Semi-classical spectral asymptotics of Toeplitz operators for lower energy forms on non-degenerate compact CR manifolds

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Zusammenfassung

H. Herrmann, C.-Y. Hsiao, G. Marinescu und der Autor dieser Arbeit beschreiben in einem kürzlich erschienen Artikel die vollständige asymptotische Entwicklung des Schwartz-Kerns bestimmter spektraler Projektionen von semiklassischen Toeplitz-Operatoren auf streng pseudokonvexen, kompakten, einbettbaren CR-Mannigfaltigkeiten. Eine solche asymptotische Expansion führt zu vielen neuen Anwendungen in der CR-Geometrie, und das Ziel dieser Arbeit ist es, dieses Ergebnis zu verallgemeinern.

Dazu betrachten wir eine kompakte, nicht-entartete CR Mannigfaltigkeit $(X, T^{1,0}X)$ mit konstanter Signatur (n_-, n_+) und einen selbstadjungierten, klassischen Pseudodifferentialoperator P erster Ordnung, der den Raum der $(0, q)$-Formen auf X auf sich selbst abbildet. Wir definieren und untersuchen dann sogenannte Levi-elliptische Toeplitz-Operatoren für Formen mit niedriger Energie. Genauer betrachten wir für jedes $\lambda > 0$ den Operator $T^{(q)}_{P,\lambda} := \Pi^{(q)}_\lambda \circ P \circ \Pi^{(q)}_\lambda$. Dabei bezeichnet $\Pi^{(q)}_\lambda$ - in Analogie zur Szegő-Projektion - die orthogonale Projektion auf Formen niedrigerer Energie ist $\mathbb{1}_{[0,\lambda]}(\Box^{(q)}_b)$ bezü-glich des Kohn-Laplace-Operators $\Box^{(q)}_b$. Im Fall $q = n_-$, nehmen wir an, dass P levi-elliptisch ist, was bedeutet, dass das matrixwertige Hauptsymbole von P bezogen auf die Signatur (n_-, n_+) einer abgeschwächten Elliptizitätsbedingung genügt. Wir betrachten dann den durch die Helffer–Sjöstrand-formel definierten Spektraloperator $\chi(k^{-1}T^{(q)}_{P,\lambda})$, $q = n_-$, $k > 0$, bezüglich einer glatten Funktion $\chi\colon \mathbb{R} \to \mathbb{C}$ mit kompaktem Träger in $\mathbb{R} \setminus \{0\}$. Unter Verwendung mikroanalytischer Methoden von Hsiao–Marinescu und Galasso–Hsiao untersuchen wir das asymptotische Verhalten von $\chi(k^{-1}T^{(q)}_{P,\lambda})$ für $k \to +\infty$. Unser Hauptergebnis beschreibt die vollständige asymptotische Entwicklung des Schwartz-Kerns $\chi(k^{-1}T^{(q)}_{P,\lambda})(x, y)$ als Summe zweier semi-klassischer oszillierender Integrale komplexer Phasen. Dabei sind die entsprechenden komplexwertigen Phasenfunktionen mit den beiden Zusammenhangskomponenten Σ^- bzw. Σ^+ des charakteristischen Kegels des Kohn-Laplace-Operators assoziiert.

Der letzte Teil dieser Arbeit umfasst die gemeinsame Arbeit von C.-Y. Hsiao und dem dieser Autor über den zweiten Koeffizienten für die Expansion von Boutet de Monvel und Sjöstrand. Dieses Ergebnis könnte in Zukunft für die Berechnung von $A^{\mp}_1(x)$ in der semi-klassischen Expansion von $\chi(k^{-1}T^{(q)}_{P,\lambda})(x, x)$ bei $q = n_- = 0$ hilfreich sein.

Abstract

A recent result of H. Herrmann, C.-Y. Hsiao, G. Marinescu and the author establishes the full asymptotic expansion of the Schwartz kernel of certain spectral projection of semi-classical Toeplitz operators on strictly pseudoconvex, compact and embeddable CR manifolds. Such asymptotic expansion leads to many new applications in CR geometry, and the aim of the thesis is to generalize this result.

For this propose, we consider any compact CR manifold $(X, T^{1,0}X)$ whose Levi form is non-degenerate and has a constant signature (n_-, n_+). We introduce a specific type of operators called Levi-elliptic Toeplitz operators for lower energy forms. For any $\lambda > 0$, these operators, represented as $T^{(q)}_{P,\lambda} := \Pi^{(q)}_\lambda \circ P \circ \Pi^{(q)}_\lambda$, is the pre- and post- composition of certain classical pseudodifferential operators P of order one and the orthogonal projection $\Pi^{(q)}_\lambda$. We assume that P maps the space of $(0,q)$-forms on X to itself and to be formally self-adjoint. When $q = n_-$, we assume that the matrix-valued principal symbol of P satisfy some relaxed elliptic conditions corresponding to the pair (n_-, n_+). The orthogonal projection $\Pi^{(q)}_\lambda$ is an analogue of the Szegő projection and defined by $\mathbb{1}_{[0,\lambda]}(\Box^{(q)}_b)$, which is the spectral projection to lower energy forms associated with the Kohn Laplacian at degree $(0,q)$. For any smooth function $\chi : \mathbb{R} \to \mathbb{C}$ compactly supported on $\mathbb{R} \setminus \{0\}$ and $q = n_-$, our primary focus is the spectral operator $\chi(k^{-1}T^{(q)}_{P,\lambda})$ defined by the Helffer–Sjöstrand formula in spectral theory. Starting from the microlocal analysis of Hsiao–Marinescu and Galasso–Hsiao, as $k \to +\infty$ we develop a semi-classical microlocal analysis of $\chi(k^{-1}T^{(q)}_{P,\lambda})$, and our main result describes the full asymptotic expansion of the Schwartz kernel $\chi(k^{-1}T^{(q)}_{P,\lambda})(x,y)$ as the sum of two semi-classical oscillatory integrals of complex phases. The canonical relation of the two oscillatory integrals corresponds to Σ^- and Σ^+, respectively, where $\Sigma^\mp$ are the only two connected components of the characteristic cone of $\Box^{(q)}_b$.

The last part of the thesis is the joint work of C.-Y. Hsiao and the author about second coefficient for the expansion of Boutet de Monvel and Sjöstrand. Our result and method should be helpful in the future for the calculation of $A^\mp_1(x)$ in the semi-classical expansion of $\chi(k^{-1}T^{(q)}_{P,\lambda})(x,x)$ when $q = n_- = 0$.

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Contents

CHAPTER 1

Introduction

1.1. Semi-classical spectral asymptotics of Toeplitz operators

The theory of Toeplitz operators is a classical subject in several complex variables. For a bounded strictly pseudoconvex domain $M \subset \mathbb{C}^{n+1}$, $n \geq 1$, we let X be the boundary of M and $H_b^0(X)$ the closure in $L^2(X)$ of the space of boundary values of holomorphic functions on M. We call operators of the form

$$T_P := \Pi \circ P \circ \Pi \tag{1.1.1}$$

Toeplitz operators, where Π is the orthogonal projection of $L^2(X)$ onto $H_b^0(X)$ and P is a pseudodifferential operator on X. Inspired by the earlier results of Melin–Sjöstrand [72] and Boutet de Monvel–Sjöstrand [13], respectively on Fourier integral operators with complex phase and the asymptotic expansion of Szegő kernel $\Pi(x,y)$, in [10] Boutet de Monvel proves that these operators admit symbolic calculus as pseudodifferential operators, and he gives a famous result about a variant of the Atiyah–Singer index theorem in this context. It turns out that Boutet's general theory of Toeplitz operators is a powerful tool not only in index theory, but also in complex geometry and topics in deformation quantization [11, §§ 6,7,8]. We mention some works inspired by such point of view [2,5,6,14,16–18,20,24,28,29,32,34,55,59,67,68,70,76,77,81,82], to quote just a few.

Our interests for Toeplitz operators is their spectral theory on non-degenerate CR manifolds. These CR manifolds play a fundamental role in the context of microlocal analysis, see Boutet de Monvel [8], Hörmander [42], and Kohn [63]. We provide an alternative method comparing to the calculus in the main text of Boutet de Monvel–Guillemin [12] to study Toeplitz operators. In this thesis, on any compact CR manifold with a non-degenerate Levi form of constant signature (n_-, n_+), for $q = n_-$ and any number $\lambda > 0$ we introduce the Levi-elliptic Toeplitz operator on lower energy forms $T^{(q)}_{P,\lambda}$. For any $\chi \in \mathscr{C}^\infty_0(\mathbb{R} \setminus \{0\})$ we consider the spectral operator $\chi(k^{-1}T^{(q)}_{P,\lambda})$, and under the semi-classical limit $k \to +\infty$ our main result establishes the full asymptotic expansion of the Schwartz kernel $\chi(k^{-1}T^{(q)}_{P,\lambda})(x,y)$ as the sum of two semi-classical oscillatory integrals. In the realm of CR geometry, this work is a continuation of research into the semi-classical spectral asymptotics of Toeplitz operators. Our method heavily relies on microlocal analysis of [13, 28, 45, 56, 72] and especially the semi-classical microlocal approaches introduced by Herrmann, Hsiao, Marinescu and the author in [35], where these operators are examined on compact, strictly pseudoconvex, and embeddable CR manifold. The study employs the approach of Melin–Sjöstrand, Boutet de Monvel–Sjöstrand, Hsiao–Marinescu and Galasso–Hsiao, utilizing a calculus of specific complex phase Fourier integral operators, cf. §3. Additionally, it incorporates a semi-classical analysis on a distinct integral, as defined by the Helffer–Sjöstrand formula, cf. §4. This kind of analysis was well-studied for order zero Toeplitz operators [28] and for the order one situation [35].

From now on, we work on the following set-up. We let $(X, T^{1,0}X)$ be a compact Cauchy–Riemann manifold of hypersurface type (CR manifold for short) of real dimension $2n+1$, $n \geq 1$, and assume that there is a contact form $\boldsymbol{\alpha}$ on X such that the Levi form $\mathcal{L} := \frac{i}{2} d\boldsymbol{\alpha}|_{T^{1,0}X}$ induced by $\boldsymbol{\alpha}$ is non-degenerate on whole X. This means that the numbers of the negative and positive eigenvalues of $\mathcal{L}$ on X is always the constant and we denote them by n_- and n_+, respectively. The pair (n_-, n_+) is called the signature of X. For any classical pseudodifferential operator $P \in L^1_{\mathrm{cl}}(X; T^{*0,q}X)$ of first order such that $P : \mathscr{C}^\infty(X, T^{*0,q}X) \to \mathscr{C}^\infty(X, T^{*0,q}X)$, we consider the operator

$$T^{(q)}_{P,\lambda} := \Pi^{(q)}_\lambda \circ P \circ \Pi^{(q)}_\lambda, \tag{1.1.2}$$

where $\lambda > 0$ is any number, $\Pi^{(q)}_\lambda : L^2_{0,q}(X) \to E([0,\lambda])$ is the orthogonal projection called *Szegő projection on lower energy forms*, $L^2_{0,q}(X)$ is the square integrable $(0,q)$ forms on X, and the subspace of lower energy forms $E([0,\lambda]) :=$ Range $\mathbb{1}_{[0,\lambda]}(\Box^{(q)}_b)$ is the image of the spectral projection of the Kohn laplacian $\Box^{(q)}_b$ (extended by Gaffney extension). We always assume that P is formally self-adjoint. When $q = n_-$ we assume the following *Levi-ellipticity conditions*. First of all, we let $\{W_j\}_{j=1}^n$ be an orthonormal frame of $T^{1,0}X$ in a neighbourhood of x such that $\mathcal{L}_x(W_j, \overline{W}_s) = \delta_{j,s}\mu_j$, $j,s = 1,\cdots,n$, and $\mu_1 \leq \cdots \leq \mu_{n_-} < 0 < \mu_{1+n_-} \leq \cdots \leq \mu_n$. We take the dual basis $\{\omega_j\}_{j=1}^n$ of $T^{*0,1}X$ with respect to $\{\overline{W}_j\}_{j=1}^n$ and we consider the subspace $\mathcal{N}^{n_-}_x := \{c\, \omega_1(x) \wedge \cdots \wedge \omega_{n_-}(x) :\ c \in \mathbb{C}\}$ and $\mathcal{N}^{n_+}_x := \{c\, \omega_{1+n_-}(x) \wedge \cdots \wedge \omega_n(x) :\ c \in \mathbb{C}\}$ of $T^{*0,q}_x X$. With respect to the hermitian metric induced by the given one on $\mathbb{C}TX$, we can define the orthogonal projections

$$\tau^{n_\mp}_{x_0} : T^{*0,q}_x X \to \mathcal{N}^{n_\mp}_{x_0}. \tag{1.1.3}$$

Second, we use the notation $p_0 \in \mathscr{C}^\infty\left(T^*X, \mathscr{L}(T^{*0,q}X, T^{*0,q}X)\right)$ for the matrix-valued principal symbol of P. For all $x \in X$, we require

$$\tau_x^{n_-} p_0(-\boldsymbol{\alpha}_x)\tau_x^{n_-} > 0 \ \text{ when } q = n_-, \tag{1.1.4}$$

and we additionally need

$$\tau_x^{n_+} p_0(\boldsymbol{\alpha}_x)\tau_x^{n_+} < 0 \ \text{ when } q = n_- = n_+. \tag{1.1.5}$$

We will see in Theorem 3.8 that $T^{(q)}_{P,\lambda}$ has a self-adjoint L^2-extension through maximal extension. In fact, for $q \notin \{n_-, n_+\}$, $T^{(q)}_{P,\lambda}$ is a compact operator. When $q = n_-$, we will see in Theorem 3.9 that the set $\operatorname{Spec} T^{(q)}_{P,\lambda} \subset \mathbb{R}$ is also discrete and the accumulation points of the spectrum is the subset of $\{-\infty, +\infty\}$. Roughly speaking, the conditions (1.1.4) and (1.1.5) are responsible for the part of eigenvalues accumulated at $+\infty$ and $-\infty$, respectively. We will discuss Theorem 3.9 more in §3.2.

Next, for any function $\chi \in \mathscr{C}^\infty_0(\mathbb{R} \setminus \{0\})$, our main result describes the operator $\chi(k^{-1}T^{(q)}_{P,\lambda})$ as the sum of two semi-classical Fourier integral operators modulo some k-negligible operator when $q = n_-$ and $k \to +\infty$. From the spectral theorem of $T^{(q)}_{P,\lambda}$, we denote $\{\lambda_j\}_{j\in J}$ be the non-zero eigenvalues of $T^{(q)}_{P,\lambda}$, and $\{f_j\}_{j\in J}$ be the correposnding eigenforms such that $(f_j|f_k) = \delta_{jk}$ with respect to L^2-inner product $(\cdot|\cdot)$. We then have the formula

$$\chi(k^{-1}T^{(q)}_{P,\lambda})(x,y) = \sum_{k^{-1}\lambda_j \in \operatorname{supp}\chi} \chi(k^{-1}\lambda_j) f_j(x) \otimes f_j^*(y). \tag{1.1.6}$$

However, to state the precise semi-classical spectral asymptotics of this finite sum, we need more detail for the fundamental theorem [56, Theorem 4.1] about the microlocal structure of $\Pi^{(q)}_\lambda$, cf. also Theorems 2.3 and 2.4. For $q = n_-$ and any coordinate patch (Ω, x) on X, there is some complex-valued function $\varphi_\mp(x,y) \in \mathscr{C}^\infty(\Omega \times \Omega, \mathbb{C})$ with the properties

$$\operatorname{Im} \varphi_\mp(x,y) \geq 0, \tag{1.1.7}$$

$$\varphi_\mp(x,y) = 0 \text{ if and only if } x = y, \tag{1.1.8}$$

$$d_x\varphi_\mp(x,x) = -d_y\varphi_\mp(x,x) = \mp\boldsymbol{\alpha}(x), \tag{1.1.9}$$

and some Hörmander symbol $s^\mp(x,y,t)$ with the asymptotic expansion

$$s^\mp(x,y,t) \sim \sum_{j=0}^{+\infty} s_j^\mp(x,y)t^{n-j} \text{ in } S^n_{1,0}\left(\Omega \times \Omega \times \mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X)\right) \tag{1.1.10}$$

such that for the Foureier integral operators $S_\mp$ determined by

$$S_\mp(x,y) = \int_0^{+\infty} e^{it\varphi_\mp(x,y)} s^\mp(x,y,t)dt \tag{1.1.11}$$

we have

$$\Pi_\lambda^{(q)} = S_- + S_+ + F \text{ on } \Omega \tag{1.1.12}$$

for some F smoothing on Ω. Here $\mathbb{R}_+ := \mathbb{R}_{>0}$. In fact, $s^\mp(x,y,t)$ and $s_j^\mp(x,y)$ are properly supported in the variables (x,y) for all $j \in \mathbb{N}_0$ and $s^+(x,y,t) = 0$ when $n_- \neq n_+$. When $q \notin \{n_-, n_+\}$, $\Pi_\lambda^{(q)}$ is a smoothing operator.

THEOREM 1.1. *We let $(X, T^{1,0}X)$ be a compact and non-degenerate CR manifold and α be the contact form on X such that the Levi form induced by α has the constant signature (n_-, n_+). We denote $\dim_{\mathbb{R}} X := 2n+1$ and $n \geq 1$. For $q \in \{0, \cdots, n\}$, any formally self-adjoint $P \in L^1_{\mathrm{cl}}(X; T^{*0,q}X)$ such that when $q = n_-$ we have the Levi-ellipticity conditions (1.1.4) and (1.1.5), and any $\lambda > 0$, we consider the Toeplitz operator $T^{(q)}_{P,\lambda} := \Pi^{(q)}_\lambda \circ P \circ \Pi^{(q)}_\lambda$ by $\Pi^{(q)}_\lambda := \mathbb{1}_{[0,\lambda]}(\Box_b^{(q)})$. For any function $\chi \in \mathscr{C}_0^\infty(\mathbb{R}\setminus\{0\}, \mathbb{C})$ such that $\chi \not\equiv 0$, the spectral operator $\chi(k^{-1}T^{(q)}_{P,\lambda})$ has the following semi-classical limit as $k \to +\infty$:*

$$\chi(k^{-1}T^{(q)}_{P,\lambda}) = 0 \text{ on } X \text{ if } q \notin \{n_-, n_+\}, \tag{1.1.13}$$

and for each coordinate patch (Ω, x) in X we have the full asymptotic expansion of the Schwartz kernel

$$\chi(k^{-1}T^{(q)}_{P,\lambda})(x,y) = \int_0^{+\infty} e^{ikt\varphi_-(x,y)} A^-(x,y,t;k)dt + \int_0^{+\infty} e^{ikt\varphi_+(x,y)} A^+(x,y,t;k)dt + O\left(k^{-\infty}\right) \text{ on } \Omega\times\Omega \text{ if } q = n_-, \tag{1.1.14}$$

where $\varphi_\mp(x,y) \in \mathscr{C}^\infty(\Omega\times\Omega, \mathbb{C})$ are the functions with the properties (1.1.7), (1.1.8), (1.1.9) and we have the following smooth data:

$$A^\mp(x,y,t;k) \sim \sum_{j=0}^{+\infty} A_j^\mp(x,y,t)k^{n+1-j} \text{ in } S^{n+1}_{\mathrm{loc}}\left(1; \Omega\times\Omega\times\mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X)\right), \tag{1.1.15}$$

and $A^+(x,y,t;k) = 0$ when $n_- \neq n_+$. In fact, when $\operatorname{supp}\chi \cap \mathbb{R}_+ \neq \emptyset$, there is an interval $I_- \Subset \mathbb{R}_+$ such that when $A_j^-(x,y,t) \neq 0$ and $A^-(x,y,t) \neq 0$ we have $t \in I_-$ for all $j \in \mathbb{N}_0$; when $n_- = n_+$ and $\operatorname{supp}\chi \cap \mathbb{R}_- \neq \emptyset$, there is also an interval $I_+ \Subset \mathbb{R}_+$ such that when $A_j^+(x,y,t) \neq 0$ and $A^+(x,y,t) \neq 0$ we have $t \in I_+$ for all $j \in \mathbb{N}_0$. Moreover, for any $\tau_1, \tau_2 \in \mathscr{C}^\infty(X)$ such that $\operatorname{supp}\tau_1 \cap \operatorname{supp}\tau_2 = \emptyset$, we have

$$\tau_1 \circ \chi_k(T^{(q)}_{P,\lambda}) \circ \tau_2 = O(k^{-\infty}) \text{ on } X, \tag{1.1.16}$$

where τ_1 and τ_2 are seen as the multiplication operator.

The main results (1.1.13) and (1.1.14) can be seen as the analogue on CR manifolds of the famous result of Andreotti–Grauert vanishing theorem and Bergman kernel expansion for high powers of line bundles of the mixed curvature type, respectively. We will discuss this in §4.4.

We also have the description of leading term of our main result through the local picture (1.1.3). We let $m(x)dx$ be the given volume form on X and $v(x)dx$ be the volume form induced by the Hermitian metric on $\mathbb{C}TX$ which is compatible with $\boldsymbol{\alpha}$. For the expansion (1.1.10) of $s^{\mp}(x,y,t)$, by [56, Theorem 3.5] (cf. Theorem 2.11 for detail), for $x \in \Omega$ we have

$$s_0^-(x,x) = \frac{|\det \mathcal{L}_x|}{2\pi^{n+1}} \frac{v(x)}{m(x)} \tau_x^{n_-}, \tag{1.1.17}$$

$$s_0^+(x,x) = \frac{|\det \mathcal{L}_x|}{2\pi^{n+1}} |\det \mathcal{L}_x| \frac{v(x)}{m(x)} \tau_x^{n_+} \text{ when } n_- = n_+. \tag{1.1.18}$$

By our assumption, the eigenvalues $\{\mu_j(x)\}_{j=1}^n$ of the Levi form $\mathcal{L}_x$ on X satisfy

$$\mu_j(x) < 0 : \ 1 \le j \le n_-, \tag{1.1.19}$$

$$\mu_j(x) > 0 : \ n_- + 1 \le j \le n, \tag{1.1.20}$$

and we let

$$I_0 := \{1, \cdots, n_-\}, \tag{1.1.21}$$

$$J_0 := \{n_- + 1, \cdots, n\}. \tag{1.1.22}$$

We can write the principal symbol $p_0(x,\eta)$ of P by

$$p_0(x,\eta) = \sum_{|\mathbf{I}|=|\mathbf{J}|=q} p_{\mathbf{I},\mathbf{J}}(x,\eta)\omega_{\mathbf{I}}^{\wedge} \otimes \omega_{\mathbf{J}}^{\wedge,*}, \tag{1.1.23}$$

where $p_{\mathbf{I},\mathbf{J}}(x,\eta) \in \mathscr{C}^\infty(T^*X, \mathbb{C})$ and $\mathbf{I}, \mathbf{J}$ are strictly increasing index sets. Then the Levi-elliptic condition (1.1.4) and (1.1.5) now become

$$p_{I_0,I_0}(-\boldsymbol{\alpha}_x) > 0 : \ q = n_- \ , \tag{1.1.24}$$

and

$$p_{J_0,J_0}(\boldsymbol{\alpha}_x) < 0 \ : \ q = n_- = n_+ \ , \tag{1.1.25}$$

respectively. The result for the leading coefficient of our expansion is as follows.

THEOREM 1.2. *Following Theorem 1.1 and the above local picture, if $q = n_-$, the leading term $A_0^-(x,y,t)$ in the expansion (1.1.15) has the form*

$$A_0^-(x,x,t) = t^n \chi(p_{I_0,I_0}(-\boldsymbol{\alpha}_x)t) \frac{|\det \mathcal{L}_x|}{2\pi^{n+1}} \frac{v(x)}{m(x)} \tau_x^{n_-}. \tag{1.1.26}$$

In addition, if $n_- = n_+$ the leading term $A_0^+(x,y,t)$ in the expansion (1.1.15) also has the form

$$A_0^+(x,x,t) = t^n \chi(p_{J_0,J_0}(\boldsymbol{\alpha}_x)t)\frac{|\det \mathcal{L}_x|}{2\pi^{n+1}}\frac{v(x)}{m(x)}\tau_x^{n_+}. \tag{1.1.27}$$

We have an immediate corollary to Theorems 1.1 and 1.2.

COROLLARY 1.3. *With the same notations and assumptions in Theorem 1.1, we have the asymptotic expansion for $q = n_-$*

$$\chi(k^{-1}T_{P,\lambda}^{(q)})(x,x) \sim \sum_{j=0}^{+\infty} k^{n+1-j}\left(A_j^-(x) + A_j^+(x)\right) \quad \text{in } S^{n+1}_{\mathrm{loc}}(1;X,\mathscr{L}(T^{*0,q}X,T^{*0,q}X)), \tag{1.1.28}$$

where for all $j \in \mathbb{N}_0$ we have

$$A_j^{\mp}(x) = \int_0^{+\infty} A_j^{\mp}(x,x,t)dt \in \mathscr{C}^\infty(X,\mathscr{L}(T^{*0,q}X,T^{*0,q}X)) \tag{1.1.29}$$

and locally on (Ω,x) we have

$$A_0^-(x) = \left(\int_0^{+\infty} t^n\chi(p_{I_0,I_0}(-\boldsymbol{\alpha}_x)t)dt\right)\frac{|\det \mathcal{L}_x|}{2\pi^{n+1}}\frac{v(x)}{m(x)}\tau_x^{n_-}, \tag{1.1.30}$$

and when $n_- = n_+$ we also have

$$A_0^+(x) = \left(\int_0^{+\infty} t^n\chi(p_{J_0,J_0}(\boldsymbol{\alpha}_x)t)dt\right)\frac{|\det \mathcal{L}_x|}{2\pi^{n+1}}\frac{v(x)}{m(x)}\tau_x^{n_+}. \tag{1.1.31}$$

We will also give an elementary proof of this theorem on circle bundles in §4.4.

We also have the following picture of the distribution of the eigenvalues of $T_{P,\lambda}^{(q)}$ when $q = n_-$. It can be regarded as a Szegő type limit theorem, cf. also [12, §13] and [35, Theorem 1.3]. With respect to the notations used in (1.1.6), we consider the scaled spectral measures $\mu_k^{(q)}$ which is defined on $\mathbb{R}$ and given by

$$\mu_k^{(q)} := k^{-n-1}\sum_{j\in J}\delta\big(t - k^{-1}\lambda_j\big). \tag{1.1.32}$$

COROLLARY 1.4. *In the situation of Theorem 1.1, for $q = n_-$ the scaled spectral measures $\mu_k^{(q)}$ converges weakly as $k \to +\infty$ to the continuous measure*

$$\mu_{+\infty}^{(q)} = C_P^{(q)}\, t^n dt, \tag{1.1.33}$$

where dt is the Lebesgue measure on $\mathbb{R}$ and

$$C_P^{(q)} := \frac{1}{2\pi^{n+1}}\left(\int_X \frac{|\det \mathcal{L}_x|v(x)dx}{p_{I_0,I_0}^{n+1}(-\boldsymbol{\alpha})} + \mathbb{1}_{\{n_+\}}(q)\int_X \frac{|\det \mathcal{L}_x|v(x)dx}{p_{J_0,J_0}^{n+1}(\boldsymbol{\alpha})}\right). \tag{1.1.34}$$

Let us talk more about the motivation of our main result. When $(n_-, n_+) = (0, n)$, which means that X is stirctly (or strongly) pseudoconvex, and when $q = n_- = 0$ and $\Box_b^{(0)}$ has L^2-closed range, Theorem 1.1 was obtained by [35, Theorem 1.1]. We remark that for a compact strictly pseudoconvex CR manifold, the L^2-closed range condition of Kohn Laplacian is equivalent to the global CR embeddability of X. Such condition holds automatically when $\dim_{\mathbb{R}} X$ is at least five, cf. the argument and results in [8, 9, 62, 63], and conditionally when $\dim_{\mathbb{R}} X = 3$ and X has transversal CR $\mathbb{R}$-action, cf. [64, 71]. In this context, by standard spectral theory [21] and the spectrum result of $\Box_b^{(q)}$ [56, Theorem 1.7], in particular there is a number $\lambda > 0$ such that $\Pi_\lambda^{(q)} = \Pi^{(0)} = \Pi$, which is the Szegő projection of $(0,0)$-forms on X, and $T_{P,\lambda}^{(q)} = T_P^{(0)} = T_P := \Pi \circ P \circ \Pi$, which is the Toeplitz operator on X as we mention in the beginning of this section. The Levi ellipticity condition in this case is equivalent to the condition that the principal symbol of P restricted at Σ^- is positive, where

$$\Sigma^{\mp} := \{\mp c\boldsymbol{\alpha} : \ c \in \mathbb{R}_+\} \subset T^*X. \tag{1.1.35}$$

One important example is the differential operator $P = \frac{1}{2}\left(-i\boldsymbol{T} + (-i\boldsymbol{T})^*\right)$, where $\boldsymbol{T}$ is the Reeb vector field associated with $\boldsymbol{\alpha}$ and $(-i\boldsymbol{T})^*$ is the formal adjoint of $-i\boldsymbol{T}$. Following the argument and results in [12, §2], cf. also pseudodifferential calculus of Hermite type [8], there is a special pseudodifferential operator $\mathscr{P}$ which has the same principal symbol as P at Σ^- and is elliptic everywhere on T^*X such that

$$T_P = \Pi \circ \mathscr{P} \circ \Pi, \quad \mathscr{P} \circ \Pi = \Pi \circ \mathscr{P}. \tag{1.1.36}$$

So one can reduce the estimate of T_P to the estimate of $\mathscr{P}$, and one can prove that T_P has a self-adjoint L^2-extension and the spectrum $\operatorname{Spec}(T_P) \subset \mathbb{R}$ of T_P consists only of isolated eigenvalues bounded from below and it has only $+\infty$ as a point of accumulation. Moreover, for every $\mu \in \operatorname{Spec}(T_P) \setminus \{0\}$, the eigenspace $\operatorname{Ker}(T_P - \mu I)$ is a finite dimensional subspace of the space of smooth CR functions, cf. also [12, Proposition 2.14]. We recall that for any orthonormal basis $\{F_j\}_{j=1}^{+\infty}$ of $H_b^0(X)$, we have

$$\Pi(x,y) = \sum_{j=1}^{+\infty} F_j(x)\overline{F}_j(y). \tag{1.1.37}$$

On the other hand, if we denote the non-zero orthonormal eigenfunctions of T_P corresponding to eigenvalues $\{\lambda_j\}_{j=1}^{+\infty}$ by $\{f_j\}_{j=1}^{+\infty}$, then for $\chi \in \mathscr{C}_0^\infty(\mathbb{R} \setminus \{0\})$ the operator $\chi(k^{-1}T_P)$ defined by functional calculus has the Schwartz kernel

$$\chi(k^{-1}T_P)(x,y) = \sum_{j=1}^{+\infty} \chi\left(\frac{\lambda_j}{k}\right) f_j(x)\overline{f}_j(y) \tag{1.1.38}$$

as a finite sum. Heuristically, as $k \to +\infty$, $\chi(k^{-1}T_P)$ should capture many CR functions and thus can be regarded as the semi-classical weighted Szegő projection. In this point of view, it is observed and proved in [35] that from the microlocal description of the Szegő projection Π by Boutet de Monvel–Sjöstrand we can get the semi-classical version Boutet de Monvel–Sjöstrasnd theorem for $\chi(k^{-1}T_P) = \chi(k^{-1}\Pi \circ P \circ \Pi)$.

Since this thesis is heavily influenced by the result we just mention, in the following we give a sketch the proof of this theorem. Later we will also point out the difference between the general case of $n_- > 0$ which we consider and the case $n_- = 0$ which was already studied. We let $\widetilde{\chi} \in \mathscr{C}_0^\infty(\mathbb{C} \setminus \{0\})$ be an almost analytic extension of χ. By construction, for all $N \in \mathbb{N}$ there exists $C_N > 0$ such that for all $z \in \mathbb{C}$ we have

$$\left|\partial_{\bar{z}}\widetilde{\chi}(k^{-1}z)\right| \leq C_N k^{-N} |\mathrm{Im}\, z|^N . \tag{1.1.39}$$

We apply the general idea of Helffer–Sjöstrand formula

$$\chi(k^{-1}T_P) = \chi(k^{-1}T_P) \circ \Pi = \int_{\mathbb{C}} \frac{\partial \widetilde{\chi}(\frac{z}{k})}{\partial \bar{z}} (z - T_P)^{-1} \circ \Pi \, \frac{dz \wedge d\bar{z}}{2\pi i} , \tag{1.1.40}$$

and solve $(z - T_P)^{-1} \circ \Pi$ by the microlocal analysis of Melin–Sjöstrand [72] and Boutet de Monvel–Sjöstrand [13]. We treat T_P as a classical Fourier integral operator with complex phase and we notice that the identity element of the algebra of Toeplitz operators is the Szegő projection Π. One of the main contribution of our proof is utilizing the variant of [28, Lemma 4.1] to simplify the construction of the pamatetrix of $z - T_P$. Heuristically, we need to formally set some Hörmander symbol $a(x, y, t, z)$, which depends smoothly on z, and examine the relation

$$(z - T_P) \int e^{it\varphi(x,y)} a(x, y, t, z) u(y) dm(y) dt = \Pi u, \;\; u \in \mathscr{C}_0^\infty(\Omega), \tag{1.1.41}$$

where (Ω, x) is any coordinate patch and dm is a volume of X. In general, we can get some transport equations to determine $a(x, y, t, z)$ and $(z - T_P)^{-1} \circ \Pi$, cf. [45, 73] and [22, §10]. However, this method is usually applied when T_P is a regular pseudoddifferential operator, which is not our situation. The main difficulty is to solve the off-diagonal behavior of these transport equations, and our solution revealed that we can simply determine the complicated off-diagonal information from the one on the diagonal, cf. also §4.1. We remark that this argument is independent of the construction of the special pseudodifferential operator $\mathscr{P}$ of (1.1.36) by Boutet de Monvel–Guillemin.

We also remark that the method (1.1.41) was first used in [29, Theorem 1.1] for the Toeplitz operator $T_P^{(q)} := \Pi^{(q)} \circ P \circ \Pi^{(q)}$ on the same CR manifold $(X, T^{1,0}X)$ considered in Theorem 1.1, but with the further assumptions that the Kohn laplacian on X always have the L^2-closed range and $P \in L^0_{\mathrm{cl}}(X; T^{0,n_-}X)$

is of zero order, self-adjoint and has the scalar principal symbol. The assumption of the order of the pseudodifferential operator P significantly influences the spectral behavior of the Toeplitz operator decided by P. Roughly speaking, in this situation the ansatz (1.1.41) can be completely solved by developing a relatively standard Hörmander symbol space theory because the singularities of $a(x,y,t,z)$ in z is determined by the term $(z-a_0(x,x))^{-1}$, which does not contribute any error to t. Here $a_0(x,y)$ is the leading coefficient of the Fourier integral operator $T_P^{(q)}$ and one can also refer the notation $a_0(x,y)$ to Theorem 3.5. By this method, in [29] one can use standard Cauchy–Pompieu formula argument for functional calculus to give an alternative proof of the result that the functional calculus of a Toeplitz operator is still a Toeplitz operator. Although the semi-classical limit (1.1.40) is not considered for this situation for the order reason, by considering the CR manifolds with group action, this result still have many applications in geometric quantization.

Let us return to the case of $P \in L^1_{\mathrm{cl}}(X)$ in [35]. In this context, the ansatz (1.1.41) can only be approximated by a series of $a_j(x,y,t,z)$, where the singularities of $a_j(x,y,t,z)$ in z is in the form of the power of $(z-t)^{-2j-1}$, $j \in \mathbb{N}_0$ and $t \in \mathbb{R}_+$. We refer such approximation to [35, §3.2] and also Theorem 4.7. Comparing to the case order zero Toeplitz operators, here it is hard to define the symbol space theory to simultaneously estimate the growth of $|z|$ and t in a suitable way. However, if we look at the more refined microlocal structure, that is, the semi-classical consideration (1.1.40), we will see that the approximation of the resolvent type operator we just mention is enough. On one hand, one can decide the principal terms of $\chi(k^{-1}T_P)$ again by the Cauchy–Pompeiu formula, cf. also Lemma 4.10. On the other hand, we make a contribution in [35, §4] that only in the semi-classical limit of (1.1.40), given any $\ell_1, \ell_2 \in \mathbb{N}$, by taking the high N-order of the microlocal approximation of $(z-T_P)^{-1} \circ \Pi$, the C^ℓ-norm of the remainders in (1.1.40) on $X \times \Omega$ is bounded by $k^{-\ell_2}$. This is achieved by a suitable semi-classical truncation, utilizing the fact that T_P has discrete spectrum, and the estimates of (1.1.39) and [12, Proposition 12.1]. We hence prove that $\chi(k^{-1}T_P)$ is a semi-classical Fourier integral operator of complex phase. We finally remark that from this formalism, one can not only gives an analogue of classical results about Weyl law and Szegő type limit theorems for Toeplitz operators considered by Boutet–Guillemin, but also helps us a lot to understand more about the CR structure. Our formulation through phase function provides a CR Kodaira embedding (the embedding by weighted non-zero eigenfuncitons of T_P), the semi-classical approximation of contact forms and Reeb vector fields (an analogue of Tian's approximation of Bergman metric in complex geometry), and the almost spherical CR embedding (the CR embeddability into the finite sphere fails in general but can be achieved if the concept of infinite sphere is introduced. This result shows that the CR embeddability into the finite sphere only just fails). We refer the detailed discussion of these results to the paper we just mention, and we also refer to [36, 59] for the related results.

Let us return to the situation considered in Theorem 1.1. The concept of non-degenerate CR manifolds is a natural generalization of the strictly pseudoconvex of CR manifolds. Now, the L^2-closed range condition for the Kohn Laplacian holds whenever $|n_- - n_+| \neq 1$. To include the case $|n_- - n_+| = 1$ we have to apply the Szegő projection on lower energy forms $\Pi^{(q)}_\lambda$ instead of the Szegő projection $\Pi^{(q)}$ and apply the microlocal analysis [56] to study our Toeplitz operators. The main contribution in this work is the semi-classical microlocal analysis under the Levi ellipticity condition of $T^{(n_-)}_{P,\lambda}$, which is inspired by [28, Lemma 4.1]. We notice that this condition is clearly a generalization of the concept of elliptic Toeplitz operators because we allow mild degeneracy of the principal symbol of the pseudodiffential operator P used to define $T^{(n_-)}_{P,\lambda}$. On the other hand, our relatively general ellipticity assumption restrains us from arguing directly as in the case of $(0,0)$-forms by Boutet de Monvel–Guillemin. But we can still construct the parametrix of $T^{(n_-)}_{P,\lambda}$ in the space of lower energy CR harmonic forms in this context. In fact, such parametrix is also in the form of Toeplitz operators we consider, cf. Theorem 3.6. Another difference of this work from [35] is that we need to first establish the asymptotic expansion of $\chi(k^{-1}T^{(n_-)}_{P,\lambda})$ in the operator level and get an estimate for the number of eigenvalues λ_j's of $T^{(n_-)}_{P,\lambda}$ such that $k^{-1}\lambda_j \in \operatorname{supp}\chi$, cf. Theorem 4.20 and Lemma 4.21. Because P is not necessarily elliptic, such estimate can not be obtained directly as in the case of $n_- = 0$ by the method of Boutet de Monvel–Guillemin [12, Proposition 12.1]. Last but not the least, comparing to [36, §4], we give a slight different approach in the final step of the proof of Theorem 1.1. We also apply some suitable semi-classical truncation and utilize the discrete spectrum property of $T^{(n_-)}_{P,\lambda}$. However, in Theorem 4.24, we will directly apply Theorem 4.20 and Lemma 4.21 to obtain the semi-classical expansion of $\chi(k^{-1}T^{(n_-)}_{P,\lambda})$ in the level of Schwartz kernel.

Our results should have more applications in the following problems on non-degenerate compact CR manifolds: the analogue of Kodaira embedding and Tian's almost isometry theorem as [7, 27, 35, 36, 69, 77, 79] and the analogue of geometric quantization as [49, 50, 53]. It could also be helpful for the analysis and geometry of the space of CR harmonic sections of high power of line bundles as [48, 52, 57] and for the theory of Weyl law and eta invariant of Toeplitz operators.

1.2. Second coefficient of Boutet de Monvel–Sjöstrand expansion

The second result we present in this thesis is the joint work of C.-Y. Hsiao and the author [61]. The study of the coefficients for asymptotic expansion of projection in several complex variables is always interesting and important. The leading coefficient for the asymptotics of Bergman kernel on bounded strictly pseudoconvex domains appear for the first time in Hörmander [38], cf. also the historical remark [41]. In this context, the asymptotic expansion of singularities of Bergman kernel restricted on diagonal was obtained by Fefferman [25] through formal expansion of the defining function in terms of pole type and log term singularities. The off-diagonal expansion of Bergman kernel is established by Boutet de Monvel–SJöstrand [9] by a different approach with a shorter proof. We also refer to [45, 58, 60] for various generalities and other microlocal analysis on Bergman kernel. Depending on the geometric data on the boundary of the domain, the coefficients of the expansion may vary. When the contact from on the boundary is pseudo-Einstein and the defining function is chosen to be the Monge–Ampère solution, these coefficients are studied by Fefferman [26], cf. also [37]. On the side of complex geometry, the Bergman kernel has also been studied by many mathematicians, in various generalities, establishing the asymptotic expansion for high powers of line bundles. Moreover, it was discovered that the coefficients in the asymptotic expansion encode geometric information about the underlying complex projective manifolds. Besides the references we mentioned before, which are influenced by the development of Boutet–Sjöstrand and Boutet de Monvel-Guillemin, we also refer the results of semi-classical Bergman asymptotic expansion and the corresponding leading coefficient in this context to [3, 4, 75, 79]. For the first fewer coefficients of the expansion, readers can find in [20, 33, 46, 47, 51, 65–68] for example. We refer the role of these coefficients in complex geometry to [23, 80] for instance.

The coefficients for the asymptotic expansion of singularities of Szegő kernel, like its cousin Bergman kernel, is also fundamental in CR geometry. Different from the solution of Fefferman, which also works for Szegő kernel, we want to study the coefficient problem by the microlocal analysis of the asymptotic expansion of Boutet de Monvel–Sjöstrand [13]. Let us explain our set-up. We let $(X, T^{1,0}X)$ be a compact CR manifold of hypersurface type. We assume there is a global non-vanishing 1-form $\boldsymbol{\alpha} \in \mathscr{C}^\infty(X, T^*X)$ so that

$$\boldsymbol{\alpha}(u) = 0 \text{ for every } u \in T^{1,0}X \oplus T^{0,1}X, \tag{1.2.1}$$

$$-\frac{1}{2i} d\boldsymbol{\alpha}|_{T^{1,0}X} \text{ is positive definite.} \tag{1.2.2}$$

This implies that X is a strictly pseudoconvex CR manifold. For every $x \in X$ and for all $u, v \in T_x^{1,0}X$, we recall that the Levi form at x is the Hermitian quadratic form on $T^{1,0}_x X$ given by

$$\mathcal{L}_x(u,\overline{v}) := -\frac{1}{2i}\langle\, d\boldsymbol{\alpha}(x)\,,\, u\wedge\overline{v}\,\rangle. \tag{1.2.3}$$

We let $T \in \mathscr{C}^\infty(X, TX)$ be the global real vector field determined by

$$\boldsymbol{\alpha}(T) \equiv -1,\ \ d\boldsymbol{\alpha}(T,\cdot) \equiv 0. \tag{1.2.4}$$

The Levi form $\mathcal{L}_x$ induces a Hermitian metric called Levi metric $\langle\,\cdot\,|\,\cdot\,\rangle$ on $\mathbb{C}TX$ determined by

$$\langle\, u\,|\,v\,\rangle = \mathcal{L}_x(u,\overline{v}),\ \ \text{for every } u,v \in T^{1,0}_x X,\, x\in X, \tag{1.2.5}$$

$$\langle\, \overline{u}\,|\,\overline{v}\,\rangle = \overline{\langle\, u\,|\,v\,\rangle},\ \ \text{for every } u,v \in T^{1,0}_x X,\, x\in X, \tag{1.2.6}$$

$$T^{1,0}X \perp T^{0,1}X,\ \ T \perp (T^{1,0}X \oplus T^{0,1}X), \tag{1.2.7}$$

$$\langle\, T\,|\,T\,\rangle = 1. \tag{1.2.8}$$

We let $\dim_{\mathbb{R}} X = 2n+1$, $n \geq 1$. We denote $\det \mathcal{L}_x := \mu_1(x)\cdots\mu_n(x)$, where $\mu_j(x)$, $j = 1,\cdots,n$, are the eigenvalues of the Levi form with respect to the given Hermitian metric on $\mathbb{C}TX$. If we take Levi metric on $\mathbb{C}TX$, then

$$\det \mathcal{L}_x \equiv 1. \tag{1.2.9}$$

We let $(\,\cdot\,|\,\cdot\,)$ be the L^2-inner product on $\Omega^{0,q}(X)$ induced by $\langle\,\cdot\,|\,\cdot\,\rangle$ and let $L^2_{0,q}(X)$ be the completion of $\Omega^{0,q}(X)$ with respect to $(\,\cdot\,|\,\cdot\,)$. We write $L^2(X) := L^2_{0,0}(X)$. We denote $\overline{\partial}_b : \mathscr{C}^\infty(X) \to \Omega^{0,1}(X)$ to be the tangential Cauchy-Riemann operator on X. We recall that the orthogonal projection

$$\Pi : L^2(X) \to \operatorname{Ker}\overline{\partial}_b \tag{1.2.10}$$

is called the *Szegő projection*, and its distribution kernel denoted by $\Pi(x,y)$ is called Szegő kernel. When $\overline{\partial}_b : \operatorname{Dom}\overline{\partial}_b \subset L^2(X) \to L^2_{0,1}(X)$ has closed range, then under this natural assumption for any fixed $p \in X$, by [9, Theorem at pp. IX.5], there is an open set U of p and an injective immersion F given by

$$F : U \to \mathbb{C}^{n+1}, \tag{1.2.11}$$

$$x \mapsto (F_1(x),\cdots,F_{n+1}(x)), \tag{1.2.12}$$

where $F_1,\cdots,F_{n+1} \in \mathscr{C}^\infty(X)\cap \operatorname{Ker}\overline{\partial}_b$. From now on, we identify U with $\partial M \bigcap \Omega$, where

$$\partial M := \{z \in \mathbb{C}^{n+1} : r(z) = 0\}, \tag{1.2.13}$$

$$r(z) \in \mathscr{C}^\infty(\mathbb{C}^{n+1}, \mathbb{R}), \tag{1.2.14}$$

$$|J(dr)| = 1 \text{ on } \partial M,\ J \text{ is the standard complex structure on } \mathbb{C}^{n+1}, \tag{1.2.15}$$

$$\Omega \text{ is an open set of } p \text{ in } \mathbb{C}^{n+1}. \tag{1.2.16}$$

From the standard Chern–Moser trick, cf also [42, Lemma 3.2], we can find local holomorphic coordinates $x = (x_1,\cdots,x_{2n+2}) = z = (z_1,\cdots,z_{n+1})$, $z_j =$

$x_{2j-1} + ix_{2j}$, $j = 1, \cdots, n+1$, defined on Ω (we assume that Ω is small enough) such that

$$z(p) = 0, \tag{1.2.17}$$

$$r(z) = 2\mathrm{Im}\, z_{n+1} + \sum_{j=1}^{n} |z_j|^2 + O(|(z_1, \cdots, z_{n+1})|^4). \tag{1.2.18}$$

Under such coordinates, by [13, Proposition 1.1 & Theorem 1.5]) we can take an almost analytic extension of $r(z)$ denoted by $\rho(z,w)$ such that

$$\rho(z,w) = \frac{1}{i} \sum_{\alpha,\beta \in \mathbb{N}_0^{n+1}, |\alpha|+|\beta| \le N} \frac{\partial^{\alpha+\beta} r}{\partial z^\alpha \partial \overline{z}^\beta}(0) \frac{z^\alpha}{\alpha!} \frac{\overline{w}^\beta}{\beta!} + O(|(z,w)|^{N+1}) \quad \text{for every } N \in \mathbb{N}, \tag{1.2.19}$$

and for the function

$$\phi(x,y) := \rho(z,w)|_{U \times U} \tag{1.2.20}$$

we have the following expansion

$$\Pi(x,y) \equiv \int_0^{+\infty} e^{it\phi(x,y)} a(x,y,t) dt \mod \mathscr{C}^\infty(U \times U) \tag{1.2.21}$$

called Boutet de Monvel–Sjöstrand expansion, where

$$\phi(x,y) \in \mathscr{C}^\infty(U \times U), \tag{1.2.22}$$

$$\mathrm{Im}\, \phi \ge 0, \tag{1.2.23}$$

$$\phi(x,y) = 0 \text{ if and only if } x = y, \tag{1.2.24}$$

$$d_x \phi(x,x) = -d_y \phi(x,x) = -\boldsymbol{\alpha}(x), \text{ for every } x \in U, \tag{1.2.25}$$

and

$$a(x,y,t) \sim \sum_{j=0}^{+\infty} a_j(x,y) t^{n-j} \text{ in } S^n_{1,0}(U \times U \times \mathbb{R}_+), \tag{1.2.26}$$

$$a_0(x,x) = \frac{1}{2\pi^{n+1}}, \text{ for every } x \in U. \tag{1.2.27}$$

Moreover, we can check that the function ϕ constructed from ρ satisfies

$$\overline{\partial}_{b,x}(\phi(x,y)) \text{ vanishes to infinite order at } x = y, \tag{1.2.28}$$

$$\overline{\partial}_{b,y}(-\overline{\phi}(y,x)) \text{ vanishes to infinite order at } x = y. \tag{1.2.29}$$

Nevertheless, we will see in Theorem 2.6 that for any Szegő phase function $\varphi \in \mathscr{C}^\infty(U \times U)$ which has the properties that

$$\operatorname{Im} \varphi(x,y) \geq 0, \tag{1.2.30}$$

$$\varphi(x,y) = 0 \text{ if and only if } x = y, \tag{1.2.31}$$

$$d_x\varphi(x,x) = -d_y\varphi(x,x) = -\boldsymbol{\alpha}(x), \tag{1.2.32}$$

but not necessarily satisfies (1.2.28) and (1.2.29), we can always find a symbol $s(x,y,t) \sim \sum_{j=0}^{+\infty} s_j(x,y)t^{n-j}$ in $S^n_{1,0}(U \times U \times \mathbb{R}_+)$ such that

$$\Pi(x,y) \equiv \int_0^{+\infty} e^{it\varphi(x,y)} s(x,y,t)dt. \tag{1.2.33}$$

This implies for each Fourier integral operator of Boutet de Monvel–Sjöstrand microlocally approximated the Szegő projection, the coefficients $s_j(x,y)$ naturally depend on the choice of $\varphi(x,y)$. We will prove a known result in Theorem 3.3 that the leading coefficient $s_0(x,x)$ on diagonal is independent of the choice of $\varphi(x,y)$. However, the sub-leading coefficient $s_1(x,x)$ on diagonal not only depends on the choice of the Szegő phase function, but also relies on the choice of the off-diagonal behavior of the leading coefficient $s_0(x,y)$. Let us see an example. If we let

$$\sigma_0(x,y) := s_0(x,y) + \kappa(x,y)\varphi(x,y), \;\; \kappa(x,y) \in \mathscr{C}^\infty(U \times U), \tag{1.2.34}$$

$$\sigma_1(x,y) := s_1(x,y) - in\kappa(x,y) \tag{1.2.35}$$

and set

$$\sigma(x,y,t) \sim t^n\sigma_0(x,y) + t^{n-1}\sigma_1(x,y) + \sum_{j=0}^{+\infty} t^{n-j}s(x,y) \text{ in } S^n_{\mathrm{cl}}\left(U \times U \times \mathbb{R}_+\right), \tag{1.2.36}$$

then it is not difficult to check that

$$\int_0^{+\infty} e^{it\varphi(x,y)} s(x,y,t)dt \equiv \int_0^{+\infty} e^{it\varphi(x,y)} \sigma(x,y,t)dt \mod \mathscr{C}^\infty(U \times U), \tag{1.2.37}$$

and it is clear that $s_1(x,x) \neq \sigma_1(x,x)$ may happen.

If we want to determine the value of $s_1(x,x)$, such ambiguity will leads to a huge difficulty. To settle up the issue, we need to find a suitable class of phase function and the corresponding leading term to understand $a_1(x,x)$ in the expansion of Boutet–Sjöstrand. We now formulate our main result. We let $x = (x_1, \cdots, x_{2n+1})$ be a local coordinates of X defined on an open set $D \subset X$ with $T = -\frac{\partial}{\partial x_{2n+1}}$ on D. By the Malgrange preparation theorem [40, Theorem 7.5.5], for any Szegő phase function φ, we may assume that

$$\varphi(x,y) = f(x,y)(-x_{2n+1} + g(x',y)) \;\text{ on } D, \tag{1.2.38}$$

where $f, g \in \mathscr{C}^\infty(D \times D)$ and $f(x,x) = 1$ for every $x \in D$. We let

$$\Phi := -x_{2n+1} + g(x', y). \tag{1.2.39}$$

It is not difficult to see that $\Phi(x,y)$ satisfies (1.2.25), $\Phi(x,y)t$ and $\phi(x,y)t$ are equivalent in the sense of Melin–Sjöstrand. Moreover, in this context we can check that

$$(T^2\Phi)(x,x) = 0 \text{ at every } x \in D. \tag{1.2.40}$$

Under the above supplementary conditions, we have the following.

THEOREM 1.5 ([61, Lemma 1.1]). *We let X be a compact strictly pseudoconvex embeedable CR manifold of $\dim_{\mathbb{R}} X = 2n+1$, $n \geq 1$, and $D \subset X$ be any open coordinate patch with local coordinates $x = (x_1, \cdots, x_{2n+1})$. For Szegő phase functions $\varphi_1(x,y), \varphi_2(x,y) \in \mathscr{C}^\infty(D \times D)$, we assume that φ_1, φ_2 satisfy* (1.2.40). *If we have*

$$\int_0^{+\infty} e^{it\varphi_2(x,y)} \alpha(x,y,t)dt \equiv \int_0^{+\infty} e^{it\varphi_1(x,y)} \beta(x,y,t)dt \mod \mathscr{C}^\infty(D \times D), \tag{1.2.41}$$

where $\alpha(x,y,t), \beta(x,y,t) \in S^n_{\mathrm{cl}}(D \times D \times \mathbb{R}_+)$ have the properties that for all $x \in D$ we have

$$\alpha_0(x,x) = \beta_0(x,x), \tag{1.2.42}$$

$$(T\alpha_0)(x,x) = (T\beta_0)(x,x) = 0, \tag{1.2.43}$$

then we get

$$\alpha_1(x,x) = \beta_1(x,x). \tag{1.2.44}$$

We will present the proof of above statement in Lemma 5.5, which is the local version of this theorem. Accordingly, we know which class of phase function and leading coefficient we should seek for determining the sub-leading coefficient, and we will see that the assumption on the leading coefficient in the previous theorem is always possible by Lemma 5.6. Furthermore, by applying Lemma 5.5 and Theorem 5.7, we can demonstrate that the pointwise value of the sub-leading term can be calculated. Specifically, we can use the special Szegő phase function, denoted as ϕ as before and constructed by (1.2.19). From Taylor expansion of ϕ, we can represent some pseudohermitian data by derivatives of ϕ, and by combining Hörmander stationary phase formula, we can read the recursive formula of the leading and sub-leading coefficients from the relation $\Pi = \Pi \circ \Pi$. Our result for the second coefficient of Boutet–Sjöstrand expansion then follows as below.

THEOREM 1.6 ([61, Theorem 1.2]). *We let $(X, T^{1,0}X, \boldsymbol{\alpha})$ be a compact strictly pseudoconvex embeedable CR manifold of $\dim_{\mathbb{R}} X = 2n+1$, $n \geq 1$, and $D \subset X$ be any open coordinate patch with local coordinates $x = (x_1, \cdots, x_{2n+1})$. We let T be the Reeb vector field on X induced by strict psedoconvexity of $(X, T^{1,0}X, \boldsymbol{\alpha})$. The set* *of Szegő phase functions* $\{\varphi \in \mathscr{C}^\infty(D \times D) : (T^2\varphi)(x,x) = 0$ *for all* $x \in D\}$ *is non-empty, and for any element* φ *in the set we can find a symbol*

$$s(x,y,t) \sim \sum_{j=0}^{+\infty} s_j(x,y)t^{n-j} \text{ in } S^n_{1,0}(D \times D \times \mathbb{R}_+) \tag{1.2.45}$$

such that for all $(x,y) \in D \times D$ *we have*

$$s_0(x,x) = \frac{1}{2\pi^{n+1}}, \tag{1.2.46}$$

$$T_x \circ s_0(x,y) = 0, \tag{1.2.47}$$

$$s_1(x,x) = \frac{1}{4\pi^{n+1}} R_{\mathrm{scal}}(x), \tag{1.2.48}$$

and

$$\Pi(x,y) \equiv \int_0^{+\infty} e^{it\varphi(x,y)} s(x,y,t)dt \mod \mathscr{C}^\infty(D \times D), \tag{1.2.49}$$

where R_{scal} *is the Tanaka–Webster scalar curvature on* X, *cf.* (5.1.15).

The proof of this theorem will be conducted in §5.3. The consideration for the existence of Φ such that $(T^2\Phi)(x,x) = 0$ is motivated by the case of that X admits a transversal CR circle action, cf. [33]. We give an elementary example which our result does not apply. We let $z, w \in \partial\mathbb{D}^n \subset \mathbb{C}^{n+1}$, where $D := \{z \in \mathbb{C}^{n+1} : |z|^2 = 1\}$. By Riesz–Fischer theorem argument, it is known in classical several complex variables that

$$\Pi(z,w) = \frac{n!}{2\pi^{n+1}} \frac{1}{(1 - \langle z, \overline{w}\rangle)^{n+1}} = \int_0^{+\infty} e^{it(i(1-\langle z,\overline{w}\rangle))} \frac{1}{2\pi^{n+1}} t^n dt. \tag{1.2.50}$$

as a distribution and an oscillatory integral. However, we can check that for $z, w \in \partial\mathbb{D}^n$ we have $T_z^2(1 - \langle z, \overline{w}\rangle) \neq 0$ and $R_{\mathrm{scal}}(z) = \frac{n(n+1)}{2} \neq 0$. This example shows that the the auxiliary condition in our theorem is necessary.

CHAPTER 2

Preliminaries

We use the following notations and conventions throughout this article. $\mathbb{Z}$ is the set of integers, $\mathbb{N} = \{1, 2, 3, \cdots\}$ is the set of natural numbers and we put $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$; $\mathbb{R}$ is the set of real numbers, $\mathbb{R}_+ := \{x \in \mathbb{R} : x > 0\}$ and $\dot{\mathbb{R}} := \mathbb{R} \setminus \{0\}$; $\mathbb{C}$ is the set of complex numbers and $\dot{\mathbb{C}} := \mathbb{C} \setminus \{0\}$. For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n$ and $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, we set

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \tag{2.0.1}$$

$$\partial_{x_j} = \frac{\partial}{\partial x_j}, \quad \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x^\alpha}. \tag{2.0.2}$$

Let $z = (z_1, \cdots, z_n)$, $z_j = x_{2j-1} + i x_{2j}$, $j = 1, \cdots, n$, be coordinates on $\mathbb{C}^n$. We write

$$z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}, \quad \overline{z}^\alpha = \overline{z}_1^{\alpha_1} \cdots \overline{z}_n^{\alpha_n}, \tag{2.0.3}$$

$$\partial_{z_j} = \frac{\partial}{\partial z_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_{2j-1}} - i\frac{\partial}{\partial x_{2j}}\Big), \quad \partial_{\overline{z}_j} = \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_{2j-1}} + i\frac{\partial}{\partial x_{2j}}\Big), \tag{2.0.4}$$

$$\partial_z^\alpha = \partial_{z_1}^{\alpha_1} \cdots \partial_{z_n}^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial z^\alpha}, \quad \partial_{\overline{z}}^\alpha = \partial_{\overline{z}_1}^{\alpha_1} \cdots \partial_{\overline{z}_n}^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial \overline{z}^\alpha}. \tag{2.0.5}$$

For $j, s \in \mathbb{Z}$, we set $\delta_{js} = 1$ if $j = s$, $\delta_{js} = 0$ if $j \neq s$. All the smooth manifolds in this work are assumed to be paracompact.

2.1. Elements of microlocal and semi-classical analysis

In this section we recall basic notions of microlocal and semi-classical analysis, and we refer to [22,31,39,40,72] for detail.

For a $\mathscr{C}^\infty$-orientable manifold W, we let TW and T^*W denote the tangent bundle of W and the cotangent bundle of W respectively. The complexified tangent bundle of W and the complexified cotangent bundle of W will be denoted by $\mathbb{C}TW$ and $\mathbb{C}T^*W$ respectively. We write $\langle\,\cdot\,,\cdot\,\rangle$ to denote the pointwise duality between TW and T^*W. We extend $\langle\,\cdot\,,\cdot\,\rangle$ bilinearly to $\mathbb{C}TW \times \mathbb{C}T^*W$. We let E be a $\mathscr{C}^\infty$-vector bundle over W. The fiber of E at $x \in W$ will be denoted by E_x. With respect to the base manifold W, the spaces of smooth sections of E will be denoted by $\mathscr{C}^\infty(W, E)$, and we let $\mathscr{C}_0^\infty(W, E)$ be the subspace of $\mathscr{C}^\infty(W, E)$ whose elements have compact support in W; the spaces of distribution sections of E will be denoted by $\mathscr{D}'(W, E)$, and we let $\mathscr{E}'(W, E)$ be the subspace of $\mathscr{D}'(W,E)$ whose elements have compact support in W. We denote I to be the identity map on W. For an open set $V \subset W$, $f \in \mathscr{C}^\infty(V \times V, E)$ and a number $N \in \mathbb{N}$, we write $f = O(|x-y|^N)$ if f vanishes to $(N-1)$-order at the diagonal (when $E = \mathbb{C}$, this means that $(\partial_x^\alpha \partial_y^\beta f)(x,x) = 0$ for all $x \in V$ and $|\alpha| + |\beta| \leq N-1$). We also write $f = O(|x-y|^{+\infty})$ if the previous relation holds for arbitrary $N \in \mathbb{N}$. Similarly, at a point $p \in V$ we write $f = O(|(p,p)|^N)$ if f vanishes to $(N-1)$- order at (p,p) and we write $f = O(|(p,p)|^{+\infty})$ if the previous relation holds for arbitrary $N \in \mathbb{N}$.

We recall the Schwartz kernel theorem [40, §5.2]. We let E and F be $\mathscr{C}^\infty$-vector bundles over orientable $\mathscr{C}^\infty$-manifolds W_1 and W_2, respectively, equipped with smooth densities of integration. If $A : \mathscr{C}_0^\infty(W_2, F) \to \mathscr{D}'(W_1, E)$ is continuous, we write $A(x,y)$ to denote the Schwartz kernel of A. The following two statements are equivalent

(i) A is continuous: $\mathscr{E}'(W_2, F) \to \mathscr{C}^\infty(W_1, E)$,
(ii) $A(x,y) \in \mathscr{C}^\infty(W_1 \times W_2, \mathscr{L}(F,E))$.

Here we write $\mathscr{L}(F,E)$ to denote the vector bundle with fiber over $(x,y) \in W_1 \times W_2$ consisting of the linear maps $\mathscr{L}(F_y, E_x)$ from F_y to E_x. If A satisfies (i) or (ii), we say that A is smoothing on $W_1 \times W_2$. For continuous operators $A, B : \mathscr{C}_0^\infty(W_2, F) \to \mathscr{D}'(W_1, E)$, we write

$$A \equiv B \text{ on } W_1 \times W_2 \tag{2.1.1}$$

if $A - B$ is a smoothing operator. If (2.1.1) holds when $W_1 = W_2 = W$, we simply write $A \equiv B$ on W or just $A \equiv B$. For an open set $V \subset W$, we say that a distributional section $A(x,y) \in \mathscr{D}'(V \times V, \mathscr{L}(E,E))$, which possibly smoothly depends on some other parameter, is properly supported (in the variables (x,y)) if the restrictions of the two projections $(x,y) \mapsto x$, $(x,y) \mapsto y$ to $\operatorname{supp} A(x,y)$ are proper maps, and we say an operator A is properly supported if the Schwartz kernel $A(x,y)$ is properly supported.

For a $\mathscr{C}^\infty$-vector bundle E over a $\mathscr{C}^\infty$-orientable compact manifold W and any number $s \in \mathbb{R}$, with respect to the standard L^2-norm $\|\cdot\|$ for the section of E we let $H^s(W,E)$ to be the standard Sobolev space of order s for sections of E with the Sobolev norm $\|\cdot\|_s$. We let $H^s_{\text{comp}}(W,E)$ be the subspace of $H^s(W,E)$ whose elements have compact support in W. For a relatively compact open set $U \Subset W$, we put

$$H^s_{\text{loc}}(U,E) = \left\{ u \in \mathscr{D}'(U,E) : \chi u \in H^s_{\text{comp}}(U,E), \forall \chi \in \mathscr{C}_0^\infty(U) \right\}. \tag{2.1.2}$$

For smooth vector bundles E, F over W and an operator $F_z : H^{s_1}_{\text{comp}}(W_1, E) \to H^{s_2}_{\text{loc}}(W_2, F)$ smoothly depending on some parameter $z \in \mathbb{C}$, we write

$$F_z = O(g(z)) \text{ in } \mathscr{L}\left(H^{s_1}_{\text{comp}}(W_1, E), H^{s_2}_{\text{loc}}(W_2, F)\right) \tag{2.1.3}$$

if for every $z \in \mathbb{C}$ the operator $F_z : H^{s_1}_{\mathrm{comp}}(W_1, E) \to H^{s_2}_{\mathrm{loc}}(W_2, F)$ is continuous and for any $\chi_j \in \mathscr{C}^\infty_0(W_j)$, $j = 1, 2$, $\tau_1 \in \mathscr{C}^\infty_0(W_1)$, $\tau_1 \equiv 1$ on $\operatorname{supp} \chi_1$, there is a constant $c > 0$ independent of z such that

$$\|\chi_2 F_z \chi_1 u\|_{s_2} \leq c \cdot |g(z)| \cdot \|\tau_1 u\|_{s_1}, \forall u \in H^{s_1}_{\mathrm{loc}}(W_1, E). \tag{2.1.4}$$

For any $m \in \mathbb{R}$, $0 \leq \delta \leq \rho \leq 1$, $N \in \mathbb{N}$ and the smooth vector bundle $E = \mathbb{C}$ over an open set $V \subset \mathbb{R}^n$, the Hörmander symbol sapce $S^m_{\rho,\delta}\left(V \times \mathbb{R}^N, E\right)$ of order m with type (ρ, δ) is defined by the set collecting all $a(x, \eta) \in \mathscr{C}^\infty\left(V \times \mathbb{R}^N, E\right)$ such that for all compact sets $K \Subset V$ and all multi-indices α, β, there is a constant $C = C_{K,\alpha,\beta}(a) > 0$ such that

$$\left|\partial^\alpha_x \partial^\beta_\eta a(x, \eta)\right| \leq C(1 + |\eta|)^{m-\rho|\beta|+\delta|\alpha|}, \text{ for all } (x, \eta) \in K \times \mathbb{R}^N, |\eta| \geq 1. \tag{2.1.5}$$

The space of symbol of order minus infinity is denoted by $S^{-\infty}_{\rho,\delta}\left(V \times \mathbb{R}^N, E\right)$, which collects all $a(x, \eta) \in \mathscr{C}^\infty\left(V \times \mathbb{R}^N, E\right)$ such that for all compact sets $K \Subset V$, all multi-indices α, β, and each $M > 0$, there is a constant $C = C_{K,\alpha,\beta}(a) > 0$ such that

$$\left|\partial^\alpha_x \partial^\beta_\eta a(x, \eta)\right| \leq C(1 + |\eta|)^{-M}, \text{ for all } (x, \eta) \in K \times \mathbb{R}^N, |\eta| \geq 1. \tag{2.1.6}$$

We can check that for any fixed $(\rho, \delta) \in [0, 1] \times [0, 1]$,

$$S^{-\infty}\left(V \times \mathbb{R}^N, E\right) = \bigcap_{m \in \mathbb{R}} S^m_{\rho,\delta}\left(V \times \mathbb{R}^N, E\right). \tag{2.1.7}$$

For $a_j \in S^{m_j}_{\rho,\delta}\left(V \times \mathbb{R}^N, E\right)$, $j = 0, 1, 2, \cdots$ with $m_j \to -\infty$ as $j \to +\infty$, by Borel construction, there always exists $a \in S^{m_0}_{\rho,\delta}\left(V \times \mathbb{R}^N, E\right)$ unique modulo $S^{-\infty}\left(V \times \mathbb{R}^N, E\right)$ such that for all $\ell = 1, 2, \cdots$,

$$a - \sum_{j=0}^{\ell-1} a_j \in S^{m_\ell}_{\rho,\delta}\left(V \times \mathbb{R}^N, E\right). \tag{2.1.8}$$

If a and a_j have the properties above, we call a is the asymptotic sum of a_j and write

$$a \sim \sum_{j=0}^{+\infty} a_j \text{ in } S^{m_0}_{\rho,\delta}\left(V \times \mathbb{R}^N, E\right). \tag{2.1.9}$$

We recall the classical symbol space $S^m_{\mathrm{cl}}\left(V\times\mathbb{R}^N,E\right)$ is the set collecting all elements $a\in S^m_{1,0}\left(V\times\mathbb{R}^N,E\right)$ such that

$$a\sim\sum_{j=0}^{+\infty}(1-\chi)a_j \text{ in } S^m_{1,0}\left(V\times\mathbb{R}^N,E\right), \tag{2.1.10}$$

$$\chi\in\mathscr{C}^\infty(\mathbb{R}^N),\ \chi\equiv 1 \text{ near } 0, \tag{2.1.11}$$

$$a_j\in\mathscr{C}^\infty\left(V\times\dot{\mathbb{R}}^N,E\right),\ a_j(x,\lambda\eta)=\lambda^{m-j}a_j(x,\eta),\ \forall\lambda>0. \tag{2.1.12}$$

We recall the local definition of oscillatory integrals. For open sets $V_1\subset\mathbb{R}^{n_1}$, $V_2\subset\mathbb{R}^{n_2}$, $V:=V_1\times V_2$, $0<\rho\leq 1$, $0\leq\delta<1$, $\varphi(x,\eta)\in\mathscr{C}^\infty(V\times\dot{\mathbb{R}}^N)$ such that

$$\operatorname{Im}\varphi(x,y,\eta)\geq 0, \tag{2.1.13}$$

$$\varphi(x,y,\lambda\eta)=\lambda\varphi(x,y,\eta),\ \forall\,\lambda>0, \tag{2.1.14}$$

$$\sum_{j=1}^{n_1}\frac{\partial\varphi}{\partial x_j}dx_j+\sum_{j=1}^{n_2}\frac{\partial\varphi}{\partial y_j}dy_j+\sum_{\ell=1}^{N}\frac{\partial\varphi}{\partial\eta_\ell}d\eta_\ell\neq 0, \tag{2.1.15}$$

then for $a(x,y,\eta)\in S^m_{\rho,\delta}(V\times\mathbb{R}^N,E)$ and $m+\ell<-N$, $\ell\in\mathbb{N}$, we always have

$$\int e^{i\varphi(x,y,\eta)}a(x,y,\eta)d\eta\in\mathscr{C}^\ell(V,E). \tag{2.1.16}$$

Moreover, using partial integration, for any $a(x,y,\eta)\in\bigcup_{m\in\mathbb{R}}S^m_{\rho,\delta}(V\times\mathbb{R}^N,E)$, we can check that for any $\tau\in\mathscr{C}^\infty_0(\mathbb{R}^N)$ such that $\tau(0)=1$ the notation

$$\int e^{i\varphi(x,y,\eta)}a(x,y,\eta)d\eta:=\lim_{\epsilon\to 0}\int e^{i\varphi(x,y,\eta)}\tau(\epsilon\eta)a(x,y,\eta)d\eta \tag{2.1.17}$$

defined by the limit in the sense of distribution is in fact also an element in $\mathscr{D}'(V,E)$ and this notation coincides with the relation (2.1.16) when the order of m is suitably small. The above form of integral is called an oscillatory integral, and a continuous operator $I_\varphi(a):\mathscr{C}^\infty_0(V_2)\to\mathscr{C}^\infty(V_1)$ determined by the Schwartz kernel in the form of (2.1.17) is called a Fourier integral operator of order

$$\left(m+\frac{N}{2}-\frac{n_1+n_2}{4}\right). \tag{2.1.18}$$

By properly supported partition of unity [31, pp. 29], if a Fourier integral operator is smoothing away from the diagonal, it can be decomposed into the sum of a properly supported Fourier integral operator and a smoothing operator. When $\rho+\delta=1$ and $\rho>\delta$, by the theory of the equivalence of phase functions, cf. [39, §2.3] and [31, §11] when $\operatorname{Im}\varphi=0$ and [72, §4] for the general case of $\operatorname{Im}\varphi\geq 0$, we can define global Fourier integral operators on smooth manifolds. All the notations above about Hörmander symbols and Fourier integral operators are also well-defined for abstract smooth vector bundles and arbitrary smooth manifolds, where the reader can refer to [43, §18] and [44, §25].

Next, for an open set $W \subset \mathbb{C}^{n_1}$ such that $W_{\mathbb{R}} := W \cap \mathbb{R}^{n_1} \neq \emptyset$, we say that a function $f \in \mathscr{C}^\infty(W)$ is almost analytic if, for any compact subset $K \subset W$ such that $K \cap \mathbb{R}^{n_1} \neq \emptyset$ and any $N \in \mathbb{N}_0$, there is a constant $C_N > 0$ such that

$$\left|\frac{\partial f}{\partial \bar{z}}(z)\right| \leq C_N |\mathrm{Im} z|^N \text{ for all } z \in K. \tag{2.1.19}$$

We say two almost analytic functions f_1 and f_2 are equivalent if for any compact subset $K \subset W$ such that $K \cap \mathbb{R}^{n_1} \neq \emptyset$ and any $N \in \mathbb{N}_0$, there is a constant $C_N > 0$ such that $|(f_1 - f_2)(z)| \leq C_N |\mathrm{Im} z|^N$ for all $z \in K$. For any $f \in \mathscr{C}^\infty(W_{\mathbb{R}})$, f always admits an almost analytic extension on W up to equivalence. We need the following result from [72, Lemma 2.1].

THEOREM 2.1. *We assume $f(x,w)$ is a smooth complex-valued function in a neighborhood of $(0,0) \in \mathbb{R}^{n_1+n_2}$ and that $\mathrm{Im} f \geq 0$, $\mathrm{Im} f(0,0) = 0$, $f'_x(0,0) = 0$, $\det f''_{xx}(0,0) \neq 0$. We let $\widetilde{f}(z,\widetilde{w})$ be an almost analytic extension of f to a complex neighborhood of $(0,0)$, where $z \in \mathbb{C}^{n_1}$ and $\widetilde{w} \in \mathbb{C}^{n_2}$. Then the equation*

$$\frac{\partial \widetilde{f}}{\partial z}(z,\widetilde{w}) = 0 \tag{2.1.20}$$

has a solution of the form $z = Z(\widetilde{w})$ in a neighborhood of $0 \in \mathbb{C}^{n_2}$, and there is a constant $C > 0$ such that

$$\mathrm{Im} \widetilde{f}(Z(\widetilde{w}),\widetilde{w}) \geq C|\mathrm{Im} Z(\widetilde{w})|^2, \text{ for } \widetilde{w} \in \mathbb{R}^{n_2} \text{ near } 0. \tag{2.1.21}$$

We now present the complex stationary phase formula by Melin–Sjöstrand [72, Theorem 2.3], which will be used several times in our work. For convenience, we use the notation $\widetilde{f}'_z = \partial \widetilde{f} / \partial z$.

THEOREM 2.2. *Let $f(x,w)$ be as in Theorem 2.1. Then there are real neighborhoods U and V of the origin of $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively, and differential operators $C_{w,j} := C_{w,j}(D_x)$, $j \in \mathbb{N}_0$, of order less or equal to $2j$ with smooth coefficients in $w \in V$, such that for each compact set $K_1 \subset U$ and any $u(x,w) \in \mathscr{C}^\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ with support in $K_1 \times V$ we have the following estimate: For all compact set $K_2 \subset V$, for every $N \in \mathbb{N}$, for any $\alpha \in \mathbb{N}_0^{n_2}$ and $\beta \in \mathbb{N}_0$, there is a constant $C = C_{K_1,K_2,N,\alpha,\beta}(u,f) > 0$ so that* *for* $(w,t) \in K_2 \times [1,+\infty)$*,*

$$\begin{aligned}(2.1.22)\quad &\Bigg|\partial_w^\alpha \partial_t^\beta \Bigg(\int e^{itf(x,w)} u(x,w) dx \\ &- \left(\det\left(\frac{t\widetilde{f}''_{zz}(Z(w),w)}{2\pi i}\right)\right)^{-\frac{1}{2}} e^{it\widetilde{f}(Z(w),w)} \sum_{j=0}^{N-1} (C_{w,j}\widetilde{u})(Z(w),w)t^{-j}\Bigg)\Bigg| \\ &\leq C\, t^{-N-\frac{n_1}{2}+|\alpha|},\end{aligned}$$

where $C_{w,0} = 1$*. Here* $\widetilde{u}(z,\widetilde{w})$ *is an almost analytic extension of* $u(x,w)$ *on* $U \times V$ *and*

$$(2.1.23)\qquad \left(\det\left(\frac{t\widetilde{f}''_{zz}(Z(w),w)}{2\pi i}\right)\right)^{-\frac{1}{2}}$$

is the branch of the square root of the

$$(2.1.24)\qquad \left(\det\left(\frac{t\widetilde{f}''_{zz}(Z(w),w)}{2\pi i}\right)\right)^{-1},$$

which is continuously deformed into 1 *under the homotopy*

$$(2.1.25)\qquad s \in [0,1] \mapsto -i(1-s)\widetilde{f}''_{zz}(Z(w),w) + sI \in \mathrm{GL}(n_1,\mathbb{C}).$$

We recall some notations and properties of pseudodifferential operators used in this work. We let $U \subset \mathbb{R}^n$ be an open set and E be a smooth vector bundle over U. By $P \in L^m_{\rho,\delta}(U;E)$ we mean a pseudodifferential operator P of order m of type (ρ,δ) sending sections of E to sections of E, where $\rho+\delta = 1$. This means that the operator P has the Schwartz kernel given by the oscillatory integral

$$(2.1.26)\qquad P(x,y) := \int_{\mathbb{R}^n} e^{i\langle x-y,\eta\rangle} p(x,y,\eta) \frac{d\eta}{(2\pi)^n},$$

where $p(x,y,\eta) \in S^m_{\rho,\delta}(U \times U \times \mathbb{R}^n, \mathscr{L}(E,E))$. It is straightforward to check that

$$(2.1.27)\quad F : \mathscr{E}'(U,E) \to \mathscr{C}^\infty(U,E) \text{ is continuous if and only if } F \in L^{-\infty}(U;E),$$

and from now on we also us the notation $L^{-\infty}(U;E)$ for the space of smoothing operator on U acting on sections of E. We recall that any pseudodifferential operator locally can be decomposed into the sum of a properly supported pseudodifferential operator and a smoothing operator. In this context, directly by the trick of Kuranishi, cf. [31, pp. 34-35] for example, we can extend the definition of pseudodifferential operators to smooth manifolds. When P is properly supported and the type of P satisfies $\rho > \delta$, we say that P is regular, and in this context the complete symbol of P defined by

$$\sigma_P(x,\eta) := e^{-i\langle x,\eta\rangle} P(e^{i\langle y,\eta\rangle}) \tag{2.1.28}$$

can be constructed by asymptotic sums

$$\sigma_P(x,\eta) \sim \sum_{\alpha\in\mathbb{N}^n} \frac{i^{-|\alpha|}}{\alpha!} (\partial_\eta^\alpha \partial_y^\alpha p(x,y,\eta))|_{y=x} \tag{2.1.29}$$

in $S^m_{\rho,\delta}(U\times\mathbb{R}^n, \mathscr{L}(E,E))$ such that

$$P(x,y) \equiv \int_{\mathbb{R}^n} e^{i\langle x-y\rangle} \sigma_P(x,\eta) \frac{d\eta}{(2\pi)^n}. \tag{2.1.30}$$

By the asymptotic expansion (2.1.29) of $\sigma_P(x,\eta)$, people develop the calculus of regular pseudodifferential operators. In this context, the standard pseudodifferential calculus enable one to directly check that if $P \in L^m_{\rho,\delta}(U;E)$ is regular, then P is a continuous operator on Sobolev spaces that

$$P : H^s_{\rm loc}(U,E) \to H^{s-m}_{\rm loc}(U,E), \tag{2.1.31}$$

$$P : H^s_{\rm comp}(U,E) \to H^{s-m}_{\rm comp}(U,E), \tag{2.1.32}$$

for every $s\in\mathbb{R}$. Also, by the asymptotic expansion of $\sigma_P(x,\eta)$ we can define the principal symbol $p_0(x,\eta)$ by the image of $\sigma_P(x,\eta)$ in the quotient $S^m_{\rho,\delta}/S^{m-(\rho-\delta)}_{\rho,\delta}$, and the principal symbol $p_0(x,\eta)$ turns out to be globally-defined on T^*U. We denote $L^m_{\rm cl}(U:E) \subset L^m_{1,0}(U;E)$ to be the space of classical pseudodifferential operators, where for $P \in L^m_{\rm cl}(U;E)$ we have $\sigma_P(x,\eta) \in S^m_{\rm cl}(U\times\mathbb{R}^n, E)$, and through the asymptotic expansion of $\sigma_P(x,\eta)$ for $P\in L^m_{\rm cl}(U,E)$ we may assume that $p_0(x,\eta)$ satisfies $p_0(x,\lambda\eta) = \lambda^m p_0(x,\eta)$ for all $\lambda\geq 1$ in this situation. We say a regular pseudodifferential operator $P\in L^m_{\rho,\delta}(U;E)$ is elliptic at a point $(x_0,\eta_0)\in T^*U$ if there is a constant $C>0$ such that the complete symbol $|\sigma_P(x,\eta)| \geq \frac{1}{C}(1+|\eta|)^m$ in a conic neighborhood of (x_0,η_0) when $|\eta|\geq C$. When the principal symbol $p_0(x,\eta)$ is positively homogeneous of order m, this condition is equivalent to $p_0(x_0,\eta_0)$ is invertible. We say a regular pseudodifferential operator P is elliptic on U if P is elliptic at all point $(x,\eta)\in T^*U$. If P is elliptic, then for every $u\in\mathscr{D}'(U,E)$ we have $u\in H^s_{\rm loc}(U,E)$ if and only if $Pu\in H^{s-m}_{\rm loc}(U,E)$. We say P is hypoelliptic if for every $u\in\mathscr{D}'(W,E)$ such that $Pu\in\mathscr{C}^\infty(W,E)$ we have $u\in\mathscr{C}^\infty(W,E)$. When the type of P satisfies $\rho=\delta=\frac{1}{2}$, it is more difficult to construct the asymptotic expansion of the complete symbol σ_P and the symbolic calculus of such pseudodifferential operators. In our work, we apply the classical theory of Calderon and Vaillancourt, cf. [22, pp. 50-51] or [43, Theorem 18.6.6], which implies that when $P\in L^m_{\frac{1}{2},\frac{1}{2}}(U;E)$ we still have the continuous maps between Sobolev spaces that

$$P : H^s_{\rm loc}(U, E) \to H^{s-m}_{\rm loc}(U, E), \tag{2.1.33}$$

$$P : H^s_{\rm comp}(U, E) \to H^{s-m}_{\rm comp}(U, E), \tag{2.1.34}$$

for every $s \in \mathbb{R}$.

Finally, we recall some notations in semi-classical analysis. We let W_1, W_2 be bounded open subsets of $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively. Let E and F be smooth complex vector bundles over W_1 and W_2, respectively. Let $s_1, s_2 \in \mathbb{R}$ and $n_0 \in \mathbb{Z}$. Let W be an open set in $\mathbb{R}^N$ and E be a vector bundle over W, we define the space

$$S(1; W, E) := \{a \in \mathscr{C}^\infty(W, E) : \sup_{x\in W} |\partial^\alpha_x a(x)| < +\infty \text{ for all } \alpha \in \mathbb{N}^N_0\}, \tag{2.1.35}$$

where the number 1 in the notation for symbol space we mean the order function equals to 1 (cf. [22, Definition 7.4] for example). We then consider the space $S^0_{\rm loc}(1; W)$ containing all smooth functions $a(x, k)$ with real parameter k such that for all multi-index $\alpha \in \mathbb{N}^N_0$, any cut-off function $\chi \in \mathscr{C}^\infty_0(W)$, we have

$$\sup_{\substack{k\in\mathbb{R} \\ k\geq 1}} \sup_{x\in W} |\partial^\alpha_x(\chi(x)a(x, k))| < +\infty. \tag{2.1.36}$$

For general $m \in \mathbb{R}$, we can also consider

$$S^m_{\rm loc}(1; W, E) := \{a(x, k) : k^{-m}a(x, k) \in S^0_{\rm loc}(1; W, E)\}. \tag{2.1.37}$$

In other words, $S^m_{\rm loc}(1; W, E)$ takes all the smooth function $a(x, k)$ with parameter $k \in \mathbb{R}$ satisfying the following estimate. For any compact set $K \Subset W$, any multi-index $\alpha \in \mathbb{N}^n_0$, there is a constant $C_{K,\alpha} > 0$ independent of k such that

$$|\partial^\alpha_x(a(x, k))| \leq C_{K,\alpha}k^m, \text{ for all } x \in K,\ k \geq 1. \tag{2.1.38}$$

For a sequence of $a_j \in S^{m_j}_{\rm loc}(1; W, E)$ with m_j decreasing, $m_j \to -\infty$, and $a \in S^{m_0}_{\rm loc}(1; W, E)$, we denote

$$a(x, k) \sim \sum_{j=0}^{+\infty} a_j(x, k) \text{ in } S^{m_0}_{\rm loc}(1; W, E) \tag{2.1.39}$$

if for all $\ell \in \mathbb{N}$ we have

$$a - \sum_{j=0}^{\ell-1} a_j \in S^{m_\ell}_{\rm loc}(1; W, E). \tag{2.1.40}$$

In fact, for all sequence a_j above, there always exists an element a as the asymptotic sum, which is unique up to the elements in

$$S^{-\infty}_{\rm loc}(1; W, E) := \bigcap_m S^m_{\rm loc}(1; W, E). \tag{2.1.41}$$

We use the notation $S^m_{\mathrm{loc,\,cl}}(1;W,E)$ to denote the subspace of $S^m_{\mathrm{loc}}(1;W,E)$ collecting the elements $a(x,k)$ with the asymptotic expansion

$$a(x,k) \sim \sum_{j=0}^{+\infty} a_j(x) k^{m-j} \text{ in } S^m_{\mathrm{loc}}(1;W,E). \tag{2.1.42}$$

We recall the concept of k-negligible operators. For $k \in \mathbb{R}_+$ and an operator $F_k : \mathscr{C}^\infty_0(V,E) \to \mathscr{D}'(U,E)$ depending on the parameter k, we write

$$F_k = O\left(k^{-\infty}\right) \text{ in } \mathscr{L}\left(H^{s_1}_{\mathrm{comp}}(W_1,E) \to H^{s_2}_{\mathrm{loc}}(W_2,F)\right) \tag{2.1.43}$$

if

$$F_z = O\left(k^{-N}\right) \text{ in } \mathscr{L}\left(H^{s_1}_{\mathrm{comp}}(W_1,E) \to H^{s_2}_{\mathrm{loc}}(W_2,F)\right), \ \forall N \in \mathbb{N}_0. \tag{2.1.44}$$

Also, we say a kernel $F_k(x,y)$ is k-negligible and write

$$F_k(x,y) = O(k^{-\infty}) \text{ on } U \times V \tag{2.1.45}$$

or just

$$F_k = O(k^{-\infty}) \text{ on } U \times V \tag{2.1.46}$$

if for all $k > 0$ large enough, F_k is a smoothing operator, and for any compact set K in $U \times V$, for all multi-index $\alpha \in \mathbb{N}_0^{n_1}$, $\beta \in \mathbb{N}_0^{n_2}$ and $N \in \mathbb{N}_0$, there exists a constant $C_{K,\alpha,\beta,N} > 0$ such that

$$\left|\partial^\alpha_x \partial^\beta_y F_k(x,y)\right| \le C_{K,\alpha,\beta,N} k^{-N} \tag{2.1.47}$$

for all $x, y \in K$. For k-dependent operators F_k and G_k, sometimes we also write

$$F_k = G_k \ \text{ on } W_1 \times W_2 \tag{2.1.48}$$

if $F_k - G_k = O\left(k^{-\infty}\right)$ on $W_1 \times W_2$. By straightforward generalization, all the notations introduced above can be defined on smooth manifolds.

2.2. Non-degenerate Cauchy–Riemann manifolds

Let us first recall some essential material about geometry of Cauchy–Riemann manifolds we will use. We let X be a connected, smooth and orientable manifold of real dimension $2n+1$, $n \ge 1$. We say a pair $(X, T^{1,0}X)$ is a codimension one or hypersurface type Cauchy–Riemann manifold if there is a subbundle $T^{1,0}X \subset \mathbb{C}TX$, such that

(i) $\dim_{\mathbb{C}} T^{1,0}_p X = n$ for any $p \in X$.
(ii) $T^{1,0}_p X \cap T^{0,1}_p X = \{0\}$ for any $p \in X$, where $T^{0,1}_p X := \overline{T^{1,0}_p X}$.
(iii) For vector fields $V_1, V_2 \in \mathscr{C}^\infty(X, T^{1,0}X)$, then $[V_1, V_2] \in \mathscr{C}^\infty(X, T^{1,0}X)$, where $[\cdot,\cdot]$ stands for the Lie bracket between vector fields.

We will use the phrase *CR manifold* in this work to abbreviate the hypersurface type Cauchy–Riemann manifold. For the above subbundle $T^{1,0}X$, we call it a *CR structure* of the CR manifold X. From now on, we always discuss on a CR manifold $(X, T^{1,0}X)$ of real dimension $2n+1$, $n \geq 1$.

We denote by $T^{*1,0}X$ and $T^{*0,1}X$ the dual bundles of $T^{1,0}X$ and $T^{0,1}X$, respectively. We define the vector bundle of $(0,q)$-forms by $T^{*0,q}X := \Lambda^q\, T^{*0,1}X$.

The Levi distribution (or holomorphic tangent space) HX of the CR manifold X is the real part of $T^{1,0}X \oplus T^{0,1}X$, i.e., the unique sub-bundle HX of TX such that $\mathbb{C}HX = T^{1,0}X \oplus T^{0,1}X$. We let $J : HX \to HX$ be the complex structure given by $J(u+\overline{u}) = iu - i\overline{u}$, for every $u \in T^{1,0}X$. If we extend J complex linearly to $\mathbb{C}HX$ we have $T^{1,0}X = \{V \in \mathbb{C}HX;\ JV = iV\}$. The annihilator $(HX)^0 \subset T^*X$ of HX is called the characteristic bundle of the CR manifold. Since X is orientable, the characteristic bundle $(HX)^0$ is a trivial real line subbundle. We fix a global frame of $(HX)^0$, that is, a real non-vanishing 1-form $\boldsymbol{\alpha} \in \mathscr{C}^\infty(X, T^*X)$ such that $(HX)^0 = \mathbb{R}\boldsymbol{\alpha}$, called *characteristic one form*. We have

$$\langle\, \boldsymbol{\alpha}(x), u\,\rangle = 0 \text{ for any } u \in H_xX,\ x \in X. \tag{2.2.1}$$

It turns out that the restriction of $d\boldsymbol{\alpha}$ on HX is a $(1,1)$-form. The *Levi form* of X at $x \in X$ is the Hermitian quadratic form on $T^{1,0}_xX$ given by

$$\mathcal{L}_x(u,\overline{v}) = -\frac{1}{2i}\langle\, d\boldsymbol{\alpha}(x), u \wedge \overline{v}\,\rangle = -\frac{1}{2i} d\boldsymbol{\alpha}(u,\overline{v}) \text{ for } u, v \in T^{1,0}_xX. \tag{2.2.2}$$

A CR manifold X is said to be *non-degenerate* if for every $x \in X$ the Levi form $\mathcal{L}_x$ is non-degenerate. It is clear that this definition does not depend on the choice of the characteristic one form $\boldsymbol{\alpha}$. If X is non-degenerate then $\boldsymbol{\alpha}$ is a contact form and the Levi distribution HX is a contact structure. Locally, there exists an orthonormal basis $\{\mathcal{Z}_1, \cdots, \mathcal{Z}_n\}$ of $T^{1,0}X$ with respect to the Hermitian metric $\langle\,\cdot\,|\,\cdot\,\rangle$ such that $\mathcal{L}_p$ is diagonal in this basis, $\mathcal{L}_p(\mathcal{Z}_j, \overline{\mathcal{Z}}_\ell) = \delta_{j,\ell}\mu_j(p)$. The entries $\mu_1(p)\cdots, \mu_n(p)$ are called the *eigenvalues of the Levi form* at $p \in X$ with respect to $\langle\,\cdot\,|\,\cdot\,\rangle$. We notice that the sign of the eigenvalues does not depend on the choice of the metric $\langle\,\cdot\,|\,\cdot\,\rangle$. From now on, we use n_- to denote the number of negative eigenvalues and n_+ for the number of positive eigenvalues of the Levi form on X, respectively. In our context, $n_- + n_+ = n$ and the pair (n_-, n_+) is called the signature (of the Levi form) of the CR manifold $(X, T^{1,0}X)$, which is in fact independent of the choice of Hermitian metric on $\mathbb{C}TX$. A strongly pseudoconvex CR manifold of real dimension $2n+1$ has a constant signature $(n_-, n_+) = (0, n)$. In fact, it is known that for each $j \in \{1, \cdots, n\}$ the function $p \mapsto \mu_j(p)$ is a continuous function on X, so by intermediate value theorem we know that the Levi form on a non-degenerate CR manifold must have the constant signature. Finally, we let $\boldsymbol{T} \in \mathscr{C}^\infty(X, TX)$ be a vector field, called *characteristic vector field*, such that

$$\mathbb{C}TX = T^{1,0}X \oplus T^{0,1}X \oplus \mathbb{C}\boldsymbol{T} \text{ and } \iota_{\boldsymbol{T}}\boldsymbol{\alpha} = -1, \tag{2.2.3}$$

and we let $\langle\cdot|\cdot\rangle$ be a Hermitian metric on $\mathbb{C}TX$ such that the decomposition of $\mathbb{C}TX$ is orthogonal.

2.3. Szegő projections for lower energy forms

Let us recall some essential material about analysis on Cauchy–Riemann manifolds. By linear algebra, the Hermitian metric $\langle\cdot|\cdot\rangle$ induces a Hermitian metric on $\Lambda^r\mathbb{C}T^*X$ given by

$$\langle u_1 \wedge \cdots \wedge u_r | v_1 \wedge \cdots \wedge v_r\rangle = \det\left(\left(\langle u_j|u_k\rangle\right)_{j,k=1}^r\right) \tag{2.3.1}$$

where $u_j, v_k \in \mathbb{C}T^*X,\ j,k = 1,\cdots,r$. We can take the orthogonal projection

$$\pi^{(0,q)} : \Lambda^q\mathbb{C}T^*X \to T^{*0,q}X := \Lambda^q(T^{*0,1}X). \tag{2.3.2}$$

The *tangential Cauchy–Riemann operator* is defined by

$$\overline{\partial}_b := \pi^{(0,q+1)} \circ d : \mathscr{C}^\infty(X, T^{*0,q}X) \to \mathscr{C}^\infty(X, T^{*0,q+1}X). \tag{2.3.3}$$

By Cartan's magic formula, we can check that $\overline{\partial}_b^2 = 0$. We take the L^2-inner product $(\cdot|\cdot)$ on $\mathscr{C}^\infty(X, T^{*0,q}X)$ induced by $\langle\cdot|\cdot\rangle$ via

$$(f|g) := \int_X \langle f|g\rangle dm(x),\ f,g \in \mathscr{C}^\infty(X, T^{*0,q}X), \tag{2.3.4}$$

where

$$dm(x) = m(x)dx \tag{2.3.5}$$

is the given volume form on X. We also recall that there is another volume form

$$dv(x) = v(x)dx \tag{2.3.6}$$

which is induced by the Hermitian metric and compatible with $\boldsymbol{\alpha}$ such that

$$v(x) := \sqrt{\det g}, \tag{2.3.7}$$

$$g := (g_{jk})_{j,k=1}^{2n+1}, \tag{2.3.8}$$

$$g_{jk} := \langle \frac{\partial}{\partial x_j} | \frac{\partial}{\partial x_k} \rangle. \tag{2.3.9}$$

We let $L^2_{0,q}(X) := L^2(X, T^{*0,q}X)$ be the completion of $\Omega^{0,q}(X) := \mathscr{C}^\infty(X, T^{*0,q}X)$ with respect to $(\cdot|\cdot)$. We extend the closed and densely-defined operator $\bar{\partial}_b$ to $L^2_{0,q}(X)$, $q \in \{0, 1, \cdots, n\}$, by

$$\bar{\partial}_b : \mathrm{Dom}\,\bar{\partial}_b \subset L^2_{0,q}(X) \to L^2_{0,q+1}(X)\,, \tag{2.3.10}$$

where

$$\operatorname{Dom}\bar\partial_b := \{u \in L^2_{0,q}(X) : \bar\partial_b u \in L^2_{0,q+1}(X)\} \tag{2.3.11}$$

and for any $u \in L^2_{0,q}(X)$ we define $\bar\partial_b u$ in the sense of distributions. We also write

$$\bar\partial^*_{b,H} : \operatorname{Dom}\bar\partial^*_{b,H} \subset L^2_{0,q+1}(X) \to L^2_{0,q}(X), \tag{2.3.12}$$

where

$$\begin{aligned}&\operatorname{Dom}\bar\partial^*_{b,H}\\ &:= \{v \in L^2_{0,q}(X) : \exists!\, w \in L^2_{0,q+1}(X) \text{ so that } \left(\bar\partial_b u|v\right) = (u|w),\ \forall u \in \operatorname{Dom}\bar\partial_b\},\end{aligned} \tag{2.3.13}$$

to denote the Hilbert space adjoint of $\bar\partial_b$ in the L^2 space with respect to $(\cdot|\cdot)$. We let $\Box^{(q)}_b$ denote the *Kohn Laplacian* (extended by Gaffney extension) given by

$$\operatorname{Dom}\Box^{(q)}_b = \left\{s \in \operatorname{Dom}\bar\partial_b \cap \operatorname{Dom}\bar\partial^*_{b,H} : \bar\partial_b s \in \operatorname{Dom}\bar\partial^*_{b,H},\ \bar\partial^*_{b,H} s \in \operatorname{Dom}\bar\partial_b\right\}, \tag{2.3.14}$$

$$\Box^{(q)}_b s = \bar\partial_b\bar\partial^*_{b,H} s + \bar\partial^*_{b,H}\bar\partial_b s \ \text{ for } s \in \operatorname{Dom}\Box^{(q)}_b. \tag{2.3.15}$$

For every $q \in \{0, 1, \cdots, n\}$, $\Box^{(q)}_b$ is a positive self-adjoint operator. We refer this fact to the functional analysis argument [68, Proposition 3.1.2]. We also notice that $\Box^{(q)}_b$ is never an elliptic differential operator for its principal symbol vanishes on the set Σ, where

$$\Sigma := \Sigma^- \cup \Sigma^+, \tag{2.3.16}$$

$$\Sigma^- := \left\{(x,\eta) \in T^*X : \sum_{j=1}^{2n+1} \eta_j(x)dx_j = c\boldsymbol{\alpha}(x),\ c < 0\right\}, \tag{2.3.17}$$

$$\Sigma^+ := \left\{(x,\eta) \in T^*X : \sum_{j=1}^{2n+1} \eta_j(x)dx_j = c\boldsymbol{\alpha}(x),\ c > 0\right\}. \tag{2.3.18}$$

From now on, we also assume X is compact. In our context, when $q \notin \{n_-, n_+\}$, $\Box^{(q)}_b$ is hypoelliptic with loss of one derivative and has L^2-closed range [45, Part I, §6]. For the concerning result in a more general set-up called *Y(q) condition*, we consult to the [19]. When $q \in \{n_-, n_+\}$, $\Box^{(q)}_b$ may not even be hypoelliptic, i.e, $\Box^{(q)}_b u \in \mathscr{C}^\infty(X, T^{*0,q}X)$ might not imply that $u \in \mathscr{C}^\infty(X, T^{*0,q}X)$. When $q \in \{n_-, n_+\}$ and $\Box^{(q)}_b$ has L^2-closed range in $L^2_{0,q}(X)$, the Szegő projection $\Pi^{(q)}$ on $(0,q)$-forms, which is defined by the orthogonal projection

$$\Pi^{(q)} : L^2_{0,q}(X) \to \operatorname{Ker}\Box^{(q)}_b, \tag{2.3.19}$$

is the sum of two Fourier integral operators of order zero with complex-valued phase functions [45, Part I, Theorem 1.2]. The Schwartz kernel $\Pi^{(q)}(x,y) \in \mathscr{D}'(X\times X, \mathscr{L}(T_y^{*0,q}X, T_x^{*0,q}X))$ called Szegő kernel has the singularities described by Hörmander's wavefront set (cf. [40, §8] or [31, §7]):

$$\mathrm{WF}(\Pi^{(q)}(x,y)) = \{(x,\eta,x,-\eta) : (x,\eta) \in \widehat{\Sigma}\}, \tag{2.3.20}$$

$$\widehat{\Sigma} := \Sigma^- \text{ when } q = n_-,\ n_- \neq n_+, \tag{2.3.21}$$

$$\widehat{\Sigma} := \Sigma^+ \text{ when } q = n_+,\ n_+ \neq n_-, \tag{2.3.22}$$

$$\widehat{\Sigma} := \Sigma \text{ when } q = n_- = n_+. \tag{2.3.23}$$

This kind of microlocal analysis for Szegő projecition and kernel was first introduced in Boutet de Monvel–Sjöstrand [13] when $(n_-,n_+) = (0,n)$. It was speculated in [41, 42] that it can be applied to general non-degenerate (n_-,n_+) after careful modification. Hsiao [45] uses a different approach than [13] by developing the microlocal heat equation method (see also [73]) together with Witten's trick (see also [4]).

We notice that the L^2-closed range condition only holds automatically when $|n_- - n_+| \neq 1$ by [63]. A classical counter example is the Rossi's nonembeddable example [19, §12.4], where $(n_-,n_+) = (0,1)$. However, we always have the following local result [56, Theorem 3.1 & Theorem 3.2], which is essentially from [45, Part I].

THEOREM 2.3. *We let $(Y, T^{1,0}Y)$ be a connected orientable CR manifold with real dimension $2n+1$, $n \geq 1$, and assume that the Levi form of Y is non-degenerate of constant signature (n_-,n_+) on a relatively compact set $\Omega \Subset Y$ with respect to some characteristic form $\boldsymbol{\alpha}$. If $q \notin \{n_-,n_+\}$, then there is a properly supported pseudodifferential operator $\mathbf{G} \in L^{-1}_{\frac12,\frac12}(\Omega; T^{*0,q}Y)$ such that*

$$\Box_b^{(q)}\mathbf{G} \equiv I \text{ on } \Omega. \tag{2.3.24}$$

If $q = n_-$, then there are properly supported operators $G \in L^{-1}_{\frac12,\frac12}(\Omega; T^{*0,q}Y)$ and $S_-, S_+ \in L^{0}_{\frac12,\frac12}(\Omega; T^{*0,q}Y)$ such that on Ω we have

$$\Box_b^{(q)}G + S_- + S_+ \equiv I, \tag{2.3.25}$$

$$\Box_b^{(q)}S_- \equiv \Box_b^{(q)}S_+ \equiv 0, \tag{2.3.26}$$

$$G \equiv G^*,\ S_-G \equiv S_+G \equiv 0, \tag{2.3.27}$$

$$S_- \equiv S_-^* \equiv S_-^2, \tag{2.3.28}$$

$$S_+ \equiv S_+^* \equiv S_+^2, \tag{2.3.29}$$

$$S_-S_+ \equiv S_+S_- \equiv 0. \tag{2.3.30}$$

where G^*, S_-^* and S_+^* are the formal adjoints of G, S_- and S_+ with respect to the given L^2-inner product on X, respectively, and the Schwartz kernel $S_\mp(x,y)$ are oscillatory integrals given by

$$S_\mp(x,y) = \int_0^{+\infty} e^{it\varphi_\mp(x,y)} s^\mp(x,y,t)dt \tag{2.3.31}$$

with the full symbols $s^\mp(x,y,t) \in S^n_{\mathrm{cl}}\left(\Omega\times\Omega\times\mathbb{R}_+, \mathscr{L}(T^{*0,q}_y X, T^{*0,q}_x X)\right)$ such that

(2.3.32)

$$s^\mp(x,y,t) \sim \sum_{j=0}^{+\infty} s^\mp_j(x,y)t^{n-j} \text{ in } S^n_{1,0}\left(\Omega\times\Omega\times\mathbb{R}_+, \mathscr{L}(T^{*0,q}_y X, T^{*0,q}_x X)\right),$$

(2.3.33)

$$s^\mp(x,y,t) \text{ and } s^\mp_j(x,y) \text{ are properly supported in the variables } (x,y),\ \forall j\in\mathbb{N}_0,$$

(2.3.34)

$$s^+(x,y,t) = 0 \text{ when } n_- \neq n_+,$$

and with the complex-valued functions $\varphi_\mp(x,y) \in \mathscr{C}^\infty(\Omega\times\Omega)$ we call Szegő phase functions such that

$$\operatorname{Im}\varphi_\mp(x,y) \geq 0, \tag{2.3.35}$$

$$\varphi_\mp(x,y) = 0 \text{ if and only if } x = y, \tag{2.3.36}$$

$$d_x\varphi_\mp(x,x) = -d_y\varphi_\mp(x,x) = \mp\boldsymbol{\alpha}(x). \tag{2.3.37}$$

We recall some notation in microlocal analysis here. Similar to the concept we introduce in §2.1, for any $m\in\mathbb{R}$, we denote

$$S^m_{1,0}(\Omega\times\Omega\times\mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X)) \tag{2.3.38}$$

to be the space collecting all $a(x,y,t) \in \mathscr{C}^\infty(\Omega\times\Omega\times\mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X))$ such that for all compact sets $K \Subset \Omega\times\Omega$, all $\alpha,\beta\in\mathbb{N}_0^{2n+1}$ and $\gamma\in\mathbb{N}_0$, there is a constant $C_{K,\alpha,\beta,\gamma} > 0$ satisfying the estimate

$$\left|\partial_x^\alpha\partial_y^\beta\partial_t^\gamma a(x,y,t)\right| \leq C_{K,\alpha,\beta,\gamma}(1+|t|)^{m-|\gamma|} \quad \text{for all } (x,y,t)\in K\times\mathbb{R}_+,\ |t|\geq 1. \tag{2.3.39}$$

We put

$$S^{-\infty}(\Omega\times\Omega\times\mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X)) := \bigcap_{m\in\mathbb{R}} S^m_{1,0}(\Omega\times\Omega\times\mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X)). \tag{2.3.40}$$

It is clear that for Szegő phase functions $\varphi_\mp(x,y)$, the functions $\varphi_\mp(x,y)t$ satisfy (2.1.13)-(2.1.15), and for $a(x,y,t)\in\mathscr{C}^\infty(\Omega\times\Omega\times\mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X))$, in the limit of distribution we define the oscillatory integral and can check that

$$\begin{aligned}(2.3.41)\quad \int_0^{+\infty} e^{it\varphi_\mp(x,y)}a(x,y,t)dt &:= \lim_{\epsilon\to 0}\int e^{it\varphi_\mp(x,y)}\tau(\epsilon t)a(x,y,t)dt \\ &= \lim_{\epsilon\to 0}\int_0^{+\infty} e^{it(\varphi_\mp(x,y)+i\epsilon)}a(x,y,t)dt,\end{aligned}$$

where $\tau\in\mathscr{C}_0^\infty(\mathbb{R})$ is arbitrary and satisfies $\tau(0)=1$.

We also recall that when $\Box_b^{(q)}$ has L^2-closed range, from the spectral theory for self-adjoint operators [21] and the spectrum of $\Box_b^{(q)}$ [56, Theorem 1.7] there is some $\lambda>0$ such that $\Pi^{(q)}=\Pi_\lambda^{(q)}$.

The above pure analytic result corresponds to the following global operator [56, Theorem 4.1].

THEOREM 2.4. *With the same notations and assumptions in Theorem 2.3, we consider the orthogonal projection $\Pi_\lambda^{(q)}: L^2_{0,q}(X)\to E([0,\lambda])$ called Szegő projections for lower energy forms, where $E([0,\lambda]):=\mathrm{Range}\,\mathbb{1}_{[0,\lambda]}(\Box_b^{(q)})$ is the image of the spectral projection of the self-adjoint and positive operator $\Box_b^{(q)}$. Then if $q=n_-$, on $\Omega\times\Omega$ we have*

$$(2.3.42)\qquad \Pi_\lambda^{(q)}(x,y)\equiv S_-(x,y)+S_+(x,y).$$

We list some important information which we will use later. The first one we want to mention is the following coordinates and the corresponding Taylor expansion of the tangential Hessian of phase functions [56, Theorem 3.4].

THEOREM 2.5. *Following Theorem 2.3, for a given point $x_0\in\Omega$, let $\{W_j\}_{j=1}^n$ be an orthonormal frame with respect to $\langle\cdot\,|\,\cdot\rangle$ of $T^{1,0}X$ in a neighbourhood of x_0 such that the Levi form is diagonal at x_0, i.e., $\mathcal{L}_{x_0}(W_j,\overline{W}_s)=\delta_{j,s}\mu_j$, $j,s=1,\cdots,n$. We can take local coordinates $x=(x_1,\cdots,x_{2n+1})$, $z_j=x_{2j-1}+ix_{2j}$, $j=1,\cdots,n$, defined on some neighbourhood of x_0 such that $x(x_0)=0$,*

$$(2.3.43)\qquad \boldsymbol{\alpha}(x_0)=dx_{2n+1},$$

and for some $c_j\in\mathbb{C}$, $j=1,\cdots,n$,

$$(2.3.44)\quad W_j=\frac{\partial}{\partial z_j}-i\mu_j\overline{z}_j\frac{\partial}{\partial x_{2n+1}}-c_jx_{2n+1}\frac{\partial}{\partial x_{2n+1}}+O(|x|^2),\ j=1,\cdots,n-1.$$

We set $y = (y_1, \cdots, y_{2n+1})$, $w_j = y_{2j-1} + iy_{2j}$, $j = 1, \cdots, n$, *then for* φ_- *in Theorem 2.3, under the above coordinates we have*

$$\operatorname{Im}\varphi_-(x,y) \geq c\sum_{j=1}^{2n}\left|x_j - y_j\right|^2, \ \ c > 0, \tag{2.3.45}$$

in some neighbourhood of $(0,0)$ *and*

(2.3.46)

$$\begin{aligned}
&\varphi_-(x,y)\\
&= -x_{2n+1} + y_{2n+1} + i\sum_{j=1}^{n}\left|\mu_j\right|\left|z_j - w_j\right|^2\\
&+\sum_{j=1}^{n}\Big(i\mu_j(\overline{z}_j w_j - z_j\overline{w}_j) + c_j(-z_j x_{2n+1} + w_j y_{2n+1}) + \overline{c}_j(-\overline{z}_j x_{2n+1} + \overline{w}_j y_{2n+1})\Big)\\
&+(x_{2n+1} - y_{2n+1})f(x,y) + O(|(x,y)|^3),
\end{aligned}$$

where f *is smooth and satisfies* $f(0,0) = 0$, $f(x,y) = \overline{f}(y,x)$.

We remark that in the above theorem at x_0 the volume form $v(x)dx$ induced by Hermitian metric in our convention satisfies $v(x_0) = 2^n$. We also remark that for such small enough coordinate patch, there exist a constant $C > 0$ such that

$$\operatorname{Im}\varphi_-(x,y) \geq C|z - w|^2. \tag{2.3.47}$$

We refer to [45, Part I, Proposition 7.16] for a proof of (2.3.47).

We have the following [56, Theorem 5.4] about equivalence class of Szegő phase functions.

THEOREM 2.6. *With the same notations and assumptions in Theorem 2.3, for any function* $\psi_\mp(x,y)$ *satisfying* (2.3.35), (2.3.36) *and* (2.3.37) *we can find a symbol* $s^{\psi_\mp}(x,y,t)$ *in* $S^n_{\mathrm{cl}}\left(\Omega\times\Omega\times\mathbb{R}_+, \mathscr{L}(T^{*0,q}_y X, T^{*0,q}_x X)\right)$ *such that*

$$S_\mp(x,y) \equiv \int_0^{+\infty} e^{it\psi_\mp(x,y)} s^{\psi_\mp}(x,y,t)dt \tag{2.3.48}$$

on $\Omega\times\Omega$. *Moreover, when* $S_\mp$ *is not smoothing there is some* $f_\mp(x,y) \in \mathscr{C}^\infty(\Omega\times\Omega)$ *satisfying* $f_\mp(x,x) \neq 0$ *such that*

$$\varphi_\mp(x,y) - f_\mp(x,y)\psi_\mp(x,y) = O(|x-y|^{+\infty}). \tag{2.3.49}$$

PROOF. For the stream of reading, we present the proof from [56, §8]. First of all, using the notations and results of [72, Theorem 3.6], the positive Lagrangian manifold $\Lambda_{\varphi_\mp t}$ associated to the non-degenerate phase function $\varphi_\mp(x,y)t$ is given by

$$\left\{(\widetilde{x},\widetilde{y},\frac{\partial\widetilde{\varphi}_{\mp}}{\partial\widetilde{x}}(\widetilde{x},\widetilde{y})\widetilde{t},\frac{\partial\widetilde{\varphi}_{\mp}}{\partial\widetilde{y}}(\widetilde{x},\widetilde{y})\widetilde{t}):\widetilde{\varphi}(\widetilde{x},\widetilde{y})=0\right\} \subset(\Omega^{\mathbb{C}}\times\mathbb{C}^{2n+1})\times(\Omega^{\mathbb{C}}\times\mathbb{C}^{2n+1}). \tag{2.3.50}$$

We also refer the precise meaning of the notation above to [45, Part I, Remark 7.17]. From (2.3.35), (2.3.36) and (2.3.37), we can check that

$$\Lambda_{\varphi_{\mp}t}=\Lambda_{\psi_{\mp}t}\ \text{at}\ \mathrm{diag}\left((\Sigma^{\mp}\bigcap T^*\Omega)\times(\Sigma^{\mp}\bigcap T^*\Omega)\right), \tag{2.3.51}$$

hence we can apply Melin–Sjöstrand global theory of Fourier integral operators [72, Definition 4.1 & Theorem 4.2] and apply [31, Proposition 7.3] for example to get the following: for any given $\ell\in\mathbb{R}$ and any element $b^{\varphi_{\mp}}(x,y,t)$ in $S^{\ell}_{\mathrm{cl}}(\Omega\times\Omega\times\mathbb{R}_+;\mathscr{L}(T^{*0,q}X,T^{*0,q}X))$, we can find an element $b^{\psi_{\mp}}(x,y,t)$ in $S^{\ell}_{\mathrm{cl}}(\Omega\times\Omega\times\mathbb{R}_+;\mathscr{L}(T^{*0,q}X,T^{*0,q}X))$ such that

$$\int_0^{+\infty}e^{it\varphi_{\mp}(x,y)}b^{\varphi_{\mp}}(x,y,t)dt\equiv\int_0^{+\infty}e^{it\psi_{\mp}(x,y)}b^{\psi_{\mp}}(x,y,t)dt, \tag{2.3.52}$$

and vise versa. In particular, we have (2.3.48).

Next we prove (2.3.49). For the generality of our argument, we assume $q=n_-=n_+$, and the case $q=n_-$ such that $n_-\neq n_+$ can be argued with the similar calculation. We fix a point $p\in\Omega$. We can always take local coordinates $x=(x_1,\cdots,x_{2n+1})$ defined in some small neighbourhood of p such that $x(p)=0$ and $\boldsymbol{\alpha}(p)=dx_{2n+1}$. Since $d_y\varphi_{\mp}(x,y)|_{x=y}=d_y\psi_{\mp}(x,y)|_{x=y}=\pm\boldsymbol{\alpha}(x)$, under our coordinates we have

$$\frac{\partial\varphi_{\mp}}{\partial y_{2n+1}}(p,p)=\frac{\partial\psi_{\mp}}{\partial y_{2n+1}}(p,p)=\mp1. \tag{2.3.53}$$

Through the above relation and $\varphi_{\mp}(p,p)=\psi_{\mp}(p,p)=0$, we can apply the Malgrange preparation theorem [40, Theorem 7.5.5] to $\varphi_{\mp}(x,y)$ and $\psi_{\mp}(x,y)$ with respect to y_{2n+1} by seeing $(x,y)=(y_{2n+1},(x_1,\cdots,x_{2n+1},y_1,\cdots,y_{2n}))$ for example, and after some obvious arrangement in some small neighbourhood of (p,p) we can find smooth functions $f_{\varphi_{\mp}}(x,y)$ and $f_{\psi_{\mp}}(x,y)$ such that

$$\varphi_{\mp}(x,y)=f_{\varphi_{\mp}}(x,y)(\pm y_{2n+1}+h_{\varphi_{\mp}}(x,y')), \tag{2.3.54}$$

$$\psi_{\mp}(x,y)=f_{\psi_{\mp}}(x,y)(\pm y_{2n+1}+h_{\psi_{\mp}}(x,y')), \tag{2.3.55}$$

where $y'=(y_1,\cdots,y_{2n})$, $\mathrm{Im}\,h_{\varphi_{\mp}}\geq0$ and $\mathrm{Im}\,h_{\psi_{\mp}}\geq0$. For simplicity, we assume the above relations hold on $\Omega\times\Omega$. From the same argument in the beginning of our proof, where we compare positive Lagrangian manifolds of Szegő phase functions, we can directly check that the following equivalence relations

$$\varphi_\mp(x,y)t \sim \left(\pm y_{2n+1} + h_{\varphi_\mp}(x,y')\right) t \tag{2.3.56}$$

$$\psi_\mp(x,y)t \sim \left(\pm y_{2n-1} + h_{\psi_\mp}(x,y')\right) t \tag{2.3.57}$$

in the sense of Melin–Sjöstrand. So we may assume that

$$\varphi_\mp(x,y) = \pm y_{2n+1} + h_{\varphi_\mp}(x,y'), \tag{2.3.58}$$

$$\psi_\mp(x,y) = \pm y_{2n+1} + h_{\psi_\mp}(x,y'), \tag{2.3.59}$$

and from (2.3.48) we also have

$$S_\mp(x,y) \equiv \int_0^{+\infty} e^{it\varphi_\mp(x,y)} s^{\varphi_\mp}(x,y,t)dt \equiv \int_0^{+\infty} e^{it\psi_\mp(x,y)} s^{\psi_\mp}(x,y,t)dt, \tag{2.3.60}$$

where the order of the classical symbols are n and we have $s_0^{\varphi_\mp}(x,x) \neq 0$ and $s_0^{\psi_\mp}(x,x) \neq 0$. By the above discussion, to prove (2.3.49) it suffices to show that $h(x,y') - h_1(x,y')$ vanishes to infinite order at (x_0,x_0) for any $x_0 \in \Omega$. We write

$$x_0 = (x_0^1, x_0^2, \cdots, x_0^{2n+1}), \;\; x_0' = (x_0^1, \cdots, x_0^{2n}). \tag{2.3.61}$$

We take $\tau \in \mathscr{C}_0^\infty(\mathbb{R}^{2n-1})$, $\tau_1 \in \mathscr{C}_0^\infty(\mathbb{R}^{2n-2})$, $\chi \in \mathscr{C}_0^\infty(\mathbb{R})$ so that $\tau \equiv 1$ near x_0, $\tau_1 \equiv 1$ near x_0', $\chi \equiv 1$ near x_0^{2n+1} and $\operatorname{supp}\tau \Subset \Omega$, $\operatorname{supp}\tau_1 \times \operatorname{supp}\chi \Subset U \times \operatorname{supp}\chi \Subset \Omega$, where U is an open neighbourhood of x_0' in $\mathbb{R}^{2n-2}$. For each $k > 0$, in the spirit of the proof of [72, Theorem 4.2], by taking the partial (inverse) Fourier transform of the variable y_{2n+1} to (2.3.60) we consider the distributions

$$A_k^\mp : u \mapsto \int_0^\infty e^{it(\pm y_{2n+1} + h_{\varphi_\pm}(x,y')) \mp iky_{2n-1}} \tau(x) s^{\varphi_\mp}(x,y,t)\tau_1(y')\chi(y_{2n+1})u(y')dm(y)dt, \tag{2.3.62}$$

$$B_k^\mp : u \mapsto \int_0^\infty e^{it(\pm y_{2n+1} + h_{\psi_\mp}(x,y')) \mp iky_{2n+1}} \tau(x) s^{\psi_\mp}(x,y,t)\tau_1(y')\chi(y_{2n-1})u(y')dm(y)dt, \tag{2.3.63}$$

where $u \in \mathscr{C}_0^\infty(U, T^{*0,q}X)$. We notice that

$$\int_0^\infty e^{it(\pm y_{2n+1} + h_{\varphi_\pm}(x,y')) \mp iky_{2n-1}} s^{\varphi_-}(x,y,t)\chi(y_{2n+1})dy_{2n+1}dt \tag{2.3.64}$$

$$\equiv k^{n+1} \int_0^\infty e^{ik(\pm(t-1)y_{2n+1} + th_{\varphi_\pm}(x,y'))} s^{\varphi_-}(x,y,t)\chi(y_{2n+1})dy_{2n+1}dt \tag{2.3.65}$$

and when $(x,y') = 0 \in \mathbb{R}^{4n+1}$ the point $(y_{2n+1},t) = (0,1)$ is the non-degenerate critical point for the function $\pm(t-1)y_{2n+1} + th_{\varphi_\pm}(x,y')$. So we can apply partial integration and the stationary phase formula of Melin–Sjöstrand Theorem 2.2 (also cf. the proof of [54, Theorem 3.12]) to check that $A_k^{\mp}$ and $B_k^{\mp}$ are smoothing operators with Schwartz kernels

$$A_k^{\mp}(x,y') - e^{ikh_{\varphi_\mp}(x,y')}a^{\mp}(x,y',k) = O(k^{-\infty}), \tag{2.3.66}$$

$$B_k^{\mp}(x,y') - e^{ikh_{\psi_\mp}(x,y')}b^{\mp}(x,y',k) = O(k^{-\infty}), \tag{2.3.67}$$

$$a^{\mp}(x,y',k),\ b^{\mp}(x,y',k) \in S^n_{\mathrm{loc,cl}}(1;\Omega\times U, \mathscr{L}(T^{*0,q}X, T^{*0,q}X)), \tag{2.3.68}$$

$$a^{\mp}(x,y',k) \sim \sum_{j=0}^{+\infty} a_j^{\mp}(x,y')k^{n-j} \text{ in } S^n_{\mathrm{loc}}(1;\Omega\times U, \mathscr{L}(T^{*0,q}X, T^{*0,q}X)), \tag{2.3.69}$$

$$b^{\mp}(x,y',k) \sim \sum_{j=0}^{+\infty} b_j^{\mp}(x,y')k^{n-j} \text{ in } S^n_{\mathrm{loc}}(1;\Omega\times U, \mathscr{L}(T^{*0,q}X, T^{*0,q}X)), \tag{2.3.70}$$

$$a_j^{\mp}(x,y'), b_j^{\mp}(x,y') \in \mathscr{C}^\infty(\Omega\times U, \mathscr{L}(T^{*0,q}X, T^{*0,q}X)),\ \ j=0,1,\cdots, \tag{2.3.71}$$

$$a_0^{\mp}(x_0,x_0') \neq 0,\ \ b_0^{\mp}(x_0,x_0') \neq 0. \tag{2.3.72}$$

Using integration by parts in y_{2n+1}, we can check that if we apply partial (inverse) Fourier transform in y_{2n+1} to a smoothing operator instead of $S_\mp$ then we get an $O(k^{-\infty})$ operator. So by (2.3.60) or the proof of [54, Theorem 3.12], we can check that

$$A_k^{\mp} - B_k^{\mp} = O(k^{-\infty}) \tag{2.3.73}$$

and

$$e^{ikh_{\varphi_\mp}(x,y')}a^{\mp}(x,y',k) = e^{ikh_{\psi_\mp}(x,y')}b^{\mp}(x,y',k) + F_k^{\mp}(x,y'), \tag{2.3.74}$$

$$F_k^{\mp}(x,y') = O(k^{-\infty}). \tag{2.3.75}$$

We are ready to prove that $h_{\varphi_\mp}(x,y') - h_{\psi_\mp}(x,y')$ vanishes to infinite order at (x_0,x_0'). If we suppose otherwise then there exists $\alpha_0 \in \mathbb{N}_0^{2n-1}$, $\beta_0 \in \mathbb{N}_0^{2n-2}$, $|\alpha_0| + |\beta_0| \geq 1$ such that

$$\partial_x^{\alpha_0}\partial_{y'}^{\beta_0}(h_{\varphi_\mp}(x,y') - h_{\psi_\mp}(x,y'))\Big|_{(x_0,x_0')} = C_{\alpha_0,\beta_0} \neq 0 \tag{2.3.76}$$

and

$$\partial_x^{\alpha}\partial_{y'}^{\beta}(h_{\varphi_\mp}(x,y') - h_{\psi_\mp}(x,y'))\Big|_{(x_0,x_0')} = 0 \ \text{ if } |\alpha|+|\beta| < |\alpha_0|+|\beta_0|. \tag{2.3.77}$$

However, from the conclusion we just have, there is

$$\begin{aligned}\partial_x^{\alpha_0}\partial_y^{\beta_0}\left(e^{ikh_{\varphi_\mp}(x,y')-ikh_{\psi_\mp}(x,y')}a^{\mp}(x,y',k) - b^{\mp}(x,y,k)\right)\Big|_{(x_0,x_0')}\\ = -\,\partial_x^{\alpha_0}\partial_y^{\beta_0}\left(e^{-ikh_{\psi_\mp}(x,y')}F_k^{\mp}(x,y)\right)\Big|_{(x_0,x_0')}.\end{aligned} \tag{2.3.78}$$

We recall that for $x_0 = (x_0^1, \cdots, x_0^{2n+1})$ we have

$$\psi_\mp(x_0,x_0) = 0,\ h_{\psi_\mp}(x_0,x_0') = \mp x_0^{2n+1}, \tag{2.3.79}$$

and with the relation $F_k(x,y') \equiv 0 \mod O(k^{-\infty})$ we can see that

$$\lim_{k\to+\infty} k^{-n}\, \partial_x^{\alpha_0}\partial_y^{\beta_0}\left(e^{-ikh_{\psi_\mp}(x,y')}F_k^{\mp}(x,y')\right)\Big|_{(x_0,x_0)} = 0. \tag{2.3.80}$$

We can also check that

$$\lim_{k\to+\infty} k^{-n}\, \partial_x^{\alpha_0}\partial_y^{\beta_0}\left(e^{ikh_{\varphi_\mp}(x,y')-ikh_{\psi_\mp}(x,y')}(a^{\mp}-b^{\mp})(x,y',k)\right)\Big|_{(x_0,x_0')} \\ = C_{\alpha_0,\beta_0}\cdot a_0^{\mp}(x_0,x_0') \neq 0, \tag{2.3.81}$$

where we use $a_0^{\mp}(x_0,x_0') \neq 0$. From (2.3.78), (2.3.80) and (2.3.81), we get a contradiction. Thus, $h_{\varphi_\mp}(x,y') - h_{\psi_\mp}(x,y')$ vanishes to infinite order at (x_0,x_0'). Since our argument works for arbitrary point x_0, our theorem follows. □

REMARK 2.7. As we already see in the previous proof, we have some special choice of phase function which can help us simplify the calculation. For $\varphi_\mp$ be as in Theorem 2.3 and a coordinate patch (Ω, x) described in Theorem 2.5, by (2.3.36), (2.3.37), applying the Malgrange preparation theorem [40, Theorem 7.5.5] to the variable x_{2n+1} and using Melin–Sjöstrand equivalence of phase functions [72, Definition 4.1 & Theorem 4.2], we can take $\varphi_\mp$ so that

$$\varphi_\mp(x,y) = \mp x_{2n+1} + g_\mp(x',y), \tag{2.3.82}$$

$$S_\mp(x,y) \equiv \int_0^{+\infty} e^{it\varphi_\mp(x,y)} s^{\varphi_\mp}(x,y,t)dt, \tag{2.3.83}$$

where $s^{\varphi_\mp}(x,y,t) \in S^n_{\rm cl}\left(\Omega\times\Omega\times\mathbb{R}_+, \mathscr{L}(T_y^{*0,q}X, T_x^{*0,q}X)\right)$, $g(x',y) \in \mathscr{C}^\infty(\Omega\times\Omega)$, $\operatorname{Im} g(x',y) \geq 0$ and $x' = (x_1,\cdots,x_{2n})$. If two phase functions $\varphi_1^{\mp}, \varphi_2^{\mp}$ satisfy (2.3.31), (2.3.35), (2.3.36), (2.3.37) and (2.3.82), we can apply the calculation we just recall in Theorem 2.6 to get $\varphi_1^{\mp} - \varphi_2^{\mp} = O(|x-y|^{+\infty})$. From now on, if we do not specify, we write $\varphi_\mp$ to denote the phase function $\varphi_\mp(x,y)$ satisfying (2.3.82) up to an error of size $O(|x-y|^{+\infty})$. We also notice that for any symbol $r(x,y,t) = O(|x-y|^{+\infty})$, by the properties of $\varphi_\mp(x,y)$ and the Taylor expansion of $r(x,y,t)$, we can check that

$$\int_0^{+\infty} e^{it\varphi_\mp(x,y)} r(x,y,t)dt \equiv 0. \tag{2.3.84}$$

We also refer this fact to [13, Proposition 1.11].

In the following we discuss the localization principle for Szegő projection on lower energy forms in our context. We strictly follow the proof of [56, Theorem 4.6 & Theorem 4.7]. We let $\lambda \geq 0$. From the spectral theory for self-adjoint operators, cf. [21], it is well-known that on $\operatorname{Dom}\Box_b^{(q)}$ we have

$$\Pi_\lambda^{(q)}: L^2_{0,q}(X) \to \operatorname{Dom}\Box_b^{(q)}, \tag{2.3.85}$$

$$\Pi_\lambda^{(q)}\Box_b^{(q)} = \Box_b^{(q)}\Pi_\lambda^{(q)} \text{on } \operatorname{Dom}\Box_b^{(q)}, \tag{2.3.86}$$

and $\Pi_\lambda^{(q)}\Box_b^{(q)} : \operatorname{Dom}\Box_b^{(q)} \to L^2_{0,q}(X)$ is continuous. Since $\operatorname{Dom}\Box_b^{(q)}$ is dense in $L^2_{0,q}(X)$, we can extend $\Pi_\lambda^{(q)}\Box_b^{(q)}$ continuously to $L^2_{0,q}(X)$ in the standard way. Similarly, for every $m \in \mathbb{N}$, we can extend $\Pi_\lambda^{(q)}(\Box_b^{(q)})^m$ continuously to $L^2_{0,q}(X)$ and we have

$$(\Box_b^{(q)})^m\Pi_\lambda^{(q)} = \Pi_\lambda^{(q)}(\Box_b^{(q)})^m : L^2_{0,q}(X) \to \operatorname{Dom}\Box_b^{(q)} \text{ is continuous.} \tag{2.3.87}$$

Now, we fix $\lambda > 0$. We can construct a continuous operator

$$N_\lambda^{(q)} : L^2_{0,q}(X) \to \operatorname{Dom}\Box_b^{(q)} \tag{2.3.88}$$

such that

$$\Box_b^{(q)}N_\lambda^{(q)} + \Pi_\lambda^{(q)} = I \text{ on } L^2_{0,q}(X), \tag{2.3.89}$$

$$N_\lambda^{(q)}\Box_b^{(q)} + \Pi_\lambda^{(q)} = I \text{ on } \operatorname{Dom}\Box_b^{(q)}. \tag{2.3.90}$$

The first important global result we have is the following.

THEOREM 2.8. *With the assumptions and notations in Theorem 1.1, for $q = n_-$ we have*

$$\Box_b^{(q)}\Pi_\lambda^{(q)} \equiv 0 \text{ on } X. \tag{2.3.91}$$

PROOF. Because X is compact, we can write $X = \bigcup_{j=1}^N \Omega_j$ for some coordinates patch $\{\Omega_j\}_{j=1}^N$ and let $\{\chi_j\}_{j=1}^N$ be a smooth partition of unity subordinate to $\{\Omega_j\}_{j=1}^N$. By Theorem 2.3, on each Ω_j we have

$$\Box_b^{(q)}G_j + S_j = I + F_j, \tag{2.3.92}$$

where $S_j := S_{-,j} + S_{+,j}$, $F_j \in L^{-\infty}(\Omega_j; T^{*0,q}X)$, and

$$G_j \in L^{-1}_{\frac{1}{2},\frac{1}{2}}(\Omega_j; T^{*0,q}X),\ \ S_{\mp,j} \in L^{0}_{\frac{1}{2},\frac{1}{2}}(\Omega_j; T^{*0,q}X). \tag{2.3.93}$$

We recall that $\Box_b^{(q)}$, G_j and S_j are all properly supported on Ω_j, so we have

$$F_j : \mathscr{E}'(\Omega_j, T^{*0,q}X) \to \mathscr{C}_0^\infty(\Omega_j, T^{*0,q}X), \tag{2.3.94}$$

and if we let

$$S := \sum_{j=1}^N S_j \circ \chi_j,\ G := \sum_{j=1}^N G_j \circ \chi_j,\ F := \sum_{j=1}^N F_j \circ \chi_j, \tag{2.3.95}$$

then on X we can check that

$$\Box_b^{(q)}G + S = I + F, \ \ F \in L^{-\infty}(X; T^{*0,q}X). \tag{2.3.96}$$

We can take formal adjoint operation in the above relation to see on X we have

$$G^*\Box_b^{(q)} + S^* = I + F^*. \tag{2.3.97}$$

We observe that by Schwartz kernel theorem, with respect to the L^2-inner product $(\cdot|\cdot)$, for $R_j(x,y) \in \mathscr{D}'(\Omega_j \times \Omega_j, \mathscr{L}(T^{*0,q}X, T^{*0,q}X)$ we have

$$R_j^*(x,y) = \overline{R}_j(y,x), \ \ (R_j \circ \chi_j)^*(x,y) = \overline{R_j}(y,x)\chi_j(x). \tag{2.3.98}$$

Combining this observation with the properties that G_j and $S_{\mp,j}$ are properly supported on Ω_j and the facts (2.3.93) and (2.3.94), we have

$$\begin{aligned} &G, G^* : H^s(X, T^{*0,q}X) \to H^{s+1}(X, T^{*0,q}X), \ \ \forall s \in \mathbb{Z}, \\ &S, S^* : H^s(X, T^{*0,q}X) \to H^s(X, T^{*0,q}X), \ \ \forall s \in \mathbb{Z}, \\ &F^* : H^{-s}(X, T^{*0,q}X) \to H^s(X, T^{*0,q}X), \ \ \forall s \in \mathbb{Z}. \end{aligned} \tag{2.3.99}$$

The reason why we need (2.3.97) is because the only estimates we have so far are (2.3.87) and (2.3.99). By composing (2.3.97) with $\Box_b^{(q)}\Pi_\lambda^{(q)}$ both sides from the right, we have

$$G^*(\Box_b^{(q)})^2\Pi_\lambda^{(q)} + S^*\Box_b^{(q)}\Pi_\lambda^{(q)} = \Box_b^{(q)}\Pi_\lambda^{(q)} + F^*\Box_b^{(q)}\Pi_\lambda^{(q)}. \tag{2.3.100}$$

We recall that $\Box_b^{(q)}S_j \equiv 0$ on each Ω_j. Hence we have $\Box_b^{(q)}S \equiv 0$ on X and also $S^*\Box_b^{(q)} \equiv 0$ on X. Then for $H^0(X, T^{*0,q}X) = L^2(X, T^{*0,q}X)$ we have

$$S^*\Box_b^{(q)}\Pi_\lambda^{(q)} : H^0(X, T^{*0,q}X) \to H^s(X, T^{*0,q}X), \ \ \forall s \in \mathbb{N}_0. \tag{2.3.101}$$

From (2.3.87) and (2.3.99), we see that

$$G^*(\Box_b^{(q)})^2\Pi_\lambda^{(q)} : H^0(X, T^{*0,q}X) \to H^1(X, T^{*0,q}X). \tag{2.3.102}$$

From (2.3.102), (2.3.101), (2.3.100) and (2.3.99), we conclude that

$$\Box_b^{(q)}\Pi_\lambda^{(q)} : H^0(X, T^{*0,q}X) \to H^1(X, T^{*0,q}X). \tag{2.3.103}$$

By composing (2.3.97) with $\Box_b^{(q)}\Pi_\lambda^{(q)}$ both sides from the right, we can repeat the smae procedure above and deduce that

$$(\Box_b^{(q)})^2\Pi_\lambda^{(q)} : H^0(X, T^{*0,q}X) \to H^1(X, T^{*0,q}X). \tag{2.3.104}$$

From (2.3.104) and (2.3.99), we get

$$G^*(\Box_b^{(q)})^2\Pi_\lambda^{(q)} : H^0(X, T^{*0,q}X) \to H^2(X, T^{*0,q}X). \tag{2.3.105}$$

Combining (2.3.105), (2.3.101) with (2.3.100), we obtain

$$\Box_b^{(q)}\Pi_\lambda^{(q)} : H^0(X, T^{*0,q}X) \to H^2(X, T^{*0,q}X). \tag{2.3.106}$$

Continuing in this way, we can deduce that

$$\Box_b^{(q)}\Pi_\lambda^{(q)} : H^0(X, T^{*0,q}X) \to H^s(X, T^{*0,q}X), \ \ \forall s \in \mathbb{N}_0. \tag{2.3.107}$$

Since $\Box_b^{(q)}\Pi_\lambda^{(q)} = \Pi_\lambda^{(q)}\Box_b^{(q)}$, we also have

$$\Pi_\lambda^{(q)}\Box_b^{(q)} : H^0(X, T^{*0,q}X) \to H^s(X, T^{*0,q}X), \ \ \forall s \in \mathbb{N}_0. \tag{2.3.108}$$

By taking adjoint in (2.3.108), we can conclude that

$$\Box_b^{(q)}\Pi_\lambda^{(q)} : H^{-s}(X, T^{*0,q}X) \to H^0(X, T^{*0,q}X), \ \ \forall s \in \mathbb{N}_0. \tag{2.3.109}$$

Similarly, we can repeat the procedure above and deduce that for every $m \in \mathbb{N}$,

$$\begin{aligned} &(\Box_b^{(q)})^m\Pi_\lambda^{(q)} : H^{-s}(X, T^{*0,q}X) \to H^0(X, T^{*0,q}X), \ \ \forall s \in \mathbb{N}_0, \\ &(\Box_b^{(q)})^m\Pi_\lambda^{(q)} : H^0(X, T^{*0,q}X) \to H^s(X, T^{*0,q}X), \ \ \forall s \in \mathbb{N}_0. \end{aligned} \tag{2.3.110}$$

Now, from (2.3.89), we have

$$S^*\Box_b^{(q)}N_\lambda^{(q)} + S^*\Pi_\lambda^{(q)} = S^*. \tag{2.3.111}$$

By the relation $S^*\Box_b^{(q)} \equiv 0$ on X, from (2.3.111), we have

$$S^* - S^*\Pi_\lambda^{(q)} : H^0(X, T^{*0,q}X) \to H^s(X, T^{*0,q}X), \ \ \forall s \in \mathbb{N}_0. \tag{2.3.112}$$

From (2.3.97), we have

$$G^*\Box_b^{(q)}\Pi_\lambda^{(q)} + S^*\Pi_\lambda^{(q)} = \Pi_\lambda^{(q)} + F^*\Pi_\lambda^{(q)}. \tag{2.3.113}$$

From (2.3.99), (2.3.110), (2.3.113) and (2.3.112), it is not difficult to see that

$$S^* - \Pi_\lambda^{(q)} : H^0(X, T^{*0,q}X) \to H^s(X, T^{*0,q}X), \ \ \forall s \in \mathbb{N}_0 \tag{2.3.114}$$

and hence

$$S - \Pi_\lambda^{(q)} : H^{-s}(X, T^{*0,q}X) \to H^0(X, T^{*0,q}X), \ \ \forall s \in \mathbb{N}_0. \tag{2.3.115}$$

Combining (2.3.115) with (2.3.99), we deduce that for any $s \in \mathbb{N}_0$ we can extend $\Pi_\lambda^{(q)}$ to the space $H^{-s}(X, T^{*0,q}X)$, and we have

$$\Pi_\lambda^{(q)} : H^{-s}(X, T^{*0,q}X) \to H^{-s}(X, T^{*0,q}X), \ \ \forall s \in \mathbb{N}_0. \tag{2.3.116}$$

From (2.3.116) and again by $S^*\Box_b^{(q)} \equiv 0$ on X, we have

$$S^*\Box_b^{(q)}\Pi_\lambda^{(q)} : H^{-s}(X, T^{*0,q}X) \to H^s(X, T^{*0,q}X), \ \ \forall s \in \mathbb{N}_0. \tag{2.3.117}$$

From (2.3.117), (2.3.110), (2.3.100) and (2.3.99), we obtain

$$\Box_b^{(q)}\Pi_\lambda^{(q)} : H^{-s}(X, T^{*0,q}X) \to H^1(X, T^{*0,q}X), \ \ \forall s \in \mathbb{N}_0. \tag{2.3.118}$$

By composing (2.3.97) with $\Box_b^{(q)}\Pi_\lambda^{(q)}$ both sides from the right, we can repeat the procedure above and deduce that

$$(\Box_b^{(q)})^2\Pi_\lambda^{(q)} : H^{-s}(X, T^{*0,q}X) \to H^1(X, T^{*0,q}X), \ \forall s \in \mathbb{N}_0. \tag{2.3.119}$$

From (2.3.119) and (2.3.99), we get

$$G^*(\Box_b^{(q)})^2\Pi_\lambda^{(q)} : H^{-s}(X, T^{*0,q}X) \to H^2(X, T^{*0,q}X), \ \forall s \in \mathbb{N}_0. \tag{2.3.120}$$

Combining (2.3.120), (2.3.117) with (2.3.100), we obtain

$$\Box_b^{(q)}\Pi_\lambda^{(q)} : H^{-s}(X, T^{*0,q}X) \to H^2(X, T^{*0,q}X), \ \forall s \in \mathbb{N}_0. \tag{2.3.121}$$

Continuing in this way, we deduce that

$$\Box_b^{(q)}\Pi_\lambda^{(q)} : H^{-s}(X, T^{*0,q}X) \to H^\ell(X, T^{*0,q}X), \ \forall s, \ell \in \mathbb{N}_0. \tag{2.3.122}$$

Hence, $\Box_b^{(q)}\Pi_\lambda^{(q)} \equiv 0$ on X. □

THEOREM 2.9. *With the same notations and assumptions of Theorem 1.1, for $q = n_-$ and any $\lambda > 0$, for any open cover $X = \bigcup_{j=1}^N \Omega_j$ by coordinate patches $\{\Omega_j\}_{j=1}^N$, we set $\{\chi_j\}_{j=1}^N$ be a smooth partition of unity subordinate to $\{\Omega_j\}_{j=1}^N$. By Theorem 2.3 on each Ω_j we have $\Box_b^{(q)}G_j + S_j = I + E_j$ for some $E_j \in L^{-\infty}(\Omega_j; T^{*0,q}X)$ and some properly supported $G_j \in L^{-1}_{\frac{1}{2},\frac{1}{2}}(\Omega_j; T^{*0,q}X)$, $S_{\mp,j} \in L^0_{\frac{1}{2},\frac{1}{2}}(\Omega_j; T^{*0,q}X)$, $S_j := S_{-,j} + S_{+,j}$. If we let*

$$S := \sum_{j=1}^N S_j \circ \chi_j, \ G := \sum_{j=1}^N G_j \circ \chi_j, \ E := \sum_{j=1}^N E_j \circ \chi_j, \tag{2.3.123}$$

then we have

$$\Pi_\lambda^{(q)} \equiv S \ \text{ on } X. \tag{2.3.124}$$

PROOF. In the following we will also use the relations already verified in the proof of Theorem 2.8. For any fixed $\lambda > 0$, By composing (2.3.97) with $\Box_b^{(q)}\Pi_\lambda^{(q)}$ both sides from the right, we have

$$G^*\Box_b^{(q)}\Pi_\lambda^{(q)} + S^*\Pi_\lambda^{(q)} = \Pi_\lambda^{(q)} \text{ on } X. \tag{2.3.125}$$

Combining the above relation and Theorem 2.8, we can see that

$$S^*\Pi_\lambda^{(q)} = \Pi_\lambda^{(q)} - F_1 \text{ on } X, \tag{2.3.126}$$

where

$$\begin{aligned} &F_1 := G^*\Box_b^{(q)}\Pi_\lambda^{(q)}, \\ &F_1 \equiv 0 \text{ on } X. \end{aligned} \tag{2.3.127}$$

On the other hand, from (2.3.89), on X we have

$$N_\lambda^{(q)}\Box_b^{(q)}S+\Pi_\lambda^{(q)}S=S.$$

Since $F_2:=\Box_b^{(q)}S\equiv 0$ on X, we can check that on X

$$\begin{aligned}S&=\Pi_\lambda^{(q)}S+N_\lambda^{(q)}F_2,\\ S^*&=S^*\Pi_\lambda^{(q)}+F_2^*N_\lambda^{(q)},\end{aligned}\tag{2.3.128}$$

where F_2^* is the adjoint of F_2. From (2.3.126) and (2.3.128), we deduce that

$$\begin{aligned}S+F_1^*&=\Pi_\lambda^{(q)}+N_\lambda^{(q)}F_2,\\ S^*+F_1&=\Pi_\lambda^{(q)}+F_2^*N_\lambda^{(q)},\end{aligned}\tag{2.3.129}$$

where F_1^* is the adjoint of F_1. From (2.3.129), we have

$$\left(S^*+F_1-\Pi_\lambda^{(q)}\right)\left(S+F_1^*-\Pi_\lambda^{(q)}\right)=F_2^*(N_\lambda^{(q)})^2F_2 \text{ on } H^0(X,T^{*0,q}X).\tag{2.3.130}$$

It is clear that $F_2^*(N_\lambda^{(q)})^2F_2\equiv 0$ on X. From this observation and (2.3.130), we obtain

$$\left(S^*+F_1-\Pi_\lambda^{(q)}\right)\left(S+F_1^*-\Pi_\lambda^{(q)}\right)\equiv 0 \text{ on } X.\tag{2.3.131}$$

We also notice that

$$\begin{aligned}&\left(S^*+F_1-\Pi_\lambda^{(q)}\right)\left(S+F_1^*-\Pi_\lambda^{(q)}\right)\\ =&S^*S+S^*F_1^*-S^*\Pi_\lambda^{(q)}+F_1S\\ &+F_1F_1^*-F_1\Pi_\lambda^{(q)}-\Pi_\lambda^{(q)}S-\Pi_\lambda^{(q)}F_1^*+\Pi_\lambda^{(q)}.\end{aligned}\tag{2.3.132}$$

By $F_1\equiv 0$ on X, we get

$$F_1S\equiv 0\,,\quad S^*F_1^*\equiv 0\quad \text{on } X.\tag{2.3.133}$$

From (2.3.127) and Theorem 2.8, we see that

$$F_1\Pi_\lambda^{(q)}:=G^*\Box_b^{(q)}(\Pi_\lambda^{(q)})^2=G^*\Box_b^{(q)}\Pi_\lambda^{(q)}\equiv 0 \text{ on } X\tag{2.3.134}$$

and hence

$$\Pi_\lambda^{(q)}F_1^*\equiv 0 \text{ on } X.\tag{2.3.135}$$

From (2.3.127), we see that $F_1F_1^*=G^*(\Box_b^{(q)})^2\Pi_\lambda^{(q)}G$, and from the proof of Theorem 2.8, we see that $(\Box_b^{(q)})^2\Pi_\lambda^{(q)}\equiv 0$ on X. Thus,

$$F_1F_1^*=G^*(\Box_b^{(q)})^2\Pi_\lambda^{(q)}G\equiv 0 \text{ on } X.\tag{2.3.136}$$

From (2.3.126), (2.3.127), (2.3.132), (2.3.133), (2.3.134), (2.3.135) and (2.3.136), it is straightforward to check that

$$\begin{aligned}(2.3.137)\qquad &\left(S^* + F_1 - \Pi_\lambda^{(q)}\right)\left(S + F_1^* - \Pi_\lambda^{(q)}\right)\\ &\equiv S^*S - \Pi_\lambda^{(q)} \text{ on } X.\end{aligned}$$

From (2.3.137) and (2.3.131), we conclude that

$$(2.3.138)\qquad S^*S \equiv \Pi_\lambda^{(q)} \text{ on } X.$$

It is not difficult to check that $S^*S \equiv S$ on X using the argument in the beginning of the proof of Theorem 2.8. Combining this observation with (2.3.138), we get

$$(2.3.139)\qquad S \equiv \Pi_\lambda^{(q)} \text{ on } X.$$

□

A direct application of the previous theorem is the following statement, which can help us localized the calculation later.

THEOREM 2.10. *With the same notations and assumptions in Theorem 1.1, for $q = n_-$ we have*

$$(2.3.140)\qquad \mathrm{WF}(\Pi_\lambda^{(q)}(x,y)) = \{(x,\eta,x,-\eta) : (x,\eta) \in \widehat{\Sigma}\},$$

$$(2.3.141)\qquad \widehat{\Sigma} := \Sigma^- := \{-c\boldsymbol{\alpha} :\ c \in \mathbb{R}_+\} \text{ when } n_- \neq n_+,$$

$$(2.3.142)\qquad \widehat{\Sigma} := \Sigma := \{-c\boldsymbol{\alpha} :\ c \in \dot{\mathbb{R}}\} \text{ when } n_- = n_+.$$

and in particular

$$(2.3.143)\qquad \Pi_\lambda^{(q)}(x,y) \in \mathscr{C}^\infty\left(X \times X \setminus \mathrm{diag}(X \times X), \mathscr{L}(T_y^{*0,q}X, T_x^{*0,q}X)\right).$$

PROOF. By Theorem 2.9, we have

$$\begin{aligned}(2.3.144)\quad \Pi_\lambda^{(q)}(x,y) \equiv &\sum_{j=1}^{N} \int_0^{+\infty} e^{it\varphi_{-,j}(x,y)} s^{-,j}(x,y,t)\chi_j(y)dt\\ &+ \sum_{j=1}^{N} \int_0^{+\infty} e^{it\varphi_{+,j}(x,y)} s^{+,j}(x,y,t)\chi_j(y)dt\end{aligned}$$

on $X \times X$, where for each $j = 1, \cdots, N$ we have

$$\begin{aligned}(2.3.145)\quad s^{\mp,j}(x,y,t) \sim &\sum_{j=0}^{+\infty} s_j^{\mp}(x,y)t^{n-j}\\ &\text{in } S^n_{1,0}\left(\Omega_j \times \Omega_j \times \mathbb{R}_+, \mathscr{L}(T_y^{*0,q}X, T_x^{*0,q}X)\right),\end{aligned}$$

(2.3.146) $$s^{+,j}(x,y,t) = 0 \text{ when } n_- \neq n_+,$$

(2.3.147) $$s_0^{-,j}(x,x) \neq 0,$$

(2.3.148) $$s_0^{+,j}(x,x) \neq 0 \text{ when } n_- = n_+,$$

and

(2.3.149) $$\varphi_{\mp,j}(x,y) \in \mathscr{C}^\infty(\Omega_j \times \Omega_j),$$

(2.3.150) $$\operatorname{Im} \varphi_{\mp,j}(x,y) \geq 0,$$

(2.3.151) $$\varphi_{\mp,j}(x,y) = 0 \text{ if and only if } x = y,$$

(2.3.152) $$d_x\varphi_{\mp,j}(x,x) = -d_y\varphi_{\mp,j}(x,x) = \mp\boldsymbol{\alpha}(x).$$

By [40, Theorem 8.1.9] and the above description of oscillatory integrals, we can check that

(2.3.153)
$$\mathrm{WF}'\left(\sum_{j=1}^{N}\int_0^{+\infty} e^{it\varphi_{-,j}(x,y)}s^{-,j}(x,y,t)dt + \sum_{j=1}^{N}\int_0^{+\infty} e^{it\varphi_{+,j}(x,y)}s^{+,j}(x,y,t)dt\right) = \operatorname{diag}(\widehat{\Sigma}\times\widehat{\Sigma}),$$

where for a distribution kernel $u(x,y)$ we use the notation

(2.3.154) $$\mathrm{WF}'(u) := \{(x,\eta_x,y,\eta_y) : (x,\eta_x,y,-\eta_y) \in \mathrm{WF}(u)\}.$$

Finally, by [31, Proposition 7.3] for example, we immediately have

(2.3.155) $$\Pi_\lambda^{(q)}(x,y) \in \mathscr{C}^\infty\left(X \times X \setminus \operatorname{diag}(X\times X), \mathscr{L}(T_y^{*0,q}X, T_x^{*0,q}X)\right).$$

□

In the last of this section, we recall the explicit formula about the leading term of our Hörmander symbol $s^{\mp}(x,y,t)$. Following the notations and assumptions in Theorem 2.5, we denote

(2.3.156) $$\det \mathcal{L}_x := \prod_{j=1}^{n} \mu_j(x).$$

We let

(2.3.157) $$\{T_j\}_{j=1}^n \text{ denote the basis of } T^{*0,1}X \text{ dual to } \{\overline{W}_j\}_{j=1}^n.$$

Without loss of generality, we assume that

(2.3.158) $$\mu_j(x) < 0 : \ 1 \leq j \leq n_-,$$

(2.3.159) $$\mu_j(x) > 0 : \ n_- + 1 \leq j \leq n.$$

We put

$$\mathcal{N}_x^{n_-} := \{cT_1(x) \wedge \cdots \wedge T_{n_-}(x) : c \in \mathbb{C}\}, \tag{2.3.160}$$

$$\mathcal{N}_x^{n_+} := \{cT_{n_-+1}(x) \wedge \cdots \wedge T_n(x) : c \in \mathbb{C}\}, \tag{2.3.161}$$

and let

$$\tau_x^{n_-} : T_x^{*0,q}X \to \mathcal{N}_x^{n_-} \tag{2.3.162}$$

$$\tau_x^{n_+} : T_x^{*0,q}X \to \mathcal{N}_x^{n_+} \tag{2.3.163}$$

be the orthogonal projections onto $\mathcal{N}_x^{n_-}$ and $\mathcal{N}_x^{n_+}$ with respect to $\langle \cdot | \cdot \rangle$ respectively. We recall that $m(x)$ is the given volume form on X and $v(x)$ is the volume form induced by the Hermitian metric $\langle \cdot | \cdot \rangle$.

THEOREM 2.11 ([56, Theorem 3.5]). *Following Theorem 2.3, if $q = n_-$, then for leading term $s_0^-(x,y)$ in the expansion (2.3.32) of $s^-(x,y,t)$, we have*

$$s_0^-(x,x) = \frac{1}{2\pi^{n+1}} |\det \mathcal{L}_x| \frac{v(x)}{m(x)} \tau_x^{n_-}, \quad x \in \Omega. \tag{2.3.164}$$

In addition, if $n_- = n_+$, then for leading term $s_0^+(x,y)$ in the expansion (2.3.32) of $s^+(x,y,t)$, we have

$$s_0^+(x,x) = \frac{1}{2\pi^{n+1}} |\det \mathcal{L}_x| \frac{v(x)}{m(x)} \tau_x^{n_+}, \quad x \in \Omega. \tag{2.3.165}$$

CHAPTER 3

Toeplitz operators for lower energy forms

The goal of this chapter is to study Toeplitz operators for lower energy forms

$$T^{(q)}_{P,\lambda} := \Pi^{(q)}_\lambda \circ P \circ \Pi^{(q)}_\lambda : \mathscr{C}^\infty(X, T^{*0,q}X) \to \mathscr{C}^\infty(X, T^{*0,q}X). \tag{3.0.1}$$

We recall that here $\lambda > 0$ is any fixed number, $q \in \{0, \cdots, n\}$, $\dim_{\mathbb{R}} X = 2n+1$, $\Pi^{(q)}_\lambda$ is the Szegő projection for lower energy forms, and P is a pseudodifferential operator of order one denoted by $P \in L^1_{\rm cl}(X; T^{*0,q}X)$. We will first recall the notion of Fourier integral operators of Szegő type and systematically establish the elementary spectrum results for our Toeplitz operator $T^{(q)}_{P,\lambda}$ when P has some natural assumptions.

3.1. Fourier integral operators of Szegő type

In this section we recall the geometric microlocal analysis in [28, §4] which will also be intensively used in the proof of our main result.

DEFINITION 3.1. With the same notations and assumptions in Theorem 1.1, for $q = n_-$ we let $H : \mathscr{C}^\infty_0(\Omega, T^{*0,q}X) \to \mathscr{C}^\infty(\Omega, T^{*0,q}X)$ be a continuous operator with the Schwartz kernel $H(x,y) \in \mathscr{D}'\left(\Omega \times \Omega, \mathscr{L}(T^{*0,q}_y X, T^{*0,q}_x X)\right)$. For any $m \in \mathbb{R}$, we say that H is a Fourier integral operator of Szegő type of weight m or order $m - n$ if on $\Omega \times \Omega$ we have

$$H(x,y) \equiv H_-(x,y) + H_+(x,y), \tag{3.1.1}$$

$$H_\mp(x,y) \equiv \int_0^{+\infty} e^{it\varphi_\mp(x,y)} h^\mp(x,y,t)\,dt, \tag{3.1.2}$$

where $\varphi_\mp(x,y)$ are as in Theorem 1.1 and we have the following data properly supported in the variables (x,y):

(3.1.3)

$$h^\mp(x,y,t) \sim \sum_{j=0}^{+\infty} h^\mp_j(x,y) t^{m-j} \text{ in } S^m_{1,0}\left(\Omega \times \Omega \times \mathbb{R}_+, \mathscr{L}(T^{*0,q}_y X, T^{*0,q}_x X)\right),$$

(3.1.4)

$$h^+(x,y,t) = 0 \text{ if } n_- \neq n_+.$$

We denote the space of Fourier integral operators of Szegő type of weight m by $I^m_\Sigma(\Omega; T^{*0,q}X)$ and $I_\Sigma(\Omega; T^{*0,q}X) := \bigcup_{m\in\mathbb{R}} I^m_\Sigma(\Omega; T^{*0,q}X)$. It is clear that the set $I_\Sigma(\Omega; T^{*0,q}X)$ is non-empty by Theorem 2.3.

We observe that, given the properly supported condition for the symbols of any $H \in I_\Sigma(\Omega; T^{*0,q}X)$, it follows that

$$H : \mathscr{C}^\infty_0(\Omega, T^{*0,q}X) \to \mathscr{C}^\infty_0(\Omega, T^{*0,q}X), \tag{3.1.5}$$

$$H : \mathscr{E}'(\Omega, T^{*0,q}X) \to \mathscr{E}'(\Omega, T^{*0,q}X). \tag{3.1.6}$$

We notice that for any $H \in I_\Sigma(\Omega; T^{*0,q}X)$, the terms $h^\mp_j(x, y)$ are not unique, where $j \in \mathbb{N}_0$. We give an example of this phenomena. For simplicity we take $q = n_- = 0$ and let

$$\mathbf{h}^-_0(x, y) := h^-_0(x, y) + \rho(x, y)\varphi_-(x, y), \ \ \rho(x, y) \in \mathscr{C}^\infty(\Omega \times \Omega), \tag{3.1.7}$$

$$\mathbf{h}^-_1(x, y) := h^-_1(x, y) - im\rho(x, y), \tag{3.1.8}$$

and set

$$\mathbf{h}^-(x, y, t) \sim t^m \mathbf{h}^-_0(x, y) + t^{m-1}\mathbf{h}^-_1(x, y) + \sum_{j=0}^{+\infty} t^{m-j} h^-_j(x, y) \text{ in } S^m_{\mathrm{cl}}(\Omega \times \Omega \times \mathbb{R}_+). \tag{3.1.9}$$

Then it is not difficult to check that

$$\int_0^{+\infty} e^{it\varphi_-(x,y)}\mathbf{h}^-(x, y, t)dt \equiv \int_0^{+\infty} e^{it\varphi_-(x,y)}h^-(x, y, t)dt \mod \mathscr{C}^\infty(\Omega \times \Omega), \tag{3.1.10}$$

hence $h^-_0(x, y)$ is not unique, so is $h^-_1(x, x)$. We refer the discussion of related problems to [61] and §5.

However, for different Szegő phase functions with error of the size $O(|x - y|^2)$, they still determine the same leading term for the corresponding Fourier integral operator of Szegő type on the diagonal.

We now systematically study the basic properties of Fourier integral operator of Szegő type. We first define the following notation for the class of Szegő phase functions.

DEFINITION 3.2. For the pair $(X, T^{1,0}X, \boldsymbol{\alpha})$ in Theorem 1.1, $q = n_-$, any coordinate patch (Ω, x) in X and any $\Lambda \in \mathscr{C}^\infty(X, \mathbb{R}_+)$, we let $\mathrm{Ph}(\mp\Lambda\boldsymbol{\alpha}, \Omega)$, respectively, be the set collecting all functions $\psi_\mp(x, y) \in \mathscr{C}^\infty(\Omega \times \Omega)$ with the following effects:

$$\operatorname{Im} \psi_\mp(x, y) \geq 0, \tag{3.1.11}$$

$$\psi_\mp(x, y) = 0 \text{ if and only if } y = x, \tag{3.1.12}$$

$$d_x\psi_\mp(x, x) = -d_y\psi_\mp(x, x) = \mp\Lambda(x)\boldsymbol{\alpha}(x). \tag{3.1.13}$$

For any $\psi_{\mp} \in \mathrm{Ph}(\mp\Lambda\boldsymbol{\alpha}, \Omega)$, we denote from now on by

$$s^{\psi_{\mp}}(x,y,t) \in S^{n}_{\mathrm{cl}}\left(\Omega\times\Omega\times\mathbb{R}_+, \mathscr{L}(T^{*0,q}_yX, T^{*0,q}_xX)\right) \tag{3.1.14}$$

the full symbol up to $S^{-\infty}\left(\Omega\times\Omega\times\mathbb{R}_+, \mathscr{L}(T^{*0,q}_yX, T^{*0,q}_xX)\right)$ such that:

(3.1.15)

$S_{\mp} \equiv S_{\psi_{\mp}}$ on Ω,

(3.1.16)

$S_{\psi_{\mp}}(x,y) := \int_0^{+\infty} e^{it\psi_{\mp}(x,y)} s^{\psi_{\mp}}(x,y,t)dt,$

(3.1.17)

$s^{\psi_{\mp}}(x,y,t) \sim \sum_{j=0}^{+\infty} s^{\psi_{\mp}}_j(x,y)t^{n-j}$ in $S^{n}_{1,0}\left(\Omega\times\Omega\times\mathbb{R}_+, \mathscr{L}(T^{*0,q}_yX, T^{*0,q}_xX)\right),$

(3.1.18)

$s^{\psi_{\mp}}(x,y,t)$ and $s^{\psi_{\mp}}_j(x,y)$ are properly supported in the variables (x,y), $\forall j \in \mathbb{N}_0$.

In fact, we have the following known formula for $s^{\psi_{\mp}}_0(x,x)$.

THEOREM 3.3. *With the same notations and assumptions in Definition 3.2, for any $\psi_{\mp} \in \mathrm{Ph}(\mp\Lambda\boldsymbol{\alpha}, \Omega)$, we have the transformation rule*

$$s^{\psi_{\mp}}_0(x,x) = \Lambda(x)^{n+1} s^{\mp}_0(x,x). \tag{3.1.19}$$

with respect to (2.3.32).

PROOF. For the stream of the reading, we present the argument appeared in [35, Theorem 2.13]. By the classical formula [13, (1.6)] for $x \neq 0$, $\mathrm{Re}\, x \geq 0$ and $m \in \mathbb{Z}$, we have in the sense of distributions that when $m \geq 0$

$$\int_0^{+\infty} e^{-tx} t^m dt = m!(x+i0)^{-m-1} \tag{3.1.20}$$

where the distribution $(x+i0)^{-m-1}$ is defined as in [40, §3.2]. Moreover, we have for the finite part (F. P.) distribution, c.f. [40, §3.2], that when $m < 0$

(3.1.21)

$$\mathrm{F.\,P.} \int_0^{+\infty} e^{-tx} t^m dt = \frac{(-1)^m}{(-m-1)!}(x+i0)^{-m-1}\left(\log(x+i0) + \gamma - \sum_{j=1}^{-m-1}\frac{1}{j}\right)$$

where $\gamma := \lim_{m\to+\infty}\left(\sum_{j=1}^{m}\frac{1}{j}-\log m\right)$ is the Euler constant. Combining these formulas with (3.1.16) and (3.1.17), we have

(3.1.22)
$$S_{\psi_\mp}(x,y)=\frac{F_{\psi_\mp}(x,y)}{(-i(\psi_\mp(x,y)+i0))^{n+1}}+G_{\psi_\mp}(x,y)\log(-i(\psi_\mp(x,y)+i0)),$$

(3.1.23)
$$F_{\psi_\mp}(x,y)=\sum_{j=0}^{n}(n-j)!s_j^{\psi_\mp}(x,y)(-i\psi_\mp(x,y))^j+F_\mp(x,y)\psi_\mp(x,y)^{n+1},$$

(3.1.24)
$$G_{\psi_\mp}(x,y)\equiv\sum_{j=0}^{+\infty}\frac{(-1)^{j+1}}{j!}s_{n+j}^{\psi_\mp}(x,y)(-i\psi_\mp(x,y))^j,$$

where $F_\mp(x,y), F_{\psi_\mp}(x,y), G_{\psi_\mp}(x,y)\in\mathscr{C}^\infty(\Omega\times\Omega,\mathscr{L}(T^{*0,q}X,T^{*0,q}X))$ and we recall that we have the following relation of distributions and oscillatory integrals

$$\frac{1}{\psi_\mp(x,y)+i0}:=\lim_{\theta\to0}\frac{1}{\psi_\mp(x,y)+i\theta}=\lim_{\theta\to0}\frac{1}{i}\int_0^{+\infty}e^{i(\psi_\mp(x,y)+i\theta)t}dt=\frac{1}{i}\int_0^{+\infty}e^{i\psi_\mp(x,y)t}dt. \tag{3.1.25}$$

From Theorem 2.6, we have some smooth function $f_\mp(x,y)$ with $f_\mp(x,x)\neq0$ such that

$$\varphi_\mp(x,y)-f_\mp(x,y)\psi_\mp(x,y)=O(|x-y|^{+\infty}). \tag{3.1.26}$$

For any point $x_0\in\Omega$, we can take a local coordinates x around x_0 such that the characteristic vector field $\boldsymbol{T}$ satisfies $\boldsymbol{T}=-\frac{\partial}{\partial x_{2n+1}}$. Identifying x_0 as $0\in\mathbb{R}^{2n+1}$, then (2.3.37) and (3.1.26) imply that

$$f((0,x_{2n+1}),0)=\Lambda(0)^{-1}+O(|x_{2n+1}|). \tag{3.1.27}$$

After writing (2.3.31) and (2.3.32) in the same form of (3.1.22), (3.1.23) and (3.1.24), we can compare $S_\mp\equiv S_{\psi_\mp}$ through the relations (3.1.26), (3.1.27) and

$$0\neq\psi((0,x_{2n+1}),0)=O(|x_{2n+1}|) \tag{3.1.28}$$

to find that

$$s_0^\mp((0,x_{2n+1}),0)=\Lambda(0)^{-n-1}s_0^{\psi_\mp}((0,x_{2n+1}),0)+O(|x_{2n+1}|). \tag{3.1.29}$$

Because $s_0^\mp$ are continuous, we can see that

$$s_0^\mp(0,0)=\Lambda(0)^{-n-1}s_0^{\psi_\mp}(0,0). \tag{3.1.30}$$

For the argument above works for all point in Ω, we have

$$s_0^{\psi_\mp}(x,x) = \Lambda(x)^{n+1} s_0^{\mp}(x,x). \tag{3.1.31}$$

□

The most important material in this section is the following variant of [28, Lemma 4.1].

THEOREM 3.4. *With the same notations and assumptions in Definition 3.1, we consider the certain operator* $H \in I_\Sigma^m(\Omega; T^{*0,q}X)$ *with the assumption that*

$$H \equiv (S_- + S_+) \circ H \equiv H \circ (S_- + S_+) \ \text{ on } \Omega, \tag{3.1.32}$$

$$\tau_x^{n_-} h_0^-(x,x) \tau_x^{n_-} = 0, \ \ \forall x \in \Omega, \tag{3.1.33}$$

and when $n_- = n_+$ *we additionally require*

$$\tau_x^{n_+} h_0^+(x,x) \tau_x^{n_+} = 0, \ \ \forall x \in \Omega. \tag{3.1.34}$$

Then if we write

$$h_0^\mp(x,y) = \sum_{|\mathbf{I}|=|\mathbf{J}|=q} h_{\mathbf{I},\mathbf{J}}^\mp(x,y) \omega_{\mathbf{I}}^\wedge(x) \otimes \omega_{\mathbf{J}}^{\wedge,*}(y) \tag{3.1.35}$$

in the strictly increasing index, cf. (1.1.23), *we have*

$$h_{\mathbf{I},\mathbf{J}}^\mp(x,y) - \rho_{\mathbf{I},\mathbf{J}}^\mp(x,y)\varphi_\mp(x,y) = O(|x-y|^{+\infty}) \tag{3.1.36}$$

for some $\rho_{\mathbf{I},\mathbf{J}}^\mp(x,y) \in \mathscr{C}^\infty(\Omega \times \Omega)$.

PROOF. For the stream of reading, we give a proof here following [28, Lemma 4.1]. The basic idea is that this theorem is independent of coordinates, and we can take a good coordinates $x = (x_1, \cdots, x_{2n+1})$, $y = (y_1, \cdots, y_{2n+1})$ such that by the properties (2.3.36) and (2.3.37) we can apply the Malgrange preparation theorem [40, Theorem 7.5.6] to write

$$h_0^\mp(x,y) = \rho_\mp(x,y)\varphi_\mp(x,y) + r_\mp(x,y') \tag{3.1.37}$$

for some smooth $r(x,y) = r(x,y')$, where $y' = (y_1, \cdots, y_{2n})$. However, to show $r_\mp(x,y')$ is a matrix with all entries vanishing to infinite order on the diagonal, we need to reduce our theorem to some special situation.

Let us first prove our theorem in such special situation. At each point $x_0 \in \Omega$ identified as $0 \in \mathbb{R}^{2n+1}$, we can always take a coordinates $x = (x_1, \cdots, x_{2n+1})$ and $y = (y_1, \cdots, y_{2n+1})$ near x_0 such that

$$\boldsymbol{T} = -\frac{\partial}{\partial x_{2n+1}}, \tag{3.1.38}$$

$$\boldsymbol{\alpha}(x) = dx_{2n+1} \text{ at } x_0, \tag{3.1.39}$$

$$\overline{\partial}_b = d\overline{z} \wedge \frac{\partial}{\partial \overline{z}} \text{ at } x_0. \tag{3.1.40}$$

We denote $x' = z = (x_1, \cdots, x_{2n})$ and $y' = w = (y_1, \cdots, y_{2n})$. For the same $H \in I^m_\Sigma(\Omega; T^{*0,q}X)$ in our theorem, we additionally assume that

$$H = H^* \text{ is formally self-adjoint,} \tag{3.1.41}$$

$$\varphi_\mp(x,y) = \mp x_{2n+1} \pm y_{2n+1} + g_\mp(x,y'), \tag{3.1.42}$$

$$h_0^\mp(x,y) = h_0^\mp(x,y'). \tag{3.1.43}$$

Our first goal is to show that in this case

$$h_0^\mp(x,y) = O(|(x,y)|^{+\infty}). \tag{3.1.44}$$

Now we fix $\Omega_0 \Subset \Omega$, where Ω_0 is an open set of $0 \in \mathbb{R}^{2n+1}$ and $0 \in \mathbb{R}^{2n+1}$ is identified as a point in Ω. We let $\tau \in \mathscr{C}_0^\infty(\mathbb{R}, [0,1])$, $\tau \equiv 1$ on $[-\frac{1}{2}, \frac{1}{2}]$. We put $\varepsilon > 0$ be a small constant so that $\varepsilon^{-1}(x_{2n+1} - y_{2n+1}) \notin \operatorname{supp}\tau$ for every $(x', x_{2n+1}) \in \Omega_0$ and every $(y', y_{2n+1}) \notin \Omega$. From our assumption we have

$$H \equiv (H_- + H_+) \circ (S_- + S_+). \tag{3.1.45}$$

So for any $g \in \mathscr{C}_0^\infty(\Omega_0, T^{*0,q}X)$, we have on Ω_0 that

$$\begin{aligned} Hg(x) = {} & H_- \circ S_- g(x) + H_+ \circ S_+ g(x) \\ & + H_- \circ S_+ g(x) + H_+ \circ S_- g(x) + Fg(x), \end{aligned} \tag{3.1.46}$$

where F is a smoothing operator on Ω_0,

$$\begin{aligned} H_\mp \circ S_\mp g(x) = {} & \int_0^{+\infty} \int_\Omega \int_0^{+\infty} \int_\Omega e^{i\gamma\varphi_\mp(x,w) + it\varphi_\mp(w,y)} \\ & h^\mp(x,w',\gamma) \circ s^\mp(w,y',t) g(y) dm(y) dt dm(w) d\gamma, \end{aligned} \tag{3.1.47}$$

and

$$\begin{aligned} H_\mp \circ S_\pm g(x) = {} & \int_0^{+\infty} \int_\Omega \int_0^{+\infty} \int_\Omega e^{i\gamma\varphi_\mp(x,w) + it\varphi_\pm(w,y)} \\ & h^\mp(x,w',\gamma) \circ s^\pm(w,y',t) g(y) dm(y) dt dm(w) d\gamma. \end{aligned} \tag{3.1.48}$$

We can take the change of variable $\gamma = t\sigma$ and switch the order of integration in the above oscillatory integrals and we have

$$\begin{aligned} (H_\mp \circ S_\mp)(x,y) = {} & \int_0^{+\infty} \int_0^{+\infty} \int_\Omega e^{it(\sigma\varphi_\mp(x,w) + \varphi_\mp(w,y))} \\ & h^\mp(x,w',\gamma) \circ s^\mp(w,y',t) t\, m(w) dw d\sigma dt, \end{aligned} \tag{3.1.49}$$

and we also have

$$\begin{aligned} (H_\mp \circ S_\pm)(x,y) = {} & \int_0^{+\infty} \int_0^{+\infty} \int_\Omega e^{it(\sigma\varphi_\mp(x,w) + \varphi_\mp(w,y))} \\ & h^\mp(x,w',\gamma) \circ s^\pm(w,y',t) t\, m(w) dw d\sigma dt. \end{aligned} \tag{3.1.50}$$

We notice that we can write

$$(H_{\mp}\circ S_{\pm})(x,y)=\mathbf{I}^{\mp}_{\varepsilon}(x,y)+\mathbf{II}^{\mp}_{\varepsilon}(x,y), \tag{3.1.51}$$

where

$$\mathbf{I}^{\mp}_{\varepsilon}(x,y)=\int_0^{+\infty}\int_0^{+\infty}\int_{\Omega}e^{it(\varphi_{\mp}(x,w)\sigma+\varphi_{\pm}(w,y))}\tau(\frac{x_{2n+1}-w_{2n+1}}{\varepsilon})h^{\mp}(x,w,\sigma t)\circ s^{\pm}(w,y,t)t\, m(w)dwd\sigma dt, \tag{3.1.52}$$

and

$$\mathbf{II}^{\mp}_{\varepsilon}(x,y)=\int_0^{+\infty}\int_0^{+\infty}\int_{\Omega}e^{it(\varphi_{\mp}(x,w)\sigma+\varphi_{\pm}(w,y))}\Big(1-\tau(\frac{x_{2n+1}-w_{2n+1}}{\varepsilon})\Big)\,h^{\mp}(x,w,\sigma t)\circ s^{\pm}(w,y,t)t\, m(w)dwd\sigma dt. \tag{3.1.53}$$

By (2.3.35), for $\mathbf{II}^{\mp}_{\varepsilon}$ we can integrate by parts with respect to σ and conclude that $\mathbf{II}^{\mp}_{\varepsilon}(x,y)$ is smooth. Moreover, since both $H_{\mp}(x,y)$ and $S_{\pm}(x,y)$ are smoothing away from the diagonal on Ω, along with the fact $\mathbf{II}^{\mp}_{\varepsilon}(x,y)$ is smooth we just see, we know that $\mathbf{I}^{\mp}_{\varepsilon}(x,y)$ is also smoothing away from the diagonal on Ω and we may assume that $|x-y|<\varepsilon$. From (2.3.37), we can see that

$$d_w(\varphi_{\mp}(x,w)\sigma+\varphi_{\pm}(w,y))|_{w=x=y}=(\sigma+1)\boldsymbol{\alpha}_x \tag{3.1.54}$$

is non-vanishing. So when $\varepsilon>0$ is suitably small, in $\mathbf{I}^{\mp}_{\varepsilon}$ we can integrate by parts with respect to w and conclude that $\mathbf{I}^{\mp}_{\varepsilon}(x,y)$ is smooth. In conclusion, we have

$$H_{\mp}\circ S_{\pm}\equiv 0 \text{ on } \Omega. \tag{3.1.55}$$

Since when $w=y=x$ and $\sigma=1$ we have

$$d_w(\sigma\varphi_{\mp}(x,w)+\varphi_{\mp}(w,y))=d_\sigma(\sigma\varphi_{\mp}(x,w)+\varphi_{\mp}(w,y)))=0, \tag{3.1.56}$$

using the integration by parts argument we just use before with some minor change, we can also write

$$(H_{\mp}\circ S_{\mp})(x,y)=\int_0^{+\infty}\int_0^{+\infty}\int_{\Omega}e^{it(\sigma\varphi_{\mp}(x,w)+\varphi_{\mp}(w,y))}\tau(\frac{1-\sigma}{\varepsilon})\tau(\frac{x_{2n+1}-w_{2n+1}}{\varepsilon})h^{\mp}(x,w',\gamma)\circ s^{\mp}(w,y',t)t\, m(w)dwd\sigma dt \tag{3.1.57}$$

Our next step is to apply Melin–Sjöstrand complex stationary phase formula Theorem 2.2 to (3.1.57) to study the leading term $h_0^{\mp}$. We notice that

$$\begin{aligned} & e^{it(\sigma\varphi_{\mp}(x,w)+\varphi_{\mp}(w,y))} \\ =& e^{it(\sigma\varphi_{\mp}(x,w)+(\mp w_{2n+1}\pm y_{2n+1}+g_{\mp}(w,y')))} \\ =& e^{\pm ity_{2n+1}}e^{it\Phi_{\mp}(w,\sigma;x,y')}, \end{aligned} \tag{3.1.58}$$

where

$$\Phi_{\mp}(w,\sigma;x,y') := \sigma\varphi_{\mp}(x,w) \mp w_{2n+1} + g_{\mp}(w,y'). \tag{3.1.59}$$

We will use the following notations. For an open set $W \subset \mathbb{R}^m$ and for every $f \in \mathscr{C}^\infty(W)$ we write $\widetilde{f} \in \mathscr{C}^\infty(W^{\mathbb{C}})$ to denote any almost analytic extension [22, pp. 93-94] of f, where $W^{\mathbb{C}}$ is an open set of $\mathbb{C}^m$ with $W^{\mathbb{C}} \bigcap \mathbb{R}^m = W$. Now we take $\widetilde{\Phi_{\mp}}(\widetilde{w},\widetilde{\sigma};x,y')$ so that

$$\widetilde{\Phi_{\mp}}(\widetilde{w},\widetilde{\sigma};\widetilde{x},\widetilde{y}') = \widetilde{\sigma}\widetilde{\varphi_{\mp}}(\widetilde{x},\widetilde{w}) \mp \widetilde{w_{2n+1}} + \widetilde{g_{\mp}}(\widetilde{w},\widetilde{y}'). \tag{3.1.60}$$

We denote $U := \{y \in \mathbb{R}^{2n} : \exists\, y_{2n+1} \in \mathbb{R} \text{ such that } (y', y_{2n+1}) \in \Omega\}$, and we have the critical points

$$\beta^{\mp}(\widetilde{x},\widetilde{y}') = (\beta_1^{\mp}(\widetilde{x},\widetilde{y}'),\cdots,\beta_{2n+1}^{\mp}(\widetilde{x},\widetilde{y}')) \in \mathscr{C}^\infty(\Omega^{\mathbb{C}} \times U^{\mathbb{C}}, \mathbb{C}^{2n+1}), \tag{3.1.61}$$

$$\gamma^{\mp}(\widetilde{x},\widetilde{y}') \in \mathscr{C}^\infty(\Omega^{\mathbb{C}} \times U^{\mathbb{C}}, \mathbb{C}), \tag{3.1.62}$$

which is the solution by implicit function theorem for the system of equations

$$\begin{aligned} & \frac{\partial\widetilde{\Phi_{\mp}}}{\partial\widetilde{w}_j}(\beta^{\mp}(\widetilde{x},\widetilde{y}'),\gamma^{\mp}(\widetilde{x},\widetilde{y}');\widetilde{x},\widetilde{y}') \\ & \qquad = \gamma^{\mp}(\widetilde{x},\widetilde{y}')\frac{\partial\widetilde{\varphi}_{\mp}}{\partial\widetilde{w}_j}(\widetilde{x},\beta^{\mp}(\widetilde{x},\widetilde{y}')) + \frac{\partial\widetilde{g}_{\mp}}{\partial\widetilde{w}_j}(\beta^{\mp}(\widetilde{x},\widetilde{y}'),\widetilde{y}') = 0, \end{aligned} \tag{3.1.63}$$

where, $j = 1,\cdots,2n+1$, and

$$\frac{\partial\widetilde{\Phi_{\mp}}}{\partial\widetilde{\sigma}}(\beta^{\mp}(\widetilde{x},\widetilde{y}'),\gamma^{\mp}(\widetilde{x},\widetilde{y}');\widetilde{x},\widetilde{y}') = \widetilde{\varphi}_{\mp}(\widetilde{x},\beta^{\mp}(\widetilde{x},\widetilde{y}')) = 0. \tag{3.1.64}$$

Now, by applying complex stationary phase formula of Melin–Sjöstrand Theorem 2.2 and Theorem 2.5 to

$$\begin{aligned} & \int_0^{+\infty}\int_\Omega e^{it\Phi_{\mp}(w,\sigma;x,y')} \\ & \qquad \tau(\frac{1-\sigma}{\varepsilon})\tau(\frac{x_{2n+1}-w_{2n+1}}{\varepsilon})h^{\mp}(x,w',\gamma)\circ s^{\mp}(w,y',t)t\, m(w)dwd\sigma, \end{aligned} \tag{3.1.65}$$

along with (3.1.55) and (3.1.57) we get

$$H(x,y) \equiv \int_0^{+\infty} e^{it\varphi_1^-(x,y)}f^-(x,y,t)dt + \int_0^{+\infty} e^{it\varphi_1^+(x,y)}f^+(x,y,t)dt \tag{3.1.66}$$

on Ω_0, where

(3.1.67)
$$\varphi_1^{\mp}(x,y)=\widetilde{\varphi_1^{\mp}}(\beta^{\mp}(x,y'),y)=\mp\beta_{2n+1}^{\mp}(x,y')\pm y_{2n+1}+\widetilde{g_{\mp}}(\beta(x,y'),y'),$$

and

(3.1.68)
$$f^{\mp}(x,y,t)\sim\sum_{j=0}^{\infty}f_j^{\mp}(x,y)t^{k-j}\text{ in }S^m_{1,0}(\Omega_0\times\Omega_0\times\mathbb{R}_+,\mathscr{L}(T^{*0,q}X,T^{*0,q}X)),$$

(3.1.69)
$$f_j^{\mp}(x,y)\in\mathscr{C}^{\infty}(\Omega_0\times\Omega_0,\mathscr{L}(T^{*0,q}X,T^{*0,q}X)),\ j\in\mathbb{N}_0.$$

We also have

$$f_0^{\mp}(x,y)=f_0^{\mp}(x,y')=c(x,y')\widetilde{h_0^{\mp}}(x,\beta^{\mp}(x,y'))\circ\widetilde{s_0^{\mp}}(\beta^{\mp}(x,y'),y'),\tag{3.1.70}$$

where $c(x,y')\in\mathscr{C}^{\infty}(\Omega_0\times\Omega_0,\mathbb{C})$ is the term of determinant of tangential Hessian of $t\widetilde{\Phi}_{\mp}(\widetilde{w},\widetilde{\sigma};\widetilde{x},\widetilde{y}')$ at $(\widetilde{w},\widetilde{\sigma})=(\beta^{\mp}(x,y'),\gamma^{\mp}(x,y'))$ and is nowhere vanishing for every $(x,y)\in\Omega_0\times\Omega_0$.

We notice that by treating $S^{\mp}\circ S^{\mp}$ with the same order of integration (as oscillatory integrals) as we just calculated, again by Melin–Sjöstrand stationary phase formula Theorem 2.2, the Szegő phase function of the Fourier integral operator $S^{\mp}\circ S^{\mp}$ in this manner is also $\varphi_1^{\mp}$. Thus, from (3.1.42), (3.1.67), $S^{\mp}\circ S^{\mp}\equiv S^{\mp}$ and Theorem 2.6, we can check that

$$\varphi_{\mp}(x,y)-\varphi_1^{\mp}(x,y)=O(|x-y|^{+\infty}).\tag{3.1.71}$$

Hence, we can replace $\varphi_1^{\mp}$ by $\varphi_{\mp}$ and we have

(3.1.72)
$$\int_0^{+\infty}e^{it\varphi_-(x,y)}(h^--f^-)(x,y,t)dt+\int_0^{+\infty}e^{it\varphi_+(x,y)}(h^+-f^+)(x,y,t)dt\equiv 0$$

on Ω_0. From the proof of Theorem 2.6 and induction, we can check that

$$h_0^{\mp}(x,y)-f_0^{\mp}(x,y)=O(|x-y|^{+\infty}).\tag{3.1.73}$$

From this observation and (3.1.70), we get

$$h_0^{\mp}(x,y)-c(x,y)\widetilde{h_0^{\mp}}(x,\beta(x,y))\circ\widetilde{s_0^{\mp}}(\beta(x,y),y)=O(|x-y|^{+\infty}).\tag{3.1.74}$$

From Theorem 2.5, Theorem 2.11 and (3.1.74), we can deduce

$$h_0^{\mp}(x,y)(I-\tau_y^{n_{\mp}})=O(|x-y|).\tag{3.1.75}$$

Similarly, by the assumption that

$$(H_-+H_+)\circ(S_-+S_+)\equiv H,\tag{3.1.76}$$

we can repeat the procedure above and deduce that

$$(I-\tau_x^{n_{\mp}})h_0^{\mp}(x,y)=O(|x-y|).\tag{3.1.77}$$

From (3.1.75), (3.1.77) and the assumption that

$$\tau_x^{n_-} h_0^-(x,x)\tau_x^{n_-} = 0, \tag{3.1.78}$$

$$\tau_x^{n_+} h_0^+(x,x)\tau_x^{n_+} = 0 \text{ when } n_- = n_+, \tag{3.1.79}$$

we can conclude that

$$h_0^{\mp}(x,y) = O(|x-y|). \tag{3.1.80}$$

We now show we can use induction to prove our theorem in our special situation. We recall that now for the given point $(x_0,y_0) \in \Omega \times \Omega$ we identify it as $(0,0) \in \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1}$. We assume that

$$h_0^{\mp}(x,y) = O(|(x,y)|^{N_0}) \tag{3.1.81}$$

for some $N_0 \in \mathbb{N}$. To check the our goal at (N_0+1)-order, we notice that from the assumption

$$H \equiv (S_- + S_+) \circ H \equiv H \circ (S_- + S_+), \tag{3.1.82}$$

and from the previous argument, we already have

$$h_0^{\mp}(x,y)(I - \tau_y^{n_{\mp}}) = O(|(x,y)|^{N_0+1}), \tag{3.1.83}$$

$$(I - \tau_x^{n_{\mp}})h_0^{\mp}(x,y) = O(|(x,y)|^{N_0+1}). \tag{3.1.84}$$

So we only need to check whether

$$\tau_x^{n_{\mp}} h_0^{\mp}(x,y)\tau_y^{n_{\mp}} = O(|(x,y)|^{N_0+1}). \tag{3.1.85}$$

Following the convention in our theorem, we recall that we suppose the negative eigenvalues $\mu_j < 0$ of Levi form are from $j = 1, \cdots, n_-$ and the positive eigenvalues $\mu_j > 0$ are from $j = n_- + 1, \cdots, n$. We write

$$h_0^{\mp}(x,y) = \sum_{|\mathbf{I}|=|\mathbf{J}|=q} h_{\mathbf{I},\mathbf{J}}^{\mp}(x,y)\omega_{\mathbf{I}}(x)^{\wedge} \otimes \omega_{\mathbf{J}}^{\wedge,*}(y) \tag{3.1.86}$$

in the strictly increasing index and $I_0 = \{1, \cdots, q\}$, $J_0 = \{q+1, \cdots, n\}$, $q = n_-$. To prove (3.1.85), we only need to prove that

$$h_{I_0,I_0}^{-}(x,y) = O(|(x,y)|^{N_0+1}), \tag{3.1.87}$$

$$h_{J_0,J_0}^{+}(x,y) = O(|(x,y)|^{N_0+1}). \tag{3.1.88}$$

From (2.3.26), we have

$$\Box_b^{(q+1)} \circ \overline{\partial}_b \circ (S_- + S_+) = \overline{\partial}_b \circ \Box_b^{(q)} \circ (S_- + S_+) \equiv 0, \tag{3.1.89}$$

and moreover along with (2.3.24) we can see that

$$\overline{\partial}_b \circ (S_- + S_+) \equiv \mathbf{G} \circ \Box_b^{(q+1)} \circ \overline{\partial}_b \circ (S_- + S_+) \equiv 0. \tag{3.1.90}$$

This implies that

$$\int_0^{+\infty} e^{it\varphi_-(x,y)}(\bar\partial_{b,x}\varphi_-)s_0^-(x,y)t^{n+1}dt + \int_0^{+\infty} e^{it\varphi_-(x,y)}s_1^-(x,y,t)dt + \int_0^{+\infty} e^{it\varphi_+(x,y)}(\bar\partial_{b,x}\varphi_+)s_0^+(x,y)t^{n+1}dt + \int_0^{+\infty} e^{it\varphi(x,y)}s_1^+(x,y,t)dt \equiv 0, \tag{3.1.91}$$

where $s_1^{\mp}(x,y,t) \in S^n_{\mathrm{cl}}(\Omega\times\Omega\times\mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X))$. We notice that from (3.1.42) we can see that $\bar\partial_{b,x}\varphi_{\mp}(x,y)$ are independent of y_{2n+1}. Then we can apply the partial (inverse) Fourier transform argument in the proof of Theorem 2.6 and find some symbols $\mathbb{S}_0(x,y)$ and $\mathbb{S}_1(x,y',k)$ and an operator $F_1^{\mp}$ such that

$$k^{n+1}(\bar\partial_{b,x}\varphi_-)(x,y')\mathbb{S}_0^-(x,y') + \mathbb{S}_1^-(x,y',k) = e^{-ikg_-(x,y')}F_k^-(x,y'), \tag{3.1.92}$$

$$k^{n+1}(\bar\partial_{b,x}\varphi_+)(x,y')\mathbb{S}_0^+(x,y') + \mathbb{S}_1^+(x,y',k) = e^{-ikg_+(x,y')}F_k^+(x,y'), \tag{3.1.93}$$

where

$$\mathbb{S}_0^{\mp}(x,y')|_{y=x} = s_0^{\mp}(x,x) \neq 0, \tag{3.1.94}$$

$$\mathbb{S}_1^{\mp}(x,y',k) \in S^n_{\mathrm{loc,cl}}(\Omega\times\Omega, \mathscr{L}(T^{*0,q}X, T^{*0,q}X)), \tag{3.1.95}$$

$$F_k^{\mp} = O\left(k^{-\infty}\right). \tag{3.1.96}$$

We recall that $\varphi_{\mp}(x,x) = 0$, which implies that here $g(x,y')|_{y=x} = 0$. Then by multiplying k^{-n-1} both sides in (3.1.92) and (3.1.93) it is clear that

$$(\bar\partial_{b,x}\varphi_{\mp})(x,y')\mathbb{S}_0^{\mp}(x,y') = O(|x-y|^{+\infty}), \tag{3.1.97}$$

and we can apply $\mathbb{S}_0^{\mp}(x,y')|_{y=x} \neq 0$, Leibniz rule and induction to show that

$$(\bar\partial_{b,x}\varphi_{\mp})(x,y') = O(|x-y|^{+\infty}). \tag{3.1.98}$$

Also, by the assumption (3.1.32) of H we can check that

$$\bar\partial_b \circ H \equiv \bar\partial_b \circ (S_- + S_+) \circ H \equiv 0. \tag{3.1.99}$$

This implies that

$$\int_0^{+\infty} e^{it\varphi_-(x,y)}(\bar\partial_{b,x}\varphi_-)h^-(x,y,t)dt + \int_0^{+\infty} e^{it\varphi_-(x,y)}\bar\partial_{b,x}h_0^-(x,y,t)dt + \int_0^{+\infty} e^{it\varphi_-(x,y)}(\bar\partial_{b,x}\varphi_+)h^+(x,y,t)dt + \int_0^{+\infty} e^{it\varphi_+(x,y)}\bar\partial_{b,x}h_0^+(x,y,t)dt \equiv 0, \tag{3.1.100}$$

and by (3.1.98) we can rewrite the above relation by

$$\int_0^{+\infty} e^{it\varphi_-(x,y)}h_0^-(x,y)t^n dt + \int_0^{+\infty} e^{it\varphi_-(x,y)}h_1^-(x,y,t)dt + \int_0^{+\infty} e^{it\varphi_+(x,y)}h_0^+(x,y)t^n dt + \int_0^{+\infty} e^{it\varphi_+(x,y)}h_1^+(x,y,t)dt \equiv 0, \tag{3.1.101}$$

where $h_1^{\mp}(x,y,t) \in S^{n-1}_{\mathrm{cl}}(\Omega\times\Omega\times\mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X))$. By the above relation, the assumption here that $h_0^{\mp}(x,y) = h_0^{\mp}(x,y')$ and the same partial (inverse) Fourier transform argument in the proof of Theorem 2.6, we can then check that

$$\overline{\partial}_{b,x} h_0^{\mp}(x,y) = O(|x-y|^{+\infty}). \tag{3.1.102}$$

Then, by combining (3.1.84), (3.1.83) and (3.1.102), we can check that

$$\overline{\partial}_{b,x}\tau_x^{n_\mp} h_0^{\mp}(x,y)\tau_y^{n_\mp} = O(|(x,y)|^{N_0}), \tag{3.1.103}$$

which is equivalent to

$$\frac{\partial}{\partial \overline{z}_j} h^-_{I_0,I_0}(x,y) = O(|(x,y)|^{N_0}),\ \ j=q+1,\cdots,n, \tag{3.1.104}$$

$$\frac{\partial}{\partial \overline{z}_j} h^+_{J_0,J_0}(x,y) = O(|(x,y)|^{N_0}),\ \ j=1,\cdots,q. \tag{3.1.105}$$

Similarly, from the argument before we can also check that

$$\overline{\partial}_b^* \circ (S_- + S_+) \equiv 0, \tag{3.1.106}$$

$$\overline{\partial}_b^* \circ H \equiv \overline{\partial}_b^*(S_- + S_+)\circ H \equiv 0, \tag{3.1.107}$$

and we can repeat the same procedure before with some minor change to deduce that

$$\overline{\partial}_{b,x}^*\tau_x^{n_\mp} h_0^{\mp}(x,y)\tau_y^{n_\mp} = O(|(x,y)|^{N_0}), \tag{3.1.108}$$

which is equivalent to

$$\frac{\partial}{\partial z_j} h^-_{I_0,I_0}(x,y) = O(|(x,y)|^{N_0}),\ \ j=1,\cdots,q, \tag{3.1.109}$$

$$\frac{\partial}{\partial z_j} h^+_{J_0,J_0}(x,y) = O(|(x,y)|^{N_0}),\ \ j=q+1,\cdots,n. \tag{3.1.110}$$

From our temporary assumption $H = H^*$, we have

$$\begin{aligned}\int_0^{+\infty} e^{it\varphi_-(x,y)}h^-(x,y,t)dt + \int_0^{+\infty} e^{it\varphi_+(x,y)}h^+(x,y,t)dt\\ \equiv \int_0^{+\infty} e^{it\varphi_-^*(x,y)}h^{-,*}(x,y,t)dt + \int_0^{+\infty} e^{it\varphi_+^*(x,y)}h^{+,*}(x,y,t)dt.\end{aligned} \tag{3.1.111}$$

We recall that $\varphi_\mp(x,y) \in \mathrm{Ph}(\mp\alpha,\Omega)$ and $S_\mp \equiv S_\mp^*$. Accordingly, from Theorem 2.6 we have

$$\varphi_\mp^*(x,y) - f_\mp(x,y)\varphi_\mp(x,y) = O(|x-y|^{+\infty}), \tag{3.1.112}$$

where $f_{\mp}(x,x) \neq 0$. From the above relation and (3.1.20), we can write the oscillatory integral

$$\int_0^{+\infty} e^{it\varphi^*_{\mp}(x,y)} h_0^{\mp,*}(x,y) t^m dt \equiv \int_0^{+\infty} e^{it\varphi_{\mp}(x,y)} \frac{h_0^{\mp,*}(x,y)}{f_{\mp}^m(x,y)} t^m dt. \tag{3.1.113}$$

We denote

$$F_{\mp}(x,y) := \frac{h_0^{\mp,*}(x,y)}{f_{\mp}^m(x,y)}, \tag{3.1.114}$$

$$F_{\mp}(x,y') := \widetilde{F}_{\mp}(x,y',x_{2n+1} \mp g_{\mp}(x,y')). \tag{3.1.115}$$

By almost analytic extension, we have $F_{\mp}(x,y) = \widetilde{F}_{\mp}(x,y',y_{2n+1})$. Also, by Taylor formula, we have

$$\begin{aligned} &\widetilde{F}_{\mp}(x,y',\widetilde{y}_{2n+1}) \\ &= \widetilde{F}_{\mp}(x,y',x_{2n+1} \mp g_{\mp}(x,y')) + (-x_{2n+1} + \widetilde{y}_{2n+1} \pm g_{\mp}(x,y'))r_{\mp}(x,y). \end{aligned} \tag{3.1.116}$$

By the above relation and along with our special choice of $\varphi_{\mp}$ here, we can write

$$\begin{aligned} &\int_0^{+\infty} e^{it\varphi_{\mp}(x,y)} F_{\mp}(x,y) t^m dt \\ &= \int_0^{+\infty} e^{it\varphi_{\mp}(x,y)} F_{\mp}(x,y') t^m dt + \int_0^{+\infty} e^{it\varphi_{\mp}(x,y)} \varphi_{\mp}(x,y) r(x,y) t^m dt \\ &= \int_0^{+\infty} e^{it\varphi_{\mp}(x,y)} F_{\mp}(x,y') t^m dt + \int_0^{+\infty} e^{it\varphi_{\mp}(x,y)} i r_{\mp}(x,y) (\frac{d}{dt} t^m) dt, \end{aligned} \tag{3.1.117}$$

where we apply integration by parts of oscillatory integrals. From the proof of Theorem 2.6, we can see that

$$F_{\mp}(x,y') - h_0(x,y') = O(|x-y|^{+\infty}), \tag{3.1.118}$$

which implies that

$$\widetilde{h}_0^{\mp,*}(x,y',x_{2n+1} \mp g(x,y')) - \widetilde{f}_{\mp}^m(x,y',x_{2n+1} \mp g(x,y')) h_0^{\mp}(x,y') = O(|x-y|^{+\infty}). \tag{3.1.119}$$

We recall that by Schwarz kernel theorem we have $H^*(x,y) = \overline{H}(y,x)$, so we can take $h_0^{*,\mp}(x,y) = \overline{h_0^{\mp}}(y,x)$. We also recall that here $\boldsymbol{\alpha}(0) = dx_{2n+1}$ and $d_x\varphi_{\mp}(x,x) = d_y\varphi_{\mp}(x,x) = \mp\boldsymbol{\alpha}(x,x)$. So from (3.1.104), (3.1.105), (3.1.109), (3.1.110), (3.1.119) and induction hypothesis, we can check that

$$\frac{\partial}{\partial w_j}h^-_{I_0,I_0}(x,y) = O(|(x,y)|^{N_0}),\ \ j = q+1,\cdots,n, \tag{3.1.120}$$

$$\frac{\partial}{\partial w_j}h^+_{J_0,J_0}(x,y) = O(|(x,y)|^{N_0}),\ \ j = 1,\cdots,q, \tag{3.1.121}$$

$$\frac{\partial}{\partial \overline{w}_j}h^-_{I_0,I_0}(x,y) = O(|(x,y)|^{N_0}),\ \ j = 1,\cdots,q, \tag{3.1.122}$$

$$\frac{\partial}{\partial \overline{w}_j}h^+_{J_0,J_0}(x,y) = O(|(x,y)|^{N_0}),\ \ j = q+1,\cdots,n. \tag{3.1.123}$$

For simplicity, in the later discussion we will only demonstrate the case for $h^-_{I_0,I_0}$, and the case for $h^+_{J_0,J_0}$ can be calculated similarly. From our assumption that $h^-_{I_0,I_0}(x,x) = 0$ and Taylor formula, for a fixed $j \in \{q+1,\cdots,n\}$ and fixed $\alpha,\beta \in \mathbb{N}_0$ such that $\alpha+\beta = N_0$, we have

$$\left(\left((\frac{\partial}{\partial z_j}+\frac{\partial}{\partial w_j})^{\alpha}(\frac{\partial}{\partial \overline{z}_j}+\frac{\partial}{\partial \overline{w}_j})^{\beta}\right)h^-_{I_0,I_0}\right)(0,0) = 0. \tag{3.1.124}$$

This relation implies that

$$\left((\frac{\partial}{\partial z_j})^{\alpha}(\frac{\partial}{\partial \overline{w}_j})^{\beta}h^-_{I_0,I_0}\right)(0,0) = \sum_{\substack{\alpha_1,\alpha_2,\beta_1,\beta_2\in\mathbb{N}_0\\ \alpha_1+\alpha_2=\alpha\\ \beta_1+\beta_2=\beta\\ \alpha_2+\beta_1>0}} c_{\alpha_1,\alpha_2,\beta_1,\beta_2} \times\left((\frac{\partial}{\partial z_j})^{\alpha_1}(\frac{\partial}{\partial w_j})^{\alpha_2}(\frac{\partial}{\partial \overline{z}_j})^{\beta_1}(\frac{\partial}{\partial \overline{w}_j})^{\beta_2}h^-_{I_0,I_0}\right)(0,0), \tag{3.1.125}$$

where each $c_{\alpha_1,\alpha_2,\beta_1,\beta_2}$ is a constant. When $\alpha_2+\beta_1 > 0$, from (3.1.104) and (3.1.122) we get

$$\left((\frac{\partial}{\partial z_j})^{\alpha_1}(\frac{\partial}{\partial w_j})^{\alpha_2}(\frac{\partial}{\partial \overline{z}_j})^{\beta_1}(\frac{\partial}{\partial \overline{w}_j})^{\beta_2}h^-_{I_0,I_0}\right)(0,0) = 0 \tag{3.1.126}$$

for every $\alpha_1,\alpha_2,\beta_1,\beta_2 \in \mathbb{N}_0$, $\alpha_1+\alpha_2 = \alpha$, $\beta_1+\beta_2 = \beta$. From this observation and (3.1.125), we get

$$\left((\frac{\partial}{\partial z_j})^{\alpha}(\frac{\partial}{\partial \overline{w}_j})^{\beta}h^-_{I_0,I_0}\right)(0,0) = 0 \text{ for every } \alpha,\ \beta\in\mathbb{N}_0,\ \alpha+\beta = N_0. \tag{3.1.127}$$

Similarly, for a fixed $j \in \{1, \cdots, q\}$, we can repeat the procedure above and deduce that

$$\left((\frac{\partial}{\partial \overline{z}_j})^{\alpha} (\frac{\partial}{\partial w_j})^{\beta} h^{-}_{I_0,I_0} \right)(0,0) = 0 \text{ for every } \alpha,\ \beta \in \mathbb{N}_0,\ \alpha + \beta = N_0. \tag{3.1.128}$$

Also, again by $h_{I_0,I_0}(x,x) = 0$, we have

$$\left(\frac{\partial}{\partial x_{2n+1}} + \frac{\partial}{\partial y_{2n+1}} \right)^{N} h^{-}_{I_0,I_0}(0,0) = 0 \text{ for every } N \in \mathbb{N} \text{ with } |N| \leq N_0. \tag{3.1.129}$$

Since here the special $h_{I_0,I_0}(x,y) = h_{I_0,I_0}(x,y')$ is independent of y_{2n+1}, we have

$$\frac{\partial}{\partial y_{2n+1}} h^{-}_{I_0,I_0}(x,y) = O(|(x,y)|^{+\infty}) \tag{3.1.130}$$

and with the above observation we can deduce that

$$\left(\frac{\partial^N}{\partial x^N_{2n+1}} h^{-}_{I_0,I_0} \right)(0,0) = 0 \text{ for every } N \in \mathbb{N} \text{ with } |N| \leq N_0. \tag{3.1.131}$$

From this relation, we can repeat the argument above with minor change and deduce that for $j \in \{q+1, \cdots, n\}$, $\ell \in \{1, \cdots, q\}$, we have

$$\left((\frac{\partial}{\partial z_j})^{\alpha_0} (\frac{\partial}{\partial \overline{w}_j})^{\beta_0} (\frac{\partial}{\partial \overline{z}_\ell})^{\alpha} (\frac{\partial}{\partial w_\ell})^{\beta} (\frac{\partial}{\partial x_{2n+1}})^{\gamma} h^{-}_{I_0,I_0} \right)(0,0) = 0, \tag{3.1.132}$$

for every $\alpha_0, \beta_0, \alpha, \beta \in \mathbb{N}^n_0$, $\gamma \in \mathbb{N}_0$, $|\alpha_0| + |\beta_0| + |\alpha| + |\beta| + |\gamma| = N_0$. Combining all the above argument, we prove by induction that in this special case $h^{-}_{I_0,I_0}(x,y)$ vanishes to infinite order at (p,p). We can show so does $h^{+}_{J_0,J_0}(x,y)$ by the similar method.

For the purpose of reducing general case of our theorem to the special case we just demonstrate, we consider H^* be the formal adjoint with respect to the given L^2-inner product on $\Omega^{0,q}(X)$. As before, by Schwartz kernel theorem we can write

$$H^*(x,y) \equiv \int_0^{+\infty} e^{it\varphi^*_-(x,y)} h^{-,*}(x,y,t)dt + \int_0^{+\infty} e^{it\varphi^*_+(x,y)} h^{+,*}(x,y,t)dt, \tag{3.1.133}$$

$$\varphi^*_\mp(x,y) = -\overline{\varphi^*_\mp}(y,x), \tag{3.1.134}$$

$$h^{\mp,*}(x,y,t) = \overline{h^{\mp,*}}(y,x,t). \tag{3.1.135}$$

Under the same notations and coordinates before, again by the relation

$$d_x \varphi_\mp(x,x) = -d_y \varphi_\mp(x,x) = \mp \boldsymbol{\alpha}(x), \tag{3.1.136}$$

and Malgrange preparation theorem [40, Theorem 7.5.5], after some arrangement we can write

$$\varphi_{\mp}(x,y) = f_{1,\mp}(x,y)(\mp x_{2n+1} \pm y_{2n+1} + g_{1,\mp}(x,y')),\ \mathrm{Im}\, g_{1,\mp} \geq 0, \tag{3.1.137}$$

$$-\overline{\varphi_{\mp}}(y,x) = f_{2,\mp}(x,y)(\mp x_{2n+1} \pm y_{2n+1} + g_{2,\mp}(x,y')),\ \mathrm{Im}\, g_{2,\mp} \geq 0, \tag{3.1.138}$$

where $(f_{1,\mp}f_{2,\mp})(x,x) \neq 0$. By (2.3.36) and (2.3.37) we can check that the complex-valued phase functions $\varphi_{\mp}(x,y)t$ and $-\overline{\varphi_{\mp}}(y,x)t$ generate the same equivalent canonical relations as almost analytic manifolds [72, Definition 4.1 & Theorem 4.2], so we have the Melin–Sjöstrand's equivalent of phase functions

$$\varphi_{\mp}(x,y)t \sim \left(\mp x_{2n+1} \pm y_{2n+1} + g_{1,\mp}(x,y')\right) t, \tag{3.1.139}$$

$$-\overline{\varphi_{\mp}}(y,x)t \sim \left(\mp x_{2n+1} \pm y_{2n+1} + g_{2,\mp}(x,y')\right) t. \tag{3.1.140}$$

By the proof of Theorem 2.6 we can check that

$$g_{1,\mp}(x,y') = g_{2,\mp}(x,y') + O\left(|x-y|^{+\infty}\right). \tag{3.1.141}$$

For simplicity, from now on we denote

$$\phi_{\mp}(x,y) = \mp x_{2n+1} \pm y_{2n+1} + g_{\mp}(x,y'), \tag{3.1.142}$$

$$g_{\mp}(x,y) := g_{1,\mp}(x,y), \tag{3.1.143}$$

and we already have

$$\varphi_{\mp}(x,y) = f_{1,\mp}(x,y)\phi_{\mp}(x,y). \tag{3.1.144}$$

We also notice that (3.1.141) and Remark 2.7 imply we may write

$$\varphi^*_{\mp}(x,y) = f_{2,\mp}(x,y)\phi_{\mp}(x,y). \tag{3.1.145}$$

Now, for

$$\begin{aligned} &(H+H^*)(x,y) \\ &\equiv \int_0^{+\infty} e^{it\varphi_-(x,y)}h^-(x,y,t)dt + \int_0^{+\infty} e^{it\varphi^*_-(x,y)}h^{-,*}(x,y,t)dt \\ &+ \int_0^{+\infty} e^{it\varphi_+(x,y)}h^+(x,y,t)dt + \int_0^{+\infty} e^{it\varphi^*_+(x,y)}h^{+,*}(x,y,t)dt, \end{aligned} \tag{3.1.146}$$

we can apply the trick of (3.1.113), (3.1.116) and integration by parts of oscillatory integrals to check, up to some Fourier integral operators of Szegő type of weight $n-1$, we have

$$
\begin{aligned}
&(H+H^*)(x,y)\\
&\equiv \int_0^{+\infty} e^{it\phi_-(x,y)}t^n\\
&\times\left(\frac{\widetilde{h}_0^-\,(x,y',x_{2n+1}-g_-(x,y'))}{\widetilde{f}_{1,-}^{n+1}\,(x,y',x_{2n+1}-g_-(x,y'))}+\frac{\overline{\widetilde{\widetilde{h}_0^-}}(y',x_{2n+1}-g_-(x,y'),x)}{\widetilde{f}_{2,-}^{n+1}\,(x,y',x_{2n+1}-g_-(x,y'))}\right)dt\\
&+\int_0^{+\infty} e^{it\phi_+(x,y)}t^n\\
&\times\left(\frac{\widetilde{h}_0^+\,(x,y',x_{2n+1}+g_+(x,y'))}{\widetilde{f}_{1,+}^{n+1}\,(x,y',x_{2n+1}+g_+(x,y'))}+\frac{\overline{\widetilde{\widetilde{h}_0^+}}(y',x_{2n+1}+g_+(x,y'),x)}{\widetilde{f}_{2,+}^{n+1}\,(x,y',x_{2n+1}+g_+(x,y'))}\right)dt.
\end{aligned}
\tag{3.1.147}
$$

Similarly, up to some Fourier integral operators of Szegő type of weight $n-1$, we also have

$$
\begin{aligned}
&(iH-iH^*)(x,y)\\
&\equiv \int_0^{+\infty} e^{it\phi_-(x,y)}t^n\\
&\times\left(i\frac{\widetilde{h}_0^-\,(x,y',x_{2n+1}-g_-(x,y'))}{\widetilde{f}_{1,-}^{n+1}\,(x,y',x_{2n+1}-g_-(x,y'))}-i\frac{\overline{\widetilde{\widetilde{h}_0^-}}(y',x_{2n+1}-g_-(x,y'),x)}{\widetilde{f}_{2,-}^{n+1}\,(x,y',x_{2n+1}-g_-(x,y'))}\right)dt\\
&+\int_0^{+\infty} e^{it\phi_+(x,y)}t^n\\
&\times\left(i\frac{\widetilde{h}_0^+\,(x,y',x_{2n+1}+g_+(x,y'))}{\widetilde{f}_{1,+}^{n+1}\,(x,y',x_{2n+1}+g_+(x,y'))}-i\frac{\overline{\widetilde{\widetilde{h}_0^+}}(y',x_{2n+1}+g_+(x,y'),x)}{\widetilde{f}_{2,+}^{n+1}\,(x,y',x_{2n+1}+g_+(x,y'))}\right)dt.
\end{aligned}
\tag{3.1.148}
$$

We also notice that both $H+H^*$ and $iH-iH^*$ are formally self-adjoint, and by our assumption we have

$$(H+H^*)\equiv(S_-+S_+)\circ(H+H^*)\equiv(H+H^*)\circ(S_-+S_+),\tag{3.1.149}$$

$$(iH-iH^*)\equiv(S_-+S_+)\circ(iH-iH^*)\equiv(iH-iH^*)\circ(S_-+S_+).\tag{3.1.150}$$

We notice that at $(x,y)=(0,0)$ we have the following relations

$$\widetilde{\widetilde{h}_0^{\mp}}\left(x,x',x_{2n+1}\mp g_{\mp}(x,x')\right)=h_0^{\mp}(x,x)\text{ is a zero map },\tag{3.1.151}$$

$$\widetilde{f}_1^{n+1}\left(x,x',x_{2n+1}\mp g_{\mp}(x,x')\right)=f_1^{n+1}(x,x)\neq 0.\tag{3.1.152}$$

So at $(x,y)=(0,0)$ we also have

$$\frac{\widetilde{\widetilde{h_0^{\mp}}}(x,y',x_{2n+1}\mp g_{\mp}(x,x'))}{\widetilde{f_1^{n+1}}(x,y',x_{2n+1}\mp g_{\mp}(x,x'))}+\frac{\widetilde{\widetilde{\widetilde{h_0^{\mp}}}}(y',x_{2n+1}\mp g_{\mp}(x,x'),x)}{\widetilde{f_2^{n+1}}(x,y',x_{2n+1}\mp g_{\mp}(x,x'))}=0, \tag{3.1.153}$$

$$i\frac{\widetilde{\widetilde{h_0^{\mp}}}(x,y',x_{2n+1}\mp g_{\mp}(x,x'))}{\widetilde{f_1^{n+1}}(x,y',x_{2n+1}\mp g_{\mp}(x,x'))}-i\frac{\widetilde{\widetilde{\widetilde{h_0^{\mp}}}}(y',x_{2n+1}\mp g_{\mp}(x,x'),x)}{\widetilde{f_2^{n+1}}(x,y',x_{2n+1}\mp g_{\mp}(x,x'))}=0. \tag{3.1.154}$$

Thus, the Fourier integral operators $H+H^*$ and $iH-iH^*$ are exactly in the form as we already studied in the special case, and we immediately have

$$\begin{aligned}\frac{\widetilde{\widetilde{h_0^{\mp}}}(x,y',x_{2n+1}\mp g_{\mp}(x,y'))}{\widetilde{f_1^{n+1}}(x,y',x_{2n+1}\mp g_{\mp}(x,y'))}&+\frac{\widetilde{\widetilde{\widetilde{h_0^{\mp}}}}(y',x_{2n+1}\mp g_{\mp}(x,y'),x)}{\widetilde{f_2^{n+1}}(x,y',x_{2n+1}\mp g_{\mp}(x,y'))}\\&=O\left(|(x,y)|^{+\infty}\right),\end{aligned} \tag{3.1.155}$$

$$\begin{aligned}i\frac{\widetilde{\widetilde{h_0^{\mp}}}(x,y',x_{2n+1}\mp g_{\mp}(x,y'))}{\widetilde{f_1^{n+1}}(x,y',x_{2n+1}\mp g_{\mp}(x,y'))}&-i\frac{\widetilde{\widetilde{\widetilde{h_0^{\mp}}}}(y',x_{2n+1}\mp g_{\mp}(x,y'),x)}{\widetilde{f_2^{n+1}}(x,y',x_{2n+1}\mp g_{\mp}(x,y'))}\\&=O\left(|(x,y)|^{+\infty}\right).\end{aligned} \tag{3.1.156}$$

Accordingly, we get

$$\frac{\widetilde{\widetilde{h_0^{\mp}}}(x,y',x_{2n+1}\mp g_{\mp}(x,y'))}{\widetilde{f_1^{n+1}}(x,y',x_{2n+1}\mp g_{\mp}(x,y'))}=O\left(|(x,y)|^{+\infty}\right). \tag{3.1.157}$$

From the above relation, it is not hard to check that

$$\widetilde{\widetilde{h_0^{\mp}}}(x,y',x_{2n+1}\mp g_{\mp}(x,y'))=O\left(|(x,y)|^{+\infty}\right). \tag{3.1.158}$$

As we mention in the beginning of the proof, by applying the Malgrange preparation theorem [40, Theorem 7.5.6] to the variable y_{2n+1} we can write

$$h_0^{\mp}(x,y)=\rho_{\mp}(x,y)\phi_{\mp}(x,y)+r_{\mp}(x,y') \tag{3.1.159}$$

for some $r(x,y')\in\mathscr{C}^\infty(\Omega\times U,\mathscr{L}(T^{*0,q}X,T^{*0,q}X))$. In particular, we can take almost analytic extension of (3.1.159) in the variable y_{2n+1} such that

$$\widetilde{h_0^{\mp}}(x,y',\widetilde{y}_{2n+1})=\widetilde{\rho}_{\mp}(x,y',\widetilde{y}_{2n+1})\left(\mp x_{2n+1}\pm\widetilde{y}_{2n+1}+g_{\mp}(x,y')\right)+r_{\mp}(x,y'). \tag{3.1.160}$$

We can put $\widetilde{y}_{2n+1}=x_{2n+1}\mp g_{\mp}(x,y')$ in the above equation, and with the fact that $r(x,y')$ is independent of y_{2n+1} and (3.1.157) we immediately see that

$$r_{\mp}(x,y')=O\left(|(x,y)|^{+\infty}\right). \tag{3.1.161}$$

The above argument holds for any point. So by (3.1.137), (3.1.159) and (3.1.161) we finish the verification of our theorem. ☐

3.2. Microlocal analysis of Toeplitz operators

With the same notations and assumptions in Theorem 1.1, we consider the Toeplitz operator for lower energy forms

$$T^{(q)}_{P,\lambda} := \Pi^{(q)}_{\lambda} \circ P \circ \Pi^{(q)}_{\lambda} : \Omega^{0,q}(X) \to \Omega^{0,q}(X) \tag{3.2.1}$$

associated by a formally self-adjoint $P \in L^1_{\mathrm{cl}}(X; T^{*0,q}X)$. We denote the principal symbol of P by

$$p_0 \in \mathscr{C}^\infty(T^*X, \mathscr{L}(T^{*0,q}X, T^{*0,q}X)), \tag{3.2.2}$$

and we assume that the following Levi-elliptic condition holds:

$$\tau^{n_-}_x p_0(-\boldsymbol{\alpha}_x)\tau^{n_-}_x > 0: \ q = n_-, \tag{3.2.3}$$

and additionally

$$\tau^{n_+}_x p_0(\boldsymbol{\alpha}_x)\tau^{n_+}_x < 0: \ q = n_- = n_+. \tag{3.2.4}$$

In this case we say that $T^{(q)}_{P,\lambda}$ is a *Levi-elliptic Toeplitz operator* and denote its Schwartz kernel by $T^{(q)}_{P,\lambda}(x,y)$. Our definition coincides with the one for elliptic Toeplitz operators [12, §2] on CR manifolds for $q = n_- = 0$.

We have the following microloal structure of Toeplitz operators on lower energy forms, which can be deduced from the method of complex stationary phase of Melin–Sjöstrand [72, pp. 156], Theorems 2.3 and 2.10. Reader can also refer the proof in [28, Theorem 4.4].

THEOREM 3.5. *In the situation of Theorem 1.1 and for $q = n_-$, $T^{(q)}_{P,\lambda}$ is the sum of Fourier integral operators*

$$T^{(q)}_{P,\lambda} = T_{\varphi_-} + T_{\varphi_+} + F \ \text{ on } \Omega, \tag{3.2.5}$$

where $F : \mathscr{E}'(\Omega; T^{*0,q}X) \to \mathscr{C}^\infty(X; T^{*0,q}X)$ is continuous and we have the Schwartz kernel

$$T_{\varphi_\mp}(x,y) = \int_0^{+\infty} e^{it\varphi_\mp(x,y)} t a^\mp(x,y,t) dt \tag{3.2.6}$$

associated with the Szegő phase function $\varphi_\mp(x,y)$ in Remark 2.7 and the symbol

$$a^\mp(x,y,t) \in S^n_{\mathrm{cl}}(\Omega \times \Omega \times \mathbb{R}_+, \mathscr{L}(T^{*0,q}_y X, T^{*0,q}_x X)) \tag{3.2.7}$$

with the properties that

$$a^{\mp}(x,y,t) \sim \sum_{j=0}^{+\infty} a_j^{\mp}(x,y) t^{n-j} \text{ in } S^n_{1,0}(\Omega\times\Omega\times\mathbb{R}_+, \mathscr{L}(T^{*0,q}_y X, T^{*0,q}_x X)), \tag{3.2.8}$$

$$a_0^{+}(x,y,t) = 0 \text{ when } n_- \neq n_+, \tag{3.2.9}$$

$a^{\mp}(x,y,t)$ and $a_j^{\mp}(x,y)$ are properly supported in the variables (x,y) for all $j\in\mathbb{N}_0$,

$$a_0^{-}(x,x) = \frac{|\det \mathcal{L}_x|}{2\pi^{n+1}} \frac{v(x)}{m(x)} \tau_x^{n_-} p_0(-\boldsymbol{\alpha}_x) \tau_x^{n_-}, \tag{3.2.10}$$

and when $n_- = n_+$ we additionally have

$$a_0^{+}(x,x) = \frac{|\det \mathcal{L}_x|}{2\pi^{n+1}} \frac{v(x)}{m(x)} \tau_x^{n_+} p_0(\boldsymbol{\alpha}_x) \tau_x^{n_+}. \tag{3.2.11}$$

In fact, given any $m\in\mathbb{R}$ and $\widetilde{P}\in L^m_{\mathrm{cl}}(X;T^{*0,q}X)$, since in our context we have $\Pi^{(q)}_\lambda \in L^0_{\frac12,\frac12}(X;T^{*0,q}X)$, by Calderon–Vaillancourt theorem we have that $T^{(q)}_{\widetilde{P},\lambda} := \Pi^{(q)}_\lambda \circ \widetilde{P}\circ \Pi^{(q)}_\lambda$ is also a bounded operator between the Sobolev spaces $H^{s+m}_{0,q}(X)$ and $H^s_{0,q}(X)$, where $H^s_{0,q}(X) := H^s(X,T^{*0,q}X)$, for all $s\in\mathbb{R}$. We denote this fact by

$$T^{(q)}_{\widetilde{P},\lambda} = O(1) \text{ in } \mathscr{L}(H^{s+m}_{0,q}(X), H^s_{0,q}(X)),\ \forall\, s\in\mathbb{R}. \tag{3.2.12}$$

Also, we can still construct the parametrix for Levi-elliptic Toeplitz operators although they are defined by the elliptic pseudodifferential operator.

THEOREM 3.6. *In the situation of Theorem 1.1 and for $q=n_-$, we can always find a formally self-adjoint pseudodifferential operator $Q\in L^{-1}_{\mathrm{cl}}(X;T^{*0,q}X)$ such that*

$$T^{(q)}_{Q,\lambda}\circ T^{(q)}_{P,\lambda} \equiv T^{(q)}_{P,\lambda}\circ T^{(q)}_{Q,\lambda} \equiv \Pi^{(q)}_\lambda \text{ on } X. \tag{3.2.13}$$

PROOF. First of all, we notice that if we can find some $Q\in L^{-1}_{\mathrm{cl}}(X;T^{*0,q}X)$ such that $T^{(q)}_{Q,\lambda}\circ T^{(q)}_{P,\lambda} \equiv T^{(q)}_{P,\lambda}\circ T^{(q)}_{Q,\lambda} \equiv \Pi^{(q)}_\lambda$, then we can replace Q be $\frac12(Q+Q^*)$ which is clearly formally self-adjoint.

For the generality of our argument, we demonstrate the case $n_-=n_+$, and the case $n_-\neq n_+$ follows from the same argument with some minor change. In the following we always use the convention (1.1.23). When $q=n_-$, we have $p_{I_0,I_0}(-\boldsymbol{\alpha}_x) > 0$ and we can find a conic neighborhood $\mathcal{C}^-_1$ of Σ^- such that $p_{I_0,I_0}(-\boldsymbol{\alpha}_x) >$ on the closure of $\mathcal{C}^-_1$. We take a function $\rho(x,\eta)\in\mathscr{C}^\infty(T^*X)$ that ρ vanishes for small $|\eta|$, ρ is positively homogeneous in η of degree zero when $|\eta|\geq 1$, ρ equals to one when (x,η) in a conic neighborhood of Σ^- and ρ has support in the closure of $\mathcal{C}_1^-$. We also take any $r(x,\eta) \in S^{-1}_{\rm cl}(T^*X \setminus \mathcal{C}_1^-)$ and we can see that for $\ell_{I_0,I_0} := \rho p_{I_0,I_0} + (1-\rho)r$ we have $\ell_{I_0,I_0} \in S^{-1}_{\rm cl}(T^*X)$ and

$$\ell_{I_0,I_0}(-\boldsymbol{\alpha}_x)p_{I_0,I_0}(-\boldsymbol{\alpha}_x) = 1. \tag{3.2.14}$$

We can also similarly construct $\ell_{J_0,J_0} \in S^{-1}_{\rm cl}(T^*X)$ such that

$$\ell_{J_0,J_0}(\boldsymbol{\alpha}_x)p_{J_0,J_0}(\boldsymbol{\alpha}_x) = 1. \tag{3.2.15}$$

By the above argument, we can find $L^{(0)} \in L^{-1}_{\rm cl}(X; T^{*0,q}X)$ with the principal symbol $\ell_0^{(0)} = \sum'_{\mathbf{I},\mathbf{J}} \ell_{\mathbf{I},\mathbf{J}}\omega^\wedge_{\mathbf{I}} \otimes \omega^{\wedge,*}_{\mathbf{J}} \in S^{-1}_{\rm cl}(T^*X, \mathscr{L}(T^{*0,q}X, T^{*0,q}X))$ such that (3.2.14) and (3.2.15) holds. Then for any coordinate patch $\Omega \subset X$, by combining Theorem 3.5, Melin–Sjöstrand stationary phase method Theorem 2.2, Theorem 2.5, (1.1.17), (1.1.18) and Theorem 2.10 we can check that

$$T^{(q)}_{L^{(0)},\lambda} \circ T^{(q)}_{P,\lambda} = I_0^- + I_0^+ + R_0 \quad \text{on } \Omega. \tag{3.2.16}$$

where $R_0 : \mathscr{E}'(\Omega, T^{*0,q}X) \to \mathscr{C}^\infty(X, T^{*0,q}X)$ is continuous and the Schwartz kernels

$$I_0^\mp(x,y) = \int_0^{+\infty} e^{it\varphi_\mp(x,y)}\mathscr{S}^\mp(x,y,t)dt \tag{3.2.17}$$

have the same $\varphi_\mp$ before and the following data properly supported in (x,y): $\mathscr{S}^\mp(x,y,t) \sim \sum_{j=0}^{+\infty} \mathscr{S}_j(x,y)t^{n-j}$ in $S^n_{1,0}\left(\Omega\times\Omega\times\mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X)\right)$,

(3.2.18)

$$\mathscr{S}_0^\mp(x,y) = \sum_{|\mathbf{I}|=|\mathbf{J}|=q} \mathscr{S}^\mp_{\mathbf{I},\mathbf{J}}(x,y)\omega^\wedge_{\mathbf{I}} \otimes \omega^{\wedge,*}_{\mathbf{J}} \text{ for the strictly increasing index sets,}$$

(3.2.19)

$$\mathscr{S}^-_{I_0,I_0}(x,x) = \mathscr{S}^+_{J_0,J_0}(x,x) = \frac{|\det \mathcal{L}_x|}{2\pi^{n+1}}\frac{v(x)}{m(x)}.$$

By Theorem 3.4 and the above relations, we can deduce that $T^{(q)}_{L^{(0)},\lambda} \circ T^{(q)}_{P,\lambda} - \Pi^{(q)}_\lambda = H_0 + G_0$ on Ω, where $H_0 \in I^{n-1}_\Sigma(\Omega; T^{*0,q}X)$ and $G_0 : \mathscr{E}'(\Omega, T^{*0,q}X) \to \mathscr{C}^\infty(X, T^{*0,q}X)$ is continuous. Then for any $N \in \mathbb{N}$ and $j = 1, \cdots, N-1$, with the same method we can construct $L^{(j)} \in L^{-1-j}_{\rm cl}(X; T^{*0,q}X)$ such that

$$\sum_{j=0}^{N-1} T^{(q)}_{L^{(j)},\lambda} \circ T^{(q)}_{P,\lambda} - \Pi^{(q)}_\lambda = H_N + G_N \quad \text{on } \Omega, \tag{3.2.20}$$

where $H_N \in I^{n-N}_\Sigma(\Omega; T^{*0,q}X)$ and $G_N : \mathscr{E}'(\Omega, T^{*0,q}X) \to \mathscr{C}^\infty(X, T^{*0,q}X)$ is continuous. We can then construct the symbol $\ell \in S^{-1}_{\rm cl}(T^*X, \mathscr{L}(T^{*0,q}X, T^{*0,q}X))$ from the asymptotic sums of the complete symbol of $L^{(j)}$, $j = 0, 1, \cdots,$ and we can define $L \in L^{-1}_{\mathrm{cl}}(X; T^{*0,q}X)$ by the symbol ℓ. Since the above argument holds for arbitrary Ω and X is compact, we can check that $T^{(q)}_{L,\lambda} \circ T^{(q)}_{P,\lambda} - \Pi^{(q)}_{\lambda} \equiv 0$ on X. By the same method above, we also have an $R \in L^{-1}_{\mathrm{cl}}(X; T^{*0,q}X)$ such that $T^{(q)}_{P,\lambda} \circ T^{(q)}_{R,\lambda} - \Pi^{(q)}_{\lambda} \equiv 0$ on X. Then we have that $T^{(q)}_{L,\lambda} = T^{(q)}_{L,\lambda} \circ \Pi^{(q)}_{\lambda} \equiv T^{(q)}_{L,\lambda} \circ \left(T^{(q)}_{P,\lambda} \circ T^{(q)}_{R,\lambda}\right) \equiv \left(T^{(q)}_{L,\lambda} \circ T^{(q)}_{P,\lambda}\right) \circ T^{(q)}_{R,\lambda} \equiv \Pi^{(q)}_{\lambda} \circ T^{(q)}_{R,\lambda} = T^{(q)}_{R,\lambda}$ on X and we conclude our theorem. ☐

We have the following type of elliptic estimates which now easily follows from Theorem 3.6. For convenience, we denote $\mathcal{H}^{(q)}_{b,\lambda}(X) := \mathrm{Ker}\,(I - \Pi^{(q)}_{\lambda})$.

THEOREM 3.7. *In the context of Theorem 1.1, for every $s \geq 0$ there exists a constant $C_s > 0$ such that*

$$\|u\|_{s+1} \leq C_s \left(\|T^{(q)}_{P,\lambda} u\|_s + \|u\|_s\right), \quad \forall u \in \mathcal{H}^{(q)}_{b,\lambda}(X). \tag{3.2.21}$$

In particular, given a non-zero eigenvalue $\mu \in \mathbb{R}$ of $T^{(q)}_{P,\lambda}$ and for any $s \in \mathbb{N}_0$, there exists a constant $c_s > 0$ such that

$$\|u\|_s \leq c_s (1 + |\mu|)^s \|u\|, \quad \forall u \in \mathrm{Ker}\,(T^{(q)}_{P,\lambda} - \mu I). \tag{3.2.22}$$

In other words, $\mathrm{Ker}(T^{(q)}_{P,\lambda} - \mu I) \subset \Omega^{0,q}(X)$ *for all* $\mu \in \dot{\mathbb{R}}$. *Moreover, (3.2.22) holds with $\mu = 0$ for all $u \in \mathrm{Ker}\, T^{(q)}_{P,\lambda} \cap \mathcal{H}^{(q)}_{b,\lambda}(X)$.*

PROOF. By Theorem 3.6, when $u \in \mathcal{H}^{(q)}_{b,\lambda}(X)$ we have a $Q \in L^{-1}_{\mathrm{cl}}(X; T^{*0,q}X)$ and a smoothing operator F such that

$$u = \Pi^{(q)}_{\lambda} u = (\Pi^{(q)}_{\lambda} \circ Q \circ \Pi^{(q)}_{\lambda}) \circ T^{(q)}_{P,\lambda} u + Fu. \tag{3.2.23}$$

By the continuity of Q, $\Pi^{(q)}_{\lambda}$ and F, we have a constant $C_s > 0$ such that

$$\|u\|_{s+1} \leq \|T^{(q)}_{Q,\lambda} \circ T^{(q)}_{P,\lambda} u\|_{s+1} + \|Fu\|_{s+1} \tag{3.2.24}$$

$$\leq C_s \left(\|T^{(q)}_{P,\lambda} u\|_s + \|u\|_s\right) \tag{3.2.25}$$

holds for all $u \in \mathcal{H}^{(q)}_{b,\lambda}(X)$.

In particular, for an eigenform u of $T^{(q)}_{P,\lambda}$ corresponding to an eigenvalue $\mu \neq 0$, we have

$$\mu u = T^{(q)}_{P,\lambda} u = \Pi^{(q)}_{\lambda} \circ T^{(q)}_{P,\lambda} u = \mu \Pi^{(q)}_{\lambda} u. \tag{3.2.26}$$

Thus $u \in \mathcal{H}^{(q)}_{b,\lambda}(X)$. By applying the above estimate inductively, we have

$$\|u\|_{s+1} \leq C_s (1 + |\mu|) \|u\|_s \leq \cdots \leq c_s (1 + |\mu|)^s \|u\|, \tag{3.2.27}$$

where $C_s, c_s > 0$ are some constants. For the case of $\mu = 0$, if $u \in \mathcal{H}^{(q)}_{b,\lambda}(X)$ then the same argument still works. ☐

From Theorem 3.7, we immediately have the following self- adjoint extension of $T^{(q)}_{P,\lambda}$ in $L^2_{0,q}(X) := L^2(X, T^{*0,q}X)$.

THEOREM 3.8. *In the context of Theorem 1.1, the maximal extension*

$$T^{(q)}_{P,\lambda} : \mathrm{Dom}\, T^{(q)}_{P,\lambda} \subset L^2_{0,q}(X) \to L^2_{0,q}(X), \tag{3.2.28}$$

$$\mathrm{Dom}\, T^{(q)}_{P,\lambda} := \left\{ u \in L^2_{0,q}(X) : \, T^{(q)}_{P,\lambda} u \in L^2_{0,q}(X) \right\}, \tag{3.2.29}$$

is a self-adjoint extension of $T^{(q)}_{P,\lambda}$. *In particular,* $\mathrm{Spec}\, T^{(q)}_{P,\lambda} \subset \mathbb{R}$.

PROOF. Let $(T^{(q)}_{P,\lambda})^*_H : \mathrm{Dom}(T^{(q)}_{P,\lambda})^*_H \subset L^2_{0,q}(X) \to L^2_{0,q}(X)$ be the Hilbert space adjoint of $T^{(q)}_{P,\lambda}$. We first show that $\mathrm{Dom}\, T^{(q)}_{P,\lambda} \subset \mathrm{Dom}(T^{(q)}_{P,\lambda})^*_H$ and $(T^{(q)}_{P,\lambda})^*_H = T^{(q)}_{P,\lambda}$ on $\mathrm{Dom}\, T^{(q)}_{P,\lambda}$. Let $v \in \mathrm{Dom}\, T^{(q)}_{P,\lambda}$ and let $w = \Pi^{(q)}_\lambda v$. From Theorem 3.7, we can see that $w \in H^1_{0,q}(X)$. We can take a sequence $\{w_j\}_{j=1}^{+\infty}$ in $\Omega^{0,q}(X)$ such that $w_j \to w$ in $H^1_{0,q}(X)$ as $j \to +\infty$, and by (3.2.12) we also have $T^{(q)}_{P,\lambda} w_j \in \Omega^{0,q}(X)$ such that $T^{(q)}_{P,\lambda} w_j \to T^{(q)}_{P,\lambda} w$ in $L^2_{0,q}(X)$. Now, for all $u, v \in \mathrm{Dom}\, T^{(q)}_{P,\lambda}$, on one hand

$$(T^{(q)}_{P,\lambda} u | v) = (\Pi^{(q)}_\lambda \circ T^{(q)}_{P,\lambda} u | v) = (T^{(q)}_{P,\lambda} u | \Pi^{(q)}_\lambda v) = (T^{(q)}_{P,\lambda} u | w). \tag{3.2.30}$$

On the other hand $(T^{(q)}_{P,\lambda} u | w) = \lim_{j \to +\infty} (T^{(q)}_{P,\lambda} u | w_j)$. We notice that we can take a sequence $\{u_\ell\}_\ell$ in $\mathscr{C}^\infty(X)$ such that $u_\ell \to u$ in $L^2(X)$ and $T^{(q)}_{P,\lambda} u_\ell \to T^{(q)}_{P,\lambda} u$ in $H^{-1}(X)$, as $\ell \to +\infty$. Along with $w_j \in \mathscr{C}^\infty(X)$ for each j, we have

$$\lim_{\ell \to +\infty} |(T^{(q)}_{P,\lambda}(u - u_\ell) | w_j)| \leq \lim_{\ell \to +\infty} \|(T^{(q)}_{P,\lambda}(u - u_\ell)\|_{-1} \|w_j\|_1 = 0, \tag{3.2.31}$$

which implies that

$$(T^{(q)}_{P,\lambda} u | w_j) = \lim_{\ell \to +\infty} (T^{(q)}_{P,\lambda} u_\ell | w_j) = \lim_{\ell \to +\infty} (u_\ell | T^{(q)}_{P,\lambda} w_j) = (u | T^{(q)}_{P,\lambda} w_j). \tag{3.2.32}$$

We have

$$\begin{aligned} |(T^{(q)}_{P,\lambda} u | w)| &= \lim_{j \to +\infty} |(T^{(q)}_{P,\lambda} u | w_j)| = \lim_{j \to +\infty} |(u | T^{(q)}_{P,\lambda} w_j)| \\ &\leq \lim_{j \to +\infty} \|u\| \cdot \|T^{(q)}_{P,\lambda} w_j\| = \|T^{(q)}_{P,\lambda} w\| \cdot \|u\|. \end{aligned} \tag{3.2.33}$$

Combining all the estimate above we see that there exists a constant $C > 0$ such that

$$|(T^{(q)}_{P,\lambda} u | v)| \leq C \|u\| \tag{3.2.34}$$

for all $u, v \in \operatorname{Dom} T^{(q)}_{P,\lambda}$. This implies that $v \in \operatorname{Dom}(T^{(q)}_{P,\lambda})^*_H$ and $(T^{(q)}_{P,\lambda})^*_H v = T^{(q)}_{P,\lambda} v$.

Conversely, for all $v \in \operatorname{Dom}(T^{(q)}_{P,\lambda})^*_H$, by the Riesz representation theorem we can find a $w \in L^2_{0,q}(X)$ such that

$$(T^{(q)}_{P,\lambda} u | v) = (u | w), \tag{3.2.35}$$

for all $u \in \operatorname{Dom} T^{(q)}_{P,\lambda}$ with $(T^{(q)}_{P,\lambda})^*_H v = w$. In particular, for all $u \in \Omega^{0,q}(X)$, by the density argument in (3.2.32), we can check that

$$(u|w) = (T^{(q)}_{P,\lambda} u | v) = ((T^{(q)}_{P,\lambda})^*_H u | v) = (u | T^{(q)}_{P,\lambda} v), \tag{3.2.36}$$

which implies that $T^{(q)}_{P,\lambda} v = w$ (almost everywhere) as an element of $L^2_{0,q}(X)$. $\square$

We have the following analogue of [12, Proposition 2.14]. The proof is quite standard from the technique of elliptic estimate and Rellich compact embedding lemma.

THEOREM 3.9. *With the same notations and assumptions in Theorem 1.1, for $q = n_-$, the spectrum* $\operatorname{Spec}(T^{(q)}_{P,\lambda}) \subset \mathbb{R}$ *consist only by eigenvalues, where the non-zero eigenvalues all have finite multiplicity. For any $c > 0$, the set* $\operatorname{Spec} T^{(q)}_{P,\lambda} \cap [c, \infty) \cap (-\infty, -c]$ *is a discrete subset of $\mathbb{R}$. Also, the only possible accumulation points of* $\operatorname{Spec}(T^{(q)}_{P,\lambda})$ *are $\pm\infty$.*

PROOF. We first show that $\operatorname{Spec}(T^{(q)}_{P,\lambda})$ consists only by eigenvalues. If we suppose otherwise, then we have a number $\mu \in \operatorname{Spec}(T^{(q)}_{P,\lambda})$ such that $\mu - T^{(q)}_{P,\lambda}$ is injective. We claim that for this λ there is a constant $C > 0$ such that

$$\|(\mu - T^{(q)}_{P,\lambda}) u\| \geq C \|u\| \tag{3.2.37}$$

for all $u \in \operatorname{Dom} T^{(q)}_{P,\lambda}$. We notice that $u = \Pi^{(q)}_\lambda u + (I - \Pi^{(q)}_\lambda) u$ and $T^{(q)}_{P,\lambda} \circ (I - \Pi^{(q)}_\lambda) = 0$. Thus to verify this inequality we may assume $u = \Pi^{(q)}_\lambda u$. Supposing (3.2.37) does not hold, we can find a sequence $\{u_j\}_{j=1}^{+\infty}$ in $\operatorname{Dom} T^{(q)}_{P,\lambda}$ with $u_j = \Pi^{(q)}_\lambda u_j$ and $\|u_j\| = 1$ such that $\|(\mu - T^{(q)}_{P,\lambda}) u_j\| < \frac{1}{j}\|u_j\| = \frac{1}{j}$. However, we have

$$\|u_j\|_1 \leq C_1 (\|T^{(q)}_{P,\lambda} u_j\| + \|u_j\|) \leq C_1 (\frac{1}{j} + |\mu| + 1) \tag{3.2.38}$$

for some constant $C_1 > 0$ by Theorem 3.7. From Rellich compact embedding lemma, we can find a subsequence $u_{j_\ell} \to v$ in $L^2_{0,q}(X)$ such that $\|v\| = 1$, which contradicts $v \in \operatorname{Ker}(\mu - T^{(q)}_{P,\lambda}) = \{0\}$. Now, with (3.2.37) we can check that the closed range property $\overline{\text{Range}(\mu - T^{(q)}_{P,\lambda})} = \text{Range}(\mu - T^{(q)}_{P,\lambda})$, and along with linear algebra we get

$$\begin{aligned} L^2_{0,q}(X) =& \overline{\text{Range}(\mu - T^{(q)}_{P,\lambda})} \bigoplus \text{Range}(\mu - T^{(q)}_{P,\lambda})^{\perp} \\ =& \text{Range}(\mu - T^{(q)}_{P,\lambda}) \bigoplus \text{Ker}\,(\mu - T^{(q)}_{P,\lambda}) \\ =& \text{Range}(\mu - T^{(q)}_{P,\lambda}). \end{aligned} \tag{3.2.39}$$

This leads to a contradiction because such $\lambda \in \text{Spec}(T^{(q)}_{P,\lambda})$ makes $\mu - T^{(q)}_{P,\lambda}$ become bijective and have the bounded inverse. So $\text{Spec}(T^{(q)}_{P,\lambda})$ consists only by eigenvalues, and all the eigenforms corresponding to non-zero eigenvalues are smooth by Theorem 3.7. Moreover, if we suppose that $\dim \text{Ker}\,(T^{(q)}_{P,\lambda} - \mu_j I) = +\infty$, then we can find orthonormal sequence $\{f_\ell\}_{\ell=1}^{+\infty}$ in $\text{Ker}\,(T^{(q)}_{P,\lambda} - \mu_j I)$. However, with Theorem 3.7 and Rellich lemma this implies that there is a subsequence of $\{f_j\}_{j=1}^{+\infty}$ converges in $L^2_{0,q}(X)$, which contradicts the orthonormal assumption of $\{f_j\}_{j=1}^{+\infty}$. So all the non-zero eigenvalues of $T^{(q)}_{P,\lambda}$ have finite multiplicty.

For any $0 < c_1 < c_2 < \infty$, we claim that $\text{Spec}\, T^{(q)}_{P,\lambda} \cap [c_1, c_2]$ is a discrete subset of $\mathbb{R}$. If we suppose otherwise, then we can find infinitely many $f_j \in \Omega^{0,q}(X)$, $j = 1, 2, \cdots$, such that $T^{(q)}_{P,\lambda} f_j = \mu_j f_j$, $\mu_j \in [c_1, c_2]$, $(\, f_j \,|\, f_\ell \,) = \delta_{j,\ell}$. By Theorem 3.7 and the above relations, we can see that $\{f_j\}_{j=1}^{+\infty}$ is uniform bounded in $H^1_{0,q}(X)$. So we can apply Rellich compact embedding lemma to get a subsequence $\{f_{j_\ell}\}_{\ell=1}^{+\infty}$ of $\{f_j\}_{j=1}^{+\infty}$, where $1 < j_1 < j_2 < \cdots$, such that $f_{j_\ell} \to f$ in $L^2_{0,q}(X)$ as $\ell \to +\infty$, for some $f \in L^2_{0,q}(X)$. But $(\, f_{j_\ell} \,|\, f_{j_h} \,) = 0$ if $\ell \neq h$ and we get a contradiction. We conclude that $\text{Spec}\, T^{(q)}_{P,\lambda} \cap [c_1, c_2]$ is a discrete subset of $\mathbb{R}$, for any $0 < c_1 < c_2 < \infty$. We can also use the same argument to show that $\text{Spec}\, T^{(q)}_{P,\lambda} \cap [c_1, c_2]$ is a discrete subset of $\mathbb{R}$ for any $0 > c_1 > c_2 > -\infty$. Thus, for any $c > 0$, $\text{Spec}\, T^{(q)}_{P,\lambda} \cap [c, \infty) \cap (-\infty, -c]$ is a discrete subset of $\mathbb{R}$.

Finally, by the spectral theorem of self-adjoint operator [21, Theorem 2.5.1], we know that the only possible accumulation points of $\text{Spec}(T^{(q)}_{P,\lambda})$ are $\pm\infty$. ☐

From the above spectral theorems, we have the following formula.

THEOREM 3.10. *With the same notations and assumptions in Theorem 1.1, for $q = n_-$ we can find an L^2-orthonormal system $\{f_j\}_{j\in J}$ such that $T^{(q)}_{P,\lambda} f_j = \lambda_j f_j$ and*

$$\chi(k^{-1}T^{(q)}_{P,\lambda})(x,y) = \sum_{k^{-1}\lambda_j \in \operatorname{supp}\chi} \chi(k^{-1}\lambda_j) f_j(x) \otimes f_j^*(y) \in T^{*0,q}_x X \otimes (T^{*0,q}_y X)^*. \tag{3.2.40}$$

In the end of this section, we discuss more about the spectral theorem of $T^{(q)}_{P,\lambda}$ when $q = n_- \neq n_+$. We first present a weaker statement comparing to Theorem 3.7 with a proof independent of Theorem 3.4. We will also present that when $q = n_- \neq n_+$, the spectrum of $T^{(q)}_{P,\lambda}$ could be bounded from below.

The condition we assume here is stronger than Theorem 1.1: we assume that $q = n_- \neq n_+$ and the principal symbol of P is positive definite on the set Σ^-, cf. (2.3.21). First, we notice that for any $Q \in L^{-1}_{\mathrm{cl}}(X; T^{*0,q}X)$, we always have the composition formula

$$\begin{aligned} T^{(q)}_{Q,\lambda} \circ T^{(q)}_{P,\lambda} &= \Pi^{(q)}_\lambda \circ Q \circ \Pi^{(q)}_\lambda \circ P \circ \Pi^{(q)}_\lambda \\ &= \Pi^{(q)}_\lambda \circ Q \circ P \circ \Pi^{(q)}_\lambda + \Pi^{(q)}_\lambda \circ Q \circ [P, \Pi^{(q)}_\lambda] \circ \Pi^{(q)}_\lambda. \end{aligned} \tag{3.2.41}$$

Second, we recall a fundamental result: combining [45, Part I, Theorem 7.7] and [56, §4], on $\Omega \times \Omega$ we actually have

$$\Pi^{(q)}_\lambda(x,y) \equiv \int e^{i(\psi(+\infty,x,\eta) - \langle y,\eta\rangle)} a(+\infty,x,\eta) \frac{d\eta}{(2\pi)^{2n+1}} \tag{3.2.42}$$

where

$$a(+\infty,x,\eta) \in S^0_{\mathrm{cl}}(T^*\Omega, \mathscr{L}(T^{*0,q}X, T^{*0,q}X)), \tag{3.2.43}$$

$$a(+\infty,x,\eta) \sim \sum_0^{+\infty} a_j(+\infty,x,\eta) \text{ in } S^0_{1,0}(T^*\Omega, \mathscr{L}(T^{*0,q}X, T^{*0,q}X)), \tag{3.2.44}$$

$$a_j(+\infty,x,\eta) \in \mathscr{C}^\infty(T^*\Omega, \mathscr{L}(T^{*0,q}X, T^{*0,q}X)),\ j = 0,1,\cdots, \tag{3.2.45}$$

$$a_j(+\infty,x,\lambda\eta) = \lambda^{-j} a_j(+\infty,x,\eta),\ \lambda \geq 1,\ |\eta| \geq 1,\ j = 0,1,\cdots, \tag{3.2.46}$$

and

$$\psi(+\infty,x,\eta) \in \mathscr{C}^\infty(T^*\Omega), \tag{3.2.47}$$

$$\psi(+\infty,x,\lambda\eta) = \lambda\psi(+\infty,x,\eta),\ \lambda \geq 1,\ |\eta| \geq 1, \tag{3.2.48}$$

$$d^\alpha_{x,\eta}(\psi(+\infty,x,\eta) - \langle x,\eta\rangle) = 0 \text{ on } \Sigma,\ \alpha \in \mathbb{N}_0^{2n+1},\ 0 \leq |\alpha| \leq 1,, \tag{3.2.49}$$

and

$$a_j(+\infty,x,\eta) = 0 \ \text{ in a conic neighborhood of } \Sigma^+, \forall j \in \mathbb{N}_0. \tag{3.2.50}$$

Using Melin–Sjöstrand complex stationary phase method Theorem 2.2, we can check that

$$[P,\Pi_\lambda^{(q)}](x,y) \equiv \int e^{i(\psi(+\infty,x,\eta)-\langle y,\eta\rangle)} a_P(+\infty,x,\eta)\frac{d\eta}{(2\pi)^{2n+1}}, \tag{3.2.51}$$

where $a_P(+\infty,x,\eta)$ vanishes on Σ^-. By using Taylor expansion of $a_P(+\infty,x,\eta)$ near Σ^- and (3.2.49), along with [45, Part I, Proposition 5.18] we can check that

$$[P,\Pi_\lambda^{(q)}] \in L^{-\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}}(X;T^{*0,q}X), \tag{3.2.52}$$

and with Calderon–Vaillancourt's theorem we have the continuity

$$[P,\Pi_\lambda^{(q)}] = O(1) \text{ in } \mathscr{L}(H^s(X,T^{*0,q}X),H^{s+\frac{1}{2}}(X,T^{*0,q}X)) \tag{3.2.53}$$

for all $s\in\mathbb{R}$. Accordingly, for all $s\in\mathbb{R}$ we always have

$$\begin{aligned}\Pi_\lambda^{(q)}\circ Q\circ[P,\Pi_\lambda^{(q)}]\circ\Pi_\lambda^{(q)} &= O(1)\\ &\text{in } \mathscr{L}\left(H^s(X,T^{*0,q}X)\to H^{s+\frac{1}{2}}(X,T^{*0,q}X)\right).\end{aligned} \tag{3.2.54}$$

Third, because here we assume that the principal symbol of P is positive definite at Σ^-, we can find a conic neighborhood $\mathcal{C}_1^-$ of Σ^- such that the principal symbol of P is also positive definite on the closure of $\mathcal{C}_1^-$. We let $\mathcal{C}_2$ be another conic neighborhood of Σ^- such that $\mathcal{C}_2\Subset\mathcal{C}_1$ and take a suitable $F\in L^0_{\mathrm{cl}}(X;T^{*0,q}X)$ such that

$$F\equiv 0 \text{ outside } \mathcal{C}_1, \tag{3.2.55}$$

$$F\equiv I \text{ on } \mathcal{C}_2. \tag{3.2.56}$$

By choosing a suitable $\mathcal{P}\in L^1_{\mathrm{cl}}(X;T^{*0,q}X)$ which has the symbol positive definite on T^*X, it is not difficult to find an operator $\mathscr{P}$ given by

$$\mathscr{P} := F\circ P + (I-F)\circ\mathcal{P} \tag{3.2.57}$$

such that

$$\text{the principal symbol of } \mathscr{P} \text{ is positive definite on } T^*X, \tag{3.2.58}$$

$$T^{(q)}_{P,\lambda} \equiv T^{(q)}_{\mathscr{P},\lambda}. \tag{3.2.59}$$

Hence, we may assume the principal symbol of P is positive definite every on T^*X, and if Q is the parametrix of P, that is, $Q\in L^{-1}_{\mathrm{cl}}(\Omega;T^{*0,q}X)$ such that

$$Q\circ P\equiv I, \tag{3.2.60}$$

then (3.2.41) and (3.2.54) imply that the following (non-sharp) estimate

$$T^{(q)}_{Q,\lambda}T^{(q)}_{P,\lambda}-\Pi_\lambda^{(q)} = O(1) \text{ in } \mathscr{L}(H^s(X,T^{*0,q}X),H^{s+\frac{1}{2}}(X,T^{*0,q}X)), \tag{3.2.61}$$

and for all $u \in H^{s+1}(X, T^{*0,q}X)$ with $u = \Pi_\lambda^{(q)} u$, we hence have the (non-sharp) estimate

(3.2.62)
$$\|u\|_{s+1} \leq C_s \left(\|T_{P,\lambda}^{(q)} u\|_{s+\frac{1}{2}} + \|u\|_{s+\frac{1}{2}} \right),$$

where $C_s > 0$ is a constant.

In fact, we can still get the standard elliptic estimate in Theorem 3.7 if we apply Theorem 3.4 to (3.2.41). On one hand, we have

(3.2.63)
$$\Pi_\lambda^{(q)} \circ Q \circ [P, \Pi_\lambda^{(q)}] \circ \Pi_\lambda^{(q)}$$

(3.2.64)
$$= \Pi_\lambda^{(q)} \circ \Pi_\lambda^{(q)} \circ Q \circ [P, \Pi_\lambda^{(q)}] \circ \Pi_\lambda^{(q)}$$

(3.2.65)
$$= \Pi_\lambda^{(q)} \circ ([\Pi_\lambda^{(q)}, Q] \circ [P, \Pi_\lambda^{(q)}]) \circ \Pi_\lambda^{(q)} + \Pi_\lambda^{(q)} \circ Q \circ (\Pi_\lambda^{(q)} \circ [P, \Pi_\lambda^{(q)}] \circ \Pi_\lambda^{(q)})$$

where $\Pi_\lambda^{(q)} \circ [P, \Pi_\lambda^{(q)}] \circ \Pi_\lambda^{(q)} = 0$. On the other hand, from the Fourier integral operators of Szegő type and Melin–Sjöstrand complex stationary phase formula Theorem 2.2, we can check that

(3.2.66)
$$\Pi_\lambda^{(q)} \circ [\Pi_\lambda^{(q)}, Q] \circ [P, \Pi_\lambda^{(q)}] \circ \Pi_\lambda^{(q)} \equiv \int_0^{+\infty} e^{it\varphi_-(x,y)} s^{q,p}(x,y,t) dt,$$

where φ_- is a Szegő phase function for $\Pi_\lambda^{(q)}$ and $s^{q,p}(x,y,t) \sim \sum_{j=0}^{+\infty} s_j^{q,p}(x,y) t^{n-j}$ in $S^n_{1,0}(\Omega \times \Omega \times \mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X))$ with $s_0^{q,p}(x,x) = 0$. From Theorem 3.4 and integration by parts in t, we may assume that $s^{q,p}(x,y,t) \in S^n_{1,0}(\Omega \times \Omega \times \mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X))$. We can check that the phase functions $\psi(+\infty, x, \eta) - \langle y, \eta \rangle$ and $\varphi_-(x,y)t$ are equivalent at each point of the set $\{(x,y,\eta) \in \Omega \times \Omega \times \mathbb{R}^{2n+1} : y = x,\ (x,\eta) \in \Sigma^-\}$ in the sense of Melin–Sjöstrand [72, Definition 4.1 & Theoren 4.2]. By [45, Part I, Proposition 5.18], the above argument implies that

(3.2.67)
$$\Pi_\lambda^{(q)} \circ [\Pi_\lambda^{(q)}, Q] \circ [P, \Pi_\lambda^{(q)}] \circ \Pi_\lambda^{(q)} \in L^{-1}_{\frac{1}{2},\frac{1}{2}}(X; T^{*0,q}X),$$

and again by Calderon–Vaillancourt's theorem we can hence get the continuity

(3.2.68)
$$\Pi_\lambda^{(q)} \circ Q \circ [P, \Pi_\lambda^{(q)}] = O(1) \text{ in } \mathscr{L}(H^s(X, T^{*0,q}X), H^{s+1}(X, T^{*0,q}X))$$

for all $s \in \mathbb{R}$. Then by the same parametrix argument $\mathscr{P}$ before, we have an alternative proof of Theorem 3.7 in a more restricted situation.

Finally, we show that when the principal symbol of P is positive definite at Σ^- and $n_- \neq n_+$, $\text{Spec}(T_{P,\lambda}^{(q)})$ is in fact bounded from below as the case of $n_- = 0$. The argument is as follows. From the construction we just present before, we can find a pseudodifferential operator $\mathscr{P} \in L^1(X; T^{*0,q}X)$, whose principal symbol is positive definite on whole T^*X, and a smoothing operator F such that on X we have

$$T^{(q)}_{P,\lambda} = \Pi^{(q)}_{\lambda} \circ \mathscr{P} \circ \Pi^{(q)}_{\lambda} + F. \tag{3.2.69}$$

We notice that by an equivalent definition of smoothing operators, we have a constant $c_0 > 0$ such that

$$|(Fu|u)| \leq \|Fu\| \cdot \|u\| \leq c_0 \|u\|^2 \tag{3.2.70}$$

holds for all $u \in \Omega^{0,q}(X)$. Also, because the principal symbol of $\mathscr{P}$ is positive definite on whole T^*X, we can apply Gårding inequality (cf. [**31**, pp. 51]) and find some constants $c_1, C > 0$ such that

$$(\mathscr{P}u|u) \geq \frac{1}{C}\|u\|_{\frac{1}{2}} - C\|u\| \geq -c_1\|u\| \tag{3.2.71}$$

for all $u \in \Omega^{0,q}(X)$. Now for $\mu \neq 0$ and $u \in \text{Ker }(T^{(q)}_{P,\lambda} - \mu I) \cap \Omega^{0,q}(X)$, by $u = \Pi^{(q)}_{\lambda} u$ and the above discussions, We have a constant $c_2 > 0$ such that

$$(T^{(q)}_{P,\lambda} u|u) = (\Pi^{(q)}_{\lambda} \circ \mathscr{P} \circ \Pi^{(q)}_{\lambda} u|u) + (Ru|u) \geq -c_2\|u\|^2 \tag{3.2.72}$$

for all $u \in \Omega^{0,q}(X)$, which implies that $\mu \geq -c_2 > -\infty$ in this context.

CHAPTER 4

Semi-classical asymptotic expansion for the spectral operator

In this chapter we prove Theorem 1.1 and Theorem 1.2. We recall that we assume X is compact.

When $q \notin \{n_-, n_+\}$, we recall that $\Pi^{(q)}_\lambda \in L^{-\infty}(X; T^{*0,q}X)$ and this implies that $T^{(q)}_{P,\lambda} \in L^{-\infty}(X; T^{*0,q}X)$. This is equivalent to that $T^{(q)}_{P,\lambda} : H^s(X; T^{*0,q}X) \to H^\ell(X; T^{*0,q}X)$ is continuous for all $s, \ell \in \mathbb{R}$. So we can apply Rellich compact embedding to see that $T^{(q)}_{P,\lambda}$ is a compact operator on X. We then know Spec $T^{(q)}_{P,\lambda}$, $q \notin \{n_-, n_+\}$, is a bounded set in $\mathbb{R}$, cf. [21, Theorem 4.2.2] for example. As we assume that $\chi \in \mathscr{C}^\infty_0(\dot{\mathbb{R}})$, when $k \to +\infty$ we can conclude that:

$$\text{If } q \notin \{n_-, n_+\},\ \chi(k^{-1}T^{(q)}_{P,\lambda}) = 0 \ \text{ on } X. \tag{4.0.1}$$

The main difficulty is the case $q = n_-$. Since $\chi \in \mathscr{C}^\infty_0(\dot{\mathbb{R}})$, by standard spectral theory and functional analysis, we can check that

$$\chi(k^{-1}T^{(q)}_{P,\lambda}) = \chi(k^{-1}T^{(q)}_{P,\lambda}) \circ \Pi^{(q)}_\lambda. \tag{4.0.2}$$

Our strategy is to apply Helffer–Sjöstrand formula

$$\begin{aligned}\chi(k^{-1}T^{(q)}_{P,\lambda}) &= \chi(k^{-1}T^{(q)}_{P,\lambda}) \circ \Pi^{(q)}_\lambda \\ &= \int_{\mathbb{C}} \frac{\partial \widetilde{\chi}(z)}{\partial \overline{z}} (z - k^{-1}T^{(q)}_{P,\lambda})^{-1} \circ \Pi^{(q)}_\lambda \frac{dz \wedge d\overline{z}}{2\pi i} \\ &= \int_{\mathbb{C}} \frac{\partial \widetilde{\chi}(\frac{z}{k})}{\partial \overline{z}} (z - T^{(q)}_{P,\lambda})^{-1} \circ \Pi^{(q)}_\lambda \frac{dz \wedge d\overline{z}}{2\pi i}\end{aligned} \tag{4.0.3}$$

to solve the full asymptotic expansion of the Schwartz kernel $\chi(k^{-1}T^{(q)}_{P,\lambda})(x,y)$ as $k \to +\infty$. The first difficulty is the microlocal analysis of $(z - T^{(q)}_{P,\lambda})^{-1} \circ \Pi^{(q)}_\lambda$ when $z \notin \text{Spec}(T^{(q)}_{P,\lambda})$, and the second challenge is the semi-classical analysis of the integral

$$\int_{\mathbb{C}} \frac{\partial \widetilde{\chi}(\frac{z}{k})}{\partial \overline{z}} (z - T^{(q)}_{P,\lambda})^{-1} \circ \Pi^{(q)}_\lambda \frac{dz \wedge d\overline{z}}{2\pi i},\ k \to +\infty. \tag{4.0.4}$$

4.1. Expansion of resolvent type Toeplitz operators

In this section we always assume $q = n_-$. With respect to (1.1.23), we recall that we use the convention

$$p_0(x,\eta) = \sum_{|I|=|J|=q} p_{I,J}(x,\eta)\omega_I^\wedge \otimes \omega_J^{\wedge,*} \text{ for strictly increasing } I, J\,, \tag{4.1.1}$$

$$I_0 := \{1,\cdots,q\} \leftrightarrow \mu_1 < 0,\cdots,\mu_q < 0, \tag{4.1.2}$$

$$J_0 := \{q+1,\cdots,n\} \leftrightarrow \mu_{q+1} > 0,\cdots,\mu_n > 0, \tag{4.1.3}$$

$$\{\mu_1,\cdots,\mu_n\} \text{ are the eigenvalues of the Levi form } \mathcal{L} := -\frac{d\boldsymbol{\alpha}}{2i}\bigg|_{T^{1,0}X}. \tag{4.1.4}$$

We also recall that we assume

$$p_{I_0,I_0}(-\boldsymbol{\alpha}) > 0, \tag{4.1.5}$$

and when $n_- = n_+$ we additionally assume that

$$p_{J_0,J_0}(\boldsymbol{\alpha}) < 0. \tag{4.1.6}$$

In the expansion of $(z - T^{(q)}_{P,\lambda})^{-1} \circ \Pi^{(q)}_\lambda$, we will come across various types of smoothing operators that are dependent on z. These operators will appear as the remainder of the expansion. Subsequently, we will demonstrate that when this expansion of $(z - T^{(q)}_{P,\lambda})^{-1} \circ \Pi^{(q)}_\lambda$ is incorporated into the Helffer–Sjöstrand formula, the terms that involve these operators contribute solely as k-negligible operators.

From now on, we let

$$\tau \in \mathscr{C}^\infty(\mathbb{R}),\ \tau(t) = 0 \ \text{ for } t \le 1,\ \tau(t) = 1 \ \text{ for } t \ge 2. \tag{4.1.7}$$

Definition 4.1. In the situation of Theorem 1.1, for $q = n_-$ we denote by $\mathcal{E}_z(\Omega; T^{*0,q}X)$ the set of finite linear combinations of the operators with kernels

$$\int_0^{+\infty} e(x,y,t)\frac{z^{M_2}}{(z - tp(x))^{M_1}}\tau(\varepsilon t)dt \tag{4.1.8}$$

over $\mathbb{C}$, where the symbol $e(x,y,t) \in S^{-\infty}(\Omega \times \Omega \times \mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X))$ is properly supported in the variables (x,y), $p(x) \in \mathscr{C}^\infty(X, \dot{\mathbb{R}})$, and $M_1, M_2 \in \mathbb{N}_0$.

Definition 4.2. In the situation of Theorem 1.1, for $q = n_-$ we denote by $\mathcal{F}_z(\Omega; T^{*0,q}X)$ the set of finite linear combinations of the operators with kernels

$$\int_0^{+\infty} f(x,y,t)\frac{z^{M_2}}{(z - tp(x))^{M_1}}\tau(\varepsilon t)dt \tag{4.1.9}$$

over $\mathbb{C}$, where the symbol $f(x,y,t) \in S^{-\infty}(X \times \Omega \times \mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X))$ is properly supported in the variables (x,y), $p(x) \in \mathscr{C}^\infty(X, \dot{\mathbb{R}})$, and $M_1, M_2 \in \mathbb{N}_0$.

DEFINITION 4.3. In the situation of Theorem 1.1, for $q = n_-$ we denote by $\mathcal{G}_z(\Omega; T^{*0,q}X)$ the set of finite linear combination of the operators with kernels

$$\int_0^{+\infty} e^{it\psi(x,y)} g(x,y,t) \frac{z^{M_2}}{(z - tp(x))^{M_1}} \tau(\varepsilon t) dt \tag{4.1.10}$$

where $g(x,y,t) = O(|x-y|^{+\infty}), g(x,y,t) \in S^m_{\mathrm{cl}}(\Omega \times \Omega \times \mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X))$ for some $m \in \mathbb{R}$, $g(x,y,t)$ is properly supported in the variables (x,y), $p(x) \in \mathscr{C}^\infty(X, \dot{\mathbb{R}})$, $M_1, M_2 \in \mathbb{N}_0$, and $\psi \in \mathrm{Ph}(\Lambda \boldsymbol{\alpha}, \Omega)$ for some $\Lambda \in \mathscr{C}^\infty(X, \dot{\mathbb{R}})$.

DEFINITION 4.4. In the situation of Theorem 1.1, for $q = n_-$ we denote by $\mathcal{R}_z(\Omega; T^{*0,q}X)$ the set of finite linear combination of the operators with kernels

$$\int_0^{+\infty} \int_0^{+\infty} \int_\Omega e^{it\psi_\mp(x,w) + i\sigma\psi_\pm(w,y)} r_1(x,w,t) \circ r_2(w,y,\sigma) \frac{z^{M_2}}{(z - tp(x))^{M_1}} \tau(\varepsilon t) m(w) dw d\sigma dt \tag{4.1.11}$$

or

$$\int_0^{+\infty} \int_0^{+\infty} \int_\Omega e^{it\psi_\mp(x,w) + i\sigma\psi_\pm(w,y)} r_1(x,w,t) \circ r_2(w,y,\sigma) \frac{z^{M_2}}{(z - \sigma p(w))^{M_1}} \tau(\varepsilon \sigma) m(w) dw d\sigma dt \tag{4.1.12}$$

where $r_1(x,w,t), r_2(w,y,\sigma) \in S^m_{\mathrm{cl}}(\Omega \times \Omega \times \mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X))$ for some $m \in \mathbb{R}$, $r_1(x,w,t)$ is properly supported in the variables (x,w), $r_2(w,y,\sigma)$ is properly supported in the variables (w,y), $p(x) \in \mathscr{C}^\infty(X, \dot{\mathbb{R}})$, $M_1, M_2 \in \mathbb{N}_0$, and $\psi_\mp \in \mathrm{Ph}(\mp\Lambda\boldsymbol{\alpha}, \Omega)$ for some $\Lambda \in \mathscr{C}^\infty(X, \mathbb{R}_+)$.

DEFINITION 4.5. In the situation of Theorem 1.1, for $q = n_-$ we define the notation $L_z^{-\infty}(\Omega; T^{*0,q}X)$ by the set collecting all elements of the form $\sum_{j \in J} c_j u_j$, where $c_j \in \mathbb{C}$,

$$u_j \in \mathcal{E}_z(\Omega; T^{*0,q}X) \cup \mathcal{F}_z(\Omega; T^{*0,q}X) \cup \mathcal{G}_z(\Omega; T^{*0,q}X) \cup \mathcal{R}_z(\Omega; T^{*0,q}X), \tag{4.1.13}$$

and $|J| < +\infty$.

We are ready to construct the parametrix type Fourier integral operator for the operator $(z - T^{(q)}_{P,\lambda})$.

THEOREM 4.6. *With the notations and assumptions in Theorem 1.1,* (1.1.10), (1.1.11) *and* (1.1.23), *we let* $q = n_-$, $z \notin \mathrm{Spec}(T^{(q)}_{P,\lambda}) \setminus \{0\}$, $\tau \in \mathscr{C}^\infty(\mathbb{R}_+)$ *of* (4.2.2) *and take a fixed constant* $\varepsilon > 0$ *such that* $\tau(\varepsilon t)\chi(t) = \chi(t)$. *Then the Fourier integral* *operator* $A_{z,0} : \mathscr{C}_0^\infty(\Omega, T^{*0,q}X) \to \mathscr{C}_0^\infty(\Omega, T^{*0,q}X)$ *given by*

$$A_{z,0}(x,y) := \int_0^{+\infty} e^{it\varphi_-(x,y)} \frac{s_0^-(x,y)}{z - tp_{I_0,I_0}(-\boldsymbol{\alpha}_x)} t^n \tau(\varepsilon t)dt + \int_0^{+\infty} e^{it\varphi_+(x,y)} \frac{s_0^+(x,y)}{z - tp_{J_0,J_0}(\boldsymbol{\alpha}_x)} t^n \tau(\varepsilon t)dt \tag{4.1.14}$$

depends on z smoothly and we have

$$\begin{aligned}&\left((z - T_{P,\lambda}^{(q)}) \circ (S_- + S_+) \circ A_{z,0} \circ (S_- + S_+) - \Pi_\lambda^{(q)}\right)(x,y) \\ &\equiv \int_0^{+\infty} e^{it\Psi_-(x,y)} \frac{r_1^-(x,y,t;z)}{(z-t)^2} \tau(\varepsilon t)dt + \int_0^{+\infty} e^{it\Psi_+(x,y)} \frac{r_1^+(x,y,t;z)}{(z+t)^2} \tau(\varepsilon t)dt\end{aligned} \tag{4.1.15}$$

up to a kernel associated by an element in $L_z^{-\infty}(\Omega; T^{*0,q}X)$*, where*

$$\Psi_- \in \mathrm{Ph}(p_{I_0,I_0}^{-1}(-\boldsymbol{\alpha})(-\boldsymbol{\alpha}), \Omega), \tag{4.1.16}$$

$$\Psi_+ \in \mathrm{Ph}(p_{J_0,J_0}^{-1}(-\boldsymbol{\alpha})\boldsymbol{\alpha}, \Omega), \tag{4.1.17}$$

and

$$r_1^+(x,y,t;z) = 0 \text{ when } n_- = n_+, \tag{4.1.18}$$

$$r_1^\mp(x,y,t;z) = \sum_{|\alpha|+|\beta|\leq 2} r_{1,\alpha,\beta}^\mp(x,y,t) t^\alpha z^\beta, \tag{4.1.19}$$

$$r_{1,\alpha,\beta}^\mp(x,y,t) \in S_{\mathrm{cl}}^{n-1}(\Omega \times \Omega \times \mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X)), \tag{4.1.20}$$

$$r_{1,\alpha,\beta}^\mp(x,y,t) \text{ are properly supported in the variables } (x,y). \tag{4.1.21}$$

Also, up to a kernel associated to an element in $L_z^{-\infty}(\Omega; T^{*0,q}X)$ *we have*

$$\begin{aligned}&\Big((S_- + S_+) \circ A_{z,0} \circ (S_- + S_+)\Big)(x,y) \\ &\equiv \int_0^{+\infty} e^{it\Psi_-(x,y)} \frac{\alpha^-(x,y,t)}{z-t} \tau(\varepsilon t)dt + \int_0^{+\infty} e^{it\Psi_+(x,y)} \frac{\alpha^+(x,y,t)}{z+t} \tau(\varepsilon t)dt,\end{aligned} \tag{4.1.22}$$

where

$$\alpha^\mp(x,y,t) \sim \sum_{j=0}^{+\infty} \alpha_j^\mp(x,y) t^{n-j} \text{ in } S_{\mathrm{cl}}^n(\Omega \times \Omega \times \mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X)), \tag{4.1.23}$$

$\alpha^\mp(x,y,t)$ *and* $\alpha_j^\mp(x,y)$ *are properly supported in the variables* (x,y) *for all* $j \in \mathbb{N}_0$*,*

$$\alpha^+(x,y,t) = 0 \text{ when } n_- \neq n_+, \tag{4.1.24}$$

$$\alpha_0^-(x,x) = \frac{|\det \mathcal{L}_x|}{2\pi^{n+1}} \frac{v(x)}{m(x)}\, p_{I_0,I_0}^{-n-1}(-\boldsymbol{\alpha}_x), \tag{4.1.25}$$

and when $n_- = n_+$ *we additionally have*

$$\alpha_0^+(x,x) = \frac{|\det \mathcal{L}_x|}{2\pi^{n+1}} \frac{v(x)}{m(x)}\, p_{J_0,J_0}^{-n-1}(-\boldsymbol{\alpha}_x). \tag{4.1.26}$$

PROOF. For the generality of our argument, we assume $n_- = n_+$, and the case $n_- \neq n_+$ also follows from our proof with some minor change. We first notice that for any $u \in \mathscr{C}_0^\infty(\Omega, T^{*0,q}X)$, by Theorem 2.3 and (2.3.143), there are operators $E, F : \mathscr{E}'(\Omega; T^{*0,q}X) \to \mathscr{C}^\infty(X; T^{*0,q}X)$ such that

$$\begin{aligned}(z - T_{P,\lambda}^{(q)}) \circ (S_- + S_+)u &= (z - T_{P,\lambda}^{(q)}) \circ (\Pi_\lambda^{(q)} + E)u \\ &= (z\Pi_\lambda^{(q)} - T_{P,\lambda}^{(q)})u + (z\Pi_\lambda^{(q)} - T_{P,\lambda}^{(q)}) \circ Eu \\ = \int_0^{+\infty} & e^{it\varphi_-(x,y)} \left(zs^-(x,y,t) - ta^-(x,y,t)\right) u(y)m(y)dydt \\ + \int_0^{+\infty} & e^{it\varphi_+(x,y)} \left(zs^+(x,y,t) - ta^+(x,y,t)\right) u(y)m(y)dydt \\ &\qquad + (zE - F)u + (z\Pi_\lambda^{(q)} - T_{P,\lambda}^{(q)}) \circ Eu.\end{aligned} \tag{4.1.27}$$

We notice that the operators $zE - F$ and $(z\Pi_\lambda^{(q)} - T_{P,\lambda}^{(q)}) \circ E$ are in $L_z^{-\infty}(\Omega; T^{*0,q}X)$.

On the other hand, we have

$$A_{z,0} \circ (S_- + S_+) = B_{z,0}^{-,-} + B_{z,0}^{+,+} + B_{z,0}^{-,+} + B_{z,0}^{+,-}, \tag{4.1.28}$$

where

$$\begin{aligned}B_{z,0}^{-,\mp}(x,y) &= \int_\Omega \left(\int_0^{+\infty} e^{it\varphi_-(x,w)} \frac{s_0^-(x,w)}{z - tp_{I_0,I_0}(-\boldsymbol{\alpha}_x)} t^n \tau(\varepsilon t) dt \right) \circ S_\mp(w,y)m(w)dw \\ &= \int_0^{+\infty} \frac{\tau(\varepsilon t)t^n}{z - tp_{I_0,I_0}(-\boldsymbol{\alpha}_x)} \\ &\quad \times \left(\int_0^{+\infty} \int_\Omega e^{it(\varphi_-(x,w)+\sigma\varphi_\mp(w,y))} s_0^-(x,w) \circ s^\mp(w,y,t\sigma)m(w)dwd\sigma \right) dt,\end{aligned} \tag{4.1.29}$$

and

$$\begin{aligned}B_{z,0}^{+,\mp}(x,y) &= \int_\Omega \left(\int_0^{+\infty} e^{it\varphi_+(x,w)} \frac{s_0^+(x,w)}{z - tp_{J_0,J_0}(\boldsymbol{\alpha}_x)} t^n \tau(\varepsilon t) dt \right) \circ S_\mp(w,y)m(w)dw \\ &= \int_0^{+\infty} \frac{\tau(\varepsilon t)t^n}{z - tp_{J_0,J_0}(\boldsymbol{\alpha}_x)} \\ &\quad \times \left(\int_0^{+\infty} \int_\Omega e^{it(\varphi_+(x,w)+\sigma\varphi_\mp(w,y))} s_0^+(x,w) \circ s^\mp(w,y,t\sigma)m(w)dwd\sigma \right) dt.\end{aligned} \tag{4.1.30}$$

By Definition 4.5, the operators $B_{z,0}^{-,+}$ and $B_{z,0}^{+,-}$ are in $L_z^{-\infty}(\Omega; T^{*0,q}X)$. Also, using Melin–Sjöstrand stationary phase formula Theorem 2.2 and Theorem 2.5, cf. also [13, §4] or [45, pp. 76-77], we have

$$B^{-,-}_{z,0}(x,y)=\int_0^{+\infty}e^{it\varphi_-(x,y)}\frac{\tau(\varepsilon t)t^n}{z-tp_{I_0,I_0}(-\boldsymbol{\alpha}_x)}b^{-,0}(x,y,t)dt+E^{-,0}_z(x,y) \tag{4.1.31}$$

$$B^{+,+}_{z,0}(x,y)=\int_0^{+\infty}e^{it\varphi_+(x,y)}\frac{\tau(\varepsilon t)t^n}{z-tp_{J_0,J_0}(\boldsymbol{\alpha}_x)}b^{+,0}(x,y,t)dt+E^{+,0}_z(x,y) \tag{4.1.32}$$

where $E^{\mp,0}_z\in L^{-\infty}_z(\Omega;T^{*0,q}X)$ and

$$b^{\mp,0}(x,y,t)\sim\sum_{j=0}^{+\infty}b^{\mp,0}_j(x,y)t^{-j}\text{ in }S^0_{\rm cl}(\Omega\times\Omega\times\mathbb{R}_+,\mathscr{L}(T^{*0,q}X,T^{*0,q}X)), \tag{4.1.33}$$

$$b^{\mp,0}_0(x,x)=s^{\mp}_0(x,x). \tag{4.1.34}$$

Combining the calculation before, up to some element in $L^{-\infty}_z(\Omega;T^{*0,q}X)$, we can check that

$$\Big((z-T^{(q)}_{P,\lambda})\circ(S_-+S_+)\Big)\circ(A_{z,0}\circ(S_-+S_+))=C^{-,-}_{z,0}+C^{+,+}_{z,0}, \tag{4.1.35}$$

where

$$C^{-,-}_{z,0}(x,y)=\int_0^{+\infty}\int_0^{+\infty}\int_\Omega e^{i\gamma\varphi_-(x,w)+i\beta\varphi_-(w,y)}\tau(\varepsilon\beta)\beta^n \times\frac{zs^-(x,w,\gamma)-\gamma a^-(x,w,\gamma)}{z-\beta p_{I_0,I_0}(-\boldsymbol{\alpha}_w)}\circ b^{-,0}(w,y,\beta)m(w)dwd\gamma d\beta \tag{4.1.36}$$

and

$$C^{+,+}_{z,0}(x,y)=\int_0^{+\infty}\int_0^{+\infty}\int_\Omega e^{i\gamma\varphi_+(x,w)+i\beta\varphi_+(w,y)}\tau(\varepsilon\beta)\beta^n \times\frac{zs^+(x,w,\gamma)-\gamma a^+(x,w,\gamma)}{z-\beta p_{J_0,J_0}(\boldsymbol{\alpha}_w)}\circ b^{+,0}(w,y,\beta)m(w)dwd\gamma d\beta. \tag{4.1.37}$$

For $C^{-,-}_{z,0}$, we recall that $p_{I_0,I_0}(-\alpha)>0$ and we can apply the change of variables

$$\beta=p^{-1}_{I_0,I_0}(-\boldsymbol{\alpha}_w)t,\ t\geq 0, \tag{4.1.38}$$

$$\gamma=p^{-1}_{I_0,I_0}(-\boldsymbol{\alpha}_w)t\sigma,\ \sigma\geq 0, \tag{4.1.39}$$

and we have

(4.1.40)

$$C^{-,-}_{z,0}(x,y) = \int_0^{+\infty}\int_0^{+\infty}\int_\Omega \exp\Big(it\cdot p^{-1}_{I_0,I_0}(-\boldsymbol{\alpha}_w)(\sigma\varphi_-(x,w)+\varphi_-(w,y)\Big) \times \tau(\varepsilon p^{-1}_{I_0,I_0}(-\boldsymbol{\alpha}_w)t)(p^{-1}_{I_0,I_0}(-\boldsymbol{\alpha}_w)t)^n \times \frac{z\cdot s^-(x,w,p^{-1}_{I_0,I_0}(-\boldsymbol{\alpha}_w)t\sigma)-p^{-1}_{I_0,I_0}(-\boldsymbol{\alpha}_w)t\cdot a^-(x,w,p^{-1}_{I_0,I_0}(-\boldsymbol{\alpha}_w)t\sigma)}{z-t} \circ b^{-,0}(w,y,p^{-1}_{I_0,I_0}(-\boldsymbol{\alpha}_w)t)p^{-2}_{I_0,I_0}(-\boldsymbol{\alpha}_w)t\, m(w)dwd\sigma dt.$$

We recall that $s^-, a^- \in S^n_{\rm cl}(\Omega\times\Omega\times\mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X))$, $b^{-,0}\in S^0_{\rm cl}(\Omega\times\Omega\times\mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X))$, and the leading coefficients $s^-_0, a^-_0, b^{-,0}_0$ in their symbol asymptotic expansion satisfies

$$a^-_0(x,x) = p_{I_0,I_0}(-\boldsymbol{\alpha}_x)s^-_0(x,x) = p_{I_0,I_0}(-\boldsymbol{\alpha}_x)b^{-,0}_0(x,x). \tag{4.1.41}$$

Accordingly, we can write

$$\begin{aligned}&\frac{z\cdot s^-(x,w,p^{-1}_{I_0,I_0}(-\boldsymbol{\alpha}_w)t\sigma)-p^{-1}_{I_0,I_0}(-\boldsymbol{\alpha}_w)t\cdot a^-(x,w,p^{-1}_{I_0,I_0}(-\boldsymbol{\alpha}_w)t\sigma)}{z-t} \circ b^{-,0}(w,y,p^{-1}_{I_0,I_0}(-\boldsymbol{\alpha}_w)t)\\ &= p^{-n}_{I_0,I_0}(-\boldsymbol{\alpha}_w)t^n\sigma^n\frac{(zs^-_0-p^{-1}_{I_0,I_0}(-\boldsymbol{\alpha}_w)^{-1}ta^-_0)(x,w)}{z-t}\circ b^{-,0}_0(w,y)\\ &+\frac{z\mathscr{S}^-_1(x,y,w,t,\sigma)-t\mathscr{A}^-_1(x,y,w,t,\sigma)+z\mathcal{S}^-_1(x,y,w,t,\sigma)-t\mathcal{A}^-_1(x,y,w,t,\sigma)}{z-t},\end{aligned}\tag{4.1.42}$$

where for each fixed $t_0 > 0$

$$\mathscr{S}^-_1(x,y,w,t_0,\sigma)\in S^{n-1}_{\rm cl}(\Omega\times\Omega\times\Omega\times\mathbb{R}_+,\mathscr{L}(T^{*0,q}X,T^{*0,q}X)), \tag{4.1.43}$$

$$\mathscr{A}^-_1(x,y,w,t_0,\sigma)\in S^{n-1}_{\rm cl}(\Omega\times\Omega\times\Omega\times\mathbb{R}_+,\mathscr{L}(T^{*0,q}X,T^{*0,q}X)), \tag{4.1.44}$$

$$\mathcal{S}^-_1(x,y,w,t_0,\sigma)\in S^{n}_{\rm cl}(\Omega\times\Omega\times\Omega\times\mathbb{R}_+,\mathscr{L}(T^{*0,q}X,T^{*0,q}X)), \tag{4.1.45}$$

$$\mathcal{A}^-_1(x,y,w,t_0,\sigma)\in S^{n}_{\rm cl}(\Omega\times\Omega\times\Omega\times\mathbb{R}_+,\mathscr{L}(T^{*0,q}X,T^{*0,q}X)), \tag{4.1.46}$$

and for each fixed $\sigma_0 > 0$

$$\mathscr{S}^-_1(x,y,w,t,\sigma_0)\in S^{n-1}_{\rm cl}(\Omega\times\Omega\times\Omega\times\mathbb{R}_+,\mathscr{L}(T^{*0,q}X,T^{*0,q}X)), \tag{4.1.47}$$

$$\mathscr{A}^-_1(x,y,w,t,\sigma_0)\in S^{n-1}_{\rm cl}(\Omega\times\Omega\times\Omega\times\mathbb{R}_+,\mathscr{L}(T^{*0,q}X,T^{*0,q}X)), \tag{4.1.48}$$

$$\mathcal{S}^-_1(x,y,w,t,\sigma_0)\in S^{n-1}_{\rm cl}(\Omega\times\Omega\times\Omega\times\mathbb{R}_+,\mathscr{L}(T^{*0,q}X,T^{*0,q}X)), \tag{4.1.49}$$

$$\mathcal{A}^-_1(x,y,w,t,\sigma_0)\in S^{n-1}_{\rm cl}(\Omega\times\Omega\times\Omega\times\mathbb{R}_+,\mathscr{L}(T^{*0,q}X,T^{*0,q}X)). \tag{4.1.50}$$

We notice that at $w = x = y$ we have

$$\text{(4.1.51)}\quad \frac{p^{-n}_{I_0,I_0}(-\boldsymbol{\alpha}_w)t^n\sigma^n(zs^-_0 - p^{-1}_{I_0,I_0}(-\boldsymbol{\alpha}_w)^{-1}ta^-_0)(x,w)}{z-t} \circ b^{-,0}_0(w,y) = p^{-n}_{I_0,I_0}(-\alpha_x)t^n\sigma^n s^-_0(x,x) \circ s^-_0(x,x).$$

Also, for

$$\text{(4.1.52)}\quad \Phi_-(w,\sigma;x,y) := p^{-1}_{I_0,I_0}(-\boldsymbol{\alpha}_w)(\sigma\varphi_-(x,w) + \varphi_-(w,y)),$$

by (2.3.36) and (2.3.37), at $w = x = y$ and $\sigma = 1$ we have

$$\text{(4.1.53)}\quad d_w\Phi_- = d_\sigma\Phi_- = 0.$$

Also, at $w = y = x = x_0$ and $\sigma = 1$, by Theorems 2.3 and 2.5, under the local coordinates we can check that

$$\text{(4.1.54)}\quad \det\frac{H(\Phi_-)}{2\pi i} := \det\frac{1}{2\pi i}\begin{bmatrix} \left(\frac{\partial^2\Phi_-}{\partial w_j\partial w_k}\right)^{2n+1}_{j,k=1} & \left(\frac{\partial^2\Phi_-}{\partial w_j\partial\sigma}\right)^{2n+1}_{j=1} \\ \left(\frac{\partial\Phi_-}{\partial\sigma\partial w_j}\right)^{2n+1}_{j=1} & \frac{\partial\Phi_-}{\partial\sigma^2} \end{bmatrix}$$

equals to
(4.1.55)

$$p^{-2n-2}_{I_0,I_0}(-\boldsymbol{\alpha}_{x_0})\cdot\det\begin{bmatrix} \frac{4i|\mu_1|}{2\pi i} & & & & & & * \\ & \frac{4i|\mu_1|}{2\pi i} & & & & & * \\ & & \ddots & & & & * \\ & & & \frac{4i|\mu_n|}{2\pi i} & & & * \\ & & & & \frac{4i|\mu_n|}{2\pi i} & & * \\ & & & & & * & \frac{1}{2\pi i} \\ * & * & * & * & * & \frac{1}{2\pi i} & 0 \end{bmatrix}_{(2n+2)\times(2n+2)},$$

and from direct calculation we can check that the value above is

$$\text{(4.1.56)}\quad p^{-2n-2}_{I_0,I_0}(-\boldsymbol{\alpha}_{x_0})\frac{2^{2n-2}}{\pi^{2n+2}}|\mu_1\cdots\mu_n|^2 \neq 0.$$

We also recall that under our convention and coordinates, the volume induced by Hermitian metric satisfies

$$\text{(4.1.57)}\quad v(0) = 2^n.$$

So by the above calculating results (4.1.40), (4.1.42), (4.1.51), (4.1.54), (4.1.56), (4.1.57) and Melin–Sjöstrand stationary phase formula Theorem 2.2, up to a kernel associated to an element in $L_z^{-\infty}(\Omega; T^{*0,q}X)$, we can write

$$C^{-,-}_{z,0}(x,y) = \int_0^{+\infty} e^{it\Psi_-(x,y)} \frac{z\mathbb{S}_0^-(x,y) - t\mathbb{A}_0^-(x,y)}{z-t} t^n \tau(\varepsilon t)dt + \int_0^{+\infty} e^{it\Psi_-(x,y)} \frac{z\mathbb{E}_1^-(x,y,t) - t\mathbb{F}_1^-(x,y,t)}{z-t} \tau(\varepsilon t)dt. \tag{4.1.58}$$

Here $\Psi_-(x,y)$ is the critical value for $\widetilde{\Phi}_-(\widetilde{w},\widetilde{\sigma};x,y)$ in (4.1.52), and by [72, Lemma 2.1], (2.3.36) and (2.3.37) we can check that

$$\Psi_-(x,y) \in \mathrm{Ph}\left(-p^{-1}_{I_0,I_0}(-\boldsymbol{\alpha})\boldsymbol{\alpha}\right). \tag{4.1.59}$$

Also, the symbols here satisfy

$$\mathbb{S}_0^-(x,x) = \mathbb{A}_0^-(x,x) = p^{n+1}_{I_0,I_0}(-\boldsymbol{\alpha}_x)s_0^-(x,x), \tag{4.1.60}$$

$$\mathbb{E}_1^-(x,y,t),\ \mathbb{F}_1^-(x,y,t) \in S^{n-1}_{\mathrm{cl}}(\Omega\times\Omega\times\mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X)). \tag{4.1.61}$$

Similarly, since we assume that $p_{J_0,J_0}(-\boldsymbol{\alpha}) = -p_{J_0,J_0}(\boldsymbol{\alpha}) > 0$ when $n_- = n_+$, we can also write

$$\begin{aligned} C^{+,+}_{z,0}(x,y) = \int_0^{+\infty}\int_0^{+\infty}\int_\Omega &\exp\left(it\cdot p^{-1}_{J_0,J_0}(-\boldsymbol{\alpha}_w)(\sigma\varphi_+(x,w)+\varphi_+(w,y))\right) \\ &\times \tau(\varepsilon p^{-1}_{J_0,J_0}(-\boldsymbol{\alpha}_w)t)(p^{-1}_{J_0,J_0}(-\boldsymbol{\alpha}_w)t)^n \\ \times \frac{z\cdot s^+(x,w,p^{-1}_{J_0,J_0}(-\boldsymbol{\alpha}_w)t\sigma) - p^{-1}_{J_0,J_0}(-\boldsymbol{\alpha}_w)t\cdot a^+(x,w,p^{-1}_{J_0,J_0}(-\boldsymbol{\alpha}_w)t\sigma)}{z+t}& \\ \circ\, b^{+,0}(w,y,p^{-1}_{J_0,J_0}(-\boldsymbol{\alpha}_w)t)p^{-2}_{J_0,J_0}(-\boldsymbol{\alpha}_w)t\, m(w)dwd\sigma dt.& \end{aligned} \tag{4.1.62}$$

Then by the same argument for $C^{-,-}_{z,0}$, up to a kernel associated to an element in $L_z^{-\infty}(\Omega; T^{*0,q}X)$, we can write

$$C^{+,+}_{z,0}(x,y) = \int_0^{+\infty} e^{it\Psi_+(x,y)} \frac{z\mathbb{S}_0^+(x,y) + t\mathbb{A}_0^+(x,y)}{z+t} t^n \tau(\varepsilon t)dt + \int_0^{+\infty} e^{it\Psi_+(x,y)} \frac{z\mathbb{E}_1^+(x,y,t) + t\mathbb{F}_1^+(x,y,t)}{z+t} \tau(\varepsilon t)dt, \tag{4.1.63}$$

where

$$\Psi_+(x,y) \in \mathrm{Ph}(p^{-1}_{J_0,J_0}(-\boldsymbol{\alpha})\boldsymbol{\alpha}), \tag{4.1.64}$$

$$\mathbb{S}_0^+(x,x) = \mathbb{A}_0^+(x,x) = p^{n+1}_{J_0,J_0}(\boldsymbol{\alpha}_x)s_0^+(x,x), \tag{4.1.65}$$

$$\mathbb{E}_1^+(x,y,t),\ \mathbb{F}_1^+(x,y,t) \in S^{n-1}_{\mathrm{cl}}(\Omega\times\Omega\times\mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X)). \tag{4.1.66}$$

We notice that in terms of Definition 3.2 and Theorem 3.3, we can write

$$S_{\mp}(x,y) \equiv \int_0^{+\infty} e^{it\Psi_{\mp}(x,y)} \left(s_0^{\Psi_{\mp}}(x,y)t^n + s_1^{\Psi_{\mp}}(x,y,t) \right) \tau(\varepsilon t) dt, \tag{4.1.67}$$

where $s_1^{\Psi_{\mp}}(x,y,t) \in S^{n-1}_{\mathrm{cl}}(\Omega \times \Omega \times \mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X))$ and

$$s_0^{\Psi_-}(x,x) = p^{-n-1}_{I_0,I_0}(-\boldsymbol{\alpha}_x) s_0^-(x,x). \tag{4.1.68}$$

Combining all the calculation so far, we conclude that up to a kernel associated with an element in $L_z^{-\infty}(\Omega; T^{*0,q}X)$

$$\begin{aligned}
&\left((z - T^{(q)}_{P,\lambda}) \circ (S_- + S_+) \circ A_{z,0} \circ (S_- + S_+) - \Pi^{(q)}_\lambda \right)(x,y) \\
&\equiv \left(C^{-,-}_{z,0} + C^{+,+}_{z,0} - S_- + S_+ \right)(x,y) \\
&\equiv \int_0^{+\infty} e^{it\Psi_-(x,y)} \left(\frac{z\mathbb{S}_0^-(x,y) - t\mathbb{A}_0^-(x,y)}{z-t} - s_0^{\Psi_-}(x,y) \right) t^n \tau(\varepsilon t) dt \\
&+ \int_0^{+\infty} e^{it\Psi_-(x,y)} \left(\frac{z\mathbb{S}_0^+(x,y) + t\mathbb{A}_0^+(x,y)}{z+t} - s_0^{\Psi_+}(x,y) \right) t^n \tau(\varepsilon t) dt \\
&+ \int_0^{+\infty} e^{it\Psi_-(x,y)} \frac{z(\mathbb{E}_1^- - s_1^{\Psi_-})(x,y,t) - t(\mathbb{F}_1^- - s_1^{\Psi_-})(x,y,t)}{z-t} \tau(\varepsilon t) dt \\
&+ \int_0^{+\infty} e^{it\Psi_+(x,y)} \frac{z(\mathbb{E}_1^+ - s_1^{\Psi_-})(x,y,t) + t(\mathbb{F}_1^+ - s_1^{\Psi_-})(x,y,t)}{z+t} \tau(\varepsilon t) dt,
\end{aligned} \tag{4.1.69}$$

where $\mathbb{E}_1^{\mp}$, $\mathbb{F}_1^{\mp}$, $s_1^{\Psi_{\mp}} \in S^{n-1}_{\mathrm{cl}}(\Omega \times \Omega \times \mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X))$.

The final step of the proof is to apply Theorem 3.4 to reduce the above formula. For the purpose we first let

$$\mathbb{I} := \left((z - T^{(q)}_{P,\lambda}) \circ (S_- + S_+) \circ A_{z,0} \circ (S_- + S_+) - \Pi^{(q)}_\lambda \right)\Big|_{z=0}. \tag{4.1.70}$$

From the previous calculation, we can check that up to a smoothing kernel on $\Omega \times \Omega$ we have

$$\begin{aligned}
\mathbb{I}(x,y) \equiv &\int_0^{+\infty} e^{it\Psi_-(x,y)} (\mathbb{A}_0^- - s_0^{\Psi_-})(x,y) t^n \tau(\varepsilon t) dt \\
+ &\int_0^{+\infty} e^{it\Psi_+(x,y)} (\mathbb{A}_0^+ - s_0^{\Psi_+})(x,y) t^n \tau(\varepsilon t) dt \\
+ &\int_0^{+\infty} e^{it\Psi_-(x,y)} (\mathbb{F}_1^- - s_1^{\Psi_-})(x,y,t) \tau(\varepsilon t) dt \\
+ &\int_0^{+\infty} e^{it\Psi_+(x,y)} (\mathbb{F}_1^+ - s_1^{\Psi_-})(x,y,t) \tau(\varepsilon t) dt.
\end{aligned} \tag{4.1.71}$$

By (4.1.70), (4.1.60), (4.1.65), (4.1.68) and the above formula, we can see that I_0 satisfies all the assumptions in Theorem 2.6 and we have

$$(\mathbb{A}_0^- - s_0^{\Psi_-})(x,y) - f_0^-(x,y)\Psi_-(x,y) = O(|x-y|^{+\infty}), \tag{4.1.72}$$

$$(\mathbb{A}_0^+ - s_0^{\Psi_+})(x,y) - f_0^+(x,y)\Psi_+(x,y) = O(|x-y|^{+\infty}). \tag{4.1.73}$$

On the other hand, if we let

$$\mathbb{II} := \frac{\partial}{\partial z}\left((z - T_{P,\lambda}^{(q)}) \circ (S_- + S_+) \circ A_{z,0} \circ (S_- + S_+) - \Pi_\lambda^{(q)}\right)\Big|_{z=0}, \tag{4.1.74}$$

then directly from (4.1.69) we have

$$\begin{aligned} \mathbb{II}(x,y) \equiv & \int_0^{+\infty} e^{it\Psi_-(x,y)}(-\mathbb{S}_0^- + \mathbb{A}_0^-)(x,y)t^{n-1}\tau(\varepsilon t)dt \\ & + \int_0^{+\infty} e^{it\Psi_+(x,y)}(\mathbb{S}_0^+ - \mathbb{A}_0^+)(x,y)t^{n-1}\tau(\varepsilon t)dt \\ & + \int_0^{+\infty} e^{it\Psi_-(x,y)}(-\mathbb{E}_1^- + \mathbb{F}_1^-)(x,y,t)t^{-1}\tau(\varepsilon t)dt \\ & + \int_0^{+\infty} e^{it\Psi_+(x,y)}(\mathbb{E}_1^+ - \mathbb{F}_1^+)(x,y,t)t^{-1}\tau(\varepsilon t)dt. \end{aligned} \tag{4.1.75}$$

Again by (4.1.70), (4.1.60), (4.1.65), (4.1.68) and the above formula, we can see $\mathbb{II}$ satisfies all the assumptions in Theorem 2.6 and we have

$$(-\mathbb{S}_0^- + \mathbb{A}_0^-)(x,y) - f_1^-(x,y)\Psi_-(x,y) = O(|x-y|^{+\infty}), \tag{4.1.76}$$

$$(\mathbb{S}_0^+ - \mathbb{A}_0^+)(x,y) - f_1^+(x,y)\Psi_+(x,y) = O(|x-y|^{+\infty}). \tag{4.1.77}$$

Then by (4.1.72), (4.1.73), (4.1.76) (4.1.77) and (2.3.84), up to a kernel associated to an element in $L_z^{-\infty}(\Omega; T^{*0,q}X)$ we can write

$$\begin{aligned} & \int_0^{+\infty} e^{it\Psi_-(x,y)}\left(\frac{z\mathbb{S}_0^-(x,y) - t\mathbb{A}_0^-(x,y)}{z-t} - s_0^{\Psi_-}(x,y)\right)t^n\tau(\varepsilon t)dt \\ & + \int_0^{+\infty} e^{it\Psi_-(x,y)}\left(\frac{z\mathbb{S}_0^+(x,y) + t\mathbb{A}_0^+(x,y)}{z+t} - s_0^{\Psi_+}(x,y)\right)t^n\tau(\varepsilon t)dt \\ \equiv & \int_0^{+\infty}\left(\frac{1}{i}\frac{d}{dt}e^{it\Psi_-(x,y)}\right)\frac{f_1^-(x,y)t^n}{z-t}\tau(\varepsilon t)dt \\ & + \int_0^{+\infty}\left(\frac{1}{i}\frac{d}{dt}e^{it\Psi_+(x,y)}\right)\frac{f_1^+(x,y)t^n}{z+t}\tau(\varepsilon t)dt \\ \equiv & \int_0^{+\infty} e^{it\Psi_-(x,y)}\frac{g_1^-(x,y,t;z)}{(z-t)^2}t^{n-1}\tau(\varepsilon t)dt \\ & + \int_0^{+\infty} e^{it\Psi_-(x,y)}\frac{g_1^+(x,y,t;z)}{(z+t)^2}t^{n-1}\tau(\varepsilon t)dt, \end{aligned} \tag{4.1.78}$$

where

$$g_1^{\mp}(x,y,t;z) = \sum_{|\alpha|+|\beta|\leq 1} g_{1,\alpha,\beta}^{\mp}(x,y)t^{\alpha}z^{\beta}. \tag{4.1.79}$$

Combining all the calculation above, we can conclude our theorem. ☐

Now we can state and prove the most important result in this section.

THEOREM 4.7. *With the assumptions and notations of Theorem 4.6, for every* $N \in \mathbb{N}_0$ *and* $j = 0, \cdots, N$, *we can construct Fourier integral operators* $A_{z,j}, R_{z,N+1} : \mathscr{C}_0^{\infty}(\Omega; T^{*0,q}X) \to \mathscr{C}_0^{\infty}(\Omega; T^{*0,q}X)$, *which smoothly depends on* z, *such that up to an element in* $L_z^{-\infty}(\Omega; T^{*0,q}X)$ *we have*

$$(z - T_{P,\lambda}^{(q)}) \circ \sum_{j=0}^{N}(S_- + S_+) \circ A_{z,j} \circ (S_- + S_+) \equiv \Pi_{\lambda}^{(q)} + R_{z,N+1}. \tag{4.1.80}$$

In fact, up to a kernel associated to an element in $L_z^{-\infty}(\Omega; T^{*0,q}X)$, *we have*

$$\begin{aligned}(S_- + S_+) \circ A_{z,j} \circ (S_- + S_+)(x,y) &\equiv \int_0^{+\infty} e^{it\Psi_-(x,y)} \frac{\sum_{|\beta|+|\gamma|\leq 2j} \alpha_{\beta,\gamma}^{-,j}(x,y,t)t^{\beta}z^{\gamma}}{(z-t)^{2j+1}}\tau(\varepsilon t)dt \\ &\quad + \int_0^{+\infty} e^{it\Psi_+(x,y)} \frac{\sum_{|\beta|+|\gamma|\leq 2j} \alpha_{\beta,\gamma}^{+,j}(x,y,t)t^{\beta}z^{\gamma}}{(z+t)^{2j+1}}\tau(\varepsilon t)dt,\end{aligned} \tag{4.1.81}$$

where $\alpha_{\beta,\gamma}^{\mp}(x,y,t) \in S_{\mathrm{cl}}^{n-j}\left(\Omega \times \Omega \times \mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X)\right)$, $\alpha_{\beta,\gamma}^{\mp}(x,y,t)$ *is properly supported in the variables* (x,y) *and* $\alpha_{\beta,\gamma}^{+}(x,y,t) = 0$ *when* $n_- \neq n_+$. *Also, up to a kernel associated to an element in* $L_z^{-\infty}(\Omega; T^{*0,q}X)$, *we have*

$$\begin{aligned}R_{z,N+1}(x,y) &\equiv \int_0^{+\infty} e^{it\Psi_-(x,y)} \frac{\sum_{|\beta|+|\gamma|\leq 2N+2} R_{\beta,\gamma}^{-,N+1}(x,y,t)t^{\beta}z^{\gamma}}{(z-t)^{2N+2}}\tau(\varepsilon t)dt \\ &\quad + \int_0^{+\infty} e^{it\Psi_+(x,y)} \frac{\sum_{|\beta|+|\gamma|\leq 2N+2} R_{\beta,\gamma}^{+,N+1}(x,y,t)t^{\beta}z^{\gamma}}{(z+t)^{2N+2}}\tau(\varepsilon t)dt,\end{aligned} \tag{4.1.82}$$

where

$$R_{\beta,\gamma}^{\mp,N+1}(x,y,t) \in S_{\mathrm{cl}}^{n-N-1}(\Omega \times \Omega \times \mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X)), \tag{4.1.83}$$

$$R_{\beta,\gamma}^{\mp,N+1}(x,y,t) \text{ is properly supported in the variables } (x,y), \tag{4.1.84}$$

$$R_{\beta,\gamma}^{+,N+1}(x,y,t) = 0 \text{ when } n_- \neq n_+. \tag{4.1.85}$$

PROOF. From Theorem 4.6, we already have Fourier integral operators $A_{z,0}$ and $R_{z,1}$ with all the properties we need. Especially, the properties of (4.1.81) are verified from the calculation of Melin–Sjöstrand stationary phase method applied in (4.1.58) and (4.1.63). This suggests us to use induction to prove our theorem. Now we assume our theorem holds for some $N = N_0 \in \mathbb{N}_0$. We denote

$$\begin{aligned}(4.1.86)\qquad & R_{z,N_0+1}(x,y)\\ &:= \int_0^{+\infty} e^{it\Psi_-(x,y)} \frac{\sum_{|\beta|+|\gamma|\le 2N_0+2} R^{-,N_0+1}_{\beta,\gamma}(x,y,t)t^\beta z^\gamma}{(z-t)^{2N_0+2}}\tau(\varepsilon t)dt\\ &+ \int_0^{+\infty} e^{it\Psi_+(x,y)} \frac{\sum_{|\beta|+|\gamma|\le 2N_0+2} R^{+,N_0+1}_{\beta,\gamma}(x,y,t)t^\beta z^\gamma}{(z+t)^{2N_0+2}}\tau(\varepsilon t)dt\end{aligned}$$

and

$$(4.1.87)\qquad R^{-,N_0+1}_{\beta,\gamma}(x,y,t) \sim \sum_{\ell=0}^{+\infty} R^{-,N_0+1}_{\beta,\gamma,\ell}(x,y)t^{n-N_0-1-\ell}$$

$$(4.1.88)\qquad R^{+,N_0+1}_{\beta,\gamma}(x,y,t) \sim \sum_{\ell=0}^{+\infty} R^{+,N_0+1}_{\beta,\gamma,\ell}(x,y)t^{n-N_0-1-\ell}$$

in $S^{n-N-1}_{1,0}(\Omega\times\Omega\times\mathbb{R}_+,\mathscr{L}(T^{*0,q}X,T^{*0,q}X))$. For $z\notin \mathrm{Spec}(T^{(q)}_{P,\lambda})\setminus\{0\}$ we consider the operator $A_{z,N_0+1} : \mathscr{C}^\infty_0(\Omega)\to\mathscr{C}^\infty_0(\Omega)$ determined by the oscillatory integral

$$\begin{aligned}(4.1.89)\qquad & A_{z,N_0+1}(x,y)\\ &:= \int_0^{+\infty} e^{it\varphi_-(x,y)} \frac{\sum_{|\beta|+|\gamma|\le 2N_0+2} \alpha^{-,N_0+1}_{\beta,\gamma}(x,y,t)t^\beta z^\gamma}{(z-tp_{I_0,I_0}(-\alpha_x))^{2N_0+3}}\tau(\varepsilon t)dt\\ &+ \int_0^{+\infty} e^{it\varphi_+(x,y)} \frac{\sum_{|\beta|+|\gamma|\le 2N_0+2} \alpha^{+,N_0+1}_{\beta,\gamma}(x,y,t)t^\beta z^\gamma}{(z-tp_{J_0,J_0}(\alpha_x))^{2N_0+3}}\tau(\varepsilon t)dt,\end{aligned}$$

where we have the following symbols properly supported in the variables (x,y):

$$(4.1.90)\qquad \alpha^{-,N_0+1}_{\beta,\gamma}(x,y,t) := -R^{-,N_0+1}_{\beta,\gamma,0}(x,y)\cdot p^{(n-N_0-1+\beta)+1}_{I_0,I_0}(-\alpha_x)t^{n-N_0-1},$$

$$(4.1.91)\qquad \alpha^{+,N_0+1}_{\beta,\gamma}(x,y,t) := -R^{+,N_0+1}_{\beta,\gamma,0}(x,y)\cdot p^{(n-N_0-1+\beta)+1}_{J_0,J_0}(-\alpha_x)t^{n-N_0-1}.$$

From our construction, it is clear that

$$(4.1.92)\qquad \alpha^{+,j}_{\beta,\gamma}(x,y,t) = 0 \text{ when } n_-\neq n_+,$$

and by the same stationary phase method of Melin–Sjöstrand applied in (4.1.58) and (4.1.63), up to a kernel associated to an element in $L^{-\infty}_z(\Omega;T^{*0,q}X)$ we can check that

(4.1.93)

$$\Big((z-T^{(q)}_{P,\lambda})\circ(S_-+S_+)\circ A_{z,N_0+1}\circ(S_-+S_+)+R_{z,N_0+1}\Big)(x,y)$$

$$\equiv\int_0^{+\infty}e^{it\Psi_-(x,y)}\tau(\varepsilon t)$$

$$\times\frac{\sum_{|\beta|+|\gamma|\leq 2N_0+2}\Big(z(\mathbb{S}^{-,N_0+1}_{\beta,\gamma}+R^{-,N_0+1}_{\beta,\gamma})-t(\mathbb{A}^{-,N_0+1}_{\beta,\gamma}+R^{-,N_0+1}_{\beta,\gamma})\Big)(x,y,t)t^{\beta}z^{\gamma}}{(z-t)^{2N_0+3}}dt$$

$$+\int_0^{+\infty}e^{it\Psi_+(x,y)}\tau(\varepsilon t)$$

$$\times\frac{\sum_{|\beta|+|\gamma|\leq 2N_0+2}\Big(z(\mathbb{S}^{+,N_0+1}_{\beta,\gamma}+R^{+,N_0+1}_{\beta,\gamma})+t(\mathbb{A}^{+,N_0+1}_{\beta,\gamma}+R^{+,N_0+1}_{\beta,\gamma})\Big)(x,y,t)t^{\beta}z^{\gamma}}{(z+t)^{2N_0+3}}dt,$$

where

$$\mathbb{S}^{\mp,N_0+1}_{\beta,\gamma}(x,y,t)\sim\sum_{\ell=0}^{+\infty}\mathbb{S}^{\mp,N_0+1}_{\beta,\gamma,\ell}(x,y)t^{n-N_0-1-\ell}$$
$$\text{in } S^{n-N_0-1}_{1,0}(\Omega\times\Omega\times\mathbb{R}_+,\mathscr{L}(T^{*0,q}X,T^{*0,q}X)),\tag{4.1.94}$$

and

$$\mathbb{A}^{\mp,N_0+1}_{\beta,\gamma}(x,y,t)\sim\sum_{\ell=0}^{+\infty}\mathbb{A}^{\mp,N_0+1}_{\beta,\gamma,\ell}(x,y)t^{n-N_0-1-\ell}$$
$$\text{in } S^{n-N_0-1}_{1,0}(\Omega\times\Omega\times\mathbb{R}_+,\mathscr{L}(T^{*0,q}X,T^{*0,q}X)),\tag{4.1.95}$$

and we have the leading term relation

$$\mathbb{S}^{\mp,N_0+1}_{\beta,\gamma,0}(x,x)=\mathbb{A}^{\mp,N_0+1}_{\beta,\gamma,0}(x,x)=-R^{\mp,N_0+1}_{\beta,\gamma,0}(x,x).\tag{4.1.96}$$

This implies that up to a kernel associated to an element in $L^{-\infty}_z(\Omega;T^{*0,q}X)$ we have

$$\Big((z-T^{(q)}_{P,\lambda})\circ(S_-+S_+)\circ A_{z,N_0+1}\circ(S_-+S_+)+R_{z,N_0+1}\Big)(x,y)\tag{4.1.97}$$

$$\equiv\int_0^{+\infty}e^{it\Psi_-(x,y)}\tau(\varepsilon t)\frac{\sum_{|\beta'|+|\gamma'|\leq 2N_0+3}\mathbb{B}^{-,N_0+1}_{\beta',\gamma'}(x,y,t)t^{\beta'}z^{\gamma'}}{(z-t)^{2N_0+3}}dt\tag{4.1.98}$$
$$+\int_0^{+\infty}e^{it\Psi_+(x,y)}\tau(\varepsilon t)\frac{\sum_{|\beta'|+|\gamma'|\leq 2N_0+3}\mathbb{B}^{+,N_0+1}_{\beta',\gamma'}(x,y,t)t^{\beta'}z^{\gamma'}}{(z+t)^{2N_0+3}}dt,$$

where

$$\mathbb{B}^{\mp,N_0+1}_{\beta',\gamma'}(x,y,t) \sim \sum_{\ell=0}^{+\infty} \mathbb{B}^{\mp,N_0+1}_{\beta',\gamma',\ell}(x,y)t^{n-N_0-1-\ell} \quad \text{in } S^{n-N_0-1}_{1,0}(\Omega\times\Omega\times\mathbb{R}_+,\mathscr{L}(T^{*0,q}X,T^{*0,q}X)). \tag{4.1.99}$$

By (4.1.96), in the above asymptotic expansion we also have

$$\mathbb{B}^{\mp,N_0+1}_{\beta',\gamma',0}(x,x)=0 \tag{4.1.100}$$

for all $(\beta',\gamma')\in\mathbb{N}_0^2$ such that $|\beta'|+|\gamma'|\leq 2N_0+3$. On the other hand, we notice that by induction hypothesis we have

$$(z-T^{(q)}_{P,\lambda})\circ\sum_{j=0}^{N_0}(S_-+S_+)\circ A_{z,j}\circ(S_-+S_+)-\Pi^{(q)}_\lambda\equiv R_{z,N_0+1} \tag{4.1.101}$$

up to an element in $L^{-\infty}_z(\Omega;T^{*0,q}X)$. Thus, we consider the operator

$$\mathbb{I}_0:=\left((z-T^{(q)}_{P,\lambda})\circ(S_-+S_+)\circ A_{z,N_0+1}\circ(S_-+S_+)+R_{z,N_0+1}\right)\Big|_{z=0}. \tag{4.1.102}$$

We notice that up to a kernel associated with an element in $L^{-\infty}_z(\Omega;T^{*0,q}X)$, we have

$$\begin{aligned}&\mathbb{I}_0(x,y)\\ &\equiv\int_0^{+\infty}e^{it\Psi_-(x,y)}\frac{\sum_{|\beta'|\leq 2N_0+3}\mathbb{B}^{-,N_0+1}_{\beta',0}(x,y,t)t^{\beta}}{(-t)^{2N_0+3}}\tau(\varepsilon t)dt\\ &\quad+\int_0^{+\infty}e^{it\Psi_+(x,y)}\frac{\sum_{|\beta'|\leq 2N_0+3}\mathbb{B}^{+,N_0+1}_{\beta',0}(x,y,t)t^{\beta}}{t^{2N_0+3}}\tau(\varepsilon t)dt,\end{aligned} \tag{4.1.103}$$

and from (4.1.96) and (4.1.101) we can check that $\mathbb{I}_0$ satisfies all the assumptions in Theorem 3.4. This implies that

$$\mathbb{B}^{-,N_0+1}_{2N_0+3,0,0}(x,y)-f^{-,N_0+1}_{2N_0+3,0}(x,y)\Psi_-(x,y)=O(|x-y|^{+\infty}), \tag{4.1.104}$$

$$\mathbb{B}^{+,N_0+1}_{2N_0+3,0,0}(x,y)-f^{+,N_0+1}_{2N_0+3,0}(x,y)\Psi_+(x,y)=O(|x-y|^{+\infty}). \tag{4.1.105}$$

Next, we consider

$$\mathbb{I}_1:=\left(\frac{\partial}{\partial z}(z-T^{(q)}_{P,\lambda})\circ(S_-+S_+)\circ A_{z,N_0+1}\circ(S_-+S_+)+\frac{\partial}{\partial z}R_{z,N_0+1}\right)\Big|_{z=0}. \tag{4.1.106}$$

we can apply the same argument for $\mathbb{I}_1$ and check that

$$\left(\mathbb{B}^{-,N_0+1}_{2N_0+2,1,0}+(2N_0+3)\mathbb{B}^{-,N_0+1}_{2N_0+3,0,0}\right)(x,y)-g^{-,N_0+1}_{2N_0+2,1}(x,y)\Psi_-(x,y)=O(|x-y|^{+\infty}), \tag{4.1.107}$$

$$\left(\mathbb{B}^{+,N_0+1}_{2N_0+2,1,0}+(2N_0+3)\mathbb{B}^{+,N_0+1}_{2N_0+3,0,0}\right)(x,y)-g^{+,N_0+1}_{2N_0+2,1}(x,y)\Psi_+(x,y)=O(|x-y|^{+\infty}). \tag{4.1.108}$$

From (4.1.104) and (4.1.105), we immediately have

$$\mathbb{B}^{-,N_0+1}_{2N_0+2,1,0}(x,y)-f^{-,N_0+1}_{2N_0+2,1}(x,y)\Psi_-(x,y)=O(|x-y|^{+\infty}), \tag{4.1.109}$$

$$\mathbb{B}^{+,N_0+1}_{2N_0+2,1,0}(x,y)-f^{+,N_0+1}_{2N_0+2,1}(x,y)\Psi_+(x,y)=O(|x-y|^{+\infty}). \tag{4.1.110}$$

In the order of $j=2,3,\cdot,2N_0+3$, we also consider

$$\mathbb{I}_\ell:=\left(\frac{\partial^j}{\partial z^j}(z-T^{(q)}_{P,\lambda})\circ(S_-+S_+)\circ A_{z,N_0+1}\circ(S_-+S_+)+\frac{\partial^j}{\partial z^j}R_{z,N_0+1}\right)\Big|_{z=0}. \tag{4.1.111}$$

We can then use the same method above to recursively verify that for $j=2,3,\cdot,2N_0+3$ we also have

$$\mathbb{B}^{-,N_0+1}_{2N_0+3-j,j,0}(x,y)-f^{-,N_0+1}_{2N_0+3-j,j}(x,y)\Psi_-(x,y)=O(|x-y|^{+\infty}), \tag{4.1.112}$$

$$\mathbb{B}^{+,N_0+1}_{2N_0+3-j,j,0}(x,y)-f^{+,N_0+1}_{2N_0+3-j,j}(x,y)\Psi_+(x,y)=O(|x-y|^{+\infty}). \tag{4.1.113}$$

These relations enable us to apply integration by parts in t in (4.1.98), and after some straightforward arrangement we can see that up to a kernel associated to an element in $L^{-\infty}_z(\Omega;T^{*0,q}X)$ we have

$$\begin{aligned}
&\left((z-T^{(q)}_{P,\lambda})\circ\sum_{j=0}^{N_0+1}(S_-+S_+)\circ A_{z,j}\circ(S_-+S_+)-\Pi^{(q)}_\lambda\right)(x,y)\\
&\equiv\left((z-T^{(q)}_{P,\lambda})\circ(S_-+S_+)\circ A_{z,N_0+1}\circ(S_-+S_+)+R_{z,N_0+1}\right)(x,y)\\
&\equiv\int_0^{+\infty}e^{it\Psi_-(x,y)}\frac{\sum_{|\beta|+|\gamma|\leq 2N_0+4}R^{-,N_0+2}_{\beta,\gamma}(x,y,t)t^\beta z^\gamma}{(z-t)^{2N_0+4}}\tau(\varepsilon t)dt\\
&+\int_0^{+\infty}e^{it\Psi_+(x,y)}\frac{\sum_{|\beta|+|\gamma|\leq 2N_0+4}R^{+,N+1}_{\beta,\gamma}(x,y,t)t^\beta z^\gamma}{(z+t)^{2N_0+4}}\tau(\varepsilon t)dt
\end{aligned} \tag{4.1.114}$$

for some

$$R^{\mp,N_0+2}_{\beta,\gamma}(x,y,t)\in S^{n-N_0-2}_{\mathrm{cl}}(\Omega\times\Omega\times\mathbb{R}_+,\mathscr{L}(T^{*0,q}X,T^{*0,q}X)), \tag{4.1.115}$$

$$R^{\mp,N_0+2}_{\beta,\gamma}(x,y,t)\text{ properly supported in the variables }(x,y), \tag{4.1.116}$$

$$R^{+,N_0+2}_{\beta,\gamma}(x,y,t)=0\text{ when }n_-\neq n_+. \tag{4.1.117}$$

This completes the induction and the proof of our theorem. □

4.2. Helffer–Sjöstrand formula and the semi-classical estimates

In this section, we establish the semi-classical estimate for Helffer–Sjöstrand integral

$$\chi(k^{-1}T^{(q)}_{P,\lambda}) = \int_{\mathbb{C}} \frac{\partial \widetilde{\chi}(\frac{z}{k})}{\partial \overline{z}}(z - T^{(q)}_{P,\lambda})^{-1} \circ \Pi^{(q)}_{\lambda} \frac{dz \wedge d\overline{z}}{2\pi i} \tag{4.2.1}$$

in the operator level as $k \to +\infty$. To simplify the discussion we define some notations.

DEFINITION 4.8. We let

$$\tau \in \mathscr{C}^{\infty}(\mathbb{R}),\ \tau(t) = 0 \ \text{ for } t \leq 1,\ \tau(t) = 1 \ \text{ for } t \geq 2. \tag{4.2.2}$$

For $N \in \mathbb{N}_0$, we use the notation $S^{-N}_{\Sigma,z}(\Omega; T^{*0,q}X)$ to denote the space of Szegő type Fourier integral operators H_z smoothly dependent on z and associated by the kernels

$$\begin{aligned} H_z(x,y) := \int_0^{+\infty} & e^{it\psi_-(x,y)} \frac{\sum_{\alpha+\gamma\leq\beta} z^{\gamma} h^-_{\alpha,\gamma}(x,y,t)}{(z-t)^{\beta}} \tau(\varepsilon t) dt \\ & + \int_0^{+\infty} e^{it\psi_+(x,y)} \frac{\sum_{\alpha+\gamma\leq\beta} z^{\gamma} h^+_{\alpha,\gamma}(x,y,t)}{(z+t)^{\beta}} \tau(\varepsilon t) dt, \end{aligned} \tag{4.2.3}$$

where $\beta \in \mathbb{N}_0$ is arbitrary, $\alpha, \gamma \in \mathbb{N}_0$ and

$$h^{\mp}_{\alpha,\gamma}(x,y,t) \in S^{n-N+\alpha}_{\mathrm{cl}}(\Omega \times \Omega \times \mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X)), \tag{4.2.4}$$

$$h^{\mp}_{\alpha,\gamma}(x,y,t) \text{ is properly supported in the variables } (x,y), \tag{4.2.5}$$

$$h^{+}_{\alpha,\gamma}(x,y,t) = 0 \text{ when } n_- \neq n_+. \tag{4.2.6}$$

The consideration of function τ above is to avoid the possible blow-up of the negative power of t near 0 when we define $H_z(x,y)$. When $z = 0$, H_z is nothing but a Fourier integral operator of Szegő type.

DEFINITION 4.9. With the same notations and assumptions in Theorem 1.1, for $q = n_-$ and any $m \in \mathbb{Z}$, we let $\mathcal{I}^{-m}_{\Sigma,k}(\Omega; T^{*0,q}X)$ be the set of all k-dependent continuous operators $H_{(k)} : \mathscr{C}^{\infty}_0(\Omega, T^{*0,q}X) \to \mathscr{C}^{\infty}(X, T^{*0,q}X)$ such that the distribution kernel of $H_{(k)}$ satisfies

$$\begin{aligned} H_{(k)}(x,y) = \int_0^{+\infty} & e^{ikt\psi_-(x,y)} h^-(x,y,t,k) dt \\ & + \int_0^{+\infty} e^{ikt\psi_+(x,y)} h^+(x,y,t,k) dt + G_k(x,y), \end{aligned} \tag{4.2.7}$$

where $G_k = O(k^{-\infty})$ on $X \times \Omega$, $\psi_{\mp} \in \mathrm{Ph}(\mp\Lambda\boldsymbol{\alpha}, \Omega)$, $\Lambda \in \mathscr{C}^\infty(X, \mathbb{R}_+)$,

$$h^{\mp}(x,y,t,k) \sim \sum_{j=0}^{+\infty} h_j^{\mp}(x,y,t,k) \quad \text{in } S^{n+1-m}_{\mathrm{loc}}(1; \Omega \times \Omega \times \mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X)), \tag{4.2.8}$$

$$h_j^{\mp}(x,y,t,k) \in S^{n+1-m-j}_{\mathrm{loc}}(1; \Omega \times \Omega \times \mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X)), \tag{4.2.9}$$

$$h_j^{\mp}(x,y,t,k_0) \in S^{n-m-j}_{1,0}(\Omega \times \Omega \times \mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X)) \quad \text{for each } k_0 > 0, \tag{4.2.10}$$

$h^{\mp}(x,y,t,k)$ and $h_j^{\mp}(x,y,t,k)$ are properly supported in the variables (x,y) for all $j \in \mathbb{N}_0$, and $h^+(x,y,t,k) = 0$ if $n_- \neq n_+$.

We need the following lemma.

LEMMA 4.10. *For $H_z \in S^{-N}_{\Sigma,z}(\Omega; T^{*0,q}X)$ in Definition 4.8, we actually have*

$$\int_{\mathbb{C}} \frac{\partial \widetilde{\chi}(\frac{z}{k})}{\partial \bar{z}} H_z \frac{dz \wedge d\bar{z}}{2\pi i} \in \mathcal{I}^{-(N-1)}_{\Sigma,k}(\Omega; T^{*0,q}X). \tag{4.2.11}$$

PROOF. Without loss of generality, we take $q = n_- \neq n_+$, and the case $n_- = n_+$ can be deduced from the same argument with some minor changes. By using integration by parts to the variable t in the oscillatory integral, we can write

$$\int_{\mathbb{C}} \frac{\partial \widetilde{\chi}(\frac{z}{k})}{\partial \bar{z}} H_z(x,y) \frac{dz \wedge d\bar{z}}{2\pi i} \tag{4.2.12}$$

by

$$\begin{aligned} &\int_{\mathbb{C}} \frac{\partial \widetilde{\chi}(\frac{z}{k})}{\partial \bar{z}} \left(\int_0^{+\infty} e^{it\psi_-(x,y)} \frac{\sum_{\alpha+\gamma\leq\beta} z^\gamma h^-_{\alpha,\gamma}(x,y,t)}{(z-t)^\beta} \tau(\varepsilon t) dt \right) \frac{dz \wedge d\bar{z}}{2\pi i} \\ &= \int_{\mathbb{C}} \frac{\partial \widetilde{\chi}(\frac{z}{k})}{\partial \bar{z}} \frac{(-1)^{\beta-1}}{(\beta-1)!} \\ &\int_0^{+\infty} \left(\frac{\partial}{\partial t}\right)^{\beta-1} \left(e^{it\psi_-(x,y)} \sum_{\alpha+\gamma\leq\beta} z^\gamma h^-_{\alpha,\gamma}(x,y,t)\tau(\varepsilon t) \right) \frac{1}{z-t} dt \frac{dz \wedge d\bar{z}}{2\pi i}. \end{aligned} \tag{4.2.13}$$

Then, by the oscillatory integral version of Fubini theorem, we can write the last integral by

$$\begin{aligned}(4.2.14)\quad & \int_0^{+\infty}\frac{(-1)^{\beta-1}}{(\beta-1)!}\int_{\mathbb{C}}\frac{\partial\widetilde{\chi}(\frac{z}{k})}{\partial\overline{z}} \\ & \left(\frac{\partial}{\partial t}\right)^{\beta-1}\left(e^{it\psi_-(x,y)}\sum_{\alpha+\gamma\leq\beta}z^\gamma h^-_{\alpha,\gamma}(x,y,t)\tau(\varepsilon t)\right)\frac{1}{z-t}\frac{dz\wedge d\overline{z}}{2\pi i}dt.\end{aligned}$$

By Cauchy–Pompeiu formula, we have

$$\begin{aligned}(4.2.15)\quad & \int_{\mathbb{C}}\frac{\partial\widetilde{\chi}(\frac{z}{k})}{\partial\overline{z}}\left(\frac{\partial}{\partial t}\right)^{\beta-1}\left(e^{it\psi_-(x,y)}\sum_{\alpha+\gamma\leq\beta}\tau(\varepsilon t)h^-_{\alpha,\gamma}(x,y,t)z^\gamma\right)\frac{1}{z-t}\frac{dz\wedge d\overline{z}}{2\pi i} \\ & =\chi\left(\frac{t}{k}\right)\sum_{\alpha+\gamma\leq\beta}t^\gamma\left(\frac{\partial}{\partial t}\right)^{\beta-1}\left(e^{it\psi_-(x,y)}\tau(\varepsilon t)h^-_{\alpha,\gamma}(x,y,t)\right).\end{aligned}$$

So we know (4.2.14) equals to

$$\begin{aligned}(4.2.16)\quad & \frac{(-1)^{\beta-1}}{(\beta-1)!}\int_0^{+\infty}\chi\left(\frac{t}{k}\right)\sum_{\alpha+\gamma\leq\beta}t^\gamma\left(\frac{\partial}{\partial t}\right)^{\beta-1}\left(e^{it\psi_-(x,y)}\tau(\varepsilon t)h^-_{\alpha,\gamma}(x,y,t)\right)dt \\ & =\frac{(-1)^{2\beta-2}}{(\beta-1)!}\int_0^{+\infty}e^{it\psi_-(x,y)}\sum_{\alpha+\gamma\leq\beta-1}\tau(\varepsilon t)h^-_{\alpha,\gamma}(x,y,t)\frac{\partial^{\beta-1}}{\partial t^{\beta-1}}\left(t^\gamma\chi\left(\frac{t}{k}\right)\right)dt \\ & =\int_0^{+\infty}e^{ikt\psi_-(x,y)}\sum_{\alpha+\gamma\leq\beta-1}\frac{\tau(\varepsilon kt)h^-_{\alpha,\gamma}(x,y,kt)}{(\beta-1)!}k^{1+\gamma-(\beta-1)}\frac{\partial^{\beta-1}}{\partial t^{\beta-1}}(t^\gamma\chi(t))dt.\end{aligned}$$

By $h^-_{\alpha,\gamma}(x,y,t)\in S^{n-N+\alpha}_{\mathrm{cl}}\left(\Omega\times\Omega\times\mathbb{R}_+,\mathscr{L}(T^{*0,q}X,T^{*0,q}X)\right)$, we have the asymptotic expansion

$$\begin{aligned}(4.2.17)\quad & k^{1+\gamma-(\beta-1)}h^-_{\alpha,\gamma}(x,y,kt)\sim\sum_{j=0}^{+\infty}k^{n+1-(N-1)-j}h^-_{\alpha,\gamma,j}(x,y,t) \\ & \text{in } S^{n+1-(N-1)}_{\mathrm{loc}}\left(1;\Omega\times\Omega\times\mathbb{R}_+,\mathscr{L}(T^{*0,q}X,T^{*0,q}X)\right).\end{aligned}$$

We recall that $\varepsilon>0$ is a fixed number such that $\tau(\varepsilon t)\chi(t)=\chi(t)$ whenever $t\in\operatorname{supp}\chi\cap\mathbb{R}_+$. By our assumption on χ, when $k>0$ is large enough we can see the products between $\tau(\varepsilon kt)$ and derivatives of $\chi(t)$ are always nonzero. We also have $(\tau(\varepsilon t)-\tau(k\varepsilon t))\chi(t)\in S^{-\infty}_{\mathrm{loc}}(1;\mathbb{R}_+)$. By the definition of $\mathcal{I}^{-(N-1)}_{\Sigma,k}(\Omega;T^{*0,q}X)$, we hence complete the proof of our theorem. □

We can now prove the following important estimate.

THEOREM 4.11. *For any* $L_z \in L_z^{-\infty}(\Omega; T^{*0,q}X)$ *in Definition 4.5, we have*

$$\int_{\mathbb{C}} \frac{\partial \widetilde{\chi}(\frac{z}{k})}{\partial \overline{z}} L_z \frac{dz \wedge d\overline{z}}{2\pi i} = O\left(k^{-\infty}\right) \quad \text{on } X \times \Omega. \tag{4.2.18}$$

PROOF. First of all, when

$$L_z(x,y) = \int_0^{+\infty} e(x,y,t) \frac{1}{(z-t)^{M_1}} \tau(\varepsilon t) dt, \tag{4.2.19}$$

where $M_1 \in \mathbb{N}$ and the symbol $e(x,y,t) \in S^{-\infty}(\Omega \times \Omega \times \mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X))$ is properly supported in the variables (x,y), from the proof of Lemma 4.10 and especially the first line of (4.2.16), we can find some $E(x,y,t) \in S^{-\infty}(\Omega \times \Omega \times \mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X))$ properly supported in the variables (x,y) such that

(4.2.20)

$$\int_{\mathbb{C}} \frac{\partial \widetilde{\chi}(\frac{z}{k})}{\partial \overline{z}} L_z(x,y) \frac{dz \wedge d\overline{z}}{2\pi i} = \int_0^{+\infty} E(x,y,t) \chi(k^{-1}t) dt = O\left(k^{-\infty}\right) \quad \text{on } X \times \Omega.$$

After applying some minor modification, this method also works for any $L_z \in \mathcal{E}_z(\Omega; T^{*0,q}X)$ and any $L_z \in \mathcal{F}_z(\Omega; T^{*0,q}X)$.

Second, we consider

$$L_z(x,y) = \int_0^{+\infty} e^{it\psi(x,y)} g(x,y,t) \frac{1}{(z-t)^{M_1}} \tau(\varepsilon t) dt, \tag{4.2.21}$$

where $g(x,y,t) = O(|x-y|^{+\infty})$, $g(x,y,t) \in S^m_{\mathrm{cl}}(\Omega \times \Omega \times \mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X))$ for some $m \in \mathbb{R}$, $g(x,y,t)$ is properly supported in the variables (x,y), $M_1 \in \mathbb{N}_0$, and $\psi \in \mathrm{Ph}(-\Lambda\boldsymbol{\alpha}, \Omega)$ for some $\Lambda \in \mathscr{C}^\infty(X, \mathbb{R}_+)$. Again by the first line of (4.2.16), we can can find a $G(x,y,t) = O(|x-y|^{+\infty})$ properly supported in the variables (x,y) and $G(x,y,t) \in S^{m_1}_{\mathrm{cl}}(\Omega \times \Omega \times \mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X))$ for some $m_1 \in \mathbb{R}$ such that

$$\int_{\mathbb{C}} \frac{\partial \widetilde{\chi}(\frac{z}{k})}{\partial \overline{z}} L_z(x,y) \frac{dz \wedge d\overline{z}}{2\pi i} = \int_0^{+\infty} e^{it\psi(x,y)} G(x,y,t) \chi(k^{-1}t) dt. \tag{4.2.22}$$

We notice that for any point $p \in \Omega$, we have $\frac{\partial \psi}{\partial y_{2n+1}}(p,p) > 0$ from our assumption. From the Malgrange preparation theorem, we can check that

$$\psi(x,y) = f(x,y)(y_{2n+1} + \psi_0(x,y')) \tag{4.2.23}$$

near (p,p), where ψ_0 and g are smooth functions near (p,p), $f(p,p) > 0$, $\mathrm{Im}\psi \geq 0$ around (p,p) and $y' := (y_1, \cdots, y_{2n})$. When Ω is small enough, we may assume that (4.2.23) holds on $\Omega \times \Omega$ and as (2.3.47) we also have

$$\mathrm{Im}\, \psi(x,y) \geq C|x'-y'|^2 \quad \text{on } \Omega \times \Omega, \tag{4.2.24}$$

where $C > 0$ is a constant. We let $\widetilde{G}(x,y,t)$ be an almost analytic extension of $g(x,y,t)$ in the y_{2n+1} variables. For every $N \in \mathbb{N}$, by using Taylor expansion at

$y_{2n+1} = -\psi_0(x,y')$, we have

$$G(x,y,t) = \widetilde{G}(x,(y',y_{2n+1}),t) = \sum_{j=0}^{N} G_j(x,y',t)(y_{2n+1}+\psi_0(x,y'))^j + (y_{2n+1}+\psi_0(x,y'))^{N+1}R_{N+1}(x,y,t), \tag{4.2.25}$$

where

$$G_j(x,y',t) \in S^{m_1}_{\mathrm{cl}}(\Omega\times\Omega\times\mathbb{R}_+,\mathscr{L}(T^{*0,q}X,T^{*0,q}X)),\ j=1,\cdots,N, \tag{4.2.26}$$

$$R_{N+1}(x,y',t) \in S^{m_1}_{\mathrm{cl}}(\Omega\times\Omega\times\mathbb{R}_+,\mathscr{L}(T^{*0,q}X,T^{*0,q}X)). \tag{4.2.27}$$

On one hand, since $G(x,y,t) = O(|x-y|^{+\infty})$, by taking $(N+1)$-times derivatives of y_{2n+1} in (4.2.25) we can first check that

$$R_{N+1}(x,y,t) = O(|x-y|^{+\infty}). \tag{4.2.28}$$

Then similarly we can check that

$$G_j(x,y',t) = O(|x'-y'|^{+\infty}),\ \ j=N,N-1,\cdots,0. \tag{4.2.29}$$

We then let

$$\mathbb{G}_N(x,y,t) := e^{it\psi(x,y)}\sum_{j=0}^{N}(y_{2n+1}+\psi_0(x,y'))^j G_j(x,y',t), \tag{4.2.30}$$

and consider the operator $\mathbb{G}_{(k,N)}$ defined by kernel

$$\mathbb{G}_{(k,N)}(x,y) := \int_0^{+\infty}\mathbb{G}_N(x,y,t)\chi(k^{-1}t)dt. \tag{4.2.31}$$

From (4.2.24) and (4.2.29), we can check that

$$e^{it\psi(x,y)}G_j(x,y',t) \in S^{-\infty}(\Omega\times\Omega\times\mathbb{R}_+,\mathscr{L}(T^{*0,q}X,T^{*0,q}X)),\ j=1,\cdots,N, \tag{4.2.32}$$

$$\mathbb{G}_N(x,y,t) \in S^{-\infty}(\Omega\times\Omega\times\mathbb{R}_+,\mathscr{L}(T^{*0,q}X,T^{*0,q}X)). \tag{4.2.33}$$

Then by the result we just have proved in the first step we have

$$\mathbb{G}_{(k,N)} = O(k^{-\infty})\ \text{ on } X\times\Omega. \tag{4.2.34}$$

Also, we notice that by $(N+1)$-times of partial integration we have

$$\int_0^{+\infty} e^{it\psi(x,y)}(y_{2n+1}+\psi_0(x,y'))^{N+1}R_{N+1}(x,y,t)\chi(k^{-1}t)dt = \sum_{j=0}^{N+1}\int_0^{+\infty} e^{it\psi(x,y)}\gamma_{N+1,j}(x,y,t)k^{-(N+1-j)}\frac{\partial^{N+1-j}\chi}{\partial t}(k^{-1}t)dt, \tag{4.2.35}$$

where $\gamma_{N+1,j}(x,y,t) \in S^{m-N-1+j}_{\mathrm{cl}}(\Omega\times\Omega\times\mathbb{R}_+,\mathscr{L}(T^{*0,q}X),T^{*0,q}X)$ for each $j=0,\cdots,N+1$. By taking the Taylor expansion of $g(x,y,t)$ to arbitrary high order N and by the condition that $G(x,y,t)$ is properly supported in (x,y), the above arguments imply that

$$\int_{\mathbb{C}}\frac{\partial\widetilde{\chi}(\frac{z}{k})}{\partial\overline{z}}L_z(x,y)\frac{dz\wedge d\overline{z}}{2\pi i}=O\left(k^{-\infty}\right)\quad\text{on } X\times\Omega \tag{4.2.36}$$

for L_z in the form of (4.2.21). With some minor change of the argument we just used, this method also works for any $L_z\in\mathcal{G}_z(\Omega;T^{*0,q}X)$.

Finally, we notice that for $L_z\in\mathcal{R}_z(\Omega;T^{*0,q}X)$ of the form

$$L_z(x,y)=\int_0^{+\infty}\int_0^{+\infty}\int_\Omega e^{it\psi_-(x,w)+i\sigma\psi_+(w,y)}r_1(x,w,t)\circ r_2(w,y,\sigma)\frac{z^{M_2}}{(z-t)^{M_1}}\tau(\varepsilon t)m(w)dwd\sigma dt, \tag{4.2.37}$$

where $\psi_\mp\in\mathrm{Ph}(\mp\Lambda\boldsymbol{\alpha},\Omega)$ for some $\Lambda\in\mathscr{C}^\infty(X,\mathbb{R}_+)$ and r_1,r_2 are Hörmander symbols, by the properties that $\psi(x,w)=0$ when $x=w$, $\psi(w,y)=0$ when $w=y$, $d_w\psi_-(x,w)=d_w\psi_+(w,y)$ at $w=x=y$, $t\geq 0$ and $\sigma\geq 0$, we have the following observation: given a suitably small $\delta>0$, when $|x-w|>\delta$ we can apply arbitrary times of partial integration in t; when $|w-y|>\delta$ we can apply arbitrary times of partial integration in σ; when $|x-y|<2\delta$ we can apply arbitrary times of partial integration in w. Then by this observation and the proof of the previous lemma we can check that

$$\int_{\mathbb{C}}\frac{\partial\widetilde{\chi}(\frac{z}{k})}{\partial\overline{z}}L_z(x,y)\frac{dz\wedge d\overline{z}}{2\pi i}=O\left(k^{-\infty}\right)\quad\text{on } X\times\Omega, \tag{4.2.38}$$

and again with some minor changes the same argument also works for general $L_z\in\mathcal{R}_z(\Omega;T^{*0,q}X)$. □

The next Theorem follows directly from Lemma 4.10 and Theorems 4.7 and 4.11.

THEOREM 4.12. *With the same notations and assumptions in Theorem 4.7, for any* $m\in\mathbb{N}_0$ *and* $\mathscr{A}_{z,m}:=(S_-+S_+)\circ A_{z,m}\circ(S_-+S_+)$, *we have*

$$A_{(k,m)}:=\int_{\mathbb{C}}\frac{\partial\widetilde{\chi}(\frac{z}{k})}{\partial\overline{z}}\mathscr{A}_{z,m}\frac{dz\wedge d\overline{z}}{2\pi i}\in\mathcal{I}^{-m}_{\Sigma,k}(\Omega;T^{*0,q}X). \tag{4.2.39}$$

In fact, up to an k-negligible kernel on $X\times\Omega$ *we have*

$$A_{(k,m)}(x,y)\equiv\int_0^{+\infty}e^{ikt\Psi_-(x,y)}a^{-,m}(x,y,t,k)dt+\int_0^{+\infty}e^{ikt\Psi_+(x,y)}a^{+,m}(x,y,t,k)dt, \tag{4.2.40}$$

where

$$\Psi_- \in \mathrm{Ph}(p^{-1}_{I_0,I_0}(-\boldsymbol{\alpha})(-\boldsymbol{\alpha}),\Omega), \tag{4.2.41}$$

$$\Psi_+ \in \mathrm{Ph}(p^{-1}_{J_0,J_0}(-\boldsymbol{\alpha})\boldsymbol{\alpha},\Omega), \tag{4.2.42}$$

and we have the following data are properly supported in (x,y)*:*

$$a^{\mp,m}(x,y,t,k) \sim \textstyle\sum_{j=0}^{+\infty} a_j^{\mp,m}(x,y,t)k^{n+1-m-j} \quad \text{in } S^{n+1-m}_{\mathrm{loc}}(1;\Omega\times\Omega\times\mathbb{R}_+,\mathscr{L}(T^{*0,q}X,T^{*0,q}X)), \tag{4.2.43}$$

$$\forall j\in\mathbb{N}_0,\ a_j^{\mp,m}(x,y,t)\neq 0 \implies t\in\operatorname{supp}\chi\,, \tag{4.2.44}$$

$$a^{\mp,m}(x,y,t)\neq 0 \implies t\in\operatorname{supp}\chi\,, \tag{4.2.45}$$

and

$$a^{+}(x,y,t,k)=0 \text{ when } n_-\neq n_+\,. \tag{4.2.46}$$

Moreover, for $m=0$*,*

$$a_0^{-,0}(x,x,t)=\frac{|\det\mathcal{L}_x|}{2\pi^{n+1}}\frac{v(x)}{m(x)}\,p^{-n-1}_{I_0,I_0}(-\boldsymbol{\alpha}_x)\,\chi(t)t^n, \tag{4.2.47}$$

$$a_0^{+,0}(x,x,t)=\frac{|\det\mathcal{L}_x|}{2\pi^{n+1}}\frac{v(x)}{m(x)}\,p^{-n-1}_{J_0,J_0}(-\boldsymbol{\alpha}_x)\,\chi(-t)t^n \text{ when } n_-=n_+\,. \tag{4.2.48}$$

From Theorem 4.12, now we have

$$\begin{aligned}
&\chi(k^{-1}T^{(q)}_{P,\lambda})\\
&=\int_{\mathbb{C}}\frac{\partial\widetilde{\chi}(\frac{z}{k})}{\partial\bar z}(z-T^{(q)}_{P,\lambda})^{-1}\circ\Pi^{(q)}_{\lambda}\frac{dz\wedge d\bar z}{2\pi i}\\
&=\int_{\mathbb{C}}\frac{\partial\widetilde{\chi}(\frac{z}{k})}{\partial\bar z}\left(\sum_{m=0}^{N}\mathscr{A}_{z,m}+(z-T^{(q)}_{P,\lambda})^{-1}(R_{z,N+1}+F_{z,N+1})\right)\frac{dz\wedge d\bar z}{2\pi i}\\
&=\sum_{m=0}^{N}A_{(k,m)}+R_{(k,N+1)}+F_{(k,N+1)},
\end{aligned}\tag{4.2.49}$$

where $R_{z,N+1}$ is as described in Theorem 4.7, $F_{z,N+1}\in L^{-\infty}_z(\Omega;T^{*0,q}X)$, and

$$R_{(k,N+1)}:=\int_{\mathbb{C}}\frac{\partial\widetilde{\chi}(\frac{z}{k})}{\partial\bar z}(z-T^{(q)}_{P,\lambda})^{-1}\circ R_{z,N+1}\frac{dz\wedge d\bar z}{2\pi i}, \tag{4.2.50}$$

$$F_{(k,N+1)}:=\int_{\mathbb{C}}\frac{\partial\widetilde{\chi}(\frac{z}{k})}{\partial\bar z}(z-T^{(q)}_{P,\lambda})^{-1}\circ F_{z,N+1}\frac{dz\wedge d\bar z}{2\pi i}. \tag{4.2.51}$$

We are going to show that for any $N\in\mathbb{N}_0$ we have

$$F_{(k,N_0+1)}=O(k^{-N}) \text{ in } \mathscr{L}(H^{-N}_{\mathrm{comp}}(\Omega,T^{*0,q}X),H^{N}(X,T^{*0,q}X)) \tag{4.2.52}$$

and for any $N_1, N_2 \in \mathbb{N}_0$ we can find an $N_0 > 0$ large enough such that

$$R_{(k,N_0+1)} = O(k^{-N_1}) \text{ in } \mathscr{L}(H^{-N_2}_{\mathrm{comp}}(\Omega, T^{*0,q}X), H^{N_2}(X, T^{*0,q}X)). \tag{4.2.53}$$

To proceed further, we need the following resolvent estimate.

THEOREM 4.13. *For $z \notin \mathrm{Spec}(T^{(q)}_{P,\lambda})$ and any $s \in \mathbb{N}_0$, we have*

$$\Pi^{(q)}_\lambda \circ (z - T^{(q)}_{P,\lambda})^{-1} = O\left(\frac{|z|^s}{|\mathrm{Im}\, z|}\right) \text{ in } \mathscr{L}(H^s(X, T^{*0,q}X), H^{s+1}(X, T^{*0,q}X)). \tag{4.2.54}$$

PROOF. From Theorem 3.6, we have some $Q \in L^{-1}_{\mathrm{cl}}(X; T^{*0,q}X)$, $T^{(q)}_{Q,\lambda} := \Pi^{(q)}_\lambda \circ Q \circ \Pi^{(q)}_\lambda$ and $F \in L^{-\infty}(X; T^{*0,q}X)$ such that

$$T^{(q)}_{Q,\lambda} \circ (z - T^{(q)}_{P,\lambda}) = zT^{(q)}_{Q,\lambda} - \Pi^{(q)}_\lambda + F. \tag{4.2.55}$$

This implies that

$$\Pi^{(q)}_\lambda \circ (z - T^{(q)}_{P,\lambda})^{-1} = -T^{(q)}_{Q,\lambda} + zT^{(q)}_{Q,\lambda} \circ (z - T^{(q)}_{P,\lambda})^{-1} + F \circ (z - T^{(q)}_{P,\lambda})^{-1}. \tag{4.2.56}$$

From Theorem 3.8 and the spectral theory of self-adjoint operators, we have

$$(z - T^{(q)}_{P,\lambda})^{-1} = O\left(\frac{1}{|\mathrm{Im}\, z|}\right) \text{ in } \mathscr{L}(L^2(X, T^{*0,q}X), L^2(X, T^{*0,q}X)). \tag{4.2.57}$$

By the above estimate, (3.2.12) and (4.2.56), we immediately have

$$\Pi^{(q)}_\lambda \circ (z - T^{(q)}_{P,\lambda})^{-1} = O\left(\frac{|z|}{|\mathrm{Im}\, z|}\right) \text{ in } \mathscr{L}(L^2(X, T^{*0,q}X), H^1(X, T^{*0,q}X)). \tag{4.2.58}$$

We can put this estimate back to (4.2.56), then by $T^{(q)}_{Q,\lambda} \circ (z - T^{(q)}_{P,\lambda})^{-1} = T^{(q)}_{Q,\lambda} \circ \Pi^{(q)}_\lambda \circ (z - T^{(q)}_{P,\lambda})^{-1}$ and the same argument and estimate we just used, we have

$$\Pi^{(q)}_\lambda \circ (z - T^{(q)}_{P,\lambda})^{-1} = O\left(\frac{|z|^2}{|\mathrm{Im}\, z|}\right) \text{ in } \mathscr{L}(H^1(X, T^{*0,q}X), H^2(X, T^{*0,q}X)). \tag{4.2.59}$$

We can hence bootstrap and get our theorem. □

Now we can prove the following.

THEOREM 4.14. *With the same notations and assumptions in Theorem 4.7, for any operator $E_z \in \mathcal{E}_z(\Omega; T^{*0,q}X)$ and $N \in \mathbb{N}_0$, we have*

$$\begin{aligned} &\int_{\mathbb{C}} \frac{\partial \widetilde{\chi}(\frac{z}{k})}{\partial \bar{z}} (z - T^{(q)}_{P,\lambda})^{-1} \circ E_z \frac{dz \wedge d\bar{z}}{2\pi i} \\ &\qquad = O(k^{-N}) \text{ in } \mathscr{L}\left(H^{-N}_{\mathrm{comp}}(\Omega, T^{*0,q}X), H^N(X, T^{*0,q}X)\right). \end{aligned} \tag{4.2.60}$$

PROOF. For simplicity, we assume that the kernel of E_z is given by

$$E_z(x,y) = \int_0^{+\infty} e(x,y,t)\frac{z^{M_2}}{(z-t)^{M_1}}\tau(\varepsilon t)dt, \tag{4.2.61}$$

where $e(x,y,t) \in S^{-\infty}(\Omega\times\Omega\times\mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X))$ is properly supported in the variables (x,y), and $M_1, M_2 \in \mathbb{N}_0$. The general situation can be deduced from some straightforward modification of the below argument.

We notice that when $z \notin \operatorname{Spec}(T^{(q)}_{P,\lambda})$ we have

$$\begin{aligned}(z-T^{(q)}_{P,\lambda})^{-1} &= \frac{T^{(q)}_{P,\lambda}}{z}\circ(z-T^{(q)}_{P,\lambda})^{-1} + \frac{I}{z} \\ &= \cdots = \frac{T^{(q),M}_{P,\lambda}}{z^M}\circ(z-T^{(q)}_{P,\lambda})^{-1} + \sum_{j=0}^{M-1}\frac{T^{(q),j}_{P,\lambda}}{z^{1+j}}\end{aligned} \tag{4.2.62}$$

where $M \in \mathbb{N}$ is arbitrary,

$$T^{(q),j}_{P,\lambda} := T^{(q)}_{P,\lambda}\circ\cdots\circ T^{(q)}_{P,\lambda} \tag{4.2.63}$$

is the j-times composition between $T^{(q)}_{P,\lambda}$ and $T^{(q),0}_{P,\lambda} := I$. Also, we notice that $z \neq 0$ when $z \in \operatorname{supp}\widetilde{\chi}(\frac{\cdot}{k})$, and from $\Pi^{(q)}_\lambda\circ\Pi^{(q)}_\lambda = \Pi^{(q)}_\lambda$ and $[\Pi^{(q)}_\lambda, T^{(q)}_{P,\lambda}] = 0$ we can also check that

$$\left[\Pi^{(q)}_\lambda, (z-T^{(q)}_{P,\lambda})^{-1}\right] = 0 \tag{4.2.64}$$

when $z \in \operatorname{supp}\widetilde{\chi}(\frac{\cdot}{k})$.

Then, on one hand, for the integral

$$\int_{\mathbb{C}}\frac{\partial\widetilde{\chi}(\frac{z}{k})}{\partial\bar{z}}z^{-1-j}\,T^{(q),j}_{P,\lambda}\circ E_z\frac{dz\wedge d\bar{z}}{2\pi i} = T^{(q),j}_{P,\lambda}\circ\int_{\mathbb{C}}\frac{\partial\widetilde{\chi}(\frac{z}{k})}{\partial\bar{z}}z^{-1-j}\,E_z\frac{dz\wedge d\bar{z}}{2\pi i}, \tag{4.2.65}$$

we can apply Fubini theorem in the sense of oscillatory integrals so that

$$\begin{aligned}&\int_{\mathbb{C}}\frac{\partial\widetilde{\chi}(\frac{z}{k})}{\partial\bar{z}}z^{-1-j}\,E_z(x,y)\frac{dz\wedge d\bar{z}}{2\pi i} \\ &= \int_{\mathbb{C}}\frac{\partial\widetilde{\chi}(\frac{z}{k})}{\partial\bar{z}}z^{-1-j}\int_0^{+\infty} e(x,y,t)\frac{z^{M_2}}{(z-t)^{M_1}}\tau(\varepsilon t)dt\frac{dz\wedge d\bar{z}}{2\pi i}.\end{aligned} \tag{4.2.66}$$

Using (M_1-1)-times of integration by parts to t, we can find the above integral equals to

$$\int_0^{+\infty}\int_{\mathbb{C}}\frac{\partial}{\partial\bar{z}}\left(\widetilde{\chi}\left(\frac{z}{k}\right)z^{-1-j+M_2}\right)(z-t)^{-1}\frac{dz\wedge d\bar{z}}{2\pi i}\delta(x,y,t)dt, \tag{4.2.67}$$

where $\delta(x,y,t) \in S^{-\infty}(\Omega\times\Omega\times\mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X))$ is properly supported in the variables (x,y) and $\delta(x,y,t) \neq 0$ if and only if $\varepsilon t \in \operatorname{supp}\tau$.

So we can apply Cauchy–Pompeiu formula and get

$$\text{(4.2.68)}\quad \int_0^{+\infty}\int_{\mathbb{C}} \frac{\partial}{\partial \bar z}\left(\widetilde{\chi}\left(\frac{z}{k}\right) z^{-1-j+M_2}\right)(z-t)^{-1}\frac{dz\wedge d\bar z}{2\pi i}\delta(x,y,t)dt = \int_0^{+\infty}\chi\left(\frac{t}{k}\right)t^{-1-j+M_2}\delta(x,y,t)dt = O\left(k^{-\infty}\right).$$

By the Sobolev-boundedness of $T^{(q)}_{P,\lambda}$, we know that this part of integral satisfies the estimate we want.

On the other hand, for arbitrary $M\in\mathbb{N}_0$ such that $M\equiv 0 \mod 4$, we have the integral

$$\text{(4.2.69)}\quad \int_{\mathbb{C}} \frac{\partial\widetilde{\chi}(\frac{z}{k})}{\partial \bar z} z^{-M}\, T^{(q),M}_{P,\lambda}\circ(z-T^{(q)}_{P,\lambda})^{-1}\circ E_z\frac{dz\wedge d\bar z}{2\pi i} = T^{(q),\frac{M}{2}}_{P,\lambda}\circ\int_{\mathbb{C}} \frac{\partial\widetilde{\chi}(\frac{z}{k})}{\partial \bar z} z^{-M}\,\Pi^{(q)}_{\lambda}\circ(z-T^{(q)}_{P,\lambda})^{-1}\circ T^{(q),\frac{M}{2}}_{P,\lambda}\circ E_z\frac{dz\wedge d\bar z}{2\pi i}.$$

By Theorem 4.13, the continuity of $T^{(q)}_{P,\lambda}$ between Sobolev spaces and the direct estimate of E_z, we can check that for any $M\in\mathbb{N}_0$ we have

$$\text{(4.2.70)}\quad T^{(q),\frac{M}{2}}_{P,\lambda}\circ\int_{\mathbb{C}} \frac{\partial\widetilde{\chi}(\frac{z}{k})}{\partial \bar z} z^{-M}\,\Pi^{(q)}_{\lambda}\circ(z-T^{(q)}_{P,\lambda})^{-1}\circ T^{(q),\frac{M}{2}}_{P,\lambda}\circ E_z\frac{dz\wedge d\bar z}{2\pi i}$$
$$= O\left(\sup_{k^{-1}z\in\operatorname{supp}\widetilde{\chi}} k^2\cdot\frac{|\mathrm{Im}\, z|^{1+M_1}}{k^{1+M_1}}\cdot|z|^{-M}\cdot\frac{|z|^{\frac{M}{2}+\frac{M}{4}-1}}{|\mathrm{Im}\, z|}\cdot\frac{|z|^{M_2}}{|\mathrm{Im}\, z|^{M_1}}\right)$$
$$= O(k^{-\frac{M}{4}+M_2-M_1}) \text{ in } \mathscr{L}\left(H^{-\frac{M}{4}}_{\mathrm{comp}}(\Omega,T^{*0,q}X),H^{M}(X,T^{*0,q}X)\right).$$

Combining all the estimates above we complete the proof. □

We would like to note that during the proof of the previous theorem, the step where we split $T^{M}_{P,\lambda}$ into $T^{\frac{M}{2}}_{P,\lambda}\circ T^{\frac{M}{2}}_{P,\lambda}$ is crucial. This step is designed to prevent the argument from breaking down when we apply Theorem 4.13. Specifically, it helps us avoid a situation where the term $|z|^s$ contributes an excessive power of k, which can occur when s is too large.

With the same proof, we also have the following.

THEOREM 4.15. *With the same notations and assumptions in Theorem 4.7, for any operator $F_z\in\mathcal{F}_z(\Omega;T^{*0,q}X)$ and $N\in\mathbb{N}_0$ we have*

$$\text{(4.2.71)}\quad \int_{\mathbb{C}} \frac{\partial\widetilde{\chi}(\frac{z}{k})}{\partial \bar z}(z-T^{(q)}_{P,\lambda})^{-1}\circ F_z\frac{dz\wedge d\bar z}{2\pi i} = O(k^{-N}) \text{ in } \mathscr{L}\left(H^{-N}_{\mathrm{comp}}(\Omega,T^{*0,q}X),H^{N}(X,T^{*0,q}X)\right).$$

The next kind of remainder estimate needs more work.

THEOREM 4.16. *With the same notations and assumptions in Theorem 4.7, for any operator* $G_z \in \mathcal{G}_z(\Omega; T^{*0,q}X)$ *and* $N \in \mathbb{N}_0$, *we have*

$$\int_{\mathbb{C}} \frac{\partial \widetilde{\chi}(\frac{z}{k})}{\partial \overline{z}}(z - T^{(q)}_{P,\lambda})^{-1} \circ G_z \frac{dz \wedge d\overline{z}}{2\pi i} = O(k^{-N}) \text{ in } \mathscr{L}\left(H^{-N}_{\mathrm{comp}}(\Omega, T^{*0,q}X), H^N(X, T^{*0,q}X)\right). \tag{4.2.72}$$

PROOF. For simplicity, we prove the case for $q = n_- = 0$ and

$$G_z(x,y) = \int_0^{+\infty} e^{it\psi(x,y)} g(x,y,t) \frac{z^{M_2}}{(z-t)^{M_1}} \tau(\varepsilon t) dt, \tag{4.2.73}$$

where $g(x,y,t) = O(|x-y|^{+\infty})$ is in $S^m_{\mathrm{cl}}(\Omega \times \Omega \times \mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X))$ for some $m \in \mathbb{R}$ and is properly supported in the variables (x,y), $M_1, M_2 \in \mathbb{N}_0$, and $\psi \in \mathrm{Ph}(-\Lambda\boldsymbol{\alpha}, \Omega)$ for some $\Lambda \in \mathscr{C}^\infty(X, \mathbb{R}_+)$. The general situation can be deduced from some straightforward modification of the following argument.

As in the proof of Theorem 4.11, we may assume that

$$\psi(x,y) = f(x,y)(y_{2n+1} + \psi_0(x,y')) \text{ on } \Omega \times \Omega. \tag{4.2.74}$$

Also, as (2.3.47) we may assume that

$$\mathrm{Im}\, \psi(x,y) \geq C|x'-y'|^2 \text{ on } \Omega \times \Omega, \tag{4.2.75}$$

where $C > 0$ is a constant. We let $\widetilde{g}(x,y,t)$ be an almost analytic extension of $g(x,y,t)$ in the y_{2n+1} variables. For every $N \in \mathbb{N}$, by Taylor expansion we have

$$g(x,y,t) = \sum_{j=0}^{N} g_j(x,y',t)(y_{2n+1} + \psi_0(x,y'))^j + (y_{2n+1} + \psi_0(x,y'))^{N+1} r_{N+1}(x,y,t), \tag{4.2.76}$$

where

$$g_j(x,y',t) \in S^m_{\mathrm{cl}}(\Omega \times \Omega \times \mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X)), \ j = 1, \cdots, N, \tag{4.2.77}$$

$$r_{N+1}(x,y',t) \in S^m_{\mathrm{cl}}(\Omega \times \Omega \times \mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X)). \tag{4.2.78}$$

On one hand, since $g(x,y,t) = O(|x-y|^{+\infty})$, we can check that

$$r_{N+1}(x,y,t) = O(|x-y|^{+\infty}). \tag{4.2.79}$$

Then, we also have

$$g_j(x,y',t) = O(|x'-y'|^{+\infty}), \ j = N, N-1, \cdots, 0. \tag{4.2.80}$$

We let

$$\mathbb{G}_N(x,y,t) := e^{it\psi(x,y)} \sum_{j=0}^{N} (y_{2n+1} + \psi_0(x,y'))^j g_j(x,y',t), \tag{4.2.81}$$

and consider the operator $G_{z,N}$ by kernel

$$\mathbb{G}_{z,N}(x,y) := \int_0^{+\infty} e^{it\psi(x,y)}\mathbb{G}_N(x,y,t)\frac{z^{M_2}}{(z-t)^{M_1}}\tau(\varepsilon t)dt. \tag{4.2.82}$$

From (4.2.75) and (4.2.80), we can check that

$$\mathbb{G}_N(x,y,t) \in S^{-\infty}(\Omega\times\Omega\times\mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X)). \tag{4.2.83}$$

This implies that $\mathbb{G}_{z,N} \in \mathcal{E}_z(\Omega; T^{*0,q}X)$, and Theorem 4.14 implies that on $X\times\Omega$ we have

$$\int \frac{\partial\widetilde{\chi}(\frac{z}{k})}{\partial\bar{z}}(z-T^{(q)}_{P,\lambda})^{-1}\circ\mathbb{G}_{z,N}\frac{dz\wedge d\bar{z}}{2\pi i} = O(k^{-\infty}) \ \text{ on } X\times\Omega. \tag{4.2.84}$$

On the other hand, for the operator $\zeta_{z,N}$ associated by the kernel

$$\begin{aligned}\zeta_{z,N}(x,y)& \\ := \int_0^{+\infty} & e^{it\psi(x,y)}(y_{2n+1}+\psi_0(x,y'))^{N+1}r_{N+1}(x,y,t)\frac{z^{M_2}}{(z-t)^{M_1}}\tau(\varepsilon t)dt,\end{aligned} \tag{4.2.85}$$

by

$$e^{it\psi(x,y)}(y_{2n+1}+\psi_0(x,y'))^{N+1} = (if(x,y))^{-N-1}\frac{\partial^{N+1}}{\partial t^{N+1}}e^{it\psi(x,y)} \tag{4.2.86}$$

and integration by parts in t, we can also write

$$\zeta_{z,N}(x,y) = \int_0^{+\infty} e^{it\psi(x,y)}z^{M_2}\frac{\partial^{N+1}}{\partial t^{N+1}}\Big((z-t)^{-M_1}\cdot r^f_{N+1}(x,y,t)\cdot\tau(\varepsilon t)\Big)\,dt \tag{4.2.87}$$

for some $r^f_{N+1}(x,y,t) \in S^m_{\mathrm{cl}}(\Omega\times\Omega\times\mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X))$. Now for the operator

$$\zeta_{(k,N)} := \int_{\mathbb{C}\setminus\mathrm{Spec}(T^{(q)}_{P,\lambda})} \frac{\partial\widetilde{\chi}(\frac{z}{k})}{\partial\bar{z}}(z-T^{(q)}_{P,\lambda})^{-1}\circ\zeta_{z,N}\frac{dz\wedge d\bar{z}}{2\pi i}, \tag{4.2.88}$$

we recall that when $z\notin\mathrm{Spec}(T^{(q)}_{P,\lambda})$ we have

$$(z-T^{(q)}_{P,\lambda})^{-1} = \frac{T^{(q),M}_{P,\lambda}}{z^M}\circ(z-T^{(q)}_{P,\lambda})^{-1} + \sum_{j=0}^{M-1}\frac{T^{(q),j}_{P,\lambda}}{z^{1+j}}, \tag{4.2.89}$$

where $M \in \mathbb{N}$ is arbitrary, $T^{(q),j}_{P,\lambda} := T^{(q)}_{P,\lambda} \circ \cdots \circ T^{(q)}_{P,\lambda}$ is the j-times composition between $T^{(q)}_{P,\lambda}$ and $T^{(q),0}_{P,\lambda} := I$. So we can write

$$\zeta_{(k,N)} = \int_{\mathbb{C}\setminus \mathrm{Spec}(T^{(q)}_{P,\lambda})} \frac{\partial \widetilde{\chi}(\frac{z}{k})}{\partial \overline{z}} \frac{T^{(q),M}_{P,\lambda}}{z^M} \circ (z - T^{(q)}_{P,\lambda})^{-1} \circ \zeta_{z,N} \frac{dz \wedge d\overline{z}}{2\pi i} + \sum_{j=0}^{M-1} T^{(q),j}_{P,\lambda} \circ \int_{\mathbb{C}\setminus \mathrm{Spec}(T^{(q)}_{P,\lambda})} \frac{\partial \widetilde{\chi}(\frac{z}{k})}{\partial \overline{z}} z^{-1-j}\, \zeta_{z,N} \frac{dz \wedge d\overline{z}}{2\pi i}. \tag{4.2.90}$$

By the proof of Theorem 4.11, we can check that for all $M \in \mathbb{N}$ we have

$$\sum_{j=0}^{M-1} T^{(q),j}_{P,\lambda} \circ \int_{\mathbb{C}\setminus \mathrm{Spec}(T^{(q)}_{P,\lambda})} \frac{\partial \widetilde{\chi}(\frac{z}{k})}{\partial \overline{z}} z^{-1-j}\, \zeta_{z,N} \frac{dz \wedge d\overline{z}}{2\pi i} = O\left(k^{-\infty}\right) \text{ on } X \times \Omega. \tag{4.2.91}$$

It remains to handle the estimate for the first part of the integral in (4.2.90) for large N. For this purpose we need to choose some suitably large number M, which is an arbitrary number in (4.2.90). When $N \in \mathbb{N}$ is large enough and

$$N + M_1 - m \equiv 0 \mod 4, \tag{4.2.92}$$

we take

$$M := \frac{N + M_1 - m}{2}. \tag{4.2.93}$$

Then, from the formula of $\zeta_{z,N}$ in (4.2.87), the observation that $\frac{\partial}{\partial t}\tau(\varepsilon t) = O(t^{-\infty})$, and the elementary estimate

$$\frac{1}{|z-t|^2} = \frac{|\overline{z}|^2}{|z\overline{z} - t\overline{z}|^2} \leq \frac{|z|^2}{|\mathrm{Im}\,(z\overline{z} - t\overline{z})|^2} = \frac{|z|^2}{|\mathrm{Im} z|^2\, t^2}, \tag{4.2.94}$$

up to a kernel associated by an element in $\mathcal{E}_z(\Omega; T^{*0,q}X)$ we can write

$$\zeta_{z,N}(x,y) = \int_0^{+\infty} e^{it\psi(x,y)} R^f_{N+1}(x,y,t,z)\tau(\varepsilon t)dt, \tag{4.2.95}$$

where for all multi-indices α, β, γ we have some constant $c_{K,\alpha,\beta,\gamma} > 0$ such that

$$|\partial^\alpha_x \partial^\beta_y \partial^\gamma_t R^f_{N+1}(x,y,t,z)| \leq c_{K,\alpha,\beta,\gamma} \frac{|z|^{M_1+M_2+(N+1)}}{|\mathrm{Im}\, z|^{M_1+(N+1)}} t^{m-M_1-(N+1)} \tag{4.2.96}$$

when $x, y \in K \Subset \Omega$ and $t \in \mathbb{R}_+$ such that $\tau(\varepsilon t) > 0$. By the above estimate of $R^f_{N+1}(x,y,t,z)$, we can check that up to a an element in $\mathcal{E}_z(\Omega; T^{*0,q}X)$ we have

$$\zeta_{z,N} = O\left(\frac{|z|^{M_1+M_2+(N+1)}}{|\mathrm{Im}\, z|^{M_1+(N+1)}}\right) \quad \text{in } \mathscr{L}\left(H^{-\frac{3M}{4}}_{\mathrm{comp}}(\Omega, T^{*0,q}X), H^{\frac{M}{4}+\frac{M}{2}-1+\frac{M}{2}}(X, T^{*0,q}X)\right). \tag{4.2.97}$$

Then, by combining all the above estimates above and Theorem 4.14, for any $N \in \mathbb{N}$ which is arbitrarily large enough such that $N + M_1 - m \equiv 0 \mod 4$, then the number $2M := N + M_1 - m$ is consequently arbitrarily large and we have

$$\begin{aligned}
&\int_{\mathbb{C}\setminus\mathrm{Spec}(T^{(q)}_{P,\lambda})} \frac{\partial\widetilde{\chi}(\frac{z}{k})}{\partial\overline{z}} z^{-M}\, T^{(q),M}_{P,\lambda} \circ (z - T^{(q)}_{P,\lambda})^{-1} \circ \zeta_{z,N} \frac{dz\wedge d\overline{z}}{2\pi i}\\
&= \int_{\mathbb{C}\setminus\mathrm{Spec}(T^{(q)}_{P,\lambda})} \frac{\partial\widetilde{\chi}(\frac{z}{k})}{\partial\overline{z}} z^{-M}\, T^{(q),\frac{M}{2}}_{P,\lambda} \circ \left(\Pi^{(q)}_{\lambda} \circ (z - T^{(q)}_{P,\lambda})^{-1}\right) \circ T^{(q),\frac{M}{2}}_{P,\lambda} \circ \zeta_{z,N} \frac{dz\wedge d\overline{z}}{2\pi i}\\
&= O\left(\sup_{k^{-1}z\in\mathrm{supp}\,\widetilde{\chi}} k^2 \cdot \frac{|\mathrm{Im}\, z|^{1+M_1+(N+1)}}{k^{1+M_1+(N+1)}} \cdot k^{-M} \cdot \frac{|z|^{\frac{M}{4}+\frac{M}{2}-1}}{|\mathrm{Im}\, z|} \cdot \frac{|z|^{M_1+M_2+(N+1)}}{|\mathrm{Im}\, z|^{M_1+(N+1)}}\right)\\
&= O(k^{-\frac{M}{4}+M_2}) \ \text{ in } \mathscr{L}\left(H^{-\frac{3M}{4}}_{\mathrm{comp}}(\Omega, T^{*0,q}X), H^{\frac{M}{2}}(X, T^{*0,q}X)\right).
\end{aligned} \tag{4.2.98}$$

Here we recall that for $\chi \in \mathscr{C}^\infty_0(\dot{\mathbb{R}})$ we can take $\widetilde{\chi}$ such that $\widetilde{\chi} \in \mathscr{C}^\infty_0(\dot{\mathbb{C}})$, so there is a constant $C > 0$ such that $\frac{k}{C} < |z| < Ck$ when $k^{-1}z \in \mathrm{supp}\,\widetilde{\chi}$.

Combining all the estimates above, we finish our proof. □

We also have the following.

THEOREM 4.17. *With the same notations and assumptions in Theorem 4.7, for any operator $\mathscr{R}_z \in \mathcal{R}_z(\Omega; T^{*0,q}X)$ and $N \in \mathbb{N}_0$, we have*

$$\int_{\mathbb{C}} \frac{\partial\widetilde{\chi}(\frac{z}{k})}{\partial\overline{z}} (z - T^{(q)}_{P,\lambda})^{-1} \circ \mathscr{R}_z \frac{dz\wedge d\overline{z}}{2\pi i} = O(k^{-N}) \ \text{in } \mathscr{L}\left(H^{-N}_{\mathrm{comp}}(\Omega, T^{*0,q}X), H^{N}(X, T^{*0,q}X)\right). \tag{4.2.99}$$

PROOF. For $\mathscr{R}_z$ in the form of (4.1.11) and a very small $\epsilon > 0$, we notice that for the function $it\psi_\mp(x,w) + i\sigma\psi_\pm(w,y)$, when $|x-w| > \epsilon$, $|w-y| > \epsilon$ and $|x-w|, |w-y| < \epsilon$, we can apply arbitrary times of partial integration in t, σ and w, respectively. Along with the elementary estimate that $|z-t|^{-1} \leq |z| \cdot |\mathrm{Im}\, z|^{-1} t^{-1}$ when $t > 0$, we can hence directly estimate $\mathscr{R}_z$, and we can apply the same argument in Theorem 4.14 to get our theorem. □

From Theorems 4.14, 4.15, 4.16 and 4.17, we can conclude the remainder estimates contributed by the elements of $L_z^{-\infty}(\Omega; T^{*0,q}X)$ as follows.

THEOREM 4.18. *With the same notations and assumptions in Theorem 4.7, for any* $L_z \in L_z^{-\infty}(\Omega; T^{*0,q}X)$,

$$L_{(k)} := \int_{\mathbb{C}} \frac{\partial \widetilde{\chi}(\frac{z}{k})}{\partial \overline{z}} (z - T_{P,\lambda}^{(q)})^{-1} \circ L_z \frac{dz \wedge d\overline{z}}{2\pi i}, \tag{4.2.100}$$

and any $N \in \mathbb{N}_0$, *we have*

$$L_{(k)} = O(k^{-N}) \text{ in } \mathscr{L}(H_{\text{comp}}^{-N}(\Omega, T^{*0,q}X), H^N(X, T^{*0,q}X)). \tag{4.2.101}$$

The only remainder estimate remains to be checked is the following.

THEOREM 4.19. *With the same notations and assumptions in Theorem 4.7, for the operator*

$$R_{(k,N+1)} := \int_{\mathbb{C}} \frac{\partial \widetilde{\chi}(\frac{z}{k})}{\partial \overline{z}} (z - T_{P,\lambda}^{(q)})^{-1} \circ \Pi_\lambda^{(q)} \circ R_{z,N+1} \frac{dz \wedge d\overline{z}}{2\pi i}, \tag{4.2.102}$$

and for any $N_1, N_2 \in \mathbb{N}_0$, *we can find an* $N_0 > 0$ *large enough such that*

$$R_{(k,N_0+1)} = O(k^{-N_1}) \text{ in } \mathscr{L}(H_{\text{comp}}^{-N_2}(\Omega, T^{*0,q}X), H^{N_2}(X, T^{*0,q}X)). \tag{4.2.103}$$

PROOF. For simplicity, we only prove the case for $n_- \neq n_+$, and the situation $n_- = n_+$ can be deduced from the same argument with some minor change. For all $M \in \mathbb{N}$, we recall that we can write

$$R_{(k,N+1)} \tag{4.2.104}$$

$$= \int_{\mathbb{C} \setminus \text{Spec}(T_{P,\lambda}^{(q)})} \frac{\partial \widetilde{\chi}(\frac{z}{k})}{\partial \overline{z}} z^{-M} T_{P,\lambda}^{(q),M} \circ (z - T_{P,\lambda}^{(q)})^{-1} \circ R_{z,N+1} \frac{dz \wedge d\overline{z}}{2\pi i} \tag{4.2.105}$$

$$+ \sum_{j=0}^{M-1} T_{P,\lambda}^{(q),j} \circ \int_{\mathbb{C} \setminus \text{Spec}(T_{P,\lambda}^{(q)})} \frac{\partial \widetilde{\chi}(\frac{z}{k})}{\partial \overline{z}} z^{-1-j} R_{z,N+1} \frac{dz \wedge d\overline{z}}{2\pi i}. \tag{4.2.106}$$

We recall that here we have

$$R_{z,N+1}(x,y) = \int_0^{+\infty} e^{it\Psi_-(x,y)} \frac{\sum_{|\beta|+|\gamma| \leq 2N+2} R_{\beta,\gamma}^{-,N+1}(x,y,t) t^\beta z^\gamma}{(z-t)^{2N+2}} \tau(\varepsilon t) dt, \tag{4.2.107}$$

and

$$\Psi_- \in \text{Ph}(p_{I_0,I_0}^{-1}(-\boldsymbol{\alpha})(-\boldsymbol{\alpha}), \Omega), \tag{4.2.108}$$

$$R_{\beta,\gamma}^{-,N+1}(x,y,t) \in S_{\text{cl}}^{n-N-1}(\Omega \times \Omega \times \mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X)), \tag{4.2.109}$$

$$R_{\beta,\gamma}^{-,N+1}(x,y,t) \text{ is properly supported in the variables } (x,y). \tag{4.2.110}$$

On one hand, by partial integration in t and Cauchy–Pompieu formula, we can check that

$$\begin{aligned}&\int_{\mathbb{C}\setminus\operatorname{Spec}(T^{(q)}_{P,\lambda})}\frac{\partial\widetilde{\chi}(\frac{z}{k})}{\partial\overline{z}}z^{-1-j}\,R_{z,N+1}\frac{dz\wedge d\overline{z}}{2\pi i}\\ =&\frac{1}{(2N+1)!}\sum_{\beta+\gamma\leq 2N+2}\int_0^{+\infty}e^{it\Psi_-(x,y)}R^{-,N+1}_{\beta,\gamma}(x,y,t)t^{\beta}\tau(\varepsilon t)\\ &\partial_t^{2N+1}\left(\chi\left(\frac{t}{k}\right)t^{-1-j+\gamma}\right)dt+O\left(k^{-\infty}\right),\end{aligned}\tag{4.2.111}$$

which is $O(k^{-N_1})$ in $\mathscr{L}(H^{-N_2}_{\mathrm{comp}}(\Omega,T^{*0,q}X),H^{N_2}(X,T^{*0,q}X))$ for any given $N_1,N_2\in\mathbb{N}_0$ when N is large enough.

On the other hand, for the integral

$$\int_{\mathbb{C}\setminus\operatorname{Spec}(T^{(q)}_{P,\lambda})}\frac{\partial\widetilde{\chi}(\frac{z}{k})}{\partial\overline{z}}z^{-M}T^{(q),M}_{P,\lambda}\circ(z-T^{(q)}_{P,\lambda})^{-1}\circ R_{z,N+1}\frac{dz\wedge d\overline{z}}{2\pi i},\tag{4.2.112}$$

when $M\equiv 0 \mod 2$ we can rewrite it by

$$\int_{\mathbb{C}\setminus\operatorname{Spec}(T^{(q)}_{P,\lambda})}\frac{\partial\widetilde{\chi}(\frac{z}{k})}{\partial\overline{z}}z^{-M}T^{(q),\frac{M}{2}}_{P,\lambda}\circ\Pi^{(q)}_{\lambda}\circ(z-T^{(q)}_{P,\lambda})^{-1}\circ T^{(q),\frac{M}{2}}_{P,\lambda}\circ R_{z,N+1}\frac{dz\wedge d\overline{z}}{2\pi i}.\tag{4.2.113}$$

Here the number $M\in\mathbb{N}$ is arbitrary and we can do the same estimate as in (4.2.98): by applying the Sobolev continuity estimate in the order of $T^{(q),\frac{M}{2}}_{P,\lambda}$, $\Pi^{(q)}_{\lambda}\circ(z-T^{(q)}_{P,\lambda})^{-1}$, $T^{(q),\frac{M}{2}}_{P,\lambda}$ and $R_{z,N+1}$, and using $\left|\frac{\partial\widetilde{\chi}(k^{-1}z)}{\partial\overline{z}}\right|=O\left(k^{-N}|\operatorname{Im} z|^N\right)$ and the estimate that $k<|z|<2k$ when $z\in\operatorname{supp}\widetilde{\chi}(k^{-1}z)$ and k is large, we can check that for any $N_1,N_2\in\mathbb{N}_0$, we can find a suitable and large $N_0>0$ and another large number $M>0$ depending on N_0 such that (4.2.113) is $O(k^{-N_1})$ in $\mathscr{L}(H^{-N_2}_{\mathrm{comp}}(\Omega,T^{*0,q}X),H^{N_2}(X,T^{*0,q}X))$.

Combing all the estimates above, we complete the proof of our theorem. □

By Theorems 4.12, 4.18 and 4.19 and taking the asymptotic sum of the symbols of $A_{(k,m)}$, $m\in\mathbb{N}_0$, the standard semi-classical analysis immediately implies the following result.

THEOREM 4.20. *In the situation of Theorem 1.1, for $q=n_-$ we have an $A_{(k)}\in\mathcal{I}^0_{\Sigma,k}(\Omega;T^{*0,q}X)$ such that for any $N_0\in\mathbb{N}$ we have*

$$\chi(k^{-1}T^{(q)}_{P,\lambda})-A_{(k)}=O(k^{-N_0})\text{ in }\mathscr{L}(H^{-N_0}_{\mathrm{comp}}(\Omega,T^{*0,q}X),H^{N_0}(X,T^{*0,q}X)).\tag{4.2.114}$$

In the end of this section, we use the results just proved to establish an important estimate.

LEMMA 4.21. *In the situation of Theorem 1.1, for $q = n_-$ we have a constant $C > 0$ independent of k such that*

$$\#\{\text{eigenvalues } \lambda_j \text{ of } T^{(q)}_{P,\lambda} : k^{-1}\lambda_j \in \operatorname{supp}\chi\} \leq C \cdot k^{2n+2} \tag{4.2.115}$$

where the notation $\#A$ we mean the number of the set A.

PROOF. By Theorem 4.20 and using the finite partition of unity of the compact manifold X, we see that when k is large there is a constant $c > 0$ independent of k such that

$$\|\chi(k^{-1}T^{(q)}_{P,\lambda})u\| \leq c \cdot k^{n+1}\|u\|, \quad \forall u \in \mathscr{C}^\infty(X, T^{*0,q}X), \tag{4.2.116}$$

where $\|\cdot\|$ is the L^2-norm on X. Combining the above estimate and Theorem 3.10, we can also deduce that

$$\sqrt{\sum_{j\in J}|\chi|^2\left(\frac{\lambda_j}{k}\right)} = \|\chi(k^{-1}T^{(q)}_{P,\lambda})(x,y)\| \leq C \cdot k^{n+1} \tag{4.2.117}$$

for some constant $C > 0$ independent of k. Since all our results hold for arbitrary $\chi \in \mathscr{C}^\infty_0(\dot{\mathbb{R}})$, we can take some $\rho \in \mathscr{C}^\infty_0(\dot{\mathbb{R}}, [0,1])$ such that $\rho \equiv 1$ on $\operatorname{supp}\chi$, then we have

$$\#\{\text{eigenvalues } \lambda_j \text{ of } T^{(q)}_{P,\lambda} : k^{-1}\lambda_j \in \operatorname{supp}\chi\} \leq \sum_{j\in J}\rho^2\left(\frac{\lambda_j}{k}\right) \leq C \cdot k^{2n+2} \tag{4.2.118}$$

for some constant $C > 0$ independent of k. $\square$

4.3. The full asymptotic expansion

In this section we establish the asymptotic expansion of $\chi(k^{-1}T^{(q)}_{P,\lambda})$ in the level of Schwartz kernel. To simplify our calculation for the remainder terms, we fix a small enough number $\epsilon > 0$ and let

$$\mathbb{1}_{(k)} := \mathbb{1}_{[k^{1-\epsilon},k^{1+\epsilon}]} + \mathbb{1}_{[-k^{1+\epsilon},-k^{1-\epsilon}]}, \tag{4.3.1}$$

where $\mathbb{1}_{[k^{1-\epsilon},k^{1+\epsilon}]}$ and $\mathbb{1}_{[-k^{1+\epsilon},-k^{1-\epsilon}]}$ are the indicator function of $[k^{1-\epsilon}, k^{1+\epsilon}]$ and $[-k^{1+\epsilon}, -k^{1-\epsilon}]$, respectively.

Let us start the discussion from the principal terms of our expansion. We will show in the following that the truncation $\mathbb{1}_{(k)}$ does not influence the principal terms of $\chi(k^{-1}T^{(q)}_{P,\lambda})(x,y)$.

LEMMA 4.22. *With the same notations and assumptions in Theorem 4.12, for any $m \in \mathbb{N}_0$ and $\rho \in \mathscr{C}^\infty_0(\Omega)$ we have $A_{(k,m)} \circ \rho\left(\Pi^{(q)}_\lambda - \mathbb{1}_{(k)}(T^{(q)}_{P,\lambda})\right) = O\left(k^{-\infty}\right)$ on $X \times X$.*

PROOF. For simplicity, in the following we prove our result when $q = n_- = 0$, and the general case of $q = n_-$ can be deduced from the same argument with some minor change.

We notice that $A_{(k,m)} \circ \rho \left(\Pi^{(q)}_\lambda - \mathbb{1}_{(k)}(T^{(q)}_{P,\lambda})\right) = \mathbb{I}(x,y) + \mathbb{II}(x,y)$, where

$$\mathbb{I}(x,y) = \sum_{|\lambda_j|>k^{1+\varepsilon}} \left(A_{(k,m)}(x,u)\Big|(\rho f_j)(u)\right)\overline{f_j}(y), \tag{4.3.2}$$

$$\mathbb{II}(x,y) = \sum_{|\lambda_j|<k^{1-\varepsilon}} \left(A_{(k,m)}(x,u)\Big|(\rho f_j)(u)\right)\overline{f_j}(y), \tag{4.3.3}$$

and $\{f_j\}_{j\in J}$ is an orthonormal system in $L^2_{0,q}(X)$ such that $f_j \in \operatorname{Ker}(T^{(q)}_{P,\lambda} - \lambda_j I)$.

As before, we may assume that $\frac{\partial\psi_-}{\partial y_{2n+1}}(x,y) \neq 0$ on $\Omega\times\Omega$. One one hand, by partial integration we have

$$\begin{aligned}\mathbb{II}(x,y) = &-\sum_{|\lambda_j|<k^{1-\varepsilon}} \overline{f_j}(y) \\ &\int\int e^{ikt\psi(u,y)}\frac{\partial}{\partial u_{2n+1}}\Big(\frac{1}{ikt\partial_{u_{2n+1}}\psi}a^{-,m}(x,u,t,k)\overline{\rho f_j}(u)m(u)\Big)dudt,\end{aligned} \tag{4.3.4}$$

and we notice that we can apply arbitrary times of partial integration in this way. On the other hand, by Theorem 3.7 and Sobolev embedding theorem we know there exists $n_0 \in \mathbb{N}$ such that for any $\ell \in \mathbb{N}_0$ we find a constant $C_\ell > 0$ so that

$$\|f_j(x)\|_{\mathscr{C}^\ell(X)} \leq C_\ell(1+|\lambda_j|)^{n_0+\ell}, \quad \forall j \in \mathbb{N}_0. \tag{4.3.5}$$

By the above two observations, the assumption that $|\lambda_j| < k^{1-\varepsilon}$ in $\mathbb{II}(x,y)$ and Lemma 4.21, we can deduce that $\mathbb{II}(x,y) = O(k^{-\infty})$ on $X\times X$.

Next, we fix a large enough number $N \in \mathbb{N}$, and by the fact that $T^{(q)}_{P,\lambda}$ is self-adjoint we also have

$$\mathbb{I} = \sum_{\lambda_j>k^{1+\epsilon}} \overline{f_j}(y)\lambda_j^{-N}\int(T^{(q),N}_{P,\lambda}\circ A_{(k,m)})(u,y)\overline{\rho f_j}(u)dm(u), \tag{4.3.6}$$

where $T^{(q),N}_{P,\lambda}$ is the N-times composition of $T^{(q)}_{P,\lambda}$. From (4.3.5), the direct estimate of $A_{(k,m)}$ and the Sobolev continuity of $T^{(q)}_{P,\lambda}$, the Sobolve embedding theorem, the assumption that $\lambda_j > k^{1+\epsilon}$ in $\mathbb{I}(x,y)$ and the fact that the number $N \in \mathbb{N}$ can be arbitrary large, we have $\mathbb{I} = O(k^{-\infty})$ on $X\times X$. □

THEOREM 4.23. *With the same notations and assumptions in Theorem 4.12, for any $m \in \mathbb{N}_0$ and $\rho \in \mathscr{C}^\infty_0(\Omega)$ we have $A_{(k,m)} \circ \rho\Pi^{(q)}_\lambda = \mathcal{A}_{(k,m)} \in \mathcal{I}^{-m}_{\Sigma,k}(\Omega; T^{*0,q}X)$.*

In fact, up to an k-negligible kernel on $X \times \Omega$ we have

(4.3.7)

$$\mathcal{A}_{(k,m)}(x,y) \equiv \int_0^{+\infty} e^{ikt\Psi_-(x,y)}\mathbf{a}^{-,m}(x,y,t,k)dt + \int_0^{+\infty} e^{ikt\Psi_+(x,y)}\mathbf{a}^{+,m}(x,y,t,k)dt,$$

where

$$\Psi_- \in \mathrm{Ph}(p^{-1}_{I_0,I_0}(-\boldsymbol{\alpha})(-\boldsymbol{\alpha}),\Omega),\ \Psi_+ \in \mathrm{Ph}(p^{-1}_{J_0,J_0}(-\boldsymbol{\alpha})\boldsymbol{\alpha},\Omega), \tag{4.3.8}$$

and we have the following data properly supported in (x,y):

$$\mathbf{a}^{\mp,m}(x,y,t,k) \sim \sum_{j=0}^{+\infty} \mathbf{a}_j^{\mp,m}(x,y,t)k^{n+1-m-j} \quad \text{in } S^{n+1-m}_{\mathrm{loc}}\left(1;\Omega\times\Omega\times\mathbb{R}_+,\mathscr{L}(T^{*0,q}X,T^{*0,q}X)\right), \tag{4.3.9}$$

$$\forall j \in \mathbb{N}_0,\ \mathbf{a}_j^{\mp,m}(x,y,t) \neq 0 \text{ and } \mathbf{a}^{\mp,m}(x,y,t,k) \neq 0 \implies t \in \operatorname{supp}\chi, \tag{4.3.10}$$

and we have $\mathbf{a}^+(x,y,t,k)=0$ when $n_- \neq n_+$. Moreover, for $m=0$ we have

$$\mathbf{a}_0^{-,0}(x,x,t) = \rho(x)\frac{|\det\mathcal{L}_x|}{2\pi^{n+1}}\frac{v(x)}{m(x)}\,p^{-n-1}_{I_0,I_0}(-\boldsymbol{\alpha}_x)\,\chi(t)t^n, \tag{4.3.11}$$

and when $n_- = n_+$ we also have

$$\mathbf{a}_0^{+,0}(x,x,t) = \rho(x)\frac{|\det\mathcal{L}_x|}{2\pi^{n+1}}\frac{v(x)}{m(x)}\,p^{-n-1}_{J_0,J_0}(-\boldsymbol{\alpha}_x)\,\chi(-t)t^n. \tag{4.3.12}$$

PROOF. We recall that by Theorem 2.4, we can write $\rho\Pi^{(q)}_\lambda = \rho(S_- + \rho S_+ + F)$ on $X\times\Omega$, where the kernel $F(x,y) \in \mathscr{C}^\infty(\Omega\times\Omega)$. By Theorem 4.12, the partial integration argument in the proof of Theorem 4.17, and complex stationary phase formula of Melin–Sjöstrand Theorem 2.2, we can check that $A_{(k,m)} \circ \rho(S_- + S_+) = \mathcal{A}_{(k,m)} \in \mathcal{I}^{-m}_{\Sigma,k}(\Omega; T^{*0,q}X)$ as stated in our theorem. Also, by writing

$$\begin{aligned}(A_{k,m}\circ\rho F)(x,y) &= \int e^{ikt\Psi_-(x,w)}a^{-,m}(x,w,t,k)(\rho F)(w,y)m(w)dwdt \\ &+ \int e^{ikt\Psi_+(x,w)}a^{+,m}(x,w,t,k)(\rho F)(w,y)m(w)dwdt + G_k(x,y),\end{aligned} \tag{4.3.13}$$

where $G_k = O(k^{-\infty})$ on $X\times\Omega$, we can apply [31, Lemma 1.12] for example to do partial integration in (w,t) as many times as we want and see that $(A_{k,m}\circ\rho F)(x,y) = O(k^{-\infty})$ on $X\times\Omega$. This completes the proof. ☐

We can now conclude Theorems 1.1 and 1.2 through the next result.

THEOREM 4.24. *In the situation of Theorem 1.1 and $q = n_-$, up to a k-negligible kernel on $\Omega \times \Omega$ we have*

(4.3.14)
$$\chi(k^{-1}T^{(q)}_{P,\lambda})(x,y) \equiv \int_0^{+\infty} e^{ikt\Psi_-(x,y)}\mathbf{A}^-(x,y,t,k)dt + \int_0^{+\infty} e^{ikt\Psi_+(x,y)}\mathbf{A}^+(x,y,t,k)dt,$$

where

$$\Psi_- \in \mathrm{Ph}(-\Lambda_1\boldsymbol{\alpha},\Omega),\ \Psi_+ \in \mathrm{Ph}(\Lambda_2\boldsymbol{\alpha},\Omega),\ \Lambda_1(x),\Lambda_2(x) \in \mathscr{C}^\infty(X,\mathbb{R}_+), \tag{4.3.15}$$

$$\mathbf{A}^\mp(x,y,t,k) \sim \sum_{j=0}^{+\infty} \mathbf{A}_j^\mp(x,y,t)k^{n+1-j} \quad \text{in } S^{n+1-j}_{\mathrm{loc}}\left(1;\Omega\times\Omega\times\mathbb{R}_+,\mathscr{L}(T^{*0,q}X,T^{*0,q}X)\right), \tag{4.3.16}$$

and $\mathbf{A}^+(x,y,t,k) = 0$ when $n_- \neq n_+$. In fact, when $\operatorname{supp}\chi \cap \mathbb{R}_+ \neq \emptyset$, there is an interval $I_- \Subset \mathbb{R}_+$ such that when $\mathbf{A}_j^-(x,y,t) \neq 0$ and $\mathbf{A}^-(x,y,t) \neq 0$ we have $t \in I_-$ for all $j \in \mathbb{N}_0$; when $n_- = n_+$ and $\operatorname{supp}\chi \cap \mathbb{R}_- \neq \emptyset$, there is also an interval $I_+ \Subset \mathbb{R}_+$ such that when $\mathbf{A}_j^+(x,y,t) \neq 0$ and $\mathbf{A}^+(x,y,t) \neq 0$ we have $t \in I_+$ for all $j \in \mathbb{N}_0$. Moreover, with respect to (1.1.23), we have

$$\mathbf{A}_0^-(x,x,t) = \frac{|\det \mathcal{L}_x|}{2\pi^{n+1}}\frac{v(x)}{m(x)}\,\Lambda_1^{n+1}(x)\,\chi(t\cdot p_{I_0,I_0}(-\boldsymbol{\alpha}_x)\cdot\Lambda_1(x))\,t^n, \tag{4.3.17}$$

and when $n_- = n_+$ we also have

$$\mathbf{A}_0^+(x,x,t) = \frac{|\det \mathcal{L}_x|}{2\pi^{n+1}}\frac{v(x)}{m(x)}\,\Lambda_2^{n+1}(x)\,\chi(-t\cdot p_{J_0,J_0}(-\boldsymbol{\alpha}_x)\cdot\Lambda_2(x))\,t^n. \tag{4.3.18}$$

In the last, for any $\tau_1,\tau_2 \in \mathscr{C}^\infty(X)$ such that $\operatorname{supp}(\tau_1)\cap\operatorname{supp}(\tau_2) = \emptyset$ we have

$$\tau_1 \circ \chi(k^{-1}T^{(q)}_{P,\lambda}) \circ \tau_2 = O\left(k^{-\infty}\right), \tag{4.3.19}$$

where τ_1,τ_2 are regarded as multiplication operators by the function τ_1,τ_2, respectively.

PROOF. We let $X = \bigcup_\ell \Omega_\ell$ and $\rho_\ell \in \mathscr{C}^\infty_0(\Omega_\ell)$ such that $\sum_\ell \rho_\ell = 1$ on X. Then for large enough $k > 0$ we have

$$\chi(k^{-1}T^{(q)}_{P,\lambda}) = \chi(k^{-1}T^{(q)}_{P,\lambda}) \circ \sum_\ell \rho_\ell \circ \mathbb{1}_{(k)}(T^{(q)}_{P,\lambda}). \tag{4.3.20}$$

Applying Theorem 4.20, for each $N \in \mathbb{N}_0$ we have the same operator $A_k^\ell \in \mathcal{I}^0_{\Sigma,k}(\Omega_\ell;T^{*0,q}X)$ therein and an operator $R^\ell_{k,N}$ which depends on N such that

$$\begin{aligned}\chi(k^{-1}T^{(q)}_{P,\lambda}) \circ \sum_\ell \rho_\ell \circ \mathbb{1}_{(k)}(T^{(q)}_{P,\lambda}) &= \sum_\ell \chi(k^{-1}T^{(q)}_{P,\lambda}) \circ \rho_\ell \circ \mathbb{1}_{(k)}(T^{(q)}_{P,\lambda}) \\ &\sum_\ell A_k^\ell \circ \rho_\ell \circ (\mathbb{1}_{(k)}(T^{(q)}_{P,\lambda}) - \Pi^{(q)}_\lambda + \Pi^{(q)}_\lambda) + \sum_\ell R^\ell_{k,N} \circ \rho_\ell \circ \mathbb{1}_{(k)}(T^{(q)}_{P,\lambda}),\end{aligned} \tag{4.3.21}$$

where

$$R^{\ell}_{k,N} = O(k^{-N}) \text{ in } \mathscr{L}(H^{-N}_{\mathrm{comp}}(\Omega_\ell, T^{*0,q}X), H^N(X, T^{*0,q}X)). \tag{4.3.22}$$

By Lemma 4.22, Theorems 4.23 and 3.9, we can see that

$$\chi(k^{-1}T^{(q)}_{P,\lambda}) \circ \sum_{\ell} \rho_\ell \circ \mathbb{1}_{(k)}(T^{(q)}_{P,\lambda}) - \sum_{\ell} \mathbf{A}^{\ell}_k - \sum_{\ell} \sum_{k^{1-\epsilon}<|\lambda_j|<k^{1+\epsilon}} \left(R^{\ell}_{k,N} \circ \rho f_j\right) \otimes f^*_j \tag{4.3.23}$$

is $O\left(k^{-\infty}\right)$ on $X \times X$. We recall that up to an element of $O\left(k^{-\infty}\right)$ on $X \times \Omega_\ell$ we have

$$\mathbf{A}^{\ell}_k(x,y) \equiv \int_0^{+\infty} e^{ikt\Psi^{\ell}_-(x,y)} \mathbf{A}^{-,\ell}(x,y,t,k)dt + \int_0^{+\infty} e^{ikt\Psi^{\ell}_+(x,y)} \mathbf{A}^{+,\ell}(x,y,t,k)dt \tag{4.3.24}$$

for some $\Psi^{\ell}_- \in \mathrm{Ph}(p^{-1}_{I_0,I_0}(-\boldsymbol{\alpha})(-\boldsymbol{\alpha}), \Omega_\ell)$, $\Psi^{\ell}_+ \in \mathrm{Ph}(p^{-1}_{J_0,J_0}(-\boldsymbol{\alpha})\boldsymbol{\alpha}, \Omega_\ell)$, and

$$\mathbf{A}^{\mp,\ell}(x,y,t,k) \sim \sum_{j=0}^{+\infty} \mathbf{A}^{\mp,\ell}_j(x,y,t)k^{n+1-j} \quad \text{in } S^{n+1-j}_{\mathrm{loc}}\left(1; \Omega_\ell \times \Omega_\ell \times \mathbb{R}_+, \mathscr{L}(T^{*0,q}X, T^{*0,q}X)\right) \tag{4.3.25}$$

such that $\mathbf{A}^{\mp,\ell}_j(x,y,t) \neq 0$ whenever $t \in \operatorname{supp}\chi$ for all $j \in \mathbb{N}_0$ and $\mathbf{A}^{+}(x,y,t,k) = 0$ when $n_- \neq n_+$. Moreover, with respect to (1.1.23), for all $x \in \Omega_\ell$ we have

$$\mathbf{A}^{-,\ell}_0(x,x,t) = \rho_\ell(x) \frac{|\det \mathcal{L}_x|}{2\pi^{n+1}} \frac{v(x)}{m(x)}\, p^{-n-1}_{I_0,I_0}(-\boldsymbol{\alpha}_x)\, \chi(t)t^n, \tag{4.3.26}$$

and when $n_- = n_+$ we also have

$$\mathbf{A}^{+,\ell}_0(x,x,t) = \rho_\ell(x) \frac{|\det \mathcal{L}_x|}{2\pi^{n+1}} \frac{v(x)}{m(x)}\, p^{-n-1}_{J_0,J_0}(-\boldsymbol{\alpha}_x)\, \chi(-t)t^n. \tag{4.3.27}$$

By (4.3.22) and Sobolev embedding lemma, and (4.3.5) and Lemma 4.21, we have

$$\sum_{\ell} \sum_{k^{1-\epsilon}<|\lambda_j|<k^{1+\epsilon}} \left(R^{\ell}_{k,N} \circ \rho f_j\right) \otimes f^*_j = O\left(k^{-\infty}\right) \quad \text{on } X \times X. \tag{4.3.28}$$

Since A^{ℓ}_k are properly supported on Ω_ℓ, we get (4.3.19) from (4.3.23) and the above estimate.

Finally, given any $\Omega \subset \bigcup_\ell \Omega_\ell$, for any $\Lambda_1(x), \Lambda_2(x) \in \mathscr{C}^\infty(X, \mathbb{R}_+)$ and $\Psi_- \in \mathrm{Ph}(-\Lambda_1\boldsymbol{\alpha}, \Omega)$ and $\Psi_+ \in \mathrm{Ph}(\Lambda_2\boldsymbol{\alpha}, \Omega)$, from [48, Theorem 2.4 and Lemma 6.8] (cf. also [48, §12]) we have

$$\int_0^{+\infty} e^{ikt\Psi_\mp(x,y)}\mathbf{A}^\mp(x,y,t,k)dt = \sum_\ell \int_0^{+\infty} e^{ikt\Psi^\ell_\mp(x,y)}\mathbf{A}^{\mp,\ell}(x,y,t,k)dt + O\left(k^{-\infty}\right) \quad \text{on } \Omega\times\Omega \tag{4.3.29}$$

for the desired $\mathbf{A}^\mp(x,y,t,k)$. We hence conclude our result. □

Our main results Theorem 1.1 and Theorem 1.2 follow immediately from the previous theorem, so does Corollary 1.3.

PROOF OF COROLLARY 1.3. On any coordinate patch (Ω_1, x), by Theorem 1.1 and the property that $\varphi_\mp(x,x)=0$, we have

$$\chi(k^{-1}T^{(q)}_{P,\lambda})(x,x) \sim \sum_{j=0}^{+\infty} k^{n+1-j}\left(A_j^-(x)+A_j^+(x)\right) \quad \text{in } S^{n+1}_{\mathrm{loc}}(1;\Omega_1,\mathscr{L}(T^{*0,q}X,T^{*0,q}X)), \tag{4.3.30}$$

where $A_j^\mp(x)\in\mathscr{C}^\infty(\Omega_1,\mathscr{L}(T^{*0,q}X,T^{*0,q}X))$ and

$$A_j^\mp(x) = \int_0^{+\infty} A_j^\mp(x,x,t)dt, \tag{4.3.31}$$

$$A_0^-(x) = \left(\int_0^{+\infty} t^n\chi(p_{I_0,I_0}(-\boldsymbol{\alpha}_x)t)dt\right)\frac{|\det\mathcal{L}_x|}{2\pi^{n+1}}\frac{v(x)}{m(x)}\tau_x^{n_-}, \tag{4.3.32}$$

$$A_0^+(x) = \left(\int_0^{+\infty} t^n\chi(p_{J_0,J_0}(\boldsymbol{\alpha}_x)t)dt\right)\frac{|\det\mathcal{L}_x|}{2\pi^{n+1}}\frac{v(x)}{m(x)}\tau_x^{n_+} \text{ when } n_-=n_+. \tag{4.3.33}$$

We let (Ω_2, y) be another coordinate patch with $\Omega_1\cap\Omega_2\neq\emptyset$, and by the same reasoning we have

$$\chi(k^{-1}T^{(q)}_{P,\lambda})(y,y) \sim \sum_{j=0}^{+\infty} k^{n+1-j}\left(B_j^-(y)+B_j^+(y)\right) \quad \text{in } S^{n+1}_{\mathrm{loc}}(1;\Omega_2,\mathscr{L}(T^{*0,q}X,T^{*0,q}X)). \tag{4.3.34}$$

where $B_j^\mp(x)\in\mathscr{C}^\infty(\Omega_2,\mathscr{L}(T^{*0,q}X,T^{*0,q}X))$. Then on $\Omega_1\cap\Omega_2$ we have

$$\sum_{j=0}^{+\infty}(A_j^-+A_j^+)(\cdot)k^{n+1-j} \sim \sum_{j=0}^{+\infty}(B_j^-+B_j^+)(\cdot)k^{n+1-j} \quad \text{in } S^{n+1}_{\mathrm{loc}}(1;\Omega_1\cap\Omega_2,\mathscr{L}(T^{*0,q}X,T^{*0,q}X)). \tag{4.3.35}$$

The relation (4.3.35) shows that $A_j^-+A_j^+=B_j^-+B_j^+$ on $\Omega_1\cap\Omega_2$ and hence there exists global $A_j^\mp\in\mathscr{C}^\infty(X,\mathscr{L}(T^{*0,q}X,T^{*0,q}X))$, $j\in\mathbb{N}_0$, such that our corollary holds. □

4.4. Expansion on circle bundles

In this section we explain the role of Theorem 1.1 in complex geometry by the so called circle bundle framework. We start with a very important result for Kähler geometry and quantization introduced by Boutet de Monvel–Guillemin [12, §§13-14]. We let M be a compact Kähler manifold and $(L, h^L) \to M$ be a positive holomorphic line bundle over M with respect to a smooth Hermitian metric h^L, and we assume that the Chern curvature $\frac{i}{2\pi}R^L$ of (L, h^L) represents a Kähler metric on M. Then the principal circle bundle

$$X = \{v \in L^* : |v|_{h^*} = 1\} \tag{4.4.1}$$

is a strictly pseudoconvex CR manifold called Grauert tube [30] and plays an important application to the Kodaira embedding theorem for singular spaces. The connection 1-form $\boldsymbol{\alpha}$ on X associated to the Chern connection ∇^L is a contact form on X and the corresponding Reeb vector field $\mathcal{T}$ is the infinitesimal generator ∂_ϑ of the S^1-action on X. In this case, $\boldsymbol{\alpha}(-i\partial_\vartheta) = 1$, ∂_ϑ commutes with the tangential Cauchy–Riemann operator $\overline{\partial}_b$ and the Szegő projection Π, and $\operatorname{Spec}(-i\partial_\vartheta) = \mathbb{Z}$. Thus the operator $P = -i\partial_\vartheta$ restricted to the space of L^2-Caucy–Riemann functions $H^0_b(X)$ is an elliptic self-adjoint Toeplitz operator of order one. Also, its positive spectrum consists of $m \in \mathbb{N}$ with finite multiplicity, and the eigenspace corresponding to m can be identified with the space $H^0(M, L^m)$ of holomorphic sections of L^m, the m-th power of L, cf. [12, Lemma 14.14]. Moreover, m large enough occurs in the spectrum with the multiplicity $\rho(m)$, where ρ is the Hilbert polynomial of (M, L). For related results about spectral asymptotics for Toeplitz operators on strictly pseudoconvex circle bundles, readers can refer to [6,15,74] for example. We also refer to [34] for the case of irregular Sasakian manifolds.

Now, we discuss a more general situation in complex geometry. We assume that (M, J) is a complex manifold with complex structure J and L is a holomorphic line bundle bundles over M with a smooth Hermitian metric h^L. We assume that ∇^L is the holomorphic Hermitian connections, also known as Chern connections, on (L, h^L) and moreover, with respect to the Chern curvature $R^L := (\nabla^L)^2$ the two form

$$\omega := \frac{i}{2\pi}R^L \tag{4.4.2}$$

defines a symplectic form on M. Therefore under this context the signature (n_-, n_+) of the curvature R^L (the number of negative and positive eigenvalues) with respect to any Riemannian metric compatible with J will be the same. We let g^{TX} be any Riemannian metric on TX compatible with J. We let $\overline{\partial}^{L^m,*}$ be the formal adjoint of the Dolbeault operator $\overline{\partial}^{L^m}$ on the Dolbeault complex $\Omega^{0,q}(M, L^m)$, $q = 0, \cdots, n := \dim_{\mathbb{C}} M$, with the scalar product induced by g^{TX}

and h^L. We set

$$\mathbf{D}_m := \sqrt{2}\left(\overline{\partial}^{L^m} + \overline{\partial}^{L^m,*}\right) \tag{4.4.3}$$

and denote by

$$\square^{L^m} := \overline{\partial}^{L^m,*} \circ \overline{\partial}^{L^m} + \overline{\partial}^{L^m} \circ \overline{\partial}^{L^m,*} \tag{4.4.4}$$

the Kodaira Lapalcian. It is clear that

$$\mathbf{D}_m^2 = 2\square^{L^m} \tag{4.4.5}$$

is twice the Kodaira–Laplacian and preserves the $\mathbb{Z}$-grading of $\Omega^{0,\cdot}(M, L^m)$. By standard Hodge theory, we know that for any $q, m \in \mathbb{N}$,

$$\text{Ker } \mathbf{D}_m|_{\Omega^{0,q}(M,L^m)} = \text{Ker } \mathbf{D}_m^2|_{\Omega^{0,q}(M,L^m)} = H^{0,q}(M, L^m), \tag{4.4.6}$$

where $H^{0,q}(M, L^m)$ is the Dolbeault cohomology, $q = 0 \cdots, n$. For $\mathbf{D}_m^2$, from [67, Theorem 1.5] we have the Bochner–Kodaira–Nakano type formula and we have the following vanishing theorem: when $m \in \mathbb{N}$ is large we have

$$H^{0,q}(M, L^m) = 0 \text{ for } q \neq n_-. \tag{4.4.7}$$

The vanishing result above is Andreotti–Grauert's coarse vanishing theorem [1, §23], and the original proof is by using the cohomology finiteness theorem for the disc bundle of L^*.

When $q = n_-$, the situation is more interesting from the point of view of semi-classical analysis. We let

$$B_m^{(q)} : L^2_{0,q}(M, L^m) \to H^{0,q}(M, L^m) \tag{4.4.8}$$

be the Bergman projection for m-power of L on $(0, q)$ forms. The Schwartz kernel $B_m^{(q)}(p', p'')$ associated to $B_m^{(q)}$ is called the Bergman kernel, which is a smooth kernel by standard Hodge theory. It is well known that $B_m^{(q)}(p', p'')$ admits the full asymptotic expansion [67, Theorem 1.7], also cf. [54, 68]. The main goal of this section is to apply the asymptotic expansion of $B_m^{(q)}(p') := B_m^{(q)}(p', p')$ to directly verify Corollary 1.3 on circle bundles.

To start with, by the expansion of $B_m^{(q)}(p', p'')$ and identifying $\mathscr{L}(L^{*,m}, L^m)$ as $\mathbb{C}$, for $q = n_-$ and $m \in \mathbb{N}$ large enough we can write

$$B_m^{(q)}(p') \sim \sum_{j=0}^{+\infty} m^{n-j} b_j^-(p') \text{ in } S^n_{\text{loc}}(1; M, \mathscr{L}(T^{*0,q}M, T^{*0,q}M)) \tag{4.4.9}$$

and

$$B_{-m}^{(n-q)}(p') \sim \sum_{j=0}^{+\infty} m^{n-j} b_j^+(p') \text{ in } S^n_{\text{loc}}(1; M, \mathscr{L}(T^{*0,q}M, T^{*0,q}M)) \tag{4.4.10}$$

as $m \to +\infty$, where we use the convention $L^{-m} := L^{*,m}$ for high power of dual line bundles and use the fact that R^{L^*} has the constant signature $(n-q, q)$. Also, in this context we have the torsion free relation

$$[-i\partial_\vartheta, W] \subset \Gamma(T^{0,1}X), \ \ \forall W \in \Gamma(T^{0,1}X). \tag{4.4.11}$$

Then by the differential of the flow $d\Phi^t_{-i\partial_\vartheta}$, we can extend the definition of the operator $-i\partial_\vartheta$ by

$$(-i\partial_\vartheta u)(W_1, \cdots, W_q) := \frac{\partial}{\partial t}\left(u(d\Phi^t_{-i\partial_\vartheta}W_1, \cdots, d\Phi^t_{-i\partial_\vartheta}W_q)\right)\Big|_{t=0}, \\ \forall u \in \Omega^{0,q}(X), \ \ \forall \{W_j\}_{j=1}^q \in \Gamma(T^{0,1}X). \tag{4.4.12}$$

Now, we consider the dynamical Toeplitz operator

$$T^{(q)}_\vartheta := \Pi^{(q)} \circ (-i\partial_\vartheta) \circ \Pi^{(q)} \tag{4.4.13}$$

on the circle bundle X over M. For $\chi \in \mathscr{C}^\infty_0(\mathbb{R})$, we can check that

$$\chi(k^{-1}T^{(q)}_\vartheta)(x,x) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} \chi\left(\frac{m}{k}\right) B^{(q)}_m \circ \pi_M(x), \tag{4.4.14}$$

where $\pi_M : X \to M$ is the natural projection such that $\pi_M(x) = p'$. For simplicity, from now on we write $\mathbf{B}^{(q)}_m := B^{(q)}_m \circ \pi_M$ and $\mathbf{b}^{\mp}_j := b^{\mp}_j \circ \pi_M$. Because of the term $\chi\left(\frac{m}{k}\right)$, when $k \to +\infty$, we only have to take consideration of m satisfying $|m| \to +\infty$ in (4.4.14). When $q \notin \{n_-, n_+\}$, by the vanishing theorem of Andreotti—Grauert and (4.4.14), we have

$$\chi(k^{-1}T^{(q)}_\vartheta)(x,x) = 0, \ k \to +\infty \tag{4.4.15}$$

in this situation.

When $q = n_-$, we have to split the discussion into $n_- \neq n_+$ and $n_- = n_+$. When $n_- \neq n_+$, again from the vanishing theorem of Andreotti—Grauert we have

$$\mathbf{B}^{(q)}_{-m} = 0, \ m \to +\infty, \tag{4.4.16}$$

and when $q = n_- = n_+$, however, we have expansion result (4.4.10) for $\mathbf{B}^{(n-q)}_{-m} = \mathbf{B}^{(q)}_{-m}$.

For the generality of our calculation, from now on we consider $q = n_- = n_+$. In this case we notice that for all $N \in \mathbb{N}$, we have

$$\sum_{m\in\mathbb{N}} \chi\left(\frac{m}{k}\right) \mathbf{B}_m^{(q)}(x) - \sum_{m\in\mathbb{N}} \chi\left(\frac{m}{k}\right) \sum_{j=0}^{N} m^{n-j}\mathbf{b}_j^{-}(x) = \sum_{m\in\mathbb{N}} \chi\left(\frac{m}{k}\right) \mathbf{B}_m^{(q)}(x) - \sum_{j=0}^{N} k^{n+1-j} \sum_{m\in\mathbb{N}} k^{-1}\chi\left(\frac{m}{k}\right)\left(\frac{m}{k}\right)^{n-j} \mathbf{b}_j^{-}(x) \tag{4.4.17}$$

and

$$\sum_{m\in\mathbb{Z}_{<0}} \chi\left(\frac{m}{k}\right) \mathbf{B}_m^{(q)}(x) - \sum_{m\in\mathbb{Z}_{<0}} \chi\left(\frac{m}{k}\right) \sum_{j=0}^{N} |m|^{n-j}\mathbf{b}_j^{+}(x) = \sum_{m\in\mathbb{Z}_{<0}} \chi\left(\frac{m}{k}\right) \mathbf{B}_m(x) - \sum_{j=0}^{N} k^{n+1-j} \sum_{m\in\mathbb{Z}_{<0}} k^{-1}\chi\left(\frac{m}{k}\right)\left(\frac{m}{k}\right)^{n-j} \mathbf{b}_j^{+}(x). \tag{4.4.18}$$

On one hand, as $k \to +\infty$, (4.4.9) implies that for any $\ell \in \mathbb{N}$ we have

$$\left\| \sum_{m\in\mathbb{N}} \chi\left(\frac{m}{k}\right) \mathbf{B}_m^{(q)}(x) - \sum_{m\in\mathbb{N}} \chi\left(\frac{m}{k}\right) \sum_{j=0}^{N} m^{n-j}\mathbf{b}_j^{-}(x) \right\|_{C^\ell} \leq c_{\ell,N} \sum_{m\in\mathbb{N}} \left|\chi\left(\frac{m}{k}\right)\right| m^{n-N-1} = O(k^{n-N}). \tag{4.4.19}$$

for some constant $c_{\ell,N} > 0$. Similarly, from (4.4.10), for any $\ell \in \mathbb{N}$ we also have

$$\left\| \sum_{m\in\mathbb{Z}_{<0}} \chi\left(\frac{m}{k}\right) \mathbf{B}_m^{(q)}(x) - \sum_{m\in\mathbb{Z}_{<0}} \chi\left(\frac{m}{k}\right) \sum_{j=0}^{N} |m|^{n-j}\mathbf{b}_j^{+}(x) \right\|_{C^\ell} = O(k^{n-N}). \tag{4.4.20}$$

On the other hand, by the Poisson summation formula, cf. [40, Theorem 7.2.1] for example, for any $\tau \in \mathscr{C}_0^\infty(\mathbb{R})$ we have

$$k^{-1} \sum_{m\in\mathbb{Z}} \tau\left(\frac{m}{k}\right) = \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}} e^{-it(2\pi km)}\tau(t)dt. \tag{4.4.21}$$

In the right-hand side of the above equation, when $m \neq 0$ we can apply arbitrary times of integration by parts in t, and when $m = 0$ we just have a number $\int_{\mathbb{R}} \tau(t)dt$. Accordingly, for any $N \in \mathbb{N}$, we can find a constant $C_N > 0$ such that

$$\left| k^{-1} \sum_{m\in\mathbb{Z}} \tau\left(\frac{m}{k}\right) - \int_{\mathbb{R}} \tau(t)dt \right| < C_N k^{-N}. \tag{4.4.22}$$

By (4.4.17), (4.4.18), (4.4.19), (4.4.20), (4.4.22) and triangle inequality, we immediately get the following special case of Corollary 1.3: For $\chi \in \mathscr{C}_0^\infty(\dot{\mathbb{R}})$ and

$q = n_- = n_+$, we have

$$\chi(k^{-1}T_{\vartheta}^{(q)})(x,x) \sim \sum_{j=0}^{+\infty} k^{n+1-j} \int_0^{+\infty} \chi(t) t^{n-j} \frac{dt}{2\pi} \, \mathbf{b}_j^-(x) + \sum_{j=0}^{+\infty} k^{n+1-j} \int_0^{+\infty} \chi(-t) t^{n-j} \frac{dt}{2\pi} \, \mathbf{b}_j^+(x) \tag{4.4.23}$$

in $S^{n+1}_{\text{loc}}(1; M, \mathscr{L}(T^{*0,q}M, T^{*0,q}M))$. When $q = n_- \neq n_+$, from the same argument above we can see that the component contributed by the positive eigenvalues of R^L in (4.4.23) will be $O(k^{-\infty})$.

The above discussion and calculation on Grauert tubes is modified from [35, §5.2] for the case of $q = n_-$ and $(n_-, n_+) = (0, n)$.

CHAPTER 5

The second coefficient of Boutet de Monvel–Sjöstrand expansion

In this chapter we discuss the second coefficient issue of Szegő type Fourier integral operators by presenting the joint work of C.-Y. Hsiao and the author in [61].

To unify the assumptions and notations in this relatively independent chapter, we formulate Boutet–Sjöstrand theorem again and from now on we always use such convention unless we specify..

THEOREM 5.1. *We let* $(X, T^{1,0}X, \boldsymbol{\alpha})$ *be a compact strictly pseudoconvex embeedable CR manifold of* $\dim_{\mathbb{R}} X = 2n+1$, $n \geq 1$, *and* $D \subset X$ *be any open coordinate patch with local coordinates* $x = (x_1, \cdots, x_{2n+1})$. *We assume that the Hermitian metric on* $\mathbb{C}TX$ *is given by the Levi metric. Then we have*

$$\Pi(x,y) \equiv \int_0^{+\infty} e^{it\phi(x,y)} a(x,y,t) dt \mod \mathscr{C}^\infty(D \times D), \tag{5.0.1}$$

where for every $(x,y) \in D \times D$ *we have*

$$\phi(x,y) \in \mathscr{C}^\infty(D \times D), \tag{5.0.2}$$

$$\operatorname{Im} \varphi \geq 0, \tag{5.0.3}$$

$$\phi(x,y) = 0 \text{ if and only if } x = y, \tag{5.0.4}$$

$$d_x\phi(x,x) = -d_y\phi(x,x) = -\boldsymbol{\alpha}(x), \tag{5.0.5}$$

and

$$a(x,y,t) \sim \sum_{j=0}^{+\infty} a_j(x,y) t^{n-j} \text{ in } S^n_{1,0}(D \times D \times \mathbb{R}_+), \tag{5.0.6}$$

$$a_0(x,x) = \frac{1}{2\pi^{n+1}}, \text{ for every } x \in D. \tag{5.0.7}$$

We will take the volume on X by $\lambda(x)dx = \frac{1}{n!}(-\frac{d\boldsymbol{\alpha}}{2})^n \wedge \boldsymbol{\alpha}$.

5.1. Some more background

We first recall some necessary tools and notations for our problem.

We start with the following version of stationary phase formula dues to Hörmander [40, Theorem 7.7.5].

THEOREM 5.2. *We let* $D \subset \mathbb{R}^n$ *be an open set,* $K \subset D$ *be a compact set,* $F \in \mathscr{C}^\infty(D)$, $\operatorname{Im} F \geq 0$ *in* D. *We assume that*

$$\operatorname{Im} F(0) = 0,\ F'(0) = 0,\ \det\, F''(0) \neq 0,\ F' \neq 0 \text{ in } K \setminus \{0\}. \tag{5.1.1}$$

For $u \in \mathscr{C}_0^\infty(D)$, $\operatorname{supp} u \subset K$ *and any* $k > 0$, *we have*

$$\left| \int e^{ikF(x)} u(x) dx - e^{ikF(0)} \det \left(\frac{kF''(0)}{2\pi i} \right)^{-\frac{1}{2}} \sum_{j<N} k^{-j} P_j u \right| \leq Ck^{-N} \sum_{|\alpha| \leq N} \sup_K |\partial_x^\alpha u|. \tag{5.1.2}$$

Here, C *is a bounded constant when* F *is bounded in* $\mathscr{C}^\infty(D)$, $\frac{|x|}{|F'(x)|}$ *has a uniform bound and*

$$P_j u := \sum_{v-\mu=j} \sum_{2v \geq 3\mu} i^{-j} 2^{-v} \langle F''(0)^{-1} D, D \rangle^v \frac{(h^\mu u)(0)}{v!\mu!}. \tag{5.1.3}$$

Also, $h(x) := F(x) - F(0) - \frac{1}{2}\langle F''(0)x, x\rangle$ *and* $D := \begin{pmatrix} -i\partial_{x_1} \\ \vdots \\ -i\partial_{x_n} \end{pmatrix}$.

This formula is essential for calculating the first order term of Szegő kernel expansion explicitly.

Next, we recall some fundamental facts from pseudohermitian geometry. We call

$$HX := \operatorname{Re} \left(T^{1,0}X \oplus T^{0,1}X \right) \tag{5.1.4}$$

the contact structure of X, and let J be the complex structure on HX so that $T^{1,0}X$ is the eigenspace of J corresponding to the eigenvalue i. We use the convention

$$\boldsymbol{\theta} := -\boldsymbol{\alpha}. \tag{5.1.5}$$

PROPOSITION 5.3 ([78, Proposition 3.1]). *With the same notations and assumptions, there exists an unique affine connection, called Tanaka–Webster connection,*

$$\nabla := \nabla^{\boldsymbol{\theta}} : \mathscr{C}^\infty(X, TX) \to \mathscr{C}^\infty(X, T^*X \otimes TX) \tag{5.1.6}$$

such that

(i) $\nabla_U \mathscr{C}^\infty(X, HX) \subset \mathscr{C}^\infty(X, HX)$ *for* $U \in \mathscr{C}^\infty(X, TX)$.
(ii) $\nabla T = \nabla J = \nabla d\boldsymbol{\theta} = 0$.
(iii) *The torsion* τ *of* ∇ *satisfies:* $\tau(U, V) = d\boldsymbol{\theta}(U, V)T$, $\tau(T, JU) = -J\tau(T, U)$, $U, V \in \mathscr{C}^\infty(X, HX)$.

For $U \in \mathscr{C}^\infty(X, TX), W, V \in \mathscr{C}^\infty(X, HX)$, we recall that $\nabla J \in \mathscr{C}^\infty(X, T^*X \otimes \mathscr{L}(HX, HX))$ is defined by

$$(\nabla_U J)W = \nabla_U(JW) - J\nabla_U W \tag{5.1.7}$$

and $\nabla d\boldsymbol{\theta} \in \mathscr{C}^\infty(T^*X \otimes \Lambda^2(\mathbb{C}T^*X))$ is defined by

$$\nabla_U d\boldsymbol{\theta}(W, V) = Ud\boldsymbol{\theta}(W, V) - d\boldsymbol{\theta}(\nabla_U W, V) - d\boldsymbol{\theta}(W, \nabla_U V). \tag{5.1.8}$$

Moreover, the relations $\nabla J = 0$ and $\nabla d\boldsymbol{\alpha} = 0$ imply that the Tanaka-Webster connection is compatible with the Levi metric. By definition, the torsion of ∇ is given by $\tau(W, U) = \nabla_W U - \nabla_U W - [W, U]$ for $U, V \in \mathscr{C}^\infty(X, TX)$ and $\tau(T, U)$ for $U \in \mathscr{C}^\infty(X, HX)$ is called pseudohermitian torsion.

We let $\{L_\alpha\}_{\alpha=1}^n$ be a local frame of $T^{1,0}X$ and $\{\theta^\alpha\}_{\alpha=1}^n$ be the dual frame of $\{L_\alpha\}_{\alpha=1}^n$. We use the notations $Z_{\overline{\alpha}} := \overline{L_\alpha}$ and $\theta^{\overline{\alpha}} = \overline{\theta^\alpha}$. We write

$$\nabla L_\alpha = \omega_\alpha^\beta \otimes L_\beta, \tag{5.1.9}$$

$$\nabla L_{\overline{\alpha}} = \omega_{\overline{\alpha}}^{\overline{\beta}} \otimes L_{\overline{\beta}}, \tag{5.1.10}$$

and recall that $\nabla T = 0$. We call ω_α^β the connection one form of Tanaka–Webster connection with respect to the frame $\{L_\alpha\}_{\alpha=1}^n$. We denote Θ_α^β the Tanaka–Webster curvature two form, and it is known that

$$\Theta_\alpha^\beta = d\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta. \tag{5.1.11}$$

By direct computation, we also have

$$\Theta_\alpha^\beta = R_{\alpha j \overline{k}}^\beta \theta^j \wedge \theta^{\overline{k}} + A_{\alpha j k}^\beta \theta^j \wedge \theta^k + B_{\alpha j k}^\beta \theta^{\overline{j}} \wedge \theta^{\overline{k}} + C_0 \wedge \boldsymbol{\theta}, \tag{5.1.12}$$

where C_0 is an one form. The term $R_{\alpha j \overline{k}}^\beta$ is called the pseudohermitian curvature tensor and the trace

$$R_{\alpha \overline{k}} := \sum_{j=1}^n R_{\alpha j \overline{k}}^j \tag{5.1.13}$$

is called the pseudohermitian Ricci curvature. Also, we write

$$d\boldsymbol{\theta} = i g_{\alpha \overline{\beta}} \theta^\alpha \wedge \theta^{\overline{\beta}} \tag{5.1.14}$$

and we let $g^{\overline{c}d}$ be the inverse matrix of $g_{a\overline{b}}$. By the definition

$$R_{\mathrm{scal}} := g^{\overline{k}\alpha} R_{\alpha \overline{k}} \tag{5.1.15}$$

we have the notation R_{scal} for the Tanaka–Webster scalar curvature with respect to the pseudohermitian structure $\boldsymbol{\theta}$.

5.2. Uniqueness result for sub-leading coefficient

We need the following fact for oscillatory integrals.

LEMMA 5.4. *We let $D \subset \mathbb{R}^n$ be a small enough open set near 0 and assume that*

$$F(x) \in \mathscr{C}^\infty(D), F(0) = 0, \ \mathrm{Im}\, F \geq 0, \ dF \neq 0 \ \text{ if } \mathrm{Im}\, F = 0, \tag{5.2.1}$$

and

$$G(x) \in \mathscr{C}^\infty(D), \ G(0) \neq 0, \ \mathrm{Im}(FG) \geq 0, \ d(FG) \neq 0 \ \text{ if } \mathrm{Im}\,(FG) = 0. \tag{5.2.2}$$

holds. For any $m \in \mathbb{Z}$, in the sense of oscillatory integral we have

$$\int_0^{+\infty} e^{itG(x)F(x)} t^m dt \equiv \int_0^{+\infty} e^{itF(x)} \frac{t^m}{G(x)^{m+1}} dt \ \mathit{mod}\ \mathscr{C}^\infty(D). \tag{5.2.3}$$

PROOF. First of all, by continuity, we may assume that on D

$$|G| \geq \frac{1}{2}|G(0)| > 0. \tag{5.2.4}$$

From the construction of oscillatory integral, when $m \in \mathbb{N}_0$, in the sense of distribution we can check that

$$\begin{aligned}
\int_0^{+\infty} e^{itG(x)F(x)} t^m dt &= \lim_{\epsilon \to 0} \int_0^{+\infty} e^{it(G(x)F(x)+i\epsilon)} t^m dt \\
&= \lim_{\epsilon \to 0} \frac{m!}{(-iG(x)F(x)+\epsilon)^{m+1}} \\
&= \frac{m!}{G(x)^{m+1}} \lim_{\epsilon \to 0} \frac{1}{(-iF(x)+\frac{\epsilon}{G(x)})^{m+1}} \\
&= \frac{m!}{(-iG(x))^{m+1}} \frac{1}{(F(x)+i0)^{m+1}} \\
&= \int_0^{+\infty} e^{itF(x)} \frac{t^m}{G(x)^{m+1}} dt.
\end{aligned} \tag{5.2.5}$$

For $m \in \mathbb{Z}_{<0}$, we can use the similar argument with some change to the log term singularities and verify the statement. □

We can now state and prove the following important lemma.

LEMMA 5.5. *With the same assumptions and notations in Theorem 5.1, we fix a point $p \in D$. For Szegő phase functions ϕ_1, ϕ_2 such that*

$$(T^2\phi_1)(p,p) = (T^2\phi_2)(p,p) = 0, \tag{5.2.6}$$

if we have

$$\int_0^{+\infty} e^{it\phi_2(x,y)} \alpha(x,y,t) dt \equiv \int_0^{+\infty} e^{it\phi_1(x,y)} \beta(x,y,t) dt \quad \mathrm{mod}\ \mathscr{C}^\infty(D \times D), \tag{5.2.7}$$

where $\alpha(x,y,t), \beta(x,y,t) \in S^n_{\rm cl}(D \times D \times \mathbb{R}_+)$ *satisfy*

$$\alpha_0(p,p) = \beta_0(p,p), \tag{5.2.8}$$

$$(T\alpha_0)(p,p) = (T\beta_0)(p,p) = 0, \tag{5.2.9}$$

then we get

$$\alpha_1(p,p) = \beta_1(p,p). \tag{5.2.10}$$

PROOF. We can take local coordinates $x = (x_1, \cdots, x_{2n+1})$ of X such that

$$T = -\frac{\partial}{\partial x_{2n+1}}. \tag{5.2.11}$$

We can check and may assume that

$$\phi_2(x,y) = f(x,y)\phi_1(x,y) + O(|x-y|^\infty) \tag{5.2.12}$$

for some $f(x,y) \in \mathscr{C}^\infty(D \times D)$. Without loss of generality, we can assume that

$$\phi_2(x,y) = f(x,y)\phi_1(x,y). \tag{5.2.13}$$

From (5.2.6) and (5.2.13), we can check that

$$f(x,x) = 1, \ \ \frac{\partial f}{\partial x_{2n+1}}(0,0) = 0, \tag{5.2.14}$$

and hence

$$f\left((0,x_{2n+1}),0\right) = 1 + O\left(|x_{2n+1}|^2\right). \tag{5.2.15}$$

By Lemma 5.4 and the expansion of (5.2.7), we can check that

$$\begin{aligned}\frac{\sum_{j=0}^n (n-j)!\alpha_j(x,y)(-i\phi_1 f)^j(x,y) \bmod \phi_1^{n+1}}{f^{n+1}(x,y)(-i(\phi_1(x,y)+i0))^{n+1}} \\ \equiv \frac{\sum_{j=0}^n (n-j)!\beta_j(x,y)(-i\phi_1)^j(x,y) \bmod \phi_1^{n+1}}{(-i(\phi_1(x,y)+i0))^{n+1}}\end{aligned} \tag{5.2.16}$$

up to some log term singularities. In particular, when $x \neq y$ we have

$$\begin{aligned}&\sum_{j=0}^n (n-j)!\alpha_j(x,y)(-i\phi_1 f)^j(x,y) \\ &= f^{n+1}(x,y)\sum_{j=0}^n (n-j)!\beta_j(x,y)(-i\phi_1)^j(x,y) + (-if\phi_1)^{n+1}(x,y)S(x,y).\end{aligned} \tag{5.2.17}$$

for some $S \in \mathscr{C}^\infty(D \times D)$. Now, if we take $x = (0, x_{2n+1})$ and $y = 0$ in the above equation, then from (5.2.8), (5.2.9), (5.2.14) and (5.2.15), it is straightforward to check that

$$(\alpha_1 - \beta_1)((0,x_{2n+1}),0) = O(|x_{2n+1}|). \tag{5.2.18}$$

By the continuity and taking $x_{2n+1} \to 0$, we get

$$\alpha_1(0,0) = \beta_1(0,0). \tag{5.2.19}$$

□

The condition of the leading term in the previous lemma is always achievable by the following.

LEMMA 5.6. *With the notations and assumptions in Theorem 5.1, we can take $\phi(x,y)$ and $a(x,y,t)$ such that for every $(x,y) \in D \times D$ we have*

$$T_x^2 \circ \phi(x,y) = 0, \tag{5.2.20}$$

$$T_x \circ a_0(x,y) = 0, \tag{5.2.21}$$

$$a_0(x,x) = \frac{1}{2\pi^{n+1}}. \tag{5.2.22}$$

PROOF. We can take local coordinates $x = (x_1, \cdots, x_{2n+1})$ of X so that $T = -\frac{\partial}{\partial x_{2n+1}}$ on D. By the relation

$$d_x\phi(x,x) = -\boldsymbol{\alpha}(x),$$

we know that $\frac{\partial\phi}{\partial x_{2n+1}}(x,x) = -1 \neq 0$. We can then apply Malgrange preparation theorem [40, Theorem 7.5.5] and may assume that on D we have

$$\phi(x,y) = f_1(x,y)(-x_{2n+1} + g_1(x',y)) \tag{5.2.23}$$

for some smooth functions $f_1(x,y)$, $g_1(x',y)$ with $f_1(x,x) = 1$ for every $x \in D$, where $x' = (x_1, \cdots, x_{2n})$. We note that g_1 is independent of x_{2n+1} and let

$$\varphi(x,y) = -x_{2n+1} + g_1(x',y). \tag{5.2.24}$$

It is clear that $T_x^2 \circ \varphi(x,y) = 0$. By Taylor formula, we have

$$a_0(x,y) = \widetilde{a}_0(x,y) = \widetilde{a}_0((x', g_1(x',y)), y) + \left(-x_{2n+1} + g_1(x',y)\right) r(x,y), \tag{5.2.25}$$

where $\widetilde{a}_0$ denotes an almost analytic extension of a_0 with respect to the real variable x_{2n+1} and $r(x,y) \in \mathscr{C}^\infty(D \times D)$. We let

$$\mathbf{a}_0(x',y) := \widetilde{a}_0((x', g_1(x',y)), y) \in \mathscr{C}^\infty(D \times D). \tag{5.2.26}$$

Then, by using integration by parts can check that

$$\begin{aligned}
&\int_0^{+\infty} e^{it\phi(x,y)} a_0(x,y) t^n dt \\
&= \int_0^{+\infty} e^{it\phi(x,y)} \mathbf{a}_0(x',y) t^n dt - i \int_0^{+\infty} \frac{d}{dt}\left(e^{itf_1(x,y)(-x_{2n+1}+g_1(x',y))}\right) \frac{R(x,y)}{f_1(x,y)} t^n dt \\
&= \int_0^{+\infty} e^{it\phi(x,y)} \mathbf{a}_0(x',y) t^n dt + i \int_0^{+\infty} e^{it\phi(x,y)} \frac{R(x,y)}{f_1(x,y)} n\, t^{n-1} dt.
\end{aligned} \tag{5.2.27}$$

By Lemma 5.4, we have
(5.2.28)

$$\int_0^{+\infty} e^{it\phi(x,y)}\mathbf{a}_0(x',y)t^n dt \equiv \int_0^{+\infty} e^{it\varphi(x,y)} f_1^{-n-1}((x',g(x',y),y))\mathbf{a}_0(x',y)t^n dt.$$

We recall that $\phi(x,y)t \sim \varphi(x,y)t$ in the sense of Melin–Sjöstrand theory. Hence, in the context of Theorem 5.1, we can always replace $a_0(x,y)$ by

$$f_1^{-n-1}((x',g(x',y),y))\mathbf{a}_0(x',y), \tag{5.2.29}$$

which is independent of x_{2n+1} and satisfy

$$T_x \circ f_1^{-n-1}((x',g(x',y),y))\mathbf{a}_0(x',y) = 0. \tag{5.2.30}$$

Also, it is straightforward to check that

$$f_1^{-n-1}((x',g(x',x),x))\mathbf{a}_0(x',x) = \frac{1}{2\pi^{n+1}}, \quad \text{for every } x \in D \tag{5.2.31}$$

by our construction. □

We notice that this lemma already verified the statements of Theorem 1.6 except the most important relation (1.2.48).

Without loss of generality, from now on we assume that $a_0(x,y)$ are described by Lemma 5.6. We have the following important result.

THEOREM 5.7. *With the notations and assumptions used in Theorem 5.1, if we assume that*

$$\overline{\partial}_{b,x}(\phi(x,y)) \text{ vanishes to infinite order at } x = y, \tag{5.2.32}$$

$$\overline{\partial}_{b,y}(-\overline{\phi}(y,x)) \text{ vanishes to infinite order at } x = y, \tag{5.2.33}$$

then we have
(5.2.34)

$$a_0(x,y) - \frac{1}{2\pi^{n+1}} = O(|x-y|^N), \text{ for every } (x,y) \in D \times D, \text{for every } N \in \mathbb{N}.$$

PROOF. For any fixed point $p \in X$, we let $x = (x_1, \cdots, x_{2n+1})$ be a local coordinates of X defined on D with $x(p) = 0$, $T = -\frac{\partial}{\partial x_{2n+1}}$ and

$$T^{1,0}X = \text{span}\,\{\overline{L}_j : j = 1, \cdots, n\}, \overline{L}_j = \frac{\partial}{\partial \overline{z}_j} + O(|x|), \tag{5.2.35}$$

$$\frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}(\frac{\partial}{x_{2j-1}} + i\frac{\partial}{\partial x_{2j}}), \; j = 1, \cdots, n.$$

By the relation that $\overline{\partial}_b\Pi = 0$, we can check that for each $j = 1, \cdots, n$, we have

$$\begin{aligned} 0 &\equiv \int_0^{+\infty} (\overline{L}_j)_x \circ (e^{it\phi(x,y)} a(x,y,t))dt \\ &= \int_0^{+\infty} e^{it\phi(x,y)} \overline{L}_{j,x}(it\phi)\, a\, dt + \int_0^{+\infty} e^{it\phi(x,y)} \overline{L}_{j,x}(a) dt. \end{aligned} \tag{5.2.36}$$

Since we assume that $\overline{\partial}_{b,x}\phi(x,y)$ vanishes to infinite order at $x = y$, we can check that

$$\int_0^{+\infty} e^{it\phi(x,y)} \overline{L}_{j,x}(it\phi)\, a\, dt \equiv 0. \tag{5.2.37}$$

We recall that for $a \sim \sum_{j=0}^{+\infty} a_j(x,y) t^{n-j}$ we may assume that $a_0(x,y) = a_0(x,y')$. By the Malgrange preparation theorem [40, Theorem 7.5.5] we may assume that

$$\phi(x,y) = f(x,y)\varphi(x,y), \tag{5.2.38}$$

$$\varphi(x,y) = -x_{2n+1} + g_1(x',y), \tag{5.2.39}$$

$$f(x,x) = 1. \tag{5.2.40}$$

Also, by applying the Malgrange preparation theorem [40, Theorem 7.5.6] to the variable x_{2n+1}, for each $j = 1, \cdots, n$ we also have

$$(\overline{L}_j a_0)(x,y) = h_j(x,y)(-x_{2n+1} + g_1(x',y)) + r_j(x',y) \tag{5.2.41}$$

With the help of partial integration in t, the above relations imply that

$$\int_0^{+\infty} e^{it\phi(x,y)} r_j(x',y) t^n dt + \int_0^{+\infty} e^{it\phi(x,y)} R_j(x,y,t) dt \equiv 0 \tag{5.2.42}$$

for some $R(x,y,t) \in S^{n-1}_{\mathrm{cl}}(D \times D \times \mathbb{R}_+)$. We can apply Lemma 5.4, Taylor expansion argument and Melin–Sjöstrand theory to rewrite the above relation by

$$\int_0^{+\infty} e^{it\varphi(x,y)} \frac{r_j(x',y)}{\widetilde{f}^{n+1}(x',g_1(x',y),y)} t^n dt + \int_0^{+\infty} e^{it\varphi(x,y)} \mathbf{R}_j(x,y,t) \equiv 0 \tag{5.2.43}$$

for some $\mathbf{R}_j(x,y,t) \in S^{n-1}_{\mathrm{cl}}(D \times D \times \mathbb{R}_+)$. By the argument of partial Fourier transform in the proof of Theorem 2.6, we can see that

$$\frac{r_j(x',y)}{\widetilde{f}^{n+1}(x',g_1(x',y),y)} = O(|x-y|^{+\infty}) \tag{5.2.44}$$

and accordingly

$$r_j(x',y) = O(|x-y|^{+\infty}). \tag{5.2.45}$$

Furthermore, since now we have $a_0 = a_0(x',y)$ and

$$(\overline{L}_j a_0)(x,y) = h_j(x,y)(-x_{2n+1} + g_1(x',y)) + O(|x-y|^{+\infty}) \tag{5.2.46}$$

for each $j = 1, \cdots, n$, by taking the derivative $\frac{\partial}{\partial x_{2n+1}}$ both sides of the above relation it is not difficult to see that

$$h_j(0,0) = 0. \tag{5.2.47}$$

We also have

$$\partial_x^\alpha \partial_y^\beta \partial_{x_{2n+1}} \overline{L}(a_0) = \partial_x^\alpha \partial_y^\beta \left(\frac{\partial h_j}{\partial x_{2n+1}} \varphi + (-h_j) \right) + O(|x-y|^{+\infty}) \tag{5.2.48}$$

for each $j = 1, \cdots, n$. By Leibniz rule, $\varphi(0,0) = 0$, $(d_x\varphi)(0,0) = -(d_y\varphi)(0,0) = -dx_{2n+1}$ and induction, it is not hard to check that

$$h_j(x,y) = O(|(x,y)|^{+\infty}). \tag{5.2.49}$$

So we can deduce from (5.2.46) that

$$\frac{\partial a_0}{\partial \overline{z}_j}(x', y) = O(|(x,y)|^{+\infty}), \;\; j = 1, \cdots, n, \tag{5.2.50}$$

where $\frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}(\frac{\partial}{\partial x_{2j-1}} - i\frac{\partial}{\partial x_{2j}})$. Similarly, by $0 \equiv \overline{\partial}_b \Pi \equiv \overline{\partial}_b \Pi^*$, $\Pi^*(x,y) = \overline{\Pi}(y,x)$ and the same argument above, we can also check that

$$\frac{\partial a_0}{\partial w_j}(x', y) = O(|(x,y)|^{+\infty}), \;\; j = 1, \cdots, n, \tag{5.2.51}$$

where $\frac{\partial}{\partial w_j} = \frac{1}{2}(\frac{\partial}{\partial y_{2j-1}} - i\frac{\partial}{\partial y_{2j}})$. We claim that

$$|a_0(x,y) - \frac{1}{2\pi^{n+1}}| = O(|(x,y)|^N) \text{ for every } N \in \mathbb{N}. \tag{5.2.52}$$

It is clear that (5.2.52) holds for $N = 1$. Suppose that (5.2.52) holds for $N = N_0$, $N_0 \in \mathbb{N}$. We are going to prove that (5.2.52) holds for $N = N_0 + 1$. We fix $j \in \{1, \cdots, n\}$ and fix $\alpha, \beta \in \mathbb{N}_0$, $\alpha + \beta = N_0$. From $a_0(x,x) = \frac{1}{2\pi^{n+1}}$, we have

$$\left(((\frac{\partial}{\partial z_j} + \frac{\partial}{\partial w_j})^\alpha (\frac{\partial}{\partial \overline{z}_j} + \frac{\partial}{\partial \overline{w}_j})^\beta) a_0 \right)(x,x) = 0. \tag{5.2.53}$$

From (5.2.53), we have

$$\left((\frac{\partial}{\partial z_j})^\alpha (\frac{\partial}{\partial \overline{w}_j})^\beta a_0 \right)(0,0) = \sum_{\alpha_1,\alpha_2,\beta_1,\beta_2 \in \mathbb{N}_0, \alpha_1+\alpha_2=\alpha, \beta_1+\beta_2=\beta, \alpha_2+\beta_1>0} c_{\alpha_1,\alpha_2,\beta_1,\beta_2} \left((\frac{\partial}{\partial z_j})^{\alpha_1} (\frac{\partial}{\partial w_j})^{\alpha_2} (\frac{\partial}{\partial \overline{z}_j})^{\beta_1} (\frac{\partial}{\partial \overline{w}_j})^{\beta_2} a_0 \right)(0,0), \tag{5.2.54}$$

where $c_{\alpha_1,\alpha_2,\beta_1,\beta_2}$ is a constant, for every $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{N}_0$, $\alpha_1 + \alpha_2 = \alpha$, $\beta_1 + \beta_2 = \beta$, $\alpha_2 + \beta_1 > 0$. Since $\alpha_2 + \beta_1 > 0$, from (5.2.50) and (5.2.51), we get

$$\Big((\frac{\partial}{\partial z_j})^{\alpha_1}(\frac{\partial}{\partial w_j})^{\alpha_2}(\frac{\partial}{\partial \overline{z}_j})^{\beta_1}(\frac{\partial}{\partial \overline{w}_j})^{\beta_2}a_0\Big)(0,0) = 0, \tag{5.2.55}$$
$$\text{for every } \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{N}_0,\ \alpha_1 + \alpha_2 = \alpha,\ \beta_1 + \beta_2 = \beta,\ \alpha_2 + \beta_1 > 0.$$

From this observation and (5.2.54), we get

$$\Big((\frac{\partial}{\partial z_j})^{\alpha}(\frac{\partial}{\partial \overline{w}_j})^{\beta}a_0\Big)(0,0) = 0 \ \text{ for every } \alpha, \beta \in \mathbb{N}_0,\ \alpha + \beta = N_0. \tag{5.2.56}$$

We can repeat the proof of (5.2.56) with minor change and deduce that

$$\Big((\frac{\partial}{\partial z})^{\alpha}(\frac{\partial}{\partial \overline{w}})^{\beta}a_0\Big)(0,0) = 0 \text{ for every } \alpha, \beta \in \mathbb{N}_0^n,\ |\alpha| + |\beta| = N_0. \tag{5.2.57}$$

Since a_0 is independent of x_{2n+1} and $a_0(x', x) = \frac{1}{2\pi^{n+1}}$, we have

$$\Big((\frac{\partial}{\partial y_{2n+1}})^{N}a_0\Big)(x,x) = 0, \ \text{ for every } N \in \mathbb{N}. \tag{5.2.58}$$

From (5.2.58), we can repeat the proof of (5.2.56) with minor change and deduce that

$$\Big((\frac{\partial}{\partial z})^{\alpha}(\frac{\partial}{\partial \overline{w}})^{\beta}(\frac{\partial}{\partial y_{2n+1}})^{\gamma}a_0\Big)(0,0) = 0 \tag{5.2.59}$$
$$\text{for every } \alpha, \beta, \gamma \in \mathbb{N}_0^n,\ |\alpha| + |\beta| + |\gamma| = N_0.$$

From (5.2.50), (5.2.51), (5.2.57), (5.2.59), we can check that

$$|a_0(x,y) - \frac{1}{2\pi^{n+1}}| = O(|(x,y)|^{N_0+1}).$$

By induction, we get the claim (5.2.52). From (5.2.52), our theorem follows. □

5.3. Calculation of the sub-leading coefficient

To calculate the value for sub-leading coefficient , we need to choose some other suitable coordinates and phase function. We quickly explain our strategy of the proof. On one hand, locally we already have choices $\varphi(x,y)$ and $s(x,y,t) \sim \sum_{j=0}^{+\infty} s_j(x,y)t^{n-j}$ such that

$$\Pi(x,y) \equiv \int_0^{+\infty} e^{it\varphi(x,y)} s(x,y,t)dt \tag{5.3.1}$$

with the properties

$$T_x^2 \circ \varphi(x,y) = 0, \tag{5.3.2}$$

$$T_x \circ s_0(x,y) = 0, \tag{5.3.3}$$

$$s_0(x,x) = \frac{1}{2\pi^{n+1}}. \tag{5.3.4}$$

On the other hand, given any point p in the local picture we will embed the neighborhood of p into $\mathbb{C}^{n+1}$, which of course preserves the CR structure of X, and construct a holomorphic coordinates $z_j = x_{2j-1} + x_{2j}$, $j = 1, \cdots, n$, and a special pair $\left(\phi(x,y), a(x,y,t) \sim \sum_{j=0}^{+\infty} a_j(x,y)t^{n-j}\right)$ such that

$$\Pi(x,y) \equiv \int_0^{+\infty} e^{it\phi(x,y)} a(x,y,t)dt \tag{5.3.5}$$

with the properties that at $(x,y) = (p,p)$, where p is identified at 0, we have

$$R_{\text{scal}}(0) = -2i \sum_{j,\ell=1}^{n} \frac{\partial^4 \phi}{\partial z_j \partial \overline{z}_j \partial z_\ell \partial \overline{z}_\ell}(0,0), \tag{5.3.6}$$

$$\frac{\partial^2 \lambda}{\partial z_\ell \partial \overline{z}_\ell}(0) = 2i \sum_{j=1}^{n} \frac{\partial^4 \phi}{\partial z_\ell \partial \overline{z}_\ell \partial z_j \partial \overline{z}_j}(0),\ \ell = 1, \cdots, n, \tag{5.3.7}$$

where $\lambda(x)dx$ is the volume form on X associated by $\boldsymbol{\alpha}$, and

$$T_x^2 \circ \phi(x,y) = 0, \tag{5.3.8}$$

$$a_0(x,y) = \frac{1}{2\pi^{n+1}} + O(|(x,y)|^N),\ \text{for any } N \in \mathbb{N}. \tag{5.3.9}$$

Then, we will perturb ϕ by the Malgrange preparation theorem [40, Theorem 7.5.5] and we will combine the trick in the previous section to show that actually there is also a pair $\left(\Phi(x,y), A(x,y,t) \sim \sum_{j=0}^{+\infty} A_j(x,y)t^{n-j}\right)$ such that

$$\Pi(x,y) \equiv \int_0^{+\infty} e^{it\Phi(x,y)} A(x,y,t)dt \tag{5.3.10}$$

with the properties that at $(x,y) = (p,p)$ we have

$$T_x^2 \circ \Phi(x,y) = 0, \tag{5.3.11}$$

$$\frac{\partial^4 \Phi}{\partial z_j \partial \overline{z}_j \partial z_\ell \partial \overline{z}_\ell}(0,0) = \frac{\partial^4 \phi}{\partial z_j \partial \overline{z}_j \partial z_\ell \partial \overline{z}_\ell}(0,0),\ j,\ell = 1 \cdots, n, \tag{5.3.12}$$

$$T_x \circ A_0(x,y) = 0, \tag{5.3.13}$$

$$A_0(x,x) = \frac{1}{2\pi^{n+1}} + O(|(x,y)|^3). \tag{5.3.14}$$

By the microlocal analysis of $\Pi = \Pi^2$, we will show that we have

$$A_1(0,0) = -\frac{1}{2\pi^{n+1}}\left[\sum_{j=1}^{n}\left(\frac{\partial^2\lambda}{\partial z_j\partial\overline{z}_j}\right)(0) - i\sum_{j,\ell=1}^{n}\frac{\partial^4\Phi}{\partial z_j\partial\overline{z}_j\partial z_\ell\partial\overline{z}_\ell}(0)\right], \tag{5.3.15}$$

If the above strategy works, then by Lemma 5.5 and Lemma 5.6, we can compare $s_1(p,p) = A_1(p,p)$ at all point p and we can prove Theorem 1.6 by the calculation of $A_1(p,p)$.

Let us start the proof now. We recall that in §1 we present the CR embedding theorem dues to Boutet de Monvel: for any fixed point $p \in X$ there is an open set U of p and an injective immersion F given by

$$F : U \to \mathbb{C}^{n+1}, \tag{5.3.16}$$

$$x \mapsto (F_1(x), \cdots, F_{n+1}(x)), \tag{5.3.17}$$

where $F_1, \cdots, F_{n+1} \in \mathscr{C}^\infty(X) \cap \mathrm{Ker}\,\overline{\partial}_b$. From now on, we identify U with $\partial M \bigcap \Omega$, where

$$\partial M := \{z \in \mathbb{C}^{n+1} : r(z) = 0\}, \tag{5.3.18}$$

$$r(z) \in \mathscr{C}^\infty(\mathbb{C}^{n+1}, \mathbb{R}), \tag{5.3.19}$$

$$J(dr) = 1 \text{ on } \partial M,\ J \text{ is the standard complex structure on } \mathbb{C}^{n+1}, \tag{5.3.20}$$

$$\Omega \text{ is an open set of } p \text{ in } \mathbb{C}^{n+1}. \tag{5.3.21}$$

By standard Chern–Moser argument, we can find local holomorphic coordinates $x = (x_1, \cdots, x_{2n+2}) = z = (z_1, \cdots, z_{n+1})$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \cdots, n+1$, defined on small enough Ω such that

$$z(p) = 0, \tag{5.3.22}$$

$$r(z) = 2\mathrm{Im}\, z_{n+1} + \sum_{j=1}^{n}|z_j|^2 + O(|(z_1, \cdots, z_{n+1}|)^4), \tag{5.3.23}$$

and we have a special Szegő phase function

$$\phi(x,y) := \rho(z,w)|_{U\times U} \tag{5.3.24}$$

where

$$\rho(z,w) = \frac{1}{i}\sum_{\alpha,\beta\in\mathbb{N}_0^{n+1}, |\alpha|+|\beta|\le N}\frac{\partial^{\alpha+\beta}r}{\partial z^\alpha\partial\overline{z}^\beta}(0)\frac{z^\alpha}{\alpha!}\frac{\overline{w}^\beta}{\beta!} + O(|(z,w)|^{N+1}) \quad \text{for every } N \in \mathbb{N}, \tag{5.3.25}$$

such that

$$\Pi(x,y) \equiv \int_0^{+\infty} e^{it\phi(x,y)}a(x,y,t)dt \mod \mathscr{C}^\infty(U\times U). \tag{5.3.26}$$

Now we let D be an open set of $\mathbb{R}^{2n+1}$ and $0 \in D$. From implicit function theorem, if Ω is small enough, we can find a function $R(x_1, \cdots, x_{2n+1}) \in \mathscr{C}^\infty(D)$ such that

$$\text{for every } x \in \Omega \text{ and } (x_1, \cdots, x_{2n+1}) \in D, \quad x \in U \text{ if and only if } x_{2n+2} = R(x_1, \cdots, x_{2n+1}). \tag{5.3.27}$$

From now on, we assume that Ω is small enough so that (5.3.27) holds. We let $x = (x_1, \cdots, x_{2n+1})$ be local coordinates of D given by the map

$$(x_1, \cdots, x_{2n+1}) \in D \mapsto (x_1, \cdots, x_{2n+1}, R(x_1, \cdots, x_{2n+1})) \in U. \tag{5.3.28}$$

From now on, we identify U with D and we will work with local coordinates $x = (x_1, \cdots, x_{2n+1})$ as (5.3.28). The following follows from some straightforward calculation. We omit the details.

PROPOSITION 5.8. *With the same notations and assumptions in this chapter, we have*

$$\begin{aligned} &R(x) = -\frac{1}{2}\sum_{j=1}^{2n} x_j^2 + O(|x|^4), \\ &\frac{\partial^2 R}{\partial z_j \partial \overline{z}_k}(0) = -\frac{1}{2}\delta_{j,k}, \ \ j, k = 1, \cdots, n, \end{aligned} \tag{5.3.29}$$

$$\boldsymbol{\alpha}(x) = dx_{2n+1} - i\sum_{j=1}^{n}\left(\frac{\partial R}{\partial z_j}dz_j - \frac{\partial R}{\partial \overline{z}_j}d\overline{z}_j\right) + O(|x|^4), \tag{5.3.30}$$

$$d\boldsymbol{\alpha}(x) = 2i\sum_{j=1,k}^{n}\frac{\partial^2 R}{\partial z_j \partial \overline{z}_k}dz_j \wedge d\overline{z}_k + O(|x|^3), \tag{5.3.31}$$

$$T_x^{1,0}X = \text{span}\left\{\frac{\partial}{\partial z_j} + i\frac{\partial R}{\partial z_j}\frac{\partial}{\partial x_{2n+1}} + O(|x|^4)\right\}_{j=1}^{n}, \tag{5.3.32}$$

$$T(x) = -\frac{\partial}{\partial x_{2n+1}} + O(|x|^2). \tag{5.3.33}$$

In particular, the volume form on X given by

$$\lambda(x)dx_1 \cdots dx_{2n+1} = \frac{1}{n!}\left(\frac{-d\boldsymbol{\alpha}}{2}\right)^n \wedge \boldsymbol{\alpha} \tag{5.3.34}$$

satisfies

(5.3.35)
$$\begin{aligned}&\lambda(0)=1,\\&\frac{\partial\lambda}{\partial x_j}(0)=0,\ \ j=1,\cdots,2n+1.\end{aligned}$$

We also need the following result.

PROPOSITION 5.9. *With the same notations above, we have*

(5.3.36)
$$\begin{aligned}\phi(x,y)=&-x_{2n+1}+y_{2n+1}\\&+\frac{i}{2}\sum_{j=1}^{n}\left[|z_j-w_j|^2+(\overline{z}_jw_j-z_j\overline{w}_j)\right]+O\left(|(x,y)|^4\right),\end{aligned}$$

(5.3.37)
$$\begin{aligned}&\frac{\partial^4\phi}{\partial z_j\partial z_k\partial\overline{z}_\ell\partial\overline{z}_s}(0,0)=-i\frac{\partial^4 R}{\partial z_j\partial z_k\partial\overline{z}_\ell\partial\overline{z}_s}(0),\ \ j,k,\ell,s\in\{1,\cdots,n\},\\&\frac{\partial^4\phi}{\partial w_j\partial w_k\partial\overline{w}_\ell\partial\overline{w}_s}(0,0)=-i\frac{\partial^4 R}{\partial z_j\partial z_k\partial\overline{z}_\ell\partial\overline{z}_s}(0),\ \ j,k,\ell,s\in\{1,\cdots,n\},\end{aligned}$$

where $\frac{\partial}{\partial w_j}=\frac{1}{2}(\frac{\partial}{\partial y_{2j-1}}-i\frac{\partial}{\partial y_{2j}})$, $j=1,\cdots,n$, *and*

(5.3.38)
$$T^2\phi(0,0)=0.$$

PROOF. By the construction of $\rho(z,w)$ and our choice of coordinates, we can check that

(5.3.39)
$$\begin{aligned}\phi(x,y)&=-x_{2n+1}+y_{2n+1}-i[R(x)+R(y)+\sum_{j=1}^{n}z_j\overline{w}_j]+O\left(|(x,y)|^4\right)\\&=-x_{2n+1}+y_{2n+1}+\frac{i}{2}\sum_{j=1}^{n}\left(|z_j|^2-2z_j\overline{w_j}+|w_j|^2\right)+O\left(|(x,y)|^4\right)\\&=-x_{2n+1}+y_{2n+1}+\frac{i}{2}\sum_{j=1}^{n}\left[|z_j-w_j|^2+(\overline{z}_jw_j-z_j\overline{w}_j)\right]+O\left(|(x,y)|^4\right)\end{aligned}$$

and

(5.3.40)
$$\begin{aligned}\phi(x,0)=&-x_{2n+1}-iR(x)+\frac{1}{i}\sum_{\alpha_j\in\mathbb{N}_0,j=1,\cdots,n+1,\alpha_1+\cdots+\alpha_{n+1}=4}\frac{1}{\alpha_1!\cdots\alpha_{n+1}!}\\&\times\frac{\partial^4 r}{\partial z_1^{\alpha_1}\cdots\partial z_{n+1}^{\alpha_{n+1}}}(0)z_1^{\alpha_1}\cdots z_n^{\alpha_n}(x_{2n+1}+iR(x))^{\alpha_{n+1}}\\&+O\left(|x|^5\right).\end{aligned}$$

From (5.3.40), we get

$$\frac{\partial^4\phi}{\partial z_j\partial z_k\partial\bar{z}_\ell\partial\bar{z}_s}(0,0)=-i\frac{\partial^4 R}{\partial z_j\partial z_k\partial\bar{z}_\ell\partial\bar{z}_s}(0),\ \ j,k,\ell,s\in\{1,\cdots,n\}. \tag{5.3.41}$$

Similarly,

$$\frac{\partial^4\phi}{\partial w_j\partial w_k\partial\overline{w}_\ell\partial\overline{w}_s}(0,0)=-i\frac{\partial^4 R}{\partial z_j\partial z_k\partial\bar{z}_\ell\partial\bar{z}_s}(0),\ \ j,k,\ell,s\in\{1,\cdots,n\}. \tag{5.3.42}$$

From (5.3.39), (5.3.41) and (5.3.42), we get (5.3.36) and (5.3.37).

Finally, because for all x near p we have

$$T(x)=-\frac{\partial}{\partial x_{2n+1}}+O(|x|^2), \tag{5.3.43}$$

it is clear that

$$T^2\phi(0,0)=0. \tag{5.3.44}$$

□

By the Malgrange preparation theorem [40, Theorem 7.5.5] we may assume that

$$\begin{aligned}&\phi(x,y)=f(x,y)\Phi(x,y)\ \text{ on } D,\\&\Phi(x,y)=-x_{2n+1}+g(x',y),\end{aligned} \tag{5.3.45}$$

where $f(x,y),g(x',y)\in\mathscr{C}^\infty(D\times D)$, $x'=(x_1,\cdots,x_{2n})$. From Proposition 5.9, it is straightforward to check that

$$\begin{aligned}&f(x,y)=1+O(|(x,y)|^3),\\&\Phi(x,y)\text{ satisfies (5.3.36), (5.3.37) and (5.3.38).}\end{aligned} \tag{5.3.46}$$

So we have

$$\Pi(x,y)\equiv\int_0^{+\infty}e^{it\phi(x,y)}a(x,y,t)dt\equiv\int_0^{+\infty}e^{it\Phi(x,y)}A(x,y,t)dt, \tag{5.3.47}$$

where $a(x,y,t),A(x,y,t)\in S^n_{\rm cl}(D\times D\times\mathbb{R}_+)$,

(5.3.48)

$$\begin{aligned}&a(x,y,t)\sim\sum_{j=0}^{+\infty}a_j(x,y)t^{n-j}\text{ in }S^n_{\rm cl}(D\times D\times\mathbb{R}_+),\ a_j(x,y)\in\mathscr{C}^\infty(D\times D),\\&A(x,y,t)\sim\sum_{j=0}^{+\infty}A_j(x,y)t^{n-j}\text{ in }S^n_{\rm cl}(D\times D\times\mathbb{R}_+),\ A_j(x,y)\in\mathscr{C}^\infty(D\times D),\\&a_j(x,y),A_j(x,y)\in\mathscr{C}^\infty(D\times D)\ \text{ for all }\ j\in\mathbb{N}_0.\end{aligned}$$

We can repeat the proof of Lemma 5.6 and conclude that we can take $a_j(x,y)$, $A_j(x,y)$ independent of x_{2n+1} for every $j \in \mathbb{N}_0$. From now on, we assume that $a_j(x,y)$, $A_j(x,y)$ are independent of x_{2n+1} for every $j \in \mathbb{N}_0$ and denote them by

$$\begin{aligned} a_j(x,y) &= a_j(x',y), \;\; j = 0,1,\cdots, \\ A_j(x,y) &= A_j(x',y), \;\; j = 0,1,\cdots. \end{aligned} \tag{5.3.49}$$

We recall that by the construction of $\rho(z,w)$, we can directly check that

$$\overline{\partial}_{b,x}(\phi(x,y)) \text{ vanishes to infinite order at } x = y, \tag{5.3.50}$$

$$\overline{\partial}_{b,y}(-\overline{\phi}(y,x)) \text{ vanishes to infinite order at } x = y. \tag{5.3.51}$$

So we can repeat the proof of Theorem 5.7 with minor change and deduce that

$$a_0(x',y) = \frac{1}{2\pi^{n+1}} + O(|(x,y)|^N \text{ for every } N \in \mathbb{N}, \tag{5.3.52}$$

and certainly

$$a_0(x',y)|_{y=x=0} = \frac{1}{2\pi^{n+1}}, \; \left(T_x \circ a_0(x',y)\right)|_{y=x=0} = 0 \tag{5.3.53}$$

Moreover, from Lemma 5.4 and the proof of Lemma 5.6, we can take $A_0(x',y)$ to be

$$A_0(x',y) = a_0(x',y)\frac{1}{\widetilde{f}^{n+1}((x',g(x',y)),y)}, \tag{5.3.54}$$

where $\widetilde{f}$ is an almost analytic extension of f. From (5.3.46), (5.3.52) and (5.3.54) and , it is easy to see that

$$A_0(x',y) = \frac{1}{2\pi^{n+1}} + O(|(x,y)|^3). \tag{5.3.55}$$

and of course

$$A_0(x',y)|_{y=x=0} = \frac{1}{2\pi^{n+1}}, \; \left(T_x \circ A_0(x',y)\right)|_{y=x=0} = 0 \tag{5.3.56}$$

We also notice that

$$(T^2\phi)(0,0) = 0 = (T^2\Phi)(0,0) = 0. \tag{5.3.57}$$

As we mention earlier, from Lemma 5.5 and Lemma 5.6 we can see that to prove Theorem 1.6 we only need to calculate $a_1(0,0) = A_1(0,0)$.

To compute $A_1(0,0)$, we apply the projection relation

$$\Pi = \Pi^2. \tag{5.3.58}$$

Equivalently, in the sense of oscillatory integral we have

$$\Pi(x,y) \equiv \int_D \Pi(x,w)\Pi(w,y)\lambda(w)dw \mod \mathscr{C}^\infty(D \times D), \tag{5.3.59}$$

where $\lambda(w)dw$ is the volume form on X. Now, after shrinking D if necessary, from

$$\Pi(x,y) \in \mathscr{C}^\infty(X \times X \setminus \operatorname{diag} X \times X) \tag{5.3.60}$$

we may assume that all the base variables $x, y, w \in D$ are within a compact set. Then, in the sense of oscillatory integral, we have

(5.3.61)
$$\begin{aligned}
&\int_0^{+\infty} e^{it\Phi(x,y)} A(x,y,t)dt \\
&\equiv \int_0^{+\infty}\int_0^{+\infty}\int_D e^{it\Phi(x,w)+is\Phi(w,y)} A(x,w,t)A(w,y,s)\lambda(w)dwdtds \\
&\equiv \int_0^{+\infty}\left(\int_0^{+\infty}\int_D e^{it\Phi(x,w)+it\sigma\Phi(w,y)} tA(x,w,t)A(w,y,t\sigma)\lambda(w)dwd\sigma\right)dt,
\end{aligned}$$

and the integrand over (w,σ) in the last line can be arranged into

$$\int_0^{+\infty}\int_D e^{it(-x_{2n+1}+g(x',w)+\sigma(-w_{2n+1}+g(w',y))} \times tA(x,w,t)A(w,y,t\sigma)\lambda(w)dwd\sigma. \tag{5.3.62}$$

We consider the phase function

$$\Psi(w,\sigma,x,y) := -x_{2n+1} + g(x',w) + \sigma(-w_{2n+1} + g(w',y)), \tag{5.3.63}$$

and it is clear that $\operatorname{Im}\Psi \geq 0$ for $\sigma \geq 0$. We also have

$$d_w\Psi = d_w g(x',w) + \sigma(-dw_{2n+1} + d_{x'}g(w',y)), \tag{5.3.64}$$

$$\frac{\partial\psi}{\partial\sigma} = -w_{2n+1} + g(w',y). \tag{5.3.65}$$

With respect to (w,σ) we consider the matrix $\operatorname{Hess}(\Psi)$ which is a matrix of the form

$$\operatorname{Hess}(\Psi) = \begin{bmatrix} \frac{\partial^2}{\partial w^2}(g(x',w)+\sigma g(w',y)) & \left(\frac{\partial}{\partial w}(-w_{2n+1}+g(w',y))\right)^t \\ \frac{\partial}{\partial w}(-w_{2n+1}+g(w',y)) & 0 \end{bmatrix}. \tag{5.3.66}$$

Directly, at $(w,\sigma;x,y) = (0,1,0,0) \in \mathbb{R}^{n+1} \times \mathbb{R}_+ \times \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1}$ we have

$$\Psi = 0 \text{ and } d_{w,\sigma}\Psi = 0 \tag{5.3.67}$$

and we can notice that Φ satisfies (5.3.36), (5.3.37) and (5.3.38), cf. (5.3.46)), and we can compute that at $(w,\sigma;x,y) = (0,1,0,0) \in \mathbb{R}^{n+1} \times \mathbb{R}_+ \times \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1}$,

(5.3.68)
$$\det\operatorname{Hess}(\Psi) = \det\begin{bmatrix} 2iI_{2n} & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} = -\det\begin{bmatrix} 2iI_{2n} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (-1)^{n+1}2^{2n},$$

Hence, we can find a solution $\widetilde{W}(x',y)$ and $\widetilde{\Sigma}(x',y)$ near $0 \in \mathbb{R}^{2n+1}$ and $1 \in \mathbb{R}_+$ such that

$$\frac{\partial\widetilde{\Psi}}{\partial\widetilde{w}}(\widetilde{W},\widetilde{\Sigma},x,y) = \frac{\partial\widetilde{\Psi}}{\partial\widetilde{\sigma}}(\widetilde{W},\widetilde{\Sigma},x,y) = 0. \tag{5.3.69}$$

We note that $\widetilde{W}(x,y) = \widetilde{W}(x',y)$ and $\widetilde{W}(x,y) = \widetilde{\Sigma}(x',y)$ are independent of x_{2n+1}. So by the stationary phase method of Melin–Sjöstrand Theorem 2.2 we get

$$\begin{aligned}&\int_0^{+\infty} e^{it\Phi(x,y)}A(x,y,t)dt\\ &\equiv \int_0^{+\infty}\int_0^{+\infty}\int_D e^{it\Psi(w,\sigma;x,y)}tA(x,w,t)A(w,y,t\sigma)\lambda(w)dwd\sigma dt\\ &\equiv \int_0^{+\infty} e^{it\left(-x_{2n+1}+\widetilde{g}\left(x',\widetilde{W}(x',y)\right)\right)}B(x,y,t)dt,\end{aligned} \tag{5.3.70}$$

where

$$B(x,y,t) \sim \sum_{j=0}^{\infty} B_j(x,y)t^{n-j} \text{ in } S^n_{\mathrm{cl}}(D\times D\times\mathbb{R}_+), \tag{5.3.71}$$
$$B_j(x,y) \in \mathscr{C}^\infty(D\times D), \text{ for every } j\in\mathbb{N}_0.$$

Since both $\widetilde{W}(x',y)$, $\widetilde{\Sigma}(x',y)$ and $A_j(x',y)$ are independent of x_{2n+1} for every $j \in \mathbb{N}_0$, it is straightforward to see that

$$B_j(x,y) = B_j(x',y) + O(|x-y|^N) \text{ for every } N\in\mathbb{N}, j\in\mathbb{N}_0. \tag{5.3.72}$$

Also, we observe that

$$\begin{aligned}B_0(0,0) &= \det\left(\frac{\mathrm{Hess}(\psi)}{2\pi i}\right)^{-\frac{1}{2}} A_0(0,0)^2\lambda(0)\\ &= 2\pi^{n+1}\left(\frac{1}{2\pi^{n+1}}\right)^2\\ &= \frac{1}{2\pi^{n+1}}\\ &= A_0(0,0).\end{aligned} \tag{5.3.73}$$

For now $\Pi = \Pi^2$, we have

$$\int_0^{+\infty} e^{it(-x_{2n+1}+g(x',y))}A(x,y,t)dt \equiv \int_0^{+\infty} e^{it\left(-x_{2n+1}+\widetilde{g}\left(x',\widetilde{W}(x',y)\right)\right)}B(x,y,t)dt. \tag{5.3.74}$$

Because of

$$A_0(x,y)B_0(x,y) \neq 0, \tag{5.3.75}$$

by the partial Fourier transform argument in the proof of Theorem 2.6, we can show that

$$\widetilde{g}\left(x',\widetilde{W}(x',y)\right) = g(x',y) + O(|x-y|^N) \text{ for every } N \in \mathbb{N}. \tag{5.3.76}$$

We may accordingly replace $\widetilde{g}\left(x',\widetilde{W}(x',y)\right)$ by $g(x',y)$ and we have

$$\int_0^{+\infty} e^{it\Phi(x,y)}A(x,y,t)dt \equiv \int_0^{+\infty} e^{it\Phi(x,y)}B(x,y,t)dt \bmod \mathscr{C}^\infty(D\times D). \tag{5.3.77}$$

We need the following observation.

LEMMA 5.10. *With the notations used above, we have*

$$B_j(x,y) = A_j(x,y) + O(|x-y|^N) \text{ for every } N \in \mathbb{N}, j \in \mathbb{N}. \tag{5.3.78}$$

PROOF. We can apply Malgrange preparation theorem [40, Theorem 7.5.6] and repeat the discussion of (5.2.17) to conclude that

$$\begin{aligned} B_0(x,y) - A_0(x,y) = h(x,y)(-x_{2n+1} + g(x',y)) + O(|x-y|^N) \\ \text{for every } N \in \mathbb{N}, j \in \mathbb{N}, \end{aligned} \tag{5.3.79}$$

where $h(x,y) \in \mathscr{C}^\infty(D\times D)$. After taking an almost analytic extension and $\widetilde{x}_{2n+1} = g(x',y)$ in (5.3.79), we can notice that up to $O(|x-y|^N)$, for every $N \in \mathbb{N}_0$ we have $B_0(x,y) - A_0(x,y)$ being independent of x_{2n+1}, and we conclude that

$$B_0(x,y) - A_0(x,y) = O(|x-y|^N) \text{ for every } N \in \mathbb{N}. \tag{5.3.80}$$

From (5.3.80), Malgrange preparation theorem [40, Theorem 7.5.6] and the discussion of (5.2.17) to conclude that

$$\begin{aligned} B_1(x,y) - A_1(x,y) = h_1(x,y)(-x_{2n+1} + g(x',y)) + O(|x-y|^N) \\ \text{for every } N \in \mathbb{N}, \end{aligned} \tag{5.3.81}$$

After taking an almost analytic extension and $\widetilde{x}_{2n+1} = g(x',y)$ in (5.3.81), we can notice that up to $O(|x-y|^N)$ for every $N \in \mathbb{N}_0$ we have $B_1(x,y) - A_1(x,y)$ is independent of x_{2n+1}, and we conclude this as

$$B_1(x,y) - A_1(x,y) = O(|x-y|^N) \text{ for every } N \in \mathbb{N}. \tag{5.3.82}$$

Continuing in this way for $j = 2, 3, \cdots$, the lemma follows. □

Now we look from another way for the relation $\Pi = \Pi^2$. From (5.3.70), we see that

$$\begin{aligned} & t\int_0^{+\infty}\int_D e^{it(\Phi(0,w)+\sigma\Phi(w,0))}A(0,w,t)A(w,0,t\sigma)\lambda(w)dwd\sigma \\ \sim & B_0(0,0)t^n + B_1(0,0)t^{n-1} + \cdots . \end{aligned} \tag{5.3.83}$$

We will see that from the the asymptotic expansion (5.3.83) and Lemma 5.10 there is a recursive formula between the first and the second coefficient. We let

$$F(w,\sigma) := \Phi(0,w) + \sigma\Phi(w,0) = g(0',w) + \sigma(-w_{2n+1} + g(w',0)). \tag{5.3.84}$$

As (5.3.68) and the discussion of (5.3.68), we have $(d_w F)(0,1) = 0$, $(d_\sigma F)(0,1) = 0$,

$$\det \mathrm{Hess}(F)(0,1) = \det \begin{bmatrix} 2iI_{2n} & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} = (-1)^{n+1}2^n, \tag{5.3.85}$$

$$\mathrm{Hess}(F)^{-1}(0,1) = \begin{bmatrix} \frac{1}{2i}I_{2n} & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \tag{5.3.86}$$

and

$$\left\langle \mathrm{Hess}(F)^{-1}(0,1)D, D \right\rangle = 2i\sum_{j=1}^{n} \frac{\partial^2}{\partial z_j \partial \bar{z}_j} + 2\frac{\partial^2}{\partial x_{2n+1}\partial\sigma}, \tag{5.3.87}$$

where $D := \left(-i\partial_{x_1}, \cdots, -i\partial_{x_{2n+1}}, -i\partial_\sigma\right)^t$ is the column vector. By Hörmander stationary phase formula Theorem 5.2, we have

$$B_0(0,0)t^n + B_1(0,0)t^{n-1} + \cdots \tag{5.3.88}$$

$$\sim t\int_0^{+\infty}\int_D e^{itF(w,\sigma)}A(0,w,t)A(w,0,t\sigma)\lambda(w)dwd\sigma \tag{5.3.89}$$

$$\sim e^{itF(0,1)}\det\left(\frac{t\mathrm{Hess}(F)(0,1)}{2\pi i}\right)^{-\frac{1}{2}}\sum_{j=0}^{+\infty}t^{-j}P_j \tag{5.3.90}$$

$$\sim 2\pi^{n+1}(t^nP_0 + t^{n-1}P_1 + \cdots), \tag{5.3.91}$$

where

$$P_0 = A_0(0,0)^2\lambda(0), \tag{5.3.92}$$

(5.3.93)

$$\begin{aligned} P_1 = &\sum_{0\le\mu\le 2}\frac{i^{-1}}{\mu!(\mu+1)!} \\ &\times\left(i\sum_{j=1}^{n}\frac{\partial^2}{\partial z_j\partial\bar{z}_j} + \frac{\partial^2}{\partial x_{2n+1}\partial\sigma}\right)^{\mu+1}\left(G^\mu(x,\sigma)A_0(0,x)A_0(x,0)\lambda(x)\sigma^n\right)(0,1) \\ &+2A_0(0,0)A_1(0,0)\lambda(0), \end{aligned}$$

and

$$G(x,\sigma) := F(x,\sigma) - F(0,1) - \frac{1}{2}\left\langle \mathrm{Hess}(F)(0,1)\begin{pmatrix} x \\ \sigma - 1\end{pmatrix}, \begin{pmatrix} x \\ \sigma - 1\end{pmatrix}\right\rangle$$

$$= F(x,\sigma) - \frac{1}{2}\left\langle \mathrm{Hess}(F)(0,1)\begin{pmatrix} x \\ \sigma - 1\end{pmatrix}, \begin{pmatrix} x \\ \sigma - 1\end{pmatrix}\right\rangle. \tag{5.3.94}$$

We can check that

$$\frac{\partial^\alpha G}{\partial x^{\alpha_1}\partial\sigma^{\alpha_2}}(0,1) = 0 \ \text{ for all } \alpha_1 \in \mathbb{N}_0^{2n+1}, \alpha_2 \in \mathbb{N}_0, |\alpha| = |\alpha_1| + |\alpha_2| \leq 2. \tag{5.3.95}$$

From Proposition 5.9 and (5.3.46), we can find that

$$\frac{\partial^\alpha G}{\partial x^\alpha}(0,1) = \frac{\partial^\alpha}{\partial x^\alpha}\left(\Phi(0,x) + \Phi(x,0)\right)\Big|_{x=0} = 0, \ \text{ for all } \alpha \in \mathbb{N}_0^{2n+1}, |\alpha| = 3. \tag{5.3.96}$$

Also, we observe that

$$\frac{\partial^\alpha G}{\partial\sigma^\alpha}(0,1) = 0, \ \text{ for all } \alpha \in \mathbb{N}_0, |\alpha| \geq 2. \tag{5.3.97}$$

We now calculate each terms in P_1: For $\mu = 0$ in the summation, the summand is

$$\frac{1}{i}\left(i\sum_{j=1}^n \frac{\partial^2}{\partial z_j\partial\overline{z}_j} + \frac{\partial^2}{\partial x_{2n+1}\partial\sigma}\right)\left(A_0(0,x)A_0(x,0)\lambda(x)\sigma^n\right)(0,1); \tag{5.3.98}$$

for $\mu = 1$ in the summation, the summand is

$$\frac{1}{2i}\left(-\sum_{j,k=1}^n \frac{\partial^4}{\partial z_j\partial\overline{z}_j\partial z_k\partial\overline{z}_k} + 2i\sum_{j=1}^n \frac{\partial^4}{\partial z_j\partial\overline{z}_j\partial x_{2n+1}\partial\sigma} + \frac{\partial^4}{\partial x_{2n+1}^2\partial\sigma^2}\right) \tag{5.3.99}$$

acting on

$$G(x,\sigma)A_0(0,x)A_0(x,0)\lambda(x)\sigma^n \tag{5.3.100}$$

valuing at $(x,\sigma) = (0,1)$; and for $\mu = 2$, the summand is

$$\begin{aligned}\frac{1}{12i}\Bigg(-i\sum_{j,k,l=1}^n \frac{\partial^6}{\partial z_j\partial\overline{z}_j\partial z_k\partial\overline{z}_k\partial z_\ell\partial\overline{z}_\ell} - 3\sum_{j,k=1}^n \frac{\partial^6}{\partial z_j\partial\overline{z}_j\partial z_k\partial\overline{z}_k\partial x_{2n+1}\partial\sigma} \\ +3i\sum_{j=1}^n \frac{\partial^6}{\partial z_j\partial\overline{z}_j\partial x_{2n+1}^2\partial\sigma^2} + \frac{\partial^6}{\partial x_{2n+1}^3\partial\sigma^3}\Bigg)\end{aligned} \tag{5.3.101}$$

acting on

$$G^2(x,\sigma)A_0(0,x)A_0(x,0)\lambda(x)\sigma^n \tag{5.3.102}$$

valuing at $(x,\sigma) = (0,1)$. Thus, by Proposition 5.8, Proposition 5.9, (5.3.46), (5.3.95), (5.3.96) and (5.3.97), also with (5.3.55) and Lemma 5.10, it is straightforward to check that

(5.3.103)
$$P_1 = A_0(0,0)^2\left[\left(\sum_{j=1}^{n}\frac{\partial^2\lambda}{\partial z_j\partial\overline{z}_j}\right)(0)+\frac{i}{2}\lambda(0)\sum_{j,k=1}^{n}\frac{\partial^4(g(z,0)+g(0,z))}{\partial z_j\partial\overline{z}_j\partial z_k\partial\overline{z}_k}(0',0)\right]$$
$$+2A_0(0,0)A_1(0,0)\lambda(0).$$

From (5.3.35), (5.3.37), (5.3.46) and notice that $A_0(0,0) = \frac{1}{2\pi^{n+1}}$, we can rewrite (5.3.103):

(5.3.104)
$$P_1 = \frac{1}{(2\pi^{n+1})^2}\left[\sum_{j=1}^{n}\left(\frac{\partial^2\lambda}{\partial z_j\partial\overline{z}_j}\right)(0)+\sum_{j,k=1}^{n}\frac{\partial^4 R}{\partial z_j\partial\overline{z}_j\partial z_k\partial\overline{z}_k}(0)\right]+\frac{1}{\pi^{n+1}}A_1(0,0),$$

where R is as in (5.3.37). From Lemma 5.10 and (5.3.88), we get

$$A_1(0,0) = B_1(0,0) = 2\pi^{n+1}P_1. \tag{5.3.105}$$

From this observation and (5.3.104), we get

(5.3.106)
$$\begin{aligned}A_1(0,0) &= B_1(0,0)\\ &= 2\pi^{n+1}P_1\\ &= \frac{1}{2\pi^{n+1}}\left[\sum_{j=1}^{n}\left(\frac{\partial^2\lambda}{\partial z_j\partial\overline{z}_j}\right)(0)+\sum_{j,k=1}^{n}\frac{\partial^4 R}{\partial z_j\partial\overline{z}_j\partial z_k\partial\overline{z}_k}(0)\right]+2A_1(0,0),\end{aligned}$$

and it remains to calculate

$$A_1(0,0) = -\frac{1}{2\pi^{n+1}}\left[\sum_{j=1}^{n}\left(\frac{\partial^2\lambda}{\partial z_j\partial\overline{z}_j}\right)(0)+\sum_{j,k=1}^{n}\frac{\partial^4 R}{\partial z_j\partial\overline{z}_j\partial z_k\partial\overline{z}_k}(0)\right]. \tag{5.3.107}$$

Now, we calculate each term in (5.3.107) by the geometric data on X. We will continue work with local coordinates $x = (x_1,\cdots,x_{2n+1})$ as (5.3.28). We first calculate the Tanaka–Webster scalar curvature in terms of the coordinates $x = (x_1,\cdots,x_{2n+1})$.The following data can be obtained from Proposition 5.8:

$$\boldsymbol{\alpha}(x) = dx_{2n+1} - i\sum_{j=1}^{n}\left(\frac{\partial R}{\partial z_j}dz_j - \frac{\partial R}{\partial\overline{z}_j}d\overline{z}_j\right)+O(|x|^4), \tag{5.3.108}$$

$$d\boldsymbol{\alpha}(x) = 2i\sum_{j,k=1}^{n}\frac{\partial^2 R}{\partial z_j\partial\overline{z}_k}dz_j\wedge d\overline{z}_k + O(|x|^3), \tag{5.3.109}$$

(5.3.110) $$T(x) = -\frac{\partial}{\partial x_{2n+1}} + O(|x|^2),$$

(5.3.111) $$T_x^{1,0}X = \text{span}\left\{L_j\right\}_{j=1}^n := \text{span}\left\{\frac{\partial}{\partial z_j} + i\frac{\partial R}{\partial z_j}\frac{\partial}{\partial x_{2n+1}} + O(|x|^4)\right\}_{j=1}^n,$$

(5.3.112) $$L_j = \frac{\partial}{\partial z_j} + i\frac{\partial R}{\partial z_j}\frac{\partial}{\partial x_{2n+1}} + O(|x|^4),\ \ j = 1, \cdots, n.$$

We write $\nabla_{L_i} L_j = \Gamma_{ij}^l L_l$, where ∇ denotes the Tanaka-Webster connection in Proposition 5.3. From [78, Lemma 3.2], we have

(5.3.113) $$d\boldsymbol{\alpha}\left(\nabla_{L_i} L_j, \overline{L}_k\right) = L_i(d\boldsymbol{\alpha}(L_j, \overline{L}_k)) - d\boldsymbol{\alpha}\left(L_j, [L_i, \overline{L}_k]_{T^{0,1}}\right).$$

Directly, we can check that

(5.3.114) $$d\boldsymbol{\alpha}\left(\nabla_{L_i} L_j, \overline{L}_k\right) = d\boldsymbol{\alpha}\left(\Gamma_{ij}^{\ell} L_\ell, \overline{L}_k\right) = 2i\Gamma_{ij}^l \frac{\partial^2 R}{\partial z_\ell \partial \overline{z}_k} + O\left(|x|^3\right),$$

(5.3.115) $$L_i\left(d\boldsymbol{\alpha}\left(L_j, \overline{L}_k\right)\right) = 2\left(i\frac{\partial^3 R}{\partial z_i \partial z_j \partial \overline{z}_k} - \frac{\partial R}{\partial z_i}\frac{\partial^3 R}{\partial x_{2n+1}\partial z_j \partial \overline{z}_k}\right) + O\left(|x|^3\right),$$

and

(5.3.116) $$\begin{aligned}[L_i, \overline{L}_k] &= \left[\frac{\partial}{\partial z_i} + i\frac{\partial R}{\partial z_i}\frac{\partial}{\partial x_{2n+1}} + O(|x|^4), \frac{\partial}{\partial \overline{z}_k} - i\frac{\partial R}{\partial \overline{z}_k}\frac{\partial}{\partial x_{2n+1}} + O(|x|^4)\right] \\ &= \left(\frac{\partial R}{\partial z_i}\frac{\partial^2 R}{\partial \overline{z}_k \partial x_{2n+1}} - \frac{\partial R}{\partial \overline{z}_k}\frac{\partial^2 R}{\partial z_i \partial x_{2n+1}} - 2i\frac{\partial^2 R}{\partial z_i \partial \overline{z}_k}\right)\frac{\partial}{\partial x_{2n+1}} + O(|x|^3).\end{aligned}$$

So we have

(5.3.117) $$d\boldsymbol{\alpha}\left(L_j, [L_i, \overline{L}_k]_{T^{0,1}}\right) = O(|x|^3).$$

Accordingly, by (5.3.29), for all $i, j, k = 1, \cdots, n$,

(5.3.118) $$\Gamma_{ij}^k(0) = 0.$$

Moreover, by taking $\frac{\partial}{\partial \overline{z}_h}$ both sides in (5.3.113), from (5.3.29), (5.3.114), (5.3.115) and (5.3.117), it is not difficult to check that

(5.3.119) $$\frac{\partial \Gamma_{ij}^k}{\partial \overline{z}_h}(0) = -2\frac{\partial^4 R}{\partial z_i \partial z_j \partial \overline{z}_k \partial \overline{z}_h}(0).$$

Now, we let $\{\theta^\alpha\}_{j=1}^n$ and $\{\theta^{\overline{\beta}}\}_{j=1}^n$ be the dual frame of $\{L_\alpha\}_{j=1}^n$ and $\{\overline{L}_\beta\}_{j=1}^n$, respectively. We denote

$$\nabla L_\alpha = \omega_\alpha^\beta \otimes L_\beta, \tag{5.3.120}$$

and we can check that the $(1,1)$ part of $d\omega_\alpha^\beta$ is

$$-\sum_{k,\ell=1}^n \left(\overline{L}_\ell \Gamma_{k\alpha}^\beta)\right) \theta^k \wedge \theta^{\overline{\ell}} + O(|x|), \tag{5.3.121}$$

and the $(1,1)$ part of $\Theta_\alpha^\beta = d\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta$ denoted by

$$\sum_{k,\ell=1}^n R_{\alpha k \bar{l}}^\beta \theta^k \wedge \theta^{\overline{\ell}} \tag{5.3.122}$$

equals the $(1,1)$ part ot $d\omega_\alpha^\beta$. Hence the pseudohermitian Ricci curvature tensor at origin is

$$R_{\alpha\bar{\ell}}(0) = \sum_{k=\beta=1}^n R_{\alpha k \bar{l}}^\beta(0) = -\sum_{k=\beta=1}^n \frac{\partial \Gamma_{k\alpha}^\beta}{\partial \overline{z}_\ell}(0) = 2\sum_{k=1}^n \frac{\partial^4 R}{\partial z_k \partial \overline{z}_k \partial z_\alpha \partial \overline{z}_\ell}(0). \tag{5.3.123}$$

Also, for

$$-d\boldsymbol{\alpha} = i g_{\alpha\overline{\beta}} \theta^\alpha \wedge \theta^{\overline{\beta}}, \tag{5.3.124}$$

we can find that $\theta^\alpha(0) = dz_\alpha$ and $\theta^{\overline{\beta}}(0) = d\overline{z}_\beta$, and

$$g_{\alpha\overline{\beta}}(0) = \delta_{\alpha\beta}. \tag{5.3.125}$$

We let $g^{\bar{c}d}$ be the inverse matrix of $g_{a\bar{b}}$. We have $g^{\bar{c}d}(0) = \delta_{cd}$ and the Tanaka–Webster scalar curvature at the origin is

$$R_{\text{scal}}(0) = g^{\overline{\ell}\alpha} R_{\alpha\overline{\ell}}(0) = 2\sum_{\ell=1}^n \sum_{k=1}^n \frac{\partial^4 R}{\partial z_\ell \partial \overline{z}_\ell \partial z_k \partial \overline{z}_k}(0). \tag{5.3.126}$$

Finally, for the volume form

$$\lambda(x) dx := \frac{1}{n!}\left(\left(\frac{-d\boldsymbol{\alpha}}{2}\right)^n \wedge \boldsymbol{\alpha}\right), \tag{5.3.127}$$

we have the expression

$$\begin{aligned}
&\frac{1}{n!}\left(-\sum_{j,k=1}^{n} i\frac{\partial^2 R}{\partial z_j\partial\overline{z}_k}dz_j\wedge d\overline{z}_k+O\left(|x|^3\right)\right)^n\\
&\wedge\left(dx_{2n+1}-i\sum_{j=1}^{n}\left(\frac{\partial R}{\partial z_j}dz_j-\frac{\partial R}{\partial\overline{z}_j}d\overline{z}_j\right)+O\left(|x|^4\right)\right)\\
&=\frac{1}{n!}\left(\sum_{j,k=1}^{n}-2\frac{\partial^2 R}{\partial z_j\partial\overline{z}_k}\frac{dz_j\wedge d\overline{z}_k}{-2i}+O\left(|x|^3\right)\right)^n\\
&\wedge\left(dx_{2n+1}-i\sum_{j=1}^{n}\left(\frac{\partial R}{\partial z_j}dz_j-\frac{\partial R}{\partial\overline{z}_j}d\overline{z}_j\right)+O\left(|x|^4\right)\right).
\end{aligned}\tag{5.3.128}$$

From Proposition 5.8, we can check that

(5.3.129)

$$\frac{\partial^2\lambda}{\partial z_\ell\partial\overline{z}_\ell}(0)=(-2)^n\left(-\frac{1}{2}\right)^{n-1}\sum_{j=1}^{n}\frac{\partial^4 R}{\partial z_\ell\partial\overline{z}_\ell\partial z_j\partial\overline{z}_j}(0)=-2\sum_{j=1}^{n}\frac{\partial^4 R}{\partial z_\ell\partial\overline{z}_\ell\partial z_j\partial\overline{z}_j}(0).$$

From (5.3.107), (5.3.126) and (5.3.129), we conclude that

$$A_1(0,0)=\frac{1}{4\pi^{n+1}}R_{\text{scal}}(0).\tag{5.3.130}$$

Thus, the proof of Theorem 1.6 is completed for the point-wise equation (5.3.130) holds for all $x_0\in D$ and R_{scal} is globally well-defined on whole X and in particular on D.

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