Zeros of Random Holomorphic Sections of Semipositive Line Bundles on Punctured Riemann Surfaces

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Zusammenfassung

Von frühen Arbeiten aus den 1930er Jahren an bis heute besteht ein wachsendes Interesse an der Theorie der asymptotischen Verteilungen der erwarteten Nullstellen von zufälligen Polynomen, wenn ihr Grad unendlich wächst. Eine natürliche geometrische Verallgemeinerung von zufälligen Polynomen sind zufällige Schnitte holomorpher Geradenbündel über einer komplexen Mannigfaltigkeit.

Im Jahr 1999 bewiesen Shiffman und Zelditch, dass auf einer kompakten Kähler-Mannigfaltigkeit die Nullstellen von Schnitten hoher Tensorpotenzen eines holomorphen Geradenbündels asymptotisch gleichverteilt bezüglich der normalisierten Krümmung des Bündels sind. Ihr Ergebnis hat zahlreiche Anwendungen in der mathematischen Physik und wurde seitdem in viele verschiedene Richtungen verallgemeinert.

In dieser Arbeit verallgemeinern wir ihr Ergebnis auf ein semipositiv gekrümmtes holomorphes Geradenbündel über einer punktierten Riemannschen Fläche. Dafür untersuchen wir Instrumente, deren Betrachtung sich als geeigneter Rahmen für die Untersuchung statistischer Eigenschaften von Nullstellenmengen auf komplexen Mannigfaltigkeiten erwiesen haben.

Zunächst zeigen wir die Existenz einer Spektral-Lücke für den zum Bündel gehörigen Kodaira-Laplace-Operator. Dieses Ergebnis wenden wir zusammen mit der Technik der analytischen Lokalisierung von Ma und Marinescu an, um eine punktweise globale asymptotische Entwicklung des zugehörigen Bergman-Kerns zu folgern. Zusätzlich zeigen wir lokal gleichmäßige Schranken für den Bergman-Kern und seine Ableitungen. Diese benutzen wir, um die lokal gleichmäßige Konvergenz der durch die zugehörige Kodaira-Abbildung induzierten Fubini-Study-Metriken und ihrer Potentiale zu der globalen Krümmung, beziehungsweise ihrem Potential nachzuweisen.

Abschließend zeigen wir, dass die erwarteten Nullstellen der holomorphen Schnitte bezüglich der normalisierten Krümmung der Geradenbündels gleichverteilt sind. Außerdem wenden wir die Theorie der meromorphen Transformationen von Dinh und Sibony an, um die Konvergenzgeschwindigkeit unseres Gleichverteilungsergebnisses abzuschätzen.

Abstract

With early works dating back to the 1930's until today, there is a growing interest in the theory of asymptotic distributions of expected zeros of random polynomials when their degree grows indefinitely. A natural geometric generalization of random polynomials are random sections of a holomorphic line bundle over a complex manifold. In 1999, Shiffman and Zelditch proved that on a compact Kähler manifold, the zeros of sections of high tensor powers of a holomorphic line bundle asymptotically equidistribute with respect to the normalized curvature of the line bundle. Their result has numerous applications in mathematical physics and was generalized in many different directions.

In this thesis we generalize their result to a semipositively curved holomorphic line bundle over a punctured Riemann surface.

To achieve this, we discuss many tools that have proven themselves to represent an appropriate framework to study statistical properties of ensembles of zeros on complex manifolds.

We start by proving the existence of spectral gap for the Kodaira Laplacian that is associated to the line bundle. We use this result, together with the technique of analytic localization by Ma and Marinescu, to prove a pointwise global on-diagonal asymptotic expansion of the associated Bergman kernel in our setting. Moreover, we show locally uniform estimates on the Bergman kernel and its derivatives. We use these estimates to prove the locally uniform convergence of the induced Fubini-Study metrics and their potentials to the global curvature and its potential, respectively.

We conclude by showing that the expected zeros of holomorphic sections equidistribute with respect to the normalized curvature of the line bundle. Moreover, we apply the theory of meromorphic transforms by Dinh and Sibony estimate the speed of convergence in our equidistribution result.

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Table of Contents

1. Introduction: Overview of the topic and statement of the results

For growing $p \in \mathbb{N}$, we will consider sections of line bundles $L^p \otimes E$ over a punctured Riemann surface Σ, where (E, h^E) is a smooth holomorphic Hermitian line bundle and (L, h^L) is a smooth holomorphic line bundle; we make two assumptions:

- the local geometry of (L, h^L) and Σ near the punctures are modelled by the Poincaré (punctured) disc $(\mathbb{D}^*, h_{\mathbb{D}^*})$;
- away from the punctures, the curvature R^L of h^L of L is assumed to be semipositive and it is allowed to vanish at most to finite order at every point.

We prove various results in this setting:

In Chapter 3, we show the existence of a spectral gap for the Kodaira Laplacian and prove estimates on the associated Dirac operators.

In Chapter 4 we will use the spectral gap of the Kodaira Laplacian to conclude the existence of a global pointwise on-diagonal asymptotic expansion of the Bergman kernel that is associated to h^L. Moreover, we give locally uniform estimates for the Bergman kernel from below and above as locally uniform estimates for derivatives of the Bergman kernel from above.

In Chapter 5, we will investigate the Kodaira map associated to h^L and we prove an analogue of a theorem of Tian and Ruan. Moreover, on compact subsets of Σ, we prove the convergence of the induced Fubini-Study potentials via pull-backs by this Kodaira map to the potential of h^L. A similar convergence result is shows to hold for the ∂- and $\bar{\partial}$-derivatives of the induced Fubini-Study potentials to the corresponding derivatives of the potential of h^L. All convergence results come with an estimate for the speed of convergence.

In Chapter 6 we prove the equidistribution of the currents of integration along the (support of the) zero divisors to the normalized (Chern) curvature R^L of h^L and give an estimate of the speed of convergence.

1.1 Setting: Semipositive line bundles on punctured Riemann surfaces

In this Section, we briefly explain our setting and its relations to settings that have previously been considered by different authors.

Consider a compact Riemann surface $\overline{\Sigma}$ with complex structure J. Without loss of generality, we can assume that $\overline{\Sigma}$ is connected. Fix a non-zero natural number $N \in \mathbb{N}$; for our purposes we view $0 \notin \mathbb{N}$ and set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Given a set of points $a_1, \ldots, a_N \in \overline{\Sigma}$ and

$$D := \sum_{j=1}^{N} a_j \,, \tag{1.1.1}$$

a divisor on $\overline{\Sigma}$, such that its support $\operatorname{supp} D = \{a_1, \ldots, a_N\} \subset \overline{\Sigma}$ and $\Sigma := \overline{\Sigma} \setminus \operatorname{supp} D$ is an open Riemann surface, called a punctured Riemann surface, with punctures in (the support of) D. The complex structure J on $\overline{\Sigma}$ induces a complex structure on Σ by restriction. On Σ, we take a smooth Hermitian $(1,1)$-form ω_Σ that is compatible with J.

We denote by $\mathbf{i} := \sqrt{-1}$ the imaginary unit in $\mathbb{C}$. Recall the definition of the Poincaré metric on the punctured unit disc $\mathbb{D}^* := \{z \in \mathbb{C} : 0 < |z| < 1\} \subset \mathbb{C}$, normalized as follows

$$\omega_{\mathbb{D}^*} := \frac{\mathbf{i}}{|z|^2(\log(|z|^2))^2} \, dz \wedge d\overline{z} \,; \tag{1.1.2}$$

it is a complete metric of finite volume (see (2.1.6), (2.1.7)).

Let $L \to \overline{\Sigma}$ be holomorphic line bundle equipped with a singular Hermitian metric h^L (as defined in Section 2.3).

We assume that ω_Σ and h^L are subject to the following conditions:

(α) h^L is smooth over Σ and for all $j \in \{1, \ldots, N\}$ there exists a trivialization of L in a small open neighborhood $\overline{V_j} \subset \overline{\Sigma}$, centered at a_j in $\overline{\Sigma}$ with associated complex coordinate z_j, where a_j corresponds to $z_j = 0$, such that

$$|1|^2_{h^L}(z_j) = |\log |z_j|^2| \,. \tag{1.1.3}$$

(β) The smooth (Chern) curvature R^L of h^L satisfies the following conditions:

(i) on Σ, we have $\mathbf{i}R^L \geqslant 0$,

(ii) for each $j = 1, \dots, N$, we have $\mathbf{i}R^L = \omega_\Sigma$ on $V_j := \overline{V}_j \setminus \{a_j\}$; in particular, $\omega_\Sigma = \omega_{\mathbb{D}^*}$ in the local coordinate z_j on V_j and

(iii) R^L vanishes at most to finite order at any point $x \in \Sigma$, i.e.

$$\mathrm{ord}_x(R^L) := \min\{l \in \mathbb{N} : J^l(\Sigma, \Lambda^2 T^*\Sigma) \ni j^l_x R^L \neq 0\} < \infty, \tag{1.1.4}$$

where $J^l(\Sigma, \Lambda^2 T^*\Sigma)$ denotes the l-th jet bundle over Σ (see Appendix A).

Remark 1.1.1. Note that the sets $\overline{V_j}$ are open subsets of $\overline{\Sigma}$; this notation is chosen in this way in order to emphasize that $\overline{V_j}$ are subsets of $\overline{\Sigma}$ and is not to be confused with the topological closure $V_j \cup \partial V_j$ of V_j in Σ, which we will not be interested in. Consequently, the sets $V_j := \overline{V_j} \setminus \{a_j\}$ are open subsets of Σ, for all $j = 1, \dots, N$.

Assumptions (β)(i)-(ii) imply that (Σ, ω_Σ) is complete and the total volume of Σ with respect to ω_Σ is finite.

For $x \in \Sigma$, we set

$$\rho_x := 2 + \mathrm{ord}_x(R^L) \in \mathbb{N}_{\geqslant 2}\,. \tag{1.1.5}$$

It is then evident that $\rho_x = 2$ for all $x \in V_j$, for all $j = 1, \dots, N$. Moreover, the function $\Sigma \ni x \mapsto \rho_x$ is upper semi-continuous. Since $\Sigma \setminus (\bigcup_j V_j)$ is compact, assumption (β)(iii) implies that

$$\rho_\Sigma := \max_{x \in \Sigma} \rho_x < \infty\,. \tag{1.1.6}$$

The semipositivity in assumption (β)(i) implies that ρ_x is even for all $x \in \Sigma$; consequently, ρ_Σ is an even number, as well. Moreover, we have a decomposition $\Sigma = \bigcup_{j=2}^{\rho_\Sigma} \Sigma_j$, with $\Sigma_j := \{x \in \Sigma : \rho_x = j\}$ and each $\Sigma_{\leqslant j} = \bigcup_{j'=2}^{j} \Sigma_{j'}$ is open. Consequently, the function $x \mapsto \rho_x$ is constant on each of the subsets $\Sigma_j \subset \Sigma$ and only jumps in value upon transitioning from one Σ_j to another.

When referring to condition (β), we say that L is semipositive and its curvature vanishes to at most finite order over Σ.

Now let $E \to \overline{\Sigma}$ be a holomorphic line bundle equipped with a smooth Hermitian metric h^E. We always assume the technical assumption for (E, h^E) near the punctures that (E, h^E) coincides with the trivial Hermitian line bundle on each of the neighborhoods $\overline{V_j} \subset \overline{\Sigma}$.

We will frequently consider the restrictions $L|_\Sigma$ and $E|_\Sigma$ of the line bundles L and E, respectively, to Σ. By a slight abuse of notation, we will then still denote $L|_\Sigma$ and $E|_\Sigma$

by L and E, respectively, when the context provides sufficient room to avoid confusion; otherwise an appropriate distinction is highlighted.

The results that we have mentioned in the beginning of this Chapter will be statements about the tensor

$$\left(\bigotimes_{i=1}^{p} L\right) \otimes E = L^{\otimes p} \otimes E =: L^p \otimes E \tag{1.1.7}$$

of the line bundles $L^p \to \Sigma$ and $E \to \Sigma$, for fixed $p \in \mathbb{N}$ or $p \to \infty$; the parameter $p \in \mathbb{N}$ corresponds to the semiclassical limit (see [43] for example) which originates in problems in mathematical physics.

Our results have previously been established individually in different, related settings, which we will now summarize for an appropriate context. The different results will be discussed separately in the corresponding subsections of Section 1.2. In Section 1.3, we will explain how the thesis is organized.

1.2 Overview of the results

1.2.1 Dirac and Kodaira Laplace Operators and Spectral Gap

In the book [43] of Ma and Marinescu a spectral gap for the Kodaira Laplacian was established for a positive line bundle on a compact Kähler manifold [43, Section 1.5.1] and for a modified Dirac operator on symplectic manifolds [43, Section 1.5.2]. While the spectral theory of Dirac and Laplace operators on manifolds are interesting in its own right, our view adopts the strategy mentioned in the book, where the authors show that the existence of a spectral gap can be applied to gain detailed information about asymptotic behavior of the Bergman kernel on the manifold. This application will be discussed in Chapter 4.

As mentioned in [43, Problem 4.8], Donnelly [25] presented an example of a semipositively curved line bundle on a compact complex Hermitian manifold, such that the associated Kodaira Laplacian does not possess a gap in its spectrum. However, in their papers [47] and [46], Marinescu and Savale prove that the situation is more promising on compact Riemann surfaces by imposing the additional condition to the semipositive curvature of a holomorphic Hermitian line bundle that if the order of vanishing happens at most to finite order; Marinescu and Savale achieve this by relating the condition of globally a finite order of vanishing to the bracket generating condition known from sub-Riemannian geometry [46, Proposition 11]: the authors conclude the existence of a spectral gap for the Bochner Laplacian (see [47, Proposition 6] or [46, Proposition 15]) and infer by the Lichnerowicz formula (see [47, (2.12)] or [46, (4.5)]) that a spectral gap also exists for the Kodaira Laplacian globally (see [47, Corollary 8] or [46, Corollary 21]).

In this thesis, we generalize this result to our setting by explaining how a partition of unity argument allows the statement to hold over the punctured Riemann surface Σ. To state the result, recall the definition (see Subsection 2.6.1) of the $\bar{\partial}$-operator, or Dolbeault operator, on manifolds. For each level $p \in \mathbb{N}$, the Dirac operator D_p and Kodaira Laplacian operator $\square_p$ are then defined by

$$\begin{aligned} D_p &:= \sqrt{2}(\bar{\partial}_p + \bar{\partial}_p^*)\,, \\ \square_p &:= \frac{1}{2}(D_p)^2 = \bar{\partial}_p\bar{\partial}_p^* + \bar{\partial}_p^*\bar{\partial}_p\,. \end{aligned} \tag{1.2.1}$$

For each $p \in \mathbb{N}$, consider a holomorphic section $s : \Sigma \to L^p \otimes E$ that is $\mathcal{L}^2$-integrable with respect to the $\mathcal{L}^2$-inner-product from (2.5.7). This inner product induces an $\mathcal{L}^2$-norm (see Section 2.5). We will then show that the value of $D_p s$ with respect to this norm can be estimated from above by a monotonous function of order $\mathcal{O}(p^{-2/\rho_\Sigma})$ where ρ_Σ is the maximal order of vanishing on Σ, multiplied by the norm of s. From this, we conclude that the associated Kodaira Laplacian $\square_p$ has a spectral gap. This result is summarized in the following theorem.

Theorem A (Spectral Gap and vanishing first cohomology)**.** *Let Σ be a punctured Riemann surface, and let L be a holomorphic line bundle such that L carries a singular Hermitian metric h^L satisfying conditions (α) and (β). Let E be a holomorphic line bundle on Σ equipped with a smooth Hermitian metric h^E such that (E, h^E) on each chart V_j coincides with the trivial Hermitian line bundle. Consider the Dirac and Kodaira Laplace operators as defined in* (1.2.1)*. Then there exist constants $C_1, C_2 \in \mathbb{R}_{>0}$ independent of p, such that for all $s \in \Omega^{0,1}_{\mathrm{c}}(\Sigma, L^p \otimes E)$,*

(i) the Dirac operators are bounded from below,

$$\|D_p s\|^2_{\mathcal{L}^2} \geqslant 2(C_1 p^{2/\rho_\Sigma} - C_2)\|s\|^2_{\mathcal{L}^2}\,, \tag{1.2.2}$$

(ii) for $p \in \mathbb{N}$, we have

$$\operatorname{Spec}(\square_p) \subset \{0\} \cup [C_1 p^{2/\rho_\Sigma} - C_2, \infty)\,. \tag{1.2.3}$$

In particular, the first $\mathcal{L}^2$-Dolbeault cohomology group $H^1_{(2)}(\Sigma, L^p \otimes E) = 0$ vanishes for sufficiently large $p > 0$.

As mentioned before, this result was proven in [47, Corollary 8] for the case of a semipositive line bundle on a compact Riemann surface where the curvature vanishes at most to finite order. Thus our result is a generalization to the case of a punctured Riemann surface with the Poincaré model condition (α) on the punctures.

Another known result in this direction is [43, (6.1.8)], where the authors prove a spectral gap for the Kodaira Laplacians of the associated Dirac operators on a non-compact, complete Hermitian manifold, where the line bundle is assumed to have strictly positive curvature everywhere. As a special case, this implies the existence of a spectral gap for the Kodaira Laplacian of the associated Dirac operators on a punctured Riemann surface for sections of a positive line bundle. The latter setting is also a special case of our setting, which is obtained by assuming that $\rho_\Sigma = 2$, which implies that the curvature is positive. In this sense, our result represents a generalization of this special case with an upper bound that is more precise.

Theorem A is restated as Theorem 3.2.1, and proven, in Chapter 3.

1.2.2 On-diagonal asymptotic Expansion of Bergman Kernel

The Bergman kernel is a central object to this thesis. Recall the well-known Schwarz kernel Theorem [43, Theorem B.2.7] about the existence and regularity of the Schwartz integral kernel. The Bergman kernel is the Schwartz kernel of the (unique) orthogonal projection from the space of square-integrable sections of $L^p \otimes E$ onto the space of square-integrable holomorphic sections, which is a closed subspace of the latter.

The Bergman kernel was first introduced by Stefan Bergman in 1922 [4] for domains in $\mathbb{C}^n$, for $n \geqslant 1$. Historically, the Bergman kernel has been influential in many areas of complex geometry, complex analysis, quantum physics and many others, including but not limited to partial differential equations and several complex variables [15], holomorphic embeddings and extensions of holomorphic maps (see [33] and [29], respectively), the study of domains in $\mathbb{C}^n$ whose boundaries are pseudoconvex (see [11], [32], [42]), vanishing theorems [20] and the existence and approximation of Kähler metrics [52] on complex manifolds, the theory of quantization [49], [37], [29], [12], [45] and path integrals and their computations [26]; in [43], the authors provide an extensive list of references of its importance and influences throughout the many different areas of mathematics.

One active area of research that is of particular interest in this thesis is the study of the asymptotic behavior of the Bergman kernels B_p for holomorphic Hermitian line bundles $L^p \otimes E$, for growing integer powers $p \to \infty$.

As mentioned in Subsection 1.2.1, Ma and Marinescu (see [43]) and Hsiao and Marinescu (see [34]) argue how the existence of a spectral gap of the Kodaira Laplacian for sufficiently large values of $p \in \mathbb{N}$ implies the existence of an asymptotic expansion of the Bergman kernel associated to high tensor powers of a positive holomorphic Hermitian line bundle over a compact Kähler manifold of dimension $n \in \mathbb{N}$. Let B_p be the Bergman kernel at level $p \in \mathbb{N}$ associated to a Hermitian metric h_p on a line bundle $L^p \otimes E$ and a fixed volume form on the underlying Kähler manifold. If x is a point in the underlying manifold where the Chern curvature of h^L fails to be positive, Berman [6] showed that

$$\limsup_{p\to\infty} \frac{B_p(x)}{p^n} = 0\,; \tag{1.2.4}$$

later, this result was refined by Hsiao and Marinescu [34]. Consequently, meaningful information about the asymptotic behavior of B_p can only be expected to be obtained on subsets of the manifolds where the curvature of L is positive, or at least semipositive.

In general, the asymptotic behavior of the Bergman kernel associated to high tensor powers of a positive Hermitian line bundle on a compact complex manifold has been studied by many authors. The subject of Bergman kernel asymptotics can be divided into a few parts which are oftentimes treated separately. Of particular interest are the asymptotic expansions on, near and away from the diagonal, as well as computations of specific coefficients of the asymptotic expansion, which oftentimes carry meaningful information about the line bundle, such as its curvature. In cases where no full asymptotic expansion is known, useful estimates on the modulus of the Bergman kernel can oftentimes be obtained. We review some of the relevant historical results to provide some context for Chapter 4.

In the case where L is a positive line bundle over a compact Kähler manifold, Ma and Marinescu [43, Theorem 4.1.1] prove, using the existence of a spectral gap, the finite propagation speed of solutions of the wave equation of the associated Laplacian and the technique of analytic localization, that the Bergman kernel B_p has an on-diagonal asymptotic expansion

$$\left\| B_p(x,x) - \sum_{r=0}^{k} \mathbf{b}_r(x) p^{k-r} \right\|_{\mathcal{C}^\ell(h_p)} = \mathcal{O}(p^{n-k-1}) \tag{1.2.5}$$

for any $\ell \in \mathbb{N}$, with smooth coefficients $\mathbf{b}_r$ that are polynomials in the curvatures and their derivatives. Moreover, they explicitly calculated $\mathbf{b}_0$.

The computation of the coefficients $\mathbf{b}_r$ is an interesting and active area of research in itself. For a discussion on the topic we refer the reader to the authors book.

The existence of an expansion such as (1.2.5) started with a paper of Tian [52] (in this context it is also important to mention Bouche [10] and Ruan [48]) after a suggestion of Yau in [54] and [55]. The existence of the expansion (1.2.5) was first established by Catlin [13] and Zelditch [56], where the respective authors also gave an explicit formula for the leading coefficient $\mathbf{b}_0$. For a detailed list of references we refer the reader of this thesis to [43, Section 4.3].

Of importance for our subject are also the Ma and Marinescu's result [43, Theorem 6.1.1], where they generalize (1.2.5) to the case of compact subsets of a complete Hermitian manifold, under certain conditions on the geometry of the line bundle and the manifold.

In [1] and [2], Auvray Ma and Marinescu go on to study the asymptotic behavior of the Bergman kernel near punctures of a punctured Riemann surface, when the local model is the punctured Poincaré disc, identical to our case; the line bundle in their papers is otherwise assumed to be globally positive. In their seminal papers, the authors compare the Bergman kernel on the manifold to the Bergman kernel of the local model and prove various useful estimates such as of the quotients of the two Bergman kernels; in the corresponding later chapters, the authors discuss useful applications such consequences of the estimates for the study of the metric aspect of the associated Kodaira maps and the equidistribution of zeros of random holomorphic sections.

In the present text, we will show the pointwise existence of an on-diagonal asymptotic expansion of the Bergman kernel. The result was proven by Marinescu and Savale in [47] to hold on a compact Riemann surface with a semipositive line bundle, that is, a line bundle that is equipped with an Hermitian metric the (Chern) curvature of which is a semipositive form.

We explain how their results applies to our setting to obtain a global on-diagonal pointwise asymptotic expansion on the punctured Riemann surface Σ under the same curvature conditions away from the punctures. The result is the following.

Theorem B (On-diagonal asymptotic expansion of the Bergman kernel)**.** *Let Σ, L and E be as in Section 1.1. For $\rho_0 \in \{2, 4, \ldots, \rho_\Sigma\}$ define a smooth path $W : [0,1] \ni t \mapsto W(t) \in \Sigma$ such that $W(t) \in \Sigma_{\rho_0}$ for all $t \in [0,1]$. Then for all $r \in \mathbb{N}$, there exists a smooth function $b_r(x)$ with $x \in \mathrm{range}(W)$, such that for all $k \in \mathbb{N}$ the following asymptotic expansion of the Bergman kernel function holds uniformly on* $\mathrm{range}(W)$ *in any $\mathcal{C}^\ell$-topology, with $\ell \in \mathbb{N}$:*

$$B_p(x,x) = p^{2/\rho_0} \left[\sum_{r=0}^{k} b_r(x) p^{-2r/\rho_0} \right] + \mathcal{O}(p^{-2k/\rho_0}) , \tag{1.2.6}$$

Moreover, for $x \in W$, the leading term satisfies

$$b_0(x) = B^{j_z^{\rho_0-2} R^L}(0,0) > 0 , \tag{1.2.7}$$

where the (ρ_0-2)-th jet $j_x^{\rho_0-2} R^L \in \mathbf{i} S^{\rho_0-2}\mathbb{R}^2 \otimes \Lambda^2(\mathbb{R}^2)^*$ is identified with the (ρ_0-2)-degree homogeneous part of the Taylor expansion of R^L in the geodesic normal coordinate centered at z, and $B^{j_x^{\rho_0} R^L}$ is the model Bergman projection defined in Subsection 4.4.1.

For $h \in (0,1)$, $\gamma \in (0, \frac{1}{2})$, $\ell, m \in \mathbb{N}$, and V_j described in assumption (α) with coordinate z_j, the following asymptotic expansion of the Bergman kernel function holds uniformly in any $\mathcal{C}^l$-topology, with $\ell \in \mathbb{N}$, for points z_j in the ring $\mathbb{D}^*(a_j, \frac{1}{6}) \setminus \mathbb{D}^*(a_j, he^{-p^\gamma})$:

$$B_p(z_j, z_j) = \frac{p-1}{2\pi} + \mathcal{O}(p^{-m}) . \tag{1.2.8}$$

Later, Theorem B will be restated as Theorem 4.1.1. The proof can be found in Chapter 4.

The expansion holds pointwise when one considers larger subsets on Σ, which contain points where the curvature has different orders of vanishing. One of the reasons for this is the presence of 'jumps' in the coefficients in the asymptotic expansions when one moves from a point on Σ along a path that crosses points where the order of the vanishing of the curvature changes. To establish the expansion in Chapter 4, we follow the approach of Ma and Marinescu [43, Chapter 4] in using the spectral gap property of the Kodaira Laplacian together with the analytic localization technique that was inspired by Bismut and Lebeau [7].

Our result generalizes [47, Theorem 1] to the case of a punctured Riemann surface with the Poincaré disc as the local model near the punctures and otherwise semipositive curvature that vanishes at most to finite order. The difference to the previous results authors are the existence of the punctures, since our assumptions away from the punctures are locally identical to the situation in the paper of Marinescu and Savale.

We also have the following estimates on the Bergman kernel on arbitrarily large, relatively compact subsets of Σ.

Lemma C. *Let $h \in (0,1)$ and $\gamma \in (0, \frac{1}{2})$ be arbitrary. Then for sufficiently large $p \in \mathbb{N}$ and all $x \in \Sigma^{p,h,\gamma}$, we have the estimates*

$$\begin{aligned} c(\rho_\Sigma)\,(1+o(1))\,p^{2/\rho_\Sigma} &\leqslant B_p(x,x) \\ &\leqslant \left[\sup_{x\in\Sigma} B^{j_x^0 R^L}(0,0)\right](1+o(1))\,p\,, \end{aligned} \tag{1.2.9}$$

where both constants in $o(1)$ are uniform in $x \in \Sigma^{p,h,\gamma}$, as $p \to \infty$.

Lemma C is an analogue of [47, Lemma 12] for the case of a punctured Riemann surface. It is restated as Lemma 4.1.7 and proven in Section 4.5.

Another result that we obtain is the following, which estimates the derivatives of the Bergman kernel locally uniformly.

Lemma D. *Let $h \in (0,1)$ and $\gamma \in (0, \frac{1}{2})$ be arbitrary. Then for sufficiently large $p \in \mathbb{N}$ and all $x \in \Sigma^{p,h,\gamma}$, we have the upper bound for the l-th jets of the Bergman kernels*

$$|j^l[B_p(x,x)]| \leqslant p^{l/3}[1+o(1)]\left[\sup_{x\in\Sigma}\frac{\left|j^l B^{j_x^1 R^L/j_x^0 R^L}(0,0)\right|}{B^{j_x^1 R^L/j_x^0 R^L}(0,0)}\right] B_p(x,x)\,, \tag{1.2.10}$$

where $o(1)$ is uniform in $x \in \Sigma^{p,h,\gamma}$, as $p \to \infty$.

Lemma D will be restated as Lemma 4.1.9 in Chapter 4. Its proof can be found in Section 4.5.

The locally uniform upper bound for the l-th jet of the Bergman kernel is an extension of [47, Lemma 13] to the case of a punctured Riemann surface.

1.2.3 Convergence of Fubini-Study potentials away from punctures

Since (L, h^L) is a positive holomorphic line bundle on the compact Riemann surface $\overline{\Sigma}$, the Riemann-Roch-Hirzebruch theorem (see [43, Theorem 1.4.6]) implies that the line bundles L^p have a lot of holomorphic sections when the tensor power p is sufficiently large.

This fact is used in the construction of the well-known Kodaira map over a compact manifold (see [43, Chapter 5], for a general discussion). In [43], the authors apply the asymptotic expansion of the Bergman kernel to study the metric aspect of the Kodaira map. We will follow their approach and construct a family of Fubini-Study metrics associated to the Kodaira maps in our case.

Because of our special setting, the study of the Kodaira map is a bit more delicate than for example in the case of a positive line bundle over a compact Riemann surface, in particular, two difficulties arise: first, the deletion of isolated points, i.e. puncturing a compact Riemann surface produces a manifold that is no longer compact, and second, the existence of regions on the manifold where the curvature is allowed to vanish to at most finite order.

In [2], Auvray, Ma and Marinescu consider the case of a punctured Riemann surface with the same local model as in our case and the authors proceed to explain how to overcome the arising sensitivity that their setting requires, in particular in the context of the associated Kodaira map and equidistribution of zeros.

In [47, Section 4], Marinescu and Savale study the Kodaira maps associated to a semipositive line bundle over a compact Riemann surface, with at most finite order of vanishing of the curvature.

We utilize these two works in our setting and give a discussion about the associated Kodaira maps in our case where we seek to combine previous results on this topic that deal with the arising difficulties.

One of the results that we obtain in this discussion is an analogue of the theorem of Tian-Ruan (see [43, Theorem 5.1.4], [52, Theorem A] and [48]) in our setting:

Theorem E. *Let Σ be a punctured Riemann Surface and let (L, h^L) (E, h^E) be Hermitian holomorphic line bundles on $\overline{\Sigma}$ such that conditions (α) and (β) are satisfied. Let $U \subset \Sigma$ be a relatively compact subset. Then the following statements are true. The normalized induced Fubini-Study metrics converge uniformly on U to the normalized semipositive curvature $R^L|_U$, with speed $\mathcal{O}(p^{-1/3})$; that is, for every $\ell \in \mathbb{N}_0$, there exists a constant $C_{\ell,U} \in \mathbb{R}_{>0}$, such that:*

$$\left\| \frac{1}{p}((\Phi_{p,(2)})^*(\omega_{\mathrm{FS},p,(2)}))|_U - \frac{\mathbf{i}}{2\pi} R^L|_U \right\|_{\mathcal{C}^\ell(U,h_p|_U)} \leqslant C_{\ell,U}\, p^{-\frac{1}{3}}\,, \tag{1.2.11}$$

for sufficiently large $p \in \mathbb{N}$. On compact subsets of Σ_2, where the curvature doesn't vanish, one may replace the fractional powers of p with -1, as in the classical version (see [43, Theorem 5.1.4]) of the theorem.

Later, Theorem E will be restated as Theorem 5.3.1. We give a proof in Chapter 5. The relevant norms in the statement of Theorem E will be defined in Section 2.5.

For a smooth holomorphic line bundle on a compact Kähler manifold, the induced Fubini-Study currents by the associated Kodaira maps converge to the Kähler form; in his celebrated theorem [52, Theorem A], Tian proved that this statement holds true in $\mathcal{C}^2$ topology with a speed estimate. Later, the theorem was refined by Ruan in [48] to hold in any $\mathcal{C}^\ell$-topology with an improved speed estimate.

It is worth mentioning that during that time, both Tian and Ruan didn't rely on the Bergman kernel, but instead used the peak section method. The latter is explained in [43, Definition 5.1.7].

In [17, Theorem 1.1], Coman and Marinescu prove that γ_p converges weakly to the first Chern class of a singular positive line bundle on a compact Kähler manifold under the condition that $\frac{1}{p}$ times the logarithm of the associated Bergman kernel function converges to 0 locally uniformly on the manifold, outside of the singular set of the metric.

In our next result from Chapter 5, we extend [47, Theorem 14] to our setting: In particular we show that the result of Marinescu and Savale hold when one allows the presence of singularities, or punctures of the underlying manifold, to exist. As mentioned before, a key consequence of the presence of punctures is that the manifold Σ is no longer compact, which affects our results on the convergences. The statement of our theorem is the following:

Theorem F (Local uniform convergence of induced Fubini-Study potentials)**.** *Let Σ be a punctured Riemann Surface and let (L, h^L) (E, h^E) be Hermitian holomorphic line bundles on $\overline{\Sigma}$ such that conditions (α) and (β) are satisfied. Let $U \subset \Sigma$ be a relatively compact subset. Then the following statements are true.*

(i) The normalized potentials of the Fubini-Study metric converge uniformly on U to the potential φ of h^L on K with speed $\mathcal{O}(p^{-1}\log p)$; that is, for each $\ell \in \mathbb{N}_0$, there exists a constant $C_{\ell,U} \in \mathbb{R}_{>0}$, such that:

$$\left\| \frac{1}{p}\varphi_{\mathrm{FS},p,(2)}|_U - \varphi|_U \right\|_{\mathcal{C}^\ell(U, h_p|_U)} \leqslant C_{\ell,U}\, p^{-1}\log p\,, \tag{1.2.12}$$

for all sufficiently large $p \in \mathbb{N}$.

(ii) The following ∂- and $\bar{\partial}$-derivatives of the normalized potentials of the Fubini-Study metric converge uniformly on U to the ∂- and $\bar{\partial}$-derivatives of the potential φ of h^L on U with the respective speeds; that is, for each $\ell \in \mathbb{N}_0$, there exists constants $C_{\ell,U,1}, C_{\ell,U,2}, C_{\ell,U,3} \in \mathbb{R}_{>0}$, such that:

$$\left\| \frac{1}{p}\, \partial\varphi_{\mathrm{FS},p,(2)}|_U - \partial\varphi|_U \right\|_{\mathcal{C}^\ell(U,h_p|_U)} \leqslant C_{\ell,U,1} p^{-2/3}\,, \tag{1.2.13}$$

$$\left\| \frac{1}{p}\bar{\partial}\varphi_{\mathrm{FS},p,(2)}|_U - \bar{\partial}\varphi|_U \right\|_{\mathcal{C}^\ell(U,h_p|_U)} \leqslant C_{\ell,U,2} p^{-2/3}\,, \tag{1.2.14}$$

$$\left\| \frac{1}{p}\, \partial\bar{\partial}\varphi_{\mathrm{FS},p,(2)}|_U - \partial\bar{\partial}\varphi|_U \right\|_{\mathcal{C}^\ell(U,h_p|_U)} \leqslant C_{\ell,U,3} p^{-1/3}\,, \tag{1.2.15}$$

for all sufficiently large $p \in \mathbb{N}$.

The following is true both for (i) and (ii): On compact subsets of Σ_2, where the curvature doesn't vanish, one may replace the fractional powers of p with -1, as in the classical version of the theorem.

Theorem F corresponds to Theorem 5.4.1. A proof is given in Chapter 5.

Finally, we conclude the following weak convergence of induced Fubini-Study currents to the semipositive curvature current R^L on $\overline{\Sigma}$, as an application of our analogues of the quotients of the Bergman kernels B_p and the model Bergman kernel

Theorem G. *Let Σ be a punctured Riemann Surface and let (L, h^L) (E, h^E) be Hermitian holomorphic line bundles on $\overline{\Sigma}$ such that conditions (α) and (β) are satisfied.*

Then the normalized induced Fubini-Study metrics converge weakly in the sense of currents to the normalized semipositive curvature current R^L on $\overline{\Sigma}$:

$$\frac{1}{p}(\Phi_{p,(2)})^*(\omega_{\mathrm{FS},p,(2)}) \rightharpoonup \frac{\mathbf{i}}{2\pi} R^L\,, \tag{1.2.16}$$

as $p \to \infty$.

Theorem G corresponds to Theorem 5.5.2. A proof is sketched in Section 5.5.

Our result is a generalization of [1, Theorem 4.3] to the case of a semipositive curvature and otherwise identical geometrical conditions.

1.2.4 Equidistribution

The study of equidistribution of zeros can be motivated by the following observation (see [3] for a survey): if the coefficients of a polynomial are subject to a random error, the positions its zeros will also be subject to a random error. A natural question to ask in this context is how the former error affects the latter. The appropriate framework to tackle this question is to consider polynomials whose coefficients are independent and identically distributed random variables (in a suitable space of polynomials) and then study the statistical properties of the (positions of the) zeros. One of particular interest in this thesis is the degree of uniformity of the distribution of zeros in the case of holomorphic sections $s \in H^0_{(2)}(\Sigma, L^p \otimes E)$, as $p \to \infty$.

The above circle of ideas has been applied by many authors in seminal works such as Bloch and Pólya [9], Littelwood and Offord [39], [40], [41], Erdős and Turán [28], Kac [36], [35] and Hammersley [31], to name a few. We refer the reader to the survey [3] for a more extensive list of references, as well as the relations of the study of equidistribution to other fields in mathematics and mathematical physics.

We first explain the terminology that we use: a sequence of points on the real number line $\mathbb{R}$ is equidistributed if the proportion of terms that fall into a (non-empty) interval is proportional to the size of the interval. This can be described in terms of distributions: for equidistributed sets of points, the value of integration against the counting measure of the points, when restricted to some interval I, can be estimated in terms of the integration of the indicator function on I against the Lebesgue measure.

When working over a manifold, the volume of a subset of the manifold is calculated by integrating against a volume form, in the case of a Kähler manifold, this volume form is related to the curvature of the corresponding line bundle. The equivalent problem of proving equidistribution of a set of points when these points are the support of a zero divisor of sections of a line bundle is the following: A sequence of currents of integration along a zero divisor of sections of a holomorphic line bundle is said to be equidistributed if it converges weakly to the first Chern class of the line bundle, i.e. the normalized (Chern) curvature.

In their paper [50], Shiffman and Zelditch proved that the zeros of holomorphic sections of high tensor powers of positive line bundles on a compact complex manifold converge almost surely to the volume form of the smooth metric. The same authors proceed to further consider the correlations between zeros and their variance (see Bleher, Shiffman and Zelditch [8], and Shiffman and Zelditch [51]). In [23], Dinh and Sibony delivered a new approach, using the formalism of meromorphic transforms, and used it to estimate the speed of convergence of zeros to the asymptotic distribution in the compact case. Their work improved estimates that were obtained in [50]. Dinh, Marinescu and Schmidt utilized the techniques from Dinh and Sibony to prove equidistribution hold on complete n-dimensional Hermitian manifolds, under mild conditions, including on the curvatures of the line bundle, and after assuming that the space of global holomorphic sections is finite dimensional and grows as a polynomial of degree n in the tensor power of the bundle.

In Chapter 6, we prove the following result on the almost sure convergence of the currents of integration along the zero loci of holomorphic sections $L^p \otimes E$ to the normalized curvature, as $p \to \infty$, on relatively compact subsets $U \subset \Sigma$:

Theorem H (Equidistribution of zeros of random holomorphic sections)**.** *Let Σ be a punctured Riemann surface, and let L be a holomorphic line bundle such that L carries a singular Hermitian metric h^L satisfying conditions (α) and (β). Let E be a holomorphic line bundle on Σ equipped with a smooth Hermitian metric h^E such that (E, h^E) on each chart V_j coincides with the trivial Hermitian line bundle. Then for μ-almost all $\mathbf{s} = \{s_p\}_{p \in \mathbb{N}} \in \Omega$, the sequence of currents converges weakly to the semipositive curvature form on relatively compact subsets $U \subset \Sigma$:*

$$\frac{1}{p}[\mathrm{Div}(s_p)]|_U \rightharpoonup \frac{\mathbf{i}}{2\pi} R^L|_U\,, \qquad \textit{as } p \longrightarrow \infty\,. \tag{1.2.17}$$

An analogue of our theorem has been proven in [2, Theorem 4.3] and more generally, in [27] by Drewitz, Liu and Marinescu for a larger class of probability measures. Each of the authors considered a punctured Riemann surface such as Σ, but with otherwise positive curvature globally. In our case, we only consider Gaussian probability measures (see Section 6.2). Our result extend [27, Theorem 1.5] for Gaussian measures and [2, Theorem 4.3] by allowing the curvature to vanish at most to finite order away from the punctures.

In [47, Theorem 4], Marinescu and Savale proof a similar result in the case of a compact Riemann surface with semipositive curvature that vanished at most to finite order.

As mentioned in the introductory section of Subsection 1.2.4, closely related to equidistribution is the study of the convergence speed of the weak convergence in the statement of Theorem 6.3.1. In Section 6.4, we follow and apply a method of Dinh, Marinescu and

Schmidt [22], Dinh, Ma and Marinescu [21], and Dinh and Sibony [23] to estimate the speed of convergence, dependent of the size of a subset that is cut out of the complex projective space of $\mathcal{L}^2$-holomorphic sections. The statement is the following:

Theorem I (Convergence speed of equidistribution of zeros)**.** *Let Σ be a punctured Riemann surface as above and (L, h^L) a Hermitian holomorphic line bundle with semipositive curvature which vanishes at most to finite order at any point. Then for any relatively compact open subset $U \subset \Sigma$ there exist $c_U > 0$ and $p(U) \in \mathbb{N}$ with the following property. For any sequence $(\lambda_p)_{p\in\mathbb{N}}$ of real numbers and for any $p \geqslant p(U)$ there exists a set $\Theta_p \subset \mathbb{P}(H^0_{(2)}(\Sigma, L^p \otimes E))$ such that:*

(a) $\sigma_{\mathrm{FS},p}(\Theta_p) \leqslant c_U p^2 e^{-\lambda_p/c_U}$,

(b) For any $s_p \in \mathbb{P}(H^0_{(2)}(\Sigma, L^p \otimes E)) \setminus \Theta_p$ and any relatively compact open subset $U \subset \Sigma$,

$$\left\| \frac{1}{p}\,[s_p = 0] - \frac{\sqrt{-1}}{2\pi} R^L \right\|_{U,-2} \leqslant \lambda_p p^{-1/3}\,.$$

On open sets U where the curvature is strictly positive, we can replace the term $p^{-1/3}$ by p^{-1} in the inequality above.

1.3 Organization of the thesis

The thesis is organized as follows:

In Chapter 2 we explain most of the notation that is used and the chapter also attempts to summarize common preliminaries, for example from differential geometry and functional analysis, that are needed throughout the thesis.

In Chapter 3, we introduce the Dirac and Kodaira Laplace operators and prove the existence of a gap in the spectrum of the latter.

Chapter 4 discusses the Bergman kernel and we shows that a pointwise on-diagonal asymptotic expansion holds globally on the non-compact manifold Σ.

In Chapter 5 we introduce the Kodaira map associated to a Hermitian metric in two separate cases which we will compare to each other: the original Hermitian metric h^L that has semipositive curvature R^L, and in the case of a positive Hermitian metric on Σ. Furthermore, we prove a theorem of Tian-Ruan in our setting and prove that the pull-backs by the Kodaira maps that are associated to h^L of the potentials of the Fubini-Study metrics converge globally uniformly to the potentials of h^L and we give an estimate on the speed of this convergence.

Finally in Chapter 6, after reminding the reader of the definition and some useful properties of currents and introducing terminology from probability theory that is needed for equidistribution, we state and prove that equidistribution holds. Moreover, we prove the convergence speed of the weakly convergence of currents of integration along the zero divisors to the normalized semipositive curvature by applying an approach of Dinh, Marinescu and Schmidt [22], which has its roots in ideas from Dinh and Sibony [23].

2. Geometric setting and preliminaries

In this chapter we elaborate on the geometric setting that is described in the introduction. We explain necessary preliminaries and define concepts that are central to the topics and problems explained in the forthcoming chapters.

2.1 Geometry on Σ

2.1.1 Geometric structures on Σ

We denote by $T\Sigma$ and $T^*\Sigma$ the real tangent and cotangent bundles over Σ, respectively. The complex structure J on Σ induces a splitting by bidegree on the complexified real tangent bundle of Σ:

$$T\Sigma \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)}\Sigma \otimes T^{(0,1)}\Sigma\,, \tag{2.1.1}$$

into two vector bundles that are the eigenspaces of the complexified endomorphism $J \otimes \mathrm{Id}$, associated to the eigenvalues $\mathbf{i}$ and $-\mathbf{i}$. The former is called the holomorphic tangent bundle and the latter the anti-holomorphic tangent bundle of Σ. There is a canonical choice of basis in each fiber, given by

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathbf{i}\frac{\partial}{\partial y}\right)\,, \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathbf{i}\frac{\partial}{\partial y}\right)\,, \tag{2.1.2}$$

respectively, in a local chart, where $x := \mathrm{Re}(z), y := \mathrm{Im}(z) \in \mathbb{R}$ are the real and imaginary part of $z \in \mathbb{C}$, respectively. For its corresponding dual bundles we write $T^{*(1,0)}\Sigma$ and $T^{*(0,1)}\Sigma$, respectively.

Let $g^{T\Sigma}$ denote the Riemannian metric on Σ that is associated to ω_Σ, i.e. $g^{T\Sigma}(J\cdot,\cdot) = \omega_\Sigma(\cdot,\cdot)$. This Riemannian metric $g^{T\Sigma}$ induces a Hermitian metric $h^{T^{(1,0)}\Sigma}$ on the holomorphic tangent bundle, i.e. a family of positive-definite Hermitian sesquilinear forms on each fiber. In local coordinates $(z, \bar{z})$, the Hermitian metric is a smooth map $\Sigma \to$

$(T^{1,0}\Sigma)^* \otimes (T^{0,1}\Sigma)^*)$ and satisfies

$$h^{T^{(1,0)}\Sigma} = h^{T^{(1,0)}\Sigma}_{z\bar{z}} \, \mathrm{d}z \otimes \mathrm{d}\bar{z} \tag{2.1.3}$$

for a suitable coefficient $h^{T^{(1,0)}\Sigma}_{z\bar{z}} \in \mathbb{C}$ that depends of the coordinate. Furthermore, the $(1,1)$-form ω_Σ that was defined in Section 1.1 satisfies

$$\omega_\Sigma = h^{T^{(1,0)}\Sigma}_{z\bar{z}} \, \mathrm{d}z \wedge \mathrm{d}\bar{z}, \tag{2.1.4}$$

i.e. ω_Σ is exactly the fundamental form of $h^{T^{(1,0)}\Sigma}$. The Hermitian $(1,1)$-form ω_Σ induces a Riemannian volume form, which in the case of our Riemann surface Σ is simply $\mathrm{d}v_\Sigma = \omega_\Sigma$.

Let L and E be holomorphic line bundles over $\overline{\Sigma}$. As mentioned in the Section 1.1, we equip E with a Hermitian metric h^E that is trivial near punctures and L with a singular Hermitian metric satisfying assumptions (α) and (β). For any $p \in \mathbb{N}$, we abbreviate the tensor power line bundles of L by $L^p := L^{\otimes p}$ and the Hermitian metrics h^L and h^E induce a metric on the twisted bundle $L^p \otimes E$ by

$$h_p := h^{L^p} \otimes h^E = (h^L)^{\otimes p} \otimes h^E \,. \tag{2.1.5}$$

2.1.2 Completeness and finite volume of (Σ, ω_Σ)

We wish to understand the geometry in our setting near the punctures, which is locally modelled by the Poincaré metric $\omega_{\mathbb{D}^*}$ on the punctures unit disc. This metric is complete: Fix $r_0 \in (0,1)$. Then

$$\int_0^{r_0} \frac{1}{r} \frac{1}{|\log r|} \, \mathrm{d}r = -\log\log \frac{1}{r}\Big|_{r=0}^{r_0} = \infty \,. \tag{2.1.6}$$

Consequently, by (β)(ii), the punctured Riemann surface (Σ, ω_Σ) is complete.

Denote by $\mathbb{D}^*_{r_0}$ the punctured disc of radius r_0. Then the total volume of $\mathbb{D}^*_{r_0}$ is calculated as

$$\int_0^{2\pi} \int_0^{r_0} \frac{1}{r^2} \frac{r}{|\log r|^2} \, \mathrm{d}r \, \mathrm{d}\theta = -2\pi \frac{1}{\log r}\Big|_{r=0}^{r_0} < \infty \,. \tag{2.1.7}$$

This implies that (Σ, ω_Σ) has finite volume.

Now observe that since $\overline{\Sigma} \setminus \bigcup_j \overline{V_j}$ is compact, by (2.1.6) we infer that the non-compact Riemann surface (Σ, ω_Σ) is complete.

2.2 Connections and curvature

In this section we review the concept of connections and curvature.

2.2.1 Hermitian and holomorphic connections

A connection on $L \to \Sigma$ is a complex linear map $\nabla^L : \mathcal{C}^\infty(\Sigma, L) \to \mathcal{C}^\infty(\Sigma, T^*\Sigma \otimes L)$ (see Section 2.5 for the definitions of function spaces) that is a derivation over scalar-valued functions with respect to the exterior differential operator d; that is for $s \in L$ and $\varphi \in \mathcal{C}^\infty(\Sigma, \mathbb{C})$ and $U \in T\Sigma$ the Leibnitz rule is satisfied in the following way:

$$\nabla^F(\varphi \cdot s) = \mathrm{d}\varphi \otimes s + \varphi \nabla^L s\,, \tag{2.2.1}$$

$$\nabla^L_U(\varphi \cdot s) = U(\varphi) \otimes s + \varphi \nabla^L_U s\,. \tag{2.2.2}$$

One can always construct connections by choosing an open covering together with a partition of unity and a choice of local frames, where the latter both are subordinated to the covering. This linear map ∇^L extends uniquely to differential forms.

A Hermitian connection is a connection that is compatible with the Hermitian metric: given a Hermitian metric h^L, compatibility between ∇^L and h^L means that the following relation holds

$$\mathrm{d}h^L(s_1, s_2) = h^L(\nabla^L s_1, s_2) + h^L(s_1, \nabla^L s_2)\,, \tag{2.2.3}$$

for all $s_1, s_2 \in \mathcal{C}^\infty(\Sigma, L)$. In the case of a singular Hermitian metric, we will demand the above condition (2.2.3) to hold almost everywhere.

If as in our case the line bundle L is holomorphic, we can consider holomorphic connections. These are connections, that satisfy

$$\nabla^L_U s = \iota_U(\bar{\partial}^L s)\,, \tag{2.2.4}$$

for all $U \in T^{(0,1)}\Sigma$ and $s \in \mathcal{C}^\infty(\Sigma, L)$, where ι is the contraction operator and $\bar{\partial}^L$ is the Dolbeault operator as defined in [20, Paragraph 3.C].

There exists a unique holomorphic Hermitian connection, called Chern connection (see [43, Subsection 1.1.2]).

In the case of a holomorphic connection, the following holds: in light of the splitting via bidegree, the connection decomposes as $\nabla^L = (\nabla^L)^{(1,0)} + (\nabla^L)^{(0,1)}$, where

$(\nabla^L)^{(0,1)} = \bar{\partial}^L$.

If ∇^L and ∇^E are the respective Chern connections, then the induced connection $\nabla^{L^p \otimes E}$ is exactly the Chern connection on $L^p \otimes E$ with respect to the induced Hermitian metric h_p.

2.2.2 Chern curvature

Note that for any $U, V \in T\Sigma$ and any $s \in \mathcal{C}^\infty(\Sigma, L)$, we have the relation

$$(\nabla^L)^2(U,V)s = \nabla^L_U \nabla^L_V s - \nabla^L_V \nabla^L_U s - \nabla^L_{[U,V]} s\,, \tag{2.2.5}$$

where $[U,V] = UV - VU$ is the Lie bracket. The map $(\nabla^F)^2 : L \to \Lambda^2 T^*\Sigma \otimes L$ defines a bundle morphism and there exists $R^L \in \mathcal{C}^\infty(\Sigma, \Lambda^2 T^*\Sigma)$ such that $(\nabla^L)^2 s = R^L s$ for all $s \in \mathcal{C}^\infty(\Sigma, L)$. R^L is called the curvature of ∇^L. In the case of a Hermitian connection, we also say that R^L is the curvature of h^L (or of the line bundle L).

Given our Hermitian metric h^L, by [43, Theorem 1.1.5], there exists a unique holomorphic Hermitian connection, called Chern connection. The associated curvature R^L is called Chern curvature. in our case, it is a form of bidegree $(1,1)$ such that $\mathbf{i}R^F$ is real at any point in Σ.

The first Chern form is then defined as

$$c_1(F, h^F) := \frac{\mathbf{i}}{2\pi} R^F\,. \tag{2.2.6}$$

Given a local frame, connections can locally be expressed in terms of their Christoffel symbols, which are 1-forms that describe how a frame element scales when expressed as the transport of a second frame element along a third frame element with respect to the connection. Given Hermitian line bundles $(T\Sigma, g^{T\Sigma})$ and (E, h^E) and (L, h^L) and singular Hermitian line bundle, with respective Hermitian connections $\nabla^{T\Sigma}$, ∇^E and ∇^L, locally in a suitable coordinate system, we have the Christoffel symbols

$$a_i^{\Lambda^{0,\bullet}} = \int_0^1 \rho x^j R_{ij}^{\Lambda^{0,\bullet}}(\rho x)\mathrm{d}\rho\,, \tag{2.2.7}$$

$$a_i^L = \int_0^1 \rho x^j R_{ij}^L(\rho x)\mathrm{d}\rho\,, \tag{2.2.8}$$

$$a_i^E = \int_0^1 \rho x^j R_{ij}^E(\rho x)\mathrm{d}\rho\,, \tag{2.2.9}$$

where $R^{\Lambda^{0,\bullet}} = \mathrm{d}a^{\Lambda^{0,\bullet}}$ is the curvature of the connection $\nabla^{\Lambda^{0,\bullet}}$ which is the extension of the connection $\nabla^{T\Sigma}$ to smooth forms of bidegree $(0,q)$, for all $q \in \mathbb{N}$. We will also study the product connection, which in terms of (2.2.7) - (2.2.9) reads:

$$\nabla^{\Lambda^{0,\bullet}\otimes L^p \otimes E} = \mathrm{d} + a^{\Lambda^{0,\bullet}} + pa^L + a^E \,. \tag{2.2.10}$$

2.3 Local potentials and singular Hermitian metric

For any Hermitian line bundle (L, h^L), the value of the norm of a local holomorphic frame $\mathbf{e}_L$ is given by

$$h^L(\mathbf{e}_L, \mathbf{e}_L) =: |\mathbf{e}_L|^2_{h^L} = e^{-2\varphi} \tag{2.3.1}$$

for a function $\varphi : \overline{\Sigma} \to \mathbb{R} \cup \{\pm\infty\}$ that is locally integrable on $\overline{\Sigma}$ and smooth on Σ; functions that arise as in (2.3.1) are called local weights or local potentials of the Hermitian metric in question. If the local potentials fail to be smooth on the $\overline{\Sigma}$ the metric is called a singular Hermitian metric, otherwise smooth Hermitian metric.

For any holomorphic section s of L with $s = f\mathbf{e}_L$ locally in terms of a local holomorphic frame, we then have $|s|^2_{h^L} = |f|^2 |\mathbf{e}_L|^2_{h^L}$ almost everywhere: we need to exclude points $x \in \overline{\Sigma}$, where $s(x) = 0$ and $|\mathbf{e}_L|_{h^L}(x) = \infty$, such as the puncture divisor D. Given an open covering $\overline{\Sigma} = \bigcup_\alpha U_\alpha$ and a local holomorphic frame $\mathbf{e}_L = \{(\mathbf{e}_L)_\alpha\}_\alpha$, the metric then gives rise to a family of local potentials φ_α which satisfy

$$\log |(\mathbf{e}_L)^2_\alpha|_{h^L} = \log |g_{\alpha\beta}|^2 + \log |(\mathbf{e}_L)_\beta|^2_{h^L} \,, \tag{2.3.2}$$

where $g_{\alpha\beta}$ is a cocycle on $U_\alpha \cap U_\beta$, which is holomorphic and non-vanishing. Consequently, we have $\partial\overline{\partial} \log |g_{\alpha\beta}|^2 = 0$. Therefore, because of the independence of the choice of local frames, the following relation characterizes the curvature R^L of (L, h^L) in terms of the local potentials:

$$R^L = 2\,\partial\overline{\partial}\varphi_\alpha \,. \tag{2.3.3}$$

This point of view will be used in Chapters 5 and 6 when we will calculate the asymptotic behavior from families of potentials that are associated to the line bundles $L^p \otimes E$, for growing $p \in \mathbb{N}$.

Remark 2.3.1. It is important to note that when the Hermitian metric is smooth, the associated curvature R^L is a $(1,1)$-form. This is no longer the case for singular Hermitian metrics: the appropriate perspective is to view R^L as a $(1,1)$-current on $\overline{\Sigma}$, that is, a $(1,1)$-form with coefficients that take values in the space of distributions (see Section 2.8

for a definition and elaborate discussion of currents).

2.4 Notions of positivity of line bundles

For line bundles on compact complex manifolds, there exists multiple distinct notions of positivity (see [43] for a discussion on this). Aside from notions that take into consideration the curvature of the line bundle, the so-called ampleness is of particular relevance to our subject, in particular in the discussion of the Kodaira maps associated to the Hermitian metrics in Chapter 5. We will recall the definition of ampleness in Subsection 5.2.2.

With our assumptions (α) and (β) the curvature R^L of the Hermitian holomorphic line bundle L with singular Hermitian metric h^L (as a bundle over $\overline{\Sigma}$) is allowed to vanish. On the other hand, semipositivity and an upper bound on the maximal order of vanishing of the curvature implies that there exist points where the curvature is strictly positive, i.e. $\Sigma_2 \neq \varnothing$. In fact, the set of the points Σ_2 where $\mathbf{i}R^L$ is strictly positive is an open dense subset of Σ. Consequently, when we view L as a holomorphic line bundle with singular Hermitian metric h^L over $\overline{\Sigma}$, we have

$$0 < \int_\Sigma \frac{\mathbf{i}}{2\pi} R^L \tag{2.4.1}$$

and

$$\deg_{\overline{\Sigma}}(L) := \int_{\overline{\Sigma}} c_1(L, h^L) = \int_{\overline{\Sigma}} \frac{\mathbf{i}}{2\pi} R^L = \int_\Sigma \frac{\mathbf{i}}{2\pi} R^L = \int_{\Sigma_2} \frac{\mathbf{i}}{2\pi} R^L\,, \tag{2.4.2}$$

i.e. L is said to have positive degree as a line bundle over $\overline{\Sigma}$.

2.5 Spaces of sections and topologies

In our calculation we make use of several different norms, which we will now define. Throughout this section, let $U \subset \Sigma$ be any subset.

For $k \in \mathbb{N}$, let $\mathcal{C}^k(U, L^p \otimes E)$ be the space of sections of $L^p \otimes E \to \Sigma$, whose partial derivatives of order $\leqslant k$ exist and are continuous. In all spaces of sections that we discuss, when considering functions over U, we will trim the notation by writing $\mathcal{C}^k(U) := \mathcal{C}^k(U, \mathbb{C})$. The support $\operatorname{supp}(s)$ of a section s is the topological closure of $\{x \in \Sigma : s(x) \neq 0\}$. Set $\mathcal{C}^\infty(U, L^p \otimes E) := \bigcap_{k\in\mathbb{N}_0} \mathcal{C}^k(U, L^p \otimes E)$ for the space of smooth sections and

$$\mathcal{C}^\alpha_c(U, L^p \otimes E) := \{s \in \mathcal{C}^\alpha(U, L^p \otimes E) : \operatorname{supp}(s) \text{ is compact}\}\,, \tag{2.5.1}$$

for the spaces of such sections with support contained in a compact set, where $\alpha \in \mathbb{N}_0 \cup \{\infty\}$.

Remark 2.5.1. Throughout the thesis, for all spaces of sections that we consider, the requirement of a compact support will always be indicated by adding a lower case letter c to the subscript.

Smooth sections are called test functions or test forms (with values in a line bundle), depending on the codomain. For $k \in \mathbb{N}$ and $s \in \mathcal{C}^\infty(\Sigma, L^p \otimes E)$, $x \in \Sigma$, set

$$|s|_{\mathcal{C}^k(h_p)}(x) := (|s|_{h_p} + |\nabla^{p,\Sigma} s|_{h_p,\omega_\Sigma} + \ldots + |(\nabla^{p,\Sigma})^k s|_{h_p,\omega_\Sigma})(x)\,, \tag{2.5.2}$$

where $\nabla^{p,\Sigma}$ is the connection on $(T\Sigma)^{\otimes l} \otimes L^p \otimes E$, for every $l \in \mathbb{N} \cup \{0\}$, induced by the Levi-Civita connection associated to ω_Σ and the Chern connection that corresponds to the metric h_p, and $|\cdot|_{h_p,\omega_\Sigma}$ denotes the norm of the Hermitian metric on $(T\Sigma)^{\otimes l} \otimes L^p \otimes E$ induced by $g^{T\Sigma}$ and h_p.

For any subset $U \subset \Sigma$, define the norm $\|\cdot\|_{\mathcal{C}^k(U,h_p)}$ on U as follows,

$$\|s\|_{\mathcal{C}^k(U,h_p)} := \sup_{x \in U} |s|_{\mathcal{C}^k(h_p)}(x). \tag{2.5.3}$$

If $U = \Sigma$, we abbreviate $\|s\|_{\mathcal{C}^k(h_p)} := \|s\|_{\mathcal{C}^k(\Sigma,h_p)}$.

Let

$$\Omega^m(\Sigma, L^p \otimes E) := \mathcal{C}^\infty(\Sigma, \Lambda^m(T\Sigma) \otimes L^p \otimes E) \tag{2.5.4}$$

be the spaces of smooth m-forms on Σ with values in $L^p \otimes E$, for any $p \in \mathbb{N}$. Similarly, let

$$\Omega^{r,q}(\Sigma, L^p \otimes E) := \mathcal{C}^\infty(\Sigma, \Lambda^r(T^{*(1,0)}\Sigma) \otimes \Lambda^q(T^{*(0,1)}\Sigma) \otimes L^p \otimes E) \tag{2.5.5}$$

be the spaces of smooth (r,q)-forms on Σ with values in $L^p \otimes E$, for any $p \in \mathbb{N}$. Note that there exists a splitting/Hodge decomposition as follows

$$\Omega^m(\Sigma, L^p \otimes E) = \bigoplus_{r+q=m} \Omega^{r,q}(\Sigma, L^p \otimes E)\,; \tag{2.5.6}$$

and we have $\mathcal{C}^\infty(\Sigma, L^p \otimes E) = \Omega^{0,0}(\Sigma, L^p \otimes E) = \Omega^0(\Sigma, L^p \otimes E)$. Moreover, the letters r and q in the bidigree satisfy $0 \leqslant r+q \leqslant 2$, since we work on a manifold of real dimension 2.

The choice of a Hermitian metric h_p on the line bundle $L^p \otimes E$ and a volume form $\omega_\Sigma$on the underlying manifold Σ allows us to define the space $\mathcal{L}^2(\Sigma, L^p \otimes E)$ of square-integrable sections of $L^p \otimes E$, together with an inner product/$\mathcal{L}^2$-metric via integration with respect to h_p and ω_Σ:

$$\langle s_1, s_2 \rangle_{\mathcal{L}^2(U,h_p)} := \int_U h_p(s_1, s_2)\omega_\Sigma \tag{2.5.7}$$

for all $U \subset \Sigma$ and $s_1, s_2 : \Sigma \to L^p \otimes E$, whenever the right hand side is well defined. For the induced norms we write $\langle s, s \rangle_{\mathcal{L}^2(U,h_p)} =: \|s\|_{\mathcal{L}^2(U,h_p)}$, for all $s : \Sigma \to L^p \otimes E$ such that (2.5.7) is finite, and abbreviate $\| \cdot \|_{\mathcal{L}^2(h_p)} := \| \cdot \|_{\mathcal{L}^2(\Sigma,h_p)}$.

For $p \geqslant 1$, we then denote by $\mathcal{L}^2(U, L^p \otimes E) := \mathcal{L}^2(U, \omega_\Sigma, L^p \otimes E, h_p)$ the space of sections over $U \subset \Sigma$ with values in $L^p \otimes E$ that are square integrable with respect to (2.5.7).

We define the Sobolev space and Sobolev norm as follows: for $k \geqslant 1$, let $\mathbf{H}^{2,k}(U, L^p \otimes E) := \mathbf{H}^{2,k}(U, \omega_\Sigma, L^p \otimes E, h_p)$ denote the Sobolev space of sections of $L^p \otimes E$ over U endowed with the Hermitian metric h_p over $\Sigma|_U$ whose derivatives up to order k exist and are integrable with respect to (2.5.7) over U, with respect to ω_Σ and h_p. For $s \in \mathbf{H}^{2,k}(U, L^p \otimes E)$, set

$$\|s\|^2_{\mathbf{H}^{2,k}_p(U,\omega_\Sigma,h_p)} := \int_U \left(|s|^2_{h_p} + |\nabla^{p,\Sigma} s|^2_{h_p,\omega_\Sigma} + \ldots + |(\nabla^{p,\Sigma})^k s|^2_{h_p,\omega_\Sigma} \right) \omega_\Sigma \,. \tag{2.5.8}$$

Equivalently, $\mathbf{H}^{2,k}(\Sigma, L^p \otimes E)$ is the $\| \cdot \|^2_{\mathbf{H}^{2,k}_p(\Sigma,\omega_\Sigma,h_p)}$-completion of the space of smooth sections of $L^p \otimes E \to \Sigma$ with compact support. For $k = 0$ we write $\| \cdot \|^2_{\mathbf{H}^{2,k}_p(\Sigma,\omega_\Sigma,h_p)} = \| \cdot \|^2_{\mathcal{L}^2_p(\Sigma,\omega_\Sigma,h_p)}$. Similarly, the space of restricted sections $s|_U : L|_U \to U \subset \Sigma$ is defined. We abbreviate $\| \cdot \|_{\mathbf{H}^{2,k}_p(\omega_\Sigma,h_p)} := \| \cdot \|_{\mathbf{H}^{2,k}_p(\Sigma,\omega_\Sigma,h_p)}$. When considering functions of this regularity, we allow ourselves to drop the letter p from the notation, since there is no dependence on p in these cases.

Remark 2.5.2. For all norms that we discuss, when no ambiguity arises, we drop h_p and ω_Σ from the notation for convenience.

2.6 Dolbeault cohomology and holomorphic sections

2.6.1 Dolbeault operator and Dolbeault cohomology

Denote by $\Omega^{r,q}_{(2)}(\Sigma, L^p \otimes E)$ the Hilbert space that is obtained by considering the completion of $\Omega^{r,q}_c(\Sigma, L^p \otimes E)$ with respect to $\| \cdot \|_{\mathcal{L}^2(h_p)}$. Note that for the bidegree $(0,0)$ we have the equalities $\mathcal{L}^2(\Sigma, L^p \otimes E) = \Omega^{0,0}_{(2)}(\Sigma, L^p \otimes E)$. We set $\Omega^{0,\bullet}_{(2)}(\Sigma, L^p \otimes E) = \bigoplus_q \Omega^{0,q}_{(2)}(\Sigma, L^p \otimes E)$.

Let $\overline{\partial}_p$ be the $\mathcal{L}^2$-Dolbeault operator on Σ acting on $\Omega^{0,\bullet}_{(2)}(\Sigma, L^p \otimes E)$. In general, the operator $\overline{\partial}_p$ acts on the larger space $\Omega^{0,\bullet}_c(\Sigma, L^p \otimes E)$ with range in $\Omega^{0,\bullet}_{(2)}(\Sigma, L^p \otimes E)$ and is preclosed, so there exist closed extensions of $\overline{\partial}_p$. We always identify $\overline{\partial}_p$ with its maximal extension, with domain given by

$$\mathrm{Dom}(\overline{\partial}_p) := \{s \in \Omega^{0,0}_{(2)}(\Sigma, L^p \otimes E) \,:\, \overline{\partial}_p s \in \Omega^{0,1}_{(2)}(\Sigma, L^p \otimes E)\}\,. \tag{2.6.1}$$

Then the unbounded linear operator $\overline{\partial}_p$, being its own maximal extension, is densely defined and closed (see [43, Lemma 3.1.1]).

Let $\overline{\partial}^*_p$ denote the maximal extension of the formal adjoint of $\overline{\partial}_p$ with respect to the $\mathcal{L}^2$-metric. Then, since (Σ, ω_Σ) is complete, $\overline{\partial}^*_p$ coincides with the Hilbert adjoint of $\overline{\partial}_p$ (see [43, Corollary 3.3.3]).

Recall the definition of a graph norm of a linear operator between Hilbert spaces. By Andreotti-Vesentini (see [43, Lemma 3.3.1]), $\Omega^{0,\bullet}_c(\Sigma, L^p \otimes E)$ is dense in the spaces of sections $\mathrm{Dom}(\overline{\partial}_p)$, $\mathrm{Dom}(\overline{\partial}^*_p)$, $\mathrm{Dom}(\overline{\partial}_p) \cap \mathrm{Dom}(\overline{\partial}^*_p)$ in the graph norms of $\overline{\partial}_p$ and $\overline{\partial}^*_p$ and $\overline{\partial}_p + \overline{\partial}^*_p$, respectively.

Now note that the square of the differential operator $\overline{\partial}_p$ vanishes identically. Thus there exists an associated cochain complex, the $\mathcal{L}^2$-Dolbeault complex such that $\overline{\partial}_p$ is its coboundary operator:

$$0 \to \Omega^{0,0}_{(2)}(\Sigma, L^p \otimes E) \xrightarrow{\overline{\partial}_p} \Omega^{0,1}_{(2)}(\Sigma, L^p \otimes E) \to 0\,. \tag{2.6.2}$$

Let $H^0(\Sigma, L^p \otimes E)$ be the space of sections s of $L^p \otimes E$ such that $\overline{\partial}_p s = 0$ and consider the subspace

$$H^0_{(2)}(\Sigma, L^p \otimes E) := H^0(\Sigma, L^p \otimes E) \cap \mathcal{L}^2(\Sigma, L^p \otimes E)\,, \tag{2.6.3}$$

of holomorphic sections of $L^p \otimes E$ that are square-integrable with respect to h^L.

2.6.2 Spaces of holomorphic sections

By [1, Remark 3.2], $\mathcal{L}^2$-bounded holomorphic sections of $L^p \otimes E$ on Σ extend to holomorphic sections of $L^p \otimes E$ over $\overline{\Sigma}$. We will now have a closer look from an algebraic point of view on the section space $H^0_{(2)}(\Sigma, L^p \otimes E)$.

If $p \geqslant 2$, then elements in $H^0_{(2)}(\Sigma, L^p \otimes E)$ are precisely those in $H^0(\overline{\Sigma}, L^p \otimes E)$ that vanish in D):

$$H^0_{(2)}(\Sigma, L^p \otimes E) \cong \{\sigma \in H^0(\overline{\Sigma}, L^p \otimes E) : \sigma|_D = 0\} \subset H^0(\overline{\Sigma}, L^p \otimes E)\,; \tag{2.6.4}$$

moreover, in general, $H^0(\overline{\Sigma}, L^p \otimes E)$ is strictly bigger than $H^0_{(2)}(\overline{\Sigma}, L^p \otimes E)$, as the former might contain sections that have no zero in D.

Let $\mathcal{O}_{\overline{\Sigma}}(D)$ be the holomorphic line bundle on $\overline{\Sigma}$ that is defined by the divisor $D = \sum_{j=1}^N a_j$ and let σ_D be the canonical section of $\mathcal{O}_{\overline{\Sigma}}(D)$. Now the isomorphism

$$\begin{aligned} H^0(\overline{\Sigma}, L^p \otimes E \otimes \mathcal{O}_{\overline{\Sigma}}(-D)) &\longrightarrow \{\sigma \in H^0(\overline{\Sigma}, L^p \otimes E) : \sigma|_D = 0\} \\ s &\longmapsto s \otimes \sigma_D \end{aligned} \tag{2.6.5}$$

gives an identification of vector spaces

$$H^0(\overline{\Sigma}, L^p \otimes E \otimes \mathcal{O}_{\overline{\Sigma}}(-D)) \otimes \sigma_D \cong H^0_{(2)}(\Sigma, L^p \otimes E) \subset H^0(\overline{\Sigma}, L^p \otimes E)\,. \tag{2.6.6}$$

Since the zero divisor of σ_D is D, the following sets coincide

$$\begin{aligned} &\{\sigma \in H^0_{(2)}(\Sigma, L^p \otimes E) : \sigma(x) = 0\} = \\ &\quad \{s \in H^0(\overline{\Sigma}, L^p \otimes E \otimes \mathcal{O}_{\overline{\Sigma}}(-D)) : s(x) = 0\} \otimes \sigma_D\,, \end{aligned} \tag{2.6.7}$$

or all $x \in \Sigma$. Note that by Riemann-Roch theorem the dimension is

$$\begin{aligned} d_p &:= \dim_{\mathbb{C}} H^0_{(2)}(\Sigma, L^p \otimes E) = \dim_{\mathbb{C}} H^0(\overline{\Sigma}, L^p \otimes E \otimes \mathcal{O}_{\overline{\Sigma}}(-D)) \\ &= \deg(L)p + \deg(E) - N + 1 - \mathrm{genus}(\overline{\Sigma})\,, \end{aligned} \tag{2.6.8}$$

which is a finite quantity.

Remark 2.6.1. From the above observation and the proof of [1, Lemma 3.1] we also see that the value of d_p might vary drastically when one wishes to change the underlying inner product, which is the behavior that one should expect for general non-compact Σ. This will become important to keep in mind later in Chapter 5, where we also consider different Hermitian metrics on L.

2.7 Bergman kernel

To define the Bergman kernel, we first recall that the space of $\mathcal{L}^2$-integrable holomorphic for complex domains $G \subset \mathbb{C}$ is closed and the Bergman condition (see (2.7.1) below) holds: one can show with the Cauchy integral formula and the

-inequality that

$$|f(z_0)| \leqslant \sup_{z\in K} |f(z)| \leqslant c_K \cdot \|f\|_{\mathcal{L}^2(G,\mathbb{C})} \tag{2.7.1}$$

for all z_0 in any compact subset $K \subset G$, where the constant c_K depends only on K; the norm on the right hand side will be defined in 2.5. This implies that the evaluation map

$$\mathcal{L}^2(G,\mathbb{C}) \supset H^0_{(2)}(G,\mathbb{C}) \ni f \mapsto f(z) \in \mathbb{C} \tag{2.7.2}$$

is a continuous linear map and families of holomorphic functions on domains are normal, in the sense that every sequence of such functions contains a subsequence which converges uniformly on compact subsets (on metric spaces this is equivalent to the usual definition of normal family that is a precompact subset of a set of continuous functions with respect to the compact-open topology). Hence every Cauchy-sequence (with respect to the norm coming from the $\mathcal{L}^2$-inner-product) is compact convergent and by completeness of square-integrable functions $G \to \mathbb{C}$ with respect to this $\mathcal{L}^2$-metric the limit function lies in $H^0_{(2)}(G,\mathbb{C})$. This means that these $\mathcal{L}^2$-bounded holomorphic functions form a closed subspace and hence a Hilbert space. Note that the same can be said about sections of our Riemann surfaces (which are paracompact by assumption), since they are locally represented by families of holomorphic functions with conformal transition conditions on intersections of open subsets of Σ.

Therefore, by (2.7.1) and (2.7.2), there exists a unique orthogonal projection map $\mathcal{L}^2(\Sigma, L^p \otimes E) \to H^0_{(2)}(\Sigma, L^p \otimes E)$, called the Bergman kernel. By Fischer-Riesz the Bergman kernel map has an unique Schwartz integral kernel that is a reproducing kernel for the space $H^0_{(2)}(\Sigma, L^p \otimes E)$: if $\{S_j\}_{1\leqslant j \leqslant d_p}$ is an orthonormal basis of $H^0_{(2)}(\Sigma, L^p \otimes E)$, it is given by

$$B_p(x,y) = \sum_{j=1}^{d_p} S_j^p(x) \otimes S_j^{p,*}(y) \in (L^p \otimes E)_x \otimes (L^p \otimes E)^*_y\,, \tag{2.7.3}$$

for $x, y \in \Sigma$, where $S_j^{p,*}(y)$ is the metric dual of $S_j^p(y)$ with respect to h_p. The on-diagonal Bergman kernel

$$B_p(x) := B_p(x,x) = \sum_{j=1}^{d_p} \|S_j^p(x)\|^2_{\mathcal{L}^2} \in \mathbb{R}\,, \tag{2.7.4}$$

for $x \in \Sigma$, is also called the Bergman kernel function. Since the $\overline{\partial}$-operator is elliptic and hence hypoelliptic, the Bergman kernel in two variables is of smooth regularity. As computed in [17, Lemma 3.1], one sees that it is positive and independent of the choice of basis and moreover, has the following variational characterization:

$$B_p(x) = \max_{\substack{\|S\|_{\mathcal{L}^2(h_p)}=1 \\ S\in H^0_{(2)}(\Sigma, L^p\otimes E)}} |S(x)|^2_{h_p} . \tag{2.7.5}$$

We will also be interested in the values of the logarithm of the Bergman kernel. In [17, Lemma 3.2(iii)] the authors show that $\log B_p \in \mathcal{L}^1_{\text{loc}}(\Sigma)$ is locally integrable, which will be used in multiple calculations in the later chapters of the thesis.

As mentioned in Subsection 1.2.2, given a positive line bundle over a compact Kähler manifold, Ma and Marinescu [43, Theorem 4.1.1] prove the following on-diagonal asymptotic expansion for the associated Bergman kernel B_p:

$$\left\| B_p(x,x) - \sum_{r=0}^{k} \mathbf{b}_r(x) p^{k-r} \right\|_{\mathcal{C}^\ell(h_p)} = \mathcal{O}(p^{n-k-1}) \tag{2.7.6}$$

for any $\ell \in \mathbb{N}$, with smooth coefficients $\mathbf{b}_r$ that are polynomials in the curvatures and their derivatives. Moreover, they explicitly calculated $\mathbf{b}_0$.

In the same author prove in [43, Theorem 6.1.1], under the mild assumptions [43, (6.1.1)] on a complete Hermitian manifold, that the associated Bergman kernel has the same asymptotic expansion as in (1.2.5) on compact subsets of the manifold.

Moreover, the authors go on to prove the existence of a full off-diagonal asymptotic expansion of the corresponding Bergman kernels and consider other more general cases, such as when the underlying manifold is symplectic, as well. However, in this thesis, we will only be interested in obtaining an on-diagonal expansion in our setting.

As mentioned in Subsection 1.2.2 Auvray, Ma and Marinescu study the same setting locally near the punctures of their punctures Riemann surface. The authors obtained an asymptotic expansion of B_p near the punctures in [1] by studying the asymptotic behavior of the Bergman kernel on the Poincaré punctured disc model. Moreover, the authors go on to prove an optimal global upper bound for B_p in [1, Corollary 1.4]. We argue that their bound holds in our case as well, which we summarize in our Corollary 4.1.3.

In [47, Theorem 1], Marinescu and Savale prove a pointwise on-diagonal asymptotic expansion with fractional powers in the exponents. Our result Theorem B (this is Theorem 4.1.1 in Chapter 4) is an extension of their asymptotics to our setting globally, with the same fractional exponents.

2.8 Currents on Σ

2.8.1 Spaces of currents, topologies and dual norms on Σ

Currents are differential forms with distribution coefficients. They carry the structure of a topological vector space and generalize various objects that are of interest in geometric analysis, such as functions, differential forms, measures and distributions. Furthermore, the target domain in the process of integration defines a current. Thus, currents interpolate between homological and cohomological objects. We give a rigorous construction of the space of currents.

For $0 \leqslant m \leqslant 2 = \dim_{\mathbb{R}} \Sigma$, let $\Omega_c^m(\Sigma, L^p \otimes E)$ be the vector space of compactly supported, smooth m-forms with values in $L^p \otimes E$, as defined in Chapter 2.

Let $0 \leqslant l = 2 - m \leqslant 2$, a l-current on Σ is a continuous linear functional T from $\Omega_c^m(\Sigma, L^p \otimes E)$ to $\mathbb{C}$. Equivalently, a current is a differential form that takes values in distributions.

Evaluation of a current T at some element $u \in \Omega_c^m(\Sigma, L^p \otimes E)$ will be denoted by $T(u) := (T, u)$, i.e. the natural pairing between an object and its algebraic dual.

The continuity of such linear functionals is to be understood in the following sense: for every sequence $(u_j)_{j\in\mathbb{N}} \subset \Omega_c^m(\Sigma, L^p \otimes E)$ such that there exists a compact subset $K \subset \Sigma$ with the property that $\operatorname{supp} u_j \subset K$ for all $j \in \mathbb{N}$ and the sequence $(u_j)_{j\in\mathbb{N}}$ converges to some $u \in \Omega_c^m(\Sigma, L^p \otimes E)$ uniformly as $j \to \infty$, we have $(T, u_j) \to (T, u)$, as $j \to \infty$.

An equivalent way of defining currents that says more about the involved topologies is the following: to every compact subset $K \subset \Sigma$, we associate a seminorm by

$$p_K(u) := \sup_{x\in K} \max_{|I|=m} \left| \frac{\partial^\alpha}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} u_I(x) \right| , \tag{2.8.1}$$

where $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ is a multi-index. For each K, we define $\mathcal{D}^m(K, L^p \otimes E)$ to be the space $\Omega_c^m(K, L^p \otimes E)$ equipped with the topology that induced by any finite set of seminorms p_{K_j}, such that the compact subsets K_j cover K; and set $\mathcal{D}^m(\Sigma, L^p \otimes E) := \bigcup_K \mathcal{D}^m(K, L^p \otimes E)$.

Similarly, we define $\mathcal{E}^m(\Sigma, L^p \otimes E)$ to be the space $\Omega^m(K, L^p \otimes E)$ of smooth m-forms, not necessarily with compact support, equipped with the same topology as above.

The space of currents of dimension m (or equivalently, of degree $2-m$) is the topological dual space $\mathcal{D}'_m(\Sigma, L^p \otimes E)$ of linear forms T on $\mathcal{D}^m(\Sigma, L^p \otimes E)$ with respect to the topology in which the restriction of T to all subspaces $\mathcal{D}^m(K, L^p \otimes E)$, for all compact $K \subset \Sigma$, is continuous. Currents of degree $2-m$ will also be called m-currents for brevity.

The support $\operatorname{supp} T$ of a current T is defined as the smallest closed subset $U \subset \Sigma$, such that the restriction of T to $\mathcal{D}'_m(\Sigma \setminus U, L^p \otimes E)$ is identically zero. One can drop the condition of a compact support of the differential forms that the currents are acting on if one requires the currents themselves to have compact support; the definition carries over to this case without further adjustments.

The space of currents of compact support will be denoted by $\mathcal{E}'_m(\Sigma, L^p \otimes E)$. The same definitions can be used for spaces of currents over $\overline{\Sigma}$; the requirement of a compact support of either the smooth forms or the currents themselves is not needed in this case.

The regularity of a current is described by the following lemma, the proof of which is straightforward and similar to its analogue from the theory of distributions:

Lemma 2.8.1. *A linear functional T on $\mathcal{D}^m(\Sigma, L^p \otimes E)$ is a current if and only if for every compact $K \subset \Omega$, there exists a constant $C_K =: C > 0$, such that*

$$|(T, u)| \leqslant C \|u\|_{\mathcal{C}^k_c(K, \Lambda^m T\Sigma \otimes L^p \otimes E)} \tag{2.8.2}$$

for every m-form u of regularity $\mathcal{C}^k$ with $\operatorname{supp} u \subset K$.

A current T for which the integer k in (2.8.2) can be chosen independently of K is said to have finite order. In this case, the smallest finite (non-negative) integer k such that (2.8.2) holds for all compact subsets $K \subset \Sigma$ is called the order of T.

Currents of compact support are naturally of finite order.

The vector space of real currents satisfies a Hodge decomposition

$$\mathcal{D}'_m(\Sigma, L^p \otimes E) = \bigoplus_{q+r=m} \mathcal{D}'_{q,r}(\Sigma, L^p \otimes E)\,; \tag{2.8.3}$$

elements of $\mathcal{D}'_{q,r}(\Sigma, L^p \otimes E)$ are called currents of bidimension (q, r) (or equivalently, of bidegree $(2-q, 2-r)$). Currents of compact support split similarly.

On the topological closure $\overline{\Sigma}$, which is a compact Riemann surface, one can define the following seminorms on the set of currents T of order 0 on $\overline{\Sigma}$; for $U \subset \Sigma$, we set

$$||T||_{U,-\alpha} := \sup_{x \in U} |(T, u)| \,, \tag{2.8.4}$$

where the supremum is taken over smooth test forms u with support in U, that lie on the closed unit disc with respect to the $\mathcal{C}^{\alpha}$ norm that was defined in Section 2.5.

For $\alpha = 0$ we obtain the usual notion of mass of a current, which we denote by $\|T\|_U$. From the definition (2.8.4) it is clear that

$$\|T\|_{U,-\alpha} \geqslant \|T\|_{U,-\beta} \,, \tag{2.8.5}$$

if $\beta \geqslant \alpha$.

Later, when working with the norm (2.8.4), the only case that we we will consider is $\alpha = 2$ and we will compute estimates of the $\| \cdot \|_{U,-2}$-norm of differences of certain currents (for instance, see Theorem 6.4.1). However, other cases can be obtained as a consequence of our estimates by the theory of interpolation between Banach spaces [53]: let $W \cup \Sigma$ be an relatively compact open subset of Σ, such that $U \cup W$ is a relatively compact open subset of W, then

$$\|T\|_{U,-\alpha} \leqslant c\|T\|_W^{1-\frac{\alpha}{\beta}} \|T\|_{W,-\beta}^{\frac{\alpha}{\beta}} \,, \tag{2.8.6}$$

for $0 < \beta \leqslant \alpha \leqslant 1$, where the constant $c > 0$ is independent of T (see [24]).

In the case $U = \overline{\Sigma}$ the norm in (2.8.4) and the induced topology coincides with the weak topology on any set of currents (on $\overline{\Sigma}$) with mass bounded by a fixed constant, i.e. the topology in the definition we have given above for the spaced of currents when defined over the topological closure $\overline{\Sigma}$.

2.8.2 Closed and positive currents

The wedge product of a current and a differential form is defined by duality (see [20, Paragraph 2.B.2]).

We review the definition of the exterior derivative on currents (see [20, Paragraph

2.B.1]): recall that for $0 \leqslant m \leqslant 2$, if $\beta \in \mathcal{E}^m(\Sigma, L^p \otimes E)$ and $\alpha \in \mathcal{D}^m(\Sigma, L^p \otimes E)$, then by the classical Stokes formula, we have $(\mathrm{d}\beta, \alpha) = (-1)^{m+1}(\beta, \mathrm{d}\alpha)$. This motivates the following definition for the exterior derivative on currents: the $\mathrm{d}T$ of a current T of degree l on Σ is a current of degree $l+1$ defined intrinsically by

$$(\mathrm{d}T, \alpha) := (-1)^{l+1}(T, \mathrm{d}\alpha)\,, \tag{2.8.7}$$

for $\alpha \in \mathcal{D}^{2-(l+1)}(\Sigma)$. The map $T \mapsto \mathrm{d}T$ is continuous for the topology of currents that we have defined above.

The exterior derivative is defined analogously for currents that have compact support. We say that a current is closed, if $\mathrm{d}T = 0$, i.e. if it lies in the kernel of the map d. Currents of maximal degree, i.e. distributions, are always closed currents.

A current T is called exact if there exists another current S, such that $\mathrm{d}S = T$, i.e. T is exact if and only if it lies in the image of the map d.

For a current T of bidegree (r, q), $\mathrm{d}T$ can be decomposed as a sum $\partial T + \bar{\partial} T$, where ∂T is a $(r+1, q)$-current and $\bar{\partial} T$ is a $(r, q+1)$-current. Note that $\mathrm{d}(\mathrm{d}T) = 0$ and hence $\partial(\partial T) = \bar{\partial}(\bar{\partial} T) = 0$ and $\partial\bar{\partial} T = -\bar{\partial}\,\partial T$. Furthermore,

$$(\partial T, \alpha) := (-1)^{r+q+1}(T, \partial\alpha)\,, (\bar{\partial} T, \alpha) := (-1)^{r+q+1}(T, \bar{\partial}\alpha) \qquad \text{for } \alpha \in \mathcal{D}^{2-(r+q+1)}(\Sigma)\,. \tag{2.8.8}$$

The conjugate $\overline{T}$ is defined by $(\overline{T}, \alpha) := \overline{(T, \overline{\alpha})}$ for all differential forms α of suitable degree or bidegree. If $T = \overline{T}$ holds, then T is a real current.

There is a notion of positivity for currents: First, Recall that a real $(1, 1)$-form α is positive if at any point on Σ it coincides with a linear combination of semipositive $(1, 1)$-forms with (real) positive coefficients.

A (q, r)-current is called positive (respectively weakly positive), if $(T, \alpha) \geqslant 0$ for every weakly positive (respectively positive) test form α of bidegree $(1-q, 1-r)$. Positive currents of maximal bidegree are positive measures.

If a current T is positive and of bidegree (q, q) on (Σ, ω_Σ), the following notion of mass is equivalent to the one we have defined in (2.8.4):

$$||T|| = (T, \omega_\Sigma^{1-q})\,; \tag{2.8.9}$$

if the current at hand is closed, their mass (in the sense of (2.8.9)) is invariant under addition of an exact current.

Examples 2.8.2. (i) Distributions define currents of top degree (or bidigree in the case of a complex distribution).

(ii) Differential forms u with locally $\mathcal{L}^1$-integrable coefficients define currents T_u of order 0. The map $u \mapsto T_u$ associating a $\mathcal{L}^1_{\mathrm{loc}}$-differential form to a current is injective. In the same way that $\mathcal{L}^1_{\mathrm{loc}}$-functions are usually identified to the distributions that they induce, we will identify u to the its image T_u in the space of currents.

(iii) We recall an important example of currents in the the general case of a Kähler manifold X with $\dim_{\mathbb{C}} X = m \in \mathbb{N}$. Then the following are called currents of integration on a complex submanifold $M \subset X$: let $1 \leqslant q = \dim_{\mathbb{C}} M \leqslant m$ be the dimension of the submanifold M and equip M with its canonical, induced orientation. The current of integration $[M]$ on M is defined by

$$([M], u) := \int_M u \,, \tag{2.8.10}$$

for all $u \in \Omega_c^{q,q}(\Sigma)$. This is a positive current of bidimension (q,q) on Σ, with $\operatorname{supp}[M] = M$. Moreover, by the theorem of Stokes, we have $\mathrm{d}[M] = \pm[\partial M] = 0$ (see [20, Paragraph 1.20]).

In the case of a Riemann surface, the only interesting case is the case of a current of integration along a submanifold of dimension 1: let $\Gamma \subset \Sigma$ (or $\Gamma \subset \overline{\Sigma}$) be a set of points, then the current of integration along Γ is $[\Gamma] = \sum_{x\in\Gamma} \delta_x$, where δ_x is the Dirac delta distribution. For any smooth, compactly supported function on Σ (or $\overline{\Sigma}$, we have $([\Gamma], \varphi) = \sum_{x\in\Gamma} \varphi(x)$.

3. Estimates for the Dirac Operator and Spectral Gap of the Kodaira Laplacian

3.1 Dirac and Kodaira Laplace operators

In this chapter we establish an estimate for the Dirac operator D_p and proof the spectral gap of the Kodaira Laplacian $\square_p$ (both defined in (1.2.1)).

Note that $\square_p : \Omega^{0,\bullet}_{\mathrm{c}}(\Sigma, L^p \otimes E) \to \Omega^{0,\bullet}_{\mathrm{c}}(\Sigma, L^p \otimes E)$ is essentially self-adjoint, so that it has a unique self-adjoint extension which we still denote by $\square_p$, the domain of this extension is

$$\mathrm{Dom}(\square_p) = \{s \in \Omega^{0,\bullet}_{(2)}(\Sigma, L^p \otimes E) \,:\, \square_p(s) \in \Omega^{0,\bullet}_{(2)}(\Sigma, L^p \otimes E)\}\,. \tag{3.1.1}$$

From their definition, it can be seen that D_p interchanges and $\square_p$ preserves the $\mathbb{Z}$-grading of $\Omega^{0,\bullet}_{\mathrm{c}}(\Sigma, L^p \otimes E)$. Since Σ has complex dimension 1, we have

$$\begin{aligned}
D_{p|_{\Omega^{0,0}(\Sigma,L^p\otimes E)}} &= \sqrt{2}\;\overline{\partial}_p\,,\\
D_{p|_{\Omega^{0,1}(\Sigma,L^p\otimes E)}} &= \sqrt{2}\;\overline{\partial}^*_p\,,\\
\square^0_p := \square_{p|_{\Omega^{0,0}(\Sigma,L^p\otimes E)}} &= \overline{\partial}^*_p\overline{\partial}_p\,,\\
\square^1_p := \square_{p|_{\Omega^{0,1}(\Sigma,L^p\otimes E)}} &= \overline{\partial}_p\overline{\partial}^*_p\,.
\end{aligned} \tag{3.1.2}$$

Unlike in the case of compact Kähler manifolds, it is not trivially true in the non-compact setting for $\overline{\partial}_p$ and $\overline{\partial}^*_p$ to have closed range. Since $\overline{\partial}_p$ is closed (and so is $\overline{\partial}^*_p$), however, it is enough that this holds for one of them. This condition is desirable, since we want to talk about the quotient space

$$H^q_{(2)}(\Sigma, L^p \otimes E) := {}^{\ker \overline{\partial}^q_p}\!\big/_{\mathrm{im}\overline{\partial}^{q-1}_p} \tag{3.1.3}$$

where $\overline{\partial}_p^q := \overline{\partial}_p : \Omega^{0,q}_{(2)}(\Sigma, L^p \otimes E) \to \Omega^{0,q+1}_{(2)}(\Sigma, L^p \otimes E)$. For $q = 0$ we just identify the left hand side of (3.1.3) to the space of $\mathcal{L}^2$-bounded holomorphic sections of $L^p \otimes E$, defined in (2.6.3). By completeness of (Σ, ω_Σ), $\overline{\partial}_p^1$ has closed range, so the cokernel in (3.1.3) is separable in the topological sense; furthermore, we have an isomorphism

$$\ker \square_p^q \cong H^q_{(2)}(\Sigma, L^p \otimes E)\,, \tag{3.1.4}$$

for $q = 0, 1$, which shows that the Kodaira Laplacian is well-suited for studying the spaces of holomorphic sections and $(0,1)$-forms with values in $L^p \otimes E$. One of the consequences in the existence of a gap in the spectrum of the Kodaira Laplacian is that the Bergman kernel can then be represented in terms an integral of the resolvent of $\square_p$.

3.1.1 Clifford action and Lichnerowicz formula

For $x \in \Sigma$, $v \in T_x\Sigma$, by the splitting (2.1.1), we write $v = v^{(1,0)} + v^{(0,1)} \in T_x^{(1,0)}\Sigma \oplus T_x^{(0,1)}\Sigma$; we denote by $\overline{v}^{(1,0)*} \in T_x^{(0,1)*}\Sigma$ the metric dual of $v^{(1,0)}$. The Clifford multiplication endomorphism $c : T_x\Sigma \to \mathrm{End}(\Lambda^\bullet(T_x^{*(0,1)}\Sigma))$ is then defined as

$$v \mapsto c(v) := \sqrt{2}(\overline{v}^{(1,0)*} \wedge -\iota_{v^{(0,1)}})\,. \tag{3.1.5}$$

If $\{e_1, e_2\}$ is a local orthonormal frame of $(T\Sigma, g^{T\Sigma})$, then the Dirac operators in (3.1.2) can then be written as follows:

$$D_p = \sum_{j=1}^{2} c(e_j) \nabla_{e_j}^{\Lambda^{0,\bullet} \otimes L^p \otimes E}, \tag{3.1.6}$$

where $\nabla^{\Lambda^{0,\bullet} \otimes L^p \otimes E}$ is the product connection that was introduced in (2.2.10).

Set $\omega = \frac{1}{\sqrt{2}}(e_1 - \mathbf{i}e_2)$ which is an orthonormal frame of $T^{(1,0)}\Sigma$. Let $\overline{\omega}^*$ denote the metric dual of ω. By [43, Theorem 1.4.7], let $\Delta^{\Lambda^{0,\bullet} \otimes L^p \otimes E}$ denote the Bochner Laplacian associated with $\nabla^{\Lambda^{0,\bullet} \otimes L^p \otimes E}$, i.e.

$$\Delta^{\Lambda^{0,\bullet} \otimes L^p \otimes E} := \left(\nabla^{\Lambda^{0,\bullet} \otimes L^p \otimes E}\right)^* \nabla^{\Lambda^{0,\bullet} \otimes L^p \otimes E}\,. \tag{3.1.7}$$

Then we have the following Lichnerowicz formula for $\square_p$:

$$\begin{aligned} \square_p = \frac{1}{2}\Delta^{\Lambda^{0,\bullet} \otimes L^p \otimes E} + \frac{r^\Sigma}{4}\overline{\omega}^* \wedge \iota_{\overline{\omega}} + p(R^L(\omega, \overline{\omega})\overline{\omega}^* \wedge \iota_{\overline{\omega}} - \frac{1}{2}R^L(\omega, \overline{\omega})) \\ + R^E(\omega, \overline{\omega})\overline{\omega}^* \wedge \iota_{\overline{\omega}} - \frac{1}{2}R^E(\omega, \overline{\omega})\,, \end{aligned} \tag{3.1.8}$$

where $r^\Sigma = 2R^{T^{(1,0)}\Sigma}(\omega,\overline{\omega})$ is the scalar curvature of $(\Sigma, g^{T\Sigma})$. Note that r^Σ is a bounded function on Σ which is constant near punctures. In particular, near the punctures, we have

$$R^E(\omega,\overline{\omega})\overline{\omega}^* \wedge \iota_{\overline{\omega}} - \frac{1}{2}R^E(\omega,\overline{\omega}) = 0\,. \tag{3.1.9}$$

3.2 Estimates and Spectral Gap

Now we consider the action of $\square_p$ on $\Omega^{0,1}_{\mathrm{c}}(\Sigma, L^p\otimes E)$. Since we assume that $\mathbf{i}R^L$ is nonnegative, i.e., $R^L(\omega,\overline{\omega}) \geqslant 0$, then, on $(0,1)$-forms,

$$p\left(R^L(\omega,\overline{\omega})\overline{\omega}^* \wedge \iota_{\overline{\omega}} - \frac{1}{2}R^L(\omega,\overline{\omega})\right) \geqslant \frac{1}{2}pR^L(\omega,\overline{\omega}) \geqslant 0\,. \tag{3.2.1}$$

For the points where R^L does not vanish, the above term therefore admits a local lower bound growing linearly in p.

We allow R^L to vanish up to a finite order and with this assumption, in [47, Proposition 6], Marinescu and Savale proved that for a compact subset $K \subset \Sigma$, there exist constants $C_1 \in \mathbb{R}_{>0}$, $C_2 \in \mathbb{R}_{>0}$ such that for sufficiently large $p > 1$ and for $s \in \Omega^{0,1}_{\mathrm{c}}(\Sigma, L^p\otimes E)$ with $\mathrm{supp}(s) \subset K$,

$$(C_1 p^{2/\rho_\Sigma} - C_2)\|s\|_{\mathcal{L}^2} \leqslant \left\|\frac{1}{2}\Delta^{\Lambda^{0,\bullet}\otimes L^p\otimes E}s\right\|_{\mathcal{L}^2}. \tag{3.2.2}$$

The key idea to obtain the above inequality is the sub-elliptic estimate for the sub-Riemannian Laplacian on the circle bundle of $L^p\otimes E$ on Σ .

We will combine the above considerations to prove the following theorem that establishes the spectral gap for the Kodaira Laplacian.

Theorem 3.2.1 (Spectral Gap and vanishing first cohomology)**.** *Let Σ be a punctured Riemann surface, and let L be a holomorphic line bundle such that L carries a singular Hermitian metric h^L satisfying conditions (α) and (β). Let E be a holomorphic line bundle on Σ equipped with a smooth Hermitian metric h^E such that (E, h^E) on each chart V_j coincides with the trivial Hermitian line bundle. Consider the Dirac and Kodaira Laplace operators as defined in* (1.2.1)*. Then there exist constants $C_1, C_2 \in \mathbb{R}_{>0}$ independent of p, such that for all $s \in \Omega^{0,1}_{\mathrm{c}}(\Sigma, L^p\otimes E)$,*

(i) the Dirac operators are bounded from below,

$$\|D_p s\|^2_{\mathcal{L}^2} \geqslant 2(C_1 p^{2/\rho_\Sigma} - C_2)\|s\|^2_{\mathcal{L}^2}\,, \tag{3.2.3}$$

(ii) for $p \in \mathbb{N}$, we have

$$\mathrm{Spec}(\Box_p) \subset \{0\} \cup [C_1 p^{2/\rho_\Sigma} - C_2, \infty)\,. \tag{3.2.4}$$

In particular, the first $\mathcal{L}^2$-Dolbeault cohomology group $H^1_{(2)}(\Sigma, L^p \otimes E) = 0$ vanishes for sufficiently large $p > 0$.

Proof of Theorem 3.2.1. In scope of this proof, for $s \in \Omega^{0,1}_{\mathrm{c}}(\Sigma, L^p \otimes E)$ and a domain $A \subset \Sigma$, set

$$\|s\|_A^2 := \int_A |s|^2_{h_p} \omega_\Sigma\,; \tag{3.2.5}$$

observe that $A \subset B$ implies $\|\cdot\|_A \leqslant \|\cdot\|_B$. We fix a compact subset K of Σ such that outside of K we have $\mathbf{i}R^L > c_K \omega_\Sigma$ with some constant $c_K \in \mathbb{R}_{>0}$. Then R^L can only vanish at the points in K. Let $U \subset \Sigma$ be an open relatively compact neighborhood of K. Take smooth functions $\phi_1, \phi_2 : \Sigma \to [0,1]$ such that

$$\mathrm{supp}\,(\phi_1) \subset U\,, \tag{3.2.6}$$

$$\mathrm{supp}\,(\phi_2) \subset \Sigma \setminus K\,, \tag{3.2.7}$$

with $\phi_1 \equiv 1$ on K and $\phi_1^2 + \phi_2^2 \equiv 1$ on Σ. Note that near the punctures, ϕ_2 takes the constant value 1, then $|\overline{\partial}\phi_2|^2_{h_p, g^{T^{*(0,1)}\Sigma}} < \infty$, where the norm is defined in (2.5.3) The assumption on (E, h^E) that it is the trivial line bundle near punctures implies that there exists a constant $c_0 \in \mathbb{R}_{>0}$ such that for $x \in \Sigma$, we have

$$R^E(\omega, \overline{\omega})\overline{\omega}^* \wedge \iota_{\overline{\omega}} - \frac{1}{2} R^E(\omega, \overline{\omega}) \geqslant -c_0 \mathrm{Id}_{T^{*(0,1)}\Sigma \otimes L^p \otimes E}\,. \tag{3.2.8}$$

At first, we apply (3.2.2) to the sections with support contained in U. Then by (3.1.8), (3.2.1), (3.2.8) and using the same arguments as in [47, Proposition 7], we get that there exist constant $c_1, c_2 \in \mathbb{R}_{>0}$ such that for $s \in \Omega^{0,1}_{\mathrm{c}}(\Sigma, L^p \otimes E)$,

$$(c_1 p^{2/\rho_\Sigma} - c_2)||\phi_1 s||_U^2 \leqslant ||\overline{\partial}_p^*(\phi_1 s)||_U^2\,. \tag{3.2.9}$$

On the other hand, since $\mathbf{i}R^L(\omega, \overline{\omega}) > c_K \omega_\Sigma$ on the support of ϕ_2, then by (3.2.8) and [43, Theorem 6.1.1, (6.1.7)], there exists a constant $c_3 \in \mathbb{R}_{>0}$, such that for sufficiently large $p \in \mathbb{N}$

$$c_3 p ||\phi_2 s||^2_{\Sigma\setminus K} \leqslant ||\overline{\partial}_p^*(\phi_2 s)||^2_{\Sigma \setminus K}\,. \tag{3.2.10}$$

Let $\nabla^{\Lambda^{0,\bullet} \otimes L^p \otimes E}$ be the connection on $\Lambda^\bullet(T^{*(0,1)}\Sigma) \otimes L^p \otimes E$ that is induced by the holomorphic Hermitian connection $\nabla^{T^{(1,0)}\Sigma}$ and $\nabla^{L^p\otimes E}$, and let $0 \neq w \in T^{(1,0)}\Sigma$ be any local orthonormal frame, defined on some open set V. Because our Riemann surface Σ is a Kähler manifold, by [43, Lemma 1.4.4], we have locally $\bar{\partial}_p^* = -\iota_{\overline{w}}\nabla_{\overline{w}}^{\Lambda^{0,\bullet}\otimes L^p\otimes E}$ for $p \in \mathbb{N}$. As a consequence, we get, for $(\star, j) = (U, 1)$ or $(\Sigma \setminus K, 2)$ and $p \in \mathbb{N}$,

$$\|\bar{\partial}_p^*(\phi_j s)\|_\star^2 \leqslant |\bar{\partial}\phi_j|^2_{h_p, g^{T^*(0,1)\Sigma}} \cdot \|s\|^2_{\mathcal{L}^2} + \|\phi_j \bar{\partial}_p^* s\|^2_{\mathcal{L}^2} \,. \tag{3.2.11}$$

Combining (3.2.9) - (3.2.11), for sufficiently large $p \in \mathbb{N}$,

$$\left(\min\left\{c_1 p^{2/\rho_\Sigma} - c_2, c_3 p\right\} - |\bar{\partial}\phi_1|^2_{h_p, g^{T^*(0,1)\Sigma}} - |\bar{\partial}\phi_2|^2_{h_p, g^{T^*(0,1)\Sigma}}\right) \|s\|^2_{\mathcal{L}^2} \leqslant \|D_p s\|^2_{\mathcal{L}^2} \,. \tag{3.2.12}$$

Since $\rho_\Sigma \geqslant 2$, the above inequality implies that there exist constants $C_1, C_2 \in \mathbb{R}_{>0}$ such that for $p \in \mathbb{N}$,

$$||D_p s||^2_{\mathcal{L}^2} \geqslant 2(C_1 p^{2/\rho_\Sigma} - C_2)||s||^2_{\mathcal{L}^2} \,. \tag{3.2.13}$$

This proves the desired inequality (3.2.3) for the Dirac operators.

Observe that $\mathrm{Spec}(\Box_p) = \mathrm{Spec}(\Box_p^0) \cup \mathrm{Spec}(\Box_p^1) \subset \mathbb{R}_{\geqslant 0}$. For $s \in \Omega_{\mathrm{c}}^{0,1}(\Sigma, L^p \otimes E)$,

$$\|D_p s\|^2_{\mathcal{L}^2} = 2\langle \Box_p s, s\rangle. \tag{3.2.14}$$

Then by (3.2.3), we get $\mathrm{Spec}(\Box_p^1) \subset [C_1 p^{2/\rho_\Sigma} - C_2, \infty)$, and $H^1_{(2)}(\Sigma, L^p \otimes E) = 0$ for $p \gg 0$. For $s \in \Omega_{\mathrm{c}}^{(0,0)}(\Sigma, L^p \otimes E)$, applying (3.2.3) to $\bar{\partial}_p s$ gives

$$\|\Box_p^0 s\|^2_{\mathcal{L}^2} \geqslant (C_1 p^{2/\rho_\Sigma} - C_2)\langle \Box_p^0 s, s\rangle \,. \tag{3.2.15}$$

As a consequence, $\mathrm{Spec}(\Box_p^0) \subset \{0\} \cup [C_1 p^{2/\rho_\Sigma} - C_2, \infty)$, so that we get (3.2.4). This concludes the proof. $\square$

4. Asymptotic expansion of the Bergman kernel

4.1 Strategy and results

In this chapter, we investigate the Bergman kernel on Σ, which will be used in the proof of the equidistribution phenomenon of zeros of holomorphic sections of $L^p \otimes E$ over Σ, as p grows indefinitely. In particular, we will prove that the on-diagonal Bergman kernel $B_p(x)$ exhibits a pointwise asymptotic expansion when $p \to \infty$, for every $x \in \Sigma$. Our strategy is to estimate the difference of our Bergman kernel in a neighborhood of a fixed point to the Bergman kernel of two different model situations, depending on the position of the point in question. As a consequence, we treat both cases separately and conclude a global asymptotic expansion after having justified control on the Bergman kernel functions on any open subset of Σ.

Close to the punctures, the Bergman kernel has been shown to have an asymptotic expansion via this method in [1], [2], where the authors make use of the existence of the asymptotic expansion of the Bergman kernel $B_p^{\mathbb{D}^*}$ on the punctured unit disc $(\mathbb{D}^*, \omega_{\mathbb{D}^*})$. We will follow this approach in these regions.

Away from the punctures, our local model is a model Bergman kernel that is defined in the complex plane. Here the curvature is allowed to vanish. An asymptotic expansion in this geometry is proven in [47].

Finally, in order to compare the asymptotic expansions of the model Bergman kernels in the above cases, we utilize the method of analytic localization that was developed by Ma and Marinescu [43], which is inspired by Bismut-Lebeau [7].

The following theorem extends [47, Theorem 1] (equivalently, [46, Theorem 3]).

Theorem 4.1.1 (On-diagonal asymptotic expansion of the Bergman kernel)**.** *Let Σ, L and E be as in Theorem 3.2.1. For $\rho_0 \in \{2, 4, \dots, \rho_\Sigma\}$ define a smooth path $W : [0,1] \ni t \mapsto W(t) \in \Sigma$ such that $W(t) \in \Sigma_{\rho_0}$ for all $t \in [0,1]$. Then for all $r \in \mathbb{N}$, there* *exists a smooth function $b_r(x)$ with $x \in \mathrm{range}(W)$, such that for all $k \in \mathbb{N}$ the following asymptotic expansion of the Bergman kernel function holds uniformly on* range(W) *in any $\mathcal{C}^\ell$-topology, with $\ell \in \mathbb{N}$:*

$$B_p(x,x) = p^{2/\rho_0} \left[\sum_{r=0}^{k} b_r(x) p^{-2r/\rho_0}\right] + \mathcal{O}(p^{-2k/\rho_0}), \tag{4.1.1}$$

Moreover, for $x \in W$, the leading term satisfies

$$b_0(x) = B^{j_z^{\rho_0-2} R^L}(0,0) > 0, \tag{4.1.2}$$

where the (ρ_0-2)-th jet $j_x^{\rho_0-2}R^L \in \mathbf{i}S^{\rho_0-2}\mathbb{R}^2 \otimes \Lambda^2(\mathbb{R}^2)^*$ is identified with the (ρ_0-2)-degree homogeneous part of the Taylor expansion of R^L in the geodesic normal coordinate centered at z, and $B^{j_x^{\rho_0} R^L}$ is the model Bergman projection defined in Subsection 4.4.1.

For $h \in (0,1)$, $\gamma \in (0, \frac{1}{2})$, $\ell, m \in \mathbb{N}$, and V_j described in assumption (α) with coordinate z_j, the following asymptotic expansion of the Bergman kernel function holds uniformly in any $\mathcal{C}^l$-topology, with $\ell \in \mathbb{N}$, for points z_j in the ring $\mathbb{D}^*(a_j, \frac{1}{6}) \setminus \mathbb{D}^*(a_j, he^{-p^\gamma})$:

$$B_p(z_j, z_j) = \frac{p-1}{2\pi} + \mathcal{O}(p^{-m}). \tag{4.1.3}$$

for every $m \in \mathbb{N}$.

The reason why we consider the expansion along smooth paths along which the order of vanishing of the curvature does not change is to obtain a stronger statement, i.e. the expansion holds uniformly on the image of such paths. Thus, considering such paths seems to present itself to be an appropriate setting when stating the result. When applying a pointwise expansion is sufficient for the matter at hand, our result on the punctured Riemann surface identical to [47, Theorem 1] for points away from the punctures.

Remark 4.1.2. Note that the function $x \mapsto \rho_x$ may exhibit jumps with respect to $x \in \Sigma$ along regions where the order of vanishing of the curvature varies. On subsets of Σ where this happens, the function ρ is discontinuous and hence the above asymptotic expansion for B_p can not be uniform in any neighborhood such points.

As a Corollary, we get the following asymptotic expansion for the sup of the Bergmen kernel function. The proof can be found in Section 4.5. The result is an analogue of [1, Corollary 1.4] in our setting.

Corollary 4.1.3. *We have*

$$\sup_{x\in\Sigma} B_p(x,x) = \left(\frac{p}{2\pi}\right)^{3/2} + \mathcal{O}(p)\,. \tag{4.1.4}$$

Remark 4.1.4. Note that for $x \in \Sigma_2$, $B^{j_x^0 R^L}(0,0) > 0$ and moreover, the quantity $B^{j_x^0 R^L}(0,0)$ depends smoothly on x. In particular, if x is close to a puncture, then $B^{j_x^0 R^L}(0,0) = B_p^{\mathbb{D}^*}(x,x) = \frac{1}{2\pi}$. For x such that $\rho_x \geqslant 4$, we have $j_x^0 R^L = 0$ so that $B^{j_x^0 R^L}(0,0) = 0$, since a $\mathcal{L}^2$-integrable entire function on $\mathbb{C}$ has to be identically 0.

For $h \in (0,1)$ and $\gamma \in (0, \frac{1}{2})$, set

$$\Sigma^{p,h,\gamma} := \Sigma \setminus \bigcup_{j=1}^{N} \mathbb{D}^*(a_j, he^{-p^\gamma})\,, \tag{4.1.5}$$

where $\mathbb{D}^*(a_j, he^{-p^\gamma})$ denotes the punctured (open) disc of radius he^{-p^γ} centered at a puncture a_j in the coordinate $z_j \in V_j$ described in assumption (α). Set

$$\Sigma^{p,h,\gamma}_{\rho_\Sigma} := \Sigma^{p,h,\gamma} \cap \Sigma_{\rho_\Sigma}\,. \tag{4.1.6}$$

Remark 4.1.5. Observe that for any arbitrary but fixed pair of real numbers $h \in (0,1)$ and $\gamma \in (0, \frac{1}{2})$ the set-valued function $\mathbb{N} \ni p \to \Sigma^{p,h,\gamma}_{\rho_\Sigma}$ is monotonically increasing with respect to the partial order of inclusion of subsets, that is, if $p_1, p_2 \in \mathbb{N}$ are such that $p_1 < p_2$, then $\Sigma_{p_1,h,\gamma,\rho_\Sigma} \subset \Sigma_{p_2,h,\gamma,\rho_\Sigma}$. In particular, if $\rho_\Sigma > 2$, then for any fixed pair of real numbers $h \in (0,1)$ and $\gamma \in (0, \frac{1}{2})$, for all sufficiently large $p \in \mathbb{N}$, we have $\Sigma^{p,h,\gamma}_{\rho_\Sigma} = \Sigma_{\rho_\Sigma}$.

We have the following corollary, which we will prove in Section 4.5.

Corollary 4.1.6. *For any given $h \in (0,1)$, $\gamma \in (0, \frac{1}{2})$, the following constant is positive*

$$c(p,h,\gamma,\rho_\Sigma) := \inf_{x\in\Sigma^{p,h,\gamma}_{\rho_\Sigma}} B^{j_x^{\rho_\Sigma-2} R^L}(0,0) > 0\,. \tag{4.1.7}$$

Moreover, for all $h \in (0,1)$ and $\gamma \in (0, \frac{1}{2})$, for sufficiently large $p \in \mathbb{N}$, the constant $c(p,h,\gamma,\rho_\Sigma) =: c(\rho_\Sigma) \in \mathbb{R}_{>0}$ is independent of h and γ.

We obtain an analogue of [47, Lemma 12] for arbitrarily large, but strictly smaller relatively compact subsets of Σ in our setting as an application of the lower bound from Corollary 4.1.6. We will sketch a proof in Section 4.5.

Lemma 4.1.7. *Let $h \in (0,1)$ and $\gamma \in (0, \frac{1}{2})$ be arbitrary. Then for sufficiently large*

$p \in \mathbb{N}$ and all $x \in \Sigma^{p,h,\gamma}$, we have the estimates

$$
\begin{aligned}
c(\rho_\Sigma)\,(1+o(1))\,p^{2/\rho_\Sigma} &\leqslant B_p(x,x) \\
&\leqslant \left[\sup_{x\in\Sigma} B^{j^0_x R^L}(0,0)\right](1+o(1))\,p\,,
\end{aligned}
\tag{4.1.8}
$$

where both constants in $o(1)$ are uniform in $x \in \Sigma^{p,h,\gamma}$, as $p \to \infty$.

The derivatives of the Bergman kernel can be written in a coordinate-free fashion by means of an associated jet-bundle (see Appendix A).

A pointwise asymptotic expansion on the diagonal also exists for derivatives of the Bergman kernel. In our setting the proof is again analogous to [47, Theorem 10].

Theorem 4.1.8. *For all $l \in \mathbb{N}_0$, the l-th jet of the on-diagonal Bergman kernel has a pointwise asymptotic expansion*

$$
j^l[B_p(x,x)]/j^{l-1}[B_p(x,x)] = p^{(2+l)/r_x}\left[\sum_{j=0}^{N} c_j(x)p^{-2j/r_x}\right] + \mathcal{O}(p^{-(2N-l-1)/r_x}) \tag{4.1.9}
$$

for all $N \in \mathbb{N}$, in $j^l\mathrm{End}(E)/j^{l-1}\mathrm{End}(E) \cong S^lT^*\Sigma \otimes \mathrm{End}(E) \cong S^lT^*\Sigma \otimes \mathbb{C}$.

Again the expansion is not uniform with respect to the base point $x \in \Sigma$, but, similar to Lemma 4.1.7, we get the following upper-bound on the derivatives of the Bergman kernel, which is uniform on an arbitrarily large, relatively compact subset of Σ.

Lemma 4.1.9. *Let $h \in (0,1)$ and $\gamma \in (0,\frac{1}{2})$ be arbitrary. Then for sufficiently large $p \in \mathbb{N}$ and all $x \in \Sigma^{p,h,\gamma}$, we have the upper bound for the l-th jets of the Bergman kernels*

$$
|j^l[B_p(x,x)]| \leqslant p^{l/3}[1+o(1)]\left[\sup_{x\in\Sigma}\frac{\left|j^l B^{j^1_x R^L/j^0_x R^L}(0,0)\right|}{B^{j^1_x R^L/j^0_x R^L}(0,0)}\right] B_p(x,x)\,, \tag{4.1.10}
$$

where $o(1)$ is uniform in $x \in \Sigma^{p,h,\gamma}$, as $p \to \infty$.

Similarly to Lemma 4.1.7, we will sketch the proof in Section 4.5.

The method of analytic localization is explained in Section 4.2. The proof of the asymptotic expansion in Theorem 4.1.1 needs some preparation; all the necessary steps are explained in the Sections 4.3-4.4 in this chapter.

4.2 Analytic Localization Principle

In this section, we explain how to localize the computations for the Bergman kernel B_p on Σ by the technique of analytic localization. This method was inspired by the work of Bismut-Lebeau [7] in local index theory, and developed by Dai-Liu-Ma [18] and Ma-Marinescu [43], [44] to study the Bergman kernels. Now we explain how this method works for our setting.

In order to apply the method of analytic localization, elliptic estimates and the existence of a spectral gap for the Kodaira Laplacians $\square_p^0$ are needed, the latter of which we have proven in Theorem 3.2.1. The next step is to apply the finite propagation speed for solutions of the wave equation that is associated to the corresponding Laplace operator.

Near the punctures, where the curvature is strictly positive, the necessary elliptic estimates are proven by Auvray, Ma and Marinescu (see [1]). Here the finite propagation speed follows from [43, Theorem D.2.1], as explained in [1, p.32].

Away from the punctures the curvature might vanish; the necessary steps for the finite propagation speed of the wave equation associated to a model sub-Riemannian Bochner Laplacian (see [46, (2.11)] operator are explained in [46, Lemma 7], which use sub-elliptic estimates for this operator [46, (2.14)].

In each of both cases the conclusion is that the Bergman kernel localizes, which allows us to reduce the problem to its analogue in the corresponding model situations.

We will now move on to explain the mentioned steps in more detail; the following proposition is from [1, Proposition 4.2].

Proposition 4.2.1. *For any $m \in \mathbb{N}$, there exists $C = C(m, h^L)$ such that for $p \gg 1$ and all $s \in \mathbf{H}^{2m}(\Sigma, \omega_\Sigma, L^p \otimes E, h_p)$,*

$$\|s\|^2_{\mathbf{H}_p^{2m}} \leqslant C \sum_{j=0}^{m} p^{4(m-j)} \|(\square_p^0)^j s\|^2_{\mathcal{L}^2} \tag{4.2.1}$$

In the proof of this proposition from [1], the authors take (E, h^E) to be trivial line bundle on Σ and assume that (L, h^L) is strictly positive on all of Σ. But since neither the twist by E nor the positivity of (L, h^L) away from punctures play any role in the proof of this estimate, the same model situation near punctures applies in our setting, so that the arguments in the proof can be applied in the same way to our case.

We will now explain the underlying idea behind analytic localization. For this purpose, fix a small $\varepsilon > 0$. Let $\psi : \mathbb{R} \to [0,1]$ be a smooth even function such that

$$\psi(v) = \begin{cases} 1 & , \ |v| \leqslant \varepsilon/2 \\ 0 & , \ |v| \geqslant \varepsilon \end{cases} , \tag{4.2.2}$$

and define

$$\varphi(a) = \left(\int_{-\infty}^{\infty} \psi(v) \, \mathrm{d}v \right)^{-1} \cdot \int_{-\infty}^{\infty} e^{iva} \psi(v) \, \mathrm{d}v \tag{4.2.3}$$

which is an even function with $\varphi(0) = 1$ and lies in the Schwartz space $\mathcal{S}(\mathbb{R})$. The definition of the latter can be found in [43, Definition A.1.5].

For $p > 0$, set $\varphi_p(s) := \mathbf{1}_{[\frac{1}{2}\sqrt{C_1} p^{1/\rho_\Sigma}, \infty[}(|s|)\varphi(s)$, where C_1 is the constant in the spectral gap of Theorem 3.2.1.

Note that φ and φ_p are even functions. We consider the bounded linear operators $\varphi(D_p)$, $\varphi_p(D_p)$ acting on $\mathcal{L}_2^{0,0}(\Sigma, L^p \otimes E)$ defined via the functional calculus of $\square_p^0$.

In particular, we have

$$\varphi(D_p) = \frac{1}{2\pi} \int_{\mathbb{R}} \cos(\xi \sqrt{\square_p^0}) \hat{\varphi}(\xi) \mathrm{d}\xi \,, \tag{4.2.4}$$

where $\hat{\varphi}$ is a multiple of the function ψ defined in (4.2.2). Then for $p > 0$, such that $C_1 p^{2/\rho_\Sigma} - C_2 \geqslant \frac{C_1}{4} p^{2/\rho_\Sigma}$, we have

$$\varphi(D_p) - B_p = \varphi_p(D_p) \,. \tag{4.2.5}$$

Let $\varphi_p(D_p)(x, x')$ denote the Schwartz integral kernel of $\varphi_p(D_p)$, which is smooth on $\Sigma \times \Sigma$. Fix $0 < r < e^{-1}$ and introduce a smooth function $\eta : \Sigma \to [1, \infty)$ such that $\eta(z) = |\log |z|^2|$ for $z \in \mathbb{D}_r^*$ near each punctures. Let $\operatorname{dist}(x, x')$ denote the geodesic distance between two points x, x'. We have the following estimates as an extension of [1, Proposition 5.3].

Proposition 4.2.2. *For $\ell, m \in \mathbb{N}_0$ and any real number $\gamma > \frac{1}{2}$, there exists $C_{\ell,m,\gamma} \in \mathbb{R}_{>0}$ such that for all $x, x' \in \Sigma$ and any $p \in \mathbb{N}$, we have*

$$\left\| \eta(x)^{-\gamma} \eta(x')^{-\gamma} \varphi_p(D_p)(x, x') \right\|_{\mathcal{C}^m(h_p)} \leqslant C_{\ell,m,\gamma} p^{-\ell}. \tag{4.2.6}$$

Moreover, for $x, x' \in \Sigma$, $\operatorname{dist}(x, x') \geqslant \varepsilon$ with ε as in (4.2.2), we have

$$\left| \eta(x)^{-\gamma} \eta(x')^{-\gamma} B_p(x, x') \right|_{\mathcal{C}^m(h_p)} \leqslant C_{\ell,m,\gamma} p^{-\ell}. \tag{4.2.7}$$

Proof. Since φ is a Schwartz function on $\mathbb{R}$, then for any $k \in \mathbb{N}$, there exists $M_k \in \mathbb{R}_{>0}$

such that for all $s \in \mathbb{R}$,

$$|s^k \varphi(s)| \leqslant M_k \,. \tag{4.2.8}$$

This implies

$$|\varphi_p(s)| \leqslant M_k (\frac{4}{C_1})^{k/2} p^{-k/\rho_\Sigma} \,. \tag{4.2.9}$$

Combining (4.2.9) with the estimate (4.2.1) and the definition of $\varphi_p(D_p)$, we infer that for any $k, \ell \in \mathbb{N}$, there exists $C_{k,\ell} \in \mathbb{R}_{>0}$ such that for all $s \in \mathcal{L}_2^{0,0}(\Omega, L^p \otimes E)$,

$$\|\varphi_p(D_p)s\|_{\mathbf{H}_p^k} \leqslant C_{k,\ell} p^{-\ell} \|s\|_{\mathcal{L}^2} \,. \tag{4.2.10}$$

Using the above inequality and the Sobolev embeddings [1, Lemma 2.6] for the sections on Σ and $\Sigma \times \Sigma$, the proof of (4.2.6) follows from the same arguments given in the proof of [1, Proposition 5.3]

Note that by the Lichnerowicz formula (3.1.8), the second order term of $\square_p^0$ is $\frac{1}{2}\Delta^{\Lambda^{0,\bullet} \otimes L^p \otimes E}$. Thus by the finite propagation speed of solutions of the wave equations associated to the wave operators (see [43, Appendix Theorem D.2.1]) in (4.2.4) and our assumptions on ψ in (4.2.2), we get that for $x \in \Sigma$, the support of $\varphi(D_p)(x, \cdot)$ is included in $\mathbb{B}^\Sigma(x, \frac{\varepsilon}{\sqrt{2}})$, and $\varphi(D_p)(x, \cdot)$ depends only on the restriction of $\square_p^0$ on $\mathbb{B}^\Sigma(x, \frac{\varepsilon}{\sqrt{2}})$. In particular, if $x, x' \in \Sigma$ are such that $\mathrm{dist}(x, x') \geqslant \varepsilon$, then

$$\varphi(D_p)(x, x') = 0 \,, \tag{4.2.11}$$

so that (4.2.7) follows from (4.2.5) and (4.2.6). This completes our proof. $\square$

4.3 Bergman kernel near the punctures

4.3.1 Local model: $\mathbb{D}^*$ - Bergman kernel for the punctured unit disc

We consider the Poincaré punctured unit disc:

$$(\mathbb{D}^*, \omega_{\mathbb{D}^*}, \underline{\mathbb{C}}, h_{\mathbb{D}^*}) \,,$$

where $h_{\mathbb{D}^*} = |\log(|z|^2)| h_0^{\mathbb{C}}$ with $h_0^{\mathbb{C}}$ the flat Hermitian metric on the trivial line bundle $\underline{\mathbb{C}} \to \mathbb{D}^*$. Let $z \in \mathbb{D}^*$ denote the natural coordinate.

For $p \in \mathbb{N}$, consider the Hermitian metric $h_{p,\mathbb{D}^*} := |\log(|z|^2)|^p h_0^{\mathbb{C}}$ on $\underline{\mathbb{C}}$. Define

$$H^p_{(2)}(\mathbb{D}^*) := H^0_{(2)}(\mathbb{D}^*, \omega_{\mathbb{D}^*}, \underline{\mathbb{C}}, h_{p,\mathbb{D}^*}), \tag{4.3.1}$$

to be the space of $\mathcal{L}^2$-integrable holomorphic functions on $\mathbb{D}^*$ (with respect to the Hermitian metric $h_{p,\mathbb{D}^*}$). We denote by $B_p^{\mathbb{D}^*}$ the corresponding Bergman kernel.

By [1, Lemma 3.1], for $p \in \mathbb{N}_{\geqslant 2}$, the space of sections in (4.3.1) has orthonormal basis

$$\left\{ c_l^{(p)} = \left(\frac{l^{p-1}}{2\pi(p-2)!} \right)^{\frac{1}{2}} z^l \, : \, l \in \mathbb{N} \right\} . \tag{4.3.2}$$

Then the Bergman kernel $B_p^{\mathbb{D}^*}$ is given by

$$B_p^{\mathbb{D}^*}(z_1, z_2) = \frac{|\log(|z_2|^2)^p}{2\pi(p-2)!} \sum_{l=1}^{\infty} l^{p-1} z_1^l \bar{z}_2^l \tag{4.3.3}$$

and the corresponding Bergman kernel function by

$$B_p^{\mathbb{D}^*}(z) = \frac{|\log(|z|^2)^p}{2\pi(p-2)!} \sum_{l=1}^{\infty} l^{p-1} |z|^{2l} \tag{4.3.4}$$

for all $z \in \mathbb{D}^*$. The following result is due to [1, Proposition 3.3].

Proposition 4.3.1. *For any $0 < a < 1$ and any $m \geqslant 0$, there exists $c = c(a) > 0$ such that*

$$\left\| B_p^{\mathbb{D}^*}(z) - \frac{p-1}{2\pi} \right\|_{\mathcal{C}^m(\{a \leqslant |z| < 1\}, \omega_{\mathbb{D}^*})} = \mathcal{O}(e^{-cp}) , \tag{4.3.5}$$

as $p \to \infty$. More generally, for $0 < a < 1$ and $0 < \gamma < \frac{1}{2}$, there exists $c = c(a, \gamma) \in \mathbb{R}_{>0}$ such that, as $p \to \infty$,

$$\left\| B_p^{\mathbb{D}^*}(z) - \frac{p-1}{2\pi} \right\|_{\mathcal{C}^m(\{ae^{-p^\gamma} \leqslant |z| < 1\}, \omega_{\mathbb{D}^*})} = \mathcal{O}(e^{-cp^{1-2\gamma}}) . \tag{4.3.6}$$

We have added $\omega_{\mathbb{D}^*}$ in the notation of the norm in the above proposition to emphasize that we equip the corresponding subsets of $\mathbb{D}^*$ with $\omega_{\mathbb{D}^*}$.

Together with Proposition 4.3.1, we will use the following corollary (see [1, Corollary 3.6]) from the same authors on the supremum value of $B_p^{\mathbb{D}^*}(z)$.

Corollary 4.3.2. *The supremum over $\mathbb{D}^*$ of the Bergman kernel function $B_p^{\mathbb{D}^*}$ on the punctured unit disc has the asymptotic behavior for growing $p \in \mathbb{N}$:*

$$\sup_{z \in \mathbb{D}^*} B_p^{\mathbb{D}^*}(z) = \left(\frac{p}{2\pi} \right)^{3/2} + \mathcal{O}(p) , \;\; \text{as} \;\; p \longrightarrow \infty \tag{4.3.7}$$

4.3.2 Bergman kernel expansion near a puncture

For each puncture $a_j \in D$, consider an open neighborhood $V_j \subset \Sigma$ around a_j as described in (β), in the sense that $\overline{V_j} := V_j \cup \{a_j\}$ is an open neighborhood in $\overline{\Sigma} = \Sigma \cup D$.

Fix an arbitrary radius $0 < r < e^{-1}$; we view $\mathbb{D}^*_r$ as a subset of V_j with the local complex coordinate z_j on V_j. Then we have the identification of geometric data

$$(V_j, \omega_\Sigma, L^p \otimes E, h_p)|_{\mathbb{D}^*_r} \cong (\mathbb{D}^*, \omega_{\mathbb{D}^*}, \underline{\mathbb{C}}, h_{p,\mathbb{D}^*})|_{\mathbb{D}^*_r}, \tag{4.3.8}$$

where the right-hand side is the Poincaré punctured unit disc described in Subsection 4.3.1. Denote by $\square^0_{\mathbb{D}^*,p}$ the Kodaira Laplace operators and $D_{\mathbb{D}^*,p}$ the associated Dirac operators on the Poincaré punctured unit disc acting on $\Omega^{0,0}_{(2)}(\mathbb{D}^*, \omega_{\mathbb{D}^*}, \underline{\mathbb{C}}, h_{p,\mathbb{D}^*})$. After restricting both to $\mathbb{D}^*_r$, $\square^0_{\mathbb{D}^*,p}$ coincides with the operator $\square^0_p$.

By [1, Corollary 5.2], $\square^0_{\mathbb{D}^*,p}$ has a spectral gap: there exists $C' \in \mathbb{R}_{>0}$ such that for sufficiently large $p \in \mathbb{N}$,

$$\operatorname{Spec}(\square^0_{\mathbb{D}^*,p}) \subset \{0\} \cup [C'p, \infty)\,. \tag{4.3.9}$$

Therefore we can proceed for $\square^0_{\mathbb{D}^*,p}$ as in Subsection 4.2: For arbitrary but fixed $0 < \varepsilon < \frac{r}{2}$, define ψ as in (4.2.2) and the corresponding function φ. Then for $p \geqslant 1$,

$$\varphi(D_{\mathbb{D}^*,p}) - B_p^{\mathbb{D}^*} = \varphi_p(D_{\mathbb{D}^*,p})\,. \tag{4.3.10}$$

By the finite propagation speed of solutions of the corresponding wave equation, as explained in the proof of Proposition 4.2.2, for $z, z' \in \mathbb{D}^*_{r/2}$, we have

$$\varphi(D_{\mathbb{D}^*,p})(z,z') = \varphi(D_p)(z,z')\,. \tag{4.3.11}$$

This implies that on $\mathbb{D}^*_{r/2} \times \mathbb{D}^*_{r/2}$, we have

$$B_p(z,z') - B_p^{\mathbb{D}^*}(z,z') = \varphi_p(D_{\mathbb{D}^*,p})(z,z') - \varphi_p(D_p)(z,z')\,. \tag{4.3.12}$$

Now both terms on the right-hand side obey the estimate from (4.2.6) on $\mathbb{D}^*_{r/2} \times \mathbb{D}^*_{r/2}$. Since the problem is local in nature, we can proceed as in [1, Section 6], so that the results of [1, Theorems 1.1 & 1.2] still hold in our setting. We thus can conclude the following Theorems.

Theorem 4.3.3. *Fix any $\ell, m \in \mathbb{N}_0$. For any $\alpha \in \mathbb{R}_{>0}$, there exists a constant $C =$*

$C(\ell, m, \alpha) \in \mathbb{R}_{>0}$ such that on $\mathbb{D}^*_{r/2} \times \mathbb{D}^*_{r/2}$

$$|B_p(z,z') - B_p^{\mathbb{D}^*}(z,z')|_{\mathcal{C}^m} \leqslant Cp^{-\ell}|\log(|z|^2)|^{-\alpha}|\log(|z'|^2)|^{-\alpha}\,. \tag{4.3.13}$$

Moreover, for every $\delta \in \mathbb{R}_{>0}$, there exists a constant $C' = C'(\ell, m, \delta) \in \mathbb{R}_{>0}$, such that for all $p \in \mathbb{N}$ and $z_j \in \mathbb{D}^*_{r/2}$,

$$|B_p - B_p^{\mathbb{D}^*}|_{\mathcal{C}^m}(z_j) \leqslant C'p^{-\ell}|\log(|z_j|^2)|^{-\delta}\,. \tag{4.3.14}$$

Furthermore, the same authors Auvray, Ma and Marinescu obtain estimates on the behavior of the quotients of Bergman kernels, which also apply in our setting by the same argument as above.

Theorem 4.3.4. *Let Σ be a punctured Riemann surface and let (L, h^L) (E, h^E) be Hermitian holomorphic line bundles on $\overline{\Sigma}$ such that conditions (α) and (β) are satisfied. Then*

(i) for any $\ell \in \mathbb{N}$ there exists $C \in \mathbb{R}_{>0}$, such that for any $p \in \mathbb{N}$ we have

$$\sup_{z \in V_1 \cup \ldots \cup V_N} \left\| \frac{B_p}{B_p^{\mathbb{D}^*}}(z) - 1 \right\|_{\mathcal{C}^0(h_p)} \leqslant Cp^{-\ell}\,. \tag{4.3.15}$$

(ii) for all $k \in \mathbb{N}$, $D_1, \ldots, D_k \in \left\{\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right\}$ and any $\ell \in \mathbb{N}$ there exists $C_\ell \in \mathbb{R}_{>0}$, such that for any $p \in \mathbb{N}$ we have

$$\sup_{z \in \overline{V}_1 \cup \ldots \cup \overline{V}_N} \left\| (D_1 \cdots D_k) \frac{B_p}{B_p^{\mathbb{D}^*}}(z) \right\|_{\mathcal{C}^0(h_p)} \leqslant C_\ell\, p^{-\ell}\,. \tag{4.3.16}$$

The proofs for the statements in Theorem 4.3.4 about the quotients are analogous to the proofs of [2, Theorems 1.2 & 1.3].

We have described the behavior of $B_p^{\mathbb{D}^*}$ in Subsection 4.3. Together with Theorem 4.3.3, we infer the existence of an asymptotic expansion of B_p on $\mathbb{D}^*_{r/2}$ and hence on small punctured discs around the punctures, as $p \to \infty$.

4.4 Bergman kernel away from the punctures

4.4.1 Local model: $\mathbb{C}$ - Model Dirac and Kodaira Laplacian operators, Model Bergman kernel

Alongside the Kodaira Laplacians of our interest, we need to introduce certain model operators that plays an important role in our calculations. We always equip $\mathbb{R}^2$ with the standard Euclidean metric and the standard complex structure such that $\mathbb{R}^2 \simeq \mathbb{C}$. Let $z = x + \mathbf{i}y \in \mathbb{C}$ denote the usual complex coordinate, and let $\{e_1 := \frac{\partial}{\partial x}, e_2 = \frac{\partial}{\partial y}\}$ be the standard Euclidean basis of $\mathbb{R}^2$. Now fix an even integer $\rho' \geqslant 2$.

Let R be a non-trivial $(1,1)$-form on $\mathbb{R}^2$ whose coefficients with respect to the frame $\mathrm{d}z \wedge \mathrm{d}\bar{z} = -2\,\mathbf{i}\,\mathrm{d}x \wedge \mathrm{d}y$ are given by a non-negative real homogeneous polynomial of degree $\rho' - 2$, for some $\rho' \in \mathbb{N}_{\geqslant 2}$.

We define a smooth 1-form $a^R \in \Omega^1(\mathbb{R}^2)$ by

$$a^R_{v_1}(v_2) := \int_0^1 R_{tv_1}(v_2, tv_1)\,\mathrm{d}t\,, \tag{4.4.1}$$

where $v_1 \in \mathbb{R}^2$ and $v_2 \in T_{v_1}\mathbb{R}^2 \simeq \mathbb{R}^2$. Set

$$\nabla^R = \mathrm{d} - a^R\,; \tag{4.4.2}$$

it is a unitary connection on the trivial Hermitian line bundle $\underline{\mathbb{C}}$ over $\mathbb{R}^2$. Then by construction the curvature form of ∇^R is exactly given by R. Let Δ_R denote the associated Bochner Laplacian.

Take $\bar{\partial}$ to be the standard $\bar{\partial}$-operator on $\mathbb{R}^2 \cong \mathbb{C}$; then the $(0,1)$ part of the connection ∇^R is $\bar{\partial}_{\mathbb{C}} := \bar{\partial} - \left(a^R\right)^{0,1}$. Let $\bar{\partial}^*_{\mathbb{C}}$ denote the formal adjoint of $\bar{\partial}_{\mathbb{C}}$ with respect to the standard inner product on $\mathbb{R}^2$.

The following operators are called the model Dirac and model Kodaira Laplace operators on $\mathbb{R}^2$, associated to R:

$$\begin{aligned} D_R &:= \sqrt{2}\left(\bar{\partial}_{\mathbb{C}} + \bar{\partial}^*_{\mathbb{C}}\right) \\ \square_R &:= \frac{1}{2}\left(D_R\right)^2\,, \end{aligned} \tag{4.4.3}$$

This model Kodaira Laplacian is related to the model Bochner Laplacian by the Lichnerowicz formula (compare to (3.1.8)):

$$\square_R = \frac{1}{2}\Delta_R + \frac{1}{2}c\,(R) \tag{4.4.4}$$

with

$$c\,(R) = R(e_1, e_2)c(e_1)c(e_2)\,. \tag{4.4.5}$$

We identify Δ_R and $\square_R$ with their unique self-adjoint extensions acting on the $\mathcal{L}^2$-sections over $\mathbb{R}^2$.

For the restriction $\square_R^0$ of $\square_R$ onto $(0,0)$-sections, in [47, Proposition 25], the authors proved that there exists a constant $c_R \in \mathbb{R}_{>0}$ such that

$$\mathrm{Spec}(\square_R^0) \subset \{0\} \cup [c_R, \infty)\,. \tag{4.4.6}$$

Consider the following first-order differential operators

$$b = -2\frac{\partial}{\partial z} + \frac{1}{\rho'}\,\mathbf{i}R(e_1, e_2)\overline{z}\,, \qquad b^+ = 2\frac{\partial}{\partial \overline{z}} + \frac{1}{\rho'}\,\mathbf{i}R(e_1, e_2)z\,. \tag{4.4.7}$$

Then we have

$$\square_R^0 = \frac{1}{2}bb^+\,. \tag{4.4.8}$$

and, moreover, for $s \in \mathcal{L}_2^{0,0}(\mathbb{R}^2, \underline{\mathbb{C}})$, $s \in \ker \square_R^0$ if and only if $b^+ s \equiv 0$.

Consider the $\mathcal{L}^2$-orthogonal projection

$$B^R : \mathcal{L}_2^{0,0}(\mathbb{R}^2, \underline{\mathbb{C}}) \longrightarrow \ker \square_R^0\,. \tag{4.4.9}$$

For $z, z' \in \mathbb{R}^2$, let $B^R(z, z')$ denote the Schwartz integral kernel of the above projection, which is a smooth function on $\mathbb{R}^2 \times \mathbb{R}^2$.

The following lemma was already known in [47, Theorem 1], which can be understood as a consequence of the lower bound for the Bergman kernel that was proved by Catlin [14] by considering local models. Here we also give a direct proof in order to shed light on the space $\ker \square_R^0$.

Lemma 4.4.1. *For R as above, B^R is an even function, i.e. for all $z, z' \in \mathbb{R}^2$ we have $B_R(z, z') = B_R(-z, -z')$. Moreover,*

$$B^R(0, 0) > 0. \tag{4.4.10}$$

Furthermore, the quantity $B^R(0, 0)$ depends smoothly on R.

In scope of the following Corollary, consider the finite dimensional vector space that is spanned by the homogeneous monomials $\{z^\alpha \overline{z}^\beta\}_{\alpha,\beta\in\mathbb{N}^2, 0\leqslant|\alpha|,|\beta|\leqslant\rho'-2}$, equipped with the Euclidean norm which we simply denote by $|\cdot|$. This vector space is isomorphic to the complex vector space $\{R = p(z,\bar{z})\,\mathrm{d}z \wedge \mathrm{d}\bar{z} : \deg(p) = \rho' - 2\}$; we set $|R| = |p(z,\bar{z})\,\mathrm{d}z \wedge \mathrm{d}\bar{z}| := |p(z,\bar{z})|$.

Corollary 4.4.2. $\inf_{|R|=1} B^R(0,0) > 0$.

Proof. Let V be the vector space from above and let $V_1 := V \cap \{|p| = 1\}$ be the subset of vectors of unit length and $H := V \cap \{p : p(x) \geqslant 0\ \forall x\}$ the half plane with boundary, as well as $H_1 := H \cap \{|p| = 1\}$ the portion of the unit circle that lies in H. Clearly, H_1 is a compact subset of V_1. Now by Lemma 4.4.1, the function $R \mapsto B^R(0,0)$ is smooth, in particular continuous. Then $\inf_{|R|=1} B^R(0,0) = \min_{|R|=1} B^R(0,0)$ and the latter is positive, again by Lemma 4.4.1. $\square$

Proof of Lemma 4.4.1. Set $\omega = \frac{1}{\sqrt{2}}(e_1 - \mathbf{i}e_2)$. Then by assumption

$$\psi(x,y) := R(\omega,\overline{\omega}) = \mathbf{i}R(e_1,e_2) \tag{4.4.11}$$

is a nonnegative real homogenous polynomial in x, y of degree $\rho' - 2$. In particular, it is an even function in $(x,y) \in \mathbb{R}^2$. Hence, B^R is an even function. Let $\Psi(x,y)$ be a homogeneous polynomial in x, y of degree ρ' such that

$$\frac{\partial \Psi}{\partial \bar{z}}(x,y) = \frac{1}{\rho'}\psi(x,y)z\,. \tag{4.4.12}$$

Then for any fixed $\lambda \in \mathbb{C}$, $\Psi + \lambda z^{\rho'}$ also satisfies (4.4.12). Moreover, we have

$$-\frac{1}{2}\Delta^{\mathbb{R}^2}\operatorname{Re}(\Psi) = \psi(x,y) \geqslant 0\,, \tag{4.4.13}$$

where $\Delta^{\mathbb{R}^2} = -(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$. Then the real part $\varphi := \operatorname{Re}(\Psi)$ is a subharmonic, nonharmonic real homogenous polynomial in x, y of degree ρ'.

Now note that if g is an entire function on $\mathbb{C}$ such that $|g|^2 e^{-\varphi}$ is integrable on $\mathbb{C}$ (with respect to the standard Lebesgue measure), then

$$g e^{-\frac{1}{2}\Psi} \in \ker \square_R^0\,. \tag{4.4.14}$$

This way, we can translate our problem at hand to the study the weighted Bergman kernel on $\mathbb{C}$ associated to the real subharmonic function $\frac{1}{2}\varphi$ as in [16]. Now by [16, Proposition 1.10], $\ker \square_R^0$ is an infinite dimensional subspace of $\mathcal{L}_2^{0,0}(\mathbb{R}^2, \underline{\mathbb{C}})$. In particular, there exists a nontrivial entire function g on $\mathbb{C}$ such that $g e^{-\frac{1}{2}\Psi} \in \ker \square_R^0$. If $g(0) \neq 0$, then $g e^{-\frac{1}{2}\Psi}$ does not vanish at $z = 0$. On the other hand, if $g(0) = 0$, we can write $g(z) = z^k h(z)$, where $k \in \mathbb{N}$ and h is some entire function with $h(0) \neq 0$. Consequently, the integrability of $|g|^2 e^{-\varphi}$ implies the integrability of $|h|^2 e^{-\varphi}$, so that $he^{-\frac{1}{2}\Psi} \in \ker \square^0_R$ and it does not vanish at the center point $z = 0$. This implies that

$$B^R(0,0) > 0 \tag{4.4.15}$$

by the variational characterization (2.7.5) of the Bergman kernel.

Analogous to [43, (4.2.22)], by the spectral gap (4.4.6), for $t > 0$, we have the identity

$$\exp(-t\square^0_R) - B^R = \int_t^\infty \square^0_R \exp(-s\square^0_R)\,\mathrm{d}s\,. \tag{4.4.16}$$

Then

$$B^R(0,0) = \exp(-t\square^0_R)(0,0) - \int_t^\infty \{\square^0_R \exp(-s\square^0_R)\}(0,0)\,\mathrm{d}s\,. \tag{4.4.17}$$

Now we replace R by a smooth family of non-trivial $(1,1)$-forms on $\mathbb{R}^2$ whose coefficients with respect to $\mathrm{d}z \wedge \mathrm{d}\bar{z}$ are given by nonnegative real homogeneous polynomials in x, y of degree $\rho' - 2$. Then locally in the parameter space for this family of $(1,1)$-forms, the spectral gaps c_R in (4.4.6), as R varies, admit a uniform lower bound $c > 0$ (see [47, Appendix]). Combining with the smooth dependence of the heat kernels of $\square^0_R$ on R (see Duhamel's formula [5, Theorem 2.48]), $\int_t^\infty \{\square^0_R \exp(-s\square^0_R)\}(0,0)\mathrm{d}s$ depends continuously on R for any given $t > 0$. As a consequence of (4.4.17), we conclude that $B^R(0,0)$ depends smoothly on R. This completes our proof of the lemma. □

Example 4.4.3. We consider a simple but nontrivial example $R(x,y) = y^2\,\mathrm{d}z \wedge \mathrm{d}\bar{z}$, $\rho' = 4$, then we can rewrite it as

$$R(x,y) = -2\,\mathbf{i}y^2\,\mathrm{d}x \wedge \mathrm{d}y. \tag{4.4.18}$$

Then

$$a^R_z := \int_0^1 t^3(2\,\mathbf{i}y^2 x\,\mathrm{d}y - 2\,\mathbf{i}y^3\,\mathrm{d}x)dt = \frac{\mathbf{i}}{2}y^2(x\,\mathrm{d}y - y\,\mathrm{d}x), \tag{4.4.19}$$

and

$$(a^R)^{0,1}_z = -\frac{1}{4}y^2 z\,\mathrm{d}\bar{z}. \tag{4.4.20}$$

An explicit computation shows that $\bar{\partial}^*_{\mathbb{C}} = -2\iota_{\frac{\partial}{\partial\bar{z}}}\frac{\partial}{\partial z} + \frac{1}{2}y^2\bar{z}\iota_{\frac{\partial}{\partial\bar{z}}}$, and that

$$\begin{aligned}\square_R = \frac{1}{2}\Delta^{\mathbb{R}^2} - \frac{1}{2}y^2\Big(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial\bar{z}}\Big) + \frac{\mathbf{i}}{2}xy \\ + \frac{1}{8}y^4|z|^2 - y^2 + 2y^2\,\mathrm{d}\bar{z} \wedge \iota_{\frac{\partial}{\partial\bar{z}}}\,.\end{aligned} \tag{4.4.21}$$

Note that the differential operator

$$-\frac{1}{2}y^2(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}) + \frac{\mathbf{i}}{2}xy = \frac{\mathbf{i}}{2}y^2(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}) + \frac{\mathbf{i}}{2}xy \tag{4.4.22}$$

is formally self-adjoint with respect to the standard $\mathcal{L}^2$-metric on the functions over $\mathbb{R}^2$. In this example, we have

$$b = -2\frac{\partial}{\partial z} + \frac{1}{2}y^2\bar{z}\,, \qquad b^+ = 2\frac{\partial}{\partial \bar{z}} + \frac{1}{2}y^2 z\,. \tag{4.4.23}$$

Then

$$\square_R^0 = \frac{1}{2}bb^+\,. \tag{4.4.24}$$

Note that

$$\mathrm{Re}\{|z|^4 - |z|^2z^2 - \frac{1}{3}|z|^2\bar{z}^2 + \frac{1}{2}z^4\} \geqslant \frac{1}{24}x^4 + \frac{1}{6}y^4\,. \tag{4.4.25}$$

Consider the following $\mathcal{L}^2$-function on $\mathbb{C}$

$$f(z) = \exp\left(-\frac{1}{16}\left\{|z|^4 - |z|^2z^2 - \frac{1}{3}|z|^2\bar{z}^2 + \frac{1}{2}z^4\right\}\right)\,. \tag{4.4.26}$$

We have $f(0) = 1$, and $f \in \ker \square_R^0$. Moreover, we have

$$B^R(0,0) \geqslant \frac{1}{\|f\|_{\mathcal{L}^2}}\,. \tag{4.4.27}$$

The weighted Bergman kernel has been well studied, in particular, in [16, Theorem 1.13], an upper bound on $B^R(z,z')$ was obtained. As a consequence, we have the following estimate.

Proposition 4.4.4. *There exists a smooth positive function $\mathbb{C} \ni z \mapsto \alpha(z) > 0$ and a constant $\delta \in (0, \frac{1}{4}]$ such that for all $z \in \mathbb{R}^2$*

$$\int_{z' \in \mathbb{B}(z,\alpha(z))} \psi(z')\,\mathrm{d}V(z') \in [1-\delta, 1+\delta]\,. \tag{4.4.28}$$

Then there exist constants $C > 0$, $c > 0$ such that for all $z, z' \in \mathbb{R}^2$

$$|B^R(z,z')| \leqslant \frac{C}{\alpha(z)^2}\exp(-c|z-z'|)\,. \tag{4.4.29}$$

4.4.2 Bergman kernel expansion away from the punctures

In this section, we explain how to apply the method of analytic localization described in 4.2 on small open neighborhoods of points $x \in \Sigma$, i.e. away from the punctures. Similar to 4.3.3, we wish to compare the global Bergman kernel B_p to a model Bergman kernel of a suitable local model and estimate its differences for growing p. We will use a different local model than the punctured unit disc for points near the punctures.

We start by discussing a construction of a convenient coordinate system near an arbitrary fixed point in Σ.

4.4.2.1 Exponential map and normal coordinates

Note that $(\Sigma, g^{T\Sigma})$ is complete and hence by the Hopf-Rinow theorem geodesically complete. Thus the exponential map

$$T_x\Sigma \ni Z \mapsto \exp_x^\Sigma(Z) \in \Sigma$$

is well-defined for all $x \in \Sigma$. For an open subset $U \subset \Sigma$, the real number

$$\operatorname{inj}^U := \inf_{x\in U} \sup\{\varepsilon > 0 : \exp_x^U \text{ is a diffeomorphism of } \mathbb{B}^{T_x\Sigma}(0,\varepsilon) \text{ onto its image in } U\}, \tag{4.4.30}$$

is called the injectivity radius of U. If U contains any punctures, we always have $\operatorname{inj}^U = 0$ since the injective radius of a point $x \in U$ goes to 0 as x approaches a puncture in U. On the other hand, if U is relatively compact in Σ, then $\operatorname{inj}^U > 0$.

Fix a point $x_0 \in \Sigma$ away from the punctures and fix an open neighborhood $U_0 \subset \Sigma$ of x_0 which is relatively compact in Σ. Hence $\operatorname{inj}^{U_0} > 0$. Let $\{e_1, e_2\}$, $\{\mathfrak{e}\}$, and $\{\mathfrak{f}\}$ be orthonormal bases for $T_{x_0}\Sigma$, E_{x_0} and L_{x_0} respectively, and let $\{w = \frac{1}{\sqrt{2}}(e_1 - \mathbf{i}e_2)\}$ be an orthonormal basis for $T^{(1,0)}_{x_0}\Sigma$.

Fix a positive real number $0 < \varepsilon < \operatorname{inj}^{U_0}/4$ such that the vanishing order of R^L on $\mathbb{B}^\Sigma(x_0, 4\varepsilon)$ is at most $\rho_{x_0} - 2$. Since ε does not exceed the injectivity radius of U_0, the exponential map

$$T_{x_0}\Sigma \supset \mathbb{B}^{T_{x_0}\Sigma}(0, 4\varepsilon) \ni Z \mapsto \exp^\Sigma_{x_0}(Z) \in \mathbb{B}^\Sigma(x_0, 4\varepsilon) \subset \Sigma \tag{4.4.31}$$

is a diffeomorphism of open balls; it yields a local coordinate system in $T_{x_0}\Sigma$:

$$\mathbb{R}^2 \ni (Z_1, Z_2) \longmapsto Z_1e_1 + Z_2e_2 \in T_{x_0}\Sigma\,, \tag{4.4.32}$$

and thus a local chart near x_0, called the normal coordinate system (centered at x_0).

4.4.2.2 Local trivialization of bundles and local description of curvature

We always identify $\mathbb{B}^{T_{x_0}\Sigma}(0,4\varepsilon)$ with $\mathbb{B}^{\Sigma}(x_0,4\varepsilon)$ via (4.4.31). For $Z \in \mathbb{B}^{T_{x_0}\Sigma}(0,4\varepsilon)$ we identify L_Z, E_Z and $\Lambda^\bullet(T_Z^{*(0,1)}\Sigma)$ to L_{x_0}, E_{x_0} and $\Lambda^\bullet(T_{x_0}^{*(0,1)}\Sigma)$, respectively, by parallel transport with respect to ∇^L, ∇^E and $\nabla^{\Lambda^\bullet(T^{*(0,1)}\Sigma)}$ along $\gamma_Z : [0,1] \ni u \mapsto \exp_{x_0}^{\Sigma}(uZ)$. This way, we trivialize the bundles L, E, $\Lambda^\bullet(T^{*(0,1)}\Sigma)$ near x_0. In particular, we will still denote by $\{e_1, e_2\}$, $\{\mathfrak{e}\}$, and $\{\mathfrak{f}\}$ the respective orthonormal smooth frames of the vector bundles on point Z, defined as the parallel transports as above of the vectors $\{e_1, e_2\}$, $\{\mathfrak{e}\}$, and $\{\mathfrak{f}\}$ from x_0.

With the above local trivializations, we write the connection $\nabla^{\Lambda^{0,\bullet}\otimes L^p\otimes E}$ as follows

$$\nabla^{\Lambda^{0,\bullet}\otimes L^p\otimes E} = \mathrm{d} - \left(a^{\Lambda^{0,\bullet}} + pa^L + a^E\right) \tag{4.4.33}$$

where d denotes the ordinary differential operator, and $a^{\Lambda^{0,\bullet}}, a^E, a^L$ are respectively the local connection 1-forms of $\nabla^{\Lambda^{0,\bullet}}, \nabla^E, \nabla^L$ in this trivialization. Note that these connection 1-forms are purely imaginary.

In coordinate (Z_1, Z_2), we write

$$a^L = \sum_{i=1}^{2} a_i^L \mathrm{d}Z_i. \tag{4.4.34}$$

Let R_{ij}^L denote the coefficients of the curvature form R^L with respect to the (local) frame $\mathrm{d}Z_i \wedge \mathrm{d}Z_j$, $i,j = 1,2$. We have

$$R_{11}^L = R_{22}^L \equiv 0\,, \qquad\qquad R_{12}^L = -R_{21}^L\,. \tag{4.4.35}$$

Then we can write

$$R_Z^L = R_{12,Z}^L \;\mathrm{d}Z_1 \wedge \mathrm{d}Z_2 \tag{4.4.36}$$

Similarly, we define $R_{ij,Z}^{\Lambda^{0,\bullet}}$ and $R_{ij,Z}^E$. Moreover, we have the following relations for $Z \in B^{T_{x_0}\Sigma}(0,\varepsilon)$

$$a_{i,Z}^L = \sum_{j=1}^{2} \int_0^1 tZ^j R_{ij,tZ}^L \mathrm{d}t\,. \tag{4.4.37}$$

The analogous identities also hold for $a^{\Lambda^{0,\bullet}}, a^E$.

On the other hand, in these normal coordinates, we find that the curvature R^L of ∇^L

has the following Taylor expansion at the origin

$$R_Z^L = \sum_{|\alpha|=\rho_{x_0}-2} R^L_{12;\alpha} Z^\alpha \mathrm{d}Z_1 \wedge \mathrm{d}Z_2 + \mathcal{O}(|Z|^{\rho_{x_0}-1}) =: R^L_{Z,0} + \mathcal{O}(|Z|^{\rho_{x_0}-1}) \,, \qquad (4.4.38)$$

where the $(\mathrm{d}Z_1 \wedge \mathrm{d}Z_2)$-coefficient of $R^L_{Z,0}$ is the product of $-\mathbf{i}$ and a positive homogeneous even polynomial of order $\rho_{x_0} - 2$ in Z.

4.4.2.3 Local model for B_p

Now we construct the local model for B_p near the point x_0. For this purpose, set $\Sigma_0 := T_{x_0}\Sigma \cong \mathbb{R}^2$, and as before (see (4.4.32)) let $Z = (Z_1, Z_2)$ denote the natural coordinate on Σ_0. Denote by (L_0, h_0), (E_0, h^{E_0}) the trivial line bundles on Σ_0; their fibers are exactly (L_{x_0}, h_{x_0}) and $(E_{x_0}, h^E_{x_0})$, respectively. We equip Σ_0 with J_0 the almost complex structure on Σ_0 that coincides with the pull-back of the complex structure J on Σ by the exponential map (4.4.31) on $\mathbb{B}^\Sigma(x_0, 2\varepsilon)$, and equals to J_{x_0} outside $\mathbb{B}^\Sigma(x_0, 4\varepsilon)$. In a similar spirit, let $g^{T\Sigma_0}$ denote the Riemannian metric on Σ_0 that is compatible with J_0 and that coincides with Riemannian metric $g^{T\Sigma}$ on $\mathbb{B}^\Sigma(x_0, 2\varepsilon)$, and equals to $g^{T\Sigma}_{x_0}$ outside $\mathbb{B}^\Sigma(x_0, 4\varepsilon)$. It follows immediately that J_0 is integrable and the triplet $(\Sigma_0, J_0, g^{T\Sigma_0})$ becomes a Riemann surface equipped with a Kähler form, which we denote by ω_{Σ_0}, that is induced by $g^{T\Sigma_0}$.

Let $T^{*(0,1)}\Sigma_0$ denote the anti-holomorphic cotangent bundle of (Σ_0, J_0), and let $\widetilde{\nabla}^{\Lambda^{0,\bullet}}$ denote the Hermitian connection on $\Lambda^\bullet(T^{*(0,1)}\Sigma_0)$ associated with the Levi-Civita connection of $(T\Sigma_0, g^{T\Sigma_0})$. Note that on $\mathbb{B}^{T_{x_0}\Sigma}(0, 2\varepsilon)$, the pair $(\Lambda^\bullet(T^{*(0,1)}\Sigma_0), \widetilde{\nabla}^{\Lambda^{0,\bullet}})$ coincides with $(\Lambda^\bullet(T^{*(0,1)}\Sigma), \nabla^{\Lambda^\bullet(T^{*(0,1)}\Sigma)})$ via the identification (4.4.31), and outside $\mathbb{B}^{T_{x_0}\Sigma}(0, 4\varepsilon)$, the connection $\widetilde{\nabla}^{\Lambda^{0,\bullet}}$ is given by the trivial connection on the trivial bundle $\Lambda^\bullet(T^{*(0,1)}_{x_0}\Sigma)$. We can always trivialize $T^{*(0,1)}\Sigma_0$ by the parallel transport along the geodesic rays starting at 0, so that for $Z \in \Sigma_0$, we have the identification of fibers $T^{*(0,1)}_Z\Sigma_0 \simeq T^{*(0,1)}_{x_0}\Sigma$.

4.4.2.4 Modified Dirac and Kodaira Laplace operators

Fix an even smooth function $\chi \in C^\infty(\mathbb{R}, [0,1])$ with $\chi = 1$ on $[-2, 2]$ and $\operatorname{supp}\chi \subset [-4, 4]$. We defined a nonnegative curvature form by modifying R^L as follows, for $Z \in \Sigma_0$,

$$\widetilde{R}^{L_0}_Z := \chi\left(\frac{|Z|}{\varepsilon}\right) R^L_Z + \left(1 - \chi\left(\frac{|Z|}{\varepsilon}\right)\right) R^L_{0,Z} \,, \qquad (4.4.39)$$

where R_0^L is defined in (4.4.38). On Σ_0, define a 1-form (compare to (4.4.34) and (4.4.37)):

$$\tilde{a}^{L_0} = \sum_{i=1}^{2} \tilde{a}_i^{L_0} \mathrm{d}Z_i \,, \qquad \tilde{a}_i^{L_0}(Z) := \int_0^1 tZ^j \widetilde{R}^{L_0}_{ij,tZ} \mathrm{d}t \,. \tag{4.4.40}$$

Then we set

$$\begin{aligned} \widetilde{\nabla}^{E_0} &= \mathrm{d} - \chi\left(\frac{|Z|}{\varepsilon}\right) a^E \,, \\ \widetilde{\nabla}^{L_0} &= \mathrm{d} - \tilde{a}^{L_0} \,, \end{aligned} \tag{4.4.41}$$

which are Hermitian connections on the line bundles L_0 and E_0, respectively. Moreover, the curvature form of $\widetilde{\nabla}^{L_0}$ is exactly $\widetilde{R}^{L_0}$.

As in (1.1.5), we define for $Z \in \Sigma_0$,

$$\tilde{\rho}_Z := 2 + \mathrm{ord}_Z(\widetilde{R}^{L_0}) \,. \tag{4.4.42}$$

Since both the vanishing order of R^L on $\mathbb{B}^{\Sigma}(x_0, 4\varepsilon)$ and the vanishing order of R_0^L on Σ_0 are at most $\rho_{x_0} - 2$, we get

$$\tilde{\rho}_Z \leqslant \rho_{x_0} \,. \tag{4.4.43}$$

In particular, $\tilde{\rho}_0 = \rho_{x_0}$, and if $\widetilde{R}^{L_0}(Z) \neq 0$, we have $\tilde{\rho}_Z = 2$.

We now define Dirac and Kodaira Laplace operators on Σ_0. Note that we can either use the formulas in (4.4.3), or equivalently use the connections $\widetilde{\nabla}^{\Lambda^{0,\bullet}}$, $\widetilde{\nabla}^{L_0}$, $\widetilde{\nabla}^{E_0}$ to define the Dirac operator $\widetilde{D}_p$ by (3.1.6). Then we have the operators

$$\begin{aligned} \widetilde{D}_p &: \Omega_{\mathrm{c}}^{0,\bullet}(\Sigma_0, L_0^p \otimes E_0) \longrightarrow \Omega_{\mathrm{c}}^{0,\bullet}(\Sigma_0, L_0^p \otimes E_0) \,, \\ \widetilde{\square}_p := \frac{1}{2}(\widetilde{D}_p)^2 &: \Omega_{\mathrm{c}}^{0,\bullet}(\Sigma_0, L_0^p \otimes E_0) \longrightarrow \Omega_{\mathrm{c}}^{0,\bullet}(\Sigma_0, L_0^p \otimes E_0) \,. \end{aligned} \tag{4.4.44}$$

The above operators each extend to a respective unique self-adjoint operator acting on $\mathcal{L}^2$-sections over Σ_0; we will identify the operators in (4.4.44) with their respective extension. By construction, the differential operators $\widetilde{D}_p$ and $\widetilde{\square}_p$ coincide with D_p and $\square_p$, respectively, on $\mathbb{B}^{T_{x_0}\Sigma}(0, 2\varepsilon) \simeq \mathbb{B}^{\Sigma}(x_0, 2\varepsilon)$.

Let $\widetilde{\Delta}^{\Lambda^{0,\bullet} \otimes L_0^p \otimes E_0}$ denote the Bochner Laplacian associated to the connection $\widetilde{\nabla}^{\Lambda^{0,\bullet} \otimes L_0^p \otimes E_0}$. Analogous to the Lichnerowicz formula from (3.1.8), we have

$$\begin{aligned} \widetilde{\square}_p = \frac{1}{2}\widetilde{\Delta}^{\Lambda^{0,\bullet} \otimes L_0^p \otimes E_0} &+ \frac{r^{\Sigma_0}}{4}\overline{\omega}^* \wedge \iota_{\overline{\omega}} + p(\widetilde{R}^{L_0}(\omega, \overline{\omega})\overline{\omega}^* \wedge \iota_{\overline{\omega}} - \frac{1}{2}\widetilde{R}^{L_0}(\omega, \overline{\omega})) \\ &+ (\widetilde{R}^{E_0}(\omega, \overline{\omega})\overline{\omega}^* \wedge \iota_{\overline{\omega}} - \frac{1}{2}\widetilde{R}^{E_0}(\omega, \overline{\omega})) \,, \end{aligned} \tag{4.4.45}$$

where ω denotes a local frame of $T^{*(1,0)}\Sigma_0$ with $|\omega| = 1$, the function r^{Σ_0} is the scalar curvature of $(\Sigma_0, g^{T\Sigma_0})$, and $\widetilde{R}^{E_0}$ is the curvature form of $\widetilde{\nabla}^{E_0}$. Moreover, r^{Σ_0}, R^{E_0} vanishes identically outside of the region $\mathbb{B}^{T_{x_0}\Sigma}(0, 4\varepsilon)$.

By (4.4.45), $\widetilde{\square}_p$ preserves the degree of $\Lambda^\bullet(T^{*(0,1)}\Sigma)$. For $j = 0, 1$, let $\widetilde{\square}_p^j$ denote the restriction of $\widetilde{\square}_p$ onto $\Omega^{0,j}_{(2)}(\Sigma_0, L_0^p \otimes E_0)$. For $\widetilde{\Delta}^{\Lambda^{0,\bullet}\otimes L_0^p\otimes E_0}$, analogous to (3.2.4) by the same sub-elliptic estimate in [47, Proposition 6], we get that there exist constants C_1', $C_2' \in \mathbb{R}_{>0}$, such that the spectra of $\widetilde{\square}_p^0$ and $\widetilde{\square}_p^1$ satisfy

$$
\begin{aligned}
&\operatorname{Spec}(\widetilde{\square}_p^0) \subset \{0\} \cup [C_1' p^{2/\rho_{x_0}} - C_2', \infty)\,, \\
&\operatorname{Spec}(\widetilde{\square}_p^1) \subset [C_1' p^{2/\rho_{x_0}} - C_2', \infty)\,.
\end{aligned} \tag{4.4.46}
$$

Set

$$
H^0_{(2)}(\Sigma_0, L_0^p \otimes E_0) := \ker(\widetilde{\square}_p^0) \tag{4.4.47}
$$

and consider the orthogonal projection

$$
\widetilde{B}_{x_0,p} : \mathcal{L}_2^{0,0}(\Sigma_0, L_0^p \otimes E_0) \to H^0_{(2)}(\Sigma_0, L_0^p \otimes E_0)\,. \tag{4.4.48}
$$

Let $\widetilde{B}_{x_0,p}(Z, Z')$ denote the Schwartz kernel of $\widetilde{B}_{x_0,p}$ with respect to the volume form induced by $g^{T\Sigma_0}$, which is smooth on $\Sigma_0 \times \Sigma_0$.

We can now proceed as in Section 4.2. In particular, by Proposition 4.2.2, we get that for all $\ell, m \in \mathbb{N}_0$ and any $\gamma > \frac{1}{2}$, there exists $C_{\ell,m} \in \mathbb{R}_{>0}$ such that for any $p \in \mathbb{N}$, we have

$$
\left\| B_p(x, x') - \widetilde{B}_{x_0,p}(x, x') \right\|_{\mathcal{C}^m(\mathbb{B}^\Sigma(x_0,\varepsilon)\times\mathbb{B}^\Sigma(x_0,\varepsilon), h_p)} \leqslant C_{\ell,m,\gamma}\, p^{-\ell}. \tag{4.4.49}
$$

In a shorter notation, we will abbreviate the above statement by writing

$$
B_p - \widetilde{B}_{x_0,p} = \mathcal{O}(p^{-\infty})\,, \text{ on } \mathbb{B}^\Sigma(x_0, \varepsilon) \times \mathbb{B}^\Sigma(x_0, \varepsilon)\,. \tag{4.4.50}
$$

4.5 Proofs of Bergman kernel estimates

In this section, we present proof to Corollaries 4.1.3 and 4.1.6 and to Lemmas 4.1.7 and 4.1.9.

We begin by proving Corollary 4.1.3.

Proof of Corollary 4.1.3. For $x \in \Sigma$ near a puncture, (4.3.14), together with Proposition

4.3.1 and Corollary 4.3.2 implies that

$$\sup_{|x|\leqslant r\leqslant e^{-1}} |B_p(x)|_{h^p} = \left(\frac{p}{2\pi}\right)^{3/2} + \mathcal{O}(p) \qquad \text{as } p \to \infty\,. \tag{4.5.1}$$

By [43, Theorems 6.1.1, 6.2.3] the following asymptotic expansion holds for any compact set $K \subset \Sigma$ and in any $\mathcal{C}^m$-topology,

$$\frac{1}{p}B_p(x) = \frac{1}{2\pi} + \sum_{j=1}^{\infty} \mathbf{b}_j(x)p^{-j} \qquad \text{as } p \to \infty\,. \tag{4.5.2}$$

Now the variational characterization of the Bergman kernel and (4.5.1) conclude the proof. □

Proof of Corollary 4.1.6. We prove (4.1.7) in two separate cases. Suppose $h \in (0,1)$ and $\gamma \in (0, \frac{1}{2})$ are arbitrary, but fixed.

($\rho_\Sigma = 2$) In this case, the order of vanishing of the curvature is 0 on Σ, hence R^L is strictly positive on Σ, i.e. we have $\Sigma_{\rho_\Sigma} = \Sigma$ and thus, $\Sigma^{p,h,\gamma}_{\rho_\Sigma} = \Sigma^{p,h,\gamma}$ for all $p \in \mathbb{N}$. Note that for any fixed $p \in \mathbb{N}$, the set $\Sigma^{p,h,\gamma} \subset \Sigma$ is a relatively compact. Thus the number $c(p,h,\gamma,2)$ is a positive constant by Corollary 4.4.2.

($\rho_\Sigma > 2$) For all $p \in \mathbb{N}$, we have $\Sigma^{p,h,\gamma}_{\rho_\Sigma} = \Sigma_{\rho_\Sigma}$. The set $\Sigma_{\rho_\Sigma} \subset \Sigma$ is relatively compact. Thus, there exists a constant $c(\rho_\Sigma) \in \mathbb{R}_{>0}$ independent of $p \in \mathbb{N}$ (and even of h and γ), such that $c(p,h,\gamma,\rho_\Sigma) = c(\rho_\Sigma)$. Hence, $c(p,h,\gamma,\rho_\Sigma)$ is a positive constant by Corollary 4.4.2.

This proves (4.1.7).

For the second claim, first recall that the local model near any of the punctures is the same. Hence, it suffices to look at what happens near a single puncture a_j, for some $1 \leqslant j \leqslant N$, since from this we can conclude the same behavior of the Bergman kernel.

Observe that by assumption (β)(ii), the curvature R^L does not vanish in the open neighborhood $\overline{V}_j$ of a puncture a_j, for any $1 \leqslant j \leqslant N$. Hence there exists $p_0 \in \mathbb{N}$, such that $\mathbb{D}^*(a_j, he^{-p_0{}^\gamma}) \subset \overline{V}_j$ and the value of $B^{j_x^{\rho_\Sigma - 2}R^L}(0,0)$ is constant for all $x \in \mathbb{D}^*(a_j, he^{-p_0{}^\gamma})$.

Finally, the last argument together with (4.1.7), and Remark 4.1.5 imply, that for all sufficiently large $p \in \mathbb{N}$, the constant $c(p,h,\gamma,\rho_\Sigma) =: c(\rho_\Sigma) \in \mathbb{R}_{>0}$ is independent of h and γ. This concludes the proof. □

We now sketch a proof of Lemma 4.1.7. The proof mostly follows the same arguments as the proof of [47, Lemma 12], with a small adjustment in the beginning on the estimate from below.

Proof of Lemma 4.1.7. At first, as in [47, (3.30)], note that Theorem 4.1.1 implies that for all $x \in \Sigma$, there exists constants $C_{\rho_x}, c_x \in \mathbb{R}_{>0}$, such that

$$B_p(x,x) \geqslant C_{\rho_x}\left(|j^{\rho_x-2}R^L|p\right)^{2/\rho_x} - c_x\,, \tag{4.5.3}$$

with $c_x := c\left(|j^{\rho_x-2}R^L(x)|^{-1}\right)$ for some $c \in \mathbb{R}_{>0}$ and $C_{\rho_x} := B^{R_x}(0,0)$ is the constant from (4.4.10), which is positive by Lemma 4.4.1; here, B^{R_x} is the model Berman kernel for the local model centered at $x \in \Sigma$.

Now fix any pair of real numbers $h \in (0,1)$ and $\gamma \in (0,\frac{1}{2})$. For the rest of the proof, we take any $x \in \Sigma^{p,h,\gamma}$ for some arbitrary $p \in \mathbb{N}$.

Then, as argued in the proof of [47, Lemma 12], for any $\varepsilon > 0$, there exists a uniform constant $c_\varepsilon \in \mathbb{R}_{>0}$ that only depends on ε and $\|R^L\|_{\mathcal{C}^{\rho_\Sigma}}$, such that

$$|j^{\rho_x-2}R^L(\mathrm{x})| \geqslant (1-\varepsilon)|j^{\rho_x-2}R^L(x)|\,, \tag{4.5.4}$$

for all $\mathrm{x} \in \mathbb{B}_{c_\varepsilon|j^{\rho_x-2}R^L|}(x)$. The next step is to dissect the set $\mathbb{B}_{c_\varepsilon|j^{\rho_x-2}R^L|}(x)$ in terms of different orders of vanishing of the curvature and use the model Kodaira Laplacian $\tilde{\square}_p$ near x in terms of geodesic coordinates centered at x and rescaling at different levels centered at x, to successively give estimates for $B_p(\mathrm{x},\mathrm{x})$ from below, in terms of the values of the model Bergman kernel $B^{j_x^{\rho_x-2}R^L}(0,0)$ at the origin $(0,0)$. The reader can follow the arguments in the authors proof of [47, Lemma 12] for the rescalings and estimations in the associated regions in $\mathbb{B}_{c_\varepsilon|j^{\rho_x-2}R^L|}(x)$.

Then, since $\Sigma^{p,h,\gamma}$ is a relatively compact subset of Σ, there exists a finite set of points $\{x_1\}_{i=1}^m \subset \Sigma^{p,h,\gamma}$, such that the balls $\mathbb{B}_{c_\varepsilon|j^{\rho_{x_i}-2}R^L|}(x_i)$ cover the whole set $\Sigma^{p,h,\gamma}$. Hence, there exists a uniform constant $c_{h,\gamma,\varepsilon} \in \mathbb{R}_{>0}$, such that

$$B_p(x,x) \geqslant (1-\varepsilon)\left[\inf_{x\in\Sigma^{p,h,\gamma}_{\rho_\Sigma}} B^{j_x^{\rho_\Sigma-2}R^L}(0,0)\right]p^{2/\rho_\Sigma} - c_{h,\gamma,\varepsilon} \tag{4.5.5}$$

for all $x \in \Sigma^{p,h,\gamma}$ and all $\varepsilon > 0$. By Corollary 4.1.6, we conclude that for all sufficiently large $p \in \mathbb{N}$,

$$B_p(x,x) \geqslant (1+o(1))c(\rho_\Sigma)p^{2/\rho_\Sigma} \tag{4.5.6}$$

for all $x \in \Sigma^{p,h,\gamma}$ and the $o(1)$ term is uniform in $x \in \Sigma^{p,h,\gamma}$, as $p \to \infty$. The proof of the upper bound is the same as the proof of the authors upper bound in [47, Lemma 12]. $\square$

We conclude with a sketch of the proof of Lemma 4.1.9.

Proof of Lemma 4.1.9. We follow the arguments from the authors related result in [47, Lemma 13]. As in the proof of their related result, the arguments that are needed for an the upper bound (4.1.10) for the l-th jets of of Bergman kernels follow from the proof of Lemma 4.1.7.

Fix any pair of real numbers $h \in (0,1)$ and $\gamma \in (0, \frac{1}{2})$. any pair of real numbers $h \in (0,1)$ and $\gamma \in (0, \frac{1}{2})$ and let $x \in \Sigma^{p,h,\gamma}$ for some arbitrary $p \in \mathbb{N}$.

Let $\varepsilon > 0$, then as in (4.5.4), there exists a uniform constant $c_\varepsilon \in \mathbb{R}_{>0}$ that only depends on ε and $\|R^L\|_{\mathcal{C}^{\rho_\Sigma}}$, such that

$$|j^{\rho_x - 2} R^L(\mathrm{x})| \geqslant (1-\varepsilon)|j^{\rho_x - 2} R^L(x)|\,, \tag{4.5.7}$$

for all $\mathrm{x} \in \mathbb{B}_{c_\varepsilon |j^{\rho_x-2} R^L|}(x)$. As in the proof of Lemma 4.1.7 the same stepwise rescaling procedure and compactness argument imply, together with Corollary 4.1.6, that there exists $p_{h,\gamma} \in \mathbb{N}$, such that for all $x \in \Sigma^{p,h,\gamma}$ and all $\alpha \in \mathbb{N}_0^2$, there exists a uniform constant $c_{h,\gamma,\varepsilon} \in \mathbb{R}_{>0}$, such that

$$\frac{|\,\partial^\alpha B_p(x,x)|}{B_p(x,x)} \leqslant p^{|\alpha|/3}(1-\varepsilon)\left[\sup_{x\in\Sigma} \frac{\left|j^{|\alpha|} B^{j_x^1 R^L / j_x^0 R^L}(0,0)\right|}{B^{j_x^1 R^L / j_x^0 R^L}(0,0)}\right] + c_{h,\gamma,\varepsilon}\,, \tag{4.5.8}$$

for all $\varepsilon > 0$. This completes the proof. $\square$

5. Kodaira map, Tian's theorem and convergence of induced Fubini-Study currents

5.1 Hyperplane line bundles and Fubini-Study metrics

For a complex vector space V we denote by V^* its dual and by $\mathbb{P}(V^*)$ the projective space associated to V^*. We define the tautological line bundle on $\mathbb{P}(V^*)$ as

$$\mathcal{O}(-1) = \{(\mathbf{l}, f) \in \mathbb{P}(V^*) \times V^* \,:\, f \in \mathbf{l} \subset V^*\} \tag{5.1.1}$$

and by $(\mathcal{O}(-1))^* = \mathcal{O}(1)$ its dual, the hyperplane line bundle on $\mathbb{P}(V^*)$. As a sub-bundle of the trivial Hermitian line bundle $\underline{V^*}$ on $\mathbb{P}(V^*)$, $\mathcal{O}(-1)$ inherits a Hermitian metric $h^{\mathcal{O}(-1)}$ by restriction of the canonical Hermitian metric on $\underline{V^*}$. Then $h^{\mathcal{O}(-1)}$ induces a dual Hermitian metric $h^{\mathcal{O}(1)}$ on $\mathcal{O}(1)$.

Given a vector $v \in V$, we define a linear map $\sigma_v(f) := (f, v) = f(v) \in \mathbb{C}$ via the natural pairing between V and its algebraic dual V^*. The norm with respect to the Hermitian metric $h^{\mathcal{O}(1)}$ is then given by

$$|\sigma_v([f])|^2_{h^{\mathcal{O}(1)}} = \frac{|(f,v)|^2}{|f|_{h^{\mathcal{O}(1)}}} \qquad \text{for } f \in V^* \setminus \{0\}\,,\ [f] \in \mathbb{P}(V^*)\,, \tag{5.1.2}$$

where the expression in the enumerator is the standard Euclidean norm of (f, v).

In light of the above background, we define the (Kähler form of the) Fubini-Study metric on $\mathbb{P}(V^*)$ as

$$\omega_{\mathrm{FS}} = \frac{\mathbf{i}}{2\pi} R^{\mathcal{O}(1)} = -\frac{\mathbf{i}}{2\pi}\,\partial\bar{\partial} \log |\sigma_s|^2_{h^{\mathcal{O}(1)}}\,, \tag{5.1.3}$$

where $R^{\mathcal{O}(1)}$ is the curvature of the Chern connection of $h^{\mathcal{O}(1)}$, for all $f \in V^* \setminus \{0\}$.

Example 5.1.1. A common case is $V^* \cong \mathbb{C}^n$, such that $\mathbb{P}(V) \cong \mathbb{CP}^{n-1}$, for any $n \in \mathbb{N}_{\geqslant 2}$. The tautological line bundle in this case is given by

$$\mathcal{O}(-1) = \{([u], v) \in \mathbb{CP}^{n-1} \times \mathbb{C}^n : v \in [u] = \mathbb{C}u \subset \mathbb{CP}^{n-1}\}\,; \tag{5.1.4}$$

the projection $\pi : \mathcal{O}(-1) \to \mathbb{CP}^{n-1}$ is defined by $\pi([u], v) = [u]$. The usual open covering $(U_i)_{i=1}^{n-1}$ of $\mathbb{CP}^{n-1}$ is given via the open subsets

$$U_i = \{[z_0 : \ldots : z_{n-1}] \in \mathbb{CP}^{n-1} : z_i \neq 0\} \tag{5.1.5}$$

with smooth charts

$$\tau_i : U_i \times \mathbb{C} \longrightarrow \pi^{-1}(U_i)\,, \qquad ([u], z) \longmapsto \left([u], \frac{z}{u_i} u\right) \tag{5.1.6}$$

and the definition of this map is by evidently independent of the choice of representative u of the class $[u]$. The smooth inverse is $\tau_i^{-1}([u], v) = ([u], v_i)$. This yields transition functions

$$f_{ij} : U_i \cap U_j \longrightarrow \mathbb{C} \setminus \{0\} \qquad [u] \longmapsto \frac{u_j}{u_i} \tag{5.1.7}$$

which are holomorphic. Now the adjoint bundle $\mathcal{O}(1) = \mathcal{O}(-1)^*$ over $\mathbb{C}^n$ is defined by the family of transition functions $(\tilde{f}_{ij})_{i,j}$ for $\tilde{f}_{ij} := f_{ij}^{-1}$.

The Fubini-Study metric takes the form

$$\omega_{\mathrm{FS}}^n = \left(\frac{\mathbf{i}}{2}\right)^n n! \det(g_{i\bar{j}})\, \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 \wedge \ldots \wedge \mathrm{d}z_n \wedge \mathrm{d}\bar{z}_n\,, \tag{5.1.8}$$

where,

$$g_{i\bar{j}} := \frac{(1 + |z|^2)\delta_{ij} - \bar{z}_i z_j}{(1 + |z|^2)^2} \tag{5.1.9}$$

in the local coordinate $z \in U_i$ for any $U_i \cong \mathbb{C}$ and δ_{ij} is the Kronecker-delta. Observe that for any invertible complex $n \times n$ matrix A and complex vectors $u, v \in \mathbb{C}^n$, the relation of determinants

$$\det(A + uv^t) = (1 + v^t A^{-1} u) \det(A) \tag{5.1.10}$$

holds, where v^t is the transpose of v. Equation (5.1.10) implies that locally,

$$\det(g_{i\bar{j}}) = \frac{1}{(1 + |z|^2)^{n+1}}\,. \tag{5.1.11}$$

Thus, for any U_i,

$$\begin{aligned}\int_{U_i} \omega_{\mathrm{FS}}^n &= \int_{\mathbb{C}^{n+1}} \left(\frac{\mathbf{i}}{2}\right)^n n! \det(g_{i\bar{j}})\, \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 \wedge \ldots \wedge \mathrm{d}z_n \wedge \mathrm{d}\bar{z}_n \\ &= \left(\frac{\mathbf{i}}{2}\right)^n n!(-2\,\mathbf{i})^n \mathrm{vol}(S^{2n-1}) \int_0^\infty \frac{r^{2n-1}}{(1+r^2)^{n+1}} \mathrm{d}r \\ &= \pi^n 2n \int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta \cos\theta \mathrm{d}\theta = \pi^n \\ \Rightarrow \qquad \int_{\mathbb{CP}^n} \omega_{\mathrm{FS}}^n &= \pi^n\,, \end{aligned} \tag{5.1.12}$$

where $S^{2n-1} \subset \mathbb{C}^n$ is the unit sphere. The last implication holds because the complement $U_i^{\complement} \subset \mathbb{CP}^{n-1} \cong \mathbb{C}^{n-1} \cup \{0\}$ is a null set with respect to the Lebesgue measure in $\mathbb{C}^{n-1}$ that we have integrated against, for all U_i.

In Chapter 6, we will also consider Fubini-Study metrics on $\mathbb{P}(V)$, which we will also denote by ω_{FS}; these are defined in the same way as the metrics in (5.1.3). To mitigate confusion, we will highlight which Fubini-Study metric is meant in the appropriate place, if necessary.

The following point of view will simplify the definition of the Kodaira map, which we will discuss in the following sections of this chapter: Recall that the Grassmanian $G_{\dim_{\mathbb{C}} V-1}(V)$ is defined as the set of $(\dim_{\mathbb{C}} V - 1)$-dimensional complex linear subspaces, i.e. hyperplanes, of V and there is an isomorphy between complex vector spaces $G_{\dim_{\mathbb{C}} V-1}(V) = \mathbb{P}(V^*)$.

5.2 Kodaira map

In this section, we will define the Kodaira map which maps our manifold into a projective space. We want to compare the construction of the Kodaira map, as it is done in [43, Chapter 5], to our setting, step-by-step, and highlight possible difficulties that arise from our setting. In the book, the authors consider the case of a (strictly) positive holomorphic Hermitian line bundle over a compact complex manifold. Consequently, both, the possible vanishing of the curvature of our semipositive Hermitian metric h^L on L away from the puncture divisor D and the singular behavior of h^L at the punctures, as it was described in (α) and (β) in Chapter 1, pose complications.

In [1] and [2], Auvray, Ma and Marinescu have considered the same singular setting as we do in our case; in particular, the authors explain the construction and properties of the associated Kodaira map associated to a positively curved, singular Hermitian metric.

In [47], Marinescu and Savale discuss the case of a semipositive Hermitian metric on a holomorphic line bundle on a compact Riemann surface, without the presence of punctures in the manifold. The authors discuss the behavior of the Kodaira map that is associated to the semipositive Hermitian metric.

We will take advantage of these seminal works and utilize their arguments for our setting.

5.2.1 The base locus of a vector space of sections

The complex vector space from the previous section will be the space of holomorphic sections of $L^p \otimes E$: we set $V_{p,(2)} := H^0_{(2)}(\Sigma, L^p \otimes E)$. In light of the discussion in Subsection 2.6.2, set $V_p := H^0(\overline{\Sigma}, L^p \otimes E \otimes \mathcal{O}_{\overline{\Sigma}}(-D))$.

In order to define the Kodaira map, we need the following definition: The base locus of V_p is the set

$$\mathrm{Bl}_{V_p} := \{x \in \overline{\Sigma} \,:\, s(x) = 0 \text{ for all } s \in V_p\} . \tag{5.2.1}$$

The key interest in the Kodaira map lies in obtaining projective coordinates by taking global sections of a line bundle for each point on the manifold. This is possible at a fixed point where not all global sections disappear simultaneously, or, in other words, outside of Bl_{V_p}.

5.2.2 Kodaira maps associated to vector spaces of sections

5.2.2.1 Kodaira maps associated to V_p

So long as the evaluation map $V_{p,(2)} \ni s \mapsto s(x)$ is not identically zero at $x \in \overline{\Sigma}$, there exist $d_p - 1$ sections $s_j \in V_{p,(2)}$ that vanish at x. Hence their common kernel spans a hyperplane in $V_{p,(2)}$; it can be uniquely identified with a line of covectors in the algebraic dual $V^*_{p,(2)}$. We thus identify the projective space $\mathbb{P}(V^*_{p,(2)})$ of lines in $V^*_{p,(2)}$ to the Grassmanian manifold $G_{d_p-1}(V_{p,(2)})$ of $(d_p - 1)$-dimensional hyperplanes in $V_{p,(2)}$, and to the same in the case of the vector space V_p.

We define the Kodaira map of V_p at level $p \in \mathbb{N}$ is the meromorphic map

$$\begin{aligned} \Phi_p := \Phi_{V_p} : \overline{\Sigma} \dashrightarrow \mathbb{P}(V_p^*) \,, \\ \Phi_p(x) = \{s \in V_p : s(x) = 0\} \,; \end{aligned} \tag{5.2.2}$$

by definition, the set of poles of Φ_p is contained in the base locus Bl_{V_p} that we have defined in (5.2.1). Later in Subsection 5.2.3, we will give an analytic description of Φ_p, which will also show that they are well-defined as a holomorphic map outside of its set of indeterminacy.

5.2.2.2 Kodaira maps associated to $V_{p,(2)}$

Following the approach from Auvray, Ma and Marinescu in [1, Section 4], we can define another family of Kodaira maps $\Phi_{p,(2)}$ that correspond to the vector spaces $V_{p,(2)}$: in our setting, these corresponding to the meromorphic maps

$$\begin{aligned} \Phi_{p,(2)} := \Phi_{V_{p,(2)}} : \Sigma \dashrightarrow \mathbb{P}(V^*_{p,(2)})\,, \\ \Phi_{p,(2)}(x) = \{s \in V_{p,(2)} : s(x) = 0\}\,. \end{aligned} \tag{5.2.3}$$

The difference between our Kodaira maps $\Phi_{p,(2)}$ and the corresponding Kodaira maps on the punctured Riemann surface in [1] is that the ranges of the maps in (5.2.3) are the projectivizations of the complex vector spaces of holomorphic sections that are bounded with respect to the $\mathcal{L}^2$-norm that is induced by an $\mathcal{L}^2$-inner-product that is induced by the semipositive Hermitian metric h_L (and ω_Σ), instead of being induced by a positive Hermitian metric as in [1].

5.2.2.3 The relationship between Φ_p and $\Phi_{p,(2)}$ and their regularity

Demailly's seminal holomorphic Morse inequalities ([19], see also [43, Theorem 1.7.1], applied to the case where $q = 1$, where the vector bundle E in the book is the line bundle $E \otimes \mathcal{O}_{\overline{\Sigma}}(-D)$ in our case) imply that there exist $C \in \mathbb{R}_{>0}$ and $p_0 \in \mathbb{N}$, such that for all $p \geqslant p_0$, the dimension of V_p obeys $d_p \geqslant C \cdot p$. Hence, as we can find a growing number of linear independent elements in V_p (and consequently, the same can be said about $V_{p,(2)}$ as p grows, by definition of the base locus, the set Bl_{V_p} can not grow in size; in fact, in many cases of interest, it has the tendency to become progressively smaller for growing p (see [43]).

As the dimension grows at least linearly in p, it is reasonable to expect that the meromorphic Kodaira maps are eventually holomorphic on their whole domain, because we obtain an ever growing number of sections to choose a family of coordinates from.

We have the following lemma for the relationship between the two families of Kodaira maps and their respective global behavior.

Lemma 5.2.1. *For each $p \geqslant 2$,*

$$
\begin{tikzcd}
\Sigma \arrow[r, dashed, "\Phi_{p,(2)}"] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{P}(V_{p,(2)}^*) \arrow[d, "\cong"] \\
\overline{\Sigma} \arrow[r, dashed, "\Phi_p"] & \mathbb{P}(V_p^*)
\end{tikzcd}
\tag{5.2.4}
$$

Proof. The commutativity of the diagram (5.2.4) follows from the same reasoning as the commutativity of the corresponding diagram [1, (4.5)]: we have explained the relationship of the two vector spaces V_p and $V_{p,(2)}$, for $p \in \mathbb{N}_{\geqslant 2}$, in Section 2.4.

Let g denote the topological genus of the compact Riemann surface $\overline{\Sigma}$. We now follow the arguments as in [1, Section 4], where the authors refer to Griffiths and Harris [30, p. 215-216]: for sufficiently large $p \in \mathbb{N}$, $p \deg_{\overline{\Sigma}}(L) + \deg_{\overline{\Sigma}}(E) - N > 2g$ by Riemann-Hurwitz formula, hence, the Kodaira maps Φ_p are holomorphic embeddings, for sufficiently large $p \in \mathbb{N}$. In particular, these maps are well-defined on $\overline{\Sigma} \setminus \mathrm{Bl}_{V_p}$, and there exists $p_0 \in \mathbb{N}$, such that $\mathrm{Bl}_p = \varnothing$ for all $p \geqslant p_0$.

Consequently, by the commutativity of the diagram (5.2.4), the Kodaira maps $\Phi_{p,(2)}$ are holomorphic embeddings for sufficiently large $p \in \mathbb{N}$. $\square$

In this context, we recall the definition of ampleness of a line bundle:

A holomorphic line bundle over a compact Kähler manifold is called ample if for sufficiently large $p \in \mathbb{N}$, its associated base locus is empty and the associated Kodaira maps are holomorphic embeddings.

In the case of our compact Riemann surface $\overline{\Sigma}$, by a lemma of Cartan-Serre-Grothendieck (see [43, Lemma 5.1.11]) the existence of $p_0 \in \mathbb{N}$ such that $L^{p_0} \to \overline{\Sigma}$ is ample implies that $F^p \to \overline{\Sigma}$ is ample for all $p \geqslant p_0$.

By Lemma 5.2.1, we can conclude that for both holomorphic line bundles $L \to \overline{\Sigma}$ and its restriction $L|_{\Sigma} \to \Sigma$, sufficiently high tensor powers of L and $L|_{\Sigma}$ admit enough holomorphic sections of the respective bundles to produce a respective basis of homogeneous coordinates in a projective space. In the case of the compact Riemann surface $\overline{\Sigma}$, this is exactly the definition of $L \to \overline{\Sigma}$ being ample.

5.2.3 Isomorphisms induced by the Kodaira maps

We want to study the Fubini-Study metric ω_{FS} introduced in (5.1.3) as a Hermitian metric on Σ. The idea to do this is to consider the pull-back of the hyperplane line bundle

$\mathcal{O}(1)$ by the Kodaira map. For now, we will focus on understanding the pull-backs by the Kodaira maps $\Phi_{p,(2)}$ on Σ for large $p \in \mathbb{N}$.

For all $p \in \mathbb{N}$, we denote by $\omega_{\mathrm{FS},p,(2)}$ the smooth Kähler $(1,1)$-forms that are associated to the Fubini-Study metrics on $\mathbb{P}(V^*_{p,(2)})$.

In general, on the (punctured) Riemann surface Σ, the pull-back by the Kodaira maps $\Phi_{p,(2)}$ of the (Kähler $(1,1)$-forms of the) Fubini-Study metric are $(1,1)$-currents; accordingly, we call the positive closed $(1,1)$-currents $(\Phi_{p,(2)})^*(\omega_{\mathrm{FS},p,(2)})$ the induced Fubini-Study currents on Σ. Immediately, we see that the following two statements hold true:

First, by [21, Lemma 2.1], the currents $\Phi^*_{p,(2)}(\omega_{\mathrm{FS},p,(2)})$ are well-defined and in our case given by a $(1,1)$-form with $\mathcal{L}^1$-integrable coefficients. This $(1,1)$-form is smooth where $\Phi_{p,(2)}$ is holomorphic. When $p \in \mathbb{N}$ is sufficiently large, $\Phi_{p,(2)}$ is holomorphic on Σ and hence $\Phi^*_{p,(2)}(\omega_{\mathrm{FS},p,(2)})$ are given in terms if a smooth $(1,1)$-form on Σ (we denote this $(1,1)$-form in the same way as the $(1,1)$-currents $\Phi^*_{p,(2)}(\omega_{\mathrm{FS},p,(2)})$).

Second, by [17, Theorem 1.1], these $\Phi^*_{p,(2)}(\omega_{\mathrm{FS},p,(2)})$ extend to closed positive $(1,1)$-currents over $\overline{\Sigma}$, which we will gain denote in the same way, when there is enough room to avoid ambiguity, otherwise we emphasize the domain.

For each sufficiently large $p \in \mathbb{N}$, we will now also consider the Hermitian metrics $h^{(\Phi_{p,(2)})^*\mathcal{O}(1)}$ on $(\Phi_{p,(2)})^*\mathcal{O}(1)$ that are induced by $h^{\mathcal{O}(1)}$ on $\mathcal{O}(1)$ on $\mathbb{P}(V^*_{p,(2)})$.

The two Kodaira maps Φ_p and $\Phi_{p,(2)}$ each induce families of isomorphisms of line bundles; we summarize this in the Theorem 5.2.2 below. Under the isomorphisms that are associated to the Kodaira maps $\Phi_{p,(2)}$, we can view the $(1,1)$-currents that are associated to the Hermitian metrics $h^{(\Phi_{p,(2)})^*\mathcal{O}(1)}$ as $(1,1)$-currents on the line bundles $(L^p \otimes E)|_\Sigma \to \Sigma$; by slightly abusing notation, we will denote the currents under these isomorphisms by $(\Phi_{p,(2)})^*(h_{\mathrm{FS},p,(2)})$.

Recall that B_p is the Bergman kernel function from (2.7.4) and h_p is the Hermitian metric on $L^p \otimes E$ that was defined in Section 2.1. The statement of the theorem about the induced families of isomorphisms is the following.

Theorem 5.2.2. *For sufficiently large $p \in \mathbb{N}_{\geqslant 2}$, the Kodaira maps Φ_p and $\Phi_{p,(2)}$ each* *induce families of isomorphisms of vector bundles*

$$\Phi_p^*\mathcal{O}(1) \xrightarrow{\Psi_p} L^p \otimes E \otimes \mathcal{O}_{\overline{\Sigma}}(-D)\,, \tag{5.2.5}$$

$$(\Phi_{p,(2)})^*\mathcal{O}(1) \xrightarrow{\Psi_{p,(2)}} (L^p \otimes E)|_{\Sigma}\,. \tag{5.2.6}$$

Note that $\Phi_p^*\mathcal{O}(1)$ is a holomorphic line bundle over $\overline{\Sigma}$ and $(\Phi_{p,(2)})^*\mathcal{O}(1)$ is a holomophic line bundle over Σ.

Moreover, let $p \in \mathbb{N}$ be sufficiently large. Then the $(1,1)$-currents $(\Phi_{p,(2)})^*(h_{\mathrm{FS},p,(2)})$ are given in terms of smooth $(1,1)$-forms on Σ and for sufficiently large $p \in \mathbb{N}$, we have

$$(\Phi_{p,(2)})^*(h_{\mathrm{FS},p,(2)}) = \frac{1}{B_p(x,x)} h_p(x) \tag{5.2.7}$$

for all $x \in \Sigma$. Hence the $(1,1)$-forms $(\Phi_{p,(2)})^*(h_{\mathrm{FS},p,(2)})$ induce Hermitian metrics, called the induced Fubini-Study Hermitian metrics on Σ. We will denote the Hermitian metrics in the same way as their associated $(1,1)$-forms.

For sufficiently large $p \in \mathbb{N}$, the associated Hermitian metrics $(\Phi_{p,(2)})^*(h_{\mathrm{FS},p,(2)})$ are called induced Fubini-Study Hermitian metrics on Σ.

Proof of Theorem 5.2.2. We follow the arguments in [43, Theorem 5.1.3]. We begin by giving an analytic description of the Kodaira maps Φ_p.

Recall that $V_p = H^0(\overline{\Sigma}, L^p \otimes E \otimes \mathcal{O}_{\overline{\Sigma}}(-D))$ and $d_p = \dim_{\mathbb{C}} H^0(\overline{\Sigma}, L^p \otimes E \otimes \mathcal{O}_{\overline{\Sigma}}(-D))$ as in Subsection 2.6.2. Let $\{s_j\}_{j=1}^{d_p} \subset V_p$ be any basis of globally defined sections and write $\{s^j\}_{j=1}^{d_p}$ for its dual basis. Consider an arbitrary but fixed point $x_0 \in \overline{\Sigma}$ and a local chart $W \subset \overline{\Sigma}$ around x_0. Let $\mathbf{e}_L$, $\mathbf{e}_E$ and $\mathbf{e}_{\mathcal{O}_{\overline{\Sigma}}(-D)}$ be local holomorphic frames of L, E, and $\mathcal{O}_{\overline{\Sigma}}(-D)$, respectively, in the chart around x_0. For each p, we set

$$\mathbf{e}_p := \mathbf{e}_L^{\otimes p} \otimes \mathbf{e}_E \otimes \mathbf{e}_{\mathcal{O}_{\overline{\Sigma}}(-D)}\,; \tag{5.2.8}$$

this is a local holomorphic frame of $L^p \otimes E \otimes \mathcal{O}_{\overline{\Sigma}}(-D)$ in the chart W around x_0. Thus, near x_0, we can write $s_j = f_j \mathbf{e}_p$ and the f_j are holomorphic, for all $j \in \{1, \dots, d_p\}$. For all $s \in V_p$, we have

$$\left(\sum_{j=1}^{d_p} f_j(x) s^j, s \right) \mathbf{e}_p = \sum_{j=1}^{d_p} s_j(x)(s^j, s) = s(x)\,, \tag{5.2.9}$$

which implies that the Kodaira map Φ_p at level p satisfies

$$\Phi_p(x) = \left[\sum_{j=1}^{d_p} f_j(x)s^j\right] \in \mathbb{P}(V_p^*) \tag{5.2.10}$$

locally around x_0. In particular, Φ_p is well-defined on $W \setminus \mathrm{Bl}_{V_p}$ and since the f_j are holomorphic, we get that Φ_p is holomorphic on $W \setminus \mathrm{Bl}_{V_p}$. Furthermore, the same holds true on non-trivially overlapping subsets of two given local charts outside of Bl_{V_p}. Thus, we can glue together a set of local charts that cover $\overline{\Sigma}$ and conclude that Φ_p is well-defined and holomorphic on $\overline{\Sigma} \setminus \mathrm{Bl}_{V_p}$. Finally, by Lemma 5.2.1, there exists $p_0 \in \mathbb{N}$, such that $\mathrm{Bl}_{V_p} = \varnothing$ for all $p \geqslant p_0$. From this, we see directly that Φ_p are holomorphic for all $p \geqslant p_U$.

We proceed with the proof for the first claim. Let $p \in \mathbb{N}$ be a sufficiently large, but fixed positive integer and take any fixed section $s \in V_p$. As in Section 5.1, s induces a linear map $\sigma_s \in V_p^*$ via the natural pairing with its algebraic dual. For $\sum_{j=1}^{d_p} f_j(x)s^j \in \mathcal{O}(-1)_{\Phi_p(x)}$ we have

$$\left((\Phi_p^*\sigma_s)(x), \sum_{j=1}^{d_p} f_j(x)s^j\right)\left(\mathbf{e}_L^{\otimes p} \otimes \mathbf{e}_E\right)(x) = \sum_{j=1}^{d_p} f_j(x)(s, s^j)\left(\mathbf{e}_L^{\otimes p} \otimes \mathbf{e}_E\right)(x) = s(x)\,. \tag{5.2.11}$$

This implies that $(\Phi_p^*\sigma_s)(x) = 0$ if and only if, $s(x) = 0$.

By Lemma 5.2.1, for any $\zeta \in (\Phi_p^*\mathcal{O}(1))_x$, there exists $s \in V_p$, such that $\zeta = (\Phi_p^*\sigma_s)(x)$. Thus, for all $s \in V_p$, the map

$$\begin{aligned} &\Psi_p : \Phi_p^*\mathcal{O}(1) \longrightarrow L^p \otimes E \otimes \mathcal{O}_{\overline{\Sigma}}(-D)\,, \\ &\Psi_p\left((\Phi_p^*\sigma_s)(x)\right) = s(x)\,, \end{aligned} \tag{5.2.12}$$

is well-defined and moreover, Ψ_p is an isomorphism of vector bundles $\overline{\Sigma}$. To obtain the second family of isomorphisms $\Psi_{p,(2)}$ of the theorem, we simply restrict the maps Ψ_p from $\overline{\Sigma}$ and compose them with the isomorphisms from (2.6.6). The resulting family of isomorphisms are

$$\begin{aligned} &\Psi_{p,(2)} : \Phi_{p,(2)}^*\mathcal{O}(1) \longrightarrow (L^p \otimes E)|_\Sigma\,, \\ &\Psi_{p,(2)}\left(((\Phi_{p,(2)})^*\sigma_s)(x)\right) = s(x)\,. \end{aligned} \tag{5.2.13}$$

The second claim follows from the corresponding arguments in [43, Theorem 5.1.3] applied to the case of the line bundle $L^p \otimes E$, see also [1, Section 4], for reference; neither the possible vanishing, nor the existence of singularities of the Hermitian metric h^L affect the outcome of the statement. The proof is complete. $\square$

Remark 5.2.3. The requirement that $p \geqslant 2$ is only necessary for the construction of the second family of isomorphisms, because we want to use what we know about the relationship between the two vector spaces V_p and $V_{p,(2)}$ of sections.

In contrast: As one can see in [43, Theorem 5.1.3], the existence of induced isomorphisms and relation of Fubini-Study metrics to the Bergman kernel functions do not require conditions on the parameter p other than demanding that $p \geqslant p_0$ for some $p_0 \in \mathbb{N}$.

However, this observation is not relevant when one asks about the asymptotic behavior of the Kodaira maps or pull-backs of the induced Fubini-Study metrics.

The Hermitian metrics that are associated to $(\Phi_{p,(2)})^*(h_{\mathrm{FS},p,(2)})$ are called induced Fubini-Study Hermitian metrics on Σ.

We are now equipped to proceed with studying the asymptotic behavior of the induced Fubini-Study currents $(\Phi_{p,(2)})^*(\omega_{\mathrm{FS},p,(2)})$ on Σ and $(\Phi_{p,(2)})^*(h_{\mathrm{FS},p,(2)})$ on $(L^p \otimes E)|_\Sigma$, as $p \to \infty$. The next two sections will deal with each of these cases separately.

5.3 The Theorem of Tian-Ruan away from the punctures

With our tools at hand, we are equipped to prove an analogue of this the theorem of Tian-Ruan (see [43, Theorem 5.1.4]) in our setting.

Theorem 5.3.1. *Let Σ be a punctured Riemann Surface and let (L, h^L) (E, h^E) be Hermitian holomorphic line bundles on $\overline{\Sigma}$ such that conditions (α) and (β) are satisfied. Let $U \subset \Sigma$ be a relatively compact subset. Then the following statements are true. The normalized induced Fubini-Study metrics converge uniformly on U to the normalized semipositive curvature $R^L|_U$, with speed $\mathcal{O}(p^{-1/3})$; that is, for every $\ell \in \mathbb{N}_0$, there exists a constant $C_{\ell,U} \in \mathbb{R}_{>0}$, such that:*

$$\left\| \frac{1}{p}((\Phi_{p,(2)})^*(\omega_{\mathrm{FS},p,(2)}))|_U - \frac{\mathbf{i}}{2\pi} R^L|_U \right\|_{\mathcal{C}^\ell(U, h_p|_U)} \leqslant C_{\ell,U} p^{-\frac{1}{3}}\,, \tag{5.3.1}$$

for sufficiently large $p \in \mathbb{N}$. On compact subsets of Σ_2, where the curvature doesn't vanish, one may replace the fractional powers of p with -1, as in the classical version (see [43, Theorem 5.1.4]) of the theorem.

Proof of Theorem 5.3.1. As in [43, (5.1.21)], for all sufficiently large $p \in \mathbb{N}$, we have

$$\frac{1}{p}((\Phi_{p,(2)})^*(\omega_{\mathrm{FS},p,(2)})) - \frac{\mathbf{i}}{2\pi}R^L = \frac{\mathbf{i}}{2\pi p}R^E + \frac{\mathbf{i}}{2\pi p}\partial\bar{\partial}\log B_p\,. \tag{5.3.2}$$

Now let $U \subset \Sigma$ be relatively compact. Since the logarithm is monotonic, we can apply Lemma 4.1.9 to $\partial\bar{\partial}\log B_p|_U$ to infer that for all $\ell \in \mathbb{N}$ there exists a constant $C_{\ell,U} \in \mathbb{R}_{>0}$, such that

$$\left\|\frac{1}{p}((\Phi_{p,(2)})^*(\omega_{\mathrm{FS},p,(2)}))|_U - \frac{\mathbf{i}}{2\pi}R^L|_U\right\|_{\mathcal{C}^\ell(U,h_p|_U)} \leqslant C_{\ell,U}p^{-\frac{1}{3}}\,, \tag{5.3.3}$$

verifying the first claim (5.3.1).

For the second claim, we may go another route: Note that the curvature $R^L|_{\Sigma_2}$ is strictly positive and hence the same is true for $R^L|_K$ for any compact subset $K \subset \Sigma_2$. Moreover, the conditions of [43, Theorem 6.1.1] are met: therefore the Bergman kernel that is associated to the restrictions $h^L|_K$ and $\omega_\Sigma|_K$ onto the compact subset $K \subset \Sigma_2 \subset \Sigma$ exhibits a uniform on-diagonal asymptotic expansion with remainder of order p^{-1}, for all $p \in \mathbb{N}$. Hence for all compact subsets $K \subset \Sigma_2$ and all $\ell \in \mathbb{N}_0$, there exists constants $C_{\ell,K} \in \mathbb{R}_{>0}$, such that

$$\left\|\frac{1}{p}((\Phi_{p,(2)})^*(\omega_{\mathrm{FS},p,(2)}))|_K - \frac{\mathbf{i}}{2\pi}R^L|_K\right\|_{\mathcal{C}^\ell(K,h_p|_K)} \leqslant C_{\ell,K}p^{-1}\,. \tag{5.3.4}$$

□

Remark 5.3.2. Our result is local: the reason is that the upper bound from Lemma 4.1.9 is only uniform on relatively compact subsets of the form $\Sigma^{(p_{h,\gamma}),h,\gamma} \subset \Sigma$, for any $h \in (0,1)$ and $(0,\frac{1}{2})$ and not on all of Σ.

In the next section, we proceed by studying the asymptotic behavior of the induced Fubini-Study Hermitian metrics $(\Phi_{p,(2)})^*(h_{\mathrm{FS},p,(2)})$, as $p \to \infty$, on relatively compact subsets of Σ.

5.4 Convergence of induced Fubini-Study potentials away from the punctures

In this section, for any relatively compact subset $U \subset \Sigma$ and each sufficiently large $p \in \mathbb{N}$, we will consider the asymptotics of the induced Fubini-Study Hermitian metrics $((\Phi_{p,(2)})^*(h_{\mathrm{FS},p,(2)}))|_U$, which we have defined in Subsection 5.2.3.

For this purpose, at first, consider be an auxiliary positive Hermitian metric h_0^L on L; we know that such a Hermitian metric exists, as argued in Section 2.4. We can express our original semipositive singular Hermitian metric h_L in terms of a global weight φ with respect to h_0^L as follows: $h_L = e^{-2\varphi} h_0^L$; by the conditions on h^L from Section 1.1, we know that φ is $\mathcal{L}^1$-integrable function on $\overline{\Sigma}$ with values in $\mathbb{R} \cup \{\pm\infty\}$ and is smooth on Σ.

We also express the induced Fubini-Study Hermitian metric $(\Phi_{p,(2)})^*(h_{\mathrm{FS},p,(2)})$ in terms of h_0^L:

$$(\Phi_{p,(2)})^*(h_{\mathrm{FS},p,(2)}) = e^{-2\varphi_{\mathrm{FS},p,(2)}} (h_0^L)^p \otimes h^E \,. \tag{5.4.1}$$

The weight functions $\varphi_{\mathrm{FS},p,(2)}$ are smooth on Σ and are called induced Fubini-Study potentials.

Theorem 5.4.1 (Local uniform convergence of induced Fubini-Study potentials)**.** *Let Σ be a punctured Riemann Surface and let (L, h^L) (E, h^E) be Hermitian holomorphic line bundles on $\overline{\Sigma}$ such that conditions (α) and (β) are satisfied. Let $U \subset \Sigma$ be a relatively compact subset. Then the following statements are true.*

(i) The normalized potentials of the Fubini-Study metric converge uniformly on U to the potential φ of h^L on K with speed $\mathcal{O}(p^{-1}\log p)$; that is, for each $\ell \in \mathbb{N}_0$, there exists a constant $C_{\ell,U} \in \mathbb{R}_{>0}$, such that:

$$\left\| \frac{1}{p} \varphi_{\mathrm{FS},p,(2)}|_U - \varphi|_U \right\|_{\mathcal{C}^\ell(U, h_p|_U)} \leqslant C_{\ell,U} p^{-1} \log p \,, \tag{5.4.2}$$

for all sufficiently large $p \in \mathbb{N}$.

(ii) The following ∂- and $\overline{\partial}$-derivatives of the normalized potentials of the Fubini-Study metric converge uniformly on U to the ∂- and $\overline{\partial}$-derivatives of the potential φ of h^L on U with the respective speeds; that is, for each $\ell \in \mathbb{N}_0$, there exists constants $C_{\ell,U,1}, C_{\ell,U,2}, C_{\ell,U,3} \in \mathbb{R}_{>0}$, such that:

$$\left\| \frac{1}{p}\, \partial\varphi_{\mathrm{FS},p,(2)}|_U - \partial\varphi|_U \right\|_{\mathcal{C}^\ell(U, h_p|_U)} \leqslant C_{\ell,U,1} p^{-2/3} \,, \tag{5.4.3}$$

$$\left\| \frac{1}{p} \overline{\partial}\varphi_{\mathrm{FS},p,(2)}|_U - \overline{\partial}\varphi|_U \right\|_{\mathcal{C}^\ell(U, h_p|_U)} \leqslant C_{\ell,U,2} p^{-2/3} \,, \tag{5.4.4}$$

$$\left\| \frac{1}{p}\, \partial\overline{\partial}\varphi_{\mathrm{FS},p,(2)}|_U - \partial\overline{\partial}\varphi|_U \right\|_{\mathcal{C}^\ell(U, h_p|_U)} \leqslant C_{\ell,U,3} p^{-1/3} \,, \tag{5.4.5}$$

for all sufficiently large $p \in \mathbb{N}$.

The following is true both for (i) and (ii): On compact subsets of Σ_2, where the curvature doesn't vanish, one may replace the fractional powers of p with -1, as in the classical version of the theorem.

Proof. First, by (5.4.1), we have

$$\begin{aligned}R^{(L,h^L)} &= 2\,\partial\overline{\partial}\varphi + R^{(L,h_0^L)}\,,\\ R^{(L^p,h_{\mathrm{FS},p,(2)})} &= 2\,\partial\overline{\partial}\varphi_{\mathrm{FS},p,(2)} + pR^{(L,h_0^L)} + R^{(E,h^E)}\\ &= -\,\mathbf{i}2\pi p\,(\Phi_{p,(2)})^*(\omega_{\mathrm{FS},p,(2)})\,.\end{aligned} \tag{5.4.6}$$

Now let $U \subset \Sigma$ be relatively compact.

(i) By (2.3.3), (5.2.7) and (5.3.2), we infer that

$$\frac{1}{p}\varphi_{\mathrm{FS},p,(2)}(x) - \varphi(x) = \frac{1}{p}\log B_p(x,x) \tag{5.4.7}$$

for every $x \in U$. As in the proof of Theorem 5.3.1, Lemma 4.1.9 implies that for all $\ell \in \mathbb{N}_0$ there exists a constant $C_{\ell,U} \in \mathbb{R}_{>0}$, such that

$$\left\|\frac{1}{p}\varphi_{\mathrm{FS},p,(2)}|_U - \varphi|_U\right\|_{\mathcal{C}^\ell(U,h_p|_U)} \leqslant C_{\ell,U}p^{-1}\log p\,, \tag{5.4.8}$$

for all sufficiently large $p \in \mathbb{N}$. This verifies (i).

(ii) The second statement (ii) follows the same arguments as above with application of the uniform upper-bound of the jets of the Bergman kernel from Lemma 4.1.9 on the ∂-, $\overline{\partial}$- and $\partial\overline{\partial}$-derivatives of the Bergman kernel: hence for all $\ell \in \mathbb{N}_0$ there exists constants $C_{\ell,U,1}, C_{\ell,U,3}, C_{\ell,U,3} \in \mathbb{R}_{>0}$, such that

$$\left\|\frac{1}{p}\,\partial\varphi_{\mathrm{FS},p,(2)}|_U - \partial\varphi|_U\right\|_{\mathcal{C}^\ell(U,h_p|_U)} \leqslant C_{\ell,U,1}p^{-2/3}\,, \tag{5.4.9}$$

$$\left\|\frac{1}{p}\overline{\partial}\varphi_{\mathrm{FS},p,(2)}|_U - \overline{\partial}\varphi|_U\right\|_{\mathcal{C}^\ell(U,h_p|_U)} \leqslant C_{\ell,U,2}p^{-2/3}\,, \tag{5.4.10}$$

$$\left\|\frac{1}{p}\,\partial\overline{\partial}\varphi_{\mathrm{FS},p,(2)}|_U - \partial\overline{\partial}\varphi|_U\right\|_{\mathcal{C}^\ell(U,h_p|_U)} \leqslant C_{\ell,U,3}p^{-1/3}\,, \tag{5.4.11}$$

for all sufficiently large $p \in \mathbb{N}$.

Note that the exponent in p in the last equation is higher than the exponents in p in the first two equations. This comes from the fact that in this case, we use the upper bound (4.1.10) on the jets/derivatives of the Bergman kernel for $\ell = 2$, since we apply two partial derivatives $\overline{\partial}$ and ∂.

For the last statement, the argument is analogous to the one in the proof of Theorem 5.3.1. □

Remark 5.4.2. Again, our result is local, for the same reason as before: the upper bound from Lemma 4.1.9 is only uniform on relatively compact subsets of the form $\Sigma^{(p_{h,\gamma}),h,\gamma} \subset \Sigma$, for any $h \in (0,1)$ and $(0, \frac{1}{2})$ and not on all of Σ.

In the following section, we explain how the results of [1] apply in our setting, which is identical to theirs locally near the punctures in the compact Riemann surface $\overline{\Sigma}$.

5.5 Global convergence results

Near the punctures, Auvray, Ma and Marinescu study their Kodaira maps using their comparison results between the associated Bergman kernel and the Bergman kernel on the punctured disc from the local model. Since the curvature does not vanish near the punctures in our setting, their conclusions for the Kodaira maps apply directly for our Kodaira maps, on neighborhoods of punctures. In particular, the analogue of [2, Theorem 4.1] holds for our Kodaira maps Φ_p and $\Phi_{p,(2)}$ that are associated to the semipositively curved Hermitian metrics h_p. The necessary arguments for a proof are identical to the arguments made in [2, Section 4].

Theorem 5.5.1. *Let $\overline{V}_1, \ldots, \overline{V}_N \subset \overline{\Sigma}$ be open neighborhoods of the punctures $a_1, \ldots, a_N$ as in condition (α) and set $\overline{V}_{1,\ldots,N} := \overline{V}_1 \cup \ldots \cup \overline{V}_N$ for their union. Then Φ_p and $\Phi_{p,(2)}$ are well-defined and holomorphic on $\overline{V}_{1,\ldots,N}$. Then, as $p \to \infty$, the following two asymptotic identities*

$$(\Phi_p|_{\overline{V}_{1,\ldots,N}})^*(h_{\mathrm{FS},p}) = (1 + \mathcal{O}(p^{-\infty}))\frac{h_p}{B_p^{\mathbb{D}^*}}\,, \tag{5.5.1}$$

$$\frac{1}{p}(\Phi_p|_{\overline{V}_{1,\ldots,N}})^*(\omega_{\mathrm{FS},p}) = \frac{1}{2\pi}\omega_\Sigma + \frac{\mathbf{i}}{2\pi p}\,\partial\bar{\partial}\log(B_p^{\mathbb{D}^*}) + \mathcal{O}(p^{-\infty}) \tag{5.5.2}$$

hold uniformly on $\overline{V}_{1,\ldots,N}$.

Proof. As a consequence of Theorem 5.2.2, analogous arguments as in the proof of [2, Theorem 4.1] imply that

$$(\Phi_p|_\Sigma)^*(h_{\mathrm{FS},p})(x,x) = \frac{1}{B_p(x,x)}h_p(x,x) \tag{5.5.3}$$

$$\frac{1}{p}(\Phi_p|_\Sigma)^*(\omega_{\mathrm{FS},p}) = \frac{\mathbf{i}}{2\pi}R^L + \frac{\mathbf{i}}{2\pi p}\,\partial\bar{\partial}\log(B_p)\,, \tag{5.5.4}$$

for all $x \in V_1 \cup \ldots \cup V_N$. The claims follow by an application of Theorem 4.3.4 about the quotient of Bergman kernels. □

From Theorem 5.5.1, in particular, by using (5.5.2), immediately conclude the following theorem as in the proof of Theorem 5.3.1 (see [1, Theorem 4.3]).

Theorem 5.5.2. *Let Σ be a punctured Riemann Surface and let (L, h^L) (E, h^E) be Hermitian holomorphic line bundles on $\overline{\Sigma}$ such that conditions (α) and (β) are satisfied.*

Then the normalized induced Fubini-Study metrics converge weakly in the sense of currents to the normalized semipositive curvature current R^L on $\overline{\Sigma}$:

$$\frac{1}{p}(\Phi_{p,(2)})^*(\omega_{\mathrm{FS},p,(2)}) \rightharpoonup \frac{\mathbf{i}}{2\pi}R^L\,, \tag{5.5.5}$$

as $p \to \infty$.

6. Equidistribution of zeros of holomorphic sections

6.1 Poincaré-Lelong formula

Let $p \in \mathbb{N}$ and $s \in H^0_{(2)}(\Sigma, L^p \otimes E)$ be arbitrary. The divisor of zeros of s is the formal sum

$$\mathrm{Div}(s) := \sum_{s(x)=0} \mathrm{ord}_x(s) \cdot x\,, \tag{6.1.1}$$

with $\mathrm{ord}_x(s)$ the multiplicity of s at x and the sum runs over the zeros $x \in \Sigma$ of s. its support $\operatorname{supp}\mathrm{Div}(s) \subset \Sigma$ is the set of points $x \in \Sigma$ where $\mathrm{ord}_x(s) \neq 0$; we call it the zero set of s. By the identity theorem from complex analysis in $\mathbb{C}$, the zero set of a section is closed and discrete.

The measure of zeros of s will be denoted by

$$[\mathrm{Div}(s)] := \sum_{s(x)=0} \mathrm{ord}_x(s) \cdot \delta_x\,, \tag{6.1.2}$$

where δ_x is the Dirac delta distribution supported in $x \in \Sigma$; it defines a current of integration along $\operatorname{supp}\mathrm{Div}(s)$

The following theorem is due to Poincaré-Lelong. It constitutes an important link between homological and cohomological points of view, which is quintessential in the study of the distribution of zeros of holomorphic sections by the tools from complex analysis on manifolds that we have introduced in the earlier chapters.

Theorem 6.1.1 (Poincaré-Lelong)**.** *Let L be a Hermitian line bundle over $\overline{\Sigma}$, equipped with a singular Hermitian metric h^L. Then for any meromorphic section s of L we have*

$$\frac{\mathbf{i}}{2\pi}\,\partial\overline{\partial}\log|s|^2_{h^L} = [\mathrm{Div}(s)] - \frac{\mathbf{i}}{2\pi}R^{(L,h^L)}\,. \tag{6.1.3}$$

Since sections are locally functions and the relation itself is of local nature, the proof follows from the case of meromorphic functions; it can be found in [43, Theorem 2.3.3].

6.2 Gaussian measures on spaces of sections

We will first define the concept of a random holomorphic section. In general, there are multiple equivalent ways to do this. Our interest lies in the expected positions of zero sets of holomorphic sections of powers L^p, twisted by E, on the manifold. Therefore, picking a section $s \in H^0_{(2)}(\Sigma, L^p \otimes E)$ at random and asking about the expected positions of their zeros on Σ is equivalent to asking about the expected location of a subset of Σ where the subset is chosen by a given law.

In general, let V be a complex vector space of dimension $\dim_{\mathbb{C}} V =: m < \infty$, equipped with the standard Hermitian product $\langle \cdot, \cdot \rangle_V$ and the m-dimensional Lebesgue measure λ^m and let $A = (a_{ij})_{i,j=1}^m$ be a positive Hermitian matrix with complex entries. A complex Gaussian measure μ_V (complex Gaussian, or Gaussian in short) with mean 0 and covariance matrix A is the measure that is absolutely continuous with respect to λ^m, with density given by

$$\mathrm{d}\mu_V := \frac{1}{\pi^m \det A} e^{-\langle A^{-1}v, v\rangle_V} \mathrm{d}\lambda^m(v) \,. \tag{6.2.1}$$

Note that μ_V as defined by its density in (6.2.1) has total mass 1, which is assured by the fractional scaling factor before the exponential, and hence it defines a probability measure. If μ_V is a complex Gaussian on V and $\tau : V \to \tilde{V}$ is a surjective linear map, then the push-forward $\tau_* \mu_V$ is a complex Gaussian on $\tilde{V}$. In particular, if $\tilde{V} = \mathbb{C}^m$, then

$$\mathrm{d}(\tau_* \mu_V) := \frac{1}{\pi^m \det A} e^{-(A^{-1}z \cdot \bar{z})} \mathrm{d}z \,. \tag{6.2.2}$$

As defined in Appendix B.15 we write $\mathbf{E}$ for the expected value operator. Integrating against the probability measure that has density given by (6.2.2) yields the expected values $\mathbf{E}[\mathbf{z}_i] = 0$, $\mathbf{E}[\mathbf{z}_i \mathbf{z}_j] = 0$ and $\mathbf{E}[\mathbf{z}_i \bar{\mathbf{z}}_j] = a_{ij}$, for all $1 \leqslant i, j \leqslant m$, where $\mathbf{z}_i := z_i \circ \tau : V \to \mathbb{C}$ is the component of the standard coordinates in $\mathbb{C}^m$.

In our case, the complex vector space of consideration is $V = H^0_{(2)}(\Sigma, L^p \otimes E)$. We thus start by choosing an orthonormal basis $\{S^p_1, \dots, S^p_{d_p}\}$ of $H^0_{(2)}(\Sigma, L^p \otimes E)$; then any $s \in H^0_{(2)}(\Sigma, L^p \otimes E)$ can be expressed in coordinates $s = \sum_i c_i S^p_i$ for some complex numbers $c_1, \dots, c_{d_p} \in \mathbb{C}$. We equip $H^0_{(2)}(\Sigma, L^p \otimes E)$ with the complex Gaussian probability measure, that has density with respect to λ^{2d_p} the $2d_p$-dimensional Lebesgue measure given by

$$\mathrm{d}\mu_p = \frac{1}{\pi^{d_p}} e^{-|\cdot|_V^2} \mathrm{d}\lambda^{2d_p} \tag{6.2.3}$$

where $|s|_V^2 := \langle s, s\rangle_V$ and the latter denotes the standard Hermitian product for the complex vector space $V = H^0_{(2)}(\Sigma, L^p \otimes E)$, i.e.

$$\langle s, s'\rangle_V := \sum_{i=1}^{d_p} c_i \bar{c}'_i \,, \qquad \text{for} \qquad s = \sum_{i=1}^{d_p} c_i S_i^p \,, \quad s' = \sum_{i=1}^{d_p} c'_i S_i^p \,. \tag{6.2.4}$$

Then by definition, the measure μ_p is absolutely continuous with respect to the Lebesgue measure λ^{2d_p} and again its total mass is a unit. This Gaussian measure is characterized by the property that the $2d_p$ real random variables $\mathrm{Re}\, c_i$ and $\mathrm{Im}\, c_i$, for $1 \leqslant i \leqslant d_p$, are independent random variables with mean 0 and variance $\frac{1}{2}$: in expectation, we have

$$\mathbf{E}[c_i] = 0 \,, \quad \mathbf{E}[c_i c_j] = 0 \,, \quad \mathbf{E}[c_i \bar{c}_j] = \delta_{ij} \tag{6.2.5}$$

with δ_{ij} the Kronecker-delta.

Another common way of introducing random sections is the following: First we observe that the zero sets are unaffected by the process of multiplying the section at hand by a nowhere vanishing, global holomorphic function on Σ. Hence the position of the zero sets of any fixed section $s \in H^0_{(2)}(\Sigma, L^p \otimes E)$ coincides with the position of the zero sets of any representative of the class of elements $[s] \in \mathbb{P}H^0_{(2)}(\Sigma, L^p \otimes E)$.

Now, as argued before in Subsection 2.6.2 (2.6.8), we have $d_p = \dim_{\mathbb{C}} H^0_{(2)}(\Sigma, L^p \otimes E) < \infty$ and the vector space of L^2-bounded holomorphic sections $H^0_{(2)}(\Sigma, L^p \otimes E)$ carries the natural L^2-metric (2.5.7) that is associated to ω_Σ and h_p. As seen in Chapter 5, this data induces a family of Fubini-Study metrics $\omega_{\mathrm{FS},p}$, parametrized by the semiclassical parameter $p \in \mathbb{N}$, on the projective spaces $\mathbb{P}(H^0_{(2)}(\Sigma, L^p \otimes E)^*)$, and similarly a family of Fubini-Study metrics, which we will also denote by $\omega_{\mathrm{FS},p}$, on the projective spaces $\mathbb{P}(H^0_{(2)}(\Sigma, L^p \otimes E))$. First. recall that, as explained earlier in Subsection 5.2.2 of Chapter 5, we have the identification

$$\mathbb{P}(H^0_{(2)}(\Sigma, L^p \otimes E)) \ni [\xi] \mapsto H_\xi \in G_{d_p-1}(H^0_{(2)}(\Sigma, L^p \otimes E)) \tag{6.2.6}$$

between the projective space $\mathbb{P}(H^0_{(2)}(\Sigma, L^p \otimes E))$ and the Grassmanian $G_{d_p-1}(H^0_{(2)}(\Sigma, L^p \otimes E))$ of $(d_p - 1)$-dimensional hyperplanes in $H^0_{(2)}(\Sigma, L^p \otimes E)$.

Second, recall that the total volume of an d_p-dimensional projective space is π^{d_p}, as calculated in (5.1.12). Therefore, a normalization by

$$\sigma_{\mathrm{FS},p} := \frac{1}{\pi^{d_p}} \omega_{\mathrm{FS},p}^{d_p}\,, \tag{6.2.7}$$

gives a measure of total mass, i.e. its integral over the whole space, equal to 1. Hence $\sigma_{\mathrm{FS},p}$ defines a probability measure on the projective spaces $\mathbb{P}(H^0_{(2)}(\Sigma, L^p \otimes E)) \cong \mathbb{CP}^{d_p}$.

Recall the definition of a probability space (see Appendix B.2). We now consider the products of probability spaces

$$(\Omega, \mu) := \prod_{p=1}^{\infty} (\mathbb{P}(H^0_{(2)}(\Sigma, L^p \otimes E), \sigma_{\mathrm{FS},p})\,, \tag{6.2.8}$$

The underlying σ-algebra is the product-σ-algebra, as we have defined in Appendix B.13. A random section of $L^p \otimes E$ in our context will therefore just be a random variable

$$\Sigma \ni x \longmapsto [s_p(x)] \in (\mathbb{CP}^{d_p}, \mu_{d_p}) \cong (\mathbb{P}(H^0_{(2)}(\Sigma, L^p \otimes E)), \sigma_{\mathrm{FS},p})\,, \tag{6.2.9}$$

where μ_{d_p} is the probability measure on $\mathbb{CP}^{d_p}$ that we get by pulling back $\sigma_{\mathrm{FS},p}$ under this isomorphism of line bundles. The measure $\sigma_{\mathrm{FS},p}$ is identical to the complex Gaussian introduced via its density in (6.2.3): integration of the pull-backs of the normalized Fubini-Study metrics yield the same expected values, which characterize this complex Gaussian.

The authors in [43, Section 5.3] offer yet another approach for introducing the probability measure that we are looking for: if $\mathrm{d}S^{2d_p-1}$ denotes the spherical (Haar) measure, i.e. the usual volume form on the $(2d_p-1)$-dimensional unit sphere $S^{2d_p-1} = \{z_p = (z_{p,1}, \dots, z_{p,d_p}) \in \mathbb{C}^{d_p} : |z_p| = 1\}$ and

$$SH^0_{(2)}(\Sigma, L^p \otimes E) : \{s \in H^0_{(2)}(\Sigma, L^p \otimes E) : |s|_{h_p} = 1\} \tag{6.2.10}$$

is the (unit) circle bundle over Σ, then the complex Gaussian relates to the normalized Haar measure by

$$\mathrm{d}\mu_p = \frac{\mathrm{d}S^{2d_p-1}}{\mathrm{vol}(S^{2d_p-1})}\,. \tag{6.2.11}$$

Hence the total mass of (6.2.11) is equal to 1 and the corresponding expected values coincide with the expectations (6.2.5); thus (6.2.11) again defines the same complex Gaussian as in the two approaches that we have given before.

We will use the three approaches from above interchangeably and use the different points of view to our advantage, whenever possible; we call the probability measure μ in

(6.2.8) the equidistribution measure with respect to h_p and ω_{FS}.

From (6.2.3), we can also see the relevance of the Bergman kernel: functions (of two variables) of the form as the density given in (6.2.3), or (6.2.2) in general, are called covariance kernels of the probability distribution at hand; hence after specifying an orthonormal basis of $H^0_{(2)}(\Sigma, L^p \otimes E)$, the Bergman kernel (2.7.3) takes the form of the covariance kernel of the complex Gaussian on $H^0_{(2)}(\Sigma, L^p \otimes E)$.

6.3 Equidistribution of zeros of random holomorphic sections

We will now adjust our notation a little bit for matters of convenience when talking about equidistribution of zeros of holomorphic sections. In particular, we now consider $\mathbf{s} := \{s_p\}_{p\in\mathbb{N}} \in \Omega$ to be a sequence of sections $s_p \in \mathbb{P}H^0_{(2)}(\Sigma, L^p \otimes E)$; i.e. the letter p in the subscript in s_p indicates the power of the line bundle L^p, which has an influence on the dimension of the space $H^0_{(2)}(\Sigma, L^p \otimes E)$. Now for every $p \in \mathbb{N}$, we fix an orthonormal basis $\{S_{p,i}\}_{i=1}^{d_p}$ of the complex vector space $H^0_{(2)}(\Sigma, L^p \otimes E)$, such that the sphere S^{2d_p-1} gets identified to $SH^0_{(2)}(\Sigma, L^p \otimes E)$ by

$$S^{d_p-1} \ni (z_1, \ldots, z_{d_p}) \longmapsto \sum_{i=1}^{d_p} z_i S_{p,i} \in SH^0_{(2)}(\Sigma, L^p \otimes E) \tag{6.3.1}$$

and denote by $s_p = \sum_{i=1}^{d_p} \lambda_{p,i} S_{p,i}$ the representation of s_p with respect to this basis. We proceed by considering the (sequences of) currents of integration $[\mathrm{Div}(s_p)]$, for each $p \in \mathbb{N}$. In particular, we view each member s_p of the sequence $\mathbf{s} = \{s_p\}_{p\in\mathbb{N}}$ as a random variable over the corresponding probability space $(\mathbb{P}(H^0_{(2)}(\Sigma, L^p \otimes E), \omega_{\mathrm{FS},p})$ and wish to understand the expectations

$$(\mathbf{E}[\mathrm{Div}(s_p)], \varphi) = \int_{S^{2d_p-1}} \left(\left[\left(\sum_{i=1}^{d_p} \lambda_{p,i} S_{p,i} \right) = 0 \right], \varphi \right) \mathrm{d}\mu_p(\lambda_p) \tag{6.3.2}$$

for all $\varphi \in \Omega^{0,0}(\Sigma)$ and all $\lambda_p := (\lambda_{p,1}, \ldots, \lambda_{p,d_p}) \in S^{d_p-1}$.

The main result of this section is that equidistribution holds with respect to the Hermitian metrics h_p.

Theorem 6.3.1 (Equidistribution of zeros of random holomorphic sections)**.** *Let Σ be a punctured Riemann surface, and let L be a holomorphic line bundle such that L carries a singular Hermitian metric h^L satisfying conditions (α) and (β). Let E be a holomorphic* *line bundle on Σ equipped with a smooth Hermitian metric h^E such that (E, h^E) on each chart V_j coincides with the trivial Hermitian line bundle. Then for μ-almost all $\mathbf{s} = \{s_p\}_{p\in\mathbb{N}} \in \Omega$, the sequence of currents converges weakly to the semipositive curvature form on relatively compact subsets $U \subset \Sigma$:*

$$\frac{1}{p}[\mathrm{Div}(s_p)]|_U \rightharpoonup \frac{\mathbf{i}}{2\pi} R^L|_U\,, \qquad \textit{as } p \longrightarrow \infty\,. \tag{6.3.3}$$

Proof. Let (Ω, μ) be the probability space defined in (6.2.8) and let $s = \{s_p\}_{p\in\mathbb{N}} \in \Omega$ be a sequence of sections, i.e. $s_p \in H^0_{(2)}(\Sigma, L^p \otimes E)$ for all $p \in \mathbb{N}$. We will consider the random variables

$$\left(\frac{1}{p}[\mathrm{Div}(s_p)]|_U - \frac{\mathbf{i}}{2\pi} R^L|_U, \varphi\right) \tag{6.3.4}$$

for each $p \in \mathbb{N}$ and $\varphi \in \Omega^{0,0}_c(\Sigma)$, i.e. φ is a smooth function on Σ with compact support $\mathrm{supp}\,(\varphi) \subset U$. By Theorem 5.3.1, we have

$$\begin{aligned}\left(\frac{1}{p}[\mathrm{Div}(s_p)]|_U - \frac{\mathbf{i}}{2\pi} R^L|_U, \varphi\right) &= \left(\frac{1}{p}[\mathrm{Div}(s_p)]|_U - \frac{1}{p}((\Phi_{p,(2)})^*(\omega_{\mathrm{FS},p,(2)}))|_U, \varphi\right) \\ &\qquad + \mathcal{O}\left(p^{-1/3}\|\varphi\|_{\mathcal{C}^0(U)}\right) \\ &=: Y_{p,\varphi,U}(\mathbf{s}) + \mathcal{O}\left(p^{-1/3}\|\varphi\|_{\mathcal{C}^0(U)}\right)\,. \end{aligned} \tag{6.3.5}$$

The error term shrinks arbitrarily for growing $p \in \mathbb{N}$. Thus, in order to prove (6.3.3), it suffices to show that, μ-almost surely for $\mathbf{s} \in \Omega$,

$$Y_{p,\varphi,U}(\mathbf{s}) \longrightarrow 0\,, \qquad \text{as } p \longrightarrow \infty\,, \tag{6.3.6}$$

for all $\varphi \in \Omega^{0,0}_c(\Sigma)$ with $\mathrm{supp}\,\varphi \subset U$. As a first step, we show that for all $p \in \mathbb{N}$,

$$\mathbf{E}[\mathrm{Div}(s_p)]|_U = ((\Phi_{p,(2)})^*(\omega_{\mathrm{FS},p,(2)}))|_U\,, \tag{6.3.7}$$

relating our expectations to the Kodaira map that we have studied in Chapter 5. An application of the Poincaré-Lelong formula, Theorem 6.1.1, to the currents of integration of each zero set of the members s_p of our sequence $\mathbf{s} = \{s_p\}_{p\in\mathbb{N}}$ implies

$$[\mathrm{Div}(s_p)]|_U = \frac{\mathbf{i}}{2\pi}\,\partial\bar{\partial}\log|(s_p)|_U|^2_{h_p} - p\frac{\mathbf{i}}{2\pi}R^{(L,h^L)}|_U - \frac{\mathbf{i}}{2\pi}R^{(E,h^E)}|_U\,. \tag{6.3.8}$$

Then applying (6.3.8) and formula (5.3.2) yields

$$
\begin{aligned}
([\mathrm{Div}(s_p)]|_U, \varphi) &= \int_\Sigma \left(\frac{\mathbf{i}}{2\pi} \partial\bar{\partial} \log |(s_p)|_U|^2_{h_p} - p\frac{\mathbf{i}}{2\pi} R^{(L,h^L)}|_U - \frac{\mathbf{i}}{2\pi} R^{(E,h^E)}|_U \right) \varphi \\
&= \int_\Sigma ((\Phi_{p,(2)})^*(\omega_{\mathrm{FS},p,(2)}))|_U \varphi + \frac{\mathbf{i}}{2\pi} \int_\Sigma \partial\bar{\partial} \log |B_p(x,x)^{-\frac{1}{2}} s_p(x)|^2_{h_p} \varphi \\
&= \int_\Sigma ((\Phi_{p,(2)})^*(\omega_{\mathrm{FS},p,(2)}))|_U \varphi + \frac{\mathbf{i}}{2\pi} \int_\Sigma \log |B_p(x,x)^{-\frac{1}{2}} s_p(x)|^2_{h_p} \partial\bar{\partial}\varphi \, .
\end{aligned}
\tag{6.3.9}
$$

for $x \in U$. Next, for each $p \in \mathbb{N}$, we fix an orthonormal basis $\{S_{p,i}\}_{i=1}^{d_p}$ of $H^0_{(2)}(\Sigma, L^p \otimes E)$; recall the identification (6.3.1) with the sphere $S^{d_p - 1}$. We take unit vectors $e_L(x)$ of L and $e_E(x)$ of E at $x \in \Sigma$. In light of (2.7.4), set

$$
\psi(x) := \frac{1}{\sqrt{B_p(x,x)}} \frac{\big(S_{p,1}(x), \dots, S_{p,d_p}(x)\big)}{(e_L^{\otimes p} \otimes e_E)(x)} \in S^{2d_p - 1} \, , \tag{6.3.10}
$$

which is well defined up to multiplication by complex numbers of unit length. Set

$$
u \cdot v = \sum_{i=1}^{d_p} u_i v_i \, , \qquad u \cdot S_p = \sum_{i=1}^{d_p} u_i S_{p,i} \tag{6.3.11}
$$

for the usual dot product on $\mathbb{C}^{d_p} \ni u = (u_1, \dots, u_{d_p}), v = (v_1, \dots, v_{d_p})$. Then, formula (2.7.4) implies that

$$
\begin{aligned}
\int_S^{2d_p - 1} \log |B_p(x,x)^{-\frac{1}{2}} s_p(x)|^2_{h_p} \, \mathrm{d}\mu_p(\lambda) &= \int_S^{2d_p - 1} \log |B_p(x,x)^{-\frac{1}{2}} \sum_{i=1}^{d_p} \lambda_{p,i} S_{p,i}(x)|^2_{h_p} \, \mathrm{d}\mu_p(\lambda) \\
&= \int_S^{2d_p - 1} \log |\frac{\sum_{i=1}^{d_p} S_{p,i}(x)}{(\sqrt{B_p(x,x)})(\sqrt{B_p(x,x)})} \sum_{i=1}^{d_p} \lambda_{p,i} S_{p,i}(x)|^2_{h_p} \, \mathrm{d}\mu_p(\lambda) \\
&= \int_S^{2d_p - 1} \log |\lambda_p \cdot \psi|^2_{h_p} \, \mathrm{d}\mu_p(\lambda) \\
&= \int_S^{2d_p - 1} \log |\lambda_p \cdot u|^2_{h_p} \, \mathrm{d}\mu_p(\lambda) =: c_p \, ,
\end{aligned}
\tag{6.3.12}
$$

is a constant function on the manifold Σ, for $x \in U$; the last step follows by utilizing the rotational invariance of the function ψ, which allows us to exchange ψ by a constant vector $u = (u_1, \dots, u_{d_p}) \in \mathbb{C}^{d_p}$ in the formula under the integral. By definition (6.3.2) of the currents of integration along the zero sets of s_p, by (6.3.8) and (6.3.12) we finally get

$$
\begin{aligned}
(\mathbf{E}[\mathrm{Div}(s_p)|_U], \varphi) &= \int_{S^{2d_p-1}} \left(\left[\mathrm{Div} \left(\sum_{i=1}^{d_p} \lambda_{p,i} (S_{p,i})|_U \right) \right], \varphi \right) \mathrm{d}\mu_p(\lambda_p) \\
&= \int_\Sigma ((\Phi_{p,(2)})^*(\omega_{\mathrm{FS},p,(2)}))|_U \varphi + c_p \frac{\mathbf{i}}{2\pi} \int_\Sigma \partial\bar{\partial}\varphi \\
&= \int_\Sigma ((\Phi_{p,(2)})^*(\omega_{\mathrm{FS},p,(2)}))|_U \varphi \,,
\end{aligned}
\tag{6.3.13}
$$

since the total mass of S^{2d_p-1} with respect to $\mathrm{d}\mu_p$ is a unit; this holds for all $\varphi \in \Omega_c^{0,0}(U)$, for all $p \in \mathbb{N}$. This confirms (6.3.7). A consequence is that

$$
\mathbf{E}\left[|Y_{p,\varphi,U}(\mathbf{s})|^2 \right] = \frac{1}{p^2} \mathbf{E}\left[|([\mathrm{Div}(s_p)]\,, \varphi)|^2 \right] - \frac{1}{p^2} \left| \left(((\Phi_{p,(2)})^*(\omega_{\mathrm{FS},p,(2)}))|_U, \varphi \right) \right|^2 \tag{6.3.14}
$$

for all $\varphi \in \Omega_c^{(0,0)}(U)$ and all $p \in \mathbb{N}$. The next step is now to understand the first summand in the last equation. By (6.3.2), (6.3.8) and (6.3.12), we get

$$
\begin{aligned}
\mathbf{E}\left[|([\mathrm{Div}(s_p)]\,|_U, \varphi)|^2 \right] &= \left| \left(((\Phi_{p,(2)})^*(\omega_{\mathrm{FS},p,(2)}))|_U, \varphi \right) \right|^2 \\
&= \frac{1}{4\pi^2} \int_\Sigma \int_\Sigma \left(\partial\bar{\partial}\varphi(x) \right) \overline{\left(\partial\bar{\partial}\varphi(y) \right)} \int_{S^{2d_p-1}} \log \left| B_p(x,x)^{-1/2} \lambda_p \cdot S_p(x) \right|_{h_p}^2 \\
&\quad \times \log \left| B_p(y,y)^{-1/2} \lambda_p \cdot S_p(y) \right|_{h_p}^2 \mathrm{d}\mu_p(\lambda_p) \,,
\end{aligned}
\tag{6.3.15}
$$

for $(x, y) \in U \times U$. Now define a function for $u, v \in S^{2d_p-1}$ by

$$
A_p(u, v) := \int_{S^{2d_p-1}} \log(|\lambda_p \cdot u|) \log(|\lambda_p \cdot v|) \,\mathrm{d}\mu_p(\lambda_p) \,. \tag{6.3.16}
$$

By [43, Lemma 5.3.2], there exists a constant $C_p > 0$, such that $A_p(u, v) - C_p$ is uniformly bounded for $(u, v) \in S^{2d_p-1} \times S^{2d_p-1}$, for all $p \in \mathbb{N}$. Hence, equations (6.3.14) and (6.3.15) imply that

$$
\begin{aligned}
\mathbf{E}\left[|([\mathrm{Div}(s_p)]\,|_U, \varphi)|^2 \right] &= \frac{1}{4\pi^2 p^2} \int_\Sigma \int_\Sigma \left(\partial\bar{\partial}\varphi(x) \right) \overline{\left(\partial\bar{\partial}\varphi(y) \right)} A_p(\psi(x), \psi(y)) \,\mathrm{d}\mu_p(\lambda_p) \\
&= \mathcal{O}\left(p^{-2} \right) ,
\end{aligned}
\tag{6.3.17}
$$

for $x, y \in U$. Finally we conclude by (6.3.14) and (6.3.17), that

$$
\int_\Omega \sum_{p=1}^\infty |Y_{p,\varphi,U}(\mathbf{s})|^2 \,\mathrm{d}\mu(\mathbf{s}) = \sum_{p=1}^\infty \int_\Omega |Y_{p,\varphi,U}(\mathbf{s})|^2 \,\mathrm{d}\mu(\mathbf{s}) = \sum_{p=1}^\infty \mathbf{E}\left[|([\mathrm{Div}(s_p)]\,|_U, \varphi)|^2 \right] < \infty \,, \tag{6.3.18}
$$

which implies the desired μ_p-almost surely (see Appendix B.16) convergence (6.3.6). The proof is complete. $\square$

6.4 Convergence speed of equidistribution of zeros

Having established equidistribution in our setting, it is natural to ask if stronger results also hold. There are multiple ways to generalize Theorem 6.3.1.

In particular, we will proof a variant of [22, Theorem 1.5], which is an equidistribution result on relatively compact open subsets of Σ, together with an estimate on the speed of convergence.

6.4.1 Plurisubharmonic functions

Plurisubharmonic functions are a central topic to pluripotential theory and many topics that are related to the theory of currents. Plurisubharmonic functions and currents have an interwoven relationship.

Let $U \subset \mathbb{C}$ be an open set in the complex plane and $\varphi : U \to \mathbb{R} \cup \{-\infty\}$ an upper semi-continuous function. Recall that φ is called subharmonic, if it satisfies the mean value inequality (also called the sub-mean inequality)

$$\varphi(a) \leqslant \frac{1}{2\pi} \int_0^{2\pi} \varphi(a + re^{\mathbf{i}\theta}) \, \mathrm{d}\theta \,, \tag{6.4.1}$$

on any closed ball $\overline{\mathbb{B}}_r(a) \cup U$ with center $a \in U$ and radius $r > 0$ and φ is harmonic, if both φ and $-\varphi$ are subharmonic.

An upper semi-continuous function $\varphi : \Sigma \to \mathbb{R} \cup \{-\infty\}$ is called plurisubharmonic (psh for short) if for all $a \in \mathbb{C}$, $\mathbb{B}_1(a) \subset \mathbb{C}$ and for any holomorphic function $f : \mathbb{B}_1(a) \to \Sigma$, the composition $\varphi \circ f$ is subharmonic. Now it is worth noting that the definition works on manifolds of higher dimensions and not just in dimension $\dim_{\mathbb{C}} \Sigma = 1$ that we are dealing with in our case. In fact, this is what the syllable 'pluri' is meant to indicate. For the sake of consistency with secondary literature, we will still call functions on Σ that satisfy this definition plurisubharmonic.

There are many natural examples that one can construct from forms and currents. In particular (see [20, I. Theorem 5.8]), if $\varphi : \Sigma \to \mathbb{R}$ is any smooth function, then φ is psh if and only if $-\partial\overline{\partial}\varphi$ is a semipositive form. In light of this relationship we call an locally integrable, upper semi-continuous function φ strictly plurisubharmonic (spsh for short) if $-\partial\overline{\partial}\varphi$ is a positive form.

This relationship holds more generally in weaker regularity: let $\varphi : \Sigma \to \mathbb{R} \cup \{-\infty\}$ is a psh (spsh) function that is locally integrable (this time without requiring it to be upper semi-continuous). Then $-\partial\bar{\partial}\varphi$ is a positive (strictly positive) $(1,1)$-current. Conversely, if $\varphi \in \mathcal{L}^1_{\text{loc}}(\Sigma)$ such that the $(1,1)$-current $-\partial\bar{\partial}\varphi$ is positive (strictly positive), then there exists a psh (spsh) function $\psi : \Sigma \to \mathbb{R} \cup \{-\infty\}$, such that $\varphi = \psi$ almost everywhere (with respect to the standard Lebesgue measure).

More generally and also of interest to the theory of currents are quasi-plurisubharmonic functions. These are functions $\varphi : \Sigma \to \mathbb{R} \cup \{-\infty\}$ with the property that φ can locally be written as the sum of a smooth function α and some plurisubharmonic function on Σ ; we may write α-psh, i.e. quasi-psh such that $\partial\bar{\partial}\varphi \geqslant -\alpha$, when brevity is beneficial. Consequently, quasi-plurisubharmonic functions generalize plurisubharmonic functions, since if φ is α-psh for $\alpha \equiv 0$ if and only if φ is psh.

Similarly to the above relationship between positive (strictly positive) currents and psh (spsh) functions, the following equivalence holds (see [43, Proposition B.2.17]): let $\varphi : \Sigma \to \mathbb{R} \cup \{-\infty\}$ be a quasi-plurisubharmonic function that is locally integrable. Then there exists a continuous $(1,1)$-form α on Σ, such that $-\partial\bar{\partial}\varphi \geqslant \alpha$, i.e. $-\partial\bar{\partial}\varphi - \alpha$ is a positive $(1,1)$-current. Conversely, if $\varphi \in \mathcal{L}^1_{\text{loc}}(\Sigma)$ such that $-\partial\bar{\partial}\varphi \geqslant \alpha$ for some continuous $(1,1)$-form α on Σ, i.e. the $(1,1)$-current $-\partial\bar{\partial}\varphi - \alpha$ is positive, then there exists a quasi-plurisubharmonic function $\psi : \Sigma \to \mathbb{R} \cup \{-\infty\}$, such that $\varphi = \psi$ almost everywhere.

With this framework about potentials (quasi-potentials) of currents (closed and positive currents) in mind, we proceed with the next subsection, where we prove a result for the speed of convergence of equidistribution.

6.4.2 Convergence speed of equidistribution of zeros

We prove the following theorem on the speed of convergence for the equidistribution from our Theorem 6.3.1. The theorem below was inspired by Dinh and Sibony's method of meromorphic transforms.

Theorem 6.4.1 (Convergence speed of equidistribution of zeros)**.** *Let Σ be a punctured Riemann surface as above and (L, h^L) a Hermitian holomorphic line bundle with semi-positive curvature which vanishes at most to finite order at any point. Then for any relatively compact open subset $U \subset \Sigma$ there exist $c_U > 0$ and $p(U) \in \mathbb{N}$ with the following property. For any sequence $(\lambda_p)_{p\in\mathbb{N}}$ of real numbers and for any $p \geqslant p(U)$ there exists a set $\Theta_p \subset \mathbb{P}(H^0_{(2)}(\Sigma, L^p \otimes E))$ such that:*

(a) $\sigma_{\mathrm{FS},p}(\Theta_p) \leqslant c_U p^2 e^{-\lambda_p/c_U}$,

(b) For any $s_p \in \mathbb{P}(H^0_{(2)}(\Sigma, L^p \otimes E)) \setminus \Theta_p$ *and any relatively compact open subset* $U \subset \Sigma$,

$$\left\| \frac{1}{p}\,[s_p = 0] - \frac{\sqrt{-1}}{2\pi} R^L \right\|_{U,-2} \leqslant \lambda_p p^{-1/3} \,. \tag{6.4.2}$$

On open sets U *where the curvature is strictly positive, we can replace the term* $p^{-1/3}$ *by* p^{-1} *in the inequality above.*

Recall the definition of the normalized Fubini-Study measure $\sigma_{\mathrm{FS},p}$ in (6.2.7). In order to prove Theorem 6.4.1, we will apply the following Lemma, the proof of which can be found in [22, p.12] or [21, p.3091-3094].

Lemma 6.4.2. *Let* V *be a* $(d+1)$*-dimensional complex vector space, let* $\Xi \subset \mathbb{P}(V^*)$ *be a closed subset and take a function* $u \in \mathcal{L}^1(\mathbb{P}(V^*)) \cap \mathcal{C}^0(\mathbb{P}(V^*) \setminus \Xi)$ *and a positive real constant* $\gamma \in \mathbb{R}_{>0}$*. Suppose that there exists a positive closed* $(1,1)$*-current* S *of mass* 1 *(with respect to the norm defined in* (2.8.9)*), such that* $-S \leqslant \sqrt{-1}\,\partial\bar{\partial} u \leqslant S$ *and the integral of* u *with respect to the measure* $\sigma_{\mathrm{FS},p}$ *is zero. Then there exists a constant* $c > 0$ *and a Borel set* $\Theta = \Theta(S, \gamma) \subset \mathbb{P}(V^*)$*, such that*

$$\sigma_{\mathrm{FS},p}(\Theta) \leqslant c d^2 e^{-\gamma} \qquad \text{and} \qquad |u(a)| \leqslant \gamma \qquad \forall a \in \mathbb{P}(V^*) \setminus (\Xi \cup \Theta) \,. \tag{6.4.3}$$

Proof of Theorem 6.4.1. Recall the identification (6.2.6) of the projective space $\mathbb{P}(H^0_{(2)}(\Sigma, L^p \otimes E))$ with the Grassmanian $G_{d_p-1}(H^0_{(2)}(\Sigma, L^p \otimes E))$ of (d_p-1)-dimensional hyperplanes in $H^0_{(2)}(\Sigma, L^p \otimes E)$; in light of this identification, we will use the projective space and the Grassmanian interchangeably. We take a $\mathcal{L}^2$-holomorphic section $s_p \in H^0_{(2)}(\Sigma, L^p \otimes E)$, so that $[s_p] \in \mathbb{P}(H^0_{(2)}(\Sigma, L^p \otimes E))$, and denote by $H_{s_p} = \ker(s_p)$ the corresponding (unique) hyperplane in $G_{d_p-1}(H^0_{(2)}(\Sigma, L^p \otimes E))$. It defines a current of integration $(\Phi_{p,(2)})^*[H_{s_p}] := [\mathrm{Div}(s_p')]$ for any $s_p' \in [s_p]$. Observe that when the preimage $(\Phi_{p,(2)})^{-1}(H_{s_p})$ does not contain any open subset of Σ, the pull-back $(\Phi_{p,(2)})^*[H_{s_p}]$ is well-defined as a current of integration; for if any neighborhood of a point $x \in \Sigma$ gets mapped by $\Phi_{p,(2)}$ onto the same set of sections as x, these sections must be identically vanishing so their span defines no unique hyperplane. Equivalently, locally, we can write $[H_{s_p}] = \sqrt{-1}\,\partial\bar{\partial} u$ for some plurisubharmonic function u.

Then $(\Phi_{p,(2)})^*[H_{s_p}] := \sqrt{-1}\,\partial\bar{\partial}(u \circ \Phi_{p,(2)})$ is well-defined since u is equal to $-\infty$ on H_{s_p} and smooth otherwise, so that $u \circ \Phi_{p,(2)}$ is not identically $-\infty$. Define Ξ_p to be the closure of the set of points $[s_p] \in \mathbb{P}(H^0_{(2)}(\Sigma, L^p \otimes E))$ where this fails to hold.

In our case, where $\overline{\Sigma}$ is a compact Riemann Surface, Ξ_p consists of finitely many isolated points. For the sections s_p, such that the line $[s_p]$ does not intersect any point in Ξ_p, the assignment $s_p \mapsto (\Phi_{p,(2)})^*[H_{s_p}]$ to the corresponding current is then a continuous function.

Let $\Gamma \subset \Sigma \times \mathbb{P}(H^0_{(2)}(\Sigma, L^p \otimes E)^*)$ be the graph of the Kodaira map $(\Phi_{p,(2)})$ and define

$$\begin{aligned}\tilde{\Sigma} := \{(x,[s_p]) : \exists v \in \mathbb{P}(H^0_{(2)}(\Sigma, L^p \otimes E)^*) \text{ such that } (x,v) \in \Gamma, v \in H_{s_p}\} \\ \subset \Sigma \times \mathbb{P}(H^0_{(2)}(\Sigma, L^p \otimes E))\,. \qquad (6.4.4)\end{aligned}$$

This is a compact analytic subset of dimension $1 + (d_p - 1) = d_p$. Denote by π_1 and π_2 the projections of $\tilde{\Sigma}$ onto Σ and $\mathbb{P}(H^0_{(2)}(\Sigma, L^p \otimes E)^*)$ respectively.

It follows that for any open $U \subset \Sigma$, there exists an integer $p'(U) \in \mathbb{N}$ and a suitable neighborhood $W \supset U$, such that $(\Phi_{p,(2)})|_W$ is an embedding in W for all $p \geqslant p'(U)$. We fix a smooth positive $(1,1)$-form τ with compact support contained in W, such that for any real smooth $\mathcal{C}^2$-function φ with compact support in U and $||\varphi||_{\mathcal{C}^2} \leqslant 1$ we have

$$-\tau \leqslant \sqrt{-1}\,\partial\bar{\partial}\varphi \leqslant \tau\,. \qquad (6.4.5)$$

With help of the projection operators, we can write $(\Phi_{p,(2)})^*[H_{s_p}] = (\pi_1)_*([(\pi_2)^*(s_p)])$ and define

$$v := (\pi_2)_*(\pi_1)^*(\varphi) \in \mathcal{L}^1(\mathbb{P}(H^0_{(2)}(\Sigma, L^p \otimes E)) \cap \mathcal{C}^0(\mathbb{P}(H^0_{(2)}(\Sigma, L^p \otimes E)) \setminus \Xi_p)\,. \qquad (6.4.6)$$

For any $s_p \in H^0_{(2)}(\Sigma, L^p \otimes E)$, we thus have

$$v([s_p]) = \Big((\Phi_{p,(2)})^*[H_{s_p}], \varphi\Big) \qquad (6.4.7)$$

and by definition (6.2.7) of $\sigma_{\mathrm{FS},p}$, the mean value of v is given by

$$M_v := \int_{\mathbb{P}(H^0_{(2)}(\Sigma,L^p\otimes E))} v\,\mathrm{d}(\sigma_{\mathrm{FS},p}) = \Big((\Phi_{p,(2)})^*(\omega_{\mathrm{FS},p}^{d_p-1}), \varphi\Big)\,. \qquad (6.4.8)$$

Let $T := (\pi_2)_*(\pi_1)^*(\tau)$. Recall that by assumption, τ is positive and of maximal bidigree, hence closed. Since the operators $(\pi_1)^*$ and $(\pi_2)_*$ preserve these conditions, T is a positive closed $(1,1)$-current on $\mathbb{P}(H^0_{(2)}(\Sigma, L^p \otimes E))$.

Also, the relation (6.4.5) implies

$$-T \leqslant \sqrt{-1}\,\partial\bar{\partial}v \leqslant T\,. \qquad (6.4.9)$$

Let $\vartheta := ||T||$ be the mass of T with respect to the norm (2.8.9). Now on $\mathbb{P}(H^0_{(2)}(\Sigma, L^p \otimes E))$, we define a function $u := \vartheta^{-1}(v - M_v)$ to measure the deviation of the integration of our functions in question to their mean values.

We consider the normalized current $S := \vartheta^{-1}T$. For any number $\gamma > 0$ we can apply the previous Lemma 6.4.2 to S and γ/ϑ, so that there exists a set Θ'_p independent of φ, such that $\sigma_{\mathrm{FS}}(\Theta'_p) \leqslant c(d_p - 1)^2 e^{-\gamma/\vartheta} \sim cp^2 e^{-\gamma/\vartheta}$ and $|u| \leqslant \gamma/\vartheta$ outside of $\Theta_p := \Xi_p \cup \Theta'_p$. Replacing ϑ by $\max(c, \vartheta)$ verifies the first claim. For the second claim, note that the Lemma 6.4.2 also implies that

$$||(\Phi_{p,(2)})^*[H_s] - (\Phi_{p,(2)})^*(\omega_{\mathrm{FS},p})||_{U,-2} \leqslant \gamma\,. \tag{6.4.10}$$

by the definition of u and the norm (2.8.4). Now on U, the restriction of the Kodaira map $(\Phi_{p,(2)})|_W$ (onto $W \subset U$) is an embedding for all $p \geqslant p'(U)$ and hence Theorem 5.3.1 implies that $p^{-1}((\Phi_{p,(2)})^*(\omega_{\mathrm{FS},p,(2)}))|_U$ differs from $\frac{\mathbf{i}}{2\pi} R^L|_U$ by a form of norm bounded by $C_{U,2}/p^{-1/3}$, for some constant $C_{U,2} > 0$ that depends on U.

Thus, we can find a sequence $(\lambda'_p)_{p\in\mathbb{N}}$ of positive real numbers $\lambda'_p \in \mathbb{R}_{>0}$, such that for any smooth function $\varphi \in \Omega^{0,0}_c(\Sigma)$ with compact support contained in U, we have

$$\left|\left(\frac{1}{p}(\Phi_{p,(2)})^*[H_{s_p}] - \frac{1}{p}(\Phi_{p,(2)})^*(\omega_{\mathrm{FS},p,(2)}), \varphi\right)\right| \leqslant (C_{U,2} + \lambda_p)\, p^{-1/3} ||\varphi||_{\mathcal{C}^2(U)} \tag{6.4.11}$$

where $\lambda_p := \lambda'_p p^{-2/3}$. Now take $p''(U) \in \mathbb{N}$ to be the smallest integer, such that for all $p \geqslant p''(U)$ we have $\lambda_p \geqslant C_{U,2}$ and set $p(U) := \max(p'(U), p''(U))$. This implies the second claim (ii) and the proof is complete. $\square$

Appendices

A Jet-bundle and induced norms

Let (F, h^F) be a real (or complex) vector bundle on Σ with h^F a Euclidean (or Hermitian) inner product.

For $x \in \Sigma$, let $\mathcal{G}_x(F)$ denote the germs of local sections of F at x. For $l \in \mathbb{N}$, $s \in \mathcal{G}_x(F)$, the l-th jet of s at x, denoted by $j^l_x s$, is the equivalence class of s in $\mathcal{G}_x(F)$ under the equivalence relation given as follows: two germs are said to be equivalent if on some open coordinate chart containing x where the bundle F is trivialized, their Taylor expansions at x are identical in the first summands up to terms of order l. Let $J^l(F)_x$ denote the vector space of all l-th jets $j^l_x s$, $s \in \mathcal{G}_x(F)$. Then $J^l(F)_x$ is finite dimensional; moreover, the fibration $\amalg_{x\in\Sigma} J^l(F)_x \to \Sigma$ defines a smooth vector bundle on Σ, which is denoted by $J^l(F)$ and called the l-th jet bundle of F on Σ. Note that by this construction, $J^0(F)$ is just F itself.

For an integer $l > 0$, let $\pi^l_{l-1} : J^l(F) \longrightarrow J^{l-1}(F)$ denote the usual projection of vector bundles. Observe that there exists a short exact sequence of vector bundles over Σ (see [38, pp.121])

$$0 \to S^l T^*\Sigma \otimes F \xrightarrow{\text{incl}} J^l(F) \xrightarrow{\pi^l_{l-1}} J^{l-1}(F) \to 0\,, \tag{A.1}$$

where $S^l T^*\Sigma$ is the l-th symmetric tensor power of $T^*\Sigma$. The map incl is defined as follows: for $x \in \Sigma$, we fix a local chart U around x where F is trivialized as F_x; then one element ξ in $(S^l T^*\Sigma \otimes F)_x$ can be constructed as $df_1 \odot df_2 \odot \cdots \odot df_l \otimes v$, where $\odot$ denotes the symmetric tensor product, $v \in F_x$ and $f_1, \ldots, f_l$ are smooth functions on U which vanish at x. Then we define $\text{incl}(\xi) := j^l_x(f_1 f_2 \cdots f_l \otimes v)$. As a consequence, we have an identification of vector bundles over Σ as follows,

$$S^l T^*\Sigma \otimes F \simeq {J^l(F)}\big/{J^{l-1}(F)}\,. \tag{A.2}$$

We equip the vector bundle $S^l T^*\Sigma \otimes F$ with the metric induced by $g^{T\Sigma}$ and h^F. For $s \in \mathcal{G}_x(F)$, let $j^l_x s / j^{l-1}_x s \in (S^l T^*\Sigma \otimes F)_x$ be the unique element that is determined by the isomorphism (A.2), and let $|j^l_x s / j^{l-1}_x s|$ denote the corresponding norm. For $x \in \Sigma$, let $(Z_1, Z_2) \in \mathbb{R}^2 \simeq T_x\Sigma$ denote the (geodesic) normal coordinate centered at x. Then for any germ $s \in \mathcal{G}_x(F)$, we have

$$|j^l_x s / j^{l-1}_x s|^2 := \sum_{|\alpha|=l} \frac{1}{\alpha!} \left| \frac{\partial^{|\alpha|} s}{\partial Z^\alpha}(0) \right|^2_{h^F_x} \tag{A.3}$$

and set $|j^0_x s / j^{-1}_x s| := \|s(x)\|_{h^F}$; here, $\alpha \in \mathbb{N}^2$ is a multi-index and $|\alpha|$ its length.

This way, we can define a norm on $J^l(F)$ as follows: for $s \in \mathcal{G}_x(F)$, set

$$\|j^l_x s\|^2 := \sum_{k=0}^{l} |j^k_x s / j^{k-1}_x s|^2 \,, \tag{A.4}$$

where $\|j^0_x s / j^{-1}_x s\| := \|s(x)\|_{h^F}$.

B Probability Theory

We recall some definitions and elementary results from measure theory and probability theory.

Definition B.1 (σ-algebra, measurable space)**.** Let Ω be a set that is not empty. A subset $\mathcal{F} \subset 2^\Omega$ of the set 2^Ω of all subsets of Ω is called σ-algebra over Ω if the following properties hold:

(i) $\Omega \in \mathcal{F}$;

(ii) if $A \in \mathcal{F}$, then the complement $A^\complement \in \mathcal{F}$;

(iii) if $(A_j)_{j\in\mathbb{N}} \subset \mathcal{F}$, then the union $\cup_{j\in\mathbb{N}} A_j \in \mathcal{F}$.

The data $(\Omega, \mathcal{F})$ is called a measurable space.

We denote by Borel$(\mathbb{R}^d)$ the σ-algebra that is generated by the open sets in $\mathbb{R}^d$ with respect to the Euclidean topology.

Definition B.2 (Probability measure, probability space)**.** A probability measure $\mathbf{P}$ on the measurable space $(\Omega, \mathcal{F})$ is a function such that $\mathbf{P}(\Omega) = 1$ and for any sequence $(A_j)_{j\in\mathbb{N}} \subset \mathcal{F}$ of pairwise disjoint sets, i.e. $A_i \cap A_j = \varnothing$ for all $i, j \in \mathbb{N}$ with $i \neq j$, we have $\mathbf{P}(\bigcup_{j\in\mathbb{N}} A_j) = \sum_{j\in\mathbb{N}} \mathbf{P}(A_j)$.

The data $(\Omega, \mathcal{F}, \mathbf{P})$ is called a probability space and in this case $\mathcal{F}$ is called the σ-algebra of events.

Remark B.3*.* The second property in Definition B.2 is called σ-additivity; since measures are always σ-additive, the above definition can be simplified: a probability measure is a measure with total mass $= 1$.

Definition B.4 (Random variable)**.** Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and $(M, \mathcal{M})$ a measurable space. A function $X : \Omega \to M$ is called a random variable if for all $A \in \mathcal{M}$

its preimage under X is contained in $\mathcal{F}$, i.e. if $X^{-1}(A) \in \mathcal{F}$. The values $X(\omega) \in M$ for $\omega \in \Omega$ are called realizations of the random variable. The probability of an event described by the random variable X is the value of the image measure $\mathbf{P} \circ X^{-1}$.

Definition B.5 (Distribution function)**.** Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and $X : (\Omega, \mathcal{F}, \mathbf{P}) \to (\mathbb{R}^d, \mathrm{Borel}(\mathbb{R}^d))$ be a random variable. Then

$$\begin{aligned} F_X : \mathbb{R}^d &\longrightarrow [0,1] \\ t &\longmapsto \mathbf{P}(X \leqslant t) := \mathbf{P}\left(X \in \times_{j=1}^d (-\infty, t_j]\right) \end{aligned} \tag{B.1}$$

is called (cumulative) distribution function (in short cfd) of X. Similarly, if μ is a probability measure on $(R^d, \mathrm{Borel}(R^d))$, then

$$\begin{aligned} F_\mu : \mathbb{R}^d &\longrightarrow [0,1] \\ t &\longmapsto \mu\left(X \in \times_{j=1}^d (-\infty, t_j]\right) \end{aligned} \tag{B.2}$$

is called the distribution function of μ.

Recall the following basic properties of distribution functions, which can be derived by elementary methods.

Theorem B.6. *Let F be the distribution function of a real random variable or of a probability measure on $(\mathbb{R}^d, \mathrm{Borel}(\mathbb{R}^d))$. Then F obeys the following properties:*

(i) *F is non-decreasing.*

(ii) $\lim_{t\to-\infty} F(t) = 0$ *and* $\lim_{t\to\infty} F(t) = 1$.

(iii) *For all $t_0 \in \mathbb{R}$, $F(t_0) = \lim_{t\searrow} F(t)$, i.e. F is continuous from the right.*

Moreover, distribution functions are characterized in terms of these three properties, i.e. any function that satisfies these three properties is a distribution function of a real random variable (or of a probability measure on $(\mathbb{R}^d, \mathrm{Borel}(\mathbb{R}^d))$.

In light of Theorem B.6, we will call any function $F : \mathbb{R} \to [0,1]$ that satisfies properties (i)-(iii) a distribution function.

The following elementary result, together with Theorem B.6 establishes a one-to-one correspondence between (real-valued) random variables and distribution functions.

Theorem B.7. *If $F : (\mathbb{R}^d, \mathrm{Borel}(\mathbb{R}^d), \mathbf{P}) \to (\mathbb{R}^d, \mathrm{Borel}(\mathbb{R}^d))$ is any distribution function, then there exists a unique probability measure μ on $(\mathbb{R}^d, \mathrm{Borel}(\mathbb{R}^d))$, such that $F_\mu = F$.*

Recall the following elementary fact from measure theory.

Theorem B.8. *Let $(\Omega, \mathcal{F})$ and $(M, \mathcal{M})$ be a measurable spaces and equip the former with a measure μ. If $f : (\Omega, \mathcal{F}) \to (M, \mathcal{M})$ is a measurable map, then the set function $\mu \circ f^{-1} : \mathcal{M} \ni A \longmapsto \mu(f^{-1}(A)) \in [0, \infty]$ is a measure on $(M, \mathcal{M})$. If μ is a probability measure (and hence f is a random variable), then the push-forward $\mu \circ f^{-1}$ is called the distribution (or law) of f.*

Remark B.9. In terms of random variables, saying that an event happens at random with respect to a distribution μ can be expressed in the following way: Given any distribution/law μ, i.e. a probability measure on $(M, \mathcal{M})$, a random variable X with law μ can be reconstructed by choosing $(M, \mathcal{M}, \mu)$ as underlying probability space and set $X : M \ni \omega \to \omega \in M$, i.e. X is the identity on the set M. Then $X : (M, \mathcal{M}, \mu) \to (M, \mathcal{M})$ defined a random variable with law μ.

Thus, if we want to describe a random experiment whose outcome is a value in a set M, we can do so by choosing $(M, \mathcal{M}, \mu)$ as the underlying probability space, for a suitable σ-algebra $\mathcal{M}$ and law μ.

Definition B.10 (Probability density)**.** Let $(\Omega, \mathcal{F})$ and $(M, \mathcal{M})$ be a measurable spaces and equip the former with a measure μ and $f : (\Omega, \mathcal{F}) \to (\mathbb{R} \cup -\infty, \infty, \mathrm{Borel}(\mathbb{R} \cup -\infty, \infty))$ be a measurable function. We call f a probability density (with respect to μ), if $\int_{\mathbb{R}} f \, \mathrm{d}\mu = 1$.

In this thesis, we are interested in studying infinite sequences of random variables. We thus need to make sure that the underlying probability space is sufficiently rich in structure to accommodate for the wealth of information that the corresponding infinite family of probability spaces of the associated member random variables of the infinite sequence has. One suitable way to model this situation is by considering product probability spaces.

Definition B.11. Let Λ be any non-empty index set and $(\Omega_\lambda)_{\lambda \in \Lambda}$ a family of non-empty sets. The product space $\times_{\lambda \in \Lambda} \Omega_\lambda$ is defined to be the set of all maps $f : \Lambda \ni \lambda \to f(\lambda) \in \bigcup_{\lambda \in \Lambda} \Omega_\lambda$, such that $f(\lambda) \in \Omega_\lambda$ for all $\lambda \in \Lambda$. In the case where $\Omega_\lambda = \Omega$ for all $\lambda \in \Lambda$, we will denote the corresponding product space by Ω^Λ.

Definition B.12. Let Λ be any non-empty index set and $I \subset J \subset \Lambda$ any non-empty subsets. We call

$$\begin{aligned}\mathrm{proj}_I^J : \times_{\lambda \in J} \Omega_\lambda &\longrightarrow \times_{\lambda \in I} \Omega_\lambda \\ \omega &\longmapsto \omega|_I \end{aligned} \tag{B.3}$$

the projection from J onto I. If $J = \Lambda$, we write $\mathrm{proj}_I^J = \mathrm{proj}_I$ and if $I = \{\lambda\}$ for any $\lambda \in \Lambda$, we write $\mathrm{proj}_I^J = \mathrm{proj}_{\{\lambda\}}^J =: \mathrm{proj}_\lambda^J$.

Definition B.13. Let Λ be any non-empty index set and $(\Omega_\lambda, \tau_\lambda)_{\lambda \in \Lambda}$ be any family of topological spaces indexed by Λ. The product topology τ on the product (topological) space $\times_{\lambda \in \Lambda} \Omega_\lambda$ is the smallest topology on $\times_{\lambda \in \Lambda} \Omega_\lambda$ such that for each $\lambda' \in \Lambda$ the projection maps $\mathrm{proj}_{\lambda'} : \times_{\lambda \in \Lambda} \Omega_\lambda \to \Omega_{\lambda'}$ are continuous with respect to the topologies τ and τ_λ, for all $\lambda \in \Lambda$.

Definition B.14 (Product σ-algebra)**.** Let Λ be any non-empty index set and $(\Omega_\lambda \mathcal{F}_\lambda)_{\lambda \in \Lambda}$ be any family of measurable spaces indexed by Λ. The product-σ-algebra $\bigotimes_{\lambda \in \Lambda} \mathcal{F}_\lambda$ on the product (measurable) space $\times_{\lambda \in \Lambda} \Omega_\lambda$ is the smallest σ-algebra on $\times_{\lambda \in \Lambda} \Omega_\lambda$ such that for each $\lambda' \in \Lambda$ the projection maps $\mathrm{proj}_{\lambda'} : \times_{\lambda \in \Lambda} \Omega_\lambda \to \Omega_{\lambda'}$ are $\bigoplus_{\lambda \in \Lambda} \mathcal{F}_\lambda$-$\mathcal{F}_{\lambda'}$-measurable, i.e. for all $A \in \mathcal{F}_{\lambda'}$, we have $\mathrm{proj}_{\lambda'}^{-1}(A) \in \bigotimes_{\lambda \in \Lambda} \mathcal{F}_\lambda$, for all $\lambda \in \Lambda$.

We set $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ and $\overline{\mathbb{R}}^n := \overline{\mathbb{R}}^{\times n}$ for the n-times Cartesian product of $\overline{\mathbb{R}}$.

Definition B.15 (Expectation, (Co-)variance)**.** Let $X = (X_1, \dots, X_n), Y = (Y_1, \dots, Y_n) : (\Omega, \mathcal{F}, \mathbf{P}) \to (\mathbb{K}^n, \mathrm{Borel}(\mathbb{K}^n))$be two random variables, where $\mathbb{K}^n \in \{\mathbb{R}^n, \overline{\mathbb{R}}^n, \mathbb{C}^n\}$ for $n \in \mathbb{N}$. Assuming the right hand side is well-defined, the expectation (or expected value) of X is

$$\mathbf{E}[X] := \int_\Omega X \mathrm{d}\mathbf{P} \,. \tag{B.4}$$

where the right hand side is vector valued, if $n \neq 1$. Assume that the absolute value $|X|$ of X is Lebesgue integrable. Then the covariance matrix is the matrix with entries

$$\mathrm{Cov}[X_i, Y_j] := \mathbf{E}[(X_i - \mathbf{E}[X_i])(X_j - \mathbf{E}[X_j])] \in [0, \infty] \,, \quad \mathrm{Cov}[X, Y] := (\mathrm{Cov}[X_i, Y_j])_{i,j} \,. \tag{B.5}$$

For all $1 \leqslant i \leqslant n$, the variance of X_i is given by

$$\mathrm{Var}[X_i] = \mathbf{E}[(X_i - \mathbf{E}[X_i])^2] \tag{B.6}$$

i.e. the variances of the components of a vector valued random variable are the entries of the covariance matrix of the random variable with itself on the diagonal.

Definition B.16 (Almost sure convergence)**.** Let X and $(X_n)_{n\in\mathbb{N}}$ be a random variable and a sequence of random variables, each with domains a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and ranges in a metric space (S, d). Then X_n is said to converge $\mathbb{P}$-almost surely (short $\mathbb{P}$-a.s. or a.s.) to X if

$$\mathbf{P}\left(\lim_{n\to\infty} \mathrm{d}(X_n, X) = 0\right) = \mathbf{P}\left(\left\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\}\right) = 1\,. \tag{B.7}$$

In other words, the realizations $X_n(\omega)$ converge pointwise to the realization $X(\omega)$ for all $\omega \in \Omega$ except at most any set of measure zero with respect to the measure $\mathbb{P}$. The same definition can therefore be extended to any measures that are not necessarily probability measures.

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