On geodesible vector fields and related geometric structures

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Abstract

A nowhere vanishing vector field X on a manifold M is called *geodesible* if there exists a Riemannian metric on M for which X is of unit length and such that the orbits of X are geodesics. After discussing some examples of such vector fields, we extend an existence result of Gluck and Hajduk–Walczak about geodesible vector fields on odd-dimensional manifolds using open books. Furthermore, we provide a construction of geodesible vector fields on round 1-handlebodies and, as an application, prove the existence of geodesible vector fields on a certain family of manifolds not covered by the previous constructions. We provide some new conditions on the sectional or Ricci curvatures of geodesic vector fields on 3-manifolds that are necessary or sufficient for the orthogonal distribution to define a contact structure or foliation. We give sufficient conditions for a geodesible vector field on a 3-manifold to be realisable as the Reeb vector field of a contact form or stable Hamiltonian structure. Specifically, we consider geodesic vector fields on flat 3-manifolds, and show that these vector fields are tangent to a 2-dimensional totally geodesic foliation in case the underlying manifold is a nontrivial quotient of $\mathbb{E}^3$. Using this, we derive a condition in terms of induced contact structures for these vector fields to be realisable as Reeb vector fields, and we show that the underlying contact structure is always universally tight. Finally, we present a detailed proof of a theorem by Scott about the geometrisation of Seifert fibred 3-manifolds. We show further that — with respect to these geometries — the fibres are geodesics and that their orthogonal distribution defines a universally tight contact structure if and only if the Euler number is nonzero. In particular, we deduce that a contact structure admitting a Reeb vector field tangent to the fibres of a Seifert fibration is necessarily universally tight.

Zusammenfassung

Ein nirgends verschwindendes Vektorfeld X auf einer Mannigfaltigkeit M heißt *geodisierbar*, falls es eine Riemannsche Metrik auf M gibt, bezüglich welcher X von konstanter Länge 1 ist, und so dass die Bahnen von X Geodätische sind. Wir stellen zunächst einige Beispiele solcher Vektorfelder vor und erweitern anschließend ein Existenzresultat von Gluck und Hajduk–Walczak über geodisierbare Vektorfelder auf ungerade-dimensionalen Mannigfaltigkeiten mithilfe von offenen Büchern. Des Weiteren beschreiben wir eine Konstruktion geodisierbarer Vektorfelder auf runden 1-Henkelkörpern, und zeigen damit die Existenz geodisierbarer Vektorfelder auf einer gewissen Familie von Mannigfaltigkeiten, auf welche die vorherigen Konstruktionen nicht anwendbar sind. Anschließend leiten wir einige neue geometrische Bedingungen an die Schnitt- oder Ricci-Krümmungen geodätischer Vektorfelder auf 3-Mannigfaltigkeiten her, die notwendig oder hinreichend dafür sind, dass das orthogonale Ebenenfeld eine Kontaktstruktur oder eine Blätterung definiert. Wir zeigen, unter welchen Bedingungen ein geodisierbares Vektorfeld auf einer 3-Mannigfaltigkeit als Reeb-Vektorfeld einer Kontaktform oder stabilen Hamiltonschen Struktur realisiert werden kann. Insbesondere betrachten wir geodätische Vektorfelder auf flachen 3-Mannigfaltigkeiten und zeigen, dass diese tangential an eine 2-dimensionale total geodätische Blätterung sind, falls die gegebene Mannigfaltigkeit ein nichttrivialer Quotient von $\mathbb{E}^3$ ist. Daraus leiten wir eine Bedingung in Bezug auf induzierte Kontaktstrukturen dafür her, dass ein solches Vektorfeld als Reeb-Vektorfeld einer Kontaktform realisiert werden kann, und zeigen zudem, dass die zugehörige Kontaktstruktur notwendigerweise universell straff ist. Abschließend präsentieren wir einen detaillierten Beweis eines Satzes von Scott über die Geometrisierung Seifert-gefaserter 3-Mannigfaltigkeiten. Darüber hinaus zeigen wir, dass die Fasern Geodätische bezüglich dieser Geometrien sind, und dass das Ebenenfeld orthogonal zu den Fasern genau dann eine universell straffe Kontaktstruktur definiert, wenn die Euler-Zahl ungleich Null ist. Insbesondere folgern wir, dass eine Kontaktstruktur, die ein Reeb-Vektorfeld tangential an eine Seifert-Faserung zulässt, notwendigerweise universell straff ist.

Contents

Introduction

In his 1972 publication, Epstein proved the following, nowadays classical theorem: If all orbits of a flow on a compact 3-manifold are closed, then they share a common period [21]. In other words, the flow is periodic, that is to say there is an S^1-action with the same orbits. This remarkable result turns out to be false in higher dimensions; there are examples of flows on compact 5-manifolds with closed orbits only, but unbounded minimal period [68]. In 1974, Wadsley extended Epstein's result in his Ph.D. thesis for manifolds of arbitrary dimension under an additional geometric assumption. Namely, he proved that a flow all of whose orbits are closed is periodic if and only if there is a Riemannian metric for which all flow lines are geodesics [77]. A flow with this property is called *geodesible*, a notion that was introduced by Gluck in the early 1980s, who was interested in the question of whether or not a given manifold can be 'filled' by geodesics, and under which circumstances a given 1-dimensional foliation can be realised as a foliation by geodesics [33].

Apart from periodic flows, examples of geodesible vector fields (or flows) appear naturally in different areas of geometry: Reeb vector fields of contact forms, suspension flows or Killing vector fields — to name a few — are always geodesible. There is a useful characterisation of geodesible vector fields due to Wadsley and Sullivan, which we are going to prove in Section 1.1: a vector field X is geodesible if and only if there is transverse hyperplane field preserved by the flow of X (Proposition 1.1.2 in this thesis). For example, in the case of a Reeb vector field, this hyperplane field is given by the underlying contact structure.

Despite the abundance of examples, the existence question for geodesible vector fields has not been answered completely. In Chapter 1, we will address this question and discuss and extend a result originally stated by Gluck and proved by Hajduk and Walczak on the existence of geodesible vector fields on closed, orientable odd-dimensional manifolds (Theorem 1.2.2). The proof makes use of the existence of so-called open book decompositions of these manifolds. Motivated by the notion of supporting open book decompositions for contact structures due to Giroux [32], we introduce the notion of supporting open books for geodesible vector fields. In fact, the construction of Hajduk and Walczak produces geodesible vector fields supported by an open book decomposition of a given odd-dimensional manifold. Moving on, we address the existence question in even dimensions. We give a brief introduction to the theory of round handle decompositions due to Asimov to then prove the existence of geodesible vector fields on round 1-handlebodies of arbitrary dimension. The vector fields we construct turn out to be tangent to the boundary, and their restrictions to the boundary are supported by an open book decomposition. As an application, we prove the existence of geodesible vector fields on a certain family of manifolds (of arbitrary dimension) not covered by the previous constructions (Proposition 1.5.1 and Theorem 1.5.4).

In Chapter 2, we study geodesic vector fields on 3-manifolds with a given Riemannian metric. In that case, one can consider the orthogonal plane field which is preserved under the flow of the geodesic vector field. One question of interest is under which assumptions (for example, on the sectional or Ricci curvature) this plane field defines a contact structure, i.e. a maximally non-integrable plane field. In this case, we say that the given geodesic vector field *induces* a contact structure. Secondly, one can ask about the properties of contact structures arising this way, for example tightness in dimension 3. A known result in this direction is due to Gluck: a geodesic vector field on the standard round 3-sphere always induces a contact structure, and this contact structure is diffeomorphic to the standard one (Theorem 2.2.3). Similar results for geodesic vector fields on flat 3-space were obtained by Harrison (Theorem 2.2.6) and Geiges and the author (Theorem 2.2.7). The proofs of these statements rely on topological and geometrical features of fibrations by great circles or lines. We will provide alternative proofs for some of these statements in a more general setting (Theorem 2.3.5) using the notion of adapted Jacobi fields, which we will introduce in Section 2.3. Another known result is due to Aazami and, independently, Harris and Paternain, stating that a geodesic vector field X on a closed Riemannian 3-manifold M induces a contact structure if X has positive Ricci curvature everywhere (Theorem 2.2.4). We obtain a slight generalisation of this result in Section 2.2 by weakening the assumption on the Ricci curvature (Theorem 2.2.13). Then we turn our attention to geodesic vector fields whose so-called Jacobi tensor is parallel along flow lines. Examples of this type are given by Killing vector fields or geodesic vector fields on locally symmetric 3-manifolds. Given such a vector field X, we show that X induces a contact structure if it satisfies a certain nondegeneracy condition, and if the minimal sectional curvature of planes containing X is nonnegative everywhere; see Theorem 2.4.3 for the precise statement. The condition on the sectional curvature is weaker than $\operatorname{Ric} X > 0$, so that Theorem 2.4.3 can be seen as a generalisation of Theorem 2.2.13 within the class of geodesic vector fields with parallel Jacobi tensor. Then, in Section 2.5, we investigate geodesic vector fields whose orthogonal plane field is integrable, i.e. tangent to a foliation. This can be thought of as the other end of a spectrum, where one end is given by geodesic vector fields inducing contact structures (which are maximally non-integrable). In particular, we derive a bound on the 'total Ricci curvature' of a geodesic vector field whose orthogonal distribution is integrable (Theorem 2.5.6).

Now if a geodesic vector field induces a contact structure, then it is not hard to show that it defines the Reeb vector field of its dual (contact) 1-form. More generally, one can ask whether or not a given geodesible vector field X can be realised by the Reeb vector field of a contact form, where the contact structure is not necessarily orthogonal to X. We will address this question in Chapter 3. Using a modified version of the construction of geodesible vector fields on open books, we first show that every closed orientable 3-manifold admits geodesible vector fields that are not realisable as Reeb vector fields of contact forms or, more generally, stable Hamiltonian structures (Proposition 3.1.2). Then, using basic cohomology and a notion of volume for geodesible vector fields, we obtain a criterion for the 'Reebability' of geodesible vector fields (Proposition 3.3.5). As an application, we recover a result by Kegel and Lange, which states that a vector field whose flow defines a Seifert fibration can be realised by a Reeb vector field if and only if the Euler number of the Seifert fibration is nonzero (Corollary 3.4.3). After that, we focus on geodesic vector fields on flat 3-manifolds. We show in Section 3.5 that geodesic vector fields on flat 3-manifolds given by nontrivial quotients of $\mathbb{E}^3$ must be of a very simple '1-parameter' type, that is, they are tangent to a codimension-1 foliation whose leaves are totally geodesic (Theorem 3.5.1). In Section 3.6 we use this characterisation to provide a necessary and sufficient condition for the Reebability of such vector fields in terms of induced contact structures (Theorem 3.6.2), in particular proving that the underlying contact structure must lift to the standard one on $\mathbb{R}^3$ (up to diffeomorphism). Finally, we show that a geodesic vector field on $\mathbb{E}^3$ is always — up to rescaling by a positive function — given by the Reeb vector field of a contact form, and that the associated contact structure is necessarily tight (Corollary 3.7.2 and Theorem 3.7.4). The latter generalises an earlier result obtained in [5]. The contents of Sections 3.5, 3.6 and 3.7 are based on the article [4].

In the final Chapter 4, we study a specific class of geodesible fibrations given by Seifert fibrations of 3-manifolds. We provide a detailed proof of a theorem by Scott about the geometrisation of Seifert manifolds, stating that a Seifert manifold can be equipped with a locally homogeneous metric (Theorem 4.4.6). Furthermore, we show that the Seifert fibres are geodesics with respect to this metric (unless the manifold is a lens space), and that the plane field orthogonal to the fibres defines a universally tight contact structure if and only if the Euler number of the Seifert fibration is nonzero (Theorem 4.0.1). As a consequence, we deduce that if a Reeb vector field of a contact form is tangent to the Seifert fibres, then the corresponding contact structure is universally tight (Corollary 4.0.3).

1

Geodesible vector fields

In this chapter, we introduce geodesible vector fields (or foliations), the central object of this thesis. We start by discussing some important characterisations and reviewing a number of examples. In Section 1.2, we introduce open book decompositions in order to prove a theorem of Gluck on the existence of geodesible vector fields on closed, orientable, odd-dimensional manifolds (Theorem 1.2.2). This theorem was already partly proven by Hajduk and Walczak [38]. In Section 1.4, we introduce round handle decompositions (due to Asimov [2]) to prove the existence of geodesible vector fields on round 1-handlebodies in Section 1.5 (Proposition 1.5.1). As an application, we prove the existence of geodesible vector fields on a certain family of manifolds of arbitrary dimension (Theorem 1.5.4).

Throughout this thesis, all manifolds, vector fields, functions etc. are assumed to be smooth unless stated otherwise.

1.1 Definitions and examples

Definition 1.1.1. Let M be a smooth manifold, and X a nowhere-vanishing vector field on M. Then X is said to be *geodesible* if there exists a Riemannian metric g such that X is of unit length and its integral curves are geodesics for g. In other words, X has to satisfy

$$|X| = 1 \quad \text{and} \quad \nabla_X X = 0,$$

where ∇ is the Levi-Civita connection associated with g.

If the metric g is given, X is called *geodesic*. Similarly, a 1-dimensional foliation is called geodesible (respectively geodesic) if there exists a geodesible (respectively geodesic) vector field spanning it.

The following important characterisation of geodesible vector fields is due to Wadsley [77]. A purely geometric proof of the equivalence of conditions (i) and (ii) was given by Sullivan [69]. The proof presented here is based on [17].

Proposition 1.1.2 (Wadsley, Sullivan)**.** *Let X be a nowhere vanishing vector field on a manifold M. Then the following are equivalent.*

(i) X is geodesible;

(ii) There is a 1-form α on M such that $\alpha(X) = 1$ and $i_X \mathrm{d}\alpha = 0$;

(iii) There is a hyperplane field η transverse to X and invariant under the flow of X.

In this case, the 1-form α and the Riemannian metric g for which X is geodesic are related by $\alpha = i_X g$, and η is given as the orthogonal complement of X.

In order to prove Proposition 1.1.2, we need the following lemma.

Lemma 1.1.3. *Let X be a vector field on a Riemannian manifold M, and let $\alpha := i_X g$. Then*

$$\mathrm{d}\alpha(X, \cdot) = g(\nabla_X X, \cdot) - \frac{1}{2}\mathrm{d}(|X|^2).$$

Proof. Let Y be an arbitrary vector field. Then

$$\begin{aligned}
\mathrm{d}\alpha(X, Y) &= X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y]) \\
&= X(g(X, Y)) - Y(|X|^2) - g(X, [X, Y]) \\
&= g(\nabla_X X, Y) + g(X, \nabla_X Y - [X, Y]) - Y(|X|^2) \\
&= g(\nabla_X X, Y) + g(X, \nabla_Y X) - Y(|X|^2) \\
&= g(\nabla_X X, Y) - \frac{1}{2}\mathrm{d}(|X|^2)(Y),
\end{aligned}$$

hence, $\mathrm{d}\alpha(X, \cdot) = g(\nabla_X X, \cdot) - \mathrm{d}(|X|^2)/2$, as was claimed. □

Proof of Proposition 1.1.2. We start by proving the equivalence of (i) and (ii). Assume that X is geodesic for some metric g, and let $\alpha := i_X g$. Then, by Lemma 1.1.3, $i_X \mathrm{d}\alpha = 0$ (since the right-hand side of the equation vanishes), and $\alpha(X) = |X|^2 = 1$, which proves (ii).

Conversely, assume that there is a 1-form α such that $\alpha(X) = 1$ and $i_X \mathrm{d}\alpha = 0$. Define a metric g by first choosing an arbitrary metric on $\ker \alpha$, then setting $|X| \equiv 1$ and declaring X and $\ker \alpha$ to be orthogonal. Of course, there are many ways to define such a metric, but the specific choice is not important for the argument. Given such a metric g, by Lemma 1.1.3, we have that

$$0 = i_X \mathrm{d}\alpha = g(\nabla_X X, \cdot) - \frac{1}{2}\mathrm{d}(|X|^2) = g(\nabla_X X, \cdot),$$

which implies that $\nabla_X X$ must vanish identically, since g is nondegenerate. Hence, X is geodesic with respect to g.

Next, we show that (ii) implies (iii). Given α, we define η to be the hyperplane field given by $\ker \alpha$. Then, Lemma 1.1.3 implies that

$$L_X \alpha = \mathrm{d}(\underbrace{\alpha(X)}_{\equiv 1}) + i_X \mathrm{d}\alpha = 0,$$

hence, α is invariant under the flow, and so is $\eta = \ker \alpha$. Conversely, if such a hyperplane field η is given, take any 1-form α such that $\alpha(X) = 1$ and $\ker \alpha = \eta$. Then, using Cartan's formula, by the invariance of η we obtain

$$i_X \mathrm{d}\alpha = L_X \alpha = \lambda\, \alpha,$$

where λ is some function $M \to \mathbb{R}$. Plugging X into both sides of this equation, we see that $\lambda = 0$, hence $i_X \mathrm{d}\alpha = 0$. This finishes the proof. $\square$

Remark 1.1.4. Given a geodesible vector field X and a hyperplane field η as in Proposition 1.1.2, it follows from the above proof that the space of Riemannian metrics for which X is geodesic is at least as large as the space of Riemannian metrics on η.

Definition 1.1.5. Let X be a geodesible vector field. Any 1-form α satisfying $\alpha(X) = 1$ and $i_X \mathrm{d}\alpha = 0$ (as in Proposition 1.1.2) is called **connection 1-form** of X. The pair (X, α) is called **geodesible pair**.

In dimension three, there is another interesting characterisation of geodesible vector fields due to Rechtman [61]. Let (M, g) be an orientable Riemannian 3-manifold, and let μ be an arbitrary volume form on M. For a vector field X, its **curl** (with respect to μ) is defined to be the unique vector field $\operatorname{curl} X$ that satisfies the equation

$$i_{\operatorname{curl} X}\mu = \mathrm{d}(i_X g). \tag{1.1}$$

Note that if X is geodesic for g, then, by Lemma 1.1.3, we have that $i_X \mathrm{d}(i_X g) = 0$. It then follows from equation (1.1) that X must be parallel to its curl, that is, $\operatorname{curl} X = \lambda\, X$, where λ is some (perhaps vanishing) function. Conversely, suppose there is some volume form μ for which $\operatorname{curl} X$ and X are parallel. Again, by equation (1.1), this implies that $i_X \mathrm{d}(i_X g) = 0$, and therefore, by Lemma 1.1.3, X is geodesic after normalisation. Thus, we have proven the following.

Proposition 1.1.6 (Rechtman [61])**.** *Let M be an orientable* 3*-manifold, and X a nowhere vanishing vector field on M. Then X is geodesible if and only if there is a volume form μ and a function $\lambda \in C^\infty(M)$ such that* $\operatorname{curl} X = \lambda X$. $\square$

Remark 1.1.7. A divergence-free vector field that is parallel to its own curl is also called **Beltrami field**. These vector fields play an important role in the theory of hydrodynamics, where they appear as solutions of the steady-state Euler equations, cf. [23].

For the sake of completeness, let us mention another characterisation of geodesible vector fields due to Sullivan [69] in terms of so-called foliation currents. He shows that a nowhere vanishing vector field X is geodesible if and only if no foliation cycle can be arbitrarily well approximated by a 2-chain tangent to X. We refer the reader to [10] for a profound introduction to the theory of foliation currents, and to [33] for a very nice discussion of Sullivan's characterisation.

Now let us review some examples of geodesible vector fields.

Example 1.1.8 (Hopf fibration)**.** Let S^3 be the 3-sphere, thought of as the unit sphere in $\mathbb{C}^2$. Consider the quotient map $S^3 \to \mathbb{C}P^1 \cong S^2$, where $\mathbb{C}P^1$ is being identified with the quotient space of S^3 under the equivalence relation that identifies a point $(z_1, z_2) \in S^3$ with $(\lambda z_1, \lambda z_2)$ for every $\lambda \in S^1 \subset \mathbb{C}$. This quotient map is in fact a fibre bundle with S^1-fibres, and is called the **Hopf fibration**. More precisely, consider a point $z = (z_1, z_2) \in S^3$, and let C_z be the fibre through z. Then C_z can be written as

$$C_z = \{\cos\varphi\,(z_1, z_2) + \sin\varphi\,(\mathrm{i}z_1, \mathrm{i}z_2) \in S^3\colon\ \varphi \in \mathbb{R}/2\pi\mathbb{Z}\},$$

which is clearly a great circle (that is, the intersection of S^3 with some plane through the origin). Hence, the fibres of the Hopf fibration are geodesics of S^3, equipped with the standard round metric, so it defines a geodesic fibration.

Now say we are given a linear isomorphism $L \in \mathrm{GL}(4, \mathbb{R})$. For a Hopf fibre C, let P be the plane in $\mathbb{R}^4$ whose intersection with S^3 is C. Then $L(P)$ is a plane through the origin whose intersection with S^3 defines another great circle C'. The union of all such great circles yields another fibration of S^3 by oriented great circles, and fibrations of these type are called *skew Hopf fibrations* (see [35]). The space of *all* great circle fibrations of S^3, however, is much larger: In particular, there are examples of great circle fibrations which are not given as skew Hopf fibrations [35, Section 3].

Example 1.1.9 (Geodesic flows)**.** Let (M, g) be a Riemannian manifold and STM its unit tangent bundle, that is,

$$STM := \{v \in TM\colon\ |v| = 1\}.$$

The **geodesic flow** of M is a (local) flow on STM defined as follows. For any open, relatively compact neighbourhood $U \subset STM$, it is given by

$$G\colon\ U \times (-\varepsilon, \varepsilon) \longrightarrow STM, \quad (v, t) \longmapsto \dot{\gamma}_v(t),$$

where γ_v is the unique geodesic satisfying the initial conditions $\gamma_v(0) = \pi(v)$ and $\dot{\gamma}_v(0) = v$. Here, $\pi\colon\ STM \to M$ is the natural projection. Moreover, the value of $\varepsilon > 0$ is chosen small enough so that geodesics in M whose initial velocity is some vector in U exist for time at least ε. Note that this is a global flow if and only if M is geodesically complete.

The **Sasaki metric** is a natural Riemannian metric g_S on TM induced by g, which can be defined as follows. If $X, Y\colon (-\delta, \delta) \to TM$ are smooth curves, that is, smooth vector fields on M along curves $\gamma_X := \pi \circ X$ and $\gamma_Y := \pi \circ Y$, respectively, we set

$$g_S(\dot{X}, \dot{Y}) := g(\dot{\gamma}_X, \dot{\gamma}_Y) + g(\nabla_{\dot{\gamma}_X} X, \nabla_{\dot{\gamma}_Y} Y).$$

In fact, this corresponds to the decomposition of tangent vectors of TM into a vertical and a horizontal part (where the horizontal part is defined by the Levi-Civita connection); this is explained more carefully in Appendix B. Let us show now that the flow lines of the geodesic flow are geodesics with respect to the Sasaki metric. One way to see this is by showing that these flow lines are locally length-minimising. First, let $Y\colon (-\delta, \delta) \to STM$ be any curve, and let $\gamma = \pi \circ Y$ (i.e. Y is a vector field along γ). Denote by $L(Y)$ and $L(\gamma)$ the lengths of Y and γ, respectively. Then

$$L(Y) = \int_{-\delta}^{\delta} |\dot{Y}(t)|_{g_S}\, \mathrm{d}t = \int_{-\delta}^{\delta} \sqrt{|\dot{\gamma}(t)|_g^2 + |\nabla_{\dot{\gamma}(t)} Y|_g^2}\, \mathrm{d}t \geq \int_{-\delta}^{\delta} |\dot{\gamma}(t)|_g\, \mathrm{d}t = L(\gamma).$$

In particular, Y is (locally) length-minimising if γ is (locally) length-minimising in M and $\nabla_{\dot{\gamma}} Y = 0$. Now, if $t \mapsto Y(t)$ is a flow line of the geodesic flow, then $Y(t) = \dot{\gamma}(t)$ for some geodesic $\gamma = \pi \circ Y$, and therefore, $\nabla_{\dot{\gamma}} Y = \nabla_{\dot{\gamma}} \dot{\gamma} = 0$. Hence, Y is (locally) length-minimising and therefore a geodesic.

Example 1.1.10 (Reeb vector fields)**.** Let M be a $(2n+1)$-dimensional manifold, α a contact form on M and R_α its Reeb vector field (see Section 2.1 for definitions). Then R_α is geodesible by Proposition 1.1.2, simply by taking α as connection 1-form for R_α.

More generally, one can consider a **stable Hamiltonian structure** (SHS), which is a pair (ω, α) consisting of a closed 2-form ω and a 1-form α such that $\alpha \wedge \omega^n$ is nowhere vanishing, and $\ker \omega \subset \ker \mathrm{d}\alpha$. Again, there is a unique Reeb vector field R satisfying $i_R\omega = 0$ and $\alpha(R) = 1$. Now since $\ker \omega \subset \ker \mathrm{d}\alpha$, we obtain that $i_R \mathrm{d}\alpha = 0$, so that R is geodesible with connection 1-form α.

Conversely, there are geodesible vector fields on odd-dimensional manifolds that cannot be realised as Reeb vector fields of contact forms or stable Hamiltonian structures, see Section 3.1.

Example 1.1.11 (Flows-with-section)**.** A **section** of a flow is a compact hypersurface $\Sigma \subset M$ without boundary that intersects each flow line transversely and at least once. Such a flow is called **flow-with-section**. We want to show that flows-with-section on closed manifolds are geodesible, following [33]. Given a section Σ, by Theorem A.1 and the subsequent Corollary A.2 in Appendix A, we can write M as the mapping torus of some diffeomorphism $\phi\colon \Sigma \to \Sigma$, that is,

$$M \cong \Sigma(\phi) = (\Sigma \times [0,1])/(x,1) \sim (\phi(x),0).$$

Let X be the vector field on $\Sigma(\phi)$ obtained by pushing forward the vector ∂_t on $\Sigma \times [0,1]$ by the quotient map, where t is the coordinate of the second factor. In other words, the flow defined by X is the suspension flow corresponding to ϕ. Now let $\Psi\colon [0,1] \to [0,1]$ be a smooth function such that $\Psi(t) = 0$ for t near 0 and $\Psi(t) = 1$ for t near 1. Let g be any Riemannian metric on Σ, and define a metric $\tilde{g}$ on $\Sigma \times [0,1]$ by

$$\tilde{g} = (1 - \Psi(t))g + \Psi(t)\phi^* g + (\mathrm{d}t)^2.$$

Then the constant vector field ∂_t is geodesic with respect to $\tilde{g}$ (this follows from Proposition 1.1.2, as $i_{\partial t}\tilde{g} = \mathrm{d}t$). Furthermore, the metric $\tilde{g}$ descends to a (well-defined) metric on the quotient M, so that X is a geodesic vector field with respect to that metric.

Note that one could also argue without constructing an explicit metric, using Proposition 1.1.2: simply take the push-forward of the 1-form $\mathrm{d}t$ as connection form for X.

Example 1.1.12 (Killing vector fields)**.** Let (M, g) be a Riemannian manifold. A **Killing vector field** is a vector field X on M whose flow induces a 1-parameter family of isometries. The claim is that a Killing vector field of unit length is geodesic. To see this, pick any point $p \in M$ and let $v \in T_pM$ be an arbitrary tangent vector. Choose a small disc D through p and transverse to X, and identify a neighbourhood of p with $D \times (-\varepsilon, \varepsilon)$ using the flow of X. Extend v in an arbitrary way to a vector field defined on $D \times \{0\}$, and then to a vector field on U by pushing forward via the flow of X. This way, we obtain a local vector field V invariant under the flow of X. Hence $[X, V] = 0$, so that $\nabla_X V = \nabla_V X$. Then, since X is Killing, we have that

$$\begin{aligned} 0 &= X(g(X,V)) \\ &= g(\nabla_X X, V) + g(X, \nabla_X V) \\ &= g(\nabla_X X, V) + g(X, \nabla_V X) \\ &= g(\nabla_X X, V) + \frac{1}{2} V(\underbrace{|X|^2}_{\equiv 1}) \\ &= g(\nabla_X X, V). \end{aligned}$$

Evaluating this at p yields $\nabla_X X(p) = 0$, since v can be chosen arbitrarily. Hence, $\nabla_X X$ vanishes identically, and X is geodesic.

If X is a Killing vector field that is not of unit length, then after replacing the Riemannian metric g by $\tilde{g} := (1/|X|^2)\, g$, one can check easily that X is still Killing for the new metric $\tilde{g}$, but now of unit length and therefore geodesic. In particular, every Killing vector field is geodesic for a metric that is conformally equivalent to the original one.

Example 1.1.13 (Left-invariant vector fields on compact Lie groups)**.** Let G be a compact Lie group. Then G admits a bi-invariant Riemannian metric g, i.e. left and right multiplication by any element of G define isometries of (G, g) (see [16, Proposition 3.16]). Now let X be a left-invariant vector field on G (that is, $\mathrm{d}_\rho L_{\rho_1}(X_\rho) = X_{\rho_1\rho}$ for every $\rho, \rho_1 \in G$). Then X is of constant length and its flow lines are geodesics [16, Corollary 3.19]. Hence it defines (up to rescaling by a constant) a geodesic vector field.

Example 1.1.14 (Compact Lie group actions)**.** Suppose G is a compact Lie group acting on a manifold M, and X is a vector field whose flow defines a 1-parameter subgroup of G. Then X is geodesible, which can be seen as follows. First, pick an arbitrary Riemannian metric g on M. Let μ be a Haar measure on G, and define a new metric $\tilde{g}$ by

$$\tilde{g} = \int_G (L_\rho)^* g \, \mathrm{d}\mu,$$

where L_ρ denotes left-multiplication by $\rho \in G$. Then $\tilde{g}$ is left-invariant under the action of G. In particular, X is a Killing vector field for $\tilde{g}$, hence, by Example 1.1.12, X is geodesible.

The simplest instance of this example is that of an S^1-action. That is, a vector field whose flow defines an S^1-action is always geodesible. In dimension 3, by a classical result of Epstein [21], given any flow with closed orbits only, there is an S^1-action with the same orbits. Hence every vector field on a 3-manifold with closed orbits only is (up to rescaling) geodesible. In particular, every Seifert fibration is geodesible. We will discuss this class of examples more thoroughly in Chapter 4.

Example 1.1.15 (Gradient vector fields)**.** Let (M, g) be a Riemannian manifold and $f\colon M \to \mathbb{R}$ a smooth function whose gradient $\operatorname{grad} X$ (with respect to g) is nowhere vanishing (this is of course only possible if M is open). Then the vector field $X = \operatorname{grad} f / |\operatorname{grad} f|^2$ is geodesic, since its dual 1-form is given by $\alpha = \mathrm{d}f$ which satisfies $\alpha(X) = 1$ and, of course, $i_X \mathrm{d}\alpha = 0$.

1.2 Existence of geodesible vector fields

The basic question we want to address in this section is the following:

Given a manifold M, does it admit a geodesible vector field?

Despite being interesting in its own right, this question is particularly relevant in the context of so-called Reeb embeddings that were studied by Cardona et al. in [14]. Given a manifold M, an embedding $e\colon M \hookrightarrow (N, \xi)$ of M into some contact manifold (N, ξ) is called **Reeb embedding** if $e(M)$ is an invariant subset of some Reeb flow associated with ξ. In other words, there is a defining contact form α for ξ such that $R_{e(p)} \in T_{e(p)}M$ for all $p \in M$, where R denotes the Reeb vector field of α (see Section 2.1 for definitions). It was shown by Cardona et al. that if M admits a geodesible vector field, there is a Reeb embedding $M \hookrightarrow (N, \xi)$ into some contact manifold N of dimension $\geq 3 \dim M + 1$ [14, Corollary 3.9]. Conversely, if $M \hookrightarrow (N, \xi)$ is a Reeb embedding with corresponding Reeb vector field R, then the induced vector field on M is geodesible, where the connection 1-form is given by the pullback of the corresponding contact form. Hence the existence of a geodesible vector field on M is equivalent to the existence of a Reeb embedding of M.

Of course, an obvious necessary condition for the existence of a geodesible vector field is the existence of a nonsingular vector field. This is no obstruction if M is an *open* manifold (i.e. non-compact or with nonempty boundary). In this case, we can find a function $f\colon M \to \mathbb{R}$ without critical points (see [44, Theorem 4.8]), and a gradient vector field of f is (up to rescaling) geodesible by Example 1.1.15. Hence we have proven the following.

Proposition 1.2.1. *If M is an open manifold, then it admits a geodesible vector field.* □

On the other hand, if M is closed, the existence of a nowhere-vanishing vector field is equivalent to the Euler characteristic $\chi(M)$ being zero. By Poincaré duality, this is always true if M is orientable and the dimension of M is odd. For example, every orientable closed 3-manifold admits a contact form by a theorem of Martinet [57]. The corresponding Reeb vector field is geodesible (Example 1.1.10), hence every closed orientable 3-manifold admits a geodesible vector field. This argument, however, does not work in higher dimensions, since there are manifolds of odd dimension ≥ 5 that do not admit contact structures. Nevertheless, it turns out that *every* closed orientable odd-dimensional manifold admits a geodesible vector field, see Theorem 1.2.2 below. This was originally announced by Gluck [33], whose proof has never been published. A proof of the first part was given by Hajduk and Walczak [38], using the existence of open book decompositions (see Section 1.3), as suggested by Gluck. We complete it by providing a proof for the second part.

Theorem 1.2.2. *Let M be a closed orientable odd-dimensional manifold. Then:*

(i) M admits a geodesible vector field X;

(ii) If $S^1_1, \ldots, S^1_n \hookrightarrow M$ are disjointly embedded circles, X can be chosen such that the S^1_i are orbits of X.

Remark 1.2.3. By recent work of Cardona [12], one can find a vector field X as in Theorem 1.2.2 in every homotopy class of nowhere vanishing vector fields on M; furthermore, the vector field X may be chosen so as to preserve some volume form on M.

1.3 Geodesible vector fields on open books

An open book decomposition is, roughly speaking, a way to decompose a manifold as the union of (infintely many) codimension-1 submanifolds with boundary, called *pages*, that are glued together along a codimension-2 submanifold, called *binding*. A precise definition is given below.

Definition 1.3.1 (Open book decomposition)**.** Let M be an n-manifold and B an $(n-2)$-dimensional submanifold of M. Let $\pi\colon\ M \setminus B \to S^1$ be a fibration and $\nu B \cong B \times D^2$ a tubular neighbourhood of B with polar coordinates (r, θ) on D^2, such that $\pi|_{\nu B} = \theta$. Then the pair (B, π) is called an **open book decomposition** (short: OBD) of M. The submanifold B is called **binding**, and the $(n-1)$-dimensional submanifolds $\overline{\pi^{-1}(\theta)}$ (whose boundaries are given by B) are called **pages** of the open book.

An alternative, more abstract way of defining an open book is the following.

Definition 1.3.2 ((Abstract) open book)**.** Let Σ be a $(n-1)$-manifold with boundary $\partial\Sigma \neq \emptyset$, and let $\phi\colon\ \Sigma \to \Sigma$ be a diffeomorphism that restricts to the identity near $\partial\Sigma$. The **(abstract) open book** defined by ϕ is the n-manifold

$$M(\phi) := \Sigma(\phi) \cup_{\mathrm{id}} (\partial\Sigma \times D^2),$$

where $\Sigma(\phi)$ is the **mapping torus** of ϕ, that is, the quotient of $\Sigma \times [0, 2\pi]$ under the relation $(x, 2\pi) \sim (\phi(x), 0)$. Here, $\Sigma(\phi)$ and $\partial\Sigma \times D^2$ are identified along their boundary via the identity map

$$\mathrm{id}\colon\ \partial(\Sigma(\phi)) = \partial\Sigma \times S^1 \longrightarrow \partial\Sigma \times S^1 = \partial(\partial\Sigma \times D^2).$$

The codimension-1 submanifolds

$$\Sigma_\theta := \Sigma \times \{\theta\} \subset \Sigma(\phi) \subset M(\phi)$$

are called **pages**, and the codimension-2 submanifold

$$B := \partial\Sigma \times \{0\} \subset \partial\Sigma \times D^2 \subset M(\phi)$$

is called **binding** of the open book.

It is not hard to show that these definitions are equivalent; that is, every abstract open book admits an open book decomposition $\pi\colon M \setminus B \to S^1$; conversely, every open book decomposition (B, π) gives rise to an abstract open book (cf. [28, pp. 149–150]).

Example 1.3.3. (i) Consider the 3-sphere $S^3 = \mathbb{R}^3 \cup \{\infty\}$, viewed as the one-point compactification of $\mathbb{R}^3$. Let (r, θ, z) be cylindrical coordinates of $\mathbb{R}^3$ and let $B = \{r = 0\} \cup \{\infty\} \cong S^1 \subset S^3$. Then

$$\pi\colon S^3 \setminus B = \mathbb{R}^3 \setminus \{r = 0\} \longrightarrow S^1, \quad (r, \theta, z) \longmapsto \theta$$

defines an open book decomposition of S^3 with binding B, whose pages are given by $\Sigma_{\theta_0} = \{r \geq 0,\, \theta = \theta_0\} \cup \{\infty\} \cong D^2$. This is depicted in Figure 1.1 below.

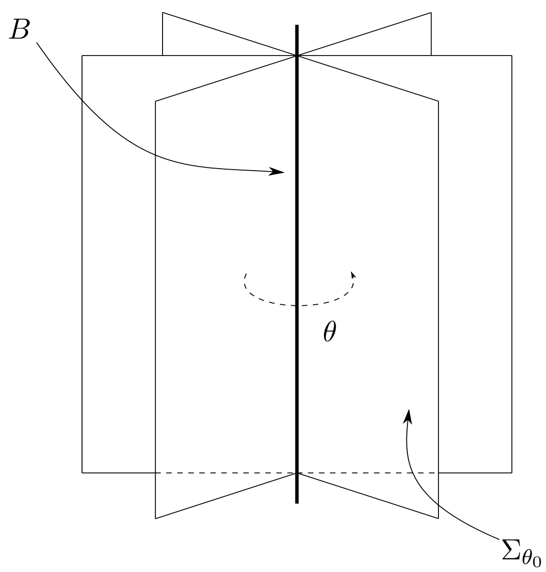

Figure 1.1: An open book decomposition of S^3.

(ii) Consider $M = S^2$ and $B = \{N, S\} \subset S^2$, where N and S are any two points. Then $S^2 \setminus B \cong S^1 \times (-1, 1)$, and the projection onto the S^1-factor clearly describes an open book decomposition with binding B.

Then given any closed manifold N, the product $M := N \times S^2$ admits an open book decomposition with binding $N \times B$, given by projection onto the second factor of $M \setminus (N \times B) \cong N \times S^1 \times (-1, 1)$.

By work of Winkelnkemper, Lawson and Quinn [78, 53, 60], we have the following existence result for open book decompositions in odd dimensions.

Theorem 1.3.4 (Winkelnkemper, Lawson, Quinn)**.** *Every closed, orientable odd-dimensional manifold admits an open book decomposition.*

For contact forms, there is a notion of *supporting* open book decompositions, going back to Giroux [32]. Let M be an oriented, odd-dimensional manifold with an open book decomposition (B, π) such that B is also oriented. Then a contact structure ξ on M is said to be **supported** by the OBD (B, π) if there is a defining contact form α whose Reeb vector field is tangent to B and positively transverse to the pages (note that this is not the standard definition, but an equivalent one, see [22, Lemma 3.5]). Similarly, we can define supporting open books for geodesible vector fields.

Definition 1.3.5. Let (X, α) be a geodesible pair on a manifold M equipped with an open book decomposition (B, π). Then (X, α) is said to be **supported** by (B, π) if the following holds true.

(1) X is tangent to the binding B;

(2) there is a tubular neighbourhood $\nu B \subset M$ of B such that $\alpha|_{M \setminus \nu B} = \pi^* \mathrm{d}\theta$, where θ is the angular coordinate of S^1.

Remark 1.3.6. The second point implies that X is positively transverse to the pages outside the tubular neighbourhood νB, as for the Reeb vector field of a contact form supported by (B, π). It should be noted, however, that our Definition 1.3.5 is not really a generalisation of the definition of supporting OBD's for contact forms, since the connection 1-form α in Definition 1.3.5 can *never* be a contact form (as it is required to be closed on $M \setminus \nu B$).

Proof of Theorem 1.2.2. The proof of the first part presented here is due to Hajduk and Walczak [38]. The proof is by induction over the dimension of M. In fact, we will inductively construct a geodesible pair supported by a given open book decomposition. If $\dim M = 1$, then M is the topological sum of copies of S^1, so M clearly admits a geodesible vector field. Now assume that $\dim M = 2n + 1$, and assume further that the claim holds for every closed orientable manifold of odd dimension $\leq 2n - 1$. By Theorem 1.3.4, M admits an open book decomposition, that is, it can be written as an abstract open book. Thus, we may write M as

$$M \cong M(\phi) = \Sigma(\phi) \cup_{\mathrm{id}} (\partial\Sigma \times D^2)$$

for some diffeomorphism $\phi\colon\ \Sigma \to \Sigma$. By the induction hypothesis, there is a geodesible pair (X_B, α_B) on the binding $B = \partial\Sigma \times \{0\} \subset M(\phi)$. We now want to extend this to a geodesible pair (X, α) on $M(\phi)$. On $\Sigma(\phi)$, we set X equal to ∂_θ and α equal to $\mathrm{d}\theta$, where θ is the coordinate of $\Sigma(\phi) = (\Sigma \times [0, 2\pi])/\sim$ corresponding to the second factor of $\Sigma \times [0, 2\pi]$. On $\partial\Sigma \times D^2$, we make the ansatz

$$X = f_1(r)\,\partial_\theta + f_2(r)\,X_B, \quad \alpha = g(r)\,\mathrm{d}\theta + (1 - g(r))\,\alpha_B,$$

where r is the radial coordinate of D^2, and f_1, f_2, $g\colon\ [0, 1] \to [0, 1]$ are smooth functions such that $f_1 = g = 0$, $f_2 = 1$ near $r = 0$, and $f_1 = g = 1$, $f_2 = 0$ near $r = 1$. Then

$$\alpha(X) = f_1\,g + f_2\,(1 - g),$$

and

$$i_X \mathrm{d}\alpha = (f_2 - f_1)\,g'\,\mathrm{d}r.$$

Now choose f_1, f_2 and g such that $f_1 = f_2 = 1$ on $\{g' \neq 0\}$, $f_2 = 1$ on $\{g = 0\}$, and $f_1 = 1$ on $\{g = 1\}$ (see Figure 1.2). Then $\alpha(X) = 1$ and $i_X \mathrm{d}\alpha = 0$. Hence (X, α) is a geodesible pair on M supported by the open book decomposition (B, π).

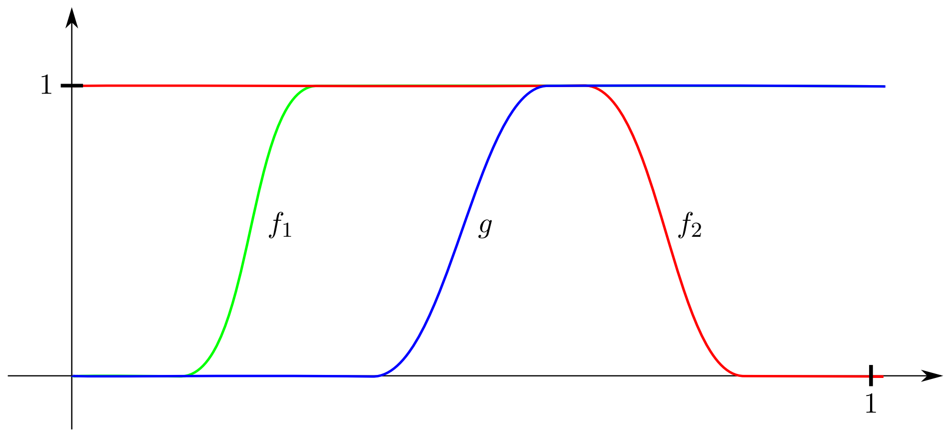

Figure 1.2: Functions f_1, f_2 and g.

Now we come to the proof of part (ii). To keep notation simple, we prove the claim for a single closed curve $\gamma\colon\ [0, 1] \to M$ only, but everything we do can be easily generalised to the case of more than one curve. We will use the notation γ synonymously for the explicit parametrisation as well as the trace of the curve as a subset of M. The proof will be carried out in two steps. The first step is to prove the claim under the assumption that γ is contained in $\Sigma(\phi)$ and is positively transverse to the pages, meaning that $\mathrm{d}\theta(\gamma') > 0$ everywhere. The second step is to prove that this can actually always be assumed, by constructing an isotopy of M that sends γ to such a curve.

Let us begin with the first step. Identify a collar neighbourhood of $\partial\Sigma$ in Σ with $[-2,0]\times\partial\Sigma$, and assume that the monodromy ϕ is given as the identity on this collar neighbourhood. Hence there is a trivially fibred collar neighbourhood $[-2,0]\times\partial\Sigma\times S^1$ of $\partial\Sigma\times S^1\subset\Sigma(\phi)$. We may choose this collar neighbourhood small enough so that

$$\gamma\subset W:=\Sigma(\phi)\setminus([-2,0]\times\partial\Sigma\times S^1)\subset\Sigma(\phi).$$

Choose open tubular neighbourhoods U and V of γ such that

$$\gamma\subset U\subset\overline{U}\subset V\subset W.$$

Let X and α be as in the first part of the proof. Let $\mu\colon M\to[0,1]$ be a bump function that is equal to 1 on U and equal to 0 outside V. Then, on V, replace X by

$$\hat{X}=\mu\,\gamma'+(1-\mu)\,X,$$

where γ' is the velocity vector field of γ, extended to a positively transverse vector field on V. Now $\hat{X}$ clearly extends to all of M so that X and $\hat{X}$ agree on $M\setminus V$. Since $\alpha|_W=\mathrm{d}\theta$ and $X|_W=\partial_\theta$ by construction, we find that

$$\alpha|_W\,(\hat{X})=\mu\underbrace{\mathrm{d}\theta(\gamma')}_{>0}+(1-\mu)\underbrace{\mathrm{d}\theta(\partial_\theta)}_{=1}>0,$$

and of course $i_{\hat{X}}(\mathrm{d}\alpha|_W)\equiv 0$, as α is closed on W. Thus, after replacing $\hat{X}$ by $(1/\alpha(\hat{X}))\hat{X}$, we obtain a geodesible pair with γ as an integral curve.

For the second step, we first show how to isotope γ so that it is contained in $\Sigma(\phi)\subset M(\phi)$. For this, we write

$$M(\phi):=\Sigma(\phi)\cup_{\mathrm{id}}(\partial\Sigma\times D^2)$$

as in Definition 1.3.2. First note that since $\dim B+\dim\gamma=\dim M-1$, we may isotope γ so that $\gamma\cap B=\emptyset$. Then an isotopy pushing γ into $\Sigma(\phi)$ can be found, for example, as the time-1 flow of a vector field on $M(\phi)$ that is equal to $2r\,\partial/\partial r$ on $\partial\Sigma\times D^2$ (extended to $M(\phi)$ in an arbitrary way). Now, after applying another isotopy, we may assume that there are only finitely many isolated points at which γ is tangent to the pages. In other words, we find a subdivision $0=t_1<\ldots<t_k=1$ such that $\gamma_i:=\gamma|_{(t_i,t_{i+1})}$ is transverse to the pages for each i. If every γ_i is positively transverse, we can apply another isotopy to remove possible saddle points and we are done. So assume there is some γ_i which is negatively transverse. Possibly after refining the subdivision we may assume that

(1) γ_i is contained in a trivial neighbourhood of the form $\Sigma\times[\theta_0,\theta_1]\subset\Sigma(\phi)$ for some angles $\theta_0<\theta_1$.

(2) The subset

$$\Sigma \setminus \mathrm{pr}_1(\gamma \cap (\Sigma \times [\theta_0, \theta_1])) \subset \Sigma$$

is path-connected, where $\mathrm{pr}_1 \colon \Sigma \times [\theta_0, \theta_1] \to \Sigma$ is the projection onto the first factor.

Now (2) implies that there is a point $q \in \partial\Sigma$ and curves

$$\beta_0,\ \beta_1 \colon\ [0,1] \longrightarrow \Sigma \setminus \mathrm{pr}_1(\gamma \cap (\Sigma \times [\theta_0, \theta_1]))$$

such that $\beta_0(0) = \gamma_i(t_{i+1})$, $\beta_1(0) = \gamma_i(t_i)$ and $\beta_0(1) = \beta_1(1) = q$. Now consider the curves $\delta_0,\ \delta_1 \colon\ [0,1] \to M$ defined by

$$\delta_0(t) = (\beta_0(1-t), \theta_0 + (t-1)\,\varepsilon), \quad \delta_1(t) = (\beta_1(t), \theta_1 + t\,\varepsilon),$$

for some small $\varepsilon > 0$, so that δ_0 and δ_1 both are positively transverse to the pages. Note that $\delta_0(0) = (q, \theta_0 - \varepsilon)$ and $\delta_1(1) = (q, \theta_1 + \varepsilon)$ are both contained in the disc $\{q\} \times D^2 \subset \partial\Sigma \times D^2$. Now choose a third curve $\delta_2 \colon\ [0,1] \to \{q\} \times D^2$, positively transverse to the pages, with $\delta_2(0) = (q, \theta_1 + \varepsilon)$ and $\delta_2(1) = (q, \theta_0 - \varepsilon)$ (for this δ_2 has to go around the binding, see Figure 1.3 below).

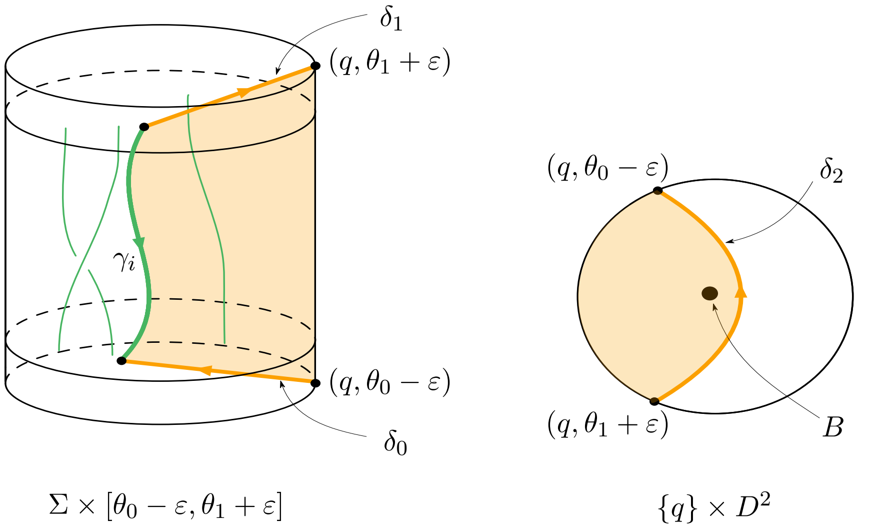

Figure 1.3: The negatively transverse part γ_i (in green) and the positively transverse curve $\delta = \delta_1\,\delta_2\,\delta_0$ (in orange) bound a disc which does not intersect γ in its interior. Thus, we can isotope γ_i to δ leaving the rest of γ invariant.

Then, using property (2) above, the curves $\gamma_i, \delta_0, \delta_1$ and δ_2 bound a disc which does not intersect γ in its interior. In particular, we find an (continuous) isotopy of γ that sends γ_i to the product curve $\delta := \delta_1\,\delta_2\,\delta_0$, and does not move any part of $\gamma \setminus \gamma_i$. The curve δ is, by construction, a piecewise smooth curve that is transverse to the pages of $M(\phi)$. Applying this procedure successively, we can isotope γ to a piecewise smooth curve whose smooth parts are positively transverse to the pages of the OBD. After smoothening and applying the isotopy extension theorem, we have constructed an isotopy of M sending γ to a smooth, positively transverse curve as desired. □

1.4 Round handle decompositions

One might ask if Theorem 1.2.2 can be extended to even-dimensional manifolds, under the additional assumption that the Euler characteristic vanishes. The first problem, however, is that these manifolds do not, in general, carry an open book decomposition. Furthermore, even if the manifold in question admits an open book decomposition, this is perhaps not the case for its binding, so that the inductive argument used in the proof of Theorem 1.2.2 does not work. Hence one has to use a different type of structure. The so-called *round handle decomposition*, introduced by Asimov in 1975, seems to be a suitable candidate. In this section, we will give a brief introduction to the theory of round handles. We refer the reader to Asimov's original article [2] or the survey article [18] for a more detailed discussion.

A round handle is simply the product of an ordinary handle with S^1, and a round handle decomposition is defined similarly as for ordinary handles. Below we will give a precise definition.

Definition 1.4.1. An **n-dimensional round handle of index k** (or **round k-handle**, for short) is a copy of $S^1 \times D^k \times D^{n-k-1}$, denoted by R_k. In other words, $R_k = S^1 \times H_k$, where $H_k = D^k \times D^{n-k-1}$ is an ordinary $(n-1)$-dimensional k-handle. We write $\partial R_k = \partial_- R_k \cup \partial_+ R_k$, where

$$\partial_- R_k := S^1 \times S^{k-1} \times D^{n-k-1}$$

and

$$\partial_+ R_k := S^1 \times D^k \times S^{n-k-2}.$$

As for ordinary handles, one says that the manifold W' is obtained from W by attaching a round k-handle if there is a smooth embedding $h\colon \partial_- R_k \hookrightarrow \partial W$ such that W' can be written as

$$W' = W \cup_h R_k := (W + R_k)/\sim,$$

where '+' denotes the topological sum, and $x \in \partial_- R_k$ is being identified with $h(x) \in \partial W \subset W$. Note that $\partial_- R_0 = \emptyset$. Hence, attaching a round 0-handle to W corresponds to taking the topological sum $W' := W + R_0$ (in particular, one can make sense of attaching R_0 to the empty set; the resulting manifold is just R_0). As in the case of ordinary handles, the manifold resulting from attaching a round handle does only depend (up to diffeomorphism) on the isotopy type of the attaching map h.

We will often omit the attaching map in the notation and simply write $W \cup R_k$ instead of $W \cup_h R_k$.

Definition 1.4.2. Let W be an n-dimensional manifold whose boundary is given by $\partial W = \partial_- W \cup \partial_+ W$. A **round handle decomposition** of W relative to $\partial_- W$ is an identification of W with

$$\partial_- W \times [0, 1] \cup \{\text{round } k\text{-handles}\}$$

where each round handle is attached on the side of $\partial_- W \times \{1\}$. W is called **round k-handlebody** if it admits a round handle decomposition relative to the empty set, consisting of round handles of index $\leq k$.

Asimov showed how to obtain round handles from ordinary ones, and vice versa, as follows.

Lemma 1.4.3 (Fundamental Lemma of Round Handles [2])**.** *Let W be a manifold with boundary $\partial W \neq \emptyset$. Assume that W' is obtained from W by attaching one (ordinary) k-handle and one $(k+1)$-handle, i.e. $W' = W \cup H_k \cup H_{k+1}$, where H_k and H_{k+1} are attached independently (that is, $\partial_+ H_k \cap \partial_- H_{k+1} = \emptyset$, where $\partial_+ H_K = D^k \times S^{n-k-1}$ and $\partial_- H_{k+1} = S^k \times D^{n-k-1}$). Then $W' \cong W \cup R_k$.* □

Clearly, every round handle (and also its boundary) has vanishing Euler characteristic. Hence if a manifold W admits a round handle decomposition relative to $\partial_- W$, then $\chi(W) = \chi(\partial_- W)$. Using his Fundamental Lemma, Asimov showed that this condition is also sufficient in dimensions ≥ 4.

Theorem 1.4.4 (Asimov [2])**.** *Let W be a compact manifold of dimension ≥ 4 with boundary $\partial W = \partial_- W \cup \partial_+ W$. Then W admits a round handle decomposition relative to $\partial_- W$ if and only if $\chi(W) = \chi(\partial_- W)$.* □

Remark 1.4.5. The cases $\partial_- W = \emptyset$ or $\partial_+ W = \emptyset$ (or both) are not excluded in the theorem above. In particular, a closed manifold W of dimension ≥ 4 admits a round handle decomposition (relative to the empty set) if and only if $\chi(W) = 0$.

Some important existence results of certain geometric structures on manifolds with vanishing Euler characteristic were proved using round handle decompositions. Asimov himself developed round handle decompositions in order to prove the existence of nonsingular Morse-Smale flows on manifolds with vanishing Euler characteristic [2]. Later, Thurston used them to prove the existence of codimension-1 foliations on these manifolds [71]. More recently, Vogel used round handles to prove the existence of Engel structures on parallelisable 4-manifolds [76].

It seems natural to try to use round handle decompositions in order to prove the existence of geodesible vector fields, as every round handle admits an obvious geodesible vector field, namely the one spanning the S^1-factor. The problematic part is of course the attachment of the handles, as it is not clear a priori if (and how) one can extend a given geodesible pair to an attached round handle. It turns out that this can be done at least in the case of round 1-handles, as we will see in the following section.

1.5 Geodesible vector fields on round 1-handlebodies

The goal of this section is to construct geodesible pairs on round 1-handlebodies. As an application, we prove the existence of geodesible vector fields on certain even-dimensional manifolds (Theorem 1.5.4).

Proposition 1.5.1. *Let N be an n-dimensional round 1-handlebody. Then N admits a geodesible pair (X, α) such that X is tangent to the boundary of N. Furthermore, $(X, \alpha)|_{\partial N}$ is supported by an open book decomposition of ∂N.*

We start by constructing suitable geodesible pairs on round 0-handles.

Lemma 1.5.2. *An n-dimensional round 0-handle R_0 admits a geodesible pair (X, α) such that $(X, \alpha)|_{\partial R_0}$ is supported by an open book decomposition of ∂R_0.*

Proof. Let $R_0 := S^1 \times D^{n-1}$ be a round 0-handle. Then $\partial R_0 = S^1 \times S^{n-2}$ admits an open book decomposition, which can be described as follows. Let φ denote the coordinate for the S^1-factor, and $(x_1, \ldots, x_{n-1})$ be (Cartesian) coordinates for the D^{n-1}-factor of R_0. Let $B := S^1 \times S^{n-4} \subset S^1 \times S^{n-2}$, where

$$S^{n-4} := \{(x_1, \ldots, x_{n-1}) \in S^{n-2} \colon\ x_1 = x_2 = 0\}.$$

Then

$$\pi \colon\ (S^1 \times S^{n-2}) \setminus B \longrightarrow S^1, \quad (\varphi, x_1, \ldots, x_{n-1}) \longmapsto (x_1^2 + x_2^2)^{-1/2}\,(x_1, x_2)$$

defines an open book decomposition of $S^1 \times S^{n-2}$ with binding B. We will now construct a geodesible pair (X_0, α_0) on R_0 whose restriction to ∂R_0 is supported by this open book decomposition. Consider smooth functions $f, g \colon \mathbb{R} \to [0, 1]$ with the following properties:

- $f(t) = g(t) = 0$ for $t \leq 1/4$, and $f(t) = g(t) = 1$ for $t \geq 3/4$;
- if $f'(t) \neq 0$ or $f(t) \neq 0$, then $g(t) = 1$;

see Figure 1.4 below.

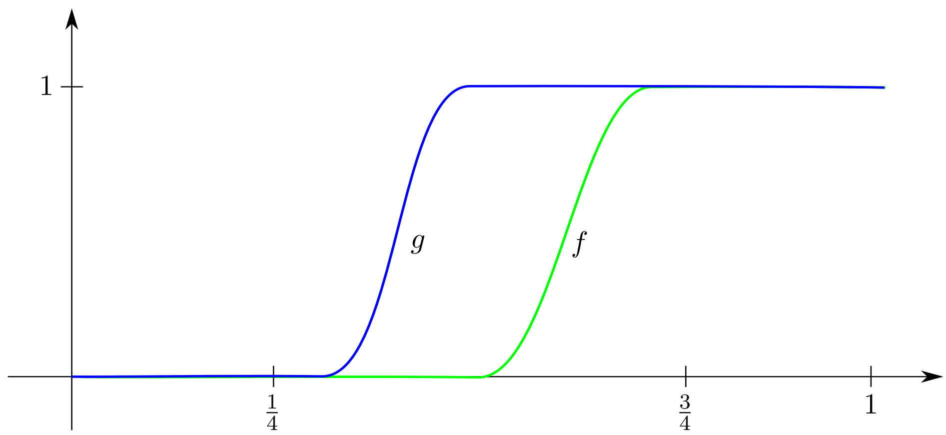

Figure 1.4: Functions f and g.

For coordinates $(\varphi, x_1, \ldots, x_{n-1})$ of $S^1 \times D^{n-1}$ as above, let

$$\rho := \sqrt{x_1^2 + x_2^2} \quad \text{and} \quad r := \sqrt{x_1^2 + \ldots x_{n-1}^2}.$$

Now define X_0 and α_0 on R_0 by

$$X_0 = \partial_\varphi + g(\rho)g(r)\,\partial_\theta, \quad \alpha_0 = (1 - f(\rho)f(r))\,\mathrm{d}\varphi + f(\rho)f(r)\,\mathrm{d}\theta,$$

where θ denotes the angular coordinate of the base S^1, pulled back to $S^1 \times S^{n-2} \setminus B$ via π. We compute

$$\mathrm{d}\alpha_0 = [f'(\rho)f(r)\mathrm{d}\rho + f(\rho)f'(r)\mathrm{d}r] \wedge (\mathrm{d}\theta - \mathrm{d}\varphi),$$

hence

$$i_{X_0}\mathrm{d}\alpha_0 = \underbrace{f'(\rho)f(r)\big(1 - g(\rho)g(r)\big)}_{\equiv 0}\,\mathrm{d}\rho + \underbrace{f(\rho)f'(r)\big(1 - g(\rho)g(r)\big)}_{\equiv 0}\,\mathrm{d}r = 0.$$

Furthermore,

$$\alpha_0(X_0) = 1 + \underbrace{f(\rho)f(r)\big(g(\rho)g(r) - 1\big)}_{\equiv 0} = 1,$$

hence α_0 is a connection form for X_0. On the boundary $\partial R_0 = \{r = 1\}$ we have that $X_0 = \partial_\varphi + g(\rho)\,\partial_\theta$ and $\alpha_0 = (1 - f(\rho))\mathrm{d}\varphi + f(\rho)\,\mathrm{d}\theta$. Thus, taking $\nu B := \{\rho \leq 3/4\}$, we see that the pair $(X_0, \alpha_0)|_{\partial R_0}$ is supported by the open book decomposition described above. □

The second step is to show how to 'geodesibly' attach round 1-handles, keeping track of the open book decomposition of the boundary.

Lemma 1.5.3. *Let N be an n-dimensional manifold with boundary $\partial N \neq \emptyset$ and (X, α) a geodesible pair on N such that $(X, \alpha)|_{\partial N}$ is tangent to ∂N and supported by an open book decomposition of ∂N with binding B. Let $\hat{N} = N \cup R_1$ be obtained from N by attaching a round 1-handle. Then $\hat{N}$ admits a geodesible pair $(\hat{X}, \hat{\alpha})$ that coincides with (X, α) outside a small open neighbourhood of R_1, and such that $(\hat{X}, \hat{\alpha})$ is tangent to $\partial\hat{N}$ and supported by an open book decomposition of $\partial\hat{N}$ with binding B.*

Proof. Write $\hat{N} = N \cup R_1$, where $R_1 = S^1 \times D^1 \times D^{n-2}$ is being attached to ∂N via an embedding

$$h\colon\ \partial_- R_1 = S^1 \times S^0 \times D^{n-2} \longrightarrow \partial N.$$

Write $S^0 = \{-1, 1\}$ and $h_j := h|_{S^1 \times \{j\} \times D^{n-2}}$ for $j = -1, 1$. Now consider the two core circles $C_j := h_j(S^1 \times \{j\} \times \{0\}) \subset \partial N$, $j = -1, 1$, with orientation given by some fixed orientation of S^1 (note that this orientation does *not* agree with the boundary orientation of $S^1 \times \{j\} \times \{0\} \subset S^1 \times D^1 \times \{0\}$!). As in the proof of Theorem 1.2.2 (ii), we may isotope the C_j in ∂N so that they are both contained in $\partial N \setminus \nu B$ and positively transverse to the pages of (B, π). In fact, by a similar argument, we may apply an additional isotopy so that C_{-1} and C_1 make the same number of $k \in \mathbb{Z}\setminus\{0\}$ turns around the binding (that is, $\int_{C_{-1}} \mathrm{d}\theta = \int_{C_1} \mathrm{d}\theta = k$). Indeed, consider a short segment γ_i of γ satisfying properties (1) and (2) as in the proof of Theorem 1.2.2 (ii). Of course in this case, γ_i is already positively transverse to the pages. As in the proof of Theorem 1.2.2, we find curves

$$\beta_0,\ \beta_1\colon\ [0,1] \longrightarrow \Sigma \setminus \mathrm{pr}_1(\gamma \cap (\Sigma \times [\theta_0, \theta_1]))$$

such that $\beta_0(0) = \gamma_i(t_i)$, $\beta_1(0) = \gamma_i(t_{i+1})$ and $\beta_0(1) = \beta_1(1) = q$. Now consider the curves $\delta_0, \delta_1\colon\ [0,1] \to M$ defined by

$$\delta_0(t) = (\beta_0(t), \theta_0 + t\varepsilon), \quad \delta_1(t) = (\beta_1(1-t), \theta_1 + (t-1)\varepsilon).$$

Choose a disc-like neighbourhood U of q in $\partial\Sigma \cong B$ and a curve δ_2 in $U \times D^2$, positively transverse to the pages, with $\delta_2(0) = (q, \theta_0 + \varepsilon)$ and $\delta_2(1) = (q, \theta_1 - \varepsilon)$ (see Figure 1.5, where the 3-dimensional case is depicted). Now we can isotope γ_i (keeping endpoints fixed) to the product curve $\delta_0\,\delta_2\,\delta_1$, so that after smoothening the resulting piecewise smooth curve, the winding number $\int_{C_j} \mathrm{d}\theta$ is increased by 1. Geometrically, we simply introduce an additional twist of C_j around one of the binding components. This way we can increase the winding numbers of C_1 and C_{-1} arbitrarily, hence we may assume that they are the same.

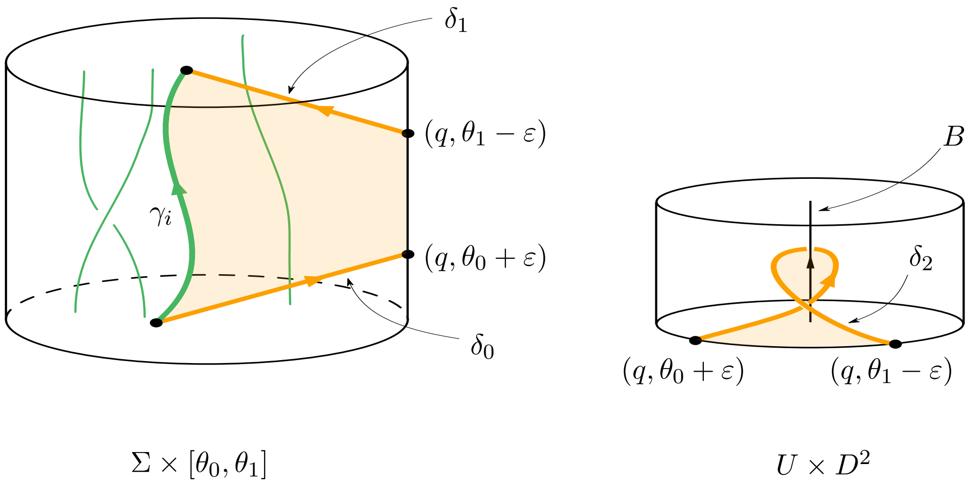

Figure 1.5: Increasing the winding number around the binding by introducing an additional twist.

We may further assume that $\operatorname{Im} h_j \cong S^1 \times D^{n-2} \subset \partial N \setminus \nu B$. Arguing now as in the proof of the Theorem 1.2.2 (ii), we may change X inside the tubular neighbourhood $\operatorname{Im} h_j$ of C_j so that

$$(h_j^{-1})_* X = \frac{1}{k}\partial_\vartheta, \quad h_j^* \alpha = k \, \mathrm{d}\vartheta, \quad j = -1, 1$$

for suitably chosen coordinates $(\vartheta, t, x_1, \ldots, x_{n-2})$ on $R_1 = S^1 \times I \times D^{n-2}$. Hence we can extend the pair (X, α) to R_1 by setting $(X, \alpha)|_{R_1} := (1/k\, \partial_\vartheta, k\, \mathrm{d}\vartheta)$. To finish the proof, let us show that this extended geodesible pair on $\hat{N}$ is supported by an open book decomposition of $\partial \hat{N}$. First note that $\partial \hat{N} = (\partial N \setminus h(\partial_- R_1)) \cup \partial_+ R_1$. Then, using the map

$$\partial_+ R_1 \longrightarrow S^1, \quad (\vartheta, t, x_1, \ldots, x_{n-2}) \longmapsto k\, \vartheta,$$

we extend $\pi \colon \partial N \setminus B \to S^1$ to a (well-defined) map

$$\hat{\pi} \colon \partial \hat{N} \setminus B = ((\partial N \setminus B) \setminus h(\partial_- R_1)) \cup \partial_+ R_1 \longrightarrow S^1,$$

which yields the required supporting open book decomposition of $\partial \hat{N}$ with binding B. □

Proof of Proposition 1.5.1. On the round 0-handles, choose a geodesible pair as in Lemma 1.5.2. Then using Lemma 1.5.3, we can successively attach the round 1-handles and extend the geodesible pair and the open book decomposition in each step. □

As an application of Proposition 1.5.1, we can prove the existence of geodesible vector fields on a certain family of manifolds (of arbitrary dimension).

Theorem 1.5.4. *For every $n, k \geq 1$, the manifold*

$$N := \Big(\mathop{\#}_{k} (S^1 \times S^{n-1}) \Big) \# \Big(\mathop{\#}_{k-1} (S^2 \times S^{n-2}) \Big)$$

admits a geodesible vector field.

Before proving Theorem 1.5.4, we need two preparatory lemmas. Recall that if M and N are two n-dimensional manifolds with nonempty boundaries, their **boundary connected sum** $M \natural N$ is defined as follows. Choose two embedded $(n-1)$-discs $D_1 \subset \partial M$ and $D_2 \subset \partial N$. Then $M \natural N := (M + N) \cup H_1$, where the 1-handle $H_1 = [0,1] \times D^{n-1}$ is attached to $M + N$ by gluing $\{0\} \times D^{n-1}$ to D_1 and $\{1\} \times D^{n-1}$ to D_2 (via the identity). As in the case of ordinary connected sums, this operation is independent of the choice of discs D_1 and D_2, and the resulting space admits a smooth structure.

Secondly, recall that the **double** $\mathcal{D}M$ of a manifold M with $\partial M \neq \emptyset$ is defined by taking two copies of M and gluing them along their boundaries via the identity map.

Lemma 1.5.5. *Let M_1 and M_2 be two n-dimensional manifolds with $\partial M_1, \partial M_2 \neq \emptyset$. Then*

$$\mathcal{D}M_1 \# \mathcal{D}M_2 \cong \mathcal{D}(M_1 \natural M_2).$$

Proof. Identify tubular neighbourhoods U_i of $\partial M_i \subset \mathcal{D}M_i$ with $[-1,1] \times \partial M_i$, for $i = 1, 2$, such that ∂M_i is identified with $\{0\} \times \partial M_i$. Choose embedded open balls $B_i^{n-1} \subset \partial M_i \subset U_i$, and let $B_i^n := (-1/2, 1/2) \times B_i^{n-1} \subset U_i$, as well as $\hat{B}_i^n := (-1/2, 0] \times B_i^{n-1}$. Write $\partial(M_i \setminus \hat{B}_i^n) = M_i^1 \cup M_i^2$, where

$$M_i^1 = \partial M_i \setminus B_i^{n-1}, \quad M_i^2 = \partial(M_i \setminus \hat{B}_i^n) \setminus M_i^1,$$

see Figure 1.6 below. Then

$$\begin{aligned}
\mathcal{D}M_1 \# \mathcal{D}M_2 &= (\mathcal{D}M_1 \setminus B_1^n) \cup_\partial (\mathcal{D}M_1 \setminus B_2^n) \\
&= \Big[(M_1 \setminus \hat{B}_1^n) \cup_{M_1^1} (M_1 \setminus \hat{B}_1^n) \Big] \cup_\partial \Big[(M_2 \setminus \hat{B}_2^n) \cup_{M_2^1} (M_2 \setminus \hat{B}_2^n) \Big] \\
&\cong \underbrace{\Big[\Big(M_1 \setminus \hat{B}_1^n \Big) \cup_{M_i^2} \Big(M_2 \setminus \hat{B}_2^n \Big) \Big]}_{\cong M_1 \natural M_2} \cup_\partial \underbrace{\Big[\Big(M_1 \setminus \hat{B}_1^n \Big) \cup_{M_i^2} \Big(M_2 \setminus \hat{B}_2^n \Big) \Big]}_{\cong M_1 \natural M_2} \\
&\cong \mathcal{D}(M_1 \natural M_2).
\end{aligned}$$

□

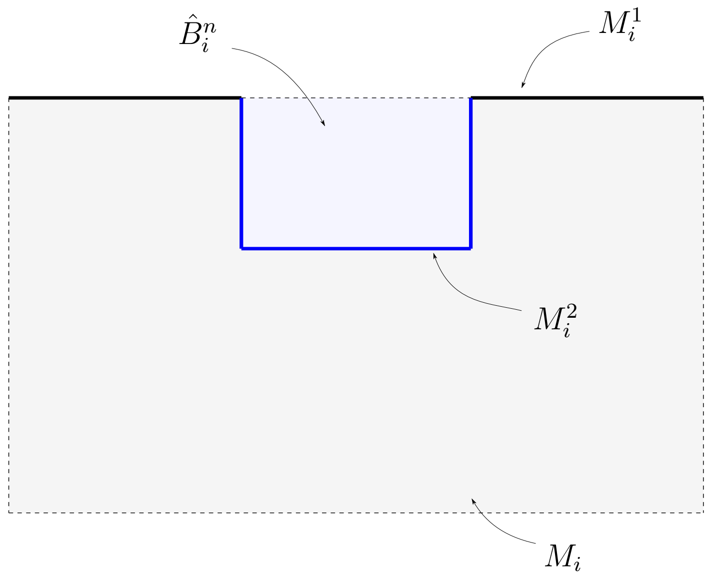

Figure 1.6: Decomposition of the boundary of $M_i \setminus \hat{B}_i^n$.

Lemma 1.5.6. *Let M be an n-dimensional manifold with $\partial M \neq \emptyset$. Then*

$$M \natural (S^1 \times D^{n-1}) \natural (S^2 \times D^{n-2}) \cong M \cup R_1.$$

Proof. Write $S^1 \times D^{n-1} = D^1_+ \times D^{n-1} \cup D^1_- \times D^{n-1}$, where D^1_+ and D^1_- denote the upper and lower hemisphere, respectively. Then

$$\begin{aligned} M \natural (S^1 \times D^{n-1}) &= M \natural (D^1_+ \times D^{n-1} \cup D^1_- \times D^{n-1}) \\ &\cong (\underbrace{M \natural (D^1_+ \times D^{n-1})}_{\cong M}) \cup D^1_- \times D^{n-1} \\ &\cong M \cup H_1, \end{aligned}$$

hence taking the boundary connected sum with $S^1 \times D^{n-1}$ amounts to attaching an (ordinary) 1-handle. Similarly, writing $S^2 \times D^{n-2} = D^2_+ \times D^{n-2} \cup D^2_- \times D^{n-2}$, we see that

$$M \natural (S^2 \times D^{n-2}) \cong M \cup (D^2_- \times D^{n-2}) \cong M \cup H_2,$$

so that taking the boundary connected sum with $S^2 \times D^{n-2}$ amounts to attaching a 2-handle. Then $M \natural (S^1 \times D^{n-1}) \natural (S^2 \times D^{n-2})$ is obtained from M by independently attaching a 1- and a 2-handle, so that $M \cong M \cup R_1$ by the fundamental lemma of round handles (Lemma 1.4.3). □

Proof of Theorem 1.5.4. Write

$$S^1 \times S^{n-1} = \mathcal{D}(S^1 \times D^{n-1}), \quad S^2 \times S^{n-2} = \mathcal{D}(S^2 \times D^{n-2}).$$

Then, by Lemma 1.5.5,

$$\begin{aligned}
N &= \Big(\mathop{\#}_{k} (S^1 \times S^{n-1}) \Big) \# \Big(\mathop{\#}_{k-1} (S^2 \times S^{n-2}) \Big) \\
&\cong \mathcal{D}\bigg(\Big(\mathop{\natural}_{k} S^1 \times D^{n-1} \Big) \natural \Big(\mathop{\natural}_{k-1} S^2 \times D^{n-2} \Big) \bigg) \\
&\cong \mathcal{D}\bigg(\underbrace{S^1 \times D^{n-1}}_{=:R_0} \natural \Big(\mathop{\natural}_{k-1} (S^1 \times D^{n-1}) \natural (S^2 \times D^{n-2}) \Big) \bigg) \\
&\cong \mathcal{D}\bigg(R_0 \cup \bigcup_{j=1}^{k-1} R_1^j \bigg),
\end{aligned}$$

where the last diffeomorphism is obtained from Lemma 1.5.6. Now $M := R_0 \cup \bigcup_j R_1^j$ admits a geodesible pair (X, α) tangent to ∂M by Proposition 1.5.1. Hence we can glue two copies of (M, X, α) in the obvious way to obtain a geodesible pair on $N = \mathcal{D}M$. $\square$

2

Induced contact structures and foliations

In Chapter 1, we have seen two basic examples of geodesic vector fields (or flows): the Hopf flow of S^3, and the geodesic flow on the unit tangent bundle of any Riemannian manifold. In both of these examples, the codimension-1 distribution $X^\perp$ consisting of tangent vectors orthogonal to X (where X is the unit vector field defining the flow) defines a contact structure; in fact, these contact structures are the standard one on S^3 and the natural one on STM. As we shall see, one may replace the Hopf fibration by any fibration of oriented great circles of S^3; the orthogonal distribution will always define a contact structure.

This motivates the question of whether or not the orthogonal distribution of a general geodesic vector field defines a contact structure. More precisely, we ask:

(1) Given a geodesic vector field X, are there suitable geometric assumptions which guarantee $X^\perp$ to define a contact structure?

(2) If $X^\perp$ defines a contact structure, what are its properties?

Here, geometric assumptions are meant to be assumptions on the geometry of M and X (for example, on the sectional curvature of M or the Ricci curvature of X).

We will mainly focus on the 3-dimensional case. The main reason is that in dimension 3, the contact condition assumes a much simpler form: A (nowhere vanishing) 1-form α on M^3 is contact if and only if the 2-form $\mathrm{d}\alpha$ is non-vanishing when restricted to $\ker\alpha$. In this case, regarding question (2), one might ask if the contact structure $\ker\alpha$ is tight or overtwisted, see Definition 2.1.11.

We will start in Section 2.1 by recalling some basic facts from contact geometry needed in the later parts. In Section 2.2, we will first review some known results concerning questions (1) and (2) above. For example, by a result of Aazami and Harris–Paternain [1, 40], if X is a geodesic vector field with positive Ricci curvature, then it always induces a contact structure (Theorem 2.2.4). We then show how to improve these results (Theorem 2.2.13 and Corollary 2.2.14). In Section 2.3, we first introduce the notion of adapted Jacobi fields and derive some basic facts about them. This allows us to reprove theorems by Gluck and Harrison about geodesic vector fields on space forms in a more general context (see Theorem 2.3.5 and Corollary 2.3.6). In Section 2.4, we define the Jacobi tensor of a geodesic vector field and prove a more general version of Theorem 2.2.13 for geodesic vector fields whose Jacobi tensor is parallel along flow lines. A particular example of these types of vector fields is given by geodesic vector fields on (locally) symmetric manifolds, which are — apart from space forms — the simplest type of Riemannian manifolds. We conclude in Section 2.5 with a discussion of geodesic vector fields whose orthogonal complement is integrable. In this case, we derive a bound on the total Ricci curvature of X with respect to an invariant measure in terms of a certain operator associated with X (Theorem 2.5.6).

2.1 Basic contact geometry

In this section we introduce some basic notions of contact geometry, mainly following [28].

Definition 2.1.1. Let M be a manifold of dimension $2n+1$. A **contact structure** is a maximally non-integrable hyperplane field $\xi \subset TM$. That is, writing ξ locally as $\xi = \ker\alpha$ for some (local) 1-form α, then $\alpha\wedge(\mathrm{d}\alpha)^n$ is a volume form, i.e. $\alpha\wedge(\mathrm{d}\alpha)^n \neq 0$ everywhere. The latter is called **contact condition**. Any such 1-form α is called **contact form**.

Remark 2.1.2. (i) In dimension 3, if α is a 1-form such that $\alpha \wedge \mathrm{d}\alpha \equiv 0$, then $\ker\alpha$ is integrable by Frobenius' theorem [26], i.e. there is a codimension-1 foliation whose leaves are everywhere tangent to ξ. In the case of a contact structure, the situation is completely opposite, since we require $\alpha \wedge \mathrm{d}\alpha$ to be *nowhere* vanishing. Therefore, a contact structure is in some sense as far from being integrable as possible, hence the terminology *maximally non-integrable.*

(ii) In general, it is not possible to find a *global* 1-form α such that $\xi = \ker\alpha$. The existence of such a 1-form is equivalent to ξ being coorientable, meaning that the line bundle TM/ξ is trivial.

Example 2.1.3. (i) Consider $\mathbb{R}^{2n+1}$ with coordinates $(x_1, y_1, \ldots, x_n, y_n, z)$ and the 1-form $\alpha_{\mathrm{st}} := \mathrm{d}z + \sum_{i=1}^n x_i\,\mathrm{d}y_i$. Then

$$\alpha_{\mathrm{st}} \wedge (\mathrm{d}\alpha_{\mathrm{st}})^n = n!\,\mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \ldots \wedge \mathrm{d}x_n \wedge \mathrm{d}y_n \wedge \mathrm{d}z,$$

hence $\ker\alpha_{\mathrm{st}}$ defines a contact structure. It is called the **standard contact structure on** $\mathbb{R}^{2n+1}$, see Figure 2.1.

(ii) Let (M, g) be a Riemannian manifold and TM its tangent bundle. Using the Riemannian metric g, there is a bundle isomorphism between TM and the cotangent bundle T^*M, namely $TM \ni u \mapsto g(u, \cdot) \in T^*M$. This induces a bundle metric g^* on T^*M by pushing forward the bundle metric g. Then one can consider the unit cotangent bundle $ST^*M := \{\lambda \in STM\colon |\lambda|_{g^*} = 1\}$. Denote by $\pi\colon ST^*M \to M$ the natural projection and $\mathrm{d}\pi\colon T(ST^*M) \to TM$ its differential. Then, the **tautological 1-form** λ on ST^*M, defined as $\lambda_u := u \circ \mathrm{d}_u\pi$ for $u \in ST^*M$, defines a contact form on ST^*M (see [28, Theorem 1.5.2]).

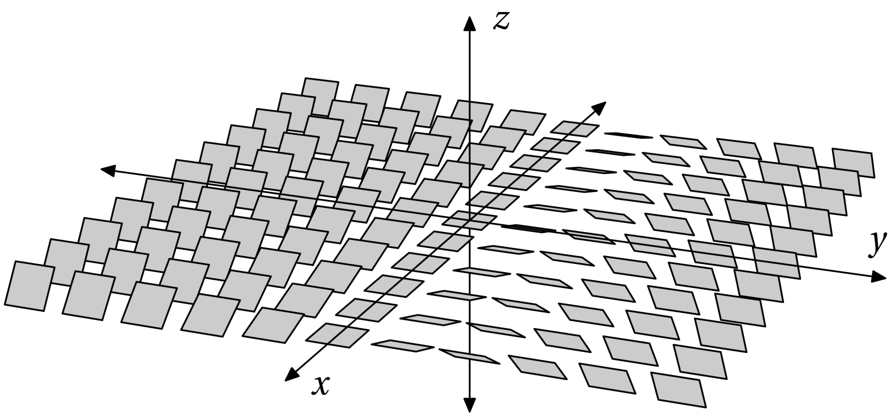

Figure 2.1: The standard contact structure on $\mathbb{R}^3$.
(Created by user Msr657, Wikipedia, retrieved 22 September 2023 at `https://en.wikipedia.org/wiki/File:Standard_contact_structure.svg`)

Definition 2.1.4. Let α be a contact form on M^{2n+1}. Then the vector field R_α, uniquely determined by the equations

$$i_{R_\alpha}\mathrm{d}\alpha = 0, \quad \alpha(R_\alpha) = 1,$$

is called the **Reeb vector field** of α.

Note that a Reeb vector field is associated with a contact form rather than a contact structure. If $\xi = \ker\alpha$ is a contact structure, and $\tilde\alpha$ is another contact form defining ξ, then $\tilde\alpha = \lambda\alpha$ for some function $\lambda\colon\ M \to \mathbb{R}\backslash\{0\}$. Then $\mathrm{d}\tilde\alpha = \mathrm{d}\lambda\wedge\alpha + \lambda\,\mathrm{d}\alpha$, so that $i_{R_\alpha}\mathrm{d}\tilde\alpha = \mathrm{d}\lambda(R_\alpha)\alpha - \mathrm{d}\lambda$, which is nonzero in general. In fact, the Reeb vector fields of α and $\tilde\alpha$ will have completely different dynamics in general, cf. [28, Example 2.2.5].

Example 2.1.5. (i) Consider the standard contact form on $\mathbb{R}^{2n+1}$ as defined in Example 2.1.3. Its Reeb vector field is given by ∂_z. In particular, the Reeb vector field spans a fibration by oriented, pairwise parallel lines.

(ii) Let

$$S^3 = \{(x_1, y_1, x_2, y_2) \in \mathbb{R}^4\colon\ x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1\} \subset \mathbb{R}^4$$

be the unit sphere in $\mathbb{R}^4$. Then the 1-form α on $\mathbb{R}^4$, defined by

$$\alpha = \frac{1}{2}\left(x_1 \mathrm{d}y_1 - y_1 \mathrm{d}x_1 + x_2 \mathrm{d}y_2 - y_2 \mathrm{d}x_2\right),$$

restricts to a contact form on TS^3, since

$$r\,\mathrm{d}r \wedge \alpha \wedge \mathrm{d}\alpha = \mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}y_2$$

is a volume form on $\mathbb{R}^4$, where $r^2 := x_1^2 + y_1^2 + x_2^2 + y_2^2$. The contact structure $\xi := \ker \alpha|_{TS^3}$ is called **standard contact structure on** S^3. Its Reeb vector field is given by

$$R = 2\left(x_1 \partial_{y_1} - y_1 \partial_{x_1} + x_2 \partial_{y_2} - y_2 \partial_{x_2}\right)$$

The claim is now that R spans the fibres of the Hopf fibration (Example 1.1.8). In particular, this means that R is (up to rescaling by a constant) a geodesic vector field. Indeed, the Hopf fibre through $(z_1, z_2) = (x_1 + \mathrm{i}y_1, x_2 + \mathrm{i}y_2) \in S^3 \subset \mathbb{C}^2$ can be parametrised by $t \mapsto (\mathrm{e}^{\mathrm{i}t} z_1, \mathrm{e}^{\mathrm{i}t} z_2) =: \gamma(t)$, so that

$$\dot\gamma(0) = \mathrm{i}\gamma(0) = (\mathrm{i}x_1 - y_1, \mathrm{i}x_2 - y_2),$$

hence $\dot\gamma(0) = \frac{1}{2}R$.

(iii) Consider the unit cotangent bundle ST^*M of a Riemannian manifold (M, g) equipped with its tautological (contact) 1-form λ (Example 2.1.3 (ii)). Using the identification $STM \cong ST^*M$ induced by g, we can think of λ as a contact form on STM. Now equip STM with the Sasaki metric g_S (see Definition B.8). Then one can show that $\lambda = g_S(\mathcal{G}, \cdot)$, where $\mathcal{G}$ denotes the (horizontal) geodesic vector field on STM, see [59, Lemma 1.37] for a proof, and Example 1.1.9 or Definition B.13 for the definition of the geodesic vector field on STM. In particular, the Reeb vector field of λ is given by $\mathcal{G}$ (see Remark 2.2.2).

Remark 2.1.6. In Examples 2.1.5 (ii) and (iii), the Reeb vector field R is geodesic, and the corresponding contact structure is given by its orthogonal complement. These are the first two examples of contact structures *induced* by geodesic vector fields, a concept that is going to be discussed in Section 2.2.

Definition 2.1.7. Two contact manifolds (M_1, ξ_1) and (M_2, ξ_2) are called **contactomorphic** if there is a **contactomorphism** between them, that is, a diffeomorphism $h\colon M_1 \to M_2$ such that $\mathrm{d}h_p((\xi_1)_p) = (\xi_2)_{h(p)}$ for every $p \in M_1$. Equivalently, if $\xi_1 = \ker \alpha_1$ and $\xi_2 = \ker \alpha_2$, then $h^*\alpha_2 = \lambda\,\alpha_1$ for some function $\lambda\colon M_1 \to \mathbb{R} \setminus \{0\}$. If $\lambda \equiv 1$, then h is called a **strict contactomorphism** between the strict contact manifolds (M_1, α_1) and (M_2, α_2).

Example 2.1.8. (i) On $\mathbb{R}^{2n+1}$ with coordinates $(x_1, y_1, \ldots, x_n, y_n, z)$, consider the contact form

$$\alpha := \mathrm{d}z + \sum_{j=1}^{n} r_j^2 \,\mathrm{d}\varphi_j = \mathrm{d}z + \sum_{j=1}^{n} (x_j \,\mathrm{d}y_j - y_j \,\mathrm{d}x_j),$$

where (r_j, φ_j) are polar coordinates of the (x_j, y_j)-plane, $j = 1, \ldots, n$. Consider the diffeomorphism

$$h\colon\ (\mathbf{x}, \mathbf{y}, z) \longmapsto \left(\frac{\mathbf{x}+\mathbf{y}}{2}, \frac{\mathbf{y}-\mathbf{x}}{2}, z + \frac{\mathbf{xy}}{2}\right),$$

where $\mathbf{x} := (x_1, \ldots, x_n)$, $\mathbf{y} := (y_1, \ldots, y_n)$ and $\mathbf{xy} := \sum_{j=1}^{n} x_j y_j$ [28, Example 2.1.3]. Then $h^*\alpha = \alpha_{\mathrm{st}}$, hence h defines a strict contactomorphism between $(\mathbb{R}^{2n+1}, \alpha)$ and $(\mathbb{R}^{2n+1}, \alpha_{\mathrm{st}})$. The contact structure $\ker \alpha$ is often called *standard cylindrically symmetric* contact structure.

(ii) Let $\theta\colon \mathbb{R} \to \mathbb{R}$ be a strictly monotone function (i.e. $\theta' \neq 0$ everywhere). Consider the contact form $\alpha := \sin\theta(z)\,\mathrm{d}x + \cos\theta(z)\,\mathrm{d}y$ on $\mathbb{R}^3$, and let $\alpha_{\mathrm{st}} := \mathrm{d}z + x\,\mathrm{d}y$ denote the standard contact form on $\mathbb{R}^3$. Then, the diffeomorphism

$$(x, y, z) \longmapsto \left(z \sin\theta(y) - \frac{x\cos\theta(y)}{\theta'(y)}, z\cos\theta(y) + \frac{x\sin\theta(y)}{\theta'(y)}, y\right).$$

pulls back α to α_{st}, hence $(\mathbb{R}^3, \alpha)$ and $(\mathbb{R}^3, \alpha_{\mathrm{st}})$ are strictly contactomorphic.

Using Moser's trick, one can show that on closed manifolds there are no non-trivial deformations of contact structures.

Theorem 2.1.9 (Gray stability)**.** *Let M be a closed manifold and ξ_t, $t \in [0,1]$ a smooth family of contact structures on M. Then there is an isotopy $(\Psi_t)_{t\in[0,1]}$ of M such that $\mathrm{d}\Psi_t(\xi_0) = \xi_t$. In particular, there is a contactomorphism between (M, ξ_0) and (M, ξ_t) for every $t \in [0,1]$.*

Proof. See [28, Theorem 2.2.2]. □

Remark 2.1.10. Note that Gray stability holds for contact structures but not for contact forms, which can be seen again in terms of the dynamics of Reeb vector fields [28, Example 2.2.5].

In dimension 3, there is an important dichotomy of *tight* and *overtwisted* contact structures, which we are now going to define.

Definition 2.1.11. Let (M, ξ) be a 3-dimensional contact manifold. An embedded disc $\Delta \subset M$ is called **overtwisted** if for all $p \in \partial\Delta$, the contact plane at p coincides with the tangent plane of Δ at p, that is, $\xi_p = T_p\Delta$.

A contact 3-manifold (M, ξ) is called **overtwisted** if it contains an embedded overtwisted disc. Otherwise, it is called **tight**.

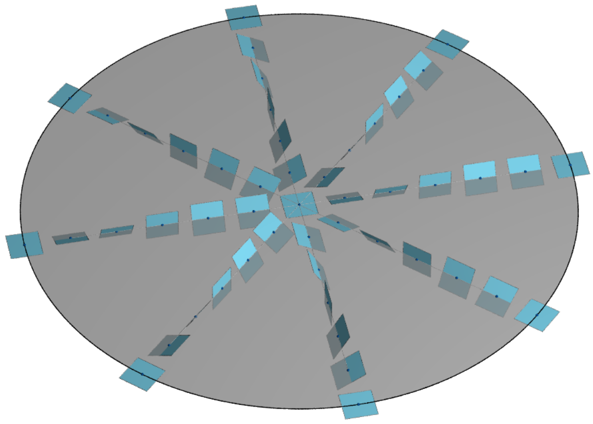

Figure 2.2: The contact structure ξ_{ot} with the overtwisted disc Δ.
(Created by user Pmassot, Wikimedia Commons, retrieved 22 September 2023 at `https://commons.wikimedia.org/wiki/File:Overtwisted_contact_structure.png`)

Example 2.1.12. (1) Consider $\mathbb{R}^3$ with cylindrical coordinates (r, θ, z). Let

$$\alpha_{\mathrm{ot}} = \cos r \,\mathrm{d}z + r \sin r \,\mathrm{d}\theta = \cos r \,\mathrm{d}z + f(r) r^2 \,\mathrm{d}\theta,$$

where f is the *smooth* function

$$f \colon r \longmapsto \begin{cases} \frac{\sin r}{r}, & \text{if } r \neq 0, \\ 1, & \text{if } r = 0. \end{cases}$$

Then α_{ot} defines a smooth 1-form on $\mathbb{R}^3$ (since the 1-form $r^2 \,\mathrm{d}\theta$ is smooth). One computes

$$\alpha_{\mathrm{ot}} \wedge \mathrm{d}\alpha_{\mathrm{ot}} = (1 + f(r)\cos r) r \,\mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}z = \underbrace{(1 + f(r)\cos r)}_{\neq 0} \,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z,$$

hence α_{ot} is a contact form defining the contact structure $\xi_{\mathrm{ot}} := \ker \alpha_{\mathrm{ot}}$. Now consider the disc $\Delta := \{z = 0, r \leq \pi\} \subset \mathbb{R}^3$. Then $\alpha_{\mathrm{ot}}|_{\partial\Delta} = -\mathrm{d}z$, hence $\xi_{\mathrm{ot}}|_{\partial\Delta} = T\partial\Delta$, so that Δ is an overtwisted disc for ξ_{ot}. See also Figure 2.2.

(2) The standard contact structures on $\mathbb{R}^3$ and S^3 (Examples 2.1.3 and 2.1.5 (ii)) are both tight. This is a highly non-trivial fact, see [28, Remark 4.6.37 and Corollary 6.5.10].

The notion of overtwistedness was introduced by Eliashberg in his seminal work [19], where he provided a complete classification of overtwisted contact structures: two overtwisted contact structures are isotopic (through contact structures) if and only if they are isotopic as plane fields. In contrast, tight contact structures are usually hard to classify. On $\mathbb{R}^3$ and S^3, however, there is only one tight contact structure up to contactomorphism. This is due to Eliashberg [20]; see also [28, Theorem 4.10.1].

Theorem 2.1.13 (Eliashberg)**.** *The manifolds S^3 and $\mathbb{R}^3$ admit a unique tight contact structure up to isotopy.*

Overtwisted discs also play an important role in the context of the prominent *Weinstein conjecture*, which asserts that the Reeb vector field of a contact form on a closed manifold always admits at least one periodic orbit. Hofer showed that the Weinstein conjecture is true for overtwisted contact structures on closed 3-manifolds; in this case, the Reeb vector field always admits a *contractible* periodic orbit [45]. In dimension 3, the Weinstein conjecture has been proven to be true in full generality by Taubes [70]; see also [47] for a nice survey on the topic.

There is also a boundary version of Hofer's result, which we record here for future reference.

Theorem 2.1.14 (Etnyre–Ghrist [24, Theorem 5.8])**.** *Let M be a 3-manifold with boundary and $\xi = \ker\alpha$ an overtwisted contact structure on M such that the Reeb vector field R_α is tangent to the boundary of M. Then R_α admits a contractible periodic orbit.*

2.2 Contact structures induced by geodesic vector fields

Definition 2.2.1. Let X be a geodesic vector field on a Riemannian manifold (M^{2n+1}, g). We say that X **induces a contact structure** if the orthogonal plane field $X^\perp$ defines a contact structure. Equivalently, if $\alpha = i_X g$, then X induces a contact structure if and only if $\alpha \wedge (\mathrm{d}\alpha)^n \neq 0$ everywhere.

Remark 2.2.2. Note that if X induces a contact structure, then X is the Reeb vector field of its dual (contact) 1-form $\alpha = i_X g$. Indeed, we have that $i_X \mathrm{d}\alpha = 0$ by the proof of Wadsley's characterisation (Proposition 1.1.2), and $\alpha(X) = |X|^2 = 1$.

As discussed at the beginning of this chapter, the question now is whether or not a given geodesic vector field X induces a contact structure, given some (geometric) information on X or the underlying manifold. To start off, we present some known results. The first one is due to Gluck.

Theorem 2.2.3 (Gluck [34])**.** *Let X be a geodesic vector field on the round 3-sphere, i.e. X spans a fibration by oriented great circles. Then X induces a contact structure which is diffeomorphic to the standard one.*

The proof of the first part of the theorem uses a description of great circle fibrations as certain submanifolds of the Grassmannian $\mathrm{Gr}_2(\mathbb{R}^4)$ found in [35]. For the second part of the theorem, one uses the fact that every fibration of S^3 by oriented great circles may be isotoped through such fibrations to a Hopf fibration [35, Theorem D]; the statement then follows from Gray stability (Theorem 2.1.9).

Concerning the first part of Theorem 2.2.3, we actually have the following more general result, which is due to Aazami and, independently, Harris and Paternain.

Theorem 2.2.4 (Aazami [1], Harris–Paternain [40])**.** *Let M be a compact Riemannian 3-manifold and X a geodesic vector field on M. If $\operatorname{Ric} X > 0$ everywhere, then X induces a contact structure.*

Remark 2.2.5. (1) Harris and Paternain show in fact that if X is a geodesic vector field and $X^\perp$ is *not* contact at some point $p \in M$, then the orbit of X through p must be free of conjugate points. This, however, is not compatible with the assumption $\operatorname{Ric} X_p > 0$ (cf. [15, Theorem 2.12]).

(2) The assumption of $\operatorname{Ric} X > 0$ is of course much weaker than having constant sectional curvature. It is not known yet whether or not, under this weaker assumption, the induced contact structure is tight. More specifically, if one assumes M^3 to have positive Ricci curvature everywhere (in which case the assumption of Theorem 2.2.4 is satisfied), then the universal cover of M is diffeomorphic to S^3 by a theorem of Hamilton [39]. Then, one might ask whether the induced contact structure lifts to the standard one on S^3. For example, if one assumes the metric to be *compatible* with the contact structure $\xi = \ker \alpha$ (i.e. $\star \mathrm{d}\alpha = c\alpha$ for some constant c, where α is the unit contact form defining ξ and $\star$ denotes the Hodge star operator), and the sectional curvature is positive and 1/4-pinched, then ξ lifts to the standard contact structure on S^3. This is known as the '1/4-pinched contact sphere theorem', which was first proven by Etnyre, Komendarczyk and Massot for a 5/9-pinched metric [25], and was later improved by Ge and Huang to the case of a 1/4-pinching constant [27]. Now it is not hard to show that the Reeb vector field R_α of a compatible (unit) contact form is geodesic and orthogonal to the contact structure [25, Lemma 2.3]. In particular, the contact structure ξ is induced by the geodesic vector field R_α. One might ask now if the 1/4-pinched contact sphere theorem is still true in the more general setting of contact structures induced by geodesic vector fields. To the best of the author's knowledge, this question is still open, even for pinching constants larger than 1/4.

For flat 3-manifolds, there is a result by Harrison similar to the one by Gluck.

Theorem 2.2.6 (Harrison [42, 43]). *Let X be a geodesic vector field on 3-dimensional Euclidean space $\mathbb{E}^3$ (or, more generally, any complete flat 3-manifold). Then X induces a contact structure if and only if for all $p \in \mathbb{E}^3$, the map $X_p^\perp \ni v \mapsto \nabla_v X(p)$ is non-zero.*

We provide a proof for Theorems 2.2.3 and 2.2.6 in Section 2.3 (see Theorem 2.3.5 and Corollary 2.3.6) that is similar to the proof given in [43]. However, our proof is in the context of more general geodesic foliations, and we avoid using special geometric features of great circle or line fibrations.

Let us remark that Theorems 2.2.3 and 2.2.6 are not true in higher odd dimensions; see [36] for the case of great circle fibrations, and [3], [43] for the case of line fibrations.

As in Theorem 2.2.3, we again have the following standardness result.

Theorem 2.2.7 (Harrison [42], Becker–Geiges [5]). *Let X be a geodesic vector field on $\mathbb{E}^3$. If X induces a contact structure ξ, then ξ is diffeomorphic to the standard contact structure on $\mathbb{R}^3$.*

Harrison proved the statement above in the case where the given line fibration admits a fibre that is not parallel to any other fibre. The general case was proved in [5]. There, the authors show that if the fibration does not have the mentioned geometric feature, then it must already be of a simple '1-parameter' type (see Section 3.5 and Figure 3.2), in which case the induced contact structure can be described explicitly (see also Example 2.1.8 (ii)).

We will in fact prove a more general statement in Section 3.7 (Theorem 3.7.4): If X is a geodesic vector field on $\mathbb{E}^3$ that is given (up to rescaling) as the Reeb vector field of a contact form, then the corresponding contact structure is tight and therefore diffeomorphic to the standard one by Theorem 2.1.13.

In order to strengthen some of the results above, we first need to derive some equivalent formulations of the contact condition for plane fields orthogonal to geodesic fields. Let X be a geodesic vector field on an orientable Riemannian 3-manifold $(M, g = \langle \cdot, \cdot \rangle)$ and $\alpha = i_X g$ its dual 1-form. To start off, consider a point $p \in M$ and tangent vectors $v, w \in X_p^\perp$. Extend v and W arbitrarily to local vector fields V and W. Then

$$\begin{aligned} \mathrm{d}\alpha(V, W) &= V(\alpha(W)) - W(\alpha(V)) - \alpha([V, W]) \\ &= V\langle X, W\rangle - W\langle X, V\rangle - \langle X, [V, W]\rangle \\ &= \langle \nabla_V X, W\rangle - \langle \nabla_W X, V\rangle + \langle X, \underbrace{\nabla_V W - \nabla_W V - [V, W]}_{=0}\rangle \\ &= \langle \nabla_V X, W\rangle - \langle \nabla_W X, V\rangle. \end{aligned}$$

Evaluating this at p yields

$$\mathrm{d}\alpha_p(v,w) = \langle (\nabla_v X)_p, w\rangle - \langle (\nabla_w X)_p, v\rangle. \tag{2.1}$$

Now, define the linear bundle morphism

$$\beta := \nabla X \colon\ X^\perp \longrightarrow X^\perp, \quad v \longmapsto \nabla_v X.$$

Note that since $\langle X, X\rangle \equiv 1$ we have that $0 = X\langle X, X\rangle = 2\langle \beta(X), X\rangle$, hence the image of β is indeed contained in $X^\perp$. Then (2.1) translates into

$$\mathrm{d}\alpha(v,w) = \langle \beta(v), w\rangle - \langle v, \beta(w)\rangle. \tag{2.2}$$

Now fix an orientation of M. For $p \in M$, consider the endomorphism $J_p\colon\ X_p^\perp \to X_p^\perp$ defined by $J_p(v) := w$ and $J_p(w) := -v$, where v, w is an oriented orthonormal basis of $X_p^\perp$. Here, the orientation of $X_p^\perp$ is chosen so that X_p, v, w form an oriented basis of T_pM.

Proposition 2.2.8. *Let X be a geodesic vector field on an oriented Riemannian 3-manifold $(M, g = \langle\cdot,\cdot\rangle)$, and $\alpha = i_X g$ its dual 1-form. Then, for any point $p \in M$, the following are equivalent.*

(1) $(\alpha \wedge \mathrm{d}\alpha)_p = 0$*;*

(2) β_p *is self-adjoint;*

(3) $\mathrm{tr}\,(\beta_p \circ J_p) = 0$*;*

(4) $\langle \mathrm{curl}\, X_p, X_p\rangle = 0$*;*

(5) $\mathrm{curl}\, X_p = 0$.

Proof. The equivalence of (1) and (2) follows directly from (2.2), since $(\alpha \wedge \mathrm{d}\alpha)_p = 0$ if and only if $\mathrm{d}\alpha_p = 0$ (since $\alpha(X) = 1$ and $i_X \mathrm{d}\alpha = 0$). By definition of J_p, we have

$$\mathrm{tr}\,(\beta_p \circ J_p) = \langle \beta_p \circ J_p(v), v\rangle + \langle \beta_p \circ J_p(w), w\rangle = \langle \beta_p(w), v\rangle - \langle \beta_p(v), w\rangle = \mathrm{d}\alpha_p(w,v),$$

from which the equivalence of (1) and (3) follows. The equivalence of (4) and (5) follows from the fact that $\mathrm{curl}\, X$ and X are multiples of each other (Proposition 1.1.6). Now, we are going to prove the equivalence of (1) and (4) which will finish the proof. Let vol_g denote the Riemannian volume form on M (which exists since M is assumed to be oriented). Write

$$\alpha \wedge \mathrm{d}\alpha = \lambda\, \mathrm{vol}_g$$

for some function $\lambda \in C^\infty(M)$. Plugging $\operatorname{curl} X = \langle \operatorname{curl} X, X\rangle X$ into both sides of this equation yields (by definition of the curl)

$$\langle \operatorname{curl} X, X\rangle \, \mathrm{d}\alpha = \lambda \, \mathrm{d}\alpha,$$

hence $\lambda = \langle \operatorname{curl} X, X\rangle$. Therefore, $(\alpha \wedge \mathrm{d}\alpha)_p = 0$ if and only if $\langle \operatorname{curl} X_p, X_p\rangle = 0$. $\square$

Remark 2.2.9. For a given point $p \in M$, denote by $\mathrm{Orb}_X(p)$ the orbit of X through p. Then the fact that $L_X\alpha = 0$ implies that $X^\perp$ is contact either at every or at none of the points in $\mathrm{Orb}_X(p)$.

Example 2.2.10. (i) Let Σ be an orientable (not necessarily closed) surface and $ST\Sigma$ its unit tangent bundle, equipped with the Sasaki metric. Let $\mathcal{V}$ be the geodesic vector field on STM spanning the (oriented) vertical fibres of STM (see Definition B.14 in Appendix B). Then $\mathcal{V}_u = (Ju)^v$ for $u \in T_pM$, where J denotes the almost complex structure on $T\Sigma$ given by rotation by $\pi/2$. Hence, by Proposition B.9,

$$\beta_u(u^h) = \left(\nabla^S_{u^h}\mathcal{V}\right)_u = \frac{1}{2}\left(R_p(u, Ju)u\right)^h_u,$$

and

$$\beta_u((Ju)^h) = \left(\nabla^S_{(Ju)^h}\mathcal{V}\right)_u = \frac{1}{2}\left(R_p(u, Ju)Ju\right)^h_u.$$

hence if $\alpha = g_s(\mathcal{V}, \cdot)$ denotes the 1-form dual to $\mathcal{V}$, then by (2.2),

$$\mathrm{d}\alpha_u(u^h, (Ju)^h) = \frac{1}{2}\left(R_p(u, Ju, u, Ju) - R_p(u, Ju, Ju, u)\right) = -K_p, \qquad (2.3)$$

where $K_p = R_p(u, Ju, Ju, u)$ denotes the Gauß curvature of Σ at p. Now let $\mathcal{G}$ denote the (horizontal) geodesic vector field on STM and consider the contact forms $\lambda_1 = g_S(\mathcal{G}, \cdot)$ and $\lambda_2 = g_S(J\mathcal{G}, \cdot)$ (where J is thought of as an almost complex structure on HM using the identification $HM \cong TM$). Recall that $\mathcal{G}_u = u^h$ for $u \in T_pM$. Then $i_{\mathcal{V}}\mathrm{d}\alpha = 0$ implies that $\mathrm{d}\alpha$ is a multiple of $\lambda_1 \wedge \lambda_2$, and (2.3) shows that in fact

$$\mathrm{d}\alpha = -\pi^* K \, \lambda_1 \wedge \lambda_2.$$

This is known as one of Cartan's structural equations, see [67, Section 7.2]. In particular, it follows that $\mathcal{V}$ induces a contact structure if and only if the Gauß curvature of Σ is nowhere vanishing.

(ii) Let G be a compact Lie group equipped with a bi-invariant metric, and X a left-invariant vector field on G. Then X is geodesic (Example 1.1.13), and the sectional curvatures of planes containing X are given by $K(X, V) =$

$1/4\big|[X,V]\big|^2$ for any vector field V (see [16, Corollary 3.19]). Now let $\rho \in G$ and $v \in X_\rho^\perp$, $|v| = 1$. Extend v to a local vector field V invariant under the flow of X as in Example 1.1.12. Then

$$K(X_\rho, v) = \frac{1}{4}\big|[X,V]_\rho\big|^2 = \frac{1}{4}\big|(\nabla_v X)_\rho\big|^2 = \frac{1}{4}\big|\beta_\rho(v)\big|^2 \geq 0.$$

Hence, using Theorem 2.2.4, X induces a contact structure if and only if $\beta_\rho \neq 0$, or equivalently, $\operatorname{Ric}(X_\rho) > 0$ for every $\rho \in G$.

The following proposition gives a relation between the map β and the Ricci curvature of X that will be crucial for the remainder of the chapter.

Proposition 2.2.11. *Assume that $X^\perp$ is not contact at $p \in M$. Then, β is self-adjoint along* $\operatorname{Orb}_X(p)$ *and*

$$X(\operatorname{tr}\beta) = -(\operatorname{Ric} X + \lambda^2 + \mu^2),$$

where λ and μ are the (real) eigenvalues of β.

For the proof we need the following result by Harris and Paternain. Here, $\beta' := \nabla_X \beta$ denotes the covariant derivative of β in the direction of X.

Proposition 2.2.12 (Harris–Paternain [40])**.** *Let X be a geodesic vector field. Then β satisfies the Riccati equation*

$$\beta' + \beta^2 + R(\cdot, X)X = 0. \tag{2.4}$$

Proof. This is a pointwise statement, hence it is enough to show that for every $p \in M$ and $v \in T_pM$ we have that $\beta'(v) + \beta^2(v) + R(v, X_p)X_p = 0$. Extend v to a parallel vector field V along the integral curve of X through p. Then, we compute

$$\begin{aligned} R(V,X)X &= \nabla_V \underbrace{\nabla_X X}_{=0} - \nabla_X\nabla_V X - \nabla_{[V,X]}X \\ &= -\nabla_X\nabla_V X - \nabla_{\nabla_V X}X + \nabla_{\underbrace{\nabla_X V}_{=0}}X \\ &= -\nabla_X(\beta(V)) - \beta^2(V) \\ &= -\beta'(V) - \beta(\underbrace{\nabla_X V}_{=0}) - \beta^2(V). \end{aligned}$$

□

Proof of Proposition 2.2.11. The first part follows from Proposition 2.2.8. Note that if β is self-adjoint, then both of its eigenvalues λ and μ are real. Taking the trace on both sides of equation (2.4), we obtain

$$\operatorname{tr}\beta' + \operatorname{tr}\beta^2 + \operatorname{Ric} X = 0.$$

Since β is self-adjoint with eigenvalues λ and μ, the eigenvalues of β^2 are given by λ^2 and μ^2, so that $\operatorname{tr}\beta^2 = \lambda^2 + \mu^2$. Hence, the only thing left to show is that $\operatorname{tr}\beta' = X(\operatorname{tr}\beta)$. Let $p \in M$ and let E_1, E_2 be parallel vector fields along $\operatorname{Orb}_X(p)$ such that $E_{1,q}$, $E_{2,q}$ form an orthonormal basis of $X_q^\perp$ for every $q \in \operatorname{Orb}_X(p)$. Then

$$\begin{aligned}\operatorname{tr}\beta' &= \langle \beta'(E_1), E_1\rangle + \langle \beta'(E_2), E_2\rangle \\ &= \langle \nabla_X(\beta(E_1)), E_1\rangle + \langle \beta(\underbrace{\nabla_X E_1}_{=0}), E_1\rangle + \langle \nabla_X(\beta(E_2)), E_2\rangle + \langle \beta(\underbrace{\nabla_X E_2}_{=0}), E_2\rangle \\ &= \langle \nabla_X(\nabla_{E_1}X), E_1\rangle + \langle \nabla_X(\nabla_{E_2}X), E_2\rangle \\ &= X(\operatorname{tr}\beta) - \langle \nabla_{E_1}X, \underbrace{\nabla_X E_1}_{=0}\rangle - \langle \nabla_{E_2}X, \underbrace{\nabla_X E_2}_{=0}\rangle. \end{aligned}$$ □

Theorem 2.2.13. *Let X be a complete geodesic vector field on a (not necessarily closed) Riemannian 3-manifold M. Assume that*

$$\operatorname{Ric} X + \frac{|\lambda - \mu|^2}{2} \geq 0$$

everywhere, where λ and μ are the (complex) eigenvalues of β. Then, if $X^\perp$ is not contact at $p \in M$, either one of the following is true:

(1) $\operatorname{Ric}(X_p) < 0$*, or*

(2) $\operatorname{Ric}(X_p) = 0$ *and* $\beta_p = 0$*.*

Corollary 2.2.14. *Let X be a complete geodesic vector field on a (not necessarily closed) Riemannian 3-manifold M. Assume that* $\operatorname{Ric}(X) \geq 0$ *everywhere and* $\beta_p \neq 0$ *if* $\operatorname{Ric}(X_p) = 0$*. Then X induces a contact structure.*

Proof of Corollary 2.2.14. This follows immediately from Theorem 2.2.13, since X satisfies the assumption in the theorem. □

Proof of Theorem 2.2.13. The idea of the proof is similar to that of [1, Proposition 1], making use of Proposition 2.2.11. Assume that $X^\perp$ is not contact at $p \in M$. Parametrise the orbit through p by $t \mapsto \gamma(t)$, where $\gamma(0) = p$. Consider the one-parameter family $t \mapsto \beta_t := \beta_{\gamma(t)}$. Then, by Proposition 2.2.11,

$$\begin{aligned} X(\operatorname{tr}\beta) &= -\left(\operatorname{Ric}(X) + \lambda^2 + \mu^2\right) \\ &= -\left(\operatorname{Ric}(X) + \frac{(\lambda-\mu)^2}{2} + \frac{(\operatorname{tr}\beta)^2}{2}\right) \\ &\leq -\frac{(\operatorname{tr}\beta)^2}{2}. \end{aligned} \tag{2.5}$$

Now, assume for the moment that there is some $t_0 \in \mathbb{R}$ such that $\operatorname{tr}\beta_{t_0} \neq 0$. After replacing X by $-X$, if necessary, we may assume that $\operatorname{tr}\beta_{t_0} < 0$. Now let f be the unique solution of the initial value problem

$$\begin{cases} f' + \frac{1}{2}f^2 = 0, \\ f(0) = \operatorname{tr}\beta_{t_0}. \end{cases} \tag{2.6}$$

Then, comparing (2.5) and (2.6), we see that $\operatorname{tr}\beta \leq f$ everywhere, and f is given by $f(t) = ((t/2) + (1/f(0)))^{-1}$. Thus $f(t) \to -\infty$ as $t \to 2$, and therefore also $\operatorname{tr}\beta_t \to -\infty$ as $t \to 2$. But this cannot happen since X is assumed to be complete. It follows that $\operatorname{tr}\beta_t \equiv 0$. Hence, by (2.5), $\operatorname{Ric}(X) + (\lambda - \mu)^2/2 = 0$ along γ. In particular we have that either $\operatorname{Ric}(X_p) < 0$ or $\operatorname{Ric}(X_p) = 0$ and $\lambda_p = \mu_p$. In the second case, since $0 = \operatorname{tr}\beta_p = \lambda_p + \mu_p$, we obtain that $\lambda_p = \mu_p = 0$, thus $\beta_p = 0$ which proves the claim. □

2.3 Adapted Jacobi fields and space forms

In this section, we present an alternative proof of Theorems 2.2.3 and 2.2.6 using a notion of *adapted Jacobi fields*, which can be described as follows. Let X be a complete geodesic vector field on a Riemannian manifold (of any dimension). Consider a geodesic variation Γ consisting of integral curves of X, i.e.

$$\Gamma\colon (-\varepsilon, \varepsilon) \times \mathbb{R} \longrightarrow M, \quad \Gamma(s,t) = \phi^t(\exp_p(sv)), \tag{2.7}$$

where $p \in M$, $v \in T_pM$ and ϕ^t is the time-t flow of X. Here, ε is chosen small enough so that $\exp_p(sv)$ is defined for $|s| < \varepsilon$.

Definition 2.3.1. A Jacobi field J is called **adapted to** X if it is given as the variational field of a variation through integral curves of X, i.e. $J(t) = \partial_s|_{s=0}\Gamma(s,t)$, with Γ as in (2.7).

Adapted Jacobi fields (although not under this name) were already studied by Godoy and Salvai in [37]. For a Jacobi field J, denote by $J' := \nabla_X J$ the covariant derivative of J in the direction of X. The following was first observed in [37].

Proposition 2.3.2. *If J is a Jacobi field adapted to a geodesic vector field, then $J' = \beta(J)$.*

Proof. Let Γ be as in (2.7) and set $\gamma := \Gamma(0, \cdot)$. Then

$$J'(t) = \mathrm{D}_t\, \partial_{s|_{s=0}}\Gamma(s,t) = \mathrm{D}_{s|_{s=0}}\, \partial_t\Gamma(s,t) = \mathrm{D}_{s|_{s=0}}\, X_{\Gamma(s,t)} = (\nabla_{J(t)}X)_{\gamma(t)} = \beta_{\gamma(t)}(J(t)),$$

where in the second equation, we used the symmetry lemma [55, Lemma 6.3]. □

Corollary 2.3.3. *If J is a Jacobi field adapted to a geodesic vector field X, then J and X commute, i.e.* $[J, X] = 0$.

Proof. Using Proposition 2.3.2, we obtain $\nabla_X J = \beta(J) = \nabla_J X$, hence $[J, X] = \nabla_J X - \nabla_X J = 0$. □

Corollary 2.3.4. *Let J be a Jacobi field adapted to the geodesic vector field X. Then $J \equiv 0$ if and only if $J(t_0) = 0$ for some $t_0 \in \mathbb{R}$.*

Proof. The Jacobi field J solves the differential equation $J'' + R(J, X)X = 0$. In particular, J is determined completely by the values of J and J' at any given point. From Proposition 2.3.2 it follows that J' is determined by J, which proves the claim. □

Now, assume that M is a 3-dimensional space form, i.e. M has constant sectional curvature. Then we obtain the following.

Theorem 2.3.5. *Let X be a geodesic vector field on a Riemannian 3-manifold M of constant sectional curvature c, and $\beta_p = \nabla X\colon X_p^{\perp} \to X_p^{\perp}$, $p \in M$. Then:*

- *If $c > 0$, then β_p does not admit any real eigenvalues.*
- *If $c = 0$ and λ is a real eigenvalue of β_p, then $\lambda = 0$.*
- *If $c < 0$ and λ is a real eigenvalue of β_p, then $|\lambda| \leq \sqrt{|c|}$.*

Proof. Assume that β_p has a real eigenvalue λ for some $p \in M$, and let $v \in X_p^{\perp}$ be a corresponding eigenvector of unit length. Let γ be the integral curve of X through p. Let J be the Jacobi field along γ adapted to X with initial condition $J(0) = v$. Then, by Proposition 2.3.2, $J'(0) = \beta(J(0)) = \lambda v$. Now, since M has constant sectional curvature equal to c, the Jacobi equation translates into $J'' + cJ = 0$. Hence, for $c > 0$, J is given by

$$J(t) = \left(\cos(\sqrt{c}t) + \frac{\lambda}{\sqrt{c}}\sin(\sqrt{c}t)\right)E(t),$$

where E is the parallel vector field along γ satisfying $E(0) = v$. We should stress at this point that the λ above is a constant in this case (it is the eigenvalue of β_p), rather than a function of t as in the proof of Theorem 2.2.13. Now we have $J(t_0) = 0$ for

$$t_0 = \frac{1}{\sqrt{c}}\operatorname{arccot}\left(\frac{-\lambda}{\sqrt{c}}\right),$$

hence $J \equiv 0$ by Corollary 2.3.4, contradicting the fact that $J(0) = v \neq 0$.

If $c = 0$, then J is given by $J(t) = (1 + \lambda t)E(t)$ with E as before. If $\lambda \neq 0$, then $J(-1/\lambda) = 0$, which contradicts Corollary 2.3.4 again.

Finally, assume that $c < 0$. Then J is given by

$$J(t) = \left(\cosh(\sqrt{|c|}t) + \frac{\lambda}{\sqrt{|c|}}\sinh(\sqrt{|c|}t)\right)E(t),$$

with E as before. Then, if $|\lambda| > \sqrt{|c|}$, we have that $J(t_0) = 0$ for

$$t_0 = \frac{1}{\sqrt{|c|}}\operatorname{arcoth}\left(\frac{-\lambda}{\sqrt{|c|}}\right)$$

which contradicts Corollary 2.3.4 again. Therefore, we must have $|\lambda| \leq \sqrt{|c|}$. This finishes the proof. $\square$

Together with Proposition 2.2.8, we obtain the following statement, (partly) reproducing Theorems 2.2.3 and 2.2.6.

Corollary 2.3.6. *Let M be a Riemannian 3-manifold of constant sectional curvature c and let X be a geodesic vector field on M. Then:*

- *If $c > 0$, then X induces a contact structure.*
- *If $c = 0$, then X induces a contact structure if and only if β is nowhere vanishing.*
- *If $c < 0$, then X induces a contact structure if and only if for all $p \in M$, there is no orthonormal basis of $X_p^\perp$ consisting of eigenvectors of β_p with corresponding eigenvalues of absolute value $\leq \sqrt{|c|}$.*

Proof. If $X^\perp$ is not contact at p, then β_p is self-adjoint by Proposition 2.2.8, hence there is an orthonormal basis of $X_p^\perp$ consisting of eigenvectors of β_p corresponding to real eigenvalues λ_p and μ_p. If $c > 0$ then this contradicts Theorem 2.3.5, hence $X^\perp$ is contact in this case. If $c = 0$ then $\lambda_p = \mu_p = 0$ by Theorem 2.3.5, hence $\beta_p = 0$. If $c < 0$, then $|\lambda|, |\mu| \leq \sqrt{|c|}$.

In the latter two cases, the conditions are clearly necessary by (2.2). $\square$

2.4 Geodesic vector fields with parallel Jacobi tensor

Given a geodesic vector field X, consider the **Jacobi tensor** R_X defined by

$$R_X\colon X^\perp \longrightarrow X^\perp, \quad v \longmapsto R_X(v) := R(v, X)X.,$$

where R denotes the Riemann curvature tensor. We begin with the following simple observation.

Lemma 2.4.1. *R_X is a self-adjoint operator. In particular, for every $p \in M$, there is an orthonormal basis e_1, e_2 of $X_p^\perp$ consisting of eigenvectors of R_{X_p} with corresponding eigenvalues*

$$\Delta = \max_{u \in X_p^\perp \setminus \{0\}} K(u, X_p), \quad \delta = \min_{u \in X_p^\perp \setminus \{0\}} K(u, X_p).$$

Proof. Write $R(v,w,y,z) := \langle R(v,w)y, z\rangle$. From the symmetry properties of the curvature tensor R (cf. [55, Proposition 7.4]) we see that

$$\begin{aligned} \langle R_X(v), w\rangle = R(v,X,X,w) &= R(X,w,v,X) \\ &= -R(w,X,v,X) \\ &= R(w,X,X,v) \\ &= \langle R_X(w), v\rangle, \end{aligned}$$

hence R_X is self-adjoint. The rest follows from basic linear algebra. □

Now assume that R_X is parallel along orbits of X, i.e. if $\nabla_X R_X = 0$. Then the Jacobi equations assume a much simpler form.

Lemma 2.4.2. *Let X be a geodesic vector field such that R_X is parallel along orbits of X, and let J be a Jacobi field adapted to X. Let $e_1, e_2 \in X_p^\perp$ and $\Delta, \delta \in \mathbb{R}$ be as in Lemma 2.4.1. Extend e_1 and e_2 to vector fields E_1 and E_2 parallel along the integral curve of X through p. Then writing $J = J_1E_1 + J_2E_2$, we have that*

$$\begin{cases} J_1'' + \Delta J_1 = 0, \\ J_2'' + \delta J_2 = 0. \end{cases} \tag{2.8}$$

Proof. Using the fact that R_X is parallel we obtain

$$0 = (\nabla_X R_X)(E_i) = \nabla_X(R_X(E_i)) - R_X(\underbrace{\nabla_X E_i}_{=0}) = \nabla_X(R_X(E_i)),$$

hence the vector fields $R_X(E_i)$ are parallel, too. Since $R_X(e_1) = \Delta e_1$, it follows that $R_X(E_1) = \Delta E_1$, and similarly, $R_X(E_2) = \delta E_2$. Now given any adapted Jacobi field $J = J_1E_1 + J_2E_2$ through p, we have that $R_X(J) = \Delta J_1 E_1 + \delta J_2 E_2$, hence the Jacobi equations become

$$0 = J'' + R_X(J) = (J_1'' + \Delta J_1)\,E_1 + (J_2'' + \delta J_2)\,E_2,$$

hence $J_1'' + \Delta J_1 = 0$ and $J_2'' + \delta J_2 = 0$. □

We are now ready to prove the main result of this section, generalising Theorem 2.2.13 and Corollary 2.2.14 for geodesic vector fields whose Jacobi tensor is parallel along flow lines.

Theorem 2.4.3. *Let X be a complete geodesic vector field on a Riemannian 3-manifold M. Assume that $\nabla_X R_X = 0$ and for every $p \in M$, either one of the following holds:*

(i) $\max_{v \in X_p^\perp \setminus \{0\}} K(v, X_p) > 0$, *or*

(ii) $\max_{v \in X_p^\perp \setminus \{0\}} K(v, X_p) = 0$ *and* $\operatorname{rank} \beta_p = 2$,

where $K(\cdot, \cdot)$ denotes the sectional curvature. Then X induces a contact structure.

Remark 2.4.4. The condition $\nabla_X R_X = 0$ in the theorem above is satisfied for example if X is a Killing vector field, or if M is a locally symmetric space, i.e. if $\nabla R = 0$. In the latter case, however, we do not really produce new examples. This is because the universal cover of a complete 3-dimensional locally symmetric space is either a space form (in which case Theorem 2.3.5 applies), or equal to $S^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$ (see [7, Corollary 7.74 and Theorem 7.76]). In the case of $S^2 \times \mathbb{R}$, we do not know whether there are non-trivial geodesic foliations (see Proposition 2.4.7 and the remark thereafter). In the case of $\mathbb{H}^2 \times \mathbb{R}$, a geodesic vector field X satisfying $\max K(\cdot, X_p) \geq 0$ for every p would need to be tangent to the $\mathbb{H}^2$-fibres; however, in that case $\operatorname{rank} \beta \leq 1$, so that Theorem 2.4.3 does not apply.

Proof of Theorem 2.4.3. We argue by contradiction. Assume that $X^\perp$ is not contact at p. Let Δ and δ be as in Lemma 2.4.1. Then, by assumption, $\Delta \geq 0$. We may also assume that $\delta < 0$, for otherwise $\operatorname{Ric} X_p \geq 0$, in which case Corollary 2.2.14 applies. As before, $X_p^\perp$ admits an orthonormal basis v, w consisting of eigenvectors of β_p corresponding to real eigenvalues λ and μ. Now let J and $\tilde{J}$ be the unique Jacobi fields through p adapted to X such that $J(0) = v$ and $\tilde{J}(0) = w$. Then, by Proposition 2.3.2,

$$J'(0) = \beta_p(v) = \lambda v$$

and

$$\tilde{J}'(0) = \beta_p(w) = \mu w.$$

Now, let us assume first that $\Delta > 0$. Writing $J = J_1 E_1 + J_2 E_2$ and $\tilde{J} = \tilde{J}_1 E_1 + \tilde{J}_2 E_2$ as in Lemma 2.4.2, we obtain (by the same lemma)

$$\begin{cases} J_1 = v_1 \left(\cos(\sqrt{\Delta} t) + \frac{\lambda}{\sqrt{\Delta}} \sin(\sqrt{\Delta} t) \right), \\ J_2 = v_2 \left(\cosh(\sqrt{|\delta|} t) + \frac{\lambda}{\sqrt{|\delta|}} \sinh(\sqrt{|\delta|} t) \right), \end{cases}$$

and similarly,

$$\begin{cases} \tilde{J}_1 = w_1 \left(\cos(\sqrt{\Delta} t) + \frac{\mu}{\sqrt{\Delta}} \sin(\sqrt{\Delta} t) \right), \\ \tilde{J}_2 = w_2 \left(\cosh(\sqrt{|\delta|} t) + \frac{\mu}{\sqrt{|\delta|}} \sinh(\sqrt{|\delta|} t) \right). \end{cases}$$

Now consider the function

$$\mathbb{R} \ni t \longmapsto \phi(t) := \det \begin{pmatrix} J_1(t) & \tilde{J}_1(t) \\ J_2(t) & \tilde{J}_2(t) \end{pmatrix}. \tag{2.9}$$

Using $\sinh(t) = (\mathrm{e}^t - \mathrm{e}^{-t})/2$ and $\cosh(t) = (\mathrm{e}^t + \mathrm{e}^{-t})/2$, we compute

$$\phi(t) = \frac{\mathrm{e}^{\sqrt{|\delta|} t}}{2} \left(c_1^+ \cos(\sqrt{\Delta} t) + c_2^+ \sin(\sqrt{\Delta} t) \right) + \frac{\mathrm{e}^{-\sqrt{|\delta|} t}}{2} \left(c_1^- \cos(\sqrt{\Delta} t) + c_2^- \sin(\sqrt{\delta} t) \right),$$

where

$$c_1^{\pm} = 1 \pm \frac{v_1 w_2 \mu - v_2 w_1 \lambda}{\sqrt{|\delta|}}, \quad c_2^{\pm} = \frac{v_1 w_2 \lambda - v_2 w_1 \mu}{\sqrt{\Delta}} \pm \frac{\lambda \mu}{\sqrt{\Delta |\delta|}}.$$

Note that since ϕ does not vanish identically (since $\phi(0) = 1$), one of the constants $c_1^{\pm}, c_2^{\pm}$ is nonzero. Say $c_1^+ > 0$ (the other cases being similar), then $\phi(t_n) < 0$ for $t_n := (\pi + 2\pi n)/\sqrt{\Delta}$ and $n > 0$ large. Hence, by the intermediate value theorem, there is $t_0 \in \mathbb{R}$ with $\phi(t_0) = 0$, i.e. $J(t_0)$ and $\tilde{J}(t_0)$ are linearly dependent. But then it follows from Corollary 2.3.4 that $J(t)$ and $\tilde{J}(t)$ are linearly dependent for every t, which is a contradiction.

Now consider the case $\Delta = 0$. In this case, it follows from Lemma 2.4.2 that

$$\begin{cases} J_1 = v_1 (1 + \lambda t), \\ J_2 = v_2 \left(\cosh(\sqrt{|\delta|} t) + \frac{\lambda}{\sqrt{|\delta|}} \sinh(\sqrt{|\delta|} t) \right), \end{cases}$$

as well as

$$\begin{cases} \tilde{J}_1 = w_1 (1 + \mu t), \\ \tilde{J}_2 = w_2 \left(\cosh(\sqrt{|\delta|} t) + \frac{\mu}{\sqrt{|\delta|}} \sinh(\sqrt{|\delta|} t) \right). \end{cases}$$

Then

$$\phi(t) = \frac{\mathrm{e}^{\sqrt{|\delta|} t}}{2} \left(c_1^+ + c_2^+ t \right) + \frac{\mathrm{e}^{-\sqrt{|\delta|} t}}{2} \left(c_1^- + c_2^- t \right),$$

where now

$$c_1^{\pm} = 1 \pm \frac{v_1 w_2 \mu - v_2 w_1 \lambda}{\sqrt{|\delta|}}, \quad c_2^{\pm} = v_1 w_2 \lambda - v_2 w_1 \mu \pm \frac{\lambda \mu}{\sqrt{|\delta|}}.$$

It follows that $c_2^+ \geq 0$ and $c_2^- \leq 0$ (otherwise, one can argue again that ϕ has to take negative values, which yields a contradiction). Hence

$$\frac{2\lambda\mu}{\sqrt{\Delta|\delta|}} = c_2^+ - c_2^- \geq 0 \Longrightarrow \lambda\mu \geq 0.$$

Now let $t \mapsto \gamma(t)$ be the parametrised orbit of X through p. We know that $X^\perp$ is not contact along all of γ (Remark 2.2.9). Hence, if λ_t, μ_t denote the eigenvalues of $\beta_t = \beta_{\gamma(t)}$, then the argument above shows that

$$\lambda_t\mu_t \geq 0 \quad \text{for all } t \in \mathbb{R}. \tag{2.10}$$

On the other hand, by Lemma 2.4.5 below, ϕ is given by $\phi(t) = \mathrm{e}^{B(t)}$, where B is a primitive of $t \mapsto \operatorname{tr}\beta_t$. Comparing these two description yields

$$\frac{\mathrm{e}^{\sqrt{|\delta|}t}}{2}\left(c_1^+ + c_2^+ t\right) + \frac{\mathrm{e}^{-\sqrt{|\delta|}t}}{2}\left(c_1^- + c_2^- t\right) = \mathrm{e}^{B(t)}. \tag{2.11}$$

Taking derivatives on both sides, we obtain

$$\frac{\mathrm{e}^{\sqrt{|\delta|}t}}{2}\left(\sqrt{|\delta|}\left(c_1^+ + c_2^+ t\right) + c_2^+\right) + \frac{\mathrm{e}^{-\sqrt{|\delta|}t}}{2}\left(c_2^- - \sqrt{|\delta|}\left(c_1^- + c_2^- t\right)\right) = (\operatorname{tr}\beta_t)\,\mathrm{e}^{B(t)},$$

hence

$$\operatorname{tr}\beta_t = \frac{\frac{\mathrm{e}^{\sqrt{|\delta|}t}}{2}\left(\sqrt{|\delta|}\left(c_1^+ + c_2^+ t\right) + c_2^+\right) + \frac{\mathrm{e}^{-\sqrt{|\delta|}t}}{2}\left(c_2^- - \sqrt{|\delta|}\left(c_1^- + c_2^- t\right)\right)}{\mathrm{e}^{B(t)}}. \tag{2.12}$$

Recall that $c_2^+ \geq 0$ and $c_2^- \leq 0$. At this point, we have to distinguish several cases.

Case 1: $c_2^+ > 0$ and $c_2^- < 0$. In that case, by (2.12), $\operatorname{tr}\beta_t$ takes positive values for large positive t and negative values for large negative t. In particular, $\operatorname{tr}\beta_t = 0$ for some t. But then $\lambda_t = -\mu_t$, hence, using (2.10),

$$0 \leq \lambda_t\mu_t = -\lambda_t^2 \Longrightarrow \lambda_t = \mu_t = 0,$$

so $\beta_t = 0$ which yields a contradiction.

Case 2: $c_2^+ = 0$ and $c_2^- < 0$. In that case we must have $c_1^+ > 0$ for otherwise ϕ takes negative values for large positive t, which gives a contradiction as before. But then again we have that $\operatorname{tr}\beta_t > 0$ for $t > 0$ large, and $\operatorname{tr}\beta_t < 0$ for $t < 0$ large, which gives the same contradiction as in the first case.

Case 3: $c_2^+ > 0$ and $c_2^- = 0$. This is completely analogous to Case 2.

Case 4: $c_2^+ = 0$ and $c_2^- = 0$. In that case we have that

$$0 = c_2^+ - c_2^- = \frac{2\lambda\mu}{\sqrt{\Delta}\sqrt{\delta}} \Longrightarrow \lambda\mu = 0,$$

hence $\lambda = 0$ or $\mu = 0$, so that $\operatorname{rank}\beta \leq 1$, contradicting the assumption.

These are all possible cases, thus the proof is finished. □

Lemma 2.4.5. *Assume that $X^\perp$ is not contact at p and let ϕ be as in (2.9). Then* $\phi(t) = \exp\left(\int_0^t \operatorname{tr}\beta(s)\mathrm{d}s\right)$.

Proof. Since α is invariant under the flow of X, the plane field $X^\perp$ is not contact along the whole orbit γ through p. Hence, by Proposition 2.2.8, for every $t \in \mathbb{R}$ there is an orthonormal basis $V = V_t$, $W = W_t$ of $X^\perp_{\gamma(t)}$ consisting of eigenvectors of β_t corresponding to real eigenvalues $\lambda = \lambda_t$ and $\mu = \mu_t$, respectively. Let us assume for the moment that we can choose V and W to depend smoothly on t. Then

$$0 = X\langle V, X\rangle = \langle \nabla_X V, X\rangle$$

as well as

$$0 = X\langle V, V\rangle = 2\langle \nabla_X V, V\rangle.$$

It follows that $\nabla_X V = aW$ for some function $t \mapsto a(t)$. Since

$$0 = X\langle V, W\rangle = a + \langle V, \nabla_X W\rangle$$

we find that $\nabla_X W = -aV$. Now write

$$J = J_1 V + J_2 W, \quad \tilde{J} = \tilde{J}_1 V + \tilde{J}_2 W.$$

Then

$$\lambda J_1 V + \mu J_2 W = \beta(J) = J' = (J_1' - aJ_2)V + (J_2' + aJ_1)W,$$

so

$$\begin{cases} J_1' = \lambda J_1 + aJ_2. \\ J_2' = -aJ_1 + \mu J_2. \end{cases} \tag{2.13}$$

Similarly, we obtain

$$\begin{cases} \tilde{J}_1' = \lambda \tilde{J}_1 + a\tilde{J}_2. \\ \tilde{J}_2' = -a\tilde{J}_1 + \mu \tilde{J}_2. \end{cases}$$

Therefore,

$$\begin{aligned} \phi' &= J_1'\tilde{J}_2 + J_1\tilde{J}_2' - J_2'\tilde{J}_1 - J_2\tilde{J}_1' \\ &= (\lambda + \mu)\phi(t) \\ &= (\operatorname{tr}\beta)\phi. \end{aligned} \tag{2.14}$$

At this point we still have to deal with the issue of smoothness of the vector fields V and W. It turns out that we cannot, in general, choose V and W to depend smoothly on t for all $t \in \mathbb{R}$, but only on an open, dense subset $U \subset \mathbb{R}$ (which we are going to define in a second). But this is already good enough for our purpose, because in this case, by continuity, ϕ satisfies (2.14) on *all* of $\mathbb{R}$. Then, since $\phi(0) = 1$, we must have

$$\phi(t) = \exp\left(\int_0^t \operatorname{tr}\beta(s)\mathrm{d}s\right)$$

globally, as claimed. In order to find the subset U, let $A := \{t \in \mathbb{R} \colon \lambda(t) \neq \mu(t)\}$. Then, for all $t \in A$, there is a unique decomposition of $X^{\perp}_{\gamma(t)}$ into eigenspaces of β. Then V and W can be chosen to depend smoothly on t for all $t \in A$ (see [54, Chapter 9, Theorem 8]). On the open subset $B := \mathbb{R} \setminus \overline{A}$ we have that $\lambda(t) = \mu(t)$ and $\beta = \lambda \,\mathrm{id}$, so that every vector is an eigenvector. So on B, too, we can choose V and W smoothly. Then $U := A \sqcup B$ does the job. □

The following two examples show that we cannot, in general, drop the assumptions $\max K(v, X) \geq 0$ or $\operatorname{rank} \beta = 2$ if $K(v, X) = 0$ in Theorem 2.4.3.

Example 2.4.6. (i) Consider $M = \mathbb{H}^2 \times \mathbb{R}$, where we use the Poincaré half-plane model for $\mathbb{H}^2$, that is, $\mathbb{H}^2 = \{(x_1, x_2) \in \mathbb{R}^2 \colon x_2 > 0\}$ with the metric

$$g = \frac{\mathrm{d}x_1^2 + \mathrm{d}x_2^2}{x_2^2},$$

and M is equipped with the product metric. Let x_3 be the $\mathbb{R}$-coordinate and consider the geodesic vector field $X = x_2 \partial_{x_2}$. Then $X^{\perp}$ is spanned by ∂_{x_1} and ∂_{x_3}, hence it is tangent to the fibration of affine planes given by $\{x_2 = \text{const.}\}$. Note that $K(\partial_{x_3}, X) = 0$, so X satisfies the condition $\max K(X, \cdot) = 0$ in Theorem 2.4.3 (ii). Now an easy computation shows that with respect to the basis x_1, x_2, x_3, the relevant Christoffel symbols are given by $\Gamma^k_{i3} = 0$ for $i \neq k$ or $i = k = 3$ and $\Gamma^1_{12} = -x_1^{-1}$. Hence

$$\beta(\partial_{x_1}) = -x_2^{-1} \partial_{x_1}, \quad \beta(\partial_z) = 0,$$

so that $\operatorname{rank} \beta = 1$. Now since $X^{\perp}$ does not define a contact structure, this shows that we do need β to have full rank in Theorem 2.4.3 (ii).

(ii) Consider the half-space model of hyperbolic 3-space, that is,

$$\mathbb{H}^3 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \colon x_3 > 0\}$$

equipped with the metric

$$g := \frac{\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2}{x_3^2}.$$

Let X be the geodesic vector field on $\mathbb{H}^3$ defined by $X := x_3 \, \partial_{x_3}$. Then $X^{\perp}$ is spanned by ∂_{x_1} and ∂_{x_2} and hence tangent to the affine planes $\{x_3 = \text{const.}\}$. Furthermore, one computes $\beta(\partial_{x_1}) = -\partial_{x_1}$ and $\beta(\partial_{x_2}) = -\partial_{x_2}$, hence $\beta = -\,\mathrm{id}$ (in particular, $\operatorname{rank} \beta = 2$). However, X does not satisfy the assumption on the sectional curvature in Theorem 2.4.3, since $K(\partial_{x_2}, X) = K(\partial_{x_3}, X) = -1 < 0$.

We conclude this section with the following proposition.

Proposition 2.4.7. *Consider $M = S^2 \times S^1$, equipped with the standard product metric, and let X be a geodesic vector field on M. Then X is everywhere tangent to the S^1-factor. In other words, the only geodesic foliation of $S^2 \times S^1$ is given by the trivial S^1-fibration whose fibres are of the form $\{p\} \times S^1$.*

Proof. Let X be a geodesic vector field on M. Using the splitting $TM = TS^2 \oplus TS^1$, we can write X uniquely as $X = X_1 + X_2$, where $X_1(p,\theta) \in T_pS^2$ and $X_2(p,\theta) \in T_\theta S^1$ for every $(p,\theta) \in S^2 \times S^1$. Consider the (continuous) functions

$$\ell_1 \colon M \longrightarrow [0,1], \quad \ell_1(p,\theta) = |X_1(p,\theta)|$$

and

$$\ell_2 \colon M \longrightarrow [0,1], \quad \ell_2(p,\theta) = |X_2(p,\theta)|.$$

Note that $\ell_1^2 + \ell_2^2 \equiv 1$, since X is of unit length. Now it suffices to show that ℓ_1 (and then also ℓ_2) is constant. Indeed, if $\ell_1 \equiv c_1$ and $\ell_2 \equiv c_2$, then c_1 must be equal to zero; otherwise, the vector field $\pi_*(X_2|_{S^2 \times \{*\}})$ (where $\pi \colon S^2 \times S^1 \to S^2$ is the projection on the first factor) would give a nowhere vanishing vector field on S^2. Hence $\ell_2 \equiv 1$ and the claim would follow. Therefore, for the sake of contradiction, assume that ℓ_1 and ℓ_2 are not constant. Then, by the intermediate value theorem, there is a point $(p,\theta) \in S^2 \times S^1$ such that $\ell_1(p,\theta)/\ell_2(p,\theta) \in \mathbb{R} \setminus \mathbb{Q}$. Let $\gamma = (\gamma_1, \gamma_2)$ be the orbit of X through (p,θ). Then γ_1 parametrises the great circle through $p \in S^2$ with initial velocity $X_1(p,\theta)$. Now consider the torus $T := C \times S^1 \subset S^2 \times S^1$. Since $\ell_1(p,\theta)/\ell_2(p,\theta) \in \mathbb{R} \setminus \mathbb{Q}$, the geodesic γ does not close up; in fact, it sits densely inside T. Hence, by continuity, T is invariant under the flow of X, and every orbit contained in T is dense in T. Note that T divides $S^2 \times S^1$ into two solid tori. Let U denote the interior of one of these solid tori. Consider a point $(\tilde{p}, \tilde{\theta}) \in U$, and let $\tilde{\gamma} = (\tilde{\gamma}_1, \tilde{\gamma}_2)$ denote the corresponding orbit of X. Then $\tilde{\gamma}_1$ is either constant or parametrises another great circle $\tilde{C}$ in S^2. In the latter case, note that the two great circles C and $\tilde{C}$ intersect; hence $\tilde{\gamma}$ must intersect T transversely in some point, which is a contradiction (since X is tangent to T). Therefore, $\tilde{\gamma}_1$ is constant, and $\tilde{\gamma}$ is of the form $\tilde{\gamma}(t) = (0,t)$. That is, $X|_U = \partial_\theta$. But then also $X|_T = \partial_\theta$, which is a contradiction. □

Remark 2.4.8. It does not seem to be clear whether or not $S^2 \times \mathbb{R}$ (equipped with the product metric) admits a nontrivial fibration by geodesics (compare this also to Theorem 3.5.1 and Remark 3.5.3).

2.5 Geodesic vector fields with integrable orthogonal distribution

So far, we have discussed criteria for the orthogonal distribution of a geodesic vector field to define a contact structure, in other words, being maximally non-integrable. In this section, we will discuss the other extreme case; namely, we will consider geodesic vector fields whose orthogonal distribution is integrable, that is, there is a codimension-1 foliation tangent to $X^\perp$. It turns out that this has strong implications on the topology and geometry of the underlying manifold. The first one we are going to prove is a direct consequence of a theorem by Tischler.

Proposition 2.5.1. *Let X be a geodesic vector field on a closed Riemannian manifold whose orthogonal distribution $X^\perp$ is integrable. Then, there is a fibration $\pi\colon M \to S^1$ such that X is transverse to the fibres of π.*

Proof. Denote by $\alpha = i_X g$ the 1-form dual to X. Then, since $i_X \mathrm{d}\alpha \equiv 0$, the integrability of $X^\perp$ is equivalent to $\mathrm{d}\alpha|_{X^\perp} = 0$, hence $\mathrm{d}\alpha = 0$, which means that α is a closed, nowhere vanishing 1-form. Then by Tischler's theorem [74, Theorem 1], for every $\varepsilon > 0$ there is a fibration $\pi\colon M \to S^1$ such that $|\pi^*\mathrm{d}\varphi - \alpha| < \varepsilon$, where φ is the S^1-coordinate. In particular, if $\varepsilon < 1$, this implies that $\pi^*\mathrm{d}\varphi(X) > 0$, which means that X is transverse to the fibres of π. □

Turning to the geometric side, by Remark 2.2.5 (i), we have the following.

Corollary 2.5.2. *If X is a geodesic vector field with integrable orthogonal distribution, then every orbit of X is free of conjugate points.* □

Remark 2.5.3. By results of Ruggiero [63], the statements of Proposition 2.5.1 and Corollary 2.5.2 remain true if the vector field X is assumed to be continuous rather than smooth (or C^1).

Recall that a **Riemannian foliation** $\mathcal{F}$ of a Riemannian manifold (M, g) is a foliation such that g is *bundle-like* for the tangent bundle $T\mathcal{F}$, which means that the local submersions defining $\mathcal{F}$ can be chosen to be Riemannian submersions; see [62] or [75] for the precise definition. It turns out that there is a duality between geodesic and Riemannian foliations, as follows.

Theorem 2.5.4 (cf. [49, Theorem 1.6])**.** *Let X be a geodesic vector field on a Riemannian manifold (M, g) whose orthogonal distribution $X^\perp$ is integrable. Then, the foliation spanned by $X^\perp$ is Riemannian. Conversely, if $\mathcal{F}$ is an oriented Riemannian foliation of codimension 1, then the (positively) orthogonal unit vector field X is geodesic.* □

The second statement actually follows from the following observation of Reinhart [62, Proposition 2]: If $\mathcal{F}$ is a Riemannian foliation and γ a geodesic that is somewhere orthogonal to $\mathcal{F}$, then γ is everywhere orthogonal to $\mathcal{F}$. Now let X be a unit vector field orthogonal to $\mathcal{F}$, and take any point $p \in M$. Then the integral curve of X through p is, of course, everywhere orthogonal to $\mathcal{F}$; on the other hand, the unique geodesic γ with $\gamma(0) = p$ and $\dot\gamma(0) = X(p)$ must be orthogonal to $\mathcal{F}$ as well. Hence these two curves coincide, and it follows that X is geodesic.

Remark 2.5.5. In fact Theorem 2.5.4 is true for totally geodesic and Riemannian foliations of any (co-)dimension. For a further discussion of the geometric consequences of this duality, we refer the reader to [49].

As another geometric consequence of the integrability of $X^\perp$, we will prove the following, which is in contrast to Theorems 2.2.4 and 2.2.13.

Theorem 2.5.6. *Let X be a geodesic vector field on a closed Riemannian manifold whose orthogonal distribution $X^\perp$ is integrable. Let m be a finite measure on M invariant under the flow of X. Then $\int_M (\operatorname{Ric} X + \operatorname{tr}\beta^2)\,\mathrm{d}m = 0$.*

Proof. Since $X^\perp$ is integrable, Proposition 2.2.8 tells us that β_p is self-adjoint for every $p \in M$. Denote by λ_p and μ_p the eigenvalues of β_p and consider the function $p \mapsto \psi(p) := \operatorname{Ric} X_p + \lambda_p^2 + \mu_p^2$. First note that $\psi(p) = \operatorname{Ric} X_p + \operatorname{tr}\beta_p^2$. Indeed, take an orthonormal basis of eigenvectors of β, so that β can be represented by the matrix $\operatorname{diag}(\lambda, \mu)$, hence $\beta^2 = \operatorname{diag}(\lambda^2, \mu^2)$ and $\operatorname{tr}\beta^2 = \lambda^2 + \mu^2$. Now, by Proposition 2.2.11, the function

$$p \longmapsto (\operatorname{tr}\beta)'(p) + \psi(p) \tag{2.15}$$

vanishes identically (where $(\operatorname{tr}\beta)' := X(\operatorname{tr}\beta)$). It follows that, for every $p \in M$ and $\delta > 0$,

$$0 = \frac{1}{\delta}\int_0^\delta \left((\operatorname{tr}\beta)'(\phi^t(p)) + \psi(\phi^t(p))\right)\,\mathrm{d}t \tag{2.16}$$

$$= \frac{1}{\delta}\left(\operatorname{tr}\beta(\phi^\delta(p)) - \operatorname{tr}\beta(p) + \int_0^\delta \psi(\phi^t(p))\,\mathrm{d}t\right) \tag{2.17}$$

Now let m be an finite measure on M invariant under ϕ. Then integrating (2.16) against m yields

$$0 = \underbrace{\frac{1}{\delta}\int_M \left(\operatorname{tr}\beta\circ\phi^\delta - \operatorname{tr}\beta\right)\,\mathrm{d}m}_{=0} + \int_M \left(\frac{1}{\delta}\int_0^\delta \psi(\phi^t(\cdot))\,\mathrm{d}t\right)\mathrm{d}m, \tag{2.18}$$

where $\int_M \left(\operatorname{tr}\beta\circ\phi^\delta - \operatorname{tr}\beta\right)\,\mathrm{d}m = 0$ since m is invariant under the flow of X. Now let $\varepsilon > 0$. By uniform continuity, there is a $\delta > 0$ such that

$$\sup_{p\in M}\left|\psi(p) - \frac{1}{\delta}\int_0^\delta \psi(\phi^t(p))\,\mathrm{d}t\right| < \varepsilon.$$

Using this and (2.18), it follows that

$$\int_M \psi \,\mathrm{d}m = \int_M \left(\psi - \frac{1}{\delta}\int_0^\delta \psi(\phi^t(\cdot))\,\mathrm{d}t\right)\mathrm{d}m < \varepsilon \int_M \mathrm{d}m$$

Taking $\varepsilon \to 0$ yields $\int_M \psi \,\mathrm{d}m = 0$, which was the claimed identity. □

Remark 2.5.7. (1) Using Theorem 2.2.13, we know that if $X^\perp$ is integrable, then $\operatorname{Ric} X + |\lambda - \mu|^2/2 \leq 0$ everywhere. But since

$$\operatorname{Ric} X + \operatorname{tr}\beta^2 = \operatorname{Ric} X + |\lambda - \mu|^2/2 + |\lambda + \mu|^2/2,$$

we do not know whether or not $\operatorname{Ric} X + \operatorname{tr}\beta^2$ is positive or negative. Hence, the statement in Theorem 2.5.6 gives more than what we can deduce solely from Theorem 2.2.13.

(2) If M is locally isometric to a product $U \times I$, and X a geodesic vector field given as ∂_t in this local description (where t is the coordinate of the I-factor), then $X^\perp$ is integrable and $\operatorname{Ric} X \equiv 0$. In that case, of course, the statement of Theorem 2.5.6 is trivially satisfied. There are, however, examples of geodesic vector fields with integrable orthogonal distribution and strictly negative total Ricci curvature (see Example 2.4.6).

3

Geodesible versus Reeb vector fields

We have seen in Example 1.1.10 that the Reeb vector field of a contact form (or, more generally, a stable Hamiltonian structure) is always geodesible. In this chapter, we want to address the converse question:

Given a geodesible vector field on an odd-dimensional manifold, can it be realised as the Reeb vector field of a contact form or a stable Hamiltonian structure?

In other words, the question is whether or not a geodesible vector field is 'Reebable'. This is not always the case, not even in the 3-dimensional case, as we will see in Section 3.1 below.

The rest of the chapter is organised as follows. In Section 3.2, we discuss the notion of *volume* of a geodesible vector field, following [30]. For example, the Reeb vector field of a contact form always has nonzero volume, which is not true for geodesible vector fields in general. Then, we will introduce *basic cohomology* in Section 3.3, which is the cohomology of differential forms adapted to a given (geodesible) foliation in a certain way, and use that in Section 3.4 to give an answer to the question above for some special types of geodesible vector fields: Killing vector fields (in particular, periodic vector fields) and vector fields whose flow is transitive. In Sections 3.5, 3.6 and 3.7 (which are mainly based on the article [4]), we focus on geodesic vector fields on flat 3-manifolds. We will show that every fibration of $\mathbb{E}^3$ by oriented lines can be realised as a Reeb fibration (Corollary 3.7.2), and that every such contact structure is necessarily tight (Theorem 3.7.4). Furthermore, we will show that geodesic vector fields on flat 3-manifolds not equal to $\mathbb{E}^3$ are always tangent to a codimension-1 foliation whose leaves are totally geodesic, and use that to derive a criterion for the Reebability of geodesic vector fields on closed flat 3-manifolds (Theorem 3.6.2).

3.1 Constructing non-Reeb geodesible vector fields

Let M be a $(2n+1)$-dimensional manifold. Recall that a stable Hamiltonian structure is a pair (ω, α), where ω a closed 2-form and α a 1-form such that $\alpha \wedge \omega^n \neq 0$ everywhere, and $\ker \omega \subset \ker \mathrm{d}\alpha$ (Example 1.1.10). Its Reeb vector field is, by definition, the unique vector field R satisfying $i_R\omega = 0$ (in particular, $i_X \mathrm{d}\alpha = 0$) and

$\alpha(R) = 1$. If $\omega = \mathrm{d}\alpha$, then α is a contact form, hence stable Hamiltonian structures can be viewed as generalisations of contact forms. Now consider the following sets of vector fields on M:

$$\begin{aligned} \mathcal{R}(M) &:= \{\text{Reeb vector fields of contact forms}\} \\ \mathcal{RS}(M) &:= \{\text{Reeb vector fields of stable Hamiltonian structures}\} \\ \mathcal{G}(M) &:= \{\text{Geodesible vector fields}\} \end{aligned}$$

Then we have inclusions $\mathcal{R}(M) \subset \mathcal{RS}(M) \subset \mathcal{G}(M)$. Note that these inclusions are strict in general. For example, consider $M = S^2 \times S^1$ and $X = \partial_\theta$, where θ is the S^1-coordinate. Then X is clearly geodesible; in fact it is the Reeb vector field of the stable Hamiltonian structure $(\omega, \mathrm{d}\theta)$, where ω is some area form on S^2. But X cannot be realised as the Reeb vector field of a contact form, not even up to rescaling: If there were a contact form α on M such that $X = R_\alpha$, then R_α would be positively transverse to $S^2 \times \{\theta\}$ for every $\theta \in S^1$, hence $\mathrm{d}\alpha$ would restrict to a positive area form on $S^2 \times \{\theta\}$, which is not possible by Stokes' theorem. More generally, a flow-with-section (or suspension flow, see Example 1.1.11) is always geodesible, but never the Reeb vector field of a contact form.

In Proposition 3.1.2 below, we will prove that in fact for *every* closed orientable 3-manifold, both inclusions $\mathcal{R}(M) \subset \mathcal{RS}(M)$ and $\mathcal{RS}(M) \subset \mathcal{G}(M)$ are strict. We begin with the following simple observation.

Proposition 3.1.1 ([17, Corollary 2.3]). *Let M be a closed orientable 3-manifold, and let $X \in \mathcal{G}(M)$ be a geodesible vector field. Then $X \in \mathcal{RS}(M)$ if and only if there is volume form μ on M such that the flow of X preserves μ, that is, $L_X\mu = 0$.*

Proof. If X is the Reeb vector field of the stable Hamiltonian structure (ω, α), then the flow of X preserves the volume form $\mu := \alpha \wedge \omega$. Conversely, if μ is a volume form preserved by the flow of X, set $\omega := i_X\mu$ and let α be a connection 1-form for X. Writing $\alpha \wedge \omega = f\mu$ for some function $f\colon M \to \mathbb{R}$ and contracting both sides with X, we see that $f \equiv 1$; in particular, $\alpha \wedge \omega$ is a volume form. Furthermore, $X \in \ker \mathrm{d}\alpha$ spans the 1-dimensional kernel of ω, hence $\ker \omega \subset \ker \mathrm{d}\alpha$. Thus (ω, α) is a stable Hamiltonian structure with Reeb vector field X. □

Proposition 3.1.2. *Let M be a closed orientable 3-manifold. Then*

$$\emptyset \neq \mathcal{R}(M) \subsetneq \mathcal{RS}(M) \subsetneq \mathcal{G}(M).$$

Proof. By Martinet's theorem [57], the set of contact forms on M is nonempty; hence $\mathcal{R}(M) \neq \emptyset$. The inclusions hold by the discussion in Example 1.1.10. To see that they are strict, we are going to adjust the construction in the proof of Theorem

1.2.2 to provide vector fields $X_1 \in \mathcal{RS}(M) \setminus \mathcal{R}M$ and $X_2 \in \mathcal{G}(M) \setminus \mathcal{RS}(M)$. We begin with the construction of X_1. As in the proof of Theorem 1.2.2, consider an open book decomposition (B, π) of M, where in this case, the binding B consists of a number of disjointly embedded circles. To simplify notation, we assume that B is connected, hence $B \cong S^1$. Let φ be the angular coordinate for B, and let θ be the angular coordinate corresponding to the fibration $\pi\colon\ M \setminus B \to S^1$. Consider a tubular neighbourhood $\nu B \cong S^1 \times D^2$ on which π is given by the angular coordinate of the D^2-factor whose radial coordinate we denote by r. As in the construction of Theorem 1.2.2, let $X_1 = f_1(r)\,\partial_\varphi + f_2(r)\,\partial_\theta$ and $\alpha_1 = (1 - g(r))\,\mathrm{d}\varphi + g(r)\,\mathrm{d}\theta$, where this time f_1, f_2 and g are as depicted in Figure 3.1. On the interval $[1/3-\varepsilon, 1/3+\varepsilon]$, f_1 and f_2 satisfy the relation $f_2 = (1 + f_1)/2$. Note that $f_1 = f_2$ on the set $\{g' \neq 0\}$, hence $i_{X_1}\mathrm{d}\alpha_1 = (f_1 - f_2)g'\mathrm{d}r = 0$. Furthermore, one can easily check that $\alpha_1(X_1) = (1-g)f_1 + g f_2 = 1$ (this is clear if $f_1(r) = f_2(r) = 1$; everywhere else, g is constantly equal to 0, 1 or 2, and the condition translates into $f_1 = 1$, $f_2 = 1$ or $f_2 = (1+f_1)/2$, respectively). Thus (X_1, α_1) defines a geodesible pair. To see that $X_1 \in \mathcal{RS}(M)$, we claim that X_1 preserves a volume form, arguing as in [12, Theorem 3.10]. On νB, the volume form $\mu := r\,\mathrm{d}r \wedge \mathrm{d}\varphi \wedge \mathrm{d}\theta$ is clearly invariant under the flow of X_1. Now pick a page $\Sigma_{\theta_0} := \pi^{-1}(\theta_0)$, and extend the area form $r\,\mathrm{d}r \wedge \mathrm{d}\varphi$ on $\Sigma_{\theta_0} \cap \nu B$ to an area form defined on all of Σ_{θ_0}. This area form extends to a θ-invariant 2-form ω on M such that $\omega = r\,\mathrm{d}r \wedge \mathrm{d}\varphi$ on νB. Thus μ extends as $\omega \wedge \mathrm{d}\theta$ to $M \setminus \nu B$ which is clearly invariant under the flow of X_1. Therefore, $X_1 \in \mathcal{RS}(M)$ by Proposition 3.1.1. However, X_1 cannot be realised as the Reeb vector field of a contact form, because in νB, when going in radial direction, X_1 makes half a twist in one direction, and then half a twist in the other direction (as $X_1(r = 0) = \partial_\theta$, $X_1(r = 1/3) = -\partial_\theta$ and $X_1(r = 2/3) = \partial_\theta$), which is incompatible with the property of being Reeb for a contact form; for the precise argument we refer to the proof of Proposition 3.6.6, $(i) \Rightarrow (ii)$, or Remark 3.6.7.

For the construction of X_2, start with X_1 as above (or any geodesible vector field of the form $X = f_1\,\partial_\varphi + f_2\,\partial_\theta$, for that matter) and introduce a small perturbation on νB by setting $X_2 := X_1 - \Psi(r)\,\partial_r$, where $\Psi\colon\ [0, 1] \to [0, 1]$ is a smooth function equal to 0 near $r = 0$ and $r = 1$, and positive for $r \in [\varepsilon, 2\varepsilon]$, for some $\varepsilon > 0$. Then, for ε chosen small enough, α_1 is still a connection form for X_2, hence $X_2 \in \mathcal{G}(M)$. To see that $X_2 \notin \mathcal{RS}(M)$, consider the solid torus $S^1 \times D^2_\varepsilon \subset S^1 \times D^2 \cong \nu B$, where D^2_ε denotes the disc of radius ε. If ϕ_1 denotes the time-1 flow of X, then $\phi_1(S^1 \times D^2_\varepsilon) \subset S^1 \times D^2_r$ for $r < \varepsilon$. This is of course not compatible with the preservation of a volume form, hence $X_2 \notin \mathcal{RS}(M)$ by Proposition 3.1.1. □

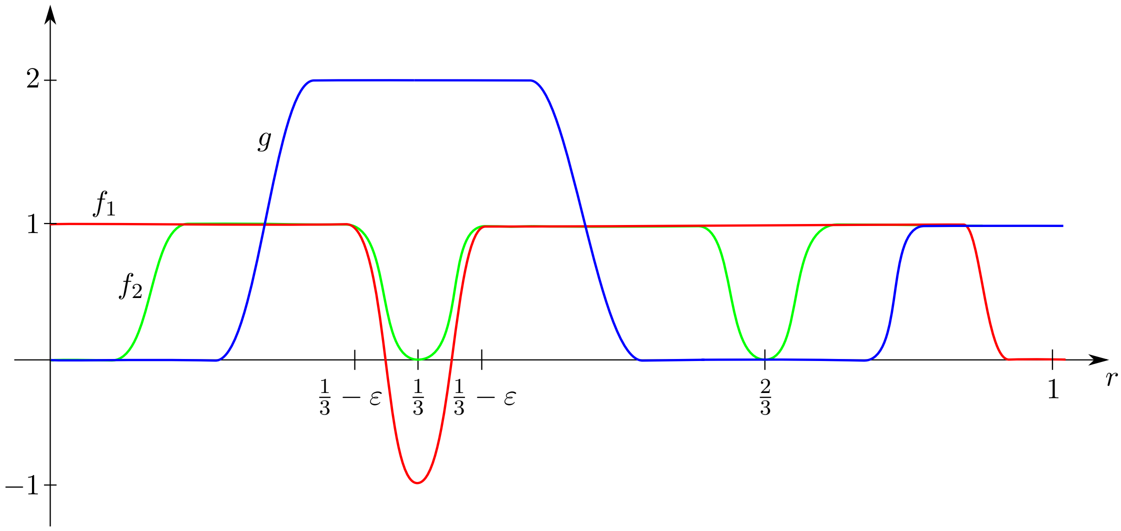

Figure 3.1: Interpolation functions f_1, f_2 and g for constructing non-Reeb geodesible fields.

Note that in the proof above we have constructed X_2 by a C^∞-small perturbation of an element of $\mathcal{RS}(M)$ or $\mathcal{R}(M)$ (by choosing Ψ C^∞-small). Hence we have the following.

Proposition 3.1.3. *For any given closed* 3*-manifold* M*, the subsets* $\mathcal{R}(M)$ *and* $\mathcal{RS}(M)$ *are* ***not*** C^∞*-open in* $\mathcal{G}(M)$*.* □

In comparison, Cardona has recently proven the following.

Theorem 3.1.4 (Cardona [13])**.** *For any given closed* 3*-manifold* M*,* $\mathcal{R}(M)$ *is not* C^1*-dense in* $\mathcal{RS}(M)$ *(or, more generally, the space of vector fields preserving a volume form).*

3.2 The volume of a geodesible vector field

In this section, we introduce the notion of volume for a geodesible vector field, following [30].

Definition/Lemma 3.2.1. Let X be a geodesible vector field on a closed $(2n+1)$-dimensional manifold M with connection form α. Then, the real number

$$\operatorname{vol}_X := \int_M \alpha \wedge (\mathrm{d}\alpha)^n$$

does not depend on the choice of α. It is called the **volume** of X.

Proof. The following equation holds for arbitrary 1-forms α and β (see [30, Lemma 1.1]):

$$\begin{aligned}\alpha\wedge&(\mathrm{d}\alpha)^n - \beta \wedge (\mathrm{d}\beta)^n \\ &= (\alpha-\beta)\wedge\sum_{j=0}^{n}(\mathrm{d}\alpha)^j\wedge(\mathrm{d}\beta)^{n-j} + \mathrm{d}\Big(\alpha\wedge\beta\wedge\sum_{j=1}^{n-1}(\mathrm{d}\alpha)^j\wedge(\mathrm{d}\beta)^{n-1-j}\Big).\end{aligned} \tag{3.1}$$

Now if α and β are connection 1-forms for X, then the first n summands of the right-hand side of the equation above vanish (which can be seen by contracting with X), and the remainder is an exact form which integrates to zero by Stokes' Theorem. It follows that

$$\int_M \alpha\wedge(\mathrm{d}\alpha)^n = \int_M \beta\wedge(\mathrm{d}\beta)^n.$$ □

Remark 3.2.2. In dimension 3, if X is a geodesic vector field on an orientable Riemannian manifold (M, g), we have seen in the proof of Proposition 2.2.8 that $\alpha\wedge\mathrm{d}\alpha = \langle \operatorname{curl} X, X\rangle \operatorname{vol}_g$. Hence $\operatorname{vol}_X = \int_M \langle \operatorname{curl} X, X\rangle \operatorname{vol}_g$. This is also known as the **helicity** of X, a quantity that appears in plasma physics and fluid dynamics (see, for example, [6, 11, 58]).

Example 3.2.3. (i) Let $\pi\colon M \to \Sigma$ be a Seifert fibration with Euler number e (see Definition 4.1.6), defined by a (geodesible) vector field X whose period along the regular fibres is equal to 1. Then $\operatorname{vol}_X = -e$ [30, Corollary 6.3].

(ii) Let α be a contact form on a $(2n+1)$-dimensional manifold and R_α its Reeb vector field. Then R_α is geodesible and thus has a well-defined volume, which is necessarily non-zero since $\alpha\wedge(\mathrm{d}\alpha)^n$ is a volume form.

The following proposition is a slight generalisation of Proposition 2.1 in [30], in the sense that in our case the two Reeb vector fields need not be the same, only up to rescaling by a positive function. Before giving the statement, let us introduce some notation.

Notation. For two nowhere vanishing vector fields X and Y on a manifold M, write $X \sim Y$ if there is a function $\lambda\colon M \to \mathbb{R}^+$ such that $Y = \lambda X$.

Proposition 3.2.4. *Let M be a closed 3-manifold and let α_0 and α_1 be two contact forms on M such that $R_{\alpha_0} \sim R_{\alpha_1}$. Then, the volumes of R_{α_0} and R_{α_1} have the same sign. Furthermore, the contact structures $\ker\alpha_0$ and $\ker\alpha_1$ are diffeomorphic.*

Proof. The first statement is equivalent to saying that the orientations defined by the volume forms $\alpha_0\wedge\mathrm{d}\alpha_0$ and $\alpha_1\wedge\mathrm{d}\alpha_1$ are the same. For the sake of contradiction, assume that they are not. Since $R_{\alpha_0} \sim R_{\alpha_1}$, the 2-forms $\mathrm{d}\alpha_0$ and $\mathrm{d}\alpha_1$ must be multiples of each other, so we may write $\mathrm{d}\alpha_1 = \mu\,\mathrm{d}\alpha_0$, where $\mu\colon M \to \mathbb{R}^-$. Also, set $\lambda := \alpha_1(R_{\alpha_0}) \in C^\infty(M, \mathbb{R}^+)$. Now $\alpha_1 \wedge \mathrm{d}\alpha_1 = f\,\alpha_0 \wedge \mathrm{d}\alpha_0$ for some function $f\colon M \to \mathbb{R}^-$. By contracting both sides of this equation with R_{α_0}, we see that $f = \lambda\mu$. Similarly, one computes that

$$(\alpha_0 - \alpha_1) \wedge (\mathrm{d}\alpha_0 + \mathrm{d}\alpha_1) = (1-\lambda)(1+\mu)\,\alpha_0 \wedge \mathrm{d}\alpha_0.$$

Then, identity (3.1) implies that

$$\begin{aligned}\int_M (1-\lambda\mu)\,\alpha_0 \wedge \mathrm{d}\alpha_0 &= \int_M \alpha_0 \wedge \mathrm{d}\alpha_0 - \int_M \alpha_1 \wedge \mathrm{d}\alpha_1 \\ &= \int_M (\alpha_0 - \alpha_1) \wedge (\mathrm{d}\alpha_0 + \mathrm{d}\alpha_1) \\ &= \int_M (1-\lambda)(1+\mu)\,\alpha_0 \wedge \mathrm{d}\alpha_0.\end{aligned}$$

But then $\int_M (\mu-\lambda)\,\alpha_0 \wedge \mathrm{d}\alpha_0$ must vanish, which is impossible since $\mu - \lambda$ is assumed to be negative everywhere. Hence, we arrive at a contradiction, and the first statement is proven. Now consider the family of 1-forms $\alpha_t := (1-t)\,\alpha_0 + t\,\alpha_1$. Again, write $\lambda := \alpha_1(R_{\alpha_0}) \in C^\infty(M, \mathbb{R}^+)$ and $\mathrm{d}\alpha_1 = \mu\,\mathrm{d}\alpha_0$, where $\mu\colon M \to \mathbb{R}^+$ is now a positive function. Then

$$\alpha_t \wedge \mathrm{d}\alpha_t = \underbrace{\left[(1-t)^2 + t(1-t)(\lambda+\mu) + t^2\lambda\mu\right]}_{>0}\,\alpha_0 \wedge \mathrm{d}\alpha_0,$$

hence α_t is a contact form for every $t \in [0,1]$. Then $\ker \alpha_0$ and $\ker \alpha_1$ are diffeomorphic by Gray stability (Theorem 2.1.9). □

Lemma 3.2.5. *Let $\pi\colon M \to N$ be a k-fold covering, X a geodesible vector field on N and Y its lift to M. Then Y is geodesible and the volumes of X and Y are related as*

$$\operatorname{vol}_Y = k \operatorname{vol}_X.$$

Proof. Let α be a connection 1-form for X. Then $\pi^*\alpha$ clearly defines a connection 1-form for Y, hence Y is geodesible. Furthermore,

$$\operatorname{vol}_Y = \int_M \pi^*\alpha \wedge (\mathrm{d}(\pi^*\alpha))^n = \int_M \pi^*\left(\alpha \wedge (\mathrm{d}\alpha)^n\right) = k \int_N \alpha \wedge (\mathrm{d}\alpha)^n = k \operatorname{vol}_X. \quad \square$$

3.3 Basic cohomology

In this section, we introduce a special kind of cohomology adapted to a foliation of a manifold, called *basic cohomology*. It is given as the de Rham cohomology of the subcomplex of so-called *basic* differential forms. Here, we discuss the concept for 1-dimensional foliations only; a more general introduction can be found in [75].

Definition 3.3.1. Let $\mathcal{F}$ be a 1-dimensional foliation of a manifold M, spanned by a non-singular vector field X. A k-form β is called **basic** with respect to $\mathcal{F}$ if

$$i_X\beta = 0 \quad \text{and} \quad i_X \mathrm{d}\beta = 0.$$

The vector space of basic k-forms is denoted by $\Omega^k_B(\mathcal{F})$. By definition, the exterior derivative preserves basic forms. In particular, we have an induced complex

$$\dots \xrightarrow{\mathrm{d}_B} \Omega^{k-1}_B(\mathcal{F}) \xrightarrow{\mathrm{d}_B} \Omega^k_B(\mathcal{F}) \xrightarrow{\mathrm{d}_B} \Omega^{k+1}_B(\mathcal{F}) \xrightarrow{\mathrm{d}_B} \dots$$

called the **basic de Rham complex**. The cohomology of this complex is called the **basic cohomology** associated with the foliation $\mathcal{F}$, and the cohomology groups are denoted by $H^k_B(\mathcal{F})$.

Example 3.3.2. (i) Let $\mathcal{F}$ be a 1-dimensional foliation whose leaves are given as the fibres of a fibration $\pi\colon M \to B$. Then, any basic k-form $\beta \in \Omega^k_B(\mathcal{F})$ gives rise to a k-form $\eta \in \Omega^k(B)$, as follows. For a point $p \in B$ and tangent vectors $v_1, \dots, v_k \in T_pB$, set

$$\eta_p(v_1, \dots, v_k) := \beta_q(\tilde{v}_1, \dots, \tilde{v}_k),$$

where $q \in \pi^{-1}(p)$ and $\tilde{v}_1, \dots, \tilde{v}_k$ are some lifts of $v_1, \dots, v_k$. To see that this is well defined, note that

$$L_X\beta = \mathrm{d}(i_X\beta) + i_X\mathrm{d}\beta = 0,$$

which implies that the above definition does not depend on the choice of $q \in \pi^{-1}(p)$. Furthermore, since $i_X\beta = 0$, it does not depend on the choice of lifts $\tilde{v}_1, \dots, \tilde{v}_k$ either, since any two lifts of a given tangent vector of B differ by a multiple of X. Conversely, given a k-form $\eta \in \Omega^k(B)$, the pull-back $\beta := \pi^*\eta$ is basic. Clearly, these two operations are inverse to each other. Hence, we obtain a one-to-one correspondence between basic k-forms on $(M, \mathcal{F})$ and ordinary k-forms on B. One may easily check that this correspondence preserves exterior derivatives. Thus, the basic cohomology groups of $\mathcal{F}$ are isomorphic to the de Rham cohomology groups of the basis B. This justifies the name 'basic' cohomology.

(ii) The following is an example of a foliation $\mathcal{F}$ whose second basic cohomology is infinite-dimensional (cf. [17, Proposition 3.39]). Let $M = T^2 \times [0,1]$ with coordinates (φ, θ, r) and let $X = \sin f(r)\,\partial_\varphi + \cos f(r)\,\partial_\theta$, where $f\colon [0,1] \to \mathbb{R}$ is a smooth function with $f' \neq 0$ everywhere. Let $\mathcal{F}$ be the 1-dimensional foliation spanned by X. Note that the leaves of $\mathcal{F}$ are tangent to the tori

$T^2 \times \{*\}$. If r is such that $f(r) \in 2\pi\,\mathbb{Q}$, then $\mathcal{F}$ gives rise to a fibration of $T^2 \times \{r\}$ by circles, whereas if $f(r) \in 2\pi\,(\mathbb{R} \setminus \mathbb{Q})$, then every leave of $\mathcal{F}$ contained in $T^2 \times \{r\}$ is dense in $T^2 \times \{r\}$.

The claim is now that $H^2_B(\mathcal{F})$ is infinite-dimensional. To see this, consider the basic 2-form $\mathrm{d}\alpha$, where $\alpha = \sin f(r)\,\mathrm{d}\varphi + \cos f(r)\,\mathrm{d}\theta$ is the 1-form dual to X. Then $\mathrm{d}\alpha|_{X^\perp}$ is non-vanishing. Hence, if β is an arbitrary basic 2-form, then $\beta = \lambda\,\mathrm{d}\alpha$ for some function $\lambda\colon\ M \to \mathbb{R}$. Now, since β is basic, we have that

$$\begin{aligned} 0 = L_X\beta &= i_X(\mathrm{d}(\lambda\,\mathrm{d}\alpha)) + \mathrm{d}(i_X(\lambda\,\mathrm{d}\alpha)) \\ &= \mathrm{d}\lambda(X)\,\mathrm{d}\alpha - \mathrm{d}\lambda \wedge (\underbrace{i_X\mathrm{d}\alpha}_{=0}) + \mathrm{d}\lambda \wedge (\underbrace{i_X\mathrm{d}\alpha}_{=0}) + \lambda\,\mathrm{d}(\underbrace{i_x\mathrm{d}\alpha}_{=0}) \\ &= \mathrm{d}\lambda(X)\,\mathrm{d}\alpha, \end{aligned}$$

so that $\mathrm{d}\lambda(X) = 0$, i.e. λ is invariant under the flow of X. Hence λ is constant on the tori $T^2 \times \{r\}$ for r satisfying $f(r) \in 2\pi\,(\mathbb{R} \setminus \mathbb{Q})$. Since these tori are dense in M (as $f' \neq 0$ everywhere), it follows that f depends on r only.

Now consider a basic 1-form $\eta \in \Omega^1_B(\mathcal{F})$. Write

$$\eta = a\,\mathrm{d}r + b\,\mathrm{d}\varphi + c\,\mathrm{d}\theta,$$

for some functions $a, b, c \in C^\infty(M)$. Then, by our observation above, $\mathrm{d}\eta = \lambda\,\mathrm{d}\alpha$ for some function $\lambda = \lambda(r)$. This translates into the following system of equations:

$$\begin{aligned} \text{(I)}\ \ &\partial_r b - \partial_\varphi a = \lambda f' \cos f, \\ \text{(II)}\ \ &\partial_r c - \partial_\theta a = -\lambda f' \sin f, \\ \text{(III)}\ \ &\partial_\varphi c - \partial_\theta b = 0. \end{aligned}$$

On the other hand, we know that

$$0 = \eta(X) = b \sin f(r) + c \cos f(r). \tag{3.2}$$

Differentiating this equation with respect to φ yields

$$\partial_\varphi b \sin f(r) + \partial_\varphi c \cos f(r) = 0.$$

Using this and (III), we obtain

$$\begin{aligned} X(b) &= \partial_\varphi b \sin f(r) + \partial_\theta b \cos f(r) \\ &= \partial_\varphi b \sin f(r) + \partial_\varphi c \cos f(r) \\ &= 0. \end{aligned}$$

Hence, b is invariant under the flow of X and therefore (using the same argument as before) it is a function of r only. The same is true for c. Now from equations (II) and (III) it follows that $\partial_\varphi a$ and $\partial_\theta a$ depend on r only, and since a is periodic with respect to φ and θ, we conclude that a, too, must be a function of r only. Hence, equations (I) and (II) above become

$$\text{(I)}\ \ \partial_r b = \lambda f' \cos f,$$

$$\text{(II)}\ \ \partial_r c = -\lambda f' \sin f.$$

Differentiating equation (3.2) with respect to r and plugging in equations (I) and (II), we obtain

$$b\, f' \cos f - c\, f' \sin f = 0.$$

Since $f' \neq 0$ everywhere, this is equivalent to $b \cos f - c \sin f = 0$. But now this, together with (3.2), yields

$$\begin{pmatrix} b & c \\ -c & b \end{pmatrix} \begin{pmatrix} \sin f \\ \cos f \end{pmatrix} = 0,$$

hence $b = c = 0$, and then also $\lambda = 0$ by (I) and (II). This means that $\mathrm{d}\eta = \lambda\, \mathrm{d}\alpha = 0$. In particular, since η was an arbitrary basic 1-form, it follows that

$$\operatorname{Im}(\mathrm{d}\colon\ \Omega^1_B \to \Omega^2_B) = 0.$$

This implies that

$$\begin{aligned} H^2_B(\mathcal{F}) = \ker(\mathrm{d}\colon\ \Omega^2_B \to \Omega^3_B) = \Omega^2_B &= \{\lambda\, \mathrm{d}\alpha\colon\ \lambda = \lambda(r) \in C^\infty([0,1])\} \\ &\cong C^\infty([0,1]), \end{aligned}$$

hence $H^2_B(\mathcal{F})$ is infinite-dimensional.

Now let X be a geodesible vector field and $\mathcal{F}_X$ the 1-dimensional foliation spanned by X. Given any connection 1-form α for X, the 2-form $\mathrm{d}\alpha$ is basic, hence it defines a basic cohomology class $[\mathrm{d}\alpha]_B \in H^2_B(\mathcal{F}_X)$. This class does not depend on the specific choice of connection form α. Indeed, if β is another connection form for X, then $\alpha - \beta$ is basic, hence $\mathrm{d}\alpha - \mathrm{d}\beta \in \operatorname{Im} \mathrm{d}_B$. This allows us to make the following definition, following [30].

Definition 3.3.3. Let X be a geodesible vector field with connection form α. Then, the basic cohomology class $e_X := [\mathrm{d}\alpha]_B \in H^2_B(\mathcal{F}_X)$ is called the **Euler class** of X.

Proposition 3.3.4 ([30, Proposition 5.5])**.** *A geodesible vector field X on a $(2n+1)$-dimensional manifold is the Reeb vector field of a contact form if and only if e_X has an odd-symplectic representative, that is, there is a basic closed 2-form ω such that $[\omega]_B = e_X$ and $\omega^n \neq 0$.*

Proof. If X is the Reeb vector field of a contact form α, then $e_X = [\mathrm{d}\alpha]_B$ and $\mathrm{d}\alpha$ is odd-symplectic. Conversely, if there is such an odd-symplectic form ω, then

$$\mathrm{d}\alpha = \omega + \mathrm{d}\beta$$

for some basic 1-form β. Then the 1-form $\tilde{\alpha} := \alpha - \beta$ is contact, and its Reeb vector field is given by X. □

In particular, a necessary condition for a geodesible vector field to be Reeb is the existence of an odd-symplectic basic 2-form ω. In that case, the volume form $\mu := \alpha \wedge \omega^n$ is nowhere vanishing and invariant under the flow of X, since

$$L_X\mu = \mathrm{d}(i_X(\alpha \wedge \omega^n)) = \mathrm{d}\omega^n = 0.$$

In dimension 3, the converse is also true: If μ is an invariant volume form, then $\omega := i_X\mu$ is an odd-symplectic basic 2-form.

Proposition 3.3.5. *Let X be a geodesible vector field on a 3-manifold such that*

- *the flow of X preserves a volume form;*
- $\mathrm{vol}_X \neq 0$*;*
- $H^2_B(\mathcal{F}_X) \cong \mathbb{R}$.

Then X is the Reeb vector field of a contact form.

Proof. Since the flow of X preserves a volume form, there is an odd-symplectic basic 2-form ω defining a basic cohomology class $[\omega]_B \in H^2_B(\mathcal{F}_X)$. Note that since $\mathrm{vol}_X \neq 0$, the Euler class $e_X \in H^2_B(\mathcal{F}_X)$ is nontrivial. Indeed, if there were a basic 1-form β such that $\mathrm{d}\alpha = \mathrm{d}\beta$, then

$$0 \neq \mathrm{vol}_X = \int_M \alpha \wedge \mathrm{d}\alpha = \int_M \alpha \wedge \mathrm{d}\beta = \underbrace{\int_M \mathrm{d}(\alpha \wedge \beta)}_{=0} - \int_M \beta \wedge \mathrm{d}\alpha.$$

But $i_X(\beta \wedge \mathrm{d}\alpha) = 0$, hence $\beta \wedge \mathrm{d}\alpha = 0$ which gives a contradiction. For the same reason, the class $[\omega]_B$ is nontrivial. Since $H^2_B(\mathcal{F}_X) = \mathbb{R}$, there is a basic 1-form β and a constant $c \neq 0$ such that $\mathrm{d}\alpha - c\omega = \mathrm{d}\beta$. Then $\alpha - \beta$ is a contact form whose Reeb vector field is given by X. □

The third condition in Proposition 3.3.5 will not be satisfied in general, since $H^2_B(\mathcal{F}_X)$ can be infinite-dimensional, as we saw in Example 3.3.2 (ii). However, there are certain examples where the condition holds, and we will discuss these in the next section.

3.4 Isometric and transitive flows

In this section, we want to discuss some examples of geodesible vector fields for which Proposition 3.3.5 applies. Consider a Riemannian manifold (M, g) and X a Killing vector field on M. Recall that if X is of unit length, then it is a geodesic vector field (Example 1.1.12). If the underlying 3-manifold M is orientable, then the Riemannian volume form vol_g is defined, and the flow of X preserves this volume form. Hence X satisfies condition (1) of Proposition 3.3.5. It also satisfies condition (3), as the next lemma says.

Lemma 3.4.1 ([75, (6.15)]). *Let X be a Killing vector field of unit length on some Riemannian 3-manifold. Then $H^2_B(\mathcal{F}_X) \cong H^3_{dR}(M)$. In particular, $H^2_B(\mathcal{F}_X) \cong \mathbb{R}$ if M is closed.* □

Together with Proposition 3.3.5, we immediately obtain the following.

Corollary 3.4.2. *Let X be a Killing vector field of unit length on some closed Riemannian 3-manifold. Then X is the Reeb vector field of a contact form if and only if* $\mathrm{vol}_X \neq 0$. □

Recall that if X is a periodic vector field defining a Seifert fibration, then X can be realised as a Killing vector field of unit length for a suitable Riemannian metric (Example 1.1.14). Now we have seen in Example 3.2.3 that the volume of X is given by $\mathrm{vol}_X = c^{-1}e$, where e is the Euler number of the Seifert fibration and c is the regular period of X. Together with Corollary 3.4.2, this yields the following statement that was first proven by Kegel and Lange [50, Theorem 1.4].

Corollary 3.4.3. *Let X be a periodic vector field defining a Seifert fibration with Euler number e. Then X is the Reeb vector field of a contact form if and only if $e \neq 0$.* □

A second situation in which $H^2_B(\mathcal{F})$ is finite-dimensional is that of a transitive flow, i.e. a flow that admits a dense orbit.

Proposition 3.4.4. *Let X be a geodesible vector field on a 3-manifold M preserving some volume form (i.e., X is the Reeb vector field of a stable Hamiltonian structure). If* $\mathrm{vol}_X \neq 0$ *and X has a dense orbit, then $H^2_B(\mathcal{F}_X) \cong \mathbb{R}$, and X is the Reeb vector field of a contact form.*

Proof. Let μ be a volume form preserved by X, and let $\omega := i_X\mu$. Then ω defines a nontrivial basic cohomology class $[\omega]_B \in H^2_B(\mathcal{F}_X)$. If η is another basic 2-form, then $\eta = \lambda\,\omega$ for some function $\lambda\colon\ M \to \mathbb{R}$. Then $\mathrm{d}\eta = \mathrm{d}\lambda \wedge \omega$, hence η is closed if and only if $\mathrm{d}\lambda(X) = 0$, which means that λ is invariant under the flow of X. Since X

has a dense orbit, every such function must be constant. Hence every closed basic 2-form is a constant multiple of ω, implying that $H^2_B(\mathcal{F}_X) \cong \mathbb{R}$. The statement then follows from Proposition 3.3.5. □

3.5 Geodesic vector fields on flat 3-manifolds

The contents of this section and the remainder of this chapter are mainly based on the article [4]. We now want to address the question asked in the beginning of this chapter in a more geometric setting. That is, we consider geodesic vector fields on manifolds with a given Riemannian metric. Of course the simplest Riemannian manifolds to consider are space forms (i.e. spaces of constant curvature). For instance, we have seen in Section 2.2 that every fibration of the round 3-sphere by oriented great circles induces a contact structure (Theorem 2.2.3); in particular, by Remark 2.2.2, every unit vector field spanning a great circle fibration is the Reeb vector field of a contact form, and the corresponding contact structure must be diffeomorphic to the standard one by Proposition 3.2.4. Hence in the case of positive constant curvature, there is nothing more to do. Similarly, using Harrison's result (Theorem 2.2.6), a geodesic vector field X on a flat 3-manifold induces a contact structure if and only if $\operatorname{rank} \nabla X \geq 1$, and X is also Reeb in this case. However, consider for example the constant geodesic vector field ∂_z on $\mathbb{E}^3$. This clearly does not induce a contact structure (the orthogonal complement being a constant plane field), but it is the Reeb vector field of a contact form, namely the standard one given by $\mathrm{d}z + x\,\mathrm{d}y$. That is, unlike in the case of positive constant curvature, the class of geodesic Reeb vector fields on flat 3-manifolds is larger than the class of geodesic vector fields that induce contact structures. In fact it is not hard to show that *every* geodesic vector field on $\mathbb{E}^3$ can be realised as the Reeb vector field of a contact form, diffeomorphic to the standard one (Corollary 3.7.2).

For *closed* flat 3-manifolds, the situation is different again. For example, the geodesic vector field ∂_z on T^3 (equipped with flat coordinates (x, y, z)) cannot be realised as the Reeb vector field of a contact form, for the reason discussed in the beginning of Section 3.1, as it admits a transverse 2-torus. The question is now whether one can find a reasonable criterion for the Reebability of such vector fields. We will provide an answer to that question in Section 3.6 (Theorem 3.6.2). The proof of Theorem 3.6.2 is based on the following characterisation, which says that if the given flat 3-manifold is not equal to $\mathbb{E}^3$, any foliation by oriented geodesics is of a very simple type.

Theorem 3.5.1. *Let M be a complete flat 3-manifold not equal to $\mathbb{E}^3$. Then any 1-dimensional oriented geodesic foliation of M is tangent to a 2-dimensional totally geodesic foliation.*

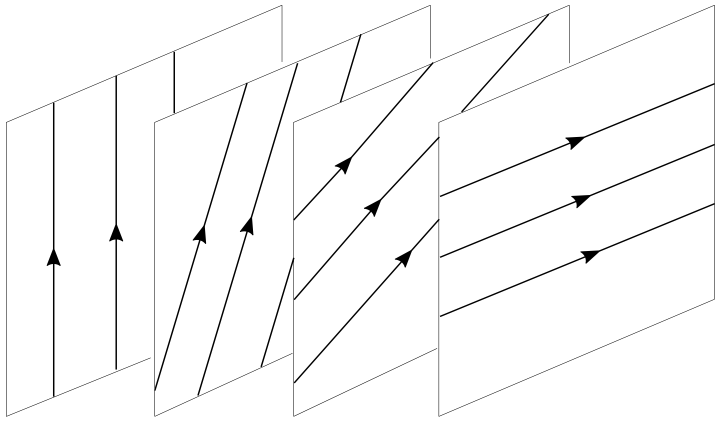

Figure 3.2: A 1-parameter fibration by lines.

Remark 3.5.2. Note that if M is a complete flat 3-manifold, then M can be identified with the quotient space $\mathbb{E}^3/\Gamma$, where $\Gamma < \mathrm{Isom}(\mathbb{E}^3)$ is some discrete subgroup of isometries acting freely (cf. [55, Corollary 11.13]). In particular, every geodesic vector field X on M lifts to a geodesic vector field $\tilde{X}$ on $\mathbb{E}^3$. Theorem 3.5.1 then says that if $M \neq \mathbb{E}^3$, then $\tilde{X}$ is tangent to a fibration by planes, see Figure 3.2. That is, the vector field $\tilde{X}$ can be described by a single angular function $\theta\colon \mathbb{E}^3 \to S^1$. In this case, the line fibration spanned by $\tilde{X}$ is called **1-parameter** (see [42]).

Remark 3.5.3. We do not assume M to be oriented or closed in Theorem 3.5.1. Furthermore, the statement is false for geodesic foliations of $\mathbb{E}^3$. In fact, there exist fibrations of $\mathbb{E}^3$ by pairwise non-parallel oriented lines (so-called *skew fibrations*), which are of course far from being 1-parameter. One way to construct such a fibration is the following (cf. [41]). Start with a single oriented line. The complement of this line in $\mathbb{E}^3$ is fibred by nested one-sheeted hyperboloids, all of which are ruled surfaces and can therefore be written as the union of (oriented) lines. This yields a fibration of $\mathbb{E}^3$ by oriented lines, and one may check that no two of these lines are parallel to each other. See Figure 3.3 below.

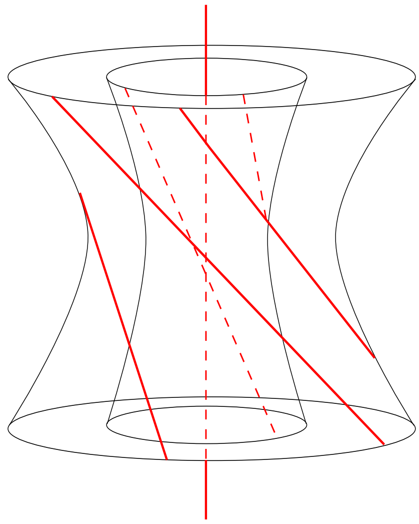

Figure 3.3: A skew fibration.

Proof of Theorem 3.5.1. Let M be a complete flat 3-manifold not equal to $\mathbb{E}^3$. Then M can be identified with $\mathbb{E}^3/\Gamma$, where $\Gamma < \mathrm{Isom}(\mathbb{E}^3)$ is a nontrivial discrete subgroup of isometries acting freely (see Remark 3.5.2). Now let $\mathcal{F}$ be a (1-dimensional, oriented) geodesic foliation of M, spanned by a geodesic vector field X. Then $\mathcal{F}$ lifts to a geodesic foliation $\tilde{\mathcal{F}}$ of $\mathbb{E}^3$, spanned by the lifted vector field $\tilde{X}$. Here, we view $\tilde{\mathcal{F}} = \{\ell\}$ just as a set of lines. For a point $p \in \mathbb{E}^3$, denote by $\ell_p \in \tilde{\mathcal{F}}$ the fibre through p. Note that $\tilde{\mathcal{F}}$ must be invariant under the action of Γ, that is, $\ell_{\gamma(p)} = \gamma(\ell_p)$ for every $\gamma \in \Gamma$ and $p \in \mathbb{E}^3$. Now it clearly suffices to prove the statement for the lifted foliation $\tilde{\mathcal{F}}$, since the covering map $\pi\colon \mathbb{E}^3 \to M$ is locally isometric. That is, we have to show that the fibration $\tilde{\mathcal{F}}$ of $\mathbb{E}^3$ by oriented lines is tangent to a fibration by affine planes, i.e. $\tilde{\mathcal{F}}$ is 1-parameter. To do so, let us take a closer look at the group $\Gamma < \mathrm{Isom}(\mathbb{E}^3)$. It is well known that every isometry of $\mathbb{E}^3$ (also called *Euclidean motion*) is given by the composition of a reflection in a plane or rotation about some axis, and some (perhaps trivial) translation. Then one can easily see that any fixed-point free Euclidean motion must be one of the following three:

- a translation;
- a screw motion, i.e. rotation about some axis followed by translation in the direction of this axis;
- a glide reflection, i.e. reflection in some plane followed by translation parallel to this plane.

Note that applying a glide reflection twice yields a (pure) translation again. Hence, we may assume that the group Γ contains a nontrivial translation or screw motion.

We will treat these two cases separately.

<u>**First case (Γ contains a translation):**</u> Assume that there is some $T_v \in \Gamma$, where T_v is the translation by some vector $v \in \mathbb{R}^3$. If $\tilde{X}$ is constant, there is nothing to prove. Otherwise, there is a point $p_0 \in \mathbb{E}^3$ such that ℓ_{p_0} does not point in the direction of $\pm v$. Let $P \subset \mathbb{E}^3$ be the affine plane through p_0 spanned by v and the cross product $\tilde{X}_{p_0} \times v$. Then P is transverse to ℓ_{p_0}, so we can consider the projection $\pi \colon \mathbb{E}^3 \to P$ onto P in the direction of ℓ_{p_0}. Define a vector field Y on P by

$$Y_p := \mathrm{d}\pi_p(\tilde{X}_p), \quad p \in P,$$

and denote by ℓ_p^Y the line in P spanned by Y_p. Note that ℓ_p^Y is just given by the projected line $\pi(\ell_p)$. The $\mathbb{Z}$-action on $\mathbb{E}^3$ generated by the translation T_v restricts to a $\mathbb{Z}$-action on P, and Y is invariant under this action. Now partition P as $P = A \sqcup B$, where $A = \{Y \neq 0\}$ and $B = \{Y = 0\}$. Note that B is precisely the set of points $p \in P$ for which ℓ_p is parallel to ℓ_{p_0}. Therefore, we may assume that $A \neq \emptyset$, for otherwise, $\tilde{X}$ is constant and therefore trivially 1-parameter. Also, if $Y_p \neq 0$ at some point $p \in P$, then ℓ_p^Y must be disjoint from B. Indeed, if there were a point $q \in B \cap \ell_p^Y$, then the fibre ℓ_p would intersect ℓ_q transversely, which is of course not possible.

Now we consider two cases. First, assume that there is a point $q \in A$ for which Y_q is parallel to v. Then Y must be parallel to Y_q on the whole line ℓ_q^Y. Indeed, if that were not the case, then the set of lines $\{\ell_p^Y \colon p \in \ell_q^Y\}$ would fill out a cone that intersects $p_0 + \mathbb{Z}\,v$, see Figure 3.4 below. In particular, there would be some line ℓ_p^Y intersecting a point in B, which is not possible, as we have seen above. For the same reason, Y must be non-vanishing on ℓ_q^Y (in fact, we have that $Y_p = Y_q$ for all $p \in \ell_q^Y$). It follows that the affine plane spanned by ℓ_q^Y and $\tilde{X}_q$ is fibred by pairwise parallel lines in $\mathcal{F}$. The same holds for every parallel translate of that plane, and we conclude that $\tilde{\mathcal{F}}$ is 1-parameter.

Thus, we may assume that Y is nowhere parallel to v. Let $Q \subset \mathbb{E}^3$ be the affine plane through p_0 spanned by v and $\tilde{X}_{p_0}$. Then Q contains infinitely many fibres of $\mathcal{F}$ parallel to ℓ_{p_0}, namely, the fibres through points in $p_0 + \mathbb{Z}\,v$. Note that these points are contained in B. If Q' is another affine plane parallel to Q, then there must be fibres contained in Q' as well. To see this, denote by U_Q and $U_{Q'}$ the set of points in Q and Q', respectively, where $\tilde{X}$ is transverse to Q (resp. Q'). Then, the flow of $\tilde{X}$ maps U_Q diffeomorphically to $U_{Q'}$. But since there is a $\mathbb{Z}$-family of fibres tangent to Q, we see that U_Q is either empty or disconnected, so the same must be true for $U_{Q'}$. In particular, $U_{Q'} \neq Q'$, so that there must be fibres in $\mathcal{F}$ tangent to Q'. All of these fibres must be parallel to ℓ_{p_0}, for otherwise Y is parallel to v (and non-zero) somewhere, and we are in the first case again. Furthermore, the translates of these

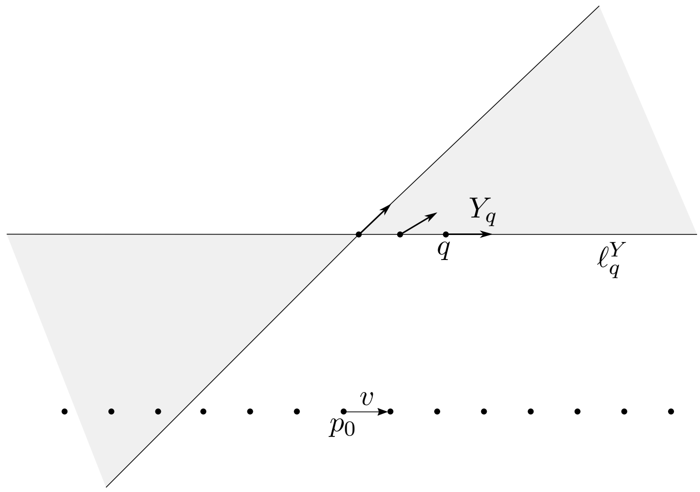

Figure 3.4: The set of lines spanned by Y contains the grey cone, which intersects the set of points $\{p_0 + \mathbb{Z}\, v\}$.

fibres by integer multiples of v are again fibres of $\mathcal{F}$ contained in Q'. But then every disc of radius $> |v|$ in P must intersect B in at least one point. Now by an argument similar to the one in the first case, we see that for every point $q \in A$ and $p \in \ell_q^Y$, we have that $Y_p = Y_q$, and we conclude that $\tilde{\mathcal{F}}$ is 1-parameter.

Second case (Γ contains a screw motion): Assume that Γ contains a screw motion γ, where γ is given by some rotation followed by translation by some vector $v \in \mathbb{R}^3$. We may assume that the angle of rotation is an irrational multiple of 2π, for otherwise, applying γ some number of k times yields a (pure) translation, and we are in the first case again.

Denote by P the plane through the origin orthogonal to v, and for $t \in \mathbb{R}$ let $P_t := P + tv$, the parallel translate of P by the vector tv. Consider the fibre ℓ_0 through the origin, and let $\ell_t := \ell_{tv}$. We need the following additional lemma.

Lemma 3.5.4. *Either $\ell_t \subset P_t$ for all t, or ℓ_0 is parallel to v.*

Proof. The statement is equivalent to saying that if ℓ_t is transverse to P_t for some t, then ℓ_t is parallel to v. Therefore, for the sake of contradiction, let us assume that there is some $t \in \mathbb{R}$ such that $\ell := \ell_t$ is transverse to P_t (and hence transverse to P) and not parallel to v. For simplicity assume that $t = 0$. Let $\pi\colon \mathbb{E}^3 \to P$ be the orthogonal projection onto P. Then ℓ projects to a line $\pi(\ell) \subset P$. Now consider the projected lines $\pi(\ell_t)$ for $t \in \mathbb{R}$. If $\pi(\ell_t) = \pi(\ell)$ for all $t \in \mathbb{R}$, then the lines ℓ_t must be pairwise parallel, thus the plane Q spanned by v and ℓ is fibred by (parallel) lines. Then γ must preserve Q in order for the fibration $\tilde{\mathcal{F}}$ to be preserved, which is only possible if γ is trivial, a contradiction. Hence, we may assume that there is some $t_0 \in \mathbb{R}$ such that $\pi(\ell_{t_0}) \neq \pi(\ell)$. We may further assume (without loss of generality) that $t_0 < 0$ and that every ℓ_t, for $t \in [t_0, 0]$, intersects P transversely (by choosing t_0 close enough to 0). Now let

$$N := \bigcup_{t\in[t_0,0]} \ell_t \subset \mathbb{E}^3.$$

Then the projection $\pi(N) \subset P$ contains the cone $K \subset P$ given by the convex hull of $\pi(\ell)$ and $\pi(\ell_{t_0})$ (see Figure 3.5). Let θ be the angle between $\pi(\ell)$ and $\pi(\ell_{t_0})$. Since the angle of rotation of γ is irrational, there is some $k \in \mathbb{N}$ such that the projection of $\gamma^k(\ell) \in \mathcal{F}$ onto P is a line obtained by rotating $-\pi(\ell)$ towards the interior of K by an angle of less than θ. In other words, $\pi(\gamma^k(\ell)) \subset \operatorname{Int} K \cup \{0\}$. From this we deduce that $\gamma^k(\ell)$ intersects N. However, since $k > 0 > t_0$, we see that $\gamma^k(\ell) \not\subset N$, hence $\gamma^k(\ell)$ intersects some line in N transversely, a contradiction. $\square$

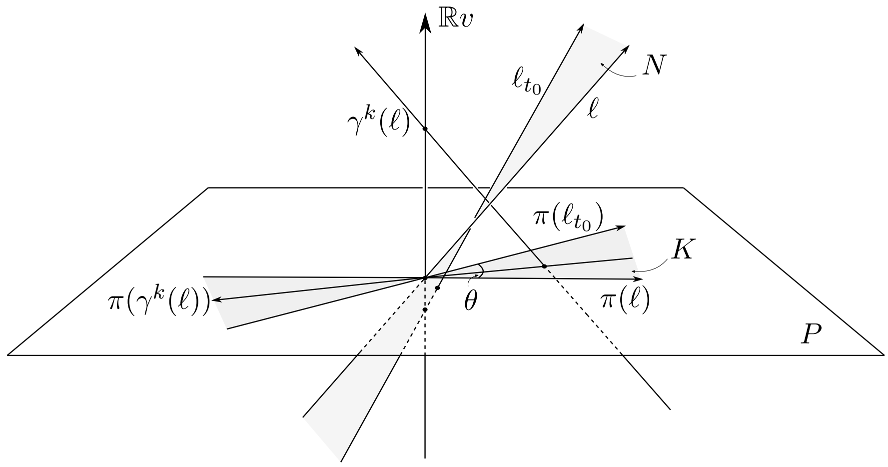

Figure 3.5: The line $\gamma^k(\ell)$ intersects N transversely.

Proof of Theorem 3.5.1 (cont.) Using Lemma 3.5.4, we now have to consider two cases. The first is that $\ell_t \subset P_t$ for all $t \in \mathbb{R}$. Let us show that, under this assumption, every fibre of $\tilde{\mathcal{F}}$ must be contained in one of the planes P_t (in particular, the fibration will be 1-parameter). To see this, note first that each of the oriented lines ℓ_t divides P_t into two open, oriented half-planes ℓ_t^+ and ℓ_t^-, where $\partial\ell_t^+ = \ell_t$ and $\partial\ell_t^- = -\ell_t$ (that is, ℓ_t with the opposite orientation). Here, the orientations of ℓ_t^+ and ℓ_t^- come from a consistently chosen orientation of the P_t. Now assume that there is a point $p \in P$ such that ℓ_p intersects P (and hence every P_t) transversely. We may assume that ℓ_p is not parallel to v, so that the orthogonal projection $\pi(\ell_p) \subset P$ of ℓ_p is a line again. Furthermore, we may assume that $\pi(\ell_p)$ intersects ℓ_0 transversely (if that is not the case, simply replace P by an appropriate P_t, and ℓ_0 by ℓ_t, for some $t \in \mathbb{R}$). Now, without loss of generality, let us assume that $p \in \ell_0^+$. Then the point p_t given by the intersection of ℓ_p with P_t must be contained in ℓ_t^+ for every $t \in \mathbb{R}$, for otherwise, the line ℓ_p intersects one of the ℓ_t transversely. But since $\pi(\ell_p)$ intersects ℓ_0 transversely, there is some $T > 0$ such that $\pi(p_t) \in \ell_0^-$ for all $t > T$. Again, since the angle of rotation of γ is irrational, we can approximate ℓ_0 arbitrarily well by $\pi(\gamma^k(\ell_0)) = \pi(\ell_k)$ for large enough $k \in \mathbb{N}$, hence we can approximate ℓ_0^- by $\pi(\gamma^k(\ell_0))^-$. In particular, there is some $k \geq T$ such that $\pi(p_k) \in \ell_0^- \cap \pi(\ell_k)^-$. But then $p_k \in \ell_k^-$, a contradiction.

The other case is that ℓ_0 is parallel to v (and then, in particular, $\ell_t = \ell_0$ for all t). We will show that in this case, every fibre must be parallel to v, and so the fibration is trivially 1-parameter. Arguing again by contradiction, we assume that there are fibres that are not parallel to v. Choose a small closed disc $D \subset P$ such that

(i) for every $p \in D$, the fibre ℓ_p is transverse to D;

(ii) for every $p \in \partial D$, the fibre ℓ_p is not parallel to v.

Such a disc can be found as follows. First, take a disc $D = D_r(0)$ (the closed disc about 0 of radius $r > 0$) that satisfies (i). Now if (ii) does not hold, then there is some $p_0 \in \partial D$ such that ℓ_{p_0} is parallel to v. By applying γ successively (once again using the fact that its rotational angle is irrational) we find that for a dense subset of ∂D, the corresponding fibres must be parallel to v. Then by continuity, this must hold for every fibre through points in ∂D. But then the set of all fibres through ∂D form a straight cylinder parallel to v, and thus every fibre inside that cylinder must be parallel to v as well. In other words, the fibration is constant over D. But since the fibration is assumed to be globally non-constant, we find a larger disc, again called D, so that (i) is still satisfied and the fibration is not constant over D. Then D has to satisfy (ii) as well.

Now let $\Sigma := \{\ell_p\colon\ p \in \partial D\}$ be the surface consisting of all fibres through points in ∂D. Let $\Sigma_t := \Sigma \cap P_t$, with $P_t = P + tv$ as before, and let $\pi(\Sigma_t)$ be its projection to P. We shall prove that there is some $T > 0$ such that for all $t \in \mathbb{R}$ with $|t| > T$ we have that

$$D \subset \operatorname{Int} \pi(\Sigma_t), \tag{3.3}$$

where $\operatorname{Int} \pi(\Sigma_t)$ denotes the interior of $\pi(\Sigma_t)$, that is, the connected component of $P_t \setminus \pi(\Sigma_t)$ bounded by $\pi(\Sigma_t)$ with compact closure. Indeed, $\pi(\Sigma_t)$ is obtained from $\pi(\Sigma_0) = \partial D$ by flowing in the direction of the projected lines $\pi(\ell_p)$, $p \in \partial D$. Denote this flow by Φ. For T large enough and $|t| > T$, the set $\Phi_t(\partial D)$ lies outside of D, that is, $\Phi_t(\partial D) \subset P \setminus D$. The fact that we can write D instead of $\operatorname{Int} D$ here is because the ℓ_p project to lines and not points, due to property (ii) above; hence no point on ∂D is fixed under the flow Φ. Since none of the projected lines point to the origin (due to Lemma 3.5.4), the origin stays in the interior while applying the flow, from which (3.3) follows. This is illustrated in Figure 3.6.

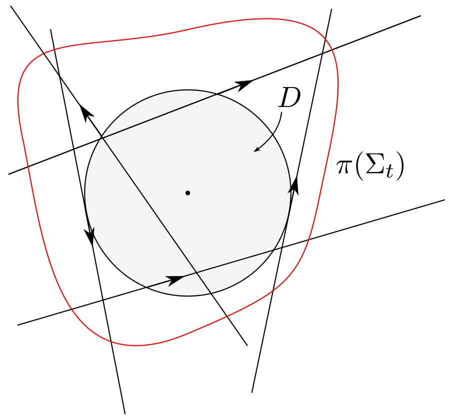

Figure 3.6: D is contained in the interior of $\pi(\Sigma_t)$.

Now let $k > T$ and consider the surface $\tilde{\Sigma} := \gamma^k(\Sigma)$. Then, since $\tilde{\mathcal{F}}$ is invariant under the action of Γ, we see that $\tilde{\Sigma}$, too, is a union of fibres of $\tilde{\mathcal{F}}$. Hence, either Σ and $\tilde{\Sigma}$ are disjoint, or they intersect in a set of common fibres. In particular, the intersection $\Sigma \cap \tilde{\Sigma}$ is either empty or there is a non-empty intersection in every t-level, that is, $\Sigma_t \cap \tilde{\Sigma}_t \neq \emptyset$ for every t. On the other hand, from (3.3) we deduce that $\pi(\tilde{\Sigma}_k) = \partial D \subset \operatorname{Int} \pi(\Sigma_k)$, hence $\tilde{\Sigma}_k \subset \operatorname{Int} \Sigma_k$. Similarly, one can show that $\Sigma_0 \subset \operatorname{Int}(\tilde{\Sigma}_0)$, see Figure 3.7.

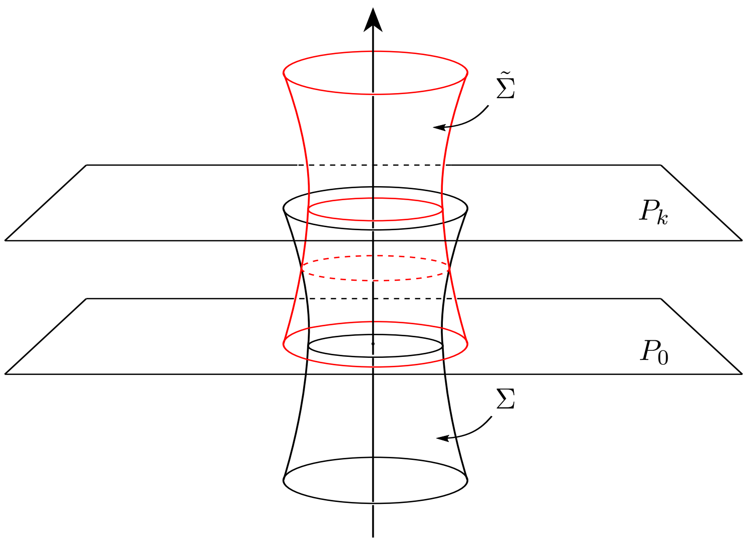

Figure 3.7: The surfaces Σ and $\tilde{\Sigma}$ intersect transversely.

But this means that $\Sigma \cap \tilde{\Sigma} \neq \emptyset$ while $\Sigma_k \cap \tilde{\Sigma}_k = \emptyset$, a contradiction. This finishes the proof of Theorem 3.5.1. □

3.6 Reebability in the flat case

The goal of this section is to prove a necessary and sufficient criterion for the Reebability of geodesic vector fields on closed flat 3-manifolds. Before giving the statement, let us make the following definition.

Definition 3.6.1. A geodesible vector field X on an odd-dimensional manifold M is called **conformally Reeb** if there is a contact form α with Reeb vector field R_α such that $X \sim R_\alpha$. In other words, there is a function $\lambda\colon M \to \mathbb{R}^+$ such that $X = \lambda R_\alpha$.

Secondly, recall that a **flat torus** is the manifold given by $T^n = \mathbb{E}^n/\mathbb{Z}^n$ equipped with the induced (flat) Riemannian metric, where $\mathbb{Z}^n$ is a lattice in $\mathbb{E}^n$ acting by translations. For the remainder of this chapter, by T^3 we shall always mean a flat 3-torus.

Theorem 3.6.2. *Let X be a geodesic vector field on a closed orientable complete flat 3-manifold M. Then X is conformally Reeb for a contact form α if and only if there is a geodesic vector field Y on M inducing a contact structure ξ such that X is everywhere transverse to ξ.*

In this case, writing M as $M = T^3/\Gamma$, where T^3 is some flat 3-torus and $\Gamma <$ $\mathrm{Isom}(T^3)$, there is a fibration $\zeta\colon T^3 \to S^1$ whose fibres are totally geodesic 2-tori such that the lifted vector fields X_T and Y_T are tangent to the fibres of ζ. Furthermore, the lifted contact structures $\ker \alpha_T$ *and ξ_T on T^3 are both diffeomorphic to*

$$\ker\left(\sin\left(\frac{\mathrm{vol}_X|\Gamma|}{A}\zeta\right)\mathcal{E}^1 + \cos\left(\frac{\mathrm{vol}_X|\Gamma|}{A}\zeta\right)\mathcal{E}^2\right),$$

where

- *$\mathcal{E}^1$ and $\mathcal{E}^2$ are 1-forms dual to a global orthonormal parallel frame (E_1, E_2) spanning the fibres of ζ,*
- *$A := \int_{\zeta^{-1}(a)} \mathcal{E}^1 \wedge \mathcal{E}^2$ is the (Euclidean) area of a typical fibre.*

In particular, we obtain the following standardness result, which follows also from Theorem 3.7.4; however, the proof of Corollary 3.6.3 below is by a direct argument and does not use sophisticated results like Hofer's theorem on overtwisted contact structures or Eliashberg's classification of tight contact structures on $\mathbb{R}^3$.

Corollary 3.6.3. *Let X be a geodesic vector field on a closed flat 3-manifold. If X is conformally Reeb for a contact form α, then the lifted contact structure* $\ker\widetilde{\alpha}$ *on $\mathbb{R}^3$ is diffeomorphic to the standard contact structure* $\ker(\mathrm{d}z + x\,\mathrm{d}y)$.

In order to prove Theorem 3.6.2 and Corollary 3.6.3, we first show how to reduce the problem to the special case of $M = T^3$, a flat 3-torus. For this let M be a closed orientable complete flat 3-manifold. By the classical Bieberbach theorems [8, 9] (see also [73, Theorem 4.2.2]), M can be written as $M = T^3/\Gamma$, where $T^3 = \mathbb{E}^3/\mathbb{Z}^3$ is some flat 3-torus and $\Gamma < \mathrm{Isom}(T^3)$ is a finite subgroup of isometries of T^3 acting freely and orientation-preservingly.

Proposition 3.6.4. *Let X be a geodesic vector field on $M = T^3/\Gamma$ and X_T its lift to T^3. Then X is conformally Reeb if and only if X_T is conformally Reeb.*

Proof. Assume first that X is conformally Reeb. That is, there is a contact form α on M such that $X \sim R_\alpha$. Let $\pi\colon T^3 \to M$ be the natural projection. Then $p^*\alpha$ is again a contact form, and clearly $R_{p^*\alpha} \sim X_T$.

Conversely, assume that $X_T \sim R_{\alpha_T}$ for some contact form α_T on T^3. Since $|\Gamma| < \infty$, we can average under the action of Γ to obtain a 1-form

$$\alpha := \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma^* \alpha_T.$$

Then α is again a contact form, since $\gamma_* X_T = X_T$ and $\mathrm{d}\gamma$ maps the hyperplane field $X_T^\perp$ orientation-preservingly to itself, for every $\gamma \in \Gamma$. Here we are using the fact that in dimension 3, a 1-form β with $\beta(X_T) > 0$ is contact if and only if $\mathrm{d}\beta$ is non-vanishing on any hyperplane field transverse to X_T. It also follows that $R_\alpha = R_{\alpha_T} \sim X_T$. Now since $\gamma^*\alpha = \alpha$ for every $\gamma \in \Gamma$, the contact form α descends to a contact form on M whose Reeb vector field is a multiple of X. □

We may now, for the remainder of the section, assume that $M = T^3$. We may further assume that the geodesic vector field X is not constant, for otherwise, there is an embedded 2-torus transverse to X and so by Stokes' theorem, X cannot be (conformally) Reeb. Let $\tilde{X}$ be the lift of the geodesic vector field X to $\mathbb{E}^3$. By Theorem 3.5.1, $\tilde{X}$ is tangent to a fibration $\mathcal{P}$ of affine planes. Now choose a parallel orthonormal frame (E_1, E_2, E_3) of $\mathbb{E}^3$ such that E_1 and E_2 span the fibres of $\mathcal{P}$. This frame descends to an orthonormal frame of T^3, which we call (E_1, E_2, E_3) again. Then E_1 and E_2 span the leaves of the totally geodesic foliation $\mathcal{P}_T$ of T^3 covered by $\mathcal{P}$. Let us see that $\mathcal{P}_T$ is in fact a T^2-fibration over S^1. First note that the leaves are embedded copies of T^2. Indeed, each leaf $P_T \in \mathcal{P}_T$ is covered by a plane $P \in \mathcal{P}$, hence P_T is either a 2-torus, or a dense immersed cylinder $S^1 \times \mathbb{R}$, or a dense immersed copy of $\mathbb{R}^2$. But since X is constant on each leaf, the existence of dense leaves would force X_T to be globally constant, which we already ruled out.

Next, we want to define a map $\zeta\colon T^3 \to S^1$ whose fibres are the elements of $\mathcal{P}_T$. Fix a 2-torus $P_T \in \mathcal{P}_T$ covered by a plane $P \in \mathcal{P}$. The claim is now that the orbit of P under the action of $\mathbb{Z}^3$ is an infinite discrete set of equally spaced affine planes. That is, the distance between any two adjacent planes in $\mathbb{Z}^3(P)$ is the same. To see this, choose global coordinates (x_1, x_2, x_3) of $\mathbb{E}^3$ corresponding to the global orthonormal frame (E_1, E_2, E_3), and consider the projection

$$\pi\colon\ \mathbb{E}^3 \longrightarrow \mathbb{R}, \quad (x_1, x_2, x_3) \longmapsto x_3.$$

Let $A \subset \mathbb{R}$ be the image of $\mathcal{P}$ under this projection, i.e. every point in A corresponds to a plane in $\mathcal{P}$, and the distance of two points in A equals the distance of the corresponding planes in $\mathbb{E}^3$. Hence the claim is that A is a discrete set of equally spaced points. Note that the $\mathbb{Z}^3$-action on $\mathbb{E}^3$ induces a $\mathbb{Z}^3$-action on $\mathbb{R} = \pi(\mathbb{E}^3)$, and A is invariant under this action. Furthermore, the action of $\mathbb{Z}^3$ on $\mathbb{E}^3$ commutes with the action of $\mathbb{R}^3$ by translations, which implies that the action of $\mathbb{Z}^3$ on $\mathbb{R}$ commutes with the action of $\mathbb{R}$. In particular, we have the following.

Lemma 3.6.5. *Let $a, b \in A$ and $h \in \mathbb{R}$ such that $a + h \in A$. Then $b + h \in A$.*

Proof. Write $b = \gamma(a)$ for some $\gamma \in \mathbb{Z}^3$. Then

$$b + h = \gamma(a) + h = \gamma(a + h) \in A. \qquad \square$$

Using this lemma, we can now show that A is a discrete set. Indeed, if $a = \lim_{n\to\infty} \in A$ were an accumulation point, then writing $a_n = a + h_n$, where $h_n = a_n - a$, Lemma 3.6.5 implies that every point $b \in A$ is an accumulation point, namely $b = \lim_{n\to\infty}(b + h_n)$. But then A must be a dense set, which would mean that the fibre P_T is dense in T^3, which is not possible. Now consider three consecutive points $a, b, c \in A$. Write $b = a + h$, then $b + h \in A$ by Lemma 3.6.5, hence $\mathrm{dist}(b, c) \geq \mathrm{dist}(a, b)$, and similarly $\mathrm{dist}(a, b) \geq \mathrm{dist}(b, c)$. Hence $\mathrm{dist}(a, b) = \mathrm{dist}(b, c)$, so that A is a discrete set of equally spaced points. A similar argument shows that the minimal distance of two distinct points in A does not depend on the choice of fibre $P_T \in \mathcal{P}_T$.

We are now ready to define the map $\zeta\colon T^3 \to S^1$. Pick a fibre $P_T \in \mathcal{P}_T$ and let $\Phi = \Phi_t$ denote the flow of E_3. Let t_0 denote the minimal distance of two distinct points in A as above. Then, for $q \in T^3$, define $\zeta(q) \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$ as

$$\zeta(q) := 2\pi - \frac{2\pi\, t_q}{t_0} \mod 2\pi, \tag{3.4}$$

where $t_q > 0$ is the smallest positive number such that $\Phi_{t_q}(q)$ lies in the fibre P_T. It follows from the discussion above that this is a well-defined map; in fact, ζ defines a fibration of T^3 whose fibres are the elements of $\mathcal{P}_T$. Now we can write X as

$$X = \sin\theta(\zeta)\, E_1 + \cos\theta(\zeta)\, E_2 \tag{3.5}$$

for some function $\theta\colon S^1 \to S^1$. Using the identification $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, we may think of θ (or any function $S^1 \to S^1$) as a function $\mathbb{R} \to \mathbb{R}$, such that $\theta(t+2\pi)-\theta(t) \in 2\pi\,\mathbb{Z}$ for all $t \in \mathbb{R}$. As usual, one defines the **degree** of θ as

$$\deg\theta = \frac{1}{2\pi}(\theta(2\pi) - \theta(0)).$$

By θ' we mean the usual derivative of θ when viewed as a function defined on $\mathbb{R}$.

The following proposition will be crucial for the proof of Theorem 3.6.2.

Proposition 3.6.6. *Let X be a geodesic vector field on T^3. Then the following are equivalent.*

(i) X is conformally Reeb.

(ii) $\deg\theta \neq 0$ *and for any $a, b \in \mathbb{R}$ with $a < b$ we have that*

$$\theta(b) - \theta(a) > -\pi, \quad \textit{if } \deg\theta > 0,$$

and

$$\theta(b) - \theta(a) < \pi, \quad \textit{if } \deg\theta < 0.$$

(iii) The set

$$\mathcal{B} := \left\{\varphi\colon\ \mathbb{R}/2\pi\mathbb{Z} = S^1 \to S^1\colon\ \varphi' \neq 0,\ d(\varphi,\theta) < \frac{\pi}{2}\right\}$$

is non-empty. Here, $d(\varphi,\theta)$ is the maximum Euclidean distance of φ and θ (modulo 2π), that is,

$$d(\varphi,\theta) := \max_{x\in S^1}\big(|\varphi(x) - \theta(x)| \mod 2\pi\big).$$

Proof. We first show that (iii) implies (ii). So assume that (iii) holds, and choose some $\varphi \in \mathcal{B}$. Note that $\deg\theta = \deg\varphi \neq 0$. If $\deg\theta > 0$, then φ' must be positive everywhere. Then, for $a < b$,

$$\theta(a) - \frac{\pi}{2} < \varphi(a) < \varphi(b) < \theta(b) + \frac{\pi}{2},$$

which implies that $\theta(b) - \theta(a) > -\pi$. A similar argument applies for the case of $\deg\theta$ being negative.

Conversely, if (ii) holds, we need to show that $\mathcal{B} \neq \emptyset$. We will do so by constructing some $\varphi \in \mathcal{B}$ explicitly. Assume that $\deg\theta > 0$ (the case $\deg\theta < 0$ is analogous). Let

$$\varepsilon := \min_{a<b}\big(\theta(b) - \theta(a)\big) + \pi > 0.$$

Choose a function $\tilde{\theta}\colon\ S^1 \to \mathbb{R}$ such that $d(\tilde{\theta}, \theta) < \varepsilon/8$, and such that $\tilde{\theta}$ has finitely many local minima and maxima, respectively, and no other critical points. That is, there is a subdivision

$$0 < a_1 < b_1 < \ldots < a_n < b_n < 2\pi,$$

such that $\tilde{\theta}$ has a local maximum at every a_k and a local minimum at every b_k. Note that, since $d(\tilde{\theta}, \theta) < \varepsilon/8$,

$$\min_{a<b} \left(\tilde{\theta}(b) - \tilde{\theta}(a)\right) + \pi > \frac{7}{8}\varepsilon. \tag{3.6}$$

Now define intervals I_k by

$$I_k := \left[\tilde{\theta}(a_k) - \frac{\pi}{2} + \frac{\varepsilon}{4}, \tilde{\theta}(b_k) + \frac{\pi}{2} - \frac{\varepsilon}{4}\right].$$

Note that it follows from (3.6) that every I_k defines an interval with non-empty interior. Moreover, we have that

$$\max I_l > \min I_k \tag{3.7}$$

for every $l = 1, \ldots, n$ and $k \leq l$. Now we want to find some numbers

$$c_1 \leq c_2 \leq \ldots \leq c_n,$$

such that $c_k \in I_k$ for every k. Given such numbers, we can find a function $\phi\colon\ S^1 \to S^1$ with the following properties:

- $d(\phi, \tilde{\theta}) < \pi/2 - \varepsilon/4$;
- ϕ is constantly equal to c_k on $[a_k, b_k]$;
- ϕ is non-decreasing.

This is illustrated in Figure 3.8. This construction is possible since $\tilde{\theta}$ is strictly increasing on (b_k, a_{k+1}). Once we have constructed ϕ, we can choose a strictly increasing function $\varphi\colon\ S^1 \to S^1$ such that $d(\varphi, \phi) < \varepsilon/8$. Then, by the triangle inequality,

$$d(\varphi, \theta) \leq d(\varphi, \phi) + d(\phi, \tilde{\theta}) + d(\tilde{\theta}, \theta) < \frac{\varepsilon}{8} + \frac{\pi}{2} - \frac{\varepsilon}{4} + \frac{\varepsilon}{8} = \frac{\pi}{2},$$

hence $\varphi \in \mathcal{B}$, so that $\mathcal{B} \neq \emptyset$. Therefore, all we are left to do is to find numbers c_k as above. This is best done reversely, starting with c_n. Set $c_n := \max I_n$. The remaining c_k are defined inductively as

$$c_k := \min\{c_{k+1}, \max I_k\} \leq c_{k+1}.$$

Note that $c_k \in I_k$ since c_k is given by the maximum of some I_l, $l \geq k$, so that $c_k > \min I_k$ by (3.7). Clearly $c_1 \leq \ldots \leq c_n$, and this concludes the proof of the equivalence of (ii) and (iii).

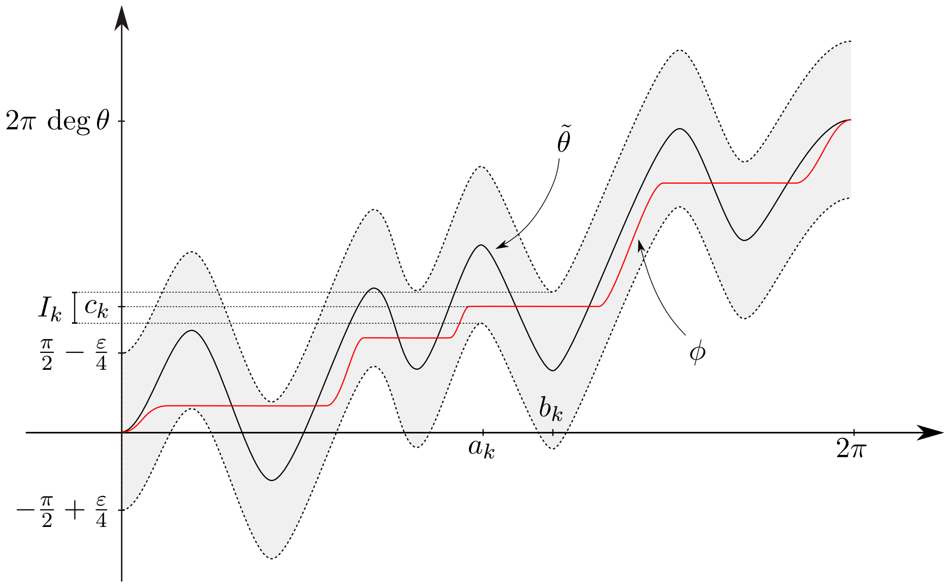

Figure 3.8: Construction of the function ϕ.

Now, let us see how (iii) implies (i). Given $\varphi \in \mathcal{B}$ as in (iii), consider the 1-form

$$\alpha = \sin\varphi(\zeta)\,\mathcal{E}^1 + \cos\varphi(\zeta)\,\mathcal{E}^2,$$

where $(\mathcal{E}^1, \mathcal{E}^2, \mathcal{E}^2)$ is the dual frame to (E_1, E_2, E_3). A simple calculation shows that

$$\mathrm{d}\alpha = \varphi'(\zeta)\cos\varphi(\zeta)\,\mathcal{E}^3 \wedge \mathcal{E}^1 - \varphi'(\zeta)\sin\varphi(\zeta)\,\mathcal{E}^3 \wedge \mathcal{E}^2,$$

hence

$$\alpha \wedge \mathrm{d}\alpha = \varphi'(\zeta)\,\mathcal{E}^1 \wedge \mathcal{E}^2 \wedge \mathcal{E}^3 \neq 0,$$

so α is a contact form. Its Reeb vector field is given by

$$R_\alpha = \sin\varphi(\zeta)\,E_1 + \cos\varphi(\zeta)\,E_2.$$

We claim that, for a suitably chosen $\varphi \in \mathcal{B}$, there are functions $f, g\colon\ S^1 \to \mathbb{R}^+$ such that

$$f(\zeta)\,X = R_{(1/g(\zeta))\alpha}. \tag{3.8}$$

Generally, for $h\colon\ T^3 \to \mathbb{R}^+$, we have that $R_{(1/h)\alpha} = hR_\alpha + Y$, where Y is the unique vector field satisfying $\alpha(Y) = 0$ and

$$i_Y\mathrm{d}\alpha = \mathrm{d}h(R_\alpha)\,\alpha - \mathrm{d}h. \tag{3.9}$$

Now, if $h = g \circ \zeta$ for some function $g\colon\ S^1 \to \mathbb{R}^+$, then $\mathrm{d}h(R_\alpha) = \mathrm{d}g \circ \mathrm{d}\zeta(R_\alpha) = 0$, so (3.9) translates into

$$i_Y\mathrm{d}\alpha = -\mathrm{d}h = -\mathrm{d}g \circ \mathrm{d}\zeta = -g'(\zeta)\,\mathcal{E}^3,$$

where we again think of g as a 2π-periodic function $\mathbb{R} \to \mathbb{R}^+$, with g' being its usual derivative. Then, to solve equation (3.8), we need to find functions f and g such that $Y := f(\zeta)X - g(\zeta)R_\alpha$ satisfies

$$0 = \alpha(Y) = f(\zeta)\,\alpha(X) - g(\zeta), \tag{3.10}$$

as well as

$$i_Y \mathrm{d}\alpha = -g'(\zeta)\,\mathcal{E}^3. \tag{3.11}$$

Now (3.10) is equivalent to $g = f\cos(\varphi - \theta)$, which is positive iff f is positive, since $d(\varphi, \theta) < \pi/2$. Thus we only need to find a suitable function f and then define g by the equation $g = f\cos(\varphi - \theta)$. In view of this equation, (3.11) translates into

$$f\varphi' \sin(\varphi - \theta)\,\mathcal{E}^3 = -\left(f' \cos(\varphi - \theta) - f(\varphi' - \theta')\sin(\varphi - \theta)\right)\,\mathcal{E}^3,$$

where we refrained from writing ζ in the arguments for simplicity. This, in turn, reduces to

$$f' \cos(\varphi - \theta) + f\theta' \sin(\varphi - \theta) = 0.$$

This differential equation is being solved by

$$f(x) := \exp\left(-\int_0^x \tan(\varphi(t) - \theta(t))\theta'(t)\,\mathrm{d}t\right) > 0.$$

However, for a generic choice of φ, the function f is not 2π-periodic, hence it does not define a function on S^1. Note that f is 2π-periodic if and only if

$$I(\varphi) := \int_0^{2\pi} \tan(\varphi(t) - \theta(t))\theta'(t)\,\mathrm{d}t$$

vanishes. Therefore, we need to show that the function $I\colon \mathcal{B} \to \mathbb{R}$ has a zero. First observe that since

$$\int_0^{2\pi} \tan(\varphi(t) - \theta(t))(\varphi'(t) - \theta'(t))\,\mathrm{d}t = \int_{x_0}^{x_0} \tan(u)\,\mathrm{d}u = 0 \quad (\text{where } u = \varphi - \theta),$$

we can write

$$I(\varphi) = \int_0^{2\pi} \tan(\varphi(t) - \theta(t))\,\varphi'(t)\,\mathrm{d}t.$$

Now, it is easy to see that $\mathcal{B}$ is convex. Hence, it suffices to find functions $\varphi^+, \varphi^- \in \mathcal{B}$ such that $I(\varphi^+) \geq 0$ and $I(\varphi^-) \leq 0$. For then we can simply interpolate between φ^+ and φ^- to find a zero of I. To achieve this, one can adjust the construction of ϕ in the proof of (ii) $\Rightarrow$ (iii) so that $\phi < \tilde{\theta}$ wherever ϕ is not constant (in fact, the function ϕ drawn in Figure 3.8 has this property). By approximating this function with a function in $\mathcal{B}$, we obtain a function $\varphi^- \in \mathcal{B}$ with $I(\varphi^-) \leq 0$. The function φ^+ is constructed similarly. This proves that (iii) implies (i).

To finish the proof, we show that (i) implies (ii). Assume that $X \sim R_\alpha$ for some contact form α of T^3. Suppose, for the sake of contradiction, that (ii) does not hold. Assume for the moment that $\deg\theta > 0$. Then (ii) being false means that there are $a, b \in [0, 2\pi]$ with $a < b$ such that $\theta(b) - \theta(a) = -\pi$, as well as $c, d \in [0, 2\pi]$ with $b < c < d$ such that $\theta(c) = \theta(b)$ and $\theta(d) = \theta(a)$ (since $\deg\theta > 0$). Furthermore, we may choose a, b and c, d so that

$$\theta(x) \in [\theta(b), \theta(a)] \quad \text{for all} \quad x \in [a, b] \cup [c, d]. \tag{3.12}$$

Now choose a point $p \in \mathbb{E}^3$ that projects to a point in $\zeta^{-1}(a) \subset T^3$ and let P be the affine plane in $\mathbb{E}^3$ through p spanned by E_3 and $\widetilde{X}_p$. Assume for the moment that P covers a 2-torus in T^3, which we call Σ. Consider the two subsets

$$\Sigma_1 := \Sigma \cap \{a \le \zeta \le b\}, \quad \Sigma_2 := \Sigma \cap \{c \le \zeta \le d\}.$$

Both Σ_1 and Σ_2 are diffeomorphic to cylinders, and each of their boundaries consists of two integral curves of X. Choose an orientation of Σ and orient Σ_1 and Σ_2 accordingly as submanifolds of Σ. Denote the (oriented) boundary curves of Σ_1 and Σ_2 by

$$\partial\Sigma_1 = \gamma_a \sqcup \gamma_b, \quad \partial\Sigma_2 = \gamma_c \sqcup \gamma_d.$$

We may choose the orientation of Σ so that γ_a and γ_b are negatively tangent to X, whereas γ_c and γ_d are positively tangent, see Figure 3.9. It follows that

$$\int_{\Sigma_1} \mathrm{d}\alpha = \int_{\gamma_a} \alpha + \int_{\gamma_b} \alpha < 0,$$

and

$$\int_{\Sigma_2} \mathrm{d}\alpha = \int_{\gamma_c} \alpha + \int_{\gamma_d} \alpha > 0.$$

However, it follows from (3.12) that X (and then also R_α) is positively transverse to the interiors of both Σ_1 and Σ_2. Then, since Σ_1 and Σ_2 are oriented consistently, $\int_{\Sigma_1} \mathrm{d}\alpha$ and $\int_{\Sigma_2} \mathrm{d}\alpha$ must have the same sign, and we arrive at a contradiction.

We are left to deal with the case of P covering some dense infinite cylinder in T^3 (instead of a 2-torus). Parametrise P using coordinates s and t such that ∂_s is parallel to $\widetilde{X}_p$. Consider subsets of the form

$$P_{s_0} := P \cap \{-s_0 \le s \le s_0\} \subset P$$

for some $s_0 > 0$. Then P_{s_0} covers a cylinder in T^3, which we call $\Sigma = \Sigma^{s_0}$. Let $\Sigma_1 = \Sigma_1^{s_0} = \Sigma \cap \{a \le \zeta \le b\}$ as before. Then

$$\partial\Sigma_1 = \partial_v\Sigma_1 \cup \partial_h\Sigma_1,$$

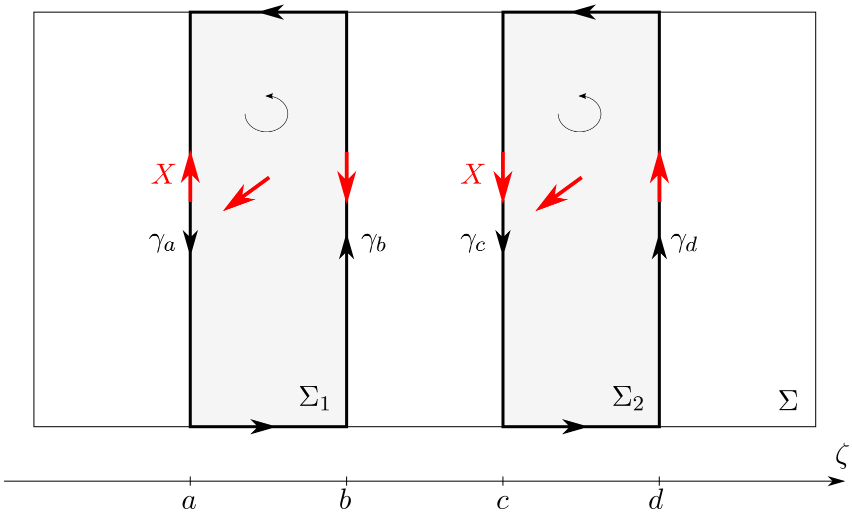

Figure 3.9: Σ_1 and Σ_2.

where $\partial_v\Sigma_1 = \gamma_a \sqcup \gamma_b$ and $\partial_h\Sigma_1 = \partial\Sigma_1 \cap \partial\Sigma$. In other words, $\partial_v\Sigma_1$ and $\partial_h\Sigma_1$ are the 'vertical' and 'horizontal' part of $\partial\Sigma_1$, respectively. Note that, since α is non-zero on the vertical boundary components, we have that

$$\left|\int_{\partial_v\Sigma_1^{t_0}} \alpha\right| > \left|\int_{\partial_v\Sigma_1^{s_0}} \alpha\right| \tag{3.13}$$

for $t_0 > s_0$. Now let

$$C := \left|\int_{\partial_v\Sigma_1^{s_0}} \alpha\right|$$

for some s_0, and choose $t_0 > s_0$ large enough so that

$$\left|\int_{\partial_h\Sigma_1^{t_0}} \alpha\right| < C.$$

This can be done due to the fact that P covers a dense cylinder in T^3. Then (3.13) implies that

$$\operatorname{sgn}\left(\int_{\partial\Sigma_1^{t_0}} \alpha\right) = \operatorname{sgn}\left(\int_{\partial_v\Sigma_1^{t_0}} \alpha\right),$$

and the same may be assumed for $\Sigma_2^{t_0}$. Then, using the same reasoning as in the first case, we arrive at a contradiction again.

The case $\deg\theta < 0$ is analogous. Now we are still left to show that $\deg\theta$ is indeed nonzero. Note that if $\deg\theta = 0$ and the image of θ is contained in an open interval of length at most π, then there is an embedded 2-torus transverse to X, so that X cannot be conformally Reeb. Therefore, we may again assume that there are $a < b < c < d$ with $\theta(b) - \theta(a) = \mp\pi$, $\theta(c) = \theta(b)$ and $\theta(d) = \theta(a)$, and we arrive at a contradiction using the same argument as before. $\square$

Remark 3.6.7. In the proof of (i) $\Rightarrow$ (ii), the surfaces Σ_1 and Σ_2 form what is called a *negative* and *positive partial section*, respectively. This notion was introduced by Cardona in his preprint [13], which was uploaded to the arXiv after the acceptance of the article [4]. Amongst other things, he proves that a vector field admitting both a negative and a positive partial section cannot be conformally Reeb [13, Lemma 12], and the argument is similar to the one given above.

Proof of Theorem 3.6.2. Assume first that there is a geodesic vector field Y on M inducing a contact structure ξ such that X is everywhere transverse to ξ. In other words, X and Y are nowhere orthogonal. Then the same is true for the lifted vector fields X_T and Y_T on T^3. In particular, both X_T and Y_T are non-constant, so by the discussion prior to Proposition 3.6.6, there are fibrations $\mathcal{T}_X$ and $\mathcal{T}_Y$ of T^3 by 2-tori tangent to X_T and Y_T, respectively. These two fibrations must coincide: Indeed, if this were not the case, we could consider a loop in some $T \in \mathcal{T}_X$ that is transverse to X_T and also transverse to every fibre in $\mathcal{T}_Y$. Along this loop, X_T is constant, whereas Y_T must make at least one complete turn (since Y_T induces a contact structure), hence X_T and Y_T are orthogonal somewhere, a contradiction. Therefore, writing X_T as

$$X_T = \sin\theta(\zeta)\, E_1 + \cos\theta(\zeta)\, E_2$$

as in (3.5), we find that Y_T is of the form

$$Y_T = \sin\varphi(\zeta)\, E_1 + \cos\varphi(\zeta)\, E_2 \tag{3.14}$$

for some function $\varphi\colon\ S^1 \to \mathbb{R}$ with $\varphi' \neq 0$. Since X_T and Y_T are nowhere orthogonal, we have that $d(\varphi, \theta) < \pi/2$. Thus, it follows from Propositions 3.6.4 and 3.6.6 that X is conformally Reeb.

Conversely, assume that X is conformally Reeb, that is, $X \sim R_\alpha$ for some contact form α on $M = T^3/\Gamma$. As before, write $X_T = \sin\theta(\zeta)\, E_1 + \cos\theta(\zeta)\, E_2$ for the lift of X to T^3. Then, by Proposition 3.6.6, there is a function $\varphi\colon\ S^1 \to S^1$ such that $\varphi' \neq 0$ and $d(\varphi, \theta) < \pi/2$. Hence the geodesic vector field $Y_T := \sin\varphi(\zeta)\, E_1 + \cos\varphi(\zeta)\, E_2$ induces a contact structure and is nowhere orthogonal to X_T. If $M = T^3$, then $Y = Y_T$ and we are done. So suppose that M is not equal to T^3, that is, $M = T^3/\Gamma$, where Γ is a nontrivial subgroup of $\mathrm{Isom}(T^3)$. We want to adjust the construction of Y_T (resp. φ) so that it is invariant under the action of Γ, and therefore descends to a geodesic vector field Y on M. First note that every element of Γ must be a screw motion of finite order in Γ, since glide reflections are not orientation-preserving. If $\gamma \in \Gamma$ is such a screw motion, then γ must preserve the fibration $\mathcal{T}$ of 2-tori defined by ζ. Indeed, if there were some $T \in \mathcal{T}$ such that $\gamma(T) \notin \mathcal{T}$, then $\gamma(T)$ would intersect every fibre of $\mathcal{T}$ transversely. But then X_T is constant along each fibre of

$\mathcal{T}$ and also constant along $\gamma(T)$ (since $\gamma_* X_T = X_T$), thus X_T is globally constant. In particular, X_T cannot be Reeb, a contradiction. But this means that the axis of rotation of γ (and consequently its translational part) must be orthogonal to $\mathcal{T}$. In other words, the translation vector of γ is a multiple of E_3. Now choose $\gamma_0 \in \Gamma$ so that the absolute value of its translational part is minimal among all elements of Γ. Then γ_0 generates Γ (in particular, Γ is cyclic). Write γ_0 as $\gamma_0 = T_{\lambda E_3} \circ R_\psi$, where R_ψ is rotation about the axis spanned by E_3 of angle ψ, and $T_{\lambda E_3}$ is the translation by the vector λE_3 for some real number λ. Then, it suffices to choose φ such that $\varphi(t+\lambda) = \varphi(t) + \psi$ for all t, for then $\varphi \circ \zeta \circ \gamma_0 = \varphi \circ \zeta + \psi$ which implies that $(\gamma_0)_* Y_T = Y_T$. To find an appropriate φ, we can construct φ first on the interval $[0, \lambda]$ as in the proof of Proposition 3.6.6, and then extend it to the whole real line via the rule $\varphi(t+\lambda) := \varphi(t) + \psi$. Here, one has to be a little careful to ensure smoothness of φ at points $k\lambda$, $k \in \mathbb{Z}$, but this can be arranged easily. The vector field Y_T we end up with is invariant under Γ. Note that if X_T and Y_T are not orthogonal on $\{0 \leq t \leq \lambda\}$, then they are nowhere orthogonal, since X_T and Y_T are both invariant under the action of Γ. Then Y_T descends to a vector field Y on M with the desired properties.

To prove the second statement of the theorem, note that in Proposition 3.6.6 it is actually shown that if X_T is conformally Reeb for some contact form α_T, then it is also conformally Reeb for a multiple of the contact form $\alpha_\varphi = \sin\varphi(\zeta)\mathcal{E}^1 + \cos\varphi(\zeta)\mathcal{E}^2$ whose kernel defines the contact structure ξ_T. Then, by Proposition 3.2.4, $\ker \alpha_T$ and ξ_T are diffeomorphic. Now set

$$n := 2\pi \deg \varphi = 2\pi \deg \theta = \theta(2\pi) - \theta(0).$$

Denoting by Φ the flow of E_3 again, consider the diffeomorphism

$$h\colon\ T^3 \longrightarrow T^3, \quad p \longmapsto (\Phi_{f(p)})(p),$$

where $f(p) := t_p + (t_0/2\pi)\varphi^{-1}(n\zeta(p))$, with t_0, t_p as in (3.4) and $\varphi^{-1}(n\zeta(p)) \in [0, 2\pi)$. Then h pulls α_φ back to

$$\alpha_n := \sin(n\zeta)\,\mathcal{E}^1 + \cos(n\zeta)\,\mathcal{E}^2.$$

On the other hand, denoting by $\beta_T = \sin\theta(\zeta)\mathcal{E}^1 + \cos\theta(\zeta)\mathcal{E}^2$ the 1-form dual to X_T, we have that

$$\begin{aligned} |\Gamma| \operatorname{vol}_X = \operatorname{vol}_{X_T} &= \int_{T^3} \beta_T \wedge \mathrm{d}\beta_T \\ &= \int_{T^3} \theta'(\zeta)\mathcal{E}^1 \wedge \mathcal{E}^2 \wedge \mathcal{E}^3 \\ &= \theta(2\pi) \int_{\zeta^{-1}(2\pi)} \mathcal{E}^1 \wedge \mathcal{E}^2 - \theta(0) \int_{\zeta^{-1}(0)} \mathcal{E}^1 \wedge \mathcal{E}^2 \\ &= nA, \end{aligned}$$

where the first equation follows from Lemma 3.2.5. Hence, $n = |\Gamma|\mathrm{vol}_X/A$. □

Proof of Corollary 3.6.3. Choose global coordinates (x, y, z) for $\mathbb{R}^3$ such that the frame (E_1, E_2, E_3) on T^3 is covered by the coordinate frame $(\partial_x, \partial_y, \partial_z)$. Then, by Theorem 3.6.2, $\ker\tilde{\alpha}$ is diffeomorphic to the kernel of

$$\tilde{\alpha}_n = \sin(nz)\mathrm{d}x + \cos(nz)\mathrm{d}y,$$

which, in turn, is diffeomorphic to $\ker\alpha_{\mathrm{st}}$ (see Example 2.1.8). □

3.7 Open flat 3-manifolds

We start with the following general result.

Proposition 3.7.1. *Let M be an orientable 3-manifold with $H^2_{dR}(M) = 0$, and X a nowhere vanishing vector field on M whose flow induces a free, proper $\mathbb{R}$-action. Then X is conformally Reeb.*

Proof. Since X induces a free and proper $\mathbb{R}$-action that is also orientation-preserving, the orbit space $B = M/\mathbb{R}$ is an orientable 2-dimensional manifold, and the projection $\pi\colon M \to B$ defines a principal line bundle which is necessarily trivial. That is, we can identify M with $B \times \mathbb{R}$, where the $\mathbb{R}$-fibres correspond to the integral curves of X. Now B is a deformation retract of M, so we have that $H^2_{dR}(B) = H^2_{dR}(M) = 0$. Hence, there is an exact area form $\omega = \mathrm{d}\beta$ on B. Let t denote the coordinate of the $\mathbb{R}$-factor of $M = B \times \mathbb{R}$. Then the 1-form $\alpha := \mathrm{d}t + \pi^*\beta$ is contact, and $R_\alpha \sim \partial_t = X$. □

Corollary 3.7.2. *Let X be an aperiodic geodesic vector field on $\mathbb{E}^3$ or $\mathbb{R}^2 \times S^1$. Then X is conformally Reeb.* □

Remark 3.7.3. Of course, in the case of $\mathbb{E}^3$, every geodesic vector field is aperiodic; hence, Corollary 3.7.2 implies that every geodesic vector field on $\mathbb{E}^3$ is conformally Reeb.

In the case of a line fibration of $\mathbb{R}^3$ (or generally $\mathbb{R}^n$), the fact that the induced $\mathbb{R}$-bundle is trivial can be seen more explicitly, by describing a specific section as follows. Denote the fibration by $\mathcal{F} = \{\ell\}$ as before, and let

$$\Sigma := \{p \in \mathbb{R}^n \colon \langle p, X\rangle = 0\},$$

where X is the unit vector field defining the (oriented) fibration $\mathcal{F}$. Note that Σ contains exactly one point in every fibre ℓ, and this point is characterised by minimising the distance to the origin among all points on ℓ. Now write Σ as $\Sigma = h^{-1}(0)$, where $h\colon \mathbb{R}^3 \to \mathbb{R}$, $h(p) = \langle p, X\rangle$. The differential of h is given by

$$\mathrm{d}h_p(v) = \langle v, X(p)\rangle + \langle p, \nabla_v X(p)\rangle, \quad v \in T_pM.$$

Now, since X is geodesic, $\mathrm{d}h_p(X) = |X|^2 = 1$, which implies in particular that h has regular values only. Hence, the level sets $h^{-1}(*)$ are embedded submanifolds which are transverse to X everywhere. This holds in particular for $\Sigma = h^{-1}(0)$. It remains to prove that Σ is an embedded copy of $\mathbb{R}^{n-1}$. To see this, consider the Morse function

$$\Psi\colon \mathbb{R}^n \longrightarrow \mathbb{R}, \quad \Psi(p) := |p|^2$$

Its only critical point (a minimum) is the origin. Furthermore, since $\mathrm{d}\Psi_p(v) = 2\langle p, v\rangle$, it follows that $\mathrm{d}\Psi_p(X) = 0$ for $p \in \Sigma$. Since $T_pM = T_p\Sigma \oplus (\mathbb{R}\cdot X)$, this means that Ψ restricts to a Morse function $\Psi|_\Sigma$ whose critical points are also critical points of Ψ. Hence $\Psi|_\Sigma$ is a Morse function with a single critical point (a minimum), which implies that Σ is an embedded copy of $\mathbb{R}^{n-1}$.

Next, we prove the following generalisation of Theorem 2.2.7. This is again stated in a slightly more general way, but applies in particular for geodesic vector fields on $\mathbb{E}^3$.

Theorem 3.7.4. *Let X a nowhere vanishing vector field on $\mathbb{R}^3$ whose flow induces a free, proper $\mathbb{R}$-action. Assume that there is a contact form α such that $X \sim R_\alpha$. Then, the contact structure $\ker\alpha$ is tight.*

Proof. Assume, for the sake of contradiction, that $(M, \xi = \ker\alpha)$ contains an overtwisted disc Δ. Identify $\mathbb{R}^3$ with $\mathbb{R}^2\times\mathbb{R}$ as in the proof of Proposition 3.7.1, with coordinates (x, y, z), so that $R_\alpha \sim X \sim \partial_z$. Let $\pi\colon \mathbb{R}^2\times\mathbb{R} \to \mathbb{R}^2$ denote the projection onto the first factor. Choose a disc $D^2 \subset \mathbb{R}^2$ such that $\pi(\Delta) \subset D^2$, and let $c > 0$ big enough so that $\Delta \subset D\times[-c, c] \subset \mathbb{R}^2\times\mathbb{R}$. Denote by ϕ_t the time-t-flow of X (which is a contactomorphism of (M, ξ) for all t), and choose $T > 0$ big enough so that $\phi_T(p) \notin \Delta$ for all $p \in \Delta$. Now consider the quotient space $M := (D\times\mathbb{R})/\langle\phi_T\rangle$, where $\langle\phi_T\rangle$ is the group of contactomorphisms generated by ϕ_T. Then $M \cong D^2\times S^1$ admits an induced contact form $\hat\alpha$, and the contact structure $\ker\hat\alpha$ is still overtwisted, since no two points on Δ are being identified by the action of $\langle\phi_T\rangle$. However, the Reeb vector field of $\hat\alpha$ is tangent to the S^1-fibres; in particular, there is no contractible periodic Reeb orbit. This is a contradiction to Theorem 2.1.14. □

Note that the contact structure $\ker\alpha$ in Theorem 3.7.4 is then diffeomorphic to the standard one by Theorem 2.1.13.

The following two examples are to show that Theorem 3.6.2 is not true in general for non-closed manifolds.

Example 3.7.5. (i) Let M be equal to $S^1 \times \mathbb{R}^2$ or $T^2 \times \mathbb{R}$ with coordinates (x, y, z) and consider the geodesic vector field

$$X = \sin\theta(z)\,\partial_x + \cos\theta(z)\,\partial_y,$$

where $\theta\colon \mathbb{R} \to \mathbb{R}$ is a smooth function defined as follows. Set $\theta(0) = \theta(2\pi) = 0$, $\theta(\pi) = -\pi$, and

$$\theta(z) \approx \begin{cases} -z, & 0 \le z \le \pi, \\ z - 2\pi, & \pi \le z \le 2\pi, \end{cases}$$

where the approximation is C^0-close. Then extend θ to a 2π-periodic function defined on $\mathbb{R}$. Since $\theta(\pi) - \theta(0) = -\pi$, the condition of Proposition 3.6.6 (or Theorem 3.6.2) is not satisfied. However, X is still conformally Reeb. To see this, consider the 1-form $\beta = F(z)\,\mathrm{d}x + y\sin\theta(z)\,\mathrm{d}z$, where

$$F(z) := \int_0^z \cos\theta(t)\,\mathrm{d}t - 1.$$

Then $\mathrm{d}\beta = \cos\theta(z)\,\mathrm{d}z \wedge \mathrm{d}x + \sin\theta(z)\,\mathrm{d}y \wedge \mathrm{d}z$ is non-degenerate on the plane field η spanned by ∂_z and $\cos\theta(z)\,\partial_x - \sin\theta(z)\,\partial_y$, and $i_X\mathrm{d}\beta = 0$. Furthermore,

$$\beta(X) = F(x)\sin\theta(x) \approx \begin{cases} \sin x(1 - \sin x) \ge 0, & \text{if } 0 \le x \le \pi, \\ \sin x(\sin x - 1) \ge 0, & \text{if } \pi < x \le 2\pi. \end{cases}$$

That is, $\beta(X) \ge -\varepsilon$ for some arbitrarily small $\varepsilon > 0$. Now choose ε so that $1 + 2\varepsilon\theta' > 0$ everywhere, and consider the 1-form

$$\alpha := \beta + 2\varepsilon\,\alpha_\theta,$$

where $\alpha_\theta = \sin\theta(z)\,\mathrm{d}x + \cos\theta(z)\,\mathrm{d}y$. Then

$$\mathrm{d}\alpha = \underbrace{(1 + 2\varepsilon\theta')}_{>0}\,\mathrm{d}\beta$$

is again non-degenerate on η, and $\alpha(X) = \beta(X) + 2\varepsilon \ge \varepsilon > 0$. Therefore, as X is transverse to η and $i_X\mathrm{d}\alpha = 0$, it follows that α is a contact form with Reeb vector field $R_\alpha = (1/\alpha(X))X$.

(ii) Let $M = T^2 \times \mathbb{R}$ with coordinates (x, y, z) and choose a diffeomorphism $\varphi\colon \mathbb{R} \xrightarrow{\cong} (-\pi/4, \pi/4)$. Define geodesic vector fields X and Y on M by

$$X = \partial_y + \partial_z, \quad Y = \sin\varphi(z)\,\partial_x + \cos\varphi(z)\,\partial_y.$$

Then Y induces a contact structure and $\langle X, Y\rangle = \cos\varphi(z) > 0$. But X is transverse to the 2-torus $\{z = 0\}$, hence X cannot be conformally Reeb.

4

Seifert fibrations

In this section, we investigate a very particular class of geodesible foliations, the so-called **Seifert fibrations**. A Seifert fibration of a 3-manifold is, roughly speaking, a fibration by circles such that each fibre has a tubular neighbourhood that either looks like a trivially fibred solid torus, or one that has been cut open along a meridional disc, twisted by some angle $2\pi r$ (where $r \in \mathbb{Q}$), and then glued back together (the fibration here being the one obtained from the trivial one under this operation). Thus, Seifert fibrations can be viewed as generalisations of honest S^1-fibrations.

We have already seen that every Seifert fibration is geodesible (see the remark following Example 1.1.14). In this chapter, we will see how to describe explicit metrics on Seifert manifolds that turn the Seifert fibres into geodesics. Namely, by a theorem of Scott [65], every Seifert manifold can be equipped with a locally homogeneous Riemannian metric for which the Seifert fibres are geodesics (with the exception of lens spaces, see Theorem 4.4.6). The goal of this chapter is to work out Scott's proof of Theorem 4.4.6, including some details missing in the original exposition. Furthermore, we observe that with respect to these metrics, a (geodesic) vector field X spanning the Seifert fibres induces a contact structure if and only if the Euler number of the Seifert fibration is nonzero. In this case, the vector field X will be the Reeb vector field of its dual contact form, and the contact structure will be universally tight (i.e. the pullback to its universal cover is tight). In particular, using Proposition 3.2.4, we conclude that a contact structure that admits a Reeb vector field spanning a Seifert fibration is universally tight.

More precisely, we will present a proof of the following statements.

Theorem 4.0.1. *Let M be a Seifert fibred 3-manifold not equal to a lens space. Then M admits a locally homogeneous Riemannian metric for which the Seifert fibres are geodesics. Furthermore, denoting by ξ the 2-plane field orthogonal to the fibration, the following holds true.*

- *If the Euler number e is nonzero, the plane field ξ defines a universally tight contact structure, and there is a contact form defining ξ whose Reeb vector field is tangent to the fibres of the Seifert fibration.*

- *If $e = 0$, the plane field ξ is integrable.*

Remark 4.0.2. (i) Note that a lens space can also be equipped with a locally homogeneous Riemannian metric (namely, the one induced by S^3, see Example 4.1.2 below). However, a Seifert fibration of a lens space equipped with this metric is not necessarily geodesic. In fact, the (k_1, k_2)-fibration in Example 4.1.2 is geodesic if and only if $k_1 = k_2 = 1$.

(ii) We should note at this point that although the statement of Theorem 4.0.1 is not given in [65], it essentially follows from the proof of the main theorem in that article. Hence the contents of this chapter are of a more expository nature.

In particular, we recover Corollary 3.4.3, and (using Proposition 3.2.4) the following statement, which is probably known but nowhere to be found in the literature.

Corollary 4.0.3. *Let M be a Seifert fibred 3-manifold and α a contact form whose Reeb vector field is tangent to the Seifert fibres. Then the contact structure $\ker\alpha$ is universally tight.* □

4.1 Definitions and examples

In this section, we give a brief introduction to the theory of Seifert fibrations, following [48] and [31].

Definition 4.1.1. Let M be a closed, oriented 3-manifold. A **Seifert fibration** of M is a smooth map $\pi\colon M \to \Sigma$ onto a closed surface Σ (which may be non-orientable), with the following property: Every point $x \in \Sigma$ admits a neighbourhood $D^2 \subset \Sigma$ (where x is identified with $0 \in D^2$) such that $\pi^{-1}(D^2) \cong D^2 \times S^1$, and, choosing the diffeomorphism in a suitable way, the map $\pi\colon D^2 \times S^1 \to D^2$ is given by

$$(re^{i\phi}, e^{i\theta}) \longmapsto re^{i(p\phi+q\theta)}, \tag{4.1}$$

where p and q are some coprime integers with $p \neq 0$. The number $|p|$ is called the **multiplicity** of the central fibre $\{0\} \times S^1$. If $p > 1$, the central fibre is called **singular**. We write (M, π, Σ) for the Seifert fibred manifold M whose Seifert fibration is given by π.

Note that, in the local model above, the central fibre is the only one that can be singular. In particular, since M is compact, every Seifert fibrations admits only finitely many singular fibres. Given the standard model $\pi\colon D^2 \times S^1 \to D^2$ around a singular fibre as above, we see that a typical nonsingular fibre $\pi^{-1}(re^{i\psi})$ can be parametrised by

$$t \longmapsto (re^{i(\psi/p+tq)}, e^{-itp}).$$

It follows that every nonsingular fibre goes $-p$ times along the longitudinal direction and q times along the meridional direction. That is, if we make one whole turn in the direction of the longitude, a nonsingular fibre makes $-q/p$ turns in meridional direction. In particular, a solid torus fibred that way is p-fold covered by a trivially fibred one.

Example 4.1.2. Consider S^3, viewed as the unit sphere in $\mathbb{C}^2$ with complex coordinates (z_1, z_2). Let k_1, k_2 be a pair of coprime integers, and consider the S^1-action

$$\theta(z_1, z_2) = (\mathrm{e}^{\mathrm{i}k_1\theta} z_1, \mathrm{e}^{\mathrm{i}k_2\theta} z_2), \quad \theta \in S^1 = \mathbb{R}/2\pi\mathbb{Z}. \tag{4.2}$$

This defines a Seifert fibration of S^3 with singular fibres $S^1 \times \{0\}$ and $\{0\} \times S^1$ of multiplicity k_1 and k_2, respectively. If $k_1 = k_2 = 1$, this defines the Hopf fibration of S^3 (Example 1.1.8).

Now consider the free $\mathbb{Z}_p$-action on S^3 generated by

$$(z_1, z_2) \longmapsto (\mathrm{e}^{2\pi\mathrm{i}/p} z_1, \mathrm{e}^{2\pi\mathrm{i}q/p} z_2),$$

where p and q are coprime integers. The quotient space $L(p, q) := S^3/\mathbb{Z}_p$ is called **lens space**. Note that $L(1, 0) = S^3$. Furthermore, we set $L(0, 1) := S^2 \times S^1$.

One can easily check that the $\mathbb{Z}_p$-action commutes with the S^1-action (4.2), so that the corresponding Seifert fibration of S^3 descends to $L(p, q)$. It can be shown that these Seifert fibrations determine all Seifert fibrations of lens spaces up to Seifert isomorphism (see below for the definition), cf. [31, Theorem 5.1].

Definition 4.1.3. An **isomorphism** between two Seifert fibrations $\pi\colon M \to \Sigma$ and $\pi'\colon M' \to \Sigma'$ is an orientation-preserving diffeomorphism $f\colon M \to M'$ that preserves fibres. In other words, there is a diffeomorphism $\overline{f}\colon \Sigma \to \Sigma'$ of the bases such that the following diagram is commutative:

$$\begin{array}{ccc} M & \xrightarrow{\;f\;} & M' \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi'} \\ \Sigma & \xrightarrow{\;\overline{f}\;} & \Sigma'. \end{array}$$

Next, we will define the so-called *Seifert invariants*, which consist of a collection of integers that describe every Seifert fibration in a unique way. Say we are given an integer g and pairs $(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)$ of coprime integers with $\alpha_i \neq 0$. For the moment, let us assume that $g \geq 0$. Then let Σ_0 denote the orientable surface of genus g with n disjoint discs removed, i.e. $\Sigma_0 := \Sigma_g \setminus (\coprod_{i=1}^n D_i^n)$. Consider the trivial S^1-bundle over Σ_0 with total space $M_0 = \Sigma_0 \times S^1$, whose boundary is given by

$$\partial M_0 = S_1^1 \times S^1 \cup \ldots \cup S_n^1 \times S^1,$$

where $S^1_i = \partial D^2_i$. Let

$$\begin{aligned} R &= \Sigma_0 \times \{1\}, \\ q_i &= S^1_i \times \{1\} \text{ (oriented as a component of } -\partial R), \\ h_i &= \{1\} \times S^1 \subseteq S^1_i \times S^1. \end{aligned}$$

Now, for each $i = 1, \ldots, n$, take a solid torus $T_i = D^2 \times S^1$ with respective meridian and longitude given by

$$\mu_i = \partial D^2 \times \{1\}, \quad \lambda_i = \{1\} \times S^1 \subset T_i.$$

Consider the manifold

$$M(g; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)) := M_0 \cup_\partial \coprod_i T_i, \tag{4.3}$$

where the T_i are glued along their boundary to $S^1_i \times S^1$ via the identifications

$$\mu_i = \alpha_i q_i + \beta_i h_i, \quad \lambda_i = \alpha_i' q_i + \beta_i' h_i, \tag{4.4}$$

where integers α_i', β_i' are chosen such that

$$\det \begin{pmatrix} \alpha_i & \alpha_i' \\ \beta_i & \beta_i' \end{pmatrix} = 1.$$

Note that the result of the gluing depends only on α_i and β_i, and not on the specific choice of α_i' and β_i'. The above identifications can be written equivalently as

$$h_i = -\alpha_i' \mu_i + \alpha_i \lambda_i, \quad q_i = \beta_i' \mu_i - \beta_i \lambda_i.$$

The trivial S^1-fibration of $M_0 = \Sigma_0 \times S^1$ then extends to the T_i via

$$\pi \colon T_i = D^2 \times S^1 \longrightarrow D^2, \quad (r\mathrm{e}^{\mathrm{i}\phi}, \mathrm{e}^{\mathrm{i}\theta}) \longmapsto r\mathrm{e}^{\mathrm{i}(\alpha_i \phi + \alpha_i' \theta)}.$$

In other words, $p = \alpha_i$ and $q = \alpha_i'$ in the local model (4.1). Hence, the manifold $M(g; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$ defined in (4.3) is Seifert fibred.

If $g < 0$, consider the non-orientable surface of genus g, which — by definition — is given by the connected sum of $|g|$ copies of $\mathbb{R}P^2$ and denoted by Σ_g again. Now Σ_g can be written as the connected sum of a Klein bottle or $\mathbb{R}P^2$ with some orientable surface. Then we can do the same construction as above by doing everything over the orientable part of Σ_g. The resulting Seifert manifold is again denoted by $M(g; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$.

Conversely, by reversing this process, one can show that every Seifert fibration can be written (up to isomorphism) as $M(g; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$ for some $g \in \mathbb{Z}$

and pairs (α_i, β_i) of coprime integers. In particular, the Seifert invariants describe a Seifert fibration up to isomorphism. However, this description is not unique. For example, permuting the (α_i, β_i) or adding or deleting any pairs of the type $(1, 0)$ (which corresponds to gluing in trivially fibred solid tori) do not change the isomorphism type of the resulting Seifert fibration. The next theorem tells us precisely which operations on the Seifert invariants preserve the isomorphism type.

Theorem 4.1.4. *Two sets of Seifert invariants determine isomorphic Seifert fibrations if and only if one can be changed into the other using the following operations:*

(1) Permute the pairs (α_i, β_i).

(2) Add or delete any pair $(\alpha, \beta) = (1, 0)$.

(3) Replace each (α_i, β_i) by $(\alpha_i, \beta_i + k_i\alpha_i)$, where $\sum_{i=1}^n k_i = 0$.

(4) Replace any (α_i, β_i) by $(-\alpha_i, -\beta_i)$.

Proof. See [48, Theorem 1.5]. □

It turns out that most Seifert manifolds admit a unique Seifert fibration (up to Seifert isomorphism). The only exceptions are the following.

Theorem 4.1.5. *We have the following diffeomorphisms of Seifert manifolds:*

(1) $M(-1; (\alpha, \beta)) \cong M(0; (2, 1), (2, -1), (-\beta, \alpha))$ (called 'prism manifolds');

(2) $M(-2; (1, 0)) \cong M(0; (2, 1)(2, 1), (2, -1), (2, -1))$.

Together with the lens spaces (Example 4.1.2), these are the only Seifert manifolds admitting at least two non-isomorphic Seifert fibrations.

Proof. See [48, Theorem 5.1]. □

Definition 4.1.6. Let $\pi \colon M \to \Sigma$ be a Seifert fibration with invariants given by $(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)$. Then the **Euler number** of π is defined as

$$e = e(\pi) = -\sum_{i=1}^{n} \frac{\beta_i}{\alpha_i}.$$

One can show that if the Seifert fibration defines an honest S^1-fibration (i.e. $\alpha_i = 1$ for all i), then $e = -\sum_i \beta_i \in \mathbb{Z}$ coincides with the usual Euler number of the S^1-fibration.

4.2 Orbifolds

Let $\pi\colon\ M \to \Sigma$ be an (honest) S^1-fibration. Then, for every point $x \in \Sigma$, there is a neighbourhood $U \cong D^2$ such that $\pi^{-1}(U) \cong D^2 \times S^1$ is a trivially fibred solid torus. In $D^2 \times S^1$, any meridional disc intersects every fibre exactly once, so that the fibre space obtained by identifying every fibre to a point is given by D^2. On the other hand, if we are given a Seifert fibration, we have seen that in the standard model $\pi\colon\ D^2 \times S^1 \to D^2$ around a singular fibre of multiplicity p, the nonsingular fibres intersect the meridional discs p times. Therefore, the fibre space in this case is given by the quotient of D^2 under the $\mathbb{Z}_p$-action generated by a rotation about the origin by an angle of $2\pi/p$. Hence the orbit space of a Seifert fibration can be thought of as a space that locally looks like the quotient of D^2 under some finite group action (in this case, rotation about the origin). Such spaces will be called **orbifolds**. The general definition is given below.

Definition 4.2.1. Let $\mathcal{O}$ be a Hausdorff, paracompact space together with a covering by open subsets U_i. Associated with each U_i is

- an open subset $V_i \subset \mathbb{R}^n$,
- a finite group Γ_i, defining an action on V_i,
- a homeomorphism $\varphi_i\colon\ V_i/\Gamma_i \longrightarrow U_i$.

The tuple $(U_i, \Gamma_i, V_i, \varphi_i)$ is called **orbifold chart** for $\mathcal{O}$. Furthermore, if $U_i \subset U_j$, there is to be an inclusion $\Gamma_i \subset \Gamma_j$ and an embedding $V_i \hookrightarrow V_j$ such that the following diagram commutes:

$$
\begin{array}{ccc}
V_i & \hookrightarrow & V_j \\
\downarrow & & \downarrow \\
V_i/\Gamma_i & \hookrightarrow & V_j/\Gamma_i \\
 & & \downarrow \\
\Big\downarrow \scriptstyle{\varphi_i} & & V_j/\Gamma_j \\
 & & \Big\downarrow \scriptstyle{\varphi_j} \\
U_i & \hookrightarrow & U_j
\end{array}
$$

An atlas $\mathcal{U} = \{(U_i, \Gamma_i, V_i, \varphi_i)\}$ of orbifold charts is said to be **maximal** if every orbifold chart for $\mathcal{O}$ that is compatible (in the above sense) with every other chart of $\mathcal{U}$ is contained in $\mathcal{U}$. Finally, an **orbifold** $\mathcal{O}$ is a Hausdorff, paracompact space together with a maximal orbifold atlas.

By the discussion at the beginning of this chapter, it is now evident that if $\pi\colon M \to \Sigma$ is a Seifert fibration, then Σ admits a natural orbifold structure, where a neighbourhood of $x \in \Sigma$ is being identified with the quotient $D^2/\mathbb{Z}_p$, where p is the multiplicity of the fibre $\pi^{-1}(x)$.

Let us now review some examples of orbifolds.

Example 4.2.2. (i) (Quotient spaces) Let M a manifold and Γ be a group acting properly discontinuously on M, i.e. for every $x \in M$ there is an open neighbourhood U such that $|\{\gamma \in \Gamma\colon\ \gamma(U) \cap U \neq \emptyset\}| < \infty$. Then the quotient space M/Γ is Hausdorff and paracompact (the latter follows from the fact that the quotient map $M \to M/\Gamma$ is open). Furthermore, M/Γ admits a natural orbifold structure, as follows. Given a point $[x] \in M/\Gamma$ and a lift $x \in M$, consider the stabiliser subgroup

$$\Gamma_x := \{\gamma \in \Gamma\colon\ \gamma(x) = x\} \leq \Gamma$$

of x. Since Γ acts properly discontinuously, Γ_x is finite and there is a neighbourhood $V \subset M$ about x, diffeomorphic to an open subset of $\mathbb{R}^n$, such that $\gamma(V) \cap V \neq \emptyset$ if and only if $\gamma \in \Gamma_x$. Let

$$\tilde{V} := \bigcap_{\gamma \in \Gamma_x} \gamma(V) \subset M,$$

which can be seen as an open subset $\tilde{V} \subset V \subset \mathbb{R}^n$. Then Γ_x acts on $\tilde{V}$, and since $\gamma(\tilde{V}) \cap \tilde{V} = \emptyset$ for $\gamma \notin \Gamma_x$, there is a homeomorphism

$$\varphi\colon\ \tilde{V}/\Gamma_x \xrightarrow{\cong} \pi(\tilde{V}) =: U,$$

where $\pi\colon\ M \to M/\Gamma$ is the quotient map. Thus the tuple $(U, \Gamma_x, \tilde{V}, \varphi)$ defines an orbifold chart. Now it is not hard to see that one can cover M/Γ by such orbifold charts that are compatible with each other in the sense of Definition 4.2.1. This defines the orbifold structure of M/Γ.

(ii) Let D_1^2, D_2^2 be two copies of the closed unit disc in $\mathbb{R}^2$. Consider the $\mathbb{Z}_p$-action on D_1^2 generated by rotation about the origin by an angle of $2\pi/p$. Similarly, we define an $\mathbb{Z}_q$-action on D_2^2 (here p and q are any two integers). The quotient spaces $D_1^2/\mathbb{Z}_p$ and $D_2^2/\mathbb{Z}_q$ are topologically discs again. Hence we can glue them along their boundary (via the identity map) to obtain

$$S^2(p,q) := D_1^2/\mathbb{Z}_p \cup_\partial D_2^2/\mathbb{Z}_q,$$

which is, topologically, a copy of S^2. However, the actions of $\mathbb{Z}_p$ and $\mathbb{Z}_q$ define an orbifold structure on $S^2(p,q)$ with two orbifold points given by the centres of D_1^2 and D_2^2. If $p = q$, then $S^2(p, p)$ is orbifold isomorphic to $S^2/\mathbb{Z}_p$, with $\mathbb{Z}_p$ the action induced by rotation about some axis by an angle of $2\pi/p$. If $p \neq q$, then $S^2(p, q)$ cannot be written as a quotient orbifold (see Proposition 4.2.10). The orbifold $S^2(p, q)$ is also called the (p, q)-*football*. A special case is if $q = 1$ and $p \neq 1$. This orbifold is written as $S^2(p) = S^2(p, 1)$ and called *teardrop orbifold*.

Notation. Given an orbifold $\mathcal{O}$, we write $|\mathcal{O}|$ for its underlying topological space. In dimension 2, $|\mathcal{O}|$ is always a manifold (cf. [65]); in this case, we will think of $|\mathcal{O}|$ as the topological space underlying $\mathcal{O}$ together with its manifold structure.

There is also a concept of coverings for orbifolds.

Definition 4.2.3. A map $\pi\colon\ \mathcal{O}' \to \mathcal{O}$ between orbifolds is called **orbifold covering** if, for every point $x \in \mathcal{O}$, there is an orbifold chart $x \in U \cong V/\Gamma$ such that $\pi^{-1}(U)$ is the disjoint union of orbifold charts $U_i' \cong V/\Gamma_i'$, $i \in I$, where $\Gamma_i' \subset \Gamma_i$ and I is a discrete set (finite or infinite), and each restriction $\pi|_{U_i'}\colon\ V/\Gamma_i' \to V/\Gamma_i$ is given by the natural projection. For a nonsingular point $x \in \mathcal{O}$, the cardinality $|\pi^{-1}(x)|$ does not depend on the choice of x and is called **degree** of the covering. An orbifold covering $\tilde{\pi}\colon\ \tilde{\mathcal{O}} \to \mathcal{O}$ is called **universal** if for every orbifold covering $\pi\colon\ \mathcal{O}' \to \mathcal{O}$ there is an orbifold covering $\pi'\colon\ \tilde{\mathcal{O}} \to \mathcal{O}'$ such that the following diagram commutes:

$$\begin{array}{ccc} \tilde{\mathcal{O}} & \xrightarrow{\pi'} & \mathcal{O}' \\ & \searrow^{\tilde{\pi}} & \downarrow{\scriptstyle \pi} \\ & & \mathcal{O} \end{array}$$

For example, if M is a manifold with G a finite group acting on it and $H \subset G$ is a subgroup, then the natural projection $M/H \to M/G$ is an orbifold covering. The universal orbifold covering in this case is the projection $\tilde{M} \to M \to M/G$, where $\tilde{M}$ is the universal cover of M. It is important to distinguish between orbifold coverings and coverings (in the usual sense) of the underlying topological spaces. For example, if $M = S^2$ and $G \cong \mathbb{Z}_n$ is the group acting on S^2 by rotation about some axis by an angle of $2\pi/n$, then $|S^2/G| \cong S^2$, so $|S^2/G|$ does not admit any nontrivial covering. However, the orbifold S^2/G does admit a nontrivial orbifold covering: namely, the natural projection $S^2 \to S^2/G$.

At this point, for simplicity, we will only consider 2-dimensional orbifolds that are given as the base of some Seifert fibration. To be precise, we make the following definition.

Definition 4.2.4. Given a 2-dimensional orbifold, an orbifold point whose local model is given by $D^2/\mathbb{Z}_p$ (where $\mathbb{Z}_p$ is generated by rotation about 0 by an angle of

$2\pi/p$) is called **cone point of order** p. A closed 2-dimensional orbifold all of whose orbifold points are cone points is called **Seifert orbifold**.

We have already seen that the base of a Seifert fibration naturally has the structure of a Seifert orbifold. Conversely, given a Seifert orbifold $\mathcal{O}$, one can construct a Seifert fibration whose base orbifold is given by $\mathcal{O}$ (see the discussion following Definition 4.1.3).

The following statement is true for arbitrary orbifolds of any dimension [72, Proposition 13.2.4]; we will, however, only prove it for Seifert orbifolds.

Proposition 4.2.5. *Every Seifert orbifold admits a universal orbifold covering.*

Proof. We follow the idea given in [65]. Assume that $\mathcal{O}$ is a Seifert orbifold and $\pi\colon \mathcal{O}' \to \mathcal{O}$ is an orbifold covering, where $\mathcal{O}'$ is some 2-dimensional orbifold. In order to simplify notation, we assume that $\mathcal{O}$ has a single cone point of order p only (the general case is completely similar). Denote this orbifold point by x, and let $U \cong B^2/\mathbb{Z}_p$ be a (small) orbifold chart about x, where $B^2 \subset \mathbb{C}$ is an open disc and $\mathbb{Z}_p$ acts by rotation. Let $N = \mathcal{O} \setminus U$ and $N' = \pi^{-1}(N) \subset \mathcal{O}'$. Then both N and N' are topological surfaces with boundary and without orbifold points. Hence, the restriction

$$\pi|_{N'} : N' \longrightarrow N$$

defines a covering in the usual sense. Now by the definition of an orbifold covering, the preimage $\pi^{-1}(U)$ is the union of disjoint orbifold charts $U_1, \ldots, U_n$, where $U_i \cong B^2/\mathbb{Z}_{k_i}$ and k_i divides p, i.e. $p = k_i l_i$ for some $l_i \in \mathbb{N}$. Then, identifying both U and U_i with B^2 in the natural way, the restriction of π to U_i is given by

$$\mathbb{C} \supset B^2 \longrightarrow B^2, \quad z \longmapsto z^{l_i}.$$

Now denote by C the boundary curve of $U \subset \mathcal{O}$ and let $C_i := (\pi|_{U_i})^{-1}(C)$. Let $[C_i] \in \pi_1(N)$ denote the corresponding elements of the fundamental group of N. Then

$$\pi_*([C_i]^{k_i}) = [C]^{k_i l_i} = [C]^p,$$

hence the subgroup $\pi_*(\pi_1(N')) \le \pi_1(N)$ contains the normal subgroup K generated by $[C]^p$. Now let $\tilde{\pi}\colon \tilde{N} \to N$ denote the covering of N determined by the subgroup K (that is, $\tilde{\pi}_*(\pi_1(\tilde{N})) = K$). Note that in particular, $\tilde{N}$ covers N'. Now the covering $\tilde{\pi}$ extends to an orbifold covering $\tilde{\mathcal{O}} \to \mathcal{O}$, as follows. Given a component $\tilde{C}_i \subset \tilde{N}$ of $\tilde{\pi}^{-1}(C)$, the projection $\tilde{C}_i \to C$ is $\tilde{l}_i$-fold for some natural number $\tilde{l}_i$. Since $\tilde{\pi}_*(\pi_1(\tilde{N})) = K$, there is some number $\tilde{k}_i$ such that

$$[C]^{\tilde{k}_i \tilde{l}_i} = \pi_*([\tilde{C}_i]^{\tilde{k}_i}) = [C]^p.$$

In particular, $\tilde{k}_i$ divides p. Now let $\tilde{\mathcal{O}}$ be the orbifold obtained from $\tilde{N}$ by attaching $D^2/\mathbb{Z}_{\tilde{k}_i}$ along $\tilde{C}_i$ (for each $\tilde{C}_i$ in the preimage of C), equipped with the natural orbifold structure as in Example 4.2.2 (ii). Then, since $\tilde{k}_i$ divides p, this extends to an orbifold covering $\tilde{\mathcal{O}} \to \mathcal{O}$ via the natural projection maps $D^2/\mathbb{Z}_{\tilde{k}_i} \to D^2/\mathbb{Z}_p$. This orbifold covering is universal, since by construction, the covering $\tilde{N} \to N$ is universal among all coverings of N that extend to orbifold coverings. □

Definition 4.2.6. Let $\mathcal{O}$ be a (Seifert) orbifold and $\pi\colon \tilde{\mathcal{O}} \to \mathcal{O}$ be its universal covering. The **orbifold fundamental group** of $\mathcal{O}$ is the group of deck transformations of π (i.e. diffeomorphisms $\varphi\colon \tilde{\mathcal{O}} \to \tilde{\mathcal{O}}$ such that $\pi \circ \varphi = \pi$), and denoted by $\pi_1^{\mathrm{orb}}(\mathcal{O})$.

Remark 4.2.7. In the proof of Proposition 4.2.5, the deck transformation group of the universal covering $\tilde{\mathcal{O}} \to \mathcal{O}$ coincides with the deck transformation group of $\tilde{N} \to N$, since every deck transformation of $\tilde{N}$ extends to a deck transformation of $\tilde{\mathcal{O}}$, simply by extending the diffeomorphism of the boundary components of $\tilde{N}$ to the attached discs. Now the deck transformation group of $\tilde{N}$ is given by $\pi_1(N)/K$. Hence we obtain the following finite presentation for the orbifold fundamental group of a Seifert orbifold.

Proposition 4.2.8. *Let $\mathcal{O}$ be a Seifert orbifold whose underlying surface has genus g, with cone points $x_1, \ldots, x_n$ of orders $\alpha_1, \ldots, \alpha_n$. Then $\pi_1^{\mathrm{orb}}(\mathcal{O})$ has the following finite presentation:*

$$\pi_1^{\mathrm{orb}}(\mathcal{O}) \cong \begin{cases} \langle a_i, b_i, q_j \,|\, q_1 \cdots q_n\, [a_1, b_1] \cdots [a_g, b_g],\, q_j^{\alpha_j} \rangle, & \text{if } g \geq 0 \\ \langle a_i, q_j \,|\, q_1 \cdots q_n\, a_1^2 \cdots a_g^2,\, q_j^{\alpha_j} \rangle, & \text{if } g < 0., \end{cases}$$

where $i = 1, \ldots, |g|$ and $j = 1, \ldots, n$. □

Here, when writing a relation as a word w, we mean that $w = 1$ (for example, $q_j^{\alpha_j} = 1$ for all $j \in \{1, \ldots, n\}$).

Definition 4.2.9. An orbifold is called **good** if it is covered by a manifold (or, equivalently, if its universal cover is a manifold). Otherwise it is called **bad**.

An example of a bad orbifold is given by $S^2(p, q)$ for $p \neq q$ (see Example 4.2.2 (ii)). Indeed, up to taking a finite cover, we may assume that p and q are coprime. Then

$$\pi_1^{\mathrm{orb}}(S^2(p, q)) \cong \langle c_1, c_2 \,|\, c_1 c_2,\, c_1^p,\, c_2^q \rangle \cong \langle c \,|\, c^p,\, c^{-q} \rangle = \{1\},$$

since for every $k \in \mathbb{Z}$, there are $m, n \in \mathbb{Z}$ such that $pm - qn = k$. But this means that $S^2(p, q)$ does not admit any nontrivial orbifold covering, so in particular, it is not covered by a manifold.

It turns out that among Seifert orbifolds, these are the only examples of bad orbifolds.

Proposition 4.2.10. *The only bad Seifert orbifolds are* $S^2(p)$ *and* $S^2(p,q)$*, where* $p, q \in \mathbb{Z}$ *and* $p \neq q$*.*

Proof. See [65, Theorem 2.3]. □

Now let us see how to define a notion of Euler characteristic for a Seifert orbifold $\mathcal{O}$. A first attempt would be to define the Euler characteristic of $\mathcal{O}$ simply as the Euler characteristic of its underlying surface $|\mathcal{O}|$. However, consider for example the d-fold orbifold covering $D^2 \to D^2/\mathbb{Z}_d$, where $\mathbb{Z}_d$ acts by rotation. Then the Euler characteristic does not see this nontrivial covering, as the surface underlying $D^2/\mathbb{Z}_p$ is homeomorphic to D^2. Thus one has to adjust the definition of orbifold Euler characteristic, as follows.

Definition 4.2.11. Let $\mathcal{O}$ be a Seifert orbifold whose orbifold points are cone points of order $p_1, \ldots, p_k$, respectively. Then the **orbifold Euler characteristic** of $\mathcal{O}$ is defined as

$$\chi(\mathcal{O}) := \chi(|\mathcal{O}|) - \sum_{i=1}^{k} \left(1 - \frac{1}{p_i}\right).$$

This definition is chosen in a way so as to behave well under coverings. Indeed, one can show that if $\mathcal{O}' \to \mathcal{O}$ is a d-fold orbifold covering, then $\chi(\mathcal{O}') = d\chi(\mathcal{O})$ (see [56, Proposition 6.2.9]).

If a 2-orbifold is good, it is covered by a surface, hence its universal cover is (topologically) S^2 or $\mathbb{R}^2$. Now every surface admits a geometry modelled on one of the three model spaces $S^2, \mathbb{E}^2$ or $\mathbb{H}^2$ (written M), that is, it can be written as a quotient M/Γ, where $\Gamma \leq \mathrm{Isom}(M)$ is a discrete subgroup of isometries acting freely and properly discontinuously on M. It turns out that the same is true for a good orbifold, with Γ not acting freely but still properly discontinuously.

Theorem 4.2.12. *Let* $\mathcal{O}$ *be a good* 2*-dimensional orbifold, and let*

$$\tilde{\mathcal{O}} := \begin{cases} S^2, & \text{if } \chi(\mathcal{O}) > 0 \\ \mathbb{E}^2, & \text{if } \chi(\mathcal{O}) = 0 \\ \mathbb{H}^2, & \text{if } \chi(\mathcal{O}) < 0. \end{cases}$$

Then there is a subgroup $\Gamma \leq \mathrm{Isom}(\tilde{\mathcal{O}})$ *acting properly discontinuously on* $\tilde{\mathcal{O}}$ *such that* $\mathcal{O} = \tilde{\mathcal{O}}/\Gamma$*.*

Proof. See [56, Theorem 6.2.10]. □

4.3 The fundamental group of a Seifert manifold

In this section, our goal is to derive a presentation of the fundamental group of a Seifert manifold in terms of its invariants. We will compare this presentation to the presentation of the fundamental group of the base orbifold in Proposition 4.2.8, and show that most Seifert manifolds are determined, up to Seifert isomorphism, by their fundamental group and their base orbifold.

Proposition 4.3.1. *The fundamental group of $M = M(g; (\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n))$ has the finite presentation*

$$\pi_1(M) \cong \begin{cases} \langle a_i, b_i, q_j, h \mid h \textit{ central}, \; q_1 \cdots q_n \, [a_1, b_1] \cdots [a_g, b_g], \; q_j^{\alpha_j} h^{\beta_j} \rangle, & \textit{if } g \geq 0, \\ \langle a_i, q_j, h \mid a_i h a_i^{-1} h, \; [q_j, h], \; q_1 \cdots q_n \, a_1^2 \cdots a_{|g|}^2, \; q_j^{\alpha_j} h^{\beta_j} \rangle, & \textit{if } g < 0, \end{cases}$$

where $i = 1, \dots, |g|$ and $j = 1, \dots, n$.

Proof. We present the argument given in [48], by computing the fundamental group of M using the theorem of Seifert and van Kampen. Let $\pi \colon M \to \mathcal{O}$ be the corresponding Seifert fibration, where $\mathcal{O}$ is the base orbifold with orbifold points $x_1, \dots, x_n$. Consider small open discs B_i about the x_i and let $\Sigma_0 := |\mathcal{O}| \setminus (\coprod_{i=1}^n B_i)$. Then Σ_0 is a topological surface of genus g with n boundary components. Let us assume that $g \geq 0$ (the case $g < 0$ is similar). Then the fundamental group of Σ_0 is given by

$$\pi_1(\Sigma_0) = \langle a_i, b_i, q_j \mid q_1 \cdots q_n \, [a_1, b_1] \cdots [a_g, b_g] \rangle.$$

Hence

$$\pi_1(\Sigma_0 \times S^1) = \langle a_i, b_i, q_j, h \mid h \text{ central}, \; q_1 \cdots q_n \, [a_1, b_1] \cdots [a_g, b_g] \rangle.$$

The claim is now that the gluing of the T_i as in (4.4) adds the relations $q_j^{\alpha_j} h^{\beta_j} = 1$. Indeed, using (4.4) and the Seifert-van Kampen theorem, the gluing of T_i amounts to adding the generator λ_i to π_1, together with the relations

$$\lambda_i = q_i^{\alpha'_i} \, h^{\beta'_i}, \quad 1 = q_i^{\alpha_i} \, h^{\beta_i}$$

Now since λ_i can be expressed using the generators q_i and h, it can be deleted, leaving us only with the relations $q_i^{\alpha_i} \, h^{\beta_i}$ which proves the claim. □

Note that if $\pi \colon M \to \mathcal{O}$ is a Seifert fibration with underlying base orbifold $\mathcal{O}$, then using the presentations of the fundamental group of M in 4.3.1 and the orbifold fundamental group of $\mathcal{O}$ in Proposition 4.2.8, we have the relation

$$\pi_1^{\text{orb}}(\mathcal{O}) \cong \pi_1(M) / \langle h \rangle,$$

which should not come as a surprise, since the generator h in $\pi_1(M)$ corresponds to a typical fibre of M.

The following statement is probably not new, but we could not find it anywhere in the literature, hence we give a short proof.

Proposition 4.3.2. *Let $p_1\colon M_1 \to \mathcal{O}$ and $p_2\colon M_2 \to \mathcal{O}$ be two Seifert fibrations over the same base orbifold $\mathcal{O}$. Assume that M_1 and M_2 are not diffeomorphic to lens spaces, and that $\pi_1(M_1) \cong \pi_1(M_2)$. Then, perhaps after reversing the orientation of fibres, M_1 and M_2 are Seifert isomorphic.*

Proof. First note that M_1 and M_2 are both irreducible (that is, every embedded 2-sphere bounds a ball). Indeed, a Seifert manifold which is not irreducible is diffeomorphic to $S^2 \times S^1$ or the nontrivial S^1-bundle over $\mathbb{R}P^2$ (see [56, Corollary 10.3.40]), both of which count as lens spaces according to our definition (the latter one is diffeomorphic to $L(4,1)$, see [51, Theorem 1.1]). By a theorem of Scott [66, Theorem 3.1], the homeomorphism type of an irreducible Seifert manifold is determined by its fundamental group; in particular, M_1 and M_2 are homeomorphic. Hence, by Theorem 4.1.5, either M_1 and M_2 are Seifert isomorphic, or each of them is Seifert isomorphic to one of the exceptional Seifert manifolds in Theorem 4.1.5. In the latter case, since M_1 and M_2 have the same base, there are only two possibilities:

1. $M_1 = M(-1;(\alpha,\beta_1))$ and $M_2 = M(-1,(\alpha,\beta_2))$, or
2. $M_1 = M(0;(2,1),(2,-1),(-\beta,\alpha_1))$ and $M_2 = M(0;(2,1),(2,-1),(-\beta,\alpha_2))$,

where α and β_i are coprime, $i = 1,2$. Now in the first case, using Proposition 4.3.1, the fundamental groups of M_1 and M_2 are given by

$$\begin{aligned}\pi_1(M_1) &\cong \langle a_1, q_1, h_1 \,|\, a_1h_1a_1^{-1}h_1,\ [q_1,h_1],\ q_1a_1^2,\ q_1^{\alpha}h_1^{\beta_1}\rangle \\ &\cong \langle a_1, h_1 \,|\, a_1h_1a_1^{-1}h_1,\ a_1^{2\alpha} = h_1^{\beta_1}\rangle\end{aligned}$$

and

$$\pi_1(M_2) \cong \langle a_2, h_2 \,|\, a_2h_2a_2^{-1}h_2,\ a_2^{2\alpha} = h_2^{\beta_2}\rangle.$$

The subgroup $H \leq \pi_1(M_1)$ generated by h_1 is a normal subgroup in $\pi_1(M_1)$. From the relations in $\pi_1(M_1)$ we obtain

$$h_1^{\beta_1} = a_1^{2\alpha} = a_1h_1^{\beta_1}a_1^{-1} = h_1^{-\beta_1},$$

hence $h_1^{2\beta_1} = 1$. Thus H is isomorphic to $\mathbb{Z}_{2\beta_1}$. Furthermore, the quotient group $\pi_1(M_1)/H$ is isomorphic to $\mathbb{Z}_{2\alpha}$. Using Lagrange's theorem, we obtain

$$|\pi_1(M_1)| = |\pi_1(M_1)/H||H| = 4|\alpha\beta_1|.$$

Similarly, $|\pi_1(M_2)| = 4|\alpha\beta_2|$. In particular, since $\pi_1(M_1) \cong \pi_1(M_2)$, we must have $\beta_1 = \pm\beta_2$. Perhaps after reversing the orientation of the fibres of M_1, we may assume that $\beta_1 = \beta_2$, hence M_1 and M_2 are Seifert isomorphic. A similar argument applies in the second case. □

4.4 Geometrisation of Seifert manifolds

In this section, we want to describe how to 'geometrise' Seifert manifolds, following Scott's paper [65]. We first introduce the notion of model geometry due to Thurston [73].

Definition 4.4.1. A **model geometry** (G, X) consists of a manifold X and a Lie group G, acting on X by diffeomorphisms, such that

(i) X is connected and simply connected;

(ii) G acts transitively on X, and the stabiliser subgroups $G_x := \{g \in G\colon\ gx = x\}$ are compact for every $x \in X$;

(iii) if $G \subset H$ and H satisfies (ii), then $H = G$;

(iv) there exists at least one compact manifold M *modelled* on (G, X), i.e. M can be written as the quotient space M/Γ, where Γ is a discrete subgroup of G acting freely on X.

Note that condition (ii) implies that X admits a homogeneous Riemannian metric invariant under G (see [73, Lemma 3.4.11]). That is, G is a subgroup of the isometry group $\mathrm{Isom}(X)$ of X. It then follows from condition (iii) that G is the whole isometry group of X, since the stabiliser subgroups of $\mathrm{Isom}(X)$ are always compact. Therefore, an equivalent way of defining a model geometry is the following.

Definition 4.4.2 (Alternative definition of model geometry)**.** A model geometry is a homogeneous, connected and simply connected Riemannian manifold (X, g) such that there is at least one compact manifold modelled on X, i.e. M can be written as a quotient space M/Γ, where $\Gamma \leq \mathrm{Isom}(X)$ is a discrete subgroup of isometries acting freely on X.

This alternative definition is the one we will use throughout the remainder of this chapter. Note that if M is modelled on (X, g), then M carries an induced Riemannian metric such that the projection $X \to M$ becomes a local isometry. Furthermore, if X is a model geometry as in Definition 4.4.2 and $\Gamma \leq \mathrm{Isom}(X)$ is a discrete subgroup, then the action of Γ on X is properly discontinuous [73, Cor. 3.5.11]. In particular, if Γ acts freely on X, then the quotient space X/Γ is a manifold [73, Proposition 3.5.7].

For 3-manifolds, the model geometries have been classified by Thurston (see [73] for definitions of these spaces and their isometry groups):

Theorem 4.4.3 (Thurston [73, Theorem 3.8.4])**.** *There are eight 3-dimensional model geometries, given by:*

- *The three model geometries of constant curvature S^3, $\mathbb{H}^3$ and $\mathbb{E}^3$;*
- *the product geometries $S^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$;*
- *the geometry of $\widetilde{\mathrm{SL}}_2(\mathbb{R})$, the universal cover of the special linear group $\mathrm{SL}_2(\mathbb{R})$;*
- *the solvegeometry* Sol*;*
- *and the nilgeometry* Nil*.*

We will only briefly describe the model geometries relevant for our needs; we refer the reader to [73] for a more thorough discussion.

Thurston also showed that if a closed 3-manifold can be modelled on one of the eight geometries above, then the model geometry is unique.

Theorem 4.4.4 (Thurston [73, Theorem 4.7.8])**.** *If M is a closed 3-manifold modelled on one of the eight geometries as in Theorem 4.4.3, then it is not modelled on any of the other geometries.*

Remark 4.4.5. This statement is clearly wrong is M if not assumed to be closed; for example, $\mathbb{R}^3$ admits Euclidean and hyperbolic geometry.

It turns out that among 3-manifolds, the Seifert fibred ones are in one-to-one correspondence to manifolds modelled on six of the eight geometries, according to the following theorem by Scott.

Theorem 4.4.6 (Scott [65])**.** *Every closed Seifert manifold can be modelled on one of six model geometries, according to the following table:*

	$\chi > 0$	$\chi = 0$	$\chi < 0$
$e = 0$	$S^2 \times \mathbb{R}$	$\mathbb{E}^3$	$\mathbb{H}^2 \times \mathbb{R}$
$e \neq 0$	S^3	Nil	$\widetilde{\mathrm{SL}}_2(\mathbb{R})$

Here, e is the Euler number of the Seifert fibration and χ the orbifold Euler characteristic of the base orbifold.

Conversely, every closed 3-manifold that can be modelled on one of the six geometries above admits a Seifert fibration.

Definition 4.4.7. A model geometry equal to one of the six given in Theorem 4.4.6 is called **Seifert model geometry**. The geometries $S^2 \times \mathbb{R}$, $\mathbb{E}^3 \cong \mathbb{E}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ are called **product geometries**, whereas S^3, Nil and $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ are called **twisted geometries**.

We will provide a proof for one direction of the statement of Theorem 4.4.6, namely, that every Seifert manifold can be modelled on the according geometry. This part is only briefly discussed in [65]. The more detailed discussion we give here also allows for the proof of Theorem 4.0.1. The proof of the other direction amounts to showing that every discrete cocompact subgroup of the isometry group of a Seifert model geometry preserves some fibration by lines or circles; this is discussed in Scott's article [65].

Let us now take a look at the six Seifert model geometries in Theorem 4.4.6. The spaces $\mathbb{E}^3$, S^3, $\mathbb{H}^2 \times \mathbb{R}$ and $S^2 \times \mathbb{R}$ are each equipped with their standard metric. The two remaining model spaces, Nil and $\widetilde{\mathrm{SL}}_2(\mathbb{R})$, are perhaps less familiar, so we give a brief description.

The geometry of Nil. This is the geometry of the **Heisenberg group** Nil, which is the nilpotent Lie group consisting of upper 3×3 triangular matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

where $x, y, z \in \mathbb{R}$. In particular, Nil can naturally be identified with $\mathbb{R}^3$. Under this identification, the product induced by matrix multiplication is given by

$$(x', y', z') \cdot (x, y, z) := (x + x', y + y', z + x'y + z'), \tag{4.5}$$

and the Riemannian metric on Nil is defined as

$$\mathrm{d}x^2 + \mathrm{d}y^2 + (\mathrm{d}z - x\,\mathrm{d}y)^2.$$

It is easy to see that left multiplication by an element of Nil defines an isometry of Nil (in particular, Nil is homogeneous). This way, we can view Nil as a subgroup of Isom(Nil). Examples of 3-manifolds modelled on Nil can be obtained by taking the quotient by the discrete subgroup $\Gamma_k \leq \mathrm{Nil} \leq \mathrm{Isom}(\mathrm{Nil})$ generated by the elements $(k, 0, 0), (0, 1, 0), (0, 0, 1)$, where $k \in \mathbb{Z} \setminus \{0\}$. The resulting quotient space Nil/Γ_k is then the total space of a torus bundle over the circle.

The geometry of $\widetilde{\mathrm{SL}}_2(\mathbb{R})$. Let $\mathrm{SL}_2(\mathbb{R})$ denote the special linear group of degree 2, i.e. the group of all (2×2)-matrices with determinant equal to 1. Let $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ denote its universal cover. Since $\mathrm{SL}_2(\mathbb{R})$ is a Lie group, so is $\widetilde{\mathrm{SL}}_2(\mathbb{R})$. Let us see how to define a left-invariant metric on $\widetilde{\mathrm{SL}}_2(\mathbb{R})$. Consider the projective special linear group $\mathrm{PSL}_2(\mathbb{R})$, defined as $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{I, -I\}$, where I is the identity matrix. Since $\mathrm{PSL}_2(\mathbb{R})$ is doubly covered by $\mathrm{SL}_2(\mathbb{R})$, the universal cover of $\mathrm{PSL}_2(\mathbb{R})$ is given by $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ as well. Now, it is well known that $\mathrm{PSL}_2(\mathbb{R})$ is isomorphic to the group of orientation preserving isometries of the 2-dimensional hyperbolic plane $\mathbb{H}^2$, acting by Möbius transformations. Furthermore, $\mathbb{H}^2$ is homogeneous and isotropic with respect to orientation preserving isometries, which means that $\mathrm{PSL}_2(\mathbb{R})$ acts transitively and freely on $ST\mathbb{H}^2$, the unit tangent bundle of $\mathbb{H}^2$. Thus, we may identify $\mathrm{PSL}_2(\mathbb{R})$ with $ST\mathbb{H}^2$. We equip $ST\mathbb{H}^2$ with the Sasaki metric (see Definition B.8), which clearly defines a left-invariant metric on $ST\mathbb{H}^2$. By pulling this metric back via the covering $\widetilde{\mathrm{SL}}_2(\mathbb{R}) \to \mathrm{PSL}_2(\mathbb{R}) = ST\mathbb{H}^2$, we obtain a left-invariant (and, in particular, homogeneous) metric on $\widetilde{\mathrm{SL}}_2(\mathbb{R})$.

Examples of 3-manifolds modelled on $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ are given by unit tangent bundles of hyperbolic surfaces, equipped with the Sasaki metric.

Besides Proposition 4.3.2 above, the following proposition will be crucial for the proof of Theorems 4.4.6 and 4.0.1.

Proposition 4.4.8. *Let $\tilde{M}$ be one of the six model geometries in Theorem 4.4.6. Then there exists a submersion $\pi\colon \tilde{M} \to \Sigma$, where Σ is one of S^2, $\mathbb{E}^2$ or $\mathbb{H}^2$, such that the fibres of π are geodesics. The horizontal distribution $(\ker d\pi)^\perp \subset T\tilde{M}$ is integrable if $\tilde{M}$ is a product geometry, and defines a tight contact structure if $\tilde{M}$ is a twisted geometry. Given an isometry φ of Σ, there is an isometry Φ of $\tilde{M}$ covering φ, i.e. $\pi \circ \Phi = \varphi \circ \pi$. Furthermore, the subgroup*

$$\mathrm{Isom}_\pi(\tilde{M}) := \{\Psi \in \mathrm{Isom}(\tilde{M})\colon\ \pi \circ \Psi = \pi\} \leq \mathrm{Isom}(\tilde{M})$$

of isometries preserving every fibre is isomorphic to $\mathbb{R}$ or S^1, and generated by the flow of a Killing vector field.

Remark 4.4.9. One can show that — with the exception of the Hopf fibration — the submersion $\tilde{M} \to \Sigma$ is in fact a Riemannian submersion (for the Hopf fibration, this is only true up to a conformal change of the metric on S^2 by a constant factor). We will, however, not appeal to this fact.

Proof of Proposition 4.4.8. This is clear if $\tilde{M}$ is a product geometry (that is, $\tilde{M} = \Sigma \times \mathbb{R}$); in this case, π is just given as the projection onto the first factor, and $G \cong \mathbb{R} \leq \mathrm{Isom}(\tilde{M})$ is the group of isometries consisting of translations along the $\mathbb{R}$-fibre. The twisted geometries are treated separately.

Case $\tilde{M} = S^3$. Consider the Hopf fibration $\pi\colon\ S^3 \to S^2$ as defined in (C.3), that is,

$$\pi(x_1, \ldots, x_4) = \big(2(x_1x_3 + x_2x_4), 2(x_3x_4 - x_1x_2), 1 - 2(x_2^2 + x_3^2)\big)\,, \tag{4.6}$$

which defines a submersion with geodesic fibres, and whose horizontal distribution defines the standard (tight) contact structure on S^3 (see Example 2.1.5 (ii) and Remark 2.1.6).

Now let φ be an isometry of S^2. Since $\mathrm{Isom}(S^2) = O(3)$, it is enough to consider the cases of φ defining a rotation, or reflection in the yz-plane (every other element of $O(3)$ can be written as a composition of these). In the latter case, φ is given by $\varphi(x, y, z) = (-x, y, z)$, and the isometry $\Phi(x_1, x_2, x_3, x_4) := (-x_1, -x_2, x_3, x_4)$ of S^3 covers φ (i.e. $\pi \circ \Phi = \varphi \circ \pi$). On the other hand, if φ defines a rotation, then consider the quaternionic description of the Hopf fibration as in (C.4). That is, we identify S^3 with the set of unit quaternions and S^2 with the set of purely imaginary unit quaternions, so that $\pi(u) = f_u(\mathrm{k})$, where f_u is the rotation defined by $u \in S^3$ (see Appendix C for definitions). Now let $u_0 \in S^3$ correspond to the rotation defined by φ (i.e. $f_{u_0} = \varphi$), and let $\Phi\colon\ S^3 \to S^3$, $\Phi(u) = u_0 u$. Then Φ is an isometry of S^3, and

$$\pi \circ \Phi(u) = \pi(u_0 u) = f_{u_0 u}(\mathrm{k}) = u_0 u\, \mathrm{k}\, \overline{u_0 u} = u_0 (u \mathrm{k} \overline{u}) \overline{u_0} = f_{u_0}(\pi(u)) = \varphi \circ \pi(u),$$

hence Φ covers φ. Now consider the subgroup $G \cong S^1 \leq \mathrm{Isom}(S^3)$ consisting of isometries of the form $(z_1, z_2) \mapsto (\mathrm{e}^{2\pi \mathrm{i} t} z_1, \mathrm{e}^{2\pi \mathrm{i} t} z_2)$, $t \in \mathbb{R}$ (here, S^3 is viewed again as the unit sphere in $\mathbb{C}^2$). Then G defines the Hopf action, so in particular elements of G cover the identity on S^2. Since G is generated by the flow of a Killing vector field, we have that $G = \mathrm{Isom}_\pi(S^3)$ by Lemma 4.4.10 below.

<u>**Case** $\tilde{M} = \mathrm{Nil}$.</u> Consider the natural projection

$$\pi\colon\ \mathrm{Nil} \longrightarrow \mathbb{E}^2, \quad (x, y, z) \longmapsto (x, y),$$

which defines a submersion with geodesic fibres. The horizontal distribution is given by $\partial_z^\perp = \ker(\mathrm{d}z + x\mathrm{d}y)$ which defines the standard (tight) contact structure on $\mathbb{R}^3$.

Now given an element $\varphi \in \mathrm{Isom}(\mathbb{E}^2)$, then φ is either a translation, a rotation about the origin, the reflection in the x-axis, or a composition of such isometries. Hence it suffices to show that these isometries lift. If φ is given by translation by some vector (x', y'), then left multiplication by $(x', y', 0)$ defines the required isometry of Nil. If φ is the rotation about the origin by some angle $\theta \in [0, 2\pi)$, then a lift is given by

$$\Phi(x, y, z) := \left(\varphi(x, y), z + \frac{1}{2}(x^2 - y^2) \sin\theta \cos\theta - xy \sin^2\theta\right),$$

see [56, Section 12.5.3]. Finally, if φ is the reflection in the x-axis, that is, $\varphi(x, y) = (x, -y)$, then $\Phi(x, y, z) := (x, -y, -z)$ is the corresponding lift to Nil.

Now consider the subgroup $G \leq \mathrm{Isom}_\pi(\mathrm{Nil})$ given by translations in z-direction, i.e. isometries of the form

$$(x, y, z) \mapsto (x, y, z + t), \quad t \in \mathbb{R}.$$

Then $G \leq \mathrm{Isom}_\pi(\tilde{M})$ and G is generated by the flow of a Killing vector field, hence $G = \mathrm{Isom}_\pi(\mathrm{Nil})$ by Lemma 4.4.10.

Case $\tilde{M} = \widetilde{\mathrm{SL}}_2(\mathbb{R})$. The projection $\pi\colon \widetilde{\mathrm{SL}}_2(\mathbb{R}) \to \mathbb{H}^2$ is given by the composition

$$\widetilde{\mathrm{SL}}_2(\mathbb{R}) \longrightarrow ST\mathbb{H}^2 \longrightarrow \mathbb{H}^2,$$

where $ST\mathbb{H}^2 \to \mathbb{H}^2$ is the natural projection, and $\widetilde{\mathrm{SL}}_2(\mathbb{R}) \to ST\mathbb{H}^2$ is a locally isometric covering. This clearly defines a submersion whose fibres are geodesics, since the fibres of $ST\mathbb{H}^2 \to \mathbb{H}^2$ are geodesics, spanned by the vertical geodesic vector field $\mathcal{V}$ (Definition B.14). The horizontal subbundle HM of $ST\mathbb{H}^2$ is given by the orthogonal complement of $\mathcal{V}$, hence it defines a contact structure (see Example 2.2.10), and the same is true for the induced horizontal subbundle $\mathcal{H} \subset T\widetilde{\mathrm{SL}}_2(\mathbb{R})$. To see that the contact structure $\mathcal{H}$ is tight, one can choose suitable global coordinates on $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ (recall that $\widetilde{\mathrm{SL}}_2(\mathbb{R}) \cong \mathbb{R}^3$ as topological spaces) and give an explicit description of the contact structure to see that it coincides with the standard one on $\mathbb{R}^3$ (see [56, Section 12.6]). Alternatively, one could argue using Theorem 3.7.4.

Now given an isometry φ of $\mathbb{H}^2$, its differential defines an isometry of $ST\mathbb{H}^2$ which lifts to an isometry of $\widetilde{\mathrm{SL}}_2(\mathbb{R})$, covering φ and preserving fibres (Corollary B.12). Hence every isometry lifts. The subgroup $G \cong \mathbb{R}$ is given by the flow of $\mathcal{V}$, which is isometric since $\mathcal{V}$ is a Killing vector field (Proposition B.15). In particular, using Lemma 4.4.10 again, $G = \mathrm{Isom}_\pi(\widetilde{\mathrm{SL}}_2(\mathbb{R}))$. This finishes the proof. □

Lemma 4.4.10. *Let M be a Riemannian manifold and $\pi\colon M \to B$ a fibre bundle with 1-dimensional fibres. Consider the group Isom_π as in Proposition 4.4.8. Assume that $\mathrm{Isom}_\pi(M)$ contains a 1-parameter subgroup G generated by the flow of some Killing vector field X. Then $\mathrm{Isom}_\pi(M) = G$.*

Proof. Denote by ϕ_t the time-t flow of X. Let $\varphi \in \mathrm{Isom}_\pi(M)$ and consider the 1-parameter subgroup $G_\varphi = \{\phi_t \circ \varphi\colon t \in \mathbb{R}\} \leq \mathrm{Isom}_\pi(M)$. Then G_φ is generated by the flow of a Killing vector field Y. Since the fibres of π are 1-dimensional, we have that $Y = \lambda X$ for some smooth function $\lambda\colon M \to \mathbb{R}$. Let us show that λ has to be constant. Indeed, recall that Y being a Killing vector field means that $L_Y g = 0$. That is, given two arbitrary vector fields Z_1 and Z_2,

$$\begin{aligned} 0 = (L_Y g)(Z_1, Z_2) &= Y(g(Z_1, Z_2)) - g(L_Y Z_1, Z_2) - g(Z_1, L_Y Z_2) \\ &= g(\nabla_Y Z_1, Z_2) + g(Z_1, \nabla_Y Z_2) - g([Y, Z_1], Z_2) - g(Z_1, [Y, Z_2]) \\ &= -\big(g(\nabla_{Z_1} Y, Z_2) + g(Z_1, \nabla_{Z_2} Y)\big), \end{aligned}$$

where we used the fact that the Levi-Civita connection is metric and torsion-free.

Now since X and $Y = \lambda X$ are both assumed to be Killing, this implies that

$$\begin{aligned} 0 &= g(\nabla_{Z_1}(\lambda X), Z_2) + g(Z_1, \nabla_{Z_2}(\lambda X)) \\ &= \lambda \underbrace{((g(\nabla_{Z_1} X, Z_2) + g(Z_1, \nabla_{Z_2} X))}_{=0} + Z_1(\lambda) g(X, Z_2) + Z_2(\lambda) g(Z_1, X). \end{aligned}$$

Now taking $Z_1 = Z_2 = X$ yields $X(\lambda) = 0$. Then taking Z_1 arbitrary and $Z_2 = X$ yields $Z_1(\lambda) = 0$. It follows that λ is constant. In particular, $\phi_t \circ \varphi = \phi_{\lambda t}$, or equivalently, $\varphi = \phi_{(\lambda-1)t} \in G$. Since $\varphi \in \mathrm{Isom}_\pi(M)$ was chosen arbitrarily, it follows that $\mathrm{Isom}_\pi(M) = G$. □

4.5 Proof of Scott's Theorem

We prove only one direction of Theorem 4.4.6, namely, that every Seifert fibred manifold can be modelled on one of the six model geometries. We follow the original proof by Scott [65], but we provide some important details left out in the original proof. For the other direction, see [65] or [56].

Let $\pi\colon M \to \mathcal{O}$ be a Seifert fibration with Euler number $e = e(\pi)$, where

$$M = M(g; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)),$$

and $\mathcal{O}$ is the underlying base orbifold. First note that if $\mathcal{O}$ is bad, then $\mathcal{O} = S^2(p, q)$, $p \neq q$, by Proposition 4.2.10. Then $\chi(\mathcal{O}) > 0$, and it is not hard to see that M admits a Heegaard splitting of genus 1 (arguing as in the proof of Proposition 4.3.1), hence M is a lens space. In particular, M is modelled on S^3. As a Seifert manifold, M is of the form $M = M(0; (p, \beta_1))$ or $M = M(0; (p, \beta_1), (q, \beta_2))$. In both cases we have that $e \neq 0$. Hence we have proven the statement in case $\mathcal{O}$ is bad.

Now assume that $\mathcal{O}$ is good. Furthermore, let us assume that $|\mathcal{O}|$ is an orientable surface of genus g (the non-orientable case works analogously). Write $\mathcal{O} = \tilde{\mathcal{O}}/\pi_1^{\mathrm{orb}}(\mathcal{O})$ as in Theorem 4.2.12, where $\tilde{\mathcal{O}}$ is one of $S^2, \mathbb{E}^2$ or $\mathbb{H}^2$ and $\pi_1^{\mathrm{orb}}(\mathcal{O})$ acts by isometries and properly discontinuously on $\tilde{\mathcal{O}}$. Recall that by Proposition 4.2.8, $\pi_1^{\mathrm{orb}}(\mathcal{O})$ has the finite presentation

$$\pi_1^{\mathrm{orb}}(\mathcal{O}) \cong \langle a_1, b_1, \ldots, a_g, b_g, q_1, \ldots, q_n \,|\, q_1 \cdots q_n \,[a_1, b_1] \cdots [a_g, b_g],\, q_j^{\alpha_j} \rangle,$$

whose generators are identified with the corresponding isometries. Let $\tilde{M}$ be the corresponding model geometry according to the table in Theorem 4.4.6. At this point, we have to distinguish two cases. The first case is that the Euler number of the Seifert fibration vanishes, i.e. $e = 0$. Then $\tilde{M} = \tilde{\mathcal{O}} \times \mathbb{R}$. Let h denote the isometry of $\tilde{M}$ given by translation by 1 in $\mathbb{R}$-direction, and let $\tilde{a}_i, \tilde{b}_i$, $i = 1, \ldots, g$, denote the product of a_i, b_i with the identity in the second factor of $\tilde{M}$. Choose lifts $\tilde{q}_1, \ldots, \tilde{q}_n$

of $q_1, \ldots, q_n$ satisfying $\tilde{q}_j{}^{\alpha_j} h^{\beta_j} = 1$, $j = 1, \ldots, n$. That is, the $\tilde{q}_j$ are given as the product of q_j in the first factor, and $h^{-\beta_j/\alpha_j}$ in the second factor. Then

$$\tilde{q}_1 \cdots \tilde{q}_n \, [\tilde{a}_1, \tilde{b}_1] \cdots [\tilde{a}_g, \tilde{b}_g] = h^{-\sum_j \beta_j/\alpha_j} = h^e = \mathrm{id}\,.$$

Hence the group

$$\Gamma := \langle \tilde{a}_i, \tilde{b}_i, \tilde{q}_j, h \,|\, h \text{ central}, \, \tilde{q}_1 \cdots \tilde{q}_n \, [\tilde{a}_1, \tilde{b}_1] \cdots [\tilde{a}_g, \tilde{b}_g], \, \tilde{q}_j{}^{\alpha_j} h^{\beta_j} \rangle \cong \pi_1(M)$$

acts properly discontinuously and isometrically on $\tilde{M}$. The action is also free, which can be seen as follows. Assume that there is an element $\tilde{g} \in \Gamma$ and a point $\tilde{x} \in \tilde{M}$ such that $\tilde{g}(\tilde{x}) = \tilde{x}$. Consider the image g of $\tilde{g}$ in $\pi_1^{\mathrm{orb}}(\mathcal{O})$, and let $x = \pi(\tilde{x}) \in \tilde{\mathcal{O}}$. Then x is a fixed point of g. The claim is now that g must be conjugate to some power of some q_j, i.e. there is $\rho \in \pi_1^{\mathrm{orb}}(\mathcal{O})$ and $k \in \mathbb{Z}$ such that $g = \rho q_j^k \rho^{-1}$. Indeed, since x is a fixed point of g, its projection to $\mathcal{O}$ is an orbifold point, say x_j. That is, there is $x' \in \tilde{\mathcal{O}}$ covering x_j such that $q_j(x') = x'$. Now x and x' lie in the same fibre of x_j, hence there is an element $\rho \in \pi_1^{\mathrm{orb}}(\mathcal{O})$ such that $\rho(x') = x$ (recall that $\pi_1^{\mathrm{orb}}(\mathcal{O})$ is, by definition, the deck transformation group of the covering $\tilde{\mathcal{O}} \to \mathcal{O}$). Then $\rho^{-1} g \rho(x') = x'$, which means that $\rho^{-1} g \rho$ must be equal to some power of q_j, as these are the only elements of Γ fixing x'. Hence $g = \rho q_j^k \rho^{-1}$ for some $k \in \mathbb{Z}$, as claimed. Now this implies that $\tilde{g}$ must be equal to $\tilde{g} = h^m \tilde{\rho}^{-1} \tilde{q}_j^k \tilde{\rho}$ for some lift $\tilde{\rho}$ of ρ and $m \in \mathbb{Z}$. Along the $\mathbb{R}$-fibres, $\tilde{g}$ acts by translation by $m - k\beta_j/\alpha_j$. Now since α_j and β_j are assumed to be coprime, it follows that $m - k\beta_j/\alpha_j \neq 0$, which means that $\tilde{g}$ acts nontrivially in fibre direction, so it cannot have fixed points. This yields a contradiction. Hence, Γ defines a free and properly discontinuous action on $\tilde{M}$ by isometries, and the quotient space $\tilde{M}/\Gamma$ is a manifold modelled on $\tilde{M}$. Now consider the induced projection $\pi' \colon \tilde{M}/\Gamma \to \mathcal{O}$, so that the following diagram commutes:

$$\begin{array}{ccc}
\tilde{M} & \longrightarrow & \tilde{M}/\Gamma \\
{\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi'} \\
\tilde{\mathcal{O}} & \longrightarrow & \mathcal{O}
\end{array}$$

Then π' defines a Seifert fibration of $\tilde{M}/\Gamma$ with base orbifold $\mathcal{O}$ and fundamental group $\pi_1(\tilde{M}/\Gamma) = \Gamma \cong \pi_1(M)$. Then, by Proposition 4.3.2 (perhaps after reversing the orientation of the fibres of $\tilde{M}/\Gamma$), the Seifert manifolds $\tilde{M}/\Gamma$ and M are isomorphic (and, in particular, homeomorphic). Hence M can be modelled on $\tilde{M}$.

For the second case, assume that $e \neq 0$. In this case $\tilde{M}$ is a twisted model geometry, and we let $h = \phi_1$ be the time-1 flow of the Killing vector field whose flow generates $\mathrm{Isom}_\pi(\tilde{M})$ (see Proposition 4.4.8). Choose arbitrary lifts of the a_i, b_i

to isometries $\tilde{a}_i, \tilde{b}_i$ of $\tilde{M}$ (which exist due to Proposition 4.4.8). Furthermore, let $\tilde{q}_j$ be lifts of the q_j such that $\tilde{q}_j^{\alpha_j} h^{\beta_j} = \mathrm{id}$. Then

$$\tilde{q}_1 \cdots \tilde{q}_n \, [\tilde{a}_1, \tilde{b}_1] \cdots [\tilde{a}_g, \tilde{b}_g] = \phi_\lambda \tag{4.7}$$

for some $\lambda \in \mathbb{R}$. If we replace h by $\tilde{h} = \phi_u$ for some $u \in \mathbb{R}$, then we have to replace $\tilde{q}_j$ by $\tilde{q}_j \phi_{-(u-1)\beta_j/\alpha_j}$ (which we again call $\tilde{q}_j$), so that $\tilde{q}_j^{\alpha_j} \tilde{h}^{\beta_j} = 1$ is still satisfied. Then

$$\tilde{q}_1 \cdots \tilde{q}_n \, [\tilde{a}_1, \tilde{b}_1] \cdots [\tilde{a}_g, \tilde{b}_g] = \phi_{\tilde{\lambda}},$$

where

$$\tilde{\lambda} = \lambda - (u-1) \sum_j \beta_j/\alpha_j = \lambda + (u-1)e. \tag{4.8}$$

Now we can solve the equation $\tilde{\lambda} = 0$ for u by setting $u = (e - \lambda)/e$ (here we use that $e \neq 0$). If $u \neq 0$, then we can replace h by $\tilde{h} = \phi_u$, and the group

$$\Gamma := \langle \tilde{a}_i, \tilde{b}_i, \tilde{q}_j \,|\, \tilde{h} \text{ central}, \, \tilde{q}_1 \cdots \tilde{q}_n \, [\tilde{a}_1, \tilde{b}_1] \cdots [\tilde{a}_g, \tilde{b}_g], \, \tilde{q}_j^{\alpha_j} \tilde{h}^{\beta_j} \rangle \cong \pi_1(M)$$

again acts isometrically and freely (by the same argument as before) on $\tilde{M}$. Then, arguing as in the first case, we see that $\tilde{M}/\Gamma$ is Seifert fibred and—perhaps after changing the orientation of fibres—Seifert isomorphic to M. Hence the only thing left to show is that $u \neq 0$. Suppose $u = 0$, so $\lambda = e$. Consider the Seifert manifold

$$N := M(g; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n), (1, e))$$

and let $\tilde{\pi} \colon N \to \Sigma_g$ be the corresponding Seifert fibration. Note that N has the same base orbifold as M. Furthermore, setting $\tilde{q}_{n+1} := \phi_{-e}$, the group G of isometries spanned by $\tilde{a}_1, \tilde{b}_1, \ldots, \tilde{a}_g, \tilde{b}_g, \tilde{q}_1, \ldots, \tilde{q}_n, \tilde{q}_{n+1}$ acts freely on $\tilde{M}$ and is isomorphic to the fundamental group of N. Indeed, the long relation holds since

$$\tilde{q}_1 \cdots \tilde{q}_n \, [\tilde{a}_1, \tilde{b}_1] \cdots [\tilde{a}_g, \tilde{b}_g] = \phi_e = \tilde{q}_{n+1}^{-1}$$

by equation (4.7). It follows that the quotient $\tilde{M}/G$ is a Seifert fibred manifold with fundamental group $G \cong \pi_1(N)$ and base orbifold $\mathcal{O}$, hence $\tilde{M}/G$ is Seifert isomorphic to N by Proposition 4.3.2. In particular, N is homeomorphic to $\tilde{M}/G$, hence it can be modelled on the twisted geometry $\tilde{M}$. On the other hand, $e(\tilde{\pi}) = 0$, hence N can be modelled on a product geometry by the previous discussion. That, however, contradicts the fact that closed 3-manifolds admit at most one model geometry (Theorem 4.4.4). In conclusion, we must have $u \neq 0$, which finishes the proof of Theorem 4.4.6. □

Proof of Theorem 4.0.1. If M is a Seifert manifold not equal to a lens space, then M can be modelled on one of the six Seifert model geometries by Theorem 4.4.6. In fact, we have seen in the proof of Theorem 4.4.6 that if $\tilde{M}$ denotes the corresponding model geometry and $\pi\colon\ \tilde{M} \to \tilde{\mathcal{O}}$ the submersion as in Proposition 4.4.8, then M is Seifert isomorphic to $\tilde{M}/\Gamma$, where $\Gamma \cong \pi_1(M)$ acts isometrically on $\tilde{M}$, preserving the fibres of π. By the same proof, the Seifert fibres of $\tilde{M}/\Gamma$ lift to the fibres of π in $\tilde{M}$ which are geodesics by Proposition 4.4.8. Now denote by $\xi \subset T\tilde{M}$ the plane distribution orthogonal to this geodesic fibration. Then, again using Proposition 4.4.8, ξ is integrable if $e(\pi) = 0$, and it defines a tight contact structure if $e(\pi) \neq 0$. That is, if X denotes a unit vector field spanning the Seifert fibres of $\tilde{M}/\Gamma$, then X induces a universally tight contact structure, and (by Remark 2.2.2) X is the Reeb vector field of the (contact) 1-form dual to X. □

Appendix A

Poincaré first return map

Theorem A.1. Let M be a closed manifold and $\phi\colon\ M \times \mathbb{R} \to M$ a global flow on it. Let $\Sigma \subset M$ be a closed global section for ϕ, that is, a compact hypersurface without boundary that is transverse to the flow and meets every orbit at least once. Then, for every $x \in M$, there is a smallest positive time $\tau^+(x) > 0$ and a largest negative time $\tau^-(x) < 0$ such that $\phi(x, \tau^\pm(x)) \in \Sigma$. Furthermore, the functions $\tau^\pm\colon\ \Sigma \to \mathbb{R}$ are smooth if ϕ is smooth. In particular, the **Poincaré first return map**

$$P\colon\ \Sigma \longrightarrow \Sigma, \quad x \longmapsto \phi(x, \tau^+(x))$$

is smooth.

Proof. We prove the statement for τ^+ (the argument for τ^- is completely analogous). If x is a point in M, consider its *ω-limit set*

$$\omega(x, \phi) := \{y \in M\colon\ \exists\, t_n \longrightarrow \infty\colon\ \phi(x, t_n) \longrightarrow y \text{ as } n \to \infty\}.$$

First notice that, as M is compact, $\omega(x, \phi)$ is always nonempty. Furthermore, it is invariant under ϕ: If $y \in \omega(x, \phi)$, there is a sequence $t_n \to \infty$ such that $\phi(x, t_n) \to y$ for $n \to \infty$. Then, $\phi(x, t_n + t) = \phi(\phi(x, t_n), t) \to \phi(y, t)$ as $n \to \infty$, which means that every point $\phi(y, t)$ (that is, the whole orbit of y) is also contained in $\omega(x, \phi)$. Those two observations imply the existence of an orbit of ϕ that is contained in the ω-limit set of x. By assumption, this orbit must intersect Σ transversely at some point $z \in \Sigma$. Choose a parametrisation

$$\mathbb{R} \longrightarrow M, \quad t \longmapsto \gamma(t)$$

of that orbit, such that $\gamma(0) = z$. Now choose a small open ball $B \subset \Sigma$ around z and identify a small neighbourhood of z in M with $U := B \times [-\varepsilon, \varepsilon]$ using the flow of X, where $\gamma \cap U$ is being identified with $\{0\} \times [-\varepsilon, \varepsilon]$. Choose an increasing sequence $(t_n) \subset \mathbb{R}$ such that $\phi(x, t_n) \to z$ as $n \to \infty$. Then $\phi(x, t_n + \varepsilon) \to \phi(z, \varepsilon) = (0, \varepsilon) \in U$ and $\phi(x, t_n - \varepsilon) \to \phi(z, -\varepsilon) = (0, -\varepsilon)$. In particular, for n large, $\phi(x, t_n + \varepsilon) \in B \times (0, \varepsilon) \subset U$ and $\phi(x, t_n - \varepsilon) \in B \times (-\varepsilon, 0) \subset U$. Hence, by the intermediate value theorem, there is $t_n - \varepsilon < t < t_n + \varepsilon$ such that $\phi(x, t) \in B \times \{0\} \subset \Sigma$. It follows that the set

$$R_x = \{t \in (0, \infty)\colon\ \phi_t(x) \in \Sigma\}$$

is nonempty. Since every orbit of ϕ intersects Σ transversely, the set $R_x \cup \{0\}$ is discrete, hence the (well-defined) infimum of R_x is strictly positive. Set $\tau^+(x) := \inf R_x$. Then, by continuity, $\phi(x, \tau^+(x)) \in \Sigma$ and $\phi(x,t) \notin \Sigma$ for $0 < t < \tau^+(x)$.

Note that by continuity of the flow, it is not hard to see that the function $x \mapsto \tau^+(x)$ is continuous. To see that it is in fact smooth, we will apply the implicit function theorem. First, let $s\colon M \to \mathbb{R}$ be a smooth function such that $\Sigma = s^{-1}(0)$ and 0 is a regular value of s. Now, consider the function

$$F\colon M \times \mathbb{R} \longrightarrow \mathbb{R}, \quad (x,t) \longmapsto s(\phi(x,t)).$$

We see that, for $(x,t) \in F^{-1}(0)$,

$$\frac{\mathrm{d}}{\mathrm{dt}} F(x,t) = \mathrm{d}s(X_{\phi(x,t)}) \neq 0,$$

as X is transverse to $\Sigma = s^{-1}(0)$ and 0 is a regular value of s. Since F is smooth, using the implicit function theorem, we can write t locally as a smooth function of x (for points $(x,t) \in F^{-1}(0)$). This function, however, must equal τ^+ around any point $(x, \tau(x))$, since τ^+ is continuous. Hence, τ is smooth, and the proof is finished. □

Now given the (smooth) Poincaré return map $P\colon \Sigma \to \Sigma$, consider the mapping torus $\Sigma(P) := (\Sigma \times [0,1]) / (x,1) \sim (P(x), 0)$.

Corollary A.2. In the setting of Theorem A.1, there is a diffeomorphism

$$M \xrightarrow{\cong} \Sigma(P)$$

mapping orbits of ϕ to orbits of the suspension flow on $\Sigma(P)$.

Proof. By Theorem A.1, the map

$$\Sigma \times [0,1] \longrightarrow M, \quad (x,t) \longmapsto \phi(x, t\tau^+(x))$$

is smooth and maps vertical line segments to flow lines of ϕ. Then this map induces a diffeomorphism $\Sigma(P) \xrightarrow{\cong} M$ whose inverse is given by

$$M \longrightarrow (\Sigma \times [0,1]) / \sim, \quad x \longmapsto \left[\phi(x, \tau^-(x)), \frac{-\tau^-(x)}{\tau^+(\phi(x, \tau^-(x)))}\right].$$

□

Appendix B
Geometry of tangent bundles

Let M be a smooth n-dimensional manifold and TM its tangent bundle, endowed with the natural differentiable structure. We will describe how to obtain a Riemannian metric on TM from a metric on M, which will be called *Sasaki metric* on TM, introduced by and named after S. Sasaki [64]. The idea will be to write the double tangent bundle TTM as the direct sum of two n-dimensional subbundles HM and VM, called *horizontal* and *vertical* subbundle. The horizontal and vertical subspaces both turn out to be naturally isomorphic to the corresponding tangent space of M, and these isomorphisms define metrics on each of these subspaces (by pulling back the metric on TM). This defines Riemannian metrics on the subbundles HM and VM, which are then declared to be orthogonal. This completely determines a Riemannian metric on TM.

The vertical subbundle VM of TTM can be defined without any reference to the Riemannian metric on M.

Definition B.1. Let M be a smooth n-dimensional manifold. The **vertical subbundle** is the n-dimensional subbundle VM of the ($2n$-dimensional) double tangent bundle TTM, given by the kernel of the linear map $\mathrm{d}\pi\colon TTM \to TM$, where $\pi\colon TM \to M$ is the natural projection. In other words,

$$VM := \ker \mathrm{d}\pi = \bigcup_{w \in TM} \{\mathfrak{u} \in T_w TM\colon \ \mathrm{d}_w\pi(\mathfrak{u}) = 0\}.$$

In contrast, if there is no Riemannian metric specified on M, there is no well-defined notion of a 'horizontal' subbundle of TTM. However, if we are given a Riemannian metric g on M, a horizontal subbundle may be constructed as follows.

Definition B.2. Let $w \in TM$ and $p = \pi(w)$. For a vector $u \in T_pM$, define its **horizontal lift** to T_wTM as follows. Choose a curve $t \mapsto \gamma(t)$ in M such that $\gamma(0) = p$ and $\dot\gamma(0) = u$. Let $t \mapsto W(t)$ denote the (unique) parallel transport of w along γ, viewed as a curve in TM. Then

$$u_w^h := \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} W(t) \in T_wTM.$$

Note that this does not depend on the choice of γ. The **horizontal subspace** H_wM at $w \in TM$ is then defined as the n-dimensional subspace of T_wTM consisting of all horizontal lifts. The **horizontal subbundle** HM is the union of all horizontal subspaces, equipped with the obvious bundle structure.

Remark B.3. If it is clear from the context to which fibre u is being lifted, we simply write u^h instead of u^h_w.

Lemma B.4. Let (M, g) be a Riemannian manifold and $HM \subset TTM$ the horizontal subbundle of TTM. Then, for every $w \in TM$ and $p = \pi(w)$, the map

$$\mathrm{d}_w\pi|_{H_wM} : \; H_wM \longrightarrow T_pM$$

is a linear isomorphism.

Remark B.5. Note that the above lemma implies that $H_wM \cap V_wM = \{0\}$ for every $w \in TM$. Hence, as both H_wM and V_wM are n-dimensional, it follows that $TTM = HM \oplus VM$.

Proof of Lemma B.4. Let $w \in TM$ and $p = \pi(w)$. By definition, every horizontal tangent vector u^h at w is tangent to a curve $t \to W(t)$, where W is a parallel vector field along the curve $\gamma = \pi \circ W$, with $\gamma(0) = p$ and $\dot{\gamma}(0) = u$. Thus,

$$\mathrm{d}_w\pi \left(u^h\right) = \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} \pi(W(t)) = \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} \gamma(t) = \dot{\gamma}(0) = u,$$

hence $\mathrm{d}_w\pi|_{H_wM} : \; H_wM \to T_pM$ is an isomorphism with inverse $u \mapsto u^h_w$. ☐

So far, we have seen that the double tangent bundle TTM does indeed split into a direct sum of its horizontal and vertical subbundle, and we have given linear isomorphisms between the horizontal subspaces and the tangent spaces of M, namely, the restriction of $\mathrm{d}_w\pi$ to H_wM. Now, we want to find similar isomorphisms for the vertical subspaces.

Definition B.6. For a smooth manifold M, define the **connection map**

$$K\colon \; TTM \longrightarrow TM$$

as follows. Let $w \in TM$ and $\mathfrak{u} \in T_wTM$. Let W be a curve adapted to $\mathfrak{u}$, i.e., $W(0) = w$ and $\dot{W}(0) = \mathfrak{u}$. Let $\gamma = \pi \circ W$. Then, define

$$K(\mathfrak{u}) := (D_tW)(0) \in T_{\pi(w)}M,$$

where D_t denotes the covariant derivative along γ.

Lemma B.7. For every $w \in TM$ and $p = \pi(w)$, the restriction

$$K|_{V_wM} : \; V_wM \longrightarrow T_pM$$

is a linear isomorphism.

Proof. Given $u \in T_pM$, consider the curve $t \mapsto W(t) = w + tu$ in TM, and let $\mathfrak{u} = \frac{\mathrm{d}}{\mathrm{dt}}\big|_{t=0} W(t)$. Note that $\mathfrak{u} \in V_wM$. Then

$$K(\mathfrak{u}) = D_t|_{t=0} W(t) = u,$$

hence $K|_{V_wM}$ is surjective. Now clearly $H_wM \subset \ker K$ and $\dim H_wM = n$, hence by the rank-nullity theorem, $H_wM = \ker K_w$. In particular, as $T_wTM = H_wM \oplus V_wM$, the restriction $K|_{V_wM}$ is a linear isomorphism. □

Given $u, w \in T_pM$, we can now define the **vertical lift** of u to T_wTM as

$$u^v = u^v_w := (K|_{V_wM})^{-1}(u).$$

Note that

$$u^v_w = \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} (w + tu) \tag{B.1}$$

Lemma B.7 also yields another characterisation of the horizontal subbundle, namely, $HM = \ker K$. This is often used as the definition of HM (see e.g. [59]).

We are now ready to define the Sasaki metric.

Definition B.8. Let (M, g) be a Riemannian manifold. The **Sasaki metric** of TM is the Riemannian metric g_S on TM given as follows. For $w \in TM$ and $\mathfrak{u}_1, \mathfrak{u}_2 \in T_wTM$, set

$$g_S(\mathfrak{u}_1, \mathfrak{u}_2) := g(\mathrm{d}_w\pi(\mathfrak{u}_1), \mathrm{d}_w\pi(\mathfrak{u}_2)) + g(K(\mathfrak{u}_1), K(\mathfrak{u}_2)).$$

If $t \mapsto W_1(t)$ and $t \mapsto W_2(t)$ are curves tangent to $\mathfrak{u}_1$ and $\mathfrak{u}_2$, respectively, with $\gamma_1 = \pi \circ W_1$ and $\gamma_2 = \pi \circ W_2$, this translates into

$$g_S(\mathfrak{u}_1, \mathfrak{u}_2) = g(\dot{\gamma_1}(0), \dot{\gamma_2}(0)) + g(D^1_t W_1(0), D^2_t W_2(0)),$$

where D^1_t and D^2_t denote the covariant derivative along γ_1 and γ_2, respectively.

Using the Koszul formula, one can compute the Levi-Civita connection associated with g_S as follows.

Proposition B.9. Let X and Y be vector fields on (M, g), and let ∇^S denote the Levi-Civita connection of (TM, g_S). Then, for $w \in T_pM$,

$$\begin{aligned}
\left(\nabla^S_{X^v} Y^v\right)_w &= 0, \\
\left(\nabla^S_{X^h} Y^v\right)_w &= (\nabla_X Y)^v_w + \frac{1}{2}\left(R_p(w, Y_p)X_p\right)^h_w, \\
\left(\nabla^S_{X^v} Y^h\right)_w &= \frac{1}{2}\left(R_p(w, X_p)Y_p\right)^h_w, \\
\left(\nabla^S_{X^h} Y^h\right)_w &= (\nabla_X Y)^h_w - \frac{1}{2}\left(R_p(X_p, Y_p)w\right)^v_w.
\end{aligned}$$

Here, R denotes the Riemann curvature tensor of (M, g).

Proof. See [52, p. 125]. □

From this description we immediately obtain the following.

Corollary B.10. The vertical subspaces $V_wM \subset TTM$ are totally geodesic with respect to the Sasaki metric. □

Next, we want to show that differentials of isometries define isometries of the corresponding tangent bundles. We need the following preparatory lemma.

Lemma B.11. Let $\varphi\colon\ M \to M$ be an isometry. Let $u, w \in TM$, $p = \pi(u)$. Then

$$\mathrm{d}_w\mathrm{d}\varphi(u^h_w) = (\mathrm{d}_p\varphi(u))^h_{\mathrm{d}\varphi(w)} \quad \text{and} \quad \mathrm{d}_w\mathrm{d}\varphi(u^v_w) = (\mathrm{d}_p\varphi(u))^v_{\mathrm{d}\varphi(w)}.$$

In other words, applying the differential commutes with taking horizontal and vertical lifts.

Proof. By definition, $u^h_w := \frac{\mathrm{d}}{\mathrm{dt}}\big|_{t=0} W(t)$, where W is a parallel vector field along the curve $\gamma = \pi \circ W$. Hence

$$\mathrm{d}_w\mathrm{d}\varphi(u^h_w) = \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} \mathrm{d}\varphi(W(t)) =: \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} \widetilde{W}(t).$$

Now since φ is an isometry, the vector field $\widetilde{W}$ is parallel along the curve $\varphi\circ\gamma$, hence $\frac{\mathrm{d}}{\mathrm{dt}}\big|_{t=0} \widetilde{W}(t) = (\widetilde{W}(0))^h = (\mathrm{d}_p\varphi(u))^h$. This gives us the first identity.

On the other hand, by (B.1), the vertical lift of u can be written as $u^v = \frac{\mathrm{d}}{\mathrm{dt}}\big|_{t=0} W(t)$, where now $W(t) = w + tu$. Hence

$$\mathrm{d}_w\mathrm{d}\varphi(u^v) = \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} \mathrm{d}\varphi(W(t)) = \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} \mathrm{d}_p\varphi(w) + t\,\mathrm{d}_p\varphi(u) = (\mathrm{d}_p\varphi(u))^v_{\mathrm{d}\varphi(x)}. \quad \square$$

Corollary B.12. Let M be a Riemannian manifold and TM be equipped with the Sasaki metric. Then, if $\varphi\colon\ M \to M$ is an isometry of M, the differential $\mathrm{d}\varphi\colon\ TM \to TM$ is an isometry of TM.

Proof. Let $w \in TM$, $p = \pi(w)$, and $\mathfrak{u} \in T_wTM$. It suffices to show that $|\mathrm{d}_w\mathrm{d}\varphi(\mathfrak{u})| = |\mathfrak{u}|$. Using the splitting $T_wTM = H_wM \oplus V_wM$, we can write $\mathfrak{u} = u^h_1 + u^v_2$, where $u_1, u_2 \in T_pM$. Then, by Lemma B.11,

$$\mathrm{d}_w\mathrm{d}\varphi(\mathfrak{u}) = \mathrm{d}_w\mathrm{d}\varphi(u^h_1) + \mathrm{d}_w\mathrm{d}\varphi(u^v_2) = (\mathrm{d}\varphi(u_1))^h + (\mathrm{d}\varphi(u_2))^v,$$

hence

$$\begin{aligned}
|\mathrm{d}_w\mathrm{d}\varphi(\mathfrak{u})|_{TM} &= |(\mathrm{d}\varphi(u_1))^h|_{TM} + |(\mathrm{d}\varphi(u_2))^v|_{TM} \\
&= |\mathrm{d}\varphi(u_1)|_M + |\mathrm{d}\varphi(u_2)|_M \\
&= |u_1|_M + |u_2|_M \\
&= |u^h_1|_{TM} + |u^v_2|_{TM} \\
&= |\mathfrak{u}|_{TM}.
\end{aligned}$$

□

Now consider the unit tangent bundle (or sphere bundle) of M, given as

$$STM := \{u \in TM\colon\ |u| = 1\}.$$

As a codimension-1 submanifold of (TM, g_S), it inherits a Riemannian metric, again denoted by g_S. Then the natural projection $\pi\colon STM \to M$ is a Riemannian submersion whose fibres are totally geodesic $(n-1)$-spheres by Corollary B.10.

Definition/Lemma B.13. The horizontal vector field $\mathcal{G}$ on STM given by $\mathcal{G}_u := u^h$ is geodesic. It is called the **horizontal geodesic vector field** or simply **geodesic vector field** on STM. Its flow is called the **geodesic flow**.

Proof. Clearly $|\mathcal{G}| \equiv 1$. Now if $u \in ST_pM$ and γ is a geodesic in M with $\gamma(0) = p$ and $\dot\gamma(0) = u$, then $\left(\nabla^S_{\mathcal{G}}\mathcal{G}\right)_u = (D_t\dot\gamma)^h = 0$, where D_t denotes the covariant derivative along γ. Hence the flow lines of $\mathcal{G}$ are geodesics, and $\mathcal{G}$ is a geodesic vector field. □

If $\dim M = 2$, then the fibres of STM are geodesic circles by Corollary B.10. If M is oriented, these geodesics carry an induced orientation, hence there is a unit vector field $\mathcal{V}$ on STM tangent to the vertical geodesics.

Definition B.14. The geodesic vector field $\mathcal{V}$ spanning the fibres of STM is called the **vertical geodesic vector field**.

Proposition B.15. The vertical geodesic vector field $\mathcal{V}$ is Killing.

Proof. Recall that the vector field $\mathcal{V}$ is Killing if and only if

$$g_S(\nabla^S_Y\mathcal{V}, Z) + g_S(Y, \nabla^S_Z\mathcal{V}) = 0$$

for all vector fields Y and Z. If, for example, $Y = \mathcal{V}$, we compute

$$g_S\big(\underbrace{\nabla^S_{\mathcal{V}}\mathcal{V}}_{=0}, Z\big) + \underbrace{g_S\left(\mathcal{V}, \nabla^S_Z\mathcal{V}\right)}_{=(1/2)Z(|\mathcal{V}|^2)=0} = 0.$$

On the other hand, if Y and Z are both horizontal vector fields, then by Proposition B.9,

$$g_S\left(\nabla^S_Y\mathcal{V}, Z\right) + g_S\left(Y, \nabla^S_Z\mathcal{V}\right) = \frac{1}{2}\big(R(u, Ju, Y, Z) + R(u, Ju, Z, Y)\big) = 0.$$

It follows that $\mathcal{V}$ is indeed a Killing vector field. □

Appendix C
Quaternions and the Hopf fibration

The space of **quaternions** is the 4-dimensional vector space $\mathbb{H}$ whose elements are of the form $a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}$ for $a, c, b, d \in \mathbb{R}$, where the addition is componentwise, i.e.

$$(a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}) + (a' + b'\mathrm{i} + c'\mathrm{j} + d'\mathrm{k}) := a + a' + (b + b')\mathrm{i} + (c + c')\mathrm{j} + (d + d')\mathrm{k}.$$

Hence a basis of $\mathbb{H}$ is given by the elements 1, i, j and k. One defines a multiplication on $\mathbb{H}$ by the rules

$$\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{ijk} = -1,$$

which turns $\mathbb{H}$ into a non-commutative, associative real division algebra (i.e. every element except 0 has an multiplicative inverse). Generally, the product of two elements $q := a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}$ and $q' := a' + b'\mathrm{i} + c'\mathrm{j} + d'\mathrm{k}$ of $\mathbb{H}$ is given by

$$\begin{aligned} q \cdot q' =& (aa' - bb' - cc' - dd') + (ab' + ba' + cd' - dc')\mathrm{i} \\ &+ (ac' - bd' + ca' + db')\mathrm{j} + (ad' + bc' - cb' + da')\mathrm{k}. \end{aligned} \tag{C.1}$$

As for complex numbers, one defines the **conjugate** of $q = a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}$ as

$$\overline{q} = a - b\mathrm{i} - c\mathrm{j} - d\mathrm{k}.$$

An easy computation shows that $\overline{pq} = \overline{q}\,\overline{p}$ for $p, q \in \mathbb{H}$. Now consider the standard inner product on $\mathbb{H} \cong \mathbb{R}^4$ for which 1, i, j and k form an orthonormal basis. Then the induced norm is given by

$$|q| = \sqrt{a^2 + b^2 + c^2 + d^2} = \sqrt{q\overline{q}}.$$

Since $q\overline{q} = |q|^2 \neq 0$ for $q \neq 0$, the multiplicative inverse of q is given by $q^{-1} = \overline{q}/|q|^2$.

Now consider the set of **unit quaternions**

$$S^3 := \{q \in \mathbb{H} \colon\ |q| = 1\} \subset \mathbb{H}.$$

For $u \in S^3$, the map $L_u \colon\ \mathbb{H} \to \mathbb{H}$, $q \mapsto uq$ preserves norms, since

$$|L_u(q)|^2 = L_u(q)\overline{L_u(q)} = uq\overline{uq} = u\,q\,\overline{q}\,\overline{u} = |q|^2|u|^2 = |q|^2.$$

Hence L_u also preserves the inner product on $\mathbb{H}$, using polarisation. Similarly, right multiplication by an element of S^3 preserves the inner product, so that these maps are isometries of $\mathbb{H}$.

Now consider

$$\mathbb{R}^3 := \{x\mathrm{i} + y\mathrm{j} + z\mathrm{k}\colon\ x, y, z \in \mathbb{R}\} \subset \mathbb{H},$$

the set of purely imaginary quaternions. Note that $q \in \mathbb{R}^3$ if and only if $\overline{q} = -q$. Then, for any $u \in S^3$, consider the map

$$f_u\colon\ \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \quad f_u(q) := uqu^{-1} = uq\overline{u}.$$

The image of f_u is indeed contained in $\mathbb{R}^3$: if $q \in \mathbb{R}^3$, then

$$\overline{f_u(q)} = \overline{uq\overline{u}} = u\,\overline{q}\,\overline{u} = -u\,q\,\overline{u} = -f_u(q),$$

hence $f_u(q) \in \mathbb{R}^3$. Note that f_u defines an isometry of $\mathbb{R}^3$; in fact, writing u (uniquely) as $u = \cos(\theta/2) + \sin(\theta/2)w$ for $\theta \in [0, 2\pi)$ and $w \in \mathbb{R}^3$, $|w| = 1$, then f_u is given as rotation by an angle of θ about the axis spanned by w (cf. [29, Theorem 10.9]). In terms of the basis $\mathrm{i}, \mathrm{j}, \mathrm{k}$ of $\mathbb{R}^3$, the rotation matrix associated with f_u, for $u = a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}$, is then given by

$$\begin{pmatrix} 1 - 2(c^2 + d^2) & 2(bc - ad) & 2(ac + bd) \\ 2(ad + bc) & 1 - 2(b^2 + d^2) & 2(cd - ab) \\ 2(bd - ac) & 2(ab + cd) & 1 - 2(b^2 + c^2) \end{pmatrix}. \tag{C.2}$$

Now let us consider the Hopf fibration and see how to describe it in a quaternionic setting. Originally, the Hopf fibration was given by Hopf [46] (up to permutation of coordinates) as the map $S^3 \to S^2$ defined by

$$(x_1, x_2, x_3, x_4) \longmapsto \big(2(x_1x_3 + x_2x_4), 2(x_3x_4 - x_1x_2), 1 - 2(x_2^2 + x_3^2)\big)\,, \tag{C.3}$$

where S^3 and S^2 are thought of as the unit spheres in $\mathbb{R}^4$ and $\mathbb{R}^3$, respectively. Using the standard identification $\mathbb{C}P^1 \cong S^2$ via stereographic projection, one observes that this map coincides with the one given in Example 1.1.8. Now consider S^3 as the set of unit quaternions in $\mathbb{H}$, and S^2 the set of unit purely imaginary quaternions in $\mathbb{R}^3 \subset \mathbb{H}$. Then, comparing (C.2) and (C.3), we see that the map

$$\pi\colon\ S^3 \longrightarrow S^2, \quad \pi(u) := f_u(\mathrm{k}) \tag{C.4}$$

defines the Hopf fibration. This has now a nice geometric interpretation. Namely, we fix a base point in S^2 (in this case, k), and associate a rotation with an element $u \in S^3$. The image $\pi(u)$ is then the result of this rotation applied to the base point.

Bibliography

[1] A. B. Aazami, The Newman-Penrose Formalism for Riemannian 3-manifolds, *J. Geom. Phys.* **94** (2015), 1–7.

[2] D. Asimov, Round handles and non-singular Morse-Smale flows, *Ann. of Math. (2)* **102** (1975), 41–54.

[3] T. Becker, Geraden- und Großkreisfaserungen und induzierte Kontaktstrukturen, M.Sc. thesis, Universität zu Köln (2019).

[4] T. Becker, Geodesic and conformally Reeb vector fields on flat 3-manifolds, *Differential Geom. Appl.* **89** (2023), Paper No. 102013, 18 pp.

[5] T. Becker and H. Geiges, The contact structure induced by a line fibration of $\mathbb{R}^3$ is standard, *Bull. Lond. Math. Soc.* **53** (2021), 104–107.

[6] M. Berger, Introduction to magnetic helicity, *Plasma Phys. Control. Fusion* **41** (1999), B167.

[7] A. L. Besse, *Einstein manifolds*, Ergeb. Math. Grenzgeb. (3) **10**, Springer-Verlag, Berlin (1987).

[8] L. Bieberbach, Über die Bewegungsgruppen der Euklidischen Räume, *Math. Ann.* **70** (1911), 297–336.

[9] L. Bieberbach, Über die Bewegungsgruppen der Euklidischen Räume (Zweite Abhandlung). Die Gruppen mit einem endlichen Fundamentalbereich, *Math. Ann.* **72** (1912), 400–412.

[10] A. Candel and L. Conlon, *Foliations I*, Grad. Stud. Math. **23**, American Mathematical Society, Providence, RI (2000).

[11] J. Cantarella, D. DeTurck, H. Gluck and M. Teytel, Influence of geometry and topology on helicity, *Geophys. Monogr. Ser.* **111** (1999), 17–24.

[12] R. Cardona, Steady Euler flows and Beltrami fields in high dimensions, *Ergodic Theory Dynam. Systems* **41** (2021), 3610–3633.

[13] R. Cardona, Stability is not open or generic in symplectic four-manifolds, `arXiv:2305.13158`.

[14] R. Cardona, E. Miranda, D. Peralta-Salas and F. Presas, Universality of Euler flows and flexibility of Reeb embeddings, *Adv. Math.* **428** (2023), Paper No. 109142, 40 pp.

[15] I. Chavel, *Riemannian Geometry: A Modern Introduction*, Cambridge Tracts in Mathematics **108**, Cambridge University Press, Cambridge (1993).

[16] J. Cheeger and D. Ebin, *Comparison Theorems in Riemannian Geometry*, North-Holland Math. Library, Vol. 9, North-Holland Publishing Co., Amsterdam-Oxford; American Elsevier Publishing Co., Inc., New York (1975).

[17] K. Cieliebak and E. Volkov, First steps in stable Hamiltonian topology, *J. Eur. Math. Soc.* **17** (2015), 321–404.

[18] S. Durst, Round handle decompositions of 1-connected 5-manifolds, *Expo. Math.* **34** (2016), 43–61.

[19] Ya. Eliashberg, Classification of overtwisted contact structures on 5-manifolds, *Invent. Math.* **98** (1989), 623–637.

[20] Ya. Eliashberg, Contact 3-manifolds twenty years since J. Martinet's work, *Ann. Inst. Fourier (Grenoble)* **42** (1992), 165–192.

[21] D. B. A. Epstein, Periodic flows on three-manifolds, *Ann. of Math. (2)* **95** (1972), 66–82.

[22] J. Etnyre, Lectures on open book decompositions and contact structures, `arXiv:math/0409402`.

[23] J. Etnyre and R. Ghrist, Contact topology and hydrodynamics I: Beltrami fields and the Seifert conjecture, *Nonlinearity* **13** (2000), 441–458.

[24] J. Etnyre and R. Ghrist, Contact topology and hydrodynamics II: solid tori, *Ergodic Theory Dynam. Systems* **22** (2002), 819–833.

[25] J. Etnyre, R. Komendarczyk and P. Massot, Tightness in contact metric 3-manifolds, *Invent. Math.* **188** (2012), 621–657.

[26] G. Frobenius, Ueber das Pfaffsche Problem, *J. Reine Angew. Math.* **82** (1877), 230–315.

[27] J. Ge and Y. Huang, 1/4-pinched contact sphere theorem, *Asian J. Math.* **20** (2016), 893–901.

[28] H. Geiges, *An Introduction to Contact Topology*, Cambridge Stud. Adv. Math. **109**, Cambridge University Press, Cambridge (2008).

[29] H. Geiges, *The Geometry of Celestial Mechanics*, London Math. Soc. Stud. Texts **83**, Cambridge University Press, Cambridge (2016).

[30] H. Geiges, What does a vector field know about volume?, *J. Fixed Point Theory Appl.* **24** (2022), Paper No. 23.

[31] H. Geiges and C. Lange, Seifert fibrations of lens spaces, *Abh. Math. Semin. Univ. Hambg.* **88** (2018), 1–22.

[32] E. Giroux, Géométrie de contact: de la dimension trois vers les dimensions supérieures, in: *Proceedings of the International Congress of Mathematicians, Vol. II* (Beijing, 2002), Higher Education Press, Beijing (2002), 405–414.

[33] H. Gluck, Dynamical behavior of geodesic fields, in: *Global Theory of Dynamical Systems*, Lecture Notes in Math. **819**, Springer-Verlag, Berlin (1980), 190–215.

[34] H. Gluck, Great circle fibrations and contact structures on the 3-sphere, *Geom. Dedicata* **216** (2022), Paper No. 72, 15 pp.

[35] H. Gluck and F. Warner, Great circle fibrations of the three-sphere, *Duke Math. J.* **59** (1983), 107–132.

[36] H. Gluck and J. Yang, Great Circle Fibrations and Contact Structures on Odd-Dimensional Spheres, `arXiv:1901.06370`.

[37] Y. Godoy and M. Salvai, Global smooth geodesic foliations of the hyperbolic space, *Math. Z.* **281** (2015), 43–54.

[38] B. Hajduk and R. Walczak, On vector fields having properties of Reeb fields, *Topol. Methods Nonlinear Anal.* **41** (2013), 401–408.

[39] R. Hamilton, Three-manifolds with positive Ricci curvature, *J. Differential Geometry* **17** (1982), 255–306.

[40] A. Harris and G. Paternain, Conformal great circle flows on the 3-sphere, *Proc. Amer. Math. Soc.* **144** (2016), 1725–1734.

[41] M. Harrison, Skew flat fibrations, *Math. Z.* **282** (2016), 203–221.

[42] M. Harrison, Contact structures induced by skew fibrations of $\mathbb{R}^3$, *Bull. Lond. Math. Soc.* **51** (2019), 887–899.

[43] M. Harrison, Fibrations of $\mathbb{R}^3$ by oriented lines, *Algebr. Geom. Topol.* **21** (2021), 2899–2928.

[44] M. W. Hirsch, On Imbedding Differentiable Manifolds in Euclidean Space, *Ann. of Math.* **73** (1961), 566–571.

[45] H. Hofer, Pseudoholomorphic curves in symplectizations with applications to the Weinstein conjecture in dimension three, *Invent. Math.* **114** (1993), 515–563.

[46] H. Hopf, Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche, *Math. Ann.* **104** (1931), 637–665.

[47] M. Hutchings, Taubes's proof of the Weinstein conjecture in dimension three, *Bull. Amer. Math. Soc. (N. S.)* **47** (2010), 73–125.

[48] M. Jankins and W. Neumann, *Lectures on Seifert manifolds*, Brandeis Lecture Notes **2**, Brandeis University, Waltham, MA (1983); available at `http://www.math.columbia.edu/~neumann/preprints/`.

[49] D. L. Johnson, L. B. Whitt, Totally geodesic foliations, *J. Differential Geometry* **15** (1980), 225–235.

[50] M. Kegel and C. Lange, A Boothby-Wang theorem for Besse contact manifolds, *Arnold Math. J.* **7** (2021), 225–241.

[51] T. Konno, Unit tangent bundle over two-dimensional real projective space, *Nihonkai Math. J.* **13** (2002), 57–66.

[52] O. Kowalski, Curvature of the induced Riemannian metric on the tangent bundle of a Riemannian manifold, *J. Reine Angew. Math.* **250** (1971), 124–129.

[53] T. Lawson, Open book decompositions for odd dimensional manifolds, *Topology* **17** (1978), 189–192.

[54] P. D. Lax, *Linear Algebra*, Pure Appl. Math. (N. Y.), John Wiley & Sons, Inc., New York (1997).

[55] J. M. Lee, *Riemannian Manifolds - An Introduction to Curvature*, Grad. Texts Math. **176**, Springer, Berlin (1997).

[56] B. Martelli, An Introduction to Geometric Topology, `arXiv:1610.02592`.

[57] J. Martinet, Formes de contact sur les variétés de dimension 3, in: *Proc. Liverpool Singularities Symp. II*, Lecture Notes in Math. **209**, Springer-Verlag, Berlin (1971), 142–163.

[58] K. Moffatt, Degree of knottedness of tangled vortex lines, *J. Fluid. Mech.* **35** (1969), 117–129.

[59] G. Paternain, *Geodesic flows*, Progr. Math **180**, Birkhäuser Verlag, Basel (1999).

[60] F. Quinn, Open book decompositions, and the bordism of automorphisms, *Topology* **18** (1979), 55–73.

[61] A. Rechtman, Existence of periodic orbits for geodesible vector fields on closed 3-manifolds, *Ergodic Theory Dynam. Systems* **30** (2010), 1817–1841.

[62] B. L. Reinhart, Foliated manifolds with bundle-like metrics, *Ann. of Math. (2)* **69** (1959), 119–132.

[63] R. O. Ruggiero, On manifolds admitting continuous foliations by geodesics, *Geom. Dedicata* **78** (1999), 161–170.

[64] S. Sasaki, On the differential geometry of tangent bundles of Riemannian manifolds, *Tohoku Math. J. (2)* **10** (1958), 338–354.

[65] P. Scott, The geometries of 3-manifolds, *Bull. Lond. Math. Soc.* **15** (1983), 401–487.

[66] P. Scott, There are no fake Seifert fibre spaces with infinite π_1, *Ann. of Math. (2)* **117** (1983), 35–70.

[67] I. M. Singer and J. A. Thorpe, *Lecture notes on elementary topology and geometry*, Scott, Foresman & Co., Glenview, IL (1967).

[68] D. Sullivan, A counterexample to the periodic orbit conjecture, *Inst. Hautes Études Sci. Publ. Math.* **46** (1976), 5–14.

[69] M. Sullivan, A foliation of geodesics is characterized by having no "tangent homologies", *J. Pure Appl. Algebra* **13** (1978), 101–104.

[70] C. H. Taubes, The Seiberg-Witten equations and the Weinstein conjecture, *Geom. Topol.* **11** (2007), 2117–2202.

[71] W. P. Thurston, Existence of codimension-one foliations, *Ann. of Math. (2)* **104** (1976), 249–268.

[72] W. P. Thurston, *The Geometry and Topology of Three-Manifolds*, lecture notes (1979), electronic version 1.1 available at: `http://library.msri.org/books/gt3m/PDF/Thurston-gt3m.pdf`.

[73] W. P. Thurston, *Three-dimensional geometry and topology. Vol. 1* (S. Levy, ed.), Princeton Math. Ser. **35**, Princeton University Press, Princeton (1997).

[74] D. Tischler, On fibering certain foliated manifolds over S^1, *Topology* **9** (1970), 153–154.

[75] P. Tondeur, *Geometry of foliations*, Monogr. Math. **90**, Birkhäuser Verlag, Basel (1997).

[76] T. Vogel, Existence of Engel structures, *Ann. of Math.* (2) **169** (2009), 79–137.

[77] A. W. Wadsley, Geodesic foliations by circles, *J. Differential Geometry* **10** (1975), 541–549.

[78] H. E. Winkelnkemper, Manifolds as open books, *Bull. Amer. Math. Soc.* **79** (1973), 45–51.

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Teilpublikationen

1. T. Becker, Geodesic and conformally Reeb vector fields on flat 3-manifolds, *Differential Geom. Appl.* **89** (2023), Paper No. 102013, 18 pp.

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