in \mathbb{N}$. Moreover, under $P_u^Z := \mathbb{E}_u^Z[\mathbb{P}^{\mathrm{G}}_{\mathcal{T}^Z} \otimes \mathbb{P}^{\mathrm{E}}_{p(u)}(\cdot)]$, define A_u^Z similarly as in (II.5.23), but with $\mathcal{T}^{\mathbf{W}}$ replaced by $\mathcal{T}^Z$, and for each $k \in \mathbb{N}$, take $F_k^{g1} = \{a \in F_k^{g0} : x_k \cdot \mathbf{W}_k^{a^-} \subset A_u^Z\}$, similarly as in (II.5.29). Then by Markov's property for the Gaussian free field, conditionally on φ_{x_k}, F_k^{g1} is independent of F_i^{g1}, $i < k$, and thus for each $u \in (0, u_0)$ we have

$$P_u^Z\big(|F_k^{g1}| = \infty \,|\, F_i^{g1}, i < k\big) = E_u^Z\big[P_u^Z(|F_k^{g1}| = \infty \,|\, \varphi_{x_k}) \,|\, F_i^{g1}, i < k\big] \geqslant \delta; \qquad \text{(II.5.32)}$$

here, the last inequality follows from the fact that, for each $a \in \mathbb{R}$, the law of F_k^{g1} conditionally on $\varphi_{x_k} = a$ under P_u^Z is the same as the law of F^{g1} conditionally on $\varphi_\varnothing = a$ under $\mathbf{E}^{\mathbf{W}}_{L(u),\tilde{u}(u)}[\mathbb{P}^{\mathrm{G}}_{\mathcal{T}^{\mathbf{W}}} \otimes \mathbb{P}^{\mathrm{E}}_{p(u)}(\cdot)]$, and δ is the constant introduced below (II.5.30). Using the tower property recursively on $k \in \mathbb{N}$, one can easily show that (II.5.32) implies that there exists P_u^Z-a.s. $k_0 \in \mathbb{N}$ such that $|F_{k_0}^{g1}| = \infty$. Note moreover that one can use Proposition II.4.2 similarly as in the proof of Proposition II.5.6, to obtain an interlacements $\mathcal{I}^u$ on $\mathcal{T}^Z$ with $x_k \cdot \mathbf{W}_k^a \subset \mathcal{I}^u$ for each $a \in F_k^{g0}$ and $k \in \mathbb{N}$. To this effect, note in particular that (II.5.22) still holds on $\mathcal{T}^Z$ since for each $k \in \mathbb{N}$ and $a \in F_k^{g0}$, the subtree $\mathcal{T}^Z_{x_k \cdot \hat{a}1}$ of $\mathcal{T}^Z$ below $x_k \cdot \hat{a}1$ is the copy $\mathcal{T}_k^{\hat{a}1}$ of $\mathcal{T}^{\hat{a}1}$ associated to Z_k, translated by x_k. Therefore, for each $u \in (0, u_0)$, the set $F_{k_0}^{g1}$ is P_u^Z-a.s. infinite and its associated watersheds $\mathbf{W}_{k_0}^a$, $a \in F_{k_0}^{g1}$, are included in $\mathcal{I}^u \cap A_u^Z$, and we can conclude.

□

In order to deduce Theorem II.1.1 from Proposition II.5.7, we are going to use the isomorphism (II.2.21) between the Gaussian free field and random interlacements. We first show that condition (II.2.20) – which entails the validity of the isomorphism (II.2.21) by Proposition II.2.5 – holds $\mathbb{P}^{\mathrm{GW}}$-a.s. for the Galton–Watson tree $\mathcal{T}$.

Proposition II.5.8. $\mathbb{P}^{\mathrm{GW}}$*-almost surely we have that for all* $x \in \mathcal{T}$*,*

$$P_x^{\mathcal{T}}(\,\cdot\,|\, H_{x^-} = \infty)\text{-almost surely,} \quad \mathrm{cap}_{\mathcal{T}}(\{X_i, i \in \mathbb{N}\}) = \infty.$$

Proof. Let $x \in \mathcal{X}$ and $L \in \mathbb{N}$. Under some probability $\widetilde{\mathbf{Q}}_x^L$, we now define a tree $\widetilde{\mathbf{T}}$, with weights denoted by $\widetilde{\boldsymbol{\lambda}}_{y,z}$, $y, z \in \widetilde{\mathbf{T}}$, $y \sim z$, as some extension of the tree $\mathbf{T}_{V_L}$ starting at x from Section II.4.1, by completing its remaining ends so that $\widetilde{\mathbf{T}}$ is a Galton–Watson tree conditioned on $x \in \widetilde{\mathbf{T}}$. More precisely, first define $\widetilde{\mathbf{T}} \backslash \widetilde{\mathbf{T}}_x$, that is the part of the tree $\widetilde{\mathbf{T}}$ which is not below x, with the same law as $\mathcal{T}\backslash\mathcal{T}_x$ under $\mathbb{P}^{\mathrm{GW}}(\,\cdot\,|\, x \in \mathcal{T})$, endowed with the corresponding weights. Then, attach to x a copy of the tree $\mathbf{T}_{V_L}$ with the same law as under $\mathbf{Q}_x^{\widetilde{\boldsymbol{\lambda}}_{x^-,x},L}$, as defined in Section II.4.1. With a slight abuse of notation, we see $\mathbf{T}_{V_L}$ as a subset of $\widetilde{\mathbf{T}}$. Finally for each remaining point $y \in \partial\mathbf{T}_{V_L}$, attach to y an independent copy of $y \cdot \mathcal{T}$. Let $\widetilde{\mathbf{X}}$ be a process with the same law as $(\mathbf{X}_{k\wedge V_L})_{k\in\mathbb{N}_0}$ under $\mathbf{Q}_x^{\widetilde{\lambda}_{x^-,x},L}$, it follows easily from Proposition II.4.1 that $(\widetilde{\mathbf{T}},\widetilde{\mathbf{X}})$ under $\widetilde{\mathbf{Q}}_x^L$ has the same law as $(\mathcal{T},(X_{k\wedge V_L})_{k\in\mathbb{N}_0})$ under $\mathbb{E}^{\mathrm{GW}}[P_x^{\mathcal{T}}(\cdot)\,|\,x\in\mathcal{T}]$.

Similarly as in the proof of Lemma II.5.2 iv), one can show that there exist positive constants c_λ and c_f so that, for each $\varepsilon>0$, if L is large enough, then

$$\widetilde{\mathbf{Q}}_x^L\Big(\big|\{y\in\partial\mathbf{T}_{V_L}:\widetilde{\boldsymbol{\lambda}}_{y,y^-}\geqslant c_\lambda\}\big|<c_fL,\,V_L(\widetilde{\mathbf{X}})<H_{x^-}(\widetilde{\mathbf{X}})\Big)\leqslant\varepsilon.$$

Indeed, this follows easily from (II.4.6) and a reasoning similar to the one in (II.5.4), (II.5.5) and (II.5.6), replacing $\{\sum_{i\in\mathbb{N}}\boldsymbol{\lambda}_i^{(k)}\leqslant C_\Lambda\}$ by $\{\exists\, i\in\mathbb{N}:\boldsymbol{\lambda}_i^{(k)}\geqslant c_\Lambda\}$.

Since, conditionally on $\mathbf{T}_{V_L}$, $g^{\widetilde{\mathbf{T}}_y}(y,y)$, $y\in\partial\mathbf{T}_{V_L}$, are i.i.d. with the same law as $g^{\mathcal{T}}(\varnothing,\varnothing)$, by the law of large number and the bound on the Green function from Lemma II.5.2 ii) we deduce that for L large enough

$$\widetilde{\mathbf{Q}}_x^L\Big(\big|\{y\in\partial\mathbf{T}_{V_L}:\widetilde{\boldsymbol{\lambda}}_{y,y^-}\geqslant c_\lambda,\ g^{\widetilde{\mathbf{T}}_y}(y,y)\leqslant C_g\ \}\big|<\frac{c_f}{4}L,V_L(\widetilde{\mathbf{X}})<H_{x^-}(\widetilde{\mathbf{X}})\Big)\leqslant 2\varepsilon.$$

Note that the event $\{\widetilde{\boldsymbol{\lambda}}_{y,y^-}\geqslant c_\lambda,\,g^{\widetilde{\mathbf{T}}_y}(y,y)\leqslant C_g\}$ implies by a similar reasoning to above (II.5.22) that $g^{\widetilde{\mathbf{T}}_{y^-}}(y^-,y^-)\leqslant C_g+\frac{1}{c_\lambda}$. Let $\widetilde{\mathbf{W}}=\{\widetilde{\mathbf{X}}_0,\dots,\widetilde{\mathbf{X}}_{V_L}\}$. Recalling the definition of the equilibrium measure from (II.2.12), we moreover have that $e_{\widetilde{\mathbf{W}},\widetilde{\mathbf{T}}}(z)=e_{\{z\},\widetilde{\mathbf{T}}_z}(z)=(g^{\widetilde{\mathbf{T}}_z}(z,z))^{-1}$ for each $z\in\partial\widetilde{\mathbf{W}}\backslash\{x\}$. Since $y^-\in\partial\widetilde{\mathbf{W}}$ for each $y\in\partial\mathbf{T}_{V_L}$ by construction, we deduce that for L large enough

$$\widetilde{\mathbf{Q}}_x^L\Big(\mathrm{cap}_{\widetilde{\mathbf{T}}}(\widetilde{\mathbf{W}})<\frac{c_f}{4(C_g+1/c_\lambda)}L,V_L(\widetilde{\mathbf{X}})<H_{x^-}(\widetilde{\mathbf{X}})\Big)\leqslant 2\varepsilon.$$

Since $\widetilde{\mathbf{W}}$ has the same law under $\widetilde{\mathbf{Q}}_x^L(\cdot,\,V_L(\widetilde{\mathbf{X}})<H_{x^-}(\widetilde{\mathbf{X}}))$ as the first L points visited by X under $\mathbb{E}^{\mathrm{GW}}[P_x^{\mathcal{T}}(\cdot,\,V_L(X)<H_{x^-}(X))\,|\,x\in\mathcal{T}]$, letting first $L\to\infty$ and then $\varepsilon\to0$, and noting that $\{V_L(X)<H_{x^-}(X)\}$ decreases to $\{H_{x^-}(X)=\infty\}$, we readily obtain (II.5.8). □

We can now deduce Theorem II.1.1 from Proposition II.5.7 using the isomorphism from Proposition II.2.5 combined with Proposition II.5.8.

Proof of Theorem II.1.1. Consider the probability space $\mathbb{Q}_{\mathcal{T}}^u$ from Proposition II.2.5. Abbreviating $\mathcal{E}_x:=\mathcal{E}_x^{(1)}$, we have $\ell_{x,u}\geqslant\lambda_x^{-1}\mathcal{E}_x$ for all $x\in\mathcal{I}^u$ by (II.2.19). In view of Proposition II.5.8, we can apply the isomorphism (II.2.21), and we get $\mathbb{Q}_{\mathcal{T}}^u$-a.s. for all $x\in\mathcal{I}^u\cap A_u$

$$\gamma_x=-\sqrt{2u}+\sqrt{2\ell_{x,u}+\varphi_x^2}\geqslant-\sqrt{2u}+\sqrt{2\lambda_x^{-1}\mathcal{E}_x+\varphi_x^2}\overset{\text{(II.1.9)}}{\geqslant}-\sqrt{2u}+2\sqrt{2u}=\sqrt{2u}.$$

This yields (II.1.8) by defining $\widehat{E}^{\geqslant\sqrt{2u}}=\{x\in\mathcal{T}:\gamma_x\geqslant\sqrt{2u}\}$. By Proposition II.5.7, for all $u\in(0,u_0)$ there is $\mathbb{Q}_{\mathcal{T}}^u$-a.s. an unbounded component for $A_u\cap\mathcal{I}^u$, and so also for the level set $\widehat{E}^{\geqslant\sqrt{2u}}$. This readily implies $h_*>0$ since $\widehat{E}^{\geqslant\sqrt{2u}}$ has the same law as $E^{\geqslant\sqrt{2u}}$. □

Remark II.5.9. Rather surprisingly, our proof does not work anymore if one tries to replace the inclusion (II.1.8) by any of the simpler inclusions $\mathcal{I}^u\cap\{x\colon\mathcal{E}_x>4u\lambda_x\}\subset\widehat{E}^{\geqslant\sqrt{2u}}$ or $\mathcal{I}^u\cap\{x\colon|\varphi_x|>2\sqrt{2u}\}\subset\widehat{E}^{\geqslant\sqrt{2u}}$. In other words, we need to use both the local times of random interlacements and the Gaussian free field φ in the isomorphism (II.2.21), and not just one of the two. Indeed, in view of Proposition II.5.5, one needs to take L at least equal to C/u for some large constant $C < \infty$ in order for F^g to percolate. For instance for constant conductances and small enough u, the probability that $\mathbf{W}^a \subset \{x : \mathcal{E}_x > 4u\lambda_x\}$ is at least $1 - CuL$, and the probability that $\mathbf{W}^a \subset \{x : |\varphi_x| > 2\sqrt{2u}\lambda_x\}$ is of order $1 - C\sqrt{u}L$ in view of (II.5.24), for some constant $C < \infty$. These bounds are not interesting for the previous choice of $L = C/u$. However combining them gives that the probability that $\mathbf{W}^a \subset A_u$ is of order $1 - Cu^{3/2}L$, see (II.5.25), which goes to one for the previous choice of L when $u \to 0$.

Proof of Theorem II.1.2. The statement for random interlacements follows trivially from Proposition II.5.7 for $u \leqslant u_0$ by the inclusion $\mathcal{I}^u \cap A_u \cap B_p \subseteq \mathcal{I}^u \cap B_p$. Using the monotonicity in u of interlacements we obtain the statement for all $u > 0$. The statement for the Gaussian free field also follows from Propositions II.5.7, II.2.5 and II.5.8 similarly as in the proof of Theorem II.1.1. □

Remark II.5.10. An interesting open question is whether Theorem II.1.2 is true in the whole supercritical phase of the Gaussian free field, that is for each $h < h_*$, does there exist $p \in (0, 1)$ such that $E^{\geqslant h} \cap B_p$ percolates, or is transient even?

II.6 Transience of the level sets

In this section we prove Theorem II.1.3, that is that both, the interlacements set and the level sets of the Gaussian free field above small positive levels, are transient – even when intersected with a small Bernoulli noise. More precisely, we prove that the random walk on the tree of very good watersheds is transient, see Proposition II.6.3, and use arguments similar to the proof of Theorem II.1.1 to conclude. The notion of very goodness we use here is a refinement of the one introduced in Definition II.5.1, see (*iv*)′) below, and is adapted in order to ensure that the random walk on the tree of very good watersheds can be compared to a random walk on a Galton–Watson with a constant drift, see (II.6.4). We then follow the strategy of the proof of [Col06, Theorem 1] in order to deduce transience. In addition to the usual assumption (II.1.2), we assume throughout this section that, conditionally on the non-weighted tree $\mathcal{T}$, the family $(\lambda_{x,y})_{x\sim y\in\mathcal{T}}$ is i.i.d. and has compact support. In terms of the construction of the Galton–Watson tree in Section II.2.1, this is equivalent to assuming that, under ν and conditionally on $\pi((\lambda_j)_{j\in\mathbb{N}})$, the family $(\lambda_i)_{1\leqslant i\leqslant\pi((\lambda_j)_{i\in\mathbb{N}})}$ is i.i.d., that the law of λ_1 does not depend on $\pi((\lambda_j)_{j\in\mathbb{N}})$, and that there exist $0 < \overline{c}_\lambda < \overline{C}_\Lambda < \infty$ such that ν-a.s.

$$\overline{c}_\lambda < \lambda_i < \overline{C}_\Lambda \text{ for all } 1 \leqslant i \leqslant \pi((\lambda_j)_{j\in\mathbb{N}}). \tag{II.6.1}$$

We use the independence of the conductances when referring to [Gan+12] in the proof of Lemma II.6.1, and the assumption (II.6.1) in (II.6.4). Note that (II.2.3) and (II.6.1) imply that the mean offspring distribution m is finite.

Let us now define a notion of goodness which is stronger than the one introduced in Definition II.5.1: in this section, we say that a point $a \in F$ is (L, B, C_g, c_f, c_L)-*very good* if it verifies the conditions i) to iii) with $c_\lambda = \overline{c}_\lambda$ and $C_\Lambda = \overline{C}_\Lambda$ (which simplifies these conditions in view of (II.6.1)), and v) of Definition II.5.1, as well as

$iv)'$ the set of children of the vertex a in the tree of free points F satisfies

$$\left|\{a' \in G_a^F : d_{\mathcal{T}^{\mathbf{W}}}(\widehat{a}, \widehat{a'}) \geqslant c_L L\}\right| \geqslant \frac{c_f L}{2},$$

where we recall that $d_{\mathcal{T}^{\mathbf{W}}}$ denotes the graph distance within $\mathcal{T}^{\mathbf{W}}$. Note that the inequality $\lambda^F_{a,a'} \leqslant C_\Lambda = \overline{C}_\Lambda$ is trivially satisfied under (II.6.1) by taking $C_\Lambda = \overline{C}_\Lambda$, and thus $iv)'$ is stronger than iv) in Definition II.5.1 (up to changing the constant c_f). We now follow a strategy inspired by that of Section II.5 in order to show that the tree of very good free points contains a d-ary tree. We first evaluate the probability for a point to verify the property $iv)'$, analogously to Lemma II.5.2 iv). Recall the construction of the trees $\mathbf{T}_k$, $k \in \mathbb{N}_0$, under the probability measure $\mathbf{Q}_x^{\kappa,L}$ from Section II.4.1, as well as the stopping time $V_L(\mathbf{X})$ and $\widetilde{V}_L(\mathbf{X})$ from (II.4.2) and (II.4.3). In what follows we abbreviate $V_L = V_L(\mathbf{X})$ to simplify notation.

Lemma II.6.1. *Let c_f be as in Lemma II.5.2. There exists $c_L > 0$ such that for all $\varepsilon > 0$, there exists $L_0 = L_0(\varepsilon) \in \mathbb{N}$ such that for all $x \in \mathcal{X}$, $L \geqslant L_0$ and $\kappa \leqslant \overline{C}_\Lambda$,*

$$\mathbf{Q}_x^{\kappa,L}\Big(\left|\{y \in \partial \mathbf{T}_{V_L} \backslash \{x1, \mathbf{X}_{V_L}\} : d_{\mathbf{T}_{V_L}}(x,y) \geqslant c_L L\}\right| < c_f L/2, \widetilde{V}_L(\mathbf{X}) = \infty\Big) \leqslant \varepsilon.$$

Proof. It is known, see [LP16, Theorem 17.13], that the speed of a random walk on a Galton–Watson tree $\mathcal{T}$ with unit conductances is $\mathbb{P}^{\mathrm{GW}}$-a.s. strictly positive and deterministic; i.e., the limit $v := \lim_{k\to\infty} \frac{d_{\mathcal{T}}(\varnothing, X_k)}{k} > 0$ exists and is a constant. This result was generalized in [Gan+12] to Galton–Watson trees with finite mean for the offspring distribution and i.i.d. conductances verifying (II.1.2). In view of Proposition II.4.1, the process $\mathbf{X}$ under $\mathbf{Q}_x^{\kappa,L}(\,\cdot\,, \widetilde{V}_L(\mathbf{X}) = \infty)$ has the same law as a random walk X on $\mathcal{T}$ under $P_x^{\mathcal{T}}(\,\cdot\,, \widetilde{V}_L(X) = \infty \,|\, \lambda_{x,x^-} = \kappa)$. Therefore, for all $\varepsilon > 0$ we can find a $k_0 = k_0(\varepsilon)$ such that for all $k > k_0$, $L \in \mathbb{N}$, $x \in \mathcal{X}$ and $\kappa \leqslant \overline{C}_\Lambda$, we have

$$\mathbf{Q}_x^{\kappa,L}\Big(\exists\, n \geqslant k : d_{\mathbf{T}_{\widetilde{V}_L}}(\mathbf{X}_n, x) \leqslant vk/2, \widetilde{V}_L(\mathbf{X}) = \infty\Big) \leqslant \varepsilon/3. \tag{II.6.2}$$

In order to find enough vertices in $\mathfrak{F}_a$ at distance at least c_L from x, we note that $|\mathbf{T}_k| \leqslant |\mathbf{T}_{V_k}| = \sum_{x\in\{\mathbf{X}_1,\dots,\mathbf{X}_{V_k}\}} |\{x\} \cup G_x^{\mathbf{T}_{V_k}}|$, and that $\{G_x^{\mathbf{T}_{V_k}} : x \in \{X_1, \dots, X_{V_k}\}\}$ is an i.i.d. family of cardinality k if $\widetilde{V}_L = \infty$, $k \leqslant L$, similarly as in (II.4.6). Since $m < \infty$, by the weak law of large number we can find $C_P > 0$ such that for all $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that for all $k > k_0$, $L \geqslant k$, $x \in \mathcal{X}$ and $\kappa > 0$

$$\mathbf{Q}_x^{\kappa,L}(|\mathbf{T}_k| \geqslant C_P k, \widetilde{V}_L(\mathbf{X}) = \infty) \leqslant \varepsilon/3. \tag{II.6.3}$$

Applying (II.6.2) and (II.6.3) with $k = \frac{c_f}{2C_P} L$, for L large enough so that $k \geqslant k_0$, we obtain that with probability at most $2\varepsilon/3$, on the event $\widetilde{V}_L(\mathbf{X}) = \infty$, there are more than $c_f L/2$ points in $\mathbf{T}_{V_L}$ at distance less than $c_L L$ from x, where $c_L := \frac{vc_f}{4C_P}$. We can then conclude by combining this with Lemma II.5.2 iv) for $\varepsilon/3$.

□

Recall the definition of $A_u^{\mathbf{W}}$ in (II.5.23). We can now prove analogously to the proof of Proposition II.5.7 that (L, B, C_g, c_f, c_L)-*very good* points, whose associated watershed is included in $A_u^{\mathbf{W}}$, contain a supercritical Galton–Watson tree.

Proposition II.6.2. *Let $c_\lambda = \overline{c}_\lambda$, C_g and c_f be as in Lemma II.5.2, c_e as in (II.5.19), and c_L as in Lemma II.6.1. For each $d \in \mathbb{N}$, there exist $B > 0$ and $u_0 > 0$,* *such that, for each* $u \in (0, u_0)$*, there exist* $L \in \mathbb{N}$ *and* $p \in (0,1)$*, so that under* $\mathbf{E}^W_{L,\tilde{u}}[\mathbb{P}^{\mathrm{G}}_{\mathcal{T}^{\mathbf{W}}} \otimes \mathbb{P}^{\mathrm{E}}_p(\,\cdot\,|\,\varphi_\varnothing)]$*, with* $\tilde{u} = uc_e$*, the tree*

$$F^{g1'} := \{\varnothing\} \cup \Big\{a \in F \backslash \{\varnothing\} : a^- \text{ is } (L, B, C_g, c_f, c_L)\text{-very good}, \\ d_{\mathcal{T}^{\mathbf{W}}}(\hat{a}, \widehat{a^-}) \geqslant c_L L \text{ and } \mathbf{W}^{a^-} \subseteq A_u^{\mathbf{W}}\Big\}$$

contains with positive probability, not depending on $\varphi_\varnothing$*, a* d*-ary tree.*

Proof. Using Lemma II.6.1 in place of Lemma II.5.2 iv), and adding the condition $d_{\mathcal{T}^{\mathbf{W}}}(\hat{a}, \widehat{a^-}) \geqslant c_L L$ in the definition (II.5.15) – which is possible in view of the condition *iv)'* – one can easily prove similarly as below (II.5.30) that for each $B > 0$ there exists $u_0 = u_0(B)$, such that for all $u \in (0, u_0)$, there exists $L = L(u, B)$ and $p = p(u, B)$ as in (II.5.26), so that $F^{g1'}$ contains a $d\big(d(c_d B^{-2/3})/4\big)$-ary tree, and we can conclude in view of (II.5.31). □

We prove now transience using the argument of [Col06, Theorem 1].

Proposition II.6.3. *There exists* $B > 0$*,* $u > 0$*,* $L \in \mathbb{N}$ *and* $p \in (0,1)$*, such that under* $\mathbf{E}^W_{L,uc_e}[\mathbb{P}^{\mathrm{G}}_{\mathcal{T}^{\mathbf{W}}} \otimes \mathbb{P}^{\mathrm{E}}_p(\,\cdot\,|\,\varphi_\varnothing)]$*, the connected component of* $\varnothing$ *in the tree with vertex set*

$$\mathcal{T}^{g1'} := \bigcup_{a \in F^{g1'}} \mathbf{W}^a$$

is transient with positive probability, not depending on $\varphi_\varnothing$*.*

Proof. Consider a random walk X on $\mathcal{T}^{g1'}$ starting in $\varnothing$. We proceed by contradiction, and assume that $\mathcal{T}^{g1'}$ is recurrent, that is, the walk X comes back to the root almost surely. We introduce the following color scheme: $\varnothing$ is white, and a vertex $ai \in F^{g1'}$ is white if a is white and $\widehat{ai}$ is visited by X in the interval $[H_{\hat{a}}, \inf\{k \geqslant H_{\hat{a}} : X_k = \widehat{a^-}\}]$. We want to show that there is an infinite number of white vertices with positive probability; indeed, since then there would in particular be an infinite connected component of white vertices, this would constitute a contradiction as the watershed associated to each white vertex in the connected component of $\varnothing$ is visited by X in the interval $[H_{(\mathbf{W}^\varnothing)^c}, \inf\{k \geqslant H_{(\mathbf{W}^\varnothing)^c} : X_k = \varnothing\}]$ by definition.

For a fixed vertex $ai \in F^{g1'}$, we evaluate the probability, starting from $\hat{a}$, to visit $\widehat{ai}$ before returning to $\widehat{a^-}$. Because of recurrence, for the computation of this probability, we can restrict ourselves to the only path connecting $\widehat{a^-}$ to $\widehat{ai}$ and we compute its effective conductance $\mathcal{C}$ (see [LP16, (2.4)]). Both the distances between $\widehat{a^-}$ and $\hat{a}$, and the one between $\hat{a}$ and $\widehat{ai}$ are at least $c_L L$ by definition of $F^{g1'}$, and at most L by definition of watersheds, see in particular (II.4.2) and (II.4.9). Therefore, using the series law (see [LP16, Subsection 2.3.I]) we obtain that the probability of a random walk starting from $\hat{a}$, to visit $\widehat{ai}$ before returning to $\widehat{a^-}$, is equal to

$$\frac{\mathcal{C}(\hat{a} \leftrightarrow \widehat{ai}\,)}{\mathcal{C}(\widehat{a^-} \leftrightarrow \hat{a}) + \mathcal{C}(\hat{a} \leftrightarrow \widehat{ai}\,)} = \frac{\Big(\sum_{x \in (\hat{a}, \widehat{ai}\,]} \frac{1}{\lambda_{x^-,x}}\Big)^{-1}}{\Big(\sum_{x \in (\widehat{a^-}, \hat{a}]} \frac{1}{\lambda_{x^-,x}}\Big)^{-1} + \Big(\sum_{x \in (\hat{a}, \widehat{ai}\,]} \frac{1}{\lambda_{x^-,x}}\Big)^{-1}} \overset{\text{(II.6.1)}}{\geqslant} \frac{\overline{c}_\lambda}{\overline{C}_\Lambda} \frac{c_L}{2}, \tag{II.6.4}$$

where $(x, y]$ denotes the unique path connecting x to y, minus x. For each $d \in \mathbb{N}$, it follows from Proposition II.6.2 that for an appropriate choice of B, u, L and p, the tree of white vertices contains with positive probability a weightless Galton–Watson tree with mean offspring distribution larger than $d\frac{\bar{c}_\lambda}{\overline{C}_\Lambda}\frac{c_L}{2}$. Taking $d = \lceil 4\frac{\overline{C}_\Lambda}{\bar{c}_\lambda c_L}\rceil$, this tree of white vertices is infinite with positive probability, which concludes the proof. □

Proof of Theorem II.1.3. Similarly to the proofs of Theorems II.1.1 and II.1.2 at the end of Section II.5, one can use the isomorphism (II.2.21), which holds by Proposition II.5.8 similarly as in the proof of Theorem II.1.1, as well as Proposition II.5.6 to show that the component of $\varnothing$ in the tree $\mathcal{T}^{g1'}$ from Proposition II.6.3 can be included in $\mathcal{I}^u \cap B_p$ or $\widehat{E}^{\geqslant\sqrt{2u}} \cap B_p$, proving the transience of those sets with positive probability by Rayleigh's Monotonicity Principle (see [LP16, Section 2.4]). To show that transience occurs almost surely for some component, one can proceed similarly to the end of the proof of Theorem II.5.7 by considering the Galton–Watson tree $\mathcal{T}^Z$ on which there are infinitely many conditionally independent copies of $\mathcal{T}^{g1'}$, and thus one of these copies is transient a.s. □

II.A The critical parameters are deterministic

In this section we prove that:

Theorem II.A.1. *Under* $\mathbb{P}^{\mathrm{GW}}$

$$\mathcal{T} \mapsto u_*(\mathcal{T}) \text{ is a.s. constant} \tag{II.A.1}$$

$$\mathcal{T} \mapsto h_*(\mathcal{T}) \text{ is a.s. constant} \tag{II.A.2}$$

Those results are known from [Tas10] and [AS18] in the case of deterministic unit conductances. We provide here the generalizations for the case of random conductances.

The proofs are based on the 0-1 law for inherited properties from [LP16, Chapter 5], which we will shortly recall here. For this purpose, we start with introducing the following definition.

Definition II.A.2. A property $\mathcal{P}$ is called *inherited* if

- All finite trees satisfy property $\mathcal{P}$, and
- if a tree $\mathcal{T}$ with root x has the property $\mathcal{P}$, then all the descendant trees $\mathcal{T}_y$ with $y \in G_x^{\mathcal{T}}$ have $\mathcal{P}$.

Since we are dealing with different trees, in this section we underline the dependence on the graph writing $\mathbb{P}^{\mathrm{G}}_{\mathcal{T}}$ and $\mathbb{P}^{\mathrm{RI}}_{\mathcal{T}}$ for the law of the Gaussian free field and random interlacements on the tree $\mathcal{T}$.

There exists a 0-1 law for surviving Galton-Watson trees [LP16, Proposition 5.6]. We generalize it here for our context of Galton-Watson trees with random conductances verifying (SA).

Theorem II.A.3. *If $\mathcal{P}$ is an inherited property, then*

$$\mathbb{P}^{\mathrm{GW}}(\mathcal{T} \text{ has } \mathcal{P}) \in \{0, 1\}.$$

Proof. We write $\mathbb{P}^{\mathrm{GW}}_*$ for the law of the unpruned weighted Galton-Watson tree under ν, while we coherently use $\mathbb{P}^{\mathrm{GW}}$ for the pruned tree conditioned on surviving. Denote by A the set of trees satisfying property $\mathcal{P}$, and by G_1 we denote the first generation's size of the Galton-Watson tree, which is a random variable with law μ (cf. (II.2.4)). Then

$$\mathbb{P}^{\mathrm{GW}}_*(\mathcal{T} \in A) = \mathbb{E}^{\mathrm{GW}}_*[\mathbb{P}^{\mathrm{GW}}_*(\mathcal{T} \in A \,|\, G_1)] \leqslant \mathbb{E}^{\mathrm{GW}}_*[\mathbb{P}^{\mathrm{GW}}_*(\forall i = 1, \dots, G_1,\ \mathcal{T}_i \in A \,|\, G_1)],$$

where the inequality follows from the fact that $\mathcal{P}$ is inherited. Now since conditionally on G_1, the subtrees $\mathcal{T}_i$, $i = 1, \dots, G_1$, are independent and have the same law as $\mathcal{T}$, we can continue the above to get

$$\mathbb{P}^{\mathrm{GW}}_*(\mathcal{T} \in A) \leqslant \mathbb{E}^{\mathrm{GW}}_*[\mathbb{P}^{\mathrm{GW}}_*(\mathcal{T} \in A)^{G_1}] = f(\mathbb{P}^{\mathrm{GW}}_*(\mathcal{T} \in A)),$$

where $f(s) := \mathbb{E}^{\mathrm{GW}}_*[s^{G_1}]$, for $s \in [0, 1]$, is the probability generating function of the tree under ν.

It is known that if $m > 1$ the function f is strictly convex with two fixed points $q = \mathbb{P}^{\mathrm{GW}}_*(\mathcal{T} \text{ is finite})$ and 1. Together with the inequality $\mathbb{P}^{\mathrm{GW}}_*(\mathcal{T} \in A) \leqslant f(\mathbb{P}^{\mathrm{GW}}_*(\mathcal{T} \in A))$ this implies that $\mathbb{P}^{\mathrm{GW}}_*(\mathcal{T} \in A) \in [0, q] \cup \{1\}$. But since all finite trees are in A, we infer that $\mathbb{P}^{\mathrm{GW}}_*(\mathcal{T} \in A) \in \{q, 1\}$. Rewriting this in terms of the pruned measure, we infer

$$\mathbb{P}^{\mathrm{GW}}(\mathcal{T} \in A) = \frac{\mathbb{P}^{\mathrm{GW}}_*(\mathcal{T} \in A,\, |\mathcal{T}| = \infty)}{\mathbb{P}^{\mathrm{GW}}_*(|\mathcal{T}| = \infty)} \in \{0, 1\},$$

and this finishes the proof. ∎

II.A.1 The critical parameter u_* is constant

We generalize now the proof from [Tas10] of (II.A.1).

For $u > 0$, define the property $\mathcal{P}^u$ as follows: we say that $\mathcal{T}$ has the property $\mathcal{P}^u$ if either

$$\mathcal{T} \text{ is finite, or} \qquad \mathbb{P}^{\mathrm{RI}}_{\mathcal{T},u}(|\mathcal{C}_\varnothing| = \infty) = 0$$

where $\mathcal{C}_\varnothing$ is the cluster of the root $\varnothing$ in $\mathcal{V}^u$.

Lemma II.A.4. *For each $u > 0$ the property $\mathcal{P}^u$ is inherited.*

Proof. Finite trees have $\mathcal{P}^u$ by definition, so we show that

$$\exists x \sim \varnothing\colon \mathcal{T}_x \text{ has } \mathcal{P}_u \qquad \Rightarrow \qquad \mathcal{T}_\varnothing \text{ has } \mathcal{P}_u.$$

Let $x \sim \varnothing$ such that the subtree $\mathcal{T}_x$ has not $\mathcal{P}_u$, i.e.

$$\mathbb{P}^{\mathrm{RI}}_{\mathcal{T}_x,u}(|\mathcal{C}_x| = \infty) > 0. \tag{II.A.3}$$

We use, as in [Tas10], a result from Teixeira:

Proposition II.A.5 ([Tei09, Theorem 5.1])**.** *Let T be a transient weighted tree with locally bounded degree. Define for a fixed $x \in T$ the functon $h\colon T_x \to [0,1]$*

$$h^x_{T_x}(z) = P^{T_x}_z(\widetilde{H}_{z^-} = \infty) P^{T_x}_z(H_z = \infty)\lambda_z \mathbf{1}_{\{z \neq x\}}.$$

Then, conditionally on $\{x \in \mathcal{V}^u\}$, $\mathcal{C}_x \cap T_x$ under $\mathbb{P}^{\mathrm{RI}}_u$ has the same law as open cluster containing x of independent Bernoulli percolation with parameter $p_u(z) = e^{-uh^x_{T_x}(z)}$.

By definition, for all $z \in \mathcal{T}_x \backslash \{x\}$, $h^x_{\mathcal{T}_x}(z) = h^x_{\mathcal{T}}(z)$, and this implies that

$$\mathbb{P}^{\mathrm{RI}}_{\mathcal{T},u}(|\mathcal{C}_x \cap \mathcal{T}_x| = \infty \,|\, x \in \mathcal{V}^u) = \mathbb{P}^{\mathrm{RI}}_{\mathcal{T}_x,u}(|\mathcal{C}_x| = \infty \,|\, x \in \mathcal{V}^u). \tag{II.A.4}$$

Again, by definition, for all $z \in \mathcal{T}_x \backslash \{x\}$, $h^x_{\mathcal{T}}(z) = h^\varnothing_{\mathcal{T}}(z)$, and the last Proposition implies that the law of $\mathcal{C}_\varnothing \cap \mathcal{T}_x$ under $\mathbb{P}^{\mathrm{RI}}_{\mathcal{T},u}(\cdot \,|\, \varnothing, x \in \mathcal{V}^u)$ is the same as the law of $\mathcal{C}_x \cap \mathcal{T}_x$ under $\mathbb{P}^{\mathrm{RI}}_{\mathcal{T},u}(\cdot \,|\, x \in \mathcal{V}^u)$ and this implies that

$$\mathbb{P}^{\mathrm{RI}}_{\mathcal{T},u}(|\mathcal{C}_\varnothing \cap \mathcal{T}_x| = \infty \,|\, \varnothing, x \in \mathcal{V}^u) = \mathbb{P}^{\mathrm{RI}}_{\mathcal{T},u}(|\mathcal{C}_x \cap \mathcal{T}_x| = \infty \,|\, x \in \mathcal{V}^u). \tag{II.A.5}$$

Then, using the capacity of sets, it holds

$$\mathbb{P}^{\mathrm{RI}}_{\mathcal{T},u}(\varnothing, x \in \mathcal{V}^u) = e^{-u \operatorname{cap}_{\mathcal{T}}(\{\varnothing, x\})} > 0. \tag{II.A.6}$$

Altogether

$$\begin{aligned}
\mathbb{P}^{\mathrm{RI}}_{\mathcal{T},u}(|\mathcal{C}_\varnothing| = \infty) &\geqslant \mathbb{P}^{\mathrm{RI}}_{\mathcal{T},u}(|\mathcal{C}_x \cap \mathcal{T}_x| = \infty \,|\, \varnothing, x \in \mathcal{V}^u)\mathbb{P}^{\mathrm{RI}}_{\mathcal{T},u}(\varnothing, x \in \mathcal{V}^u) \\
&= \mathbb{P}^{\mathrm{RI}}_{\mathcal{T},u}(|\mathcal{C}_\varnothing \cap \mathcal{T}_x| = \infty \,|\, \varnothing, x \in \mathcal{V}^u)\mathbb{P}^{\mathrm{RI}}_{\mathcal{T},u}(\varnothing, x \in \mathcal{V}^u) \\
&\overset{\text{(II.A.5)}}{=} \mathbb{P}^{\mathrm{RI}}_{\mathcal{T},u}(|\mathcal{C}_x \cap \mathcal{T}_x| = \infty \,|\, x \in \mathcal{V}^u)\mathbb{P}^{\mathrm{RI}}_{\mathcal{T},u}(\varnothing, x \in \mathcal{V}^u) \\
&\overset{\text{(II.A.4)}}{=} \mathbb{P}^{\mathrm{RI}}_{\mathcal{T}_x,u}(|\mathcal{C}_x| = \infty \,|\, x \in \mathcal{V}^u)\mathbb{P}^{\mathrm{RI}}_{\mathcal{T},u}(\varnothing, x \in \mathcal{V}^u) \\
&\overset{\text{(II.A.3),(II.A.6)}}{>} 0
\end{aligned}$$

which proves that $\mathcal{P}_u$ is inherited. ∎

With the previous 0-1 law and the hereditary property $\mathcal{P}^u$, we can prove (II.A.1): since $\mathcal{P}^u$ is inherited, by Theorem II.A.3, $\mathbb{P}^{\mathrm{GW}}(\mathcal{T} \text{ has } \mathcal{P}_u) \in \{0,1\}$. Hence for every s in $\mathbb{Q}^+$, there exist a set A_s with $\mathbb{P}^{\mathrm{GW}}(A_s) = 1$, where $\mathbf{1}_{\{\mathbb{P}^{\mathrm{RI}}_u(|\mathcal{C}_\varnothing|=\infty)=0\}}$ is constant on A_s. Taking intersection over $\mathbb{Q}^+$, on the set $A := \bigcap_{s\in\mathbb{Q}^+}$, all the functions $\mathbf{1}_{\{\mathbb{P}^{\mathrm{RI}}_{\mathcal{T},u}(|\mathcal{C}_\varnothing|=\infty)=0\}}$ are constant.

Now, since the function $u \mapsto \mathbb{P}^{\mathrm{RI}}_{\mathcal{T},u}(|\mathcal{C}_\varnothing| = \infty)$ is decreasing, the function

$$\mathcal{T} \mapsto \inf_{s\in\mathbb{Q}^+} \{\mathbb{P}^{\mathrm{RI}}_{\mathcal{T},s}(|\mathcal{C}_\varnothing| = \infty) = 0\}$$

is well defined and constant on A. Hence

$$\begin{aligned} u_*^{\mathcal{T}} &= \inf_{s\in\mathbb{Q}^+} \{\mathbb{P}^{\mathrm{RI}}_{\mathcal{T},s}(|\mathcal{C}_\varnothing| = \infty) = 0\} \\ &= \inf_{s\in\mathbb{R}^+} \{\mathbb{P}^{\mathrm{RI}}_{\mathcal{T},s}(|\mathcal{C}_\varnothing| = \infty) = 0 \end{aligned}$$

is $\mathbb{P}^{\mathrm{GW}}$-a.s. constant.

II.A.2 The critical parameter h_* is constant

We show (II.A.2) in a similar way to what done for u_*. Define for each $h \in \mathbb{R}$ the property $\mathcal{P}^h$ by saying that a tree T rooted at x satisfies $\mathcal{P}^h$ if T_y is transient for all $y \in T$ and

$$\mathbb{P}^{\mathrm{G}}_T\big(\big|E_x^{\geqslant h}\big| = \infty\big) = 0,$$

where for $y \in T$ we denote by $E_y^{\geqslant h}$ the connected component of y in $\{z \in T : \varphi_z \geqslant h\}$. We now need to prove that the property $\mathcal{P}^h$ is inherited, which has been done in the setting of unit conductances in [AS18, Lemma 5.1]. For the reader's convenience we now present a proof in our setting inspired by [Tas10].

Lemma II.A.6. *For each $h \in \mathbb{R}$, the property $\mathcal{P}^h$ is inherited.*

Proof. Assume that T is a tree rooted at x verifying $\mathcal{P}^h$. For any $y \in T$ with $y \in G_x^T$ we have

$$\mathbb{P}^{\mathrm{G}}_T\big(\big|E_x^{\geqslant h}\big| = \infty\big) \geqslant \mathbb{P}^{\mathrm{G}}_T\big(\big|E_y^{\geqslant h} \cap T_y\big| = \infty, \varphi_x \geqslant h\big) \geqslant \mathbb{P}^{\mathrm{G}}_T\big(\big|E_y^{\geqslant h} \cap T_y\big| = \infty\big)\mathbb{P}^{\mathrm{G}}_T(\varphi_x \geqslant h),$$

where the second inequality is a consequence of the finite dimensional FKG inequality for Gaussian fields, see [Pit82], and a classical limiting procedure. Since the second factor on the right-hand side is non-zero, $\mathbb{P}^{\mathrm{G}}_T\big(\big|E_x^{\geqslant h}\big| = \infty\big) = 0$ implies for each $y \in G_x^T$

$$\mathbb{P}^{\mathrm{G}}_T\big(\big|E_y^{\geqslant h} \cap T_y\big| = \infty\big) = 0.$$

What is left to do is to show that the previous equation holds also for the Gaussian free field on the subtree T_y. By disintegration, we observe that for λ-almost all $b \in \mathbb{R}$ we have

$$\mathbb{P}^{\mathrm{G}}_T\Big(\big|E_y^{\geqslant h} \cap T_y\big| = \infty \,\big|\, \varphi_y = b\Big) = 0.$$

From the Markov property applied to the set $K = \{y\}$, it follows that the restriction of the Gaussian free field under $\mathbb{P}^{\mathrm{G}}_T(\,\cdot \mid \varphi_y = b)$ to T_y has the same law as the Gaussian free field under $\mathbb{P}^{\mathrm{G}}_{T_y}(\,\cdot \mid \varphi_y = b)$. Hence we obtain that for each $y \in G_x^T$ and λ-almost all $b \in \mathbb{R}$ we have

$$\mathbb{P}^{\mathrm{G}}_{T_y}\Big(\big|E_y^{\geqslant h}\big| = \infty \,\big|\, \varphi_y = b\Big) = 0.$$

Integrating again we obtain $\mathbb{P}^{\mathrm{G}}_{T_y}\big(\big|E_y^{\geqslant h}\big| = \infty\big) = 0$, proving that $\mathcal{P}^h$ is inherited. $\square$

With the previous 0-1 law and the inherited property $\mathcal{P}^h$, we can prove (II.A.2).

Proof of (II.A.2). Since the property $\mathcal{P}^h$ is inherited by Lemma II.A.6, it follows from Theorem II.A.3 that $\mathbb{P}^{\mathrm{GW}}(\mathcal{T} \text{ has } \mathcal{P}_h) \in \{0,1\}$ for each $h \in \mathbb{R}$. Moreover by Proposition II.2.1 and since $\mathcal{T}_x$ has the same law as $x \cdot \mathcal{T}$ under $\mathbb{P}^{\mathrm{GW}}$, see (SA), $\mathcal{T}_x$ is transient for all $x \in \mathcal{T}$ $\mathbb{P}^{\mathrm{GW}}$-a.s. Hence for every $s \in \mathbb{Q}$, there exists an event A_s with $\mathbb{P}^{\mathrm{GW}}(A_s) = 1$ such that $\mathcal{T} \mapsto \mathbf{1}_{\{\mathbb{P}^{\mathrm{G}}_{\mathcal{T}}(|E^{\geqslant s}_{\varnothing}|=\infty)=0\}}$ is constant on A_s. Thus on the event $A := \bigcap_{s\in\mathbb{Q}} A_s$, all the functions $\mathbf{1}_{\{\mathbb{P}^{\mathrm{G}}_{\mathcal{T}}(|E^{\geqslant s}_{\varnothing}|=\infty)=0\}}$, $s \in \mathbb{Q}$, are constant. Now, since the function $h \mapsto \mathbb{P}^{\mathrm{G}}_{\mathcal{T}}(|E^{\geqslant h}_{\varnothing}| = \infty)$ is decreasing, the function

$$\mathcal{T} \mapsto \inf_{s\in\mathbb{Q}}\{\mathbb{P}^{\mathrm{G}}_{\mathcal{T}}(|E^{\geqslant s}_{\varnothing}| = \infty) = 0\} = \inf_{h\in\mathbb{R}}\{\mathbb{P}^{\mathrm{G}}_{\mathcal{T}}(|E^{\geqslant h}_{\varnothing}| = \infty) = 0\}$$

is well defined and constant on A, and we can conclude by (II.1.4) and FKG inequality. $\square$

Chapter III

The Lipschitz cutset on fractal graphs

III.1 Introduction

Consider a collection of particles on an infinite (connected) graph, where a Poisson distributed number of particles are placed at each vertex of the graph. Then, over time, each particle performs an independent continuous time simple random walk on the graph. Assume that at time 0 a single additional infected particle is placed somewhere on the graph and consider the infection dynamics to be as follows: whenever a particle shares a vertex with an infected particle, it instantaneously becomes infected itself. Infected particles can also recover and become healthy/susceptible again, which occurs independently for each infected particle at some exponentially distributed random time. Due to the infection mechanism outlined above, a particle can only truly recover when it is the sole particle at a vertex - otherwise it gets reinfected straight away by one of the other particles sharing its location.

This problem has been studied in various forms among others by Kesten and Sidoravicius. In [KS05] the authors consider the graph to be the nearest neighbour square lattice $\mathbb{Z}^d$ and treat the case where infected particles never recover. They show that for large times and with high probability, the sites of $\mathbb{Z}^d$ that have already been visited by an infected particle contain a ball of radius proportional to time around the site where the infection started. They also prove that these sites are themselves completely contained in a bigger ball of radius that is also proportional to time, again with high probability. In [KS08] they refine this result and prove a shape theorem for the infection under suitable rescaling of space. In a parallel paper [KS06], they study the case of infection with recovery on $\mathbb{Z}^d$ and prove the existence of a phase transition with respect to the recovery rate of the particles - for rates higher than a critical threshold, the infection will almost surely go extinct (i.e. no infected particle remains after some finite time), whereas for rates below this threshold, the infection will with positive probability survive indefinitely.

More recently, Gracar and Stauffer [GS19a; GS19b] have developed a general framework with which they were able to prove that on the weighted graph $(\mathbb{Z}^d, \lambda)$, with edges equipped with uniformly elliptic conductances $\lambda_{x,y}$, the infection still spreads with positive speed. They also showed that in the case of infection with recovery, the infection not only survives indefinitely with positive probability, but also spreads with positive speed - a question that was left unanswered previously. A further application of this framework can be found in [BS23], where it is shown that in the case of infection with recovery, conditioned on the infection surviving, the origin of $\mathbb{Z}^d$ (i.e. where the infection is started) is visited by an infected particle at arbitrarily large times. The key benefit of the framework used in these works is that it can be applied to different variations of the Poisson random walks and infection models, and that the multi-scale analysis which is done in order to set up the framework does not need to be redone from scratch when the type of event studied changes. Given a local, translation invariant, increasing event with a high enough probability, the framework gives the existence of a connected surface in space-time where the event holds and which acts as a cutset in space-time, separating the origin from infinity, so that any particle which visits the origin has to intersect the surface at some later time.

In this work, we adapt the framework to an entirely new class of graphs — sub-diffusive fractal lattice graphs. In particular, we study the behaviour of a particle system on the Sierpiński gasket and on generalized Sierpiński carpets. Intuitively, these are the graphs of the famous triangle and square based fractals, where instead of repeating the construction recursively inwards, one instead expands outwards, by attaching copies of the current stage of the graph recursively. A key difference between the standard Euclidean lattice ($\mathbb{Z}^d$ as well as for example the triangle or hexagonal lattice nearest neighbour graph) and the graphs we study is that random walks on the latter exhibit subdiffusive behaviour. I.e., random walks move through the graph much more slowly than e.g. on the Euclidean lattice, and it takes on average r^{d_w} amount of time to leave a ball of radius r, where $d_w > 2$ is a constant that depends on the dimension of the graph, and on which parts of the graph are missing. Compared to Euclidean lattices, where this average is of order r^2 regardless of the dimension of the lattice, this shows that on such fractal graphs random walks exhibit a quantitatively different behaviour. Crucially, this slower movement of the particles makes it unclear whether the dynamics of the infection process remain unchanged or whether the infection has a harder or easier time surviving over time. Our main result provides an answer to this question.

In order to state it, we quickly formalize some of the concepts above. Let G be either the Sierpiński gasket graph or a generalized Sierpiński carpet graph (defined precisely in Sections III.2.1 and III.8. See also the corresponding Figures III.1 and III.8). In our first result we adapt the so-called Lipschitz surface framework from [GS19a] to the fractal graph case. Notably, although the framework remains the same in spirit, it requires changes across the board due to the significantly changed geometry of the graph, starting with the analogue of the Lipschitz surface for fractal graphs. On $\mathbb{Z}^d$, the framework gives rise to a discrete, Lipschitz connected surface in (a coarse-grained) space-time graph $\mathbb{Z}^{d+1}$. On the fractal graphs we study we cannot hope for such a strong connectivity property. However, as we define in Subsection III.2.5 and prove in Section III.3, the corresponding object still acts as a cutset on the coarse-graining of the space-time graph, meaning that any path escaping toward infinity must intersect this cutset (cf. Definition III.2.9). Furthermore it is in some sense minimal and still retains the Lipschitz connectivity property along the time dimension (cf. Corollary III.3.5). We call this object the Lipschitz cutset. We prove in Theorem III.2.12 that such a Lipschitz cutset exists a.s. and in Theorem III.2.13 that it surrounds the origin within a finite distance a.s..

The Lipschitz cutset retains the flexibility of the Lipschitz surface and can be used to prove various statements; we present one as an example. Consider the infection process with recovery as outlined above, where at the beginning there is an independent Poisson distributed with intensity μ_0 number of particles at each vertex of the graph, and γ is the rate at which infected particles recover. We say that the infection survives if for every time $t \geqslant 0$ there exists at least one infected particle somewhere on the graph. We then have the following result.

Theorem III.1.1. *For any $\mu_0 > 0$ there exists $\gamma_0 > 0$ such that for all $\gamma \in (0, \gamma_0)$ the infection with recovery on G survives with positive probability.*

Theorem III.1.1 is a direct consequence of Proposition III.9.1 which gives the above statement even in the case where the fractal graph is equipped with uniformly elliptic conductances, and it is proven as an application of the Lipschitz cutset from Theorems III.2.12 and III.2.13, and the property of “Lipschitz in the time dimension” from Corollary III.3.5.

This paper is structured as follows. In Section III.2 we define the Sierpiński gasket graph and formalize the definitions and basic properties outlined above. We also state the two main technical Theorems III.2.12 and III.2.13 which give the existence and key properties of the Lipschitz cutset. In Section III.3 we construct the Lipschitz cutset and provide a sufficient condition for its existence, as well as prove its key geometric properties. Section III.4 covers the tool used in our multi-scale analysis, a mixing theorem that allows us under the right conditions to resample particles independently. In Section III.5 we define the multi-scale tessellation of the space-time graph and its properties which lead to the proof of Theorem III.2.12 in Section III.6. We prove Theorem III.2.13 in Section III.7 with an extension of the multi-scale argument developed before. Section III.8 covers the adaptation of the results which are written with the Sierpiński gasket graph in mind to the case of generalized Sierpiński carpet graphs. The paper concludes with Section III.9 with the application of Theorems III.2.12 and III.2.13 in order to prove Theorem III.1.1.

Throughout this work we will denote constant with $c_0, c_1, \dots$ and $C_1, C_2, \dots$. Important constant that should be kept track off will be denoted differently: this includes C_λ, $\mathrm{C}_{\mathrm{mix}}$, C_ψ and the constants in mixing Theorem III.4.6: $M_1, M_2, M_3, M_4, \Theta$.

III.2 Settings and definitions

We start by defining the Sierpiński graph and the coarse-graining which we will use throughout the paper. We then proceed to formally define the particle system we will be studying before stating the two main results of this paper.

III.2.1 The Sierpiński gasket graph

The Sierpiński gasket is a fractal which was introduced in [Sie15]. Here we define the Sierpiński graph or Sierpiński prefractal based on the Sierpiński fractal with a recursive construction as presented in [Del02]. Consider any of the graphs obtained from the d-dimensional unit side-length regular simplex in $\mathbb{R}^d$, $d \geqslant 2$, by placing one vertex in the origin. Fix such a graph and denote it with $\triangle^d$. More precisely, $\triangle^d := (V, E)$ where V are the $d+1$ vertices corresponding to the corners of the simplex and E is the set of all undirected pairs of vertices which share an edge in the simplex. For $d = 2$, this is the graph induced by the equilateral triangle with unit length sides, motivating the notation $\triangle^d$. In $d = 3$, the graph is induced by the equilateral tetrahedron. We furthermore assume the graph to be weighted with conductances $\lambda := (\lambda_{x,y})_{\{x,y\} \in E}$, which are positive symmetric and we assume the existence of a constant C_λ such that the conductances are uniformly elliptic, i.e.

$$\frac{1}{C_\lambda} \leqslant \lambda_{x,y} \leqslant C_\lambda. \tag{III.2.1}$$

Define now $\triangle_0^d := \triangle^d$ and iteratively the graph of scale n, for $n \geqslant 1$, as

$$\triangle_n^d := \bigcup_{x \in 2^{n-1} \triangle_0^d} \left(x + \triangle_{n-1}^d\right), \tag{III.2.2}$$

taking care of identifying overlapping vertices at the junctions; edges carry the same conductance as in $\triangle_0^d$, i.e. for any $n \geqslant 1$, $z \in 2^{n-1}\triangle_0^d$ and $x, y \in \triangle_0^d$, the conductance on the edge $(z+x, z+y)$ is $\lambda_{x,y}$. The d-dimensional Sierpiński graph $\mathbb{G}^d$ is the graph obtained by taking the union of $\triangle_n^d$ over $n \in \mathbb{N}_0$. We write $x \sim y$ if there is an edge between x and y, and let $\lambda_x := \sum_{y \sim x} \lambda_{x,y}$. We will denote by $(\mathbb{G}^d, (\lambda_{x,y})_{x \sim y})$ the weighted graph $\mathbb{G}^d$ with conductances $(\lambda_{x,y})_{x \sim y}$.

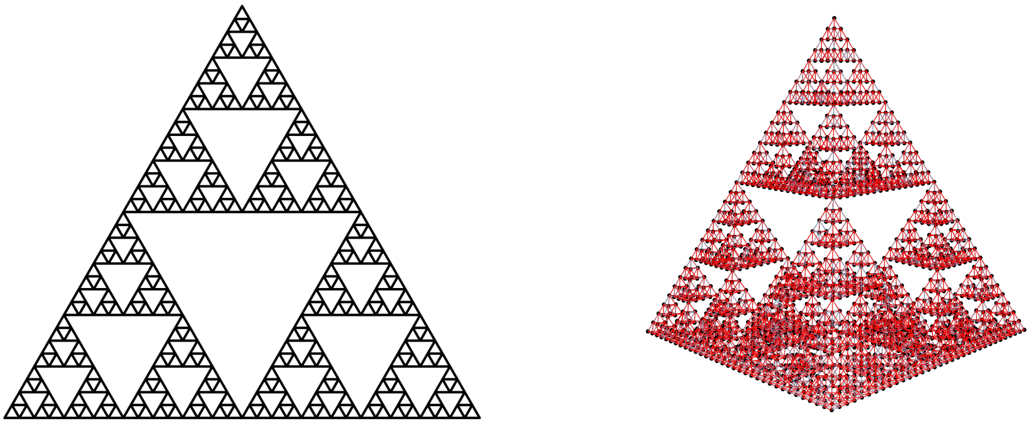
We introduce the set

$$\mathbb{B}^d := \left\{\iota \in \mathbb{G}^d \colon \iota + \triangle_0^d \text{ is a subgraph of } \mathbb{G}^d\right\}, \tag{III.2.3}$$

which intuitively contains those vertices in $\mathbb{G}^d$ which are the "lower left" corner of some translation of the simplex $\triangle^d$ in $\mathbb{G}^d$. Note that this set is stable under multiplication with powers of 2 in the sense that for all $m \in \mathbb{N}_0$ and $\iota \in \mathbb{B}^d$,

$$\iota 2^m + \triangle^d_m \text{ is a subgraph of } \mathbb{G}^d. \tag{III.2.4}$$

We consider the natural graph distance $d(\cdot,\cdot)$ on $\mathbb{G}^d$ and define the distance between sets as the usual minimum of the distances between vertices contained therein. For a finite set A we define the volume $\mathrm{Vol}(A) := |A|$ as the cardinality of the set A. Define the ball of radius $r \geqslant 0$ with center $x \in \mathbb{G}^d$ as $B_r(x) := \{y \in \mathbb{G}^d \colon d(x,y) \leqslant r\}$, and the volume of such balls $\mathrm{Vol}_r(x) := \mathrm{Vol}(B_r(x))$ as the number of vertices contained in it. Note that the conductances do not affect $d(\cdot,\cdot)$ or the volume.



(a) $d = 2$ (b) $d = 3$

Figure III.1: The first six stages of the Sierpiński Gasket.

It can be shown that for each $d \geqslant 2$, there exist constants $\mathrm{c_{vol}}, \mathrm{C_{Vol}} > 0$ (depending on the dimension) such that for all $x \in \mathbb{G}^d$ and $r \geqslant 1$

$$\mathrm{c_{vol}}\, r^{d_v} \leqslant \mathrm{Vol}_r(x) \leqslant \mathrm{C_{Vol}}\, r^{d_v}, \tag{$\mathrm{Vol}(d_v)$}$$

and we call d_v the *volume dimension* of the graph. We refer to the discussion below ($\mathrm{E}(d_w)$) for a brief list of the different names of d_v in the literature. It is well-known that in dimension two $d_v = \log_2(3)$. To show that ($\mathrm{Vol}(d_v)$) holds in any dimension d, it is not hard to generalize the proof in [Bar98] in order to obtain that $d_v = \log_2(d+1)$.

We now present a regular coarse graining—referred to as *tesselation*—of the space-time space $\mathbb{G}^d \times \mathbb{Z}$ which we need in order to state the theorems. This definition will be in line with the more complex tessellation presented in Subsection III.5.1.

III.2.2 First level tessellation

Definition III.2.1. For a given value $\ell \in \mathbb{N}_0$, we tessellate the graph $\mathbb{G}^d$ into *tiles* $S_1(\iota) := \iota 2^\ell + \triangle^d_\ell$, for $\iota \in \mathbb{B}^d$, so that each tile is indexed by ι and has side length equal to 2^ℓ.

For a given value $\beta > 0$, we tessellate $\mathbb{R}$ (which will play the role of time) into intervals $T_1(\tau) := [\tau\beta, (\tau+1)\beta)$, indexed by $\tau \in \mathbb{Z}$.

We then define the (space-time) *cell* indexed by (ι,τ) as $R_1(\iota,\tau) := S_1(\iota) \times T_1(\tau)$.

When referring to subsets of the spatial graph in general, such as tiles, unions of tiles or balls on the graph, we will refer to them as *regions* or *subregions* when the distinction between the kind of subset does not play a role.

Later on, we might refer to cells with shorter notation such as simply u or v when we do not need to specify the indices of the cell. This will usually be in conjunction with some set of cells, where we will write $u \in A$ as a shorthand for $R_1(\iota, \tau) \in A$ (see for example (III.2.8) and the text immediately thereafter).

Definition III.2.2. We say two cells $R_1(\iota_1, \tau_1) \neq R_1(\iota_2, \tau_2)$ are *adjacent* if either $\iota_1 = \iota_2$ and $|\tau_1 - \tau_2| \leqslant 1$ or else if $d(S_1(\iota_1), S_1(\iota_2)) = 0$ and $\tau_1 = \tau_2$.

Remark III.2.3. We could alternatively define $S_1(\iota)$ to be "half-open" in the sense that only the "corner" corresponding to ι is in $S_1(\iota)$ while all other corners are not, making the tiles disjoint. This distinction makes no difference for the combinatorial arguments we will use; it could however be important for the lowest level events one could consider (cf. Definition III.2.8) in the application of our framework.

We will use this space-time tessellation in order to define a dependent percolation model where space-time cells will be *good* or *bad* depending on whether a given event dependent on the particle behaviour occurs roughly in the region defined by the corresponding $S_1(\iota)$ during the time interval $T_1(\tau)$. More precisely however, the events that we will consider will not be limited to events localized entirely within $S_1(\iota)$. Instead, they will involve larger regions which in particular may intersect for different pairs (ι, τ) and (ι', τ'). To this end we introduce the following extension.

Definition III.2.4. Let $\eta \in \mathbb{N}$. For $\iota \in \mathbb{B}^d$ we define the *super-tile*

$$S_1^\eta(\iota) := \bigcup_{\iota' \in \mathbb{B}^d :\ d(\iota, \iota') \leqslant \eta} S_1(\iota'),$$

and for $\tau \in \mathbb{Z}$ the *super-interval* $T_1^\eta(\tau) := [\tau\beta_1, (\tau + \eta)\beta_1)$, as well as the *super-cell* $R_1^\eta(\iota, \tau)$ as $S_1^\eta(\iota) \times T_1^\eta(\tau)$.

III.2.3 Random walks on the Sierpiński graph

We will study Poisson random walks and for this purpose we start by analyzing properties of the simple random walk on Sierpiński gaskets. We call a stochastic process $(X_t)_{t \geqslant 0}$ taking values in $\mathbb{G}^d$ a (continuous time simple) random walk on $\mathbb{G}^d$ under the probability measure P_{x_0}, if $X_0 = x_0$ holds P_{x_0}-a.s., and while at $x \in \mathbb{G}^d$, it jumps to $y \sim x$ with rate $\lambda_{x,y}/\lambda_x$. We say that a function $f \colon \mathbb{G}^d \times \mathbb{R} \to \mathbb{R}$ is *caloric* if satisfies the discrete heat equation

$$\frac{\partial}{\partial t} f(x,t) = \sum_{y \sim x} \frac{\lambda_{x,y}}{\lambda_x} (f(y,t) - f(x,t))$$

and it is easy to verify that the heat kernel $p_t(x,y) := \frac{1}{\lambda_y} \mathbb{P}_x(X_t = y)$ seen as a function of y and t, with x fixed, satisfies it.

It is well known that the transition probabilities for a random walk on $\mathbb{Z}^d$ satisfy Gaussian estimates. Instead, the Sierpiński gasket falls into the class of nested fractals studied in [HK04, Corollary 4.13], which shows the validity of sharp upper and lower bounds for the heat kernel: denoting by $p_n(x,y) := \frac{1}{\lambda_y} \mathbb{P}_x(X_n = y)$ the heat kernel for the discrete time random walk, it holds

$$p_n(x,y) \asymp n^{-\frac{d_v}{d_w}} \exp\Big(- \Big(\frac{d(x,y)^{d_w}}{c_3 n} \Big)^{1/(d_w - 1)} \Big) \tag{III.2.5}$$

for $n > d(x,y)$, where $\asymp$ indicates that the ratio of the two sides is bounded from above and below by positive constants independent of x, y and n. The result was first shown on $\mathbb{G}^2$ in [Jon96]. Using the fact that the continuous time random walk X_t has jump rate 1, one can generalize the proof of [LL10, Theorem 2.5.6] to obtain a continuous time version of (III.2.5): for all $x, y \in \mathbb{G}^d$ and $t > 0$ with $d(x,y) < t$ it holds that

$$p_t(x,y) \asymp t^{-\frac{d_v}{d_w}} \exp\Big(-\Big(\frac{d(x,y)^{d_w}}{c_4 t}\Big)^{1/(d_w-1)}\Big). \qquad (\mathrm{HKB}(d_v,d_w))$$

We say that the Parabolic Harnack inequality holds for the graph $\mathbb{G}^d$ if there exists a constant $C_3 > 0$ such that for all $x \in \mathbb{G}^d$, $R \geqslant 1$ and non-negative $h\colon \mathbb{G}^d \times \mathbb{R} \to \mathbb{R}$ caloric in $B_{2R}(x) \times (0, 4R^{d_w})$ satisfies

$$\sup_{B_R(z)\times[R^{d_w},2R^{d_w}]} h(x,t) \leqslant C_3 \inf_{B_R(z)\times[3R^{d_w},4R^{d_w}]} h(x,t). \qquad (\mathrm{PH}(d_w))$$

Next, we introduce the walk dimension, and for this purpose, for any subset B of the graph $\mathbb{G}^d$ we write $H_B := \inf\{t > 0\colon X_t \in B\}$. We say that the graph has *walk dimension* d_w, if

$$E_x[H_{B_r(x)^c}] \asymp r^{d_w} \qquad (\mathrm{E}(d_w))$$

for all $x \in \mathbb{G}^d$. In the literature, the volume dimension (Vol(d_v)) and walk dimension (E(d_w)) are often referred to by different symbols: for example [Bar98] uses d_f and d_w respectively, [Bar04] uses α and β, [Jon96] $\frac{d_s d_w}{2}$ and d_w, and [Del02] uses d_f for the volume dimension.

It is proven that the gasket in dimension $d = 2$ has walk dimension $d_w = \log_2(5)$ (see for example [Bar98] or [GY18]).

For any dimension, the validity of (E(d_w)) and (PH(d_w)) follows from the following: from Theorem 3.1 of [GT02] the following implications hold:

$$(\mathrm{Vol}(d_v)) + (\mathrm{HKB}(d_v,d_w)) \Longleftrightarrow (\mathrm{PH}(d_w)) \Longrightarrow (\mathrm{E}(d_w)),$$

and in particular the Sierpiński gasket $\mathbb{G}^d$ satisfies (PH(d_w)) and (E(d_w)) for some value d_w (dependent on the dimension d).

Volume and walk dimensions are related: a simple inequality, which is valid for any graph which satisfies (Vol(d_v)) and (E(d_w)), is given by

$$2 \leqslant d_w \leqslant d_v + 1; \qquad (\mathrm{III.2.6})$$

and a proof can e.g. be found in [Bar04, Theorem 1].

We will also need the following folklore estimate on the confinement probability, which is a direct consequence of the estimates on the exit probability $\Psi_n(x,R)$ in [GT01, Proposition 7.1] on a graph with arbitrary random walk dimension.

Lemma III.2.5. *Let (X_t) be a random walk on (G,λ) starting from x_0 and $\mathbf{\Delta}, z > 0$ such that (E(d_w)) holds true. Then there exist $c_5, c_6, c_7 > 0$ such that for all $\mathbf{\Delta} > c_7 z$ the event*

$$\mathrm{Conf}(B_z,\mathbf{\Delta}) := \big\{X_t \in B_z(x_0) \text{ for all } t \in [0,\mathbf{\Delta}]\big\}$$

satisfies

$$\mathbb{P}\big(\mathrm{Conf}(B_z,\mathbf{\Delta})\big) \geqslant 1 - c_5 e^{-c_6\left(\frac{z^{d_w}}{\mathbf{\Delta}}\right)^{\frac{1}{d_w-1}}}. \qquad (\mathrm{Conf}(d_w))$$

III.2.4 Poisson particle system

We define a particle configuration Π as a random function in $(\mathbb{N}_0)^{\mathbb{G}^d}$, where $\Pi(x)$ is to be interpreted as the number of particles at $x \in \mathbb{G}^d$. We denote by φ_x the coordinate map of Π defined by $\varphi_x(\Pi) = \Pi(x)$ and call $\mathcal{F}$ the σ-algebra generated by the coordinate maps.

We define a particle system as a family of particle configurations $(\Pi_t)_{t\in\mathbb{R}} \in (\Omega, \mathcal{F}')$, with $\Omega := \{f : (-\infty, +\infty) \to (\mathbb{N}_0)^{\mathbb{G}^d}\}$ and $\mathcal{F}' := \mathcal{F}^{\otimes\mathbb{R}}$ the product sigma algebra of $\mathcal{F}$ over $\mathbb{R}$. We define (Π_t) under a probability measure $\nu_{c\mu}$ as a Poisson point process of random walkers with intensity given by $c\mu(x) := c\mu_0\lambda_x$ for $x \in \mathbb{G}^d$ and some $\mu_0 > 0, c \in (0, 1]$. It is easy to verify that the particle system is stationary (in fact, even reversible) in the sense that at any time $t \in \mathbb{R}$, the particles remain distributed according to a Poisson point process with intensity $c\mu$. This system is often referred to as *Poisson random walks.*

We say that an event $E \in \mathcal{F}'$ is increasing for the particle system $(\Pi_t)_{t\in\mathbb{R}}$ if the fact that E holds for $(\Pi_t)_{t\in\mathbb{R}}$ implies that E holds for all particle systems $(\Pi'_t)_{t\in\mathbb{R}}$ with $\Pi'_s \geqslant \Pi_s$ for all $s \geqslant 0$, where $\Pi'_s \geqslant \Pi_s$ indicates that $\Pi'_s(x) \geqslant \Pi_s(x)$ for all $x \in \mathbb{G}^d$.

We now define what it means for an event to be measurable with respect to a particle system. Although one could define this for an arbitrary particle system, we will consider events that are measurable with respect to the more restrictive Poisson random walks particle system from above. In particular, this means that we will consider events that are measurable with respect not only to the locations of the particles at different times, but also their movements over time.

Definition III.2.6. Let $A \subseteq \mathbb{G}^d$ and $t_0 \in \mathbb{R}$ and $t_1 > 0$. Denoting with $P_{x,t_0} := (p_{x,t_0,i})_{i=1}^{\Pi(x)}$ the set of particles (including their movements over time) that are located at x at time t_0 and with $P_{x,t_0,i}(t)$ the position of particle $p_{x,t_0,i}$ at time t, we say that an event E is *restricted* to A and a time interval $[t_0, t_0 + t_1]$ if it is measurable with respect to $\sigma\{P_{x,t_0,i}(t), i \in \{1, \ldots, \Pi(x)\}, x \in A, t \in (t_0, t_0 + t_1)\}$.

Definition III.2.7. Let $r > 0$. We say that a particle is *confined* inside B_r during $[t_0, t_0 + t_1]$ if during the time interval $[t_0, t_0 + t_1]$ it stays inside the ball $B_r(x)$, where x is the location of the particle at time t_0.

The probability of being confined has been estimated in (Conf(d_w)). We define now the probability associated to an event E.

Definition III.2.8. For $c \in (0, 1]$, $\mu(x) = \mu_0\lambda_x$ and an increasing event E restricted to $A \subseteq \mathbb{G}^d$ and $[0, t]$, we define

$$\nu_E(c\mu, A, B_r, t) := \nu_{c\mu}\big(E \big| \text{the particles in } A \text{ at } 0 \text{ are confined inside } B_r \text{ during } [0, t]\big).$$

III.2.5 Main results

We now provide the final definitions necessary to state the main theorems. For each $(\iota, \tau) \in \mathbb{B}^d \times \mathbb{Z}$ we will call $E(\iota, \tau)$ an increasing event restricted to the super-cell $R_1^\eta(\iota, \tau)$. We will call the cell $R_1(\iota, \tau)$ *bad* if the event $E(\iota, \tau)$ does not hold, and *good* otherwise. We next introduce a base of the space-time graph $\mathbb{G}^d \times \mathbb{Z}$. Recalling the definition of the gasket via $\triangle^d$ in (III.2.2), we consider the $d-1$-dimensional subgraph $\triangle^{d-1}$ of $(d-1)$ points including the origin defined in the same way, and letting $n \to \infty$ we obtain the $(d-1)$-dimensional Sierpiński gasket $\mathbb{G}^{d-1}$, which by construction is a subgraph of $\mathbb{G}^d$. Intuitively, this corresponds to the Euclidean space identification of the square lattice $\mathbb{Z}^2$ with the subgraph $\mathbb{Z}^2 \times \{0\}$ of $\mathbb{Z}^3$ and the origin of $\mathbb{Z}^2$ with the origin in $\mathbb{Z}^3$. Just like in the square lattice case, the choice of which subgraph of $\mathbb{G}^d$ to identify with $\mathbb{G}^{d-1}$ is not unique and can be chosen arbitrarily among the ones admissible.

We now define the *base* of the space-time tessellation as

$$L_0 := \mathbb{G}^{d-1} \times \mathbb{Z} \tag{III.2.7}$$

seen as a subgraph of $\mathbb{G}^d \times \mathbb{Z}$ as explained above, and the *base of cells*

$$L_1 := \bigcup_{(\iota,\tau)\in L_0 \cap (\mathbb{B}^d \times \mathbb{Z})} \{R_1(\iota,\tau)\}. \tag{III.2.8}$$

We will often consider the distance

$$d(R_1(\iota,\tau), L_0) := \min_{x\in R_1(\iota,\tau),\ y\in L_0} d(x,y)$$

between a cell $R_1(\iota,\tau) \subseteq \mathbb{G}^d \times \mathbb{Z}$ and the base L_0, which we will refer to as the *height* of the cell; it helps to visualize the base L_0 to lie "horizontally" as a subgraph of $\mathbb{G}^d \times \mathbb{Z}$. We can now finally define the Lipschitz cutset. Recall the definition of adjacent cells from Definition III.2.2.

Definition III.2.9. A Lipschitz cutset F is a set of cells in $\mathbb{G}^d \times \mathbb{Z}$ such that the following property is fulfilled: any sequence $\{R_1(\iota_j,\tau_j)\}_{j\in\mathbb{N}}$ inside $\mathbb{G}^d \times \mathbb{Z}$ of adjacent cells, which we will refer to from now on as a *path*, starting in any cell $v \in L_1$, with $d(R_1(\iota_j,\tau_j), L_0) \to \infty$, intersects F.

Definition III.2.9 is stable under taking unions, and in particular the entire graph $\mathbb{G}^d \times \mathbb{Z}$ seen as a union of all cells satisfies the definition. To prevent such undesired examples, we introduce the following condition.

Definition III.2.10. We say that a Lipschitz cutset F is *minimal* if, for each $F' \subset F$ we have that F' is not a Lipschitz cutset.

Remark III.2.11. The minimal Lipschitz cutset we will end up constructing is the analogue for fractal graphs of the "Lipschitz surface" in the lattice settings of $\mathbb{Z}^d$, see [Dir+10; DSW15; GS19a]. There, a Lipschitz surface is $*$-connected, or equivalently, for any point $(b, 0)$ in the base of $\mathbb{Z}^d$ one finds the corresponding height $h = F(b)$ of the Lipschitz surface, which satisfies a Lipschitz condition of type $|F(b_2)-F(b_1)| \leqslant 1$ whenever $\|b_2 - b_1\|_1 \leqslant 1$.

For the geometry of the fractal, we cannot hope for such a strong connectivity property of the surface. Seeing the fractal graph as a subset of the triangular lattice, we could define the height h as the coordinate of one dimension of the lattice; in this case however, not every cell (b, h) in the triangular lattice would belong to the fractal graph $\mathbb{G}^d$, since it may lie in one of the "holes". In particular we cannot require for any b_2 such that $\|b_2 - b_1\|_1 \leqslant 1$ that $|F(b_2) - F(b_1)| \leqslant 1$, since not every point (b_2, h_2) is in $\mathbb{G}^d$. In other words, this property remains true for the points belonging to the cutset, but not everywhere because of the "holes" in the fractal. However the key property which remains true is that an appropriately[1] constructed minimal Lipschitz cutset F separates the origin $(0,0) \in \mathbb{G}^d \times \mathbb{Z}$ from infinity in the sense of Definition III.2.9 in the fashion of a cutset and it retains some mild Lipschitz continuity properties, so we opted to use the name *Lipschitz cutset.*

[1] See Proposition III.3.4 and Corollary III.3.5.

We can now state our first technical result.

Theorem III.2.12. *Let $\mathbb{G}^d$ be the $d-$dimensional Sierpiński gasket with conductances satisfying* (III.2.1). *Let $\ell \in \mathbb{N}$ and let $\beta \in \mathbb{N}$ be large enough. Furthermore, let $\eta \in \mathbb{N}$, $\varepsilon \in (0,1)$ and $\zeta \in (0,\infty)$ such that*

$$\zeta \geqslant \frac{1}{\ell}\sqrt[d_w]{\left[\tfrac{1}{c_6}\log\left(\tfrac{8c_5}{3\varepsilon}\right)\right]^{d_w-1}\eta\beta},$$

and tessellate $\mathbb{G}^d \times \mathbb{Z}$ into space-time cells as described above. Let $E := E(\iota,\tau)$ be an increasing event restricted to the super cell $R_1^\eta(\iota,\tau)$ whose associated probability $\nu_E\big((1-\varepsilon)\mu, S_1^\eta(\iota,\tau), B_{\zeta\ell}, \eta\beta\big)$ has a uniform lower bound across all $(\iota,\tau) \in \mathbb{B}^d \times \mathbb{Z}$ denoted with

$$\nu_E\big((1-\varepsilon)\mu, S_1^\eta, B_{\zeta\ell}, \eta\beta\big).$$

Then there exists $\alpha_0 \in (0,\infty)$ such that if

$$\psi_1(\varepsilon,\mu_0,\ell,\eta) := \min\left\{\frac{\varepsilon^2\mu_0 2^{d_v\ell}}{C_\lambda}, -\log\Big(1-\nu_E\big((1-\varepsilon)\lambda, S_1^\eta, B_{\zeta\ell}, \eta\beta\big)\Big)\right\} \geqslant \alpha_0,$$

there exists almost surely a minimal Lipschitz cutset F with the property that $E(\iota,\tau)$ occurs for all $R_1(\iota,\tau) \in F$.

We can prove a further property of the Lipschitz cutset, which gives us control on the distance of F from any cell $R_1(\iota,\tau) \in L_1$, without loss of generality and in particular from the $R_1(0,0)$, the cell containing the origin: for a fixed radius r we look if the Lipschitz cutset F at distance r surrounds the origin. More precisely, for a Lipschitz cutset F and $r > 0$, we say that the event $S(F,r)$ holds if any path $\{v_j\}_{j=1}^n$ of adjacent cells from $R_1(0,0)$ with $d\big(v_n, R_1(0,0)\big) > r$ intersects with F. Note that this event is considerably more restrictive than the one in Definition III.2.9; if $S(F,r)$ holds, it implies in particular that the Lipschitz cutset does not only have finite distance from L_0, but essentially "surrounds" the cell $R_1(0,0)$ and prevents paths from obtaining arbitrary lengths while keeping their distance to L_0 small.

Theorem III.2.13. *Under the conditions of Theorem III.2.12, let F be the Lipschitz cutset from Theorem III.2.12 on which, in particular, the event E holds. Then there exists $C_4 > 0$ such that for r_0 large enough*

$$\mathbb{P}\big(S(F,r_0)^{\mathsf{c}}\big) \leqslant \sum_{r\geqslant r_0} r^{d_v+1}\exp\big\{-C_4 r^{c_s}\big\},$$

with $0 < c_s < \frac{d_v}{d_v+1} - \frac{1}{2}$.

The theorem implies that in particular, the Lipschitz cutset surrounds $R_1(0,0)$ at an almost surely finite distance, or equivalently, any path of cells starting from $R_1(0,0)$ that contains a cell u with $d(R_1(0,0),u)$ larger than some almost surely finite value intersects the Lipschitz cutset, even if all cells of the path have their distance to L_0 small.

Strategy of the proof. The existence of a Lipschitz cutset of good cells is equivalent to having all paths of bad cells having only finite lengths. However, simply estimating the number and the probability of bad paths does not work; even in the simplest case where $\eta = 0$ (i.e. the super-cells would be just the cells themselves and therefore non-intersecting), two events $E(\iota, \tau)$ and $E(\iota', \tau')$ can be heavily correlated whenever $\tau \neq \tau'$ and especially if the two corresponding tiles are close to each other. As an example, knowing that there were no particles present in the tile ι during the time interval τ increases the probability that all spatially close tiles will have fewer than expected particles for some time to come. On the other hand, as long as the occurrence of $E(\iota, \tau)$ depends principally on the particle system behaving "typically", it becomes more probable that the event will occur if the cells are all made bigger. Just blowing everything up is not enough however, since this would not resolve the correlation and combinatorial issues, so we adopt a multi-scale argument. For each scale we estimate the probability of a cell of that scale to be "multi-scale bad", knowing that at a larger scale the particles were behaving typically up until shortly before; this property is defined precisely in (III.5.35). For a given time horizon we choose a maximal scale κ, the largest scale that we will consider, and show that the probability to be "multi-scale good" is exponentially close to 1 at this large scale κ and consequentially, as long as there are only sub-exponentially many cells of scale κ within the space-time region we consider, we have that at this largest scale, all cells are "multi-scale good" with arbitrarily large probability. By partitioning space-time into cells of ever smaller scale until reaching scale 1, this gives rise to a space-time dependent fractal percolation problem on which we want to count the number of paths of bad cells. Using the fractal percolation nature of the setup and the alluded property that large cells are much less likely to be bad than even all of their "descendant" cells being bad at once, we consider paths of bad cells across multiple scales. This makes the combinatorial arguments more involved, but gives much better bounds on the probabilities of individual paths existing. After some additional path surgery to consider only the most vital cells of a path and the use of a mixing result to decouple the remaining space-time cells of a path, combined with a clever union bound for the probability of finding a path of cells of various scales then gives the result.

III.3 Constructing the Lipschitz cutset

Recall the definitions of adjacent cells from Definition III.2.2, of L_0 and L_1 from (III.2.7), (III.2.8) and of bad cells at the very start of subsection III.2.5, where we considered a cell $R_1(\iota, \tau)$ *bad* if the event $E(\iota, \tau)$ does not hold. To construct the Lipschitz cutset we will make use of the concept of d-paths of cells, hills and mountains which we now define.

Definition III.3.1 (d-path)**.** A *d-path* in $\mathbb{G}^d \times \mathbb{Z}$ is a sequence $\{u_k\}_{k \in \mathbb{N}}$ of adjacent cells starting with a bad cell $u_0 \in L_1$ such that for each $k \in \mathbb{N}$ one of the following holds true:

- *increasing move*: u_{k+1} is bad and $d(L_0, u_{k+1}) \geqslant d(L_0, u_k)$
- *diagonal move*: $d(L_0, u_{k+1}) < d(L_0, u_k)$

A d-path is defined in a way that it can increase or maintain the distance to the base L_0 only by moving to a bad cell in the next step, and otherwise can go "down" towards L_0 with the so-called *diagonal move*, independently of the state of the cell it is moving to.

Remark III.3.2. We kept the name *diagonal move* as in the lattice setting of [GS19a] for consistency and in order to distinguish a connection in the path that can only go toward L_0 regardless of the state of the cell. Furthermore, in the carpet setting it will revert to a $*-$neighbors connection (cf. Definition III.8.3), thus rendering the term *diagonal* more meaningful.

To describe the set of cells which can be reached via d-paths we introduce hills and mountains.

Definition III.3.3 (Hill and Mountain)**.** For any two cells $u, v \subseteq \mathbb{G}^d \times \mathbb{Z}$, we write $u \to v$ if u is a *bad* cell and there is a d-path from u to v. For a cell $u \in L_1$ define the hill H_u and mountain M_u around $u \in L_1$ as

$$\mathrm{H}_u := \bigcup_{v\colon u \to v} \{v\} \quad \text{ and } \quad \mathrm{M}_u := \bigcup_{v \in L_1\colon u \in \mathrm{H}_v} \mathrm{H}_v,$$

with the convention that if u is good, then the hill H_u is defined to be the empty set.

For a set of cells S, i.e. of the form $S = \bigcup_{i\in I}\{R_1(\iota_i, \tau_i)\}$ for some index set I, define for $u \in S$

$$\mathrm{rad}_u(S) := \sup\{d(u,v)\colon v \in S\}, \tag{III.3.1}$$

and

$$\partial_{\mathrm{ext}} S := \bigcup_{\substack{u \in S^{\mathsf{c}}\colon \exists v \in S \\ v \text{ adjacent to } u}} \{u\},$$

where S^{c} is the set of all cells not belonging to S. We then obtain the following result.

Proposition III.3.4. *If for all $u \in L_1$,*

$$\sum_{r \geqslant 1} r^{d_v+1} P(\mathrm{rad}_u(\mathrm{H}_u) > r) < \infty, \tag{III.3.2}$$

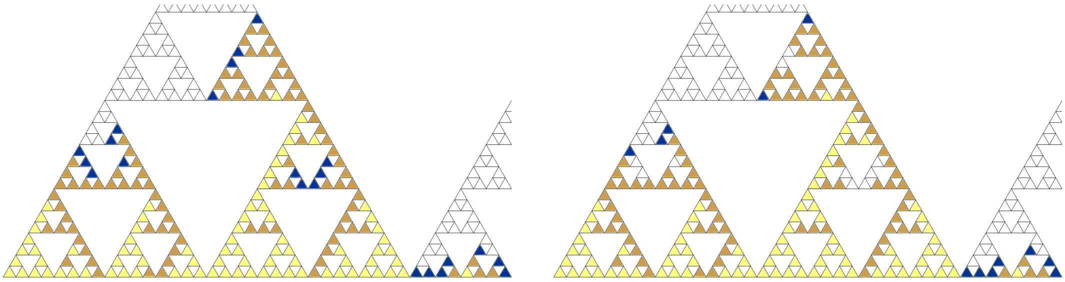
then the set

$$F := \partial_{\mathrm{ext}}\Big(\bigcup_{u \in L_1} \mathrm{M}_u\Big) \quad \cup \quad L_1 \backslash (\cup_{u \in L_1} \mathrm{M}_u)$$

is a.s. within a finite distance from L_0, is a Lipschitz cutset and all cells $u \subseteq F$ are good.

Proof. $L_1 \backslash (\cup_{u\in L_1} \mathrm{M}_u)$ is trivially within finite distance from L_0 and the cells in it contained are good since they would otherwise be contained in some hills and therefore not in $L_1 \backslash (\cup_{u\in L_1} \mathrm{M}_u)$.

Next, we prove that cells in $\partial_{\mathrm{ext}}(\bigcup_{u\in L_1} \mathrm{M}_u)$ are good. Suppose by contradiction that for some $u \in L_1$, a cell $v \in \partial_{\mathrm{ext}} \mathrm{M}_u$ is bad. By definition of $\partial_{\mathrm{ext}} \mathrm{M}_u$ there exist a cell $v' \in \mathrm{M}_u$ adjacent to v and v' can be reached by a d-path since it lies in the mountain M_u. If $d(L_0, v) \geqslant d(L_0, v')$, since v is bad, the d-path reaching v' can be extended to v with an *increasing move*. Otherwise, if $d(L_0, v) < d(L_0, v')$, v can be reached by a *diagonal move* from v' (independently of the state of v), and in both cases therefore $v \notin \partial_{\mathrm{ext}} \mathrm{M}_u$.



(a) An illustration of possible mountains (in yellow) with bad cells highlighted with a darker tone. In dark blue the cells belonging to the Lipschitz cutset F.

(b) The resulting minimal Lipschitz cutset F^o as obtained in Corollary III.3.5. The removed cells are left blank as, even though they are good, we are ignoring this information.

Figure III.2: Constructing the minimal Lipschitz cutset: a slab in $\mathbb{G}^2 \times \{0\}$.

To prove that $\partial_{\text{ext}}(\bigcup_{u\in L_1} \mathrm{M}_u)$ is within a finite distance from L_0, it is sufficient to show that for any cell $u \in L_1$ we have $\mathrm{rad}_u(M_u) < \infty$, since, by construction of mountains with the diagonal moves, if the radius of a mountain was infinite, then it would be infinite for all mountains. We therefore calculate

$$\begin{aligned} P(\mathrm{rad}_u(\mathrm{M}_u) > r) &\leqslant \sum_{v\in L_1} P\big(u \in \mathrm{H}_v,\ \mathrm{rad}_v(\mathrm{H}_v) > r - d(u,v)\big) \\ &= \sum_{\substack{v\in L_1:\\ d(u,v)\leqslant r/2}} P\big(u \in \mathrm{H}_v,\ \mathrm{rad}_v(\mathrm{H}_v) > r - d(u,v)\big) + \sum_{\substack{v\in L_1:\\ d(u,v)\geqslant r/2}} P(u \in \mathrm{H}_v). \end{aligned}$$

Writing $\mathbf{B}_r(x)$ for the ball of radius r and center x inside $L_0 \subseteq \mathbb{G}^{d-1} \times \mathbb{Z}$, we can upper bound the previous by

$$\mathrm{Vol}(\mathbf{B}_{r/2}(u))P(\mathrm{rad}_v(\mathrm{H}_v) > r/2) + \sum_{s\geqslant r/2} \mathrm{Vol}(\partial\mathbf{B}_s(u))P(\mathrm{rad}_v(\mathrm{H}_v) > s).$$

Since by ($\mathrm{Vol}(d_v)$) the volume of a ball in $\mathbb{G}^d \times \mathbb{Z}$ can be upper bounded by $\mathrm{C}_{\mathrm{Vol}}\, r^{d+1}$, by the assumption in the proposition both summands tend to 0 as r increases.

It remains to show that F is a Lipschitz cutset, i.e. it intersects any path $\{u_j\}_{j\in\mathbb{N}}$ of cells starting from L_1 with $d(u_j, L_0) \to \infty$. Note that $L_1\backslash(\cup_{u\in L_1} \mathrm{M}_u)$ and a fortiori F intersects any path that starts in a cell contained in $L_1\backslash(\cup_{u\in L_1} \mathrm{M}_u)$, so it remains to argue the case of paths that start in $L_1 \cap (\cup_{u\in L_1} \mathrm{M}_u)$. The claim is a consequence of the definition of external boundary. Since F is a.s. within finite distance from L_0, a path starting in a cell in L_1 and distance from L_0 going to infinity contains a cell u_j which is the first cell outside $\cup_{u\in L_1} \mathrm{M}_u$. In particular, for some $v \in L_1$, $u_{j-1} \in \mathrm{M}_v$, $u_j \notin \cup_{u\in L_1} \mathrm{M}_u$, and $u_j \sim u_{j-1}$ so $u_j \in \partial_{\text{ext}}(\bigcup_{u\in L_1} \mathrm{M}_u)$, i.e. the path intersects the Lipschitz cutset F. □

Before turning to the multi-scale arguments, we prove a further property of the Lipschitz cutset. We already highlighted in Remark III.2.11 that on a fractal graph we cannot hope for a general Lipschitz condition. However, a Lipschitz connectivity property holds in the "time dimension" in the following sense.

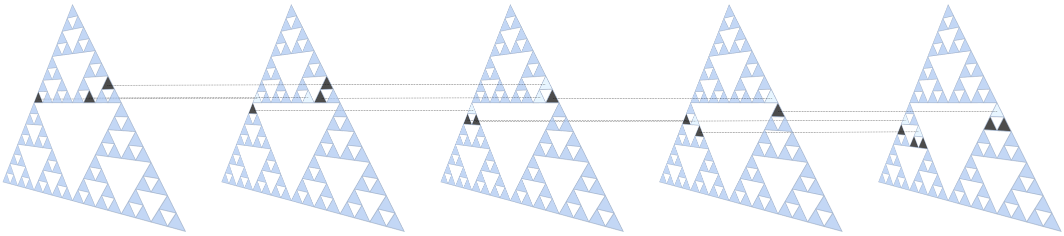


Figure III.3: A possible evolution of the minimal Lipschitz cutset F^{o} over 5 sequential time steps. Black tiles represent the cells of the minimal Lipschitz cutset at the current time index τ, the light blue tiles represent the cells of the minimal Lipschitz cutset during at the previous time index $\tau - 1$. The two are connected with dashed lines to help visualize the relationship.

Corollary III.3.5. *Let F be as in Proposition III.3.4 and consider*

$$F^{\mathrm{o}} := \bigcap_{\substack{F' \subseteq F: \\ F' \text{ is a Lipschitz cutset}}} F'.$$

Then F^{o} is a minimal Lipschitz cutset and for all $R_1(\iota, \tau) \in F^{\mathrm{o}}$, there exist $\iota_{-1}, \iota_{+1} \in \mathbb{B}^d$ such that $S_1(\iota_{-1})$ and $S_1(\iota_{+1})$ are individually either adjacent or equal to $S_1(\iota)$, and

$$R_1(\iota_{-1}, \tau - 1) \text{ and } R_1(\iota_{+1}, \tau + 1) \in F^{\mathrm{o}}$$

An example of cells of F which were removed in F^{o} is depicted in Figure III.2(b). The Lipschitz continuity in the time dimension is illustrated in Figure III.3.

Proof. F^{o} is a Lipschitz cutset as a consequence of the definition of F as we now argue. Let $\pi := \{u_i\}_{i \in \mathbb{N}}$ be any path of cells starting from L_1 such that $d(u_i, L_0) \to \infty$ as $i \to \infty$. We construct a path π' with the help of π as follows. Let u be the last cell in the intersection of π and F. Such a cell u exists, since every cell of F is either in L_1 or it is part of the external boundary of some mountain, which is by Proposition III.3.4 a.s. finite. Define now π' to be the part of π from the last visit of u (including u) onward; and by the definition of F as external boundary of a union of mountains, we can extend π' before u by some arbitrary (finite) path of cells from L_1 to u which does not intersect F: for example we can use a d-path that ends in a cell neighbouring u. Since any Lipschitz cutset $F' \subset F$ needs to intersect any such path and in particular π' and $F' \subseteq F$ we have $u \in F'$, and thus $u \in F^{\mathrm{o}}$. Since π was an arbitrary path starting in L_1 with $d(\pi_i, L_0) \to \infty$ as $i \to \infty$, we obtain that F^{o} is a Lipschitz cutset.

The minimality is straightforward due to the definition of F^{o} and it remains to show the temporal Lipschitz connectivity claim.

For this purpose, let $R_1(\iota, \tau) \in F^{\mathrm{o}}$ be arbitrary, and we show the claim only for ι_{+1} and $\tau + 1$, the other case being identical. Suppose that such ι_{+1} does not exists. We show now that it would be possible to construct a sequence $\{u_j\}_j$ of adjacent cells which includes some of the cells in

$$\overline{R_1(\iota, \tau)} := \left\{ \begin{array}{l} R_1(\bar{\iota}, \bar{\tau}) \colon \bar{\tau} \in \{\tau, \tau + 1\}, \bar{\iota} = \iota \\ \text{ or such that } S_1(\iota) \text{ is adjacent to } S_1(\bar{\iota}) \end{array} \right\} \setminus \{R_1(\iota, \tau)\},$$

starts from L_1, with $d(u_j, L_0) \to \infty$ and does not intersect F^{o}. Note that by our supposition, none of the cells in $\overline{R_1(\iota, \tau)}$ are in F^{o}.

We construct the sequence of adjacent cells $\{u_j\}_j$ so that it starts from L_1 and reaches $\overline{R_1(\iota,\tau)}$ without intersecting F^o; otherwise this would contradict the assumption of minimality of F^o. Similarly, the sequence $\{u_j\}_j$ can be extended from $\overline{R_1(\iota,\tau)}$ without intersecting F^o and with $d(u_j, L_0) \to \infty$. Since all of the cells in $\overline{R_1(\iota,\tau)}$ are adjacent, the resulting sequence $\{u_j\}_j$ contradicts the definition of Lipschitz cutset, and proves the claim. $\square$

The next three sections are devoted to the multi-scale argument which will establish the assumption (III.3.2).

III.4 Mixing Theorem

We begin by proving that when (PH(d_w)) holds, random walks started from vertices close to each other have similar probability distributions at sufficiently large times. More precisely, we have the following *fluctuation inequality.* Recall the definition of the weighted graph $(\mathbb{G}^d, (\lambda_{x,y})_{x\sim y})$ from Subsection III.2.1.

Proposition III.4.1. *Let $x_0 \in \mathbb{G}^d$ be arbitrary and suppose that* (PH(d_w)) *holds with constant $C_3 > 1$ for $Q(x_0, R) := B_{2R}(x_0) \times (0, 4R^{d_w})$ for all $R \geqslant 1$. Let $\Theta := \log_2(C_3/(C_3-1))$ and define for $x, y \in \mathbb{G}^d$*

$$\rho(x_0, x, y) := d(x_0, x) \vee d(x_0, y).$$

Then, there exists a constant $C_5 > 0$ such that the following holds. Let $r_0 \geqslant 2$ and suppose that u is caloric in $Q(x_0, r_0)$. Then, for any $x_1, x_2 \in B_{r_0/2}(x_0)$ and any t_1, t_2 for which $r_0^{d_w} - \rho(x_0, x_1, x_2)^{d_w} \leqslant t_1, t_2 \leqslant r_0^{d_w}$, we have that

$$|u(x_1,t_1) - u(x_2,t_2)| \leqslant C_5 \left(\rho(x_0,x_1,x_2)/r_0\right)^{\Theta} \sup_{(t,x)\in Q_+(x_0,r_0)} |u(x,t)|,$$

where $Q_+(x_0, r_0) := B_{r_0}(x_0) \times [3r_0^{d_w}, 4r_0^{d_w}]$.

Proof. In addition to Q and Q_+, we define $Q_-(x_0, r_0) := B_{r_0}(x_0) \times [r_0^{d_w}, 2r_0^{d_w}]$. Next, define $r_k := 2^{-k} r_0$ and set

$$\begin{aligned} Q(k) &:= 4(r_0^{d_w} - r_k^{d_w}) + Q(x_0, r_k), \\ Q_+(k) &:= 4(r_0^{d_w} - r_k^{d_w}) + Q_+(x_0, r_k), \text{ and} \\ Q_-(k) &:= 4(r_0^{d_w} - r_k^{d_w}) + Q_+(x_0, r_k), \end{aligned}$$

where the summation is to be seen as a shift of the time interval of Q (resp. Q_+ and Q_-). A quick calculation using that $d_w \geqslant 2$ then yields that $Q(k) \subset Q_+(k-1)$. Take now $k \geqslant 1$ small enough so that $r_k \geqslant 2$. We can without loss of generality consider the shifted interval $Q(k)$ with the functions $-u + \sup_{Q(k)} u$ and $u - \inf_{Q(k)} u$. To see why, note that under the change of time variable $\hat{t} := t + 4(r_0^{d_w} - r_k^{d_w})$, the function $\hat{u}(x,t) := u(x, \hat{t})$ remains caloric. Since (PH(d_w)) holds for any non-negative caloric function on $Q(x_0, r_k)$, it therefore holds for $-\hat{u} + \sup_{Q(k)} u$ and $\hat{u} - \inf_{Q(k)} u$, and in particular also for $-u + \sup_{Q(k)} u$ and $u - \inf_{Q(k)} u$ on $Q(k)$. Applying (PH(d_w)) to these two functions then gives the inequalities

$$-\inf_{Q_-(k)} u + \sup_{Q(k)} u \leqslant C_3(-\sup_{Q_+(k)} u + \sup_{Q(k)} u)$$

and

$$\sup_{Q_-(k)} u - \inf_{Q(k)} u \leqslant C_3(\inf_{Q_+(k)} u - \inf_{Q(k)} u),$$

respectively. Adding the two together and using that $\sup_{Q_-(k)} u - \inf_{Q_-(k)} u \geqslant 0$ leads to

$$\sup_{Q(k)} u - \inf_{Q(k)} u \leqslant C_3(\sup_{Q(k)} u - \inf_{Q(k)} u) - C_3(\sup_{Q_+(k)} u - \inf_{Q_+(k)} u).$$

If we define now the *oscillation* of u inside A as $\mathrm{Osc}(u, A) := \sup_A u - \inf_A u$ and set $\delta := C_3^{-1} \in (0, \infty)$, we get

$$\mathrm{Osc}(u, Q_+(k)) \leqslant (1 - \delta) \mathrm{Osc}(u, Q(k)).$$

Take now the largest m such that $r_m \geqslant \rho(x_0, x_1, x_2)$. Applying the above oscillation inequality on $Q(1) \supset Q(2) \supset \cdots \supset Q(m)$, we get since $(x_1, t_1), (x_2, t_2) \in Q(m)$ that

$$|u(x_1, t_1) - u(x_2, t_2)| \leqslant \mathrm{Osc}(u, Q(m)) \leqslant (1 - \delta)^{m-1} \mathrm{Osc}(u, Q(1)).$$

Using that $(1 - \delta)^m = 2^{-m\Theta} \leqslant (2\rho(x_0, x_1, x_2)/r_0)^{\Theta}$ we get the claim. $\square$

Next, we state a result of Popov and Teixeira [PT15], which will let us couple the locations of our particle system after they have moved with an independent Poisson point process on $\mathbb{G}$.

Proposition III.4.2 (Soft local times)**.** *Let $J \in \mathbb{N}$ and let $(Z_j)_{j \leqslant J}$ be a collection of J independent points distributed on $\mathbb{G}^d$ according to a family of probability density functions $g_j : \mathbb{G}^d \to \mathbb{R}$, $j \leqslant J$. Define for all $y \in \mathbb{G}^d$ the soft local time function $H_J(y) = \sum_{j=1}^{J} \xi_j g_j(y)$, where the ξ_j are i.i.d. exponential random variables of mean 1. Let ψ be a Poisson point process on $\mathbb{G}^d$ with intensity measure $\rho : \mathbb{G}^d \to \mathbb{R}$ and define the event $E := \{$the particles belonging to ψ are a subset of $(Z_j)_{j \leqslant J}\}$. Then there exists a coupling between $(Z_j)_{j \leqslant J}$ and ψ, such that*

$$\mathbb{P}(E) \geqslant \mathbb{P}\left(H_J(y) \geqslant \rho(y),\ \forall y \in \mathbb{G}^d\right).$$

Proof. The coupling is introduced in [PT15, Section 4] and proven in [PT15, Corollary 4.4]. A reformulation of the construction for particles on a graph can be found in [Hil+15, Appendix A], and our claim corresponds to [Hil+15, Corollary A.3]. $\square$

Proposition III.4.3. *Consider elliptic conductances $\lambda_{x,y}$ satisfying (III.2.1) for some $C_\lambda > 0$. For each $M_1 > 0$ there exist constants $M_2, M_3, M_4, \Theta \in (0, \infty)$ such that the following holds.*

Let $K > l > 0$ and $\bar{\varepsilon} > 0$. Given a region S_K tessellated into sub-regions S_i^l of side length l such that at time 0 there is a collection of particles where each sub-region S_l contains at least $\delta \sum_{y \in S_{\ell_i}} \lambda_y > M_1$ particles for some $\delta > 0$. Let $\mathbf{\Delta}, K' > 0$ with

$$\mathbf{\Delta} \geqslant \mathbf{\Delta}_0 := M_2 l^{d_w} \bar{\varepsilon}^{-\frac{4}{\Theta}} \tag{III.4.1}$$

$$K - K' \geqslant M_3(\mathbf{\Delta})^{\frac{1}{d_w}}, \tag{III.4.2}$$

and denote by Y_j the location of the j−th particle at time $\mathbf{\Delta}$.

Then, there exists a coupling $\mathbb{Q}$ of a Poisson Point Process Ξ with intensity measure $\delta(1 - \bar{\varepsilon})\lambda_y$, $y \in S_{K'}$, and $(Y_j)_j$ such that

$$\mathbb{Q}(\Xi \subseteq (Y_j)_j) \geqslant 1 - \sum_{y \in S_{K'}} e^{-M_4 \delta \lambda_y \bar{\varepsilon}^2 \mathbf{\Delta}^{\frac{d_v}{d_w}}}. \tag{III.4.3}$$

Proof. Using Proposition III.4.2, there exists a coupling $\mathbb{Q}$ of an independent Poisson point process Ψ on $\mathbb{G}$ with intensity measure $\zeta(y) = \delta(1-\bar{\varepsilon})\lambda_y$ and the locations of the particles Y_j, which are distributed according to the density functions $f_{\mathbf{\Delta}}(x_j, y) := p_{\mathbf{\Delta}}(x_j, y)\lambda_y$, such that the particles belonging to Ψ are a subset of $(Y_j)_j$ with probability at least

$$\mathbb{Q}(H_J(y) \geqslant \delta\lambda_y(1-\bar{\varepsilon}),\, \forall y \in S_{K'}),$$

where $H_J(y) = \sum_{j=1}^{J} \xi_j f_{\mathbf{\Delta}}(x_j, y)$, $(\xi_j)_{j\leqslant J}$ are i.i.d. exponential random variables with parameter 1, and J is the number of particles inside $S_{K'}$ at time Δ.

We first observe that the probability of the converse event is

$$\begin{aligned}\mathbb{Q}(\exists y \in S_{K'} :\, H_J(y) < \delta\lambda_y(1-\bar{\varepsilon})) &\leqslant \sum_{y\in S_{K'}} \mathbb{Q}\big(H_J(y) < \delta\lambda_y(1-\bar{\varepsilon})\big)\\ &\leqslant \sum_{y\in S_{K'}} e^{\gamma\lambda_y\delta(1-\bar{\varepsilon})}\mathbb{E}_{\mathbb{Q}}[\exp\{-\gamma H_J(y)\}],\end{aligned}$$

for any $\gamma > 0$ by a simple application of the exponential Chebychev inequality.

Let M_3 now be a large positive constant that we will fix later and set

$$R := M_3\mathbf{\Delta}^{1/d_w}\bar{\varepsilon}^{-\frac{d_w-1}{d_w}}. \tag{III.4.4}$$

Next, let J' be any subset of $\{1, \ldots, J\}$ such that exactly $\lceil\sum_{y\in S_i^l}\delta\lambda_y\rceil$ particles from J' are inside S_i^l for every sub-region S_i^l of S_K. For $y \in G$, define also $J'(y) \subseteq J'$ to be the set of all indices $j \in J'$ for which $d(x_j, y) \leqslant R$ and define $H'(y)$ as $H_J(y)$, but with the sum in the definition restricted to the indices $j \in J'(y)$. By definition, $H_J(y) \geqslant H'(y)$ and therefore

$$\mathbb{E}_{\mathbb{Q}}[\exp\{-\gamma H_J(y)\}] \leqslant \mathbb{E}_{\mathbb{Q}}[\exp\{-\gamma H'(y)\}].$$

Since the ξ_j in the definition of H are independent exponential random variables of parameter 1, we can calculate further

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}[\exp\{-\gamma H'(y)\}] &= \prod_{j\in J'(y)} \mathbb{E}_{\mathbb{Q}}\big[\exp\{-\gamma\xi_j f_{\mathbf{\Delta}}(x_j, y)\}\big]\\ &= \prod_{j\in J'(y)} (1+\gamma f_{\mathbf{\Delta}}(x_j, y))^{-1}.\end{aligned}$$

Furthermore, by setting the constant M_2 large enough, we have by (HKB(d_v, d_w)) that for all x with $d(x, y) \leqslant R$, $p_{\mathbf{\Delta}}(x, y) \leqslant c_8\mathbf{\Delta}^{-d_v/d_w}$ for some constant c_8. In particular this holds for all $y \in S_{K'}$ and all $x \in \bigcup S_i^l$, where the union runs across all S_i^l for which there exists $j \in J'(y)$ such that $x_j \in S_i^l$. Setting now $\gamma = \frac{1}{4c_8C_\lambda}\bar{\varepsilon}\mathbf{\Delta}^{d_v/d_w}$ gives

$$\sup_{x\in S_R(y)} \gamma f_{\mathbf{\Delta}}(x, y) = \sup_{x\in S_R(y)} \gamma\lambda_y p_{\mathbf{\Delta}}(x, y) \leqslant c_8C_\lambda\gamma\mathbf{\Delta}^{-d_v/d_w} < \bar{\varepsilon}/4. \tag{III.4.5}$$

For this value of γ and using that for $|x| \leqslant \frac{1}{2}$ we have by Taylor's expansion that

$\log(1+x) \geqslant x - x^2$, it further holds that

$$\begin{aligned}\prod_{j\in J'(y)} (1+\gamma f_{\mathbf{\Delta}}(x_j,y))^{-1} &\leqslant \prod_{j\in J'(y)} \exp\{-\gamma f_{\mathbf{\Delta}}(x_j,y)(1-\gamma f_{\mathbf{\Delta}}(x_j,y))\} \\ &\leqslant \exp\Big\{-\Big(1-\sup_{x\in B_R(y)} \gamma f_{\mathbf{\Delta}}(x,y)\Big)\sum_{j\in J'(y)} \gamma f_{\mathbf{\Delta}}(x_j,y)\Big\} \\ &\overset{\text{(III.4.5)}}{\leqslant} \exp\Big\{-\gamma(1-\bar{\varepsilon}/4)\sum_{j\in J'(y)} f_{\mathbf{\Delta}}(x_j,y)\Big\}.\end{aligned}$$

We claim now (and prove below) that

$$\sum_{j\in J'(y)} f_{\mathbf{\Delta}}(x_j,y) \geqslant \delta\lambda_y(1-\bar{\varepsilon}/2), \tag{III.4.6}$$

which then gives us that

$$\begin{aligned}\mathbb{Q}\big(\exists y\in S_{K'}:\, H_J(y) < \delta\lambda_y(1-\bar{\varepsilon})\big) &\leqslant \exp\big\{\gamma\lambda_y\delta(1-\bar{\varepsilon}) - \gamma(1-\bar{\varepsilon}/4)\delta\lambda_y(1-\bar{\varepsilon}/2)\big\} \\ &\leqslant \exp\{-\gamma\delta\lambda_y\bar{\varepsilon}/4\}.\end{aligned}$$

Using the definition γ then yields the claim. We therefore proceed to prove (III.4.6).

For each S_i^l and each particle $x_j \in S_i^l$, let $x_j' \in S_i^l$ be such that $f_{\mathbf{\Delta}}(x_j',y) = \max_{w\in S_i^l} f_{\mathbf{\Delta}}(w,y)$. Then, we can bound

$$\sum_{j\in J'(y)} f_{\mathbf{\Delta}}(x_j,y) \geqslant \sum_{j\in J'(y)} \left(f_{\mathbf{\Delta}}(x_j',y) - |f_{\mathbf{\Delta}}(x_j',y) - f_{\mathbf{\Delta}}(x_j,y)|\right).$$

We will look at the first summand: for each S_i^l, it holds that

$$\sum_{\substack{j\in J'(y)\\ x_j\in S_i^l}} f_{\mathbf{\Delta}}(x_j',y) = \max_{w\in S_i^l} f_{\mathbf{\Delta}}(w,y) \sum_{\substack{j\in J'(y)\\ x_j\in S_i^l}} 1$$

which by definition of J' can be lower bounded by

$$\max_{w\in S_i^l} f_{\mathbf{\Delta}}(w,y)\Big\lceil \sum_{z\in S_i^l} \delta\lambda_z\Big\rceil \geqslant \sum_{z\in S_i^l} \delta\lambda_z f_{\mathbf{\Delta}}(z,y).$$

Set $R(y)$ to be the set of all sites z of S_K for which $d(z,y) \leqslant R$. Note that the right side of this equation is always positive since R is by its definition in (III.4.4) proportional to l and M_3 is assumed to be large. Furthermore, note that if $z \in R(y)$ then for all particles x_j with $x_j' = z$ and $j \in J'$ we have that $j \in J'(y)$. It also holds that $\lambda f_{\mathbf{\Delta}}(z,y) = \lambda f_{\mathbf{\Delta}}(y,z)$, which combined with the preceding calculation yields for each S_i^l

$$\begin{aligned}\sum_{j\in J'(y)} f_{\mathbf{\Delta}}(x_j',y) &\geqslant \sum_{z\in R(y)} \delta\lambda_z f_{\mathbf{\Delta}}(z,y) \\ &= \delta\lambda_y \sum_{z\in R(y)} f_{\mathbf{\Delta}}(y,z) \\ &\geqslant \delta\lambda_y \mathbb{P}(\mathrm{Conf}(R,\mathbf{\Delta})).\end{aligned}$$

By Lemma III.2.5 we have that there exists constants c_5 and c_6 so as to lower bound the previous expression by

$$\delta\lambda_y\big(1 - c_5 e^{-c_6\left(\frac{R^{d_w}}{\mathbf{\Delta}}\right)^{\frac{1}{d_w-1}}}\big) \geqslant \delta\lambda_y(1-\bar{\varepsilon}/4),$$

where the last inequality holds by setting R (cf. (III.4.4)) through M_3 large enough with respect to c_5 and c_6.

It remains to find an upper bound for the second addend $\sum_{j\in J'(y)} |f_{\mathbf{\Delta}}(x'_j, y) - f_{\mathbf{\Delta}}(x_j, y)|$. Let I be the set of all i for which Q_i^l contains a particle x_j from the set $(x_j)_{j\in J'(y)}$. Then

$$\begin{aligned}\sum_{j\in J'(y)} |f_{\mathbf{\Delta}}(x'_j, y) - f_{\mathbf{\Delta}}(x_j, y)| &= \sum_{i\in I}\sum_{\substack{j\in J'(y)\\ x_j\in S_i^l}} |f_{\mathbf{\Delta}}(x'_j, y) - f_{\mathbf{\Delta}}(x_j, y)| \\ &= \lambda_y \sum_{i\in I}\sum_{\substack{j\in J'(y)\\ x_j\in S_i^l}} |p_{\mathbf{\Delta}}(x'_j, y) - p_{\mathbf{\Delta}}(x_j, y)|.\end{aligned}$$

Since the heat kernel $p_t(x,\cdot)$ is caloric, the parabolic Harnack inequality and consequently Proposition III.4.1 with $r_0^{d_w} = \mathbf{\Delta}$ can be applied. We can also use the upper heat kernel bound (HKB(d_v, d_w)) to the resulting supremum term. Writing C_5 for the constant from the application of Proposition III.4.1 and C_6 for the constant resulting from upper bounding the supremum term, we get

$$\begin{aligned}&\lambda_y \sum_{i\in I}\sum_{\substack{j\in J'(y)\\ x_j\in S_i^l}} \frac{C_6 l^\Theta}{\mathbf{\Delta}^{\Theta/d_w}} C_5 \mathbf{\Delta}^{-d_v/d_w} \\ &\leqslant \lambda_y \sum_{i\in I}\sum_{x\in S_i^l} \frac{C_6 \delta\lambda_x l^\Theta}{\mathbf{\Delta}^{\Theta/d_w}} C_5 \mathbf{\Delta}^{-d_v/d_w} \\ &= \delta\lambda_y C_6 C_5 \sum_{i\in I}\sum_{x\in S_i^l} \lambda_x l^\Theta \mathbf{\Delta}^{-(d_v+\Theta)/d_w} \\ &\overset{(\mathrm{Vol}(d_v))}{\leqslant} \delta\lambda_y C_6 C_5\, \mathrm{C_{Vol}}\, C_\lambda R^{d_v} l^\Theta \mathbf{\Delta}^{-(d_v+\Theta)/d_w} \\ &\leqslant \delta\lambda_y \bar{\varepsilon}/4,\end{aligned} \tag{III.4.7}$$

where the last inequality follows from the assumption that $\mathbf{\Delta} \geqslant \mathbf{\Delta}_0$, the definitions of $\mathbf{\Delta}_0$ and R, and by setting M_3 sufficiently large with respect to the constants $C_6, C_5, \mathrm{C_{Vol}}$, and C_λ. Combining all of the stated inequalities completes the proof. □

The statement of Proposition III.4.3 does not depend on particles located outside of the region S_K at time 0. However, since the particles can move in an unrestricted way, repeated applications of the theorem across multiple regions of time and space (cf. Sections III.2.2 and III.3) still exhibit long distance correlations that we would like to avoid. To that end, we will prove a version of Proposition III.4.3 also for particle systems conditioned on having the particle movement confined (cf. Lemma III.2.5). The main difficulty is that by conditioning the particles in this way, their transition probabilities do not necessarily satisfy (HKB(d_v, d_w)) and by extension (PH(d_w)) any longer. It turns out however that these probabilities are still quantitatively the same under some mild modifications of the assumptions, which we prove in the following lemma.

Lemma III.4.4. *Let $\lambda_{x,y}$ satisfy (III.2.1). Then there exist constants c_9 and c_{10} so that the following holds. Consider a region S_l with $l > 0$. Let $\mathbf{\Delta} > c_9 l^{d_w}$ and $\rho \geqslant c_{10}(\mathbf{\Delta}\log_2^{d_w-1}(\mathbf{\Delta}))^{1/d_w}$. Consider a random walk Y that moves along $\mathbb{G}^d$ for time $\mathbf{\Delta}$ conditioned on being confined to $B_{\rho/2}$ during the entire time interval $[0,\mathbf{\Delta}]$. Let $x, y \in S_l$ with x being the starting point of the random walk, and define*

$$g(x,y) := \mathbb{P}_x(Y_{\mathbf{\Delta}} = y \,|\, Y \text{ is confined to } B_{\rho/2} \text{ during } [0,\mathbf{\Delta}]).$$

Then there exists a constant $C > 2$ such that for $x, y, z \in S_l$ we have

$$\left|\frac{g(x,y)}{\lambda_y} - \frac{g(z,y)}{\lambda_y}\right| \leqslant C l^{\Theta} \mathbf{\Delta}^{-(d_v+\Theta)/d_w}.$$

Remark III.4.5. It is important to note that the above bound is of the same form as the bound we used in (III.4.7) for the unconditioned random walk. Consequently, we will use this lemma to prove a conditioned version of Proposition III.4.3 without having to directly use (PH(d_w)), which as mentioned above might not necessarily hold in this case.

Proof. Denote by $p_E(\rho)$ the probability that a random walk started at x is confined to $B_{\rho/2}$ during $[0,\mathbf{\Delta}]$. Using Lemma III.2.5, we have for some positive constants c_5, c_6 that

$$1 - p_E(\rho) \leqslant c_5 e^{-c_6(\rho^{d_w}/\mathbf{\Delta})^{\frac{1}{d_w-1}}}.$$

Next, writing $h(x,y) := \mathbb{P}_x(Y_{\mathbf{\Delta}} = y \,|\, Y \text{ exits } B_{\rho/2}(x) \text{ during } [0,\mathbf{\Delta}])$ and $f_{\mathbf{\Delta}}(x,y) = \mathbb{P}_x(Y_\Delta = y)$, we can write

$$f_{\mathbf{\Delta}}(x,y) = g(x,y)p_E(\rho) + h(x,y)(1 - p_E(\rho)).$$

From this, we can immediately obtain the bound

$$g(x,y) \leqslant f_{\mathbf{\Delta}}(x,y)\frac{1}{p_E(\rho)}.$$

We can then write

$$\begin{aligned}
\left|\frac{g(x,y)}{\lambda_y} - \frac{g(z,y)}{\lambda_y}\right| &= \mathbb{1}_{\{g(x,y)>g(z,y)\}}\left(\frac{g(x,y)}{\lambda_y} - \frac{g(z,y)}{\lambda_y}\right) \\
&\quad + \mathbb{1}_{\{g(x,y)\leqslant g(z,y)\}}\left(\frac{g(z,y)}{\lambda_y} - \frac{g(x,y)}{\lambda_y}\right) \\
&\leqslant \mathbb{1}_{\{g(x,y)>g(z,y)\}}\left(\frac{f_{\mathbf{\Delta}}(x,y)}{\lambda_y p_E(\rho)} - \frac{f_{\mathbf{\Delta}}(z,y)}{\lambda_y p_E(\rho)} + \frac{h(z,y)(1-p_E(\rho))}{p_E(\rho)\lambda_y}\right) \\
&\quad + \mathbb{1}_{\{g(x,y)\leqslant g(z,y)\}}\left(\frac{f_{\mathbf{\Delta}}(z,y)}{\lambda_y p_E(\rho)} - \frac{f_{\mathbf{\Delta}}(x,y)}{\lambda_y p_E(\rho)} + \frac{h(x,y)(1-p_E(\rho))}{p_E(\rho)\lambda_y}\right) \\
&\leqslant \frac{|p_{\mathbf{\Delta}}(y,x) - p_{\mathbf{\Delta}}(y,z)|}{p_E(\rho)} + \frac{\max\{h(x,y),h(z,y)\}(1-p_E(\rho))}{p_E(\rho)\lambda_y}.
\end{aligned} \tag{III.4.8}$$

Next, observe that we can write $h(x,y)$ as $\mathbb{E}_x[f_{\mathbf{\Delta}-\tau}(w,y)\,|\,\tau < \mathbf{\Delta}]$ with τ being the first time Y exits $B_{\rho/2}(x)$ and w the random vertex at the boundary of $B(x,\rho/2)$ where Y is at time τ. Since the weights $\lambda_{x,y}$ satisfy (III.2.1) we can bound $\frac{f_{\mathbf{\Delta}-\tau}(w,y)}{\lambda_y}$

from above by some positive constant C_7. This is because either $\Delta - \tau$ is larger than $d(w, y)$, which allows us to use (HKB(d_v, d_w)), or $\Delta - \tau$ is smaller than $d(w, y)$, so that $f_{\mathbf{\Delta}}(w, y)$ is bounded above by the probability that a random walk jumps at least $d(w, y)$ steps in time $\mathbf{\Delta} - \tau$, which is small enough since $d(w, y)$ is large. Therefore we have that $\frac{\max\{h(x,y),h(z,y)\}(1-p_E(\rho))}{p_E(\rho)\lambda_y}$ is at most C_8. This together with the bound on $1 - p_E(\rho)$ yields

$$\begin{aligned} \frac{\max\{h(x,y),h(z,y)\}(1-p_E(\rho))}{p_E(\rho)\lambda_y} &\leqslant \frac{C_8 \cdot c_5}{p_E(\rho)} \exp\{-c_6(\rho^{d_w}/\mathbf{\Delta})^{\frac{1}{d_w-1}}\} \\ &\leqslant \frac{C_8 \cdot c_5}{p_E(\rho)} \exp\big\{-c_6(c_{10}^{\frac{1}{d_w-1}} \log_2(\mathbf{\Delta}))\big\}. \end{aligned}$$

We now return to (III.4.8). By setting c_{10} (and by extension ρ) large enough and using the bound for $1 - p_E(\rho)$, $p_E(\rho)$ can be bounded from below by $1/2$. Applying Proposition III.4.1 to the term $|p_{\mathbf{\Delta}}(y, x) - p_{\mathbf{\Delta}}(y, z)|$, using (HKB($d_v, d_w$)) to bound the resulting supremum term, and finally setting c_{10} even larger if necessary for $\exp\{-c_6(c_{10}^{\frac{1}{d_w-1}} \log_2(\mathbf{\Delta}))\}$ to be smaller than $\mathbf{\Delta}^{-d_v/d_w}$ concludes the proof. □

We now state the version of Proposition III.4.3 for particles that are confined. Note that the statement remains essentially unchanged, other than having a stronger condition on $K - K'$ than before. This is also the statement of the result that we will rely on to conduct our multi-scale analysis (cf. Lemma III.6.1).

Theorem III.4.6. *Consider elliptic conductances $\lambda_{x,y}$ satisfying (III.2.1) for some $C_\lambda > 0$. For each $M_1 > 0$ there exist M_2, M_3, M_4, Θ such that the following holds.*

Let $K > l > 0$ and $\bar{\varepsilon} > 0$. Given a region S_K tessellated into sub-regions S_i^l of side length l such that at time 0 there is a collection of particles where each sub-region S_l contains at least $\delta \sum_{y \in S_{\ell_i}} \lambda_y > M_1$ particles for some $\delta > 0$. Let $\mathbf{\Delta}$ and $K' > 0$ with

$$\mathbf{\Delta} \geqslant \mathbf{\Delta}_0 := M_2 l^{d_w} \bar{\varepsilon}^{-\frac{4}{\Theta}} \tag{III.4.9}$$

$$K - K' \geqslant M_3(\mathbf{\Delta}(\log_2 \mathbf{\Delta})^{d_w-1})^{\frac{1}{d_w}}, \tag{III.4.10}$$

and denote with Y_j the location of the $j-$th particle at time $\mathbf{\Delta}$ conditioned on being confined to $S_{(K-K')}$ during $[0, \mathbf{\Delta}]$.

Then, there exists a coupling $\mathbb{Q}$ of a Poisson Point Process Ξ with intensity measure $\delta(1 - \bar{\varepsilon})\lambda_y$ for $y \in S_{K'}$ and the family $(Y_j)_j$ such that

$$\mathbb{Q}\big(\Xi \subseteq (Y_j)_j\big) \geqslant 1 - \sum_{y \in S_{K'}} e^{-M_4 \delta \lambda_y \bar{\varepsilon}^2 \mathbf{\Delta}^{\frac{d_v}{d_w}}}. \tag{III.4.11}$$

Proof. Using Lemma III.4.4 and the upper bound on $g(x, y)$ from its proof when setting γ, the proof proceeds the same as in Proposition III.4.3. The independence from the graph outside of $S_{(2K-K')}$ follows from the fact that we consider only particles which are confined in $B_{(K-K')}$ and ended in S'_K, so they never left S_K during $[0, \mathbf{\Delta}]$. □

III.5 Multi-scale setup

In this section we define the multi-scale set-up for the construction. For some (large) $\kappa \in \mathbb{N}$, we will define for each $1 \leqslant k \leqslant \kappa$ cells at scale k: in the fractal graph, spatial tiles will be denoted by $S_k(\iota)$ and indexed by some $\iota \in \mathbb{B}^d$; the time line $\mathbb{R}$ will be subdivided into intervals $T_k(\tau)$ and indexed by $\tau \in \mathbb{Z}$. The space-time cells $R_k(\iota, \tau)$ will simply be the Cartesian product $S_k(\iota) \times T_k(\tau)$. We will also need to introduce, for each scale k, extensions of the cells which do not need to be of the same scale. Those cells will be necessary to work with the dependencies between adjacent cells. Scale 1 will correspond to and agree with the first tessellation introduced in Definition III.2.1. The value κ instead is the largest scale that we will consider. The reader might want to think of κ to be fixed for the moment. It will be determined later in the proof of Proposition III.6.5: roughly speaking, if the paths we consider have to leave the region $B_t(0) \times [-t, t]$, then we will consider $\kappa = \mathcal{O}(\sqrt{\log(t)})$.

III.5.1 Multi-scale tessellation

Space tessellation. We start by defining the space tessellation on the graph $\mathbb{G}^d$. After the full definition of all relevant tiles and intervals and a statement of useful properties, we refer for the end of this paragraph for a short motivation and intuition regarding the roles of the different tiles introduced here.

Let $\varepsilon \in (0, 1)$ and ℓ, m, a be positive (large) integers which we will fix later. Set $\ell_0 := \ell - m$ and let

$$\ell_k := a(k-1)^2 + m(k-1) + \ell. \tag{III.5.1}$$

Define the space tiles at scale $k \in \mathbb{N}$ indexed by $\iota \in \mathbb{B}^d$ (cf. (III.2.3)) as the subgraphs of $\mathbb{G}^d$ with vertex sets

$$S_k(\iota) := \iota 2^{\ell_k} + \triangle^d_{\ell_k}, \tag{III.5.2}$$

and induced edges, which are well-defined in view of (III.2.4). We say that two cells $S_k(\iota_1) \neq S_k(\iota_2)$ are *adjacent* if $d(S_k(\iota_1), S_k(\iota_2)) = 0$. It is easy to verify that

$$S_k(\iota) \text{ has side length of } 2^{\ell_k} \tag{III.5.3}$$

$$S_{k+1}(\iota) \text{ is the union of exactly } 2^{d_v(\ell_{k+1}-\ell_k)} = (d+1)^{2ak-a+m} \text{ tiles of scale } k. \tag{III.5.4}$$

Next, we introduce a hierarchy of the space tiles. We define for $k, j \geqslant 0$ the function $\pi_k^{(j)}$ by

$$\pi_k^{(j)}(\iota) = \iota' \qquad \Leftrightarrow \qquad S_k(\iota) \subseteq S_{k+j}(\iota'), \tag{III.5.5}$$

and we say that $S_{k'}(\iota')$ is an ancestor of $S_k(\iota)$ (or equivalently that $S_k(\iota)$ is a descendant of $S_{k'}(\iota')$) if $\pi_k^{(k'-k)}(\iota) = \iota'$. Note the map is well-defined by (III.5.5), and that any cell is also a descendant and an ancestor of itself.

We define for $k \geqslant 0$ and $b(k) := ak^{2+\frac{8}{\Theta d_w}} m 2^m$ the *base*, the *area of influence*, and for $k \geqslant 1$ the *extension*, the *support* and the *extended support* as

$$S_k^{\mathrm{base}}(\iota) := \bigcup_{\iota' :\, d(S_k(\iota'),S_k(\iota))\leqslant b(k)} S_k(\iota'), \tag{III.5.6}$$

$$S_k^{\mathrm{inf}}(\iota) := \bigcup_{\iota' :\, d(S_k(\iota'),S_k(\iota))\leqslant 2b(k)} S_k(\iota'), \tag{III.5.7}$$

$$S_k^{\mathrm{ext}}(\iota) := \bigcup_{\iota' :\, \pi_{k-1}^{(1)}(\iota')=\iota} S_{k-1}^{\mathrm{base}}(\iota'), \tag{III.5.8}$$

$$S_k^{\mathrm{sup}}(\iota) := \bigcup_{\iota' :\, d(S_{k+1}(\iota'),S_{k+1}(\pi_k^{(1)}(\iota)))\leqslant m} S_{k+1}(\iota'), \tag{III.5.9}$$

$$S_k^{\mathrm{Esup}}(\iota) := \bigcup_{\iota' :\, d(S_{k+1}(\iota'),S_{k+1}(\pi_k^{(1)}(\iota)))\leqslant 3m+1} S_{k+1}(\iota'). \tag{III.5.10}$$

The choice of $b(k)$ will be made clear later in (III.6.3). Recalling the value η from Definition III.2.4, we also assume that $b(1) \geqslant \eta$, which holds if we choose a large enough. See Figure III.4 for an illustration of how the different tile extensions relate to each other.

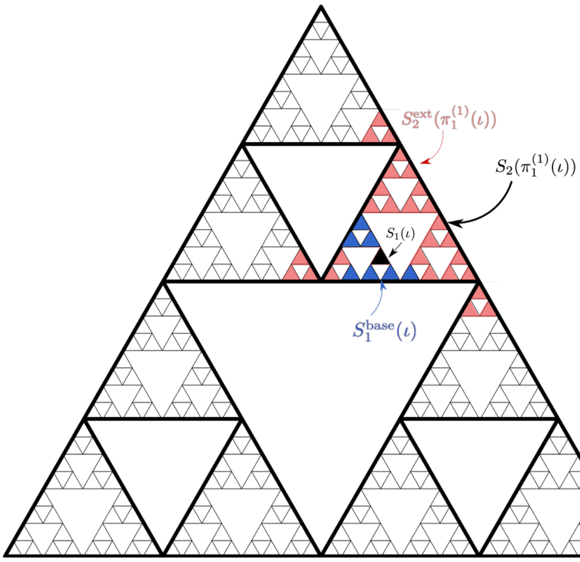


Figure III.4: Illustration of $S_1^{\mathrm{base}}(\iota)$ and $S_2^{\mathrm{ext}}(\pi_1^{(1)}(\iota))$. The thin line triangles represent the many tiles S_1 of scale 1, the thick black line triangles are tiles S_2 of scale 2. The black triangle represents the specific tile $S_1(\iota)$, while the dark blue region is $S_1^{\mathrm{base}}(\iota)$ and the light red is $S_2^{\mathrm{ext}}(\pi_1^{(1)}(\iota))$. $S_1^{\mathrm{inf}}(\iota)$ is not represented in order to keep the image legible.

We now state some properties of the above defined sets and the relations of the different tiles. It is easy to check that for all $(k,\iota) \in \mathbb{N}_0 \times \mathbb{B}^d$ it holds $S_k(\iota) \subseteq S_k^{\mathrm{base}}(\iota) \subseteq S_k^{\mathrm{inf}}(\iota)$ and

$$S_k^{\mathrm{base}}(\iota) \subseteq S_{k+1}^{\mathrm{ext}}(\pi_k^{(1)}(\iota)). \tag{III.5.11}$$

Since $b(k)$ is increasing in k, it also holds that

$$S_k^{\mathrm{ext}}(\iota) \subseteq S_k^{\mathrm{base}}(\iota).$$

Further simple properties of space tiles can easily be inferred: we will use later that

$$S_k^{\text{base}}(\iota) \text{ contains at most } \mathrm{C}_{\text{Vol}}\, b(k)^{d_v} \text{ tiles of scale } k; \text{ and} \tag{III.5.12}$$

$$S_k^{\text{ext}}(\iota) \text{ contains at most } \mathrm{C}_{\text{Vol}}(b(k-1)+2^{\ell_k})^{d_v} \text{ tiles of scale } k-1, \tag{III.5.13}$$

which both follow from (Vol(d_v)).

We now look at the properties of the larger scales. Comparing the exponential growth of S_k in (III.5.4) with the polynomial growth of $b(k)$ in (III.5.7), one sees that for a, m large enough, for all k and ι, it holds that

$$S_k^{\text{inf}}(\iota) \subseteq S_k^{\text{sup}}(\iota). \tag{III.5.14}$$

Remark III.5.1. The assumption $b(k) \geqslant \eta$ implies that $S_1^{\text{base}}(\iota)$, and a fortiori $S_1^{\text{ext}}(\iota)$, contains the *super-tile* $S_1^{\eta}(\iota)$ defined in Definition III.2.4.

We now quickly motivate the introduction of the different tiles. The tiles $S_k(\iota)$ constitute the basic tiles at each scale. The introduction of the multi-scale argument suggests that we will introduce a notion of goodness for every scale k: this is related to $S_k^{\text{base}}(\iota)$ and $S_{k+1}^{\text{ext}}(\pi_1^{(1)}(\iota))$, as well as to the events D^{base} and D^{ext} which we are going to define in (III.5.28) and (III.5.27).

Furthermore, S^{inf}, which is defined as S^{base} but with a slightly larger border, will help us to keep tiles apart: if for two tiles the *areas of influence* do not intersect, we will call these tiles *well-separated* and we will be able to treat the tiles as essentially independent. Finally, we introduced S^{sup} and S^{Esup} so that tiles whose (extended) supports intersect each other, even if otherwise well-separated, are still close enough to be part of a very general kind of path, the ScD-path (see Definition III.5.5).

Temporal tessellation. We now turn to the temporal tessellation of $\mathbb{R}$. The tessellation itself is easier than the previous one introduced for space, and it corresponds to the one in [GS19a]. Define for $k \geqslant 2$

$$\beta_k := \mathrm{C}_{\text{mix}}\left(\tfrac{k^2}{\varepsilon}\right)^{\frac{4}{\Theta}}\left(2^{\ell_{k-1}}\right)^{d_w} \tag{III.5.15}$$

where C_{mix} is a constant larger than $8^{4/\Theta}M_2$, Θ and M_2 are constants from Theorem III.4.6 and ϵ is from the beginning of Subsection III.5.1. Set as well $\beta := \beta_1 := \mathrm{C}_{\text{mix}}\frac{2^{d_w(\ell-m)}}{\varepsilon^{4/\Theta}}$, assuming m large enough so that $\mathrm{C}_{\text{mix}} \geqslant 8^{4/\Theta}M_2$ still holds. On first reading, one should not be distracted by the constant C_{mix} or the fine-tuning power $k^{8/\Theta}$ in β_k and instead focus on the leading term $2^{\ell_{k-1}}$ which is raised to the power d_w. As discussed before, the term d_w represent the power scaling between time and space from the perspective of the random walkers. That is a major difference from the lattice $\mathbb{Z}^d$ where the "walk dimension" d_w equals 2 for every dimension d of the lattice. Note in particular that ratios between two consecutive time-scales satisfy

$$\frac{\beta_{k+1}}{\beta_k} = \left(\tfrac{k+1}{k}\right)^{8/\Theta}\left(2^{2ak-3a+m}\right)^{d_w}. \tag{III.5.16}$$

Define the time intervals at scale $k \in \mathbb{N}$ as the intervals

$$T_k(\tau) = [\tau\beta_k, (\tau+1)\beta_k), \quad \tau \in \mathbb{Z}, \tag{III.5.17}$$

and we say that two intervals $T_k(\tau_1) \neq T_k(\tau_2)$ with $\tau_1, \tau_2 \in \mathbb{Z}$ are *adjacent* if $|\tau_1 - \tau_2| \leqslant 1$. We now introduce a hierarchy over time, which is more complex than the spatial one. While for space, a parent contains its children and descendants, since "time flows forward", parents with respect to time will still have larger intervals than their children, but will lie to the left (i.e. "before"): see Figure III.5. Formally, let $\gamma_k^{(0)}(\tau) = \tau$, and for $j \geqslant 1$ define

$$\gamma_k^{(j)}(\tau) := \tau' \quad \text{if} \quad \gamma_k^{(j-1)}(\tau)\beta_{k+j-1} \in T_{k+j}(\tau'+1),$$

see Figure III.5 for visualization. In analogy with the terminology introduced in the spatial setting, we say that $T_{k'}(\tau')$ is an ancestor of $T_k(\tau)$ or equivalently that $T_k(\tau)$ is a descendant of $T_{k'}(\tau')$ if $\gamma_k^{(k'-k)}(\tau) = \tau'$ and it still holds that any time interval is also a descendant and an ancestor of itself. Note that due to the "time drift" it does not contain its own descendants of any scale as subintervals.

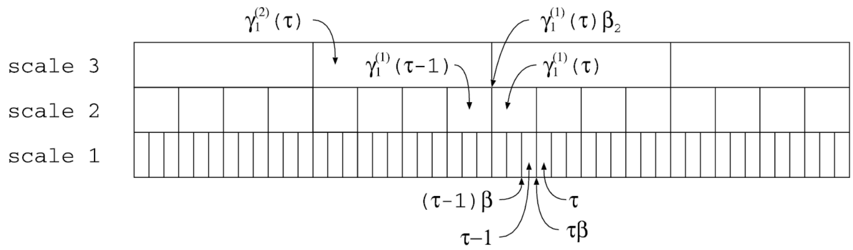


Figure III.5: Temporal tessellation and its hierarchy structure. Image from [GS19a].

As we did for space, we define for each scale k larger intervals that we will need:

$$T_1^{\text{inf}}(\tau) := [\gamma_1^{(1)}(\tau)\beta_2, (\tau + \eta \wedge 2)\beta_1], \tag{III.5.18}$$

$$T_k^{\text{inf}}(\tau) := [\gamma_1^{(1)}(\tau)\beta_2, (\tau + 2)\beta_k], \tag{III.5.19}$$

$$T_k^{\text{sup}}(\tau) := \bigcup_{i=0}^{8} T_{k+1}(\gamma_k^{(1)}(\tau) - 3 + i), \tag{III.5.20}$$

$$T_k^{\text{Esup}}(\tau) := \bigcup_{i=0}^{26} T_{k+1}(\gamma_k^{(1)}(\tau) - 12 + i). \tag{III.5.21}$$

We now claim and prove that the time analogue of (III.5.14) still holds true.

Lemma III.5.2. *Let $T_{k'}(\tau')$ be a descendant of $T_k(\tau)$, and let $T_{k'}(\tau'')$ be adjacent to $T_{k'}(\tau')$. Then for a, m large enough*

$$T_{k'}^{\text{inf}}(\tau'') \subseteq T_k^{\text{sup}}(\tau).$$

Proof. Recall that $T_{k'}^{\text{inf}}(\tau'') \subseteq [\gamma_{k'}^{(1)}(\tau'')\beta_{k'+1}, (\tau'' + 2 \wedge \eta)\beta_{k'}]$, the definition of $T_k^{\text{sup}}(\tau) = ((\gamma_k^{(1)}(\tau) - 3)\beta_{k+1}, (\gamma_k^{(1)}(\tau) + 5)\beta_{k+1}))$, in (III.5.20), and $|\tau'' - \tau'| \leqslant 1$ by adjacency.

It is easy to verify the inequality $(\gamma_k^{(1)}(\tau) - 3)\beta_{k+1} \leqslant \gamma_{k'}^{(1)}(\tau' - 1)\beta_{k'+1}$, so we concentrate on the right delimiters of the intervals. To prove the other inequality, note that for any interval $T_{k'}(\tau')$, we have $\tau'\beta_{k'} \leqslant \gamma_{k'}^{(1)}(\tau')\beta_{k'+1} + 2\beta_{k'+1}$ so iterating this $k - k'$ times we obtain

$$\tau'\beta_{k'} \leqslant \gamma_{k'}^{(k-k')}(\tau')\beta_k + 2\sum_{j=1}^{k-k'} \beta_{k'+j}.$$

We can bound using that $k' \geqslant 1$

$$\sum_{j=1}^{k-k'} \beta_{k'+j} \leqslant \sum_{j=2}^{k} \beta_j = \mathrm{C}_{\mathrm{mix}} \sum_{j=2}^{k} \Big(\frac{j^2}{\varepsilon}\Big)^{4/\Theta} 2^{d_w \ell_{j-1}} = \mathrm{C}_{\mathrm{mix}}\, \varepsilon^{-4/\Theta} \sum_{j=2}^{k} j^{8/\Theta} 2^{d_w(a(j-2)^2+m(j-2)+\ell)}$$

which by induction is smaller than

$$\mathrm{C}_{\mathrm{mix}}\, \varepsilon^{-4/\Theta} 2k^{8/\Theta} 2^{d_w(a(k-1)^2+m(k-1)+\ell)} = 2\beta_k.$$

Hence, we have

$$\begin{aligned}(\tau'' + 2 \vee \eta)\beta_{k'} &\leqslant (\tau' + 1 + 2 \vee \eta)\beta_{k'} \\ &\leqslant \gamma_{k'}^{(k-k')}(\tau')\beta_k + 2\sum_{j=1}^{k} \beta_{k'+j} + (1 + 2 \vee \eta)\beta_{k'} \\ &\leqslant \tau\beta_k + 4\beta_k + (1 + 2 \vee \eta)\beta_{k'},\end{aligned}$$

and since $4\beta_k + (1 + 2 \vee \eta)\beta_{k'} \leqslant (5 + 2 \vee \eta)\beta_k \leqslant \beta_{k+1}$ for a, m large enough, this is further smaller than

$$\tau\beta_k + \beta_{k+1} \leqslant (\gamma_k^{(1)}(\tau) + 5)\beta_{k+1},$$

proving the lemma. □

Space-time tessellation. We can now define the space-time tessellation at different scales via the Cartesian products

$$\begin{aligned} R_k(\iota,\tau) &:= S_k(\iota) \times T_k(\tau), \\ R_k^{\mathrm{inf}}(\iota,\tau) &:= S_k^{\mathrm{inf}}(\iota) \times T_k^{\mathrm{inf}}(\tau), \\ R_k^{\mathrm{sup}}(\iota,\tau) &:= S_k^{\mathrm{sup}}(\iota) \times T_k^{\mathrm{sup}}(\tau), \\ R_k^{\mathrm{Esup}}(\iota,\tau) &:= S_k^{\mathrm{Esup}}(\iota) \times T_k^{\mathrm{Esup}}(\tau). \end{aligned}$$

We say two cells $R_k(\iota_1,\tau_1)$ and $R_k(\iota_2,\tau_2)$ of same scale are *adjacent* if either $d(S_k(\iota_1), S_k(\iota_2)) = 0$ and $\tau_1 = \tau_2$, or else if $\iota_1 = \iota_2$ and $|\tau_1 - \tau_2| \leqslant 1$. We extend the mappings π and γ to a hierarchy of space-time cells. We say that $R_k(\iota,\tau)$ is an ancestor of $R_{k'}(\iota',\tau')$ if $S_k(\iota)$ is an ancestor of $S_{k'}(\iota')$ and $T_k(\tau)$ is an ancestor of $T_{k'}(\tau')$.

We observe, combining (III.5.14) and Lemma III.5.2, for any cell $R_k(\iota,\tau)$ and any cell $R_{k'}(\iota'',\tau'')$ which is adjacent to a descendant of $R_k(\iota,\tau)$ of scale k', it holds that

$$R_{k'}^{\mathrm{inf}}(\iota'',\tau'') \subseteq R_k^{\mathrm{sup}}(\iota,\tau). \tag{III.5.22}$$

In particular, for any two cells $R_k(\iota,\tau)$ and $R_{k'}(\iota',\tau')$,

$$R_k^{\mathrm{inf}}(\iota,\tau) \cap R_{k'}^{\mathrm{inf}}(\iota',\tau') \neq \varnothing \quad \Rightarrow \quad R_k^{\mathrm{sup}}(\iota,\tau) \cap R_{k'}^{\mathrm{sup}}(\iota',\tau') \neq \varnothing, \tag{III.5.23}$$

which means that if the areas of influence of two cells intersect then also the supports intersect.

Note that we defined the extended supports (III.5.10) and (III.5.21) in a way that it holds for two cells $R_{k_1}(\iota_1,\tau_1)$ and $R_{k_2}(\iota_2,\tau_2)$ with $k_1 \leqslant k_2$,

$$R_{k_1}^{\mathrm{sup}}(\iota_1,\tau_1) \cap R_{k_2}^{\mathrm{sup}}(\iota_2,\tau_2) \neq \varnothing \quad \Rightarrow \quad R_{k_2}^{\mathrm{Esup}}(\iota_2,\tau_2) \supseteq R_{k_1}^{\mathrm{sup}}(\iota_1,\tau_1), \tag{III.5.24}$$

which means that if the supports of two cells intersect, then the bigger extended support contains the smaller support.

III.5.2 Fractal percolation

We now introduce several events to define new notions of goodness for each scale k. Having multi-scale levels of goodness is the link to the theory of fractal percolation. We will provide details about the analogy and an intuitive explanation of the following definitions at the end of the subsection.

Let $\varepsilon > 0$ as in Theorem III.2.12, we define the sequence

$$\mathfrak{d}_1 := \varepsilon, \qquad \mathfrak{d}_{k+1} := \mathfrak{d}_k - \frac{\varepsilon}{2k^2}, \quad k \geqslant 1. \tag{III.5.25}$$

Recalling the definition of S^{base} and S^{ext} in (III.5.6) and (III.5.8), as well as the particle system under consideration (see Section III.2.4), define the following indicator random variables:

$$D_k(\iota,\tau) = 1 \qquad \begin{array}{l}\text{if all tiles } S_{k-1}(\iota') \subseteq S_k(\iota) \text{ contain at least} \\ (1-\mathfrak{d}_k)\mu_0 \sum_{y\in S_{k-1}(\iota')} \lambda_y \text{particles at time } \tau\beta_k,\end{array} \tag{III.5.26}$$

$$D_k^{\text{ext}}(\iota,\tau) = 1 \qquad \begin{array}{l}\text{if all tiles } S_{k-1}(\iota') \subseteq S_k^{\text{ext}}(\iota) \text{ contain at least} \\ (1-\mathfrak{d}_k)\mu_0 \sum_{y\in S_{k-1}(\iota')} \lambda_y \text{particles at time } \tau\beta_k \\ \text{that are confined during } [\tau\beta_k, (\tau+2)\beta_k] \\ \text{inside } B_{b(k-1)2^{\ell_{k-1}}},\end{array} \tag{III.5.27}$$

$$D_k^{\text{base}}(\iota,\tau) = 1 \qquad \begin{array}{l}\text{if all tiles } S_{k}(\iota') \subseteq S_k^{\text{base}}(\iota) \text{ contain at least} \\ (1-\mathfrak{d}_{k+1})\mu_0 \sum_{y\in S_{k}(\iota')} \lambda_y \text{particles at time } \gamma_k^{(1)}(\tau)\beta_{k+1} \\ \text{that are confined during } [\gamma_k^{(1)}(\tau)\beta_{k+1}, \tau\beta_k] \\ \text{inside } B_{b(k)2^{\ell_{k}}}.\end{array} \tag{III.5.28}$$

Since $S_k \subseteq S_k^{\text{ext}}$, trivially $D_k^{\text{ext}}(\iota,\tau) = 1$ implies $D_k(\iota,\tau) = 1$. Noting that $S_k^{\text{base}}(\iota) \subseteq S_{k+1}^{\text{ext}}(\pi_k^{(1)}(\iota))$ as mentioned in (III.5.11) and that $[\gamma_k^{(1)}(\tau)\beta_{k+1}, \tau\beta_k] \subset [\gamma_k^{(1)}(\tau)\beta_{k+1}, (\gamma_k^{(1)}(\tau)+2)\beta_{k+1}]$ we have by definition

$$D_{k+1}^{\text{ext}}\big(\pi_k^{(1)}(\iota), \gamma_k^{(1)}(\tau)\big) = 1 \qquad \Rightarrow \qquad D_k^{\text{base}}(\iota,\tau) = 1 \qquad \forall (k,\iota,\tau) \in \mathbb{N}\times\mathbb{B}^d\times\mathbb{Z}, \tag{III.5.29}$$

and the goal of Lemma III.6.1 below will be to show that with exponentially large probability, $\{D_k^{\text{base}}(\iota,\tau) = 1\}$ implies $\{D_k^{\text{ext}}(\iota,\tau) = 1\}$. To this end, we define

$$A_1(\iota,\tau) := \max\{\mathbf{1}_{E(\iota,\tau)}, 1 - D_1^{\text{base}}(\iota,\tau)\}, \tag{III.5.30}$$
$$A_k(\iota,\tau) := \max\{D_k^{\text{ext}}(\iota,\tau), 1 - D_k^{\text{base}}(\iota,\tau)\}, \tag{III.5.31}$$
$$A_\kappa(\iota,\tau) := D_\kappa^{\text{ext}}(\iota,\tau), \tag{III.5.32}$$

and

$$A(\iota,\tau) := \prod_{k=1}^{\kappa} A_k(\pi_1^{(k-1)}(\iota), \gamma_1^{(k-1)}(\tau)). \tag{III.5.33}$$

The first-time reader should think that $A_k(\iota,\tau) = 0$ intuitively indicates that “in the chain of space-time cells that are ancestors of $R_1(\iota,\tau)$, the particles misbehaved at scale k”: more precisely, $A_k(\iota,\tau) = 0$ if, even despite the favorable event $D_k^{\text{base}}(\iota,\tau) = 1$, according to which the particle were in a good state inherited from higher scales, it resulted in $D_k^{\text{ext}}(\iota,\tau) = 0$. As already mentioned above (III.5.30), we will prove that the previous situation happens with small probability in Lemma III.6.1.

We can now define the notions of goodness that we will consider. Recall that we defined at the very start of subsection III.2.5 that

$$\text{a cell } R_1(\iota,\tau) \text{ is } \textit{bad} \text{ if } \mathbf{1}_{E(\iota,\tau)} = 0. \tag{III.5.34}$$

We consider now a stronger notion of bad cells for any scale $1 \leqslant k \leqslant \kappa$:

$$\text{a cell } R_k(\iota,\tau) \text{ is } \textit{multi-scale bad} \text{ if } A_k(\iota,\tau) = 0. \tag{III.5.35}$$

Note that for the scale 1 this definition is stricter then the definition of being *bad*: as a simple consequence of (III.5.30), a *multi-scale bad* cell is also *bad*. Finally, we say for scale 1 cells that that

$$\text{a cell } R_1(\iota,\tau) \text{ has } \textit{bad ancestry} \text{ if } A(\iota,\tau) = 0, \tag{III.5.36}$$

or equivalently that the cell has a *multi-scale bad* ancestor.

In particular, a *bad* cell of scale 1 has *bad ancestry*, as we prove in the following lemma.

Lemma III.5.3. *For a cell $R_1(\iota,\tau)$ it holds $\mathbf{1}_{E(\iota,\tau)} \geqslant A(\iota,\tau)$. Equivalently, a scale 1 cell which is* bad*, in particular has* bad ancestry*.*

Proof. Suppose that $A(\iota,\tau) = 1$. By (III.5.33), it therefore holds for all $1 \leqslant k \leqslant \kappa$, that

$$A_k\big(\pi_1^{(k-1)}(\iota), \gamma_1^{(k-1)}(\tau)\big) = 1.$$

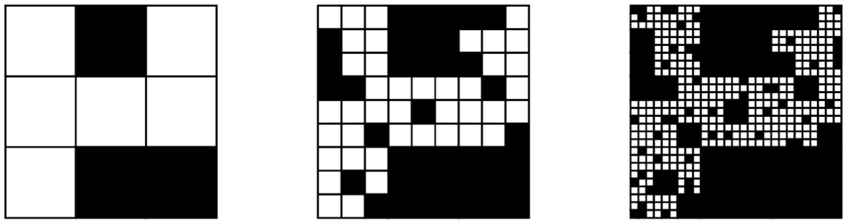
In particular $D_\kappa^{\text{ext}}\big(\pi_1^{(\kappa-1)}(\iota), \gamma_1^{(\kappa-1)}(\tau)\big) = 1$, so applying the property in (III.5.29) we obtain $D_{\kappa-1}^{\text{base}}\big(\pi_1^{(\kappa-2)}(\iota), \gamma_1^{(\kappa-2)}(\tau)\big) = 1$. Since $A_{\kappa-1}\big(\pi_1^{(\kappa-2)}(\iota), \gamma_1^{(\kappa-2)}(\tau)\big) = 1$ and it is defined as a maximum, the first argument need to be a 1, and we obtain $D_{\kappa-1}^{\text{ext}}\big(\pi_1^{(\kappa-2)}(\iota), \gamma_1^{(\kappa-2)}(\tau)\big) = 1$.

Repeating this argument for all scales down to scale 1, we need the first argument in the maximum of $A_1(\iota,\tau)$ to be 1, i.e it must hold that $\mathbf{1}_{E(\iota,\tau)} = 1$. □

Intuition. We conclude this subsection by explaining the analogy of our setup to fractal percolation, whose framework has inspired this proof. For simplicity, we will explain the arguments on $\mathbb{R}^d$ instead of the Sierpiński gasket.

Fix some value $r \in \mathbb{N}$. Consider the unit hyper-cube and subdivide it into r^d cubes of side length $\frac{1}{r}$. Then, for some value $p \in [0,1]$, declare them open independently with probability p and closed otherwise. Then, subdivide again each of the *open* cubes into r^d cubes of side length $\frac{1}{r^2}$, and each of the second-level cubes is open with probability p and closed otherwise. Note that each level-1 cube that was closed is not further subdivided and so it is entirely closed. One can then repeat the above procedure with further subdivisions, see Figure III.6. This recursive construction introduces correlations into the system that one would not see in standard Bernoulli percolation - whether two different cubes of some arbitrary size are both simultaneously open is heavily influenced by how far back in the subdivisions their common "ancestor" cube that was open is.

The similarity with our case is straightforward. To obtain $A(\iota,\tau) = 1$ we need a cell and all its ancestors to be multi-scale good, similarly to the fractal percolation where the cubes must be open at every level-k in order to be open at the last and smallest level. In view of Lemma III.5.3, a cell with $A(\iota,\tau) = 1$ is then *good*, in the



(a) (b) (c)

Figure III.6: An example of fractal percolation in $\mathbb{R}^2$. Image from [GS19a].

sense below Definition III.2.8. It may seem now that directly performing a single-level percolation at scale $k = 1$ might be easier, but unlike the fractal percolation described above, cells in our setting have further dependencies beyond the ones introduced by the subdivisions. In particular, note that knowing a cell of some scale k is bad reveals information not only about its descendent cells, but also any other cells that are spatially and temporally close enough to be affected by the behaviour of the particles from the cell in question. The other difference is that the percolation parameter p will not be kept constant: in our case the probability to be a multi-scale good cell $\mathbb{P}(A_k(\iota, \tau) = 1)$ is higher at larger scales, as we will prove in Lemma III.6.1. The proof there involves the events D_k^{ext} and D_k^{base} defined above in (III.5.27) and (III.5.28), and in particular the strategy is as follows: assuming the favorable event $D_k^{\text{ext}}(\iota, \tau) = 1$, using the mixing Theorem III.4.6, if we restrict to a slightly smaller cell (so from $S_k^{\text{ext}}(\iota)$ to $S_k^{\text{base}}(\iota)$) and "wait a bit", we are able to resample the particles according an independent Poisson point process with only a slightly smaller intensity. This resampling allows us to essentially treat the configuration of the particles in the space-time cell in question as independent of the configuration elsewhere, thus roughly recovering the fractal percolation setup outlined above and taking care of both types of correlations mentioned at once.

III.5.3 Paths of cells

We next define the two notions of "paths of cells" that we will consider. As we will see momentarily, both notions are strongly related to d-paths from Definition III.3.1.

Recall that, in line with Definition III.2.2, two cells $R_k(\iota_1, \tau_1) \neq R_k(\iota_2, \tau_2)$ of same scale are called *adjacent* if either $d(S_k(\iota_1), S_k(\iota_2)) = 0$ and $\tau_1 = \tau_2$, or $\iota_1 = \iota_2$ and $|\tau_1 - \tau_2| \leqslant 1$. We now extend this to cells of different scales. Two cells $R_{k_1}(\iota_1, \tau_1)$, and $R_{k_2}(\iota_2, \tau_2)$ with scales $k_1 > k_2$ are called *adjacent* if $R_{k_1}(\iota_1, \tau_1)$, is adjacent to $R_{k_1}\big(\pi_{k_2}^{k_1-k_2}(\iota_2), \gamma_{k_2}^{k_1-k_2}(\tau_2)\big)$. Note that in particular, a cell is not adjacent to any of its ancestors.

We say for two scale 1 cells $R_1(\iota, \tau)$ and $R_1(\iota', \tau')$ that $R_1(\iota, \tau)$ is *diagonally connected* to $R_1(\iota', \tau')$ if there exists a sequence of adjacent cells $\{R_1(\iota_1, \tau_1), \ldots, R_1(\iota_n, \tau_n)\}$ of scale 1 such that $R_1(\iota, \tau) = R_1(\iota_1, \tau_1)$, for all $j \in \{1, \ldots, n-1\}$, $d(R_1(\iota_{j+1}, \tau_{j+1}), L_0) < d(R_1(\iota_j, \tau_j), L_0)$ and $R_1(\iota_n, \tau_n)$ is either equal or adjacent to $R_1(\iota', \tau')$. When referring to the cells $R_1(\iota_j, \tau_j)$, $j \in \{1, \ldots, n-1\}$ (and $R_1(\iota_n, \tau_n)$ if it differs from $R_1(\iota', \tau')$) we will call them *diagonal steps*.

For two cells $R_{k_1}(\iota_1, \tau_1)$ and $R_{k_2}(\iota_2, \tau_2)$ of not necessarily different scales we say that $R_{k_1}(\iota_1, \tau_1)$ is *diagonally connected* to $R_{k_2}(\iota_2, \tau_2)$ if there exist two cells $R_1(\tilde{\iota}_1, \tilde{\tau}_1)$ and $R_1(\tilde{\iota}_2, \tilde{\tau}_2)$ of scale 1, respectively descendants of $R_{k_1}(\iota_1, \tau_1)$ and $R_{k_2}(\iota_2, \tau_2)$, so that $R_1(\tilde{\iota}_1, \tilde{\tau}_1)$ is diagonally connected to $R_1(\tilde{\iota}_2, \tilde{\tau}_2)$.

Definition III.5.4. We define a *D-path* as a sequence of cells of arbitrary scale, where each cell is either adjacent or diagonally connected to the next cell in the sequence.

The reader will note the analogy to the definition of d-path in Definition III.3.1. Fix a cell $v = R_1(\iota_v, \tau_v) \in L_1$ and define for any (large) $t > 0$

$$\Omega_1(v \to t) \tag{III.5.37}$$

the set of all D-paths of cells of scale 1 for which the first cell of the path is v and the last cell is the only cell not contained in $B_t(S_1(\iota_v)) \times [-t + \tau_v, \tau_v + t]$, where $B_t(S_1(\iota_v)) := \cup_{x \in S_1(\iota_v)} B_t(x)$.

The next notion of path involves instead cells of multiple scales.

Definition III.5.5. We define as *ScD-path* (support connected with diagonal paths) a sequence of cells of possibly different scales $\{R_{k_1}(\iota_1, \tau_1), \ldots, R_{k_z}(\iota_z, \tau_z)\}$ for some $z \in \mathbb{N}$, with the following properties:

- each pair of cells is *well-separated*, meaning that their areas of influence do not intersect; i.e. for any pair $R_{\tilde{k}}(\tilde{\iota}, \tilde{\tau})$, $R_{\hat{k}}(\hat{\iota}, \hat{\tau})$
$$R^{\inf}_{\tilde{k}}(\tilde{\iota}, \tilde{\tau}) \cap R^{\inf}_{\hat{k}}(\hat{\iota}, \hat{\tau}) = \varnothing,$$
- two consecutive cells $R_{k_j}(\iota_j, \tau_j)$ and $R_{k_{j+1}}(\iota_{j+1}, \tau_{j+1})$ are either
$$\textit{support adjacent}: \quad R^{\mathrm{Esup}}_{k_j}(\iota_j, \tau_j) \cap R^{\mathrm{Esup}}_{k_{j+1}}(\iota_{j+1}, \tau_{j+1}) \neq \varnothing$$
or

  support connected with diagonals: there exist two scale 1 cells, respectively subsets of the extended supports of $R_{k_j}(\iota_j, \tau_j)$ and $R_{k_{j+1}}(\iota_{j+1}, \tau_{j+1})$, so that the first cell is diagonally connected to the second.

For $v \in L_1$ and $t > 0$, we define

$$\Omega^{\sup}_{\kappa}(v \to t) \tag{III.5.38}$$

as the set of all ScD-paths of cells of scale at most κ so that the extended support of the first cell of the path contains v and the last cell is the only cell whose extended support is not contained in $B_t(S_1(\iota_v)) \times [-t + \tau_v, \tau_v + t]$ with ι_v, τ_v as before.

Define the bad cluster around $v \in L_1$ as

$$K_v := \big\{R_1(\tilde{\iota}, \tilde{\tau})\colon \text{there exists a D-path of bad cells from } v \text{ to } R_1(\tilde{\iota}, \tilde{\tau})\big\}. \tag{III.5.39}$$

We can relate D-paths and ScD-paths via the following technical lemma.

Lemma III.5.6. *For any $t > 0$ and $v \in L_1$, it holds that*

$$\begin{aligned}&\mathbb{P}\big(\exists P \in \Omega_1(v \to t) \text{ of cells with bad ancestry}\big)\\ &\leqslant \mathbb{P}\big(\exists P \in \Omega^{\sup}_{\kappa}(v \to t) \text{ of multi-scale bad cells}\big).\end{aligned}$$

Remark III.5.7. Note that for a path $P \in \Omega_1(v \to t)$ of cells with bad ancestry, the property of having a bad ancestor is required only for the cells of P and not for the cells constituting the diagonal steps in the diagonal connections of P. This is in line with Definition III.3.1, where diagonal moves of d-paths do not impose any requirements on the state of the cells. The same is of course true also for $P \in \Omega_\kappa^{\text{sup}}(v \to t)$, where being multi-scale bad is not required for the cells constituting diagonal connections.

Proof. We split the proof into two steps. Defining $\Omega_\kappa(v \to t)$ as the set of D-paths of cells of scale at most κ, where the first cell is an ancestor of v and the last cell is the only cell whose support is not contained in $B_t(S_1(\iota_v)) \times [-t+\tau_v, \tau_v+t]$, we prove in the two steps that

$$\begin{aligned}&\mathbb{P}\big(\exists P \in \Omega_1(v \to t) \text{ of cells with bad ancestry}\big)\\ &\leqslant \mathbb{P}\big(\exists P \in \Omega_\kappa(v \to t) \text{ of multi-scale bad cells}\big)\\ &\leqslant \mathbb{P}\big(\exists P \in \Omega_\kappa^{\text{sup}}(v \to t) \text{ of multi-scale bad cells}\big).\end{aligned}$$

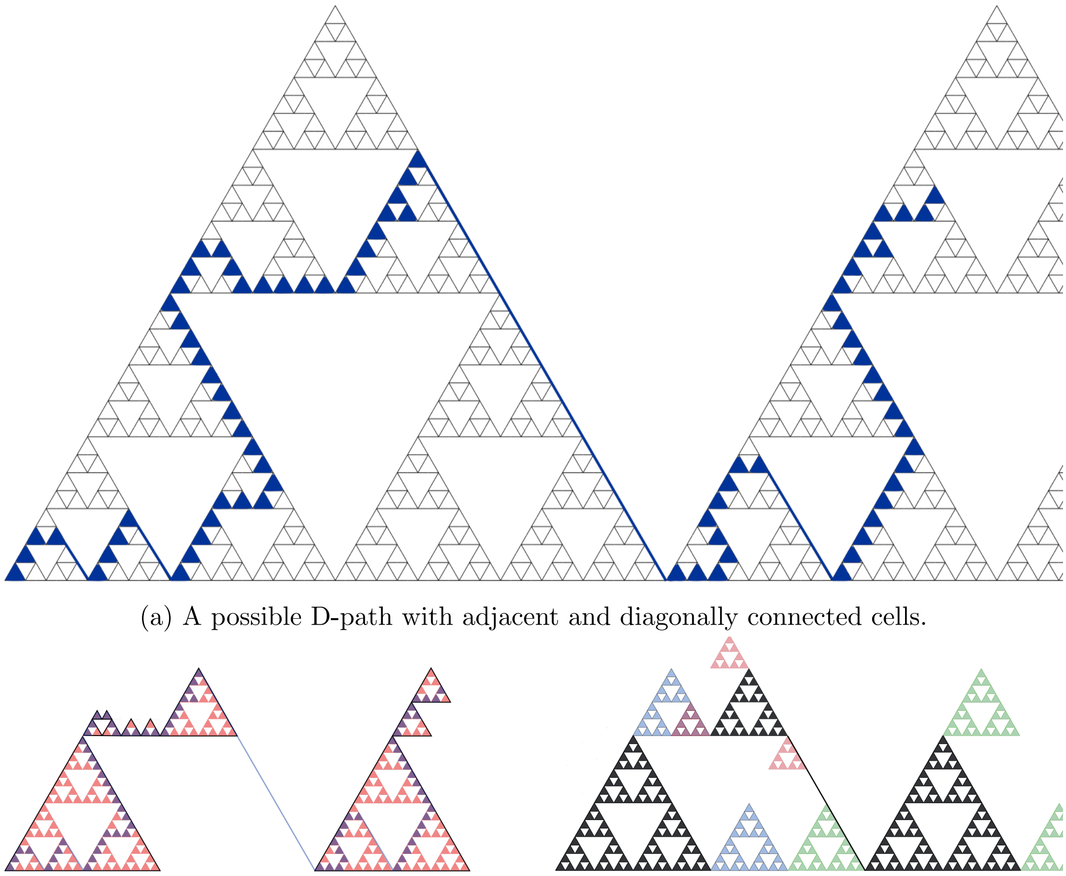
Step 1. Consider a D-path $P = \big(R_1(\iota_j, \tau_j)\big)_{j=1}^{z} \in \Omega_1(v \to t)$ of cells with bad ancestry. By definition, for each cell of P it holds that $A(\iota_j, \tau_j) = 0$, so there exists k_j such that $A_{k_j}(\pi_1^{k'_j-1}(\iota_j), \gamma_1^{k'_j-1}(\tau_j)) = 0$, so that $R_{\widetilde{k}_j}(\widetilde{\iota}_j, \widetilde{\tau}_j) := R_{k_j}\big(\pi_1^{k'_j-1}(\iota_j), \gamma_1^{k'_j-1}(\tau_j)\big)$ is a multi-scale bad cell. From the sequence $P' := \{R_{\widetilde{k}_j}(\widetilde{\iota}_j, \widetilde{\tau}_j)\}_{j=1}^{z}$ construct a subsequence $P'' := \{R_{k''_j}(\iota''_j, \tau''_j)\}_{j=1}^{z''}$ taking in the same order of the cells from P' but removing all cells indexed by $\hat{j}$ which are the descendant of some other cell in the path P' with index j_0, with $j_0 < \hat{j}$. Furthermore, if there is a cell $R_{\widetilde{k}_j}(\widetilde{\iota}_j, \widetilde{\tau}_j)$ before the last one whose support is not contained in $B_t(S_1(\iota_v)) \times [-t+\tau_v, \tau_v+t]$, we remove from P'' all following cells.

We claim that $P'' \in \Omega_\kappa(v \to t)$, which will conclude step 1. This path starts with an ancestor of v and by construction the last cell's support is not contained in $B_t(S_1(\iota_v)) \times [-t+\tau_v, \tau_v+t]$. Note that every cell in P has exactly 1 ancestor in P''. Consider now two cells $R_1(\iota_j, \tau_j)$ and $R_1(\iota_{j+1}, \tau_{j+1})$ with different ancestors in P''. If $R_1(\iota_j, \tau_j)$ is diagonally connected to $R_1(\iota_{j+1}, \tau_{j+1})$, then the ancestor of $R_1(\iota_j, \tau_j)$ is either diagonally connected or adjacent to the ancestor of $R_1(\iota_{j+1}, \tau_{j+1})$; if $R_1(\iota_j, \tau_j)$ and $R_1(\iota_{j+1}, \tau_{j+1})$ are adjacent, then their ancestors are adjacent, since two non-adjacent cells cannot have two adjacent descendants. Finally, every cell of P'' is multi-scale bad by how P'' was constructed.

Step 2. We now prove the second inequality, that is, starting from P'' we can obtain a path $\hat{P}$ of multi-scale bad cells which are well-separated and in which every sequential pair of cells is either support adjacent or the first cell of the pair is support connected with diagonals to the second.

First define a sequence L of cells from P'', but where the cells are ordered in the following way: we first order cells by scale, where cells of bigger scale come first, and within cells of the same scale we maintain the original order of P''. We construct $\hat{P}$ and create a relation between P'' and $\hat{P}$ in the following way. Following the order of L, and in particular starting with scale κ, we perform the following operations. Assuming the first cell of scale k in the list L is $R_k(\hat{\iota}, \hat{\tau})$ we

- add $R_k(\hat{\iota}, \hat{\tau})$ to $\hat{P}$;
- remove $R_k(\hat{\iota}, \hat{\tau})$ from L;



(a) A possible D-path with adjacent and diagonally connected cells.

(b) A D-path of multi-scale bad cells (in red with a thicker border) in comparison with the D-path (in blue) of the previous image. Note that many cells of scale 1 correspond to the same cell in this image.

(c) The corresponding ScD-path (in black), where some cells were discarded as they were not *well-separated*. We highlight (respectively in blue, red and green) the extended supports and (in black) the diagonal of 2 cells which are *support connected with diagonal*.

Figure III.7: From D-paths to ScD-paths. Note that this example is on $\mathbb{G}$ without the time component in order to make the visualisation easier. In practice, the procedure is conducted on cells of $\mathbb{G} \times \mathbb{Z}$.

- associate $R_k(\hat{\iota}, \hat{\tau})$ in P'' with itself in $\hat{P}$;
- remove from L all cells $R_{\tilde{k}}(\tilde{\iota}, \tilde{\tau})$ which are not well-separated from $R_k(\hat{\iota}, \hat{\tau})$ and associate them all with $R_k(\hat{\iota}, \hat{\tau})$ in $\hat{P}$.

Repeating this procedure until L is empty, we obtained a sequence of cells $\hat{P}$, and all cells in P'' are associated to some cell in $\hat{P}$. Before proceeding, we reorder $\hat{P}$ according to the ordering in P'', thus making $\hat{P}$ a path (which we will verify below). In particular, a cell v in $\hat{P}$ appears before a different cell u of $\hat{P}$ if according to the ordering of P'', there exists a cell of P'' associated to v that appears before any cell of P'' associated to u. Since the multi-scale bad property follows trivially from P'', we are only left to show that

$$\hat{P} \in \Omega_\kappa^{\sup}(v \to t). \tag{III.5.40}$$

First, let $R_{\hat{k}_1}(\hat{\iota}_1, \hat{\tau}_1) \in \hat{P}$ be the cell which $R_{k_1''}(\iota_1'', \tau_1'') \in P''$ is associated to. In the non-trivial case, $R_{\hat{k}_1}(\hat{\iota}_1, \hat{\tau}_1)$ is not associated to itself, so $R_{\hat{k}_1}(\hat{\iota}_1, \hat{\tau}_1)$ and $R_{k_1''}(\iota_1'', \tau_1'')$ are not well-separated and therefore their areas of influence intersect. By (III.5.23) their supports intersect as well. By (III.5.24), $R^{\mathrm{Esup}}_{\hat{k}_1}(\hat{\iota}_1, \hat{\tau}_1) \supseteq R^{\sup}_{k_1''}(\iota_1'', \tau_1'')$, and since $R_{k_1''}(\iota_1'', \tau_1'')$ contains v by definition of P'', we obtain that $R^{\mathrm{Esup}}_{\hat{k}_1}(\hat{\iota}_1, \hat{\tau}_1)$ contains v as desired.

Secondly, we can argue in the same way to show that the extended support of the cell which $R_{k_{z''}''}(\iota_{z''}'', \tau_{z''}'')$ is associated to is not contained in the space-time ball $B_t(S_1(\iota_v)) \times [-t + \tau_v, \tau_v + t]$.

Finally, we need to show that sequential pairs of cells of $\hat{P}$ are either support adjacent or the first cell of the pair is support connected with diagonals to the second. Consider $R_{\hat{k}_j}(\hat{\iota}_j, \hat{\tau}_j) \in \hat{P}$, and let $R_{k_{j''},}(\iota_{j''}, \tau_{j''})$ be the first cell of P'' (in the original ordering of P'') which is associated to $R_{\hat{k}_j}(\hat{\iota}_j, \hat{\tau}_j)$. Next, take $R_{k_{j''-1}}(\iota_{j''-1}, \tau_{j''-1}) \in P''$ and let $R_{\hat{k}_{j-1}}(\hat{\iota}_{j-1}, \hat{\tau}_{j-1}) \in \hat{P}$ be the cell which it is associated to. We claim that $R_{\hat{k}_j,}(\hat{\iota}_j, \hat{\tau}_j)$ and $R_{\hat{k}_{j-1}}(\hat{\iota}_{j-1}, \hat{\tau}_{j-1})$ are either support adjacent or $R_{\hat{k}_{j-1}}$ is support connected with diagonals to $R_{\hat{k}_j,}(\hat{\iota}_j, \hat{\tau}_j)$ based on whether $R_{k_{j''}}(\iota_{j''}, \tau_{j''})$ and $R_{k_{j''-1}}(\iota_{j''-1}, \tau_{j''-1}) \in P''$ are adjacent or whether $R_{k_{j''-1}}(\iota_{j''-1}, \tau_{j''-1})$ is connected with diagonals to $R_{k_{j''}}(\iota_{j''}, \tau_{j''})$.

If $R_{k_{j''-1}}(\iota_{j''-1}, \tau_{j''-1})$ and $R_{k_{j''}}(\iota_{j''}, \tau_{j''})$ are adjacent, we can suppose without loss of generality that $k_{j''-1} \leqslant k_{j''}$, and by definition there exists a cell $R_{k_{j''}}(\widetilde{\iota}_{j''-1}, \widetilde{\tau}_{j''-1})$, which is an ancestor of $R_{k_{j''-1}}(\iota_{j''-1}, \tau_{j''-1})$ and adjacent to $R_{k_{j''}}(\iota_{j''}, \tau_{j''})$. Hence applying (III.5.22) twice we obtain that

$$\begin{gathered} R^{\inf}_{k_{j''}}(\iota_{j''}, \tau_{j''}) \subseteq R^{\sup}_{k_{j''}}(\widetilde{\iota}_{j''-1}, \widetilde{\tau}_{j''-1}) \\ \text{and} \\ R^{\inf}_{k_{j''-1}}(\iota_{j''-1}, \tau_{j''-1}) \subseteq R^{\sup}_{k_{j''}}(\widetilde{\iota}_{j''-1}, \widetilde{\tau}_{j''-1}). \end{gathered} \tag{III.5.41}$$

Since $R_{k_{j''}}(\iota_{j''}, \tau_{j''})$ is associated to $R_{\hat{k}_j}(\hat{\iota}_j, \hat{\tau}_j)$, they are not well-separated and thus their areas of influence intersect. Therefore (III.5.41) implies that $R^{\sup}_{k_{j''}}(\widetilde{\iota}_{j''-1}, \widetilde{\tau}_{j''-1})$ intersects $R^{\inf}_{\hat{k}_j}(\hat{\iota}_j, \hat{\tau}_j)$ and by (III.5.22) intersects $R^{\sup}_{\hat{k}_j}(\hat{\iota}_j, \hat{\tau}_j)$; since $\hat{k} \geqslant k_{j''}$, applying (III.5.24), we have $R^{\mathrm{Esup}}_{\hat{k}_j}(\hat{\iota}_j, \hat{\tau}_j) \supseteq R^{\sup}_{k_{j''}}(\widetilde{\iota}_{j''-1}, \widetilde{\tau}_{j''-1}) \supseteq R^{\inf}_{k_{j''-1}}(\iota_{j''-1}, \tau_{j''-1})$ where the last inclusion is due to (III.5.41). Since the cells $R_{k_{j''-1}}(\iota_{j''-1}, \tau_{j''-1})$ and $R_{\hat{k}_{j-1}}(\hat{\iota}_{j-1}, \hat{\tau}_{j-1})$ are not well-separated, repeating the same argument below (III.5.40) we have $R^{\mathrm{Esup}}_{\hat{k}_{j-1}}(\hat{\iota}_{j-1}, \hat{\tau}_{j-1}) \supseteq R^{\sup}_{k_{j''-1}}(\iota_{j''-1}, \tau_{j''-1}) \supseteq R^{\inf}_{k_{j''-1}}(\iota_{j''-1}, \tau_{j''-1})$, where the last inclusion follows from (III.5.22). This shows that the two extended supports intersect.

If instead $R_{k_{j''-1}}(\iota_{j''-1}, \tau_{j''-1})$ is connected with diagonals to $R_{k_{j''}}(\iota_{j''}, \tau_{j''})$, then by definition they contain respectively two cells $R_1(\widetilde{\iota}_{j''-1}, \widetilde{\tau}_{j''-1})$ and $R_1(\widetilde{\iota}_{j''}, \widetilde{\tau}_{j''})$ such that $R_1(\widetilde{\iota}_{j''-1}, \widetilde{\tau}_{j''-1})$ is connected with diagonals to $R_1(\widetilde{\iota}_{j''}, \widetilde{\tau}_{j''})$. Additionally, since $R_{k_{j''}}(\iota_{j''}, \tau_{j''})$ is associated to $R_{\hat{k}_j}(\hat{\iota}_j, \hat{\tau}_j)$, they are not well-separated and by the argument below (III.5.40) we have $R_1(\widetilde{\iota}_{j''}, \widetilde{\tau}_{j''}) \subseteq R^{\sup}_{k_{j''}}(\iota_{j''}, \tau_{j''}) \subseteq R^{\mathrm{Esup}}_{\hat{k}_j}(\hat{\iota}_j, \hat{\tau}_j)$. With the same argument, $R_1(\widetilde{\iota}_{j''-1}, \widetilde{\tau}_{j''-1}) \subseteq R^{\mathrm{Esup}}_{\hat{k}_{j-1}}(\hat{\iota}_{j-1}, \hat{\tau}_{j-1})$. This shows that $R_{\hat{k}_{j-1}}(\hat{\iota}_{j-1}, \hat{\tau}_{j-1})$ is support connected with diagonals to $R_{\hat{k}_j}(\hat{\iota}_j, \hat{\tau}_j)$, which concludes the proof. $\square$

III.6 Multi-scale analysis

We will now use the multi-scale set-up introduced above in order to bound the probability of having paths of multi-scale bad cells. Recall that $\zeta \in (0, \infty)$ is defined in Theorem III.2.12 as the value used at the scale 1 tessellation that imposes the confinement of particle movement at that scale. We now define what will essentially be the "weight" of a cell as

$$\begin{aligned} \psi_1(\varepsilon, \mu_0, \ell) &:= \min\left\{\frac{\varepsilon^2 \mu_0 2^{d_v \ell}}{C_\lambda}, -\log\Big(1 - \nu_E\big((1-\varepsilon)\lambda, S_1^\eta, B_{\zeta\ell}, \eta\beta\big)\Big)\right\}, \\ \psi_k(\varepsilon, \mu_0, \ell) &:= \frac{\varepsilon^2 \mu_0 2^{d_v \ell_{k-1}}}{k^4}, \qquad k \geqslant 2, \end{aligned} \tag{III.6.1}$$

which we will use as a reference for both the probability of a cell of scale k to be bad, and for the number of ScD-paths which contain a cell of scale k.

III.6.1 Probability of a multi-scale bad ScD-path

We want to estimate the probability for a cell to be multi-scale bad. As close cells are heavily dependent on each other, we want to obtain a bound even conditioning on cells which are "not too close", in a spatial or temporal sense. Recall the definitions of $S_k^{\inf}$ and $T_k^{\inf}$ in (III.5.7) and (III.5.19). We define $\mathcal{F}_k(\iota, \tau)$ be the σ-algebra generated by all the $A_{k'}(\iota', \tau')$ for which either:

(a) $T_{k'}^{\inf}(\tau') \cap [\gamma_k^{(1)}(\tau)\beta_{k+1}, \infty) = \varnothing$, or

(b) $\tau'\beta_{k'} \leqslant \tau\beta_k$ and $S_k^{\inf}(\iota) \cap S_{k'}^{\inf}(\iota') = \varnothing$.

Intuitively, this is information about the behaviour of particles in space-time cells that are either far enough in the past so that we can ignore them due to the starting assumptions guaranteed by $\{D_k^{\text{base}}(\iota, \tau) = 1\}$, or which are happening roughly concurrently, but far enough away not to be able to influence the occurrence of the event $\{A_k(\iota, \tau) = 0\}$ due to the confinement of the random walks under consideration. Recall that the intensity of the Poisson point process is $\mu_x = \mu_0 \lambda_x$.

Lemma III.6.1. *Let ε, ζ, η be as in Theorem III.2.12 with*

$$\zeta \geqslant \frac{1}{\ell} \sqrt[d_w]{\left[\tfrac{1}{c_6} \log\left(\tfrac{8c_5}{3\varepsilon}\right)\right]^{d_w - 1} \eta\beta}. \tag{III.6.2}$$

If a and m are large enough, then there exist C_ψ and $\alpha_0 = \alpha_0(\varepsilon, \ell, \mu_0)$ such that if $\psi_1 > \alpha_0$, then for all $k = 1, \ldots, \kappa$, all cells $R_k(\iota, \tau)$ and any $F \in \mathcal{F}_k(\iota, \tau)$

$$\mathbb{P}(A_k(\iota, \tau) = 0 \,|\, F) \leqslant e^{-C_\psi \psi_k}.$$

Furthermore, we have for scale κ that

$$\mathbb{P}(A_\kappa(\iota, \tau) = 0) \leqslant e^{-C_\psi \psi_\kappa}.$$

Proof. We start by proving the result for $2 \leqslant k \leqslant \kappa - 1$. Let $F \in \mathcal{F}_k(\iota, \tau)$. Since

$$\mathbb{P}(A_k(\iota, \tau) = 0 \,|\, F) = \mathbb{P}(D_k^{\text{ext}}(\iota, \tau) = 0, D_k^{\text{base}}(\iota, \tau) = 1 \,|\, F),$$

if $\{D_k^{\mathrm{base}}(\iota,\tau)=1\}\cap F=\varnothing$, such probability is 0 and the lemma trivially holds, so we can assume $\{D_k^{\mathrm{base}}(\iota,\tau)=1\}\cap F\neq\varnothing$ and obtain

$$\mathbb{P}(A_k(\iota,\tau)=0 \,\big|\, F)\leqslant \mathbb{P}(D_k^{\mathrm{ext}}(\iota,\tau)=0 \,\big|\, F, D_k^{\mathrm{base}}(\iota,\tau)=1),$$

Recall that the event $D_k^{\mathrm{base}}(\iota,\tau)=1$ (see (III.5.28)) ensures that there are enough particles in $S_k^{\mathrm{base}}(\iota)$ confined in $B_{b(k)2^{\ell_k}}$ during $[\gamma_k^{(1)}(\tau)\beta_{k+1},\tau\beta_k]$. By definition F does not reveal further information about those particles because either

- by (a), $(\tau'+2)\beta_{k'}\leqslant\gamma_k^{(1)}\beta_{k+1}$ and so the time interval relevant to $A_{k'}(\iota',\tau')$ does not intersect $A_k(\iota,\tau)$, or
- by (b), $S_{k'}^{\inf}(\iota')\cap S_k^{\inf}(\iota)=\varnothing$, so the particles in $S_{k'}^{\mathrm{base}}(\iota')$ confined in $B_{b(k)2^{\ell_k}}$ cannot leave $S_{k'}^{\inf}(\iota')$ and thus cannot enter $S_k^{\inf}(\iota)$ before $\tau'\beta_{k'}$.

Conditioned on the event $D_k^{\mathrm{base}}(\iota,\tau)=1$ (defined in (III.5.28)), we apply Theorem III.4.6 to $S_k^{\mathrm{base}}(\iota,\tau)$, with the choices

$$\begin{aligned}
&K := \text{side length of } S_k^{\mathrm{base}}(\iota)=2b(k)2^{\ell_k}+2^{\ell_k},\\
&K' \text{ such that } K-K'=b(k)2^{\ell_k},\\
&l := 2^{\ell_k},\\
&\delta := (1-\mathfrak{d}_{k+1})\mu_0,\\
&\boldsymbol{\Delta} := \mathrm{length}\big([\gamma_k^{(1)}(\tau)\beta_{k+1},\tau\beta_k]\big)=\tau\beta_k-\gamma_k^{(1)}(\tau)\beta_{k+1}\in[\beta_{k+1},2\beta_{k+1}], \text{ and}\\
&\bar\varepsilon := \frac{\varepsilon}{8k^2}.
\end{aligned}$$

We check now that the conditions of Theorem III.4.6 are satisfied, starting with checking that $K-K'\geqslant M_3\big(\boldsymbol{\Delta}(\log_2\boldsymbol{\Delta})^{d_w-1}\big)^{\frac{1}{d_w}}$. Since $K-K'=b(k)2^{\ell_k}$ and $\boldsymbol{\Delta}\leqslant 2\beta_{k+1}$ we need to verify that

$$b(k)2^{\ell_k}\geqslant M_3\big(\tfrac{\beta_{k+1}}{2}(\log_2\tfrac{\beta_{k+1}}{2})^{d_w-1}\big)^{\frac{1}{d_w}},$$

which by definition of β_k in (III.5.15) is implied by $b(k)2^{\ell_k}\geqslant C_9 2^{\ell_k}\ell_k k^{\frac{8}{\Theta d_w}}$ for some constant C_9. Comparing it to the definition of ℓ_k in (III.5.1) it holds true if we set

$$b(k):=ak^{2+\frac{8}{\Theta d_w}}m2^m, \tag{III.6.3}$$

and assume a and m are large enough. To check that $\boldsymbol{\Delta}\geqslant M_2 l^{d_w}\bar\varepsilon^{-4/\Theta}$, we use that $\boldsymbol{\Delta}\geqslant\beta_{k+1}=\mathrm{C_{mix}}\big(\frac{(k+1)^2}{\varepsilon}\big)^{4/\Theta}\big(2^{\ell_k}\big)^{d_w}$ by definition of β_{k+1} in (III.5.15), and the inequality holds as $\mathrm{C_{mix}}\geqslant M_2 8^{4/\Theta}$. We finally note that

$$\begin{aligned}
K' &= K-b(k)2^{\ell_k}\\
&= b(k)2^{\ell_k}+2^{\ell_k}\\
&\geqslant \big(2b(k-1)\big)2^{\ell_{k-1}}+2^{\ell_k},
\end{aligned}$$

which is the side length of $S_k^{\mathrm{ext}}(\iota)$.

We can therefore apply Theorem III.4.6 and we obtain a coupling between the particle system at time $\tau\beta_k$ inside $S_k^{\mathrm{ext}}(\iota)$ and a Poisson point process Ξ with intensity $\big(1-\mathfrak{d}_{k+1}\big)\mu_0(1-\varepsilon)\lambda_y$ where the inclusion of Theorem III.4.6 holds with probability at least

$$1-\sum_{y\in S_k^{\mathrm{ext}}(\iota)}e^{-M_4(1-\mathfrak{d}_k)\mu_0\lambda_y\bar\varepsilon^2\boldsymbol{\Delta}^{d_v/d_w}}.$$

Using that $\boldsymbol{\Delta} \geqslant \beta_{k+1} > \mathrm{C}_{\mathrm{mix}}\, 2^{d_w \ell_k}$ and the definitions of β_k from (III.5.15), the quantity in the previous display is bigger than

$$\begin{aligned}
&1 - \sum_{y \in S_k^{\mathrm{ext}}(\iota)} e^{-C_{10}(1-\mathfrak{d}_k)\mu_0 \lambda_y \bar{\varepsilon}^2 2^{d_v \ell_k}} \\
&\geqslant 1 - \left(2b(k-1)2^{\ell_{k-1}} + 2^{\ell_k}\right)^{d_v} e^{-C_{11}(1-\mathfrak{d}_k)\mu_0 C_\lambda^{-1} \frac{\varepsilon^2}{k^4} 2^{d_v \ell_k}} \\
&\geqslant 1 - 2^{d_v(1+\ell_k)} e^{-C_{12}(1-\varepsilon)\mu_0 C_\lambda^{-1} \frac{\varepsilon^2}{k^4} 2^{d_v \ell_k}} \\
&\geqslant 1 - \frac{1}{2} e^{-\mathrm{C}_\psi \psi_k}.
\end{aligned} \tag{III.6.4}$$

The last step holds for $k = 2$ since $\psi_1(\varepsilon, \mu_0, \ell)$ and therefore also $\psi_2(\varepsilon, \mu_0, \ell)$ is large enough by assumption; the inequality for $k > 2$ follows from it by setting a, m large enough.

To obtain $D_k^{\mathrm{ext}}(\kappa, \iota) = 1$ we need to check the confinement requirement. To this end, define a Poisson point process Ξ' made of the particles of Ξ that are confined during the time $[\tau\beta_k, (\tau+2)\beta_k]$ inside $B_{b(k-1)2^{\ell_{k-1}}}$. Using the definition of confinement from Lemma III.2.5, this happens for each particle independently with probability $\mathbb{P}(\mathrm{Conf}(B_{b(k-1)2^{\ell_{k-1}}}, 2\beta_k))$. By the thinning property of Poisson processes, Ξ' is therefore a Poisson point process with intensity measure

$$\mathbb{P}\big(\,\mathrm{Conf}(B_{b(k-1)2^{\ell_{k-1}}}, 2\beta_k)\big)(1-\mathfrak{d}_{k+1})\mu_0(1-\bar{\varepsilon})\lambda_y$$

which we can estimate using (Conf(d_w)) as being bigger than

$$\begin{aligned}
&\Big(1 - c_5 e^{-c_6\left(\frac{(b(k-1)2^{\ell_{k-1}})^{d_w}}{2\beta_k}\right)^{\frac{1}{d_w-1}}}\Big)(1-\mathfrak{d}_{k+1})\mu_0(1-\tfrac{\varepsilon}{8k^2})\lambda_y \\
&\overset{\text{(III.5.15)}}{=} \Big(1 - c_5 e^{-c_6\left(\frac{b(k-1)^{d_w}}{2\,\mathrm{C}_{\mathrm{mix}}}(\frac{\varepsilon}{k^2})^{4/\Theta}\right)^{\frac{1}{d_w-1}}}\Big)(1-\mathfrak{d}_{k+1})\mu_0(1-\tfrac{\varepsilon}{8k^2})\lambda_y \\
&\overset{\text{(III.6.3)}}{=} \Big(1 - c_5 e^{-C_{13}\left(\frac{\left(a(k-1)^{2+\frac{8}{\Theta d_w}} m 2^m\right)^{d_w} \varepsilon^{4/\Theta}}{k^{8/\Theta}\,\mathrm{C}_{\mathrm{mix}}}\right)^{\frac{1}{d_w-1}}}\Big)(1-\mathfrak{d}_{k+1})\mu_0(1-\tfrac{\varepsilon}{8k^2})\lambda_y
\end{aligned}$$

and using that $\mathrm{C}_{\mathrm{mix}} = \frac{\beta}{2^{d_w \ell}} \varepsilon^{4/\Theta} 2^{m d_w}$ which can be obtained by setting $\beta_1 = \beta$ in (III.5.15), this is bigger than

$$\Big(1 - c_5 e^{-C_{14}\left(\frac{2^{d_w \ell}}{\beta} a^{d_w}(k-1)^{2d_w} m\right)^{\frac{1}{d_w-1}}}\Big)(1-\mathfrak{d}_{k+1})\mu_0(1-\tfrac{\varepsilon}{8k^2})\lambda_y.$$

Setting m large enough with respect to ε, ℓ and β, this is then bigger than

$$\begin{aligned}
&(1-\tfrac{\varepsilon}{8k^2})(1-\mathfrak{d}_{k+1})\mu_0(1-\tfrac{\varepsilon}{8k^2})\lambda_y \\
&\geqslant (1-\tfrac{\varepsilon}{4k^2})(1-\mathfrak{d}_{k+1})\mu_0\lambda_y.
\end{aligned}$$

Conditioning on the coupling above, we obtain using a union bound that the probability that all $S_{k-1}(i')$ inside $S_k^{\mathrm{ext}}(\iota)$ have at least $(1-\mathfrak{d}_k)\mu_0 \sum_{y \in S_{k-1}(i')} \lambda_y$ particles which are confined during $[\tau\beta_k, (\tau+2)\beta_k]$ inside $B_{b(k-1)2^{\ell_{k-1}}}$ is at least

$$1 - \sum_{S_{k-1}(i') \subseteq S_k^{\mathrm{ext}}(\iota)} \mathbb{Q}\big(\Xi'(S_{k-1}(i')) \leqslant (1-\mathfrak{d}_k)\mu_0 \textstyle\sum_{y \in S_{k-1}(i')} \lambda_y\big). \tag{III.6.5}$$

Using the Chernov bound (III.A.1) with χ given by

$$\begin{aligned}
1 - &\frac{(1-\mathfrak{d}_k)\mu_0 \sum_{y\in S_{k-1}(i')} \lambda_y}{(1-\frac{\varepsilon}{4k^2})(1-\mathfrak{d}_{k+1})\mu_0 \sum_{y\in S_{k-1}(i')} \lambda_y} \\
&= \frac{(1-\frac{\varepsilon}{4k^2})(1-\mathfrak{d}_{k+1}) - (1-\mathfrak{d}_k)}{(1-\frac{\varepsilon}{4k^2})(1-\mathfrak{d}_{k+1})} \\
&\geqslant (1-\tfrac{\varepsilon}{4k^2})(1-\mathfrak{d}_{k+1}) - (1-\mathfrak{d}_k) \\
&\geqslant (\mathfrak{d}_k - \mathfrak{d}_{k+1}) - \frac{\varepsilon}{4k^2} = \frac{\varepsilon}{4k^2}
\end{aligned}$$

we obtain the following lower bound for (III.6.5):

$$\begin{aligned}
&1 - \sum_{S_{k-1}(i')\subseteq S_k^{\mathrm{ext}}(\iota)} \exp\Big\{-\frac{1}{2}(\tfrac{\varepsilon}{4k^2})^2(1-\tfrac{\varepsilon}{4k^2})(1-\mathfrak{d}_{k+1})\mu_0 \textstyle\sum_{y\in S_{k-1}(i')} \lambda_y\Big\} \\
&\overset{\text{(III.5.3)}}{\geqslant} 1 - \sum_{S_{k-1}(i')\subseteq S_k^{\mathrm{ext}}(\iota)} \exp\Big\{-\frac{\varepsilon^2}{32k^4}(1-\tfrac{\varepsilon}{4})(1-\mathfrak{d}_2)\mu_0 C_\lambda{}^{-1}(2^{\ell_{k-1}})^{d_v}\Big\} \\
&\geqslant 1 - \mathrm{C}_{\mathrm{Vol}}\big(b(k-1)+2^{\ell_k-\ell_{k-1}}\big)^{d_v} \exp\Big\{-\frac{\varepsilon^2}{32k^4}(1-\tfrac{\varepsilon}{4})(1-\tfrac{\varepsilon}{2})\mu_0 C_\lambda{}^{-1}2^{d_v\ell_{k-1}}\Big\} \\
&\geqslant 1 - \frac{1}{2}e^{-C_\psi\psi_k},
\end{aligned} \tag{III.6.6}$$

where the last inequality follows from the same argument as after (III.6.4) since ψ_1 is assumed large enough.

Combining (III.6.4) and (III.6.6) proves the claim for $1<k<\kappa$.

For $k=\kappa$ the argument is easier, as there is no need to use the mixing theorem and one can simply use (III.6.6), and prove both the conditional and unconditional statements.

For $k=1$, we recall that the event $A_1(\iota,\tau)$ was defined differently (cf. (III.5.30)) We use again the mixing Theorem to obtain a coupling with a Poisson point process Ξ which succeeds with probability (III.6.4) with the choice $k=1$. To obtain $\mathbf{1}_{E(\iota,\tau)}=1$, we recall that the event $E(\iota,\tau)$ is measurable with respect to the σ-algebra of particles inside $S_1^\eta(\iota)$, which is contained in $S_1^{\mathrm{base}}(\iota)$ by Remark III.5.1, and the particles are confined in $B_{\zeta\ell_1}$ during $[\tau\beta_1,(\tau+\eta)\beta_1]$. Using Lemma III.2.5 we obtain

$$\mathbb{P}\big(\mathrm{Conf}(B_{\zeta\ell_1},\eta\beta_1)\big) \geqslant 1 - c_5 e^{-c_6\left(\frac{(\zeta\ell_1)^{d_w}}{\eta\beta_1}\right)^{\frac{1}{d_w-1}}} \overset{\text{(III.6.2)}}{\geqslant} 1-\tfrac{3\varepsilon}{8}.$$

Hence, the Poisson point process Ξ' of the particles with $\mathrm{Conf}(B_{\zeta\ell_1},\eta\beta_1)$ has intensity at least

$$\mathbb{P}\big(\mathrm{Conf}(B_{b(k-1)2^{\ell_{k-1}}},2\beta_k)\big)(1-\mathfrak{d}_2)\mu_0(1-\bar{\varepsilon})\lambda_y \geqslant (1-\tfrac{3\varepsilon}{8})(1-\tfrac{\varepsilon}{2})\mu_0(1-\tfrac{\varepsilon}{8})\lambda_y \geqslant (1-\varepsilon)\mu_0\lambda_y,$$

and since $E(\iota,\tau)$ is increasing, we have

$$\mathbb{P}(\mathbf{1}_{E(i,\tau)}=1 \,\big|\, F, D_1^{\mathrm{base}}(\iota,\tau)=1) \leqslant 1-\nu_E\big((1-\varepsilon)\lambda, S_1^\eta, B_{\zeta\ell},\eta\beta_1\big) \leqslant e^{-\alpha_0},$$

which concludes the proof. □

Now that we have a bound on the probability that a single cell $R_k(\iota,\tau)$ is multi-scale bad, we can obtain an upper bound on the probability that all multi-scale cells in a given ScD-path are multi-scale bad. Recall the definition of the weights ψ_k in (III.6.1) and the value α_0 defined in Lemma III.6.1.

Corollary III.6.2. *Let ζ as in (III.6.2), $\psi_1 > \alpha_0$ and consider an ScD-path $\{R_{k_1}(\iota_1, \tau_1), \ldots, R_{k_z}(\iota_z, \tau_z)\}$. Then*

$$\mathbb{P}\Big(\bigcap_{j=1}^{z} \{A_{k_j}(\iota_j, \tau_j) = 0\} \Big) \leqslant e^{-C_\psi \sum_{j=1}^{z} \psi_{k_j}}.$$

where C_ψ is the constant from Lemma III.6.1.

Proof. We first need to order the cells in a temporal order. To this end, consider any order $\prec$ of the indices of the cells $1, \ldots, z$ such that if $j_1 \prec j_2$ then $\tau_{j_1}\beta_{k_{j_1}} \leqslant \tau_{j_2}\beta_{k_{j_2}}$. The corollary will be a simple consequence of Lemma III.6.1 once we prove that for every $1 \leqslant \bar{j} \leqslant z$, the cells $R_{k_j}(\iota_j, \tau_j)$ with $j \prec \bar{j}$ are $\mathcal{F}_{k_{\bar{j}}}(\iota_{\bar{j}}, \tau_{\bar{j}})$-measurable.

We therefore consider two cells $R_{k_{j_1}}(\iota_{j_1}, \tau_{j_1})$ and $R_{k_{j_1}}(\iota_{j_2}, \tau_{j_2})$ with $j_1 \prec j_2$, so that $\tau_{j_1}\beta_{k_{j_1}} \leqslant \tau_{j_2}\beta_{k_{j_2}}$. By definition of an ScD-path cells are well-separated, so $R^{\inf}_{k_{j_1}}(\iota_{j_1}, \tau_{j_1}) \cap R^{\inf}_{k_{j_2}}(i_{j_2}, \tau_{j_2}) = \varnothing$, meaning that:

- either $T^{\inf}_{k_{j_1}}(\tau_{j_1}) \cap T^{\inf}_{k_{j_2}}(\tau_{j_2}) = \varnothing$ and thus (a) is satisfied;
- or $S^{\inf}_{k_{j_1}}(\iota_{j_1}) \cap S^{\inf}_{k_{j_2}}(\iota_{j_2}) = \varnothing$ and thus (b) is satisfied.

Here, (a) and (b) are as they appear at the beginning of this subsection. Hence, using the standard chain conditioning and applying Lemma III.6.1 z-many times we obtain that

$$\mathbb{P}\Big(\bigcap_{j=1}^{z} \big\{ A_{k_j}(\iota_j, \tau_j) = 0 \big\} \Big) \leqslant \prod_{j=1}^{z} \mathbb{P}\Big(A_{k_j}(\iota_j, \tau_j) = 0 \Big| \bigcap_{\bar{j} \prec j} \big\{ A_{k_{\bar{j}}}(\iota_{\bar{j}}, \tau_{\bar{j}}) = 0 \big\} \Big) \leqslant e^{-C_\psi \sum_{j=1}^{z} \psi_{k_j}},$$

which is the desired claim. □

III.6.2 Number of ScD-paths

In the previous section we established the probability for a given path of z cells of scales $k_1, \ldots, k_z$ to be made of multi-scale bad cells. We want now to count the number of such paths. Recall the definition of ScD-path in Definition III.5.5, and of $\Omega^{\sup}_\kappa(v \to t)$ in (III.5.38). We will now give an upper bound for the number of paths in $\Omega^{\sup}_\kappa(v \to t)$, given a fixed number of cells and their scales. As we will see, κ and t are going to be linked with each other, so our first bound can omit these two values, as we are for the time working with given scales.

Lemma III.6.3. *For a fixed length $z \in \mathbb{N}$, fixed scales $k_1, \ldots, k_z$ and $v \in L_1$, the number of ScD-paths of cells of scales $k_1, \ldots, k_z$ where the extended support of the first cell contains v is at most*

$$\exp\Big\{ \frac{C_\psi}{2} \sum_{j=1}^{z} \psi_{k_j} \Big\},$$

where C_ψ is the same constant as in Lemma III.6.1.

Proof. Recall that two consecutive cells $R_{k_1}(\iota_1, \tau_1)$ and $R_{k_2}(\iota_2, \tau_2)$ in a ScD-path are either support adjacent or $R_{k_1}(\iota_1, \tau_1)$ is support connected with diagonals to $R_{k_2}(\iota_2, \tau_2)$. We will prove the result in three steps: first, we will bound the number of ScD-paths where each cell is support adjacent to the next one, i.e. we don't allow diagonal connections. In the second step, we will show the result for the case in which the beginning and end of the (scale 1) diagonal steps are fixed relative to each other; in the third step we will obtain the bound where this last restriction is removed.

Step 1. We define the maximum number of scale k' cells which are support adjacent to a cell of scale k

$$\Phi_{k,k'} := \max_{(\iota,\tau)} \left|\{R_{k'}(\iota',\tau')\colon R_k(\iota,\tau) \text{ is support adjacent to } R_{k'}(\iota',\tau')\}\right| \tag{III.6.7}$$

and the number of cells of scale k whose extended support (defined in (III.5.10) and (III.5.21)) contains v

$$\chi_k := \left|\{R_k(\iota,\tau)\colon R_k^{\mathrm{Esup}}(i,\tau) \supseteq v\}\right|, \tag{III.6.8}$$

so clearly the number of support adjacent only D-paths in $\Omega_\kappa^{\mathrm{sup}}(v \to t)$ of cells with scales $k_1, \dots k_z$ is bounded above by

$$\chi_{k_1} \prod_{j=2}^{z} \Phi_{k_{j-1},k_j}.$$

We start by deriving a bound for χ_k. Since the extended support of a cell of scale k contains at most $27\,\mathrm{C_{Vol}}(3m+1)^{d_v}$ cells of scale $k+1$, there exist at most $27\,\mathrm{C_{Vol}}(3m+1)^{d_v}$ different extended supports of a cell of scale k that contain the distinct cell of scale $k+1$ containing v, and thus v itself. By (III.5.4) and (III.5.16) each cell of scale $k+1$ contains $\frac{\beta_{k+1}}{\beta_k}2^{d_v(\ell_{k+1}-\ell_k)} \leqslant 2^{8+d_w(2ak-3a+m)+d_v(2ak-a+m)}$ cells of scale k, which is therefore also the number of scale k cells that share the same extended support. We therefore have

$$\chi_k \leqslant 27\,\mathrm{C_{Vol}}(3m+1)^{d_v} 2^{8+d_w(2ak-3a+m)+d_v(2ak-a+m)} \leqslant \exp\left\{\frac{C_\psi}{16}\psi_k\right\} \tag{III.6.9}$$

where the last inequality holds trivially for m, a and α_0 large enough.

We now bound $\Phi_{k,k'}$. A cell of scale k' can only be support adjacent to a cell $R_k(\iota_1,\tau_1)$ if it is inside $B_r(p)\times A$, where $p \in S_k(\iota_1)$, $r := (3m+2)2^{\ell_{k+1}} + (3m+2)2^{\ell_{k'+1}}$ and A an interval centered around $T_k(\tau_1)$ of width $28(\beta_{k+1}+\beta_{k'+1})$. Consequently, $\Phi_{k,k'}$ can be bounded by the number of scale k' cells inside this Cartesian product. If $k \geqslant k'$ then the terms $2^{\ell_{k'+1}}$ and $\beta_{k'+1}$ are negligible (or of the same size) in comparison to $2^{\ell_{k+1}}$ and β_{k+1}, and the spatial region contains at most $\mathrm{C_{Vol}}(2(3m+2))^{d_v}$ cells of scale $k+1$, and by (III.5.4), each one of those contains exactly $2^{d_v(\ell_{k+1}-\ell_{k'})}$ cells of scale k', so

$$\text{if } k \geqslant k' \qquad \Phi_{k,k'} \leqslant \Big(\mathrm{C_{Vol}}(2(3m+2))^{d_v} 2^{d_v(\ell_{k+1}-\ell_{k'})}\Big)\Big(56\frac{\beta_{k+1}}{\beta_{k'}}\Big)$$

If instead $k < k'$ we have similarly

$$\text{if } k < k' \qquad \Phi_{k,k'} \leqslant \Big(\mathrm{C_{Vol}}(2(3m+2))^{d_v} 2^{d_v(\ell_{k'+1}-\ell_{k'})}\Big)\Big(56\frac{\beta_{k'+1}}{\beta_{k'}}\Big).$$

Combining the two and using (III.5.16) we have that

$$\Phi_{k,k'} \leqslant C_{15} 2^{d_v(6m+4)} 2^{d_v\left(a(k\vee k')^2+m(k\vee k')\right)} 2^{d_w 2a(k\vee k')+d_w m}$$

and for a, m, α_0 large it holds trivially that this is further smaller than

$$\exp\Big\{\frac{C_\psi}{16}\psi_{(k\vee k')}\Big\}.$$

Hence we obtain with (III.6.9)

$$\chi_{k_1}\prod_{j=2}^{z}\Phi_{k_{j-1},k_j} \leqslant \prod_{j=1}^{z}\Big(e^{\frac{C_\psi}{16}\psi_{k_j}}\Big)^2 \leqslant e^{\frac{C_\psi}{8}\sum_{j=1}^{z}\psi_{k_j}}.$$

Step 2. In this step, we consider $R_{k_1}(\iota_1,\tau_1)$ to be support connected with diagonals to $R_{k_2}(\iota_2,\tau_2)$, which, as defined in Definition III.5.5, means that there exist two cells $R_1(\widetilde{\iota}_1,\widetilde{\tau}_1)$ and $R_1(\widetilde{\iota}_2,\widetilde{\tau}_2)$ contained in their respective extended supports such that $R_1(\widetilde{\iota}_1,\widetilde{\tau}_1)$ is diagonally connected to $R_1(\widetilde{\iota}_2,\widetilde{\tau}_2)$. We denote by $(\widetilde{\iota}_1-\widetilde{\iota}_2,\widetilde{\tau}_1-\widetilde{\tau}_2)$ the relative position of the cell $R_1(\widetilde{\iota}_1,\widetilde{\tau}_1)$ with respect to $R_1(\widetilde{\iota}_2,\widetilde{\tau}_2)$ and write $(0,0)$ for the relative position of the cells $R_{k_1}(\iota_1,\tau_1)$ and $R_{k_2}(\iota_2,\tau_2)$ when they are adjacent. In this step we consider the relative positions to be fixed, and we will show a bound for the number of different possible relative positions in the next step. In analogy with step 1, we define

$$\Phi^*_{k_1,k_2} := \max_{(\iota_1,\tau_1)}\left|\left\{R_{k_2}(\iota_2,\tau_2)\colon \begin{array}{l} R_{k_1}(\iota_1,\tau_1) \text{ is support adjacent or support connected} \\ \text{with diagonals to } R_{k_2}(\iota_2,\tau_2) \text{ with fixed relative} \\ \text{position of } R_1(\widetilde{\iota}_1,\widetilde{\tau}_1) \text{ with respect to } R_1(\widetilde{\iota}_2,\widetilde{\tau}_2)\end{array}\right\}\right|. \tag{III.6.10}$$

The case when the relative position is $(0,0)$ was treated in the previous step, so in that case we have

$$\Phi^*_{k_1,k_2} \leqslant e^{\frac{C_\psi}{16}\psi_{k_1\vee k_2}}.$$

In the case of diagonally connected cells, since the relative position is fixed, the possible combinations are determined by the product of all the possible positions of the cell $R_1(\widetilde{\iota}_1,\widetilde{\tau}_1)$ inside the extended support of the cell $R_{k_1}(\iota_1,\tau_1)$ and the number of cells of scale 1 contained in the extended support of $R_{k_2}(\iota_2,\tau_2)$. Using the bound from the previous step we have

$$\Phi^*_{k_1,k_2} \leqslant e^{\frac{C_\psi}{16}\psi_{k_1}}e^{\frac{C_\psi}{16}\psi_{k_2}}.$$

Combining the two equations yields

$$\Phi^*_{k_1,k_2} \leqslant e^{\frac{C_\psi}{16}\psi_{k_1}}e^{\frac{C_\psi}{16}\psi_{k_2}} + e^{\frac{C_\psi}{16}\psi_{k_1\vee k_2}}.$$

Hence the number of ScD-path where the z cells have fixed relative position is bounded by

$$\chi_{k_1}\prod_{j=2}^{z}\Phi^*_{k_{j-1},k_j} \leqslant \exp\Big\{\frac{C_\psi}{4}\sum_{j=1}^{z}\psi_{k_j}\Big\} \tag{III.6.11}$$

Step 3. In the final step, we bound the number of combinations of different relative positions in a ScD-path. For two given cells of scales k_j and k_{j+1} where the first is support connected with diagonals to the second, let $R_1(\iota_1,\tau_1)$ and $R_1(\iota_2,\tau_2)$ be the corresponding two scale 1 cells for which $R_1(\iota_1,\tau_1)$ is diagonally connected to $R_1(\iota_2,\tau_2)$ with relative position $(\iota_1-\iota_2,\tau_1-\tau_2)$. Let h be the (absolute) difference between the distances of $R_1(\iota_1,\tau_1)$ and $R_1(\iota_2,\tau_2)$ from L_0, which we refer to as "difference in height"; see the discussion below (III.2.8). Define $A(h)$ to be the number of cells that $R_1(\iota_1,\tau_1)$ can be diagonally connected to, where the "difference in height" is h. More precisely, define

$$A(h) := \max_{(\iota_1,\tau_1)} \left| \left\{ R_1(\iota_2,\tau_2) \colon \begin{array}{c} R_1(\iota_1,\tau_1) \text{ is diagonally connected to } R_1(\iota_2,\tau_2) \\ \text{with } \left| d\left(L_0, R_1(\iota_1,\tau_1)\right) - d\left(L_0, R_1(\iota_2,\tau_2)\right) \right| = h \end{array} \right\} \right|.$$

As defined, $A(h)$ is also an upper bound on the number of different relative positions $(\iota_1-\iota_2,\tau_1-\tau_2)$ which result in a height difference of h.

We next note that, by definition of the diagonal steps, we can bound $A(h)$ by the number of cells of scale 1 at distance h from a given cell of scale 1. Recalling ($\mathrm{Vol}(d_v)$), we can therefore use the very generous bound

$$A(h) < \mathrm{C_{Vol}}\, h^{d_v+1}, \tag{III.6.12}$$

where the $+1$ term comes from having to also consider the time dimension.

Recall from Subsection III.5.3 that when a scale 1 cell is diagonally connected to another scale 1 cell, the height of the second cell can be at most that of the first cell. We can thus obtain easily an upper bound on the number of diagonal steps and equivalently on the total height difference. Define H_k as the side length of S_k^{Esup} divided by the side length of S_1, that is

$$H_k := (3m+1)2^{ak^2+mk}. \tag{III.6.13}$$

Then, using that a diagonal step by definition leads to a decrease of the distance to L_0, the maximum number of diagonal steps in an ScD-path of cells of scales $k_1,\dots,k_z$ is at most the combined distance from L_0 that the cells of scales $k_1,\dots,k_z$ can contribute to an ScD-path, i.e.

$$H = \sum_{i=1}^{z} H_{k_i}.$$

Hence, the number of different configurations of the diagonal steps, and in particular different relative positions, is at most

$$\sum_{l=0}^{H} \sum_{\substack{h_2,\dots h_z \\ h_2+\cdots+h_z=l}} A(h_2+1)A(h_3+1)\dots A(h_z+1),$$

where h_i represent the (absolute) height difference between the i-th and $(i-1)$-th cell; the $+1$ accounts for the fact that the final scale 1 cell of a diagonal connection might be adjacent and not equal to the next cell of the path, as per definition of being diagonally connected. Using the method of Lagrange multipliers, this is smaller than

$$\sum_{l=0}^{H} \sum_{\substack{h_2,\dots h_z \\ h_2+\cdots+h_z=l}} \left(A\Big(\frac{l}{z-1}+1\Big)\right)^{z-1}.$$

Using (III.6.12) and that the total number of combinations of $z-1$ values $h_i \geqslant 0$ which sum to l is $\binom{l+z-2}{z-2}$, this is smaller still than

$$\begin{aligned} &\sum_{l=0}^{H} \binom{l+z-2}{z-2} \mathrm{C_{Vol}} \left(\tfrac{l}{z-1}+1\right)^{(z-1)(d_v+1)} \\ \leqslant &\sum_{l=0}^{H} \binom{l+z-1}{z-1} \mathrm{C_{Vol}} \left(\tfrac{l}{z-1}+1\right)^{(z-1)(d_v+1)} \end{aligned}$$

and using repeatedly Pascal's rule we can further bound this by

$$\begin{aligned}&\binom{z+H}{z} \mathrm{C}_{\mathrm{Vol}} \left(\tfrac{H}{z-1}+1\right)^{(z-1)(d_v+1)}\\ &\leqslant \frac{(z+H)^z}{z!} C_{16}\left(\tfrac{H}{z-1}+1\right)^{(z-1)(d_v+1)}.\end{aligned}$$

Since $\frac{H}{z}$ is big by the assumption that ψ_1 is large enough, we finally get that this is smaller than

$$\begin{aligned}&\frac{(z+H)^z}{(z/3)^z} C_{16}\left(\tfrac{3H}{z}\right)^{(z-1)(d_v+1)}\\ &\leqslant (3+3H/z)^z C_{16}\left(\tfrac{3H}{z}\right)^{z(d+1)}\\ &\leqslant \left(C_{17}\tfrac{H}{z}\right)^{2z(d+1)}\end{aligned}$$

for some constant $C_{17}>0$ depending only on d; we used in the first inequality that $d_v \leqslant d$, which is a simple consequence of the fact that the graph can be embedded into the d-dimensional triangular lattice which has volume growth dimension d. To obtain that $(C_{17}H/z)^{2z(d+1)} \leqslant \exp(\frac{C_\psi}{8}\sum_{j=1}^z \psi_{k_j})$ and thus to conclude *Step 3* and the proof, we can equivalently show that

$$(d+1)(\log(C_{17}H/z) \leqslant \frac{1}{z}\frac{C_\psi}{8}\sum_{j=1}^{z}\psi_{k_j}. \tag{III.6.14}$$

Comparing H_k from (III.6.13) and ψ_k from (III.6.1) and setting m and α_0 (and thus ℓ) large enough we can obtain $H_k \leqslant \frac{C_\psi}{8(d+1)C_{17}}\psi_k$ for all k, and therefore (III.6.14) holds. □

In the previous two lemmas, we showed the relationship between ScD-paths and the sum of the weights ψ_k. We show now that if we consider an ScD-path in $\Omega_\kappa^{\sup}(v \to t)$ (defined in (III.5.38)) of cells of scales $k_1,\ldots,k_z$ for some $t>0$, then the sum of the weights ψ_k is at least of order t^{c_s}.

Lemma III.6.4. *Suppose that the largest scale κ we consider satisfies $\kappa = \mathcal{O}\big(\sqrt{\log(t)}\big)$. Then, if ψ_1 is large enough, there exist t_0 and $C_{18}>0$ such that for any $t>t_0$, $v \in L_1$ and any path $\{R_{k_j}(\iota_j,\tau_j)\}_{j=1}^z \in \Omega_k^{\sup}(v\to t)$*

$$\sum_{j=1}^{z}\psi_{k_j} \geqslant C_{18}t^{c_s},$$

where the positive constant c_s is as defined in Theorem III.2.13.

Proof. Let diam_k denote the diameter of the extended support of a cell of scale k.

The key observation to prove the lemma is that

$$\sum_{j=1}^{z}\mathrm{diam}_{k_j} \geqslant \frac{t}{2} \tag{III.6.15}$$

since by definition of $\Omega_\kappa^{\sup}(v\to t)$ in (III.5.38) the path exits from $B_t(S_1(\iota_v)) \times [-t+\tau_v,\tau_v+t]$ and with an argument similar to the one surrounding (III.6.13), the distance that can be covered by diagonal steps is at most the sum of the side lengths of the cells. Therefore, we only need to compare diam_k with ψ_k.

For the geometry of the fractal, the diameter of the tile is equal to the side length; hence, for $1 \leqslant k \leqslant \kappa$, we note that

$$\begin{aligned}\mathrm{diam}_k &\leqslant (6m+3)2^{\ell_{k+1}} + 27\beta_{k+1} \\ &\leqslant (6m+3)2^{ak+a+m}2^{\ell_k} + (\mathrm{C_{mix}}\, 2^{\ell_k})^{d_w} \\ &\leqslant C_{19}2^{2m}\ 2^{ak}2^{d_w\ell_k} \\ &\leqslant C_{19}2^{2m+ak}2^{(d_v+1)\ell_k},\end{aligned}$$

where in the last step we made use of (III.2.6). For $k \geqslant 2$

$$\begin{aligned}\psi_k &= \frac{\varepsilon^2\mu_0 2^{d_v\ell_{k-1}}}{k^4} \\ &= \frac{\varepsilon^2\mu_0 2^{d_v\ell_k}}{k^4 2^{d_v(ak-a+m)}} \\ &= \frac{\varepsilon^2\mu_0}{k^4 2^{d_v(ak-a+m)}}\frac{1}{\left(C_{19}2^{2m+ak}\right)^{\frac{d_v}{d_v+1}}}\left(C_{19}2^{2m+ak}2^{(d_v+1)\ell_k}\right)^{\frac{d_v}{d_v+1}} \\ &\geqslant \frac{\varepsilon^2\mu_0}{C_{20}k^4 2^{d_v(ak+2m)}}\left(\mathrm{diam}_k\right)^{\frac{d_v}{d_v+1}}.\end{aligned}$$

For $k = 1$ we can fix a constant $c_{11} > 0$ depending on $\varepsilon, \mu_0, a, m, \ell$ and ν_E, but crucially not on t, such that $\psi_1 \geqslant c_{11}(\mathrm{diam}_1)^{d_v/(d_v+1)}$.

Since we assumed that $\kappa = \mathcal{O}(\sqrt{\log(t)})$, we have that there exists c_{12} such that $k \leqslant c_{12}\sqrt{\log(t)}$ for all $k \leqslant \kappa$ and thus summing over all cells of the path, (III.6.15) gives

$$\sum_{j=1}^{z}\psi_j \geqslant C_{21}\frac{\varepsilon^2\mu_0}{\log^2(t)2^{d_v a\sqrt{\log(t)}}}t^{\frac{d_v}{d_v+1}}$$

which for t large is larger than $C_{18}t^{c_s}$. □

III.6.3 Size of bad clusters

Let $t > 0$ large, $v \in L_1$ and define

$$\mathbf{S}_k^t(v) := \left\{S_k(\iota') \colon \iota' \in \mathbb{B}^d,\ S_k(\iota') \cap B_t(S_1(\iota_v)) \neq \varnothing\right\},$$

$$\mathbf{T}_k^t(v) := \left\{T_\kappa(\tau') \colon \tau' \in \mathbb{Z},\ \exists \bar{\tau} \in \mathbb{Z} \colon \gamma_1^{(k-1)}(\bar{\tau}) = \tau',\ T_1(\bar{\tau}) \cap [\tau, \tau+t] \neq \varnothing\right\}$$

and

$$\mathbf{R}_k^t(v) := \left\{S \times T \colon S \in \mathbf{S}_k^t(v), T \in \mathbf{T}_k^t(v)\right\},$$

where ι_v, τ_v and $B_t(S_1(\iota_v))$ are as defined previously below (III.5.37). Recall also the definition of the bad cluster K_v from (III.5.39).

Proposition III.6.5. *Let ζ as in (III.6.2), α_0 as in Lemma III.6.1 and t_0 as in Lemma III.6.4. Then there exists a constant C_{22} independent of t such that for any $v \in L_1$*

$$\mathbb{P}\big(K_v \nsubseteq \mathbf{R}_1^t(v)\big) \leqslant e^{-C_{22}t^{c_s}}, \tag{III.6.16}$$

for all $t > t_0$.

Proof. Using Lemma III.5.3

$$\begin{aligned}\mathbb{P}\big(K_v \notin \mathbf{R}_1^t(v)\big) &\leqslant \mathbb{P}\big(\exists P \in \Omega_1(v \to t) \text{ of bad cells}\big)\\ &\leqslant \mathbb{P}\big(\exists P \in \Omega_1(v \to t) \text{ of cells with bad ancestry}\big)\end{aligned}$$

and by Lemma III.5.6 this is smaller than

$$\mathbb{P}\big(\exists P \in \Omega_\kappa^{\sup}(v \to t) \text{ of multi-scale bad cells}\big),$$

for any arbitrary choice of κ; we will fix it momentarily.

Define now the event H_κ to be the event that $A_\kappa(\iota,\tau)$ holds for all cells in $\mathbf{R}_\kappa^t(v)$, i.e.

$$H_\kappa := \bigcap_{R_\kappa(\iota,\tau)\in\mathbf{R}_\kappa^t(v)} \{A_\kappa(\iota,\tau) = 1\}$$

Recalling how the event $A_\kappa(\iota,\tau)$ is defined in (III.5.32) for the largest scale κ, using a union bound and Lemma III.6.1 we obtain directly that

$$\mathbb{P}\big(H_\kappa(v)\big) \geqslant 1 - \big|\mathbf{R}_\kappa^t(v)\big| e^{-C_\psi \psi_\kappa}.$$

We choose now κ to be the smallest integer such that $\psi_\kappa \geqslant t$. Using the definition of ψ_k in (III.6.1) one can see that $\kappa = \mathcal{O}(\sqrt{\log(t)})$; note that this choice satisfies the assumption of Lemma III.6.4. Since the cardinality of $\mathbf{R}_\kappa^t(v)$ satisfies

$$\big|\mathbf{R}_\kappa^t(v)\big| \leqslant C_{23}\Big(\frac{t}{2^{\ell_k}}\Big)^{d_v}\Big(\frac{t}{\beta_k}\Big),$$

we can use this to find some constant c_{13} such that

$$\mathbb{P}\big(H_\kappa(v)\big) \geqslant 1 - e^{c_{13}t}.$$

We now continue the previous chain of inequalities

$$\begin{aligned}&\mathbb{P}(\exists P \in \Omega_\kappa^{\sup}(v \to t) \text{ of multi-scale bad cells})\\ &\leqslant \mathbb{P}(\exists P \in \Omega_\kappa^{\sup}(v \to t) \text{ of multi-scale bad cells} \cap H_\kappa(v)) + \mathbb{P}\big(H_\kappa(v)^{\mathsf{c}}\big) \qquad \text{(III.6.17)}\\ &\leqslant \mathbb{P}(\exists P \in \Omega_{\kappa-1}^{\sup}(v \to t) \text{ of multi-scale bad cells}) + e^{-c_{13}t}.\end{aligned}$$

Since $c_s < \frac{d_v}{d_v+1} - \frac{1}{2} < 1$, the term $e^{-c_{13}t}$ is of a smaller order than the claimed bound of $e^{-C_{22}t^{c_s}}$, so we can ignore it from here on out.

We now want to bound the remaining probability. If we fix the length of the path $z \in \mathbb{N}$ and the scales $k_1,\dots,k_z$ we can use Corollary III.6.2 and Lemma III.6.3 to obtain

$$\begin{aligned}&\mathbb{P}\big(\exists P \in \Omega_{\kappa-1}^{\sup}(v \to t) \text{ of } z \text{ multi-scale bad cells of scales } k_1,\dots,k_z\big)\\ &\leqslant\ e^{-C_\psi \sum_{j=1}^z \psi_{k_j}} e^{\frac{C_\psi}{2}\sum_{j=1}^z \psi_{k_j}} = e^{-\frac{C_\psi}{2}\sum_{j=1}^z \psi_{k_j}} \leqslant e^{-\frac{C_\psi}{2}C_{18}t^{c_s}},\end{aligned}$$

where the last step follows from Lemma III.6.4 since κ and therefore also $\kappa - 1 = \mathcal{O}(\sqrt{\log(t)})$.

It only remains to estimate the number of different possible lengths and weights of a path. We rewrite the weight of a path as the sum of the weights of cells of different scales, namely $\sum_{j=1}^z \psi_{k_j} = \sum_{k=1}^{\kappa-1} h_k\psi_k$, where h_k is the number of cells of scale k. Hence, for fixed $h_1, \dots, h_{\kappa-1}$, the number of possible ways to order the cells is

$$\frac{(h_1 + \cdots + h_{\kappa-1})!}{h_1! h_2! \dots h_{\kappa-1}!} = \binom{h_1 + \cdots + h_{\kappa-1}}{h_1}\binom{h_2 + \cdots + h_{\kappa-1}}{h_2} \cdots \binom{h_{\kappa-1}}{h_{\kappa-1}}. \qquad \text{(III.6.18)}$$

By the bounds provided by Lemma III.6.4, there exists $k \in \{1, \dots, \kappa - 1\}$ such that $h_k \geqslant \frac{C_{18} t^{c_s}}{(\kappa-1)\psi_k}$. Define now

$$\mathcal{H} := \Big\{(h_1, \dots, , h_{\kappa-1}) \in (\mathbb{N}_0)^{\kappa-1} \colon \exists l \in \{1, \dots, \kappa - 1\}\ h_l \geqslant \tfrac{C_{18} t^{c_s}}{(\kappa-1)\psi_l}\Big\}.$$

We can then write

$$\begin{aligned}
&\mathbb{P}(\exists P \in \Omega^{\sup}_{\kappa-1}(v \to t) \text{ of multi-scale bad cells}) \\
&\leqslant \sum_{\mathcal{H}} \mathbb{P}\Big(\exists P \in \Omega^{\sup}_{\kappa-1}(v \to t) \colon \begin{array}{l}\text{such that for each } k = 1, \dots \kappa - 1,\ P \text{ is made} \\ \text{of } h_k \text{ multi-scale bad cells of scale } k\end{array}\Big) \\
&\leqslant \sum_{\mathcal{H}} e^{-\frac{C_\psi}{2}\sum_{k=1}^{\kappa-1} h_k \psi_k} \frac{(h_1 + \cdots + h_{\kappa-1})!}{h_1! h_2! \dots h_{\kappa-1}!}.
\end{aligned}$$

Applying (III.A.2) $\kappa - 1$ times to the right-hand side of (III.6.18) we can bound this further by

$$\sum_{\mathcal{H}} e^{-\sum_{k=1}^{\kappa-1} h_k (\frac{C_\psi}{2}\psi_k - k)},$$

and using that α_0, a, m are large enough twice, this is finally smaller than

$$\sum_{\mathcal{H}} e^{-\frac{C_\psi}{3}\sum_{k=1}^{\kappa-1} h_k \psi_k} \leqslant e^{-C_{22} t^{c_s}},$$

which concludes the proof. □

III.6.4 Proof of Theorem III.2.12

Proof. By Proposition III.3.4 we need to show for all $v \in L_1$ that

$$\sum_{r \geqslant 1} r^{d_v+1} P(\mathrm{rad}_v(\mathrm{H}_v) > r) < \infty.$$

Recalling the definition of $\mathbf{R}^t_k(v)$ above Proposition III.6.5 and letting $v = R_1(\iota_v, \tau_v)$, we note that $\mathbf{R}^t_1(v)$ contains only cells $R_1(\iota', \tau')$ with $d(S_1(\iota_v), S_1(\iota')) \leqslant \frac{t}{2^\ell}$ and $|\tau_v - \tau'| \leqslant \frac{t}{\beta}$. Hence, if r, T satisfy

$$T\Big(\frac{1}{2^\ell} + \frac{1}{C_{24} 2^{d_w \ell}}\Big) \leqslant r$$

for some constant C_{24}, it holds that

$$\mathbf{R}^T_1(v) \subseteq \big\{R_1(\iota', \tau') \colon (\iota', \tau') \in \mathbb{B}^d \times \mathbb{Z},\ d\big(R_1(\iota', \tau'), v\big) \leqslant r\big\}.$$

Define therefore $T(r) := (\frac{1}{2^\ell} + \frac{1}{C_{24}2^{d_w\ell}})^{-1}r$, and let t_0 be as in Lemma III.6.4 and r_0 such that $T(r_0) > t_0$. Then

$$\begin{aligned}\sum_{r\geqslant r_0} r^{d_v+1}P(\mathrm{rad}_v(\mathrm{H}_v) > r) &\leqslant \sum_{r\geqslant r_0} r^{d_v+1}P\left(\mathrm{H}_v \nsubseteq \mathbf{R}_1^{T(r)}(v)\right)\\ &\leqslant \sum_{r\geqslant r_0} r^{d_v+1}P\left(K_v \nsubseteq \mathbf{R}_1^{T(r)}(v)\right)\\ &\overset{\text{(III.6.16)}}{\leqslant} \sum_{r\geqslant r_0} r^{d_v+1}\exp\big\{-C_{22}T(r)^{c_s}\big\}.\end{aligned}$$

Since this series converges, the Lipschitz cutset exists almost surely as stated in Proposition III.3.4. □

III.7 Proof of Theorem III.2.13

The main tool for the proof of Theorem III.2.13 are DↃ-paths, which we define next. They are in essence a symmetric version of D-paths, in the sense that diagonal connections can go "backwards"; equivalently, being connected by a DↃ-path is a symmetric relationship unlike before.

III.7.1 DↃ-paths

Recall from Subsection III.5.3 the definitions of adjacent cells; we repeat the definition of being diagonally connected: we say for two scale 1 cells $R_1(\iota,\tau)$ and $R_1(\iota',\tau')$ that $R_1(\iota,\tau)$ is *diagonally connected* to $R_1(\iota',\tau')$ if there exists a sequence of scale 1 cells $\{R_1(\iota_1,\tau_1),\dots,R_1(\iota_n,\tau_n)\}$ such that $R_1(\iota,\tau) = R_1(\iota_1,\tau_1)$, for all $j \in \{1,\dots,n-1\}$, $d(R_1(\iota_{j+1},\tau_{j+1}),L_0) < d(R_1(\iota_j,\tau_j),L_0)$ and $R_1(\iota_n,\tau_n)$ is either equal or adjacent to $R_1(\iota',\tau')$. In addition, we define here two cells to be *diagonally linked* if the first case occurs, i.e. if $R_1(\iota_n,\tau_n) = R_1(\iota',\tau')$.

We say that two scale 1 cells $R_1(\iota,\tau)$ and $R_1(\iota',\tau')$ are *single diagonally connected* if $R_1(\iota,\tau)$ is diagonally connected to $R_1(\iota',\tau')$ or if $R_1(\iota',\tau')$ is diagonally connected to $R_1(\iota,\tau)$. We say that two scale 1 cells $R_1(\iota,\tau)$ and $R_1(\iota',\tau')$ are double diagonally connected if there exists $R_1(\widetilde{\iota},\widetilde{\tau})$ such that $R_1(\iota,\tau)$ is diagonally connected to $R_1(\widetilde{\iota},\widetilde{\tau})$, $R_1(\iota',\tau')$ is diagonally connected to $R_1(\widetilde{\iota},\widetilde{\tau})$, and either $R_1(\iota,\tau)$ or $R_1(\iota',\tau')$ is diagonally linked to $R_1(\widetilde{\iota},\widetilde{\tau})$. Note that being single diagonally connected or double diagonally connected is a symmetric relationship.

As done in Subsection III.5.3, we extend these new definitions to cells of arbitrary scale $R_{k_1}(\iota_1,\tau_1)$ and $R_{k_2}(\iota_2,\tau_2)$ by requiring that they respectively contain two scale 1 cells which satisfy the corresponding definition of the connectedness above. In analogy to Definition III.5.4 we introduce a new type of paths.

Definition III.7.1. We define a DↃ-path as a sequence $\{R_{k_j}(\iota_j,\tau_j)\}_{j=1}^n$ of cells where for all $j = 2,\dots,n$, the cells $R_{k_{j-1}}(\iota_{j-1},\tau_{j-1})$ and $R_{k_j}(\iota_j,\tau_j)$ are either adjacent, single diagonally connected or double-diagonally connected.

Similarly to (III.5.37), for some $t > 0$ and $v \in L_1$, we define

$$\Omega Ↄ_1(v \to t) \tag{III.7.1}$$

to be the set of all DↃ-paths of cells of scale 1 for which the first cell of the path is v or v is single diagonally connected to the first cell, and the last cell is the only cell not contained in the space-time ball $B_t(S_1(\iota_v)) \times [-t+\tau_v, \tau_v + t]$. We stress that, contrary to $\Omega_1(v \to t)$, v must not necessarily be part of the Dᗡ-path; it can be that v is only single diagonally connected to the path and not an actual cell of the Dᗡ-path.

We define now ScDᗡ-paths, the support connected version of Dᗡ-paths. Recall the definition of well-separated cells and support adjacent cells from Definition III.5.5. We say that two cells $R_{k_1}(\iota_1, \tau_1)$ and $R_{k_2}(\iota_2, \tau_2)$ are *support connected with single diagonal* if there exist two scale 1 cells respectively contained in the extended supports of $R_{k_1}(\iota_1, \tau_1)$ and $R_{k_2}(\iota_2, \tau_2)$ which are single diagonally connected. Similarly, we say that two cells $R_{k_1}(\iota_1, \tau_1)$ and $R_{k_2}(\iota_2, \tau_2)$ are *support connected with double diagonal* if there exist two scale 1 cells respectively contained in the extended supports of $R_{k_1}(\iota_1, \tau_1)$ and $R_{k_2}(\iota_2, \tau_2)$ which are double diagonally connected.

Definition III.7.2. We define as ScDᗡ-path (support connected Dᗡ-path) a sequence of well-separated cells $\{R_{k_j}(\iota_j, \tau_j)\}_{j=1}^z$ for some $z \in \mathbb{N}$ where for all $j = 2, \ldots, z$ the cells $R_{k_{j-1}}(\iota_{j-1}, \tau_{j-1})$ and $R_{k_j}(\iota_j, \tau_j)$ are either support adjacent, support connected with single diagonal or support connected with double-diagonals.

For $t > 0$ and $v \in L_1$, we define

$$\Omega\text{ᗡ}_\kappa^{\text{sup}}(v \to t)$$

the set of all ScDᗡ-paths of cells of scale at most κ so that the extended support of the first cell of the path contains v or v is single diagonally connected to a scale 1 cell that is contained in the extended support of the first cell, and the last cell is the only cell whose extended support is not contained in the space-time ball $B_t(S_1(\iota_v)) \times [-t+\tau_v, \tau_v+t]$. Again, we highlight the difference with $\Omega_\kappa^{\text{sup}}(v \to t)$, where instead v must be contained in the extended support, whereas here it can be only single diagonally connected to it.

Finally we define the analogue of the bad cluster K_v from (III.5.39):

$$K_v^* := \{R_1(\iota', \tau') \colon \text{there exists a Dᗡ-path of bad cells from } v \text{ to } R_1(\iota', \tau')\} \tag{III.7.2}$$

Repeating the arguments of Lemma III.5.6, we can easily obtain its analogue for Dᗡ-paths.

Lemma III.7.3. *It holds that*

$$\begin{aligned}&\mathbb{P}\big(\exists P \in \Omega\text{ᗡ}_1(v \to t) \text{ of cells with bad ancestry}\big)\\ &\leqslant \mathbb{P}\big(\exists P \in \Omega\text{ᗡ}_\kappa^{\text{sup}}(v \to t) \text{ of multi-scale bad cells}\big).\end{aligned}$$

III.7.2 Multi-scale analysis of Dᗡ-paths

We want to show that the Lipschitz cutset intersects the *base* L_0 within distance r from the origin with high probability. If the opposite was true, then we would be able to find a nearest-neighbor path in $L_1 \backslash F$ which leaves a ball of radius r. We will show that this implies the existence of a Dᗡ-path from the origin that exits such a ball and we will use similar arguments to before to prove such Dᗡ-paths are improbable.

We follow the structure of Section III.6 and write in detail only the parts where the proofs for Dᗡ-paths differ from the ones for D-paths. Lemma III.6.1 and Corollary III.6.2 still hold and and can be applied unchanged. We need to show the analogue of Lemma III.6.3.

Lemma III.7.4. *For a fixed length $z \in \mathbb{N}$, fixed scales $k_1, \dots, k_z$ and $v \in L_1$, the number of ScDⱭ-paths of cells of scale $k_1, \dots, k_z$ where the first cells either contains v or is v is single diagonally connected to a scale 1 cell contained in the extended support of the first cell, is at most*

$$\exp\Big\{\frac{C_\psi}{2}\sum_{j=1}^{z}\psi_{k_j}\Big\},$$

where C_ψ is the same constant as in Lemma III.6.1.

Proof. We follow the proof of Lemma III.6.3. For *Step 1*, we need to make a small change. Compare the definitions of $\Omega_\kappa^{\mathrm{sup}}(v \to t)$ and $\Omega\text{Ɑ}_\kappa^{\mathrm{sup}}(v \to t)$: in the latter we also allow v to be single diagonally connected to a scale 1 cell contained in the extended support of the first cell in the DⱭ-path. To account for this, note that we can fix the relative position of v and the scale 1 cell in the extended support of the first cell in the DⱭ-path, and we are only left to control the number of the possible relative position which is done in *Step 3.*

Step 2 remains unchanged, and we can turn to *Step 3.*

Consider two consecutive cells in the DⱭ-path which are single diagonally connected. We can define similarly to before

$$A(h) := \max_{(\iota_1,\tau_1)}\left|\left\{R_1(\iota_2,\tau_2) \colon \begin{array}{c} R_1(\iota_1,\tau_1) \text{ is single diagonally connected to } R_1(\iota_2,\tau_2) \\ \text{with } \big|d(L_0, R_1(\iota_1,\tau_1)) - d(L_0, R_1(\iota_2,\tau_2))\big| = h \end{array}\right\}\right|.$$

For two cells $R_1(\iota_1,\tau_1)$ and $R_1(\iota_2,\tau_2)$ in the DⱭ-path which are double diagonally connected, let $R_1(\tilde\iota,\tilde\tau)$ be the cell of the double diagonal that $R_1(\iota_1,\tau_1)$ or $R_1(\iota_2,\tau_2)$ is diagonally linked to. Letting h_1 be the height difference between $R_1(\iota_1,\tau_1)$ and $R_1(\tilde\iota,\tilde\tau)$ and h_2 the height difference between $R_1(\iota_2,\tau_2)$ and $R_1(\tilde\iota,\tilde\tau)$, we can upper bound the number of different relative positions between $R_1(\iota_1,\tau_1)$ and $R_1(\iota_2,\tau_2)$ for which the respective height differences to $R_1(\tilde\iota,\tilde\tau)$ are h_1 and h_2 by $A(h_1{+}1)A(h_2{+}1)$.

Let H_k be as in (III.6.13); similarly to what was done for D-paths, we can bound the total number of diagonal steps in a DⱭ-path with the maximal attainable distance from L_0, within the path, i.e. by

$$H = 2\sum_{i=1}^{z} H_{k_i},$$

where we added the factor 2 to account for the diagonal step to the *previous* and the *following* cell. For simplicity, when two cells are double diagonally connected we consider also the cell $R_1(\tilde\iota,\tilde\tau)$, to which both cells are diagonally connected as part of the path. So, letting h_i, $i = 1, \dots, 2z-1$ be the height difference between two diagonally connected cells, the number of diagonal steps is at most

$$\sum_{l=0}^{H}\sum_{\substack{h_1,\dots h_{2z-1} \\ h_1+\dots+h_{2z-1}=l}} A(h_1+1)A(h_2+1)\dots A(h_{2z-1}+1).$$

We can then repeat the remaining calculations as in Lemma III.6.3, substituting z with $2z$ and obtain the same result. $\square$

We also have the analogue of Lemma III.6.4:

Lemma III.7.5. *Suppose that the largest scale κ satisfies $\kappa = \mathcal{O}(\sqrt{\log(t)})$. Then if ψ_1 is large enough, there exist t_0 and $C_{18} > 0$ such that for any $t > t_0$ and any $v \in L_1$ and any ScD$\mathcal{C}$-path $\{R_{k_j}(\iota_j, \tau_j)\}_{j=1}^z \in \Omega\mathcal{C}_k^{\sup}(v \to t)$*

$$\sum_{j=1}^{z} \psi_{k_j} \geqslant C_{18} t^{c_s}.$$

Proof. The proof is unchanged from the one of Lemma III.6.4 except that in (III.6.15) we have to substitute $t/2$ with $t/3$ since we now consider 2 diagonals for each cell instead of only one. The rest remains identical. □

Recall now the definition of K_v^* in (III.7.2). The analogue of Proposition III.6.5 is then argued in the same way.

Proposition III.7.6. *Let ζ as in (III.6.2), α_0 as in Lemma III.6.1 and t_0 as in Lemma III.6.4. Then there exists a constant C_{22} independent of t such that for any $v \in L_1$*

$$\mathbb{P}(K_v^* \nsubseteq \mathbf{R}_1^t(v)) \leqslant e^{-C_{22} t^{c_s}},$$

for all $t > t_0$, with c_s as in Theorem III.2.13.

Recall the concept of hills from Definition III.3.3. In the following, we will say that two hills H_{v_1} and H_{v_2} are adjacent if there exist $v_j' \in \mathrm{H}_{v_j}$, $j = 1, 2$ that are adjacent, and call them *intersecting* if there exists $\tilde{v} \in \mathrm{H}_{v_1} \cap \mathrm{H}_{v_2}$.

Lemma III.7.7. *Let F be the Lipschitz cutset from Theorem III.2.12. Let $\pi = \{u_j\}_{j=0}^n$ with $u_j \in L_1 \backslash F$ be a sequence of sequentially pairwise adjacent cells.*

Then there exists a sequence of hills $\mathcal{H} := \{\mathrm{H}_{v_j}\}_{j=0}^k$, $k \leqslant n$, such that every u_j is contained in some hill $\mathrm{H}_{j'}$ and two consecutive hills of the sequence are either adjacent or intersecting.

Furthermore there exists a D$\mathcal{C}$-path which starts in u_0 and ends in u_n.

Proof. We start with the first claim. For each $u_j \in \pi$, we have by assumption that $u_j \notin F$, so there exists a hill $\mathrm{H}_{v_j} \ni u_j$. Furthermore, for all $j = 1, \ldots, n$, u_{j-1} and u_j are adjacent and so the respective hills $\mathrm{H}_{v_{j-1}}$ and H_{v_j} are either adjacent or they intersect. The sequence of hills $\{\mathrm{H}_{v_j}\}_{j=0}^n$ may contain repetitions of the same hills, so by removing all but the first appearance of those which appears multiple times, we end up with a sequence of $k \leqslant n$ different elements.

We prove now the existence of the D$\mathcal{C}$-path. Consider the sequence of hills $\{\mathrm{H}_{v_j}\}_{j=0}^k$ from the previous step, and denote with $\overleftarrow{v}_j \in \mathrm{H}_{v_j}$ for $j = 1, \ldots, n$ the cell (chosen in some arbitrary manner, for example lexicographically) that is either contained in or adjacent to a cell contained in $\mathrm{H}_{v_{j-1}}$. By definition of a hill, there exist a d-path from v_0 to u_0 and a d-path from v_0 to either $\overleftarrow{v}_1$ or to a cell adjacent to it. Similarly, there exist a d-path from v_j to $\overleftarrow{v}_j$ and a d-path from v_j to $\overleftarrow{v}_{j+1}$ (or a cell adjacent to it). Repeating this, we obtain a sequence of cells

$$u_0, v_0, \overleftarrow{v}_1, v_1, \overleftarrow{v}_2, \ldots, v_k, u_n$$

where for each pair of consecutive cells there exists a d-path from the first to the second or from the second to the first.

Note that, just like D-paths, d-paths are also D$\mathcal{C}$-paths. Secondly, if a certain sequence is a D$\mathcal{C}$-path, then the reverse sequence is also a D$\mathcal{C}$-path, as a simple consequence of the fact that being adjacent, single diagonally connected or double diagonally connected is a symmetric relation. Thirdly, if there exist a DꟷD-path from a cell u_1 to u_2 and one from u_2 to u_3 we can concatenate them and obtain a DꟷD-path from u_1 to u_3.

We can thus construct a DꟷD-path for the sequence $u_0, v_0, \overleftarrow{v}_1, v_1, \overleftarrow{v}_2, \dots, v_k, u_n$, concluding the lemma. $\square$

We can now prove Theorem III.2.13.

Proof of Theorem III.2.13. By Theorem III.2.12, a Lipschitz cutset F exists a.s., so we need to show that it surrounds the origin at some distance r. Suppose the converse.

This means that there exists a sequence of cells $\{u_j\}_{j=0}^n$ with $u_j := R_1(\iota_j, \tau_j) \in L_1 \backslash F$ and such that $u_0 = R_1(0,0)$ and $d(u_n, u_0) > r$. Applying Lemma III.7.7 we obtain the existence of a DꟷD-path from $R_1(0,0)$ to u_n.

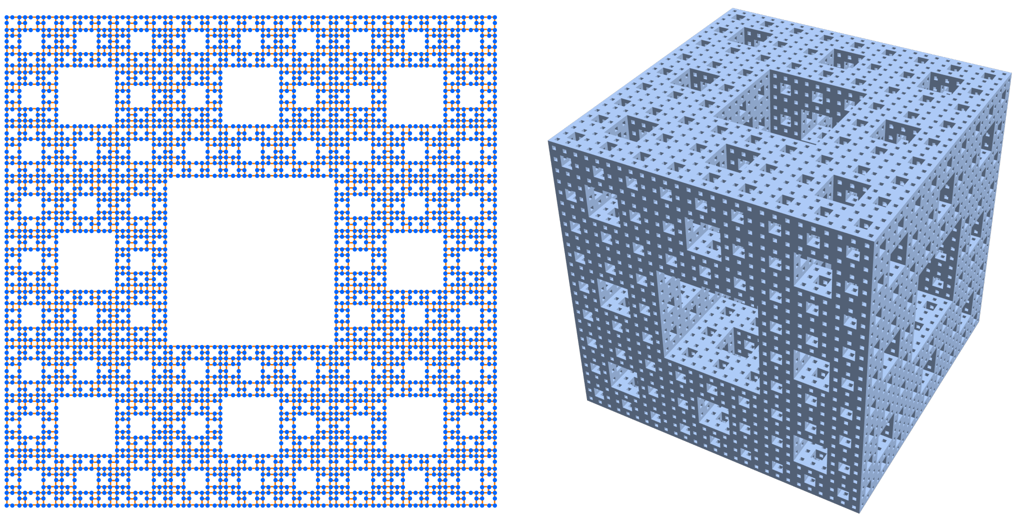
By Proposition III.7.6, for $t > t_0$, the probability that such a path exists is smaller than

$$\mathbb{P}(K^*_{(0,0)} \nsubseteq \mathbf{R}_1^t(0,0)) \leqslant e^{-C_{22}t^{c_s}}.$$

Setting again $t = (\frac{1}{2^\ell} + \frac{1}{C_{24}2^{d_w \ell}})^{-1} r$ as in the proof in Subsection III.6.4 concludes the proof for $r_0 := (\frac{1}{2^\ell} + \frac{1}{C_{24}2^{d_w \ell}}) t_0$. $\square$

III.8 Generalized Sierpiński carpets

In this section we show how to adapt the previous arguments for the Sierpiński gasket to a further class of fractal graphs, the Sierpiński carpets. We start by introducing the graph and then stating the results. As we will see, other than changes to constants and parameters, the work done for the gasket can be applied mostly without further changes necessary, so we will only highlight selected statements to show how they work in the carpet case.



(a) $d = 2$ (b) $d = 3$

Figure III.8: Examples of generalized Sierpiński Carpet.

III.8.1 Setup and statement

We consider the class of fractal graphs of [BB99b]. We state the definition for completeness and refer to [BB99b] for more details.

Let $d \geqslant 2$, $l_F \geqslant 3$, and $1 \leqslant m_F \leqslant (l_F)^d$. Next, let $F_0 := [0,1]^d$ and for $n \in \mathbb{Z}$ $\mathcal{S}_n$ be the collection of closed cubes of side $(l_F)^n$ and corner vertices in the lattice $(l_F)^n\mathbb{Z}^d$. For $A \subseteq \mathbb{R}^d$ let $\mathcal{S}_n(A) := \{S \in \mathcal{S}_n : S \subseteq A\}$. For $S \in \mathcal{S}_n$, let Ψ_S be the orientation preserving affine map which maps F_0 onto S.

Let F_1 be the union of m_F distinct cubes of $\mathcal{S}_{-1}(F_0)$ satisfying the following conditions:

(H1) *Symmetry*: F_1 is preserved by all the isometries of F_0.

(H2) *Connectedness*: the interior $\mathrm{Int}(F_1)$ is connected, and contains a path connecting the hyperplane $\{x_1 = 0\}$ and $\{x_1 = 1\}$.

(H3) *Non-diagonality*: For any cube B in F_0 which is the union of 2^d distinct elements of $\mathcal{S}_{-1}$, if $\mathrm{Int}(F_1 \cap B)$ is non-empty, it is connected.

(H4) *Borders included:* F_1 contains the segment $\{x : 0 \leqslant x_1 \leqslant 1, x_2 = \dots, = x_d = 0\}$.

Given F_n, F_{n+1} is obtained by removing the same pattern from each of the squares in $\mathcal{S}_{-n}(F_n)$, so that F_{n+1} is the union of $(m_F)^n$ squares in $\mathcal{S}_{-n}(F_0)$; formally

$$F_{n+1} := \bigcup_{S \in \mathcal{S}_{-n}(F_n)} \Psi_S(F_1)$$

and $F := \bigcap_{n=0}^{\infty} F_n$ is called a *generalized Sierpiński carpet.* The Hausdorff dimension of F is $d_v := \frac{\log(m_F)}{\log(l_F)}$ (see [BB99b]) and references therein). We now define the pre-fractal graph.

For any cube S_{-n}, call the *lower-left corner* the vertex x with $x_i \leqslant y_i$ for each $i = 1, \dots, d$ and $y \in S_{-n}$. Let $\square_n$ be the collection of lower-left corners of the cubes in $(l_F)^n F_n$, and

$$V := \bigcup_{n=0}^{\infty} \square_n,$$

see Figure III.9.

We define the *generalized Sierpiński carpet graph* $\mathbb{SC}^d := \mathbb{SC}^d(l_F, m_F)$ as the graph with vertex set V and edges $E := \{\{x,y\} \in V \times V : \|x-y\|_1 = 1\}$.

Similarly to Sierpiński gaskets, one can easily prove the volume estimate

$$\mathrm{c}_{\mathrm{vol}}\, r^{d_v} \leqslant \mathrm{Vol}_r(x) \leqslant \mathrm{C}_{\mathrm{Vol}}\, r^{d_v}, \tag{III.8.1}$$

with $d_v := \frac{\log(m_F)}{\log(l_F)}$. Theorem 1.5 in [BB99b] shows that upper and lower bounds for the heat kernels (HKB(d_v, d_w)) hold for some value d_w. Similarly to gaskets, applying [GT02, Theorem 3.1] gives that the mean exit time satisfies

$$E_x[H_{B_r(x)^c}] \asymp r^{d_w}, \tag{III.8.2}$$

and that the parabolic Harnack inequality (PH(d_w)) with parameter d_w holds. Furthermore Lemma III.2.5 also holds due to the above.

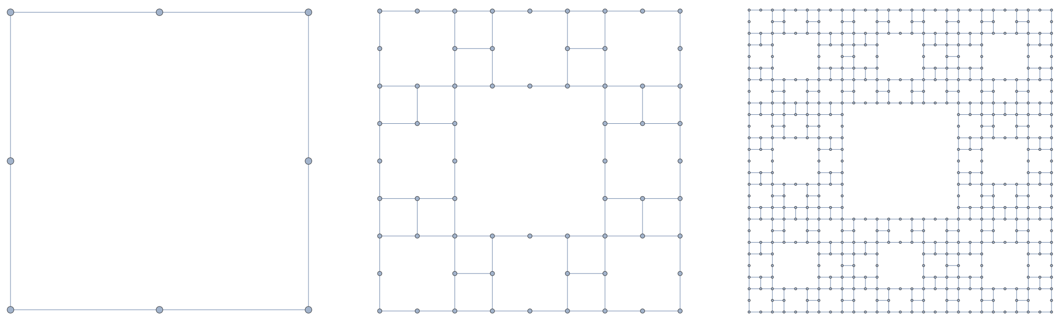


Figure III.9: $\square_1$, $\square_2$ and $\square_3$ with $d = 2, l_F = 3$, $m_F = 8$ and corresponding edges. Note that the first 2 pictures are scaled up by a factor of 3^2 and 3^1 respectively.

Now that the graph has been defined, we can define the tessellation of the carpets, in order to formulate the analogues of Theorems III.2.12 and III.2.13. We define the tiles $S_k(\iota)$ as

$$S_k(\iota) := \iota(l_F)^{\ell_k} + \square_{\ell_k},$$

$\iota \in \mathbb{SC}^d$ which is the union of $l_F^{d_v(\ell_k - \ell_{k-1})}$-many $(k-1)$ tiles.

Define just like before β_k to be

$$\beta_k := \mathrm{C}_{\mathrm{mix}}(\tfrac{k^2}{\varepsilon})^{\frac{4}{\Theta}} \big(l_F^{\ell_{k-1}}\big)^{d_w}$$

with the walk dimension d_w from (III.8.2) and we define the time interval $T_k(\tau)$, $\tau \in \mathbb{Z}$, as before. Similarly, we define space-time cells as the cross product of spatial tiles with the time intervals.

Like in the gasket case, we define L_0 and L_1 as in (III.2.7) and (III.2.8) to be the "hyperplane" subgraph and its corresponding collection of cells. Note that in order to define L_0, one needs to consider a subgraph $\mathbb{SC}^{d-1}(l_F, m_F)$ with the same l_F but an appropriately changed m_F. As an example, in the case of the 3 dimensional Sierpiński carpet from Figure III.8, m_F must be changed from 20 in $d = 3$ to $m_F = 8$ in $d = 2$.

We define two scale 1 cells $R_1(\iota_1, \tau_1)$ and $R_1(\iota_2, \tau_2)$ to be *adjacent* if $d(\iota_1, \iota_2) + |\tau_1 - \tau_2| = 1$. With this adaptation, we can define the Lipschitz cutset F as in Definition III.2.9, and state the main theorem.

Remark III.8.1. The change in how adjacency is defined is due to the "disjoint" nature of how the pre-fractal is constructed (recall that with the gasket, the corners of the triangles were shared). With this new definition of adjacency, we recover the same behaviour in the sense that two cells are adjacent if either they are spatially the same and only one time interval away from each other, or if they share the time interval and are spatially nearest neighbours, i.e. have norm 1 distance equal to 1.

Theorem III.8.2. *Let $d \geqslant 2$, $l_F \geqslant 3$, $1 \leqslant m_F \leqslant (l_F)^d$ and $\mathbb{SC}^d(l_F, m_F)$ be a $d-$dimensional generalized Sierpiński carpet. Let $\ell \in \mathbb{N}$ and let $\beta \in \mathbb{N}$ be large enough. Furthermore, let $\eta \in \mathbb{N}$, $\varepsilon \in (0,1)$ and $\zeta \in (0,\infty)$ such that*

$$\zeta \geqslant \frac{1}{\ell} \sqrt[d_w]{\big[\tfrac{1}{c_6} \log\big(\tfrac{8c_5}{3\varepsilon}\big)\big]^{d_w - 1} \eta\beta},$$

and tessellate $\mathbb{G}^d \times \mathbb{Z}$ into space-time cells as described above, and let $E(\iota, \tau)$ be an increasing event restricted to the super cell $R_1^\eta(\iota, \tau)$ whose associated probability

$\nu_E\big((1-\varepsilon)\mu, S_1^\eta(\iota,\tau), B_{\zeta\ell}, \eta\beta\big)$ *has a uniform lower bound across all* $(\iota,\tau) \in \mathbb{SC}^d \times \mathbb{Z}$ *denoted with*

$$\nu_E\big((1-\varepsilon)\mu, S_1^\eta, B_{\zeta\ell}, \eta\beta\big).$$

Then there exists α_0 *such that if*

$$\psi_1(\varepsilon,\mu_0,\ell) := \min\left\{\frac{\varepsilon^2\mu_0 2^{d_v\ell}}{C_\lambda}, -\log\Big(1-\nu_E\big((1-\varepsilon)\lambda, S_1^\eta, B_{\zeta\ell}, \eta\beta\big)\Big)\right\} \geqslant \alpha_0$$

there exists almost surely a Lipschitz cutset F *where the event* $E(\iota,\tau)$ *holds for all* $(\iota,\tau)\in F$.

Furthermore there exists $C_4 > 0$ *such that for* r_0 *large enough*

$$\mathbb{P}\big(S(F,r_0)^{\mathsf{c}}\big) \leqslant \sum_{r\geqslant r_0} r^{d_v+1} e^{-C_4 r^{c_s}},$$

for $c_s \in \big(0, \frac{d_v}{d_v+1} - \frac{1}{2}\big)$ *and* $S(F,r_0)$ *was defined above Theorem III.2.13.*

III.8.2 Proof of Theorem III.8.2

To adapt the proof, only a single notable change beyond the changes in the preceding definitions is necessary. Similar to those, this change is essentially substituting the base 2 that appeared in the gasket case with l_F, as we have seen in the definitions of $S_k(\iota)$ and β_k. From here onward we will repeatedly:

$$\text{Substitute every base 2 exponential with a base } l_F \text{ exponential.} \qquad \text{(Subst)}$$

Recall the definition of adjacent scale 1 cells above Remark III.8.1. We generalize this to cells of arbitrary scale: two cells $R_k(\iota_1,\tau_1)$ and $R_k(\iota_2,\tau_2)$ of the same scale are called *adjacent* if $d(\iota_1,\iota_2) + |\tau_1-\tau_2| \leqslant 1$, where $d(\cdot,\cdot)$ is as before the graph distance. Seeing $\mathbb{SC}^d\times\mathbb{Z}$ as a subgraph of $\mathbb{Z}^{d+1}$, we define two cells $R_k(\iota_1,\tau_1)$ and $R_k(\iota_2,\tau_2)$ to be $*$*-neighbors* if $\|(\iota_1,\tau_1)-(\iota_2,\tau_2)\|_\infty \leqslant 1$. We next define d-paths for carpets.

Definition III.8.3 (d-path)**.** A *d-path* in $\mathbb{G}^d\times\mathbb{Z}$ is a sequence $\{u_k\}_{k=0}^n$ of $*-$neighboring cells in $\mathbb{SC}^d\times\mathbb{R}$ from a bad cell $u_0\in L_1$ such that for each u_k and u_{k+1} one of the following holds:

- *increasing move*: u_{k+1} is bad and $d(L_0,u_{k+1}) \geqslant d(L_0,u_k)$
- *diagonal move*: $d(L_0,u_{k+1}) < d(L_0,u_k)$

We say for two scale 1 cells $R_1(\iota,\tau)$ and $R_1(\iota',\tau')$ that $R_1(\iota,\tau)$ is *diagonally connected* to $R_1(\iota',\tau')$ if there exists a sequence of $*$-neighbor scale 1 cells $\{R_1(\iota_1,\tau_1),\dots,R_1(\iota_n,\tau_n)\}$ such that $R_1(\iota,\tau)=R_1(\iota_1,\tau_1)$, for all $j\in\{1,\dots,n-1\}$, $d(R_1(\iota_{j+1},\tau_{j+1}),L_0) < d(R_1(\iota_j,\tau_j),L_0)$ and $R_1(\iota_n,\tau_n)$ is either equal or adjacent to $R_1(\iota',\tau')$.

Lemma III.5.6 then still applies using the change (Subst). Similarly, the definition of ψ_k is subject to (Subst). In this way, Lemma III.6.1 can be proven in the same way by again applying the mixing Theorem III.4.6 with the choices

$$\begin{aligned}
K &:= \text{side length of } S_k^{\text{base}}(\iota) = 2b(k)l_F^{\ell_k} + l_F^{\ell_k},\\
K' &\text{ such that } K - K' = b(k)l_F^{\ell_k},\\
l &:= l_F^{\ell_k},\\
\delta &:= (1 - \mathfrak{d}_{k+1})\mu_0,\\
\boldsymbol{\Delta} &:= \text{length}([\gamma_k^{(1)}(\tau)\beta_{k+1}, \tau\beta_k]) = \tau\beta_k - \gamma_k^{(1)}(\tau)\beta_{k+1}, \text{ and}\\
\bar{\varepsilon} &:= \frac{\varepsilon}{8k^2}.
\end{aligned}$$

Lemmas III.6.3 and III.6.4, Proposition III.6.5 and the proof of Theorem III.2.12 go through by applying (Subst), and therefore the first half of Theorem III.8.2 is shown.

Similarly, Section III.7 can be proven in the same way after using (Subst) and in particular we obtain the bound

$$\mathbb{P}\big(S(F, r_0)^{\mathsf{c}}\big) \leqslant \sum_{r \geqslant r_0} r^{d_v+1} e^{-C_4 r^{c_S}}.$$

III.9 Survival of the infection

We now give an application of the Lipschitz cutset framework to show that for an infection with recovery on a particle system as defined in Subsection III.2.4, the infection survives indefinitely with positive probability.

Consider either the Sierpiński gasket $\mathbb{G}^d$ or a generalized Sierpiński carpet $\mathbb{SC}^d(l_F, m_F)$ and the particle system defined in Subsection III.2.4 given by a Poisson point process with intensity $\mu(x) := \mu_0\lambda_x$. Assume furthermore that at time 0, there is an infected particle at the origin of the graph[2]. We next describe the dynamics of the infection.

Any particle of the process gets instantaneously infected when it shares a site with an infected particle. For a second parameter $\gamma > 0$, suppose that an infected particle recovers independently at rate γ, but can get infected again afterwards. In particular, we allow for a particle to get immediately reinfected if it recovers while sharing a site with an infected particle, i.e. recovery is impossible when a particle shares a site with a different particle. However, our application works also in the case where infections can only occur when particles change sites, i.e. when a healthy particle jumps to a site with an infected particle or vice versa. To model recovery, consider a collection of Poisson point processes $(R_\gamma^{x,n})_{x\in\mathbb{G}^d, n\in\mathbb{N}}$ on $\mathbb{R}^+$ with intensity γ, which we refer to as the *recovery marks*. As in [BS23], we view the process $R_\gamma^{x,n}$ as the recovery marks of the random walk $(X_t^{x,n})_{t\geqslant 0}$, where $X_t^{x,n}$ is the location of n-th particle located at x at time 0 at time t. A particle $(X_t^x)_t$ recovers at time s if it is alone i.e. $\Pi_s(X_s^x) = 1$ and $s \in R_\gamma^{x,n}$.

We say that the infection survives if for every $t > 0$ there exists at least one infected particle at time t somewhere on the graph. We denote with P_μ^γ the distribution of the process with intensity μ and recovery rate γ.

Proposition III.9.1. *For any $\mu_0 > 0$ there exists $\gamma_0 > 0$ such that for all $0 < \gamma < \gamma_0$ the infection survives with positive probability.*

[2]The choice of the site where the infection starts is arbitrary as all of the bounds we use are uniform across the graph. Note however that the local geometry of the origin is in fact different from that of any other site in the graph.

We will follow the approach introduced in [GS19b] and refined in [BS23]. To prove the result we will define a suitable event $E(\iota,\tau)$ and apply Theorem III.2.12. We will then be able to infer from the definition of $E(\iota,\tau)$ and the connectivity properties of the Lipschitz cutset that the infection survives indefinitely almost surely once the infection has entered the Lipschitz cutset, therefore surviving indefinitely as long as the infection does not recover before this. The event $E(\iota,\tau)$ will then consist of two phases - in the first phase we will use (some) of the already infected particles to infect a sufficiently large number of the particles in the cell $R_1(\iota,\tau)$. In the second phase, we will use these newly infected particles to propagate the infection to the surrounding cells.

Fix the value $\ell \in \mathbb{N}$ and consider a value β, depending on ℓ, so that the ratio $\frac{2^{d_w \ell}}{\beta}$ is fixed. We define $T := 2^{\ell(d_w - \frac{1}{3})}$ the time point between the two phases.

Define the following condition: we say that a cell $R_1(\iota,\tau)$ is *acceptable* if

(A1) for every $x \in S_1(\iota,\tau)$ with $\Pi_{\tau\beta}(x) > 0$ there exists a path denoted with π^x, which starts at x, and does not exit the super-tile $S_1^3(\iota)$ and has no recovery marks up to time $\tau\beta + T$.

(A2) for each $S_1(\iota') \subseteq S_1^3(\iota)$ and each $x \in S_1(\iota)$ with $\Pi_{\tau\beta}(x) > 0$, there exists a particle which stays inside the super-tile $S_1^3(\iota)$ and does not have any recovery marks up to time $(\tau+1)\beta$, is inside $S_1(\iota')$ at time $(\tau+1)\beta$ and intersects[3] the path π^x during the time interval $[\tau\beta, \tau\beta + T]$.

We now claim

$$\mathbb{P}_\mu^\gamma\big(R_1(\iota,\tau) \text{ satisfies (A1), (A2)}\big) \geqslant 1 - \exp\left\{C_{25}\mu_0 e^{-\gamma\beta} 2^{\frac{\ell/3}{d_w-1}}\right\}, \qquad \text{(III.9.1)}$$

the proof of which we relegate to Appendix III.C since it is an easy adaptation of the work done in [GS19b; BS23].

Remark III.9.2. One might be tempted to think that using the event

$$E(\iota,\tau) := \{R_1(\iota,\tau) \text{ is acceptable}\}$$

and Theorem III.2.12 would yield our claim. This would be true if the infection were to enter the Lipschitz cutset from the time dimension[4]. Then by definition of acceptable, the infection enters from the time dimension in all cells in $R_1^3(\iota,\tau)$ appearing in (A2), including the one in the Lipschitz cutset due to Corollary III.3.5, and thus survives indefinitely. The next definition takes care of the case in which the infection does not enter from the time dimension when it first enters the Lipschitz cutset.

For each cell $R_1(\iota,\tau)$ and each $x \in S_1(\iota)$ fix an independent realization of a random walk path $(\pi_s^x)_{s\in[0,\tau\beta]}$ with $\pi_0^x = x$. We say that a cell $R_1(\iota,\tau)$ is *decent* if

(D3) for every $x \in S_1(\iota)$ the path π_s^x has no recovery marks and for every jump time t of $(\pi_s^x)_{s\in[0,\tau\beta]}$ there exists a tile $S_1(\iota') \subseteq S_1^1(\iota)$ such that

[3] We say that a particle intersects a path if the path and the particle path intersect in space-time, i.e. have the same position at the same time at least once.

[4] The infection enters a cell $R_1(\iota,\tau)$ from the time dimension if there is an infected particle in $S_1(\iota)$ at time $\tau\beta$. We say that the infection enters the cell $R_1(\iota,\tau)$ from the spatial dimension, if there are no infected particles inside $S_1(\iota)$ at time $\tau\beta$ and there is an infected particle which enters $S_1(\iota)$ at some time $t \in (\tau\beta, (\tau+1)\beta)$.

(D3a) if $t < (\tau+1)\beta - T$ there exists a particle which has no recovery marks and stays inside $R_1^1(\iota,\tau)$, is at time $(\tau+1)\beta$ inside $S_1(\iota')$ and intersects the path $(\pi^x_{s-t})_{s\in[t,t+T]}$ during the time interval $[t, t+T]$;

(D3a) if $(\tau+1)\beta - T \leqslant t \leqslant (\tau+1)\beta$ it holds $\pi^x_{(\tau+1)\beta-t} \in S_1(\iota')$.

We refer again to Appendix III.C for the proof of

$$\mathbb{P}^\gamma_\mu(R_1(\iota,\tau) \text{ is decent}) \geqslant 1 - \exp\{-C_{26}\beta\} - \exp\{-C_{27}\gamma\beta\} - \exp\left\{C_{28}\mu_0 e^{-\gamma\beta} 2^{\frac{\ell/3}{d_w-1}}\right\}, \tag{III.9.2}$$

as the arguments remain very similar to [BS23].

Remark III.9.3. We note that unlike done in [BS23], where the authors introduce a single random walk path π^0 for each space-time cell, which they then translate to x as needed, our graphs lack translation invariance and we must therefore consider different paths for each x. This however has no bearing on the rest of the argument.

Proof of Theorem III.9.1. We introduce an alternative construction of the process using the additional paths $(\pi^x_s)_s$. We fix the tessellation and observe a cell $R_1(\iota,\tau)$. If at time $\tau\beta$ there are infected particles inside $S_1(\iota)$, we do not use the paths (π^x_s), $x \in S_1(\iota)$. If instead there are no infected particles in $S_1(\iota)$ at $\tau\beta$, we observe the process on adjacent tiles and consider the first infected particle which enters the tile $S_1(\iota)$ at some site y during $T_1(\tau)$, if it exists, and let this particle follow the path π^y_s until $(\tau+1)\beta$ or until it the same rule applies for some adjacent cell, whichever happens first. Then, as simple concatenations of random walks, with this new construction the process maintains the same distribution as the original process.

We can now define the event

$$E(\iota,\tau) := \{\text{all cells } R_1(\iota',\tau') \text{ adjacent to } R_1(\iota,\tau) \text{ are acceptable and decent}\}$$

Then the event $E(\iota,\tau)$ is increasing, restricted to the super-cell $R_1^4(\iota,\tau)$ and using the volume estimates (Vol(d_v)) for ℓ large enough and γ small enough we can find α_0 such that $P^\gamma_\mu(E(\iota,\tau)) \geqslant 1 - e^{-\alpha_0}$.

Then Theorem III.2.12 gives the existence of a Lipschitz cutset F° where the event $E(\iota,\tau)$ holds and Theorem III.2.13 gives that it surrounds the origin at some finite distance r almost surely, hence an initially infected particle starting at the origin has a positive probability of entering a cell in F° before recovery.

Suppose that this infected particle enters the Lipschitz cutset from the time dimension: then it suffices to consider (A1) and (A2) to obtain that the infection spreads to all cells in $R_1^1(\iota,\tau)$. Since by Corollary III.3.5 for every cell $R_1(\iota,\tau)$ in F° there exists a cell $R_1(\iota',\tau+1) \subseteq F^\circ$ with $d(S_1(\iota), S_1(\iota')) = 0$, by definition of *acceptable* cells once the infection enters the Lipschitz cutset it spreads to neighboring cells inside F°. Since this observation can then be inductively repeated, the infection now survives almost surely by spreading along cells of F°.

Suppose instead that the infected particle enters a *decent* cell $R_1(\iota,\tau)$ from the spatial dimension. Since the cell is *decent*, the infection spreads to at least one cell $R_1(\iota',\tau') \subseteq R_1^1(\iota,\tau)$ which is *acceptable* by the definition of $E(\iota,\tau)$. Note that this cell might not necessarily be part of F°. However, since it is *acceptable* it spreads the infection to all cells $R_1(\iota'',\tau'') \subseteq R_1^3(\iota',\tau')$. By Corollary III.3.5 and since $\eta = 3$ there exists in particular at least cell $R_1(\iota'',\tau'') \subseteq R_1^3(\iota',\tau')$ that is inside F°. By definition of acceptable cells, the infection enters this cell from the time dimension, and the infection survives indefinitely by the previous argument.

Since every cell of F^{o} is acceptable and decent by construction and the Lipschitz cutset surrounds the origin at almost surely finite distance, this yields the claim. $\square$

III.A Standard Results

Lemma III.A.1 (Chernoff Bound)**.** *Let P a Poisson random variable with parameter λ. Then, for $\delta \in (0,1)$*

$$\mathbb{P}(P(\lambda) < (1-\delta)\lambda) < e^{-\lambda\frac{\delta^2}{2}} \tag{III.A.1}$$

Lemma III.A.2. *Let $x, y \in \mathbb{N}$. Then, for any $a, b > 1$*

$$\binom{x+y}{y} e^{-ax-by} \leqslant e^{-(a-1)x-(b-1)y} \tag{III.A.2}$$

III.B Volume estimates for Sierpiński gasket graph

Lemma III.B.1. *Let $\mathbb{G}^d$, $d \geqslant 2$ the Sierpiński gasket. There exists $\mathrm{c}_{\mathrm{vol}}, \mathrm{C}_{\mathrm{Vol}} > 0$, such that for all $x \in \mathbb{G}^d$, $r \geqslant 1$ it holds*

$$\mathrm{c}_{\mathrm{vol}}\ r^{d_v} \leqslant \mathrm{Vol}_r(x) \leqslant \mathrm{C}_{\mathrm{Vol}}\ r^{d_v}, \tag{III.B.1}$$

Proof. We generalize the proof in dimension 2 from [Bar98]. For notation convenience, call any translation of $\triangle^d_n$ a "n-triangle".

First observe that any n-triangle contains

$$\binom{d+1}{2}(d+1)^n + \binom{d+1}{2}$$

vertices, which can be verified by induction observing that $\triangle^d_0 := \triangle^d$ has $d+1$ vertices, and when constructing $\triangle^d_{n+1}$ from $\triangle^d_n$ we place $d+1$ copies of $\triangle^d_n$ but we identify $\binom{d+1}{2}$ couples of them since they are in the same position.

For $r \geqslant 1$, let n such that $2^n < r \leqslant 2^{n+1}$.

We start with the upper bound. For any x, $B_r(x)$ can intersect at most $(d+2)$ $(n+1)$-triangles, which are the $(n+1)$-triangle which contains x and its $d+1$ neighbors. So

$$\begin{aligned}|B_r(x)| \leqslant (d+2)\ |\ \triangle^d_{n+1}\ | \leqslant (d+2)\Big(\big(\frac{d+1}{2}\big)(d+1)^{n+1} + \big(\frac{d+1}{2}\big)\Big)\\ < (d+2)^3(d+1)^n \leqslant (d+2)^3 2^{d_v n} < (d+2)^3 r^d_v\end{aligned}$$

For the lower bound, since every n-triangle has diameter 2^n, $B_r(x)$ must contain at least one n-triangle. So

$$|B_r(x)| \geqslant |\ \triangle^d_n\ | > \frac{1}{2}(d+1)^{n+1}\frac{1}{2}2^{d_v(n+1)} \geqslant \frac{1}{2}r^{d_v},$$

which complete the proof. □

III.C Probability of acceptable and decent cells

In this appendix we prove equations (III.9.1) and (III.9.2) adapting the proofs of [GS19b; BS23]. Recall that the ratio $\frac{2^{d_w \ell}}{\beta}$ is fixed and that $T := 2^{\ell(d_w - \frac{1}{3})}$.

Acceptable. We start by showing (III.9.1).

Lemma III.C.1 ([GS19b, Lemma 2]). *Assume that the particles in $S_1(\iota)$ are a Poisson point process of intensity $c_{14}\mu_0\lambda_x$ for some $c_{14} > 0$. For $x \in S_1(\iota)$, let π^x a path of an (infected) particle which starts in x and stays inside $S_1^3(\iota)$ during $[\tau\beta, \tau\beta + T]$. Then, for ℓ large enough, the number of particles in $S_1^3(\iota)$ at time $\tau\beta$ which intersect $\pi^x$by time $\tau\beta + T$ is a Poisson random variable with mean at least $C_{29}\mu_0 2^{\ell(\frac{1/3}{d_w-1})}$*

Proof. The proof is a simple adaptation of [GS19b, Lemma 2], using (HKB(d_v, d_w)) and splitting time into sub-intervals of length $W := 2^{\ell(d_w - \frac{1}{3} - \frac{1/3}{d_w-1})}$. □

Lemma III.C.2 ([GS19b, Lemma 3]). *Given a set of $N \in \mathbb{N}$ particles in $S_1^3(\iota)$ at time $\tau\beta + T$ and a tile $S_1(\iota') \subseteq S_1^3(\iota)$, the probability that at least one of the N particles is in $S_1(\iota')$ at time $(\tau + 1)\beta$ is at least $1 - \exp\{-Nc_p\}$ for some constant $c_p > 0$ and ℓ large enough.*

Proof. One can define a suitable binomial variable B with parameters N and $p \in (0, 1)$, the latter being the minimal probability for a particle to be in $S_1(\iota')$ after moving for $\beta - T$ amount of time, so that the probability in the statement is at least $\mathbb{P}(B \geqslant 1) \geqslant 1 - \exp\{-Np\}$. The estimate $p > c_p$ then follows from applying (HKB(d_v, d_w)) in the time interval $[T, (\tau + 1)\beta]$. □

With the help of Lemma III.2.5, we can combine the previous two statements with the help of Chernoff's bound into the following result.

Lemma III.C.3 ([GS19b, Lemma 4]). *Assume that the particles inside $S_1^3(\iota)$ at time $\tau\beta$ are a Poisson process of intensity $c_{14}\mu_0\lambda_x$ and let π^x be the path from Lemma III.C.1. The probability that at time $(\tau + 1)\beta$ there is at least one particle in every tile $S_1(\iota') \subseteq S_1^3(\iota)$ which intersected π^x during $[\tau\beta, \tau\beta + T]$ is at least $1 - \exp\{-C_{30}\mu_0 2^{\frac{\ell/3}{d_w-1}}\}$.*

Lemma III.C.3 with the use of a simple union bound across all paths π^x for $x \in S_1^3(\iota)$ and (Conf(d_w)) for (A1) yields

$$\begin{aligned}\mathbb{P}^0_\mu(R_1(\iota,\tau) \text{ satisfies (A1), (A2)}) &\geqslant 1 - \sum_{x\in S_1^3(\iota)} \Big(c_5 \exp\Big\{ - c_6 2^{\frac{\ell/3}{d_w-1}}\Big\} + \exp\Big\{C_{30}\mu_0 2^{\frac{\ell/3}{d_w-1}}\Big\}\Big)\\ &\geqslant 1 - \exp\Big\{ - C_{31}\mu_0 2^{\frac{\ell/3}{d_w-1}}\Big\}.\end{aligned}$$

Applying a further thinning on all of the particles appearing in the previous arguments (as done in detail in [BS23, Lemma 3.1]), preventing them from recovering during the time interval $[\tau\beta, (\tau+1)\beta]$, one obtains the analogous result with recovery (III.9.1).

Decent. We now bound the probability of a cell to be decent and show (III.9.2). The probability that a path has no recovery marks during an interval of length β is $e^{-\gamma\beta}$ and it holds for any random walk that

$$\mathbb{P}\big(\operatorname{Conf}(B_R, \Delta)\big) \geqslant 1 - C_{32}R^{d_v} \exp\Big\{ - C_{33}\frac{R^2}{\Delta}\Big\}, \tag{III.C.1}$$

(see for example [GT01, (4.1)]).

We now evaluate the probability of (D3b) for fixed x, t. We observe the time interval $[t, (\tau+1)\beta]$: if the length $(\tau+1)\beta - t$ is bigger then 2^ℓ we can apply Lemma III.2.5; if instead $(\tau+1)\beta - t < 2^\ell$ then we can apply (III.C.1) with $R = 2^\ell$ and $\Delta \leqslant 2^\ell$, which yields a lower bound of $1 - \exp\{-C_{34} 2^\ell\}$. All together

$$\mathbb{P}\big(\text{the pair } \pi^x, t \text{ satisfy (D3b)}\big) \geqslant 1 - c_5 \exp\{-c_6 2^{\frac{\ell/3}{d_w - 1}}\} - \exp\{-C_{34} 2^\ell\}.$$

For (D3a), we adapt a strategy similar to acceptable cells. Lemma III.C.1 still applies. Lemma III.C.2 still holds as before if $(\tau+1)\beta - t - T > 2^\ell$, if instead $(\tau+1)\beta - t - T < 2^\ell$, we need to use (III.C.1) instead of (HKB(d_v, d_w)) in the proof of Lemma III.C.2. Then Lemma III.C.3 applies with appropriately modified exponential bounds. Hence, for fixed x and t the probability of (D3a) under $\mathbb{P}^0_\mu$ is at least $\exp\{C_{35}\mu_0 2^{\frac{\ell/3}{d_w-1}}\}$.

Note now that the probability that a path has no recovery marks during an interval of length β is $e^{-\gamma\beta}$. The probability that a path jumps more than 3β times during a time interval of length β is bounded by $e^{-\beta}$ by a simple Poisson bound. Combined, we obtain

$$\begin{aligned}\mathbb{P}^0_\mu(R_1(\iota,\tau) \text{ is decent}) \geqslant 1 - \sum_{x \in S_1(\iota)} \Big(& e^{-\gamma\beta} + e^{-\beta} + 3\beta \exp\big\{-C_{35}\mu_0 2^{\frac{\ell/3}{d_w-1}}\big\} \\ & + 3\beta c_5 \exp\{-c_6 2^{\frac{\ell/3}{d_w-1}}\} + 3\beta \exp\{-C_{34} 2^\ell\}\Big).\end{aligned}$$

With the thinning property of Poisson point processes we can adapt the calculation for the recovery marks as in [BS23], and (Vol(d_v)) then yields (III.9.2) for ℓ large enough since the ratio $\frac{\ell^{d_w}}{\beta}$ is fixed.

Chapter IV

Conclusion

In this thesis we investigated mostly two models, namely the Gaussian free field on supercritical Galton–Watson trees and Poisson random walks on fractal graphs.

The statements about the critical parameter h_* and the stability under small perturbation are interesting but not at all exhaustive. In this precise setting of weighted Glaton–Watson trees further research may investigate the speed of the random walk on the infinite cluster and the critical parameters h_{**} and $\bar{h}$. An other possible research path is the question about the relation of independent and dependent percolation, as we sketched in Figure I.1, and it would be extremely interesting to pursue the rigorous proof of the mantra "positive correlation makes percolation easier", in both current and related settings: first of all to show the conjecture that the critical parameter associated to the independent field is smaller than the one relative to the positive correlated field for all values of m, and more ambitiously to understand if this holds on other graphs.

For the model of Poisson random walks we constructed the Lipschitz cutset, the analogous of the Lipschitz Surface for fractal graph. This object allowed us to obtain the survival of the infection for small intensity of the recovery parameter, but we believe that other consequences could be inferred from it. For example, in $\mathbb{Z}^d$ it was possible to obtain (see [BS23]) that the infection survives locally, i.e. the set of times for which the origin contains an infected particle is unbounded. Still on $\mathbb{Z}^d$, it was shown in [GS19b] that the infection spreads with positive speeds. It would be thus interesting to obtain those results for the fractal graphs we considered, and we believe that the Lipschitz cutset could yield both statement if some further connectivity properties within L_1 could be shown.

Erklärung

Hiermit versichere ich an Eides statt, dass ich die vorliegende Dissertation selbstständig und ohne die Benutzung anderer als der angegebenen Hilfsmittel und Literatur angefertigt habe. Alle Stellen, die wörtlich oder sinngemäß aus veröffentlichten und nicht veröffentlichten Werken dem Wortlaut oder dem Sinn nach entnommen wurden, sind als solche kenntlich gemacht. Ich versichere an Eides statt, dass diese Dissertation noch keiner anderen Fakultät oder Universität zur Prüfung vorgelegen hat; dass sie — abgesehen von unten angegebenen Teilpublikationen und eingebundenen Artikeln und Manuskripten – - noch nicht veröffentlicht worden ist sowie, dass ich eine Veröffentlichung der Dissertation vor Abschluss der Promotion nicht ohne Genehmigung des Promotionsausschusses vornehmen werde. Die Bestimmungen dieser Ordnung sind mir bekannt. Darüber hinaus erkläre ich hiermit, dass ich die Ordnung zur Sicherung guter wissenschaftlicher Praxis und zum Umgang mit wissenschaftlichem Fehlverhalten der Universität zu Köln gelesen und sie bei der Durchführung der Dissertation zugrundeliegenden Arbeiten und der schriftlich verfassten Dissertation beachtet habe und verpflichte mich hiermit, die dort genannten Vorgaben bei allen wissenschaftlichen Tätigkeiten zu beachten und umzusetzen. Ich versichere, dass die eingereichte elektronische Fassung der eingereichten Druckfassung vollständig entspricht.

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A multilingual chaos, as my stay in Cologne. Switching languages, from person to person, from sentence to sentence.

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