Multiprojective Seshadri stratifications for Schubert varieties and standard monomial theory

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Zusammenfassung

Mittels normaler Seshadri-Stratifizierungen erhielten Chirivì, Fang und Littelmann eine Standardmonomentheorie (SMT) auf dem homogenen Koordinatenring bestimmter eingebetteter projektiver Varietäten, d. h. eine Basis aus sogenannten Standardmonomen. Im Fall von Schubert-Varietäten wurde eine SMT bereits kombinatorisch von Lakshmibai, Musili und Seshadri entwickelt. Wir verallgemeinern den Begriff der Seshadri-Stratifizierung auf abgeschlossene Untervarietäten in einem Produkt projektiver Räume und konstruieren solche Stratifizierungen auf Schubert-Varietäten in jedem Dynkin-Typ. Unter Verwendung des Littelmann-Pfadmodells zeigen wir, dass diese Stratifizierungen eine geometrische Erklärung für die SMT von Hodge und Young durch semistandard Young-Tableaus liefern, sowie für die SMT von Lakshmibai, Musili und Seshadri und allgemeiner, für eine SMT, die durch Sequenzen von LS-Pfaden indiziert wird.

Abstract

Via normal Seshadri stratifications, Chirivì, Fang and Littelmann obtained a standard monomial theory (SMT) on the homogeneous coordinate ring of certain embedded projective varieties, that is to say a basis of so called standard monomials. In the case of Schubert varieties such a SMT was already developed combinatorially by Lakshmibai, Musili and Seshadri. We generalize the notion of a Seshadri stratification to closed subvarieties in a product of projective spaces and construct such stratifications on Schubert varieties in every Dynkin type. Using the Littelmann path model, we show that these stratifications provide a geometric explanation of the SMT of Hodge and Young indexed by semistandard Young tableaux, the SMT of Lakshmibai, Musili and Seshadri and more general, of a SMT indexed by sequences of LS-paths.

Contents

1. Introduction

In the 1940s, Hodge described a basis of the homogeneous coordinate ring of a Grassmann variety $\mathrm{Gr}(d,n)$ via certain products of Plücker coordinates ([Hod], [HP]). The basis vectors correspond to semistandard Young tableaux with exactly d rows and entries in $\{1,\dots,n\}$. This is the first example of what is known as a *standard monomial theory* (SMT). However there still exists no clear definition what a standard monomial theory really is, this term rather refers to specific examples, which usually come from the representation theory of semisimple algebraic groups or Lie algebras. Given an algebra generated by a finite set S, the set of all monomials in S generate this algebra as a vector space. One tries to extract a basis from this generating set via combinatorial methods. The basis vectors are then called *standard* and every monomial in S not belonging to this basis is called *non-standard*.

In their series of papers ([Ses1], [LS2], [LMS3], [LMS4], [LS5], ...) Lakshmibai, Musili and Seshadri generalized the work of Hodge to Schubert varieties in classical Dynkin types. They found a standard monomial basis of the multihomogeneous coordinate ring of the Schubert variety X with respect to the embedding

$$X \hookrightarrow \prod_{i=1}^{m} \mathbb{P}(V(\omega_i)),$$

where ω_i are certain fundamental weights. This basis is indexed by sequences of Weyl group cosets, which can be lifted to a weakly decreasing sequence in the Weyl group, called *defining chain*.

The path model of Littelmann – more specifically the path model of LS-paths – developed in [Lit94] and [Lit95] provided a suitable language for this index set, such that the SMT of Lakshmibai, Musili and Seshadri could be generalized to arbitrary Dynkin types [Lit96]). To each LS-path one associates a function called *path vector*, which Littelmann constructed in [Lit98] using quantum Frobenius splitting. Standard monomials in these path vectors are indexed by sequences of LS-paths which admit a weakly decreasing lift to the Weyl group. This leads to the notion of what we call an *LS-tableau* (see Section 4.2), a generalization of Young tableaux.

Since the discovery of this combinatorially defined standard monomial basis, it has attracted much attention and a large amount of citations and applications. As the multihomogeneous coordinate ring is an algebraic-geometric object, it is a natural question whether the SMT can also be derived using geometric methods. This leads to the main theorem of this thesis.

Theorem (Theorem 5.11 and Proposition 5.5). There exists a quasi-valuation $\mathcal{V}$ with at most one-dimensional leaves on the multihomogeneous coordinate ring $\mathbb{K}[X]$ of X, such that the elements in the image of $\mathcal{V}$ correspond to certain standard LS-tableaux.

We now explain the meaning of the objects appearing in this theorem. Note that there may exist different quasi-valuations on $\mathbb{K}[X]$, which therefore give geometrical interpretations of different SMTs.

The geometric interpretation of LS-tableaux is based on the connection between standard monomial theory and the vanishing ideals of unions of Schubert varieties. This connection was formalized by Chirivì, Fang and Littelmann in [CFL] and [CFL2]. They introduced the concept of a *Seshadri stratification* on an embedded projective variety $X \subseteq \mathbb{P}(V)$. It consists of a family $(X_p)_{p\in A}$ of closed subvarieties $X_p \subseteq X$ indexed by a graded poset A and a homogeneous function f_p in the homogeneous coordinate ring $\mathbb{K}[X]$ of X (called *extremal function*) for each $p \in A$. Every variety X_p has to be irreducible and smooth in codimension one. The grading on A needs to be compatible with the dimensions of the subvarieties, i. e. X_q is a divisor in X_p, if and only if $q < p$ is a covering relation in A.

Seshadri stratifications use a web of subvarieties in contrast to the Newton-Okounkov theoretical approach ([KK], [LM]), which uses a flag of subvarieties. By taking successive vanishing multiplicities along this web, every Seshadri stratification induces a quasi-valuation $\mathcal{V} : \mathbb{K}[X]\backslash\{0\} \to \mathbb{Q}^A$, which can be thought of as a filtration of the homogeneous coordinate ring $\mathbb{K}[X]$. In general, the quasi-valuation $\mathcal{V}$ is not quite canonical, as it depends on the choice of a total order $\geq^t$ linearizing the partial order on A. The subquotients (called *leaves*) of the filtration on $\mathbb{K}[X]$ are at most one-dimensional and they are indexed by the image Γ of $\mathcal{V}$, which is a union of finitely generated semigroups $\Gamma_{\mathfrak{C}}$ over all maximal chains $\mathfrak{C}$ in the poset A. Hence Γ is called the *fan of monoids* to the stratification. The projective variety X degenerates into a reduced union of the toric varieties to these semigroups $\Gamma_{\mathfrak{C}}$ via a Rees algebra construction. To each semigroup $\Gamma_{\mathfrak{C}}$ one can also associate a Newton-Okounkov body, which turns out to be a simplex. Hence for Seshadri stratifications, the Newton-Okounkov body of a flag of subvarieties is replaced by a simplicial complex.

For each *normal* Seshadri stratification, i. e. every semigroup $\Gamma_{\mathfrak{C}}$ is saturated, the fan of monoids Γ defines a standard monomial theory on the homogeneous coordinate ring $\mathbb{K}[X]$. Every element in Γ can be uniquely decomposed as a sum of indecomposable elements. When choosing a function $x_{\underline{a}}$ for each indecomposable element $\underline{a} \in \Gamma$ then all monomials in these functions generate $\mathbb{K}[X]$ as a vector space and a monomial $x_{\underline{a}^1} \cdots x_{\underline{a}^s}$ is standard, if and only if $\underline{a}^1 + \cdots + \underline{a}^s$ is contained in Γ.

In both [CFL2] and [CFL4], Chirivì, Fang and Littelmann already constructed a normal Seshadri stratification on every Schubert variety X_τ, embedded into a projective space over a Demazure module. Hence they obtain a SMT on the associated homogeneous coordinate ring. Note that the SMT by Lakshmibai, Musili and Seshadri mentioned above, gives rise to a basis of a different coordinate ring, namely the multihomogeneous coordinate ring of X_τ with respect to the embedding into the product $\prod_\omega \mathbb{P}(V(\omega))$, where ω runs over certain fundamental weights. This raises the following question.

Question. Is there a normal Seshadri stratification on X_τ with respect to the multiprojective embedding, such that one obtains the SMT of Lakshmibai, Musili and Seshadri, or more general, the SMT indexed by LS-tableaux?

In this thesis we show that such a stratification exists under certain combinatorial conditions. We now give an overview over the different chapters in this thesis. Answering the question above first requires generalizing the notion of a Seshadri stratification to projective varieties X embedded into a product $\prod_{i=1}^m \mathbb{P}(V_i)$ of projective spaces, which we call *multiprojective varieties*. In the first chapter, we therefore introduce multiprojective Seshadri stratifications. In contrast to the ordinary Seshadri stratifications in [CFL], the variety X_p need not be a subvariety of X itself, but of a projection X_{I_p} of X into a product $\prod_{i \in I_p} \mathbb{P}(V_i)$ indexed by a non-empty subset $I_p \subseteq \{1, \ldots, m\}$. The collection $\mathcal{I} = \{I_p \mid p \in A\}$ of these sets is called the *index poset* of the stratification, which is an additional structure not visible for ordinary stratifications. By taking the affine multicones $\hat{X}_p$ of the stratum X_p, one can still view $\hat{X}_q$ as a closed subvariety of $\hat{X}_p$, if and only if $q \leq p$. The extremal functions f_p are chosen to be multihomogeneous elements of the multihomogeneous coordinate ring $\mathbb{K}[X]$.

For multiprojective stratifications one can still define a quasi-valuation $\mathcal{V} : \mathbb{K}[X] \setminus \{0\} \to \mathbb{Q}^A$ inducing a filtration on $\mathbb{K}[X]$ with at most one-dimensional leaves and a fan Γ of finitely generated monoids. The big difference to the original Seshadri stratifications lies in the Newton-Okounkov theory. Instead of a simplicial complex, we obtain a family of polytopal complexes, which is parametrized by the elements $\underline{d} \in \mathbb{N}_0^m$. Each semigroup $\Gamma_\mathfrak{C}$ to a maximal chain $\mathfrak{C}$ in A defines a polytope $\Delta_\mathfrak{C}^{(\underline{d})}$ and its faces correspond to certain subchains of $\mathfrak{C}$. Almost all of these polytopal complexes carry information about the variety X, e. g. its dimension, but in certain edge cases the dimension of the polytopal complex can be smaller than $\dim X$. Similar to [CFL, Theorem 13.6], the volume of these polytopal complexes with respect to certain lattices computes the leading term of the multivariate Hilbert polynomial.

In chapter 3 we construct a multiprojective Seshadri stratification on each partial flag variety in Dynkin type $\mathtt{A}$ using the combinatorial ideas from [LMS4] and [Ses2]. As expected, the elements in the fan of monoids Γ correspond to certain semistandard Young tableaux. In addition, the stratification is normal and balanced, i. e. Γ does not depend on the choice of the total order $\geq^t$ on the poset A. The resulting standard monomial theory coincides with the classical Hodge-Young theory of standard monomials in Plücker coordinates (see [Ses2, Chapter 2]).

We attempt to generalize this construction in chapter 4 to Schubert varieties $X_\tau \subseteq G/Q$ in arbitrary Dynkin types. To achieve this, we define the tableau model of LS-tableaux, which we hope to find in the associated fan of monoids to the stratification. These tableaux depend on two choices: First, a sequence $\underline{\lambda} = (\lambda_1, \ldots, \lambda_m)$ of dominant weights which fixes the multiprojective embedding

$$X_\tau \hookrightarrow \prod_{i=1}^{m} \mathbb{P}(V(\lambda_i)_\tau)$$

and second, a subposet $\mathcal{I}$ of the power set poset $\mathcal{P}(\{1,\dots,m\})\setminus\{\varnothing\}$, ordered by inclusion. One can think of the elements in $\mathcal{I}$ as the possible shapes of the columns in the LS-tableaux. Each column $\pi_1,\dots,\pi_s$ is an LS-path and their shapes need to follow a weakly decreasing sequence $I_1 \supseteq \cdots \supseteq I_s$ in $\mathcal{I}$. To get the classical Young tableaux for the group $\mathrm{SL}_n(\mathbb{K})$ one would choose the poset $\mathcal{I}$ of all sets $\{1,\dots,i\}$ for $i = 1,\dots,n-1$. The shapes of the columns in a Young tableau correspond to their length, so the set $\{1,\dots,i\}$ represents a column of length i. Semistandard Young tableaux are generalized in the following way: An LS-tableau is called *τ-standard*, if one can lift the Weyl group cosets of its columns to a *defining chain*, i. e. a weakly decreasing sequence in W/W_Q.

To a fixed choice of $\underline{\lambda}$ and $\mathcal{I}$ we associate a graded poset $D(\underline{\lambda},\tau)$, called *defining chain poset*, which can hopefully be used as the underlying poset for the desired Seshadri stratification. This poset is constructed from the idea that every defining chain of a τ-standard LS-tableau should be contained in a chain of $D(\underline{\lambda},\tau)$. However, only certain index posets $\mathcal{I}$ induce a well-defined stratification. First, every two non-comparable elements need to satisfy the condition (4.2) assuring the existence of specific covering relations. Second, the poset is required to be τ-standard. These are exactly the index posets, where the τ-standardness of an associated LS-tableau can be verified locally, by comparing consecutive columns (which is known as *weak standardness*).

Theorem (Theorem 4.30). If $\mathcal{I}$ is τ-standard and satisfies the condition (4.2), then there exists a multiprojective Seshadri stratification on X_τ with underlying poset $D(\underline{\lambda},\tau)$ and index poset $\mathcal{I}$.

Fortunately, there always exists a τ-standard poset $\mathcal{I}$, namely the full power set $\mathcal{P}(\{1,\dots,m\})\setminus\{\varnothing\}$, but this is a rather unwieldy choice for computations. The author was not able to find an combinatorial characterization of τ-standard posets in full generality. When τ is the unique maximal element in W/W_Q, then τ-standardness is characterized by the existence of certain paths in the Dynkin diagram of G (see Theorem 4.24). If the Dynkin diagram is a line (i. e. in the types `A`, `B`, `C`, `F` and `G`), one can always choose a totally ordered poset $\mathcal{I}$ and the associated model of LS-tableaux is similar to classical Young tableaux. We give an example for τ-standard posets for the partial flag varieties in all Dynkin types (Section 4.4).

In the last chapter, we compute the fan of monoids Γ for the previously constructed stratifications. The elements in Γ correspond to LS-tableaux, where the shapes of their columns follow a weakly decreasing sequence in $\mathcal{I}$. To each of these tableaux we associate a monomial in the path vectors defined by Littelmann. The set of monomials corresponding to τ-standard LS-tableaux form a basis of the multihomogeneous coordinate ring of X_τ.

2. Multiprojective Seshadri stratifications

Throughout this chapter we fix an algebraically closed field $\mathbb{K}$ and a *multiprojective* variety X, i.e. a (Zariski-)closed subset $X \subseteq \mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_m)$, where $V_1, \ldots, V_m$ are finite-dimensional vector spaces over $\mathbb{K}$. We included a section about multiprojective varieties in the Appendix A.2, but for the most part they behave analogously to embedded projective varieties.

2.1. Definitions and examples

The multicone $\hat{X}$ of X is a closed subvariety of the affine space $V = V_1 \times \cdots \times V_m$. Let $R = \mathbb{K}[X] = \mathbb{K}[\hat{X}]$ be the multihomogeneous coordinate ring of X. We write $[k]$ for the set of all integers between 1 and $k \in \mathbb{N}$. Each subset $I \subseteq [m]$ comes with the two natural projections

$$\pi_I : \prod_{i \in [m]} \mathbb{P}(V_i) \twoheadrightarrow \prod_{i \in I} \mathbb{P}(V_i) \quad \text{and} \quad \hat{\pi}_I : \prod_{i \in [m]} V_i \twoheadrightarrow \prod_{i \in I} V_i \tag{2.1}$$

as well as the multiprojective variety $X_I = \pi_I(X)$. Note that the multicone $\hat{X}_I$ of X_I coincides with the image of $\hat{X}$ under the map $\hat{\pi}_I$. The surjection $\hat{X} \twoheadrightarrow \hat{X}_I$ induces an embedding of the multihomogeneous coordinate ring $\mathbb{K}[X_I]$ onto a graded subalgebra of R, namely the direct sum of all homogeneous components $R_{\underline{d}} \subseteq R$ for tuples $\underline{d} = (d_1, \ldots, d_m) \in \mathbb{N}_0^m$ where $d_j = 0$ for all $j \notin I$.

Analogous to the definition of a Seshadri stratification in [CFL], we fix a finite set A, a collection $\{X_p \mid p \in A\}$ of irreducible projective varieties, which are smooth in codimension one, and a collection of functions $\{f_p \in R \mid p \in A\}$ called **extremal functions**. The main difference to the original definition is that X_p no longer needs to be a subvariety or even a subset of X. Instead we fix a third collection $\{I_p \subseteq [m] \mid p \in A\}$ of non-empty subsets of $[m]$ and require that X_p is a closed subvariety of $X_{I_p} = \pi_{I_p}(X)$. If we view the affine space $\prod_{i \in I_p} V_i$ as a closed subvariety of V via the linear embedding $\prod_{i \in I_p} V_i \hookrightarrow V$, then $\hat{X}_p$ can be seen as a closed subvariety of $\hat{X}$. This allows us to equip the set A with the partial order $\leq$, such that $q \leq p$ if and only if $\hat{X}_q \subseteq \hat{X}_p$. The function f_p needs to be non-constant, multihomogeneous and included in the subring $\mathbb{K}[X_{I_p}] \subseteq R$.

Definition 2.1 (Multiprojective Seshadri stratification)**.** These three collections of varieties, extremal functions and index sets are called a **(multiprojective) Seshadri stratification**, if there exists an element $p_{\max} \in A$ with $I_{p_{\max}} = [m]$ and $X_{p_{\max}} = X$ and the following three conditions are fulfilled:

(S1) If $q < p$ is a covering relation, then $\hat{X}_q \subseteq \hat{X}_p$ is a codimension one subvariety (where both are seen as subvarieties of V);

(S2) The function f_q vanishes on $\hat{X}_p$, if $q \not\leq p$;

(S3) For each $p \in A$ holds the set-theoretic equality

$$\{x \in \hat{X} \mid f_p(x) = 0\} \cap \hat{X}_p = \{0\} \cup \bigcup_{p \text{ covers } q} \hat{X}_q. \tag{2.2}$$

Notice, that for $m = 1$ the notion of a multiprojective Seshadri stratification coincides with notion of a Seshadri stratification introduced in [CFL]. In this case all strata X_p for $p \in A$ are closed subvarieties of X, since $I_p = \{1\}$.

The affine multicone $\hat{X} \subseteq V$ of X is the affine cone of a projective variety $\widetilde{X} \subseteq \mathbb{P}(V)$. Hence every multiprojective Seshadri stratification on $X \subseteq \prod_{i=1}^m \mathbb{P}(V_i)$ can also be seen as a Seshadri stratification on $\widetilde{X}$. Therefore one can informally say that every result in *loc. cit.*, where the grading on R is not involved, does also hold in the multiprojective case. As a first example: The poset A is a graded poset of length $\dim \widetilde{X} = \dim \hat{X} - 1$, that is to say all maximal chains have length $\dim \widetilde{X}$. The rank of an element $p \in A$ is given by $r(p) = \dim \hat{X}_p - 1$.

Proposition 2.2. *Every multiprojective variety $X \subseteq \mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_m)$ admits a Seshadri stratification.*

Proof. We embed the variety X into the projective space over $W = V_1 \otimes \cdots \otimes V_m$ via the Segre embedding, so we have two different coordinate rings of X: The multihomogeneous coordinate ring $R = \mathbb{K}[X]$ and the homogeneous coordinate ring S of the embedding $X \hookrightarrow \mathbb{P}(W)$. Now choose any Seshadri stratification of $X \subseteq \mathbb{P}(W)$, which exists by [CFL, Proposition 2.11]. Hence for each $p \in A$ we have the closed, irreducible subvariety $X_p \subseteq X$ which is smooth in codimension one and the extremal function $f_p \in S$. This function can be pulled back to a multihomogeneous function in R and its degree is a multiple of $(1, \dots, 1)$. Clearly the conditions (S2) to (S3) are preserved under the pullback. We also need to define a subset of $[m]$ for all $p \in A$: Here we take $I_p = [m]$.

However, for $m \geq 2$ we do not obtain a multiprojective Seshadri stratification on X in this way, as the dimension of the multicones $\hat{X}_p$ for $p \in A$ minimal is greater than 1. Indeed, this multicone $\hat{X}_p$ is of the form

$$\hat{X}_p = L_1^{(p)} \times \cdots \times L_m^{(p)},$$

where $L_i^{(p)}$ is a one-dimensional linear subspace of V_i. Therefore we need to extend the graded poset A by $m - 1$ additional ranks. Set-theoretically this extension is of the form

$$\overline{A} = A \cup \{L_1^{(p)} \times \cdots \times L_i^{(p)} \mid p \in A \text{ minimal and } 1 \leq i \leq m - 1\}.$$

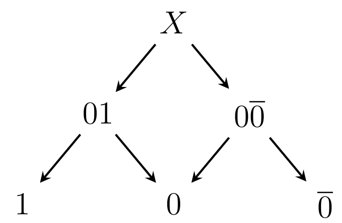
For each $q \in \overline{A} \setminus A$ of the form $L_1^{(p)} \times \cdots \times L_i^{(p)}$ we define the subset $I_q = [i]$ and the projective variety $X_q = \mathbb{P}(L_1^{(p)}) \times \cdots \times \mathbb{P}(L_i^{(p)}) \subseteq X_{I_q}$. Note that we mentioned at the beginning that in general X_p does not need to be a subvariety of X, but there exists a subset $I \subseteq [m]$ such that X_p is a closed subvariety of $X_I = \pi_I(X)$. This case did not occur so far because the stratification is induced by an ordinary Seshadri stratification (as it was defined in [CFL]). The variety X_q only consists of one point, so it is irreducible and smooth in codimension one. The affine variety q is the multicone over X_q. The partial order on $\overline{A}$ is now determined by the equivalence of $q \leq p$ and $\hat{X}_q \subseteq \hat{X}_p$.

It remains to define the extremal functions for the elements in $\overline{A} \setminus A$. Let $\mathcal{L}_i$ be the set of all lines $L_i^{(p)}$ for $p \in A$ minimal. For each $i \in [m-1]$ and $L \in \mathcal{L}_i$ we choose a linear function $h_L \in V_i^*$ which vanishes on L and that does not vanish on all other lines in $\mathcal{L}_i$. To an element $q \in \overline{A} \setminus A$ of the form $L_1^{(p)} \times \cdots \times L_i^{(p)}$ we then associate the function

$$f_q = \prod_{j \in [i]} \prod_{\substack{L \in \mathcal{L}_i \\ L \neq L_i^{(p)}}} h_L.$$

This definition ensures that both conditions (S2) and (S3) are fulfilled. The extremal functions f_p for $p \in A$ also vanish on all varieties $\hat{X}_q$ for $q \in \overline{A} \setminus A$. We therefore have constructed a multiprojective Seshadri stratification on X. □

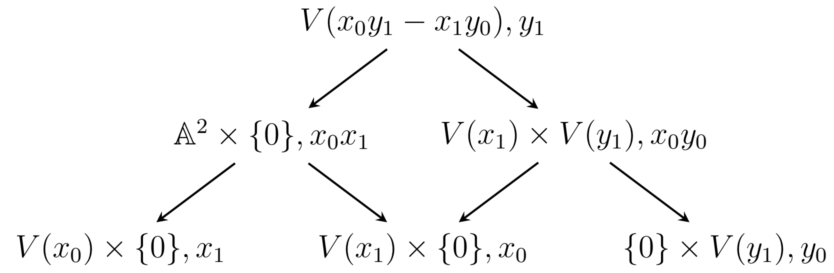
Example 2.3. Let X be the image of the closed diagonal embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}(V_1) \times \mathbb{P}(V_2)$ for $V_1 = V_2 = \mathbb{K}^2$. The coordinate ring $\mathbb{K}[V] = \mathbb{K}[x_0, x_1, y_0, y_1]$ of $V = V_1 \times V_2$ is graded with $\deg x_0 = \deg x_1 = (1,0)$ and $\deg y_0 = \deg y_1 = (0,1)$ and the vanishing ideal of X is equal to $I_{\mathbb{P}}(X) = (x_0 y_1 - x_1 y_0)$. We write $I(-)$ and $I_{\mathbb{P}}(-)$ to distinguish between affine and projective vanishing ideals (see Appendix A.2). Analogously, we differentiate between the affine vanishing set $V(-)$ and the projective vanishing set $V_{\mathbb{P}}(-)$. The multicone $\hat{X}$ of X is given by the vanishing set $V(x_0 y_1 - x_1 y_0) \subseteq V = \mathbb{A}^2 \times \mathbb{A}^2$. We define a poset A via the Hasse-diagram



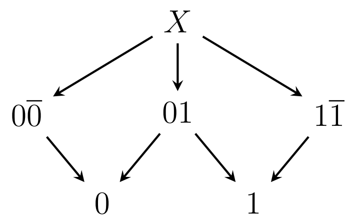
and choose the following index sets, strata and extremal functions:

$p \in A$	I_p	X_p	f_p
X	$\{1,2\}$	$X \subseteq \mathbb{P}(V_1) \times \mathbb{P}(V_2)$	y_1
01	$\{1\}$	$\mathbb{P}(V_1)$	$x_0 x_1$
$0\overline{0}$	$\{1,2\}$	$V_{\mathbb{P}}(x_1) \times V_{\mathbb{P}}(y_1) \subseteq \mathbb{P}(V_1) \times \mathbb{P}(V_2)$	$x_0 y_0$
1	$\{1\}$	$V_{\mathbb{P}}(x_0) \subseteq \mathbb{P}(V_1)$	x_1
0	$\{1\}$	$V_{\mathbb{P}}(x_1) \subseteq \mathbb{P}(V_1)$	x_0
$\overline{0}$	$\{2\}$	$V_{\mathbb{P}}(y_1) \subseteq \mathbb{P}(V_2)$	y_0

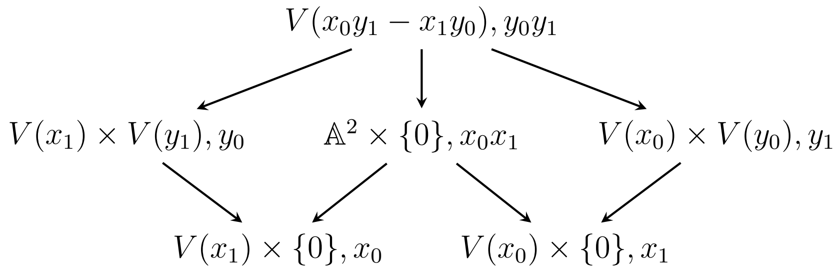
This data defines a Seshadri stratification on X, which can be summarized by a diagram of all multicones $\hat{X}_p$ and f_p for $p \in A$:



Example 2.4. Of course, for each multiprojective variety there can exist many different Seshadri stratifications. For example, there is another stratification on the variety $X = V_{\mathbb{P}}(x_0y_1 - x_1y_0) \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ with underlying poset



that is defined via the following diagram of multicones and extremal functions:



In contrast to the Seshadri stratifications introduced in [CFL], their multiprojective generalizations have an additional underlying structure, namely the poset

$$\mathcal{I} = \{I_p \subseteq [m] \mid p \in A\}, \tag{2.3}$$

which is ordered by inclusion. We call it the **index poset**.

Lemma 2.5. *The map $A \to \mathcal{I}$, $p \mapsto I_p$ is monotone and has the following properties:*

(a) Let $q < p$ be a covering relation in A. Then $I_p \setminus I_q$ contains at most one element. In the case $I_q \neq I_p$ it holds $\pi_{I_q}(X_p) = X_q$.

(b) If $p \in A$ is a minimal element, then I_p is a one-element set.

In particular, $\mathcal{I}$ is a graded poset of length $m - 1$.

Proof. The map $A \to \mathcal{I}$ is monotone, since for all $q \leq p$ in A we have the inclusion $\hat{X}_q \subseteq \hat{X}_p$ of their multicones and this implies $I_q \subseteq I_p$.

(a) Let $q < p$ be a covering relation and suppose that I_q is a proper subset of I_p. For every subset $J \subseteq I_p$ we have the natural linear projection $\hat{\pi}_J$ (see (2.1)). If

$$I_q = J_0 \subsetneq J_1 \subsetneq \cdots \subsetneq J_s = I_p$$

is a chain in $\mathcal{I}$ then we have the closed, irreducible subvarieties

$$\hat{X}_q \subseteq \hat{\pi}_{J_0}(\hat{X}_p) \subsetneq \cdots \subsetneq \hat{\pi}_{J_s}(\hat{X}_p) = \hat{X}_p,$$

which we see as subvarieties of V. As $\hat{X}_q$ is of codimension one in $\hat{X}_p$, it follows $I_p \setminus I_q = \{i\}$ for some $i \in I_p$ and $\hat{X}_q = \hat{\pi}_{I_q}(\hat{X}_p)$, because their dimensions agree.

(b) The condition (S3) implies, that an element $p \in A$ is minimal, if and only if the vanishing set of f_p inside of $\hat{X}_p$ is just the point $0 \in V$, because this is the only point in $\hat{X}_p$, that does not belong to a projective subvariety of X_J for some non-empty $J \subseteq I_p$. But as f_p is multihomogeneous and non-constant, its vanishing set $V(f_p) \subseteq \hat{X}$ is a multicone (i. e. stable under the $(\mathbb{K}^\times)^m$-action) and its irreducible components have codimension one in $\hat{X}$. Hence $V(f_p) \cap \hat{X}_p$ can only be zero, when $\dim X_p = 0$ and $|I_p| = 1$. It now follows from (a) and (b) that every maximal chain in $\mathcal{I}$ contains exactly m elements, so $\mathcal{I}$ is graded of length $m - 1$. □

For multiprojective stratifications we have the following new kind of covering relations in A which do not appear for $m = 1$.

Lemma 2.6. *Let $q < p$ be a covering relation in A with $I_p \setminus I_q = \{i\}$.*

(a) The algebra $\mathbb{K}[\hat{X}_q]$ can be seen as an $\mathbb{N}_0^{I_q}$-graded subalgebra of $\mathbb{K}[\hat{X}_p]$ and it holds

$$\mathbb{K}[\hat{X}_q] \cong \bigoplus_{\substack{\underline{d} \in \mathbb{N}_0^m \\ d_i = 0}} \mathbb{K}[\hat{X}_p]_{\underline{d}} \quad \textit{and} \quad I(\hat{X}_q) = \bigoplus_{\substack{\underline{d} \in \mathbb{N}_0^m \\ d_i > 0}} \mathbb{K}[\hat{X}_p]_{\underline{d}}.$$

(b) The vanishing multiplicity of a multihomogeneous function $g \in \mathbb{K}[\hat{X}_p] \setminus \{0\}$ along the prime divisor $\hat{X}_q \subseteq \hat{X}_p$ is equal to the i-th component of $\deg g \in \mathbb{N}_0^m$. In particular, the i-th component of $\deg f_p$ is non-zero.

Proof. (a) This first statement is immediate from the equality $X_q = \pi_{I_q}(X_p)$.

(b) The discrete valuation ring $\mathcal{O}_{\hat{X}_p, \hat{X}_q} \subseteq \mathbb{K}(\hat{X}_p)$ is isomorphic to the localization of $\mathbb{K}[\hat{X}_p]$ at the prime ideal $I(\hat{X}_q)$. By viewing g as an element of $\mathcal{O}_{\hat{X}_p, \hat{X}_q} \supseteq \mathbb{K}[\hat{X}_p]$, one can characterize the vanishing multiplicity of g along $\hat{X}_q$ as the unique integer

$n \in \mathbb{N}_0$ with $(g) = \mathfrak{m}^n$, where $\mathfrak{m}$ denotes the unique maximal ideal in $\mathcal{O}_{\hat{X}_p,\hat{X}_q}$. As the algebra $\mathbb{K}[\hat{X}_p]$ is generated in total degree one, it follows

$$\mathfrak{m}^n = \bigoplus_{\substack{\underline{d}\in\mathbb{N}_0^m \\ d_i \geq n}} \mathbb{K}[\hat{X}_p]_{\underline{d}} \subseteq \mathcal{O}_{\hat{X}_p,\hat{X}_q}.$$

Let $\deg g = (c_1, \dots, c_m)$. Clearly we have $(g) \subseteq \mathfrak{m}^{c_i}$ but $(g) \not\subseteq \mathfrak{m}^{c_i+1}$. Hence $(g) = \mathfrak{m}^{c_i}$, since every ideal of $\mathcal{O}_{\hat{X}_p,\hat{X}_q}$ is a power of $\mathfrak{m}$. □

2.2. The quasi-valuation and its associated graded algebra

We summarize some constructions and results from [CFL], since they are crucial for this thesis. Among these results are the quasi-valuation $\mathcal{V}$ and the properties of the associated graded algebra. It is strongly recommended to read the original papers, as we cannot do justice to their results on just a few pages and this section mainly serves as a reminder for all the notation introduced for Seshadri stratifications.

We fix the following notation: If K is any field of characteristic zero and S is a finite set, then we write K^S for the vector space over K with basis $\{e_s \mid s \in S\}$ indexed by S. Let $\mathbb{N}_0^S$ be the monoid generated by these basis elements and $\mathbb{Z}^S \subseteq K^S$ be the smallest group containing $\mathbb{N}_0^S$. For each element $x = \sum_{s\in S} x_s e_s \in K^S$ with coefficients $x_s \in K$ the set

$$\operatorname{supp} x = \{s \in S \mid x_s \neq 0\}$$

is called the *support* of x.

By definition, the multicone $\hat{X}_q$ is a prime divisor of $\hat{X}_p$ for every covering relation $q < p$ in A. If one extends the poset A by a unique minimal element p_{-1} with associated index set $I_{p_{-1}} = \varnothing$, then the multicone $\hat{X}_{p_{-1}} = \{0\}$ is a prime divisor of $\hat{X}_p \cong \mathbb{A}^1$ for each minimal element $p \in A$. To each covering relation $p > q$ in the extended poset $\hat{A} = A \cup \{p_{-1}\}$ we have an associated valuation, namely the discrete valuation

$$\nu_{p,q} : \mathbb{K}(\hat{X}_p) \setminus \{0\} \to \mathbb{Z},$$

sending a non-zero, rational function g to its vanishing multiplicity at the prime divisor $\hat{X}_q \subseteq \hat{X}_p$. Its value

$$b_{p,q} = \nu_{p,q}(f_p\big|_{\hat{X}_p}) \in \mathbb{N}$$

at the extremal function f_p is called the *bond* of the covering relation $q < p$. If p is minimal in A, then $b_{p,p_{-1}}$ coincides with the total degree $|\deg f_p|$, which is the sum of all entries in the degree $\deg f_p \in \mathbb{N}_0^m$.

Every Seshadri stratification gives rise to a collection of valuations on R, one for each maximal chain $\mathfrak{C}$ in A. Let $p_r > \cdots > p_0$ be the elements of $\mathfrak{C}$. To a regular function $g \in R \setminus \{0\}$ one associates a sequence $g_{\mathfrak{C}} = (g_r, \ldots, g_0)$ of rational functions inductively via $g_r := g$ and

$$g_{i-1} = \left. \frac{g_i^{b_{p_i,p_{i-1}}}}{f_{p_i}^{\nu_{p_i,p_{i-1}}(g_i)}} \right|_{\hat{X}_{p_{i-1}}} \in \mathbb{K}(\hat{X}_{p_{i-1}}).$$

for $i = r, \ldots, 1$. Further one defines the element

$$\mathcal{V}_{\mathfrak{C}}(g) = \sum_{j=0}^{r} \frac{\nu_{p_j,p_{j-1}}(g_j)}{\prod_{k=j}^{r} b_{p_k,p_{k-1}}} e_{p_j} \in \mathbb{Q}^{\mathfrak{C}}.$$

By this definition, each extremal function f_p for $p \in \mathfrak{C}$ is mapped to the vector $\mathcal{V}_{\mathfrak{C}}(f_p) = e_p$. We equip the abelian group $\mathbb{Q}^{\mathfrak{C}}$ with the lexicographic order induced by the total order on the maximal chain $\mathfrak{C}$, i. e. for all elements $\underline{a} = \sum_{i=0}^{r} a_i e_{p_i}, \underline{b} = \sum_{i=0}^{r} b_i e_{p_i}$ in $\mathbb{Q}^{\mathfrak{C}}$ it holds

$$\underline{a} \geq \underline{b} \quad \Longleftrightarrow \quad \underline{a} = \underline{b} \text{ or } a_i > b_i \text{ for the maximal index } i \in \{0, \ldots, r\} \text{ with } a_i \neq b_i.$$

Then the map $\mathcal{V}_{\mathfrak{C}} : R \setminus \{0\} \to \mathbb{Q}^{\mathfrak{C}}$ is a valuation. Chirivì, Fang and Littelmann also gave another, equivalent definition in [CFL], which we do not use here, as it is less suited for computations. Note that, by its definition, $\mathcal{V}_{\mathfrak{C}}$ takes values in the lattice

$$L^{\mathfrak{C}} = \{(a_r, \ldots, a_0) \in \mathbb{Q}^{\mathfrak{C}} \mid b_r \cdots b_{i+1} b_i a_i \in \mathbb{Z} \ \forall i = 0, \ldots, r\}. \tag{2.4}$$

In general, the lattice $L^{\mathfrak{C}}_{\mathcal{V}}$ generated by the valuation monoid $\mathbb{V}_{\mathfrak{C}}(X) = \{\mathcal{V}_{\mathfrak{C}}(g) \in \mathbb{Q}^{\mathfrak{C}} \mid g \in R \setminus \{0\}\}$ is not equal to the lattice $L^{\mathfrak{C}}$. With the following results from [CFL] one can determine the lattice $L^{\mathfrak{C}}_{\mathcal{V}}$.

Proposition 2.7 ([CFL, Lemma 6.12, Propositions 6.13 and 6.14])**.** *There exist rational functions $F_r, \ldots F_0 \in \mathbb{K}(\hat{X}) \setminus \{0\}$, such that their valuations are of the form*

$$\mathcal{V}_{\mathfrak{C}}(F_j) = \sum_{i=0}^{r} a_{i,j} e_{p_j}$$

with coefficients $a_{i,j} \in \mathbb{K}$, $a_{j,j} = b_{p_j,p_{j-1}}^{-1}$, $a_{i,j} = 0$ for all $i > j$. For each such choice of functions $F_r, \ldots, F_0$ the matrix $(a_{i,j})_{i,j=0,\ldots,r}$ is invertible and the entries of its inverse matrix $B_{\mathfrak{C}}$ are integers. Furthermore, an element $v = a_r e_{p_r} + \cdots + a_0 e_{p_0} \in \mathbb{Q}^{\mathfrak{C}}$ is contained in the lattice $L^{\mathfrak{C}}_{\mathcal{V}}$, if and only if

$$B_{\mathfrak{C}} \cdot \begin{pmatrix} a_r \\ \vdots \\ a_0 \end{pmatrix} \in \mathbb{Z}^{\mathfrak{C}}.$$

Remark 2.8. Every element $p \in A$ induces a Seshadri stratification on the multiprojective variety $X_p \subseteq \prod_{i \in I_p} \mathbb{P}(V_i)$ via the poset $A_p = \{q \in A \mid q \leq p\}$, where we take the same strata, extremal functions and index sets as in the stratification on X. By its definition, the valuation $\mathcal{V}_{\mathfrak{C}}$ is compatible in the following sense with the valuation $\mathcal{V}_{\mathfrak{C}_p}$ of the induced stratification along the maximal chain $\mathfrak{C}_p = \mathfrak{C} \cap A_p$: For every $g \in R \setminus \{0\}$, that does not vanish identically on $\hat{X}_p$, the valuation $\mathcal{V}_{\mathfrak{C}_p}(g|_{\hat{X}_p}) \in \mathbb{Q}^{\mathfrak{C}_p}$ coincides with $\mathcal{V}_{\mathfrak{C}}(g)$, when extended by zeros to an element of $\mathbb{Q}^{\mathfrak{C}}$.

The collection of all valuations $\mathcal{V}_{\mathfrak{C}}$ define a quasi-valuation $\mathcal{V}$, which respects the structure of the whole poset A, not just of one maximal chain. A **quasi-valuation** is defined similar to valuation, only the condition $\mathcal{V}(gh) = \mathcal{V}(g) + \mathcal{V}(h)$ for all $g, h \in R$ with $gh \neq 0$ is replaced by the inequality $\mathcal{V}(gh) \geq \mathcal{V}(g) + \mathcal{V}(h)$. To obtain this quasi-valuation one needs to extend $\mathcal{V}_{\mathfrak{C}}$ to a valuation $R \setminus \{0\} \to \mathbb{Q}^{\mathfrak{C}} \hookrightarrow \mathbb{Q}^A$, such that all valuations take values in the same abelian group. In order to make sense of this, we need a total order on $\mathbb{Q}^A$ such that each linear inclusion $\mathbb{Q}^{\mathfrak{C}} \hookrightarrow \mathbb{Q}^A$ is monotone. In general, there is no natural candidate for this total order. For this reason, one needs to choose and fix a total order $\geq^t$ on A linearizing the partial order, i. e. for each elements $p, q \in A$ the relation $p \geq q$ implies $p \geq^t q$. This total order induces the lexicographic order on $\mathbb{Q}^A$ and each map $\mathcal{V}_{\mathfrak{C}} : R \setminus \{0\} \to \mathbb{Q}^A$ is a valuation. One obtains the quasi-valuation $\mathcal{V}$ by taking their minimum with respect to this total order on $\mathbb{Q}^A$:

$$\mathcal{V} : R \setminus \{0\} \to \mathbb{Q}^A, \quad g \mapsto \min\{\mathcal{V}_{\mathfrak{C}}(g) \mid \mathfrak{C} \text{ maximal chain in } A\}.$$

Hence the quasi-valuation depends on the choice of this total order $\geq^t$ on A.

There is also the following inductive way of describing the quasi-valuation $\mathcal{V}$. Let p be any element in A, $g \in \mathbb{K}(\hat{X}_p)$ be a non-zero rational function. We write $\mathcal{V}_p$ for the quasi-valuation on the induced Seshadri stratification on X_p with underlying poset $A_p = \{q \in A \mid q \leq p\}$. Then it holds

$$\mathcal{V}_p(g) = \frac{\nu_{p,q}(g)}{b_{p,q}} e_p + \frac{1}{b_{p,q}} \mathcal{V}_q\Big(\frac{g^{b_{p,q}}}{f_p^{\nu_{p,q}(g)}}\big|_{\hat{X}_q}\Big), \tag{2.5}$$

where q is the unique minimal element covered by p with respect to the total order $\geq^t$, such that it holds

$$\frac{\nu_{p,q}(g)}{b_{p,q}} = \min\left\{ \frac{\nu_{p,q'}(g)}{b_{p,q'}} \;\middle|\; q' \in A \text{ covered by } p \right\}.$$

The quasi-valuation $\mathcal{V}$ has the following important properties, which we use many times throughout this thesis without mention (see [CFL, Section 8]).

- The values of $\mathcal{V}$ have non-negative entries, i. e. the quasi-valuation $\mathcal{V}(g)$ of every function $g \in R \setminus \{0\}$ is contained in the non-negative orthant $\mathbb{Q}^A_{\geq 0}$.

- One can characterize combinatorially for which maximal chains $\mathfrak{C}$ the quasi-valuation attains its minimum. For each $g \in R \setminus \{0\}$ it holds $\mathcal{V}_{\mathfrak{C}}(g) = \mathcal{V}(g)$, if and only if the support $\operatorname{supp} \mathcal{V}(g) \subseteq A$ lies in $\mathfrak{C}$. As a consequence: If $g, h \in R$ are non-zero and there exists a maximal chain $\mathfrak{C}$ containing both $\operatorname{supp} \mathcal{V}(g)$ and $\operatorname{supp} \mathcal{V}(h)$, then the quasi-valuation is additive, i. e. we have $\mathcal{V}(gh) = \mathcal{V}(g) + \mathcal{V}(h)$.

- Every extremal function f_p for $p \in A$ has the quasi-valuation $\mathcal{V}(f_p) = e_p$, so the support is given by $\operatorname{supp} \mathcal{V}(f_p) = \{p\}$. In particular: If $p_1, \dots, p_s \in A$ are contained in a chain in A and $n_1, \dots, n_s \in \mathbb{N}_0$, then it follows

$$\mathcal{V}(f_{p_1}^{n_1} \cdots f_{p_s}^{n_s}) = \sum_{i=1}^{s} n_i e_{p_i}.$$

Example 2.9. The regular function $g = x_0 y_1 \in \mathbb{K}[X]$ in the Seshadri stratification from Example 2.4 has the quasi-valuation $\mathcal{V}(g) = \frac{1}{2} e_X + \frac{1}{2} e_{01}$, independent of the choice of the total order $\geq^t$. To see this, we choose the following three parametrizations of open subsets of $\hat{X} = V(x_0 y_1 - x_1 y_0)$:

$$\begin{aligned}
\phi_{0\bar{0}} &: \mathbb{K}^\times \times \mathbb{K}^\times \times \mathbb{K} \to \hat{X}, \quad (x, y, t) \longmapsto ((x, tx), (y, ty)), \\
\phi_{01} &: \mathbb{K}^\times \times \mathbb{K}^\times \times \mathbb{K} \to \hat{X}, \quad (x, y, t) \longmapsto ((x, y), (tx, ty)), \\
\phi_{1\bar{1}} &: \mathbb{K}^\times \times \mathbb{K}^\times \times \mathbb{K} \to \hat{X}, \quad (x, y, t) \longmapsto ((tx, x), (ty, y)).
\end{aligned}$$

They are defined such that $\phi_q(\mathbb{K}^\times \times \mathbb{K}^\times \times \{0\})$ is equal to the intersection of the image of ϕ_q with the multicone $\hat{X}_q$ for each covering relation $q < X$ in A. The vanishing multiplicity of g at the divisor $\hat{X}_q$ then agrees with the exponent of t in the Laurent polynomial $g \circ \phi_q \in \mathbb{K}[x^{\pm 1}, y^{\pm 1}, t]$. We therefore have

$$\frac{\nu_{X,0\bar{0}}(g)}{b_{X,0\bar{0}}} = 1 > \frac{\nu_{X,01}(g)}{b_{X,01}} = \frac{1}{2} < \frac{\nu_{X,1\bar{1}}(g)}{b_{X,1\bar{1}}} = 1.$$

By the characterization of the quasi-valuation $\mathcal{V}$ from equation (2.5) it now follows $\mathcal{V}(g) = \frac{1}{2} e_X + \frac{1}{2} \mathcal{V}_{01}(g_1)$, where g_1 is the rational function

$$g_1 = \left(\frac{g^{b_{X,01}}}{f_X} \right) \bigg|_{\hat{X}_{01}} = \left(\frac{x_0^2 y_1^2}{y_0 y_1} \right) \bigg|_{\hat{X}_{01}} = x_0 x_1$$

on $\hat{X}_{01}$. As g_1 is the restriction of the extremal function f_{01} to $\hat{X}_{01}$, we have $\mathcal{V}_{01}(g_1) = e_{01}$.

The image of the quasi-valuation is denoted by $\Gamma = \{\mathcal{V}(g) \in \mathbb{Q}^A \mid g \in R \setminus \{0\}\}$. For each (not necessarily maximal) chain C in A the subset

$$\Gamma_C = \{\underline{a} \in \Gamma \mid \operatorname{supp} \underline{a} \subseteq C\}$$

is a finitely generated monoid. In [CFL] this was only shown if C is a maximal chain, but it implies that Γ_C is finitely generated as well, since its elements have non-negative entries. The set Γ is called the *fan of monoids* of the Seshadri stratification, since it is the union of all monoids Γ_C and the cones in $\mathbb{R}^A$ generated by these monoids form a fan.

The quasi-valuation $\mathcal{V} : R \setminus \{0\} \to \mathbb{Q}^m$ induces a filtration on R by the subrings

$$R_{\geq \underline{a}} = \{g \in R \setminus \{0\} \mid \mathcal{V}(g) \geq \underline{a}\} \cup \{0\}$$

for $\underline{a} \in \Gamma$. Since $\mathcal{V}(g)$ only has non-negative entries for all $g \in R \setminus \{0\}$, these subrings are ideals in R. The quotient of $R_{\geq \underline{a}}$ by the ideal $R_{>\underline{a}} = \{g \in R \setminus \{0\} \mid \mathcal{V}(g) > \underline{a}\} \cup \{0\}$ is one-dimensional for every $\underline{a} \in \Gamma$. They are called the *leaves* of the quasi-valuation $\mathcal{V}$. Let

$$\mathrm{gr}_{\mathcal{V}} R = \bigoplus_{\underline{a} \in \Gamma} R_{\geq \underline{a}} / R_{>\underline{a}}$$

be the associated graded algebra. For each chain C in A it contains the subalgebra

$$\mathrm{gr}_{\mathcal{V},C} R = \bigoplus_{\underline{a} \in \Gamma_C} R_{\geq \underline{a}} / R_{>\underline{a}} \subseteq \mathrm{gr}_{\mathcal{V}} R,$$

which is isomorphic to the semigroup algebra $\mathbb{K}[\Gamma_C]$ as a Γ_C-graded algebra. It is a finitely generated integral domain, so it gives rise to a toric variety $\operatorname{Spec} \mathrm{gr}_{\mathcal{V},C} R$. The fact that the associated graded algebra is the union of all these subalgebras $\mathrm{gr}_{\mathcal{V},C} R \cong \mathbb{K}[\Gamma_C]$ suggests that there is also a combinatorial way of describing the associated graded algebra by gluing the semigroup algebras $\mathbb{K}[\Gamma_C]$ into the *fan algebra* of Γ. It is defined as the algebra

$$\mathbb{K}[\Gamma] = \mathbb{K}[x_{\underline{a}} \mid \underline{a} \in \Gamma] / I(\Gamma),$$

where $I(\Gamma)$ is the ideal generated by all elements of the form

$$\begin{cases} x_{\underline{a}} x_{\underline{b}} - x_{\underline{a}+\underline{b}}, & \text{if there exists a chain } C \text{ in } A \text{ containing } \operatorname{supp} \underline{a} \text{ and } \operatorname{supp} \underline{b}, \\ x_{\underline{a}} x_{\underline{b}}, & \text{else} \end{cases}$$

with $\underline{a}, \underline{b} \in \Gamma$. For each chain C in A the fan algebra contains the subalgebra

$$\bigoplus_{\underline{a} \in \Gamma_C} \mathbb{K} x_{\underline{a}} \subseteq \mathbb{K}[\Gamma],$$

which is isomorphic to the semigroup algebra $\mathbb{K}[\Gamma_C]$.

Since the leaves of the quasi-valuation $\mathcal{V}$ are at most one-dimensional, choosing a regular function $g_{\underline{a}} \in R$ with $\mathcal{V}(g_{\underline{a}}) = \underline{a}$ for each $\underline{a} \in \Gamma$ yields a basis

$$\mathbb{B} = \{g_{\underline{a}} \mid \underline{a} \in \Gamma\}$$

of R as a vector space over $\mathbb{K}$ and the elements $\overline{g}_{\underline{a}}$ from a basis of the associated graded algebra $\mathrm{gr}_{\mathcal{V}} R$.

Theorem 2.10 ([CFL, Theorem 11.1]). *There exist scalars $c_{\underline{a}} \in \mathbb{K}^\times$ such that the map*

$$\mathbb{K}[\Gamma] \to \mathrm{gr}_{\mathcal{V}} R, \quad x_{\underline{a}} \mapsto c_{\underline{a}} \overline{g}_{\underline{a}}$$

is an isomorphism of algebras.

The concepts of normal and balanced Seshadri stratifications were introduced in [CFL, Sections 13, 15]. They can also be used in the multiprojective case.

Definition 2.11. A multiprojective Seshadri stratification is called

(a) **normal**, if $\Gamma_{\mathfrak{C}}$ is saturated for every maximal chain $\mathfrak{C}$, i. e. it is equal to the intersection of the lattice $\mathcal{L}^{\mathfrak{C}}$ generated by $\Gamma_{\mathfrak{C}}$ with the positive orthant $\mathbb{Q}^{\mathfrak{C}}_{\geq 0}$;

(b) **balanced**, when the fan of monoids Γ is independent of the choice of the total order $\geq^t$.

Every normal Seshadri stratification defines a standard monomial theory on R in the sense of the next proposition. When the stratification is balanced as well, then the normality and its associated standard monomial theory do not depend on the choice if the total order $\geq^t$.

An element $\underline{a} \in \Gamma$ is called *decomposable*, if it is 0 or it can be written in the form $\underline{a} = \underline{a}^1 + \underline{a}^2$ for two elements $\underline{a}^1, \underline{a}^2 \in \Gamma \setminus \{0\}$ with $\min \operatorname{supp} \underline{a}^1 \geq \max \operatorname{supp} \underline{a}^2$. Otherwise $\underline{a}$ is called *indecomposable*. Note that the minima and maxima exist, since the support of each element in Γ is totally ordered. Let $\mathbb{G}$ be the set of all indecomposable elements in Γ. For each $\underline{a} \in \mathbb{G}$ we fix a regular function $x_{\underline{a}} \in R \setminus \{0\}$ with $\mathcal{V}(x_{\underline{a}}) = \underline{a}$ and let $\mathbb{G}_R = \{x_{\underline{a}} \mid \underline{a} \in \mathbb{G}\}$ be the set of these functions.

We assume that the stratification is normal. In this case every element $\underline{a} \in \Gamma$ has a unique decomposition into a sum $\underline{a} = \underline{a}^1 + \cdots + \underline{a}^s$ of indecomposable elements $\underline{a}^k \in \Gamma$, such that $\min \operatorname{supp} \underline{a}^k \geq \max \operatorname{supp} \underline{a}^{k+1}$ holds for all $k = 1, \ldots, s-1$. With the choice of the set $\mathbb{G}_R$ one can therefore associate a regular function to every element $\underline{a} \in \Gamma$ via

$$x_{\underline{a}} := x_{\underline{a}^1} \cdots x_{\underline{a}^s} \in R.$$

A monomial in the functions in $\mathbb{G}_R$ is called *standard*, if it is of the form $x_{\underline{a}}$ for some element $\underline{a} \in \Gamma$.

Proposition 2.12 ([CFL, Proposition 15.6]). *If the stratification is normal and $\mathbb{G}_R$ and $x_{\underline{a}}$ are chosen as above, then the following statements hold:*

(a) The set $\mathbb{G}_R$ generates R as a $\mathbb{K}$-algebra.

(b) The set of all standard monomials in $\mathbb{G}_R$ is a basis of R as a vector space.

(c) If $\underline{a} = \underline{a}^1 + \cdots + \underline{a}^s$ is the unique decomposition of $\underline{a}$ into indecomposables, then $x_{\underline{a}} := x_{\underline{a}^1} \cdots x_{\underline{a}^s}$ is a standard monomial with $\mathcal{V}(x_{\underline{a}}) = \underline{a}$.

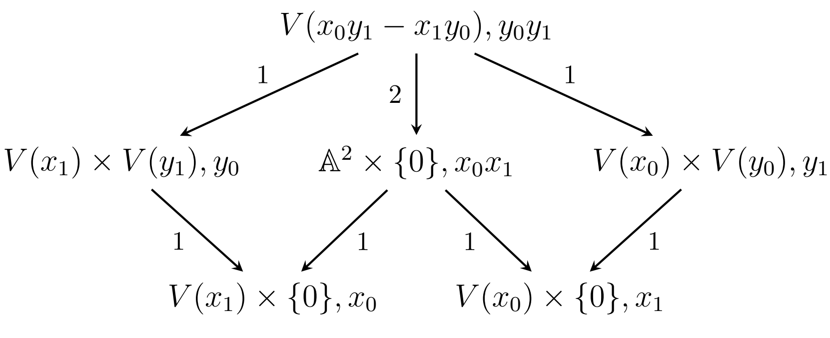
(d) For each non-standard monomial $x_{\underline{a}^1} \cdots x_{\underline{a}^s}$ in $\mathbb{G}_R$ there exists a straightening relation

$$x_{\underline{a}^1} \cdots x_{\underline{a}^s} = \sum_{\underline{b} \in \Gamma} u_{\underline{b}} x_{\underline{b}}$$

expressing it as a linear combination of standard monomials, where $u_{\underline{b}} \neq 0$ only if $\underline{b} \geq^t \underline{a}^1 + \cdots + \underline{a}^s$.

Example 2.13. The monoid $\Gamma_{\mathfrak{C}}$ to a maximal chain $\mathfrak{C}$ always contains the set $\mathbb{N}_0^{\mathfrak{C}}$, as every extremal function f_p for $p \in \mathfrak{C}$ has the quasi-valuation e_p. The stratification is called of **Hodge type**, if all its bonds $b_{p,q}$ are equal to 1. In this case every monoid $\Gamma_{\mathfrak{C}}$ coincides with $\mathbb{N}_0^m$, since $\Gamma_{\mathfrak{C}}$ is contained in the lattice $L^{\mathfrak{C}} = \mathbb{Z}^{\mathfrak{C}}$ from equation (2.4). For instance, the stratification we defined in Example 2.3 is of Hodge type. Seshadri stratifications of Hodge-type are always normal and balanced. More of their properties can be found in [CFL, Section 16].

Example 2.14. We return to the Seshadri stratification from Example 2.4. It has the following bonds:



There are four maximal chains in A, which we denote by $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_4$ from left to right. In two of these chains all bonds are equal to 1. By Example 2.13 we get the associated monoids $\Gamma_{\mathfrak{C}_1} = \mathbb{N}_0^{\mathfrak{C}_1}$ and $\Gamma_{\mathfrak{C}_4} = \mathbb{N}_0^{\mathfrak{C}_4}$. The monoid $\Gamma_{\mathfrak{C}_2}$ is contained in the intersection of the lattice

$$L^{\mathfrak{C}_2} = \{a_X e_X + a_{01} e_{01} + a_0 e_0 \mid a_X, a_X + a_{01}, a_X + a_{01} + a_0 \in \tfrac{1}{2}\mathbb{Z}\} = (\tfrac{1}{2}\mathbb{Z})^{\mathfrak{C}_2}$$

with the positive orthant $\mathbb{Q}_{\geq 0}^{\mathfrak{C}_2}$.

Using Proposition 2.7 one can even find a smaller lattice containing $\Gamma_{\mathfrak{C}_2}$. We choose the regular functions $F_2 = x_0y_1$, $F_1 = f_{01}$ and $F_0 = f_0$. Two of those are extremal functions and we already computed the quasi-valuation of F_2 in Example 2.9. Then the matrix $B_{\mathfrak{C}}$ in Proposition 2.7 is given by

$$B_{\mathfrak{C}} = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It thus follows that $\Gamma_{\mathfrak{C}_2}$ is contained in the lattice

$$L_{\mathcal{V}}^{\mathfrak{C}_2} = \{a_Xe_X + a_{01}e_{01} + a_0e_0 \mid 2a_X, a_{01} - a_X, a_0 \in \mathbb{Z}\}.$$

On the other hand, every element in $\mathcal{L}^{\mathfrak{C}_2} \cap \mathbb{Q}_{\geq 0}^{\mathfrak{C}_2}$ actually lies in $\Gamma_{\mathfrak{C}_2}$, since it can be written as a sum of the elements $e_X, e_{01}, e_0, \frac{1}{2}e_X + \frac{1}{2}e_{01} \in \Gamma_{\mathfrak{C}_2}$. Analogously, one can determine the monoid $\Gamma_{\mathfrak{C}_3}$. Summarizing our computations, we have:

$$\begin{aligned}
\Gamma_{\mathfrak{C}_1} &= \{ae_X + be_{0\bar{0}} + ce_0 \mid a, b, c \in \mathbb{N}_0\},\\
\Gamma_{\mathfrak{C}_2} &= \{ae_X + be_{01} + ce_0 \mid a, b \in \tfrac{1}{2}\mathbb{N}_0, c \in \mathbb{N}_0, a + b \in \mathbb{N}_0\},\\
\Gamma_{\mathfrak{C}_3} &= \{ae_X + be_{01} + ce_1 \mid a, b \in \tfrac{1}{2}\mathbb{N}_0, c \in \mathbb{N}_0, a + b \in \mathbb{N}_0\},\\
\Gamma_{\mathfrak{C}_4} &= \{ae_X + be_{1\bar{1}} + ce_1 \mid a, b, c \in \mathbb{N}_0\}.
\end{aligned}$$

As all these monoids are saturated, the stratification is normal. It also is balanced: Every element in Γ is a sum of the elements e_p for $p \in A$ and $\mathcal{V}(F_2) = \frac{1}{2}e_X + \frac{1}{2}e_{01}$ and the quasi-valuation of F_2 is independent of the choice of the total order $\geq^t$ (see Example 2.9).

2.3. Multidegrees and multigradings

The $\mathbb{N}_0^m$-grading on the multihomogeneous coordinate ring $R = \mathbb{K}[X]$ corresponds to an action of the torus $T = (\mathbb{K}^\times)^m$ on the multicone $\hat{X} \subseteq V$ by scaling in each factor V_i. This also induces an T-action on the multihomogeneous coordinate ring itself: If g is a function in R and $\underline{t} \in T$, then $g^{\underline{t}} := \underline{t} \cdot g$ is defined by $(g^{\underline{t}})(x) = g(\underline{t}^{-1} \cdot x)$ for all $x \in \hat{X}$.

Lemma 2.15.

(a) For all $g \in R \setminus \{0\}$ and $\underline{t} \in (\mathbb{K}^\times)^m$ it holds $\mathcal{V}_{\mathfrak{C}}(\underline{t} \cdot g) = \mathcal{V}_{\mathfrak{C}}(g)$.

(b) If $h = \sum_{\underline{d} \in \mathbb{N}_0^m} h_{\underline{d}} \in R$ is the decomposition of $h \neq 0$ into its multihomogeneous components $h_{\underline{d}} \in R_{\underline{d}}$, then

$$\mathcal{V}_{\mathfrak{C}}(h) = \min\{\mathcal{V}_{\mathfrak{C}}(h_{\underline{d}}) \mid \underline{d} \in \mathbb{N}_0^m \text{ such that } h_{\underline{d}} \neq 0\}.$$

Proof. The statements can be proved analogously to [CFL, Lemma 6.15]. One has to replace the $\mathbb{K}^\times$-action by the $(\mathbb{K}^\times)^m$-action and use Lemma A.12. □

Definition 2.16. We define the **degree map** to be the $\mathbb{Q}$-linear map

$$\deg : \mathbb{Q}^A \to \mathbb{Q}^m, \quad e_p \mapsto \deg f_p$$

and call $\deg \underline{a}$ the **degree** of an element $\underline{a} \in \mathbb{Q}^A$.

Lemma 2.17. *If $g \in R \setminus \{0\}$ is multihomogeneous, then $\deg g = \deg \mathcal{V}(g)$.*

Proof. Let $\mathfrak{C} : p_r > \cdots > p_0$ be a maximal chain in A with $\mathcal{V}_{\mathfrak{C}}(g) = \mathcal{V}(g)$. We write $\mathcal{V}(g)$ in the form $a_r e_{p_r} + \cdots + a_0 e_{p_0}$ with coefficients $a_i \in \mathbb{Q}$ and fix a positive integer N, such that $N\mathcal{V}(g) \in \mathbb{Z}^{\mathfrak{C}}$. Then we have

$$\mathcal{V}(g^N) = N\mathcal{V}(g) = \mathcal{V}(\prod_{i=0}^{r} f_p^{Na_i}).$$

Suppose that g^N and $f \coloneqq \prod_{i=0}^{r} f_{p_i}^{Na_i}$ have different multidegrees. Since the leaves of the quasi-valuation are one-dimensional, the quasi-valuation of the non-zero function $h = g^N - f$ is strictly larger than $N\mathcal{V}(g)$. On the other hand, h consists of the two multihomogeneous components g^N and f, so Lemma 2.15 implies $\mathcal{V}(h) = N\mathcal{V}(g)$, which contradicts our assumption. Thus g^N and f have the same multidegree and it therefore follows:

$$\deg g = \frac{1}{N} \deg \prod_{i=0}^{r} f_{p_i}^{Na_i} = \sum_{i=1}^{r} a_i \deg f_{p_i} = \deg \mathcal{V}(g). \qquad \square$$

For every chain C in A the degree map $\deg : \Gamma \to \mathbb{N}_0^m$ defines an $\mathbb{N}_0^m$-grading on the monoid Γ_C via the subsets $\Gamma_{C,\underline{d}}$ of all elements of degree $\underline{d}$. We write $\Gamma_{\underline{d}}$ for the elements in Γ of degree $\underline{d}$. This induces an $\mathbb{N}_0^m$-grading on $\mathrm{gr}_{\mathcal{V}} R$ by the subgroups

$$(\mathrm{gr}_{\mathcal{V}} R)_{\underline{d}} = \bigoplus_{\underline{a} \in \Gamma_{\underline{d}}} R_{\geq \underline{a}} / R_{> \underline{a}}.$$

For each chain C in A, $\mathrm{gr}_{\mathcal{V},C} R$ is a graded subalgebra of $\mathrm{gr}_{\mathcal{V}} R$. The fan algebra $\mathbb{K}[\Gamma]$ also carries a grading by $\mathbb{N}_0^m$ induced by the degree map and there exists an isomorphism $\mathrm{gr}_{\mathcal{V}} R \cong \mathbb{K}[\Gamma]$ of $\mathbb{N}_0^m$-graded algebras. This follows from the construction of the isomorphism on basis elements (see Theorem 2.10).

Let $x_{\mathfrak{C}} = \prod_{p \in \mathfrak{C}} f_p \in R$ be the product of all extremal functions along a maximal chain $\mathfrak{C}$ in A and $I_{\mathfrak{C}} \subseteq \mathrm{gr}_{\mathcal{V}} R$ be the annihilator of the element $\overline{x_{\mathfrak{C}}} \in \mathrm{gr}_{\mathcal{V}} R$. It was shown in [CFL, Corollary 10.8] that there exists an isomorphism of algebras

$$\mathrm{gr}_{\mathcal{V}} R / I_{\mathfrak{C}} \cong \mathrm{gr}_{\mathcal{V},\mathfrak{C}} R \tag{2.6}$$

and the intersection of all ideals $I_{\mathfrak{C}}$ is the minimal prime decomposition of the zero ideal in $\mathrm{gr}_{\mathcal{V}} R$. As the associated graded algebra $\mathrm{gr}_{\mathcal{V}} R$ is finitely generated and reduced, its corresponding variety $\operatorname{Spec}\operatorname{gr}_{\mathcal{V}}R$ therefore is the scheme-theoretical union of the toric varieties $\operatorname{Spec}\operatorname{gr}_{\mathcal{V},\mathfrak{C}}R$, each of which is irreducible and of dimension $\dim\hat{X}$. Using the language of Multiproj schemes, which can be found in Appendix A, we can conclude an analogous statement about the scheme $\operatorname{Multiproj}(\operatorname{gr}_{\mathcal{V}}R)$. Its irreducible components, however, are only schemes, not necessarily projective varieties. We have already looked at an example where this happens: The stratification from Example 2.3 is of Hodge type and the algebra $\operatorname{gr}_{\mathcal{V},\mathfrak{C}}R \cong \mathbb{K}[\Gamma_{\mathfrak{C}}]$ associated to the maximal chain $\mathfrak{C} : X > 0\overline{0} > 0$ is isomorphic to the algebra from Example A.3 as an $\mathbb{N}_0^2$-graded algebra. Hence it induces a non-separated scheme.

Corollary 2.18. *The scheme* $\operatorname{Multiproj}(\operatorname{gr}_{\mathcal{V}}R)$ *is the scheme-theoretical union of the closed, integral subschemes* $\operatorname{Multiproj}(\operatorname{gr}_{\mathcal{V},\mathfrak{C}}R)$*, where* $\mathfrak{C}$ *runs over all maximal chains in* A*. Each of these subschemes is integral and of dimension* $\dim X$*.*

Proof. All the statements follow directly from Corollary 10.8 in [CFL] in combination with the Lemmas A.4 and A.5. To use these lemmas we require the ideal $I_{\mathfrak{C}} \subseteq \operatorname{gr}_{\mathcal{V},\mathfrak{C}}R$ to be homogeneous and prime, which holds by the isomorphism (2.6). We also need to show that the degrees of the homogeneous elements in $\operatorname{gr}_{\mathcal{V},\mathfrak{C}}R$ generate a sublattice of $\mathbb{Z}^m$ of full rank, i. e. the image of the degree map $\Gamma_{\mathfrak{C}} \to \mathbb{Z}^m$ generates a group of rank m. It contains the degrees of all extremal functions f_p for $p \in \mathfrak{C}$. A suitable subset of size m of these degrees is linearly independent, as they can be arranged in an upper triangular matrix with non-zero diagonal (up to permutation of the rows). This can be seen via Lemma 2.6 (b). □

2.4. Semi-toric degeneration

Every Seshadri stratification on an embedded projective variety $Y \subseteq \mathbb{P}(V)$ induces a degeneration of Y into an union of projective toric varieties. We generalize this result using an analogous approach to the construction in [CFL, Chapter 12] via Rees algebras.

Let $\mathcal{J}$ be the image of the map $\Gamma \to \mathbb{N}_0^m \times \Gamma$, $\underline{a} \mapsto (\deg\underline{a}, \underline{a})$ and let $\succeq$ be the lexicographic order on $\mathbb{N}_0^m \times \Gamma$. For each $(\underline{d}, \underline{a}) \in \mathbb{N}_0^m \times \Gamma$ we define the following multihomogeneous ideals in R:

$$\mathcal{I}_{\succeq(\underline{d},\underline{a})} = \langle\, g \in R \mid g \text{ multihomogeneous and } (\deg g, \mathcal{V}(g)) \succeq (\underline{d}, \underline{a})\,\rangle\,,$$
$$\mathcal{I}_{\succ(\underline{d},\underline{a})} = \langle\, g \in R \mid g \text{ multihomogeneous and } (\deg g, \mathcal{V}(g)) \succ (\underline{d}, \underline{a})\,\rangle\,.$$

Their quotient is given by

$$\mathcal{I}_{\succeq(\underline{d},\underline{a})}/\mathcal{I}_{\succ(\underline{d},\underline{a})} = \begin{cases} \{0\}, & \text{if } (\underline{d},\underline{a}) \notin \mathcal{J}, \\ R_{\geq\underline{a}}/R_{>\underline{a}}, & \text{if } (\underline{d},\underline{a}) \in \mathcal{J}. \end{cases}$$

By fixing an isomorphism of posets $\pi : (\mathbb{N}_0, \geq) \to (\mathcal{J}, \succeq)$ we get a descending filtration

$$R = \mathcal{I}_0 \supseteq \mathcal{I}_1 \supseteq \mathcal{I}_2 \supseteq \dots$$

where we write $\mathcal{I}_j = \mathcal{I}_{\pi(j)}$ for $j \in \mathbb{N}_0$. Since each ideal $\mathcal{I}_j$ is multihomogeneous, the Rees algebra

$$\mathcal{A} = \dots \oplus Rt^2 \oplus Rt \oplus R \oplus \mathcal{I}_1 t^{-1} \oplus \mathcal{I}_2 t^{-2} \oplus \dots$$

to this filtration is an $\mathbb{N}_0^m$-graded subalgebra of $R[t, t^{-1}] = \bigoplus_{\underline{d} \in \mathbb{N}_0^m} R_{\underline{d}}[t, t^{-1}]$.

As a submodule of $R[t, t^{-1}]$, the algebra $\mathcal{A}$ is a torsion free module over a Dedekind domain, hence flat over $\mathbb{K}[t]$. Additionally the inclusion $\mathbb{K}[t] \hookrightarrow \mathcal{A}$ maps to degree 0. These two properties imply that the induced morphism $\phi : \operatorname{Spec} \mathcal{A} \to \mathbb{A}^1$ is flat and $(\mathbb{K}^\times)^m$-equivariant (with the trivial action on $\mathbb{A}^1$). In particular, it induces a morphism $\psi : \operatorname{Multiproj} \mathcal{A} \to \mathbb{A}^1$. By Corollary 2.2.11 (iv) in [Gro], ψ inherits the flatness of ϕ, because the morphism $\operatorname{Spec} \mathcal{A} \setminus V(\mathcal{A}_+) \to \operatorname{Multiproj} \mathcal{A}$ is surjective (set-theoretically) as a geometric quotient.

The general fiber of ϕ at $t \neq 0$ is isomorphic to the multicone $\hat{X}$, because $\mathcal{A}/(t-b) \cong R$ for all $b \in \mathbb{A}^1 \setminus \{0\}$. On the other hand we have

$$\mathcal{A}/(t) \cong \bigoplus_{j \in \mathbb{N}_0} \mathcal{I}_j / \mathcal{I}_{j+1} \cong \operatorname{gr}_{\mathcal{V}} R,$$

so the special fiber is isomorphic to $\operatorname{Spec}(\operatorname{gr}_{\mathcal{V}} R)$.

Corollary 2.19. *The general fiber of the flat morphism* $\psi : \operatorname{Multiproj} \mathcal{A} \to \mathbb{A}^1$ *is isomorphic to* $\operatorname{Multiproj} R \cong X$ *and its special fiber at* $t = 0$ *is isomorphic to* $\operatorname{Multiproj}(\operatorname{gr}_{\mathcal{V}} R)$.

2.5. The Newton-Okounkov polytopal complexes

In [CFL] a Newton-Okounkov theoretical object was associated to a given Seshadri stratification. For each maximal chain $\mathfrak{C}$ one obtains a simplex, such that its lattice points describe the rate of growth for the dimensions of the graded components of $\operatorname{gr}_{\mathcal{V},\mathfrak{C}} R$. These simplices fit together to form a simplicial complex $\Delta_{\mathcal{V}}$. The dimension of X is equal to the dimension of the simplicial complex and the degree of $X \subseteq \mathbb{P}(V)$ can be extracted via the volume of $\Delta_{\mathcal{V}}$ with respect to certain lattices.

In the multiprojective setting the simplices generalize to polytopes. However, we obtain not just one polytopal complex, but a polytopal complex $\Delta_{\mathcal{V}}^{(\underline{d})}$ for each multidegree $\underline{d} \in \mathbb{N}_0^m$. This structure is not visible for $m = 1$ since the polytopal complexes are scaled versions of each other in this case. For most values of $\underline{d}$ the complex $\Delta_{\mathcal{V}}^{(\underline{d})}$ has the same dimension as the variety X, but in some edge cases it can collapse to a smaller dimension.

Throughout this section let $r \coloneqq \dim X$. For multiprojective varieties there also exists a Hilbert polynomial $H_R \in \mathbb{Q}[x_1, \dots, x_m]$. We refer to the Appendix A.2 for its properties and the definition of the multidegrees of X. Let

$$G_R = \sum_{\underline{k}} \frac{\deg_{\underline{k}}(X)}{k_1! \cdots k_m!} x_1^{k_1} \cdots x_m^{k_m}$$

be the homogeneous component of highest total degree in H_R. Since the leaves of the quasi-valuation $\mathcal{V}$ are at most one-dimensional, there exists a basis $\mathbb{B}$ of R as a vector space over $\mathbb{K}$, such that $\mathbb{B} \to \Gamma$, $g \mapsto \mathcal{V}(g)$ is a bijection. In particular:

$$\dim R_{\underline{d}} = \dim (\mathrm{gr}_{\mathcal{V}} R)_{\underline{d}} = |\Gamma_{\underline{d}}|. \tag{2.7}$$

The equality

$$G_R(\underline{d}) = \lim_{n\to\infty} \frac{\dim R_{n\underline{d}}}{n^r}$$

for all $\underline{d} \in \mathbb{N}_0^m$ suggests that we should examine the Veronese subalgebras

$$\mathrm{gr}_{\mathcal{V}}^{(\underline{d})} R = \bigoplus_{n\in\mathbb{N}_0} (\mathrm{gr}_{\mathcal{V}} R)_{n\underline{d}}$$

of the associated graded algebra $\mathrm{gr}_{\mathcal{V}} R$. First we need to fix some notation. For each chain C in A, the algebra $\mathrm{gr}_{\mathcal{V}}^{(\underline{d})} R$ contains the Veronese subalgebra

$$\mathrm{gr}_{\mathcal{V},C}^{(\underline{d})} R = \mathrm{gr}_{\mathcal{V}}^{(\underline{d})} R \cap \mathrm{gr}_{\mathcal{V},C} R$$

of $\mathrm{gr}_{\mathcal{V},C} R$. The fan of monoids contains the Veronese subfan $\Gamma^{(\underline{d})} = \bigcup_{n\in\mathbb{N}_0} \Gamma_{n\underline{d}}$ and for each monoid Γ_C we have the Veronese submonoid $\Gamma_C^{(\underline{d})} = \Gamma_C \cap \Gamma^{(\underline{d})}$. By Theorem 2.10 there are again isomorphisms of $\mathbb{N}_0$-graded algebras:

$$\mathrm{gr}_{\mathcal{V}}^{(\underline{d})} R \cong \mathbb{K}[\Gamma^{(\underline{d})}] \quad \text{and} \quad \mathrm{gr}_{\mathcal{V},C}^{(\underline{d})} R \cong \mathbb{K}[\Gamma_C^{(\underline{d})}].$$

In general, the fan of monoids Γ and the Veronese-fan of monoids $\Gamma^{(\underline{d})}$ have different combinatorial structures. Whereas the poset of all monoids Γ_C, ordered by inclusion, is isomorphic to the poset $\Delta(A)$ of all chains in A, different chains can have the same Veronese monoid $\Gamma_C^{(\underline{d})}$. For this reason we now define a map

$$\Delta(A) \to \Delta(A), \quad C \mapsto C_{\underline{d}},$$

so that $\Gamma_C^{(\underline{d})} = \Gamma_D^{(\underline{d})}$ is equivalent to $C_{\underline{d}} = D_{\underline{d}}$ for all chains $C, D \subseteq A$. The chain $C_{\underline{d}}$ depends on the cone

$$\sigma_C = \mathrm{Cone}\{\deg \underline{a} \mid \underline{a} \in \Gamma_C\} \subseteq \mathbb{R}^m. \tag{2.8}$$

Regarding (rational) polyhedral cones and (lattice) polytopes, we use the language of Cox, Little and Schenck from [CLS]. On polyhedral cones and polytopes, we always use the standard euclidean topology. The cone in $\mathbb{R}^A$ spanned by Γ_C is generated by the vectors $e_p \in \mathbb{R}^C$ for $p \in C$, as $\Gamma_C \subseteq \mathbb{Q}^C_{\geq 0}$. Therefore σ_C is a rational polyhedral cone with respect to the lattice $\mathbb{Z}^m \subseteq \mathbb{R}^m$.

If $\underline{d} \notin \sigma_C$ then we set $C_{\underline{d}} = \varnothing$. Now assume $\underline{d} \in \sigma_C$. Since every polyhedral cone is the disjoint union of the relative interiors (i. e. the interior in its closure) of its faces, there exists a unique face τ of σ_C with $\underline{d} \in \operatorname{relint} \tau$. This is also the unique minimal face containing $\underline{d}$. Now each convex cone, which is generated by a finite set S is generated by its edges, i. e. its one dimensional faces, and every generating set of the cone contains at least one non-zero element from each edge. As σ_C is generated by the set of all $\deg f_p$ with $p \in C$, every edge of σ_C is of the form $\mathbb{R}_{\geq 0}\, e_p$ for $p \in C$. We then define

$$C_{\underline{d}} = \{p \in C \mid \mathbb{R}_{\geq 0}\, e_p \text{ is an edge of } \tau\}.$$

Since $\tau = \sigma_{C_{\underline{d}}}$, it is immediate that the image $\Delta^{(\underline{d})}(A)$ of the map $\Delta(A) \to \Delta(A)$, $C \mapsto C_{\underline{d}}$ is equal to

$$\Delta^{(\underline{d})}(A) = \{C \in \Delta(A) \mid \underline{d} \in \operatorname{relint} \sigma_C\} \cup \{\varnothing\}.$$

Lemma 2.20.

(a) For any two chains $C, D \in \Delta(A)$ it holds $\Gamma_C^{(\underline{d})} \subseteq \Gamma_D^{(\underline{d})}$, if and only if $C_{\underline{d}} \subseteq D_{\underline{d}}$.

(b) The map $\Delta(A) \to \Delta^{(\underline{d})}(A)$, $C \mapsto C_{\underline{d}}$ is monotone.

(c) The following map is an isomorphism of posets:

$$\Delta^{(\underline{d})}(A) \to \{\Gamma_C^{(\underline{d})} \mid C \in \Delta(A)\}, \quad C \longmapsto \Gamma_C^{(\underline{d})}.$$

Proof. For each $C \in \Delta(A)$ the monoids $\Gamma_C^{(\underline{d})}$ and $\Gamma_{C_{\underline{d}}}^{(\underline{d})}$ coincide. Indeed, if $\underline{a}$ is an element of $\Gamma_C^{(\underline{d})}$, then $\deg \underline{a} \in \mathbb{N}\underline{d}$ lies in the face $\tau \subseteq \sigma_C$ defined by $C_{\underline{d}}$. For every $p \in C \setminus C_{\underline{d}}$ with $p \in \operatorname{supp} \underline{a}$ we can write the degree of $\underline{a}$ in the form

$$\deg \underline{a} = c \deg f_p + \sum_{q \in C} c_q \deg f_q$$

with real numbers $c_q \geq 0$ and $c > 0$. All elements $\deg f_q$ for $q \in C$ lie in σ_C but $\deg f_p$ is not contained in the face τ. This is impossible, as $\deg \underline{a} \in \tau$.

Let C, D be two chains in A. If $C_{\underline{d}} \subseteq D_{\underline{d}}$, then we clearly have $\Gamma_C^{(\underline{d})} \subseteq \Gamma_D^{(\underline{d})}$. Now suppose that $\Gamma_C^{(\underline{d})} \subseteq \Gamma_D^{(\underline{d})}$ and fix an element $p \in C_{\underline{d}}$. By the definition of $C_{\underline{d}}$, the multidegree $\deg f_p$ lies in the relative interior of the face τ corresponding to $C_{\underline{d}}$. This allows us to use the following argument, which appears multiple times throughout this section: By the properties of the relative interior, there exists an element in the intersection of the translated cone $\deg f_p + \tau \subseteq \mathbb{R}^m$ with the set $\mathbb{N}\underline{d}$. This holds for every convex polyhedral cone. As the cone σ_C is generated by lattice points in $\mathbb{Z}^m \subseteq \mathbb{R}^m$, we can therefore find non-negative, rational numbers $a_q \in \mathbb{Q}$ and $N \in \mathbb{N}$, such that

$$N\underline{d} = \deg f_p + \sum_{q \in C_{\underline{d}}} a_q \deg f_q.$$

By multiplying with a common denominator of all a_q, we can assume that these rational numbers are non-negative integers. Hence we see that

$$\mathcal{V}(f_p \cdot \prod_{q \in C_{\underline{d}}} f_q^{a_p}) = e_p + \sum_{q \in C_{\underline{d}}} a_q e_q \in \Gamma_C^{(\underline{d})} = \Gamma_{D_{\underline{d}}}^{(\underline{d})},$$

which implies $C_{\underline{d}} \subseteq D_{\underline{d}}$. Finally, the parts (b) and (c) follow from the first statement. □

Lemma 2.21. *The monoid $\Gamma_C^{(\underline{d})}$ is finitely generated for each chain $C \subseteq A$ and $\underline{d} \in \mathbb{N}_0^m$.*

Proof. Choose finitely many generators $\underline{a}^{(1)}, \ldots, \underline{a}^{(s)} \in \Gamma_C$ and consider the map

$$\phi : \mathbb{Z}^s \to \mathbb{Z}^m / \mathbb{Z}\underline{d}, \quad (n_1, \ldots, n_s) \longmapsto \sum_{i=1}^{s} n_i \deg \underline{a}^{(i)}.$$

It is sufficient to show, that the monoid $M = \mathbb{N}_0^s \cap \ker \phi \subseteq \mathbb{Z}^s$ is finitely generated, since its image under $\mathbb{N}_0^s \to \mathcal{L}^C$ coincides with $\Gamma_C^{(\underline{d})}$, where $\mathcal{L}^C \subseteq \mathbb{Q}^A$ denotes the lattice generated by Γ_C. The set $\mathbb{R}_{\geq 0}^s \cap \operatorname{span}_{\mathbb{R}}(\ker \phi)$ is a rational polyhedral cone w. r. t. the lattice $\ker \phi$. The intersection of this cone with the kernel of ϕ is exactly M. By Gordan's Lemma, M is finitely generated. □

Analogous to Corollary 2.18, the irreducible components of the projective variety $\operatorname{Proj}(\operatorname{gr}_{\mathcal{V}}^{(\underline{d})} R)$ are determined by the maximal elements in the poset $\Delta^{(\underline{d})}(A)$. Clearly, every maximal element in $\Delta^{(\underline{d})}(A)$ is of the form $\mathfrak{C}_{\underline{d}}$ for a maximal chain $\mathfrak{C} \in \Delta(A)$, but the converse is false. There can also exist two different maximal chains $\mathfrak{C}$ and $\mathfrak{D}$ in A with $\mathfrak{C}_{\underline{d}} = \mathfrak{D}_{\underline{d}}$. Fortunately, in most cases, the maximal elements in $\Delta^{(\underline{d})}(A)$ are easy to describe: When $\underline{d}$ does not lie on the boundary of the cone $\sigma_{\mathfrak{C}}$, then $\mathfrak{C}_{\underline{d}}$ is maximal in $\Delta^{(\underline{d})}(A)$, if and only if $\underline{d} \in \sigma_{\mathfrak{C}}$.

Lemma 2.22. *The projective variety $\operatorname{Proj}(\operatorname{gr}_{\mathcal{V}}^{(\underline{d})} R)$ is scheme-theoretically the irredundant union of the toric subvarieties $\operatorname{Proj}(\operatorname{gr}_{\mathcal{V},C}^{(\underline{d})} R)$, where C runs over all maximal elements in the poset $\Delta^{(\underline{d})}(A)$.*

Proof. The proof of this statement is mostly analogous to the proof of Proposition 10.7 in [CFL]. Recall that for each maximal chain $\mathfrak{C}$ in A we defined the product

$x_{\mathfrak{C}} = \prod_{p\in\mathfrak{C}} f_p \in R$ of all extremal functions along $\mathfrak{C}$ and the annihilator $I_{\mathfrak{C}} \subseteq \mathrm{gr}_{\mathcal{V}} R$ of the element $\overline{x_{\mathfrak{C}}} \in \mathrm{gr}_{\mathcal{V}} R$. In *loc. cit.* it was shown that the ideal $I_{\mathfrak{C}}$ is given by

$$I_{\mathfrak{C}} = \bigoplus_{\underline{a}\in\Gamma\setminus\Gamma_{\mathfrak{C}}} R_{\geq \underline{a}}/R_{>\underline{a}}.$$

The intersection $I_{\mathfrak{C}}^{(\underline{d})} = I_{\mathfrak{C}} \cap \mathrm{gr}_{\mathcal{V}}^{(\underline{d})} R$ is a prime ideal in $\mathrm{gr}_{\mathcal{V}}^{(\underline{d})} R$ and it can be written as

$$I_{\mathfrak{C}}^{(\underline{d})} = \bigoplus_{\underline{a}\in\Gamma^{(\underline{d})}\setminus\Gamma_{\mathfrak{C}}^{(\underline{d})}} R_{\geq \underline{a}}/R_{>\underline{a}}. \tag{2.9}$$

It follows that the intersection of the ideals $I_{\mathfrak{C}}^{(\underline{d})}$ over all maximal chains is equal to the zero ideal. On the other hand, $I_{\mathfrak{C}}^{(\underline{d})}$ does not depend on $\mathfrak{C}$ but only on the monoid $\Gamma_{\mathfrak{C}}^{(\underline{d})}$. Hence we can choose a subset $\mathcal{C}$ of all maximal chains in A, which maps bijectively to the maximal elements in $\Delta^{(\underline{d})}(A)$, such that

$$\bigcap_{\mathfrak{C}\in\mathcal{C}} I_{\mathfrak{C}}^{(\underline{d})} = (0).$$

This intersection is irredundant, since the Veronese fan of monoids $\Gamma^{(\underline{d})}$ is the irredundant union of the monoids $\Gamma_C^{(\underline{d})}$ over all maximal elements $C \in \Delta^{(\underline{d})}(A)$. By (2.9), we have $(\mathrm{gr}_{\mathcal{V}}^{(\underline{d})} R)/I_{\mathfrak{C}}^{(\underline{d})} \cong \mathrm{gr}_{\mathcal{V},\mathfrak{C}}^{(\underline{d})} R$ for every $\mathfrak{C} \in \mathcal{C}$.

Finally, we need to show that $I_{\mathfrak{C}}^{(\underline{d})}$ is a minimal prime ideal in $\mathrm{gr}_{\mathcal{V}}^{(\underline{d})} R$ for all $\mathfrak{C} \in \mathcal{C}$. If I was an ideal properly contained in $I_{\mathfrak{C}}^{(\underline{d})}$, then there exists a non-zero function $g \in R$ with $\mathcal{V}(g) \notin \Gamma_{\mathfrak{C}}^{(\underline{d})}$ and $\overline{g} \in I_{\mathfrak{C}}^{(\underline{d})} \setminus I$. Then we have $\overline{g} \cdot \overline{x}_{\mathfrak{C}} = 0$ in $\mathrm{gr}_{\mathcal{V}} R$. We now wish to multiply $\overline{x}_{\mathfrak{C}}$ with a suitable element $\overline{h} \in \mathrm{gr}_{\mathcal{V},\mathfrak{C}} R$ such that their product lies in $\mathrm{gr}_{\mathcal{V}}^{(\underline{d})} R$. Then I cannot be prime since both $\overline{g}$ and $\overline{x}_{\mathfrak{C}}\overline{h}$ are non-zero in $\mathrm{gr}_{\mathcal{V}}^{(\underline{d})} R/I$, but their product is zero. The multidegree of $x_{\mathfrak{C}}$ lies in the cone $\sigma_{\mathfrak{C}} = \sigma_{\mathfrak{C}_{\underline{d}}}$ and the tuple $\underline{d}$ is contained in its relative interior. Therefore we can find natural numbers n_p, $p \in \mathfrak{C}$, such that

$$\deg(x_{\mathfrak{C}} \cdot \prod_{p\in\mathfrak{C}} f_p^{n_p}) = \deg x_{\mathfrak{C}} + \sum_{p\in\mathfrak{C}} n_i \deg f_p \in \mathbb{N}\underline{d},$$

which gives us the desired function $h = \sum_{p\in\mathfrak{C}} f_p^{n_p}$. □

Example 2.23. Let $\underline{d} = (0,1)$ in the Seshadri stratification from Example 2.14. The Veronese submonoids are given by

$$\Gamma_{\mathfrak{C}_1}^{(\underline{d})} = \mathbb{N}_0 e_X + \mathbb{N}_0 e_{0\overline{0}}, \;\; \Gamma_{\mathfrak{C}_2}^{(\underline{d})} = \Gamma_{\mathfrak{C}_3}^{(\underline{d})} = \mathbb{N}_0 e_X, \;\; \Gamma_{\mathfrak{C}_4}^{(\underline{d})} = \mathbb{N}_0 e_X + \mathbb{N}_0 e_{1\overline{1}}.$$

Hence the set $\mathcal{C} = \{\mathfrak{C}_1, \mathfrak{C}_4\}$ in the proof of Lemma 2.22, so the projective variety $\mathrm{Proj}(\mathrm{gr}_{\mathcal{V}}^{(\underline{d})} R)$ is the irredundant union of the two irreducible components $\mathrm{Proj}(\mathrm{gr}_{\mathcal{V},\mathfrak{C}_1}^{(\underline{d})} R)$ and $\mathrm{Proj}(\mathrm{gr}_{\mathcal{V},\mathfrak{C}_4}^{(\underline{d})} R)$. For $\underline{d} = (d_1, d_2)$ with $d_1, d_2 \geq 1$, however, every maximal chain in A

is maximal in $\Delta^{(\underline{d})}(A)$ and $\mathrm{Proj}(\mathrm{gr}_{\mathcal{V}}^{(\underline{d})}R)$ consists of four irreducible components.

In the same way Kaveh and Khovanskii associate a Newton-Okounkov convex body to a pair of a semigroup and an admissable rational half-space (see [KK]), we define the set

$$\Delta_C^{(\underline{d})} = \overline{\bigcup_{n\in\mathbb{N}} \tfrac{1}{n}\Gamma_{C,n\underline{d}}} \subseteq \mathbb{R}^A$$

for the monoid $\Gamma_C^{(\underline{d})}$ and the half-space of all elements of degree $\mathbb{R}_{\geq 0}\underline{d}$ in its span. When we extend the degree map to $\mathbb{R}^A \to \mathbb{R}^m$, $\Delta_C^{(\underline{d})}$ can be written in the form

$$\Delta_C^{(\underline{d})} = \mathbb{R}_{\geq 0}^C \cap \{\underline{a} \in \mathbb{R}^C \mid \deg \underline{a} = \underline{d}\}. \tag{2.10}$$

Remember that $\mathbb{R}_{\geq 0}^C$ is exactly the cone spanned by Γ_C. To see this equality (2.10), it suffices to show that every rational conical combination $\underline{a}$ of elements in Γ_C with $\deg \underline{a} = \underline{d}$ lies in $\Delta_C^{(\underline{d})}$. So let $\underline{a} \in \mathbb{R}^C$ be of degree $\underline{d}$ and of the form $\underline{a} = \lambda_1 \underline{a}^{(1)} + \cdots + \lambda_s \underline{a}^{(s)}$ with $\lambda_i \in \mathbb{Q}_{\geq 0}$ and $\underline{a}^{(i)} \in \Gamma_C$. Since Γ_C is a subset of $\mathbb{Q}_{\geq 0}^C$, there exists a natural number N, such that $N\underline{a}$ is a $\mathbb{Z}$-linear combination of the unit vectors $e_p \in \mathbb{R}^C$ with non-negative coefficients. In particular, $N\underline{a}$ is contained in Γ_C and $\underline{a} \in \Delta_C^{(\underline{d})}$. The other inclusion is immediate from the definition of $\Delta_C^{(\underline{d})}$.

The cone generated by Γ_C is also compatible with the Veronese submonoids:

$$\mathrm{Cone}\,\Gamma_C^{(\underline{d})} = \mathrm{Cone}\,\Gamma_C \cap \{x \in \mathbb{R}^C \mid \deg x \in \mathbb{R}\underline{d}\}$$

Again, one can show this equality by looking at the rational conical combinations of Γ_C and taking the closure. In particular, the set $\Delta_C^{(\underline{d})}$ can be described by all elements of degree $\underline{d}$ in $\mathrm{Cone}\,\Gamma_C^{(\underline{d})}$.

We denote the lattice generated by Γ_C by $\mathcal{L}^C \subseteq \mathbb{Q}^A$ and the lattice generated by $\Gamma_C^{(\underline{d})}$ by $\mathcal{L}^{C,(\underline{d})} \subseteq \mathbb{Q}^A$. Equation (2.10) implies that $\Delta_C^{(\underline{d})}$ is a polytope and we will see shortly that its vertices are contained in the $\mathbb{Q}$-span of the lattice $\mathcal{L}^{C,(\underline{d})}$. Clearly this lattice is contained in $\mathcal{L}^C$, but in general not every element $\underline{a} \in \mathcal{L}^C$ with $\deg \underline{a} \in \mathbb{Z}\underline{d}$ is contained in $\mathcal{L}^{C,(\underline{d})}$. However, this statement is true if C is an element of $\Delta^{(\underline{d})}(A)$.

Lemma 2.24. *For each chain C in A and $\underline{d} \in \mathbb{N}_0^m$ the lattice $\mathcal{L}^{C,(\underline{d})}$ is equal to the $\underline{d}$-th Veronese sublattice of $\mathcal{L}^{C_{\underline{d}}}$:*

$$\mathcal{L}^{C,(\underline{d})} = \{\underline{a} \in \mathcal{L}^{C_{\underline{d}}} \mid \deg \underline{a} \in \mathbb{Z}\underline{d}\}.$$

If $\underline{d} \in \sigma_C$, then it is of rank $|C_{\underline{d}}| - \dim \sigma_{C_{\underline{d}}} + 1$. Additionally, this number is bounded from above by $r + 1 = \dim X + 1$.

Proof. The statement is trivial when $\underline{d}$ is not contained in σ_C. Now let $\underline{d} \in \sigma_C$. Since the lattice $\mathcal{L}^{C,(\underline{d})}$ does not change, when we replace C by $C_{\underline{d}}$, we can assume that $\underline{d}$ lies in the relative interior of σ_C.

Clearly $\mathcal{L}^{C,(\underline{d})}$ is contained in $\{\underline{a} \in \mathcal{L}^C \mid \deg \underline{a} \in \mathbb{Z}\underline{d}\}$. To show the other inclusion, let $\underline{a}$ be an element of $\mathcal{L}^C$ with $\deg \underline{a} \in \mathbb{Z}\underline{d}$. We can write $\underline{a}$ in the form $\underline{b}-\underline{c}$ for $\underline{b}, \underline{c} \in \Gamma_C$. Since $\underline{d} \in \operatorname{relint} \sigma_C$, again we can use an argument similar to the proof of Lemma 2.20: There exists an element $\underline{b}' \in \Gamma_C$ such that $\deg(\underline{b}+\underline{b}') \in \mathbb{N}_0\underline{d}$. Then we have $\underline{a} = (\underline{b}+\underline{b}') - (\underline{c}+\underline{b}')$ and since $\deg \underline{a} \in \mathbb{Z}\underline{d}$, the degree of $\underline{c}+\underline{b}'$ is a multiple of $\underline{d}$ as well. Hence $\underline{a} \in \mathcal{L}^{C,(\underline{d})}$.

Therefore the lattice $\mathcal{L}^{C,(\underline{d})}$ coincides with the kernel of the map $\phi : \mathcal{L}^C \to \mathbb{Z}^m/\mathbb{Z}\underline{d}$, $\underline{a} \mapsto \deg \underline{a} + \mathbb{Z}\underline{d}$. The image of the degree map $\mathcal{L}^C \to \mathbb{Z}^m$ is free and its rank is equal to the dimension of the cone σ_C. This implies that the rank of $\mathcal{L}^{C,(\underline{d})}$ is given by $\operatorname{rank}(\mathcal{L}^C) - (\dim \sigma_C - 1)$. We get the desired formula, as the lattice $\mathcal{L}^C$ is of rank $|C|$, because it contains all unit vectors e_p for $p \in C$.

Finally, the rank of the lattice $\mathcal{L}^{C,(\underline{d})}$ is at most $r+1$. To see this, let $\mathfrak{C}$ be a maximal chain containing C. Of course, we have

$$\mathcal{L}^{C,(\underline{d})} \subseteq \mathcal{L}^{\mathfrak{C},(\underline{d})} \subseteq \{\underline{a} \in \mathcal{L}^{\mathfrak{C}} \mid \deg \underline{a} \in \mathbb{Z}\underline{d}\}.$$

By a similar argument, the lattice on the right hand side is of rank $|\mathfrak{C}| - \dim \sigma_{\mathfrak{C}} + 1$. We have seen in the proof of Corollary 2.18, that the degrees of the extremal functions along $\mathfrak{C}$ span a group of rank m, so the dimension of $\sigma_{\mathfrak{C}}$ is equal to m and it follows

$$\operatorname{rank} \mathcal{L}^{C,(\underline{d})} \leq |\mathfrak{C}| - \dim \sigma_{\mathfrak{C}} + 1 = \dim \hat{X} - m + 1 = \dim X + 1. \qquad \square$$

Example 2.25. The polytope $\Delta_C^{(\underline{d})}$ of a chain C is of a particularly nice form, namely a product of simplices, when the support of $\deg f_p$ is a one-element set for each $p \in A$. In this case, we get a partition of the poset A into the subsets

$$A_i = \{p \in A \mid \deg f_p \in \mathbb{N}e_i\}.$$

We fix a chain $C \in \Delta(A)$ and a degree $\underline{d} = (d_1, \dots, d_m) \in \mathbb{N}_0^m$. For each $i \in [m]$ we have the subchain $C_i = C \cap A_i$ and by equation (2.10), $\Delta_C^{(\underline{d})}$ is equal to the direct product of the polytopes

$$P_i = \{x \in \mathbb{R}^{A_i}_{\geq 0} \mid \operatorname{supp} x \subseteq C_i, (\deg x)_i = d_i\}$$

where $(\deg x)_i$ denotes the i-th component of $\deg x$. We write $|\underline{c}| = c_1 + \dots + c_m$ for the total degree of a tuple $\underline{c} \in \mathbb{N}_0^m$. We show that each of these polytopes is given by $d_i \Delta_{C_i} \subseteq \mathbb{R}^{A_i}$ with

$$\Delta_{C_i} = \begin{cases} \operatorname{Conv}\left\{ \frac{1}{|\deg f_p|} e_p \;\middle|\; p \in C_i \right\}, & \text{if } C_i \neq \varnothing, \\ \{0\}, & \text{if } C_i = \varnothing \text{ and } d_i = 0, \\ \varnothing, & \text{if } C_i = \varnothing \text{ and } d_i > 0. \end{cases}$$

For $C_i = \varnothing$, we clearly have $P_i = d_i \Delta_{C_i}$. If $C_i \neq \varnothing$, then each point $x \in P_i$ can be written in the form

$$x = \sum_{p \in C_i} \lambda_p \cdot \frac{d_i}{|\deg f_p|} e_p$$

with coefficients $\lambda_p \in \mathbb{R}_{\geq 0}$. Since $d_i = \deg x = \sum_{p \in C_i} \lambda_p d_i$, this is a convex combination of elements in $d_i \Delta_{C_i}$. The other inclusion $d_i \Delta_{C_i} \subseteq P_i$ is immediate from the definition of the two polytopes.

Now let us return to the general case. We are ready to describe the face lattice of the polytope $\Delta_C^{(\underline{d})}$, i. e. the poset

$$L(\Delta_C^{(\underline{d})}) = \{F \mid F \text{ face of } \Delta_C^{(\underline{d})}\}$$

ordered by inclusion. It is well known that $L(\Delta_C^{(\underline{d})})$ is a graded lattice of length $\dim \Delta_C^{(\underline{d})} + 1$.

Proposition 2.26. *The following statements hold for each maximal chain $\mathfrak{C}$ in A:*

(a) For any two subchains $C, D \subseteq \mathfrak{C}$ we have $\Delta_C^{(\underline{d})} \subseteq \Delta_D^{(\underline{d})}$, if and only if $C_{\underline{d}} \subseteq D_{\underline{d}}$.

(b) The map

$$F_{\mathfrak{C}}^{(\underline{d})} : \{C \in \Delta^{(\underline{d})}(A) \mid C \subseteq \mathfrak{C}\} \to L(\Delta_{\mathfrak{C}}^{(\underline{d})}), \quad C \longmapsto \Delta_C^{(\underline{d})}$$

is an isomorphism of posets.

(c) For all $C \subseteq \mathfrak{C}$ the face $\Delta_C^{(\underline{d})}$ is of dimension $|C_{\underline{d}}| - \dim \sigma_{C_{\underline{d}}}$ and its vertices lie in the $\mathbb{Q}$-span of the lattice $\mathcal{L}^{C,(\underline{d})}$.

Proof. For each $p \in \mathfrak{C}$ the set $\Delta_{\mathfrak{C}\setminus\{p\}}^{(\underline{d})} = \Delta_{\mathfrak{C}}^{(\underline{d})} \cap \{x \in \mathbb{R}^{\mathfrak{C}} \mid p \notin \operatorname{supp} x\}$ is a face of $\Delta_{\mathfrak{C}}^{(\underline{d})}$. As an intersection of these faces, $\Delta_C^{(\underline{d})}$ is a face as well for each $C \subseteq \mathfrak{C}$. By Lemma 2.20 it coincides with the face $\Delta_{C_{\underline{d}}}^{(\underline{d})}$.

(a) For any two subsets $C, D \subseteq \mathfrak{C}$ the polytopes $\Delta_C^{(\underline{d})}$ and $\Delta_D^{(\underline{d})}$ agree if $C_{\underline{d}} = D_{\underline{d}}$. Conversely, suppose that $\Delta_C^{(\underline{d})} \subseteq \Delta_D^{(\underline{d})}$. We have seen in the proof of Lemma 2.20 that the monoid $\Gamma_C^{(\underline{d})}$ contains an element $\underline{a}^{(p)}$ with $p \in \operatorname{supp} \underline{a}^{(p)}$ for every $p \in C_{\underline{d}}$, which implies $C_{\underline{d}} \subseteq D_{\underline{d}}$.

(b) First, we prove that every face of $\Delta_{\mathfrak{C}}^{(\underline{d})}$ is induced by a subchain $C \subseteq \mathfrak{C}$. We set $D := \mathfrak{C}_{\underline{d}}$. By part (a), we have $\Delta_D^{(\underline{d})} = \Delta_{\mathfrak{C}}^{(\underline{d})}$ and $\Delta_{D\setminus\{p\}}^{(\underline{d})}$ is a facet of $\Delta_D^{(\underline{d})}$ for all $p \in D$, i. e. a face of codimension 1. Let F be any facet of $\Delta_D^{(\underline{d})}$. Clearly it holds

$$F \subseteq \partial \Delta_D^{(\underline{d})} \subseteq \bigcup_{p \in D} \{x \in \mathbb{R}^D_{\geq 0} \mid p \notin \operatorname{supp} x\}.$$

If K is any convex set in $\mathbb{R}^D_{\geq 0}$ and $x, y \in K$, then there is a point $z \in K$ on the line through x and y with $\operatorname{supp} z = \operatorname{supp} x \cup \operatorname{supp} y$. This implies that there must exist an element $p \in D$ with

$$F \subseteq \Delta^{(\underline{d})}_D \cap \{x \in \mathbb{R}^D_{\geq 0} \mid p \notin \operatorname{supp} x\} = \Delta^{(\underline{d})}_{D \setminus \{p\}}.$$

and since $\Delta^{(\underline{d})}_{D\setminus\{p\}}$ is a facet of $\Delta^{(\underline{d})}_D$, it follows $F = \Delta^{(\underline{d})}_{D\setminus\{p\}}$. Because every face of a polytope is an intersection of facets, each face of $\Delta^{(\underline{d})}_{\mathfrak{C}}$ is equal to $\Delta^{(\underline{d})}_C$ for some $C \subseteq \mathfrak{C}$. Hence the map $F^{(\underline{d})}_{\mathfrak{C}}$ is surjective. By (a), it is injective as well.

(c) Fix a subchain $C \subseteq \mathfrak{C}$. We know that the polytope $\Delta^{(\underline{d})}_C$ is given by the intersection of a cone with an affine hyperplane of codimension one:

$$\Delta^{(\underline{d})}_C = \operatorname{Cone} \Gamma^{(\underline{d})}_C \cap \{x \in \operatorname{span}_{\mathbb{R}}(\Gamma^{(\underline{d})}_C) \mid \deg x = \underline{d}\}.$$

In particular, Lemma 2.24 implies:

$$\dim \Delta^{(\underline{d})}_C = \dim \operatorname{Cone} \Gamma^{(\underline{d})}_C - 1 = \operatorname{rank} \mathcal{L}^{C,(\underline{d})} - 1 = |C_{\underline{d}}| - \dim \sigma_{C_{\underline{d}}}.$$

The vertices of $\Delta^{(\underline{d})}_C$ are of the form $\Delta^{(\underline{d})}_D$ for a subchain $D \subseteq C$. By the definition of these polytopes, we have $\Delta^{(\underline{d})}_D = \{\frac{1}{n}\underline{a}\}$ for each element $\underline{a} \in \Gamma^{(\underline{d})}_D$ of degree $n\underline{d}$. Hence the vertices of $\Delta^{(\underline{d})}_C$ lie in the rational span of the lattice $\mathcal{L}^{C,(\underline{d})}$. □

As the face lattice of every polytope is a graded poset, it follows that $\Delta^{(\underline{d})}(A)$ is the union of graded posets. In general, not all maximal chains in $\Delta^{(\underline{d})}$ have the same length, but there still exists a rank function $r : \Delta^{(\underline{d})}(A) \to \mathbb{N}_0$, where the rank of a chain $C \in \Delta^{(\underline{d})}(A)$ is given by

$$r(C) = \operatorname{rank} \mathcal{L}^{C,(\underline{d})} = |C| - \dim \sigma_C + 1.$$

Definition 2.27. A **polytopal complex** in a finite-dimensional real vector space W is a set $\mathcal{K}$ of polytopes in W that satisfies the following properties:

(a) If $P \in \mathcal{K}$ and Q is a (possibly empty) face of P, then $Q \in \mathcal{K}$;

(b) the intersection of any two polytopes $P, Q \in \mathcal{K}$ is a face of both P and Q.

Definition 2.28. Let $\underline{d} \in \mathbb{N}^m_0$. We define the **Newton-Okounkov polytopal complex** of the Veronese subalgebra $R^{(\underline{d})} \subseteq R$ as the union

$$\Delta^{(\underline{d})}_{\mathcal{V}} = \bigcup_{\mathfrak{C}} \Delta^{(\underline{d})}_{\mathfrak{C}} \subseteq \mathbb{R}^A$$

running over all maximal chains $\mathfrak{C}$ in A.

By the description of the polytopes from equation (2.10) we have $\Delta_C^{(\underline{d})} \cap \Delta_D^{(\underline{d})} = \Delta_{C\cap D}^{(\underline{d})}$ for all chains $C, D \subseteq A$. Therefore the set

$$\mathcal{K}_{\mathcal{V}} = \{\Delta_C^{(\underline{d})} \mid C \in \Delta^{(\underline{d})}(A)\}$$

is a polytopal complex. Technically, $\Delta_{\mathcal{V}}^{(\underline{d})}$ is not a polytopal complex, but rather the geometric realization of the polytopal complex $\mathcal{K}_{\mathcal{V}}$.

Remark 2.29. The Newton-Okounkov polytopal complex $\Delta_{\mathcal{V}}^{(\underline{d})}$ can also be interpreted as a slice of the fan of cones:

$$\Delta_{\mathcal{V}}^{(\underline{d})} = \Big(\bigcup_{\substack{\mathfrak{C}\subseteq A \\ \text{max. chain}}} \operatorname{Cone}\Gamma_{\mathfrak{C}} \Big) \cap \left\{x \in \mathbb{R}^A \mid \deg x = \underline{d}\right\}.$$

In the literature, the cone generated by $\Gamma_{\mathfrak{C}}$ is known as the *global Newton-Okounkov body* of the algebra $\operatorname{gr}_{\mathcal{V},\mathfrak{C}} R$ as it captures the behaviour of the Newton-Okounkov bodies of all its Veronese subalgebras. These global bodies were examined in [CMM] and [LM].

Let $\mathfrak{C}$ be a maximal chain in A. The semigroup

$$\widetilde{\Gamma}_{\mathfrak{C}}^{(\underline{d})} = \mathcal{L}^{\mathfrak{C},(\underline{d})} \cap \operatorname{Cone}\Gamma_{\mathfrak{C}}^{(\underline{d})} = \mathcal{L}^{\mathfrak{C},(\underline{d})} \cap \operatorname{Cone}\Gamma_{\mathfrak{C}}$$

is called the *saturation* of $\Gamma_{\mathfrak{C}}^{(\underline{d})}$, as it is equal to the monoid of all $\underline{a} \in \mathcal{L}^{\mathfrak{C},(\underline{d})}$, such that there exists a natural number k with $k\underline{a} \in \Gamma_{\mathfrak{C}}^{(\underline{d})}$. Gordan's Lemma implies that the saturation is finitely generated. By definition, its elements are given by the lattice points in the scaled polytopes $n\Delta_{\mathfrak{C}}^{(\underline{d})}$:

$$(\widetilde{\Gamma}_{\mathfrak{C}}^{(\underline{d})})_n = \{\underline{a} \in \widetilde{\Gamma}_{\mathfrak{C}}^{(\underline{d})} \mid \deg \underline{a} = n\} = n\Delta_{\mathfrak{C}}^{(\underline{d})} \cap \mathcal{L}^{\mathfrak{C},(\underline{d})}. \tag{2.11}$$

This links our problem of describing the leading function G_R of the Hilbert polynomial to Ehrhart theory. The growth rate of an Ehrhart polynomial is determined by the dimension and the volume of the polytope. But as $\Delta_{\mathfrak{C}}^{(\underline{d})}$ is not full-dimensional in the span of the lattice $\mathcal{L}^{\mathfrak{C},(\underline{d})}$, we first need to find a suitable rational structure.

Definition 2.30. Let P be a polytope in a real vector space $\mathbb{R}^d$. An **integral structure** (respectively **rational structure**) on P is an affine embedding $\iota : P \hookrightarrow \mathbb{R}^{\dim P}$ together with a collection of subsets $P(n) \subseteq P$ for all $n \in \mathbb{N}$, such that the following conditions are fulfilled:

(a) The vertices of $\iota(P)$ have integral (respectively rational) coordinates;

(b) for each $n \in \mathbb{N}$ it holds

$$\iota(P(n)) = \{x \in \iota(P) \mid nx \in \mathbb{Z}^{\dim P}\}.$$

Having a rational structure on a given polytope P allows the use of Ehrhart theory, even if P is not full-dimensional in its ambient space: The cardinality

$$|P(n)| = |\iota(P) \cap \tfrac{1}{n}\mathbb{Z}^{\dim P}| = |n\iota(P) \cap \mathbb{Z}^{\dim P}|$$

then is a quasi-polynomial in n of degree $\dim P$ and its leading coefficient is a constant equal to the standard Euclidean volume of $\iota(P)$.

The main obstacle for constructing a rational structure for the polytope $\Delta_{\mathfrak{C}}^{(\underline{d})}$ are the degrees appearing in the lattice $\mathcal{L}^{\mathfrak{C},(\underline{d})}$. To proceed, we need to show that $\mathcal{L}^{\mathfrak{C},(\underline{d})}$ is not empty in degree 1. Unfortunately, this statement can actually be wrong in certain edge cases, as we show in the example below. However, when $\underline{d}$ lies in the relative interior of $\sigma_{\mathfrak{C}}$, then $\mathcal{L}^{\mathfrak{C},(\underline{d})}$ has elements of degree 1: By Lemma 2.24, the lattice $\mathcal{L}^{\mathfrak{C},(\underline{d})}$ is the $\underline{d}$-th Veronese sublattice of $\mathcal{L}^{\mathfrak{C}}$, hence we only need to check whether $\mathcal{L}^{\mathfrak{C}}$ has an element of degree $\underline{d}$. This serves as the motivation for the next two lemmas.

Example 2.31. Consider the maximal chain $\mathfrak{C} := \mathfrak{C}_2 : X > 01 > 0$ in the Seshadri stratification from Example 2.4 and the degree $\underline{d} = (0,1) \in \mathbb{N}_0^2$. By Lemma 2.24, the lattice $\mathcal{L}^{\mathfrak{C},(\underline{d})}$ is generated by the monoid to the subchain $\mathfrak{C}_{\underline{d}} = \{X\}$. It follows from our computations in Example 2.14 that every element $\underline{a} \in \Gamma$ with $\operatorname{supp} \underline{a} \subseteq \{X\}$ is a multiple of e_X. Hence $\mathcal{L}^{\mathfrak{C},(\underline{d})}$ is generated by e_X and this element is of degree $2\underline{d}$.

Lemma 2.32. *For each non-zero rational function $g \in \mathbb{K}(\hat{X})$ and every maximal chain $\mathfrak{C}$ there exists a regular function $h \in \mathbb{K}[\hat{X}]$ with $\operatorname{supp} \mathcal{V}(gh^k) \subseteq \mathfrak{C}$ for all $k \in \mathbb{N}$.*

Proof. Let $p_r > \cdots > p_0$ be the elements of the chain $\mathfrak{C}$. For each covering relation $q < p$ in A we have the discrete valuation $\nu_{p,q} : \mathbb{K}(\hat{X}_p) \setminus \{0\} \to \mathbb{Z}$ of the prime divisor $\hat{X}_q \subseteq \hat{X}_p$ and the bond $b_{p,q} = \nu_{p,q}(f_p) \in \mathbb{N}$.

The function h can be constructed inductively over the length of the poset A. The statement of this lemma is trivial, when the length is zero. Otherwise, let B be the set of all $q \in A \setminus \{p_{r-1}\}$ which are covered by p_r. For each element $q \in B$ we choose a regular function $h_q \in \mathbb{K}[\hat{X}]$, such that h_q is the zero function on $\hat{X}_q$ and does not vanish identically on $\hat{X}_{p_{r-1}}$. Since $\nu_{p_r,q}(h_q) \geq 1$, we can choose natural numbers n_q, $q \in B$, fulfilling the following inequalities:

$$\frac{n_q \nu_{p_r,q}(h_q)}{b_{p_r,q}} \geq \frac{\nu_{p_r,p_{r-1}}(g)}{b_{p_r,p_{r-1}}} - \frac{\nu_{p_r,q}(g)}{b_{p_r,q}}.$$

We now define the regular function

$$h = \prod_{q \in B} h_q^{n_q} \in \mathbb{K}[\hat{X}].$$

As all functions h_q do not vanish identically on $\hat{X}_{p_{r-1}}$, we get $\nu_{p_r,p_{r-1}}(gh) = \nu_{p_r,p_{r-1}}(g)$.

The choice of the number n_q implies

$$\frac{\nu_{p_r,q}(gh)}{b_{p_r,q}} = \frac{1}{b_{p_r,q}}\Big(\nu_{p_r,q}(g) + \sum_{p\in B} n_p \nu_{p_r,q}(h_p)\Big) \geq \frac{1}{b_{p_r,q}}\Big(\nu_{p_r,q}(g) + n_q\nu_{p_r,q}(h_q)\Big)$$
$$\geq \frac{\nu_{p_r,p_{r-1}}(g)}{b_{p_r,p_{r-1}}} = \frac{\nu_{p_r,p_{r-1}}(gh)}{b_{p_r,p_{r-1}}}.$$

By the construction of the quasi-valuation, its values can be computed inductively using the induced Seshadri stratification on $X_{p_{r-1}}$ with the underlying poset $\{q \in A \mid q \leq p_{r-1}\}$. Let $\mathcal{V}_{p_{r-1}}$ denote its quasi-valuation. Then we have

$$\mathcal{V}(gh) = \frac{\nu_{p_r,p_{r-1}}(g)}{b_{p_r,p_{r-1}}}\, e_{p_r} + \frac{\mathcal{V}_{p_{r-1}}(g_1)}{b_{p_r,p_{r-1}}},$$

where g_1 is the rational function

$$g_1 = \frac{(gh)^{b_{p_r,p_{r-1}}}}{f_p^{\nu_{p_r,p_{r-1}}(gh)}}\Bigg|_{\hat{X}_{p_{r-1}}}.$$

Here we used the alternative description of the quasi-valuation from Remark 6.5 in [CFL].

By induction, there exists a non-zero function $h_1 \in \mathbb{K}[\hat{X}_{p_{r-1}}]$ with $\operatorname{supp}\mathcal{V}_{p_{r-1}}(g_1 h_1^k) \subseteq \mathfrak{C} \setminus \{p_r\}$ for all $k \in \mathbb{N}$. We choose any lift $\overline{h}_1$ of h_1 in $\mathbb{K}[\hat{X}]$. Note that we still have

$$\frac{\nu_{p_r,q}(gh^k h_1^k)}{b_{p_r,q}} \geq \frac{\nu_{p_r,p_{r-1}}(g)}{b_{p_r,p_{r-1}}} = \frac{\nu_{p_r,p_{r-1}}(gh^k h_1^k)}{b_{p_r,p_{r-1}}}.$$

The quasi-valuation of $gh^k h_1^k$ is equal to

$$\mathcal{V}(gh^k h_1^k) = \frac{\nu_{p_r,p_{r-1}}(g)}{b_{p_r,p_{r-1}}}\, e_{p_r} + \frac{\mathcal{V}_{p_{r-1}}(\widetilde{g}_1)}{b_{p_r,p_{r-1}}},$$

for the regular function

$$\widetilde{g}_1 = \frac{(gh^k h_1^k)^{b_{p_r,p_{r-1}}}}{f_p^{\nu_{p_r,p_{r-1}}(gh^k h_1^k)}}\Bigg|_{\hat{X}_{p_{r-1}}} = g_1 \cdot h_1^{k b_{p_r,p_{r-1}}}.$$

In particular, $\operatorname{supp}\mathcal{V}(gh^k h_1^k)$ is contained in $\mathfrak{C}$ for every $k \in \mathbb{N}$. □

Lemma 2.33. *The degree map $\mathcal{L}^{\mathfrak{C}} \to \mathbb{Z}^m$ is surjective for each maximal chain $\mathfrak{C}$ in A.*

Proof. Again, we prove this statement via induction over the length of A. When it is zero, then A only consists of one element, $m = 1$ and $\hat{X}$ is a line. Any linear function g on the multicone has the property $\deg \mathcal{V}(g) = 1$, which implies the surjectivity.

Now suppose that the length of A is non-zero. Let $p_r > \cdots > p_0$ be the elements of the chain $\mathfrak{C}$. If the index set $I_{p_{r-1}}$ is equal to $[m]$, then $\mathcal{L}^{\mathfrak{C}} \to \mathbb{Z}^m$ is surjective by induction. Otherwise $[m] \setminus I_{p_{r-1}}$ contains exactly one element, which we denote by j. By Lemma 2.5 the multicone $\hat{X}_{p_{r-1}}$ is equal to

$$\hat{X}_{p_{r-1}} = \{(v_1, \dots, v_m) \in \hat{X} \mid v_i \in V_i \; \forall i \in [m], v_j = 0\}.$$

Using the projection map $\hat{X} \twoheadrightarrow \hat{X}_{p_{r-1}}$ we view $\mathbb{K}[\hat{X}_{p_{r-1}}]$ as the graded subring

$$\bigoplus_{\substack{\underline{d} \in \mathbb{N}_0^m \\ d_j = 0}} \mathbb{K}[\hat{X}]_{\underline{d}} \subseteq \mathbb{K}[\hat{X}].$$

We choose any non-zero linear function $\ell \in V_j^*$. By induction, each tuple $\underline{d} \in \mathbb{Z}^m$ with $d_j = 0$ lies in the image of $\mathcal{L}^{\mathfrak{C}} \to \mathbb{Z}^m$. Our goal is to construct a function $g \in \mathbb{K}[\hat{X}_{p_{r-1}}]$ with $\operatorname{supp} \mathcal{V}(g\ell) \subseteq \mathfrak{C}$. Then, by Lemma 2.15, we can assume that g was multihomogeneous of degree $\underline{d} \in \mathbb{N}_0^m$ with $d_j = 0$ and it follows from Lemma 2.17 that the j-th component of $\deg \mathcal{V}(g\ell)$ is equal to 1. As $\mathcal{V}(g\ell) \in \Gamma_{\mathfrak{C}}$, the map $\mathcal{L}^{\mathfrak{C}} \to \mathbb{Z}^m$ must be surjective.

Let B be the set of all $q \in A \setminus \{p_{r-1}\}$ covered by p_r. For every $q \in B$ the intersection

$$\hat{X}_q \cap \hat{X}_{p_{r-1}} = \{(v_1, \dots, v_m) \in \hat{X}_q \mid v_i \in V_i \; \forall i \in [m], v_j = 0\}$$

is a proper subvariety of $\hat{X}_{p_{r-1}}$, otherwise this would imply $p_{r-1} \leq q$. Hence we can choose a non-zero regular function $h_q \in \mathbb{K}[\hat{X}_{p_{r-1}}]$, which restricts to the zero function on $\hat{X}_q \cap \hat{X}_{p_{r-1}}$. Seen as a function on $\hat{X}$, h_q vanishes on the whole multicone $\hat{X}_q$. Similar to the proof of Lemma 2.32 we choose $n_q \in \mathbb{N}$ with

$$\frac{n_q}{b_{p_r,q}} \nu_{p_r,q}(h_q) \geq \frac{\nu_{p_r,p_{r-1}}(\ell)}{b_{p_r,p_{r-1}}}$$

and define the regular function

$$g = \prod_{q \in B} h_q^{n_q} \in \mathbb{K}[\hat{X}_{p_{r-1}}].$$

By the choice of the number n_q we get

$$\frac{\nu_{p_r,q}(g\ell)}{b_{p_r,q}} \geq \frac{n_q}{b_{p_r,q}} \nu_{p_r,q}(h_q) \geq \frac{\nu_{p_r,p_{r-1}}(\ell)}{b_{p_r,p_{r-1}}} = \frac{\nu_{p_r,p_{r-1}}(g\ell)}{b_{p_r,p_{r-1}}},$$

so p_{r-1} lies in every maximal chain $\mathfrak{D} \subseteq A$ with $\operatorname{supp} \mathcal{V}(g\ell) \subseteq \mathfrak{D}$. But in general $\operatorname{supp} \mathcal{V}(g\ell)$ is not contained in $\mathfrak{C}$. To achieve this, we need to multiply $g\ell$ by another suitable function, which we get from Lemma 2.32: There exists a regular function $h \in \mathbb{K}[\hat{X}_{p_{r-1}}]$ with $\operatorname{supp} \mathcal{V}_{p_{r-1}}(g_1 h^k) \subseteq \mathfrak{C} \setminus \{p_r\}$ for all $k \in \mathbb{N}$, where g_1 is the rational function

$$g_1 = \left.\frac{(g\ell)^{b_{p_r,p_{r-1}}}}{f_p^{\nu_{p_r,p_{r-1}}(g\ell)}}\right|_{\hat{X}_{p_{r-1}}}.$$

Then we have

$$\mathcal{V}(gh\ell) = \frac{\nu_{p_r,p_{r-1}}(g\ell)}{b_{p_r,p_{r-1}}}\, e_{p_r} + \frac{\mathcal{V}_{p_{r-1}}(g_1 h^{b_{p_r,p_{r-1}}})}{b_{p_r,p_{r-1}}},$$

so the support of $\mathcal{V}_{p_{r-1}}(gh\ell)$ is contained in $\mathfrak{C}$. This completes the proof. □

With the help of the last lemma, we can now construct a rational structure on the polytope $\Delta_{\mathfrak{C}}^{(\underline{d})}$. The equation $(\widetilde{\Gamma}_{\mathfrak{C}}^{(\underline{d})})_n = \mathcal{L}_{n\underline{d}}^{\mathfrak{C}} \cap \operatorname{Cone}\Gamma_{\mathfrak{C}}$ suggests that these structures should be compatible for different $\underline{d}$ in the following sense.

Proposition 2.34. *For each maximal chain $\mathfrak{C}$ in A there exists a linear map* $\mathrm{pr} : \mathbb{R}^{\mathfrak{C}} \to \mathbb{R}^r$ *with the following property: For each $\underline{d} \in \operatorname{relint}\sigma_{\mathfrak{C}}$ the subsets*

$$\Delta_{\mathfrak{C}}^{(\underline{d})}(n) = \{\tfrac{1}{n}\underline{a} \mid \underline{a} \in (\widetilde{\Gamma}_{\mathfrak{C}}^{(\underline{d})})_n\} \subseteq \Delta_{\mathfrak{C}}^{(\underline{d})}$$

for $n \in \mathbb{N}$ form a rational structure on $\Delta_{\mathfrak{C}}^{(\underline{d})}$ together with the map $\Delta_{\mathfrak{C}}^{(\underline{d})} \hookrightarrow \mathbb{R}^{\mathfrak{C}} \to \mathbb{R}^r$.

Proof. By Lemma 2.33, we can choose elements $\underline{b}^{(1)}, \dots, \underline{b}^{(m)} \in \mathcal{L}^{\mathfrak{C}}$ with $\deg \underline{b}^{(i)} = e_i$. They define a group homomorphism

$$\mathcal{L}^{\mathfrak{C}} \to \mathcal{L}_0^{\mathfrak{C}}, \quad \underline{a} \mapsto \underline{a} - \sum_{i=1}^{m} c_i \underline{b}^{(i)} \quad \text{for } \deg \underline{a} = (c_1, \dots, c_m).$$

Note that the lattice $\mathcal{L}_0^{\mathfrak{C}}$ of degree 0 elements in $\mathcal{L}^{\mathfrak{C}}$ is of rank $|\mathfrak{C}| - m = r$. We extend the above map to an $\mathbb{R}$-linear map $\mathrm{pr} : \mathbb{R}^{\mathfrak{C}} \to U_0$, where U_0 is the real span of $\mathcal{L}_0^{\mathfrak{C}} \cong \mathbb{Z}^r$ in $\mathbb{R}^{\mathfrak{C}}$. For each $\underline{d} \in \operatorname{relint}\sigma_{\mathfrak{C}}$ the affine subspace $U_{\underline{d}} \coloneqq \{x \in \mathbb{R}^{\mathfrak{C}} \mid \deg x = \underline{d}\}$ contains the polytope $\Delta_{\mathfrak{C}}^{(\underline{d})}$ and both are of dimension r. By construction, pr induces a bijection $\mathcal{L}_{\underline{d}}^{\mathfrak{C}} \to \mathcal{L}_0^{\mathfrak{C}}$, $\underline{a} \mapsto \mathrm{pr}(\underline{a})$, so the composition $\iota_{\underline{d}} : \Delta_{\mathfrak{C}}^{(\underline{d})} \hookrightarrow \mathbb{R}^{\mathfrak{C}} \to U_0$ is an affine embedding. Furthermore it maps the set

$$\Delta_{\mathfrak{C}}^{(\underline{d})}(1) = (\widetilde{\Gamma}_{\mathfrak{C}}^{(\underline{d})})_1 = \Delta_{\mathfrak{C}}^{(\underline{d})} \cap \mathcal{L}^{\mathfrak{C}}$$

bijectively to $\iota_{\underline{d}}(\Delta_{\mathfrak{C}}^{(\underline{d})}) \cap \mathcal{L}_0^{\mathfrak{C}}$. Since $n\Delta_{\mathfrak{C}}^{(\underline{d})} = \Delta_{\mathfrak{C}}^{(n\underline{d})}$ for all $n \in \mathbb{N}$, it follows

$$\iota_{\underline{d}}(\Delta_{\mathfrak{C}}^{(\underline{d})}(n)) = \iota_{\underline{d}}\big(\tfrac{1}{n}(\Delta_{\mathfrak{C}}^{(n\underline{d})} \cap \mathcal{L}^{\mathfrak{C}})\big) = \tfrac{1}{n}\iota_{\underline{d}}(\Delta_{\mathfrak{C}}^{(n\underline{d})}) \cap \mathcal{L}_0^{\mathfrak{C}} = \iota_{\underline{d}}(\Delta_{\mathfrak{C}}^{(\underline{d})}) \cap \tfrac{1}{n}\mathcal{L}_0^{\mathfrak{C}}.$$

Lastly, we have seen in Proposition 2.26 that the vertices of $\Delta_{\mathfrak{C}}^{(\underline{d})}$ lie in the $\mathbb{Q}$-span of $\mathcal{L}^{\mathfrak{C}}$. As the map pr is compatible with the lattices, $\iota_{\underline{d}}$ defines a rational structure on $\Delta_{\mathfrak{C}}^{(\underline{d})}$. □

Proposition 2.35. *For each maximal chain $\mathfrak{C}$ in A we fix a map $\mathrm{pr}_{\mathfrak{C}} : \mathbb{R}^{\mathfrak{C}} \to \mathbb{R}^r$ as in Proposition 2.34. If $\underline{d} \in \mathbb{N}_0^m$ does not lie on the boundary of $\sigma_{\mathfrak{C}}$ for any maximal chain $\mathfrak{C}$, then it holds*

$$G_R(\underline{d}) = \sum_{\mathfrak{C}} \mathrm{vol}(\mathrm{pr}_{\mathfrak{C}}(\Delta_{\mathfrak{C}}^{(\underline{d})})),$$

where the sum runs over all maximal chains in A.

Proof. As R is finitely generated in total degree 1, its Veronese subalgebra $R^{(\underline{d})}$ is finitely generated in degree one, so the Hilbert quasi-polynomial of $R^{(\underline{d})}$ is a polynomial. By (2.7) it coincides with the Hilbert quasi-polynomial $H^{(\underline{d})}$ of $\mathrm{gr}_{\mathcal{V}}^{(\underline{d})} R$. We have seen in Lemma 2.22 that the associated projective variety of this degenerated algebra is the irredundant union of its irreducible components $\mathrm{Proj}(\mathrm{gr}_{\mathcal{V},C}^{(\underline{d})} R)$, where C runs over the set $\mathcal{C}$ of all maximal elements in $\Delta^{(\underline{d})}(A)$. Since $\underline{d}$ does not lie on the boundary of $\sigma_{\mathfrak{C}}$, we know that $\mathcal{C}$ consists exactly of the maximal chains $\mathfrak{C}$ in A with $\underline{d} \in \sigma_{\mathfrak{C}}$.

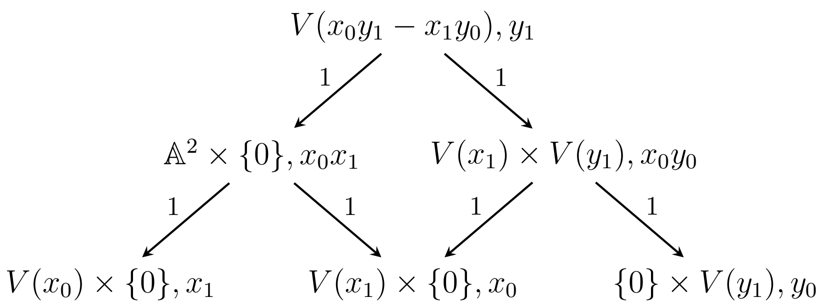
By Lemma 2.24 the component $\mathrm{Proj}(\mathrm{gr}_{\mathcal{V},\mathfrak{C}}^{(\underline{d})} R)$ for $\mathfrak{C} \in \mathcal{C}$ is a projective toric variety of dimension $|\mathfrak{C}| - \dim \sigma_{\mathfrak{C}} = r$. In particular,

$$G_R(\underline{d}) = \lim_{n \to \infty} \frac{|\Gamma_{n\underline{d}}|}{n^r}$$

computes the coefficient a of the monomial x^r in $H^{(\underline{d})} \in \mathbb{Q}[x]$. It is zero, when $\mathcal{C}$ is empty, otherwise it is the leading coefficient of $H^{(\underline{d})}$.

For $\mathfrak{C} \in \mathcal{C}$ let $H_{\mathfrak{C}}^{(\underline{d})}$ denote the Hilbert quasi-polynomial of $\mathrm{gr}_{\mathcal{V},\mathfrak{C}}^{(\underline{d})} R \cong \mathbb{K}[\Gamma_{\mathfrak{C}}^{(\underline{d})}]$. Then a is given by the sum of the leading terms $a_{\mathfrak{C}}$ of all quasi-polynomials $H_{\mathfrak{C}}^{(\underline{d})}$. Using the arguments from the Lemmas 9.9 and 9.10 in [CFL], one can prove that $a_{\mathfrak{C}}$ is the leading term of the Hilbert quasi-polynomial of $\mathbb{K}[\widetilde{\Gamma}_{\mathfrak{C}}^{(\underline{d})}]$, induced by the saturated monoid. By Proposition 2.34 this quasi-polynomial is an Ehrhart quasi-polynomial, so $a_{\mathfrak{C}}$ is constant and equal to the volume of the embedded polytope $\mathrm{pr}_{\mathfrak{C}}(\Delta_{\mathfrak{C}}^{(\underline{d})})$. This completes the proof, as $\Delta_{\mathfrak{C}}^{(\underline{d})}$ is the empty polytope for all maximal chains $\mathfrak{C} \subseteq A$ not in $\mathcal{C}$. □

Example 2.36. In the Seshadri stratification of Hodge type from Example 2.3 there are the four maximal chains $\mathfrak{C}_1, \ldots, \mathfrak{C}_4$ from left to right.



For a tuple $\underline{d}\mathbb{N}_0^2$ in the interior of $\sigma_{\mathfrak{C}_1} = \mathbb{R}^2_{\geq 0}$ the vertices of the polytope $\Delta^{(\underline{d})}_{\mathfrak{C}_1}$ are given by $d_1e_0 + d_2e_X$ and $\frac{1}{2}d_1e_{01} + d_2e_X$. By fixing the element $\underline{b}^{(1)} = e_0$ of degree $(1,0)$ and $\underline{b}^{(2)} = e_X$ of degree $(0,1)$ we get an integral structure on $\Delta^{(\underline{d})}_{\mathfrak{C}_1}$ via the proof of Proposition 2.34. It identifies the vertices with the points 0 and $d_1(\frac{1}{2}e_{01} - e_0)$ in the lattice $\mathcal{L}_0^{\mathfrak{C}_1} = \mathbb{Z} \cdot (e_{01} - 2e_0)$. The volume of the resulting polytope in the linear span of this lattice is equal to $\frac{1}{2}d_1$.

In the same way one can compute an integral structure on $\Delta^{(\underline{d})}_{\mathfrak{C}_1}$ with volume $\frac{1}{2}d_1$. For the third maximal chain $\mathfrak{C}_3$ we again have $\sigma_{\mathfrak{C}_3} = \mathbb{R}^2_{\geq 0}$. We fix the elements $\underline{b}^{(1)} = e_1$ of degree $(1,0)$ and $\underline{b}^{(2)} = e_X$ of degree $(0,1)$ for the integral structure. For $\underline{d} \in \mathbb{N}_0^2$ in the interior of this cone, one needs to distinguish between two cases. If $d_1 \geq d_2$, then the polytope $\Delta^{(\underline{d})}_{\mathfrak{C}_3}$ has the vertices $d_1e_1 + d_2e_X$ and $(d_1 - d_2)e_1 + d_2e_{0\bar{0}}$, which correspond to the points 0 and $d_2(-e_1 + e_{0\bar{0}} - e_X)$ in $\mathcal{L}_0^{\mathfrak{C}_1} = \mathbb{Z} \cdot (-e_1 + e_{0\bar{0}} - e_X)$. Hence we get the volume d_2. Analogously, we have the volume d_1 in the case $d_2 \geq d_1$.

The cone of the last maximal chain $\mathfrak{C}_4$ is spanned by $(1,1)$ and $(0,1)$. For every $\underline{d} \in \mathbb{N}_0^2$ not contained in this cone, the polytope $\Delta^{(\underline{d})}_{\mathfrak{C}_4}$ is empty. Otherwise $\Delta^{(\underline{d})}_{\mathfrak{C}_4}$ has the vertices $d_1e_{0\bar{0}} + (d_2 - d_1)e_{\bar{0}}$ and $d_1e_{0\bar{0}} + (d_2 - d_1)e_X$. Via the elements $\underline{b}^{(1)} = e_{0\bar{0}} - e_{\bar{0}}$ and $\underline{b}^{(2)} = e_{\bar{0}}$ we get the volume $d_2 - d_1$.

With these volumes, we can now compute the leading term of the Hilbert polynomial:

$$G_R(\underline{d}) = \left\{ \begin{array}{ll} \frac{1}{2}d_1 + \frac{1}{2}d_1 + d_2, & \text{for } d_1 > d_2 > 0 \\ \frac{1}{2}d_1 + \frac{1}{2}d_1 + d_1 + (d_2 - d_1), & \text{for } d_2 > d_1 > 0 \end{array} \right\} = d_1 + d_2.$$

The multidegrees of X are therefore given by $\deg_{(1,0)}(X) = \deg_{(0,1)}(X) = 1$. Indeed, the multiprojective coordinate ring $R = \mathbb{K}[x_0, x_1, y_0, y_1]/(x_0y_1 - x_1y_0)$ has a basis consisting of all monomials $x_0^a x_1^b y_0^c y_1^d$ with $a, b, c, d \in \mathbb{N}_0$ and $bc = 0$. Hence the graded component $R_{\underline{d}}$ is of dimension $(d_1 + 1) + (d_2 + 1) - 1 = d_1 + d_2 + 1$. This is already a polynomial in $\underline{d}$ and its leading term agrees with the function G_R we computed above.

2.6. Seshadri stratifications of LS-type

For suitable choices of extremal functions, the polytopes $\Delta^{(\underline{d})}_{\mathfrak{C}}$ are products of simplices for all $\underline{d}$, e. g. when the support of $\deg f_p$ is a one-element set for each $p \in A$ (see Example 2.25) or when $I_p = I_q \in \mathcal{I}$ is equivalent to $\deg f_p = \deg f_q$ for all $p, q \in A$ (see below). One might ask if there exists a rational structure as in Proposition 2.34, that is compatible with this decomposition into simplices, i. e. the map $\mathrm{pr}_{\mathfrak{C}} : \mathbb{R}^{\mathfrak{C}} \to \mathbb{R}^r$ is a product of rational structures, one for each simplex. In general, this idea is too naive: It already fails for the stratification we examined in Example 2.31, since the lattice $\mathcal{L}^{\mathfrak{C}}$ to the maximal chain $\mathfrak{C} : X > 01 > 0$ does not decompose into the product $\mathcal{L}^{\{0,01\}} \times \mathcal{L}^{\{X\}}$. However, when all the monoids $\Gamma_{\mathfrak{C}}$ are so-called LS-monoids, then such a decomposition does exist and one can compute the volumes of the polytopes explicitly via the bonds in the stratification.

Definition 2.37. Let C be a chain of covering relations in A, i. e. it consists of elements $p_s > \cdots > p_0$ such that p_i covers p_{i-1} for each $i = 1, \ldots, s$. The **Lakshmibai-Seshadri-lattice** (short: LS-lattice) associated to C is the lattice

$$\mathrm{LS}_C = \left\{ \sum_{i=0}^{s} a_i e_{p_i} \in \mathbb{Q}^C \;\middle|\; b_{p_i,p_{i-1}}(a_i + \cdots + a_s) \in \mathbb{Z} \ \forall i \in [s], a_0 + \cdots + a_s \in \mathbb{Z} \right\}.$$

and its intersection $\mathrm{LS}_C^+ = \mathrm{LS}_C \cap \mathbb{Q}_{\geq 0}^C$ with the positive orthant is called the **LS-monoid** to the chain C.

Every LS-lattice is generated by its LS-monoid, since one can shift each element in LS_C into the positive orthant via the vectors $e_{p_i} \in \mathrm{LS}_C$. LS-lattices are also compatible with subchains: If $D \subseteq C$ are two chains of covering relations, then we have $\mathrm{LS}_C \cap \mathbb{Q}^D = \mathrm{LS}_D$. This has the following consequences: If the monoid $\Gamma_{\mathfrak{C}}$ is an LS-monoid for every maximal chain $\mathfrak{C}$, then the Seshadri stratification is normal. The set $\mathbb{G} \subseteq \Gamma$ of all indecomposable elements is finite and for every $\underline{u} = \sum_{p \in A} u_p e_p \in \mathbb{G}$ the coefficients u_p add up to 1 (this follows from the proof of Lemma 3.3 in [CFL3]).

Definition 2.38. We call a Seshadri stratification on $X \subseteq \prod_{i=1}^m \mathbb{P}(V_i)$ **of LS-type**, if the following conditions are fulfilled:

(a) Each component of the multidegree $\deg f_p \in \mathbb{N}_0^m$ is at most 1 for all $p \in A$;

(b) if $I_p = I_q$ for any two elements $p, q \in A$, then $\deg f_p = \deg f_q$;

(c) the fan of monoids Γ is equal to the union $\bigcup_{\mathfrak{C}} \mathrm{LS}_{\mathfrak{C}}^+$ over all maximal chains $\mathfrak{C}$ in A.

The next remark implies that this definition generalizes the notion of a Seshadri stratification of LS-type from [CFL3, Definition 2.6]. For $m = 1$ both definitions agree.

Remark 2.39. For every stratification of LS-type, the monoid $\Gamma_{\mathfrak{C}}$ agrees with the LS-monoid $\mathrm{LS}_{\mathfrak{C}}^+$ for each maximal chain in A: We clearly have $\mathrm{LS}_{\mathfrak{C}}^+ \subseteq \Gamma \cap \mathbb{Q}^{\mathfrak{C}} = \Gamma_{\mathfrak{C}}$. For the reverse inclusion, let $p_r > \cdots > p_0$ be the elements in $\mathfrak{C}$ and $b_j = b_{p_j,p_{j-1}}$ be the bond to the covering relation $p_j > p_{j-1}$ for all $j = 0, \ldots, r$. By the definition of an LS-lattice, each element $\underline{a}^{(j)} = \frac{1}{b_j} e_{p_j} - \frac{1}{b_j} e_{p_{j-1}}$ is contained in $\mathrm{LS}_{\mathfrak{C}} \subseteq \mathcal{L}^{\mathfrak{C}}$. Hence one can find rational functions $F_r, \ldots, F_1 \in \mathbb{K}(\hat{X}) \setminus \{0\}$ with $\mathcal{V}(F_j) = \underline{a}^{(j)}$ for all $j = 1, \ldots, r$. We can now use Proposition 2.7: The matrix $B_{\mathfrak{C}}$ is given by

$$B_{\mathfrak{C}} = \begin{pmatrix} b_r^{-1} & 0 & \cdots & \cdots & 0 \\ -b_r^{-1} & b_{r-1}^{-1} & \ddots & & \vdots \\ 0 & -b_{r-1}^{-1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & b_1^{-1} & 0 \\ 0 & 0 & \cdots & -b_1^{-1} & b_0^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} b_r & 0 & \cdots & \cdots & 0 \\ b_{r-1} & b_{r-1} & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ b_1 & b_1 & \cdots & b_1 & 0 \\ b_0 & b_0 & \cdots & b_0 & b_0 \end{pmatrix}$$

Therefore $\mathcal{L}^{\mathfrak{C}}$ is contained in $\mathrm{LS}_{\mathfrak{C}}$ and by intersecting with $\mathbb{Q}^{\mathfrak{C}}_{\geq 0}$ we get $\Gamma_{\mathfrak{C}} \subseteq \mathrm{LS}^+_{\mathfrak{C}}$.

Until the end of this section we fix a Seshadri stratification of LS-type on a multiprojective variety X. The poset $\mathcal{I}$ defines a partition of A into the subsets

$$A_I = \{p \in A \mid I_p = I\}$$

for $I \in \mathcal{I}$. By definition, all extremal functions in A_I have the same multidegree. We see in the next lemma that this degree is always given by

$$e_I = \sum_{i \in \underline{I}} e_i \in \mathbb{N}_0^m$$

for a subset $\underline{I} \subseteq I$ characterized by the covering relations in the index poset $\mathcal{I}$: If I is a minimal element in $\mathcal{I}$ then it holds $\underline{I} = I$, otherwise $\underline{I}$ is the union of all sets $I \setminus J$, where $J \subsetneq I$ is a covering relation in $\mathcal{I}$.

Example 2.40. If $\mathcal{I}$ is totally ordered, one can assume w. l. o. g. that it consists of the sets $[i]$ for all $i \in [m]$. In this case, we have $\underline{[i]} = \{i\}$.

To give another example, consider the poset $\mathcal{I}$ with the elements $I = \{2\}$, $J = \{1, 2\}$, $K = \{2, 3\}$ and $L = [3]$. Here $\underline{I} = \{2\}$, $\underline{J} = \{1\}$, $\underline{K} = \{3\}$ and $\underline{L} = \{1, 3\}$.

Lemma 2.41. *For all* $p \in A$ *it holds* $\deg f_p = e_{I_p}$.

Proof. We fix an element $I \in \mathcal{I}$ and let $\underline{d} = (d_1, \ldots, d_m)$ be the multidegree of any extremal function f_p for $p \in A_I$. For all $i \in \underline{I}$ there exists a covering relation $q < p$ in A with $I_p \setminus I_q = \{i\}$ and we have $d_i \neq 0$ by Lemma 2.6 (b).

Conversely, let $d_i = 1$ for some $i \in I$ and let p be any element of A_I. Then the subvariety

$$Y = \{(v_1, \ldots, v_m) \in \hat{X}_p \mid v_j \in V_j \ \forall j \in [m], v_i = 0\}$$

of $\hat{X}_p$ is irreducible, contained in the vanishing set of f_p and the codimension of Y in $\hat{X}_p$ is at least one. If $\operatorname{codim}_Y(\hat{X}_p) = 1$, then $i \in \underline{I}$. Otherwise there exists an element $q \in A_I$ with $q < p$ and $Y \subseteq \hat{X}_q$ and we can proceed by induction over the codimension of Y. □

Fix a maximal chain $\mathfrak{C}$ in A with associated maximal chain $I_1 \subsetneq \cdots \subsetneq I_m = [m]$ in $\mathcal{I}$. It defines a decomposition of $\mathfrak{C}$ into the subchains

$$\mathfrak{C}_j = \{p \in \mathfrak{C} \mid I_p = I_j\}.$$

The covering relation $\min \mathfrak{C}_j > \max \mathfrak{C}_{j-1}$ has bond 1 by Lemma 2.6 (b). It follows from the definition of LS-lattices, that they decompose into a product of sublattices along covering relations with bond 1:

$$\mathrm{LS}_{\mathfrak{C}} = \mathrm{LS}_{\mathfrak{C}_1} \times \cdots \times \mathrm{LS}_{\mathfrak{C}_m} \subseteq \mathbb{Q}^{\mathfrak{C}}.$$

Therefore $\mathcal{L}^{\mathfrak{C}}$ is equal to the product of the sublattices $\mathcal{L}^{\mathfrak{C}_j} \subseteq \mathbb{Q}^{\mathfrak{C}_j}$ generated by $\Gamma_{\mathfrak{C}_j}$. Of course, this is compatible with the monoids as well: $\Gamma_{\mathfrak{C}} = \Gamma_{\mathfrak{C}_1} \times \cdots \times \Gamma_{\mathfrak{C}_m}$. We define the $m \times m$-matrix $M_{\mathfrak{C}}$ with entries in $\mathbb{Z}$, such that its j-th column consists of the degree vector $e_{I_j} \in \mathbb{N}_0^m$. Its follows from Lemma 2.6 (b) that this matrix is invertible over $\mathbb{Z}$, so its inverse gives rise to a group isomorphism

$$\phi^{\mathfrak{C}} : \mathbb{Z}^m \to \mathbb{Z}^m, \quad \underline{d} \mapsto M_{\mathfrak{C}}^{-1}\underline{d}$$

identifying $\sigma_{\mathfrak{C}} \cap \mathbb{N}_0^m$ with $\mathbb{N}_0^m$. For $j \in [m]$ let $\phi_j^{\mathfrak{C}} : \mathbb{Z}^m \to \mathbb{Z}$ be its projection onto the j-th component. This allows us to show

$$\mathcal{L}^{\mathfrak{C},(\underline{d})} = \{\underline{a} \in \mathcal{L}^{\mathfrak{C}} \mid \deg \underline{a} \in \mathbb{Z}\underline{d}\} \tag{2.12}$$

for all $\underline{d} \in \sigma_{\mathfrak{C}}$. Each element $\underline{a} \in \mathcal{L}^{\mathfrak{C}}$ with $\deg \underline{a} \in \mathbb{Z}\underline{d}$ can be written as $\underline{a} = \underline{b} - \underline{c}$ for $\underline{b}, \underline{c} \in \Gamma_{\mathfrak{C}}$. Using the isomorphism $\phi^{\mathfrak{C}}$ one can find an element $\underline{a}' \in \sum_{p \in \mathfrak{C}} \mathbb{N}_0 e_p$, such that $\deg(\underline{b} + \underline{a}') \in \mathbb{Z}\underline{d}$. Then both $\underline{b} + \underline{a}'$ and $\underline{c} + \underline{a}'$ lie in the Veronese monoid $\Gamma_{\mathfrak{C}}^{(\underline{d})}$ and we have $\underline{a} \in \mathcal{L}^{\mathfrak{C},(\underline{d})}$.

Let r be the dimension of X. The Newton-Okounkov polytopes of a stratification of LS-type decompose into products of simplices: For each maximal chain $\mathfrak{C}$ and $\underline{d} \in \sigma_{\mathfrak{C}}$ we can write the polytope $\Delta_{\mathfrak{C}}^{(\underline{d})}$ in the form

$$\begin{aligned}\Delta_{\mathfrak{C}}^{(\underline{d})} &= \mathbb{R}_{\geq 0}^{\mathfrak{C}} \cap \{x \in \mathbb{R}^{\mathfrak{C}} \mid \deg x = \underline{d}\} \\ &= \prod_{j=1}^{m} \mathbb{R}_{\geq 0}^{\mathfrak{C}_j} \cap \{x \in \mathbb{R}^{\mathfrak{C}_j} \mid \deg x = \phi_j^{\mathfrak{C}}(\underline{d}) e_{I_j}\} = \prod_{j=1}^{m} \Delta_{\mathfrak{C}_j}^{(\phi_j^{\mathfrak{C}}(\underline{d}) e_{I_j})}.\end{aligned}$$

Hence $\Delta_{\mathfrak{C}}^{(\underline{d})}$ is a multisimplex, since we have

$$\Delta_{\mathfrak{C}_j}^{(\phi_j^{\mathfrak{C}}(\underline{d}) e_{I_j})} = \phi_j^{\mathfrak{C}}(\underline{d}) \Delta_{\mathfrak{C}_j},$$

where $\Delta_{\mathfrak{C}_j}$ is the convex hull of all vectors e_p for $p \in \mathfrak{C}_j$. For fixed $j \in [m]$ let $p_s > \cdots > p_0$ be the elements of the subchain $\mathfrak{C}_j$ and $b_{k,k-1}$ be the bond of the covering relation $p_k > p_{k-1}$ in A for $k = 1, \ldots, s$. We define the linear map

$$\mathrm{pr}_{\mathfrak{C}_j} : \mathbb{R}^{\mathfrak{C}_j} \to \mathbb{R}^s, \quad e_{p_i} \mapsto \begin{cases} 0, & \text{if } i = 0, \\ \sum_{k=1}^{i} b_{k,k-1} e_k, & \text{if } i \geq 1. \end{cases}$$

Proposition 2.42. *For each $k \in \mathbb{N}$ the map $\mathrm{pr}_{\mathfrak{C}_j}$ and the sets*

$$(k\Delta_{\mathfrak{C}_j})(n) = \{\tfrac{1}{n}\underline{a} \mid \underline{a} \in \Gamma_{\mathfrak{C}_j, nk}\}$$

form an integral structure on the scaled polytope $k\Delta_{\mathfrak{C}_j} \subseteq \mathbb{R}^{\mathfrak{C}_j}$.

Proof. The lattice $\mathcal{L}^{\mathfrak{C}_j}$ is graded by $\mathbb{Z} \cong \mathbb{Z}e_{I_j}$. Analogous to the proof of Proposition 2.34 any element $\underline{b} \in \mathcal{L}^{\mathfrak{C}_j}$ of degree 1 defines a linear map $\overline{\mathrm{pr}}_{\mathfrak{C}} : \mathbb{R}^{\mathfrak{C}} \to U_0$ sending an element $\underline{a} \in \mathcal{L}^{\mathfrak{C}_j}$ of degree $d \in \mathbb{Z}$ to $\underline{a} - d\underline{b}$, where U_0 is the linear span of the lattice $\mathcal{L}_0^{\mathfrak{C}_j}$ of degree zero elements in $\mathcal{L}^{\mathfrak{C}_j}$. The restriction of this map to any affine subspace

$$U_d = \{x \in \mathbb{R}^{\mathfrak{C}_j} \mid \deg x = d\}$$

for $d \in \mathbb{Z}$ is bijective and identifies $d\underline{b} + \mathcal{L}_0^{\mathfrak{C}_j} = \mathcal{L}^{\mathfrak{C}_j} \cap U_d$ with $\mathcal{L}_0^{\mathfrak{C}_j}$. For every $n \in \mathbb{N}$ the polytope $kn\Delta_{\mathfrak{C}_j}$ is contained in U_{kn}, hence $\overline{\mathrm{pr}}_{\mathfrak{C}_j}$ maps the subset $(k\Delta_{\mathfrak{C}_j})(1)$ onto $\overline{\mathrm{pr}}_{\mathfrak{C}_j}(k\Delta_{\mathfrak{C}_j}) \cap \mathcal{L}_0^{\mathfrak{C}_j}$. It follows

$$\begin{aligned}\overline{\mathrm{pr}}_{\mathfrak{C}_j}\big((k\Delta_{\mathfrak{C}_j})(n)\big) &= \overline{\mathrm{pr}}_{\mathfrak{C}_j}\big(\tfrac{1}{n}(kn\Delta_{\mathfrak{C}_j})(1)\big) = \tfrac{1}{n}\big(\overline{\mathrm{pr}}_{\mathfrak{C}_j}(kn\Delta_{\mathfrak{C}_j}) \cap \mathcal{L}_0^{\mathfrak{C}_j}\big)\\ &= \overline{\mathrm{pr}}_{\mathfrak{C}_j}(k\Delta_{\mathfrak{C}_j}) \cap \tfrac{1}{n}\mathcal{L}_0^{\mathfrak{C}_j}.\end{aligned}$$

As all vertices of $k\Delta_{\mathfrak{C}_j}$ are contained in the lattice $\mathcal{L}^{\mathfrak{C}_j}$, the map $\overline{\mathrm{pr}}_{\mathfrak{C}_j}$ defines an integral structure on this polytope.

For our purposes we choose $\underline{b} = e_{p_0}$, so that the composition of $\overline{\mathrm{pr}}_{\mathfrak{C}_j}$ with

$$\psi : U_0 \hookrightarrow \mathbb{R}^{\mathfrak{C}_j} \xrightarrow{\mathrm{pr}_{\mathfrak{C}_j}} \mathbb{R}^s$$

coincides with $\mathrm{pr}_{\mathfrak{C}_j}$. By the definition of $\mathrm{pr}_{\mathfrak{C}_j}$ and the defining conditions of an LS-lattice, the map ψ restricts to a group homomorphism $\overline{\psi} : \mathcal{L}_0^{\mathfrak{C}_j} \to \mathbb{Z}^s$. To finish the proof, we need to show that $\overline{\psi}$ is an isomorphism. Its image is equal to the set of all elements $\mathrm{pr}_{\mathfrak{C}_j}(\underline{a})$ with $\underline{a} \in \mathcal{L}^{\mathfrak{C}_j}$. The lattice $\mathcal{L}^{\mathfrak{C}_j}$ contains the elements

$$\underline{a}^{(i)} = \frac{1}{b_{i,i-1}}e_{p_i} - \frac{1}{b_{i,i-1}}e_{p_{i-1}}$$

for $i = 1, \ldots, s$. The image of $\underline{a}^{(i)}$ under the map $\overline{\psi}$ is of the form $e_i + \sum_{k=1}^{i-1} \mathbb{Z}e_k$, so these images form a basis of $\mathbb{Z}^s$ and $\overline{\psi}$ is surjective. This also implies that its kernel has rank zero. □

The proposition immediately has the consequence that the product map

$$\mathrm{pr}_{\mathfrak{C}} = \mathrm{pr}_{\mathfrak{C}_1} \times \cdots \times \mathrm{pr}_{\mathfrak{C}_m} : \mathbb{R}^{\mathfrak{C}} \to \mathbb{R}^r$$

forms an integral structure on the polytope $\Delta_{\mathfrak{C}}^{(\underline{d})}$ for each $\underline{d} \in \operatorname{relint}\sigma_{\mathfrak{C}}$ together with the subsets

$$\Delta_{\mathfrak{C}}^{(\underline{d})}(n) = \prod_{j=1}^{m} \{\tfrac{1}{n}\underline{a} \mid \underline{a} \in \Gamma_{\mathfrak{C}_j, n\phi_j^{\mathfrak{C}}(\underline{d})}\} = \{\tfrac{1}{n}\underline{a} \mid \underline{a} \in \Gamma_{\mathfrak{C}, n\underline{d}}\}.$$

Since $\widetilde{\Gamma}_{\mathfrak{C}}^{(\underline{d})} = \mathcal{L}^{\mathfrak{C},(\underline{d})} \cap \operatorname{Cone}\Gamma_{\mathfrak{C}} = \Gamma_{\mathfrak{C}}^{(\underline{d})}$ follows from equation (2.12), the Veronese monoid $\Gamma_{\mathfrak{C}}^{(\underline{d})}$ is saturated. Therefore the map $\mathrm{pr}_{\mathfrak{C}}$ meets the requirements from Proposition 2.34.

The volume of $\mathrm{pr}_{\mathfrak{C}_j}(\Delta_{\mathfrak{C}_j})$ is equal to the product $\prod_{k=1}^{s} b_{k,k-1}$ of all bonds in the subchain $\mathfrak{C}_j$. Let $b_{\mathfrak{C}} = \prod_{i=1}^{r} b_{i,i-1}$ denote the product of all bonds in $\mathfrak{C}$. As all bonds connecting the chains $\mathfrak{C}_j$ are equal to 1, we get

$$\operatorname{vol}\big(\mathrm{pr}_{\mathfrak{C}}(\Delta_{\mathfrak{C}}^{(\underline{d})})\big) = b_{\mathfrak{C}} \cdot \prod_{j=1}^{m} \phi_j^{\mathfrak{C}}(\underline{d})^{|\mathfrak{C}_j|-1} \tag{2.13}$$

for every $\underline{d} \in \operatorname{relint}\sigma_{\mathfrak{C}}$. As the polytope $\Delta_{\mathfrak{C}}^{(\underline{d})}$ is empty for $\underline{d} \notin \sigma_{\mathfrak{C}}$, one needs to take care which maximal chains to consider when computing the leading term G_R of the Hilbert polynomial via Proposition 2.35.

The coefficients of G_R contain the multidegrees of X. One can compute them explicitly in the case when the poset $\mathcal{I}$ is totally ordered. W. l. o. g. we can then rearrange the numbering of the projective spaces $\mathbb{P}(V_i)$ such that the following situation applies.

Corollary 2.43. *Suppose that the poset $\mathcal{I}$ consists only of the sets $[i]$ for $i \in [m]$. Then the multidegree of the variety $X \subseteq \prod_{i=1}^{m} \mathbb{P}(V_i)$ to a tuple $\underline{k} \in \mathbb{N}_0^m$ with $k_1 + \cdots + k_m = \dim X$ is given by*

$$\deg_{\underline{k}}(X) = k_1! \cdots k_m! \sum_{\mathfrak{C}} b_{\mathfrak{C}},$$

where the sum runs over all maximal chains $\mathfrak{C}$ in A, which contain exactly $k_i + 1$ elements from $A_i = \{p \in A \mid I_p = [i]\}$ for each $i = 1, \ldots, m$ and $b_{\mathfrak{C}}$ denotes the product of all bonds in $\mathfrak{C}$.

Proof. For any maximal chain $\mathfrak{C}$ in A the matrix $M_{\mathfrak{C}}$ we defined earlier is the identity matrix and $\sigma_{\mathfrak{C}}$ coincides with the positive orthant $\mathbb{R}^m_{\geq 0}$. For all $\underline{d} \in \mathbb{N}_0^m$ we have $\phi_j^{\mathfrak{C}}(\underline{d}) = d_j$ for all $\underline{d} \in \mathbb{N}_0^m$. Using equation (2.13) we therefore get

$$G_R(\underline{d}) = \sum_{\mathfrak{C}} b_{\mathfrak{C}}\, d_1^{|\mathfrak{C}_1|-1} \cdots d_m^{|\mathfrak{C}_m|-1}.$$

This implies the claimed formula, since the coefficient of the monomial $d_1^{k_1} \cdots d_m^{k_m}$ is equal to the multidegree $\deg_{\underline{k}}(X)$ divided by $k_1! \cdots k_m!$. □

3. Multiprojective stratifications on flag varieties in type A

In this chapter we construct a multiprojective Seshadri stratification on every (partial) flag variety G/Q in Dynkin type $\mathtt{A}$. This stratification is normal and balanced and the resulting standard monomial theory (as of Proposition 2.12) is the classical Hodge-Young theory (see [Hod] and [HP]) of products of Plücker coordinates indexed by semistandard Young tableaux. This stratification on G/Q is a special case of the stratification we define in Chapter 4.

Throughout this chapter we fix the simple group $G = \mathrm{SL}_n(\mathbb{K})$ over an algebraically closed field $\mathbb{K}$ of characteristic zero, the torus T of diagonal matrices in G and the Borel subgroup B of all upper triangular matrices with determinant 1, which contains T. The Weyl group $W = N_G(T)/C_G(T)$ can be identified with the symmetric group S_n, since $C_G(T) = T$ and the normalizer of T consists of the matrices which have exactly one non-zero entry in every row and each column. Let $\varepsilon_i : T \to \mathbb{K}^\times$ be the character of T, where $\varepsilon_i(t)$ is equal to the i-th entry on the diagonal of $t \in T$. The root system Φ of G is given by all characters $\varepsilon_i - \varepsilon_j$ for $i \neq j$ in $[n]$ and the choice of the Borel subgroup corresponds to the set Φ^+ of positive roots and the set Δ of the simple roots $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i \in [n-1]$. Let Λ denote the weight lattice of the root system Φ and Λ^+ be the monoid of all dominant weights. To each $i \in [n-1]$ there is the associated fundamental weight $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i \in \Lambda^+$ and the maximal parabolic subgroup $P_i = BW_{P_i}B$, where $W_{P_i} \subseteq W$ is the stabilizer of ω_i. It is generated by the simple reflections s_α for $\alpha \in \Delta \setminus \{\alpha_i\}$.

Every weight $\lambda \in \Lambda$ can be uniquely written in the form $\lambda = c_1\varepsilon_1 + \cdots + c_{n-1}\varepsilon_{n-1}$ with coefficients $c_i \in \mathbb{Z}$. Then λ is a dominant weight, if and only if $c_1 \geq \cdots \geq c_{n-1} \geq 0$. In this way, each dominant weight λ corresponds to a partition $p(\lambda) = (c_1, \ldots, c_{n-1})$ of $n-1$ parts (which are potentially zero). The partition $p(\lambda)$ is usually visualized via a Young diagram (we use the English notation) having exactly c_i boxes in its i-th row. For each $i = 1, \ldots, n-1$ it contains exactly $\langle \lambda, \alpha_i^\vee \rangle$ columns of length i.

Definition 3.1. For each $\lambda \in \Lambda^+$ and its corresponding partition $p(\lambda)$ we define:

(a) The set $\mathrm{YT}(\lambda)$ of all Young tableaux of shape $p(\lambda)$ with entries in $[n]$;

(b) The subset $\mathrm{SSYT}(\lambda) \subseteq \mathrm{YT}(\lambda)$ of all *semistandard* Young tableaux $T \in \mathrm{YT}(\lambda)$, i. e. the entries of T increase weakly along each row (from left to right) and strictly along each column (from top to bottom).

The Grassmann varieties G/P_i for $i = 1, \ldots, n-1$ can be embedded into the projectivized fundamental representation $\mathbb{P}(V(\omega_i)) \cong \mathbb{P}(\bigwedge^i \mathbb{K}^n)$ via the usual Plücker embedding:

$$G/P_i \hookrightarrow \mathbb{P}(\textstyle\bigwedge^i \mathbb{K}^n), \quad gP_i \longmapsto [g \cdot (e_1 \wedge \cdots \wedge e_i)].$$

This representation is minuscule, that is to say the Weyl group acts transitively on the set of its weights. Hence all weight spaces are one-dimensional and the weights are in bijection to the elements of the Bruhat poset $W/W_{P_i} \cong S_n/(S_i \times S_{n-i})$. They correspond to subsets $J \subseteq [n]$ of size i and can also be identified with semistandard Young tableaux in $\mathrm{SSYT}(\omega_i)$. The weight space in $V(\omega_i)$ of weight $\theta \in W/W_{P_i}$ is generated by the vector $e_\theta = e_{j_1} \wedge \cdots \wedge e_{j_i} \in \bigwedge^i \mathbb{K}^n$, where $j_1 < \cdots < j_i$ are the elements of the subset $J \subseteq [n]$ corresponding to θ. The dual basis vectors $p_\theta \in V(\omega_i)^*$ are known as *Plücker coordinates*. It is well known that Plücker coordinates fulfill the conditions (S2) and (S3) on a Seshadri stratification (see [Ses2, pp. 1.2.10, 1.4.11]). In fact, the Grassmann varieties were one of the motivating examples for the development of Seshadri stratifications ([FL]).

Proposition 3.2. *There exists a Seshadri stratification on G/P_i with underlying poset W/W_{P_i}, where the strata X_θ are given by the Schubert varieties in G/P_i and the extremal functions $f_\theta = p_\theta$ by Plücker coordinates.*

We now go over to arbitrary parabolic subgroups. Until the end of this chapter we fix the partial flag variety $X = G/Q$ to a parabolic subgroup

$$Q = P_{k_1} \cap \cdots \cap P_{k_m}$$

with strictly ascending indices $1 \leq k_1 < \cdots < k_m \leq n-1$. Every parabolic subgroup containing B can be uniquely written in this way. Since we work with many parabolic subgroups at the same time, we have included a short chapter about Weyl groups and its parabolic subgroups in the appendix, in which we define the notation we use throughout this thesis, for example the lifting maps $\min_Q$ and $\max_Q$ and the notions of Q-minimal and Q-maximal elements. To reduce the number of indices, however, we write π_i instead of $\pi_{P_{k_i}}$ and W_i instead of $W_{P_{k_i}}$, when working in type A.

Let R denote the multihomogeneous coordinate ring of G/Q with respect to the Plücker embedding

$$G/Q \hookrightarrow \prod_{i=1}^{m} G/P_{k_i} \hookrightarrow \prod_{i=1}^{m} \mathbb{P}(V(\omega_{k_i})). \tag{3.1}$$

It is well known that this ring R contains a lot of information about the representation theory of G. It carries the structure of a G-representation and the graded component $R_{\underline{d}} \subseteq R$ of degree $\underline{d} \in \mathbb{N}_0^m$ is isomorphic to the dual representation $V(\mu)^*$ to the dominant weight $\mu = d_1\omega_{k_1} + \cdots + d_m\omega_{k_m} \in \Lambda^+$.

We view the Plücker coordinates in $V(\omega_{k_i})^*$ as elements of R via the pullback along the projection $\prod_{j=1}^m V(\omega_{k_j}) \twoheadrightarrow V(\omega_{k_i})$. For each element (θ, i) in the disjoint union $\underline{W} = \coprod_{i=1}^m W/W_i \times \{i\}$ we therefore have an associated Plücker coordinate $p_{(\theta,i)} \in R$ and R is clearly generated by these functions as a $\mathbb{K}$-algebra. Hence the monomials/products of Plücker coordinates form a generating system of R as a vector space. Each of these monomials is either called *standard* or *non-standard* and the set of standard monomials is a basis of R. Using only combinatorial methods one can determine whether a monomial is standard. This is typically called a *standard monomial theory*. For G/Q it was shown in [Ses2, Chapter 2] that this basis is given by semistandard Young tableaux in the following sense. Let $p_{\underline{\theta}} = p_{(\theta_1,i_1)} \cdots p_{(\theta_\ell,i_\ell)}$ be a product of Plücker coordinates with $i_1 \geq \cdots \geq i_\ell$. Since each element $\theta_j \in W/W_{i_j}$ can be interpreted as a tableau in $\mathrm{SSYT}(\omega_{i_j})$, the product $p_{\underline{\theta}}$ corresponds to the Young tableau

$$(\theta_1, \ldots, \theta_\ell) \in \mathrm{YT}(\omega_{i_1} + \cdots + \omega_{i_\ell}),$$

such that its j-th column is given by θ_j.

Theorem 3.3 ([Ses2, Proposition 2.3.1, Theorem 2.6.1])**.** *The multihomogeneous coordinate ring $R = \mathbb{K}[G/Q]$ has a basis consisting of the products $p_{(\theta_1,i_1)} \cdots p_{(\theta_\ell,i_\ell)}$ of Plücker coordinates with $i_1 \geq \cdots \geq i_\ell$, such that the corresponding Young tableau $(\theta_1, \ldots, \theta_\ell)$ is semistandard.*

It is our goal to construct a multiprojective stratification on G/Q, such that the associated fan of monoids is in bijection to semistandard tableaux with columns in $\underline{W}$. In order to construct such a stratification, we first need a suitable candidate for the underlying poset. Notice that the entries of every Young tableau, which only contains columns from the set $\underline{W}$, are already strictly increasing along each column, by definition. Therefore semistandardness can be seen as a local property: Such a tableau is semistandard, if and only if every two consecutive columns are semistandard (as a tableau of just 2 columns). This induces a partial order on the set $\underline{W}$.

Definition 3.4. We define a relation $\geq$ on the set $\underline{W} = \coprod_{i=1}^{m} W/W_i \times \{i\}$ via

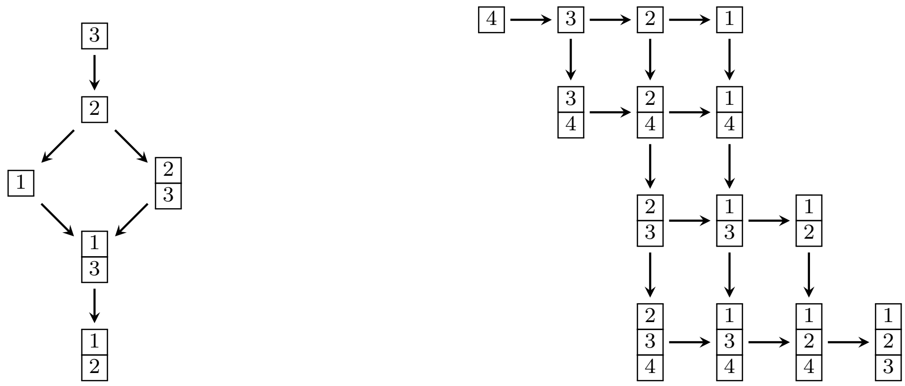
$$(\theta, i) \geq (\phi, j) \quad :\Longleftrightarrow \quad i \leq j \;\text{ and }\; \max_Q(\theta) \geq \min_Q(\phi). \tag{3.2}$$

for all $(\theta, i), (\phi, j) \in \underline{W}$.

With the interpretation of the elements of $\underline{W}$ as Young tableaux, we show in Corollary 3.8 that the relation $\geq$ can be written as

$$\begin{array}{|c|}\hline a_1 \\ \hline \vdots \\ \hline a_{k_i} \\ \hline \end{array} \geq \begin{array}{|c|}\hline b_1 \\ \hline \vdots \\ \hline \vdots \\ \hline b_{k_j} \\ \hline \end{array} \quad \Longleftrightarrow \quad i \leq j \;\text{ and }\; \begin{array}{|c|c|}\hline b_1 & a_1 \\ \hline \vdots & \vdots \\ \hline \vdots & a_{k_i} \\ \hline b_{k_j} \\ \cline{1-1} \end{array} \;\text{ is semistandard.} \tag{3.3}$$

In particular, this implies that the relation $\geq$ is a partial order. However, as we do not show this characterization of the relation right now, we carefully avoid using the transitivity of $\geq$ (the reflexivity and antisymmetry are immediate from the definition).



(a) Type $\mathtt{A}_2$ (b) Type $\mathtt{A}_3$

Figure 1: Hasse-diagrams of $\underline{W}$ for $Q = B$

Using the underlying poset $\underline{W}$ we now define a multiprojective Seshadri stratification of the partial flag variety $X = G/Q$ with respect to the Plücker embedding (3.1). For this purpose we choose the following objects for each element $(\theta, i) \in \underline{W}$:

- The subset $I_{(\theta,i)} = \{i, \dots, m\}$ of $[m]$,
- the Schubert variety $X_{\max_{Q_i}(\theta)} \subseteq G/Q_i = X_{\{i,\dots,m\}}$ as the stratum $X_{(\theta,i)}$, where Q_i is the parabolic subgroup $Q_i = \bigcap_{j=i}^{m} P_{k_j}$,
- the extremal function $f_{(\theta,i)} = p_{(\theta,i)}$.

Note that, if Q is a maximal parabolic subgroup, then G/Q is a Grassmann variety and we already know that these definitions give rise to a Seshadri stratification, namely the stratification from Proposition 3.2 of all Schubert varieties and Plücker coordinates.

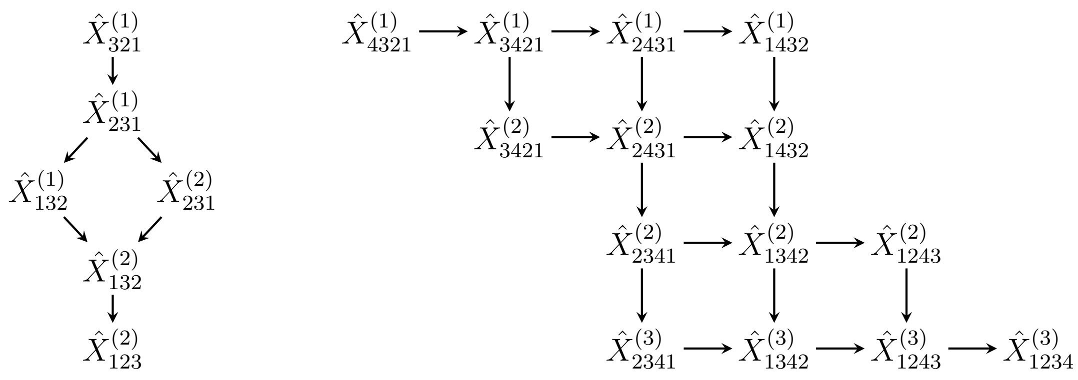
We write $\hat{X}_{(\theta,i)}$ for the multicone of $X_{\max_{Q_i}(\theta)}$, viewed as a subvariety of $\prod_{i=1}^{m} V(\omega_{k_i})$. It coincides with the intersection

$$\hat{X}_{\widetilde{\theta}}^{(i)} \coloneqq \hat{X}_{\widetilde{\theta}} \cap \{(v_1, \dots, v_m) \in \prod_{j=1}^{m} V(\omega_{k_j}) \mid v_1 = \dots = v_{i-1} = 0\},$$

where $\hat{X}_{\widetilde{\theta}}$ is the multicone of the Schubert variety $X_{\widetilde{\theta}} \subseteq G/Q$ to the element $\widetilde{\theta} = \min_Q \circ \max_{Q_i}(\theta)$.

We usually write a permutation $\sigma : [n] \to [n]$ in the Weyl group $W \cong S_n$ in the one-line notation $\sigma = \sigma(1) \cdots \sigma(n)$. Similarly, we write an element $\sigma W_{P_i} \in W/W_{P_i}$ for $i \in [n]$ in the form $\sigma(1) \cdots \sigma(i)$. With this notation the stratifications of G/B in the types $\mathtt{A}_2$ and $\mathtt{A}_3$ are shown in Figure 2.

It is well known, that the Bruhat order on W/W_{P_i} can be characterized via the one-line notation: For a tuple $\underline{j} = (j_1, \dots, j_i)$ of natural numbers, we write $\underline{j}^{\leq}$ for the permuted



(a) Type $\mathtt{A}_2$ (b) Type $\mathtt{A}_3$

Figure 2: Stratifications of G/B

tuple with weakly increasing entries (from left to right). For all $\phi = \phi(1)\cdots\phi(i), \theta = \theta(1)\cdots\theta(i) \in W/W_{P_i}$ we then have

$$\phi \leq \theta \quad \Longleftrightarrow \quad (\phi(1),\dots,\phi(i))^{\leq} \leq (\theta(1),\dots,\theta(i))^{\leq},$$

where the tuples on the right hand side are compared component-wise. The one-line notation can also be used to describe the Bruhat order on W, as $\sigma \leq \tau \in W$ is equivalent to $\pi_{P_i}(\sigma) \leq \pi_{P_i}(\tau)$ for all $i = 1,\dots,n-1$.

Theorem 3.5. *The varieties $X_{(\theta,i)}$ for $(\theta,i) \in \underline{W}$ together with the extremal functions $f_{(\theta,i)}$ form a Seshadri stratification on $X = G/Q \hookrightarrow \prod_{j=1}^{m} \mathbb{P}(V(\omega_{k_j}))$.*

Before we are able to prove this theorem, we need to show, that $\geq$ is indeed a partial order on $\underline{W}$ and establish a good understanding of this poset and its covering relations. The key ingredient is the following innocent looking lemma. Although it can be shown more abstractly, we stick to a prove using methods of type $\mathtt{A}$ for simplicity. A more general statement can be found in [LMS4, Lemma 12.4].

Lemma 3.6. *If $\theta \in W/W_{Q_i}$ is P_{k_i}-maximal, then $\pi_{Q_j}(\theta)$ is P_{k_j}-maximal for all $j \geq i$.*

Proof. The P_{k_i}-maximality of θ is equivalent to the P_{k_i}-maximality of its maximal representative $\sigma = \max_B(\theta)$ in W. We write $\sigma = \sigma(1)\cdots\sigma(n)$ and $\tau := \max_B \circ \pi_{Q_j}(\theta) = \tau(1)\cdots\tau(n)$ in one-line notation. The parabolic subgroups $P_{k_1},\dots,P_{k_m}$ partition the set $[n]$ into $m+1$ subsets $I_s = \{k_s+1,\dots,k_{s+1}\}$ for $s = 0,\dots,m$, where we set $k_0 = 0$ and $k_{m+1} = n$. Since $Q_j = \bigcup_{s=j}^{m} P_{k_s}$, the one-line notations of σ and τ agree up to permutation in the blocks $I_0 \cup \cdots \cup I_j$ and in each I_s for $s > j$. As τ is the maximal representative of $\pi_{Q_j}(\theta)$, we have $\tau s_{\alpha_\ell} < \tau$ for all $\ell \in [n] \setminus \{k_j,\dots,k_m\}$, hence the numbers $\tau(r)$ are strictly decreasing in $I_0 \cup \cdots \cup I_j$ and in all I_s for $s > j$. Additionally the P_{k_i}-maximality of σ implies, that the numbers $\sigma(r)$ are strictly decreasing in $I_0 \cup \cdots \cup I_i$

and in $I_{i+1} \cup \cdots \cup I_m$. By combining these observations, we get $\sigma(r) = \tau(r)$ for all $j+1 \leq r \leq n$ and the numbers $\tau(r)$ are strictly decreasing in $I_{j+1} \cup \cdots \cup I_m$. But this just means, that τ is P_{k_j}-maximal. □

Lemma 3.7. *For all $(\theta, i), (\phi, j) \in \underline{W}$ the relation $(\theta, i) \geq (\phi, j)$ holds, if and only if $i \leq j$ and any one of the following equivalent statements is fulfilled:*

(a) $\pi_j \circ \max_{Q_i}(\theta) \geq \phi$;

(b) there exists a parabolic subgroup $Q \subseteq Q' \subseteq P_{k_i} \cap P_{k_j}$ and lifts $\overline{\theta}$, $\overline{\phi} \in W/W_{Q'}$ of θ and ϕ respectively, such that $\overline{\theta} \geq \overline{\phi}$ in $W/W_{Q'}$;

(c) $\min_Q \circ \max_{Q_i}(\theta) \geq \min_Q \circ \max_{Q_j}(\phi)$ *in W/W_Q.*

Proof. By the definition of the partial order on $\underline{W}$, it suffices to show the following: For any two elements $(\theta, i), (\phi, j) \in \underline{W}$ with $i \leq j$ the inequality $\max_Q(\theta) \geq \min_Q(\phi)$ is equivalent to each of the three conditions (a), (b) and (c). We assume the relation $i \leq j$ for the inclusion $Q_j \subseteq Q_i$.

Let Q' be a parabolic subgroup contained in $P_{k_i} \cap P_{k_j}$ containing Q. Clearly condition (b) is equivalent to $\max_Q(\theta) \geq \min_Q(\phi)$ and (b) follows from (c). Furthermore $\max_Q(\theta) \geq \min_Q(\phi)$ implies (a), since $\pi_j \circ \max_{Q_i}(\theta) = \pi_j \circ \max_Q(\theta) \geq \pi_j \circ \min_Q(\phi) = \phi$

It remains to show, that (c) follows from (a). We write $\widetilde{\theta} = \min_Q \circ \max_{Q_i}(\theta)$ and $\widetilde{\phi} = \min_Q \circ \max_{Q_j}(\phi)$. Since both elements are Q_i-minimal, it is enough to prove the inequality $\widetilde{\theta} \geq \widetilde{\phi}$ in W/W_{Q_i}. As the element $\max_{Q_i}(\theta)$ is P_{k_i}-maximal, its projection to W/W_{Q_j} is P_{k_j}-maximal by Lemma 3.6, hence we have the equality $\pi_{Q_j} \circ \max_{Q_i}(\theta) = \max_{Q_j} \circ \pi_j \circ \max_{Q_i}(\theta)$. But this implies

$$\begin{aligned} \max_{Q_i}(\theta) &\geq \min_{Q_i} \circ \pi_{Q_j} \circ \max_{Q_i}(\theta) = \min_{Q_i} \circ \max_{Q_j} \circ \pi_j \circ \max_{Q_i}(\theta) \\ &\geq \min_{Q_i} \circ \max_{Q_j}(\phi), \end{aligned}$$

where we used condition (a) for the last inequality. This completes the proof. □

Corollary 3.8. *The characterization (3.3) of the relation on $\underline{W}$ is fulfilled. In particular, the relation is a partial order.*

Proof. Let $(\theta, i), (\phi, j)$ be two elements in $\underline{W}$ written as tableaux with one column and entries $a_1, \ldots, a_{k_i}$ and $b_1, \ldots, b_{k_j}$ respectively. The first k_i numbers in the one-line notation of $\widetilde{\theta} := \min_B \circ \max_{Q_i}(\theta) = \theta_1 \cdots \theta_n$ are strictly increasing and therefore $\theta_s = a_s$ for all $s = 1, \ldots, k_i$. The last $n - k_i$ numbers are strictly decreasing. The analogous statement holds for the one-line notation of $\widetilde{\phi} := \min_B \circ \max_{Q_j}(\phi) = \phi_1 \cdots \phi_n$.

If $i \leq j$ and $b_s \leq a_s$ holds for all $s = 1, \ldots, k_i$, then we have $b_1 \cdots b_{k_j} = \pi_j(\widetilde{\phi}) \leq \pi_j(\widetilde{\theta}) = a_1 \cdots a_{k_i} \theta_{k_i+1} \cdots \theta_{k_j}$, because the last $k_j - k_i$ numbers are the largest numbers missing in $a_1 \cdots a_{k_i}$. This implies $(\theta, i) \geq (\phi, j)$ by Lemma 3.7 (a). Conversely, if $(\theta, i) \geq (\phi, j)$, then $i \leq j$ and $\widetilde{\theta} \geq \phi$. Hence we have $b_s \leq a_s$ for all $s = 1, \ldots, k_i$, as the first k_i numbers in their one-line notation are increasingly ordered. □

We are now able to fully understand the covering relations of $\underline{W}$. If (θ, i) covers (ϕ, j), then we either have $i = j$ and $\theta > \phi$ is a covering relation in W/W_i or we have $i < j$. In the second case we have $(\theta, i) > (\pi_r \circ \max_Q(\theta), r) > (\phi, j)$ for each $i < r < j$, so it follows $j = i + 1$. Additionally, the lifts $\widetilde{\theta} = \min_Q \circ \max_{Q_i}(\theta)$ and $\widetilde{\phi} = \min_Q \circ \max_{Q_j}(\phi)$ agree, which we can show in the quotient $W/W_{Q_i} = W/W_{P_{k_i}} \cap W/W_{Q_j}$ by using Lemma B.1. Since $\widetilde{\theta} \geq \widetilde{\phi}$ holds by the previous lemma, we have

$$\begin{aligned}(\theta, i) &= (\pi_i(\widetilde{\theta}), i) \geq (\pi_i(\widetilde{\phi}), i) > (\phi, j) \quad \text{and} \\ (\theta, i) &> (\pi_j(\widetilde{\theta}), j) \geq (\pi_j(\widetilde{\phi}), j) = (\phi, j),\end{aligned}$$

hence $\widetilde{\theta}$ and $\widetilde{\phi}$ are equal in $W/W_{P_{k_i}}$ and in $W/W_{P_{k_j}}$. This yields

$$\pi_{Q_j}(\widetilde{\phi}) = \max_{Q_j}(\phi) = \max_{Q_j} \circ \pi_j(\widetilde{\theta}) = \max_{Q_j} \circ \pi_{P_{k_j}} \circ \max_{Q_i}(\theta).$$

But by Lemma 3.6 the element $\pi_{Q_j} \circ \max_{Q_i}(\theta)$ is P_{k_j}-maximal, so the right hand side is equal to $\pi_{Q_j} \circ \max_{Q_i}(\theta) = \pi_{Q_j}(\widetilde{\theta})$. Therefore $\widetilde{\theta} = \widetilde{\phi}$.

In particular, if $(\theta, i) > (\phi, j)$ is covering relation in $\underline{W}$, then $\hat{X}_{(\phi,j)}$ is of codimension one in $\hat{X}_{(\theta,i)}$. Therefore the condition (S1) on a Seshadri stratification is fulfilled and the relation $(\theta, i) \geq (\phi, j)$ implies $\hat{X}_{(\phi,j)} \subseteq \hat{X}_{(\theta,i)}$. Conversely if $\hat{X}_{(\phi,j)} \subseteq \hat{X}_{(\theta,i)}$, then the Schubert variety $X_{\max_{Q_j}(\phi)} \subseteq G/Q_j$ is contained in $X_{\pi_{Q_j} \circ \max_{Q_i}(\theta)}$. Hence $\max_{Q_j}(\phi) \leq \pi_{Q_j} \circ \max_{Q_i}(\theta)$, which implies $(\phi, j) \leq (\theta, i)$ by Lemma 3.7 (a).

Lemma 3.9. *Let* $(\theta, i) \in \underline{W}$ *and* $\widetilde{\theta} = \min_Q \circ \max_{Q_i}(\theta)$. *Then the following equality holds for all* $i < j \leq m$*:*

$$\{(v_1, \ldots, v_m) \in \hat{X}_{(\theta,i)} \mid v_i = \cdots = v_{j-1} = 0\} = \hat{X}_{(\pi_j(\widetilde{\theta}), j)}.$$

Proof. Let $v = (v_1, \ldots, v_m) \in \hat{X}_{(\theta,i)}$ with $v_i = \cdots = v_{j-1} = 0$. We choose a non-zero vector $w_r \in V(\omega_{k_r})$ for all $r = i, \ldots, m$ with $v_r \in \mathbb{K} w_r$, such that $([w_i], \ldots, [w_m])$ is an element of the Schubert variety

$$X_{(\theta,i)} = X_{\max_{Q_i}(\theta)} \subseteq \prod_{r=i}^{m} \mathbb{P}(V(\omega_{k_r})).$$

Because of the following commutative diagram, $([w_j], \ldots, [w_m])$ lies in the Schubert variety to the element $\pi_{Q_j} \circ \max_{Q_i}(\theta) \in W/W_{Q_j}$:

$$\begin{array}{ccc} X_{\pi_{Q_i}(\theta)} & \hookrightarrow & \prod_{r=i}^{m} \mathbb{P}(V(\omega_{k_r})) \\ \downarrow & & \downarrow \\ X_{\pi_{Q_j}(\theta)} & \hookrightarrow & \prod_{r=j}^{m} \mathbb{P}(V(\omega_{k_r})) \end{array}$$

But by Lemma 3.7 we have $\pi_{Q_j} \circ \max_{Q_i}(\theta) = \max_{Q_j} \circ \pi_j \circ \max_{Q_i}(\theta) = \max_{Q_j} \circ \pi_j(\widetilde{\theta})$, so v is contained in the multicone $\hat{X}_{(\pi_j(\widetilde{\theta}),j)}$.

Conversely, every element of the multicone $\hat{X}_{(\pi_j(\widetilde{\theta}),j)}$ lies in the stratum $\hat{X}_{(\theta,i)}$ because of the surjectivity of the map $X_{\max_{Q_i}(\theta)} \twoheadrightarrow X_{\pi_{Q_j} \circ \max_{Q_i}(\theta)} = X_{\pi_{Q_j}(\widetilde{\theta})}$. □

Proof of Theorem 3.5. It is well known, that Schubert-varieties are smooth in codimension one (see e. g. [CFL2, Corollary 3.5]). Their multicones $\hat{X}_{(\theta,i)} \subseteq \prod_{j=1}^m V(\omega_{k_j})$ are closed, irreducible subvarieties and they are smooth in codimension one as well by Corollary A.11.

We already proved condition (S1) and the equivalence of $(\phi, j) \leq (\theta, i)$ and the inclusion $\hat{X}_{(\phi,j)} \subseteq \hat{X}_{(\theta,i)}$ of their multicones. Next, we show (S2). Let $(\phi, j) \not\leq (\theta, i)$ in $\underline{W}$. We need to prove, that the Plücker coordinate $p_{(\phi,j)}$ vanishes identically on $\hat{X}_{(\theta,i)}$. This is trivial, if $j < i$. Now we assume $j \geq i$ and set $\kappa = \pi_j \circ \max_{Q_i}(\theta) \in W/W_j$. Note that the affine cone $\hat{X}_\kappa \subseteq V(\omega_{k_j})$ of the Schubert variety $X_\kappa \subseteq G/P_{k_j}$ coincides with the projection of $\hat{X}_{(\theta,i)}$ to $V(\omega_{k_j})$. Now Schubert varieties and Plücker coordinates form a Seshadri stratification on G/P_{k_j} by Proposition 3.2, hence (S2) is fulfilled in this case. Therefore the function $p_{(\phi,i)}$ vanishes on $\hat{X}_{(\theta,i)}$, if and only if $\phi \not\leq \kappa = \pi_j \circ \max_{Q_i}(\theta)$. By Lemma 3.7 (a) this is equivalent to $(\phi, j) \not\leq (\theta, i)$.

Lastly, we prove (S3). We fix an element $(\theta, i) \in \underline{W}$. The function $p_{(\theta,i)}$ vanishes on all multicones $\hat{X}_{(\phi,j)}$ for $(\phi, j) < (\theta, i)$. This is clearly true for $j > i$, otherwise it follows from (S3) for the stratification on G/P_{k_i}.

Conversely, let $v = (v_1, \dots, v_m) \in \hat{X}_{(\theta,i)}$ such that $p_{(\theta,i)}(v) = 0$. In the case of $v_i = 0$, the element v is contained in the multicone $\hat{X}_{(\phi,j)}$ for $j = i + 1$ and $\phi = \pi_j \circ \max_{Q_i}(\theta)$ by Lemma 3.9 and we have $(\phi, j) < (\theta, i)$. For $v_i \neq 0$, its projective class $[v_i]$ can be viewed as an element of the Schubert variety $X_\theta \subseteq G/P_{k_i}$. Again, using the Seshadri stratification on G/P_{k_i} we see that $[v_i]$ is contained in the Schubert variety X_ϕ to an element $\phi < \theta$ in W/W_i. For each $r = i, \dots, m$ we choose a non-zero vector $w_r \in V(\omega_{k_r})$ with $v_r \in \mathbb{K}w_r$ and $w := ([w_i], \dots, [w_m]) \in X_{(\theta,i)}$. Then w lies in a Schubert cell $C_\sigma \subseteq G/Q_i$ for a unique element $\sigma \in W/W_{Q_i}$. It satisfies $\pi_i(\sigma) \leq \phi < \theta$, so $\sigma \leq \max_{Q_i} \circ \pi_i(\sigma) < \max_{Q_i}(\theta)$. Hence v is contained in $\hat{X}_{(\pi_i(\sigma),i)}$, which completes the proof. □

Remark 3.10. For a fixed index $i \in [m]$ the set of lifts

$$\{\min_Q \circ \max_{Q_i}(\theta) \in W/W_Q \mid (\theta, i) \in \underline{W}\}$$

coincides with the set of all elements in W/W_Q, which are Q_i-minimal and Q^i-maximal for the parabolic subgroup $Q^i = \bigcap_{j=1}^i P_{k_i}$. We skip the proof of this statement, as it is rather lengthy and we show it in a more general setting anyway in Section 4.4.

Let SSYT_Q be the set of all semistandard Young tableaux with entries in $[n]$, where only columns of length $k_1, \dots, k_m$ may appear. Equivalently, this is the union of the sets $\mathrm{YT}(\mu)$ over all $\mu \in \mathbb{N}_0\omega_{k_1} + \dots + \mathbb{N}_0\omega_{k_m}$. A Young tableau is contained in this union, if and only if all columns come from elements in $\underline{W}$.

Corollary 3.11.

(a) The following map is a bijection:

$$\mathrm{SSYT}_Q \to \Gamma, \quad ((\theta_1, i_1), \ldots, (\theta_\ell, i_\ell)) \mapsto e_{(\theta_1,i_1)} + \cdots + e_{(\theta_\ell,i_\ell)}.$$

(b) The Seshadri stratification on G/Q is normal and balanced.

(c) The set $\mathbb{G}$ of all indecomposable elements in Γ coincides with the set

$$\Gamma(1) = \{\underline{a} \in \Gamma \mid |\deg \underline{a}| = 1\} = \{e_{(\theta,i)} \mid (\theta, i) \in \underline{W}\}$$

of all elements of total degree 1 in Γ.

(d) Let $\mathbb{G}_R$ be the set of all Plücker coordinates $p_{(\theta,i)}$ for $(\theta, i) \in \underline{W}$. Then the standard monomial basis from Proposition 2.12 agrees with the basis from Theorem 3.3.

Proof. (a) For a tableau $T \in \mathrm{SSYT}_Q$ with columns $(\theta_1, i_1), \ldots, (\theta_\ell, i_\ell)$ consider the regular function $f_T = p_{(\theta_1,i_1)} \cdots p_{(\theta_\ell,i_\ell)}$. Since T is semistandard, it follows from the equivalence (3.3) that there exists a maximal chain in $\underline{W}$ containing all elements $(\theta_1, i_1), \ldots, (\theta_\ell, i_\ell)$. Hence f_T has the quasi-valuation $\mathcal{V}(f_T) = e_{(\theta_1,i_1)} + \cdots + e_{(\theta_\ell,i_\ell)}$, so the map $\mathrm{SSYT}_Q \to \Gamma$ is well-defined. The injectivity is already contained in the definition of this map, since one can reconstruct the semistandard tableau from the coefficients of the vectors e_p, $p \in \underline{W}$. We also know from Theorem 3.3 that the functions f_T for $T \in \mathrm{SSYT}_Q$ form a basis of R. Therefore the map $\mathrm{SSYT}_Q \to \Gamma$ is surjective as well.

(b) The quasi-valuation of extremal functions does not depend on the choice of the total order $\geq^t$ on the poset $\underline{W}$ and every element in Γ is the quasi-valuation of a product of extremal functions in a common maximal chain. Hence the stratification is balanced. By part (a), the monoid $\Gamma_{\mathfrak{C}}$ to a maximal chain $\mathfrak{C}$ in $\underline{W}$ coincides with $\mathbb{N}_0^{\mathfrak{C}}$, which clearly is saturated. So the stratification is also normal.

(c) This statement is a consequence of part (a).

(d) Let $\underline{a} \in \Gamma$ and T be its corresponding tableau in SSYT_Q. Then part (d) follow from the fact, that the unique decomposition $\underline{a} = \underline{a}^1 + \ldots \underline{a}^s$ into indecomposables with $\min \operatorname{supp} \underline{a}^k \geq \max \operatorname{supp} \underline{a}^{k+1}$ for all $k = 1, \ldots, s-1$ is given by the columns of T, where $\underline{a}^1$ corresponds to the rightmost and $\underline{a}^s$ to the leftmost column. □

We show in Lemma 4.32 that the bond to a covering relation $(\theta, i) > (\phi, j)$ in $\underline{W}$ is equal to 1 for $i \neq j$, otherwise it is given by $b = |\langle \phi(\omega_{k_i}), \beta^\vee \rangle|$, where β is the unique positive root with $s_\beta \cdot \min_B(\phi) = \min_B(\theta)$. We have $b \leq 1$, since all fundamental weights are minuscule in type $\mathtt{A}$. Hence the stratification is of Hodge type.

It is not possible to get a multiprojective Seshadri stratification for Schubert varieties in the exact same manner, since Plücker coordinates have the wrong vanishing sets. As an example, let us take the Schubert variety $X_\tau \subseteq \mathrm{SL}_3(\mathbb{K})/B$ for $\tau = 312$. The Plücker coordinate $p_{(3,1)}$ on the multicone $\hat{X}_{312} \subseteq V(\omega_1) \times V(\omega_2)$ vanishes on the two subvarieties $\hat{X}_{213}$ and $\hat{X}_{132}$, which are both of codimension one. Therefore 213 and 132 should be covered by 312 in the underlying poset of the stratification. Analogously, $p_{(2,1)}$ vanishes on $\hat{X}_{123} \subseteq \hat{X}_{213}$, so 213 covers 123. But both 123 and 132 should have the same associated extremal function $p_{(1,1)}$, which is impossible due to condition (S2) on a Seshadri stratification.

4. Multiprojective stratifications on Schubert varieties

4.1. Choices and definitions

We fix a connected, simply-connected, simple algebraic group G over an algebraically closed field $\mathbb{K}$ of characteristic zero as well as a maximal torus $T \subseteq G$ and a Borel subgroup $B \subseteq G$ containing T. Let Δ be the set of all simple roots corresponding to the choice of B. The associated weight lattice shall be denoted by Λ and the monoid of dominant weights by Λ^+. Let W be the Weyl group and $W_\lambda \subseteq W$ be the stabilizer of a weight $\lambda \in \Lambda$.

Let X_τ be the Schubert variety to a Weyl group coset $\tau \in W/W_Q$, where $Q \subseteq G$ is a parabolic subgroup containing B. The flag variety G/Q can be embedded into a projective space by choosing a dominant weight λ, such that $\langle \lambda, \alpha^\vee \rangle = 0$, if and only if the simple reflection s_α is contained in W_Q. Equivalently, the stabilizer of $W_\lambda \subseteq W$ coincides with the subgroup W_Q. Let v_λ be any highest weight vector in the irreducible representation $V(\lambda)$ of G. Then the parabolic subgroup Q is the stabilizer of the highest weight space $\mathbb{K}v_\lambda$ and one obtains a closed embedding

$$G/Q \hookrightarrow \mathbb{P}(V(\lambda)), \quad gQ \mapsto [g \cdot v_\lambda].$$

For each element $\sigma \in W/W_Q$ the weight space in $V(\lambda)$ of weight $\sigma(\lambda)$ is one-dimensional. Up to a non-zero scalar, one can therefore associate a unique weight vector $v_{\sigma(\lambda)} \in V(\lambda)$ of weight $\sigma(\lambda)$. The linear span of the orbit $B \cdot v_{\tau(\lambda)}$ is known as the *Demazure module* associated to λ and τ, which we denote by $V(\lambda)_\tau$. As the Schubert variety X_τ can be written as the closure of the B-orbit $B \cdot [v_{\tau(\lambda)}] \subseteq \mathbb{P}(V(\lambda))$, one can embed X_τ as a closed subvariety of $\mathbb{P}(V(\lambda)_\tau)$.

It was shown by Chirivì, Fang and Littelmann in [CFL2], that $X_\tau \subseteq \mathbb{P}(V(\lambda)_\tau)$ admits a Seshadri stratification via its Schubert subvarieties and representation-theoretically defined extremal functions. The underlying poset $A = \{\sigma \in W/W_Q \mid \sigma \leq \tau\}$ is induced by the Weyl group and the stratum to $\sigma \in A$ is the Schubert variety $X_\sigma \subseteq G/Q$ associated to σ. The extremal functions are given by extremal weight vectors in the dual representation $V(\lambda)^*$: If one chooses a weight vector $\ell_\sigma \in V(\lambda)^*$ of weight $-\sigma(\lambda)$ for each $\sigma \in A$ (which are unique up to a non-zero scalar), then the extremal function f_σ is defined as the restriction of ℓ_σ to $X_\tau \subseteq \mathbb{P}(V(\lambda)_\tau)$. It was proved in [CFL2] that this data forms a normal and balanced Seshadri stratification of LS-type and that one can interpret the elements of degree $d \in \mathbb{N}_0$ in the associated fan of monoids Γ via the Littelmann path model $\mathbb{B}(d\lambda)$ of Lakshmibai-Seshadri-paths (LS-paths) of shape $d\lambda$. The path model was originally introduced by Littelmann in [Lit94] and then further developed in [Lit96] and [Lit95]. We can also recommend the appendix of [CFL2] as an introduction to LS-paths, which is adapted to the language of Seshadri stratifications.

The stratification on X_τ of course depends on the choice of the dominant weight λ. However, one can also consider a decomposition $\lambda = \lambda_1 + \cdots + \lambda_m$ into a sum of dominant weights, as this gives rise to the closed embedding

$$G/Q \hookrightarrow \prod_{i=1}^{m} \mathbb{P}(V(\lambda_i)), \quad gQ \mapsto ([g \cdot v_{\lambda_1}], \ldots, [g \cdot v_{\lambda_m}]), \tag{4.1}$$

where v_{λ_i} is a highest weight vector in $V(\lambda_i)$. The most well-known example is the Plücker embedding of a partial flag variety in type $\mathtt{A}$ into a product of fundamental representations, which we covered in the previous chapter. In view of the connection of the stratification on $X_\tau \subseteq \mathbb{P}(V(\lambda)_\tau)$ to LS-paths, one can hope that there also exists a multiprojective stratification such that its fan of monoids is determined by LS-paths to the weights $\lambda_1, \ldots, \lambda_m$. Unfortunately, such a stratification does not always exist, as it requires a totally ordered index poset $\mathcal{I}$. We discuss the obstacles in Section 4.3.

In order to generalize both the stratification on G/Q from Chapter 3 and the stratification on $X_\tau \subseteq \mathbb{P}(V(\lambda)_\tau)$ to multiprojectively embedded Schubert varieties in arbitrary Dynkin types we have to consider other, possibly non totally-ordered index posets $\mathcal{I}$. Therefore we choose the following objects for our construction:

- A dominant weight $\lambda \in \Lambda^+$ and a sequence $\underline{\lambda} = (\lambda_1, \ldots, \lambda_m)$ of dominant weights, that sum up to λ,
- a Schubert variety $X_\tau \subseteq G/Q$ for an element $\tau \in W/W_Q$, where $Q = BW_\lambda B$ is the parabolic subgroup associated to the stabilizer $W_\lambda \subseteq W$,
- and a subposet $\mathcal{I}$ of the power set poset $\mathcal{P}(\{1, \ldots, m\}) \setminus \{\varnothing\}$, such that $\mathcal{I}$ is a graded poset of length $m-1$ and it holds

$$\underline{J} \subseteq I \ \Rightarrow \ J \subseteq I \quad \forall J, I \in \mathcal{I}. \tag{4.2}$$

  The subset $\underline{J} \subseteq J$ is defined as in Section 2.6: If J is minimal in $\mathcal{I}$, then $\underline{J} := J$, otherwise $\underline{J}$ is the union of all $J \setminus K$, where $K \subsetneq J$ is a covering relation in $\mathcal{I}$.

The combinatorial requirement (4.2) on the poset $\mathcal{I}$ is necessary for the condition (S2) on a Seshadri stratification (see the proof of Theorem 4.30). We want to remark that there are two important cases, where this requirement is automatically satisfied, namely when $\mathcal{I}$ is totally ordered or equal to the full poset $\mathcal{P}(\{1, \ldots, m\}) \setminus \{\varnothing\}$.

Regarding Weyl groups, we use the notation in Appendix B, namely the projection maps π_Q and the lifting maps $\min_Q$ and $\max_Q$. For every $i \in [m]$ we define the parabolic subgroup

$$P_i = BW_{\lambda_i}B$$

and the projection $\tau_i = \pi_{P_i}(\tau)$ of τ to W/W_{P_i}. As the Schubert variety X_τ is the closure of the B-orbit through $w_\tau Q \in G/Q$ for a representative $w_\tau \in N_G(T)$ of τ, the map (4.1) induces an embedding of X_τ into a product of projective spaces over Demazure modules:

$$X_\tau \hookrightarrow \prod_{i=1}^{m} X_{\tau_i} \hookrightarrow \prod_{i=1}^{m} \mathbb{P}(V(\lambda_i)_{\tau_i}).$$

Here X_{τ_i} denotes the Schubert variety in G/P_i to the element $\tau_i \in W/W_{P_i}$. This is the embedding we use for the multiprojective stratification on X_τ. To obtain the construction from Chapter 3, one needs to choose the sequence $\underline{\lambda} = (\omega_{k_m}, \dots, \omega_{k_1})$ of dominant weights and the index poset $\mathcal{I} = \{[i] \mid i \in [m]\}$.

We need to fix some notation for the following chapters. For a tuple $\underline{d} \in \mathbb{N}_0^m$ we define

$$\underline{d} \cdot \underline{\lambda} := d_1\lambda_1 + \dots + d_m\lambda_m \in \Lambda^+.$$

To each index set $I \in \mathcal{I}$ we associate

- the degree $e_I = \sum_{i \in \underline{I}} e_i \in \mathbb{N}_0^m$,
- the dominant weight $\lambda_I = e_I \cdot \underline{\lambda} \in \Lambda^+$
- and the parabolic subgroup $P_I = BW_{\lambda_I}B = \cap_{i \in \underline{I}} P_i$.

It may not be intuitive to index these objects by I instead of $\underline{I}$, but helps to simplify the notation. The parabolics P_I take the role of the maximal parabolic subgroups $P_{k_1}, \dots, P_{k_m}$ from the construction in type $\mathtt{A}$ and the tuple e_I is the multidegree of all the extremal functions for strata associated to the index set I.

Let Q_τ be the unique parabolic subgroup containing Q, that is maximal with the property that τ is Q_τ-maximal. This parabolic subgroup exists: The element τ is Q'-maximal for a parabolic subgroup Q', if and only if $\ell(\tau s) < \ell(\tau)$ holds for all simple reflections $s \in W_{Q'}$ (see Corollary 2.4.5 in [BB]). Therefore Q_τ is equal to the subgroup which is generated by all parabolic subgroups Q' containing Q, such that τ is Q'-maximal.

We can now define the parabolic subgroups

$$Q_I = \bigcap_{\substack{J \in \mathcal{I} \\ J \subseteq I}} P_J \quad \text{and} \quad Q^I = Q_\tau \cap \bigcap_{\substack{J \in \mathcal{I} \\ J \supseteq I}} P_J, \tag{4.3}$$

which generalize the subgroups from Remark 3.10.

Lemma 4.1. *The following properties hold for all $I \in \mathcal{I}$:*

(a) $Q_I = \cap_{i \in I} P_i$ and W_{Q_I} is the stabilizer of $\sum_{i \in I} \lambda_i$;

(b) if $J \subsetneq I$ is a covering relation in $\mathcal{I}$, then $P_I \cap Q_J = Q_I$;

(c) $Q_I \cap Q^I = Q_\tau$.

Proof. (a) The definition of the index sets $\underline{J}$ for $J \in \mathcal{I}$ implies

$$\bigcup_{\substack{J\in\mathcal{I}\\ J\subseteq I}} \underline{J} = I.$$

In particular, we have $Q_I = \cap_{i\in I} P_i$. The Weyl subgroup W_{Q_I} thus is the intersection of the stabilizers W_{λ_i} over all $i \in I$, which is equal to the stabilizer of $\sum_{i\in I}\lambda_i$.

(b) If $J \subsetneq I$ is a covering relation, the Weyl subgroup $W_{P_I\cap Q_J}$ is the stabilizer of $\sum_{i\in \underline{I}\cup J}\lambda_i = \sum_{i\in I}\lambda_i$ and it therefore coincides with W_{Q_I}. Hence $P_I \cap Q_J = Q_I$.

(c) The equality $W_{Q_I\cap Q^I} = W_\tau$ follows from the fact that the subgroup $W_{Q_I\cap Q^I} \subseteq W$ is the intersection of W_{Q_τ} with the stabilizers W_{λ_i} over all indices in the set

$$\bigcup_{J\subseteq I} \underline{J} \;\cup\; \bigcup_{J\supseteq I} \underline{J} = [m]. \qquad \square$$

4.2. Lakshmibai-Seshadri-tableaux

To generalize the stratification from Chapter 3 we first need a suitable candidate for the underlying poset. It should again be motivated by a combinatorial model which parametrizes basis of Demazure modules. Such a model was developed by Lakshmibai, Musili and Seshadri ([LMS4], [LS5], [Ses2]) via certain sequences of Weyl group cosets, that admit a so called *defining chain*. A few years later, Littelmann generalized their tableaux to arbitrary Dynkin types using his path model of LS-paths (see [Lit96]). However, we use a slightly different notation than in *loc. cit.*: Instead of concatenations we consider tuples of LS-paths, and we call them LS-tableaux instead of LS-monomials.

Recall that an *LS-path* π of shape $\nu \in \Lambda^+$ is an element

$$\pi = (\sigma_p > \cdots > \sigma_1; 0, d_p, \ldots, d_1 = 1),$$

where $\sigma_p > \cdots > \sigma_1$ is a chain in W/W_ν and $0 < d_p < \cdots < d_1 = 1$ is a sequence of rational numbers, such that there exists a (d_i, ν)-chain in W/W_ν from σ_i to σ_{i-1} for each $i = 2, \ldots, p$. By definition, this is a chain $\sigma_i = \kappa_t > \cdots > \kappa_0 = \sigma_{i-1}$ of covering relations in W/W_ν with the following integrality property: For every $j = 1, \ldots, t$ the number $d_i\langle\kappa_j(\nu), \beta_j^\vee\rangle$ is an integer, where β_j is the unique positive root of G with $s_{\beta_j}\min_B(\kappa_{j-1}) = \min_B(\kappa_j)$. The Weyl group coset σ_p is called the *initial direction* of π and is denoted by $i(\pi)$.

The set $\mathbb{B}(\nu)$ of all LS-paths of shape ν can be interpreted in terms of the Littelmann path model (see [Lit94] or [CFL2, Appendix A]). The corresponding path model $\mathbb{B}(\pi_\nu)$ is generated by the straight-line path $\pi_\nu : [0,1] \to \Lambda \otimes_{\mathbb{Z}} R$, $t \mapsto t\nu$.

We fix a sequence $\underline{\mu} = (\mu_1, \ldots, \mu_s)$ of dominant weights with sum $\mu = \mu_1 + \cdots + \mu_s$.

Definition 4.2. A **Lakshmibai-Seshadri-tableau** (short: LS-tableau) of shape $\underline{\mu}$ is a sequence $\underline{\pi} = (\pi_1, \ldots, \pi_s)$ of LS-paths $\pi_i \in \mathbb{B}(\mu_i)$, called the **columns** of $\underline{\pi}$. Let $\sigma^{(k)}_{p_k} > \cdots > \sigma^{(k)}_1$ be the chain of cosets in W/W_{μ_k} for the LS-path π_k, $k \in [s]$. For a fixed element $\tau \in W/W_\mu$ the LS-tableau $\underline{\pi}$ is called

(a) $\boldsymbol{\tau}$**-standard**, if there exists a weakly decreasing sequence

$$\overline{\sigma}^{(1)}_{p_1} \geq \cdots \geq \overline{\sigma}^{(1)}_1 \geq \cdots \geq \overline{\sigma}^{(s)}_{p_k} \geq \cdots \geq \overline{\sigma}^{(s)}_1$$

in W/W_μ, such that $\overline{\sigma}^{(i)}_j W_{\mu_i} = \sigma^{(i)}_j \in W/W_{\mu_i}$ holds for all $i = 1, \ldots, s$ and $j = 1, \ldots, p_i$. Such a sequence is called a **defining chain**.

(b) **weakly** $\boldsymbol{\tau}$**-standard**, if the LS-tableau (π_k, π_{k+1}) of shape (μ_k, μ_{k+1}) is τ-standard for each $k = 1, \ldots, s-1$.

Note that defining chains are not unique, there can exist different defining chains for a given LS-tableau. As long as there is at least one defining chain, the tableau is τ-standard. For every parabolic subgroup Q' of G with $W_{Q'} \subseteq W_\mu$, defining chains can also be lifted via the maps $\min_{Q'}$ and $\max_{Q'}$ to weakly decreasing sequences in $W/W_{Q'}$ consisting of lifts of the columns. As the defining chain in W/W_μ is bounded by τ, its lifts to $W/W_{Q'}$ are bounded by $\max_{Q'}(\tau)$. Conversely, assume we have a weakly decreasing sequence in $W/W_{Q'}$ consisting of lifts of the columns and bounded by $\max_{Q'}(\tau)$. Such a chain clearly projects to a defining chain in W/W_μ via $W/W_{Q'} \twoheadrightarrow W/W_\mu$. Hence W_μ is the largest subgroup of W, where a defining chain is well-defined, as $W_\mu = W_{\mu_1} \cap \cdots \cap W_{\mu_s}$.

When τ is equal to the unique maximal element $w_0 W_\mu \in W/W_\mu$, we often omit τ and just talk about (weakly) standard LS-tableaux.

Example 4.3. Consider the group $G = \mathrm{SL}_4(\mathbb{K})$ and let P_i and ω_i be defined as in Chapter 3. We also use the one-line notation from this chapter for elements of W/W_{P_i} and of W. As all fundamental representations in type $\mathtt{A}$ are minuscule, LS-paths of shape ω_i correspond to Weyl group cosets in W/W_{P_i}. The tuple $\underline{\pi} = (13, 124, 3)$ is an LS-tableau of shape $(\omega_2, \omega_3, \omega_1)$. The stabilizer of $\mu = \omega_2 + \omega_3 + \omega_1$ is trivial, hence $W/W_\mu \cong W$. This tableau $\underline{\pi}$ is not standard: The element 3124 is the unique minimal lift of $3 \in W/W_{P_1}$. By Deodhar's Lemma B.3 we have unique minimal lift of 124 that is greater or equal to 3124, namely 4123. But the unique maximal lift 3142 of 13 is not greater or equal to 4123, hence there exists no defining chain for $\underline{\pi}$. However, $\underline{\pi}$ is weakly standard, since the tableaux $(13, 124)$ and $(124, 3)$ have the defining chains $1324 \geq 1243$ and $4123 \geq 3124$ respectively.

Let $\pi_{\mu_i} : [0,1] \to \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, $t \mapsto t\mu_i$ be the straight-line path to μ_i and $\mathbb{B}(\pi_{\mu_1} * \cdots * \pi_{\mu_s})$ be the path model induced by the concatenation $\pi = \pi_{\mu_1} * \cdots * \pi_{\mu_s}$, i. e. it is the smallest set of piecewise linear paths which contains π and is stable under the root operators. This path model is the connected component of the concatenation $\mathbb{B}(\pi_{\mu_1}) * \cdots * \mathbb{B}(\pi_{\mu_s})$

of the LS-path models. As the path $\pi = \pi_{\mu_1} * \cdots * \pi_{\mu_s}$ and the straight-line path $\pi_\mu : [0,1] \to \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, $t \mapsto t\mu$ both have the same end point $\pi(1) = \mu = \pi_\mu(1)$ and their images stay in the dominant Weyl chamber, there exists a unique isomorphism of crystal graphs $\phi : \mathbb{B}(\pi_{\mu_1} * \cdots * \pi_{\mu_s}) \to \mathbb{B}(\pi_\mu)$ with $\phi(\pi) = \pi_\mu$ (see [Lit95]). Using this isomorphism, Littelmann proved the following connection between τ-standard LS-tableaux and path models.

Proposition 4.4 ([Lit96, Theorems 7, 8])**.** *An LS-tableau $\underline{\pi} = (\pi_1, \ldots, \pi_s)$ of shape $\underline{\mu}$ is τ-standard, if and only if the path $\pi := \pi_1 * \cdots * \pi_s$ is contained in the connected component $\mathbb{B}(\pi_{\mu_1} * \cdots * \pi_{\mu_s}) \subseteq \mathbb{B}(\mu_1) * \cdots * \mathbb{B}(\mu_s)$ and the initial direction $i(\phi(\pi))$ of the LS-path $\phi(\pi) \in \mathbb{B}(\mu)$ is smaller or equal to τ.*

It was also proved in [Lit96] that LS-tableaux give rise to a character formula for the Demazure modules.

Theorem 4.5 ([Lit96, Corollary 4])**.** *Let $\mathbb{B}(\underline{\mu})_\tau$ denote the set of all τ-standard LS-tableaux of shape $\underline{\mu}$. Then the character of the Demazure module $V(\mu)_\tau$ is given by*

$$\operatorname{char} V(\mu)_\tau = \sum_{\underline{\pi} \in \mathbb{B}(\underline{\mu})_\tau} e^{\underline{\pi}(1)}, \tag{4.4}$$

where $\underline{\pi}(1)$ denotes the end point $(\pi_1 * \cdots * \pi_s)(1)$ of the concatenation of all paths in the LS-tableau $\underline{\pi} = (\pi_1, \ldots, \pi_s)$.

In the Appendix C we explain how LS-tableaux can be seen as a generalization of classical Young tableaux and of the Young diagrams of admissable pairs, which were defined by Lakshmibai, Musili and Seshadri (see [LMS4], [LS5]).

We have seen that the tableaux appearing in the fan of monoids to the stratification in Chapter 3 have a specific shape, which is determined by the parabolic subgroups $P_{k_1}, \ldots, P_{k_m}$ and the order $k_m > \cdots > k_1$. For the stratification on X_τ the allowed shapes are be defined by the index poset $\mathcal{I}$.

Definition 4.6. A **LS-tableau of type $(\underline{\lambda}, \mathcal{I})$** is an LS-tableau $\underline{\pi}$ of shape $(\lambda_{I_1}, \ldots, \lambda_{I_s})$, where $I_1 \supseteq \cdots \supseteq I_s$ is a (possibly empty) weakly decreasing sequence in $\mathcal{I}$. We call the tuple $\deg \underline{\pi} = e_{I_1} + \cdots + e_{I_s} \in \mathbb{N}_0^m$ the **degree** of $\underline{\pi}$.

Remark 4.7. For for each $\underline{d} = (d_1, \ldots, d_m) \in \mathbb{N}_0^m$ there exists a weakly decreasing sequence $I_1 \supseteq \cdots \supseteq I_s$ in $\mathcal{I}$, such that the LS-tableaux of shape $(\lambda_{I_1}, \ldots, \lambda_{I_s})$ have degree $\underline{d}$: This is clearly true for $m = 1$. If $m \geq 2$, we choose an index $i \in \underline{I}$ for $I = [m]$, where d_i is minimal. As the i-th entry of $\underline{d} - d_i e_I$ is zero, we can find a weakly decreasing sequence $I_1 \supseteq \cdots \supseteq I_s$ with $\sum_{k=1}^{s} e_{I_k} = \underline{d} - d_i e_I$ by induction. If we append $[m]$ exactly d_i times to the start of this sequence, we thus get a sequence for $\underline{d}$.

It is not obvious that this sequence $I_1 \supseteq \cdots \supseteq I_s$ is uniquely determined by $\underline{d}$. This follows later via Corollary 5.12. Note that the LS-tableaux for each fixed sequence to a degree $\underline{d} \in \mathbb{N}_0^m$ give rise to a character formula for the Demazure module $V(\underline{d} \cdot \underline{\lambda})_\tau$.

Definition 4.8. We define the following posets and monotone maps:

(a) Let $W(\underline{\lambda}, \tau)$ be the direct product of the posets $\{\sigma \in W/W_Q \mid \sigma \leq \tau\}$ and $\mathcal{I}$, i. e. the order relation is given by

$$(\theta, I) \geq (\phi, J) \quad :\Longleftrightarrow \quad I \supseteq J \quad \text{and} \quad \theta \geq \phi$$

for all $(\theta, I), (\phi, J) \in W(\underline{\lambda}, \tau)$.

(b) Let $\underline{W}(\underline{\lambda}, \tau) = \coprod_{I \in \mathcal{I}} \{\theta \in W/W_{P_I} \mid \theta \leq \pi_{P_I}(\tau)\} \times \{I\}$ be the poset, which partial order is given by the transitive hull of the following relation:

$$(\theta, I) \geq (\phi, J) \quad :\Longleftrightarrow \quad I \supseteq J \;\; \text{and} \;\; \max_Q(\theta) \geq \min_Q(\phi) \tag{4.5}$$

for all $(\theta, I), (\phi, J) \in \underline{W}(\underline{\lambda}, \tau)$. We also denote the order of $\underline{W}(\underline{\lambda}, \tau)$ by $\geq$.

(c) Furthermore, we define the map $\pi_{P_{\mathcal{I}}} : W(\underline{\lambda}, \tau) \to \underline{W}(\underline{\lambda}, \tau)$, $(\theta, I) \mapsto (\pi_{P_I}(\theta), I)$, which is clearly monotone.

(d) Let $\underline{\theta}$ be a chain in $\underline{W}(\underline{\lambda}, \tau)$ of elements $(\theta_\ell, I_\ell) > \cdots > (\theta_0, I_0)$. We say that $\underline{\theta}$ is ***τ*-standard**, if it has a **defining chain**, that is to say a chain $(\overline{\theta}_\ell, I_\ell) > \cdots > (\overline{\theta}_0, I_0)$ in $W(\underline{\lambda}, \tau)$ with $\pi_{P_{\mathcal{I}}}(\overline{\theta}_k, I_k) = (\theta_k, I_k)$ for all $k = 0, \ldots, \ell$.

Lemma 4.9. *For $(\theta, I), (\phi, J) \in \underline{W}(\underline{\lambda}, \tau)$ with $J \subseteq I$ the condition $\max_Q(\theta) \geq \min_Q(\phi)$ in (4.5) is equivalent to each of the following:*

(a) $\pi_{P_J} \circ \max_{Q_I}(\theta) \geq \phi$;

(b) there exists a parabolic subgroup $Q \subseteq Q' \subseteq P_I \cap P_J$ and lifts $\overline{\theta}$ and $\overline{\phi}$ in $W/W_{Q'}$ of θ and ϕ respectively, such that $\overline{\theta} \geq \overline{\phi}$ in $W/W_{Q'}$.

Proof. The inclusion $J \subseteq I$ implies $P_J \subseteq Q_I$. Projecting the condition $\max_Q(\theta) \geq \min_Q(\phi)$ to W/W_{P_J} yields $\pi_{P_J} \circ \max_{Q_I}(\theta) \geq \phi$ and this inequality in W/W_{P_J} lifts back to $\max_Q(\theta) \geq \max_Q \circ \pi_{P_J} \circ \max_{Q_I}(\theta) \geq \max_Q(\phi) \geq \min_Q(\phi)$. Clearly, $\max_Q(\theta) \geq \min_Q(\phi)$ implies condition (b). Conversely, the relation $\overline{\theta} \geq \overline{\phi}$ of lifts in $W/W_{Q'}$ gives rise to the inequality $\max_{Q'}(\theta) \geq \theta \geq \phi \geq \min_{Q'}(\phi)$, which in turn lifts to $\max_Q(\theta) \geq \min_Q(\phi)$. □

Let $\underline{\pi} = (\pi_1, \ldots, \pi_s)$ be an LS-tableau of type $(\underline{\lambda}, \mathcal{I})$ and let $\sigma_{p_k}^{(k)} > \cdots > \sigma_1^{(k)}$ denote the sequence of elements in $W/W_{P_{I_k}}$ of the LS-path $\pi_k \in \mathbb{B}(\lambda_{I_k})$ for each $k \in [s]$. Then weak τ-standardness can be described using the poset $\underline{W}(\underline{\lambda}, \tau)$:

$$\begin{aligned} &\underline{\pi} \text{ is weakly } \tau\text{-standard} \quad \Longleftrightarrow \\ &\qquad (\sigma_{p_1}^{(1)}, I_1) \geq \cdots \geq (\sigma_1^{(1)}, I_1) \geq \cdots \geq (\sigma_{p_s}^{(s)}, I_s) \geq \cdots \geq (\sigma_1^{(s)}, I_s) \;\text{ in } \underline{W}(\underline{\lambda}, \tau) \end{aligned} \tag{4.6}$$

The LS-tableau $\underline{\pi}$ is τ-standard, if and only if it is weakly τ-standard and the chain one obtains from (4.6) by erasing all duplicates is τ-standard.

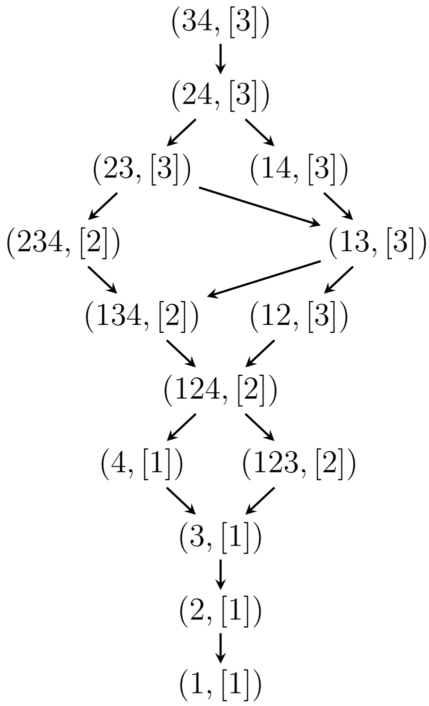


Figure 3: Hasse-diagram of $\underline{W}(\underline{\lambda}, \tau)$ in type $\mathtt{A}_3$.

The poset $\underline{W}(\underline{\lambda}, \tau)$ generalizes the poset $\underline{W}$ from Chapter 3. The relation (4.5) is again reflexive and antisymmetric, but not transitive in general. As an example, consider the sequence $\underline{\lambda} = (\omega_1, \omega_3, \omega_2)$ of fundamental weights in Dynkin type $\mathtt{A}_3$, the unique maximal element $\tau = w_0$ in W and the index poset $\mathcal{I} = \{[1], [2], [3]\}$. Then we have $(12, [3]) \geq (124, [2])$ and $(124, [2]) \geq (4, [1])$ but $(12, [3]) \not\geq (4, [1])$. Here we used the notation from Chapter 3 for elements in W/W_{P_i}. The complete poset $\underline{W}(\underline{\lambda}, \tau)$ is shown in Figure 3. Note that it cannot be the underlying poset A of a multiprojective Seshadri stratification on $X = G/B$, since the length $\ell = 9$ of the poset does not coincide with $\dim \hat{X} - 1 = 8$.

4.3. The defining chain poset

In this section we construct a poset $D(\underline{\lambda}, \tau)$, which serves as the underlying poset A for the multiprojective stratification on X_τ. This construction heavily relies on Theorem 4.12, but before we can state and prove it, we need a few more results about defining chains.

Lemma 4.10. *Every τ-standard chain $\underline{\theta} : (\theta_\ell, I_\ell) > \cdots > (\theta_0, I_0)$ in $\underline{W}(\underline{\lambda}, \tau)$ has a unique maximal and a unique minimal defining chain*

$$\overline{\theta}^{\max}\colon\ (\overline{\theta}_\ell^{\max}, I_\ell) > \cdots > (\overline{\theta}_0^{\max}, I_0) \quad \textit{and} \quad \overline{\theta}^{\min}\colon\ (\overline{\theta}_\ell^{\min}, I_\ell) > \cdots > (\overline{\theta}_0^{\min}, I_0),$$

i. e. for every defining chain $\overline{\theta} : (\overline{\theta}_\ell, I_\ell) > \cdots > (\overline{\theta}_0, I_0)$ *of* $\underline{\theta}$ *it holds* $\overline{\theta}_k^{\max} \geq \overline{\theta}_k \geq \overline{\theta}_k^{\min}$ *for all* $k = 0, \ldots, \ell$.

Proof. We only proof the statements about the unique maximal defining chain, the other statement follows analogously. Since $\underline{\theta}$ is τ-standard, there exists a defining chain $(\overline{\theta}_\ell, I_\ell) > \cdots > (\overline{\theta}_0, I_0)$. In particular, we have $\tau \geq \overline{\theta}_\ell$, so via Deodhar's Lemma B.3 we can choose a unique maximal lift $\overline{\theta}_\ell^{\max} \in W/W_Q$ of θ_ℓ that is smaller or equal to τ. Then $\overline{\theta}_\ell^{\max} \geq \overline{\theta}_\ell$. For all $k = \ell - 1, \ldots, 1$ we now iteratively choose a lift $\overline{\theta}_k^{\max}$, such that $\overline{\theta}_k^{\max} \geq \overline{\theta}_k$. Since we have $\overline{\theta}_{k+1}^{\max} \geq \overline{\theta}_{k+1} \geq \overline{\theta}_k$, there exists a unique maximal lift $\overline{\theta}_k^{\max} \in W/W_Q$ of θ_k with $\overline{\theta}_{k+1}^{\max} \geq \overline{\theta}_k^{\max}$ and this lift fulfills $\overline{\theta}_k^{\max} \geq \overline{\theta}_k$. By construction, we thus obtain the unique maximal defining chain of $\underline{\theta}$. □

Lemma 4.11. *Let* $\underline{\theta} : (\theta_\ell, I_\ell) > \cdots > (\theta_0, I_0)$ *be a* τ*-standard sequence in* $\underline{W}(\underline{\lambda}, \tau)$ *and* $(\overline{\theta}_\ell, I_\ell) > \cdots > (\overline{\theta}_0, I_0)$ *be a defining chain for* $\underline{\theta}$*. For each* $k \in \{0, \ldots, \ell\}$ *we define the parabolic subgroups*

$$Q^k = Q_\tau \cap \bigcap_{r=k}^{\ell} P_{I_r} \quad \text{and} \quad Q_k = \bigcap_{r=0}^{k} P_{I_r}$$

as well as the following elements:

$$\overline{\theta}_k^{\vartriangle} = \max\nolimits_Q \circ \pi_{Q^k}(\overline{\theta}_k) \quad \text{and} \quad \overline{\theta}_k^{\triangledown} = \min\nolimits_Q \circ \pi_{Q_k}(\overline{\theta}_k).$$

Then $(\overline{\theta}_\ell^{\vartriangle}, I_\ell) > \cdots > (\overline{\theta}_0^{\vartriangle}, I_0)$ *and* $(\overline{\theta}_\ell^{\triangledown}, I_\ell) > \cdots > (\overline{\theta}_0^{\triangledown}, I_0)$ *are also defining chains for* $\underline{\theta}$ *satisfying* $\overline{\theta}_k^{\vartriangle} \geq \overline{\theta}_k \geq \overline{\theta}_k^{\triangledown}$ *for each* $k \in \{0, \ldots, \ell\}$*. In particular, the lift of* (θ_k, I_k) *in the unique maximal/minimal defining chain of* $\underline{\theta}$ *is* Q^k*-maximal/*Q_k*-minimal respectively.*

Proof. Again, we only prove the statements about the chain $(\overline{\theta}_\ell^{\vartriangle}, I_\ell) > \cdots > (\overline{\theta}_0^{\vartriangle}, I_0)$. Since $Q^k \subseteq P_{I_k}$, the element $\overline{\theta}_k^{\vartriangle}$ is still a lift of θ_k in W/W_Q and by definition we have $\overline{\theta}_k^{\vartriangle} \geq \overline{\theta}_k$. The relation $\tau \geq \overline{\theta}_\ell$ together with the fact, that τ is Q^ℓ-maximal, implies $\tau = \max_Q \circ \pi_{Q^\ell}(\tau) \geq \max_Q \circ \pi_{Q^\ell}(\overline{\theta}_\ell) = \overline{\theta}_\ell^{\vartriangle}$. By monotony of the maps $\max_Q$ and $\pi_{Q_{I_k}}$ and the inclusion $Q^{k-1} \subseteq Q^k$ we get

$$\overline{\theta}_k^{\vartriangle} = \max\nolimits_Q \circ \pi_{Q^{I_k}}(\overline{\theta}_k) \geq \max\nolimits_Q \circ \pi_{Q^{I_k}}(\overline{\theta}_{k-1}) \geq \max\nolimits_Q \circ \pi_{Q^{I_{k-1}}}(\overline{\theta}_{k-1}) = \overline{\theta}_{k-1}^{\vartriangle}.$$

Therefore $(\overline{\theta}_\ell^{\vartriangle}, I_\ell) > \cdots > (\overline{\theta}_0^{\vartriangle}, I_0)$ is a defining chain for $\underline{\theta}$. □

Notice, that the parabolic subgroup Q_k in Lemma 4.11 coincides with the group Q_{I_k} for every $k = 0, \ldots, \ell$. The analogous statement does not hold for Q^k. In general, one just has the inclusion $Q^{I_k} \subseteq Q^k$.

Theorem 4.12. *Every maximal* τ*-standard chain* $\underline{\theta} : (\theta_\ell, I_\ell) > \cdots > (\theta_0, I_0)$ *in* $\underline{W}(\underline{\lambda}, \tau)$ *has a unique defining chain* $(\overline{\theta}_\ell, I_\ell) > \cdots > (\overline{\theta}_0, I_0)$ *and this chain is a maximal chain in*

$W(\underline{\lambda}, \tau)$. Additionally, the element $\overline{\theta}_k \in W/W_Q$ is Q_k-minimal and Q^k-maximal (using the parabolic subgroups from Lemma 4.11).

Proof. Let $(\overline{\theta}_\ell, I_\ell) > \cdots > (\overline{\theta}_0, I_0)$ be the unique maximal defining chain of $\underline{\theta}$. The largest element (θ_ℓ, I_ℓ) in $\underline{\theta}$ is equal to $(\tau, [m])$, otherwise $(\pi_{P_{[m]}}(\tau), [m]) > (\theta_\ell, I_\ell) > \cdots > (\theta_0, I_0)$ would be a longer τ-standard chain.

By Lemma 4.11 every lift $\overline{\theta}_k$ is Q^k-maximal. We first prove by descending induction over $k = \ell, \ldots, 0$, that $\overline{\theta}_k$ is Q_k-minimal as well. The element $\overline{\theta}_\ell = \tau$ is Q_ℓ-minimal, since $Q_\ell = Q$. Now suppose that $\overline{\theta}_k$ is Q_k-minimal for some $k < \ell$. We show, that $\overline{\theta}_{k-1}$ is Q_{k-1}-minimal. In order to keep indices to a minimum, we write $I = I_k$ and $J = I_{k-1}$. We need to differentiate between two cases: $I = J$ and $I \neq J$.

First, suppose that $I = J$. Let $B = [\overline{\theta}_k, \min_Q \circ \pi_{Q_I}(\overline{\theta}_{k-1})]$ be the Bruhat interval of all $\sigma \in W/W_Q$ with $\overline{\theta}_k \geq \sigma \geq \min_Q \circ \pi_{Q_I}(\overline{\theta}_{k-1})$. The image of B via π_{P_I} is exactly $\{\theta_k, \theta_{k-1}\}$. Otherwise there exists an element $\phi \in \pi_{P_I}(B)$ and a lift $\overline{\phi}$ of ϕ in W/W_Q such that $\theta_k > \phi > \theta_{k-1}$ and $\overline{\theta}_k > \overline{\phi} > \min_Q \circ \pi_{Q_I}(\overline{\theta}_{k-1})$. By inserting (ϕ, I) between (θ_k, I) and (θ_{k-1}, I) we get a longer chain in $\underline{W}(\underline{\lambda}, \tau)$, which is still τ-standard, since we can use Lemma 4.11 to construct the following defining chain:

$$(\overline{\theta}_\ell, I_\ell) > \cdots > (\overline{\theta}_k, I) > (\overline{\phi}, I) > (\min_Q \circ \pi_{Q_I}(\overline{\theta}_{k-1}), I) \geq (\overline{\theta}_{k-1}^{\triangledown}, I_{k-1}) > \cdots > (\overline{\theta}_0^{\triangledown}, I_0).$$

The image of B under the projection π_{Q_I} is equal to the Bruhat interval $[\pi_{Q_I}(\overline{\theta}_k), \pi_{Q_I}(\overline{\theta}_{k-1})]$ in W/W_{Q_I}, because both $\overline{\theta}_k$ and $\min_Q \circ \pi_{Q_I}(\overline{\theta}_{k-1})$ are Q_I-minimal. The element $\pi_{Q_I}(\overline{\theta}_{k-1})$ is the unique maximal lift of θ_{k-1} in W/W_{Q_I}, which is less or equal to $\pi_{Q_I}(\overline{\theta}_k)$. Otherwise there would exist a lift $\psi \in W/W_{Q_I}$ of θ_{k-1} such that $\pi_{Q_I}(\overline{\theta}_k) \geq \psi > \pi_{Q_I}(\overline{\theta}_{k-1})$. Taking the Q-maximum yields:

$$\overline{\theta}_k = \max_Q \circ \pi_{Q_I}(\overline{\theta}_k) \geq \max_Q(\psi) > \max_Q \circ \pi_{Q_I}(\overline{\theta}_{k-1}) \geq \overline{\theta}_{k-1}.$$

But this is a contradiction to the construction of $\overline{\theta}_{k-1}$ as it is the unique maximal lift of θ_{k-1} in W/W_Q such that $\overline{\theta}_{k-1} \leq \overline{\theta}_k$.

Combining our observations, wee see that the only element in $\pi_{Q_I}(A)$, which does not project to θ_k is $\pi_{Q_I}(\overline{\theta}_{k-1})$. Using Lemma B.4 on $\pi_{Q_I}(\overline{\theta}_k) > \pi_{Q_I}(\overline{\theta}_{k-1})$ it now follows, that this is a covering relation in W/W_{Q_I}. It lifts to the covering relation $\overline{\theta}_k > \min_Q \circ \pi_{Q_I}(\overline{\theta}_{k-1})$ in W/W_Q and since $\overline{\theta}_{k-1}$ lies in between them, it is Q_I-minimal.

It remains the case $I \neq J$. Let $K \subsetneq I$ be a covering relation in $\mathcal{I}$ such that $J \subseteq K$. Then we have $J = K$ and $\overline{\theta}_{k-1} = \overline{\theta}_k$, since $\underline{\theta}$ is maximal τ-standard and the inequalities $(\theta_k, I) > (\pi_{P_K}(\overline{\theta}_k), K) \geq (\theta_{k-1}, J)$ in $\underline{W}(\underline{\lambda}, \tau)$ can be lifted to $(\overline{\theta}_k, I) > (\overline{\theta}_k, K) \geq (\overline{\theta}_{k-1}, J)$.

The images of the two elements $\min_Q \circ \pi_{Q_J}(\overline{\theta}_k)$ and $\overline{\theta}_k$ are equal in W/W_{P_I}. Otherwise we could extend the chain $\underline{\theta}$ to the longer τ-standard chain

$$(\theta_\ell, I_\ell) > \cdots > (\theta_k, I) > (\pi_{P_I} \circ \min_Q \circ \pi_{Q_J}(\overline{\theta}_k), I) > (\theta_{k-1}, J) > \cdots > (\theta_0, I_0)$$

as it has the following defining chain:

$$(\overline{\theta}_\ell, I_\ell) > \cdots > (\overline{\theta}_k, I) > (\min_Q \circ \pi_{Q_J}(\overline{\theta}_k), I) > (\overline{\theta}_{k-1}^\triangledown, J) > \cdots > (\overline{\theta}_0^\triangledown, I_0).$$

The images of the two elements $\min_Q \circ \pi_{Q_J}(\overline{\theta}_k)$ and $\overline{\theta}_k$ are also equal in W/W_{Q_J}, hence we can lift this equality to $\pi_{Q_I} \circ \min_Q \circ \pi_{Q_J}(\overline{\theta}_k) = \pi_{Q_I}(\overline{\theta}_k)$ by Lemma B.1. Both elements $\min_Q \circ \pi_{Q_J}(\overline{\theta}_k)$ and $\overline{\theta}_k$ are Q_I-minimal. The former is even Q_J-minimal and the latter is Q_I-minimal by induction. Therefore $\min_Q \circ \pi_{Q_J}(\overline{\theta}_k) = \overline{\theta}_k$ as they are equal in W/W_{Q_I}, which shows that $\overline{\theta}_k$ is Q_J-minimal.

We still need to prove, that $\underline{\theta}$ has a unique defining chain and compute its length. We know that there is a unique minimal defining chain $(\overline{\theta}_\ell^{\min}, I_\ell) > \cdots > (\overline{\theta}_0^{\min}, I_0)$ and a unique maximal defining chain $(\overline{\theta}_\ell^{\max}, I_\ell) > \cdots > (\overline{\theta}_0^{\max}, I_0)$ for $\underline{\theta}$. It is easy to see, that $\underline{\theta}$ ends at the element $\theta_0 = \mathrm{id}W_{P_{I_0}}$. Its lift in the maximal defining chain is Q_0-minimal, hence $\overline{\theta}_0^{\max} = \mathrm{id}W_Q = \overline{\theta}_0^{\min}$. We can now work ourselves inductively through the two defining chains, showing $\overline{\theta}_k^{\max} = \overline{\theta}_k^{\min}$ for $k = 1, \ldots, \ell$. It always holds $\overline{\theta}_{k+1}^{\max} \geq \overline{\theta}_{k+1}^{\min} \geq \overline{\theta}_k^{\min} = \overline{\theta}_k^{\max}$. If $I_{k+1} = I_k$, then $\overline{\theta}_{k+1}^{\max} > \overline{\theta}_k^{\max}$ is a covering relation and $\overline{\theta}_{k+1}^{\min} \neq \overline{\theta}_k^{\min}$. For $I_{k+1} \neq I_k$ we have $\overline{\theta}_{k+1}^{\max} = \overline{\theta}_k^{\max}$. In both cases it follows $\overline{\theta}_{k+1}^{\max} = \overline{\theta}_{k+1}^{\min}$.

As the minimal and maximal defining chain coincide, there is exactly one defining chain for $\underline{\theta}$. Its first element is τ and its last element is $\mathrm{id}W_Q$. In between we only have covering relations and $m-1$ equalities representing the change of the subset $W/W_{P_I} \subseteq \underline{W}(\underline{\lambda}, \tau)$. So the chain $\underline{\theta}$ is of length $r(\tau) + m - 1$, where $r(\tau)$ denotes the rank of τ in W/W_Q, hence $\underline{\theta}$ is a maximal chain in $W(\underline{\lambda}, \tau)$. □

Definition 4.13. The **defining chain poset** $D(\underline{\lambda}, \tau) \subseteq W(\underline{\lambda}, \tau)$ consists of all elements $(\theta, I) \in W(\underline{\lambda}, \tau)$, which are contained in the unique defining chain of a maximal τ-standard chain in $\underline{W}(\underline{\lambda}, \tau)$. The order relation $\succeq$ on $D(\underline{\lambda}, \tau)$ is given by

$(\theta, I) \succeq (\phi, J) \iff (\theta, I) \geq (\phi, J)$ in $W(\underline{\lambda}, \tau)$ and there exists a maximal τ-standard chain in $\underline{W}(\underline{\lambda}, \tau)$, such that (θ, I) and (ϕ, J) are contained in its unique defining chain.

Remark 4.14. For two elements $(\theta, I), (\phi, J) \in D(\underline{\lambda}, \tau)$ the relation $(\theta, I) \succeq (\phi, J)$ is equivalent to the existence of a $\boldsymbol{P_{\mathcal{I}}}$**-chain** from (θ, I) to (ϕ, J), by which we mean a chain of covering relations in $W(\underline{\lambda}, \tau)$ from (θ, I) to (ϕ, J), which projects to a chain of the same length via the map $\pi_{P_{\mathcal{I}}}$. In particular, the relation $\succeq$ is reflexive, antisymmetric and transitive.

Example 4.15. As a first example consider the group $G = \mathrm{SL}_3(\mathbb{K})$, $\tau = 312$, $\underline{\lambda} = (\omega_2, \omega_1)$ and $\mathcal{I} = \{[1], [2]\}$. The poset $\underline{W}(\underline{\lambda}, \tau)$ looks as follows in this case:

$$(3, [2]) \longrightarrow (2, [2]) \longrightarrow (1, [2]) \longrightarrow (13, [1]) \longrightarrow (12, [1])$$

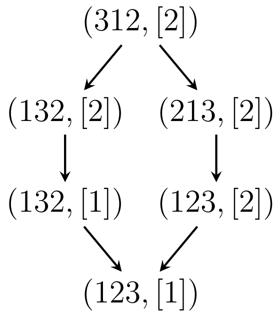


Figure 4: Defining chain poset from Example 4.15

The maximal τ-standard chains have length 4 by Theorem 4.12, so they contain all but one element of $\underline{W}(\underline{\lambda}, \tau)$. This element can only be $(2, [2])$ or $(13, [1])$. In total, we get the defining chain poset from Figure 4.

Example 4.16. If the sequence $\underline{\lambda} = (\lambda)$ only consists of one element, then we have the index poset $\mathcal{I} = \{[1]\}$, the parabolic subgroup $Q = P_{[1]}$ and the defining chain poset

$$D(\underline{\lambda}, \tau) \cong W(\underline{\lambda}, \tau) \cong \underline{W}(\underline{\lambda}, \tau) \cong \{\theta \in W/W_Q \mid \theta \leq \tau\}.$$

Hence we obtain the underlying poset of the Seshadri stratification on $X_\tau \subseteq \mathbb{P}(V(\lambda)_\tau)$ constructed in [CFL2], which we mentioned in the beginning of this chapter.

We now examine the covering relations in $D(\underline{\lambda}, \tau)$. By definition, $D(\underline{\lambda}, \tau)$ is a graded poset and of the same length as $W(\underline{\lambda}, \tau)$, so every covering relation $(\theta, I) \succ (\theta, J)$ in $D(\underline{\lambda}, \tau)$ is also a covering relation in $W(\underline{\lambda}, \tau)$. Therefore (θ, I) covers (ϕ, J) in $D(\underline{\lambda}, \tau)$ if and only if these elements are of one of the following two forms:

- $J = I$, $\theta > \phi$ is a covering relation in W/W_Q and $\pi_{P_I}(\theta) > \pi_{P_I}(\phi)$;
- $J \subsetneq I$ is a covering relation in $\mathcal{I}$ and $\theta = \phi$ in W/W_Q.

Remark 4.17. The defining chain poset is compatible with restriction: For every $(\sigma, I) \in D(\underline{\lambda}, \tau)$ the subposet

$$D(\underline{\lambda}, \tau)_{\preceq(\sigma,I)} = \{(\theta, J) \in D(\underline{\lambda}, \tau) \mid (\theta, J) \preceq (\sigma, I)\}$$

is also a defining chain poset in the following sense. Let m' be the number of elements in I and let $\kappa : [m'] \to I$ be a bijection. We define the sequence $\underline{\lambda}' = (\lambda_{\kappa(1)}, \ldots, \lambda_{\kappa(m')})$ and the index poset $\mathcal{I}' = \{\kappa^{-1}(J) \mid J \in \mathcal{I}, J \subseteq I\}$. Then the map

$$D(\underline{\lambda}, \tau)_{\preceq(\sigma,I)} \to D(\underline{\lambda}', \pi_{Q_I}(\sigma)), \quad (\theta, J) \mapsto (\pi_{Q_I}(\theta), \kappa^{-1}(J))$$

is well-defined and monotone, where $D(\underline{\lambda}', \pi_{Q_I}(\sigma))$ is the defining chain poset with respect to the index poset $\mathcal{I}'$. Since θ is Q_I-minimal for each $(\theta, J) \in D(\underline{\lambda}, \tau)_{\preceq(\sigma,I)}$, the map is injective. It is also surjective and its inverse map is monotone, because every maximal $\pi_{Q_I}(\sigma)$-standard chain can be extended to a maximal τ-standard chain by using a maximal chain from $(\tau, [m])$ to (σ, I) in $D(\underline{\lambda}, \tau)$.

Lemma 4.18. *The following are equivalent for every $(\theta, I) \in W(\underline{\lambda}, \tau)$:*

(i) $(\theta, I) \in D(\underline{\lambda}, \tau)$;

(ii) θ is Q_I-minimal and there exists a $P_{\mathcal{I}}$-chain from $(\tau, [m])$ to (θ, I);

(iii) θ is Q_I-minimal and there exists a $P_{\mathcal{I}}$-chain from an element $(\phi, J) \succeq (\theta, I)$ in $D(\underline{\lambda}, \tau)$ to (θ, I).

Proof. The implication (ii) ⇒ (iii) is obvious and (i) ⇒ (ii) follows from Theorem 4.12, since $(\tau, [m])$ is contained in every unique defining chain of a maximal τ-standard chain in $\underline{W}(\underline{\lambda}, \tau)$. Now suppose, that θ is Q_I-minimal and there exists an element $(\phi, J) \in D(\underline{\lambda}, \tau)$ and a $P_{\mathcal{I}}$-chain from (ϕ, J) to (θ, I). We choose a maximal chain $I_1 \subsetneq \cdots \subsetneq I_s = I$ in $\mathcal{I}$ from a minimal element $I_1 \in \mathcal{I}$ to I. Since the element $\theta^\triangledown := \min_Q \circ \pi_{Q_{I_{s-1}}}(\theta)$ is less or equal to θ, there exists a chain $\theta = \theta_r > \cdots > \theta_0 = \theta^\triangledown$ of covering relations in W/W_Q. Both θ and $\theta^\triangledown$ are Q_I-minimal and $\pi_{Q_{I_{s-1}}}(\theta^\triangledown) = \pi_{Q_{I_{s-1}}}(\theta)$. Since $Q_I = Q_{I_{s-1}} \cap P_I$, Lemma B.1 implies $\pi_{P_I}(\theta_k) > \pi_{P_I}(\theta_{k-1})$ for all $1 \leq k \leq r$. Hence we get the $P_{\mathcal{I}}$-chain $(\theta_r, I) > \cdots > (\theta_0, I) > (\theta_0, J)$ in $W(\underline{\lambda}, \tau)$. Analogously, we can continue this procedure by constructing $P_{\mathcal{I}}$-chains from $(\min_Q \circ \pi_{Q_{I_k}}(\theta), I_k)$ to $(\min_Q \circ \pi_{Q_{I_{k-1}}}(\theta), I_{k-1})$ for all $k = s-1, \ldots, 2$. The element $\min_Q \circ \pi_{Q_{I_1}}(\theta)$ is minimal w. r. t. $Q_{I_1} = P_{I_1}$, so there is a $P_{\mathcal{I}}$-chain in $W(\underline{\lambda}, \tau)$ from this element to $(\mathrm{id}W_Q, I_1)$. There also exists a $P_{\mathcal{I}}$-chain from $(\tau, [m])$ to (ϕ, J), as $(\phi, J) \in D(\underline{\lambda}, \tau)$. In total, we can now glue the chain from $(\tau, [m])$ to (ϕ, J) with the chain from (ϕ, J) to (θ, I) and all of our constructed chains, to obtain a $P_{\mathcal{I}}$-chain $\overline{\theta}$, which also is a maximal chain in $W(\underline{\lambda}, \tau)$. Its projection to $\underline{W}(\underline{\lambda}, \tau)$ is a maximal τ-standard chain and $\overline{\theta}$ is its unique defining chain. Therefore, we have $(\theta, I) \in D(\underline{\lambda}, \tau)$. □

Corollary 4.19. *For all $J \subseteq I$ in $\mathcal{I}$ and $(\theta, I) \in D(\underline{\lambda}, \tau)$ the element* $(\min_Q \circ \pi_{Q_J}(\theta), J)$ *lies in $D(\underline{\lambda}, \tau)$ and is less or equal to (θ, I).*

Proof. Follows from the proof of Lemma 4.18. □

Lemma 4.18 also gives an inductive procedure to compute the defining chain poset. For every $r = r(\tau) + m - 1, \ldots, 0$ we construct the set D_r of all elements in $D(\underline{\lambda}, \tau)$ of rank r, starting with the largest rank, where we clearly have $D_r = \{(\tau, [m])\}$. If D_r is known for some $r > 0$, then D_{r-1} is the union of the sets

$$\begin{aligned} D_{r-1}(\theta, I) = &\{(\theta, J) \mid \theta \text{ is } Q_J\text{-minimal and } I \text{ covers } J\} \cup \\ &\{(\phi, I) \mid \phi \text{ is } Q_I\text{-minimal, } \theta \text{ covers } \phi \text{ in } W/W_Q \text{ and } \pi_{P_I}(\theta) > \pi_{P_I}(\phi)\} \end{aligned}$$

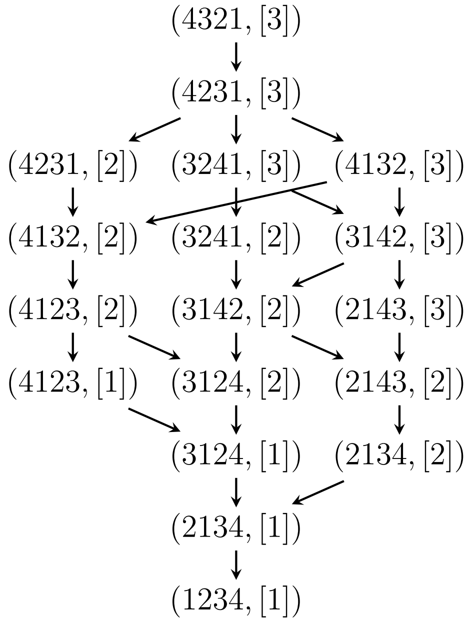


Figure 5: $D((\omega_1, \omega_3, \omega_2), w_0)$ in type $\mathtt{A}_3$ for $\mathcal{I} = \{[1], [2], [3]\}$

over all $(\theta, I) \in D_r$. Using this procedure, we compute another example of a defining chain poset, drawn in Figure 5.

The time complexity of this inductive procedure, however, scales linearly with the number of covering relations in $W(\underline{\lambda}, \tau)$, which can get out of hand quickly. Fortunately, the computation can be significantly accelerated, when $\tau = w_0 W_Q$ is the unique maximal element in W/W_Q. In this case the defining chain poset can be computed directly.

Proposition 4.20. *Suppose $\tau = w_0 W_Q$ is the unique maximal element in W/W_Q. Then an element $(\theta, I) \in W(\underline{\lambda}, \tau)$ lies in $D(\underline{\lambda}, \tau)$, if and only if θ is Q_I-minimal and there exists a chain $I = I_r \subsetneq \cdots \subsetneq I_m = [m]$ of covering relations in $\mathcal{I}$, such that θ is maximal w. r. t. the parabolic subgroup $Q^r = \bigcap_{j=r}^m P_{I_j}$ from Lemma 4.11 (since $Q_\tau = Q$).*

Proof. Let $\theta \in W/W_Q$ be Q_I-minimal and $I = I_r \subsetneq \cdots \subsetneq I_m = [m]$ be a chain of covering relations in $\mathcal{I}$ with the above property. We define the parabolic subgroups $Q^k = \bigcap_{j=k} P_{I_j}$ for $k = r, \dots, m$. If $k < m$ we have $\max_Q \circ \pi_{Q^{k+1}}(\theta) \geq \max_Q \circ \pi_{Q^k}(\theta)$ in W/W_Q and by a proof, which is completely analogous to parts of the proof of Lemma 4.18, we can construct a $P_{\mathcal{I}}$-chain between $(\max_Q \circ \pi_{Q^{k+1}}(\theta), I_{k+1})$ and $(\max_Q \circ \pi_{Q^k}(\theta), I_k)$. Since $Q^m = P_{[m]}$ there also is a $P_{\mathcal{I}}$-chain between $(\tau, [m])$ and $(\max_Q \circ \pi_{Q^m}(\theta), [m])$. Hence (θ, I) lies in $D(\underline{\lambda}, \tau)$ by Lemma 4.18. The other implication follows from Theorem 4.12. □

Although the defining chain poset can be defined in this full generality, it is not always a reasonable candidate for the underlying poset A of a Seshadri stratification on X_τ. The extremal function of an element $(\theta, I) \in D(\underline{\lambda}, \tau)$, we wish to use, is a generalization of the

Plücker coordinates in Type $\mathtt{A}$ and it only depends on the image $\rho(\theta, I) \in \underline{W}(\underline{\lambda}, \tau)$, where ρ denotes the (monotone) composition $D(\underline{\lambda}, \tau) \hookrightarrow W(\underline{\lambda}, \tau) \twoheadrightarrow \underline{W}(\underline{\lambda}, \tau)$. In combination with the condition (S2) on a Seshadri stratification, this forces us to only consider those defining chain posets, where ρ is injective, such that no two elements in $D(\underline{\lambda}, \tau)$ have the same extremal function.

Definition 4.21. We say the poset $\mathcal{I}$ is $\boldsymbol{\tau}$**-standard**, if the monotone map

$$\rho : D(\underline{\lambda}, \tau) \to \underline{W}(\underline{\lambda}, \tau), \quad (\theta, I) \mapsto (\pi_{P_I}(\theta), I)$$

is an isomorphism of posets.

The map $\rho : D(\underline{\lambda}, \tau) \twoheadrightarrow \underline{W}(\underline{\lambda}, \tau)$ is automatically an isomorphism, if it is injective. Indeed, if $(\theta, I) > (\phi, J)$ in $\underline{W}(\underline{\lambda}, \tau)$, then this is a τ-standard chain, which we can therefore extend to a maximal τ-standard chain. Its unique defining chain contains the preimages of (θ, I), and (ϕ, J) under ρ, because of the bijectivity of ρ (it is always surjective), hence these preimages are comparable in $D(\underline{\lambda}, \tau)$.

There always exists at least one τ-standard poset $\mathcal{I}$, namely $\mathcal{I} = \mathcal{P}(\{1, \ldots, m\}) \setminus \{\varnothing\}$. Here the map ρ is injective, since $P_I = Q_I$ holds for every $I \in \mathcal{I}$.

Proposition 4.22. *The poset $\mathcal{I}$ is τ-standard, if and only if every weakly τ-standard LS-tableau of type $(\underline{\lambda}, \mathcal{I})$ is τ-standard. In this case the relation (4.5) is transitive.*

Proof. The notions of weakly τ-standard and τ-standard LS-tableaux coincide, if and only if every chain in $\underline{W}(\underline{\lambda}, \tau)$ is τ-standard. This follows from the equivalence (4.6) and the fact that each element $(\theta, I) \in \underline{W}(\underline{\lambda}, \tau)$ defines an LS-path in $\mathbb{B}(\lambda_I)$, namely the straight-line path from the origin to $\theta(\lambda_I) \in \Lambda^+$.

If $\mathcal{I}$ is τ-standard and $\underline{\theta} : (\theta_\ell, I_\ell) > \cdots > (\theta_0, I_0)$ is a chain in $\underline{W}(\underline{\lambda}, \tau)$, then its unique preimage via ρ is a defining chain for $\underline{\theta}$ as $D(\underline{\lambda}, \tau) \cong \underline{W}(\underline{\lambda}, \tau)$. Additionally the relation (4.5) is transitive by Lemma 4.9, since chains in $\underline{W}(\underline{\lambda}, \tau)$ can be lifted to W/W_Q via ρ. Conversely, if $\mathcal{I}$ is not τ-standard, then there are two different preimages $(\overline{\theta}, I), (\overline{\theta'}, I) \in D(\underline{\lambda}, \tau)$ of an element $(\theta, I) \in \underline{W}(\underline{\lambda}, \tau)$, w.l.o.g. the rank of $(\overline{\theta}, I)$ in $D(\underline{\lambda}, \tau)$ is less or equal to the rank of $(\overline{\theta'}, I)$. We choose chains of covering relations

$$(\tau, [m]) = (\sigma_r, I_r) \succ \cdots \succ (\sigma_{j+1}, I_{j+1}) \succ (\overline{\theta}, I) \quad \text{and}$$
$$(\overline{\theta'}, I) \succ (\sigma_{j-1}, I_{j-1}) \succ \cdots \succ (\sigma_0, I_0)$$

in $D(\underline{\lambda}, \tau)$, where (σ_0, I_0) is a minimal element. By projecting both chains to $\underline{W}(\underline{\lambda}, \tau)$ and gluing them together at their shared element, we get a chain $\underline{\theta}$ in $\underline{W}(\underline{\lambda}, \tau)$ containing (θ, I). Its length is equal to the length of $D(\underline{\lambda}, \tau)$. In the case where $(\overline{\theta}, I)$ and $(\overline{\theta'}, I)$ have different ranks in $D(\underline{\lambda}, \tau)$, the chain $\underline{\theta}$ certainly is too long to be τ-standard.

If the ranks are equal, suppose that $\underline{\theta}$ is τ-standard. Then there exists an unique defining chain by Theorem 4.12. The beginning of this defining chain must agree with

$(\sigma_r, I_r) \succ \cdots \succ (\sigma_{j+1}, I_{j+1})$ and its end agrees with $(\sigma_{j-1}, I_{j-1}) \succ \cdots \succ (\sigma_0, I_0)$. The element in between would be a lift of (θ, I) via ρ. It is equal to $(\overline{\theta}, I)$, as $\overline{\theta}$ is the unique maximal lift of θ, which is less or equal to σ_{j+1}. Analogously this lift is equal to $(\overline{\theta'}, I)$, which is impossible. □

Example 4.23. Even in type $\mathtt{A}$ there are elements $\tau \in W/W_Q$, where no totally ordered, τ-standard poset $\mathcal{I}$ exists. One of the easiest examples is $\tau = 3412$ for $G = \mathrm{SL}_4(\mathbb{K})$ and $\underline{\lambda} = (\omega_1, \omega_2, \omega_3)$. Here we use the notation from Chapter 3. The reason for this is the following: When $\mathcal{I}$ is totally ordered, then $P_{[3]}$ is equal to a maximal parabolic subgroup P_i for $i \in [3]$. We can write τ in the form $\tau^{P_i}\tau_{P_i}$ for $\tau^{P_i} \in W^{P_i}$ and $\tau_{P_i} \in W_{P_i}$ (see Appendix B). Then the element $(\sigma\tau_{P_i}, [3])$ lies in the defining chain poset for every $\sigma \in W^{P_i}$ with $\sigma \leq \tau^{P_i}$. But for each $i \in [3]$ there is a covering relation $\sigma' < \tau$, such that $\pi_{P_i}(\sigma') < \pi_{P_i}(\tau)$ is not a covering relation: $1432 < 3412$ for $i = 1, 2$ and $3214 < 3412$ for $i = 3$. Hence $(\pi_{P_i}(\sigma'), [3])$ has multiple preimages under ρ.

Let $\mathcal{I}$ be a τ-standard index poset for $\tau = 3412$. We show that there are only two possible choices for $\mathcal{I}$. Suppose that $P_{[3]} = P_1 \cap P_2$ or $P_{[3]} = P_2 \cap P_3$. In the first case the defining chain poset would contain the chains $(3412, [3]) \succ (1432, [3])$ and $(3412, [3]) \succ (2413, [3]) \succ (1423, [3])$. But $1432 = 1423$ in $W/W_{P_1 \cap P_2}$, so $\mathcal{I}$ is not τ-standard. Similarly, in the second case we have the chains $(3412, [3]) \succ (1432, [3]) \succ (1342, [3])$ and $(3412, [3]) \succ (3142, [3])$ with $1342 = 3142$ in $W/W_{P_2 \cap P_3}$.

Therefore the parabolic $P_{[3]}$ is either equal to $P_1 \cap P_3$ or to B. If $P_{[3]} = B$, then the requirement (4.2) implies that $\mathcal{I}$ is equal to the poset $\mathcal{I} = \mathcal{P}(\{1, 2, 3\}) \setminus \{\varnothing\}$. If $P_{[3]} = P_1 \cap P_3$, the index poset $\mathcal{I}$ contains $I = \{1, 2\}$ and $J = \{2, 3\}$. Then the following elements lie in the defining chain poset: $(3412, I)$, $(3214, [3])$, $(3214, I)$, $(3412, J)$, $(1432, [3])$ and $(1432, J)$. Since $3412 = 3214$ in W/W_{P_1}, the set $\underline{I}$ cannot be equal to $\{1\}$. As $I \not\subseteq J$, we have $2 \in \underline{I}$ by (4.2). Therefore $\underline{I} = I$. Analogously, one can show $\underline{J} = J$. Hence $\mathcal{I}$ is given by $\mathcal{I} = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, [3]\}$.

For a general element τ it is difficult to tell, which posets $\mathcal{I}$ are τ-standard. However, if $\tau = w_0 W_Q$ is the unique maximal element in W/W_Q, then the injectivity of ρ translates into the absence of certain paths in the Dynkin diagram of G. To state this criterion, we define the set $\Delta_{Q'} = \{\alpha \in \Delta \mid s_\alpha \in W_{Q'}\}$ of simple roots for every parabolic subgroup $Q' \subseteq G$.

Theorem 4.24. *For $\tau = w_0 W_Q$ the poset $\mathcal{I}$ is τ-standard, if and only if one of the following equivalent conditions holds for each $I \in \mathcal{I}$ and every chain $I = I_r \subsetneq \cdots \subsetneq I_m = [m]$ of covering relations in $\mathcal{I}$. Here $Q^r = \bigcap_{j=r}^m P_{I_j}$.*

(i) The element $(\mathrm{id}W_{P_I}, I)$ has exactly one preimage via $\rho : D(\underline{\lambda}, \tau) \twoheadrightarrow \underline{W}(\underline{\lambda}, \tau)$;

(ii) $\min_Q \circ \max_{Q_I}(\mathrm{id}W_{P_I}) = \max_Q \circ \min_{Q^r}(\mathrm{id}W_{P_I})$;

(iii) $W_{P_I} \cap W^{Q_I} \subseteq W_{Q^r} \cap W^Q$.

(iv) The two parabolic subgroups Q_I and Q^r generate P_I and every path in the Dynkin diagram of G (not visiting the same vertex twice) connecting a vertex of $\Delta_{Q_I} \setminus \Delta_{Q^r}$ with a vertex of $\Delta_{Q^r} \setminus \Delta_{Q_I}$ contains a vertex not in Δ_{P_I}.

Proof. The poset $\mathcal{I}$ is τ-standard, if and only if statement (i) is fulfilled for every $I \in \mathcal{I}$. We remark, that this equivalence also holds for every $\tau \neq w_0 W_Q$. Indeed if ρ is not bijective, then there exist two different lifts $(\overline{\theta}, I), (\overline{\theta'}, I) \in D(\underline{\lambda}, \tau)$ of an element $(\theta, I) \in \underline{W}(\underline{\lambda}, \tau)$. We write $\min_B(\overline{\theta}) = \theta^{P_I}\overline{\theta}_{P_I}$ and $\min_B(\overline{\theta}) = \theta^{P_I}\overline{\theta'}_{P_I}$ for $\theta^{P_I} \in W^{P_I}$ and $\overline{\theta}_{P_I}, \overline{\theta'}_{P_I} \in W_{P_I}$. Since there are $P_{\mathcal{I}}$-chains from $(\overline{\theta}, I)$ to $(\overline{\theta}_{P_I} W_Q, I)$ and from $(\overline{\theta}, I)$ to $(\overline{\theta}_{P_I} W_Q, I)$, both $(\overline{\theta}_{P_I} W_Q, I)$ and $(\overline{\theta'}_{P_I} W_Q, I)$ are two different lifts of $(\mathrm{id}W_{P_I}, I)$ in $D(\underline{\lambda}, \tau)$ by Lemma 4.18.

Next we show the implication (i) $\Rightarrow$ (ii), by proving that both $\sigma = \min_Q \circ \max_{Q_I}(\mathrm{id}W_{P_I})$ and $\sigma' = \max_Q \circ \min_{Q^r}(\mathrm{id}W_{P_I})$ are Q_I-minimal and Q^r-maximal lifts of $(\mathrm{id}W_{P_I}, I)$. It then follows $\sigma = \sigma'$ from Proposition 4.20. The element σ is Q_I-minimal by definition and maps to $(\max_{Q_I}(\mathrm{id}W_{P_I}), \pi_{Q^r}(\sigma))$ via the map $W/W_Q \to W/W_{Q_I} \times W/W_{Q^r}$. On the other hand, $\max_Q \circ \pi_{Q^r}(\sigma)$ maps to $(\phi, \pi_{Q^r}(\sigma))$ for some lift $\phi \in W/W_{Q_I}$ of $\mathrm{id}W_{P_I}$. We clearly have $\max_{Q_I}(\mathrm{id}W_{P_I}) \geq \phi$. As $Q_I \cap Q^r = Q_\tau = Q$, it now follows $\sigma \geq \max_Q \circ \pi_{Q^r}(\sigma)$ from Lemma B.1. In particular, σ is Q^r-maximal. Analogously, one can show the Q_I-minimality of σ'.

Part (iii) follows from (ii): Since $W_{P_I} \cap W^{Q_I} \subseteq W^Q$, we only need to prove the inclusion $W_{P_I} \cap W^{Q_I} \subseteq W_{Q^r}$. Every element $\phi \in W_{P_I} \cap W^{Q_I}$ is smaller or equal to $\sigma := \min_B \circ \max_{Q_I}(\mathrm{id}W_{P_I})$ since both are Q_I-minimal and $\phi W_{Q_I} \leq \max_{Q_I}(\mathrm{id}W_{P_I}) = \sigma W_{Q_I}$. By statement (ii) we now have

$$\pi_{Q^r}(\phi) \leq \pi_{Q^r}(\sigma) = \pi_{Q^r} \circ \min_B \circ \max_Q \circ \min_{Q^r}(\mathrm{id}W_{P_I}) = \min_{Q^r}(\mathrm{id}W_{P_I}) = \mathrm{id}W_{Q^r}$$

and this inequality is equivalent to $\phi \in W_{Q^r}$.

We close the first circle of implications via (iii) $\Rightarrow$ (i): Let $(\sigma, I) \in D(\underline{\lambda}, \tau)$ be any preimage of $(\mathrm{id}W_{P_I}, I)$ via ρ. Then by Proposition 4.20 there exists a chain $I = I_r \subsetneq \cdots \subsetneq I_m = [m]$ of covering relations in $\mathcal{I}$, such that σ is Q_I-minimal and Q^r-maximal (w. r. t. this chain). By assumption we have $W_{P_I} \cap W^{Q_I} \subseteq W_{Q^r} \cap W^Q$. Notice, that the other inclusion $W_{Q^r} \cap W^Q \subseteq W_{P_I} \cap W^{Q_I}$ is always fulfilled, even if $\mathcal{I}$ is not τ-standard: Here $W_{Q^r} \cap W^Q \subseteq W_{P_I}$ follows from $Q^r \subseteq P_I$ and the Q_I-minimality of every element in $Q_{Q^r} \cap W^Q$ can again be shown via the embedding $W/W_Q \to W/W_{Q_I} \times W/W_{Q^r}$.

By the bijection in (B.1) the set W^Q can be decomposed into the product

$$W^Q = W^{Q^r} \cdot (W_{Q^r} \cap W^Q) = W^{P_I} \cdot (W_{P_I} \cap W^{Q^r}) \cdot (W_{Q^r} \cap W^Q).$$

The element $\min_B(\sigma)$ is contained in $W_{P_I} \cap W^{Q_I}$ and can therefore be (uniquely!) written in the form $\min_B(\sigma) = \mathrm{id} \cdot \theta \cdot \phi$ for $\theta \in W_{P_I} \cap W^{Q^r}$ and $\phi \in W_{Q^r} \cap W^Q$. As $\min_B(\sigma)$ is Q^r-maximal, ϕ is equal to the unique maximal element in the poset $W_{Q^r} \cap W^Q = W_{P_I} \cap W^{Q_I}$. But since $\min_B(\sigma)$ is also an element of $W^{Q_I} = W^{P_I} \cdot (W_{P_I} \cap W^{Q_I})$, it follows $\theta = \mathrm{id}$.

Therefore σ is uniquely determined: It is the maximal element of $W_{P_I} \cap W^{Q_I}$ and this is independent of the choice of the chain $I = I_r \subsetneq \cdots \subsetneq I_m = [m]$.

Next we show (iii) $\Rightarrow$ (iv). For each simple root $\alpha \in \Delta_{P_I}$ not contained in Δ_{Q_I} or Δ_{Q^r}, s_α is an element of $W_{P_I} \cap W^{Q_I}$, but does not lie in W_{Q^r}. Hence Q_I and Q^r generate P_I.

Now suppose that there exists a path $\alpha_1 \to \cdots \to \alpha_k$ in the Dynkin diagram of G, such that $\alpha_1, \ldots, \alpha_k \in \Delta_{P_I}$, $\alpha_1 \in \Delta_{Q_I} \setminus \Delta_{Q^r}$ and $\alpha_k \in \Delta_{Q^r} \setminus \Delta_{Q_I}$. Then $\sigma = s_{\alpha_1} \cdots s_{\alpha_k}$ is an element of W_{P_I} and we claim, that σ also lies in W^{Q_I}. When we have shown this Q_I-minimality, it then follows $\sigma \in W_{P_I} \cap W^{Q_I}$. But $\sigma \notin W_{Q^r}$, since $\alpha_1 \notin \Delta_{Q^r}$.

The Q_I-minimality of σ is equivalent to $\sigma s_\beta > \sigma$ for all $\beta \in \Delta_{Q_I}$. First, let $\beta \in \Delta_{Q_I}$ be a simple root, which in not contained in our chosen path. In this case $s_{\alpha_1} \cdots s_{\alpha_k} s_\beta$ is in reduced decomposition by Lemma B.2, so $\sigma s_\beta > \sigma$. Now let $\beta = \alpha_i \in \Delta_{Q_I}$ for an index $1 \leq i \leq k$. Suppose that $s_{\alpha_1} \cdots s_{\alpha_k} s_\beta$ is not in reduced decomposition. Then there exists an index j, such that $\sigma s_\beta = s_{\alpha_1} \cdots \hat{s}_{\alpha_j} \cdots s_{\alpha_k}$ (s_{α_j} is omitted). Thus $\sigma = s_{\alpha_1} \cdots \hat{s}_{\alpha_j} \cdots s_{\alpha_k} s_\beta$. The simple reflections occuring in a reduced decomposition are always uniquely determined (*not* counting with multiplicity), hence we have $\beta = \alpha_j$ and $s_{\alpha_1} \cdots s_{\alpha_k} = s_{\alpha_1} \cdots \hat{s}_{\alpha_j} \cdots s_{\alpha_k} s_{\alpha_j}$. Hence s_{α_j} commutes with $s_{\alpha_{j+1}} \cdots s_{\alpha_k}$. The simple reflection s_{α_j} commutes with all s_{α_k} for $k > j+1$, since Dynkin diagrams contain no cycles and we have a path from α_j to α_k. Therefore s_{α_j} and $s_{\alpha_{j+1}}$ must commute, but this contradicts to edge between α_j and α_{j+1} in the Dynkin diagram. The decomposition $\sigma s_\beta = s_{\alpha_1} \cdots s_{\alpha_k} s_\beta$ thus is reduced and $\sigma s_\beta > \sigma$.

Finally, (iv) implies (iii). Suppose that σ is an element in $W_{P_I} \cap W^{Q_I}$, but not in $W_{Q^r} \cap W^Q$. Since $W^{Q_I} \subseteq W^Q$, we thus have $\sigma \notin W_{Q^r}$. To each reduced decomposition $\sigma = s_{\alpha_1} \cdots s_{\alpha_k}$ we now associate a pair (p, q) of natural numbers in the following way. Since Q_I and Q^r generate P_I, all simple roots $\alpha_1, \ldots, \alpha_k$ lie in Δ_{Q_I} or Δ_{Q^r}. As $\sigma \notin W_{Q^r}$, there exists a maximal index $1 \leq p \leq k$ with $\alpha_p \in \Delta_{Q_I} \setminus \Delta_{Q^r}$. The simple root α_k is not contained in Δ_{Q_I} by the Q_I-minimality of σ. Hence there exists a minimal number $q \in \{p, \ldots, k\}$ with $\alpha_q \in \Delta_{Q^r} \setminus \Delta_{Q_I}$.

We can assume, that $\sigma = s_{\alpha_1} \cdots s_{\alpha_k}$ is a reduced decomposition, such that the associated pair (p, q) is maximal with respect to the total order

$$(p, q) \geq (p', q') \iff p > p' \text{ or } (p = p' \text{ and } q - p \leq q' - p')$$

on $\mathbb{Z} \times \mathbb{Z}$. We now partition the set $J = \{p, \ldots, q\}$ by fixing numbers $p = p_0 < p_1 < \cdots < p_t \leq q$, such that for all $p + 1 \leq j \leq q$ the simple roots α_{j-1} and α_j are disconnected in the Dynkin diagram if and only if there exists a non-zero index $i \in \{1, \ldots, t\}$ with $j = p_i$. Set $p_{t+1} \coloneqq q + 1$. For all $i = 0, \ldots, t$ we define the set $J_i = \{j \in J \mid p_i \leq j < p_{i+1}\}$, the associated set $\Delta_i = \{\alpha_j \mid j \in J_i\}$ of simple roots and the subword $\sigma_i = \prod_{j \in J_i} s_{\alpha_j}$, where the product is taken is ascending order. Let $s \in \{0, \ldots, t\}$ be the smallest integer, such that the union $\bigcup_{i=s}^t \Delta_i$ is connected in the Dynkin diagram.

By the assumption (iv), every path connecting α_p and α_q contains a simple root $\beta \notin \Delta_{P_I}$. Therefore the number s is at least 1 and by the definition of s we have

$\sigma_{s-1}\sigma_s \cdots \sigma_t = \sigma_s \cdots \sigma_t \sigma_{s-1}$. Hence, for $s \geq 2$, the number $q - p$ was not minimal and for $s = 1$ the number p was not maximal, which contradicts our choice of the reduced decomposition of σ. □

Remark 4.25. For $Q = B$ the condition (iv) simplifies as follows: The two parabolic subgroups Q_I and Q^r generate P_I and there is no edge in the Dynkin diagram of G connecting the two subsets Δ_{Q_I} and Δ_{Q^r}.

4.4. Sequences of fundamental weights

In this section we consider the special case where $\tau = w_0 W_Q$ is the unique maximal element in W/W_Q and the sequence $\underline{\lambda} = (k_1\omega_1, \ldots, k_m\omega_m)$ is given by pairwise distinct fundamental weights $\omega_1, \ldots, \omega_m$ of G and natural numbers $k_1, \ldots, k_m \in \mathbb{N}$. For a fixed parabolic subgroup $Q \subseteq G$ one can always choose a sequence consisting of the fundamental weights ω with $\langle \omega, \alpha^\vee \rangle = 0$ for all $\alpha \in \Delta_Q$. In practice, one would most likely choose $k_1 = \cdots = k_m = 1$, so that the LS-tableaux of type $(\underline{\lambda}, \mathcal{I})$ give rise to a character formula for the irreducible representation $V(\mu)$ to every dominant weight μ of the parabolic subgroup Q, i. e. for each $\mu \in \mathbb{N}_0\omega_1 + \cdots + \mathbb{N}_0\omega_m$ (see Remark 4.7).

Let $\alpha_i \in \Delta$ denote the simple root with $\langle \omega_i, \alpha_i^\vee \rangle = 1$ for all $i \in [m]$. Then the following criterion is just a reformulation of Theorem 4.24 (iv).

Corollary 4.26. *The poset $\mathcal{I}$ is τ-standard, if and only if the following two conditions hold for every $I \in \mathcal{I}$ and each chain $I = I_r \subsetneq \cdots \subsetneq I_m = [m]$ of covering relations in $\mathcal{I}$:*

(i) The intersection of the two sets I and $I' = \bigcup_{j=r}^{m} \underline{I}_j$ is equal to $\underline{I}$.

(ii) All paths from $\{\alpha_i \mid i \in I \setminus \underline{I}\}$ to $\{\alpha_i \mid i \in I' \setminus \underline{I}\}$ contain a vertex α_i for $i \in \underline{I}$.

We have seen that it has some advantages when the index poset $\mathcal{I}$ is totally ordered. For example, the cone $\sigma_{\mathfrak{C}}$ to every maximal chain $\mathfrak{C}$ is equal to $\mathbb{R}^m_{\geq 0}$ and for stratifications of LS-type one can compute the multidegrees of the variety more easily. However, there might not exist a poset $\mathcal{I}$, which is τ-standard and totally ordered at the same time.

Corollary 4.27. *There exists a τ-standard, totally ordered index poset $\mathcal{I}$, if and only if there is a path in the Dynkin diagram of G containing all vertices from the set $\Delta \setminus \Delta_Q = \{\alpha_1, \ldots, \alpha_m\}$.*

Proof. First we consider the index poset of the form $\mathcal{I} = \{[i] \mid i \in [m]\}$. Then the first condition in Corollary 4.26 is automatically fulfilled for each $I = [i] \in \mathcal{I}$ as $I' = \{i, \ldots, m\}$. Hence the Corollary implies:

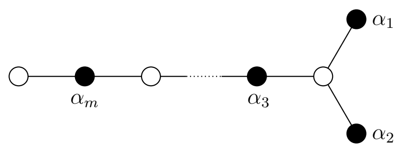
$$\mathcal{I} \text{ is } \tau\text{-standard} \iff \text{for each } i \in [m] \text{ all paths from } \{\alpha_1, \ldots, \alpha_{i-1}\} \text{ to } \{\alpha_{i+1}, \ldots, \alpha_m\} \text{ contain } \alpha_i. \tag{4.7}$$

Let $\mathcal{I} = \{[i] \mid i \in [m]\}$ be totally ordered and τ-standard. W. l. o. g. it consists of all sets $[i]$ for $i \in [m]$. Then there exists a path containing $\{\alpha_1, \ldots, \alpha_m\}$: It is given by the unique path π from α_1 to α_m. Dynkin diagrams of simple algebraic groups are connected and contain no cycles, hence π is unique. It also contains the roots $\alpha_2, \ldots, \alpha_{m-1}$ by (4.7).

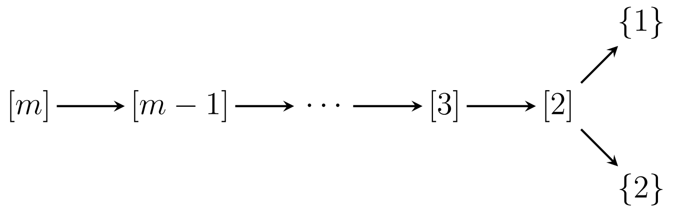
Conversely, let us assume there is a path π containing $\alpha_1, \ldots, \alpha_m$. We can rearrange the indices, such that this is also the order in which π visits these simple roots. Then the poset $\mathcal{I} = \{[i] \mid i \in [m]\}$ is τ-standard, by the equivalence in (4.7). □

We give an example of a τ-standard poset $\mathcal{I}$ for every flag variety G/Q in each Dynkin type. The cases, where the vertices in $\Delta \setminus \Delta_Q$ are all contained in one path in the Dynkin diagram, are already covered. Here we can choose a totally ordered poset $\mathcal{I}$. This of course always happens in the types A, B, C, F and G. In particular for the sequence $\underline{\lambda} = (\omega_{k_m}, \ldots, \omega_{k_1})$ we used in Chapter 3 the poset $\mathcal{I} = \{[i] \mid i \in [m]\}$ is $w_0 W_Q$-standard. In combination with Proposition 4.20 this also proves Remark 3.10.

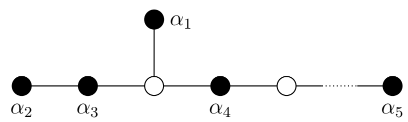
Next we look at the remaining type cases in type D, where we cannot choose a totally ordered poset $\mathcal{I}$.



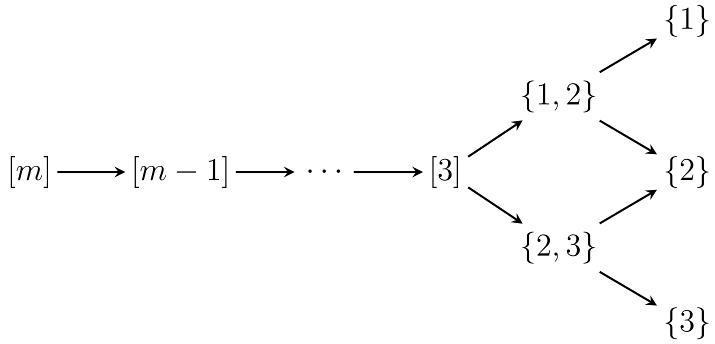
We can assume, that the roots $\alpha_1, \ldots, \alpha_m$ not belonging to Δ_Q are numbered as above. Then the following poset satisfies the condition (4.2) and is τ-standard:



In type E we can take the same poset (with suitable numbering of the roots $\alpha_1, \ldots, \alpha_m$), if in the graph one obtains by erasing the vertex of degree 3 from the Dynkin diagram there is at most one connected component, which contains two or more simple roots $\alpha_1, \ldots, \alpha_m$. In the most complicated case



we again include all $[i]$ in $\mathcal{I}$ for $i \geq 3$ as well as $\{1, 2\}$ and $\{2, 3\}$. However, Corollary 4.26 and the requirement (4.2) force us to take all possible rank one elements $\{1\}, \{2\}$ and $\{3\}$. In total, the following poset is τ-standard:



4.5. The multiprojective stratification

From now on we assume that $\mathcal{I}$ is τ-standard. Let R denote the multihomogeneous coordinate ring of $X = X_\tau$ w. r. t. the embedding fixed by the sequence $\underline{\lambda}$.

Lemma 4.28. *There exists a graded isomorphism of B-modules:*

$$\mathbb{K}[\hat{X}_\tau] = \bigoplus_{\underline{d} \in \mathbb{N}_0^m} \mathbb{K}[\hat{X}_\tau]_{\underline{d}} \cong \bigoplus_{\underline{d} \in \mathbb{N}_0^m} V(\underline{d} \cdot \underline{\lambda})^*_\tau.$$

Proof. Fix a tuple $\underline{d} \in \mathbb{N}_0^m$ and let $Q' = \bigcap_{i \in I} P_i$ be the parabolic subgroup associated to the set $I = \{i \in [m] \mid d_i \neq 0\}$. It suffices to work with the Schubert variety $X_{\tau'} \subseteq G/Q'$ for $\tau' = \pi_{Q'}(\tau)$, as the surjection $X_\tau \twoheadrightarrow X_{\tau'}$ induces an isomorphism of B-modules $\mathbb{K}[\hat{X}_{\tau'}]_{\underline{d}_I} \to \mathbb{K}[\hat{X}_\tau]_{\underline{d}}$ between the graded components of their multihomogeneous coordinate rings of degree $\underline{d}$ and $\underline{d}_I = (d_i)_{i \in I}$.

We define the G-equivariant, linear map $\phi : V(\underline{d} \cdot \underline{\lambda}) \to \bigotimes_{i=1}^m V(\lambda_i)^{\otimes d_i}$ sending a highest weight vector $v_{\underline{d} \cdot \underline{\lambda}} \in V(\underline{d} \cdot \underline{\lambda})$ to the product $v_{\lambda_1}^{\otimes d_1} \otimes \cdots \otimes v_{\lambda_m}^{\otimes d_m}$ of highest weight vectors $v_{\lambda_i} \in V(\lambda_i)$. Every weight vector $v_\tau \in V(\underline{d} \cdot \underline{\lambda})$ of weight $\tau(\underline{d} \cdot \underline{\lambda})$ is mapped to the tensor product $v_{\tau_1}^{\otimes d_1} \otimes \cdots \otimes v_{\tau_m}^{\otimes d_m}$ of weight vectors $v_{\tau_i} \in V(\lambda_i)$ of weight $\tau(\lambda_i)$. Therefore ϕ induces a morphism

$$\phi_\tau : \mathbb{P}(V(\underline{d} \cdot \underline{\lambda})_\tau) \to \mathbb{P}(\bigotimes_{i \in I} V(\lambda_i)_{\tau_i}^{\otimes d_i}).$$

It is well known that ϕ is injective for $\operatorname{char} \mathbb{K} = 0$ (which we assumed in the beginning), so ϕ_τ is injective as well.

We have the following commutative diagram of closed, B-equivariant embeddings:

$$\begin{array}{ccccc} X_{\tau'} & & \hookrightarrow & & \mathbb{P}(V(\underline{d} \cdot \underline{\lambda})_\tau) \\ \downarrow & & & & \downarrow \phi_\tau \\ \prod_{i \in I} \mathbb{P}(V(\lambda_i)_{\tau_i}) & \xhookrightarrow{\delta} & \prod_{i \in I} \mathbb{P}(V(\lambda_i)_{\tau_i})^{\times d_i} & \hookrightarrow & \mathbb{P}(\bigotimes_{i \in I} V(\lambda_i)_{\tau_i}^{\otimes d_i}) \end{array} \tag{4.8}$$

Here δ is the diagonal embedding. This diagram implies that the image of the multicone

$\hat{X}_\tau$ in $\bigotimes_{i=1}^m V(\lambda_i)_{\tau_i}^{\otimes d_i}$ is contained in $V(\underline{d}\cdot\underline{\lambda})_\tau \subseteq \bigotimes_{i\in I} V(\lambda_i)_{\tau_i}^{\otimes d_i}$. Hence there is an induced morphism $\iota : \hat{X}_{\tau'} \to V(\underline{d}\cdot\underline{\lambda})_\tau$, such that the following diagram commutes:

$$\begin{array}{ccccc} \hat{X}_{\tau'} & & \xhookrightarrow{\quad\quad\iota\quad\quad} & & V(\underline{d}\cdot\underline{\lambda})_\tau \\ \big\downarrow & & & & \big\downarrow \\ \prod_{i\in I} V(\lambda_i)_{\tau_i} & \hookrightarrow & \prod_{i\in I} V(\lambda_i)_{\tau_i}^{\times d_i} & \longrightarrow & \bigotimes_{i\in I} V(\lambda_i)_{\tau_i}^{\otimes d_i} \end{array} \tag{4.9}$$

The comorphism of the bottom row in this diagram induces a surjection from the dual space $(\bigotimes_{i\in I} V(\lambda_i)_{\tau_i}^{\otimes d_i})^*$ to the space $\mathbb{K}[\prod_{i\in I} V(\lambda_i)_{\tau_i}]_{\underline{d}_I}$. We therefore get the diagram of B-invariant surjective maps

$$\begin{array}{ccc} \mathbb{K}[\hat{X}_{\tau'}]_{\underline{d}_I} & \twoheadleftarrow & V(\underline{d}\cdot\underline{\lambda})_\tau^* \\ \uparrow & & \uparrow \\ \mathbb{K}[\prod_{i\in I} V(\lambda_i)_{\tau_i}]_{\underline{d}_I} & \twoheadleftarrow & (\bigotimes_{i\in I} V(\lambda_i)_{\tau_i}^{\otimes d_i})^* \end{array}$$

The top row of this diagram is an isomorphism: Since all maps in (4.9) are B-equivariant, the Demazure module $V(\underline{d}\cdot\underline{\lambda})_\tau$ is (linearly) spanned by the image of ι. □

For every element $I \in \mathcal{I}$ the poset

$$D(\lambda_I, \tau_I) = \{\theta \in W/W_{P_I} \mid \theta \leq \tau_I\}$$

is the defining chain poset to the (one-element) sequence (λ_I) and $\tau_I = \pi_{P_I}(\tau)$. Since the index poset is τ-standard, the restriction of the monotone map

$$D(\underline{\lambda}, \tau) \to D(\lambda_I, \tau_I), \quad (\theta, I) \mapsto \pi_{P_I}(\theta)$$

to the subset $D_I(\underline{\lambda}, \tau) = \{(\theta, J) \in D(\underline{\lambda}, \tau) \mid J = I\}$ is an isomorphism of posets. It is also compatible with covering relations: A relation $(\theta, I) \succeq (\phi, I)$ is a covering relation in $D_I(\underline{\lambda}, \tau)$, if and only if $\pi_{P_I}(\theta) \geq \pi_{P_I}(\phi)$ is a covering relation in $D(\lambda_I, \tau_I)$.

As described in the beginning this chapter, there is a Seshadri stratification on each Schubert variety $X_{\tau_I} \subseteq \mathbb{P}(V(\lambda_I)_{\tau_I})$ with underlying poset $D_I(\underline{\lambda}, \tau)$. The strata in the subset $D_I(\underline{\lambda}, \tau)$ for the multiprojective stratification should be Schubert varieties in G/Q_I and also be compatible with the strata for $X_{\tau_I} \subseteq \mathbb{P}(V(\lambda_I)_{\tau_I})$ in the sense that $X_{(\theta,I)}$ should project to $X_{\pi_{P_I}(\theta)} \subseteq G/P_I$ via the map $G/Q_I \twoheadrightarrow G/P_I$. Hence we define the stratum $X_{(\theta,I)}$ of an element $(\theta, I) \in D(\underline{\lambda}, \tau)$ to be the Schubert variety

$$X_{(\theta,I)} \coloneqq X_{\pi_{Q_I}(\theta)} \subseteq \prod_{i\in I} \mathbb{P}(V(\lambda_i)_{\tau_i}).$$

For each index $i = 1, \ldots, m$ we also have a Seshadri stratification on the Schubert variety $X_{\tau_i} \subseteq \mathbb{P}(V(\lambda_i)_{\tau_i})$ with the underlying poset

$$D(\lambda_i, \tau_i) = \{\theta \in W/W_{P_i} \mid \theta \leq \tau_i\}.$$

Its extremal functions f_θ can be pulled back to the multicone $\hat{X}_\tau$ via the linear projection

$$\prod_{j=1}^{m} V(\lambda_j)_{\tau_j} \to V(\lambda_i)_{\tau_i}.$$

Hence we can use the same extremal functions for the multiprojective stratification as well: For every $i \in [m]$ and $\phi \in D(\lambda_i, \tau_i)$ we choose a T-eigenvector ℓ_ϕ of weight $-\phi(\lambda_i)$ in the irreducible G-representation $V(\lambda_i)^*$. The extremal function $f_{(\theta,I)}$ shall then be defined as the product

$$f_{(\theta,I)} := \prod_{i \in \underline{I}} \ell_{\pi_{P_i}(\theta)}$$

It is of multidegree e_I and by Lemma 4.28 it can also be interpreted as a weight vector of weight $-\theta(\lambda_I)$ in the representation $V(\lambda_I)^*$.

Notice that this construction is a generalization of the following two stratifications:

(a) When choosing the one-element sequence $\underline{\lambda} = (\lambda)$ one obtains the original stratification on $X_\tau \subseteq \mathbb{P}(V(\lambda)_\tau)$ from [CFL2].

(b) Let $Q = P_{k_1} \cap \cdots \cap P_{k_m}$ be a parabolic subgroup in type $\mathtt{A}$ as in Chapter 3. By Corollary 4.27, the index poset $\mathcal{I} = \{[1], \ldots, [m]\}$ is τ-standard for $\underline{\lambda} = (\omega_{k_m}, \ldots, \omega_{k_1})$ and $\tau = w_0 W_Q$. Hence the defining chain poset $D(\underline{\lambda}, \tau)$ is isomorphic to $\underline{W}(\underline{\lambda}, \tau) \cong \underline{W}$.

Lemma 4.29. *For all $J \subseteq I$ in $\mathcal{I}$ and $(\theta, I) \in D(\underline{\lambda}, \tau)$, we have*

$$\{(v_1, \ldots, v_m) \in \hat{X}_{(\theta,I)} \mid v_i = 0 \ \forall i \in I \setminus J\} = \hat{X}_{(\min_Q \circ \pi_{Q_J}(\theta), J)}.$$

Proof. By Corollary 4.19, the element $(\min_Q \circ \pi_{Q_J}(\theta), J)$ indeed lies in the defining chain poset. Let $v = (v_1, \ldots, v_m) \in \hat{X}_{(\theta,I)}$ and choose $w_i \in V(\lambda_i)_{\tau_i} \setminus \{0\}$ with $v_i \in \mathbb{K}w_i$ for all $i \in I$. As the diagram

$$\begin{array}{ccc} X_{\pi_{Q_I}(\theta)} & \hookrightarrow & \prod_{i \in I} \mathbb{P}(V(\lambda_i)_{\tau_i}) \\ \downarrow & & \downarrow \\ X_{\pi_{Q_J}(\theta)} & \hookrightarrow & \prod_{i \in J} \mathbb{P}(V(\lambda_i)_{\tau_i}) \end{array}$$

commutes, the tuple $([w_j])_{j \in J} \in \prod_{j \in J} \mathbb{P}(V(\lambda_j)_{\tau_j})$ can be viewed as an element of G/Q_J and lies in the Schubert variety $X_{\pi_{Q_J}(\theta)}$. When $v_i = 0$ holds for all $i \in I \setminus J$, we have

$v \in \hat{X}_{(\min_Q \circ \pi_{Q_J}(\theta), J)}$. Conversely, if $v \in \hat{X}_{(\min_Q \circ \pi_{Q_J}(\theta), J)}$, we choose a preimage via the projection $X_{\pi_{Q_I}(\theta)} \twoheadrightarrow X_{\pi_{Q_J}(\theta)}$ and set all coordinates in $I \setminus J$ to zero. Hence v lies in the multicone $\hat{X}_{(\theta,I)}$. $\square$

Theorem 4.30. *Suppose that the poset $\mathcal{I}$ we chose in Section 4.1 is τ-standard. Then the varieties $X_{(\theta,I)}$ and extremal functions $f_{(\theta,I)}$ defined at the beginning of this section form a (multiprojective) Seshadri stratification on $X = X_\tau$. The defining chain poset $A = D(\underline{\lambda}, \tau)$ is the underlying poset of this stratification and $\mathcal{I}$ is its index poset.*

Proof. It is well known, that Schubert-varieties are smooth in codimension one (see e. g. [CFL2, Corollary 3.5]). Their multicones $\hat{X}_{(\theta,I)} \subseteq \prod_{i=1}^m V(\lambda_i)_{\tau_i}$ are closed, irreducible subvarieties of $\hat{X}_\tau$ as well as smooth in codimension one (Corollary A.11).

The relation $(\theta, I) \succeq (\phi, J)$ in $D(\underline{\lambda}, \tau)$ is equivalent to the inclusion $\hat{X}_{(\theta,I)} \supseteq \hat{X}_{(\phi,J)}$ of their corresponding multicones. Clearly $\hat{X}_{(\theta,I)} \supseteq \hat{X}_{(\phi,J)}$ holds for every covering relation $(\theta, I) \succ (\phi, J)$ in $D(\underline{\lambda}, \tau)$. Conversely if $\hat{X}_{(\theta,I)} \supseteq \hat{X}_{(\phi,J)}$, then $\hat{X}_{(\min_Q \circ \pi_{Q_J}(\theta), J)}$ lies in between these two multicones. In particular, $\pi_{Q_J}(\theta) \geq \pi_{Q_J}(\phi)$. Because of the Q_J-minimality of ϕ we can lift this to $\theta \geq \min_Q \circ \pi_{Q_J}(\theta) \geq \min_Q \circ \pi_{Q_J}(\phi) = \phi$, which implies $\rho(\theta, I) \geq \rho(\phi, J)$. Thus $(\theta, I) \succeq (\phi, J)$ since ρ is an isomorphism.

We now check the requirements (S1)-(S3) for a Seshadri stratification: The defining chain poset is a graded poset and the rank of an element $(\theta, I) \in D(\underline{\lambda}, \tau)$ is equal to the length of the subposet $W(\underline{\lambda}, \tau)_{\preceq(\theta,I)}$. By Theorem 4.12 and Remark 4.17 this length is given by $r(\theta) + |I| - 1$, which is the dimension of the multicone $\hat{X}_{(\theta,I)}$. Therefore (S1) is fulfilled.

Let $(\theta, I), (\phi, J)$ be two elements in $D(\underline{\lambda}, \tau)$ and $v = (v_1, \ldots, v_m)$ be a point in the multicone $\hat{X}_{(\theta,I)}$. By its definition, the extremal function $f_{(\phi,J)}$ vanishes on the point v, if and only if $\ell_{\pi_{P_j}(\theta)}$ vanishes on v_j for some $j \in \underline{J}$. The vanishing behaviour of $\ell_{\pi_{P_j}(\phi)}$ can be described via the Seshadri stratification on $X_{\tau_j} \subseteq \mathbb{P}(V(\lambda_j)_{\tau_j})$ with underlying poset $D(\lambda_j, \tau_j) = \{\sigma \in W/W_{P_j} \mid \sigma \leq \tau_j\}$. Since (S3) holds for this stratification and $v_j \in \hat{X}_{\pi_{P_j}(\theta)}$, it follows:

$$f_{(\phi,J)}(v) = 0 \iff v_j \in \hat{X}_\sigma \text{ for some } j \in \underline{J} \text{ and } \sigma < \pi_{P_j}(\theta) \text{ in } W/W_{P_j}. \tag{4.10}$$

For condition (S2) we assume $(\phi, J) \not\preceq (\theta, I)$. If $J \not\subseteq I$, then $f_{(\phi,J)}$ vanishes on $\hat{X}_{(\theta,I)}$ by definition of the strata and the requirement (4.2) on the poset $\mathcal{I}$. Now let $J \subseteq I$. We have $\pi_{P_J}(\phi) \not\leq \pi_{P_J}(\theta)$, otherwise $\min_Q \circ \pi_{P_J}(\phi) \leq \min_Q \circ \pi_{P_J}(\theta) \leq \theta \leq \max_Q \circ \pi_{P_I}(\theta)$ would be a contradiction to $\rho(\phi, J) \not\leq \rho(\theta, I)$. By the definition of P_J and Lemma B.1 there exists an index $j \in \underline{J}$, such that $\pi_{P_j}(\phi) \not\leq \pi_{P_j}(\theta)$, where $P_j \subseteq G$ is the parabolic subgroup associated to the dominant weight λ_j. Using the equivalence (4.10) we see that $f_{(\phi,J)}$ vanishes on $\hat{X}_{(\theta,I)}$.

Lastly we show (S3). The function $f_{(\theta,I)}$ vanishes on every point $v \in \hat{X}_{(\phi,J)}$ for $(\theta, I) \succ (\phi, J) \in D(\underline{\lambda}, \tau)$. This is clearly the case for $J \subsetneq I$. If $J = I$, there exists an index $i \in \underline{I}$, such that $\pi_{P_i}(\phi) < \pi_{P_i}(\theta)$, since $\pi_{P_I}(\phi) < \pi_{P_I}(\theta)$. Hence $f_{(\theta,I)}$ vanishes on v by (4.10).

Conversely we assume, that $f_{(\theta,I)}$ vanishes on v. Hence there exists an index $i \in \underline{I}$ and an element $\phi \in W/W_{P_i}$ with $v_i \in \hat{X}_\phi$ and $\phi < \pi_{P_i}(\theta)$. First we consider the case $v_i = 0$. If I is minimal in $\mathcal{I}$, then $v = 0$ is automatically contained in the right hand side of (2.2). Else, by Corollary 4.19 and Lemma 4.29, the point v lies in the stratum to $(\min_Q \circ \pi_{Q_J}(\theta), J)$ for $J = I \setminus \{i\}$ and this element is strictly smaller than (θ, I) in $D(\underline{\lambda}, \tau)$. It remains the case $v_i \neq 0$. For each $k \in I$ we choose an element $w_k \in V(\lambda_k)_{\tau_k} \setminus \{0\}$, such that $v_k \in \mathbb{K}w_k$. The tuple $([w_k])_{k \in I}$ lies in a unique Schubert cell C_σ for $\sigma \in W/W_{Q_I}$, viewed as a locally closed subvariety of $\prod_{k \in I} \mathbb{P}(V(\lambda_k)_{\tau_k})$. This element σ is strictly smaller than $\pi_{Q_I}(\theta)$, since $v_i \in \hat{X}_\phi$ and $\phi < \pi_{P_i}(\theta)$. By Lemma B.4 there now exists $\psi \in W/W_{Q_I}$ covered by $\pi_{Q_I}(\theta)$, such that $\pi_{Q_I}(\theta) > \psi \geq \sigma$ and $\pi_{P_I}(\theta) > \pi_{P_I}(\psi) \geq \pi_{P_I}(\sigma)$. Hence (ψ, I) lies in the defining chain poset, is strictly smaller than (θ, I) and $v \in \hat{X}_{(\psi,I)}$ is contained in the right hand side of (2.2). □

Corollary 4.31. *For $\tau = w_0 W_Q$ the stratification from Theorem 4.30 is a (multiprojective) Seshadri stratification on $X = G/Q$. One can determine combinatorially which posets $\mathcal{I}$ are τ-standard via Theorem 4.24 and therefore give rise to such a stratification.*

Similar to the Seshadri stratification on $X_\tau \subseteq V(\lambda)_\tau$ constructed in [CFL2], the bonds of the multiprojective stratification can be described via the root system combinatorics of G. In order to prove the next lemma, we therefore need more notation for root subgroups. For every root α in the root system Φ let U_α denote the associated root subgroup of G (see [Hum, Section 26.3]). Let $\mathbb{G}_\mathrm{a}$ denote the additive algebraic group $(\mathbb{K}, +)$. Up to a scalar multiple, there exists an unique isomorphism $\varepsilon_\alpha : \mathbb{G}_\mathrm{a} \to U_\alpha$, such that $t\varepsilon_\alpha(x)t^{-1} = \varepsilon_\alpha(\alpha(t)x)$ holds for all $t \in T$ and $x \in \mathbb{G}_\mathrm{a}$.

Lemma 4.32. *The bond to a covering relation $(\theta, I) \succ (\phi, J)$ in $D(\underline{\lambda}, \tau)$ is given by*

$$b_{(\theta,I),(\phi,J)} = \begin{cases} |\langle \phi(\lambda_I), \beta^\vee \rangle|, & \text{if } I = J, \\ 1, & \text{if } I \neq J, \end{cases}$$

where β is the unique positive root with $s_\beta \cdot \min_B(\phi) = \min_B(\theta) \in W$ in the case $I = J$.

Proof. The case $I \neq J$ is covered by Lemma 2.6. Now let $I = J$ and $\underline{d} \in \mathbb{N}_0^m$ be the sum of all vectors e_i for $i \in I$. We fix a weight vector v_I in the Demazure module $V(\underline{d} \cdot \underline{\lambda})_\tau$ of weight $\phi(\underline{d} \cdot \underline{\lambda})$ and a weight vector $v_i \in V(\lambda_i)_{\tau_i}$ of weight $\phi(\lambda_i)$ for each $i \in I$. Our proof relies on [CFL2, Lemma 3.3]: It states that the map

$$\overline{f} : U_\phi \times U_{-\beta} \to \mathbb{P}(V(\underline{d} \cdot \underline{\lambda})_\tau), \quad (u, v) \mapsto uv \cdot [v_I],$$

is an isomorphism onto an open subvariety of $X_\theta \subseteq \mathbb{P}(V(\underline{d} \cdot \underline{\lambda})_\tau)$, where $U_\phi \subseteq G$ is a direct product of root subgroups. The explicit construction of U_ϕ does not matter for this proof. It only matters the fact that it is generated by root subgroups to positive roots. For each $i \in I$ we also have a well defined morphism $f_i : U_\phi \times U_{-\beta} \to \mathbb{P}(V(\lambda_i)_{\tau_i})$, $(u, v) \mapsto uv \cdot [v_i]$. Together, they induce the morphism

$$f : U_\phi \times U_{-\beta} \to \prod_{i\in I} \mathbb{P}(V(\lambda_i)_{\tau_i}).$$

Since all maps in (4.8) are B-equivariant, we have the commuting diagram

$$\begin{array}{ccc} U_\phi \times U_{-\beta} & \xrightarrow{\ f\ } & \prod_{i\in I}\mathbb{P}(V(\lambda_i)_{\tau_i}) \\ \bar{f}\big\downarrow & & \big\downarrow \\ \mathbb{P}(V(\underline{d}\cdot\underline{\lambda})_\tau) & \hookrightarrow & \mathbb{P}(\bigotimes_{i\in I} V(\lambda_i)_{\tau_i}) \end{array}$$

It thus follows that f is an isomorphism onto an open subvariety of $X_\theta \subseteq \prod_{i\in I} \mathbb{P}(V(\lambda_i)_{\tau_i})$ as well. In this subvariety the divisor $\hat{X}_{(\phi,I)}$ of $\hat{X}_{(\theta,I)}$ is given by the subset $U_\phi \times \{1\}$.

As the extremal function $f_{(\theta,I)}$ can be seen as a linear function on $\bigotimes_{i\in \underline{I}} V(\lambda_i)_{\tau_i}$, we have to project to the smaller index set $\underline{I} \subseteq I$. Each weight vector $v_{\underline{I}} \in V(\lambda_I)_\tau$ of weight $\phi(\lambda_I)$ defines a morphism

$$f_{\underline{I}}\colon U_\phi \times U_{-\beta} \to \mathbb{P}(V(\lambda_I)_\tau), \quad (u, v) \mapsto uv \cdot [v_{\underline{I}}]$$

and the following diagram commutes:

$$\begin{array}{ccccc} U_\phi \times U_{-\beta} & \overset{f}{\hookrightarrow} & \prod_{i\in I}\mathbb{P}(V(\lambda_i)_{\tau_i}) & \twoheadrightarrow & \prod_{i\in \underline{I}}\mathbb{P}(V(\lambda_i)_{\tau_i}) \\ & \underset{f_{\underline{I}}}{\searrow} & & & \big\downarrow \\ & & \mathbb{P}(V(\lambda_I)_\tau) & \hookrightarrow & \mathbb{P}(\bigotimes_{i\in \underline{I}} V(\lambda_i)_{\tau_i}) \end{array}$$

We choose a parametrization of the affine spaces U_ϕ and $U_{-\beta}$ by parameters $t_\gamma \in \mathbb{K}$ for $\gamma \in \Phi_\phi^+$ and $t_\beta \in \mathbb{K}$ respectively. The action of the root subgroup $U_{-\beta} \cong \mathbb{G}_{\mathrm{a}}$ on the weight vector $v_{\underline{I}} \in V(e_I \cdot \underline{\lambda})_\tau$ is given by the polynomial

$$t_\beta \cdot v_{\underline{I}} = v_{\underline{I}} + t_\beta w_1 + t_\beta^2 w_2 + \cdots + t_\beta^b w_b,$$

where $b = |\langle \phi(\lambda_I), \beta^\vee \rangle|$ and w_i is a weight vector in $V(\lambda_I)_\tau$ of weight $\phi(\lambda_I) - i\beta$ for all $i = 1, \ldots, s$ (this was discussed in the proof of Proposition 27.2 in Humphreys' book [Hum]). Using the coordinates t_γ and t_β, the elements in the image of $f_{\underline{I}}$ are of the form

$$[t_\beta^b w_b + \text{sum of weight vectors in } V(\lambda_I)_\tau \text{ of greater weights than } \theta(\lambda_I)], \tag{4.11}$$

because U_ϕ is unipotent and generated by positive root subgroups. By the construction of the extremal function $f_{(\theta,I)}$, it is a dual vector to the extremal weight space in $V(\lambda_I)_\tau$

of weight $\theta(\lambda_I) = s_\beta(\phi(\lambda_I)) = \phi(\lambda_I) - b\beta$. Applying this to the elements in (4.11) implies that the vanishing multiplicity of $f_{(\theta,I)}$ at the divisor $\hat{X}_{(\phi,I)}$ is equal to b. $\square$

5. The LS-fan of monoids

5.1. LS-monoids and LS-tableaux

Throughout this chapter, we fix a multiprojective Seshadri stratification on a Schubert variety X_τ as in Theorem 4.30. Recall that the defining chain poset $D(\underline{\lambda}, \tau)$ is the underlying poset A of this stratification. In particular, we assume that the index poset $\mathcal{I}$ chosen in Section 4.1 is τ-standard. We show that the stratification is balanced and of LS-type and the elements in the fan of monoids Γ correspond to τ-standard LS-tableaux of type $(\underline{\lambda}, \mathcal{I})$

The stratification on X_τ was built by gluing the Seshadri stratifications on the Schubert varieties $X_{\tau_I} \subseteq \mathbb{P}(V(\lambda_I)_{\tau_I})$ for $I \in \mathcal{I}$. The disjoint union of their underlying posets

$$D(\lambda_I, \tau_I) = \{\theta \in W/W_{P_I} \mid \theta \leq \tau_I\}$$

form the defining chain poset $D(\underline{\lambda}, \tau)$. It was shown in both [CFL2] and [CFL4] that the stratification on X_{τ_I} is of LS-type: For each maximal chain $\mathfrak{C}$ in $D(\lambda_I, \tau_I)$ the lattice $\mathcal{L}^{\mathfrak{C}}$ generated by the monoid $\Gamma_{\mathfrak{C}}$ coincides with the LS-lattice $\mathrm{LS}_{\mathfrak{C},\lambda_I}$ of the chain $\mathfrak{C}$ and the monoid $\Gamma_{\mathfrak{C}}$ is given by the LS-monoid $\mathrm{LS}^+_{\mathfrak{C},\lambda_I} = \mathrm{LS}_{\mathfrak{C},\lambda_I} \cap \mathbb{Q}^{\mathfrak{C}}_{\geq 0}$. The associated fan of monoids shall be denoted by $\mathrm{LS}^+_{\lambda_I}$. We should therefore expect that the multiprojective stratification is of LS-type as well. For this reason we define the following.

Definition 5.1. For every maximal chain $\mathfrak{C}$ in $D(\underline{\lambda}, \tau)$ let $\mathrm{LS}_{\mathfrak{C},\underline{\lambda}}$ be the associated LS-lattice and let $\mathrm{LS}^+_{\mathfrak{C},\underline{\lambda}} = \mathrm{LS}_{\mathfrak{C},\underline{\lambda}} \cap \mathbb{Q}^{\mathfrak{C}}_{\geq 0}$ denote its LS-monoid. The set-theoretic union

$$\mathrm{LS}^+_{\underline{\lambda}} = \bigcup_{\mathfrak{C}} \mathrm{LS}^+_{\mathfrak{C},\underline{\lambda}}$$

over all maximal chains $\mathfrak{C}$ in $D(\underline{\lambda}, \tau)$ is called the **Lakshmibai-Seshadri-fan of monoids** corresponding to to $\underline{\lambda}$, τ and $\mathcal{I}$.

Even though the LS-fan $\mathrm{LS}^+_{\underline{\lambda}}$ does not only depend on the sequence $\underline{\lambda}$ but also on the coset $\tau \in W/W_Q$ and the index poset $\mathcal{I}$, we usually do not index the LS-fan by τ and $\mathcal{I}$ to simplify the notation.

By identifying each poset $D(\lambda_I, \tau_I)$ with $D_I(\underline{\lambda}, \tau) = \{(\theta, J) \in D(\underline{\lambda}, \tau) \mid J = I\}$, we can uniquely decompose the elements $\underline{a} \in \mathrm{LS}^+_{\underline{\lambda}}$ into a sum $\underline{a} = \sum_{I \in \mathcal{I}} \underline{a}^{(I)}$ of elements $\underline{a}^{(I)} \in \mathrm{LS}^+_{\underline{\lambda}} \cap \mathbb{Q}^{D(\lambda_I,\tau_I)}$. The definition of the LS-fan $\mathrm{LS}^+_{\underline{\lambda}}$ suggests that these elements $\underline{a}^{(I)}$ lie in the LS-fan $\mathrm{LS}^+_{\lambda_I}$.

Lemma 5.2. *For each $I \in \mathcal{I}$ the intersection $\mathrm{LS}^+_{\underline{\lambda}} \cap \mathbb{Q}^{D(\lambda_I,\tau_I)}$ coincides with the LS-fan $\mathrm{LS}^+_{\lambda_I}$ of the stratification on $X_{\tau_I} \subseteq \mathbb{P}(V(\lambda_I)_{\tau_I})$.*

Proof. We fix a maximal chain $\mathfrak{C}$ in $D(\underline{\lambda}, \tau)$ and let $I_1 \subseteq \cdots \subseteq I_m = [m]$ be the associated maximal chain in $\mathcal{I}$. As explained in Section 2.6, the LS-lattice to $\mathfrak{C}$ decomposes into

$$\mathrm{LS}_{\mathfrak{C},\underline{\lambda}} = \prod_{j=1}^{m} \mathrm{LS}_{\mathfrak{C},\underline{\lambda}} \cap \mathbb{Q}^{\mathfrak{C}_j},$$

for the subchains $\mathfrak{C}_j = \{(\theta, I) \in \mathfrak{C} \mid I = I_j\}$. This follows from the fact that all bonds connecting the subchains $\mathfrak{C}_j$ are equal to 1.

For fixed $j \in [m]$ let $J = I_j$, $(\theta, J) \succ (\phi, J)$ be a covering relation in $\mathfrak{C}_j$ and β be the unique positive root with $s_\beta \cdot \min_B(\phi) = \min_B(\theta)$. By Lemma 4.32, the corresponding covering relation $\pi_{P_J}(\theta) > \pi_{P_J}(\phi)$ in the poset $D(\lambda_J, \tau_J)$ has the bond $|\langle \phi(\lambda_J), \beta^\vee \rangle|$. Hence the bonds inside $\mathfrak{C}_j$ agree with the bonds in the corresponding chain in $D(\lambda_J, \tau_J)$. Since LS-lattices only depend on the bonds, the sublattice $\mathrm{LS}_{\mathfrak{C},\underline{\lambda}} \cap \mathbb{Q}^{\mathfrak{C}_j} \subseteq \mathrm{LS}_{\mathfrak{C},\underline{\lambda}}$ coincides with the LS-lattice $\mathrm{LS}_{\mathfrak{C}_j,\lambda_J} \subseteq \mathrm{LS}_{\lambda_J}$ to the chain $\mathfrak{C}_j \subseteq D(\lambda_J, \tau_J)$.

For each $I \in \mathcal{I}$ it now follows the equality $\mathrm{LS}^+_{\underline{\lambda}} \cap \mathbb{Q}^{D(\lambda_I,\tau_I)} = \mathrm{LS}^+_{\lambda_I}$: For every maximal chain $\mathfrak{C}$ in $D(\underline{\lambda}, \tau)$ we have

$$\mathrm{LS}^+_{\mathfrak{C},\underline{\lambda}} \cap \mathbb{Q}^{D(\lambda_I,\tau_I)} = \mathrm{LS}_{\mathfrak{C},\underline{\lambda}} \cap \mathbb{Q}^{A}_{\geq 0} \cap \mathbb{Q}^{D(\lambda_I,\tau_I)} = \mathrm{LS}_{\mathfrak{C},\lambda_I} \cap \mathbb{Q}^{\mathfrak{C}_I}_{\geq 0} \subseteq \mathrm{LS}^+_{\lambda_I}$$

for the chain $\mathfrak{C}_I = \mathfrak{C} \cap D(\lambda_I, \tau_I)$. Conversely, each maximal chain $\mathfrak{C}$ in $D(\lambda_I, \tau_I)$ is contained in a maximal chain $\mathfrak{D}$ in $D(\underline{\lambda}, \tau)$. Hence $\mathrm{LS}^+_{\mathfrak{C},\lambda_I}$ is a subset of $\mathrm{LS}^+_{\mathfrak{D},\underline{\lambda}} \subseteq \mathrm{LS}^+_{\underline{\lambda}}$. ☐

Consider the degree map $\mathbb{Q}^{D(\underline{\lambda},\tau)} \to \mathbb{Q}^m$ from Definition 2.16. By the above lemma, we can decompose any element $\underline{a} \in \mathrm{LS}^+_{\underline{\lambda}}$ into a sum of elements $\underline{a}^{(I)} \in \mathrm{LS}^+_{\lambda_I}$ over $I \in \mathcal{I}$. As $\mathrm{LS}^+_{\lambda_I}$ is the fan of monoids of the stratification on $X_{\tau_I} \subseteq \mathbb{P}(V(\lambda_I)_{\tau_I})$, we also have the degree map $\deg_I : \mathbb{Q}^{D(\lambda_I,\tau_I)} \to \mathbb{Q}$. By Lemma 2.17, the degree $\deg_I \underline{a}^{(I)}$ is a non-negative integer. We write $\mathrm{LS}^+_{\lambda_I}(d)$ for the elements of degree $d \in \mathbb{N}_0$ in $\mathrm{LS}^+_{\lambda_I}$. Therefore $\deg \underline{a} = \sum_{I\in\mathcal{I}} \deg_I(\underline{a}^{(I)})\, e_I$ is an element of $\mathbb{N}_0^m$. This provides two partitions of the LS-fan of monoids $\mathrm{LS}^+_{\underline{\lambda}}$ into the subsets

$$\mathrm{LS}^+_{\underline{\lambda}}(\underline{d}) = \{\underline{a} \in \mathrm{LS}^+_{\underline{\lambda}} \mid \deg \underline{a} = \underline{d}\} \quad \text{and} \quad \mathrm{LS}^+_{\underline{\lambda}}(d) = \{\underline{a} \in \mathrm{LS}^+_{\underline{\lambda}} \mid |\deg \underline{a}| = d\}$$

for $\underline{d} \in \mathbb{N}_0^m$ or $d \in \mathbb{N}_0$ respectively.

Lemma 5.2 also implies that the LS-fan of monoids $\mathrm{LS}^+_{\underline{\lambda}}$ is completely determined by the decomposition over the index poset $\mathcal{I}$ and the defining chain poset:

$$\mathrm{LS}^+_{\underline{\lambda}} = \left\{ \underline{a} = (\underline{a}^{(I)})_{I\in\mathcal{I}} \in \prod_{I\in\mathcal{I}} \mathrm{LS}^+_{\lambda_I} \;\middle|\; \exists\, \text{max. chain } \mathfrak{C} \subseteq D(\underline{\lambda}, \tau) \colon \operatorname{supp} \underline{a} \subseteq \mathfrak{C} \right\}. \tag{5.1}$$

This decomposition is also compatible with the degrees: For every $d \in \mathbb{N}_0$ and $I \in \mathcal{I}$ we have the inclusion $\mathrm{LS}^+_{\lambda_I}(d) \subseteq \mathrm{LS}^+_{\underline{\lambda}}(de_I)$.

To each element $\underline{a}$ in the LS-fan $\mathrm{LS}^+_{\underline{\lambda}}$ one can associate a dominant weight. If $a_{(\theta,I)} \in \mathbb{K}$

is the coefficient in $\underline{a}$ of the basis vector $e_{(\theta,I)}$, then we define

$$\operatorname{wt}\underline{a} = \sum_{(\theta,I)\in D(\underline{\lambda},\tau)} a_{(\theta,I)} \cdot \theta(\lambda_I) \in \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}.$$

By its definition, this element only lies in the rational span of the weight lattice Λ, but it follows from the bijections in Proposition 5.5 that $\operatorname{wt}\underline{a}$ is actually contained in Λ^+: Let $\underline{d}$ be the degree of $\underline{a}$ and $\underline{\pi} = (\pi_1, \dots, \pi_s) \in \mathbb{B}(\underline{\lambda}, \mathcal{I})_{\tau,\underline{d}}$ be the unique LS-tableau with $\Theta_{\underline{d}}(\underline{\pi}) = \underline{a}$. By the construction of the maps $\Theta_{\underline{d}}$ and $\Theta_d^{(I)}$, the element $\operatorname{wt}\underline{a}$ is equal to the end point $(\pi_1 * \cdots * \pi_s)(1)$, which is a dominant weight.

Remark 5.3. The LS-fan of monoids is compatible with the induced stratification on $X_{(\theta,I)}$ for a fixed $(\theta, I) \in D(\underline{\lambda}, \tau)$ in the following sense: For each $J \in \mathcal{I}$ with $J \subseteq I$ we have an induced stratification on the Schubert variety $X_{\theta_J} \subseteq \mathbb{P}(V(\lambda_J)_{\theta_J})$ to $\theta_J = \pi_{P_J}(\theta)$ with the underlying poset $D(\lambda_J, \theta_J) = \{\phi \in W/W_{P_J} \mid \phi \leq \theta_J\}$. We include additional indices to differentiate between the LS-fan $\mathrm{LS}^+_{\lambda_J,\tau_J} \subseteq \mathbb{Q}^{D(\lambda_J,\tau_J)}$ associated to the stratification on X_{τ_J} and the LS-fan $\mathrm{LS}^+_{\lambda_J,\theta_J}$ to the stratification on X_{θ_J}. Then $\mathrm{LS}^+_{\lambda_J,\theta_J}$ can naturally be seen as a subset of $\mathrm{LS}^+_{\lambda_J,\tau_J}$ via the linear map $\mathbb{Q}^{D(\lambda_J,\theta_J)} \hookrightarrow \mathbb{Q}^{D(\lambda_J,\tau_J)}$. An element $\underline{a} \in \mathrm{LS}^+_{\lambda_J,\tau_J}$ is contained in this subset, if and only if it is zero or the maximal element in its support is less or equal to θ_I in $D(\lambda_I, \tau_I)$. Analogously, the LS-fan

$$\mathrm{LS}^+_{\underline{\lambda},\theta} = \left\{ \underline{a} \in \prod_{\substack{J\in\mathcal{I} \\ J\subseteq I}} \mathrm{LS}^+_{\lambda_J,\theta_J} \;\middle|\; \exists \,\text{max. chain}\, \mathfrak{C} \subseteq D(\underline{\lambda}, \tau) : (\theta, I) \in \mathfrak{C},\ \operatorname{supp}\underline{a} \subseteq \mathfrak{C} \right\}$$

to the stratification on $X_{(\theta,I)}$ can be seen as a subset of the LS-fan $\mathrm{LS}^+_{\underline{\lambda},\tau}$ to the stratification on X_τ (see Remark 4.17). This subset contains exactly those elements, which are either zero or the maximal element in their support is less or equal to $(\theta, I) \in D(\underline{\lambda}, \tau)$.

Analogous to the stratification from Chapter 3, the fan of monoids $\mathrm{LS}^+_{\lambda_I}$ also has an interpretation in terms of LS-tableaux. For every $d \in \mathbb{N}_0$ let

$$\mathbb{B}(d\lambda_I)_{\tau_I} = \{(\sigma_p, \dots, \sigma_1; 0, a_p, \dots, a_1 = 1) \in \mathbb{B}(d\lambda_I) \mid \sigma_p \leq \tau_I\}.$$

be the set of all LS-paths in $\mathbb{B}(d\lambda)$, such that their initial direction is bounded by τ_I. Using the language of LS-tableaux: This can be viewed as the set of all τ_I-standard LS-tableaux of shape $(d\lambda_I)$. It was proved in [CFL2, Proposition A.6] that the map

$$\Theta_d^{(I)} : \mathbb{B}(d\lambda_I)_{\tau_I} \to \mathrm{LS}^+_{\lambda_I}(d), \quad (\sigma_p, \dots, \sigma_1; 0, a_p, \dots, a_1 = 1) \mapsto \sum_{j=1}^{p} (a_j - a_{j+1}) d e_{\sigma_j} \quad (5.2)$$

is a bijection, where $a_{p+1} := 0$.

It is known that every LS-path $\pi \in \mathbb{B}(d\lambda_I)_{\tau_I}$ can be uniquely decomposed (up to a reparametrization) into a concatenation $\pi = \pi_1 * \cdots * \pi_d$ of LS-paths $\pi_k \in \mathbb{B}(\lambda_I)_{\tau_I}$ with $\min \operatorname{supp} \pi_k \geq \max \operatorname{supp} \pi_{k+1}$ for all $k = 1, \ldots, d-1$. The support $\operatorname{supp} \pi$ of an LS-path $\pi = (\sigma_p, \ldots, \sigma_1; 0, a_p, \ldots, a_1 = 1)$ is the set $\operatorname{supp} \pi = \{\sigma_p, \ldots, \sigma_0\}$. The bijections $\Theta_d^{(I)}$ translate this decomposition to the fan of monoids (see [CFL2, Proposition A.5, Lemma A.8]): Every element $\underline{a} \in \mathrm{LS}^+_{\lambda_I}(d)$ can be uniquely decomposed into a sum $\underline{a} = \underline{a}^1 + \cdots + \underline{a}^d$ of elements $\underline{a}^k \in \mathrm{LS}^+_{\lambda_I}(1)$, such that $\min \operatorname{supp} \underline{a}^k \geq \max \operatorname{supp} \underline{a}^{k+1}$ holds in $D(\lambda_I, \tau_I)$ for all $k = 1, \ldots, d-1$. This property is passed to the LS-fan $\mathrm{LS}^+_{\underline{\lambda}}$ as well.

Lemma 5.4. *Every element $\underline{a} \in \mathrm{LS}^+_{\underline{\lambda}}$ has a unique decomposition $\underline{a} = \underline{a}^1 + \cdots + \underline{a}^s$ into elements $\underline{a}^k \in \bigcup_{I \in \mathcal{I}} \mathrm{LS}^+_{\lambda_I}(1) \subseteq \mathrm{LS}^+_{\underline{\lambda}}$, such that* $\min \operatorname{supp} \underline{a}^k \geq \max \operatorname{supp} \underline{a}^{k+1}$ *holds for all $j = 1, \ldots, s-1$.*

Proof. Let $I_1 \supsetneq \cdots \supsetneq I_j$ be the (unique) chain in $\mathcal{I}$ containing exactly those $I \in \mathcal{I}$ where the component $\underline{a}^{(I)}$ of $\underline{a}$ is non-zero. Then we have $\min \operatorname{supp} \underline{a}_{I_k} \geq \max \operatorname{supp} \underline{a}_{I_{k+1}}$ for all $k = 1, \ldots, r-1$. Therefore the existence and uniqueness of the claimed decomposition of $\underline{a}$ follows from the decomposition in each fan $\mathrm{LS}^+_{\lambda_I}$. □

Proposition 5.5. *Let $\mathbb{B}(\underline{\lambda}, \mathcal{I})_{\tau, \underline{d}}$ be the set of all τ-standard LS-tableaux of type $(\underline{\lambda}, \mathcal{I})$ and degree $\underline{d} \in \mathbb{N}_0^m$. Then the map*

$$\Theta_{\underline{d}} : \mathbb{B}(\underline{\lambda}, \mathcal{I})_{\tau, \underline{d}} \to \mathrm{LS}^+_{\underline{\lambda}}(\underline{d}), \quad (\pi_1, \ldots, \pi_s) \mapsto \Theta_1^{(I_1)}(\pi_1) + \cdots + \Theta_1^{(I_s)}(\pi_s)$$

is a bijection, where $(\pi_1, \ldots, \pi_s)$ is of shape $(\lambda_{I_1}, \ldots, \lambda_{I_s})$ for a weakly decreasing sequence $I_1 \supseteq \cdots \supseteq I_s$ in $\mathcal{I}$.

Proof. By the bijections in (5.2) and the τ-standardness of the tableaux, the image of $\Theta_{\underline{d}}$ is indeed contained in $\mathrm{LS}^+_{\underline{\lambda}}(\underline{d})$, so the map $\Theta_{\underline{d}}$ is well-defined.

Now let $\underline{a} = (\underline{a}_I)_{I \in \mathcal{I}} \in \mathrm{LS}^+_{\underline{\lambda}}(\underline{d})$ with the unique decomposition $\underline{a} = \underline{a}^1 + \cdots + \underline{a}^s$ from Lemma 5.4. Every element $\underline{a}^k$ corresponds to an LS-path $\pi_k \in \mathbb{B}(\lambda_{I_k})$ for some $I_k \in \mathcal{I}$. The associated LS-tableau $\underline{\pi}^{\underline{a}} = (\pi_1, \ldots, \pi_s)$ has degree $\sum_{k=1}^s e_{I_k} = \deg \underline{a} = \underline{d}$ and is τ-standard, since the support of $\underline{a}$ lies in a maximal chain of $D(\underline{\lambda}, \tau)$. The resulting map

$$\Theta_{\underline{d}}^{-1} : \mathrm{LS}^+_{\underline{\lambda}}(\underline{d}) \to \mathbb{B}(\underline{\lambda}, \mathcal{I})_{\tau, \underline{d}}, \quad \underline{a} \mapsto \underline{\pi}^{\underline{a}}$$

is inverse to $\Theta_{\underline{d}}$: By construction, $\Theta_{\underline{d}}^{-1} \circ \Theta_{\underline{d}}$ is the identity, so $\Theta_{\underline{d}}$ is injective. Furthermore, every element $\underline{a} \in \mathrm{LS}^+_{\underline{\lambda}}(\underline{d})$ is contained in its image, as $\Theta_{\underline{d}}(\underline{\pi}^{\underline{a}}) = \underline{a}$. □

5.2. Filtrations of Demazure modules

In order to show that the fan of monoids of the multiprojective stratification on X_τ coincides with the LS-fan $\mathrm{LS}^+_{\underline{\lambda}}$, we use a special set of functions called *path vectors*, which forms a basis of the leaves $R_{\geq \underline{a}}/R_{> \underline{a}}$ associated to the quasi-valuation $\mathcal{V}$. In this section we summarize the definition of path vectors and some of their important properties (see the appendix in [CFL2]). We adapt the notation to our specific situation and only consider dominant weights of the form $d\lambda_I$ for a fixed degree $d \in \mathbb{N}_0$ and index set $I \in \mathcal{I}$. There exists a canonical filtration on the Demazure module $V(d\lambda_I)_{\tau_I}$ and its dual space $V(d\lambda_I)^*_{\tau_I}$, both indexed by the set $\mathrm{LS}^+_{\lambda_I}(d)$. We refer to *loc. cit.* for an explicit construction of the vectors $v_{\underline{b},\underline{\sigma}}$ we mention below and the existence of path vectors.

We define a relation $\rhd$ on $\mathrm{LS}^+_{\lambda_I}(d)$ in the following way: Let $\underline{a}, \underline{b}$ be two elements in $\mathrm{LS}^+_{\lambda_I}(d)$, $\sigma_1 > \cdots > \sigma_p$ be the elements in $\operatorname{supp} \underline{a}$ and $\kappa_1 > \cdots > \kappa_q$ be the elements in $\operatorname{supp} \underline{b}$. The relation $\rhd$ then is defined by

$$\begin{aligned} \underline{a} \rhd \underline{b} \quad \Longleftrightarrow \quad & \sigma_1 > \kappa_1 \text{ or } (\sigma_1 = \kappa_1 \text{ and } a_{\sigma_1} > b_{\kappa_1}) \text{ or} \\ & (\sigma_1 = \kappa_1 \text{ and } a_{\sigma_1} = b_{\kappa_1} \text{ and } \sigma_2 > \kappa_2) \text{ or} \\ & (\sigma_1 = \kappa_1 \text{ and } a_{\sigma_1} = b_{\kappa_1} \text{ and } \sigma_2 > \kappa_2 \text{ and } a_{\sigma_2} = b_{\kappa_2}) \text{ or } \dots . \end{aligned}$$

We write $\underline{a} \unrhd \underline{b}$ if $\underline{a} = \underline{b}$ or $\underline{a} \rhd \underline{b}$. This relation coincides with the definition from [CFL2, Definition 6.1].

Recall that the quasi-valuation $\mathcal{V}$ of the stratification on X_τ depends on the choice of a total order $\geq^t$ on $D(\underline{\lambda}, \tau)$ linearizing the partial order $\succeq$. We also denote the associated lexicographic order on $\mathbb{Q}^{D(\underline{\lambda},\tau)}$ by $\geq^t$. Note that the relation $\unrhd$ has the following property for all $\underline{a}, \underline{b} \in \mathrm{LS}^+_{\lambda_I}(d)$: If $\underline{a} \unrhd \underline{b}$, then we have $\underline{a} \geq^t \underline{b}$ for every possible choice of the total order $\geq^t$ on $D(\underline{\lambda}, \tau)$.

To each element $\underline{a} \in \mathrm{LS}^+_{\lambda_I}(d)$ and a reduced decomposition $\underline{\sigma}$ of the unique maximal element σ in the support of $\underline{a}$ one can associate a vector $v_{\underline{a},\underline{\sigma}} \in V(d\lambda_I)_{\tau_I}$ of weight $\operatorname{wt} \underline{a}$. As σ is not an element of the Weyl group itself, a reduced decomposition of σ is a decomposition $\min_B(\sigma) = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_\ell}}$ into simple reflections with ℓ minimal. When fixing a reduced decomposition $\underline{\sigma}^{\underline{a}}$ of $\max \operatorname{supp} \underline{a}$ for each element $\underline{a} \in \mathrm{LS}^+_{\lambda_I}(d)$, then the set $\{v_{\underline{a},\underline{\sigma}^{\underline{a}}} \mid \underline{a} \in \mathrm{LS}^+_{\lambda_I}(d)\}$ is a basis of the Demazure module $V(d\lambda_I)_{\tau_I}$. This basis does depend on the chosen reduced decompositions, but there is a canonical filtration on $V(d\lambda_I)_{\tau_I}$ via the subspaces

$$\begin{aligned} V(d\lambda_I)_{\tau_I, \unlhd \underline{a}} &= \left\langle v_{\underline{b},\underline{\sigma}} \;\middle|\; \underline{b} \in \mathrm{LS}^+_{\lambda_I}(d),\ \underline{a} \unrhd \underline{b},\ \underline{\sigma} \text{ reduced decomposition of } \max \operatorname{supp} \underline{b} \right\rangle_{\mathbb{K}}, \\ V(d\lambda_I)_{\tau_I, \lhd \underline{a}} &= \left\langle v_{\underline{b},\underline{\sigma}} \;\middle|\; \underline{b} \in \mathrm{LS}^+_{\lambda_I}(d),\ \underline{a} \rhd \underline{b},\ \underline{\sigma} \text{ reduced decomposition of } \max \operatorname{supp} \underline{b} \right\rangle_{\mathbb{K}}. \end{aligned}$$

For each $\underline{a} \in \mathrm{LS}^+_{\lambda_I}(d)$ the subquotient $V(d\lambda_I)_{\tau_I, \unlhd \underline{a}} / V(d\lambda_I)_{\tau_I, \lhd \underline{a}}$ is one-dimensional.

The language of path vectors gives rise to a similar filtration on the dual space $V(d\lambda_I)^*_{\tau_I}$.

Definition 5.6 ([CFL2, Definition 6.4])**.** A **path vector** to an element $\underline{a} \in \mathrm{LS}^+_{\lambda_I}(d)$ is a weight vector $p_{\underline{a}} \in V(d\lambda_I)^*_{\tau_I}$ of weight $(-\operatorname{wt} \underline{a})$, such that

(a) there exists a reduced decomposition $\underline{\sigma}$ of $\sigma = \max \operatorname{supp} \underline{a}$ with $p_{\underline{a}}(v_{\underline{a},\underline{\sigma}}) = 1$;

(b) for each $\underline{a}' \in \mathrm{LS}^+_{\lambda_I}(d)$ and all reduced decompositions $\underline{\sigma}'$ of $\sigma' = \max \operatorname{supp} \underline{a}'$, $p_{\underline{a}}(v_{\underline{a}',\underline{\sigma}'}) \neq 0$ implies $\underline{a}' \unrhd \underline{a}$.

In [Lit98], Littelmann used quantum Frobenius splitting to associate a canonical function to every element $\underline{a} \in \mathrm{LS}^+_{\lambda_I}(d)$, which he called path vector. It satisfies the above conditions, hence the definition we use here is more general and there exists a path vector to each element in $\mathrm{LS}^+_{\lambda_I}(d)$.

Again, one obtains a basis of the dual module $V(d\lambda_I)^*_{\tau_I}$ by fixing a path vector to each element $\underline{a} \in \mathrm{LS}^+_{\lambda_I}(d)$ and the subspaces

$$V(d\lambda_I)^*_{\tau_I,\trianglerighteq\underline{a}} = \left\langle p_{\underline{b}} \;\middle|\; p_{\underline{b}} \text{ path vector to some } \underline{b} \in \mathrm{LS}^+_{\lambda_I}(d) \text{ with } \underline{b} \trianglerighteq \underline{a} \right\rangle_{\mathbb{K}},$$
$$V(d\lambda_I)^*_{\tau_I,\triangleright\underline{a}} = \left\langle p_{\underline{b}} \;\middle|\; p_{\underline{b}} \text{ path vector to some } \underline{b} \in \mathrm{LS}^+_{\lambda_I}(d) \text{ with } \underline{b} \triangleright \underline{a} \right\rangle_{\mathbb{K}}.$$

define a filtration on $V(d\lambda_I)^*_{\tau_I}$, such the subquotient $V(d\lambda_I)^*_{\tau_I,\trianglerighteq\underline{a}}/V(d\lambda_I)^*_{\tau_I,\triangleright\underline{a}}$ is one-dimensional for each $\underline{a} \in \mathrm{LS}^+_{\lambda_I}(d)$. Any path vector $p_{\underline{a}}$ to $\underline{a}$ defines a non-zero element of this subquotient.

5.3. The quasi-valuation of a path vector

Throughout this section we use the notation from Remark 5.3 for the LS-fans $\mathrm{LS}^+_{\lambda_I,\theta_I} \subseteq \mathrm{LS}^+_{\lambda_I,\tau_I}$ of an induced stratification. By Lemma 4.28, every path vector to an element $\underline{a} \in \mathrm{LS}^+_{\lambda_I}(d)$ can be seen as a multihomogeneous function in $R = \mathbb{K}[X_\tau]$ of degree de_I. Their vanishing behaviour can be described combinatorially in the following way.

Lemma 5.7. *A path vector $p_{\underline{a}}$ to an element $\underline{a} \in \mathrm{LS}^+_{\lambda_I}(1)$ vanishes identically on the multicone $\hat{X}_{(\phi,I)}$ to $(\phi, I) \in D(\underline{\lambda}, \tau)$, if and only if the unique maximal element θ in* $\operatorname{supp}\underline{a} \subseteq D(\lambda_I, \tau_I)$ *is not less or equal to $\phi_I = \pi_{P_I}(\phi)$.*

Proof. It suffices to prove this statement for the affine cone $\hat{X}_\phi \subseteq V(\lambda_I)_{\tau_I}$ instead of the multicone $\hat{X}_{(\phi,I)} \subseteq \prod_{i=1}^m V(\lambda_i)_{\tau_i}$. The same equivalence then also follows for the multicone via the diagram (4.9) for $\underline{d} = e_I$.

The Demazure module $V(\lambda_I)_\phi$ is equal to the linear span of the affine cone $\hat{X}_\phi$. Since the path vector $p_{\underline{a}}$ is linear, it vanishes identically on $\hat{X}_\phi$, if and only if it vanishes on every vector of the form $v_{\underline{a}',\underline{\sigma}'} \in V(\lambda_I)_\phi$ for $\underline{a}' \in \mathrm{LS}^+_{\lambda_I,\phi_I}(1)$ and a reduced decomposition $\underline{\sigma}'$ of $\sigma' = \max\operatorname{supp}\underline{a}'$.

If $\theta \leq \phi_I$, then the vector $v_{\underline{a},\underline{\theta}}$ is contained in $\mathrm{LS}^+_{\lambda_I,\phi_I}(1)$ for every reduced decomposition $\underline{\theta}$ of θ. Hence $p_{\underline{a}}$ does not vanish on $\hat{X}_\phi$. Conversely, if $p_{\underline{a}}(v_{\underline{a}',\underline{\sigma}'}) \neq 0$ for some element $\underline{a}' \in \mathrm{LS}^+_{\lambda_I,\phi_I}(1)$, then we have $\underline{a}' \trianglerighteq \underline{a}$ by the definition of a path vector, i. e. the maximal element σ' in $\operatorname{supp}\underline{a}'$ is larger or equal to θ. But as $\underline{a}' \in \mathrm{LS}^+_{\lambda_I,\phi_I}$, it follows $\phi_I \geq \sigma' \geq \theta$. □

For each total order $\geq^t$ on $D(\underline{\lambda}, \tau)$ and every fixed element $(\theta, I) \in D(\underline{\lambda}, \tau)$ we have an induced total order on the underlying poset $D(\underline{\lambda}, \tau)_{\preceq(\theta,I)}$ of the stratification on $\hat{X}_{(\theta,I)}$. Let $\mathcal{V}_{(\theta,I)}$ denote the associated quasi-valuation on $\mathbb{K}[X_{(\theta,I)}]$. Path vectors are compatible with the induced stratification, which allows the use of inductive arguments.

Lemma 5.8. *Let $p_{\underline{a}}$ be a path vector to an element $\underline{a} \in \mathrm{LS}^+_{\lambda_I}(1) \subseteq \mathrm{LS}^+_{\underline{\lambda}}(e_I)$ and $(\theta, I) = \max\operatorname{supp}\underline{a} \in D(\underline{\lambda}, \tau)$. Then the restriction of $p_{\underline{a}}$ to the multicone $\hat{X}_{(\theta,I)}$ is a path vector to the element $\underline{a} \in \mathrm{LS}^+_{\lambda_I,\theta_I}(1)$ and it holds*

$$\mathcal{V}(p_{\underline{a}}) = \mathcal{V}_{(\theta,I)}(p_{\underline{a}}|_{\hat{X}_{(\theta,I)}}).$$

Proof. By Lemma 6.6 in [CFL2], the restriction of $p_{\underline{a}} \in V(\lambda_I)*_\tau$ to $V(\lambda_I)_\theta$ is a path vector associated to $\underline{a} \in \mathrm{LS}^+_{\lambda_I,\theta_I}(1)$. The corresponding function in $\mathbb{K}[\hat{X}_{(\theta,I)}]$ coincides with the restriction of the function $p_{\underline{a}} \in \mathbb{K}[\hat{X}_\tau]$ to the subvariety $\hat{X}_{(\theta,I)} \subseteq \hat{X}_\tau$. This can be shown via the diagram (4.9).

If $\underline{\theta}_I$ is any reduced decomposition of $\theta_I \in W/W_{P_I}$, then $v_{\underline{a},\underline{\theta}_I}$ lies in the Demazure module $V(\lambda_I)_{\theta_I}$. Hence $p_{\underline{a}}$ restricts to a non-zero element in $\mathbb{K}[\hat{X}_{(\theta,I)}]$ and we have

$$\mathcal{V}_{(\theta,I)}(p_{\underline{a}}|_{\hat{X}_{(\theta,I)}}) = \min\{\mathcal{V}_{\mathfrak{C}}(p_{\underline{a}}) \mid \mathfrak{C} \text{ max. chain in } D(\underline{\lambda}, \tau), (\theta, I) \in \mathfrak{C}\}$$

by the definition of the quasi-valuation.

Let $\mathfrak{C}$ be any maximal chain in $D(\underline{\lambda}, \tau)$ and (ϕ, J) be the minimal element in $\mathfrak{C}$, such that the path vector $p_{\underline{a}}$ does not vanish identically on $X_{(\phi,J)}$. This means that the coefficient of the basis vector $e_{(\phi,J)}$ in $\mathcal{V}_{\mathfrak{C}}(p_{\underline{a}})$ is positive. We now show $(\phi, J) \succeq (\theta, I)$, since this implies $\mathcal{V}_{\mathfrak{C}}(p_{\underline{a}}) \geq^t \mathcal{V}_{(\theta,I)}(p_{\underline{a}}|_{\hat{X}_{(\theta,I)}})$ for every choice of the total order $\geq^t$. Therefore the quasi-valuation is given by $\mathcal{V}(p_{\underline{a}}) = \mathcal{V}_{(\theta,I)}(p_{\underline{a}}|_{\hat{X}_{(\theta,I)}})$.

As $p_{\underline{a}}$ has degree e_I in $\mathbb{K}[\hat{X}_\tau]$, this implies $\underline{I} \subseteq J$, hence we have $I \subseteq J$ by the requirement (4.2) on the poset $\mathcal{I}$. By Corollary 4.19, the element $(\phi^\triangledown, I)$ with $\phi^\triangledown = \min_Q \circ \pi_{Q_I}(\phi)$ is less or equal to (ϕ, J) in $D(\underline{\lambda}, \tau)$. The elements ϕ and $\phi^\triangledown$ are equal in W/W_{P_I}, hence the images of the multicones $\hat{X}_{(\phi,J)}$ and $\hat{X}_{(\phi^\triangledown,I)}$ under the projection map

$$\prod_{i\in J} V(\lambda_i)_{\tau_i} \twoheadrightarrow \prod_{i\in \underline{I}} V(\lambda_i)_{\tau_i}$$

coincide. We assumed that $p_{\underline{a}}$ does not vanish identically on $\hat{X}_{(\phi,J)}$, so it does not vanish identically on $\hat{X}_{(\phi^\triangledown,I)}$ as well. It now follows $(\theta, I) \preceq (\phi^\triangledown, I)$ from Lemma 5.7. This completes the proof. □

Proposition 5.9. *Let $p_{\underline{a}} \in R$ be a path vector to an element $\underline{a} \in \mathrm{LS}^+_{\lambda_I}(1)$ for some $I \in \mathcal{I}$. Then $\mathcal{V}(p_{\underline{a}}) = \underline{a}$ holds independent of the chosen total order $\geq^t$ on $D(\underline{\lambda}, \tau)$.*

Proof. We prove the statement by induction over the rank r of $(\tau, [m])$ in the defining chain poset. The case $r = 0$ is trivial. If $r \geq 1$, we consider two cases. First, when the maximal element (θ, I) in the support of $\underline{a}$ is strictly smaller than $(\tau, [m])$, we use the induced stratification on $\hat{X}_{(\theta,I)}$ and Lemma 5.8 to conclude

$$\mathcal{V}(p_{\underline{a}}) = \mathcal{V}_{(\theta,I)}(p_{\underline{a}}|_{\hat{X}_{(\theta,I)}}) = \underline{a}$$

by induction. Notice, that this holds independent of the chosen total order $\geq^t$.

Now suppose that $(\tau, [m])$ is the maximal element in $\operatorname{supp} \underline{a}$. Here we have $I = [m]$. Fix a positive integer $N \in \mathbb{N}$ with $Nu \in \mathbb{N}$, where $u \in \mathbb{Q}_{\geq 0}$ is the coefficient of the basis vector $e_{(\tau,[m])}$ in $\underline{a}$. We use Corollary C.13 in [CFL2] on the element $\underline{a} \in \mathrm{LS}^+_{\lambda_I}$: It states that $p_{\underline{a}}^N$ is – up to multiplying by a non-zero scalar – equal to $p_\tau^{Nu} p_{\underline{b}} \in V(N\lambda_I)^*_{\tau_I}$, where p_τ is a weight vector in $V(\lambda_I)^*$ of weight $-\tau(\lambda_I)$ and $p_{\underline{b}}$ is a path vector associated to the element

$$\underline{b} = N\underline{a} - Nu\, e_{\tau_I} \in \mathrm{LS}^+_{\lambda_I}(s)$$

of degree $s = Nd - Ndu$. The pullback of p_τ to the multicone $\hat{X}_\tau$ is the extremal function $f_{(\tau,[m])}$ (up to a non-zero scalar), so we have $\mathcal{V}(f^{Nu}_{(\tau,[m])}) = Nu\, e_{(\tau,[m])}$, which holds independent of the choice of $\geq^t$.

For $s = 0$ the path vector $p_{\underline{b}}$ is constant and it follows

$$\mathcal{V}(p_{\underline{a}}) = \frac{1}{N}\mathcal{V}(p_{\underline{a}}^N) = \frac{1}{N}\mathcal{V}(f^{Nu}_{(\tau,[m])}) = u\, e_{(\tau,[m])} = \underline{a}.$$

Now we assume $s \geq 1$. As the element $\underline{b}$ might not be of degree one, we have to write it in terms of path vectors of degree one to use the induction. Therefore we fix a path vector $\overline{p}_{\underline{c}}$ to each $\underline{c} \in \mathrm{LS}^+_{\lambda_I}(1)$. This defines a function $g_{\underline{c}}$ for every element $\underline{c} \in \mathrm{LS}^+_{\lambda_I}(s)$: To each $\underline{c}^k$ in the unique decomposition $\underline{c} = \underline{c}^1 + \cdots + \underline{c}^s$ from Lemma 5.4 we have the path vector $\overline{p}_{\underline{c}^k}$. It was shown in [CFL2, Proposition C.10] that the product $g_{\underline{c}} := \overline{p}_{\underline{c}^1} \cdots \overline{p}_{\underline{c}^s}$ in $\bigotimes_{i=1}^s V(\lambda_I)_{\tau_I}$ restricts to a path vector associated to $\underline{c}$, up to multiplying by a root of unity.

Using the filtration of $V(s\lambda_I)^*_{\tau_I}$ via the subspaces $V(s\lambda_I)^*_{\tau_I,\trianglerighteq \underline{c}}$, we can write the path vector $p_{\underline{b}}$ as a linear combination $p_{\underline{b}} = g_{\underline{b}} + \sum_{\underline{c} \triangleright \underline{b}} d_{\underline{c}}\, g_{\underline{c}}$ over elements $\underline{c} \in \mathrm{LS}^+_{\lambda_I}(s)$ with $\underline{c} \trianglerighteq \underline{b}$. We now show

$$\mathcal{V}(g_{\underline{c}}) > \mathcal{V}(g_{\underline{b}}) \quad \text{for all } \underline{c} \triangleright \underline{b}. \tag{5.3}$$

Let $\sigma' = \max \operatorname{supp} \underline{c}$ and $\sigma = \max \operatorname{supp} \underline{b}$. We need to distinguish between two cases.

If $\sigma' = \sigma$ we have $\max \operatorname{supp} \underline{c}^k \preceq (\sigma, I) \prec (\tau, [m])$ for each $k \in [s]$ and thus $\mathcal{V}(\overline{p}_{\underline{c}^k}) = \underline{c}^k$ by induction. Because the union of all $\operatorname{supp} \underline{c}^k$ for $k \in [s]$ lies in a maximal chain of $D(\underline{\lambda}, \tau)$, the quasi-valuation is additive:

$$\mathcal{V}(g_{\underline{c}}) = \mathcal{V}(\overline{p}_{\underline{c}^1} \cdots \overline{p}_{\underline{c}^s}) = \underline{c}^1 + \cdots + \underline{c}^s = \underline{c}.$$

This is independent of the choice of $\geq^t$. Analogously, we see $\mathcal{V}(g_{\underline{b}}) = \underline{b}$. Since $\underline{c} \triangleright \underline{b}$, it holds $\mathcal{V}(g_{\underline{c}}) = \underline{c} \geq^t \underline{b} = \mathcal{V}(g_{\underline{b}})$ for every choice of the total order $\geq^t$ on $D(\underline{\lambda}, \tau)$.

In the remaining case $\sigma' \neq \sigma$ we have $\sigma' > \sigma$, as $\underline{c} \triangleright \underline{b}$. It follows from Lemma 5.7 that $g_{\underline{c}}$ does not vanish identically on $\hat{X}_{(\sigma',I)}$, but it restricts to the zero function on the multicone $\hat{X}_{(\phi,I)}$ for each $\phi < \sigma'$ in $D(\lambda_I, \tau_I)$. The function $g_{\underline{c}}$ also vanishes identically on every multicone $\hat{X}_{(\phi,J)}$ for $(\phi, J) \prec (\sigma', I)$ in $D(\underline{\lambda}, \tau)$ and $J \subsetneq I$, because $g_{\underline{c}}$ is homogeneous of degree e_I and $\underline{I} \nsubseteq J$. Hence (σ', I) lies in the support of $\mathcal{V}(g_{\underline{c}})$. Analogously, one can show that σ is the maximal element in $\operatorname{supp} \mathcal{V}(g_{\underline{b}})$. Therefore $\mathcal{V}(g_{\underline{c}}) > \mathcal{V}(g_{\underline{b}})$.

Using the inequality (5.3) we can now conclude $\mathcal{V}(p_{\underline{b}}) = \mathcal{V}(g_{\underline{b}}) = \underline{b}$. Hence the set $\{(\tau, [m])\} \cup \operatorname{supp} \mathcal{V}(p_{\underline{b}})$ lies in a maximal chain of $D(\underline{\lambda}, \tau)$ and it follows

$$\begin{aligned}\mathcal{V}(p_{\underline{a}}) &= \frac{1}{N}\mathcal{V}(p_{\underline{a}}^N) = \frac{1}{N}\mathcal{V}(f_{(\tau,[m])}^{Nu} p_{\underline{b}}) = \frac{1}{N}\big(\mathcal{V}(f_{(\tau,[m])}^{Nu}) + \mathcal{V}(p_{\underline{b}})\big) \\ &= u\, e_{(\tau,[m])} + \big(\underline{a} - u\, e_{(\tau,[m])}\big) = \underline{a}.\end{aligned}$$

This is independent of the choice of the total order $\geq^t$. □

5.4. Standard monomial theory

For every $I \in \mathcal{I}$ the indecomposable elements in the fan $\mathrm{LS}^+_{\lambda_I}$ are exactly the elements of degree one. In case of our generalized stratification, however, not every indecomposable element in Γ needs to be of total degree one. Instead, it follows from Lemma 5.4 that the set of indecomposables in the LS-fan $\mathrm{LS}^+_{\underline{\lambda}}$ is equal to

$$\mathbb{G} = \bigcup_{I \in \mathcal{I}} \mathrm{LS}^+_{\lambda_I}(1).$$

We therefore fix a path vector $p_{\underline{a}} \in R$ to each element $\underline{a} \in \mathbb{G}$ and let $\mathbb{G}_R = \{p_{\underline{a}} \mid \underline{a} \in \mathbb{G}\}$ be the set of all these functions.

Definition 5.10. A monomial $p_{\underline{a}^1} \cdots p_{\underline{a}^s} \in R$ of path vectors in $\mathbb{G}_R$ is called **standard**, if $\min \operatorname{supp} \underline{a}^k \geq \max \operatorname{supp} \underline{a}^{k+1}$ holds for all $k = 1, \dots, s-1$.

Another way to characterize standardness comes from the language of LS-tableaux: Let $p_{\underline{a}^1} \cdots p_{\underline{a}^s}$ be a monomial of path vectors in $\mathbb{G}_R$ with $\underline{a}^k \in \mathrm{LS}^+_{\lambda_{I_k}}(1)$. Each element $\underline{a}^k$ corresponds to an LS-path π_k of shape λ_{I_k}. The monomial $p_{\underline{a}^1} \cdots p_{\underline{a}^s}$ is standard, if and only if the LS-tableau $\underline{\pi} = (\pi_1, \dots, \pi_s)$ is of type $(\underline{\lambda}, \mathcal{I})$ and τ-standard.

By Lemma 5.4 we have an associated standard monomial

$$p_{\underline{a}} \coloneqq p_{\underline{a}^1} \cdots p_{\underline{a}^s}$$

to every element $\underline{a} \in \mathrm{LS}^+_{\underline{\lambda}}$, where $\underline{a} = \underline{a}^1 + \cdots + \underline{a}^s$ is the unique decomposition into elements $\underline{a} \in \mathbb{G}$ with $\min \operatorname{supp} \underline{a}^k \geq \max \operatorname{supp} \underline{a}^{k+1}$ holds for all $k = 1, \dots, s-1$. Conversely, every standard monomial in $\mathbb{G}_R$ is of the form $p_{\underline{a}}$ for some $\underline{a} \in \mathrm{LS}^+_{\underline{\lambda}}$. The monomial $p_{\underline{a}}$ is a multihomogeneous function in R of degree $\sum_{k=1}^{s} \deg \underline{a}^k = \deg \underline{a}$.

Theorem 5.11.

(a) *For each $\underline{a} \in \mathrm{LS}^+_{\underline{\lambda}}$ holds $\mathcal{V}(p_{\underline{a}}) = \underline{a}$. Additionally, the set of all standard monomials in $\mathbb{G}_R$ forms a vector space basis of $R = \mathbb{K}[\hat{X}_\tau]$.*

(b) *The fan of monoids to this Seshadri stratification coincides with the LS-fan* $\mathrm{LS}^+_{\underline{\lambda}}$*. The stratification is balanced and of LS-type. In particular, it is normal.*

Proof. (a) For each standard monomial, the set $\operatorname{supp}\underline{a}^1 \cup \cdots \cup \operatorname{supp}\underline{a}^s$ is contained in a maximal chain of $D(\underline{\lambda}, \tau)$, so the quasi-valuation is additive and Proposition 5.9 implies

$$\mathcal{V}(p_{\underline{a}}) = \mathcal{V}(p_{\underline{a}^1} \cdots p_{\underline{a}^s}) = \underline{a}^1 + \cdots + \underline{a}^s = \underline{a}.$$

Now fix a tuple $\underline{d} \in \mathbb{N}_0^m$. The set $\{p_{\underline{a}} \mid \underline{a} \in \mathrm{LS}^+_{\underline{\lambda}}(\underline{d})\}$ is linearly independent in the graded component $\mathbb{K}[\hat{X}_\tau]_{\underline{d}}$, since the quasi-valuation $\mathcal{V}$ is injective on it. By Lemma 4.28, the cardinality of this set is therefore bounded by the dimension of the Demazure module $V(\underline{d} \cdot \underline{\lambda})_\tau$. On the other hand, we have seen in Proposition 5.5, that there is a bijection between $\mathrm{LS}^+_{\underline{\lambda}}(\underline{d})$ and the set $\mathbb{B}(\underline{\lambda}, \mathcal{I})_{\tau,\underline{d}}$ of all τ-standard LS-tableaux of type $(\underline{\lambda}, \mathcal{I})$ with degree $\underline{d}$. The degree of an LS-tableau is determined by its shape and there always exists at least one shape to each degree (see Remark 4.7). For each subset of LS-tableaux in $\mathbb{B}(\underline{\lambda}, \mathcal{I})_{\tau,\underline{d}}$ of a fixed shape $(\lambda_{I_1}, \ldots, \lambda_{I_s})$, we have the Demazure-type character formula from equation (4.4), so the size of this subset is equal to the dimension of $V(\underline{d} \cdot \underline{\lambda})_\tau$. In total, we get the following inequalities:

$$\dim V(\underline{d} \cdot \underline{\lambda})_\tau \leq |\mathbb{B}(\underline{\lambda}, \mathcal{I})_{\tau,\underline{d}}| = |\mathrm{LS}^+_{\underline{\lambda}}(\underline{d})| \leq \dim V(\underline{d} \cdot \underline{\lambda})_\tau. \tag{5.4}$$

Therefore the standard monomials of degree $\underline{d}$ form a basis of $\mathbb{K}[\hat{X}_\tau]_{\underline{d}}$.

(b) As the standard monomials in $\mathbb{G}_R$ form a basis of R, their image under $\mathcal{V}$ agrees with the fan of monoids Γ, hence $\Gamma = \mathrm{LS}^+_{\underline{\lambda}}$. This also implies that the stratification is of LS-type, since the other two requirements are fulfilled by construction. Furthermore, we have seen that the quasi-valuation $\mathcal{V}(p_{\underline{a}}) = \underline{a}$ of each standard monomial $p_{\underline{a}}$ does not depend on the choice of the total order $\geq^t$ on $D(\underline{\lambda}, \tau)$, so the stratification is also balanced. □

Corollary 5.12.

(a) *To each* $\underline{d} \in \mathbb{N}_0^m$ *there exists exactly one weakly decreasing sequence* $I_1 \supseteq \cdots \supseteq I_s$ *in* $\mathcal{I}$ *with* $\sum_{k=1}^s e_{I_k} = \underline{d}$.

(b) *For every* $I \in \mathcal{I}$ *and* $d \in \mathbb{N}_0$ *it holds* $\mathrm{LS}^+_{\lambda_I}(d) = \mathrm{LS}^+_{\underline{\lambda}}(de_I)$.

Proof. Statement (a) follows immediately from the inequalities in (5.4) and we have already seen the inclusion $\mathrm{LS}^+_{\lambda_I}(d) \subseteq \mathrm{LS}^+_{\underline{\lambda}}(de_I)$. For the reverse inclusion let $\underline{a}$ be an element in $\mathrm{LS}^+_{\underline{\lambda}}(de_I)$ and write it in the form $\underline{a} = \underline{a}^{(I_1)} + \cdots + \underline{a}^{(I_m)}$, where $I_1 \subsetneq \cdots \subsetneq I_m$ is a maximal chain in $\mathcal{I}$ and $\underline{a}^{(I_j)} \in \mathrm{LS}^+_{\lambda_{I_j}}$ for all $j \in [m]$. There exist non-negative integers $k_1, \ldots, k_m$ such that $\underline{a}^{(I_j)}$ is of degree k_j in $\mathrm{LS}^+_{\lambda_{I_j}}$. Since $\sum_{j=1}^m k_j e_{I_j} = de_I$ it now follows from part (a) that $I = I_j$ for some $j \in [m]$ and $k_j = d$. Hence $\underline{a} = \underline{a}^{(I_j)} \in \mathrm{LS}^+_{\lambda_I}(d)$. □

This completes the goal of this thesis. We have seen that there exists a normal and balanced Seshadri stratification on each multiprojective Schubert variety X_τ. The elements in its fan of monoids correspond to τ-standard LS-tableaux of type $(\underline{\lambda}, \mathcal{I})$. All tableaux of a fixed degree $\underline{d} \in N_0^m$ have the same shape $(\lambda_{I_1}, \ldots, \lambda_{I_s})$. The decomposition of $\underline{a} \in \mathrm{LS}^+_{\underline{\lambda}}(\underline{d})$ into indecomposable elements corresponds exactly to the columns $\pi_k \in \mathbb{B}(\lambda_{I_k})$, $k \in [s]$, in the corresponding LS-tableau to $\underline{a}$. We have a standard monomial theory on $\mathbb{K}[X_\tau]$ determined by the indecomposable elements $\mathbb{G}$. Each non-standard monomial in $\mathbb{G}_R$ can be written as a linear combination of standard monomials via a straightening relation as in Proposition 2.12.

Remark 5.13. In [CFL, Theorem 15.12] it was also shown that standard monomials are compatible with all strata in the stratification, in our case with all Schubert varieties $X_{(\theta,I)}$ for $(\theta, I) \in D(\underline{\lambda}, \tau)$: A standard monomial $p_{\underline{a}}$ to $\underline{a} \in \mathrm{LS}^+_{\underline{\lambda}}$ does not vanish on the multicone $\hat{X}_{(\theta,I)}$, if and only if $\max \operatorname{supp} \underline{a} \preceq (\theta, I)$. These monomials are called *standard on* $X_{(\theta,I)}$ and their restrictions to $\hat{X}_{(\theta,I)}$ form a basis of the multihomogeneous coordinate ring $\mathbb{K}[X_{(\theta,I)}]$.

Corollary 5.14. *Let* $\mathbb{B}(\underline{\lambda}, \mathcal{I})_{\tau,\underline{d}}$ *be the set of all* τ*-standard LS-tableaux* $\underline{\pi}$ *of type* $(\underline{\lambda}, \mathcal{I})$ *and degree* $\underline{d} \in \mathbb{N}_0^m$.

(a) The standard monomials in $\mathbb{G}_R$ *of degree* $\underline{d} \in \mathbb{N}_0^m$ *form a basis of the module* $V(\underline{d} \cdot \underline{\lambda})^*$*, indexed by* $\mathbb{B}(\underline{\lambda}, \mathcal{I})_{\tau,\underline{d}}$.

(b) Suppose that the fixed path vectors $p_{\underline{a}}$ *for* $\underline{a} \in \mathbb{G}$ *are constructed as in [Lit98] via quantum Frobenius splitting. Then the standard monomial basis from part (a) coincides with the standard monomial basis from Section 6 in loc. cit..*

Proof. The first statement is a consequence of Theorem 5.11 and Proposition 5.5. To show the second statement, we fix the unique weakly decreasing sequence $I_1 \supseteq \cdots \supseteq I_s$ in $\mathcal{I}$ with $e_{I_1} + \cdots + e_{I_s} = \underline{d}$. Then every LS-tableau in $\mathbb{B}(\underline{\lambda}, \mathcal{I})_{\tau,\underline{d}}$ is of shape $(\lambda_{I_1}, \ldots, \lambda_{I_s})$. We abbreviate these dominant weights by $\lambda_k := \lambda_{I_k}$. For each LS-path $\pi \in \mathbb{B}(\lambda_k)$, $k \in [s]$, let $p_\pi \in V(\lambda_k)^*_\tau$ denote the path vector constructed via quantum Frobenius splitting (see [Lit98, Section 3]). By Proposition 4.4, a monomial $p_{\pi_1} \cdots p_{\pi_s}$ in these path vectors is standard in the sense of *loc. cit.*, if and only if it is standard in the sense of Definition 5.10. □

Appendices

A. Multiproj-schemes

A.1. The Multiproj-construction

Let $R = \bigoplus_{\underline{d} \in \mathbb{Z}^m} R_{\underline{d}}$ be a (commutative) ring, which is graded by the group $\mathbb{Z}^m$. An element $r \in R$ is called *(multi-)homogeneous*, if it is contained in a subgroup $R_{\underline{d}}$. In this case $\deg r := \underline{d}$ is its *(multi-)degree*. Ideals generated by homogeneous elements are called *(multi-)homogeneous ideals*.

There is a similar construction to the Proj-construction for $\mathbb{N}_0$-graded rings, which associates a scheme $\operatorname{Multiproj} R$ to the multigraded ring R. In general, this scheme does not have all the nice properties of the usual Proj-scheme, for example it need not be projective or separated. To introduce these schemes, we follow the construction from Brenner and Schröer in [BS]. The grading on R corresponds to an action of the m-torus $\operatorname{Spec} \mathbb{Z}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$ on $\operatorname{Spec} R$. There exists a quotient $\operatorname{Quot}(R)$ of $\operatorname{Spec} R$ with respect to this action in the category of ringed spaces. However, this quotient is not a quotient in the category of schemes in general.

An element $f \in R$ is called **relevant**, if it is homogeneous and the degrees of the homogeneous elements $g \in R$, which divide some power f^k for $k \in \mathbb{N}_0$, generate a subgroup of $\mathbb{Z}^m$ of finite index. For every relevant $f \in R$ the morphism $\operatorname{Spec} R_f \to \operatorname{Spec} R_{(f)}$ is a geometric quotient (in the sense of GIT), where $R_{(f)}$ denotes the subring of R_f of all elements of multidegree zero. Therefore we have an open subset $D_+(f) \subseteq \operatorname{Quot}(R)$ isomorphic to $\operatorname{Spec} R_{(f)}$. The Multiproj-scheme of R is then defined as the locally ringed space

$$\operatorname{Multiproj} R = \bigcup_{\substack{f \in R \\ \text{relevant}}} D_+(f) \subseteq \operatorname{Quot}(R). \tag{A.1}$$

Let R_+ be the ideal in R generated by all relevant elements. It is called the **irrelevant ideal**. The induced morphism

$$\operatorname{Spec} R \setminus V(R_+) \to \operatorname{Multiproj} R$$

is then a geometric quotient with respect to the torus action.

There is also another way of realizing the scheme $\operatorname{Multiproj} R$, which directly generalizes the usual Proj-construction. We denote this scheme by $\operatorname{Multiproj} R$ as well. Set-theoretically it is given by

$$\operatorname{Multiproj} R = \{P \subseteq R \mid P \text{ multihomogeneous prime ideal}, R_+ \nsubseteq P\}$$

and the closed subsets are those of the form

$$V(I) = \{P \in \operatorname{Multiproj} R \mid P \supseteq I\},$$

where I is a multihomogeneous ideal in R. For each $P \in \operatorname{Multiproj} R$ let $R_{(P)}$ denote the subring of homogeneous elements of multidegree 0 in the localization R_P. For an open subset $U \subseteq \operatorname{Multiproj} R$ we define the ring $\mathcal{O}(U)$ of functions

$$f : U \to \coprod_{P \in U} R_{(P)},$$

which are locally given by a quotient of elements in R: To each $P \in U$ there exists an open neighborhood V of P in U and multihomogeneous elements $r, s \in R$ of the same multidegree, such that $f(P) = \frac{r}{s} \in R_{(P)}$.

The topological space $\operatorname{Multiproj} R$ together with the sheaf $\mathcal{O}$ forms a locally ringed space and the stalk at a point $P \in \operatorname{Multiproj} R$ is canonically isomorphic to the local ring $R_{(P)}$. For every relevant element $f \in R$ we have an isomorphism between the open subset $D_+(f) = \{P \in \operatorname{Multiproj} R \mid f \notin P\}$ and $\operatorname{Spec} R_{(f)}$, topologically given by

$$\chi_f : D_+(f) \to \operatorname{Spec} R_{(f)}, \quad P \mapsto \langle \phi_f(P) \rangle \cap R_{(f)},$$

where ϕ_f denotes the natural map $R \to R_f$ and $\langle \phi_f(P) \rangle$ is the ideal generated by $\phi_f(P)$. This can be seen as follows: First of all, one can use ϕ_f to construct an isomorphism $\operatorname{Multiproj} R_f \to D_+(f)$ of locally ringed spaces. On the other hand, the inclusion $\iota_f : R_{(f)} \hookrightarrow R_f$ induces an isomorphism $\operatorname{Multiproj} R_f \to \operatorname{Spec} R_{(f)}$. We can write the inclusion ι_f as the composition of the embedding $R_{(f)} \hookrightarrow R_{(f)}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$ and a ring homomorphism $R_{(f)}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}] \to R_f$ sending t_i to a homogeneous unit in R_f of multidegree e_i (which exists, since f is relevant). Both maps are graded ring homomorphisms, when we set $\deg t_i = e_i$. The first map induces an isomorphism $\operatorname{Multiproj} R_{(f)}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}] \to \operatorname{Spec} R_{(f)}$ and the second map is an isomorphism of graded rings. In total, we get the desired isomorphism $D_+(f) \cong \operatorname{Spec} R_{(f)}$. Note that the product fg of two relevant elements $f, g \in R$ is relevant as well and the inclusions we just constructed are compatible in the sense that the following diagram commutes:

$$\begin{array}{ccc} \operatorname{Spec} R_{(fg)} & \hookrightarrow & \operatorname{Spec} R_{(f)} \\ \downarrow & & \downarrow \\ \operatorname{Spec} R_{(g)} & \hookrightarrow & \operatorname{Multiproj} R. \end{array}$$

In particular, we see that $\operatorname{Multiproj} R$ is a scheme, which is isomorphic to the scheme defined in (A.1). The structure morphism $\operatorname{Spec} R \setminus V(R_+) \to \operatorname{Multiproj} R$ maps every homogeneous prime ideal not containing R_+ to itself.

Lemma A.1. *Let $A : \mathbb{Z}^m \to \mathbb{Z}^m$ be an injective group homomorphism.*

(a) For every ring homomorphism $\phi : R \to S$ between two $\mathbb{Z}^m$-graded rings with $\phi(R_{\underline{d}}) \subseteq S_{A(\underline{d})}$ for all $\underline{d} \in \mathbb{Z}^m$, the morphism $\operatorname{Spec} S \to \operatorname{Spec} R$ *induces a morphism*

$$(\operatorname{Multiproj} S) \setminus V(\langle \phi(R_+) \rangle) \to \operatorname{Multiproj} R. \tag{A.2}$$

(b) The inclusion of the graded ring

$$R^{(A)} = \bigoplus_{\underline{d} \in \mathbb{Z}^m} R_{A(\underline{d})} \subseteq R$$

into R induces an isomorphism $\operatorname{Multiproj} R \to \operatorname{Multiproj} R^{(A)}$.

Proof. Since A is injective, the image of a relevant element $f \in R$ under ϕ is relevant in S. The map $R_{(f)} \to S_{(\phi(f))}$ induces a morphism $\operatorname{Spec} S_{(\phi(f))} \to \operatorname{Spec} R_{(f)}$. For $f, g \in R$ relevant these morphisms can be glued along the inclusion $\operatorname{Spec} R_{(fg)} \hookrightarrow \operatorname{Spec} R_{(f)}$. They therefore define the desired morphism in (A.2) because the subsets $D_+(\phi(f)) \subseteq \operatorname{Multiproj} S$ for $f \in R$ relevant cover the scheme $(\operatorname{Multiproj} S) \setminus V(\langle \phi(R_+) \rangle)$.

By part (a) the inclusion $R^{(A)} \hookrightarrow R$ induces a morphism

$$\operatorname{Multiproj} R \setminus V(\langle R_+^{(A)} \rangle) \to \operatorname{Multiproj} R^{(A)}.$$

First, we show $V(\langle R_+^{(A)} \rangle) = \varnothing$. Let $P \in \operatorname{Multiproj} R$ be a homogeneous prime ideal containing $R_+^{(A)}$ and $f \in R$ be a relevant element. We fix homogeneous divisors $g_i \mid f^{n_i}$ (for $i = 1, \dots, m$), such that $\deg g_1, \dots, \deg g_m$ generate a subgroup of $\mathbb{Z}^m$ of finite index. Since $A(e_1), \dots, A(e_m) \in \mathbb{Z}^m$ generate $\mathbb{Q}^m$ as a vector space, the degree $\underline{d}$ of f can be expressed in the form

$$\underline{d} = \sum_{i=1}^{m} \frac{p_i}{q_i} \cdot A(e_i)$$

for some $p_i \in \mathbb{Z}$ and $q_i \in \mathbb{N}$. We set $N = q_1 \dots q_m$. Therefore f^N lies in the subring $R^{(A)}$. In the same way we get natural numbers N_i with $g_i^{N_i} \in R^{(A)}$. Then $g_i^{NN_i}$ divides $f^{n_i NN_i}$ and their degrees $\deg g^{NN_i}$ still generate a subgroup of $\mathbb{Z}^m$ of finite index. These degrees lie in the image of A. Using the injectivity of A we see that f^N is relevant in $R^{(A)}$. So the ideal P contains f^N and since P is prime, it contains f as well. Hence $R_+ \subseteq P$.

If $f \in R^{(A)}$ is relevant, then f is also relevant in R and the induced map $R_{(f)}^{(A)} \to R_{(f)}$ is a ring isomorphism. This proves that $\operatorname{Multiproj} R \to \operatorname{Multiproj} R^{(A)}$ is an isomorphism of schemes. □

Example A.2. The algebra $R = \mathbb{K}[x, y]$ over $\mathbb{K}$ with the grading $\deg x = (1, 0)$ and $\deg y = (1, 1)$ shows, that for $\mathbb{N}_0^m$-graded rings the irrelevant ideal does not need to agree with the ideal R'_+ generated by all homogeneous elements of degrees $\underline{d}$ with $d_i \geq 1$ for all i. By Lemma A.1 we can regrade R via the group homomorphism $A : \mathbb{Z}^2 \to \mathbb{Z}^2$, $(d_1, d_2) \mapsto (d_1 + d_2, d_2)$ and obtain an isomorphism $\operatorname{Multiproj} R \cong \operatorname{Multiproj} R^{(A)}$. The irrelevant ideal of $R^{(A)}$ is equal to (xy) and $\operatorname{Multiproj} R^{(A)}$ is isomorphic to a point. On the other hand R'_+ is generated by y and there are two multihomogeneous prime ideals in R not containing y: (0) and (x).

Example A.3. Consider the algebra $R = \mathbb{K}[x, y, z]$ over $\mathbb{K}$ with the grading $\deg x = (1, 0)$, $\deg y = (0, 1)$ and $\deg z = (1, 1)$. Its irrelevant ideal is generated by the relevant elements xy, xz and yz, hence the three open subsets defined by these elements cover $\operatorname{Multiproj} R$. We have

$$\mathbb{K}[x, y, z]_{(xy)} \cong \mathbb{K}[\tfrac{xy}{z}] \cong \mathbb{K}[x, y, z]_{(xz)} \quad \text{and} \quad \mathbb{K}[x, y, z]_{(yz)} \cong \mathbb{K}[\tfrac{z}{xy}].$$

The corresponding affine schemes are glued along their intersections

$$\mathbb{K}[x, y, z]_{(xy\,\cdot\, xz)} \cong \mathbb{K}[x, y, z]_{(xy\,\cdot\, xz)} \cong \mathbb{K}[x, y, z]_{(xy\,\cdot\, xz)} \cong \mathbb{K}[\tfrac{xy}{z}, \tfrac{z}{xy}]$$

via the obvious inclusions. Therefore $\operatorname{Multiproj} R$ is a non-separated projective line with two points at infinity.

Lemma A.4. *Let $I, J \subseteq R$ be multihomogeneous ideals.*

(a) The map $R \twoheadrightarrow R/I$ induces a closed immersion $\operatorname{Multiproj} R/I \hookrightarrow \operatorname{Multiproj} R$. *Topologically its image coincides with the closed subset $V(I)$. If I is prime, then* $\operatorname{Multiproj} R/I$ *is an integral scheme.*

(b) The scheme-theoretic intersection of $V(I)$ and $V(J)$ is given by $V(I + J)$.

(c) The scheme-theoretic union of $V(I)$ and $V(J)$ is given by $V(I \cap J)$.

Proof. All three statements can be checked in the open subschemes $\operatorname{Spec} R_{(f)}$ for $f \in R$ relevant. They are compatible with the projection $R \twoheadrightarrow R/I$ as the scheme $\operatorname{Multiproj} R/I$ is covered by all $D_+(f + I)$ for $f \in R \setminus I$ relevant.

Let $f \in R$ be relevant and ϕ_f be the natural map $R \to \mathbb{R}_f$. The second and third statement follow from the analogous statement for affine schemes and the fact that

$$\begin{aligned} \chi : \{\text{homogeneous ideals in } R\} &\to \{\text{ideals in } R_{(f)}\} \\ I &\mapsto \langle \phi_f(I) \rangle \cap R_{(f)} \end{aligned}$$

preserves sums and intersections: $\chi(I + J) = \chi(I) + \chi(J)$ and $\chi(I \cap J) = \chi(I) \cap \chi(J)$. □

Lemma A.5 ([KU, Lemma 3.6])**.** *If R is an integral domain and the degrees of its non-zero, homogeneous elements span a subgroup of maximal rank in $\mathbb{Z}^m$, then* $\operatorname{Multiproj} R$ *is an integral scheme of dimension* $\dim \operatorname{Spec} R - m$.

Lemma A.6. *If R is an $\mathbb{N}_0^m$-graded, noetherian, reduced ring, that is finitely generated in total degree 1 as an R_0-algebra, then* $\operatorname{Multiproj} R$ *is a separated, reduced scheme of finite type over R_0.*

Proof. We fix homogeneous generators $s_1, \ldots, s_k$ of R as an R_0-algebra of total degree 1. The irrelevant ideal is equal to the direct sum R'_+ of all subgroups $R_{\underline{d}}$ with $d_i \geq 1$ for all $i = 1, \ldots, m$: As R is $\mathbb{N}_0^m$-graded, every relevant element is contained in R'_+. Conversely each monomial in the generators $s_1, \ldots s_k$, that lies in R'_+, is relevant and therefore $R_+ = R'_+$. This ideal is generated by the finite set S of all monomials in the generators $s_1, \ldots s_k$ of degree $(1, \ldots, 1)$, hence $\operatorname{Multiproj} R$ is covered by the open subsets $D_+(f)$ for $f \in S$. Using Proposition 3.3 in [BS] we see that $\operatorname{Multiproj} R$ is separated. Furthermore Proposition 2.5 in *loc. cit.* implies, that the morphism $\operatorname{Multiproj} R \to \operatorname{Spec} R_0$ is of finite type. Finally, since every localization of R is reduced, $\operatorname{Multiproj} R$ is covered by reduced affine schemes and thus is reduced itself. □

A.2. Multiprojective varieties

In this section we summarize some properties of multiprojective varieties, i. e. closed subvarieties X of a product $\mathbb{P} := \mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_m)$ of projective spaces, where $V_1, \ldots, V_m$ are finite-dimensional vector spaces over an algebraically closed field $\mathbb{K}$. It is rather difficult to find the theory of multiprojective varieties in the literature, as it is a direct generalization of the theory of embedded projective varieties $Y \subseteq \mathbb{P}(V)$.

We fix a closed subvariety $X \subseteq \mathbb{P}$. The **multicone** $\hat{X} \subseteq V_1 \times \cdots \times V_m$ of X is the closure in $\mathbb{P}$ of the preimage of X under the morphism

$$\pi : (V_1 \setminus \{0\}) \times \cdots \times (V_m \setminus \{0\}) \twoheadrightarrow \mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_m).$$

A point $(v_1, \ldots, v_m) \in V_1 \times \cdots \times V_m$ is contained in the multicone, if and only if there exist non-zero vectors $w_i \in V_i$ such that $v_i \in \mathbb{K} w_i$ and $\pi(w_1, \ldots, w_m) \in X$. The coordinate ring $\mathbb{K}[X] := \mathbb{K}[\hat{X}]$ of the multicone is called the **multihomogeneous coordinate ring** of X. Its prime spectrum is isomorphic to $\hat{X}$. Note that the multihomogeneous coordinate rings of two multiprojective varieties may not be isomorphic, even if the varieties are isomorphic, so $\mathbb{K}[X]$ does depend on the embedding of X into a product of projective spaces. The $\mathbb{N}_0$-grading on the polynomial ring $\mathbb{K}[V_i] \cong \bigoplus_{d \in \mathbb{N}_0} \operatorname{Sym}^d V_i^*$ induces an $\mathbb{N}_0^m$-grading on

$$\mathbb{K}[\mathbb{P}] = \mathbb{K}[V_1] \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} \mathbb{K}[V_m],$$

which corresponds to the $(\mathbb{K}^\times)^m$-action given by component-wise scalar multiplication:

$$(t_1, \ldots, t_m) \cdot (v_1, \ldots, v_m) = (t_1 v_1, \ldots, t_m v_m)$$

for all $t_1, \dots, t_m \in \mathbb{K}^\times$ and $v_i \in V_i$. The (affine) vanishing ideal $I(\hat{X}) \subseteq \mathbb{K}[\mathbb{P}]$ is multihomogeneous, hence $\mathbb{K}[\hat{X}]$ also graded by $\mathbb{N}_0^m$.

For the rest of this section let $R = \mathbb{K}[X]$. Every multihomogeneous element $f \in R$ defines the closed subset of all $x \in X$ with $f(x) = 0$ and each closed subset of X is of the form

$$V_{\mathbb{P}}(I) = \{x \in X \mid f(x) = 0 \text{ for all } f \in I \text{ multihomogeneous}\}$$

for a multihomogeneous ideal $I \subseteq R$. Conversely, every closed subset $Y \subseteq X$ defines the multihomogeneous ideal

$$I_{\mathbb{P}}(Y) = \langle \{f \in R \mid f \text{ multihomogeneous and } f(Y) = 0\} \rangle \subseteq R.$$

In this notation the multicone of X is equal to the (affine) vanishing set $V(I_{\mathbb{P}}(X))$.

The projective Nullstellensatz can also be generalized to the multiprojective setting. It involves the ideal quotient

$$(I : J) = \{r \in R \mid rJ \subseteq I\}$$

of two ideals $I, J \subseteq R$. We say I is ***J*-saturated**, if $(I : J) = I$. As we have seen in the proof of Lemma A.6, the irrelevant ideal of R is given by

$$R_+ = \bigoplus_{\substack{\underline{d} \in \mathbb{N}_0^m \\ d_i \geq 1 \forall i}} R_{\underline{d}}.$$

Proposition A.7 (Multiprojective Nullstellensatz, [FM, Section 1.8])**.** *If $I \subseteq R$ is a multihomogeneous ideal, then*

$$I_{\mathbb{P}}(V_{\mathbb{P}}(I)) = (\sqrt{I} : R_+).$$

In particular, we have a bijection $Y \mapsto I_{\mathbb{P}}(Y)$ between the closed subvarieties $Y \subseteq X$ and all R_+-saturated, multihomogeneous radical ideals in R, which do not contain R_+. Irreducible closed subvarieties of X correspond to multihomogeneous prime ideals in R not containing R_+ (as they are automatically R_+-saturated).

Remark A.8. If R is an $\mathbb{N}_0^m$-graded, reduced $\mathbb{K}$-algebra, that is finitely generated by elements of total degree 1, then R is isomorphic to the multihomogeneous coordinate ring of a multiprojective variety X and $\operatorname{Multiproj} R \cong X$.

It was shown in [HHRT] that multiprojective varieties also have an associated Hilbert polynomial: There exists a unique polynomial $H_R \in \mathbb{Q}[x_1, \dots, x_m]$, such that $H_R(\underline{d}) = \dim \mathbb{K}[X]_{\underline{d}}$ holds for all $\underline{d} \geq \underline{d}'$ (component-wise comparison) for some $\underline{d}' \in \mathbb{N}_0^m$. Here it is essential that the algebra $\mathbb{K}[X]$ is generated by elements of total degree one, otherwise the Hilbert polynomial is replaced by a function, which is only a quasi-polynomial on certain cones and glued together along their facets. The total degree $\deg H_R$ of the Hilbert polynomial is equal to the dimension of X and it can be uniquely written in the form

$$H_R(\underline{d}) = \sum_{\underline{k}\in\mathbb{N}_0^m} a_{\underline{k}} \binom{d_1+k_1}{k_1} \cdots \binom{d_m+k_m}{k_m}$$

with coefficients $a_{\underline{k}} \in \mathbb{Z}$. For $k_1 + \cdots + k_m = \deg H_R$ these numbers are called the **multidegrees** of X and they are non-negative. We denote them by $\deg_{\underline{k}}(X) = a_{\underline{k}}$. There is a useful criterion proved in [CCL+] for determining which multidegrees are non-zero and thus actually appear in the Hilbert polynomial. It states that $\deg_{\underline{k}}(X)$ is positive, if and only if

$$\sum_{i\in I} k_i \leq \dim \pi_I(X)$$

holds for all subsets $I \subseteq [m]$, where $\pi_I : \prod_{i=1}^m \mathbb{P}(V_i) \twoheadrightarrow \prod_{i\in I} \mathbb{P}(V_i)$ is the natural projection.

Remark A.9. Let $\underline{k} \in \mathbb{N}_0^m$ with $k_1 + \cdots + k_m = \dim X$. The multidegree $\deg_{\underline{k}}(X)$ can also be interpreted as the number of points in the intersection of X in $\prod_{i=1}^m \mathbb{P}(V_i)$ with a subspace $\mathbb{P}(L_1) \times \cdots \times \mathbb{P}(L_m)$ in general position, where $L_i \subseteq V_i$ is a non-zero linear subspace of codimension k_i.

The homogeneous component G_R of the Hilbert polynomial $H_R \in \mathbb{Q}[x_1, \ldots, x_m]$ of the highest total degree it equal to

$$G_R = \sum_{\underline{k}} \frac{\deg_{\underline{k}}(X)}{k_1! \cdots k_m!} x_1^{k_1} \cdots x_m^{k_m},$$

where the sum runs over all $\underline{k} \in \mathbb{N}_0^m$ with $k_1 + \cdots + k_m = \dim X$. The value of G_R at a point $\underline{d} \in \mathbb{N}_0^m$ can also be written as

$$G_R(\underline{d}) = \lim_{n\to\infty} \frac{\dim R_{n\underline{d}}}{n^{\dim X}}.$$

This function $G_R : \mathbb{N}_0^m \to \mathbb{R}$ is sometimes called the *volume function* of R. Its connection to convex geometry via global Newton-Okounkov bodies was studied in [CMM] and we use the ideas of this paper for the Section 2.5 on Newton-Okounkov complexes.

Lemma A.10 (Multiprojective Jacobi-criterion)**.** *We identify the multihomogeneous coordinate ring of* $\mathbb{P} := \prod_{i=1}^m \mathbb{P}^{n_i}$ *with the polynomial ring* $S = \mathbb{K}[x_{i,j} \mid (i,j) \in J]$ *for* $J = \{(i,j) \in \mathbb{N}_0^2 \mid 1 \leq i \leq m, 0 \leq j \leq n_i\}$. *Let* $X \subseteq \mathbb{P}$ *be a closed subvariety and* $f_1, \ldots, f_r$ *be multihomogeneous generators of the vanishing ideal* $I_{\mathbb{P}}(X) \subseteq S$. *Then a* *point* $x = ([v_1], \dots, [v_m]) \in X$ *with* $v_i \in \mathbb{K}^{n_i+1} \setminus \{0\}$ *is smooth, if and only if the rank of the Jacobian matrix*

$$\left(\frac{\partial f_k}{\partial x_{i,j}}(v_1, \dots, v_m) \right)_{k=1,\dots,r, j \in J} \tag{A.3}$$

is at least $n_1 + \dots + n_m + m - \dim X$.

Proof. Define the affine space $V = \prod_{i=1}^m \mathbb{K}^{n_i+1}$. We can assume w.l.o.g. that x is contained in the affine patch

$$U = \{(u_1, \dots, u_m) \in V \mid x_{1,0}(u_1) = \dots = x_{m,0}(u_m) = 1\}.$$

Let $\iota : U \hookrightarrow V$ be the inclusion. The (affine) coordinate ring of U can be identified with the polynomial ring $S' = \mathbb{K}[x_{i,j} \mid (i,j) \in J']$ for $J' = \{(i,j) \in \mathbb{N}_0^2 \mid 1 \le i \le m, 1 \le j \le n_i\}$. The ideal $I(X \cap U) \subseteq S'$ is the dehomogenization of $I_{\mathbb{P}}(X)$, i.e. the image under the comorphism $\iota^* : S \to S'$ of ι. As the smoothness of x can be checked locally in U, the affine Jacobi-criterion implies that x is smooth, if and only if the rank of the matrix

$$A' = \left(\frac{\partial \circ \iota^*(f_k)}{\partial x_{i,j}}(\iota(\overline{v})) \right)_{k=1,\dots,r, j \in J'}$$

Since ι^* commutes with the partial derivatives in a coordinate of J', this is a submatrix of the matrix A in (A.3). The columns not contained in this submatrix are linearly dependent to the columns of A', as

$$\sum_{j=0}^{n_j} x_{i,j} \cdot \frac{\partial f}{\partial x_{i,j}} = \deg(f)_i \cdot f$$

holds for every multihomogeneous polynomial $f \in S$. Therefore x is smooth if and only if $\operatorname{rank} A = \operatorname{rank} A' \ge \dim V - \dim(X \cap U) = n_1 + \dots + n_m + m - \dim X$. □

Corollary A.11. *Let* $X \subseteq \prod_{i=1}^m \mathbb{P}(V_i)$ *be a closed subvariety. Then* X *is smooth in codimension one, if and only if its affine multicone* $\hat{X}$ *is smooth in codimension one.*

Proof. Let $V = V_1 \times \dots \times V_m$ and $Z_i \subseteq V$ be the preimage of $\{0\} \subseteq V_i$ under the linear projection $\hat{\pi}_i : V \twoheadrightarrow V_i$. The open subvariety $U = V \setminus (\bigcup_{i=1}^m Z_i)$ is the preimage of $\prod_{i=1}^m \mathbb{P}(V_i)$ under the natural morphism

$$\pi : \prod_{i=1}^m V_i \setminus \{0\} \twoheadrightarrow \prod_{i=1}^m \mathbb{P}(V_i).$$

By Lemma A.10, a point $x \in \hat{X} \cap U$ is smooth, if and only if its corresponding projective point $\pi(x) \in X$ is smooth, since the rank of the Jacobian matrix in (A.3) is independent of the $(\mathbb{K}^\times)^m$-action.

Now suppose that X is smooth in codimension one and let S be an irreducible component of the subvariety $\mathrm{Sing}(\hat{X})$ of singular points in the multicone $\hat{X}$. If $S \cap U \neq \varnothing$ then $S \cap U$ is an irreducible component of $\hat{X} \cap U$. Hence $\dim S = \dim S \cap U = \dim \pi(S \cap U) + m \leq \dim X - 2 + m = \dim \hat{X} - 2$, since $\mathrm{Sing}(X)$ has at least codimension 2 in X. Thus $\hat{X}$ is smooth in codimension one.

Conversely if $\hat{X}$ is smooth in codimension one and S is an irreducible component of the subvariety $\mathrm{Sing}(X)$ of singular points in X, then $\pi^{-1}(S)$ is contained in an irreducible component S' of $\mathrm{Sing}(\hat{X})$. Therefore $\dim S = \pi^{-1}(S) - m \leq \dim \hat{X} - 2 - m = \dim X - 2$ and X is smooth in codimension one. □

We close this section with a lemma which is useful for computing the quasi-valuation of a Seshadri stratification via the decomposition of R into its homogeneous components. The action of the torus $(\mathbb{K}^\times)^m$ on $\hat{X}$ induces an action on R, where an element $\underline{t} \in (\mathbb{K}^\times)^m$ acts on $g \in R$ via the left translation $\underline{t} \cdot g =: g^{\underline{t}}$, where $g^{\underline{t}}$ is the regular function on $\hat{X}$ with $g^{\underline{t}}(x) = g(\underline{t}^{-1} \cdot x)$ for all $x \in \hat{X}$.

Lemma A.12. *For every $h \in R$, the linear subspace generated by the multihomogeneous components $h_{\underline{d}}$, $\underline{d} \in \mathbb{N}_0^m$, of h coincides with the linear subspace, which is spanned by all function $h^{\underline{t}}$ for $\underline{t} \in (\mathbb{K}^\times)^m$.*

Proof. It suffices to show this statement for $X = \prod_{i=1}^m \mathbb{P}(V_i)$. By choosing a basis of every vector space V_i, we identify R with the polynomial ring in the variables $x_{i,j}$ for $i \in \{1, \ldots, m\}$ and $j \in \{0, \ldots, n_i\}$, where $n_i = \dim V_i - 1$. The torus $(\mathbb{K}^\times)^m$ acts as scalars on each subspace $R_{\underline{d}}$, $d \in \mathbb{N}_0^m$. Hence every function $h^{\underline{t}}$ for $\underline{t} \in (\mathbb{K}^\times)^m$ can be written as a linear combination of the multihomogeneous components $h_{\underline{d}}$. It remains to show the other inclusion of vector spaces.

For $\underline{c} = (c_1, \ldots, c_m) \in \mathbb{N}_0^m$ let $R_{\underline{c}} \subseteq R$ be the linear subspace of all $h \in R$, such that $h_{\underline{d}} \neq 0$ only if $d_k < c_k$ holds for all $k = 1, \ldots, m$. We prove by induction over m, that for all $\underline{c}, \underline{d} \in \mathbb{N}_0^m$ there exists a finite set $S \subseteq (\mathbb{K}^\times)^m$, such that the multihomogeneous component $h_{\underline{d}}$ of every $h \in R_{\underline{c}}$ can be written as $h_{\underline{d}} = \sum_{\underline{t} \in S} a_{\underline{t}} h^{\underline{t}}$, where the scalars $a_{\underline{t}} \in \mathbb{K}$ are independent of h.

The induction base $m = 0$ is trivial. So now let $m \geq 1$ and fix a primitive c_m-th root of unity $\zeta \in \mathbb{K}^\times$. Let B_m be the basis of the algebra $R_m = \mathbb{K}[x_{m,j} \mid 1 \leq j \leq n_m]$ of all monomials in the variables $x_{m,j}$ and let $h \in R_{\underline{c}}$, which we write in the form $h = \sum_{g \in B_m} f_g \cdot g$, where f_g lies in the ring R' of polynomials in the variables $x_{i,j}$ for $i \in \{1, \ldots, m-1\}$ and $j \in \{1, \ldots, n_i\}$. We define h_j, $j \in \mathbb{N}_0$, to be the sum of all $f_g \cdot g$, where g is of degree j in R_m. For every $i = 0, \ldots, c_m$ we have

$$h^{(1,\ldots,1,\zeta^i)} = \sum_{j=0}^{c_m} \zeta^{ij} h_j.$$

As the matrix $A = (\zeta^{ij})_{i,j=0,\ldots,c_m}$ is a Vandermonde-matrix and its determinant is non-zero by the choice of ζ, we get $\langle h^{(1,\ldots,1,\zeta^i)} \mid i = 0, \ldots, c_m \rangle_{\mathbb{K}} = \langle h_j \mid j = 0, \ldots, c_m \rangle_{\mathbb{K}}$.

Fix a tuple $\underline{d} \in \mathbb{N}_0^m$ and let $\underline{d}' = (d_1, \dots, d_{m-1})$. By induction the following equation holds for a finite set $S' \subseteq (\mathbb{K}^\times)^{m-1}$:

$$h_{\underline{d}} = \sum_{\substack{g \in B_m \\ \deg(g)=d_m}} (f_g)_{\underline{d}'}\, g = \sum_{\substack{g \in B_m \\ \deg(g)=d_m}} \sum_{\underline{s}' \in S'} a_{\underline{s}'} (f_g)^{\underline{s}'} g = \sum_{\underline{s}' \in S'} a_{\underline{s}'} \left(\sum_{\substack{g \in B_m \\ \deg(g)=d_m}} (f_g \cdot g)^{(\underline{s}',1)} \right)$$
$$= \sum_{\underline{s}' \in S'} a_{\underline{s}'} (h_{d_m})^{(\underline{s}',1)} = \sum_{\underline{s}' \in S'} \sum_{i=0}^{s} a_{\underline{s}'} a_i h^{(\underline{s}',\zeta^i)}$$

The scalars a_i are the entries in the d_m-th row of the inverse matrix of A. We see that the products $a_{\underline{s}'} a_i$ only depend on the choice of $\underline{c}$ and $\underline{d}$. $\square$

B. Weyl groups

This section contains a brief summary of the notation in this thesis and a loose collection of lemmata. All statements which we do not prove here are well known and can be found in any classical text book about Coxeter groups, for example in [BB].

We fix a semisimple algebraic group G with Weyl group W, a maximal torus T of G and a Borel subgroup B containing T. For each parabolic subgroup $Q \subseteq G$ – by which we mean a closed subgroup containing B – with Weyl subgroup $W_Q \subseteq W$ and $\sigma \in W$, there is a (unique) smallest element $\sigma^Q \in W$ in the coset σW_Q. It has the property $\ell(\sigma^Q \tau) = \ell(\sigma^Q) + \ell(\tau)$ for all $\tau \in W_Q$. We denote the set of these smallest elements by

$$W^Q = \{\sigma^Q \in W \mid \sigma \in W\}.$$

Thus the product map $W^Q \times W_Q \to W$ is a length-preserving bijection. More general: For any two parabolic subgroups $Q \subseteq Q'$ the product map

$$W^{Q'} \times (W_{Q'} \cap W^Q) \to W^Q, \quad (\sigma, \tau) \mapsto \sigma\tau \tag{B.1}$$

is a length-preserving bijection. The quotient W/W_Q is a graded poset, i. e. all maximal chains have the same length. The rank function $r : W/W_Q \to \mathbb{N}_0$ maps a coset $\theta \in W/W_Q$ to the length $\ell(\theta^Q)$ of its unique representative $\theta^Q \in W^Q$, i. e. the smallest number $\ell \in \mathbb{N}_0$ such that there exists a decomposition $\theta^Q = s_1 \cdots s_\ell$ into simple reflections. Such a minimal decomposition is usually called a *reduced decomposition*.

To every inclusion $Q \subseteq Q'$ of two parabolic subgroups there is the monotone surjection

$$\pi_{Q,Q'} \colon W/W_Q \twoheadrightarrow W/W_{Q'},\ \sigma W_Q \mapsto \sigma W_{Q'},$$

where we typically write $\pi_{Q'}$ instead, if the source is clear. Every element $\theta \in W/W_{Q'}$ has a unique minimal preimage $\min_Q(\theta)$ and a unique maximal preimage $\max_Q(\theta)$ in

W/W_Q under $\pi_{Q'}$. The corresponding two maps

$$\begin{aligned} \min_Q\colon W/W_{Q'} &\hookrightarrow W/W_Q,\ \theta \mapsto \min_Q(\theta) \quad \text{and} \\ \max_Q\colon W/W_{Q'} &\hookrightarrow W/W_Q,\ \theta \mapsto \max_Q(\theta), \end{aligned}$$

are isomorphisms of posets onto their image. We say an element $\sigma W_Q \in W/W_Q$ is **Q'-minimal/Q'-maximal**, if it lies in the image of $\min_Q$ or $\max_Q$ respectively.

Lemma B.1 ([BB, Theorem 2.6.1])**.** *Let $(Q_i)_{i\in I}$ be a finite family of parabolic subgroups over B and $Q = \bigcap_{i\in I} Q_i$. Then the following map is an isomorphism of posets onto its image:*

$$W/W_Q \hookrightarrow \prod_{i\in I} W/W_{Q_i}, \quad \sigma \mapsto (\pi_{Q_i}(\sigma))_{i\in I}.$$

Lemma B.2. *If $s_1, \ldots, s_r \in W$ are pairwise distinct simple reflections, then $s_1 \cdots s_r$ is in reduced decomposition.*

Proof. If $s_1 \cdots s_r$ was not reduced, then there are indices $1 \le i < j \le r$ with $s_1 \cdots s_r = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_r$ (where the hat indicates, that s_i and s_j are omitted). By induction over r, the prefix $s_1 \cdots s_{r-1}$ is reduced, hence $j = r$. It follows $s_1 \cdots s_{r-1} = s_1 \cdots \hat{s}_i \cdots s_r$. Both sides of this equation are in reduced decomposition. As $s_1, \ldots, s_r$ are pairwise distinct and the set of simple reflections appearing in a reduced decomposition is unique, we conclude $s_i = s_r$, which is impossible. □

The following lemma by Deodhar is used many times throughout this thesis. Our version of this lemma follows directly from [LMS4, Lemma 11.1, Lemma 11.1'] or [LG, Lemma 12.8.9] by projecting the unique lift in W to W/W_Q.

Lemma B.3 (Deodhar's Lemma)**.** *Let $Q \subseteq Q'$ be two parabolic subgroups containing B, $\theta \ge \phi$ be two elements of $W/W_{Q'}$.*

(a) If $\overline{\theta} \in W/W_Q$ is a lift of θ, then there is a unique maximal lift $\overline{\phi} \in W/W_Q$ of ϕ such that $\overline{\phi} \ge \overline{\theta}$.

(b) If $\overline{\phi} \in W/W_Q$ is a lift of ϕ, then there is a unique minimal lift $\overline{\theta} \in W/W_Q$ of θ such that $\overline{\theta} \ge \overline{\phi}$.

Lemma B.4. *Let $Q \subseteq P$ be two parabolic subgroups of G and $\theta > \phi \in W/W_Q$, such that $\pi_P(\theta) > \pi_P(\phi)$. Then there exists a covering relation $\theta > \psi$ in W/W_Q, such that $\psi \ge \phi$ and $\pi_P(\theta) > \pi_P(\psi)$.*

Proof. We show the statement by induction over difference $d = r(\theta) - r(\phi)$ of ranks in W/W_Q. For $d = 1$ there is nothing to prove. Now let $d \ge 2$ and $\overline{\phi}$ be the unique maximal lift of $\pi_P(\phi)$ in W/W_Q, that is less or equal to $\pi_Q(\theta)$. If $\pi_Q(\theta) > \overline{\phi}$ is already a covering relation, then we can take $\psi = \overline{\phi}$. Otherwise we look at the Bruhat interval $[\theta, \overline{\phi}] = \{\sigma \in W/W_Q \mid \theta \geq \sigma \geq \overline{\phi}\}$. Suppose that every element in this interval except $\overline{\phi}$ projects to $\pi_P(\theta)$. By Deodhar's Lemma (B.3) there exists a unique minimal lift $\overline{\theta} \in W/W_Q$ of $\pi_P(\theta)$ with $\overline{\theta} \geq \overline{\phi}$. Hence there is exactly one element covering $\overline{\phi}$ in $[\theta, \overline{\phi}]$. But this is false for Bruhat intervals of two elements, which lengths differ by more than 1. A proof of this statement can be found in [BB, Lemma 2.7.3].

Therefore there exists an element $\phi' \in W/W_Q$, such that $\pi_Q(\theta) > \phi' > \overline{\phi}$ and $\pi_P(\theta) > \pi_P(\phi') > \pi_P(\overline{\phi})$. Using the induction on ϕ' instead of ϕ we get an element $\psi \in W/W_Q$ covered by θ with $\pi_P(\theta) > \pi_P(\psi)$ and $\psi \geq \phi' \geq \overline{\phi} \geq \phi$. □

C. Young tableaux and other tableau models

The LS-tableaux from Section 4.2 are a generalization of more well known tableau models, like the ones of Hodge-Young in type A and of Lakshmibai-Musili-Seshadri in the types B, C and D. We fix a connected, simply-connected, simple algebraic group G, a maximal torus T and a Borel subgroup B containing T. Let Δ be the set of all simple roots corresponding to the choice of B. For each Dynkin type we order the fundamental weights $\omega_1, \dots, \omega_n$ of G (equivalently, the simple roots) in the same way as in [Bou, Plates I to IX]. Each fundamental weight ω_i corresponds to the maximal parabolic subgroup P_i stabilizing the highest weight space in $V(\omega_i)$. Furthermore, we fix a dominant weight $\mu = a_1\omega_1 + \cdots + a_n\omega_n$ for $a_1, \dots, a_n \in \mathbb{N}_0$. There exists a unique sequence $\underline{\mu} = (\omega_{i_1}, \dots, \omega_{i_s})$ of fundamental weights, such that $\omega_{i_1} + \cdots + \omega_{i_s} = \mu$ and $1 \leq i_1 \leq \cdots \leq i_s \leq n$.

Type $\mathtt{A}_n$: Let ω_i and α_i be defined as in the beginning of Chapter 3. We write s_i instead of s_{α_i}. Since all fundamental representations in type A are minuscule, each LS-path model $\mathbb{B}(\omega_i)$, for $i \in [n]$, can be set-theoretically identified with W/W_{P_i} and thus with the set $\mathrm{SSYT}(\omega_i)$ of all semistandard Young tableaux consisting of one column with exactly i boxes. Therefore the set LS-tableaux of shape $\underline{\mu}$ can be interpreted as the set $\mathrm{YT}(\mu)$ of all Young tableaux of shape μ (see Definition 3.1). Notice that the order of the columns is reverted under this bijection. For example, the LS-tableau $(\pi_1, \pi_2, \pi_3, \pi_4)$ with the columns

$$\pi_1 = (s_2 s_1 W_{P_1}; 0, 1),\ \pi_2 = (s_3 s_2 W_{P_2}; 0, 1),\ \pi_3 = (s_1 s_2 W_{P_2}; 0, 1),\ \pi_4 = (s_3 W_{P_3}; 0, 1)$$

corresponds to the Young tableau

1	2	1	3
2	3	4	
4			

A Young tableau T is semistandard, if and only if its corresponding LS-tableau is standard. We proved this in Lemma 3.7 (c): Let Q be the intersection of all maximal parabolic subgroups $P_{i_1}, \dots, P_{i_s}$. The lifts $\widetilde{\theta} = \min_Q \circ \max_{Q_i}(\theta)$ in W/W_Q of the columns (θ, i) in T are linearly ordered (from left to right), if and only if the columns (θ, i) themselves are linearly ordered and we know from Corollary 3.8 that is this equivalent to T being semistandard. In particular, the notions of standard and weakly standard LS-tableaux agree for Young tableaux.

Types $\mathtt{B}_n$ and $\mathtt{C}_n$: Instead of Young tableaux, we obtain the tableau model developed by Lakshmibai, Musili and Seshadri (see [LMS4] and [LS5]). For a maximal parabolic subgroup P_i they defined certain pairs of elements in W/W_{P_i} called *admissable pairs*.

For each $\theta \in W/W_{P_i}$ let $[X_\theta]$ denote the element in the Chow ring of G/P_i induced by the Schubert variety X_θ. Let H be the unique Schubert variety of codimension one in G/P_i. By a formula of Chevalley from [Dem] (see also [Ses2, Section 4.5]) it holds

$$[X_\theta] \cdot [H] = \sum_{\phi} d_\phi [X_\phi]$$

in the Chow ring, where the sum is taken over all elements $\phi \in W/W_{P_i}$ covered by θ. The number d_ϕ is given by $|\langle \phi(\omega_i), \beta^\vee \rangle|$, where β is the unique positive root with $s_\beta \phi^{P_i} = \theta^{P_i}$ and ϕ^{P_i} (respectively θ^{P_i}) is the unique minimal representative of ϕ (respectively θ) in the Weyl group W. This number d_ϕ is called the *(intersection) multiplicity* of X_ϕ in X_θ (sometimes also Chevalley multiplicity).

A pair (θ, ϕ) of cosets $\theta, \phi \in W/W_{P_i}$ is called an *admissable pair*, if either $\theta = \phi$ or there exists a chain $\theta = \phi_1 > \cdots > \phi_k = \phi$ covering relations in W/W_{P_i}, such that for every $j = 2, \dots, k$ the Schubert variety $X_{\phi_j} \subseteq G/P_i$ is a divisor of $X_{\phi_{j-1}}$ with intersection multiplicity 2. Note that these chains are a special case (for $a = \frac{1}{2}$) of a-chains defined by Littelmann in [Lit94, Section 2.2], which play an important role in the definition of LS-paths. An admissable pair (θ, ϕ) with $\theta > \phi$ thus corresponds to the LS-path $(\theta > \phi; 0, \frac{1}{2}, 1) \in \mathbb{B}(\omega_i)$ and an admissable pair (θ, θ) corresponds to the LS-path $(\theta; 0, 1) \in \mathbb{B}(\omega_i)$. Every LS-path in $\mathbb{B}(\omega_i)$ is of one of these two forms, since the fundamental weights ω_i in the types $\mathtt{B}$ and $\mathtt{C}$ are *classical*, i. e. $|\langle \omega_i, \beta^\vee \rangle| \leq 2$ holds for all roots β in the root system of G. Equivalently, the intersection multiplicity of $X_\phi \subseteq X_\theta$ is at least 2 for each covering relation $\theta > \phi$ in W/W_{P_i}.

A *Young diagram* of type $(a_1, \dots, a_n)$ in the sense of [LS5] can be seen as a sequence of admissable pairs (θ_j, ϕ_j) with $j = 1, \dots, s$ and $\theta_j, \phi_j \in W/W_{P_{i_j}}$. Hence these Young diagrams correspond to LS-tableaux of shape $\underline{\mu}$. Under this correspondence the notions of standard Young diagrams and standard LS-tableaux agree, as both are given via the existence of a defining chain. Lakshmibai and Seshadri even allowed other orderings of the fundamental weights, but for the explicit ordering we defined above, Littelmann showed in the Appendix of [Lit90] that one can interpret their Young diagrams via certain classical Young tableaux with entries in $\{1, \dots, 2n\}$.

Type $\mathtt{D}_n$: Since the fundamental weights in type D are classical as well (see above), the Young diagrams from the types B and C can also be used in type D and these diagrams still correspond to LS-tableaux. For the ordering of the fundamental weights we chose above, these tableaux can again be identified with certain Young tableaux, but their explicit combinatorial description is noticeably more difficult than in the types B and C. It can be found in [Lit90, Appendix A.3].

The main difference in type D is the fact that there exists no ordering of the fundamental weights, such that the notions of weakly standard LS-tableaux and standard LS-tableaux coincide (see Proposition 4.22 and Corollary 4.27). Therefore standardness cannot be verified locally by just considering consecutive columns.

Other Types: In the exceptional types not every fundamental weight is classical. There is a list of all classical fundamental weights in [LR, Section A.2.3]. Since higher intersection multiplicities can occur, one needs to replace admissable pairs by admissable quadruples for fundamental weights ω_i with $|\langle\omega_i, \beta^\vee\rangle| \leq 3$ for all roots β. The resulting tableau model was described in [Lit90, Section 3]. Again, the admissable quadruples correspond to LS-paths $(\sigma_p, \dots, \sigma_1; 0, d_p, \dots, d_1 = 1)$ with $p \leq 4$ different directions. This shows the power of LS-paths, as they provide a language suited for all intersection multiplicities.

List of notations

$\mathbb{B}(\lambda)$	path model of LS-paths to $\lambda \in \Lambda^+$
$b_{p,q}$	bond of a covering relation $p > q$
$\operatorname{Cone} S$	convex cone generated by a set S
$\operatorname{Conv} S$	convex hull generated by a set S
Δ	set of simple roots of G
$\hat{X}$	multicone of embedded projective variety X
e_I	sum of all unit vectors e_i for $i \in \underline{I}$
Γ	fan of monoids
Γ_C	monoids associated to a chain $C \subseteq A$
$\Gamma_C^{(d)}$	Veronese submonoid of Γ_C
$\mathrm{gr}_{\mathcal{V}} R$	associated graded algebra to $\mathcal{V}$
$\mathrm{gr}_{\mathcal{V},\mathfrak{C}} R$	subalgebra of $\mathrm{gr}_{\mathcal{V}} R$
$\mathrm{gr}_{\mathcal{V},\mathfrak{C}}^{(d)} R$	Veronese subalgebra of $\mathrm{gr}_{\mathcal{V},\mathfrak{C}} R$
$\underline{I}$	subset of I defined by covering relations (p. 37)
$\mathcal{L}^C$	lattice generated by Γ_C
Λ^+	monoid of dominant weights of G
Λ	weight lattice of G
λ_I	sum of all weights λ_i for $i \in \underline{I}$
$[k]$	set $\{1, \ldots, k\}$ for $k \in \mathbb{N}$
$\max_Q$	maximal lift (p. 99)
$\min_Q$	minimal lift (p. 99)
π_Q	projection map (p. 98)
P_i	parabolic subgroup to the weight λ_i
P_I	parabolic subgroup to $I \in \mathcal{I}$
Q_I	lower parabolic subgroup to I (p. 53)
Q^I	upper parabolic subgroup to I (p. 53)
Q_τ	largest parabolic over Q, where τ is Q_τ-maximal (p. 53)
$r(-)$	rank function in a graded poset
σ_C	cone of multidegrees of Γ_C
$\operatorname{supp} \underline{a}$	support of an element in some $\mathbb{Q}^A$ (p. 10)
τ_i	projection of $\tau \in W/W_Q$ to W/W_{P_i}
τ_I	projection of $\tau \in W/W_Q$ to W/W_{P_I}
$\lvert\underline{d}\rvert$	total degree of $\underline{d} \in \mathbb{N}_0^m$
V	ambient affine space of the stratification
W_λ	stabilizer of λ in the Weyl group
W^Q	set of all minimal representatives of W/W_Q in W
W_Q	Weyl subgroup of Q
X_I	projection of a multiprojective variety X to the factors in I
X_θ	Schubert variety in G/Q to $\theta \in W/W_Q$

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