Disordered System Approaches to the Yang-Mills Vacuum

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Abstract

Two novel dual descriptions of $d = 4$ U(N_c) Yang-Mills theory (YM) are constructed and studied in this dissertation. We consider a network theory inspired by Budczies-Zirnbauer model (BZ), which will be abbreviated as BZN, and a continuum field theory, Dirac-Yang-Mills model (DYM). In either BZN or DYM, the dual theory is obtained by integrating out the original gluon degrees of freedom, which leads to a strongly-disordered system of some auxiliary matter fields. We examine the possibilities of applying a modern method, superbosonization (SuB) formula for disordered systems, in the investigation of the dual theories.

In the first project, we reformulate BZN using Gaussian integral representation, and derive a master action for gluons and auxiliary matter fields, both of which live on the links of a lattice. The dual description, dual-BZN, is derived using Cayley parametrisation and a gauge-averaging trick, and the resulting dual action is a large$-N_c$ series of color-neutral composite operators. However, using SuB for a direct replacement of these operators by some supermatrix-valued fields is not possible due to the rank-deficiency in the boson-boson sector of the supermatrix. The rank-deficiency is a result of the universality condition $N_f \geq N_c$, which is necessary for BZN to flow to YM in its continuum-limit.

In the second project, we study both sides of DYM: the induced Yang-Mills (IYM) and its dual (dual-IYM). The theory of dual-IYM describes a system of massive Dirac bosons and Dirac fermions constrained by a zero-current condition (ZC). A beautiful connection between gluon condensates in IYM and matter condensates in dual-IYM inspires a low-energy effective theory (dual-EFT). We discover the relevant dual symmetry groups and assemble a Lagrangian for dual-EFT in analogy with the chiral perturbation theory. Furthermore, we explore the ZC solution space and find out dual-IYM contains all Lorentz-types components, which suggests an energy-hierarchy scheme where dual-EFT is included as the low-energy sector of dual-IYM. Dual-IYM is color gauge-invariant. However, Witten's bosonization method leads to a divergent effective action for the external field, and hence it is difficult to derive an action for some color-neutral dual-field. An attempt to directly transform the composite super-meson to the dual-field by SuB also fails because of rank-deficiency.

In the absence of successful color-neutralisation, we proceed to explore some physical aspects of dual-BZN and dual-IYM. For dual-BZN, the masses and interaction strength of the composite operators are identified. We briefly examine the dual symmetry group and the saddle-point solutions, and point out a challenge to a semi-classical approximation due to the universality condition. For dual-IYM, we present two possible applications for YM mass gap and quark confinement. Furthermore, we explain a possibility of a large$-N_c$ analysis, which might lead to a description of dual-IYM as a gravitational theory and/or a nonlinear sigma model.

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CHAPTER 1

INTRODUCTION

Quantum field theory is the primary theoretic framework for particle physics. To date, three of the four known fundamental forces, except for gravity, is best explained by the *standard model* (SM) of elementary particles. The SM is based on a specific class of relativistic field theories known as gauge theories, and it can be described in terms of a Lagrangian which is gauge invariant, i.e., unchanged under local transformations by a Lie group

$$G_{\mathrm{SM}} = \mathrm{SU}(3)_C \times \mathrm{SU}(2)_W \times \mathrm{U}(1)_Y \quad .$$

In the SM Lagrangian the matter constituents are spin-1/2 fields transforming in the (anti-)fundamental or singlet representations of the subgroups of G_{SM}, whereas the force carriers are spin-1 fields transforming in the adjoint representations[1]. These force carriers are generally known as the gauge fields of SM, and the strength for each fundamental force is determined by a gauge coupling parameter.

Among the three fundamental forces in the SM, only the electromagnetic interaction is long-range. See Table 1.1. The fact that the weak force and the strong force are short-range suggests their gauge fields have to be massive. However, a naive attempt to include mass terms for the gauge fields in the classical Lagrangian is forbidden since it would violate the gauge invariance. A promising solution to this conundrum in the electroweak sector of the SM is the *Brout–Englert–Higgs mechanism* [2–4], or in short the *Higgs mechanism*, which adds a spin-0, $\mathrm{SU}(2)_W$-doublet field (known as the Higgs field) to the model. The well-known "Mexican hat" potential of the Higgs field shifts the vacuum to a nontrivial one, and by fixing a gauge,[2] a part of the symmetries is hidden away at low temperature ($\approx 159.5 \pm 1.5$ GeV):

$$\mathrm{SU}(2)_W \times \mathrm{U}(1)_Y \mapsto \mathrm{U}(1)_{em} \quad .$$

[1] More precisely, adjoint representation of the global group transformations.

[2] A gauge-invariant description of the Higgs mechanism is explained in [5].

Force	Range (cm)	Relative strength
Strong	10^{-13}	10^{37}
Weak	10^{-15}	10^{24}
Electromagnetic	∞	10^{35}
Gravity	∞	1

TABLE 1.1. Properties of the fundamental interactions [1].

Consequently, the four massless gauge fields generating $SU(2)_W \times U(1)_Y$ are replaced by three massive spin-1 bosons: W, Z bosons are responsible for the short-range weak force, and one massless photon γ mediates the long-range electromagnetic force. In other words, only an Abelian gauge invariance of the electroweak interaction survives at low energies, and this $U(1)_{em}$ gauge theory determines the well-known *quantum electrodynamics* (QED). As a remark, the recent discovery of the Higgs boson in the Large Hadron Collider [6, 7] confirmed the Higgs mechanism.

1.1 Mass gap and color confinement

The strong force is studied in the *quantum chromodynamics* (QCD) sector of the SM, and it is mediated by the color gauge fields known as gluons. QCD is a non-Abelian gauge theory which describes the dynamics of quarks and gluons. It is founded on the *Yang-Mills theory* (YM) [8], which was originally proposed as a theory only for the non-Abelian gauge fields. A common belief today is that a gluon has a nonzero "physical" mass, and it can be dynamically generated without appealing to the Higgs mechanism [9]. At the classical level the QED photons do not interact among themselves. The QCD gluons, by contrast, have self-interactions due to the non-Abelian nature of the color gauge group $SU(3)_C$. These self-interactions are underneath the most striking features of QCD: *asymptotic freedom* and *color confinement*.

At very high energies the interactions among gluons and quarks become very weak, that is, the color charges are asymptotically free [10, 11]. In particular, the gluons are indeed massless in the ultra-violet regime as indicated by the Lagrangian. As the energy decreases the gauge coupling grows larger, and the strong force begins to bind gluons and quarks into bound-states. Up until today, no isolated color charges have been observed in the experiments conducted at accessible energies, so it is widely accepted that these bound-states must be color-neutral. This is known as the color confinement conjecture. A crucial implication of the color confinement is at long-distance the effective mass of a gluon is no longer zero, because it cannot escape from the other constituent color charges in the composite particle. This eventually explains why the strong force is also short-range.

Astonishingly, asymptotic freedom and color confinement have different fates. Asymptotic freedom was proven and hence laid the foundation of QCD as a quantum field theory defined up to arbitrarily high energies. However, it has not yet been observed because the "charge liberation" energy scale is still too high for physicists to experimentally confirm the existence of gluons and quarks. On the contrary, while color confinement is consistent with experimental evidences, there is no analytic proof for this property in QCD. Both theorists and experimentalists will be happy on the day when a next-generation instrument detects quarks and gluons, but the theorists will not be satisfied until the low-energy behaviors of QCD is understood appropriately.

In fact, for decades, much effort has been spent on studies of the underlying mechanism of quark confinement in QCD. These studies usually take place in a simplified model: QCD without dynamical quarks. The phenomenon of quark confinement is quantified by the static potential built up between two test quarks added to a purely-gluonic system. The strong force mediated by the gluons connects the quarks, and qualitatively it can be visualized as a bunch of chromoelectric flux lines. When the flux lines are squeezed into a tube between the test charges, the strong force becomes non-decaying as the static potential increases linearly with respect to the distance between the quarks. Consequently, the test charges cannot escape from each other since it would require a tremendous amount of energy. In this picture, one has to devise a mechanism of the formation of flux tubes between quarks.

1.2 Quark confinement by monopole condensation

A compelling proposal is the *dual superconductivity* description of QCD [12, 13]. In a type-II superconductor, the electrically charged Cooper pairs [14] condense when the temperature decreases below a critical value, and the external magnetic flux lines can only penetrate the condensate if they are squeezed into thin tubes [15–17]. These are known as the Abrikosov vortices [17]. By the *electric-magnetic duality* one obtains the model of dual superconductivity, where the roles of the electric fields and the magnetic fields are exchanged. It is speculated that there are chromomagnetic charges in QCD, which condense and cause the formation of the chromoelectric flux tubes. An analytic proof of the existence of a magnetic monopole condensate in QCD still remains elusive at present, but magnetic monopoles were found in several gauge theories related to QCD. For example, the compact U(1) lattice pure gauge theory can be described as a system of magnetic monopoles [18]. Probing the test electric charges in the monopole background, it was shown that in $d = 2+1$ the charges are confined at all coupling, whereas in $d = 3+1$ the confinement only occurs at strong coupling.

Magnetic monopoles were also discovered in the continuum non-Abelian gauge theories. The famous *'t Hooft-Polyakov monopole* [19, 20] was brought to light in the SU(2) Georgi-Glashow model [21] in $d = 2+1$, where the gauge field is coupled to an adjoint scalar field with a Higgs potential. Via the Higgs mechanism the scalar field induces the so-called *electromagnetic projection*, which extracts a massless component from the gauge field to serve as the U(1) photon. Originally, it was expected that the low-energy sector of the Georgi-Glashow model would correspond to the compact U(1) theory [22], which could entail the same confinement mechanism. However, it was pointed out by [23] that the remaining massive gauge bosons cannot be simply ignored at large distances; as the result, the applicability of the argument in [22] is questionable.

The Georgi-Glashow model set the stage for the famed *Montonen-Olive duality* [24], which is a version of the electric-magnetic duality, and it has a far-reaching impact on the developments in fundamental physics. In the Bogomol'nyi–Prasad–Sommerfield (BPS) limit [25, 26] the classical spectrum of this model contains electric charges and magnetic monopoles. At weak coupling, the former are point-like and light whereas the latter are heavy solitons; at strong coupling, their properties are exchanged. In light of this, by the duality it was hoped that the Georgi-Glashow model at strong coupling could be mapped to a dual theory at weak coupling, where the magnetic monopoles become the elementary particles.

The first convincing realization of the Montonen-Olive duality in a quantum theory was presented in the *Seiberg-Witten model*[27], which is a $\mathcal{N}=2$ super Yang-Mills (SYM) theory containing ‘t Hooft-Polyakov monopoles. The BPS mass spectrum is protected by the supersymmetry (SUSY), so a realization of the duality is promising. A remarkable result in the Seiberg-Witten model is that its low-energy effective theory can be written as a *dual Abelian Higgs model*, where the gauge field is a dual photon coupled to the magnetic monopoles. A soft breaking of the supersymmetry from $\mathcal{N}=2$ to $\mathcal{N}=1$ triggers a condensation of the monopoles and generates a mass term for the dual photon. Consequently, the electric flux lines are confined. It is believed that a full duality beyond the low-energy effective theories can be established in a $\mathcal{N}=4$ SYM but not the $\mathcal{N}=2$ theory. Today, the topic of interest in the $\mathcal{N}=4$ SYM has been shifted from the electric-magnetic duality to the celebrated *AdS/CFT duality* [28], also known as the *gauge/gravity duality*. This duality represents a large-N_c equivalence between the $d=3+1$ $\mathcal{N}=4$ SYM and the type IIB string theory, which is a gravitational theory living on the space $AdS_5 \times S^5$.

Can the picture of dual superconductor truly explain color confinement in QCD? The answer is positive, at least for the mechanism of the quark confinement. A successful demonstration of this picture consists of two steps. First, one has to identify magnetic monopoles in the pure YM without adjoint Higgs fields. Second, one should show that the quark confinement is indeed caused by the condensation of the monopoles. In the first attempt to extract monopole degrees of freedom from the YM [29], a purely-gluonic local operator transforming in the adjoint representation was introduced to substitute the Higgs fields in the Georgi-Glashow model. By analogy with the electromagnetic projection a specific gauge is employed to diagonalize this operator: the non-Abelian symmetry is then “broken” to its Cartan subgroup, with respect to which the gauge fields are decomposed to “photons” and “charged matter fields”. This is known as the *Abelian projection*. In this gauge, the magnetic monopoles appear as singularities in the “photon” fields, and the centers of these solitons are located at the degenerate points of the local operator.

A milestone was reached by the introduction of the *maximally Abelian gauge* [30]. It can be shown that in this gauge the linear static potential and the string tension are indeed dominated by the monopole contributions. This innovative line of research however suffers from the doubts that the monopoles and the related monopole dominance are nothing but gauge artefacts. To remedy this concern a gauge-invariant reformulation of the YM known as the *Cho–Duan–Ge–Faddeev–Niemi* (CDGFN) decomposition was developed in recent years [31–35]. The upshot of the CDGFN is that it can define the monopoles and demonstrate their dominance in the quark confinement in a manifestly gauge-invariant way. In addition, the earlier approach becomes a gauge-fixed version of this modern method. Further investigation on the properties of the monopoles and their physical effects is an active field of research today. For a thorough review, see [36].

1.3 Modern approaches to non-perturbative QCD

We now zoom out from the specific mechanism of quark confinement and look at the broader picture of the low-energy dynamics of QCD. In general, the phenomena in this regime are non-perturbative because the observables depend on the gauge coupling non-analytically. Apart from numerical explorations on lattice models, a popular theoretical approach in this regime is by solving the *Dyson-Schwinger equations* (DSEs) for the Green's functions [37–39]. For example, DSEs were used to study the infra-red properties of gluon propagators and ghost propagators [40], and the bound-states of QCD such as mesons [41] and glueballs [42]. Since the advent of functional renormalization group (FRG) [43], it has become another favorable non-perturbative tool for studying various aspects of QCD. For a comprehensive review, see [44], and a recent FRG result explaining a connection between the gluon condensates and the effective gluon mass can be found in [45].

The powerful tools such as DSEs and FRG have indeed revealed much more of the low-energy realm of QCD to us, there is however an open problem, the *Gribov ambiguity* [46]. In the studies of non-Abelian gauge theories, including the aforementioned examples, a step of gauge-fixing is usually necessary. The conventional *Faddeev-Popov* (FP) functional method [47] assumes the uniqueness of the gauge-orbit intersections, which fails to be true in general. To address this problem, the FP gauge-fixing is improved by restricting the target space of gauge fields to a so-called fundamental modular region. So far the closest implementation of this idea is the *Gribov-Zwanzigers* formalism [48], which is still not a perfect solution. The unsolved Gribov ambiguity inevitably put a limit on the applicability of the functional methods, which eventually prevents us from gaining a complete picture of the low-energy physics of QCD.

Generally speaking, modern theoretical investigations of non-perturbative QCD phenomenology can be categorized to two different approaches. The first approach is forthright: starting from the given fundamental degrees of freedom in QCD, one performs sequential functional integration to derive the correlations among local variables at a certain scale. The second approach relies on a clever change-of-variables recipe, which relates two or more descriptions of the same physical system. This feature is known as duality, and we have touched upon two exemplary kinds of duality so far. To advance along the first path, one must continue sharpening the tools like FRG, and find a way around the inherent complexity of non-Abelian gauge theories such as the Gribov ambiguity. On the contrary, the main purpose of duality is to allow us to switch to a different perspective whenever the calculations with the original variables become too involved. It is hoped that the same observables can be computed using the new variables in an easier and more transparent way.

The concept of duality have been proved useful in lower dimensions and mostly with Abelian symmetries. In addition to the compact U(1) lattice gauge theory mentioned above [18, 49, 50], there are well-known examples such as the Kramers-Wannier duality [51], the Thirring/Sine-Gordon duality [52], and the particle-vortex duality [53, 54]. In four dimensional gauge theories a concrete realization of duality usually requires SUSY, because the supersymmetric invariance restricts the quantum corrections and provides more exact results than non-supersymmetric theories.

In quantum field theories the most sought-after type of duality is the so-called *strong-weak duality* (sometimes also known as *S-duality*). The presence of S-duality in a physical system generally means there are two models of this system, and the strongly-coupled phase of one model is equivalent to the weakly-coupled phase of the other model, and vice versa. A long-standing quest in physics is to design an S-duality recipe which maps QCD, or even just the pure YM, to an equivalent model in which the large-scale physics can be accessed by perturbation methods. More specifically, we hope that such a successful dual description of YM can explain the mechanisms underlying the color confinement and the mass gap. There is even a million-dollar prize for this [55]. Can we construct a dual theory of YM without the help from SUSY? This is the question we hope to answer in this dissertation.

1.4 Induced gauge theory: a dual-pair of descriptions

The new chapter of the search for duality in YM is written in the language of *induced gauge theory*. Originally motivated by the notion of induced gravity [56–58], the central idea of this method is to conceive a master action in which the gauge fields, usually the gluons, are coupled to some auxiliary "matter" fields with purposely designed dynamics. There is no kinetic term for the gauge fields in the master action to start with; rather, the desired YM action (in the continuum) and the Wilson action (on a lattice) [59] describing the gluodynamics are "induced" by explicitly integrating out the matter fields.

The art of induced gauge theory lies in the properties of the auxiliary fields. In the first wave of studies of lattice models the common starting point is the Wilson action, and the matter fields are allowed to propagate across the lattice. Different choices for the matter fields have been tested in the pioneer works: fundamental bosons [60], fundamental fermions [61], and adjoint bosons [62]. By construction, integrating out the matter fields yields gauge-invariant composite operators in the induced action. However, in order to recover the Wilson action, one must suppress the large-loop contributions, which is achieved in these works by sending the mass and/or the number (flavors) of the auxiliary variables to infinity. In parallel with this development the idea of inducing YM in the continuum was tested in [63], where it was shown that YM is equivalent to a fermionic system with current-current interaction in the similar infinite-parameter limit. The approaches in the first wave have two shortcomings. From a practical point of view, they are difficult to examine by numerical means due to the infinite parameters. Lattice simulations for observables simply are not accessible in these proposals. From a theoretical point of view, the existence of a continuum limit of the proposed lattice models was never placed under a rigorous check, let alone a proof that they lie in the same universality class of the asymptotically-free YM.

A promising resolution to these concerns was brought forth by the *Budczies-Zirnbauer model* [64], which launched the second wave of interests towards the induced gauge theory. The major change in the master action is that the auxiliary fields can only circulate the unit plaquettes of the lattice. This enhanced "locality" is carried over to the induced gluon action such that the unwanted non-local terms are automatically suppressed. Consequently, we no longer have to tune certain parameters to infinity. At first glance, the gluon action in [64] seems very different from the Wilson action, but they both admit a continuum-limit which corresponds to YM. To assure the existence of the continuum-limit, it is necessary to use bosonic auxiliary fields in the Budczies-Zirnbauer model, while inclusion of fermions is also permissible.

In $d = 2$ and with $\mathrm{U}(N_c)$ as the color gauge group, it was proved in [64] that as long as the number of flavors of the bosons N_f exceeds the number of colors N_c, a continuum-limit can be achieved by tuning the boson mass to a finite critical value. This critical theory is the $\mathrm{U}(N_c)$ YM. A similar statement is expected to hold also in $d = 4$ based on a universality argument, but a proper proof will require a comprehensive renormalization group analysis. In recent years, the Budczies-Zirnbauer model have been tested in [65, 66], where an extension of the model to the physical gauge group $\mathrm{SU}(N_c)$ was presented. Importantly, [65] provides numerical evidences supporting the conjecture that $\mathrm{SU}(N_c)$ YM can also be induced in $d = 3$ and $d = 4$, and [66] presents agreements between Wilson's model and Budczies-Zirnbauer model in the simulation of observables.

So far, we have only mentioned one aspect of induced gauge theory: how YM can be induced from a master action. Ultimately we want to understand the other side of the story, i.e., the dual description of YM obtained by integrating out the gluons. Preliminary results in all the mentioned researches, including the Budczies-Zirnbauer model, show that a dual theory of the lattice YM is a system of color-singlet composite operators built from the auxiliary fields. While the exact steps leading to the dual theory are usually transparent, the expression of the dual action is rather complicated and lack of concrete physical meanings, unfortunately. Nevertheless, the studies of induced gauge theory did reveal a fascinating resemblance of the master action to the Hamiltonian of disordered systems. Indeed, the gluons can be viewed as the disordered-couplings among the auxiliary bosons and/or fermions in the system, and the absence of the kinetic term corresponds to the limit of infinite disorder. In view of this, the dual theory of YM should be understood as a strongly-disordered system of the matter fields!

Naturally, this motivates people to apply known analytic tools for disordered systems to the dual theory at hand. The statistical properties of a quenched disordered system, whose randomly-distributed parameters are constant in time, is characterized by its disorder ensemble. The common step of averaging the disorder is done by integrating out the random matrices in this ensemble, and a mathematically rigorous tool for this step is the *supersymmetry method* [67, 68] [3]. At weak-disorder, the average generating functional can be approximated by a nonlinear sigma model via the traditional Hubbard-Stratonovich transformation, which is most applicable in cases of Gaussian ensembles. At strong-disorder or for systems with different disorder ensembles, one must seek for alternatives. Some proposals in this direction can be found in [69–72], but for possible applications in QCD we are interested in two exact integral transformation techniques: *color-flavor transformation* [73, 74] and *superbosonization formula* [75, 76].

The color-flavor transformation (CF) was originally invented as a new tool to study systems with Dyson's Circular Unitary Ensemble [77], which is not covered by the supersymmetry method mentioned above. This transformation establishes an exact equality between an integral over U(N_c) and an integral over some supermatrix space. The most attractive feature of the CF is that it provides a practical method to directly extract color-neutral entities from lattice QCD, which could represent the physical hadrons [78, 79]. However, it was soon realized that the CF cannot be used to induce lattice YM because it is not applicable when N_f is greater than N_c.

The superbosonization (SuB) formula is used after the disorder field is integrated out. It provides a transparent expression of the resulted average generating functional as a supermatrix field theory. Complementary to the Hubbard-Stratonovich/mean-field approach, this "bosonized" theory can then be projected to a nonlinear sigma model at strong-disorder. Taking this into consideration, because the dual theory of YM is probably a strongly-disordered system of some sort, SuB seems like a promising tool to use.

1.5 Motivation and outline

The main motivation behind this dissertation is to explore a possible application of SuB to formulate a dual theory of YM. This journey of constructing a dual description began in the author's master's thesis [80]. In the previous work we have examined some lattice network models and a continuum field theory in the framework of induced gauge theory, and our main focus was to properly induce the YM action in both models.

[3] This is closely related to the supersymmetric field theories in high-energy physics, but some of the conventions used for the disordered systems are different.

On the lattice-model side, we have concluded that the Budczies-Zirnbauer model (BZ) is still the most promising lattice model to work with. On the continuum-theory side we have proposed a novel master action which yields an induced YM theory (IYM). In this dissertation this master theory will be known as Dirac-Yang-Mills model (DYM). We have also derived a dual theory using Hubbard-Stratonovich decoupling method. This dual theory remains incomprehensible to us.

Based on these discoveries, in this dissertation we will be working with a network model inspired by BZ, denoted as BZN, and the same continuum master action with minor modifications. The spotlight is now on the dual theories obtained by integrating out the gluon degrees of freedom, and the ultimate goal is to use SuB or SuB-motivated methods to transform these models to some supermatrix field theories.

Outline

This dissertation is structured as follows:

- In Chapter 2, we derive a dual description of BZN, which we call dual-BZN. After introducing the BZN partition function, we build a lattice master action by placing BZN in a Gaussian integral representation. In this representation, auxiliary bosons and fermions are introduced on the links of the lattice. We then perform the duality transformation using Cayley parametrisation and a gauge-averaging trick. The resulting dual-BZN action is a large$-N_c$ series in terms of color-neutral composite operators. We end this chapter by an early-stage discussion about some physical aspects of dual-BZN, and a possibility of an application of SuB.

- In Chapter 3, we derive a dual description of IYM, which we call dual-IYM. We first present the definition of DYM master action, and then explain its resemblance to QCD. Next, after a revelation of a connection between the gluon condensate in IYM and the matter condensates in dual-IYM, we define and examine several dual symmetry groups. Based on this qualitative knowledge, we conjecture an effective field theory (dual-EFT) from dual-IYM and end the chapter with some potential applications.

- In Chapter 4, we attempt to color-neutralise dual-IYM via some exact transformations. To derive an action for color-neutral composite variables, we test the applicability of Witten's bosonization method. Then we turn our attention to the solution space of ZC, and examine if there exists a good parametrisation of the solution space such that SuB can be used. Finally, we propose an energy-hierarchy scheme where dual-EFT is incorporated in dual-IYM, and point out a possibility to include heavier degrees of freedom. We end the chapter with a short discussion on a possible large$-N_c$ semi-classical approximation of dual-IYM.

- In Chapter 5 we summarize the discoveries from the studies of BZN and DYM, and point out some prospects of an NLσM description of dual-IYM and potential connections to a gravitational theory.

CHAPTER 2

Budczies-Zirnbauer Network Model and Its Dual

In this chapter a dual theory for Budczies-Zirnbauer Network model (BZN) is established and studied. First, the original model is reviewed and reformulated in Section 2.1 - Section 2.2. Following a remark in Section 2.3, the main steps of the duality transformation are explained in Section 2.4 - Section 2.5. Finally, we discuss some physical aspects of the dual description and explore a possibility of further simplification in Section 2.6 - Section 2.8.

Throughout this chapter we will be adopting the natural units and presenting every variable in lattice unit, i.e., setting the lattice spacing $a \equiv 1$.

2.1 Budczies-Zirnbauer lattice model

Consider a $d-$dimensional complex Λ consisting of oriented $k-$cells. An example is shown in Figure 2.1. A gluon field U is a mapping from the $1-$cells (links) of Λ to the color group $\mathrm{U}(N_c)$: $\mathbf{l} \mapsto U(\mathbf{l}) \equiv \tilde{u}_{\mathbf{l}} \in \mathrm{U}(N_c)$. We will refer to $\tilde{u}_{\mathbf{l}}$ as a gluon.

2.1.1 Set-up

In its original form, BZ is a lattice gauge theory described by the partition function

$$\mathbf{Z} = \int DU \prod_{\mathbf{p}\in\Lambda} \frac{|\mathrm{Det}(\mathbb{1}_c - \alpha_F U(\partial\mathbf{p}))|^{2N_f}}{|\mathrm{Det}(\mathbb{1}_c - \alpha_B U(\partial\mathbf{p}))|^{2N_f}}, \tag{2.1}$$

where $DU \equiv \prod_{\mathbf{l}} d\tilde{u}_{\mathbf{l}}$, $\mathbf{p} \in \Lambda$ stands for the $2-$cells (plaquettes), and $U(\partial\mathbf{p})$ is the holonomy around the boundary $\partial\mathbf{p}$. An example is given in Figure 2.2. The flavor number is denoted by N_f and $\mathbb{1}_c$ is the identity operator in the color space.

As pointed out by [64] and supported by [65, 66], we have the following important observation:

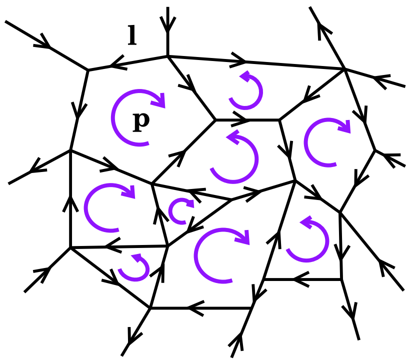


FIGURE 2.1. A $d = 2$ complex. The black arrows are the orientations of 1−cells and the blue arrows are the orientations of 2−cells.

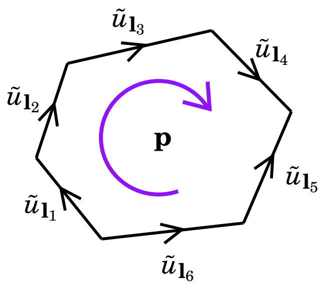


FIGURE 2.2. The holonomy around $\mathbf{p}$, which is $U(\partial\mathbf{p}) = \tilde{u}_{\mathbf{l}_1}\tilde{u}_{\mathbf{l}_2}\tilde{u}_{\mathbf{l}_3}\tilde{u}_{\mathbf{l}_4}\tilde{u}_{\mathbf{l}_5}^{-1}\tilde{u}_{\mathbf{l}_6}^{-1}$.

Conjecture 2.1. *for $d = 4$, under the conditions that $N_f \geq N_c$ and $\alpha_F \neq 1$, a continuum-limit of BZ exists at $\alpha_B \to 1$, where the correlation length of the system diverges. It is conjectured that this continuum-limit belongs to the same universality class of Yang-Mills.*

2.1.2 Reformulation of BZ to BZN

In the framework of induced gauge theory studied in [80], the first step towards a dual description of the lattice model is to find an alternative but equivalent expression of (2.1) where the gluons on different links are decoupled from one another. Instead, the gluon field couples to a bosonic field φ and a fermionic field ξ. As we will see in the subsequent sections, this new expression has an advantage because it allows us to integrate out all the gluons and eventually arrive at an equivalent description of (2.1) in terms of color-neutral composite particles built from φ and ξ.

Let us begin with a paraphrase of (2.1) by rewriting the partition function as

$$\mathbf{Z} = \int DU \prod_{\mathbf{p}\in\Lambda} \frac{|\mathrm{Det}(\mathbb{1} - \mathrm{e}^{-m}\mathscr{U}_s(\mathbf{p})\mathscr{U}_r(\mathbf{p}))|^{2N_f}}{|\mathrm{Det}(\mathbb{1} - \mathrm{e}^{-M}\mathscr{U}_s(\mathbf{p})\mathscr{U}_r(\mathbf{p}))|^{2N_f}}, \tag{2.2}$$

where we separated the circulating dynamics along each $\partial\mathbf{p}$, described by $\mathscr{U}_s(\mathbf{p})$, from the color transformation, denoted by $\mathscr{U}_r(\mathbf{p})$. The definitions of these two operators are

$$\mathscr{U}_s(\mathbf{p}) := \mathscr{S}(\mathbf{p}) \otimes \mathbb{1}_c; \qquad \mathscr{U}_r(\mathbf{p}) := \bigoplus_{\mathbf{l}\in\pm\partial\mathbf{p}} u_{\mathbf{l}}^{\epsilon(\mathbf{p},\mathbf{l})}. \tag{2.3}$$

Importantly, unlike the conventional $\tilde{u}_{\mathbf{l}}$, $u_{\mathbf{l}}$ is an endomorphism of a color space. We still call $u_{\mathbf{l}}$ a gluon. The determinant in (2.2) is now over the product of the link space and the color space, which will be called the *link-color space*. In $\mathscr{U}_r(\mathbf{p})$, each $\mathbf{l} \in \Lambda$ is included as long as either $\mathbf{l} \in \partial\mathbf{p}$ or $\mathbf{l} \in -\partial\mathbf{p}$. This slight abuse of notation means that the 1−cell $\mathbf{l}$ is a component of either the 1−chain $\partial\mathbf{p}$ or $-\partial\mathbf{p}$. Correspondingly, to ensure the correct order of the product of gluons in the holonomy along $\partial\mathbf{p}$, we introduce an *indicator* $\epsilon(\mathbf{p},\mathbf{l})$ of the relative orientation between $\mathbf{l}$ and $\mathbf{p}$, which takes value ±1 for $\mathbf{l} \in \pm\partial\mathbf{p}$.

As an illustration, we consider an example in Figure 2.3 where the plaquette $\mathbf{p}$ is surrounded by four links $\mathbf{l}_i,\ i = 1,2,3,4$. The corresponding operators are

$$\mathscr{S}(\mathbf{p}) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \quad \mathscr{U}_r(\mathbf{p}) = \begin{pmatrix} u_{\mathbf{l}_1} & 0 & 0 & 0 \\ 0 & u_{\mathbf{l}_2} & 0 & 0 \\ 0 & 0 & u_{\mathbf{l}_3} & 0 \\ 0 & 0 & 0 & u_{\mathbf{l}_4}^{-1} \end{pmatrix}. \tag{2.4}$$

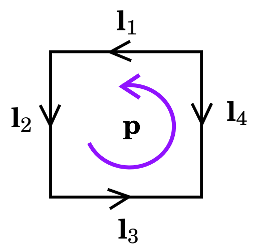


FIGURE 2.3. In this plaquette $\epsilon(\mathbf{p},\mathbf{l}_4) = -1$, so we have $u_{\mathbf{l}_4}^{-1}$ in $\mathscr{U}_r(\mathbf{p})$.

To recover the expression (2.1) in terms of holonomies, one uses the formula $|\mathrm{Det}A|^2 = \exp(\mathrm{Tr}\ln A + \mathrm{Tr}\ln A^\dagger)$, expands the logarithms, and then combines the results. We then obtain

$$\mathbf{Z} = \int DU\ \mathrm{e}^{-\mathrm{S}_{\mathrm{ind}}[U]}; \tag{2.5}$$

$$\mathrm{S}_{\mathrm{ind}}[U] = N_f \sum_{\mathbf{p}\in\Lambda} \sum_{k=1}^{\infty} \frac{\mathrm{e}^{-4mk} - \mathrm{e}^{-4Mk}}{k} [\mathrm{tr}U(\partial\mathbf{p})^k + \mathrm{tr}(U(\partial\mathbf{p})^\dagger)^k]. \tag{2.6}$$

The constituent gluons of the holonomy $U(\partial\mathbf{p})$ in (2.5) are $u_{\mathbf{l}}$. From now on, we choose Λ to be a $d = 4$ hyper-cubic lattice which explains the factor 4 in the exponent, because only the terms of the power of multiples of 4 are non-zero under the trace over the link space. This indeed equals to the same expansion of (2.1) via the identification $\alpha_F \equiv \mathrm{e}^{-4m}$, $\alpha_B \equiv \mathrm{e}^{-4M}$, and along with the substitution $u_{\mathbf{l}} \mapsto \tilde{u}_{\mathbf{l}}$. If it is not otherwise specified, "tr" stands for the color space trace whereas

"Tr" is the trace over the link-color space. Importantly, the continuum-limit in *Conjecture* 2.1 now takes place at $M \to 0$ provided that $N_f \geq N_c$ and $m \neq 0$. For simplicity, we restrict ourselves to $m > M > 0$ in this work.

2.2 Gaussian integral representation

Having convinced ourselves that (2.2) is indeed equivalent to Yang-Mills under appropriate circumstances as explained in Section 2.1.2, it is time to develop a dual description for it.

2.2.1 Fock space representation

For simplicity, let us consider a single $\mathbf{p}$ and set $N_f = 1$. Our first task is to transform the ratio of determinants

$$\frac{|\mathrm{Det}(\mathbb{1} - \mathrm{e}^{-m}\mathscr{U}_s(\mathbf{p})\mathscr{U}_r(\mathbf{p}))|^2}{|\mathrm{Det}(\mathbb{1} - \mathrm{e}^{-M}\mathscr{U}_s(\mathbf{p})\mathscr{U}_r(\mathbf{p}))|^2} \tag{2.7}$$

into a product of a supertrace of a "fermionic" operator and a trace of a "bosonic" operator.

To that end, let $\mathscr{H}$ be the link-color space with the natural index of basis $n \equiv (l, c)$, where l labels $\mathbf{l}$ and c is the color index. In the second-quantization of $\mathscr{H}$, we introduce boson creation/annihilation operators $b_n^\dagger, b_n$ and fermion creation/annihilation operators $f_n^\dagger, f_n$, which generate a bosonic Fock space and a fermionic one. Any $g = \mathrm{e}^X \in \mathrm{GL}(\mathscr{H})$ admits the following Fock space representations:

$$\sigma(\mathrm{e}^X) = \exp \sum_{n,n'} f_n^\dagger \langle n|X|n'\rangle f_{n'} \equiv \exp(f^\dagger X f) \qquad \text{(fermionic);} \tag{2.8}$$

$$\omega(\mathrm{e}^X) = \exp \sum_{n,n'} b_n^\dagger \langle n|X|n'\rangle b_{n'} \equiv \exp(b^\dagger X b) \qquad \text{(bosonic).} \tag{2.9}$$

Following [81] we know that by construction these operators satisfy $\sigma(gh) = \sigma(g)\sigma(h)\ \forall g, h \in \mathrm{GL}(\mathscr{H})$ and same for ω. Moreover, the supertrace $\mathrm{STr}\sigma(g) = \mathrm{Det}(\mathbb{1} - g)$ and $\mathrm{Tr}\omega(g) = \mathrm{Det}^{-1}(\mathbb{1} - g)$, which is true because e^{-M} guarantees $gg^\dagger < 1$. Consequently, the ratio (2.7) equals to

$$|\mathrm{STr}\sigma(\mathrm{e}^{-m}\mathscr{U}_s(\mathbf{p}))\sigma(\mathscr{U}_r(\mathbf{p}))|^2 \cdot |\mathrm{Tr}\omega(\mathrm{e}^{-M}\mathscr{U}_s(\mathbf{p}))\omega(\mathscr{U}_r(\mathbf{p}))|^2, \tag{2.10}$$

which is what we promised.

2.2.2 Gaussian integral representation

We are now ready to introduce the aforementioned bosonic field φ and the fermionic field ξ into the picture, which is essentially rooted in the Gaussian integral representations of σ and ω introduced in [81]. First, we look into the fermion sector, where one defines an operator

$$\mathrm{T}_\xi = \exp(\sum_n \bar{\xi}_n f_n + \xi_n f_n^\dagger) \equiv \exp(\bar{\xi} f + \xi f^\dagger). \tag{2.11}$$

Here, $\{\xi_n, \bar{\xi}_n\}$ are Grassmann variables, and by construction $\bar{\xi}$ transforms in the dual representation of ξ.

With (2.11), the Gaussian integral representation of σ is defined as:

$$\sigma(g) = \mathrm{Det}(\mathbb{1} - g) \int D(\xi, \bar{\xi})\ \mathrm{e}^{-\frac{1}{2}\bar{\xi}\mathrm{A}_g\xi}\mathrm{T}_\xi, \tag{2.12}$$

where we have introduced the Cauchy map:

$$\mathrm{A}(g) = \frac{\mathbb{1} + g}{\mathbb{1} - g}. \tag{2.13}$$

Applying the formula (2.12) to (2.10) and recognizing

$$|\mathrm{STr}\sigma(\mathrm{e}^{-m}\mathcal{U}_s(\mathbf{p}))\sigma(\mathcal{U}_r(\mathbf{p}))|^2 = \mathrm{STr}\sigma(\mathrm{e}^{-m}\mathcal{U}_s(\mathbf{p}))\sigma(\mathcal{U}_r(\mathbf{p})) \cdot \mathrm{STr}\sigma(\mathcal{U}_r(\mathbf{p})^\dagger)\sigma(\mathrm{e}^{-m}\mathcal{U}_s(\mathbf{p})^\dagger), \tag{2.14}$$

it can be shown that $|\mathrm{STr}\sigma(\mathrm{e}^{-m}\mathcal{U}_s(\mathbf{p}))\sigma(\mathcal{U}_r(\mathbf{p}))|^2$ equals to ($\mathbf{p}$ is omitted)

$$\Omega_F \cdot \int D(\xi_+, \bar{\xi}_+, \xi_-, \bar{\xi}_-)\ \mathrm{e}^{-\frac{1}{2}(\sum_{\alpha=\pm}\bar{\xi}_\alpha A_{m,\alpha}\xi_\alpha)} \cdot \mathrm{e}^{-\frac{1}{2}(\sum_{\alpha=\pm}\bar{\xi}_\alpha A_{r,\alpha}\xi_\alpha)}, \tag{2.15}$$

where

$$\Omega_F \equiv |\mathrm{Det}(\mathbb{1} - \mathrm{e}^{-m}\mathcal{U}_s(\mathbf{p}))|^2 |\mathrm{Det}(\mathbb{1} - \mathcal{U}_r(\mathbf{p}))|^2; \tag{2.16}$$

$$\mathrm{A}_{m,+}(\mathbf{p}) \equiv \mathrm{A}(g = \mathrm{e}^{-m}\mathcal{U}_s(\mathbf{p})),\ \mathrm{A}_{m,-}(\mathbf{p}) \equiv \mathrm{A}(g = \mathrm{e}^{-m}\mathcal{U}_s(\mathbf{p})^\dagger); \tag{2.17}$$

$$\mathrm{A}_{r,+}(\mathbf{p}) \equiv \mathrm{A}(g = \mathcal{U}_r(\mathbf{p})) = -\mathrm{A}_{r,-}(\mathbf{p}). \tag{2.18}$$

In the derivation of (2.15), we have used the fact $\mathrm{STrT}_\xi\mathrm{T}_\eta = \delta(\xi + \eta)$. Note that $\mathrm{A}_{r,\pm}(\mathbf{p})$ is singular, so one may imagine there is a small regulator $\mathrm{e}^{-\delta}$ to begin with, which eventually will be removed.

The same story can be told in the boson sector where m and ξ are replaced by M and a complex-valued field φ, respectively. The analogy of (2.11) is the following operator

$$\mathrm{T}_\varphi = \exp(\sum_n \varphi_n b_n^\dagger - \bar{\varphi}_n b_n) \equiv \exp(\varphi b^\dagger - \bar{\varphi} b), \tag{2.19}$$

which obeys $\mathrm{TrT}_\varphi\mathrm{T}_\phi = \delta(\varphi + \phi)$. The field $\bar{\varphi}$ is the hermitian conjugate of φ.

It can be shown that $|\mathrm{Tr}\omega(\mathrm{e}^{-M}\mathcal{U}_s(\mathbf{p}))\omega(\mathcal{U}_r(\mathbf{p}))|^2$ equals to

$$\Omega_B^{-1} \cdot \int D(\varphi_+, \bar{\varphi}_+, \varphi_-, \bar{\varphi}_-)\ \mathrm{e}^{-\frac{1}{2}(\sum_{\alpha=\pm}\bar{\varphi}_\alpha A_{M,\alpha}\varphi_\alpha)} \cdot \mathrm{e}^{-\frac{1}{2}(\sum_{\alpha=\pm}\bar{\varphi}_\alpha A_{r,\alpha}\varphi_\alpha)}, \tag{2.20}$$

where Ω_B and $\mathrm{A}_{M,\pm}$ are defined in the similar way as Ω_F and $\mathrm{A}_{m,\pm}$.

Finally, we combine (2.15), (2.20) and return the flavor N_f and the plaquettes $\mathbf{p}$ to the partition function. We arrive at

$$\mathbf{Z} = C \int DUD(\xi, \bar{\xi}, \varphi, \bar{\varphi})\ \mathrm{e}^{-\sum_{\mathbf{p}\in\Lambda}\Gamma_s(\mathbf{p}) + \Gamma_r(\mathbf{p})}; \tag{2.21}$$

$$\Gamma_s(\mathbf{p}) = \frac{1}{2}\left[\sum_{\alpha=\pm}\sum_{f=1}^{N_f} \bar{\varphi}_{\mathbf{p},\alpha,f}\mathrm{A}_{M,\alpha}(\mathbf{p})\varphi_{\mathbf{p},\alpha,f} + \bar{\xi}_{\mathbf{p},\alpha,f}\mathrm{A}_{m,\alpha}(\mathbf{p})\xi_{\mathbf{p},\alpha,f}\right]; \tag{2.22}$$

$$\Gamma_r(\mathbf{p}) = \frac{1}{2}\mathrm{TrA}_r(\mathbf{p})\left[\sum_{\alpha=\pm}\sum_{f=1}^{N_f} \alpha\varphi_{\mathbf{p},\alpha,f}\bar{\varphi}_{\mathbf{p},\alpha,f} - \alpha\xi_{\mathbf{p},\alpha,f}\bar{\xi}_{\mathbf{p},\alpha,f}\right]. \tag{2.23}$$

Importantly, the gluon-dependent factors in Ω_F and Ω_B cancel out and only a constant $C \neq 1$ remains. In $\Gamma_r(\mathbf{p})$, the relative minus sign comes from the reshuffling of the Grassmann variables.

2.2.3 Superanalysis consideration

The bosonic terms and the fermionic terms in the partition function (2.21) are almost identical, so one expects that the expression can be simplified further using super-variables. To that end, we re-position all the indices and introduce

$$\bar{\xi}_{\mathbf{p}}(\mathbf{l}) \equiv \left[\bar{\xi}_{\mathbf{p}}(\mathbf{l})_c\right]^{\alpha f}; \quad \xi_{\mathbf{p}}(\mathbf{l}) \equiv \left[\xi_{\mathbf{p}}(\mathbf{l})^c\right]_{\alpha f}. \tag{2.24}$$

The idea is to make the "position" information, i.e., $\mathbf{p}$, $\mathbf{l}$ explicit, which will help us develop a diagrammatic analysis later on. At each $(\mathbf{p},\mathbf{l})$, $\bar{\xi}_{\mathbf{p}}(\mathbf{l})$ is a $2N_f$ by N_c matrix and $\xi_{\mathbf{p}}(\mathbf{l})$ is N_c by $2N_f$. The same goes for $\bar{\varphi}_{\mathbf{p}}(\mathbf{l})$ and $\varphi_{\mathbf{p}}(\mathbf{l})$.

With these new notations, we introduce a set of N_c supervectors:

$$\bar{\Psi}_{\mathbf{p}}(\mathbf{l})_c \equiv \begin{pmatrix} \bar{\xi}_{\mathbf{p}}(\mathbf{l})_c \\ \bar{\varphi}_{\mathbf{p}}(\mathbf{l})_c \end{pmatrix}; \quad \Psi_{\mathbf{p}}(\mathbf{l})^c \equiv \begin{pmatrix} \xi_{\mathbf{p}}(\mathbf{l})^c & \varphi_{\mathbf{p}}(\mathbf{l})^c \end{pmatrix}. \tag{2.25}$$

Let us simplify (2.22) first. The Cauchy maps $\mathrm{A}_{M,\alpha}$, $\mathrm{A}_{m,\alpha}$ are substituted by a single $4N_f$ by $4N_f$ (super)matrix for each $(\mathbf{p},\mathbf{l},\mathbf{l}')$:

$$\mathfrak{A}(\mathbf{p})_{\mathbf{l},\mathbf{l}'} \equiv \mathrm{diag}(-\mathrm{A}_{m,+}(\mathbf{p})_{\mathbf{l},\mathbf{l}'}, -\mathrm{A}_{m,-}(\mathbf{p})_{\mathbf{l},\mathbf{l}'}, \mathrm{A}_{M,+}(\mathbf{p})_{\mathbf{l},\mathbf{l}'}, \mathrm{A}_{M,-}(\mathbf{p})_{\mathbf{l},\mathbf{l}'}) \otimes \mathbb{1}_f. \tag{2.26}$$

In terms of (2.25) and (2.26) we find

$$\Gamma_s(\mathbf{p}) = \frac{1}{2} \sum_{\mathbf{l},\mathbf{l}' \in \pm\partial\mathbf{p}} \mathrm{tr} \Psi_{\mathbf{p}}(\mathbf{l}') \mathfrak{A}(\mathbf{p})_{\mathbf{l},\mathbf{l}'} \bar{\Psi}_{\mathbf{p}}(\mathbf{l}). \tag{2.27}$$

We clarify once and for all that the color indices in the original definitions of the Cauchy maps (2.17) are no longer present in (2.26); instead, this is accounted for by the color trace "tr" in (2.27).

What remains to do is the simplification of (2.23). As we are going to integrate out the gluons $u_{\mathbf{l}}$ eventually, it is desirable to carry out the trace over the link space first. The result is

$$\Gamma_r(\mathbf{p}) = \frac{1}{2} \sum_{\mathbf{l} \in \pm\partial\mathbf{p}} \mathrm{tr} \tilde{\mathrm{A}}_r(\mathbf{p},\mathbf{l}) \left[\varphi_{\mathbf{p}}(\mathbf{l}) \mathrm{J} \bar{\varphi}_{\mathbf{p}}(\mathbf{l}) - \xi_{\mathbf{p}}(\mathbf{l}) \mathrm{J} \bar{\xi}_{\mathbf{p}}(\mathbf{l})\right], \tag{2.28}$$

where the relative orientation $\epsilon(\mathbf{p},\mathbf{l})$ and the factor α have been taken into consideration in the following elements:

$$\tilde{\mathrm{A}}_r(\mathbf{p},\mathbf{l}) \equiv \frac{\mathbb{1}_c + u_{\mathbf{l}}^{\epsilon(\mathbf{p},\mathbf{l})}}{\mathbb{1}_c - u_{\mathbf{l}}^{\epsilon(\mathbf{p},\mathbf{l})}}; \quad \mathrm{J} \equiv \mathrm{diag}(\mathbb{1}_f, -\mathbb{1}_f). \tag{2.29}$$

It is straightforward to recast this in terms of the super-variables:

$$\begin{aligned} \Gamma_r(\mathbf{p}) &= \sum_{\mathbf{l} \in \pm\partial\mathbf{p}} \mathrm{tr} \tilde{\mathrm{A}}_r(\mathbf{p},\mathbf{l}) \left[\frac{1}{2} \Psi_{\mathbf{p}}(\mathbf{l}) \begin{pmatrix} -\mathrm{J} & 0 \\ 0 & \mathrm{J} \end{pmatrix} \bar{\Psi}_{\mathbf{p}}(\mathbf{l})\right] \\ &\equiv \sum_{\mathbf{l} \in \pm\partial\mathbf{p}} \mathrm{tr} \tilde{\mathrm{A}}_r(\mathbf{p},\mathbf{l}) \tilde{\mathrm{D}}(\mathbf{p},\mathbf{l}). \end{aligned} \tag{2.30}$$

2.3 Notion of color invariance

We have successfully massaged the exponents (2.22), (2.23) into the more instructive expression (2.27), (2.30). In the framework of induced gauge theory, the next step is to integrate out the gluons to obtain a dual description of (2.21) in terms of Ψ and $\bar{\Psi}$. This only involves Γ_r, and we are going to employ the *Cayley parametrisation* of the $\mathrm{U}(N_c)$ Haar measure [82] for the gluon-integral because it will greatly simplify the Cauchy maps.

Before we delve into the calculations, it is worth pointing out an important observation about (2.21). By construction, (2.1) is invariant under

$$\tilde{u}_{\mathbf{l}} \mapsto \tilde{g}_{t(\mathbf{l})} \tilde{u}_{\mathbf{l}} \tilde{g}^{-1}_{s(\mathbf{l})}, \tag{2.31}$$

where $\tilde{g}$ is a mapping from all the $0-$cells (sites) to the color group $\mathrm{U}(N_c)$, and $s(\mathbf{l})$, $t(\mathbf{l})$ are the starting site and the ending site of $\mathbf{l}$, respectively. See Figure 2.4. We refer to $\tilde{g}$ as the *conventional color gauge-transformation*.

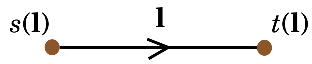


FIGURE 2.4. A link with a starting site and an ending site.

In the original work of BZ [64], the auxiliary bosons and fermions live on the sites as in most lattice QCD models, and accompanying (2.31) they transform in the (anti)fundamental representations of $\mathrm{U}(N_c)$. There, the dual description is obviously invariant under the conventional color gauge-invariance.

In BZN, the transformation (2.31) doesn't apply anymore due to the fact that $u_{\mathbf{l}}$ is an endomorphism. Instead, there exists a *new color gauge transformation* g, which is a mapping from the links to $\mathrm{U}(N_c)$. This group acts on the gluons and the auxiliary fields by

$$u_{\mathbf{l}} \mapsto g_{\mathbf{l}} u_{\mathbf{l}} g_{\mathbf{l}}^{-1}; \quad \Psi_{\mathbf{p}}(\mathbf{l}) \mapsto g_{\mathbf{l}} \Psi_{\mathbf{p}}(\mathbf{l}); \quad \bar{\Psi}_{\mathbf{p}}(\mathbf{l}) \mapsto \bar{\Psi}_{\mathbf{p}}(\mathbf{l}) g_{\mathbf{l}}^{-1}. \tag{2.32}$$

Our interest now changes to manufacturing the partition function in terms of *color-singlets* with respect to this new gauge group. As we are going to see, these color-singlets are composite particles of the form $\bar{\Psi}_{\mathbf{p}}(\mathbf{l})\Psi_{\mathbf{p}'}(\mathbf{l})$, which are invariant under (2.32). Before doing any calculations, one can already realize that the integration over gluons

$$\int DU\, \mathrm{e}^{-\sum_{\mathbf{p}\in\Lambda} \Gamma_r(\mathbf{p})} \equiv \mathrm{e}^{-\mathrm{S}_{\mathrm{eff,G}}} \tag{2.33}$$

will automatically generate the color-singlets due to the bi-invariance of the Haar measure. We will derive the effective action $\mathrm{S}_{\mathrm{eff,G}}$ in Section 2.4.3. For the Γ_s sector, however, we will be forced to make use of a *gauge-averaging* trick because different links are coupled, as one can see in (2.27).

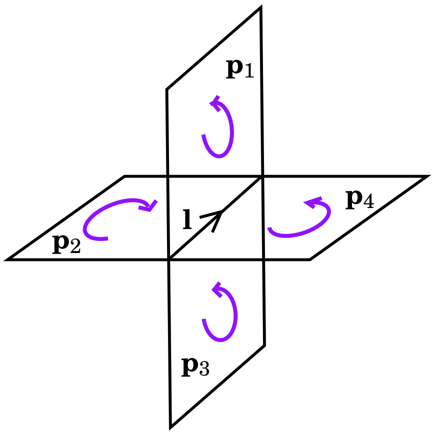


FIGURE 2.5. The plaquettes attached to a single link.

2.4 Gluon integral

Now we compute the gluon-integral (2.33).

2.4.1 Cayley parametrisation

Any link $\mathbf{l} \in \Lambda$ belongs to several plaquettes, which are denoted by $\mathbf{p} \in \mathfrak{B}(\mathbf{l})$. More precisely, the set $\mathfrak{B}(\mathbf{l})$ consists of all the plaquettes $\mathbf{p}$ whose boundary $\partial\mathbf{p}$ containing either $\mathbf{l}$ or $-\mathbf{l}$. See Figure 2.5. In (2.33) the integration involving a single link $\mathbf{l}$ is

$$\int du_{\mathbf{l}}\ \mathrm{e}^{-\sum_{\mathbf{p}\in\mathfrak{B}(\mathbf{l})} \mathrm{tr}\tilde{\mathrm{A}}_r(\mathbf{p},\mathbf{l})\tilde{\mathrm{D}}(\mathbf{p},\mathbf{l})}, \tag{2.34}$$

which can be simplified by introducing

$$\mathrm{D}(\mathbf{l}) \equiv \sum_{\mathbf{p}\in\mathfrak{B}(\mathbf{l})} \epsilon(\mathbf{p},\mathbf{l})\tilde{\mathrm{D}}(\mathbf{p},\mathbf{l}); \quad \mathrm{A}(\mathbf{l}) \equiv \frac{\mathbb{1}_c + u_{\mathbf{l}}}{\mathbb{1}_c - u_{\mathbf{l}}}. \tag{2.35}$$

With these, (2.34) becomes

$$\int du_{\mathbf{l}}\ \mathrm{e}^{-\mathrm{tr}\mathrm{A}(\mathbf{l})\mathrm{D}(\mathbf{l})}. \tag{2.36}$$

Next, applying the *Cayley transformation*, i.e., the bijection[1] (omitting the index $\mathbf{l}$):

$$u = \frac{iM + \mathbb{1}_c}{iM - \mathbb{1}_c} \Leftrightarrow iM = -\frac{\mathbb{1}_c + u}{\mathbb{1}_c - u}; \quad M \in \mathrm{Herm}_{N_c}(\mathbb{C}), \tag{2.37}$$

we can substitute the integration over $\mathrm{U}(N_c)$ in (2.36) by an integration over $\mathrm{Herm}_{N_c}(\mathbb{C})$, known as the *Cayley parametrisation* [82]:

$$\int du_{\mathbf{l}}\ \mathrm{e}^{-\mathrm{tr}\mathrm{A}(\mathbf{l})\mathrm{D}(\mathbf{l})} = C' \int \frac{dM_{\mathbf{l}}}{\mathrm{Det}^{N_c}(\mathbb{1}_c + M_{\mathbf{l}}^2)}\ \mathrm{e}^{i\mathrm{tr}M_{\mathbf{l}}\mathrm{D}(\mathbf{l})}. \tag{2.38}$$

Any constant factor of the integral such as C' will be dropped from now on, and for simplicity we are going to omit the index $\mathbf{l}$ in the following derivations. As usual, $dM \equiv \prod_i dM_{ii} \prod_{i<j} d\Re M_{ij} d\Im M_{ij}$.

[1] We restrict u such that none of its eigenvalues is 1. This won't affect the gluon integral.

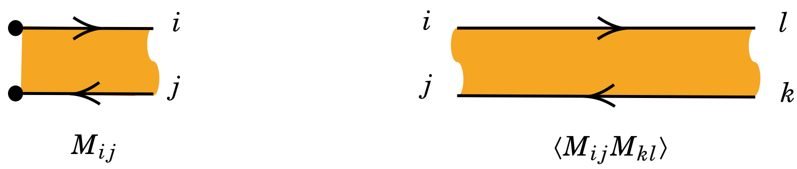


FIGURE 2.6. An external vertex and a propagator.

2.4.2 Ribbon diagrams

To proceed, one first exponentiates the determinant and rescales $M \to M/\sqrt{N_c}$ such that (2.38) reads

$$\mathscr{I}[\mathrm{D}] \equiv \int dM\ \mathrm{e}^{-N_c \mathrm{tr} \ln(\mathbb{1}_c + M^2/N_c) + i\mathrm{tr} M\mathrm{D}/\sqrt{N_c}}, \tag{2.39}$$

which is in a good shape for a large$-N_c$ analysis. To that end, let's expand the logarithm in the integrand

$$\exp\left[-\mathrm{tr}M^2 + \left(\mathrm{tr}\frac{M^4}{2N_c} - \mathrm{tr}\frac{M^6}{3N_c^2} + \dots\right) + i\mathrm{tr}M\mathrm{D}/\sqrt{N_c}\right] \tag{2.40}$$

and apply the standard machinery of *ribbon diagrams* [83]. The idea is to evaluate, order-by-order, the expectation values of the products of "vertices" with respect to the Gaussian weight $\exp(-\mathrm{tr}M^2)$, and finally assemble an effective action using only the "connected" ribbon diagrams.

The vertices in (2.40) are given by

$$\mathbf{v}[2k] \equiv \frac{(-1)^k \mathrm{tr} M^{2k}}{k N_c^{k-1}},\ k = 2, 3, \dots; \qquad \mathbf{v}[\mathrm{D}] \equiv \frac{i\mathrm{tr}M\mathrm{D}}{\sqrt{N_c}}, \tag{2.41}$$

and the following identities are used in the computations:

$$\mathbf{Z}_0 \equiv \int dM\ e^{-\mathrm{tr}M^2}; \tag{2.42}$$

$$\langle M_{ij} M_{kl} \rangle \equiv \frac{1}{\mathbf{Z}_0} \int dM\ M_{ij} M_{kl} e^{-\mathrm{tr}M^2} = \frac{1}{2}\delta_{il}\delta_{jk}; \tag{2.43}$$

$$\langle \text{odd powers of } M's \rangle = 0; \tag{2.44}$$

$$\langle \text{even powers of } M's \rangle = \sum_{\text{all pairings}} \prod_{\text{pairs}} \langle MM \rangle \cdots \langle MM \rangle. \tag{2.45}$$

By depicting each vertex $\mathbf{v}[2k]$ as a starfish, each $\mathbf{v}[D]$ as a pen, and each "propagator" $\langle MM \rangle$ as a ribbon (double-line) connecting the branches from the vertices, every term in the integral $\mathscr{I}[D]$ in (2.39) is represented by a ribbon diagram. These diagrammatic tools are depicted in Figure 2.6 and Figure 2.7. A ribbon diagram can be either *connected* or *disconnected* in the usual sense. By the linked-cluster principle, the effective action $\mathrm{S_{eff}}[\mathrm{D}]$ defined by

$$\mathbf{Z}_0^{-1} \mathscr{I}[\mathrm{D}] \equiv \mathrm{e}^{-\mathrm{S_{eff}}[\mathrm{D}]} \tag{2.46}$$

$\mathrm{tr}\left(M^4\right) = \sum_{i,j,k,l} M_{ij}M_{jk}M_{kl}M_{li}$ $\qquad$ $\mathrm{tr}(MD) = \sum_{a,b} M_{ab}D_{ba}$

FIGURE 2.7. Interaction vertices.

only contains the connected diagrams.

Our main mission is to figure out the leading terms in $\mathrm{S_{eff}[D]}$ in the large$-N_c$ expansion. A generic connected diagram has the following expression:

$$G^{(c)}(\mathbf{v}[2k_1],\ldots,\mathbf{v}[2k_p],\underbrace{\mathbf{v}[\mathrm{D}],\ldots,\mathbf{v}[\mathrm{D}]}_{2q}), \tag{2.47}$$

where $p,q \in \mathbb{Z}^+$. We only study even number ($2q$) of pen-vertices because of (2.44). First of all, as one can see in (2.41), these vertices carry a factor of

$$\frac{1}{N_c^{\sum_{i=1}^p k_i - p}} \cdot \frac{1}{N_c^q}. \tag{2.48}$$

In addition to this, the contractions of indices via the propagators defined by (2.43) result in another N_c-dependent factor. For illustration, we draw a *bridge* connecting two M's if they are contracted.

Take a single starfish-vertex $\mathbf{v}[4]$ for instance:

$$\mathrm{tr}MMMM \sim \mathcal{O}(N_c^3); \qquad \mathrm{tr}MMMM \sim \mathcal{O}(N_c). \tag{2.49}$$

When there are two (or more) vertices, we must first build at least one bridge to connect them and then do the rest of the contractions, for example:

$$\mathrm{tr}MMMM\;\mathrm{tr}MMMM \sim \mathcal{O}(N_c^4);\; \mathrm{tr}MMMM\;\mathrm{tr}MMMM \sim \mathcal{O}(N_c^2). \tag{2.50}$$

In the D-independent sector ($q = 0$), the terms like the ones on the left of both (2.49), (2.50) have the highest power in N_c. In the literature, these are known as the *planar diagrams* (Figure 2.8), whose bridges can be arranged such that they are *non-crossing*. Each planar diagram carries a factor $N_c^{\sum_{i=1}^p k_i - p + 2}$, which counters (2.48) and results in an overall factor of N_c^2 in this sector. One way to quickly derive this factor is to exploit the following rule: arbitrarily pick one bridge that connects two vertices and do the contraction, which effectively removes two M from each

FIGURE 2.8. A planar diagram (left) and a non-planar one (right).

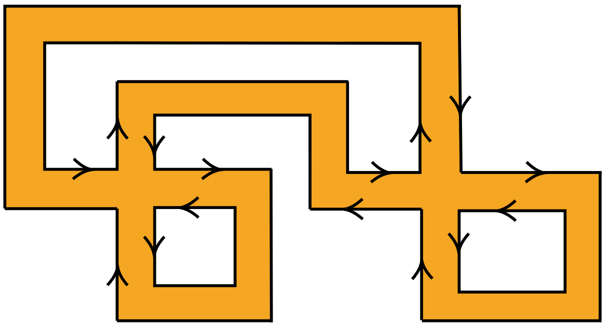


FIGURE 2.9. A planar diagram with two $\mathbf{v}[2k]$ vertices.

vertex and "merge" the remaining vertices in to a single trace. See Figure 2.9. It is straightforward to compute the factor of the resulting single-trace planar diagram, which yields the factor.

The D-dependent sector is more important. First of all, one can easily read out the factor for $p = 0,\ q = 1$, which is N_c^{-1}, and the contraction of two D's yields trD^2. Indeed, only the part from (2.48) contributes, from which we know the diagrams with larger q are of smaller order (in fact, they are all disconnected). In the presence of the $\mathbf{v}[2k]$ vertices, we understand that:

1. To stay connected, the M from every $\mathbf{v}[\mathrm{D}]$ must connect to one of the $\mathbf{v}[2k]$ vertices, but not to another $\mathbf{v}[\mathrm{D}]$.

2. Any M from a $\mathbf{v}[2k]$ vertex, as long as it is contracted with some $\mathbf{v}[\mathrm{D}]$, cannot be contracted with another M's from any $\mathbf{v}[2k]$ vertex any more.

An example is given in Figure 2.10. Consequently, in (2.47) there are $2q$ $\mathbf{v}[\mathrm{D}]$ vertices and p $\mathbf{v}[2k]$ vertices, so only $2\sum_{i=1}^{p} k_i - 2q$ "free" M's are at our disposal. Merging the $\mathbf{v}[2k]$ vertices reduces this number further by $2(p-1)$. Finally, unlike the D-independent sector, because the overall trace is absent here, the leading power of N_c is exactly half of the number of the remaining free M's. Together with (2.48) we obtain the highest power of N_c in (2.47)

$$\frac{1}{N_c^{\sum_{i=1}^{p} k_i - p}} \cdot \frac{1}{N_c^{q}} \cdot N_c^{\sum_{i=1}^{p} k_i - p - (q-1)} = \frac{1}{N_c^{2q-1}}. \tag{2.51}$$

Again, this leading-order term is described by a non-crossing diagram, and the contraction of D's yields trD^{2q}. Importantly, a term such as $(\mathrm{trD})^2$ is a crossing diagram so it is of the next-order. Take $p = q = 1$ for example, a non-crossing diagram looks like

$$\mathrm{tr} MMM \cdots MMM \quad M\mathrm{D} \quad M\mathrm{D} \sim N_c^{k-1}. \tag{2.52}$$

In comparison, a crossing diagram carries

$$\mathrm{tr} MMM \cdots MMM \quad M\mathrm{D} \quad M\mathrm{D} \sim N_c^{k-2}, \tag{2.53}$$

which indeed has the lower power of N_c.

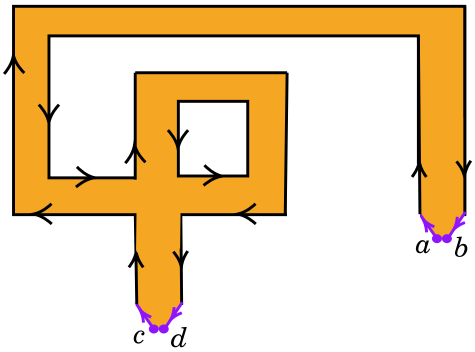


FIGURE 2.10. A non-crossing contraction of a $\mathbf{v}[2k]$ vertex with two $\mathbf{v}[\mathrm{D}]$ vertices.

2.4.3 Contribution to effective action

Combining these results, we arrive at the large$-N_c$ expansion of $\mathrm{S}_{\mathrm{eff}}[\mathrm{D}]$:

$$-\mathrm{S}_{\mathrm{eff}}[\mathrm{D}] \approx c_0 + c_1 \frac{1}{N_c} \mathrm{tr} D^2 + \mathcal{O}(\frac{1}{N_c^3}). \tag{2.54}$$

All the D$-$*independent* terms are included in the constant c_0, and it can be shown that the constant $c_1 < 0$. Finally, we substitute (2.54) for (2.36) and go back to (2.33). The effective action is

$$-\mathrm{S}_{\mathrm{eff,G}} \approx C_0 + C_1 \frac{1}{N_c} \sum_{\mathbf{l} \in \Lambda} \mathrm{trD}^2(\mathbf{l}) + \mathcal{O}(\frac{1}{N_c^3}). \tag{2.55}$$

All the terms in $\mathrm{S}_{\mathrm{eff,G}}$, as promised, are color-singlets with respect to the new gauge group (2.32). For brevity, from now on we will address (2.32) as the *color gauge group*.

2.5 Color averaging

So far we have carried out the gluon-integral in (2.21), written in terms of the supervectors:

$$\begin{aligned}\mathbf{Z} &= C\int DUD(\Psi,\bar{\Psi})\ \mathrm{e}^{-\sum_{\mathbf{p}\in\Lambda}\Gamma_s(\mathbf{p})+\Gamma_r(\mathbf{p})} \\ &= C''\int D(\Psi,\bar{\Psi})\ \mathrm{e}^{-\mathrm{S}_{\mathrm{eff,G}}}\mathrm{e}^{-\sum_{\mathbf{p}\in\Lambda}\Gamma_s(\mathbf{p})}.\end{aligned} \tag{2.56}$$

It is obvious that the integrand in (2.56) is not color-neutral yet due to the inter-link coupling in $\Gamma_s(\mathbf{p})$, as one can tell from (2.27). Fortunately, there is a trick of gauge-averaging which can help us get around this difficulty.

2.5.1 Gauge averaging

First of all, we know the effective action $\mathrm{S}_{\mathrm{eff,G}}$ defined in (2.55) and the Berezin measure $D(\xi,\bar{\xi},\varphi,\bar{\varphi})$ are both color-invariant. It follows that

$$\begin{aligned}\int D(\Psi,\bar{\Psi})\ \mathrm{e}^{-\mathrm{S}_{\mathrm{eff,G}}}\mathrm{e}^{-\sum_{\mathbf{p}\in\Lambda}\Gamma_s(\mathbf{p})} &\equiv \int D(\Psi^g,\bar{\Psi}^g)\ \mathrm{e}^{-\mathrm{S}^g_{\mathrm{eff,G}}}\mathrm{e}^{-\sum_{\mathbf{p}\in\Lambda}\Gamma^g_s(\mathbf{p})} \\ &= \int\frac{Dg}{(\int Dg)}\int D(\Psi^g,\bar{\Psi}^g)\ \mathrm{e}^{-\mathrm{S}^g_{\mathrm{eff,G}}}\mathrm{e}^{-\sum_{\mathbf{p}\in\Lambda}\Gamma^g_s(\mathbf{p})} \\ &= \int D(\Psi,\bar{\Psi})\ \mathrm{e}^{-\mathrm{S}_{\mathrm{eff,G}}}\int Dg\ \mathrm{e}^{-\sum_{\mathbf{p}\in\Lambda}\Gamma^g_s(\mathbf{p})},\end{aligned} \tag{2.57}$$

where $Dg\equiv\prod_{\mathbf{l}}dg_{\mathbf{l}}$ and we have chosen the normalization $\int dg_{\mathbf{l}}=1$ for all $\mathbf{l}$. The superscript ^{g} refers to the transformation (2.32), which turns $\Gamma_s(\mathbf{p})$ into

$$\Gamma^g_s(\mathbf{p})=\frac{1}{2}\sum_{\mathbf{l},\mathbf{l}'\in\pm\partial\mathbf{p}}\mathrm{tr}\,g_{\mathbf{l}'}\Psi_{\mathbf{p}}(\mathbf{l}')\mathfrak{A}(\mathbf{p})_{\mathbf{l},\mathbf{l}'}\bar{\Psi}_{\mathbf{p}}(\mathbf{l})g^{-1}_{\mathbf{l}}. \tag{2.58}$$

The merit of the gauge-averaging, i.e., the integrations over all $g_{\mathbf{l}}\in\mathrm{U}(N_c)$ is that the outcome only consists of color-singlets by construction. This comes with a price, however, because the averaging-integral in general cannot be solved exactly; rather, the result will be written as a large$-N_c$ series expansion. Similar to what has been done for Γ_r, we are going to present each term by a diagram, and then argue that the effective action $\mathrm{S}_{\mathrm{eff,L}}$ defined by

$$\int Dg\ \mathrm{e}^{-\sum_{\mathbf{p}\in\Lambda}\Gamma^g_s(\mathbf{p})}\equiv\mathrm{e}^{-\mathrm{S}_{\mathrm{eff,L}}} \tag{2.59}$$

is built from the *connected diagrams*.

What is the definition for a diagram? Let's first expand the exponential function in the integral:

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int Dg\left(\sum_{\mathbf{p}_1\in\Lambda}\Gamma^g_s(\mathbf{p}_1)\right)\cdots\left(\sum_{\mathbf{p}_n\in\Lambda}\Gamma^g_s(\mathbf{p}_n)\right). \tag{2.60}$$

Plugging (2.58) in it, the integral for each n can be schematically expressed as

$$\sum_{\mathbf{p}_1,\cdots,\mathbf{p}_n}\int Dg\left[\prod_{i=1}^{n}\sum_{\mathbf{l}_i,\mathbf{l}'_i\in\pm\partial\mathbf{p}}\Psi_{\mathbf{p}_i}(\mathbf{l}'_i)^{c_i}\mathfrak{A}(\mathbf{p}_i)_{\mathbf{l}_i,\mathbf{l}'_i}\bar{\Psi}_{\mathbf{p}_i}(\mathbf{l}_i)_{b_i}(g^{-1}_{\mathbf{l}_i})^{b_i}_{d_i}(g_{\mathbf{l}'_i})^{d_i}_{c_i}\right], \tag{2.61}$$

where the summations over the color indices are implied. Our task is to solve the integral:

$$\int Dg\,(g_{\mathbf{l}'_1})^{d_1}_{c_1}\cdots(g_{\mathbf{l}'_n})^{d_n}_{c_n}(g^{-1}_{\mathbf{l}_1})^{b_1}_{d_1}\cdots(g^{-1}_{\mathbf{l}_n})^{b_n}_{d_n}, \tag{2.62}$$

and the mathematical tool we will be using is the *Weingarten integral* over $\mathrm{U}(N_c)$ [84, 85]:

$$\begin{aligned}&\int dg\, g^{i_1}_{\ j_1}\cdots g^{i_k}_{\ j_k}(g^{-1})^{j'_1}_{\ i'_1}\cdots(g^{-1})^{j'_l}_{\ i'_l}=\\&\begin{cases}0, & l\neq k;\\ \sum_{\sigma,\tau\in\mathrm{S}_k}\delta^{i_1}_{i'_{\sigma(1)}}\cdots\delta^{i_k}_{i'_{\sigma(k)}}\delta^{j'_{\tau(1)}}_{j_1}\cdots\delta^{j'_{\tau(k)}}_{j_k}\mathrm{W}(\tau\sigma^{-1},N_c), & l=k.\end{cases}\end{aligned} \tag{2.63}$$

$\mathrm{W}(\tau\sigma^{-1},N_c)$ is known as the *Weingarten function*[2], and S_k denotes the permutation group for k elements.

2.5.2 Diagrammatic study

It is time to look for a diagrammatic analysis of (2.61), which takes three steps.

Partition of links: to distribute the auxiliary bosons and fermions on the links, we use a solid circle • for $\bar{\Psi}$ whereas a hollow circle ∘ for Ψ, see Figure 2.11. One of the conditions indicated by (2.63) says (2.62) is nonvanishing only if the sets $\mathbf{L}_n\equiv\{\mathbf{l}_1,\cdots,\mathbf{l}_n\}$ and $\mathbf{L}'_n\equiv\{\mathbf{l}'_1,\cdots,\mathbf{l}'_n\}$ must be identical (regardless the order). Therefore, on the same link the numbers of • and ∘ must match, i.e., they form pairs.

Next, because any link has more than one adjacent plaquette, to avoid confusion we always place the pair of circles on the side of the link indicating which plaquette it belongs to.

Finally, suppose there are n plaquettes on the exemplary $d=2$ square-lattice, as shown in Figure 2.11. The number assigned to each plaquette stands for the number of multiplicity of that plaquette, which is exactly the number of pairs of circles on that plaquette, and the sum of the multiplicities is n.

Tunnelling lines: to keep track of different products $\Psi\mathfrak{A}\bar{\Psi}$ in (2.61), we draw a line connecting one • and one ∘ for each product (Figure 2.12(a)). Obviously a tunnelling line can only connect circles on the same plaquette.

Once the lines are settled, we can apply the tool in (2.63). Diagrammatically this amounts to "binding" one • and one ∘ on the same link, which is done by putting an elliptical shade beneath them (Figure 2.12(b)). This "pig nose" represents a color-singlet.

[2] See Appendix A.

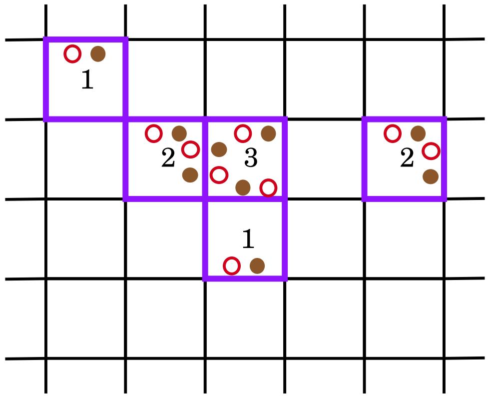
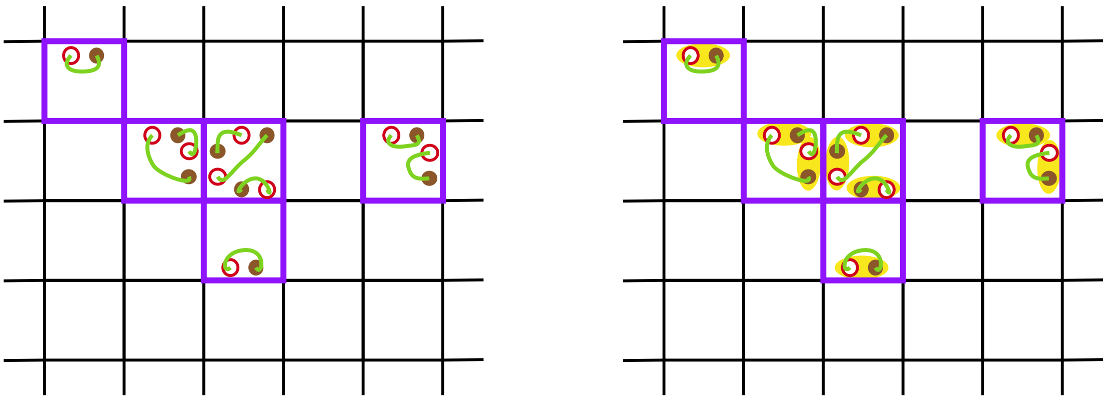


FIGURE 2.11. Plaquettes with numbers of multiplicity.



(a) (b)

FIGURE 2.12. Adding (a) tunnelling lines and (b) pig noses.

In general, as long as the circles are on the same link, all possible bindings are allowed providing they follow one rule: if the • and ∘ on the same link are connected by a tunnelling line, then they automatically form a color-singlet and hence are bound together (see the examples below).

Examples: we present here two computations for a single plaquette, one with a single-multiplicity and one with a double-multiplicity. See Figure 2.13.

For the single-multiplicity case we have

$$\begin{aligned} \int dg_{\mathbf{l}}\,(g_{\mathbf{l}}^{-1})^b_d(g_{\mathbf{l}})^d_c\Psi_{\mathbf{p}}(\mathbf{l})^c\mathfrak{A}(\mathbf{p})_{\mathbf{l},\mathbf{l}}\bar{\Psi}_{\mathbf{p}}(\mathbf{l})_b &= \Psi_{\mathbf{p}}(\mathbf{l})^b\mathfrak{A}(\mathbf{p})_{\mathbf{l},\mathbf{l}}\bar{\Psi}_{\mathbf{p}}(\mathbf{l})_b \\ &= \mathrm{STr}\left[\bar{\Psi}_{\mathbf{p}}(\mathbf{l})\Psi_{\mathbf{p}}(\mathbf{l})\right]\mathfrak{A}(\mathbf{p})_{\mathbf{l},\mathbf{l}}\,, \end{aligned} \tag{2.64}$$

where in the second equation we used the cyclicity of supertrace. Note that there is no N_c−dependent factor generated by the Weingarten integral at this order. For the double-multiplicity case we

FIGURE 2.13. Single-multiplicity (left) and double-multiplicity (right).

have a factor of N_c^{-1}:

$$\begin{aligned}\int dg_{\mathbf{l}}dg_{\mathbf{l}'}\,(g_{\mathbf{l}}^{-1})^{b_1}_{d_1}(g_{\mathbf{l}'})^{d_1}_{c_1}(g_{\mathbf{l}'}^{-1})^{b_2}_{d_2}(g_{\mathbf{l}})^{d_2}_{c_2}\Psi_{\mathbf{p}}(\mathbf{l}')^{c_1}\mathfrak{A}(\mathbf{p})_{\mathbf{l},\mathbf{l}'}\bar{\Psi}_{\mathbf{p}}(\mathbf{l})_{b_1}\Psi_{\mathbf{p}}(\mathbf{l})^{c_2}\mathfrak{A}(\mathbf{p})_{\mathbf{l}',\mathbf{l}}\bar{\Psi}_{\mathbf{p}}(\mathbf{l}')_{b_2}\\ =\frac{1}{N_c}\Psi_{\mathbf{p}}(\mathbf{l}')^{c}\mathfrak{A}(\mathbf{p})_{\mathbf{l},\mathbf{l}'}\bar{\Psi}_{\mathbf{p}}(\mathbf{l})_{b}\Psi_{\mathbf{p}}(\mathbf{l})^{b}\mathfrak{A}(\mathbf{p})_{\mathbf{l}',\mathbf{l}}\bar{\Psi}_{\mathbf{p}}(\mathbf{l})_{c}\\ =\frac{1}{N_c}\mathrm{Str}\,[\bar{\Psi}_{\mathbf{p}}(\mathbf{l}')\Psi_{\mathbf{p}}(\mathbf{l}')]\,\mathfrak{A}(\mathbf{p})_{\mathbf{l},\mathbf{l}'}\,[\bar{\Psi}_{\mathbf{p}}(\mathbf{l})\Psi_{\mathbf{p}}(\mathbf{l})]\,\mathfrak{A}(\mathbf{p})_{\mathbf{l}',\mathbf{l}}.\end{aligned}\tag{2.65}$$

2.5.3 Contribution to effective action

By the linked-cluster principle,[3] only the connected diagrams appear in $\mathrm{S}_{\mathrm{eff,L}}$. Both the examples above are connected, and they indeed build the leading-order terms in the effective action:

$$-\mathrm{S}^{(0)}_{\mathrm{eff,L}}=\frac{-1}{2}\sum_{\mathbf{p}\in\Lambda}\sum_{\mathbf{l}\in\pm\partial\mathbf{p}}\mathrm{STr}(\bar{\Psi}_{\mathbf{p}}(\mathbf{l})\Psi_{\mathbf{p}}(\mathbf{l}))\mathfrak{A}(\mathbf{p})_{\mathbf{l},\mathbf{l}};$$

$$-\mathrm{S}^{(1)}_{\mathrm{eff,L}}=\frac{1}{8N_c}\sum_{\mathbf{p}\in\Lambda}\sum_{\mathbf{l}\neq\mathbf{l}'\in\pm\partial\mathbf{p}}\mathrm{STr}(\bar{\Psi}_{\mathbf{p}}(\mathbf{l}')\Psi_{\mathbf{p}}(\mathbf{l}'))\mathfrak{A}(\mathbf{p})_{\mathbf{l},\mathbf{l}'}(\bar{\Psi}_{\mathbf{p}}(\mathbf{l})\Psi_{\mathbf{p}}(\mathbf{l}))\mathfrak{A}(\mathbf{p})_{\mathbf{l}',\mathbf{l}}.$$

For completeness, we present two higher-order terms in Figure 2.14, which carry N_c^{-2} and N_c^{-3}, respectively. There might also exist connected diagrams involving more than one plaquette.

FIGURE 2.14. Some examples of higher-order terms in the effective action.

Figure 2.15 shows an example of two adjacent plaquettes which are connected by the pig noses. However, this particular contribution was found to be zero.

[3] See Appendix B.

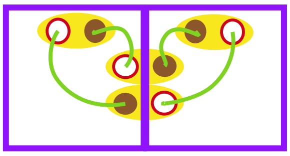


FIGURE 2.15. A connected digram with adjacent plaquettes.

2.6 Dual theory: a first look

In summary, now we have two equivalent descriptions for BZN:

$$\mathbf{Z} = \int DU\ \mathrm{e}^{-\mathrm{S}_{\mathrm{ind}}[U]} = \int D(\Psi,\bar{\Psi})\ \mathrm{e}^{-\mathrm{S}_{\mathrm{eff,G}}[\Psi,\bar{\Psi}]-\mathrm{S}_{\mathrm{eff,L}}[\Psi,\bar{\Psi}]}; \tag{2.66}$$

$$\mathrm{S}_{\mathrm{ind}}[U] = N_f \sum_{\mathbf{p}\in\Lambda}\sum_{k=1}^{\infty} \frac{\mathrm{e}^{-4mk}-\mathrm{e}^{-4Mk}}{k}[\mathrm{tr}U(\partial\mathbf{p})^k + \mathrm{tr}(U(\partial\mathbf{p})^\dagger)^k]; \tag{2.67}$$

$$\begin{aligned}\mathrm{S}_{\mathrm{eff,G}}[\Psi,\bar{\Psi}] \approx -C_0 - \frac{C_1}{4N_c}\sum_{\mathbf{l}\in\Lambda}\sum_{\mathbf{p},\mathbf{p}'\in\mathfrak{B}(\mathbf{l})} &\epsilon(\mathbf{p},\mathbf{l})\epsilon(\mathbf{p}',\mathbf{l})\mathrm{STr}(\bar{\Psi}_{\mathbf{p}'}(\mathbf{l})\Psi_{\mathbf{p}}(\mathbf{l}))\tilde{J}(\bar{\Psi}_{\mathbf{p}}(\mathbf{l})\Psi_{\mathbf{p}'}(\mathbf{l}))\tilde{J} \\ &+\mathscr{O}(\frac{1}{N_c^3});\end{aligned} \tag{2.68}$$

$$\begin{aligned}\mathrm{S}_{\mathrm{eff,L}}[\Psi,\bar{\Psi}] \approx &\frac{1}{2}\sum_{\mathbf{p}\in\Lambda}\sum_{\mathbf{l}\in\pm\partial\mathbf{p}} \mathrm{STr}(\bar{\Psi}_{\mathbf{p}}(\mathbf{l})\Psi_{\mathbf{p}}(\mathbf{l}))\mathfrak{A}(\mathbf{p})_{\mathbf{l},\mathbf{l}} \\ &-\frac{1}{8N_c}\sum_{\mathbf{p}\in\Lambda}\sum_{\mathbf{l}\neq\mathbf{l}'\in\pm\partial\mathbf{p}} \mathrm{STr}(\bar{\Psi}_{\mathbf{p}}(\mathbf{l}')\Psi_{\mathbf{p}}(\mathbf{l}'))\mathfrak{A}(\mathbf{p})_{\mathbf{l},\mathbf{l}'}(\bar{\Psi}_{\mathbf{p}}(\mathbf{l})\Psi_{\mathbf{p}}(\mathbf{l}))\mathfrak{A}(\mathbf{p})_{\mathbf{l}',\mathbf{l}} \\ &+\mathscr{O}(\frac{1}{N_c^2}).\end{aligned} \tag{2.69}$$

To obtain (2.68), we have defined $\tilde{J} \equiv \mathrm{diag}(-J,J)$ and used the cyclicity of supertrace.

2.6.1 Review of Wilson lattice theory

How is BZN related to Wilson pure gauge theory (WGT) [59]? It was demonstrated in [65, 66] that under the conditions stated in *Conjecture* 2.1, BZ and WGT share the same continuum-limit, which is believed to be YM[4]. The definition for the WGT action used in [64] is

$$\mathrm{S_W}[U] = -\frac{\beta}{2N_c}\sum_{\mathbf{p}\in\Lambda}[\mathrm{tr}U(\partial\mathbf{p}) + \mathrm{tr}(U(\partial\mathbf{p})^\dagger)]. \tag{2.70}$$

As a long-lasting exemplary model, WGT is proved to be useful when it comes to simulations of the infrared physics of QCD, such as the static quark-antiquark potential [86] and the mass-gap [87]. At the *weak-coupling limit* ($\beta \gg 1$) WGT converges to YM; however, the theory becomes non-perturbative, so it is challenging to understand the physics in this regime analytically. In comparison, at the *strong-coupling limit* ($\beta \ll 1$) one can compute the quark-antiquark potential

[4]The gauge groups $\mathrm{U}(N_c)$ and $\mathrm{SU}(N_c)$ are both investigated in various dimensions, and the universality condition is refined.

by a perturbation method, which is usually done by performing a small$-\beta$ expansion of a Wilson loop.

2.6.2 Comparison of BZN and WGT

Turning now to BZN described by (2.67). Let's try to use it to evaluate the Wilson loop expectation value for a large rectangular loop $\mathscr{C}$:

$$\langle W[\mathscr{C}]\rangle \equiv \int DU \, \left[\frac{1}{N_c}\mathrm{tr}U(\mathscr{C})\right] \mathrm{e}^{-\mathrm{S}_{\mathrm{ind}}[U]} = \int DU \, \left[\frac{1}{N_c}\mathrm{tr}U(\mathscr{C})\right] \sum_{n=0}^{\infty} \frac{(-\mathrm{S}_{\mathrm{ind}}[U])^n}{n!}. \tag{2.71}$$

Suppose the loop $\mathscr{C}$ encloses n' plaquettes, then according to the Weingarten integral, a generic n^{th}-order term in (2.71) is non-vanishing only if $n \geq n'$. This is because only at the n'^{th}-order or higher could one find enough *distinct* plaquettes to assemble a $2d$ surface enclosed by $\mathscr{C}$.

In WGT, the derivation for the leading-order term of $\langle W[\mathscr{C}]\rangle$ is straightforward, which is built from the plaquettes constituting the *minimal surface* enclosed by $\mathscr{C}$. For large $\mathscr{C}$, this term dominates the expectation value and results in the famous *area law* of $\langle W[\mathscr{C}]\rangle$, which supports the color-confinement hypothesis of a static quark-antiquark pair in QCD. In BZN, the analysis is more involving due to the series expansion in k in (2.67), and it makes an analytical study less appealing than a numerical simulation[5]. Therefore, instead of a rigorous evaluation of (2.71), we are giving a heuristic explanation below to argue how the same area law could be realized in BZN as well.

Firstly, we should make sure the series expansion in k is a sensible one. Fixing $m > M > 0$ and $N_f \geq N_c$ as suggested by *Conjecture* 2.1 , it follows that the factor (minus sign absorbed)

$$f(k) = \frac{N_f(\mathrm{e}^{-4Mk} - \mathrm{e}^{-4mk})}{k} > 0 \qquad \forall k. \tag{2.72}$$

Now, the sum of traces of the unitary matrices is bounded by $2N_c$ from above for all k, so as long as one can show that $f(k=1)$ is a finite number and $f(k)$ is strictly decreasing, the series expansion is sensible[6]. It is clear that $f(k=1)$ is finite; in addition, the derivative $f'(k)$ is strictly negative as long as $m > M > 0$, so $f(k)$ is strictly decreasing indeed.

For concreteness, from now on we fix the difference $\varepsilon = m - M$ as a positive constant, such that by tuning M alone, we are moving m, M simultaneously. As a matter of fact, $f(k)$ decreases faster when M is larger; consequently, because the continuum-limit is realized at $M \to 0$, this means the farther away BZN from the continuum, the better control we have over the series expansion in k. This same feature is shared by the conventional strong-coupling expansion of WGT.

Based on the observation above, one can imagine sending $M \gg 1$ so that the difference between the $k = 1$ terms and the $k > 1$ terms is deepened. More importantly, let us make M large

[5] For a numerical result of the Wilson loop expectation value, see [66].

[6] To a physicist, but not necessarily to a mathematician.

enough for any given N_f, ε such that $f(k=1)<1$. In this scenario, it is likely that the $k=1$ terms dominate and $f(k=1)$ plays the same role as the parameter β in WGT, and hence it seems promising that the same area-law could be derived using the conventional lattice gauge theory tools.

2.7 The dual description of Wilson loop

Our next question is what advantages does the dual description (2.68), (2.69), which will be known as *dual-BZN*, might have compared to the original one? One classic test is the computation of Wilson loop expectation value $\langle W[\mathscr{C}]\rangle$ in the dual theory, where we hope to gain better control of this observable when BZN is closer to continuum YM.

2.7.1 Computation for single link

Consider again a large loop $\mathscr{C}$ and pick the gluon u from an arbitrary link belonging to $\mathscr{C}$. By inserting this gluon into the single-gluon integral (2.36), we can compute its contribution to $\langle W[\mathscr{C}]\rangle$:

$$\mathbf{W}_{jk} \equiv \int_{\mathrm{U}(N_c)} du\ (u^{\pm})_{jk} \mathrm{e}^{-\mathrm{tr}\left(\frac{\mathbb{1}_c+u}{\mathbb{1}_c-u}\mathrm{D}\right)}, \tag{2.73}$$

where the color indices $\{j,k\}$ are written out explicitly. Note that the possible sign change $\pm$ depends on the relative orientation of the link with respect to $\mathscr{C}$, and without lost of generality it will always be set to $+$. Following the similar steps leading to (2.39), the integral is rescaled to

$$\mathbf{W}_{jk} = \int_{\mathrm{Herm}_{N_c}(\mathbb{C})} dM\ \mathrm{e}^{-N_c \mathrm{tr}\ln(\mathbb{1}_c+M^2/N_c)} \mathrm{e}^{i\mathrm{tr}M\mathrm{D}/\sqrt{N_c}} \left(\frac{i\frac{M}{\sqrt{N_c}}+\mathbb{1}_c}{i\frac{M}{\sqrt{N_c}}-\mathbb{1}_c}\right)_{jk}. \tag{2.74}$$

Firstly, we make use of the expansion

$$\left(\frac{i\frac{M}{\sqrt{N_c}}+\mathbb{1}_c}{i\frac{M}{\sqrt{N_c}}-\mathbb{1}_c}\right)_{jk} = -\delta_{jk} - 2\sum_{p=1}^{\infty}\left[\left(\frac{iM}{\sqrt{N_c}}\right)^p\right]_{jk} \tag{2.75}$$

together with the equation (summation convention implied)

$$\frac{\partial}{\partial \mathrm{D}_{kl_1}}\frac{\partial}{\partial \mathrm{D}_{l_1l_2}}\cdots\frac{\partial}{\partial \mathrm{D}_{l_{p-2}l_{p-1}}}\frac{\partial}{\partial \mathrm{D}_{l_{p-1}j}}\mathrm{e}^{i\mathrm{tr}M\mathrm{D}/\sqrt{N_c}} = \left[\left(\frac{iM}{\sqrt{N_c}}\right)^p\right]_{jk}\mathrm{e}^{i\mathrm{tr}M\mathrm{D}/\sqrt{N_c}} \tag{2.76}$$

to rewrite (2.74) as

$$\mathbf{W}_{jk} = \mathbf{Z}_0\left[-\delta_{jk} - 2\left(\frac{\partial}{\partial\mathrm{D}} + \frac{\partial}{\partial\mathrm{D}}\frac{\partial}{\partial\mathrm{D}} + \frac{\partial}{\partial\mathrm{D}}\frac{\partial}{\partial\mathrm{D}}\frac{\partial}{\partial\mathrm{D}} + \cdots\right)_{kj}\right]\mathrm{e}^{-\mathrm{S}_{\mathrm{eff}}[\mathrm{D}]}. \tag{2.77}$$

In the derivation we have used (2.42) and (2.46).

The next step is to extract the leading-order term in the large$-N_c$ expansion of $\mathbf{W}_{jk}$. An immediate observation is that $\mathbf{W}_{jk}$ doesn't have terms independent of D because of (2.73), so we only have to pay attention to the terms containing D. Now, as we've learned in Section 2.4.2 that any D−dependent term in $\mathrm{S_{eff}[D]}$ has the form trD^{2q} for some integer q, a generic term in (2.77) looks like

$$\frac{\partial}{\partial \mathrm{D}_{kl_1}} \frac{\partial}{\partial \mathrm{D}_{l_1 l_2}} \cdots \frac{\partial}{\partial \mathrm{D}_{l_{p-2}l_{p-1}}} \frac{\partial}{\partial \mathrm{D}_{l_{p-1}j}} \left[\mathrm{D}_{ab_1} \mathrm{D}_{b_1 b_2} \cdots \mathrm{D}_{b_{2q-2}b_{2q-1}} \mathrm{D}_{b_{2q-1}a} \right], \tag{2.78}$$

which is non-zero only if $p \le 2q$, and any D−dependent term must carry $p < 2q$. Carrying out the differentiations and summing over the repeated indices yield a factor of some power of N_c multiplied by a product of D's. As a rule of thumb, the highest-power of N_c comes from the $p = 2q-1$ terms where the contraction of indices result in a factor of $N_c^{p-1} = N_c^{2q-2}$, which is multiplied by D_{jk}.

Finally, we combine this factor with the one from the power-counting (2.51) and find the leading-term in (2.77):

$$\mathbf{W}_{jk} \approx \left[\frac{\text{(const)}}{N_c} \mathrm{D}_{jk} + \mathscr{O}(\frac{1}{N_c^2}) \right] \mathrm{e}^{-\mathrm{S_{eff}[D]}}. \tag{2.79}$$

2.7.2 Dual holonomy

It is now straightforward to assemble the contributions from all the gluons living on $\mathscr{C}$ to the expectation value:

$$\langle W[\mathscr{C}] \rangle \approx \frac{1}{\mathbf{Z}} \int D(\Psi, \bar{\Psi})\, \mathrm{e}^{-\mathrm{S_{eff}}[\Psi,\bar{\Psi}]} (-1)^{\rho}\, \mathrm{tr}\mathfrak{D}(\mathscr{C}) + \text{(corrections)}, \tag{2.80}$$

where $\mathrm{S_{eff}} \equiv \mathrm{S_{eff,G}} + \mathrm{S_{eff,L}}$, which can be found in (2.68), (2.69). Here, the ordered product of D's along the loop $\mathscr{C}$ is denoted by $\mathfrak{D}(\mathscr{C})$ and hence $\mathrm{tr}\mathfrak{D}(\mathscr{C})$ should be referred to as the *dual holonomy*. For simplicity, an overall factor of some power of N_c is omitted, and an factor of $(-1)^{\rho}$ is included to account for the effect of the relative orientations.

What we have achieved until (2.80) is to lay the groundwork for a computation of $W[\mathscr{C}]$ in the dual theory. However, it is still too early to say whether or not this approach can improve the existing strong-coupling results. To that end, one must take a closer look at the effective action $\mathrm{S_{eff}}$ to identify an effective "*coupling strength parameter*" in terms of M and m, and to figure out under which circumstances that this parameter permits a sensible perturbation analysis of (2.80). We will take a look at this in Section 2.8.1.

2.8 Physical aspects

The most important physical information of dual-BZN is hidden in the matrix $\tilde{J}$ in (2.68) and $\mathfrak{A}(\mathbf{p})$ in (2.69). While $\tilde{J}$ is by definition constant over the hyper-cubic lattice Λ, the matrix $\mathfrak{A}(\mathbf{p})$ acts non-trivially on the link space. As a first step towards a better understanding of dual-BZN, we will be focusing on the boson-boson sector in the subsequent discussion, and leave the other sectors for future work.

2.8.1 Mass and interaction strength

The link-space component of $\mathrm{A}_{M,\pm}$ (omitting $\mathbb{1}_c$) are:

$$\mathrm{A}_{M,+}(\mathbf{p}) = \mathrm{A}_{M,-}(\mathbf{p})^\dagger = \frac{1}{\mathrm{e}^{4M}-1}\begin{pmatrix} \mathrm{e}^{4M}+1 & 2\mathrm{e}^{M} & 2\mathrm{e}^{2M} & 2\mathrm{e}^{3M} \\ 2\mathrm{e}^{3M} & \mathrm{e}^{4M}+1 & 2\mathrm{e}^{M} & 2\mathrm{e}^{2M} \\ 2\mathrm{e}^{2M} & 2\mathrm{e}^{3M} & \mathrm{e}^{4M}+1 & 2\mathrm{e}^{M} \\ 2\mathrm{e}^{M} & 2\mathrm{e}^{2M} & 2\mathrm{e}^{3M} & \mathrm{e}^{4M}+1 \end{pmatrix}. \tag{2.81}$$

In this representation, the order of the links is determined by the orientation of $\mathbf{p}$, see for example Figure 2.3[7].

We can directly make some observations from (2.81). First of all, the diagonal elements are link-independent and share the same value

$$\mathrm{A}_{M,+}(\mathbf{p})_{\mathbf{l},\mathbf{l}} = \mathrm{A}_{M,-}(\mathbf{p})_{\mathbf{l},\mathbf{l}} = \frac{\mathrm{e}^{4M}+1}{\mathrm{e}^{4M}-1}, \tag{2.82}$$

which converges to 1 as BZN runs away from the continuum-limit, i.e., $M \gg 1$. As this is the factor of the quadratic term in (2.69), we may refer to it as the *mass*. In comparison, for $\mathbf{l} \neq \mathbf{l}'$ the off-diagonal elements are

$$\mathrm{A}_{M,+}(\mathbf{p})_{\mathbf{l},\mathbf{l}'} = \mathrm{A}_{M,-}(\mathbf{p})_{\mathbf{l}',\mathbf{l}} \in \left\{ \frac{2\mathrm{e}^{M}}{\mathrm{e}^{4M}-1}, \frac{2\mathrm{e}^{2M}}{\mathrm{e}^{4M}-1}, \frac{2\mathrm{e}^{3M}}{\mathrm{e}^{4M}-1} \right\}, \tag{2.83}$$

and all of them approach 0 as $M \to \infty$. Since these elements connect the composite particles from different links, they should be interpreted as the *interaction couplings*. Similar results can be found in $\mathrm{A}_{m,\pm}$.

Substituting (2.82), (2.83) and the parallel $\mathrm{A}_{m,\pm}$ back to the effective action (2.69), it is clear that for large M, m, dual-BZN is *weakly-coupled*. Indeed, in this regime $\mathrm{S}_{\mathrm{eff,G}}$ and the mass terms from $\mathrm{S}_{\mathrm{eff,L}}$ dominate the dual-BZN, whereas the interaction terms from $\mathrm{S}_{\mathrm{eff,L}}$ are the relatively-small corrections. As a consequence, it seems that we have a better chance to perform a perturbation analysis, say, for $\langle W[\mathscr{C}]\rangle$ when M, m are large. However, this is also true for BZN, as pointed out in Section 2.6.1. In order to argue that dual-BZN is more useful regarding the computations of some observables, one has to look into the non-perturbative regime of BZN, i.e., when M, m are close to zero.

Taking $M \to 0$ in (2.81), the matrices $\mathrm{A}_{M,\pm}$ obviously diverge due to the denominator $(\mathrm{e}^{4M} - 1)^{-1}$. Moreover, the differences among the matrix elements diminish in this limit because all numerators approach 2, which tells us that close to the continuum only the large$-N_c$ expansion is in effect in (2.69). The same feature is shared by $\mathrm{A}_{m,\pm}$.

[7]Note that the orientations of the links are irrelevant in the tunnelling, which is only dependent on the orientation of the plaquette.

In contrast to the case of large M, m, we see $\mathrm{S}_{\text{eff,G}}$ become less and less significant than $\mathrm{S}_{\text{eff,L}}$ as M, m are closer to 0. This signals the existence of a dual theory to Yang-Mills, whose feature is probably determined by the naive-continuum limit of $\mathrm{S}_{\text{eff,L}}$ alone. By construction there are no color indices in this dual theory, and we hope in the future a comprehensive study of this model can reveal more interesting information from the dual theory regarding the infra-red physics of YM.

Back to the original question, could the dual-BZN help us compute some observables? Suppressing $\mathrm{S}_{\text{eff,G}}$ means at least in the leading-order we don't have to worry about the *inter-plaqutte* couplings, such as $\bar{\Psi}_{\mathbf{p}'}(\mathbf{l})\Psi_{\mathbf{p}}(\mathbf{l})$ anymore. This might simplify the computation of $\langle W[\mathscr{C}]\rangle$ as in (2.80), for instance. Unfortunately, as for now we are not yet able to carry out this calculation. Based on the preliminary results above, it seems that we should try to "simplify" the dual theory further before we perform any realistic computation.

2.8.2 Symmetry group

There are certainly other interesting aspects of the dual-BZN that we should explore. For example, one can try to identify its global (independent of $\mathbf{l}, \mathbf{p}$) symmetry groups. Let $K \subseteq \mathrm{GL}(2N_f; \mathbb{C})$ be a symmetry group for the boson-boson part of the effective action, $\mathrm{S}_{\text{eff}}|_{BB}$, then any $k \in K$ maps $\bar{\varphi}\varphi \mapsto k^\dagger \bar{\varphi}\varphi k$ and $\forall \mathbf{l}, \mathbf{l}', \mathbf{p}$,

$$k \mathrm{A}_M(\mathbf{p})_{\mathbf{l},\mathbf{l}'} k^\dagger = \mathrm{A}_M(\mathbf{p})_{\mathbf{l},\mathbf{l}'} \qquad \text{where} \;\; \mathrm{A}_M(\mathbf{p})_{\mathbf{l},\mathbf{l}'} \equiv \begin{pmatrix} \mathrm{A}_{M,+}(\mathbf{p})_{\mathbf{l},\mathbf{l}'} & 0 \\ 0 & \mathrm{A}_{M,-}(\mathbf{p})_{\mathbf{l},\mathbf{l}'} \end{pmatrix}; \tag{2.84}$$

$$kJk^\dagger = J. \tag{2.85}$$

The identity matrix $\mathbb{1}_f$ is implied in both equations and we are using the definition (2.29) for J. Plugging (2.81) to (2.84), it is straightforward to check that $K = \mathrm{U}(N_f) \times \mathrm{U}(N_f)$, which is a compact Lie group. This can be traced back to the original partition function in (2.21). Based on our current understanding of the diagrammatic derivation for S_{eff}, it is likely that (2.84), (2.85) apply to the higher-order terms as well.

2.8.3 Saddle-point configuration

Looking at (2.66), we wonder if a large$-N_c$ analysis can open the door to a semi-classical approximation. A simple rescaling

$$\Psi \mapsto (N_c)^{1/2}\Psi, \quad \bar{\Psi} \mapsto (N_c)^{1/2}\bar{\Psi}$$

does lead to a nice form $\mathrm{S}_{\text{eff}} \equiv N_c \tilde{\mathrm{S}}_{\text{eff}}$, at least to the leading-order. Importantly, apart from the contraction of color indices within the composite operator $\bar{\Psi}\Psi$, there is no explicit N_c-dependence in the field-dependent part in $\tilde{\mathrm{S}}_{\text{eff}}$. If there exists a change-of-variable recipe which replaces the composite operators by some new variables without any color indices, then large$-N_c$ indeed favors the saddle-points of $\tilde{S}_{\text{eff}}$ in its new appearance. We will examine this possibility in Section 2.8.4.

As for now, we derive the saddle-point configurations of $S_{\text{eff}}|_{BB}$. Differentiating $S_{\text{eff}}|_{BB}$ with respect to $\varphi_{\mathbf{p}}(\mathbf{l})$, we find the equations of motion to the order $\mathcal{O}(1/N_c)$:

$$\left[\frac{1}{2}\mathrm{A}_M(\mathbf{p})_{\mathbf{l},\mathbf{l}} - \frac{1}{4N_c}\sum_{\substack{\mathbf{l}'\in\pm\partial\mathbf{p}\\ \mathbf{l}'\neq\mathbf{l}}} \mathrm{A}_M(\mathbf{p})_{\mathbf{l}',\mathbf{l}}\bar{\varphi}_{\mathbf{p}}(\mathbf{l}')\varphi_{\mathbf{p}}(\mathbf{l}')\mathrm{A}_M(\mathbf{p})_{\mathbf{l},\mathbf{l}'}\right]\bar{\varphi}_{\mathbf{p}}(\mathbf{l}) - \frac{C_1}{2N_c}\sum_{\mathbf{p}'\in\mathfrak{B}(\mathbf{l})} \epsilon(\mathbf{p};\mathbf{l})\epsilon(\mathbf{p}';\mathbf{l})J\bar{\varphi}_{\mathbf{p}}(\mathbf{l})\varphi_{\mathbf{p}'}(\mathbf{l})J\bar{\varphi}_{\mathbf{p}'}(\mathbf{l}) \approx 0, \tag{2.86}$$

where $C_1 < 0$. Now, to find a ground state it is reasonable to start with an ansatz that the $2N_f$ by N_c matrix $\bar{\varphi}_{\mathbf{p}}(\mathbf{l})$ is constant over $\mathbf{p},\mathbf{l},b$, where b is the color index. Furthermore, let's focus on the case where the last term in (2.86) is negligible, which can be realized by probing the small M regime as discussed above[8]. In this limiting case, there exists a saddle-point configuration which *breaks* the symmetry group K:

$$\bar{\varphi}_{\mathbf{p}}(\mathbf{l})_b = \sqrt{\frac{\sinh(4M)}{4+2\cosh(2M)}}\begin{pmatrix} v_+ \\ v_- \end{pmatrix}, \quad v_+^\dagger v_+ = 1 = v_-^\dagger v_- \quad \forall \mathbf{p},\mathbf{l},b; \tag{2.87}$$

where v_+, v_- are in $\mathbb{C}^{N_f}$. In general, these vectors are not invariant under $\mathrm{U}(N_f)$ and hence K doesn't preserve (2.87).

By relaxing the constraints on $\bar{\varphi}$, one expects to discover more saddle-points, and in standard practice we should compare the statistical weights of these solutions to figure out the subset of true ground-states. If it turns out a true ground-state breaks K, we can study the effect of the spontaneous symmetry breaking in the dual-BZN. However, due to the complexity of the higher-order terms in S_{eff}, we do not have a conclusion in this quest so far.

Crucially, the validity of a semi-classical approximation is still questionable. Even in a color-neutralised expression, the number of flavors is connected to the original N_c via the universality condition $N_f \geq N_c$ according to *Conjecture* 2.1. In other words, in order to keep BZN in the universality class of YM, N_f is inevitably "pushed" higher as N_c is taken to infinity.

2.8.4 Application of superbosonization

The preliminary investigations prompt again the importance of a further simplification of the dual-BZN defined by (2.68), (2.69). Indeed, S_{eff} is already written in terms of color-singlets $\bar{\Psi}_{\mathbf{p}'}(\mathbf{l})\Psi_{\mathbf{p}}(\mathbf{l})$, which are tensor-products of supervectors; so a natural question is whether or not one can rewrite the partition function (2.66) **directly** in terms of some $4N_f$ by $4N_f$ supermatrices

[8] Under the ansatz this term also vanishes identically if the numbers of $\epsilon = +1$ and of $\epsilon = -1$ are chosen to be the same. However, this particular choice should not affect the physics.

$Q_{\mathbf{p}',\mathbf{p}}(\mathbf{l})$. Precisely speaking, we would like to carry out the following change of variables:

$$\mathbf{Z} = \int D(\Psi,\bar{\Psi})\ \mathrm{e}^{-\mathrm{S}_{\mathrm{eff,G}}[\Psi,\bar{\Psi}]-\mathrm{S}_{\mathrm{eff,L}}[\Psi,\bar{\Psi}]} \overset{?}{=} \int DQ\ \mathrm{e}^{-\mathrm{S}_{\mathrm{eff}}[Q]}. \tag{2.88}$$

Here, we use the same notation for $\mathrm{S}_{\mathrm{eff}}[\Psi,\bar{\Psi}]$ and $\mathrm{S}_{\mathrm{eff}}[Q]$ to emphasize that a successful transformation of the partition function should be conducted by a direct substitution $\bar{\Psi}_{\mathbf{p}'}(\mathbf{l})\Psi_{\mathbf{p}}(\mathbf{l}) \mapsto Q_{\mathbf{p}',\mathbf{p}}(\mathbf{l})$.

There is a promising mathematical tool, the *superbosonization formula* [75], which states that given a set of supervectors $\bar{\Psi}_b,\ b = 1,2,\ldots,n$ built from p Grassmann variables and q complex variables with $n \geq q$, we have the following change-of-variables formula:

$$\int D(\Psi,\bar{\Psi})\ f(\sum_b \Psi_b \otimes \bar{\Psi}_b) = \frac{\mathrm{volU}(n)}{\mathrm{volU}(n+p-q)} \int_{\mathscr{D}} DQ\ \mathrm{SDet}^n(Q) F(Q). \tag{2.89}$$

The integration domain is $\mathscr{D} = \mathrm{U}(p) \times \mathrm{Herm}^+_q(\mathbb{C})$. For this formula to work, it is necessary that the integral on the left exists, and the function $F(Q)$ is any function that equals to $f(\sum \Psi \otimes \bar{\Psi})$ under the substitution $\sum \Psi \otimes \bar{\Psi} \mapsto Q$.

There are two obvious obstructions standing in our way. First of all, in $\mathrm{S}_{\mathrm{eff,G}}$ (and probably also in $\mathrm{S}_{\mathrm{eff,L}}$) there are color-singlets comprising variables from different plaquettes, so a naive factorization of the partition function with respect to the plaquettes $\mathbf{p}$ cannot work. Instead, one needs to extend the superspace further to include the plaquette index, in the hope that we can then apply (2.89) for each link $\mathbf{l}$ iteratively. That is, one supermatrix per link. However, our understanding of the higher-order terms in $\mathrm{S}_{\mathrm{eff}}$ is not yet complete, so it is too early to say if this could be a viable approach.

The second obstruction turns out to be more fatal. One of the necessary conditions for BZN to converge to YM in the continuum-limit is $N_f \geq N_c$. Now, the color number N_c is n in (2.89), whereas $2N_f$ is q even before we include the plaquette index; therefore, it is fundamentally impossible that $n \geq q$ in the dual-BZN.

We conclude that the superbosonization formula (2.89) is not available to us, unfortunately. At the time of writing, the author came across a recent paper [88] in which a different superbosonization formula was proposed. The case $q > n$ seems to be resolved in this work. A potential application of this new formula on dual-BZN will be one of the topics of interest in the future.

2.9 Summary and outlook for BZN

The Gaussian-integral reformulation of BZN was studied in this chapter. The master action was introduced and dual-BZN was derived by integrating out gluon degrees of freedom on the original lattice. By Cayley parametrisation and a color averaging technique, we were able to complete the duality transformation, and arrived at a dual theory written in color-neutral composite operators. The action of dual-BZN is expressed as a large$-N_c$ series; however, with only the leading-order terms it seems difficult for us to unravel the hidden physical information. Nevertheless, a few properties of dual-BZN were explored.

As a comparison with WGT, the Wilson loop expectation value was computed in BZN. In the large$-M$ limit where BZN is far away from its continuum-limit, the same area law appears. The same computation was then carried over to the dual side, where the Wilson loop was replaced by a dual-holonomy. Additionally, in the boson-boson sector, the (dual) mass parameters and the interaction strength were identified, and a preliminary investigation around its saddle-points and symmetries was conducted. Overall, it is certainly desirable to express dual-BZN as a theory of color-neutral variables. A compelling reason is a possible semi-classical analysis in the large$-N_c$ limit, but the applicability of such an approximation remains questionable due to the universality condition $N_f \geq N_c$.

Regarding the color-neutralisation, we considered a possibility of applying the SuB formula to transform dual-BZN into a more instructive form. There is, however, a conflict between the applicability of the formula (2.89) and the universality condition, because the latter results in rank-deficiency of the composite operators. We hope that future developments in the mathematical tools for SuB will help us transform dual-BZN to a supermatrix theory.

Chapter 3

Dirac-Yang-Mills Model

The construction of dual-BZN was executed by first introducing some auxiliary bosons and fermions to the partition function of BZN, and then integrating out the gluons. In this chapter, we hope to carry out the same procedure in the study of a continuum field theory called *Dirac-Yang-Mills* model (DYM).

In DYM, the essential element is a *master action* S_{mas} for a system of color gauge fields (gluons) and some auxiliary fields, which plays a similar role as the action in (2.21). Following the definition of the DYM Lagrangian in Section 3.1, it will be shown in Section 3.2 that by integrating out the auxiliary fields in S_{mas}, the YM action can be induced at least up to the leading-order. We will refer to this induced theory as *induced Yang-Mills* (IYM) and the induced action as S_{IYM}.

The main part of this chapter is devoted to a dual-description of IYM, or *dual-IYM*, which is derived by integrating out the gluons in S_{mas}. While the integration itself is straightforward, the resulting theory for the auxiliary fields is quite intriguing and a complete understanding of its physical content is still at large. To see behind the veil, we are going to study some qualitative features of dual-IYM based on its close relation to QCD and its symmetries, which are illustrated in Section 3.3 - Section 3.4. At the end, a proposal for a low-energy effective description of dual-IYM is presented.

With everything learned from this chapter, we will be able to take on a more ambitious and quantitative approach to dual-IYM in Chapter 4.

3.1 Dirac-Yang-Mills model

3.1.1 Master action for DYM

In the program of induced gauge theory, the definition for the master action S_{mas} is an art in itself. In one aspect, S_{mas} should be designed in such a way that the induced S_{IYM} is a regular functional and it indeed "approximates" YM. As a common fact, in a generic quantum field theory there are usually some ultra-violet divergences in the effective action derived by integrating out part of the field components; therefore, we expect to face some singular terms in S_{IYM}. Our proposal to "regularize" the theory is using both *bosonic* and *fermionic* auxiliary fields. See Section 3.2.

More significantly, because our main interest lies in developing a dual theory of YM, we would like to tailor S_{mas} such that the induced dual-IYM shares some of the iconic features of QCD **by construction**. One of the advantages in this approach is that the tremendous amount of existing literature on QCD and effective field theories are now at our disposal, which will assist us with the understanding of the physical aspects of dual-IYM.

Taking these into consideration, we propose a $d = 4$ Euclidean field theory [80] with the action $S_{\text{mas}} = \int d^4x \ \mathscr{L}_B + \mathscr{L}_F$, where

$$\mathscr{L}_B = i\text{Tr}[\bar{\phi}(\Gamma^\mu)^{\text{T}}(\partial_\mu + A_\mu)\phi + M_B\bar{\phi}\Omega^{\text{T}}\phi] + \epsilon\text{Tr}\bar{\phi}\phi; \tag{3.1}$$

$$\mathscr{L}_F = -i\text{Tr}[\bar{\psi}(\Gamma^\mu)^{\text{T}}(\partial_\mu + A_\mu)\psi + M_F\bar{\psi}\Omega^{\text{T}}\psi]. \tag{3.2}$$

The *gluon* is denoted by the field $A_\mu(x) \in \text{Lie}(\text{U}(N_c))$, and the auxiliary Dirac fermions and Dirac bosons[1] by $\psi, \bar{\psi}$ and $\phi, \bar{\phi}$ respectively, both of which carry mass parameters denoted by M_B, M_F. In DYM we always set $M_B > M_F$.

As usual, $\Gamma^\mu \equiv \gamma^\mu \otimes \mathbb{1}_{N_f}$ and γ^μ are the Euclidean Dirac matrices obeying

$$\{\gamma^\mu, \gamma^\nu\} = -2\delta^{\mu\nu}\mathbb{1}_4, \tag{3.3}$$

where we have chosen all γ^μ to be skew-Hermitian. In addition, we also need $\gamma_5 \equiv \gamma^0\gamma^1\gamma^2\gamma^3$. The *mass matrix* Ω is designed to be Hermitian and

$$\{\Omega, \Gamma^\mu\} = 0 \quad \forall \mu; \qquad \Omega^2 = \mathbb{1}_{4N_f}. \tag{3.4}$$

For example, $\Omega = \Gamma_5 \equiv \gamma_5 \otimes \mathbb{1}_{N_f}$ is a valid choice. The exact matrix representations for Γ^μ and Ω are not specified explicitly, and they will only be fixed when necessary.

Both ϕ and ψ live in the fundamental representation of the color group $\text{U}(N_c)$ and the dual representation[2] of the flavor group $\text{U}(N_f)$, so they carry *color* vector-indices $c = 1, \ldots, N_c$ along with *flavor* covector-indices $f = 1, \ldots, N_f$; in addition, they also have the usual Dirac *spinor* index s. The convention we follow here is to treat $\phi \equiv \phi^c{}_{sf}$ as an $N_c \times 4N_f$ matrix and accordingly its hermitian conjugate $\phi^\dagger \equiv \bar{\phi} \equiv \bar{\phi}^{sf}{}_c$ as an $4N_f \times N_c$ matrix. The fermion ψ (resp., $\bar{\psi}$) has the same

[1] This is not a physical field, so it can be interpreted as a *ghost* degrees of freedom if one prefers.

[2] The complex-conjugate of the fundamental representation of $\text{U}(N_f)$.

index structure as ϕ (resp., $\bar{\phi}$), but they are essentially two mutually independent variables, so it isn't necessary for $\bar{\psi}$ to transform in the conjugate representation of ψ. The matrices $\Gamma^\mu \equiv (\Gamma^\mu)^{sf}{}_{s'f'}$ and $\Omega \equiv \Omega^{sf}{}_{s'f'}$ are both $4N_f \times 4N_f$ and their transposes $(\Gamma^\mu)^{\mathrm{T}}$, Ω^{T} are used in (3.1), (3.2) such that the standard contraction of matrix indices applies. By definition, they act trivially on the color space. To illustrate our convention of indices, carrying out the trace "Tr" over the *spinor-flavor space* yields for instance

$$\mathrm{Tr}[\bar{\phi}(\Gamma^\mu)^{\mathrm{T}} A_\mu \phi] \equiv \bar{\phi}^{sf}{}_c [(\Gamma^\mu)^{\mathrm{T}}]_{sf}{}^{s'f'} (A_\mu)^c{}_{c'} \phi^{c'}{}_{s'f'}. \tag{3.5}$$

We observe in (3.1), (3.2) that the massive Dirac operators

$$D_B \equiv (\Gamma^\mu)^{\mathrm{T}}(\partial_\mu + A_\mu) + M_B \Omega^{\mathrm{T}}; \tag{3.6}$$

$$D_F \equiv (\Gamma^\mu)^{\mathrm{T}}(\partial_\mu + A_\mu) + M_F \Omega^{\mathrm{T}} \tag{3.7}$$

are both Hermitian, and their spectra are unbounded on both sides of the real axis. Hence, the only way to make the DYM partition function

$$\mathbf{Z}_{\mathrm{DYM}} = \int DAD\bar{\phi}D\phi D\bar{\psi}D\psi \; \mathrm{e}^{-\mathrm{S}_{\mathrm{mas}}[A,\phi,\psi]} \tag{3.8}$$

non-singular is to "Wick rotate" the Dirac operators to the imaginary axis, which is done by multiplying them by a factor of i as in (3.1), (3.2). Rigorously speaking, the integral in the bosonic sector makes sense only if a small "damping" component is added, which is the ϵ term in (3.1); this term is eventually taken to be infinitesimal so it will be dropped from now on.

3.1.2 A dual pair of theories

Having introduced all the elements in DYM, we would like to give a teaser of the main results. In Section 3.2, it will be shown that the integration over the auxiliary fields induces an effective action

$$\mathrm{S}_{\mathrm{IYM}}[A] \approx \mathrm{S}_{\mathrm{YM},g}[A] + \text{corrections}, \tag{3.9}$$

where the definition for YM action is chosen to be

$$\mathrm{S}_{\mathrm{YM},g}[A] \equiv -\frac{1}{g^2}\int d^4x \; \mathrm{tr} F_{\mu\nu}F^{\mu\nu}. \tag{3.10}$$

Here g is the gauge coupling and the field strength $F_{\mu\nu}$ is given by

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{3.11}$$

The lower-case trace "tr" is always over the color space, if not otherwise specified.

By putting back the dynamics of the gluons, i.e., adding another YM action $\mathrm{S}_{\mathrm{YM},g'}$ with a different g' to $\mathrm{S}_{\mathrm{mas}}$, we obtain a *pseudo-super QCD* action. This simply changes the gauge coupling in the induced action (3.9):

$$\mathrm{S}_{\mathrm{YM},g''}[A] + \text{corrections} \longleftarrow \mathrm{S}_{\mathrm{YM},g'}[A] + \mathrm{S}_{\mathrm{mas}}[A,\phi,\psi], \tag{3.12}$$

where $g''^{-1} = g^{-1} + g'^{-1}$. To derive a dual description of IYM, we integrate out the gluons A_μ in the pseudo-super QCD action, which results in an induced dual theory for the auxiliary fields. In analogy with the connection between QCD and some low-energy effective field theories for hadrons, we expect this dual theory to flow to a pseudo-super version of some of these effective theories in the infra-red. We will take a closer look at this in Section 3.4.

As a clarification, the word *pseudo-super* was used not only to emphasize the difference in the mass parameters $M_B > M_F$, but also the fact that (3.12) is not necessarily in the same universality class of the conventional QCD. Indeed, the additional gauge coupling g' is chosen to be **large** such that $\mathrm{S}_{\mathrm{YM},g'}$ is infinitesimal; by contrast, the QCD bare coupling must be **small** due to asymptotic freedom[3]. One should always keep this remark in mind, but throughout this work, we still assume that the effective degrees of freedom at low-energy are alike to the ones in QCD. In the subsequent sections, the gauge coupling g' will be sent to infinity; that is, we will be investigating the dual pair of theories

$$\mathrm{S}_{\mathrm{IYM}}[A] \longleftrightarrow \mathrm{S}_{\mathrm{dual}}[\phi, \psi]$$

induced from the DYM partition function (3.8), which corresponds to the *infinite-coupling* limit of the pseudo-super QCD. The main advantage of studying this limit is that the integral over A_μ can be evaluated without the need of perturbation methods and gauge-fixings because $\mathrm{S}_{\mathrm{YM},g'}[A]$ is now absent. As the result, the induced dual-IYM seems like a good prototype of a dual description of YM. Nevertheless, it is still a challenging task to build a concrete understanding of this dual theory, as we will discuss in Chapter 4.

3.2 Induced Yang-Mills

In this section, the integration over the auxiliary fields in (3.8) is presented. The original calculations for the case where the background field $A_\mu(x) \in \mathrm{Lie}(\mathrm{U}(N_c))$ was already completed in [80], but in what follows, we are going to work out a generalized case where one adds another field B_μ to the action. The reason for this generalization and the application of the result will be elaborated in Section 4.1. At the end of this section, the induced action $\mathrm{S}_{\mathrm{IYM}}$ is recovered by removing B_μ.

As we will explain in Section 4.1, the field B_μ is valued in the Lie algebra of a *dual symmetry group* which acts on the spinor-flavor space. Therefore, unlike the gluon A_μ, it doesn't commute with the matrices Γ^μ (see Section 3.2.1). This feature greatly complicates the derivation of the effective action for the new background field and leads to an intriguing singularity under certain conditions.

[3] The asymptotic freedom in the ultra-violet limit is necessary in QCD because one wishes to keep the effective coupling and/or some observables in the lower-energy regime finite. We do not ask for the same in DYM.

3.2.1 Pauli-Villars regularization

Consider $X_\mu \equiv A_\mu + B_\mu^{\mathrm{T}}$ where A_μ is the usual Lie(U(N_c))−valued color gauge field and B_μ is chosen such that $B_\mu \Gamma^\nu + \Gamma^\nu B_\mu^\dagger = 0\ \forall \mu, \nu$, $B_\mu \Omega + \Omega B_\mu^\dagger = 0\ \forall \mu$. Since A_μ commutes with $(\Gamma^\nu)^{\mathrm{T}}$ and Ω^{T} and it is skew-Hermitian, the new background field X_μ obeys

$$X_\mu^\dagger (\Gamma^\nu)^{\mathrm{T}} + (\Gamma^\nu)^{\mathrm{T}} X_\mu = 0 \quad \forall \mu, \nu; \tag{3.13}$$

$$X_\mu^\dagger \Omega^{\mathrm{T}} + \Omega^{\mathrm{T}} X_\mu = 0 \quad \forall \mu. \tag{3.14}$$

Replacing A_μ in (3.6) and (3.7) by X_μ, we are now dealing with the following Dirac operators:

$$\tilde{D}_B \equiv (\Gamma^\mu)^{\mathrm{T}} (\partial_\mu + X_\mu) + M_B \Omega^{\mathrm{T}}; \tag{3.15}$$

$$\tilde{D}_F \equiv (\Gamma^\mu)^{\mathrm{T}} (\partial_\mu + X_\mu) + M_F \Omega^{\mathrm{T}}. \tag{3.16}$$

Both operators are Hermitian as well because

$$[(\Gamma^\mu)^{\mathrm{T}} X_\mu]^\dagger = -X_\mu^\dagger (\Gamma^\mu)^{\mathrm{T}} = (\Gamma^\mu)^{\mathrm{T}} X_\mu, \tag{3.17}$$

where we have used $(\Gamma^\mu)^\dagger = -\Gamma^\mu$ and (3.13).

As reviewed in Section 3.1, the reality of the spectra allows us to integrate out ϕ, ψ in (3.8) with X_μ as the background field, and the outcome is

$$\mathrm{e}^{-\mathrm{S}_{\mathrm{ind}}[X]} \equiv \int D\bar{\phi} D\phi D\bar{\psi} D\psi \; \mathrm{e}^{-\mathrm{S}_{\mathrm{mas}}[X,\phi,\psi]} = (\text{constant}) \cdot \frac{\mathrm{Det}\tilde{D}_F}{\mathrm{Det}\tilde{D}_B} \quad (\epsilon \to 0). \tag{3.18}$$

Some comments are in order:

1. We included both Dirac fermions and Dirac bosons in DYM to achieve the ratio of determinants in (3.18), because it grants the cancellations of the ultra-violet divergences from $\tilde{D}_B$ and $\tilde{D}_F$. This recipe is in alignment with the famous *Pauli-Villars method* [89].

2. Away from the ultra-violet regime, the spectrum of $\tilde{D}_B$ differs from the one of $\tilde{D}_F$ due to the choice $M_B > M_F$. This gap between the mass parameters ensures that the induced theory described by $\mathrm{S}_{\mathrm{ind}}[X]$ is not going to be empty.

The first step in the derivation of $\mathrm{S}_{\mathrm{ind}}[X]$ is implementing a proper *ultra-violet* regulator in (3.18). Following [80], since $\tilde{D}_F^2$, $\tilde{D}_B^2$ are both positive-definite, we can make use of the heat-kernel regularization:

$$\begin{aligned} \mathrm{S}_{\mathrm{ind}}[X] &= -\frac{1}{2} (\tilde{\mathrm{T}}\mathrm{r} \ln \tilde{D}_F^2 - \tilde{\mathrm{T}}\mathrm{r} \ln \tilde{D}_B^2) \\ &= \frac{1}{2} \lim_{\delta \to 0} \int_\delta^\infty \frac{dt}{t} \, \tilde{\mathrm{T}}\mathrm{r} \left(\mathrm{e}^{-t\tilde{D}_F^2} - \mathrm{e}^{-t\tilde{D}_B^2} \right), \end{aligned} \tag{3.19}$$

where "$\tilde{\mathrm{T}}\mathrm{r}$" also contains the trace over space-time. Taking $\delta \to 0$ amounts to removing the ultra-violet regulator, which should be done after the contributions to $\mathrm{S}_{\mathrm{ind}}[X]$ from fermionic sector and bosonic sector are combined. In this small−δ limit, only the lowest-order terms in t are important, hence we are going to study the *small−t expansion* of (3.19) in the next section.

3.2.2 Small$-t$ expansion

The bosonic and the fermionic contributions are identical except for their mass parameters, so in this section, only the computation in the bosonic sector is presented. For brevity, we will skip the steps similar to the ones in [80] and only summarise the intermediate results.

Firstly, we evaluate the trace over space-time in (3.19) using the equality

$$\int d^4x \; \langle x|\, \mathrm{e}^{-t\tilde{D}_B^2}\, |x\rangle = \int d^4x \int \frac{d^4k}{(2\pi)^4}\; \mathrm{e}^{ikx}\mathrm{e}^{-t\tilde{D}_B^2}\mathrm{e}^{-ikx}. \tag{3.20}$$

The square of the Dirac operator (3.15) reads

$$\begin{aligned}\tilde{D}_B^2 &= M_B^2 - \delta^{\mu\nu}(\partial_\mu - X_\mu^\dagger)(\partial_\nu + X_\nu)\\ &+ \frac{1}{2}(\Gamma^\mu)^{\mathrm{T}}(\Gamma^\nu)^{\mathrm{T}}[\partial_\mu X_\nu - \partial_\nu X_\mu - X_\mu^\dagger\partial_\nu + X_\nu^\dagger\partial_\mu - X_\mu^\dagger X_\nu + X_\nu^\dagger X_\mu]\\ &+ 2M_B\Omega^{\mathrm{T}}(\Gamma^\mu)^{\mathrm{T}}\Re(X_\mu).\end{aligned} \tag{3.21}$$

To obtain this expression we have used the relations (3.3), (3.4), (3.13), (3.14), and denoted the *real* part of X_μ by $\Re(X_\mu) \equiv \frac{1}{2}(X_\mu + X_\mu^\dagger)$.

Following [80], the conjugation by $\exp(ikx)$ replaces any ∂_μ by $\partial_\mu - ik_\mu$, which turns the $k-$integral in (3.20) into

$$\mathrm{e}^{-tM_B^2}\cdot \int \frac{d^4k}{(2\pi)^4}\; \mathrm{e}^{-tk^2 + tF(k)}; \tag{3.22}$$

$$\begin{aligned}F(k) \equiv &-2M_B\Omega^{\mathrm{T}}(\Gamma^\mu)^{\mathrm{T}}\Re(X_\mu) + \delta^{\mu\nu}(\tilde{\nabla}_\mu\nabla_\nu - ik_\mu\nabla_\nu - ik_\nu\tilde{\nabla}_\mu)\\ &- \frac{1}{2}(\Gamma^\mu)^{\mathrm{T}}(\Gamma^\nu)^{\mathrm{T}}[\partial_\mu X_\nu - \partial_\nu X_\mu - X_\mu^\dagger\partial_\nu + X_\nu^\dagger\partial_\mu - X_\mu^\dagger X_\nu + X_\nu^\dagger X_\mu\\ &\qquad - ik_\mu X_\nu + ik_\nu X_\mu + ik_\nu X_\mu^\dagger - ik_\mu X_\nu^\dagger].\end{aligned} \tag{3.23}$$

Here, $k^2 \equiv \delta^{\mu\nu}k_\mu k_\nu$ and $\nabla_\mu \equiv \partial_\mu + X_\mu$, $\tilde{\nabla}_\mu \equiv \partial_\mu - X_\mu^\dagger$.

The integral (3.22) is computed by the standard Wick's theorem for Gaussian integrals:

$$\begin{aligned}\int \frac{d^4k}{(2\pi)^4} e^{-tk^2} &= \frac{1}{16\pi^2 t^2} =: \mathbf{C},\\ \int \frac{d^4k}{(2\pi)^4} k_\mu k_\nu e^{-tk^2} &= \frac{\mathbf{C}\delta_{\mu\nu}}{2t},\\ \int \frac{d^4k}{(2\pi)^4} k_\mu k_\nu k_\lambda k_\rho e^{-tk^2} &= \frac{\mathbf{C}}{(2t)^2}(\delta_{\mu\nu}\delta_{\lambda\rho} + \delta_{\mu\lambda}\delta_{\nu\rho} + \delta_{\mu\rho}\delta_{\nu\lambda}).\end{aligned}$$

In addition, any integral with *odd*-powers of k_μ in the integrand vanishes. Taking these into account, in order to find the $\mathcal{O}(\mathbf{C}\cdot t)$ terms in the small$-t$ expansion, we only need to evaluate the integrals up to the quadratic order in $F(k)$. To see that, one expands the $k-$integral:

$$\int \frac{d^4k}{(2\pi)^4}\; \mathrm{e}^{-tk^2 + tF(k)} \approx \mathcal{O}(F^0) + \mathcal{O}(F^1) + \mathcal{O}(F^2) + \cdots. \tag{3.24}$$

As shown in (3.23), the highest-order of k in $F(k)^n$ is n itself. Hence, the contributions to $\mathscr{O}(\mathbf{C}\cdot t)$ are either from the $\mathscr{O}(k^0)$ terms in $\mathscr{O}(F^1)$, which can be easily read out, or from the $\mathscr{O}(k^2)$ terms in $\mathscr{O}(F^2)$. As we'll see below, the main outcome is that in the general case where the background field X_μ has a non-vanishing real part $\Re(X_\mu)$, the $\mathscr{O}(\mathbf{C}\cdot t)$ terms are non-vanishing. This observation by itself is intriguing because it prevents us from taking the $\delta \to 0$ limit, and plays a crucial role in the discussion in Section 4.1.

Let us now explain the important ingredients in the derivation of the $\mathscr{O}(\mathbf{C}\cdot t)$ terms. Firstly, the integration of the $\mathscr{O}(k^2)$ terms in $\mathscr{O}(F^2)$ yields

$$\frac{\mathbf{C}t}{4}\cdot\begin{bmatrix} -\delta_{\mu\nu}(\nabla^\mu+\tilde{\nabla}^\mu)(\nabla^\nu+\tilde{\nabla}^\nu) \\ +(\Gamma^\alpha)^{\mathrm{T}}(\Gamma^\beta)^{\mathrm{T}}(\nabla^\mu+\tilde{\nabla}^\mu)(\delta_{\mu\alpha}\Re(X_\beta)-\delta_{\mu\beta}\Re(X_\alpha)) \\ +(\Gamma^\alpha)^{\mathrm{T}}(\Gamma^\beta)^{\mathrm{T}}(\delta_{\alpha\mu}\Re(X_\beta)-\delta_{\beta\mu}\Re(X_\alpha))(\nabla^\mu+\tilde{\nabla}^\mu) \\ +(\Gamma^\alpha)^{\mathrm{T}}(\Gamma^\beta)^{\mathrm{T}}(\Gamma^\gamma)^{\mathrm{T}}(\Gamma^\lambda)^{\mathrm{T}}[-\delta_{\alpha\gamma}\Re(X_\beta)\Re(X_\lambda)+(\gamma\leftrightarrow\lambda) \\ \qquad +(\alpha\leftrightarrow\beta)-(\gamma\leftrightarrow\lambda,\alpha\leftrightarrow\beta)] \end{bmatrix}, \tag{3.25}$$

where we have used $\delta^{\mu\nu}$ (resp., $\delta_{\mu\nu}$) to raise (resp., lower) the indices. The sum of covariant derivatives $\nabla_\mu+\tilde{\nabla}_\mu$ equals to $2(\partial_\mu+\Im(X_\mu))$, where $2\Im(X_\mu)\equiv X_\mu - X_\mu^\dagger$.

Using (3.13), (3.14), one observes that $\Re(X_\mu)$ *anticommutes* with $(\Gamma^\mu)^{\mathrm{T}}$, Ω^{T}, whereas $\Im(X_\mu)$ *commutes* with $(\Gamma^\mu)^{\mathrm{T}}$, Ω^{T}. This proves to be useful for permuting the factors of $(\Gamma^\mu)^{\mathrm{T}}$. More importantly, when we carry out the partial trace over spinor-flavor space (denoted by "Tr") in "$\tilde{\mathrm{Tr}}$" in (3.19), we need the fact that for any I which commutes with all $(\Gamma^\mu)^{\mathrm{T}}$, there is the following identity:

$$\mathrm{Tr}(\Gamma^\alpha)^{\mathrm{T}}(\Gamma^\beta)^{\mathrm{T}}(\Gamma^\gamma)^{\mathrm{T}}(\Gamma^\lambda)^{\mathrm{T}}I = (\delta^{\alpha\beta}\delta^{\gamma\lambda}-\delta^{\alpha\gamma}\delta^{\beta\lambda}+\delta^{\alpha\lambda}\delta^{\beta\gamma})\mathrm{Tr}I. \tag{3.26}$$

After contracting the indices, we write (3.25) as

$$\frac{\mathbf{C}t}{4}\cdot\begin{bmatrix} -(\nabla^\mu+\tilde{\nabla}^\mu)(\nabla_\mu+\tilde{\nabla}_\mu) \\ +2(\Gamma^\mu)^{\mathrm{T}}(\Gamma^\nu)^{\mathrm{T}}[\partial_\mu\Re(X_\nu)-\partial_\nu\Re(X_\mu) \\ \qquad +\Im(X_\mu)\Re(X_\nu)+\Re(X_\nu)\Im(X_\mu) \\ \qquad -\Im(X_\nu)\Re(X_\mu)-\Re(X_\mu)\Im(X_\nu)] \\ +12\Re(X^\mu)\Re(X_\mu) \end{bmatrix}. \tag{3.27}$$

The contribution from the $\mathscr{O}(k^0)$ terms in $\mathscr{O}(F^1)$ is

$$\mathbf{C}t\cdot\begin{bmatrix} -2M_B\Omega^{\mathrm{T}}(\Gamma^\mu)^{\mathrm{T}}\Re(X_\mu)+\tilde{\nabla}^\mu\nabla_\mu \\ -\frac{1}{2}(\Gamma^\mu)^{\mathrm{T}}(\Gamma^\nu)^{\mathrm{T}}(\partial_\mu X_\nu-\partial_\nu X_\mu - X_\mu^\dagger X_\nu + X_\nu^\dagger X_\mu) \end{bmatrix}. \tag{3.28}$$

Finally, after some tedious algebra, the combination of (3.27) and (3.28) gives us the $\mathscr{O}(\mathbf{C}\cdot t)$ term:

$$\mathbf{C}t\cdot\begin{bmatrix} -2M_B\Omega^{\mathrm{T}}(\Gamma^\mu)^{\mathrm{T}}\Re(X_\mu) \\ +\partial^\mu\Re(X_\mu)+\Im(X^\mu)\Re(X_\mu)-\Re(X^\mu)\Im(X_\mu)+2\Re(X^\mu)\Re(X_\mu) \\ -\frac{1}{2}(\Gamma^\mu)^{\mathrm{T}}(\Gamma^\nu)^{\mathrm{T}}(\partial_\mu\Im(X_\nu)-\partial_\nu\Im(X_\mu)+[\Im(X_\mu),\Im(X_\nu)]-[\Re(X_\mu),\Re(X_\nu)]) \end{bmatrix}. \tag{3.29}$$

Taking the partial trace "Tr" of this result, we notice that because the factor in the last line

$$\partial_\mu \Im(X_\nu) - \partial_\nu \Im(X_\mu) + [\Im(X_\mu), \Im(X_\nu)] - [\Re(X_\mu), \Re(X_\nu)] =: Y_{\mu\nu} \tag{3.30}$$

commutes with any $(\Gamma^\mu)^{\mathrm{T}}$, we have

$$\mathrm{Tr}(\Gamma^\mu)^{\mathrm{T}}(\Gamma^\nu)^{\mathrm{T}} Y_{\mu\nu} = -\delta^{\mu\nu}\mathrm{Tr}Y_{\mu\nu}. \tag{3.31}$$

By this relation and the fact that $Y_{\mu\nu}$ is anti-symmetric in $\mu \leftrightarrow \nu$, the partial trace of the last line in (3.29) turns out to be zero. Moreover, the trace of the commutator $[\Im(X^\nu), \Re(X_\nu)]$ also vanishes. However, the rest of the terms generally survive unless $\Re(X_\mu) = 0$. We will revisit this result and examine it in detail in Section 4.1.

3.2.3 Background field in Lie(U(N_c))

Back to our original task of *inducing YM*. Now, instead of X_μ we set the background field back to the color gauge field $A_\mu(x) \in \mathrm{Lie}(\mathrm{U}(N_c))$, which is skew-Hermitian. Consequently, $\Re(A_\mu)$ vanishes, which in turn sends the $\mathcal{O}(\mathbf{C}\cdot t)$ term (3.29) to **zero**. Hence, one moves forward and compute the $\mathcal{O}(\mathbf{C}\cdot t^2)$ terms from the k−integral. We have already concluded the calculations in [80], and the final result for the induced action (3.9) is

$$\begin{aligned} \mathrm{S_{IYM}}[A] &\approx (\text{constant}) - N_f \int_{M_F^2\delta}^{M_B^2\delta} \frac{dt}{48\pi^2 t} \int d^4x \ \mathrm{tr} F_{\mu\nu}F^{\mu\nu} + \mathcal{O}(t) \\ &\xrightarrow{\delta\to 0} (\text{constant}) - \frac{N_f \ln(M_B^2/M_F^2)}{48\pi^2} \int d^4x \ \mathrm{tr} F_{\mu\nu}F^{\mu\nu}. \end{aligned} \tag{3.32}$$

This is indeed a YM action, where the *induced gauge coupling* can be defined according to the convention (3.10)

$$\frac{1}{g^2} \equiv \frac{N_f \ln(M_B^2/M_F^2)}{48\pi^2}. \tag{3.33}$$

Importantly, the reason we chose $M_B > M_F$ was to ensure that $\mathrm{S_{IYM}}$ is positive semi-definite.

It is expected that in (3.32) the correction $\mathcal{O}(t)$ consists of higher-order terms in A_μ. Rigorously speaking, one needs to check if the spacetime integrals of these terms are "non-singular" enough, such that the regulator δ in the heat-kernel integral can be removed without running into any trouble. This analysis is however beyond the scope of this dissertation, and we can only give the following supportive argument.

Consider the large−N_f limit of DYM. By rescaling A_μ to $(N_f)^{-1}A_\mu$, the higher-order terms are suppressed due to the N_f−dependent factors they carry, and we hope that by sending $N_f \to \infty$, the step of taking $\delta \to 0$ becomes legitimate. It is worth pointing out that this observation is compatible with the $N_f \geq N_c$ condition in *Conjecture* 2.1 for BZ; consequently, we will always apply the same condition $N_f \geq N_c$ in the subsequent sections.

We conclude this section with a remark. So far we have been considering A_μ as a fixed background field. When IYM is promoted to a quantum field theory, a functional integration over various configurations of A_μ is introduced. In fact, if the regulator δ in DYM can be safely sent to 0, then the energy of A_μ is unbounded in the ultra-violet limit. In order to make sure IYM, like YM, is also asymptotically free, we should send the ratio M_B/M_F arbitrarily close to the infinity, which corresponds to $g \to 0$ in (3.33).

3.3 Dual theory of IYM: a first look

Having seen how YM action can be induced from the master action $\mathrm{S}_{\mathrm{mas}}$, we would like to explore the dual side of DYM. Let's return to the partition function (3.8) and try to integrate out the gluon A_μ. To that end, one introduces a *supervector*

$$\Psi \equiv \begin{pmatrix} \psi & \phi \end{pmatrix}; \qquad \bar{\Psi} \equiv \begin{pmatrix} \bar{\psi} \\ \bar{\phi} \end{pmatrix} \tag{3.34}$$

to write the *interaction part* of the DYM Lagranigan (3.1)+(3.2) as

$$i\mathrm{STr}\bar{\Psi}(\tilde{\Gamma}^\mu)^{\mathrm{T}} A_\mu \Psi = i\mathrm{tr}[\Psi\tilde{\Gamma}^\mu\bar{\Psi}]A_\mu, \tag{3.35}$$

where $(\tilde{\Gamma}^\mu)^{\mathrm{T}} \equiv \mathbb{1}_2 \otimes (\Gamma^\mu)^{\mathrm{T}}$ and the cyclicity of the supertrace was used for its alternative expression as a color trace. One then identifies $\Psi\tilde{\Gamma}^\mu\bar{\Psi} \equiv J^\mu$ as the *color current*.

Since $A_\mu \in \mathrm{Lie}(\mathrm{U}(N_c))$, the integration of A_μ yields the following *zero-current condition* (ZC):

$$[\Psi\tilde{\Gamma}^\mu\bar{\Psi}]^c_{\ c'} = (J^\mu)^c_{\ c'} \stackrel{!}{=} 0 \qquad \forall \mu, c, c'. \tag{3.36}$$

Substituting this condition back to (3.8), we arrive at the preliminary definition of dual-IYM:

$$\mathbf{Z}_{\mathrm{DYM}} \equiv \mathbf{Z}_{\mathrm{dual}} = \int_{J=0} D\bar{\phi}D\phi D\bar{\psi}D\psi \; \mathrm{e}^{-\mathrm{S}_{\mathrm{free}}[\phi,\psi]}. \tag{3.37}$$

The action $\mathrm{S}_{\mathrm{free}}$ is nothing but $\mathrm{S}_{\mathrm{mas}}$ with the gluon A_μ removed:

$$\mathrm{S}_{\mathrm{free}} = \int d^4x \; i\mathrm{Tr}[\bar{\phi}(\Gamma^\mu)^{\mathrm{T}}\partial_\mu\phi + M_B\bar{\phi}\Omega^{\mathrm{T}}\phi] - i\mathrm{Tr}[\bar{\psi}(\Gamma^\mu)^{\mathrm{T}}\partial_\mu\psi + M_F\bar{\psi}\Omega^{\mathrm{T}}\psi], \tag{3.38}$$

hence it is just a pseudo-super extension of the free Dirac action. In fact, the mystery of dual-IYM is mainly rooted in ZC, whose solution space defines the target space of the auxiliary fields ϕ, ψ.

In the past, the same ZC has already been observed in a purely fermionic system [63], but a complete picture of its solution space is still unrevealed to the best of our knowledge. In Chapter 4, we are going to undertake the derivation of the ZC solutions, which turns out to be a challenging task. Fortunately, to the extent of a qualitative study of (3.37), it seems that a comprehensive understanding of the solution space is not required. In the remaining part of this section, we will demonstrate some enlightening observations on (3.37) and use them to motivate our studies of different aspects of dual-IYM in the subsequent sections.

3.3.1 Qualitative features of dual-IYM

In Section 2.6, we have studied a possible dual description for BZN, which is known as dual-BZN. The effective action for dual-BZN was manufactured in a specific way such that it only consists of color-singlets, and one of the main reasons for doing so was to find a more attainable formalism of the dual theory. In addition to the potential computational advantage, we have more incentives in DYM to build dual-IYM as a system of color-neutral entities.

First of all, the color gauge invariance of $\mathbf{Z}_{\text{DYM}}$ is inherited by dual-IYM. Indeed, as it was pointed out in [63], any color gauge transformation in (3.37) of $S_{\text{free}}[\phi,\psi]$ vanishes because $J=0$. Therefore, intuitively we would like to find a new expression for $\mathbf{Z}_{\text{dual}}$ such that the color invariance becomes manifest. On top of this, most derivations of observables in non-Abelian gauge theories require a gauge fixing, which generally complicates the calculations. If the same observables could be studied in dual-IYM, where the color degrees of freedom are hidden away, it is likely for us to have a computational advantage and even gain some new perspectives.

In Section 3.1.2, we emphasized that dual-IYM is the infinite-coupling limit of the pseudo-super QCD. Now, in reality QCD itself does **NOT** directly describe the hadronic physics. Instead, it is generally believed that in the infra-red regime the suitable degrees of freedom are no longer the gluons and quarks in QCD, but some color-neutral fields such as mesons and baryons. Over a few decades, various theories have been developed to explain the dynamics of these color-singlets, and many of them are interpreted as low-energy effective field theories[4] of QCD. As explained in Chapter 1, we hope a dual theory of YM can take us deeper into the infra-red regime of the gluon dynamics; therefore, it is reasonable to first understand the low-energy physics of dual-IYM. In view of the resemblance of DYM to QCD, we are interested in an effective field theory for some color-neutral degrees of freedom. Heuristically speaking, this theory is expected to be a *pseudo-super* version of some effective field theory for QCD.

Condensates

To navigate ourselves in the vast ocean of the theory space, we compute the derivatives of $\ln\mathbf{Z}_{\text{DYM}}$ with respect to the mass parameters, which give us:

$$\langle \text{Tr}\bar{\phi}\Omega^{\text{T}}\phi\rangle_{\text{dual-IYM}} \propto \frac{1}{\mathbf{Z}_{\text{DYM}}}\frac{\partial \mathbf{Z}_{\text{DYM}}}{\partial M_B} \propto \langle \frac{N_f}{M_B}\text{tr}F_{\mu\nu}F^{\mu\nu}\rangle_{\text{IYM}}; \tag{3.39}$$

$$\langle \text{Tr}\bar{\psi}\Omega^{\text{T}}\psi\rangle_{\text{dual-IYM}} \propto \frac{1}{\mathbf{Z}_{\text{DYM}}}\frac{\partial \mathbf{Z}_{\text{DYM}}}{\partial M_F} \propto \langle \frac{N_f}{M_F}\text{tr}F_{\mu\nu}F^{\mu\nu}\rangle_{\text{IYM}}. \tag{3.40}$$

The brackets stand for *vacuum expectation values* (VEVs). The ones for IYM were derived using (3.32), and the ones for dual-IYM could be directly read out from (3.37). Note that $\propto$ is used here because we omitted some irrelevant constants.

A crucial observation can now be made. On the right hand side of (3.39) and (3.40), the VEV $\langle\text{tr}F_{\mu\nu}F^{\mu\nu}\rangle_{\text{IYM}}$ has been studied extensively in QCD, which is conjectured to be a nonvanishing

[4] A modern text on this is [90].

quantity known as the *gluon condensate* [91–93]. Despite the absence of an analytical proof, there are recent FRG evidences for its existence (see e.g. [94] and references therein). Under the premise that IYM flows to YM, we may assume this VEV to be nonzero in IYM as well.

Moving to the left hand side. The VEV $\langle \mathrm{Tr}\bar{\psi}\Omega^{\mathrm{T}}\psi\rangle_{\text{dual-IYM}}$ (and its boson-partner) and the well-known chiral condensate in QCD [95, 96] are only distinguished by the mass matrix Ω! Similar to the gluon condensate, there is no theoretical proof for the chiral condensate; nevertheless, its existence is well-accepted (see e.g. [97] and references therein). Motivated by this, we *presume* the VEVs in dual-IYM are nonvanishing and refer to them as the *boson (resp., fermion) condensates*. We also point out there is a theoretical analysis on a *gauged Thirring model in* $d = 3 + 1$ sharing the same current-current interaction as dual-IYM, which also supports the existence of a chiral condensate [98].

Two implications follow. Firstly, the chiral condensate in QCD breaks the (approximate) chiral symmetry down to a smaller group, and this *spontaneous symmetry breaking* (SSB) phenomenon is the foundation of one of the popular low-energy descriptions of QCD, the chiral perturbation theory (ChPT) [99]. Motivated by this, the effective field theory for dual-IYM should be assembled in the same way as how ChPT was developed, i.e., based on SSB. In the following, this is called the *chiral-type* effective theory, which will be covered in Section 3.4.

On top of this, the correspondence (3.39), (3.40) also shed light on the famous YM mass gap problem. To this day, there is no satisfying analytical derivation for the mass gap, which amounts to proving there are no massless excited states in YM. The lightest ones are known as glueballs [100]. By studying dual-IYM, we hope to provide a fresh point of view on this problem. In particular, it was argued that the value of glueball mass can be explained by the gluon condensate [45]. Therefore, if we could compute the VEVs in dual-IYM and show that either the boson condensate or the fermion condensate is formed, then the corresponding gluon condensate would support the mass gap conjecture.

As a side remark, by proving either one of the VEVs in dual-IYM is nonzero, the other VEV is automatically nonzero due to the connections between (3.39) and (3.40).

3.4 Dual Symmetries and effective field theory

In any quantum field theory, the symmetry groups always play a prominent role, if not the most important one. It is the same for dual-IYM, so the first step towards a better knowledge of this model is to understand what kinds of symmetry groups it has and their properties.

In this section, we are going to study several symmetry groups in dual-IYM. Different types of symmetries were defined and examined in Section 3.4.1 - Section 3.4.3. These symmetries are extremely useful; for example, they can be applied in the study of the solution space of ZC, which will be covered in Section 4.2. We conclude this section with another important application: a construction of a chiral-type effective theory for dual-IYM.

3.4.1 Definitions for symmetry groups

First and for most, the *dual symmetry groups* of interest to us are the ones acting on the spinor-flavor space, because according to the blueprint for dual-IYM depicted in Section 3.3, all the variables should eventually be combed into color-singlets. If not otherwise specified, we will only consider the *global* symmetry groups.

A fundamental feature of DYM is there are Dirac bosons and Dirac fermions, so apart from the usual *bosonic symmetries* which act on the *boson-boson* sector and the *fermion-fermion* sector, it is natural to expect some *fermionic symmetries* which map bosons to fermions and vice versa. In this section, only the bosonic symmetries are covered, and the *fermionic symmetries* or any topics related to *supersymmetries* will be saved for future studies. In particular, most of the discussions will be around symmetries in the *boson-boson* sector because as we will explain, they appear again in the *fermion-fermion* sector. A dual symmetry group in the *boson-boson* sector is a subgroup of $\mathrm{GL}(4N_f;\mathbb{C})$.

In a field theory, an *internal* group action only acts on the target space of the field. In this case, any $k \in \mathrm{GL}(4N_f;\mathbb{C})$ transforms the bosons by sending $\phi(x) \mapsto \phi(x)k$, $\bar{\phi}(x) \mapsto k^\dagger \bar{\phi}(x)$. Substituting this into the *boson-boson* sector of ZC, which is

$$\phi\Gamma^\mu\bar{\phi} = 0 \qquad \forall\mu, \tag{3.41}$$

it is straightforward to check that if

$$k\Gamma^\mu k^\dagger = \Lambda^\mu_{\ \nu}\Gamma^\nu \text{ for some } \Lambda^\mu_{\ \nu} \in \mathrm{GL}(4;\mathbb{R}) \qquad \forall\mu, \tag{3.42}$$

k is a symmetry transformation on the solution space of ZC. We denote the group defined in (3.42) by the symbol K. One can easily check that K possesses the group associativity and the invertibility as long as the matrix Λ is invertible as specified in the definition.

Knowing K will greatly improve our knowledge of the solution space of ZC, because the latter may be interpreted as a space of $K-$orbits. See Section 4.2. Moreover, K partially defines a symmetry of the functional measure, which in turn shapes a quantum symmetry of the theory. Note that the topic of symmetries of a functional measure is a subtle one because of *quantum anomalies*, which cannot be fully captured by a group definition such as (3.42). We will elaborate on this in Section 3.4.4.

Let's turn our attention to the free action (3.38). In the massless limit, $M_B = M_F = 0$, and S_{free} is nothing but a free action with massless Dirac bosons and Dirac fermions. It is well-known that the massless QCD action has a global symmetry $\mathrm{U}(N_f)_L \times \mathrm{U}(N_f)_R$ [90], but in order to build a dual theory, we would like to reveal additional symmetry transformations to the greatest extent. In the presence of the partial derivatives in S_{free}, it is more appropriate to consider an *external* group action $\phi(x) \mapsto \phi(k^{-1}\cdot x)k$, $\bar{\phi}(x) \mapsto k^\dagger\bar{\phi}(k^{-1}\cdot x)$, i.e., the domain of the field is transformed as well. Taking this into consideration, we propose again the same definition (3.42) for the symmetry group of S_{free} in the massless limit. As we are going to discuss in Section 3.4.2.1, by choosing the representation of K for $k^{-1} \cdot x$ properly, the factor $\Lambda^{\mu}_{\ \nu}$ can be countered such that the action is invariant. From now on, K is known as the *massless (dual) symmetry group*.

In Section 3.3, it was argued that the effective field theory of dual-IYM should be a generalization of ChPT. Briefly speaking, ChPT is defined by the SSB of its *massless* symmetry group due to the chiral condensate. This condensate is only invariant under a *massive* symmetry group, which preserves the QCD action in the presence of the non-vanishing Dirac mass. By the same strategy, in dual-IYM, we will first commit ourselves to the study of K, and then turn on the mass parameters M_B, M_F. Apart from (3.42), to preserve S_{free} with masses, the new *massive (dual) symmetry group* elements must also obey

$$k\Omega k^{\dagger} = \Omega. \tag{3.43}$$

This group is named H, which is a subgroup of K. By definition, the properties of H heavily depends on the choice for Ω, which is not unique as pointed out in Section 3.1. In Section 3.4.4 we are going to use the knowledge of K and H to construct the effective field theory, and during the process a reasonable decision for Ω will be made.

3.4.2 Massless symmetry group K

What kinds of elements does K contain? To the best of our knowledge, the definition (3.42) cannot be found in the literature, so it is worth devoting a substantial part of this section to the analysis of K. We will first present a derivation for Lie(K), followed by a discussion of the exponential map of this Lie algebra[5], and then conclude this section with some *exceptional* elements of K which are not covered by the exponential map.

Lie algebra of K We follow the standard prescription to define Lie(K). Any $X \in$ Lie(K) generates a one-parameter subgroup

$$k_X(t) \equiv \{\exp(tX) | t \in \mathbb{R}\}, \tag{3.44}$$

which represents a curve passing the identity element $\mathbb{1}_K \equiv k_X(t=0)$. The group definition (3.42) poses the following condition

$$k_X(t)\Gamma^{\mu} k_X(t)^{\dagger} = \Lambda^{\mu}_{\ \nu}(t)\Gamma^{\nu}, \tag{3.45}$$

and by linearising this equation around $t = 0$, we find that Lie(K) consists of matrices X which are subject to

$$X\Gamma^{\mu} + \Gamma^{\mu} X^{\dagger} = \dot{\Lambda}^{\mu}_{\ \nu}(0)\Gamma^{\nu} \qquad \forall \mu. \tag{3.46}$$

Note that the time derivative $\dot{\Lambda}^{\mu}_{\ \nu}(0)$ is not necessarily an invertible matrix even though $\Lambda^{\mu}_{\ \nu}$ is.

[5]More precisely, the Lie algebra of the maximal subgroup of K which is a Lie group.

In order to solve (3.46), we employ the following unique decomposition for any $X \in \mathrm{Mat}(4N_f;\mathbb{C})$:

$$X \equiv \mathbb{1}_4 \otimes X_{\mathbb{1}} + \gamma^\lambda \otimes X_\lambda + \gamma^\alpha\gamma^\beta \otimes X_{\alpha\beta}\ (\alpha < \beta) + \gamma^\sigma\gamma_5 \otimes X_{\sigma 5} + \gamma_5 \otimes X_5, \tag{3.47}$$

where each factor $X_\bullet$ is a matrix in the flavor space. In other words, the $(4N_f)^2$ components of X in the linear space $\mathrm{Mat}(4N_f;\mathbb{C})$ is divided to 16 disjoint linearly-independent subsets, labelled by the basis elements in the spinor space built from products of $\gamma-$matrices. This decomposition is useful because one can split X to its Γ_5-*even (resp.,* Γ_5-*odd)* part, denoted by X_E (resp., X_O) in the following way:

$$X_E \equiv \mathbb{1}_4 \otimes X_{\mathbb{1}} + \gamma^\alpha\gamma^\beta \otimes X_{\alpha\beta}\ (\alpha < \beta) + \gamma_5 \otimes X_5; \tag{3.48}$$

$$X_O \equiv \gamma^\lambda \otimes X_\lambda + \gamma^\sigma\gamma_5 \otimes X_{\sigma 5}. \tag{3.49}$$

By construction, X_E (resp., X_O) consists of the components of X which commute (resp., anticommute) with Γ_5. This Γ_5-*parity* divides (3.46) to the following equations,

$$X_E\Gamma^\mu + \Gamma^\mu X_E^\dagger = \dot{\Lambda}^\mu{}_\nu(0)\Gamma^\nu \qquad \forall\mu; \tag{3.50}$$

$$X_O\Gamma^\mu + \Gamma^\mu X_O^\dagger = 0 \qquad \forall\mu. \tag{3.51}$$

Lemma 3.1. $X_O = 0$.

Proof. Plugging (3.49) into (3.51), we obtain

$$\gamma^\lambda\gamma^\mu \otimes X_\lambda + \gamma^\sigma\gamma_5\gamma^\mu \otimes X_{\sigma 5} - \gamma^\mu\gamma^\lambda \otimes X_\lambda^\dagger - \gamma^\mu\gamma_5\gamma^\sigma \otimes X_{\sigma 5}^\dagger = 0 \qquad \forall\mu. \tag{3.52}$$

The linear-independence among the components guarantees that it suffices to analyse the sector for an arbitrary $\lambda-$component and the one for an arbitrary $\sigma-$component separately. For the former, we have to solve

$$\gamma^\lambda\gamma^\mu \otimes Y - \gamma^\mu\gamma^\lambda \otimes Y^\dagger = 0 \qquad \forall\mu, \tag{3.53}$$

where we used Y to denote X_λ for the fixed λ to avoid confusion. It follows that

$$\begin{cases} Y - Y^\dagger = 0, & \text{for } \mu = \lambda. \\ Y + Y^\dagger = 0, & \text{for } \mu \neq \lambda. \end{cases} \tag{3.54}$$

Since (3.53) holds for all μ, we know $Y \equiv X_\lambda \overset{!}{=} 0$ for the fixed λ. This applies to any λ; that is, $X_\lambda = 0\ \forall\lambda$. Similarly for the latter, it is necessary that $X_{\sigma 5} = 0\ \forall\sigma$ due to

$$\begin{cases} \gamma^\sigma\gamma_5\gamma^\mu = \gamma^\mu\gamma_5\gamma^\sigma, & \text{for } \mu = \sigma. \\ \gamma^\sigma\gamma_5\gamma^\mu = -\gamma^\mu\gamma_5\gamma^\sigma, & \text{for } \mu \neq \sigma. \end{cases} \tag{3.55}$$

Combining the results, we conclude that the solution to (3.51) is $X_O = 0$. ■

Lemma 3.2. *$X_E = \mathbb{1}_4 \otimes X_{\mathbb{1}} + r_{\alpha\beta}\gamma^\alpha\gamma^\beta \otimes \mathbb{1}_{N_f}$ $(\alpha < \beta) + \gamma_5 \otimes X_5$ where $X_{\mathbb{1}} + X_{\mathbb{1}}^\dagger = r\mathbb{1}_{N_f}$ for some $r \in \mathbb{R}$, $r_{\alpha\beta} \in \mathbb{R}$ and $X_5 = X_5^\dagger$.*

Proof. Following the same chain of reasoning in Lemma 3.1, we look at different components separately. If we set $X_E = \mathbb{1}_4 \otimes X_{\mathbb{1}}$ in (3.50), the only solutions are the $X_{\mathbb{1}}$ $'s$ such that $X_{\mathbb{1}} + X_{\mathbb{1}}^\dagger = r\mathbb{1}_{N_f}$ for some $r \in \mathbb{R}$. Next, the component $\gamma_5 \otimes X_5$ must obey

(3.56) $$\gamma_5\gamma^\mu \otimes X_5 + \gamma^\mu\gamma_5 \otimes X_5^\dagger = 0 \qquad \forall\mu$$

because no $\gamma^\mu\gamma_5$ terms can appear on the right hand side of (3.50). Consequently, $X_5 - X_5^\dagger \stackrel{!}{=} 0$.

Finally, for the component with arbitrarily chosen $\alpha < \beta$, $\gamma^\alpha\gamma^\beta \otimes Y'$ is subject to

(3.57) $$\gamma^\alpha\gamma^\beta\gamma^\mu \otimes Y' + \gamma^\mu\gamma^\beta\gamma^\alpha \otimes Y'^\dagger = (\dot{\Lambda}^\mu{}_\nu(0)\gamma^\nu) \otimes \mathbb{1}_{N_f}.$$

We observe that

(3.58) $$\begin{cases} Y' + Y'^\dagger = r'\mathbb{1}_{N_f} \text{ for some } r' \in \mathbb{R}, & \text{for } \mu \in \{\alpha,\beta\}; \\ Y' - Y'^\dagger = 0, & \text{for } \mu \notin \{\alpha,\beta\}, \end{cases}$$

which shows $X_{\alpha\beta} = r_{\alpha\beta}\mathbb{1}_{N_f}$ $\forall\alpha < \beta$, where $r_{\alpha\beta} \in \mathbb{R}$. ■

To clarify the structure of Lie(K), we split $X_{\mathbb{1}} \equiv \Re(X_{\mathbb{1}}) + \Im(X_{\mathbb{1}})$. The Hermitian part $\Re(X_{\mathbb{1}})$ is proportional to $\mathbb{1}_{N_f}$ according to Lemma 3.2. As summarized in Theorem 3.1, the matrix $\Re(X_{\mathbb{1}})$ represents an overall scaling, and the skew-Hermitian $\Im(X_{\mathbb{1}})$ becomes an element of Lie(U(N_f)).

Theorem 3.1. *For any N_f, the solution space to (3.46) is a real vector subspace of $Mat(4N_f;\mathbb{C})$ defined by the following direct sum*

(3.59) $$Lie(K) \equiv \mathbb{R}\cdot\mathbb{1}_{4N_f} \oplus (Cl_2(\mathbb{R}^4) \otimes \mathbb{1}_{N_f}) \oplus (\mathbb{1}_4 \otimes Lie(U(N_f))) \oplus (\gamma_5 \otimes iLie(U(N_f)))$$

where $Cl_2(\mathbb{R}^4)$ is the Lie algebra of the spin group $Spin(4)$.

3.4.2.1 Exponential map

With the complete knowledge of the Lie algebra, we can directly work out the group elements in the *identity component* of K, denoted by K_0. By definition, K_0 is the maximal connected subgroup of K which contains the unity $\equiv \mathbb{1}_{4N_f}$, and it is known that any element in K_0 is a product of some exponential maps of Lie(K). Since the first two components in (3.59) are in the center of Lie(K), we are entitled to factorize any $k \in K_0$ to

(3.60) $$k = r^+ \cdot \tilde{u}_s \cdot e^{\mathbb{1}_4\otimes a + \gamma_5\otimes b}$$

where the dot $\cdot$ is the matrix multiplication. The factors $r^+ \in \mathbb{R}^+$, $\tilde{u}_s \in \text{Spin}(4) \otimes \mathbb{1}_{N_f}$, are known as the *scaling* and *spin rotation* respectively, both of which act trivially on the flavor space. The interesting physics is mainly encoded in $\exp(\mathbb{1}_4 \otimes a + \gamma_5 \otimes b)$ with $a = -a^\dagger$, $b = b^\dagger$, which will be referred to as the *mixing part*, because in general an element in this part is indecomposable, i.e., it cannot be written as a Kronecker product of a spinor matrix and a flavor matrix.

The group K_0 defined in (3.60) is closely related to some well-known symmetry groups in the conventional Dirac field theory for massless fermions. For instance, the spin group Spin(4) is an external symmetry group for S_{eff} when we use the rotation group SO(4) to represent its action on space-time:

$$\tilde{u}_s \Gamma^\mu \tilde{u}_s^\dagger = \Lambda^\mu{}_\nu \Gamma^\nu \qquad \text{for some } \Lambda^\mu{}_\nu \in \mathrm{SO}(4);$$
$$\phi(x) \mapsto \phi'(x) := \tilde{u}_s \phi(x'),\ x'^\mu = \Lambda^\mu{}_\nu x^\nu.$$

Here, the rotation matrix $\Lambda^\mu{}_\nu$ represents $\tilde{u}_s^{-1} = \tilde{u}_s^\dagger$.

More importantly, the mixing part $\exp(\mathbb{1}_4 \otimes a + \gamma_5 \otimes b)$ forms a subgroup which resembles the global symmetry group $\mathrm{U}(N_f)_L \times \mathrm{U}(N_f)_R$ in QCD. Following the conventional language, we say a generates the *vector* symmetry and b generates the *axial* symmetry. The difference is that in QCD, another skew-Hermitian matrix a' takes place of the Hermitian matrix b. To show that the elements $\exp(\mathbb{1}_4 \otimes a + \gamma_5 \otimes a')$ constitute $\mathrm{U}(N_f)_L \times \mathrm{U}(N_f)_R$, one makes use of the spinor-space *projection operators*

(3.61) $$P_R \equiv \frac{1}{2}(\mathbb{1}_4 + \gamma_5); \qquad P_L \equiv \frac{1}{2}(\mathbb{1}_4 - \gamma_5)$$

to write

(3.62) $$\mathrm{e}^{\mathbb{1}_4 \otimes a + \gamma_5 \otimes a'} = P_R \otimes \mathrm{e}^{a+a'} + P_L \otimes \mathrm{e}^{a-a'}.$$

Let the set of a be spanned by $\{V_k\}$ and the one of a' be spanned by $\{A_k\}$, where $\{V_k\}$ and $\{A_k\}$ are two sets of basis elements of Lie(U(N_f)) associated with the same structure constants, then the $2N_f^2$ vector space can be divided to two subspaces with basis elements

(3.63) $$R_k \equiv \frac{1}{2}(V_k + A_k), \quad L_k \equiv \frac{1}{2}(V_k - A_k); \qquad k = 1, 2, \ldots, N_f^2.$$

It follows that $\{R_k\}$ (resp., $\{L_k\}$) generate Lie($\mathrm{U}(N_f)_R$) (resp., Lie($\mathrm{U}(N_f)_L$)), and these two Lie algebras commute with each other.

Back to K_0, we also have a similar "splitting":

(3.64) $$\mathrm{e}^{\mathbb{1}_4 \otimes a + \gamma_5 \otimes b} = P_R \otimes \mathrm{e}^{a+b} + P_L \otimes \mathrm{e}^{a-b}.$$

However, the same linear combinations (3.63) won't give us two mutually commuting Lie algebras because b is Hermitian. Instead, there is a more straightforward approach to the mixing part. Starting with the following observation:

Lemma 3.3. *Denote the mixing-part subgroup of K_0 by $\mathfrak{N}$. There is a Lie algebra isomorphism between Lie($\mathfrak{N}$) and Lie($GL(N_f;\mathbb{C})$), the real Lie algebra of $N_f \times N_f$ complex matrices.*

Proof. Let $\{T_l\}$ be a set of basis-elements for $\mathrm{Lie}(\mathrm{U}(N_f)) \equiv \mathfrak{h}$ and $\{t_l \equiv iT_l\}$ be a set of basis-elements for $i\mathrm{Lie}(\mathrm{U}(N_f)) \equiv \mathfrak{p}$. By definition

$$[T_l, T_k] = f_{lk}^m T_m \in \mathfrak{h}; \tag{3.65}$$

$$[t_l, t_k] = -f_{lk}^m T_m \in \mathfrak{h}; \tag{3.66}$$

$$[T_l, t_k] = f_{lk}^m t_m \in \mathfrak{p}. \tag{3.67}$$

Now, introduce $\tilde{T}_l \equiv \mathbb{1}_4 \otimes T_l$ and $\tilde{t}_l \equiv \gamma_5 \otimes t_l$, and define $\tilde{\mathfrak{h}} \equiv \mathrm{span}_{\mathbb{R}}\{\tilde{T}_l\}$ along with $\tilde{\mathfrak{p}} \equiv \mathrm{span}_{\mathbb{R}}\{\tilde{t}_l\}$. It is easy to verify that

$$[\tilde{T}_l, \tilde{T}_k] = f_{lk}^m \tilde{T}_m \in \tilde{\mathfrak{h}}; \tag{3.68}$$

$$[\tilde{t}_l, \tilde{t}_k] = -f_{lk}^m \tilde{T}_m \in \tilde{\mathfrak{h}}; \tag{3.69}$$

$$[\tilde{T}_l, \tilde{t}_k] = f_{lk}^m \tilde{t}_m \in \tilde{\mathfrak{p}}. \tag{3.70}$$

From (3.68)-(3.70) and (3.65)-(3.67), we recognize the bijection $\tilde{T}_l \leftrightarrow T_l$, $\tilde{t}_l \leftrightarrow t_l$, which in turn indicates the isomorphism between $\mathrm{Lie}(\mathrm{GL}(N_f;\mathbb{C})) = \mathfrak{h} \oplus \mathfrak{p}$ and $\mathrm{Lie}(\mathfrak{N}) = \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{p}}$. ■

Finally, since the exponential map from $\mathrm{Lie}(\mathrm{GL}(N_f;\mathbb{C}))$ to $\mathrm{GL}(N_f;\mathbb{C})$ is surjective, there is a natural group isomorphism between $\mathfrak{N}$ and $\mathrm{GL}(N_f;\mathbb{C})$. This motivates the following realization of the standard *polar decomposition*:

Corollary 3.1. *Any $A \in \mathfrak{N}$ admits a unique decomposition*

$$A = PU \qquad \textit{with some } P = e^{\gamma_5 \otimes b},\ U = e^{\mathbb{1}_4 \otimes a}. \tag{3.71}$$

This decomposition will be used in the construction of a chiral-type effective field theory in Section 3.4.4.

3.4.2.2 Exceptional elements

The identity component K_0 is going to play the central role in the subsequent sections on the topics of effective field theories and solutions of ZC. Just like the conventional Dirac field theory, there exist other symmetry transformations which are not covered by K_0. For future use, we briefly summarize two kinds of these *exceptional* elements here.

The first kind is a set of *odd*-products of $\Gamma-$matrices:

$$K_{\mathfrak{O}} := \{\Gamma^\mu, \Gamma^\mu\Gamma^\nu\Gamma^\alpha \ (\mu < \nu < \alpha)\}, \tag{3.72}$$

which is obviously not a subgroup. None of the elements in $K_{\mathfrak{O}}$ belongs to K_0 because they *anticommute* with Γ_5, by contrast, according to Theorem 3.1 all elements in K_0 commute with Γ_5. On the contrary, the *even*-products of $\Gamma-$matrices such as $\Gamma^\alpha\Gamma^\beta$ or Γ_5 are already included in $\mathrm{Spin}(4)\otimes\mathbb{1}_{N_f}$. These product-elements can be interpreted as *parity, time-reversal, charge conjugation* when their actions on ϕ are accompanied by some additional transformations.

The second kind of the exceptional elements is more *bizarre*. These elements form a group:

$$K_{\mathfrak{N}} := \{\mathbb{1}_4 \otimes F + \gamma_5 \otimes F_5\} \text{ with } F, F_5 \in \mathrm{Mat}(N_f;\mathbb{C}) \text{ that}$$
$$FF^\dagger - F_5F_5^\dagger = r\mathbb{1}_{N_f} \text{ for some } r \in \mathbb{R}; \qquad FF_5^\dagger - F_5F^\dagger = 0. \tag{3.73}$$

While the associativity of $K_{\mathfrak{N}}$ is coded in the defining relations of F and F_5, we should impose the invertibility of the group elements by embedding the group in $\mathrm{GL}(4N_f;\mathbb{C})$. Importantly, the mixing part $\mathfrak{N}$ is a proper subgroup of $K_{\mathfrak{N}}$ because any element as in (3.64) equals to

$$\mathrm{e}^{\mathbb{1}_4\otimes a+\gamma_5\otimes b} = \frac{1}{2}[\mathbb{1}_4 \otimes (h + (h^{-1})^\dagger) + \gamma_5 \otimes (h - (h^{-1})^\dagger)] \tag{3.74}$$

where $h \equiv \exp(a+b)$. One can verify that for any $h \in \mathrm{GL}(N_f;\mathbb{C})$, the corresponding element (3.74) belongs to $K_{\mathfrak{N}}$. On the contrary, there exist exceptional elements such as $\mathbb{1}_4 \otimes 2\sigma^1 + \gamma_5 \otimes \mathbb{1}_{N_f}$ in $K_{\mathfrak{N}}$, which is not contained in $\mathfrak{N}$.

It is without doubt that there are other symmetry transformations in K, which solve (3.42), but have not been identified in this work. On the one hand, knowing all the elements of K can certainly improve our understanding of the solution space to ZC as we break it down to several $K-$orbits. On the other hand, as far as an effective description of dual-IYM goes, the identity component K_0 should play the most prominent role, because the Goldstone modes come from SSB of *continuous* symmetries. In the subsequent sections, we are going to study the effect of the mass matrix Ω on the symmetry groups, and then apply the results to speculate a chiral-type effective theory for dual-IYM.

3.4.3 Massive symmetry group H

In Section 3.1, the mass matrix $\Omega \equiv \Gamma_5$ was introduced as an exemplary solution to (3.4), which in turn empowers DYM to induce YM. In this section and the next, we are going to inspect various aspects of dual-DYM with $\Omega = \Gamma_5$ and then use them to motivate an effective theory on the basis of SSB.

We start with the following observation about the massive symmetry group H defined by (3.42) and (3.43) where $\Omega = \Gamma_5$.

Lemma 3.4. *The Lie algebra of H is a Lie sub-algebra of Lie*(K) *given by*

$$Lie(H) \equiv (Cl_2(\mathbb{R}^4) \otimes \mathbb{1}_{N_f}) \oplus (\mathbb{1}_4 \otimes Lie(U(N_f))). \tag{3.75}$$

Proof. In addition to (3.46), any element X of Lie(H) is constrained by the linearisation of (3.43), which reads

$$X\Gamma_5 + \Gamma_5 X^\dagger = 0. \tag{3.76}$$

By a direct inspection of different components of X in (3.59), the result follows. ■

An immediate indication is that the identity component of H, denoted by H_0, is nothing but the unitary group $\mathrm{Spin}(4) \otimes \mathrm{U}(N_f)$.

It is worth noting that under the discrete massless symmetry operations in Section 3.4.2.2, the mass term $\bar{\phi}\Gamma_5\phi$ transforms as

$$\bar{\phi}\Gamma_5\phi \mapsto \bar{\phi}(\Gamma^\mu)^\dagger\Gamma_5(\Gamma^\mu)\phi = -\bar{\phi}\Gamma_5\phi \quad \forall\mu; \tag{3.77}$$

$$\bar{\phi}\Gamma_5\phi \mapsto \bar{\phi}(\Gamma_5\Gamma^\mu)^\dagger\Gamma_5(\Gamma_5\Gamma^\mu)\phi = -\bar{\phi}\Gamma_5\phi \quad \forall\mu. \tag{3.78}$$

Conventionally this mass term is understood as a *pseudo-scalar* because of the minus sign.

There exist other choices of Ω which also obey (3.4). For example, in the case of $N_f = 2$, one can also use $\Omega' = \gamma_5 \otimes \sigma_3$. However, it is clear that Ω' is unitarily equivalent to Γ_5, so the effect of using Ω' can be "absorbed" by going to a unitarily equivalent representaion of the $\Gamma-$matrices. Therefore, we are not very interested in testing different choices for Ω, and in the rest of the dissertation Ω will always be Γ_5.

3.4.4 Effective theory from SSB

Finally, it is time to develop a chiral-type effective theory for dual-IYM. We start with a review of some aspects of ChPT which are essential in the construction of the theory.

The global symmetry group of interest in massless QCD is $\mathrm{U}(N_f)_L \times \mathrm{U}(N_f)_R$, which we have encountered. However, this group is reduced in the construction of ChPT because one must account for the quantum anomaly and omit the $\mathrm{U}(1)_V$ for baryon numbers. The relevant anomaly is carried by the axial subgroup $\mathrm{U}(1)_A$ of the classical symmetry group $\mathrm{U}(N_f)_L \times \mathrm{U}(N_f)_R$ as it doesn't preserve the functional measure; rather, in the presence of a background color gauge field in the Dirac operator, a Jacobian factor is generated by $\mathrm{U}(1)_A$ [101][6].

Removing $\mathrm{U}(1)_V$ and $\mathrm{U}(1)_A$ from $\mathrm{U}(N_f)_L \times \mathrm{U}(N_f)_R$, it turns out $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ is the relevant massless symmetry group. The Dirac mass term in QCD is however only invariant under $\mathrm{SU}(N_f)_V$, the diagonal subgroup. Usually the elements in $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ are represented as the matrices introduced in (3.62), in which case $\mathrm{SU}(N_f)_V$ consists of the matrices with $a \equiv a'$.

Today, the existence of chiral condensate is widely-accepted in QCD, which is the VEV of the Dirac mass. Consequently, it induces the following SSB pattern

$$\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \xrightarrow{\mathrm{SSB}} \mathrm{SU}(N_f)_V. \tag{3.79}$$

Conventionally, the ChPT Lagrangian is first formulated in the massless limit, where the SSB is *exact*, and the effective field takes values in the quotient space of the two groups. This field carries the degrees of freedom of pions, which are realized as the Goldstone bosons produced by the SSB. Putting back the quark masses in QCD means the SSB becomes *soft*, and accordingly

[6] In fact, any unitary chiral transformation corresponding to (3.62) with $a = 0$ generates a Jacobian in the measure, but only $\mathrm{U}(1)_A$ has a non-trivial contribution [102].

some associated terms have to be added to the ChPT Lagrangian via a *spurion trick*, which will be discussed below.

As explained in Section 3.3.1 our objective is to transform (3.37) into an equivalent partition function in terms of some color-neutral variables, especially when we are interested in the low-energy physics. In the low-energy regime, ChPT has proven itself to be a successful phenomenological model for mesons and baryons, which are the typical color-singlets formed by quarks. In light of this, it is worthwhile to search for a SSB pattern in dual-IYM and use it to speculate a chiral-type effective theory.

In dual-IYM, the condensates $\langle \mathrm{Tr}\bar{\phi}\Gamma_5^{\mathrm{T}}\phi\rangle_{\text{dual-IYM}}$, $\langle \mathrm{Tr}\bar{\psi}\Gamma_5^{\mathrm{T}}\psi\rangle_{\text{dual-IYM}}$ are only invariant under $H_0 \subset K_0$. In analogous with (3.79), a subject of interest would be

$$K_0 \xrightarrow{\text{SSB}} H_0. \tag{3.80}$$

At the first glance, this is a reasonable proposal. First of all, K_0 is by definition a symmetry for S_{free} in the massless limit, and for the solution space of ZC as well. The main obstacle for the full K_0 group to be a *quantum symmetry* would be the anomaly.

To take a closer look, we go back to DYM. The master theory (3.8) has K_0 as a massless symmetry and H_0 as a massive symmetry. Importantly, one recognizes in (3.60) that K_0 contains a subset

$$\mathfrak{A} := \{\mathrm{e}^{\gamma_5\otimes b} | b \in i\mathrm{Lie}(\mathrm{U}(N_f))\}, \tag{3.81}$$

which is the set of chiral transformations. Any element $k \in \mathfrak{A}$ is Hermitian and sends $\phi \mapsto \phi k$, $\bar{\phi} \mapsto k\bar{\phi}$, so we expect $\mathfrak{A}$ to give rise to a Jacobian factor in the transformed boson measure. Fortunately, the same transformation in the fermion measure yields an inverse Jacobian factor which cancels out the bosonic one. We conclude that $\mathfrak{A}$ is a quantum symmetry[7] in the massless DYM and hence so is K_0, when it acts on the bosons and the fermions in the same way. This corresponds to the subgroup "diagonal" in the superspace, and within the scope of this dissertation, only this subgroup will be discussed.

At this point, it is worth noting that due to the independence between $\bar{\psi}$ and ψ, the following transformation is allowed and it is also a massless symmetry:

$$\psi \mapsto \psi \mathrm{e}^{\gamma_5\otimes a}; \quad \bar{\psi} \mapsto \mathrm{e}^{\gamma_5\otimes a}\bar{\psi}, \qquad a \in \mathrm{Lie}(\mathrm{U}(N_f)). \tag{3.82}$$

This symmetry is however anomalous because there is no counterpart in the bosonic sector. Hence, we will not consider (3.82) when constructing an effective field theory.

Collecting all the observations above, we propose a more concrete SSB pattern

$$\mathfrak{N} \xrightarrow{\text{SSB}} \mathfrak{U}; \qquad \mathfrak{U} := \mathbb{1}_4 \otimes \mathrm{U}(N_f) \tag{3.83}$$

[7] $\mathfrak{A}$ is however not a group.

because the Spin(4) group is unbroken. The scaling part of K_0 also dropped out because DYM should share the same trace anomaly in QCD[8].

We are now ready to assemble a chiral-type effective theory for dual-IYM. In the massless limit $M_B \to 0$, $M_F \to 0$, one expects to find a supersymmetry in DYM. Therefore, rigorously speaking the **full** massless dual symmetry group should be a *supergroup* represented by some supermatrices. According to the relations (3.39), (3.40), we also expect the fermionic symmetries to be broken by the condensates in dual-IYM because $M_B > M_F$ in DYM. Consequently, there might exist some *Goldstone fermions* in the **full** chiral-type effective theory as well.

Nevertheless, we believe a supersymmetry-related discussion is best postponed until a color-neutral formalism of dual-IYM is ready. In this work, we will only examine the SSB of the bosonic symmetries, which are introduced in (3.83). This SSB shall create a set of Goldstone bosons, and the corresponding effective variables are color-neutral fields valued in the coset space $\mathfrak{N}/\mathfrak{U}$. As suggested by Corollary 3.1, the natural choice for these fields is

$$P(x) = \mathrm{e}^{\gamma_5 \otimes b(x)}, \qquad b(x) \in i\,\mathrm{Lie}(\mathrm{U}(N_f)). \tag{3.84}$$

Following the conventions, we shall call $P(x)$ the *principal chiral field* as in ChPT, and refer the effective field theory in terms of $P(x)$ *dual-EFT*.

A natural description of dual-EFT is a quotient-space nonlinear sigma model (NLσM). In the massless-limit, the Lagrangian density must be invariant under $\mathfrak{N}$, which is a non-compact Lie group. In [80], we have reviewed the construction of such density, so it is straightforward to implement the result from there to write

$$\mathscr{L}_{\mathrm{EFT,kin}}[P] \equiv F^2 \mathrm{Tr}\left[(P^{-1}\partial_\mu P)|_{\tilde{\mathfrak{p}}}(P^{-1}\partial^\mu P)|_{\tilde{\mathfrak{p}}}\right], \tag{3.85}$$

where $\tilde{\mathfrak{p}}$ is a subspace of $\mathrm{Lie}(\mathfrak{N}) = \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{p}}$, which was defined in Lemma 3.3. The factor F has mass dimension +1 because P is dimensionless. $\mathscr{L}_{\mathrm{EFT,kin}}$ is well-defined as a function of the coset-valued field P, i.e.,

$$\mathscr{L}_{\mathrm{EFT,kin}}[P] = \mathscr{L}_{\mathrm{EFT,kin}}[Pu]$$

for any $\mathfrak{U}$−valued field u. It is also positive-definite since $\tilde{\mathfrak{p}}$ consists of Hermitian matrices. Furthermore, any global $\mathfrak{N}$− action which sends $P \mapsto kPu$ for constant $k \in \mathfrak{N}$, $u \in \mathfrak{U}$ is a symmetry of $\mathscr{L}_{\mathrm{EFT,kin}}$.

While the definition for $\mathscr{L}_{\mathrm{EFT,kin}}$ in (3.85) is transparent, there exists an alternative expression which comes in handy in calculations. To see that, one simply makes use of the identity

$$\begin{aligned}(P^{-1}\partial_\mu P)|_{\tilde{\mathfrak{p}}} &\equiv \frac{1}{2}\left[(P^{-1}\partial_\mu P) + (P^{-1}\partial_\mu P)^\dagger\right] \\ &= \frac{1}{2}\left[(P^{-1}\partial_\mu P) + (\partial_\mu P)P^{-1}\right]\end{aligned} \tag{3.86}$$

[8]The classical scale invariance in the chiral-limit of QCD is broken by quantum effects.

to write

$$\mathscr{L}_{\text{EFT,kin}} = -\frac{1}{4}F^2\text{Tr}\left[\partial_\mu P^2\partial^\mu P^{-2}\right]. \tag{3.87}$$

Note that P^2 transforms under K (and in particular $\mathfrak{N}$) as

$$P^2 \mapsto kP^2k^\dagger \quad \forall k \in K. \tag{3.88}$$

Our next task is to introduce a *symmetry-breaking* term to the Lagrangian of dual-EFT. Following the conventional *spurion trick* [103], we first postulate a spurion, a matrix field M which has the same transformation property as P^2, i.e., $M \mapsto kMk^\dagger$ for $k \in \mathfrak{N}$. A term such as $\text{Tr}MP^{-2}$ is certainly $\mathfrak{N}-$invariant under the postulate. From now on, M no longer has to be a spurion, but it is just an undetermined matrix. Moreover, we also want to preserve the discrete transformation property (3.77), which sends

$$P^{-2} \mapsto \Gamma^\mu P^{-2}(\Gamma^\mu)^\dagger = P^2 \tag{3.89}$$

where we have used (3.84). Taking this into consideration, we make the following proposal for the symmetry-breaking term

$$\mathscr{L}_{\text{EFT},M} \equiv F^2R\text{Tr}M\left[P^{-2} - P^2\right], \tag{3.90}$$

The trick now kicks in by setting $M \equiv M_B\Gamma_5$ in the boson-boson sector and $M \equiv M_F\Gamma_5$ in the fermion-fermion sector, such that $\mathscr{L}_{\text{EFT},M}$ is no longer $\mathfrak{N}-$invariant but only $\mathfrak{U}-$invariant. The presence of Γ_5 is compatible with (3.89) because alternatively one could use $(\Gamma^\mu)^\dagger M\Gamma^\mu = -M$ to show $\mathscr{L}_{\text{EFT},M}$ is *odd* as in (3.77). Importantly, $\mathscr{L}_{\text{EFT},M}$ is well-defined due to $P = P^\dagger$ and $[u,\Gamma_5] = 0$, $\forall u \in \mathfrak{U}$; however, it is not necessarily positive-definite[9]. To match up the dimension, we introduced another factor of mass dimension +1, R, into $\mathscr{L}_{\text{EFT},M}$.

The combined density $\mathscr{L}_{\text{EFT,kin}} + \mathscr{L}_{\text{EFT},M}$ indeed shares the same symmetry feature as dual-IYM, but certainly there are higher-order terms composed of P, M which is invariant under $\mathfrak{N}$ when M is a spurion. According to the standard Weinberg's power counting scheme (see e.g. [97]), one can assign to each term in the Lagrangian a *chiral-order*, which increases[10] along with the number of derivatives and the power in M. In the low-energy regime, the combined density dominates because it carries the lowest chiral-order:

$$\mathscr{L}_{\text{EFT},c=2} \equiv -\frac{1}{4}F^2\text{Tr}\left[\partial_\mu P^2\partial^\mu P^{-2}\right] + F^2R\text{Tr}M\left[P^{-2} - P^2\right], \tag{3.91}$$

where $c = 2$ denotes the chiral-order.

What do we mean by "low-energy regime"? On the side of IYM, we want to explore the infra-red physics, which is typically defined by $E < \Lambda_{\text{QCD}}$, where the QCD scale $\Lambda_{\text{QCD}} \approx 1$ GeV.

[9] Whether or not it is disastrous depends on the exact definition of the partition function.

[10] Conventionally, each derivative carries a chiral-order +1 whereas each M carries a chiral-order +2.

Accordingly, a *good* dual-EFT should be an effective description in the same energy regime $E < \Lambda_{\text{QCD}}$ and hence we must demand $M_B < \Lambda_{\text{QCD}}$. Now, recall the remark at the end of Section 3.2.3 that it is necessary that $M_B/M_F \gg 1$ to forge IYM to an asymptotically free theory, so it seems only reasonable for us to keep a finite $M_B < \Lambda_{\text{QCD}}$ while sending M_F arbitrarily close to 0.

The last pillar of dual-EFT is its interaction with an external gauge field. In QCD, the low-energy physics are captured by the chiral Ward identities of Green functions, which are time-ordered products of color-neutral composite operators of the light quarks. Typically these are the conserved vector current v^μ and axial-vector current a^μ associated with the massless symmetry group, along with the scalar density s and pseudo-scalar density p. In the common approach, one extracts the chiral Ward identities from a generating functional $\mathbf{Z}[v^\mu, a^\mu, s, p]$, which is by construction invariant under *local* transformations of the massless group. To ensure this local symmetry, one allows the external fields v^μ, a^μ, s, p to transform accordingly like gauge fields, thereby the name.

In dual-IYM, a similar promotion of $\mathfrak{N}$ to a gauge group can be achieved. Indeed, there are vector current and axial current generated by $\mathfrak{N}$, and we can couple both of them to a single Lie($\mathfrak{N}$)−valued external field B_μ, which reads

$$B_\mu(x) = \mathbb{1}_4 \otimes a_\mu(x) + \gamma_5 \otimes b_\mu(x); \qquad a_\mu(x), ib_\mu(x) \in \text{Lie}(\text{U}(N_f)). \tag{3.92}$$

In the presence of the source terms $\text{Tr}\bar{\phi}(\Gamma^\mu)^{\text{T}} B_\mu^{\text{T}} \phi$, $\text{Tr}\bar{\psi}(\Gamma^\mu)^{\text{T}} B_\mu^{\text{T}} \psi$ in (3.37) and taking the massless-limit ($M_B = 0 = M_F$), we have $\mathbf{Z}_{\text{dual},0}[B_\mu]$. By construction, this generating functional does not change when B_μ transforms like a gauge field under $\mathfrak{N}$,

$$B_\mu \mapsto {}^k B_\mu \equiv k B_\mu k^{-1} - k(\partial_\mu k^{-1}) \qquad \forall k \in \mathfrak{N}; \tag{3.93}$$

that is, $\mathbf{Z}_{\text{dual},0}[B_\mu] = \mathbf{Z}_{\text{dual},0}[{}^k B_\mu]$. In addition to B_μ, one can also introduce external scalar and pseudo-scalar fields, which are encapsulated into

$$B(x) = \mathbb{1}_4 \otimes a(x) + \gamma_5 \otimes b(x); \qquad a(x), ib(x) \in \text{Lie}(\text{U}(N_f)). \tag{3.94}$$

With $\text{Tr}\bar{\phi} B^{\text{T}} \phi$, $\text{Tr}\bar{\psi} B^{\text{T}} \psi$ inserted in $\mathbf{Z}_{\text{dual},0}[B_\mu]$ above, we obtain $\mathbf{Z}_{\text{dual}}[B_\mu, B]$, which is invariant under (3.93) along with

$$B \mapsto {}^k B \equiv k B k^\dagger. \tag{3.95}$$

That is, $\mathbf{Z}_{\text{dual}}[B_\mu, B] = \mathbf{Z}_{\text{dual}}[{}^k B_\mu, {}^k B]$. Heuristically, the field B "replaces" the original mass matrix.

Generally speaking, one may use different external fields in the boson-boson sector and the fermion-fermion sector; moreover, one can couple the conserved current with respect to the fermionic symmetries to some Grassmann-valued fields. These interesting aspects of dual-EFT will however only be investigated in the future. Our objective here is to incorporate B_μ and B into $\mathscr{L}_{\text{EFT}}$ by lifting $\mathfrak{N}$ to a "gauge group". This can be done straightforwardly by replacing ∂_μ by

$D_\mu \equiv \partial_\mu - B_\mu$ in (3.85) as reviewed in [80], but it requires some modifications of (3.90). To check the former, we implement a *local* transformation $P(x) \mapsto k(x)P(x)u(x)$ together with (3.93) and find

$$P^{-1}D_\mu P|_{\tilde{\mathfrak{p}}} \mapsto u^{-1}P^{-1}D_\mu P|_{\tilde{\mathfrak{p}}}u. \tag{3.96}$$

For the latter, (3.90) should be replaced by

$$BP^{-2} + R^2B^{-1}P^2 \mapsto kBP^{-2}k^{-1} + (k^\dagger)^{-1}R^2B^{-1}P^2k^\dagger, \tag{3.97}$$

which is **also** invariant under (3.89). The factor R^2 is added to balance the mass dimension.

Based on this observation, we make the following proposal for the *gauged* effective Lagrangian at the lowest chiral-order:

$$\mathscr{L}_{\text{EFT},c=2}[B_\mu, B] \equiv F^2\text{Tr}\left[(P^{-1}D_\mu P)|_{\tilde{\mathfrak{p}}}(P^{-1}D^\mu P)|_{\tilde{\mathfrak{p}}}\right] + F^2R\text{Tr}\left[BP^{-2} + R^2B^{-1}P^2\right]. \tag{3.98}$$

While (3.98) is manifestly gauge-invariant under K and in particular $\mathfrak{N}$, there exists a more convenient expression in terms of P^2. To derive such an expression, one again expands

$$(P^{-1}D_\mu P)|_{\tilde{\mathfrak{p}}} = \frac{1}{2}\left[(P^{-1}\partial_\mu P) - (P\partial_\mu P^{-1}) - P^{-1}B_\mu P - PB_\mu^\dagger P^{-1}\right] \tag{3.99}$$

and plug this into the kinetic term. The following cross term (and its Hermitian conjugate) can be simplified further as

$$\begin{aligned} &\text{Tr}\left[(P^{-1}\partial_\mu P) - (P\partial_\mu P^{-1})\right]P^{-1}B_\mu P \\ &\quad = \text{Tr}\left[(\partial_\mu P)P^{-1} - P^2(\partial_\mu P^{-1})P^{-1}\right]B_\mu \\ &\quad = \text{Tr}\{(\partial_\mu P)P^{-1} - P^2\left[(\partial_\mu P^{-2}) - P^{-1}(\partial_\mu P^{-1})\right]\}B_\mu \\ &\quad = -\text{Tr}P^2(\partial_\mu P^{-2})B_\mu. \end{aligned} \tag{3.100}$$

The final result reads

$$\begin{aligned} \mathscr{L}_{\text{EFT},c=2}[B_\mu, B] \equiv &-\frac{F^2}{4}\text{Tr}\left[\partial_\mu P^2\partial^\mu P^{-2}\right] + F^2R\text{Tr}\left[BP^{-2} + R^2B^{-1}P^2\right] \\ &+ \frac{F^2}{2}\text{Tr}\left[(P^2(\partial^\mu P^{-2})B_\mu - P^{-2}(\partial^\mu P^2)B_\mu^\dagger)\right] \\ &+ \frac{F^2}{4}\text{Tr}\left[B_\mu B^\mu + B_\mu^\dagger(B^\mu)^\dagger + 2P^2(B_\mu)^\dagger P^{-2}B^\mu\right]. \end{aligned} \tag{3.101}$$

This Lagrangian will be brought up again at the end of Section 4.1.

3.4.5 Effective theory: applications

Having spending so much time developing a reasonable effective Lagrangian for dual-IYM, we would like to inspect dual-EFT for some possible physical applications. In ChPT, the gauged Lagrangian can be used to study hadron dynamics such as pion decay and pion-pion scattering.

While it is tempting to dive deeper into dual-EFT to search for analogous dynamics, we must remind ourselves that dual-IYM itself is not a theory for real-world particle physics. More specifically, unlike ChPT, there are no experimental inputs available to us to set values for undetermined parameters such as F, R in dual-EFT.

The main reason to establish dual-EFT is to use it as a guidance for a color-neutral representation of dual-IYM. More specifically, if dual-EFT is shown to be a valid infra-red effective theory for dual-IYM, we then have a good reason to "project" dual-IYM itself as a NLσM in terms of some color-singlets. This challenging task will be the main theme in Chapter 4. We conclude this section by explaining another reason to study dual-EFT: there exist a few interesting physical applications!

YM mass gap

One of the most sought after features of YM is the mass gap. Typically one examines the decaying behaviour of gluon propagators in the infra-red regime, which usually requires sophisticated non-perturbative methods. By introducing a *source* field j^μ into the DYM partition function (3.8) and again integrating out the auxiliary fields, we find the generating functional

$$\begin{aligned}\mathbf{Z}_{\text{DYM}}[j^\mu] &\equiv \int DAD\bar{\phi}D\phi D\bar{\psi}D\psi \ \mathrm{e}^{-\mathrm{S}_{\text{mas}}[A,\phi,\psi]+\mathrm{S}_{\text{int}}[A,j]} \\ &= \int DA \ \mathrm{e}^{-\mathrm{S}_{\text{IYM}}[A]+\mathrm{S}_{\text{int}}[A,j]},\end{aligned} \tag{3.102}$$

where $\mathrm{S}_{\text{int}}[A,j] \equiv i\int d^4x \ \mathrm{tr} j^\mu A_\mu$. The dual expression of the same generating functional is obtained by integrating out the gluon field,

$$\mathbf{Z}_{\text{DYM}}[j^\mu] \equiv \int_{J=j} D\bar{\phi}D\phi D\bar{\psi}D\psi \ \mathrm{e}^{-\mathrm{S}_{\text{free}}[\phi,\psi]}. \tag{3.103}$$

In other words, the source field j^μ "lifts" the conserved current up from the ZC domain. As a side remark, while it is trivially true for the $N_c = 1$ theory that $\mathbf{Z}_{\text{DYM}}[gj^\mu g^{-1}] = \mathbf{Z}_{\text{DYM}}[j^\mu]$ for all $g \in \mathrm{U}(1)$, the same identity generally breaks down for $N_c \geq 2$, the non-Abelian theories [104]. Consequently, j^μ does **not** have to be a conserved current when $N_c \geq 2$.

The correlation functions of A_μ can be calculated by differentiating $\mathbf{Z}_{\text{DYM}}[j^\mu]$ with respect to j^μ; in particular, the gluon propagators are

$$\langle A_\nu(x)A_\rho(y)\rangle_{\text{IYM}} \propto \frac{1}{\mathbf{Z}_{\text{DYM}}[j^\mu]}\frac{\delta}{\delta j^\nu(x)}\frac{\delta}{\delta j^\rho(y)}\mathbf{Z}_{\text{DYM}}[j^\mu]\Big|_{j=0}. \tag{3.104}$$

In YM, the evaluation of (3.104) turns out to be difficult because of the non-Abelian nature of the color group. In dual-IYM, the color group is no longer around, but the same differential operators act on the domain of integration, which is the solution space to $J^\mu = j^\mu \ \forall \mu$.

For the time being, it is unclear to us how to perform such differentiations. Nevertheless, we may be able to examine the existence of a mass gap by studying dual-EFT. As a first attempt, we follow the footsteps of ChPT and introduce a (pseudo-)Goldstone boson field φ as

$$P^2(x) \equiv \exp\left(\frac{\varphi}{F}\right). \tag{3.105}$$

In terms of φ, at the leading-order (3.91) reads

$$\mathscr{L}_{\text{EFT},c=2}[\varphi] \approx -\frac{1}{4}\text{Tr}\left[\partial_\mu\varphi\partial^\mu\varphi\right] - FR\text{Tr}\left[2M\varphi\right] + \mathscr{O}(F^{-1}). \tag{3.106}$$

Importantly, the factor $(P^{-2} - P^2)$ in (3.90) forbids terms that are *even* in φ; in particular, the mass term which is quadratic in φ is missing!

This crucial observation distinguishes dual-EFT from ChPT, because it tells us even though there are masses M_B, M_F in dual-IYM, we cannot find a mass term for the color-neutral field φ in dual-EFT at the Lagrangian level. Nevertheless, it is possible that a mass term can be "generated" through quantum effects, which are captured by the higher-order terms in φ. At this early stage, we can only point out that YM is a quantum field theory with such feature: the gluon field is classically massless, but it acquires a dynamical mass through self-interactions.

Dual description of Wilson loop expectation value

An equally significant property as the mass gap is the area-law of Wilson loop expectation value in YM, which is denoted as $\langle W[\mathscr{C}]\rangle$. A naive way to compute this observable by stipulating a *loop-current* $\hat{j}^\mu$ such that

$$W[\mathscr{C}] \equiv \frac{1}{N_c}\text{tr}\left[\mathscr{P}\exp\left(\oint_{\mathscr{C}} d^\mu x\, A_\mu\right)\right] \stackrel{?}{\equiv} \frac{1}{N_c}\exp\left(\int d^4x\, \text{tr}\hat{j}^\mu A_\mu\right) \tag{3.107}$$

generally fails, because the last expression is not gauge-invariant [104]. Hence, the approach (3.103) seems like a dead end. Perhaps a better way to study $\langle W[\mathscr{C}]\rangle$ is to recall that it is determined by the static potential between a "quark" Q and an "anti-quark" $\bar{Q}$, both of which are infinitely heavy. Schematically, it equals to the following ratio

$$\langle W[\mathscr{C}]\rangle = \frac{\mathbf{Z}_{\text{DYM}}[Q,\bar{Q}]}{\mathbf{Z}_{\text{DYM}}[0]}. \tag{3.108}$$

As for now we stay close to QCD and take Q, $\bar{Q}$ to be static Grassmann-valued sources, and require the following condition:

Assumption 3.1. *In $S_{mas}[Q,\bar{Q}]$, the static fields Q, $\bar{Q}$* ***are not*** *directly coupled to the dynamical fields $\phi,\bar{\phi},\psi,\bar{\psi}$. That is, they only interact with the gluon field A_μ.*

Under this assumption, the addition of Q, $\bar{Q}$ doesn't affect the integration over the dynamical fields. Now, let the rectangular loop $\mathscr{C}$ sits on a $x^0 - x^k$ plane for any $k = 1,2,3$, where its temporal-span and spatial-span are denoted by $\mathbf{T}$, $\mathbf{R}$, respectively. On the side of IYM and in the large$-\mathbf{T}$ limit, the static potential of the $Q-\bar{Q}$ pair is

$$E(\mathbf{R}) = -\lim_{\mathbf{T}\to\infty}\frac{1}{\mathbf{T}}\ln\langle W[\mathscr{C}]\rangle, \tag{3.109}$$

Meson type	Composite fields
Light	$\bar{\phi}\phi$, $\bar{\psi}\psi$, $\bar{\phi}\psi$, $\bar{\psi}\phi$
Heavy	$\bar{Q}\phi$, $\bar{Q}\psi$, $\bar{\phi}Q$, $\bar{\psi}Q$

TABLE 3.1. Available color-neutral bound-states.

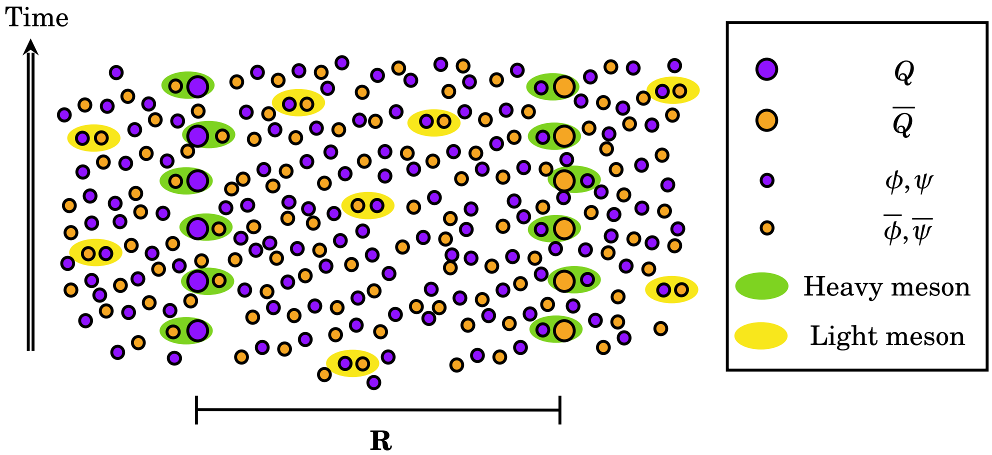


FIGURE 3.1. A simplified $d = 1+1$ illustration of color-neutral states in dual-EFT. The space should be condensely populated by light mesons but we only show a few of them.

and just as in YM, we conjecture that $E(\mathbf{R}) \propto \mathbf{R}$ when $\mathbf{R}$ is large. This is known as the area-law of $\langle W[\mathscr{C}]\rangle$ with a large loop $\mathscr{C}$ [105].

On the side of dual-IYM, the integration over A_μ should also induce interactions between the heavy field and the light field; a direct evaluation of (3.108) seems formidable. Fortunately, at the level of dual-EFT, one may speculate the effects of Q, $\bar{Q}$ as some additional color-neutral degrees of freedom added to $\mathscr{L}_{\text{EFT}}$. There have been proposals categorised as *Heavy Hadron Chiral Perturbation Theories* (HChPT) [106] as extensions of ChPT including heavy quarks, which seem helpful regarding the realization of $\langle W[\mathscr{C}]\rangle$ in dual-EFT. We provide a heuristic argument below.

Imaging we place Q and $\bar{Q}$ in the background of the dynamical fields, and separate the infinitely heavy $Q,\bar{Q}$ by the fixed distance $\mathbf{R}$. In the low-energy regime, not only the dynamical fields form *light mesons*, we should also expect to see *heavy mesons* comprised a static source and a dynamical field. In comparison with ChPT where the light mesons (e.g. pions) are Lorentz-scalar fields, the heavy mesons in HChPT also have *vector-components*. All the color-neutral effective degrees of freedom available to us are summarized in Table 3.1. The color-singlet $\bar{Q}Q$ is excluded since we are probing the behaviour of $\langle W[\mathscr{C}]\rangle$ at large $\mathbf{R}$, so it is unlikely for the far-apart $\bar{Q}$, Q to form a bound-state.

In this scenario, dual-EFT might be described by Figure 3.1, which in turn can be studied using HChPT. The heavy mesons $\bar{Q}\psi$, $\bar{\psi}Q$ are represented by additional color-singlet matrix-valued fields [106], and a direct extension to include $\bar{Q}\phi$, $\bar{\phi}Q$ seems straightforward. Even though we are not going to construct such a theory here, it is worthwhile to think about the physical implications, e.g. how the linear growth relation $E(\mathbf{R}) \propto \mathbf{R}$ can be observed in a system like Figure 3.1. This will be one of the interesting topics of dual-EFT to investigate in the future.

Chapter 4

Color Neutralisation of Dual-IYM

We took a leap of faith to conjecture the model of dual-EFT in Section 3.4. Indeed, the field contents and the Lagrangian of the effective theory are both determined by the proposed SSB pattern (3.83), which in turn was suggested by the relations among condensates (3.39), (3.40) and our knowledge of ChPT. The fundamental difference between ChPT (and its generalizations) and dual-IYM is that the former **is** a phenomenological model for hadronic physics but the latter **is not**. In fact, to consolidate the claim that "dual-EFT is an effective theory for dual-IYM in the infra-red regime", one must first understand dual-IYM itself; in particular, we have to verify the existence of the VEVs.

In this chapter, we are going to examine various aspects of dual-IYM closely. As it was pointed out in Section 3.3.1, the ultimate objective is to construct dual-IYM as a quantum theory for some color-neutral fields (the dual-field). To reach that goal, in Section 4.1 we will first discuss various attempts to color-neutralise dual-IYM and explain the challenges, and then we will present an extensive study of the solution space to ZC in Section 4.2.

4.1 Color neutralisation

4.1.1 Hubbard-Stratonovich decoupling

Our previous attempt to color-neutralise dual-IYM in [80] relies on the well-known Hubbard-Stratonovich (HS) decoupling method. Schematically, the procedure can be summarized below

$$\begin{aligned}\mathbf{Z}_{\mathrm{DYM}} &= \lim_{\sigma\to 0}\int D\bar{\Psi}D\Psi\ \mathrm{e}^{-\mathrm{S}_{\mathrm{free}}[\bar{\Psi},\Psi]-\frac{1}{\sigma^2}J\cdot J} \\ &\propto \lim_{\sigma\to 0}\int D\bar{\Psi}D\Psi\ \mathrm{e}^{-\mathrm{S}_{\mathrm{free}}[\bar{\Psi},\Psi]-\frac{1}{\sigma^2}\tilde{J}\cdot\tilde{J}} \\ &\propto \int D\bar{\Psi}D\Psi D\tilde{B}\ \mathrm{e}^{-\mathrm{S}_{\mathrm{free}}[\bar{\Psi},\Psi]+i\tilde{B}\cdot\tilde{J}} \\ &\propto \int D\tilde{B}\ \mathrm{e}^{-\mathrm{S}_{\mathrm{old}}[\tilde{B}]}.\end{aligned} \tag{4.1}$$

The Fierz rearrangement was used to "exchange" the color current J for the spinor-flavor current $\tilde{J}$ in the second line, and the quartic term $\tilde{J}\cdot\tilde{J}$ was then *decoupled* by the Hubbard-Stratonovich field $\tilde{B}$ in the third line. Finally, we integrate out the original fields $\bar{\Psi}$, Ψ to arrive at the expression in terms of $\tilde{B}$, which is a color-neutral variable by construction. In this context, $\tilde{B}$ is the dual-field.

This decoupling method is often used to study the low-energy physics of QCD, see [107] and the references therein. Formally, the integration over the gluon field in QCD yields an effective action of the color currents j^μ, which is written as a power series of j^μ. Within the approximation where only the quadratic term $j\cdot j$ is kept, which coincides with (4.1), one proceeds by rearranging $j\cdot j$ to several *channels* which transform in different representations of $\mathrm{SU}(N_c)\times\mathrm{U}(N_f)\times\mathrm{SO}(4)$[1]. The channel of particular interest is the so-called *quark-antiquark interaction*[2], which has the following form [107]

$$\bar{q}\Lambda_{\alpha A}q\bar{q}\Lambda^{\alpha A}q;\qquad \Lambda_{\alpha A}\equiv \mathbb{1}_{N_c}\otimes S_\alpha\otimes T_A, \tag{4.2}$$

where $\{T_A\}$ span Lie(U(N_f)), $S_\alpha\in\{\mathbb{1}_4, i\gamma_\mu, i\gamma_\mu\gamma_5, i\gamma_5\}$. The effective *meson* field η is exactly the HS field introduced to decouple the interaction (4.2); importantly, η has components of all Lorentz-types except for the tensor component $\propto\gamma_\mu\gamma_\nu$, as indicated by the set of the spinor matrices S_α. This feature was also discovered in [80].

In principle, the old approach (4.1) can be carried out in exact steps[3]; however, it is unlikely that the resulting $\mathrm{S}_{\mathrm{old}}$ describes a system in strong-weak duality to IYM. A quick way to see this is to recall the fact that in DYM (3.8), the integration of A_μ is by itself a "Fourier transformation", and the HS decoupling provides a second one. Consequently, we do not expect the theory of $\tilde{B}$ to be a desired dual theory of IYM.

[1] SO(4) is replaced by SO(1,3) in Minkowski spacetime.

[2] The other one is the *diquark* channel, which is suppressed in the large$-N_c$ limit.

[3] There are some non-trivial aspects of the target space of $\tilde{B}$, which was briefly discussed in [80].

4.1.2 Non-Abelian bosonization

Perhaps a more promising strategy is to color-neutralise dual-IYM by the superbosonization (SuB) formula (2.89) described in Section 2.8.4. The process of SuB is essentially a change of variables and there is no Fourier transformation involved, so the original duality is not spoiled by this manipulation on dual-IYM.

Super-meson field

Just like in dual-BZN, the first step of SuB is to identify a color-neutral matrix field in dual-IYM. The color group in DYM is $\mathrm{U}(N_c)$, so the only color-singlet composite operator available to us is the so-called *super-meson* matrix:

$$\tilde{Q}(x) \equiv \bar{\Psi}(x)\Psi(x). \tag{4.3}$$

By construction, $\tilde{Q}(x)$ is indeed color-neutral because a color gauge transformation $g(x) \in \mathrm{U}(N_c)$ maps

$$\bar{\Psi}(x)\Psi(x) \mapsto \bar{\Psi}(x)gg^{\dagger}\Psi(x) = \bar{\Psi}(x)\Psi(x).$$

Note that the general action of $\mathrm{U}(N_c)$ on the fermions $\bar{\psi} \mapsto \bar{\psi}g$, $\psi \mapsto \tilde{g}\psi$ contains the specific transformations chosen here, which correspond to $\tilde{g} \equiv g^{\dagger}$.

Our first question is which relation between N_f and N_c favors the SuB formula reviewed in Section 2.8.4. Let us denote the boson-boson block of $\tilde{Q}$ as the *meson* matrix $Q \equiv \bar{\phi}\phi$. As a $4N_f \times 4N_f$ matrix, the rank of Q is at most N_c; hence, a naive SuB of (3.37) as a functional integral over $\tilde{Q}$ is likely to fail due to the condition $N_f \geq N_c$ chosen in Section 3.2.3. To even the odds, from now on we set $N_f \equiv N_c \equiv N$ to *minimize* the rank-deficiency in Q while preserving the universality of IYM.

Witten's bosonization

The next question is if it is at least possible to rewrite the action in (3.37) using only $\tilde{Q}$. The answer is no, if one attempts to do this **directly**. An apparent difficulty can be seen in S_{free} in (3.38) where the partial derivatives prevent us from manipulating the variables to build composite operators such as $\bar{\phi}\phi$, $\bar{\psi}\psi$. In particular, a gauge-averaging trick like the one used for BZN doesn't work here. Our proposal to circumvent this obstacle is by employing Witten's method of non-Abelian bosonization [108].

The essence of Witten's non-Abelian bosonization is to make an "educated guess" of an action for some boson fields which share the same symmetries with the original fermionic theory. More precisely, the *classical* symmetry of the bosonic theory coincides with the *quantum* symmetry of the fermionic one. In $d = 1+1$, the original paper [108] proves the equivalence between a free field theory of N massless Majorana fermions and a system of $\mathrm{O}(N)-$valued bosons with a Wess-Zumino-Witten (WZW) action. Before long, the same result was demonstrated in the path-integral framework for a different symmetry group [109–111], and some applications of this technique on two-dimensional QCD were put forward [112–115].

While non-Abelian bosonization is well-established and successful in two dimensions, the same powerful method cannot be generalized to four dimensions. Nevertheless, it is still worthwhile applying the same strategy in dual-IYM. Following the path-integral approach [110], we have a three-step plan:

1. Identify all the conserved currents in dual-IYM and couple them to appropriate external background fields valued in the spinor-flavor space.

2. Integrate out the original auxiliary degrees of freedom to obtain an effective action for the external fields.

3. Speculate a “color-neutralised” partition function where some color-singlets are coupled to the same external field, such that the same effective action is produced once the color-singlets are integrated out.

The bosonization equivalence was first verified for massless fermionic theories, and the mass terms were accounted for by adding corresponding terms in the bosonized theory [108, 109]. This procedure also aligns with the spurion trick implemented in Section 3.4.4, so we are motivated to pursue the same route. Since we haven’t acquired enough knowledge of ZC for integrating over its solution space, in the first step we go back to DYM and send $M_B \to 0$, $M_F \to 0$, such that $\mathfrak{N}$ is a good dual symmetry in both the boson-boson sector and the fermion-fermion sector. We call them $\mathfrak{N}_B$ and $\mathfrak{N}_F$, and introduce external background fields to couple the conserved currents respectively.

The inclusion of the external fields was done in Section 3.2, and (3.15), (3.16) are adapted as follows

$$\tilde{D}_B \equiv (\Gamma^\mu)^{\mathrm{T}}(\partial_\mu + X_{B,\mu}) + M_B \Omega^{\mathrm{T}}; \tag{4.4}$$

$$\tilde{D}_F \equiv (\Gamma^\mu)^{\mathrm{T}}(\partial_\mu + X_{F,\mu}) + M_F \Omega^{\mathrm{T}}. \tag{4.5}$$

Here, $X_{B/F,\mu} \equiv A_\mu + B^{\mathrm{T}}_{B/F,\mu}$. In the massless limit and taking only the internal group actions into consideration, we require

$$B_{B/F,\mu}\Gamma^\nu + \Gamma^\nu B^\dagger_{B/F,\mu} = 0 \quad \forall \mu,\nu, \tag{4.6}$$

so $B_{B/F,\mu}$ contains a vector component and a axial-vector component as in (3.92). The integrations over the auxiliary fields can be carried out exactly as in Section 3.2, and we don’t expect to find an empty theory as long as $B_{B,\mu}$ is not the same as $B_{F,\mu}$.

There is one caveat in the naive massless-limit, though. Without the mass terms, the infra-red divergences from the Dirac operators are out of control, and hence the heat-kernel regularization (3.19) is not well-defined when the upper-limit of the $t-$integral is taken to ∞. To stay on the safe side, we include finite mass parameters all along but still insert the same $B_{B/F,\mu}$ to DYM. Strictly speaking the mass breaks $\mathfrak{N}$ down to $\mathfrak{U}$, or equivalently

$$B_{B/F,\mu}\Gamma_5 + \Gamma_5 B^{\dagger}_{B/F,\mu} = 0 \ \forall \mu, \tag{4.7}$$

is only solved by the vector component but not the axial-vector component. Consequently, an effective action for $B_{B/F,\mu}$ should share the same symmetry-breaking feature, which hopefully will be encoded in some "mass terms" in the color-neutralised partition function from the final step.

We proceed to the second step. In both sectors some $\mathcal{O}(\mathbf{C}\cdot t)$ terms are presented, which can be extracted from (3.29):

$$\mathbf{C}t\left[\partial^{\mu}\mathfrak{R}(X_{B/F,\mu}) + 2\mathfrak{R}(X^{\mu}_{B/F})\mathfrak{R}(X_{B/F,\mu})\right]. \tag{4.8}$$

The axial-vector component in (3.92) is Hermitian, so $\mathfrak{R}(X_{B/F,\mu}) \neq 0$; moreover, the contributions from the boson-boson sector and the fermion-fermion sector do not cancel each other out[4]. Eventually, this non-vanishing $\mathcal{O}(\mathbf{C}\cdot t)$ terms will result in a polynomial divergence as the ultraviolet regulator $\delta \to 0$, hence we cannot obtain a well-defined effective action for $B_{B/F,\mu}$ in the presence of the axial-vector components.

Perhaps the best we can do is to derive an effective action for $B_{B/F,\mu}$ with only the vector components. As one can see in (3.92), the vector component commutes with the $\Gamma-$matrices, so it is straightforward to derive the contributions from both sectors [80]:

$$\mathscr{L}_F = \lim_{\delta\to 0}\int_{\delta}^{\infty}\frac{dt}{32\pi^2 t^3}\,\mathrm{e}^{-M_F t^2}\tilde{\mathrm{tr}}\left(\mathbb{1}_{N_c} - \frac{t^2}{6}X_{F,\mu\nu}X_F^{\mu\nu} + \mathcal{O}(t^3)\right); \tag{4.9}$$

$$\mathscr{L}_B = -\lim_{\delta\to 0}\int_{\delta}^{\infty}\frac{dt}{32\pi^2 t^3}\,\mathrm{e}^{-M_B t^2}\tilde{\mathrm{tr}}\left(\mathbb{1}_{N_c} - \frac{t^2}{6}X_{B,\mu\nu}X_B^{\mu\nu} + \mathcal{O}(t^3)\right), \tag{4.10}$$

where $X_{B/F,\mu\nu}$ is the field-strength tensor of $X_{B/F,\mu}$, and $\tilde{\mathrm{tr}}$ is over the color-spinor-flavor space. Unfortunately, as long as we keep $B_{B,\mu}$ to be different from $B_{F,\mu}$, there is no way to combine $\mathscr{L}_F$, $\mathscr{L}_B$ as in Section 3.2. As a compromise, we set $B_{B,\mu} \equiv B_{F,\mu}$ by considering $\mathfrak{N}_B \equiv \mathfrak{N}_F$, which yields

$$\mathscr{L}_F + \mathscr{L}_B \approx -\frac{\ln(M_B^2/M_F^2)}{192\pi^2}\int d^4x\ \tilde{\mathrm{tr}}X_{\mu\nu}X^{\mu\nu}. \tag{4.11}$$

Finally, using the fact that B^{T}_{μ} commutes with A_{μ} one can expand

$$\tilde{\mathrm{tr}}X_{\mu\nu}X^{\mu\nu} = 4\left[N_f\mathrm{tr}F_{\mu\nu}F^{\mu\nu} + N_c\mathrm{tr}_f K_{\mu\nu}K^{\mu\nu} + 2\mathrm{tr}F_{\mu\nu}\mathrm{tr}_f K^{\mu\nu}\right] \tag{4.12}$$

where tr_f is over the flavor space and $K_{\mu\nu}$ is the flavor field-strength tensor analogous to $F_{\mu\nu}$. To derive the effective action for $B_{B,\mu} \equiv B_{F,\mu}$, we integrate out A_{μ}, which is still a non-trivial calculation due to the crossover term $\mathrm{tr}F_{\mu\nu}\mathrm{tr}_f K^{\mu\nu}$. Note that in the physical YM one uses SU(N_c)

[4]They do when $M_B = M_F$ and $B_{B,\mu}$ and $B_{F,\mu}$ are tailored to be the same field, but then the induced theory is empty.

instead of $U(N_c)$, in which case this crossover term vanishes, and the final result is nothing but a flavor YM action.

In summary, the first two step do not work out. So far, the most concrete result is obtained only when we refrain ourselves from including the axial-vector components of the external field, and consider the *diagonal* subgroup of the full $\mathfrak{N}_B \times \mathfrak{N}_F$ symmetry. It is possible to envision a dual theory which, when the dual degrees of freedom are only coupled to the vector external field, induces the same effective action derived from (4.11). Such a theory should be a supermatrix theory such that a similar Pauli-Villars regularization can take effect. Following this, one may add in axial-vector external fields to the dual theory, which will hopefully induce an effective action with the same behavior of divergence as in (4.8).

A promising starting point could be the gauged dual-EFT Lagrangian (3.101), and most likely more terms are required to fully capture the degrees of freedom in dual-IYM. It is likely that the methods for HChPT can be used to include heavier degrees of freedom (see Section 3.4.5), but this seems like a formidable task. We end this section with two observations from the polynomial divergence due to (4.8), which might assist us continuing down this path:

- In general, an attempt to induce a YM action by the method in Section 3.2 is likely to fail when the gauge group is non-compact. This is indeed the case when we insert $X_{B,\mu} \equiv X_{F,\mu}$ with a Hermitian axial-vector component. As we have reviewed in [80] and in Section 3.4, a proper action for a non-compact gauge field requires a NLσM field, which is not available to us in DYM from the very beginning. Therefore, one has to either modify DYM itself or to consider a different type of action. Interestingly, the same non-compactness doesn't cause much trouble in the conventional $d = 1+1$ models. The fermion determinant can still be computed exactly even with the presence of a Hermitian axial-vector component in the external gauge field. In $d = 1+1$, only two out of four conserved currents are independent [108]; consequently, the axial current can be included as part of the skew-Hermitian vector current.

- It is possible to conceive a partition function for some color-neutral dual-field coupled to $B_{B,\mu}$, $B_{F,\mu}$, such that the same polynomial divergence emerges when we integrate out the dual-field. A promising candidate is a gravitational theory. Usually such a theory is non-renormalisable, which in turn signals a break-down of a regularization scheme such as the one caused by (4.8). This idea is still an early-stage speculation, and a short discussion on this can be found in Chapter 5.

4.2 Zero-current condition

We faced substantial difficulties trying to analytically derive an effective action for the super-meson field $\tilde{Q}$. Nevertheless, in this section we hope to disclose as much information as possible from the partition function (3.37), where most of the intriguing physics is hidden in the target space of the fields, i.e., the solution space of ZC defined by (3.36). Our first priority is the boson-boson sector of ZC, which was defined in (3.41), and this will be the only sector studied in this dissertation[5].

4.2.1 Two different points of view

We found it convenient to explore the solution space of (3.41) through two parallel approaches:

Bottom-Up (BU)

Treating $\phi \equiv \begin{pmatrix}\phi_{+1} & \phi_{+2} & \phi_{-1} & \phi_{-2}\end{pmatrix}$ as a *full-rank*[6] $N \times 4N$ matrix ($\bar{\phi} \equiv \phi^{\dagger}$) and employ the Euclidean-Weyl representation of the $\gamma-$matrices:

$$\gamma^0 = \begin{pmatrix} 0 & i\mathbb{1}_2 \\ i\mathbb{1}_2 & 0 \end{pmatrix}; \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}; \quad \gamma_5 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix},$$

we reduce (3.41) to

$$\begin{cases} \phi_{+1}\phi^{\dagger}_{-1} + \phi_{-2}\phi^{\dagger}_{+2} = 0; & (4.13a) \\ \phi_{+2}\phi^{\dagger}_{-1} - \phi_{-2}\phi^{\dagger}_{+1} = 0. & (4.13b) \end{cases}$$

To ascertain that the solution space is not empty nor "boring", we present two exemplary solutions:

$$\phi_{\text{ch}} = \begin{pmatrix}\phi_{+1} = 0 & \phi_{+2} = 0 & \phi_{-1} & \phi_{-2}\end{pmatrix}; \tag{4.14}$$

$$\phi_{\text{nch}} = \begin{pmatrix}\phi_{+1} & \phi_{+2} = 0 & \phi_{-1} & \phi_{-2} = 0\end{pmatrix}; \qquad \phi_{+1}\phi^{\dagger}_{-1} = 0. \tag{4.15}$$

Clearly, any *chiral* mode ϕ_{ch} is non-propagating in dual-IYM because its kinetic energy in S_{free} is zero no matter the fluctuations of the nonzero submatrices.

When $N = 1$, it is easy to see that the only solutions are chiral modes, so dual-IYM is trivial in a sense. In this case the induced gauge theory might not be an appropriate tool for constructing a dual theory of the Abelian gauge theory, which is a well-understood field anyway[7]. For an interesting case where $N \geq 2$, there always exists a full-rank *non-chiral* mode ϕ_{nch} with

$$\phi_{+1} = \begin{pmatrix} \varphi_q & 0 \\ 0 & 0 \end{pmatrix}, \qquad \phi_{-1} = \begin{pmatrix} 0 & 0 \\ 0 & \varphi_{N-q} \end{pmatrix}, \tag{4.16}$$

where φ_q, φ_{N-q} are full-rank square matrices with size q, $N-q$ respectively, and $N > q > 0$. However, the kinetic energy of ϕ_{nch} given by (4.16) is again identically zero. For $N = 2$, a non-chiral

[5] We hope to explore the other sectors in future works.

[6] A common assumption made in random matrix theories because the set of rank-deficient elements has measure zero.

[7] See the literature mentioned in Chapter 1. A further study of the implications seems interesting, but they are not included in this dissertation.

mode a nonzero contribution to S_{free} is given by

$$\phi_{+1}(x) = \begin{pmatrix} a(x) & b(x) \\ 0 & 0 \end{pmatrix}, \qquad \phi_{-1}(x) = \begin{pmatrix} 0 & 0 \\ c(x) & d(x) \end{pmatrix};$$

$$a(x)\bar{c}(x) + b(x)\bar{d}(x) \overset{!}{=} 0 \quad \text{but} \quad a(x)(\partial_\mu \bar{c}(x)) + b(x)(\partial_\mu \bar{d}(x)) \neq 0. \tag{4.17}$$

We have established the fact that dual-IYM, being a dual theory of the non-Abelian YM when $N \geq 2$, is worth investigating. This is the very first step to place dual-EFT on solid ground. This program of directly solving $N \times N$ submatrices for (4.13a), (4.13b) is the *Bottom-Up* (BU) approach.

Top-Down (TD)

While BU in principle can offer us a complete set of solutions to ZC, at the moment we have no clue how to portray them in a physically meaningful way. This is where the second method, the *Top-Down* (TD) approach, comes in. The starting point is the fact that for any complex matrix X, $X = 0 \Leftrightarrow \text{Tr}XX^\dagger = 0$, which can be used to transform (3.41) to an equivalent form:

$$\text{Tr}Q\Gamma^\mu Q\Gamma^\mu = 0 \qquad \forall\mu \Longleftrightarrow \text{Tr}Q\Gamma^\mu Q\Gamma^\nu = 0 \qquad \forall\mu,\nu. \tag{4.18}$$

This *color-neutral* form of ZC is very instructive because it directly places the conditions on the meson matrix $Q \equiv \bar{\phi}\phi$, so if it turns out $\tilde{Q}$ is the appropriate dual-field, we can already use (4.18) to define the target space of its boson-boson submatrix field Q. Instead of directly dealing with ZC, we solve for Q in TD. The same procedure can also be done when the color group is SU(2), in which case there are *baryon* matrices in addition to Q. The results can be found in Appendix C. For the purpose of consistency, the subsequent discussions will be exclusively on the case where the color group is U(N).

There is however an caveat. In addition to (4.18), one has to impose another condition that Q must be a *positive-semidefinite* rank N square matrix of size $4N$. With this constraint considered, there is a simple set of rules transforming ϕ to Q and vice versa:

- Given any full-rank (N) ZC solution ϕ, the matrix $Q \equiv \bar{\phi}\phi$ is positive-semidefinite, rank N, and it solves (4.18).
- Given any positive-semidefinite rank N matrix Q solving (4.18), there exists a decomposition $Q = \bar{\phi}\phi$ where ϕ is a full-rank $N \times 4N$ matrix. Plugging this back to (4.18), it follows that ϕ is a ZC solution.

Obviously, for any Q the constituent matrix ϕ is determined up to color U(N) transformations.

Having explained the equivalence between BU and TD, we now point out that the place where TD shines is to rephrase ZC and make the hidden physical information manifest. What kinds of information are we looking for? In the real world, the hadron spectrum is very complex, partially because of the dependence of the hadron mass on multiple factors such as the constituent quark masses and the total angular momentum. In fact, there are observed hadrons which do not fit into the state-of-the-art quark models [116, 117]. The situation is much simpler in dual-IYM, where we assign the same mass to the auxiliary fields regardless which flavors they carry. This highlights the critical role played by the spin degrees of freedom, and as we will see below, the condition (4.18) is ideal for partitioning the ZC solutions according to their spinor components.

4.2.2 Lorentz components

To motivate our plan, we revisit the general scheme of effective theories for hadron physics which we have encountered in Section 3.4.4 and Section 3.4.5. Starting from the low-energy regime, the lightest mesons are Lorentz-scalars, which are captured by ChPT. Increasing the energy means the inclusion of heavier bound-states is necessary, which is done by adding more color-neutral fields to ChPT. Depending on the types of the heavy hadrons, the resulting generalized models have many different forms. Nevertheless, in the meson sector a general rule of thumb is that given the same constituent quarks, the Lorentz-vector bound-state is generally heavier than the the Lorentz-scalar one due to a spin-spin interaction[8]. From now on, we will denote this feature of spin-induced differentiation in mass the *energy-hierarchy*.

Can we observe such energy-hierarchy in dual-IYM? As a first step, recall that any $Q \in \mathrm{Mat}(4N;\mathbb{C})$ admits the unique decomposition given by (3.47):

$$Q \equiv \mathbb{1}_4 \otimes \hat{Q}_{\mathbb{1}} + i\gamma^\lambda \otimes \hat{Q}_\lambda + i\gamma^\alpha\gamma^\beta \otimes \hat{Q}_{\alpha\beta}\ (\alpha < \beta) + \gamma^\sigma\gamma_5 \otimes \hat{Q}_{\sigma 5} + \gamma_5 \otimes \hat{Q}_5. \tag{4.19}$$

For convenience, Q is already taken to be Hermitian, so the identity

$$Q = \frac{1}{2}(Q + Q^\dagger)$$

"projects" every $\hat{Q}_\bullet$ to be also Hermitian. In total there are 16 flavor matrices $\hat{Q}_\bullet$ of various *Dirac-types* (scalar, vector, tensor, axial-vector, pseudo-scalar). A quick way to justify these names is to consider

$$Q \mapsto (s \otimes \mathbb{1}_N)^\dagger Q (s \otimes \mathbb{1}_N) \qquad s \in \mathrm{Pin}(4). \tag{4.20}$$

The action of Pin(4) on the products of $\gamma-$matrices was explained in Section 3.4.2, where we discussed the continuous and the discrete transformations separately.

Plugging (4.19) back to (4.18) and making use of the basic identities

$$\mathrm{Tr}_s \gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_k} = \begin{cases} \pm 4 & \text{for } \mu_1 = \mu_2 = \cdots = \mu_k \\ 0 & \text{otherwise} \end{cases},$$

[8]An explanation can be found in [118], where the two-body system is treated in the non-relativistic limit. In this approximation the vector boson carries spin 1 whereas the scalar boson carries spin 0.

we arrive at

$$\begin{cases} -\mathbb{X}_{\mathbb{1}} + \mathbb{X}_5 + \mathbb{X}_2 + \mathbb{X}_3 - \mathbb{X}_{25} - \mathbb{X}_{35} - \mathbb{X}_{23} + \mathbb{X}_{01} = 0; & \text{(4.21a)} \\ -\mathbb{X}_{\mathbb{1}} + \mathbb{X}_5 + \mathbb{X}_0 + \mathbb{X}_1 - \mathbb{X}_{05} - \mathbb{X}_{15} - \mathbb{X}_{01} + \mathbb{X}_{23} = 0; & \text{(4.21b)} \\ -\mathbb{X}_0 + \mathbb{X}_1 + \mathbb{X}_{05} - \mathbb{X}_{15} + \mathbb{X}_{02} + \mathbb{X}_{03} - \mathbb{X}_{12} - \mathbb{X}_{13} = 0; & \text{(4.21c)} \\ -\mathbb{X}_2 + \mathbb{X}_3 + \mathbb{X}_{25} - \mathbb{X}_{35} + \mathbb{X}_{02} + \mathbb{X}_{12} - \mathbb{X}_{03} - \mathbb{X}_{13} = 0. & \text{(4.21d)} \end{cases}$$

To simplify the notations, we are using $\mathrm{Tr}_f \hat{Q}_\bullet^2 \equiv \mathbb{X}_\bullet$. We noticed an interesting pattern in these equations: any $\mathbb{X}_\bullet$ is paired with its γ_5-dual. These pairs are given by

$$\mathbb{Y}_0 \equiv -\mathbb{X}_0 + \mathbb{X}_{05}; \quad \mathbb{Y}_1 \equiv -\mathbb{X}_1 + \mathbb{X}_{15}; \quad \mathbb{Y}_2 \equiv -\mathbb{X}_2 + \mathbb{X}_{25}; \quad \mathbb{Y}_3 \equiv -\mathbb{X}_3 + \mathbb{X}_{35};$$
$$\mathbb{Y}_{\mathbb{1}} \equiv -\mathbb{X}_{\mathbb{1}} + \mathbb{X}_5; \quad \mathbb{Y}_{01} \equiv -\mathbb{X}_{01} + \mathbb{X}_{23}; \quad \mathbb{Y}_{02} \equiv -\mathbb{X}_{02} + \mathbb{X}_{13}; \quad \mathbb{Y}_{03} \equiv -\mathbb{X}_{03} + \mathbb{X}_{12},$$

and the equations (4.21a)-(4.21d) now read

$$\begin{cases} \mathbb{Y}_{\mathbb{1}} = \frac{1}{2}(\mathbb{Y}_0 + \mathbb{Y}_1 + \mathbb{Y}_2 + \mathbb{Y}_3); & \text{(4.22a)} \\ \mathbb{Y}_{01} = \frac{1}{2}(\mathbb{Y}_0 + \mathbb{Y}_1 - \mathbb{Y}_2 - \mathbb{Y}_3); & \text{(4.22b)} \\ \mathbb{Y}_{02} = \frac{1}{2}(\mathbb{Y}_0 - \mathbb{Y}_1 + \mathbb{Y}_2 - \mathbb{Y}_3); & \text{(4.22c)} \\ \mathbb{Y}_{03} = \frac{1}{2}(\mathbb{Y}_0 - \mathbb{Y}_1 - \mathbb{Y}_2 + \mathbb{Y}_3). & \text{(4.22d)} \end{cases}$$

The nice thing about rewriting (4.18) this way is that it reveals eight hyperboloids/light-cones. For example, $\mathbb{Y}_0 \equiv -\mathbb{X}_0 + \mathbb{X}_{05}$ reads

$$\mathbb{Y}_0 = (-q_1^2 - \cdots - q_N^2) + (p_1^2 + \cdots + p_N^2);$$

where $q_j, p_j \in \mathbb{R}$ are the eigenvalues of $\hat{Q}_{\mathbb{1}}$ and $\hat{Q}_5$, respectively. Depending on the value of $\mathbb{Y}_0$, the solution space has the geometry of a hyperboloid or a light cone. By analogy with the special theory of relativity, we call the eight real scalars $\mathbb{Y}_\bullet$ the "invariant masses". In summary, there is one set of equations for the invariant masses given by (4.22a)-(4.22d), and once an arbitrary solution is fixed, each component $\mathbb{Y}_\bullet$ defines a hyperboloid/light-cone of a certain Dirac-type determined by $\bullet$.

We are ready to perform a first test on the energy-hierarchy in dual-IYM. Specifically, we are going to check *if* some of the Lorentz-components of Q are suppressed by ZC. At first glance the answer is negative, because any $\hat{Q}_\bullet = 0 \Leftrightarrow \mathbb{X}_\bullet = 0$ and the hyperbolic nature of the solution spaces for $\mathbb{Y}_\bullet$ obviously allow nonvanishing $\mathbb{X}_\bullet$. More concretely, there are 4 linear equations for 16 $\mathbb{X}_\bullet$, so the best we could do it to set 4 of the components to zeros. For instance, suppose all vector components $\mathbb{X}_\lambda$ are zeros, the axial-vector components $\mathbb{X}_{5\lambda}$, which are also Lorentz-vectors, are not necessarily constrained to be zeros.

In addition to ZC, however, we still have to consider the conditions that Q must be rank N and positive-semidefinite, which did not come with the Hermiticity. Unfortunately there are no straightforward ways to add these conditions to TD. To overcome this shortcoming, we turn to

BU and simply take an exemplary solution ϕ_{nch} such as (4.17) to make

$$Q_{\text{nch}} \equiv \bar{\phi}_{\text{nch}}\phi_{\text{nch}} = \begin{pmatrix} \bar{\phi}_{+1}\phi_{+1} & 0 & \bar{\phi}_{+1}\phi_{-1} & 0 \\ 0 & 0 & 0 & 0 \\ \bar{\phi}_{-1}\phi_{+1} & 0 & \bar{\phi}_{-1}\phi_{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.23}$$

Now, every component $\hat{Q}_{\bullet}$ can be obtained by partially tracing over the spinor space of the product of Q with some $\Gamma-$matrices. For example, the vector components are

$$\hat{Q}_{\lambda} \propto \mathrm{Tr}_s \Gamma^{\lambda} Q_{\text{nch}} \propto \bar{\phi}_{-1}\phi_{+1} \pm \bar{\phi}_{+1}\phi_{-1},$$

and all of which vanish for (4.17) because $\bar{\phi}_{-1}\phi_{+1} = 0$.

There exists other solutions, of course. In fact, simply by replacing

$$\phi_{-1}(x) = \begin{pmatrix} 0 & 0 \\ c(x) & d(x) \end{pmatrix} \longrightarrow \phi_{-1}(x) = \begin{pmatrix} c(x) & d(x) \\ c(x) & d(x) \end{pmatrix}$$

in (4.17), we obtain another ZC solution but this time it is possible that $\bar{\phi}_{-1}\phi_{+1} \neq 0$. Consequently, the associated matrix Q has vector components. In the similar way, for any given Q one can check whether or not some of its components vanish. For instance, the tensor component $\hat{Q}_{12}$ could be nonzero for the case of (4.17).

What we have observed is that ZC **does not** suppress any particular Lorentz-component of the meson matrix. Hence, dual-IYM should contain color-singlets of all Lorentz-types. To elaborate on this conclusion, note that ZC should already encompass all the gluon-induced effects, which in turn play the role of the potential energy in the formulation of bound-states. Suppose the kinetic-energy effect is less significant[9], then ZC alone could indeed determine the properties of the bound-states. Eventually, this implies that because dual-EFT only contains scalar and pseudoscalar degrees of freedom, it can at most cover a "subset" of dual-IYM as a theory of color-neutral dual-fields. We close this section with a remark. As reviewed in Section 4.1.1, the color-neutral theory obtained via the HS decoupling method **does not** have Lorentz-tensor components due to the Fierz rearrangement. On the contrary, dual-IYM **does** contain the Lorentz-tensor components. This comparison signals a fundamental difference between the HS approach and the bosonization approach.

4.2.3 Parametrisation of ZC solution space by BU

Until now we have acquired some pieces of the puzzle, but they are not enough for us to assemble the complete solution space for ZC. In this section and the next, we are going to develop a parametrisation of the solution space. The general strategy adopted here is the common one: use

[9]This is no more than a guess. A direct examination is however beyond the scope of this dissertation.

the symmetries of ZC to bring every solution to a *canonical form*, which hopefully will allow us to solve the equations in a simpler setting.

Let us first practice this concept in BU by revisiting the original ZC (3.41):

$$\phi\Gamma^{\mu}\bar{\phi} = 0 \qquad \forall\mu.$$

Given any solution $\bar{\phi}$, every element in the

$$\mathrm{GL}(N;\mathbb{C}) - \mathrm{orbit} := \{\bar{\phi}(g) \equiv \bar{\phi}g | g \in \mathrm{GL}(N;\mathbb{C})\} \tag{4.24}$$

is also a solution obviously. Here we continue working with a full-rank $\bar{\phi}$, which is composed of N mutually linearly-independent column vectors in $\mathbb{C}^{4N}$. Geometrically speaking, $\bar{\phi}$ represents an $N-$plane in $\mathbb{C}^{4N}$ spanned by the column vectors, and any right-action of $\mathrm{GL}(N;\mathbb{C})$ simply maps this set of basis vectors to another set but preserves the $N-$plane. By the right-actions every $\bar{\phi}$ can be brought to its *reduced column echelon form* $\bar{\phi}_0$, where N rows of $\bar{\phi}_0$ make up an identity matrix.

For example, when $\phi_0 = \begin{pmatrix}(\phi_0)_{+1} = \mathbb{1}_N & (\phi_0)_{+2} & (\phi_0)_{-1} & (\phi_0)_{-2}\end{pmatrix}$, the equations (4.13a), (4.13b) become

$$\begin{cases} (\phi_0)^{\dagger}_{-1} + (\phi_0)_{-2}(\phi_0)^{\dagger}_{+2} = 0; & (4.25a)\\ (\phi_0)_{+2}(\phi_0)^{\dagger}_{-1} - (\phi_0)_{-2} = 0. & (4.25b)\end{cases}$$

In this simplified case, one actually has to solve only one *self-anti*†*congruence* equation:

$$(\phi_0)_{+2}(\phi_0)^{\dagger}_{-1}(\phi_0)^{\dagger}_{+2} = -(\phi_0)^{\dagger}_{-1}, \tag{4.26}$$

for $(\phi_0)^{\dagger}_{-1}$ and $(\phi_0)_{+2}$, and then automatically $(\phi_0)_{-2} = (\phi_0)_{+2}(\phi_0)^{\dagger}_{-1}$.

For arbitrary $(\phi_0)_{-1}, (\phi_0)_{+2} \in \mathrm{Mat}(N;\mathbb{C})$, there seems to be no easy way to solve (4.26), so we now seek help from the massless symmetry group K acting on the spinor-flavor space, which was introduced in Section 3.4.2. Recall that any element in the

$$K - \mathrm{orbit} := \{\bar{\phi}(k) \equiv k\bar{\phi} | k \in K\} \tag{4.27}$$

of the same solution $\bar{\phi}$ is a ZC solution. Consequently, the $\bar{\phi}_0$ from above can be further simplified by a combination of a left-action by K and a right-action by $\mathrm{GL}(N;\mathbb{C})$. To demonstrate this, we consider the mixing subgroup $\mathfrak{N}$ in (3.74) and write them out in the Euclidean-Weyl representation:

$$\mathfrak{N} \equiv \left\{ k(h) \equiv \begin{pmatrix} h & 0 & 0 & 0 \\ 0 & h & 0 & 0 \\ 0 & 0 & (h^{-1})^{\dagger} & 0 \\ 0 & 0 & 0 & (h^{-1})^{\dagger} \end{pmatrix} \middle| h \in \mathrm{GL}(N;\mathbb{C}) \right\}. \tag{4.28}$$

By combining an element $k(h) \in \mathfrak{N}$ with a color matrix $h^{-1} \in \mathrm{GL}(N;\mathbb{C})$, we transform $\bar{\phi}_0$ while preserving the identity matrix:

$$(\phi_0)^\dagger_{+1} = \mathbb{1}_N \mapsto \mathbb{1}_N; \quad (\phi_0)^\dagger_{+2} \mapsto h(\phi_0)^\dagger_{+2}h^{-1};$$
$$(\phi_0)^\dagger_{-1} \mapsto (h^{-1})^\dagger(\phi_0)^\dagger_{-1}h^{-1}; \quad (\phi_0)^\dagger_{-2} \mapsto (h^{-1})^\dagger(\phi_0)^\dagger_{-2}h^{-1}.$$

We are now free to choose h. A possibility is to use h to reduce $(\phi_0)^\dagger_{+2}$ to its Jordan normal form; or instead we can rotate $(\phi_0)^\dagger_{-1}$ along its †congruence-orbit to a canonical form [119]. Neither of the choices offers us a straightforward way to solve (4.26), unfortunately.

Looking at the broader picture. While working in the functional-integral formalism, it is important to check if the target space of the field splits to several path-connected regions. The integration over continuous field configurations in each sector has to be carried out individually. Since our understanding of the ZC solution space is incomplete, we prefer a less ambitious approach by partitioning the target space to different group-orbits. More precisely, we are interested in the orbits of some continuous symmetry groups, because the continuity assures all solutions in such an orbit must belong to the same path-connected region. Now, each solution $\bar{\phi}$ belongs to one and only one

$$\text{double-orbit} := \{\bar{\phi}(k,g) \equiv k\bar{\phi}g \mid k \in K,\ g \in \mathrm{GL}(N;\mathbb{C})\}, \tag{4.29}$$

and ultimately we hope to partition the ZC solution space to several double-orbits. Practically speaking, instead of testing all $\bar{\phi}$ against ZC and then classifying them to different orbits, it is faster to start with the set of $N-$planes and test each plane against ZC. This is because the right-action of $\mathrm{GL}(N;\mathbb{C})$ preserves every $N-$plane, while K is the group which can change the $N-$planes. Furthermore, the group $\mathrm{GL}(N;\mathbb{C})$ is path-connected, so it suffices to pick only one $\bar{\phi}$ from each $\mathrm{GL}(N;\mathbb{C})-$orbit anyway. Taking this into account, from now on we will be exclusively studying the $K-$orbits.

In Section 3.4.2 we identified the identity component K_0 and some exceptional elements from the symmetry group K. By definition, K_0 is a continuous symmetry group containing the identity, so the K_0-orbits are the orbits of interest. The existence of the exceptional elements (and other unknown symmetry transformations in K) suggests that there are many K_0-orbits populating the target space, and it is also possible that two distinct K_0-orbits belong to the same path-connected region. Nevertheless, we believe that understanding the K_0-orbits can already teach us much about the ZC solution space.

Let us first review the elements of K_0 given by (3.60). Apart from the mixing elements in (4.28), there are the scaling factor and the $\mathrm{Spin}(4) \otimes \mathbb{1}_N$ elements. Altogether, an element $k \in K_0$ looks like

$$k = \begin{pmatrix} r^+ u_s \otimes h & 0 \\ 0 & r^+ v_s \otimes (h^{-1})^\dagger \end{pmatrix}; \qquad r \in \mathbb{R}^+,\ u_s, v_s \in \mathrm{SU}(2). \tag{4.30}$$

When $N = 1$, the whole group of K is *homogeneous* (either Γ_5-even or Γ_5-odd). This is related to the fact that when $N = 1$, all ZC solutions are chiral. We present a proof in Appendix D. For $N \geq 2$, all the known elements in K are homogeneous; in particular, the K_0 elements are Γ_5-even and hence they preserve the *homogeneity*. That is, ϕ_{ch} and ϕ_{nch} must belong to different K_0-orbits.

Within the chiral sector, it is clear that K_0 also preserves the chirality. The right-handed $\phi_{\text{ch},R}$ and the left-handed $\phi_{\text{ch},L}$ are not in the same K_0-orbit; rather, they are connected via the discrete transformations in $K_{\mathfrak{D}}$ from (3.72). In the physically more interesting non-chiral sector, however, the situation is more complicated. Under the assumption that all K elements are homogeneous, the sector of block diagonal matrices and the one of block off-diagonal matrices cannot be path-connected. This is due to the fact that any path starting from the identity matrix must always be block diagonal, and hence it cannot take any element from either sector to the other one. If this were true, then we could also use $K_{\mathfrak{D}}$ to obtain more K_0-orbits in the non-chiral sector. Unfortunately, the homogeneity of K is only proven for the case of $N = 1$, so we refrain ourselves from applying the same argument here.

We point out that for $N = 2$, there exists ϕ_{nch} which doesn't belong to the K_0-orbit of the non-chiral mode in (4.17). This solution is

$$\begin{aligned} &\phi_{\text{nch}} = \begin{pmatrix} \phi_{+1} & \phi_{+2} & \phi_{-1} & \phi_{-2} = 0 \end{pmatrix}; \\ &\phi_{+1} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, \quad \phi_{+2} = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}, \quad \phi_{-1} = \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}. \end{aligned} \tag{4.31}$$

The scalars a, b, c, d are chosen to obey the same equation in (4.17) such that ϕ_{nch} solves ZC. Since we have demanded ϕ_{+1} and ϕ_{+2} to be linearly-independent, it follows that $r^+ h\bar{\phi}_{+1}$ and $r^+ h\bar{\phi}_{+2}$ are also linearly-independent for any r^+, h. As the result, because none of the $u_s \in \text{SU}(2)$ can send

$$\begin{pmatrix} \bar{\phi}_{+1} \\ \bar{\phi}_{+2} \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} \bar{\phi}_{+1} \\ 0 \end{pmatrix}$$

due to the linear independence between the submatrices, we conclude that (4.31) belongs to a different K_0-orbit. A generalization to higher N is straightforward.

Continuing down this road, one should be able to discover more K_0-orbits, but this doesn't seem like an economic way to understand the physics of dual-IYM. From the mathematical point of view, each $\text{GL}(N;\mathbb{C})-$orbit is represented by a unique point on the complex Grassmannian $\text{Gr}_{N,4N}(\mathbb{C})$. In this interpretation, the task is to figure out how to partition $\text{Gr}_{N,4N}(\mathbb{C})$ to several K_0-orbits. In the next section, however, we would like to take a practical point of view and try to designate physical meaning to the K_0-orbits.

4.2.4 Parametrisation of ZC solution space by TD

Confronting the parametrisation seems easier by TD. First and foremost, $Q = \bar{\phi}\phi$ is color-neutral by birth, so it represents an $N-$plane rather than a set of basis vectors coincide with the column vectors of $\bar{\phi}$. Therefore, we no longer need to consider the group GL(N;$\mathbb{C}$) acting on the color space in TD. Starting from the reformulation of ZC in terms of (4.22a)-(4.22d): the solution space is a $\mathbb{R}^4$ subspace embedded in $\mathbb{R}^8$. One can use the "Γ_5-odd" coordinates ($\mathbb{Y}_0, \mathbb{Y}_1, \mathbb{Y}_2, \mathbb{Y}_3$) or the "$\Gamma_5-$even" coordinates ($\mathbb{Y}_{\mathbb{1}}, \mathbb{Y}_{01}, \mathbb{Y}_{02}, \mathbb{Y}_{03}$) to describe this $\mathbb{R}^4$. They are equivalent, and we proceed with the "Γ_5-odd" one.

At any point ($\mathbb{Y}_0, \mathbb{Y}_1, \mathbb{Y}_2, \mathbb{Y}_3$), there are eight hyperboloids/light-cones defined by the invariant masses $\mathbb{Y}_\bullet$. These substructures are mutually disjoint, and the integration over the hyperboloid/light-cone for each $\mathbb{Y}_\bullet$ is straightforward. Without loss of generality, we look at the Lorentz-scalar sector:

$$\mathbb{X}_5 - \mathbb{X}_{\mathbb{1}} = \mathbb{Y}_{\mathbb{1}}; \quad \mathbb{X}_{\mathbb{1}} = \mathrm{Tr}_f \hat{Q}_{\mathbb{1}}^2,\ \mathbb{X}_5 = \mathrm{Tr}_f \hat{Q}_5^2. \tag{4.32}$$

First, by construction $\hat{Q}_{\mathbb{1}}^2$, $\hat{Q}_5^2$ are Hermitian matrices in the flavor space. The integration over, say $\hat{Q}_{\mathbb{1}}^2$, can be carried out in the standard way:

$$d\hat{Q}_{\mathbb{1}}^2 = |\Delta(\Lambda_{\mathbb{1}})|^2 d\Lambda_{\mathbb{1}} du_{\mathbb{1}},$$

where $\Lambda_{\mathbb{1}} = \mathrm{diag}(q_1, \cdots, q_N)$, $du_{\mathbb{1}}$ is the Haar measure over the flavor U(N), and $\Delta(\Lambda_{\mathbb{1}})$ is the *Vandermonde determinant*. The similar measure can be obtained for $\hat{Q}_5^2$ with $\Lambda_5 = \mathrm{diag}(p_1, \cdots, p_N)$. Putting everything together, the integration over the space of ($\hat{Q}_{\mathbb{1}}, \hat{Q}_5$) breaks down to an integration of the eigenvalues $q_1, \cdots, q_N, p_1, \cdots, p_N$ constrained by the $\mathbb{Y}_{\mathbb{1}}-$hyperboloid/light-cone and weighted by the Vandermonde determinants, plus two integrations $du_{\mathbb{1}}$, du_5 over U(N).

To summarize, as far as ZC goes, TD does provide us a clean view of the solution space. One first integrates over $\mathbb{R}^4$, and then over eight hyperboloids/light-cones by the recipe explained above. This can be done exactly. However, in order to bridge BU and TD, the final target space of Q must be the intersection of the space of positive-semidefinite, rank N matrices, with the solution space we just derived. Since neither of these conditions can be directly imposed on the decomposition (4.19), this approach seems like an dead end.

The hopes are placed on group-orbits again. In Section 4.2.3, we discussed the possibility of partitioning the target space of $\bar{\phi}$ to several K_0-orbits, and we would like to test the same strategy in TD. A challenge immediately appears when we try to transform the original integral of the supervector fields $\bar{\Psi}(x), \Psi(x)$ to an integral over the supermeson field $\tilde{Q}(x)$: the rank-deficiency in Q makes the SuB formula inapplicable, as explained in Section 4.1.2. To circumvent this obstacle, one may consider implementing SuB in each K_0-orbit, hoping that every K_0-orbit contains a representative which has one and only one $N \times N$ submatrix. If this were true, the representative matrix can be substituted by a full-rank positive-definite matrix field via SuB.

With all the knowledge of dual-IYM we have at hand, it is reasonable to check this conjecture against previous results. In the chiral-sector, the K_0-orbit of any solution $\bar{\phi}$ has a definite chirality. Take $\phi_{\mathrm{ch},R} = \begin{pmatrix}\phi_+ & \phi_- = 0\end{pmatrix}$ for example, it generates a chiral meson

$$Q_{\mathrm{ch},R} \equiv \bar{\phi}_{\mathrm{ch},R} \phi_{\mathrm{ch},R} = \begin{pmatrix} \bar{\phi}_+ \phi_+ & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.33}$$

where $\bar{\phi}_+\phi_+$ is an arbitrary $2N \times 2N$ positive-semidefinite matrix of rank N. Using (4.30), an element $k \in K_0$ sends

$$Q_{\mathrm{ch},R} \mapsto kQ_{\mathrm{ch},R}k^\dagger = (r^+)^2 \begin{pmatrix} (u_s \otimes h)(\bar{\phi}_+\phi_+)(u_s \otimes h)^\dagger & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.34}$$

For $N \geq 2$, in general $\bar{\phi}_+\phi_+$ cannot be rotated to a certain kind of desired representatives, e.g. $\bar{\phi}_+\phi_+ \xrightarrow{?} \mathrm{diag}(\mathbf{q}, 0)$, for some $\mathbf{q} \in \mathrm{Herm}_N^+(\mathbb{C})$ by a clever choice for (u_s, h). The reason is again the linear independence between the $N \times N$ submatrices of $\bar{\phi}_+\phi_+$. However, it is not immediately clear whether or not $\bar{\phi}_+\phi_+$ and a "mixing"-type representative matrix are in the same orbit. When $N = 2$, a typical mixing-type matrix is

$$\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \mathbf{q} & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \tag{4.35}$$

where $\mathbf{q} \in \mathrm{Herm}_2^+(\mathbb{C})$. A generic matrix of this type cannot be decomposed as a Kronecker product of a spin matrix and a flavor matrix, and hence it must belong to a different K_0−orbit associated with the one of $\mathrm{diag}(\mathbf{q}, 0)$, which is decomposable. Note that the other representative $\mathrm{diag}(0, \mathbf{q})$ is in the same orbit of $\mathrm{diag}(\mathbf{q}, 0)$.

All things considered, we may split the right-handed-sector to a "non-mixing" orbit of $\mathrm{diag}(\mathbf{q}, 0)$ and a "mixing" orbit of (4.35). The SuB formula can then be applied to each orbit. At present, this statement is no more than a conjecture. The water is even deeper in the non-chiral sector, where an identification of a submatrix $\mathbf{q}$ is complicated. Take Q_{nch} from (4.23) for instance, it is not clear to us if a single mixing-type representative as in (4.35) can be brought to the same orbit where the non-chiral meson belongs. This task can also be rephrased differently: being rank N and positive-semidefinite, Q is $\mathrm{U}(4N)$ equivalent to a representative, which itself contains only one diagonal $N \times N$ matrix. However, there is no guarantee that the $\mathrm{U}(4N)$ element needed for this job also belongs to K_0 (or K).

It seems that the SuB formula cannot be used here as an exact transformation of the whole ZC solution space. Instead, one may argue on physical grounds that "some" solutions $\bar{\phi}$ dominates the low-energy phenomenon of dual-IYM, and then check if the mesons built from these favourable fields can be computed with SuB. In the next section, we are going to give a brainstorming on this topic.

4.2.5 Physical aspects

In Section 3.3.1, we have pointed out the possible existence of the condensates $\langle \mathrm{Tr}\bar{\phi}\Gamma_5^{\mathrm{T}}\phi\rangle_{\text{dual-IYM}}$, $\langle \mathrm{Tr}\bar{\psi}\Gamma_5^{\mathrm{T}}\psi\rangle_{\text{dual-IYM}}$ by connecting these VEVs to the gluon condensate $\langle \mathrm{tr}F_{\mu\nu}F^{\mu\nu}\rangle_{\mathrm{IYM}}$. The dual-IYM condensates are pseudo-scalars because of the Γ_5. In the massless-limit, the action of dual-IYM coincides with QCD in the infinite-coupling limit, and we learned from QCD that there should also be chiral condensates, which are scalars. Hence, at least at the heuristic level we are convinced that Lorentz-scalar condensates, pseudo or not, exist in dual-IYM.

As usual the physical vacuum should be Lorentz-invariant, and hence there are no condensates of Lorentz-scalar type nor Lorentz-tensor type. In view of this, we expect the lowest-energy regime to be populated by the Lorentz-scalar bound-states only. As an extension of this argument, we would like to know if it makes sense to partition the target space of the fields in dual-IYM to three different *Lorentz-sectors*: Lorentz-scalar, vector, and tensor. One concern is that if there are elements in K mapping a solution in one sector to a different one, then this postulate fails. Here, we examine this property for the continuous subgroup K_0.

Since all elements in K_0 are homogeneous, we have the following result:

Lemma 4.1. *Let* $Q_{\mathbb{1}} \equiv \mathbb{1}_4 \otimes \hat{Q}_{\mathbb{1}}$ *and* $Q_5 \equiv \gamma_5 \otimes \hat{Q}_5$*. For all* $k \in K_0$*,* $kQ_{\mathbb{1}}k^\dagger$ *has no Lorentz-vector components and neither does* $kQ_5k^\dagger$*.*

Proof. The vector-component and the axial-vector component of any Q are $\mathrm{Tr}_s Q\Gamma^\lambda$, $\mathrm{Tr}_s Q\Gamma^\sigma\Gamma_5$, respectively (up to normalization). Since k preserves the homogeneity of Q, both $kQ_{\mathbb{1}}k^\dagger$ and $kQ_5k^\dagger$ are (Γ_5-)even. However, Γ^λ and $\Gamma^\sigma\Gamma_5$ are odd, and hence the partial-traces over spinor space both vanish. ■

This argument doesn't apply to the Lorentz-tensor components, though. Nevertheless, with the help of the Euclidean-Weyl representation of $\gamma-$matrices and (4.30), a second statement can be made:

Lemma 4.2. *For all* $k \in K_0$*,* $kQ_{\mathbb{1}}k^\dagger$ *has no Lorentz-tensor components and neither does* $kQ_5k^\dagger$*.*

Proof. By (4.30) we see

$$
\begin{aligned}
kQ_{\mathbb{1}}k^\dagger &\propto \begin{pmatrix} u_s \otimes h & 0 \\ 0 & v_s \otimes (h^{-1})^\dagger \end{pmatrix} \begin{pmatrix} \mathbb{1}_2 \otimes \hat{Q}_{\mathbb{1}} & 0 \\ 0 & \mathbb{1}_2 \otimes \hat{Q}_{\mathbb{1}} \end{pmatrix} \begin{pmatrix} u_s^\dagger \otimes h^\dagger & 0 \\ 0 & v_s^\dagger \otimes h^{-1} \end{pmatrix} \\
&= \begin{pmatrix} \mathbb{1}_2 \otimes h\hat{Q}_{\mathbb{1}}h^\dagger & 0 \\ 0 & \mathbb{1}_2 \otimes (h^{-1})^\dagger \hat{Q}_{\mathbb{1}} h^{-1} \end{pmatrix}.
\end{aligned}
$$

The Lorentz-tensor matrices $\gamma^\alpha\gamma^\beta$ have Pauli matrices on the diagonal, which are traceless. Therefore, $\mathrm{Tr}_s(kQ_{\mathbb{1}}k^\dagger)\Gamma^\alpha\Gamma^\beta = 0$ for any $\alpha < \beta$. The same argument obviously holds for $kQ_5k^\dagger$ as well. ■

These lemmas assure that none of the $K_0 - orbits$ runs across the border between different Lorentz-sectors. As a consequence, it is safe to parametrise the solutions in one sector by $K_0 - orbits$.

In combination with the proposal at the end of Section 4.1.2, we hereby conclude the quest for the energy hierarchy in dual-IYM with a plan to fulfil: first, we partition the target space of the dual-field to three sectors distinguished according to their Lorentz-types. Each sector is a collection of K_0−orbits. Next, we extend the dual-EFT action by including Lorentz-vector and Lorentz-tensor components systematically. A possible strategy is to follow the footsteps of HChPT.

Stationary phase approximation?

We take a last look at the color-neutralisation. For the same argument around (4.34)-(4.35), even if we drop the Lorentz-vector and tensor components in Q, in general $Q = \mathbb{1}_4 \otimes \hat{Q}_{\mathbb{1}} + \gamma_5 \otimes \hat{Q}_5$ cannot be rotated to a desired representative by K_0. Certainly, more information is needed in order to address the question posted at the end of Section 4.2.4, i.e., how to properly color-neutralise dual-IYM. We end this section by a short survey of the free action $\mathrm{S}_{\mathrm{free}}$ in dual-IYM.

Let us temporarily ignore the condition of $N_c = N_f = N$, which was devised for the benefit of color-neutralisation. By the design of DYM, $\mathrm{S}_{\mathrm{free}}$ is nothing but a free Dirac action with *tachyonic masses* $M\Gamma_5$, $m\Gamma_5$, and in our setting the weight factor $\exp(-\mathrm{S}_{\mathrm{free}})$ is oscillating. As an exercise, we solve the equations of motion in dual-IYM below, which is a tachyonic Dirac equation subject to ZC.

The boson-sector equation of motion is

$$0 \overset{!}{=} ((\Gamma^\mu)^{\mathrm{T}}\partial_\mu + M\Gamma_5^{\mathrm{T}})\phi \equiv (\partial_\mu\phi)\Gamma^\mu + M\phi\Gamma_5. \tag{4.36}$$

Note that $\phi(x)$ is a $N_c \times 4N_f$ matrix. It suffices to consider a "plane-wave":

$$\phi_p(x) = u(p)\mathrm{e}^{-ip_\mu x^\mu}, \tag{4.37}$$

where $x^\mu \in \mathbb{R}$ but we need $p_\mu \in \mathbb{C}$, because it obeys the consistency equation:

$$p_\mu p^\mu \equiv p_0p_0 + p_1p_1 + p_2p_2 + p_3p_3 \overset{!}{=} -M^2 < 0. \tag{4.38}$$

Up to normalization, the solution reads

$$u(p) = \begin{pmatrix} \xi(-p_0 - ip_k\sigma^k) & \xi M \end{pmatrix} \tag{4.39}$$

with an arbitrary $N_c \times 2N_f$ matrix ξ. Now, the Spin(4)-covariance invites us to pick a convenient reference frame and a natural choice is the "rest frame" $(p_0 = iM, 0, 0, 0)$. Plugging (4.37) back to (3.41) and let $\xi \equiv \begin{pmatrix} \xi_1 & \xi_2 \end{pmatrix}$ where ξ_1, ξ_2 are $N_c \times N_f$, the ZC equations become

$$\begin{cases} \xi_1\xi_2^\dagger = 0 = \xi_2\xi_1^\dagger; & \text{(4.40a)} \\ \xi_1\xi_1^\dagger = \xi_2\xi_2^\dagger. & \text{(4.40b)} \end{cases}$$

These equations indicate the following fact which helps us find exemplary solutions:

Lemma 4.3. *Let $r(\xi_1)$, $r(\xi_2)$ ($n(\xi_1)$, $n(\xi_2)$) be the ranks (nullities) of ξ_1, ξ_2 respectively, then*

$$r(\xi_1) = r(\xi_2) \le \frac{N_f}{2}. \tag{4.41}$$

Proof. Both ξ_1, ξ_2 are linear maps from $\mathbb{C}^{N_f} \to \mathbb{C}^{N_c}$. Equation (4.40a) tells us $\mathrm{r}(\xi_2) = \mathrm{r}(\xi_2^\dagger) \leq \mathrm{n}(\xi_1)$; by the rank-nullity theorem $\mathrm{n}(\xi_1) = N_f - \mathrm{r}(\xi_1)$ so we have $\mathrm{r}(\xi_1) + \mathrm{r}(\xi_2) \leq N_f$ (same under $1 \leftrightarrow 2$). Next, equation (4.40b) shows $\mathrm{r}(\xi_1) = \mathrm{r}(\xi_1\xi_1^\dagger) = \mathrm{r}(\xi_2\xi_2^\dagger) = \mathrm{r}(\xi_2)$, and together with the previous result we obtain (4.41). ■

There are non-trivial zero-current classical solutions. When $N_f = 2 = N_c$, there exists a solution ξ that

$$\xi_1 = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}; \quad \xi_2 = \begin{pmatrix} c & 0 \\ d & 0 \end{pmatrix}. \quad |a|^2 = |c|^2;\ |b|^2 = |d|^2;\ a\bar{b} = c\bar{d}. \tag{4.42}$$

A naive stationary phase approximation via rescaling doesn't work at this stage, unfortunately, because ϕ itself depends on both N_c and N_f. Ideally, this step should be carried out after a **properly color-neutralised** dual-IYM is obtained, and the appropriate large parameter to use will be N_c. This aligns with the similar observation made in dual-BZN, see Section 2.8.3. We would like to point out the fascinating possibility of a large$-N_c$ analysis on the both sides of DYM. In IYM, the conventional large$-N_c$ diagrammatic technique [120] is a powerful tool to study the strong-coupling dynamics of YM. In a color-neutralised dual-IYM, the large$-N_c$ may grant us a valid semi-classical approximation. This blueprint is in agreement with the celebrated gauge/gravity duality mentioned in Section 1.2. See also [121]. We would however like to emphasize again the universality condition $N_f \geq N_c$ mentioned in Section 2.8.3. If it turns out a similar condition is needed to keep IYM in the YM universality class, then the implicit N_c-dependence will certainly compromise the stationary phase approximation. On the contrary, if N_f can be "decoupled" from N_c in DYM, then such an approximation does look promising in the color-neutralised dual-IYM.

An interesting, but ambitious query would be to look at a certain subset of the gluon field A_μ in IYM. As reviewed in Section 1.2, the CDGFN decomposition of the gauge field singles out some monopole degrees of freedom, which are believed to be responsible for quark confinement. Inspired by this, instead of integrating out all possible configurations of A_μ, one can try to only integrate over the monopole-domain. Intuitively, the ZC should be replaced by a new, "softer" condition because less degrees of freedom of the Dirac fields are constrained. We haven't done anything in this direction so far.

4.3 Summary and outlook for DYM

In Chapter 3, we have reviewed the model of DYM and derived again the IYM action, and then we embarked on a research of dual-IYM. Inspired by some resemblances of dual-IYM to QCD, we envisioned that dual-IYM should describe a system of color-neutral composite fields. A remarkable connection between the gluon condensate in IYM and the boson/fermion field condensates in dual-IYM suggests a spontaneous breaking of some symmetry group. On the basis of this postulate, we first defined the relevant dual symmetry groups and revealed their properties, and then we established a theory of dual-EFT by consulting the original derivations of ChPT. Only the bosonic symmetries were taken into consideration, and the effects of fermionic symmetries were left for future studies.

In this chapter, we attempted to go beyond dual-EFT and explicitly derive a color-neutral version of dual-IYM. To color-neutralise the action, a $d = 4$ functional-integral formalism of Witten's non-Abelian bosonization method was considered. The non-compactness of the dual symmetry group inevitably results in a polynomial divergence in the effective action for the external fields. Consequently, we could not advance any further in the derivations. Nevertheless, some features of the resulting effective action might be useful references when it comes to speculating an appropriate action for some dual-field.

A natural candidate for the dual-field is the super-meson field, but due to the rank-deficiency of its boson-boson submatrix, the meson field, a direct application of the SuB formula is not viable. To resolve this, we devoted much effort to solving ZC. First, it was shown that unlike the $N = 1$ case, the $N \geq 2$ dual-IYM contains dynamical fields. More significantly, every Lorentz component of the dual-field has a role to play in dual-IYM, which suggests the limited applicability of dual-EFT. The project of parametrizing the ZC solution space however is not yet completed. As a compromise, we developed a reasonable recipe to partition the solution space to several K_0-orbits, which in turn supports the idea of energy hierarchy in dual-IYM.

The issue of rank-deficiency is still unsolved. Several proposals were tested but failed. Consequently, it seems that the ultimate goal of an exact color-neutralisation has to wait until a suitable generalization of SuB is available to us. In our vision, an ideal form of dual-IYM is an NLσM for some supermatrix fields. This NLσM should include all Lorentz-types mesons, which are arranged according to the energy-hierarchy scheme. In this scenario, the scalar-field coupling should have a negative mass dimension which makes dual-IYM non-renormalizable. This favors the possibility that the coupling, in opposition to the gauge-coupling of YM, decreases as the energy-scale is reduced. If this turns out to be true, we will be able to argue that there is a S-duality between IYM and dual-IYM. For the time being, this still seems like a dream to us, and the presence of the vector mesons most likely will complicate the situation, of course.

Chapter 5

Conclusion

In this dissertation, we have applied the concept of induced gauge theory to construct dual descriptions of U(N_c) YM in $d = 4$. We have considered a lattice-regularized model (BZN) and a heat-kernel-regularized model (DYM). In both cases, the induced actions for the gluons can be derived by integrating out auxiliary matter fields in the respective master actions. Under suitable conditions, we believe that the induced actions lie in the same universality class of YM. For either BZN or DYM, our main objective is to understand the induced theory for the matter fields, which is equivalent to YM by design. This is known as a dual theory of YM. Significantly, the master action shows resemblance to a disordered system of the matter fields, where the gluon degrees of freedom play the role of the disordered couplings. In this interpretation, the dual theory is a strongly-disordered system. Motivated by this, we have examined the dual theories in both models by some modern mathematical methods developed for strongly-disordered systems.

Master action

At the core of induced gauge theory is a thoughtfully invented master action. For the construction of a master action for BZN, we have used Gaussian-integral representation to introduce auxiliary bosons and fermions. To ensure BZN flows to YM in a continuum-limit, we have demanded $N_f \geq N_c$ and $m > M > 0$, where m and M are the fermion mass and the boson mass. The continuum-limit is attained as $M \to 0$. For DYM, the master action consists of Dirac bosons and Dirac fermions coupled to the same gluon field. Both Dirac fields have tachyonic mass terms, and as long as the boson mass M_B is greater than the fermion mass M_F, integrating out these auxiliary fields yields the YM action at least to the leading-order. This induced theory is known as IYM, and we have argued that the approximation may become exact in the large$-N_f$ limit.

Duality transformation

The concrete results in this work are two recipes for duality transformations. For BZN, the presence of both bosons and fermions makes the integration of the gluons accessible. We have demonstrated how to use the Cayley parametrisation to transform the U(N_c) Haar measure into a measure over the space of Hermitian matrices. By the standard ribbon-diagrammatic technique, the gluon-integral results in a large$-N_c$ series of composite variables in the effective action. For DYM, integrating out the gluon field projects the Dirac fields to a configuration space with zero color current in the case where U(N_c) is the color group. If the color group were SU(N_c), the color current would be valued in the one-dimensional subspace generated by the identity matrix. We have only studied the U(N_c)$-$induced theory in this dissertation, and this theory is known as dual-IYM.

Color neutralisation

To fully comprehend the dual theories in both BZN and DYM, we have tried to perform exact transformations of the original matter fields to some color-neutral variables. For BZN, we have utilized the fact that the auxiliary variables live on the links, and performed a color-averaging trick. In combination with the gluon-integral contributions, we have obtained a dual action written explicitly in terms of composite particles built from the original bosons and fermions. This theory is known as dual-BZN. For dual-IYM, while there is a natural choice for the color-neutral object, the super-meson field, so far we have not successfully turned the action into a color-neutral form. The attempt of devising a dual action using Witten's non-Abelian bosonization method leads to a divergent effective action for the external field, which makes even an educated guess for the dual action difficult.

To complete the color neutralisation program, a promising machinery to use is the SuB formula. For dual-BZN, we have explained the conflict between the universality condition $N_f \geq N_c$ and the SuB condition $q \leq n$. For dual-IYM, we have explored the boson-boson sector of the ZC solution space by two parallel methods: BU and TD. With the help from the massless symmetry group, the computations in both BU and TD can be greatly simplified, but a complete parametrisation of the solution space has not yet been found. Based on the few exemplary solutions we have discovered, it seems that the SuB formula is not directly applicable in dual-IYM either due to a similar rank-deficiency in the composite operator. Nevertheless, there is hope that in the advent of a new version of SuB, we can revisit the results obtained in this work and carry out the task of color-neutralisation.

Physical perspectives

Even without exact color neutralisation, it is possible to unravel some useful information from the dual theories. For dual-BZN, we have developed a method to compute the Wilson loop expectation value on the dual side. In the boson-boson sector, we have also identified the masses and the interaction strength from the dual action; furthermore, we have obtained some preliminary results regarding symmetry breaking. So far, our understanding of dual-BZN is quite limited due to the unknown higher-order terms in the dual action. Moreover, while the dual action suggests an application of a large$-N_c$ analysis in a properly color-neutralised version of dual-BZN, it may not be a semi-classical approximation due to the universality condition.

For dual-IYM, we have manufactured a low-energy effective field theory, known as dual-EFT, based on a chiral-type symmetry breaking. Using the DYM master action, we have postulated a possible "coexistence" of a gluon condensate in IYM and boson/fermion condensates in dual-IYM. Then by analogy with ChPT, we have recognized a pattern of spontaneous symmetry breaking in the massless symmetry group, and built an action for dual-EFT around it. There are two possible applications of dual-EFT: YM mass gap and quark confinement, and we have presented some early-stage results. Based on a further study on the massless symmetry group and ZC, we have conjectured that dual-IYM describes a system of color-neutral mesons of all Lorentz-types. Since dual-EFT only contains the Lorentz-scalar mesons, we believe that it only captures the low-energy dynamics.

Outlook

The evidences we have collected in this work are not enough to ascertain these conjectures of different aspects of dual-IYM. Nevertheless, we would like to portray the final shape of dual-IYM, which seems most fascinating to us. Being a color-neutral theory, it is unlikely that dual-IYM is a suitable model for a potential realization of the dual superconductivity. The reason is that the chromomagnetic monopoles also carry color indices. In comparison, the possibility of dual-IYM being a gravitational theory is higher.

First of all, a consideration of large$-N_c$ expansion in DYM seems straightforward. In IYM we would have a planar-diagram series expansion, whereas in dual-IYM (after color-neutralisation) a semi-classical approximation might be accessible. Under the assumption that the universality condition in DYM might be "softer" than the one in BZN, this *planar-diagram* versus *semi-classical* scenario bridges a *strongly-coupled* gauge theory to a *semi-classical* dual theory. This is compatible with the framework of AdS/CFT-duality. One should however be reminded that, in our current understanding, the prospective IYM/dual-IYM pair of theories both live in a $d = 4$ spacetime, which is different from the conventional AdS/CFT.

More specifically, if dual-IYM turns out to be an NLσM, two good things could happen. In the sector of dual-EFT, the NLσM coupling in $d = 4$ generally has a negative mass dimension. Therefore, in opposition to the YM coupling, it might decreases in the IR. In this case, there is hope that dual-IYM is in S-duality with IYM. However, the fact that dual-IYM contains all Lorentz-types composite operators would certainly introduce more terms to dual-EFT. With regard to this, we have developed an energy-hierarchy scheme, which can be improved by a further renormalisation-group analysis. On a related note, there has been a claim of some analogies between the theory of general relativity and a NLσM [122], which seems like an interesting direction to consider in the future.

We hope that our work did provide a fresh point of view for a better understanding of the low-energy physics of YM. Certainly, there is still a long road ahead of us [1].

[1] Towards the truth, and/or towards the one-million dollar prize.

Appendix A

Weingarten Functions

We borrowed the notations and examples from [123]. The Weingarten function $\mathrm{W}(\alpha, N_c)$ only depends on the cyclic structure of $\alpha \in \mathrm{S}_k$. For instance, in S_3, the permutations $\alpha = (1,2)(3)$ and $\beta = (1,3)(2)$ share the same cyclic structure "[2,1]": a transposition and a fixed point. Therefore, they contribute the same Weingarten function $\mathrm{W}([2,1], N_c)$.

The following values were originally taken from [84], but we only used the first three of them for this dissertation ($N_c \equiv N$).

$$
\begin{aligned}
\mathrm{W}([1], N) &= \frac{1}{N};\\
\mathrm{W}([1,1], N) &= \frac{1}{N^2-1};\\
\mathrm{W}([2], N) &= \frac{-1}{N(N^2-1)};\\
\mathrm{W}([1,1,1], N) &= \frac{N^2-2}{N(N^2-1)(N^2-4)};\\
\mathrm{W}([2,1], N) &= \frac{-1}{(N^2-1)(N^2-4)};\\
\mathrm{W}([3], N) &= \frac{2}{N(N^2-1)(N^2-4)}.
\end{aligned}
$$

APPENDIX B

LINKED-CLUSTER EXPANSION

Although the linked-cluster principle is well-known, we present a version tailored to BZN here. The following derivation is adapted from [124].

First we write

$$\int Dg\ \mathrm{e}^{-\sum_{\mathbf{p}\in\Lambda}\Gamma_s^g(\mathbf{p})} \equiv \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\mathbf{p}_1,\cdots\mathbf{p}_n}\mathfrak{G}(\mathbf{p}_1,\ldots,\mathbf{p}_n), \tag{B.1}$$

where $\mathfrak{G}(\mathbf{p}_1,\ldots,\mathbf{p}_n)$ stands for all diagrams involving the configuration of plaquettes $(\mathbf{p}_1,\ldots,\mathbf{p}_n)$.

Next, decompose every $\mathfrak{G}$ to a product of connected components $\mathfrak{G}^{\mathrm{c}}$ and transform (B.1) to

$$\begin{aligned}1+\sum_{n\geq 1}^{\infty}\frac{1}{n!}\sum_{k=1}^{n}\frac{1}{k!}\sum_{\substack{m_1,\ldots,m_k\geq 1;\\ m_1+\cdots+m_k=n}}\frac{n!}{m_1!\cdots m_k!}\sum_{\mathbf{p}_1,\cdots\mathbf{p}_n}\\ \mathfrak{G}^{\mathrm{c}}(\mathbf{p}_{i_1},\ldots,\mathbf{p}_{i_{m_1}})\cdots\mathfrak{G}^{\mathrm{c}}(\mathbf{p}_{i_k},\ldots,\mathbf{p}_{i_{m_k}}),\end{aligned} \tag{B.2}$$

where $1/k!$ is needed to prevent over-counting.

Finally, by the re-summation formula

$$1+\sum_{n\geq 1}^{\infty}\frac{1}{n!}\sum_{k=1}^{n}\frac{1}{k!}\sum_{\substack{m_1,\ldots,m_k\geq 1;\\ m_1+\cdots+m_k=n}} = 1+\sum_{k=1}^{\infty}\sum_{n\geq k}\sum_{\substack{m_1,\ldots,m_k\geq 1;\\ m_1+\cdots+m_k=n}} = 1+\sum_{k=1}^{\infty}\sum_{m_1\geq 1}\cdots\sum_{m_k\geq 1}$$

we arrive at

$$1+\sum_{k=1}^{\infty}\frac{1}{k!}\left[\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{\mathbf{p}_1,\cdots\mathbf{p}_m}\mathfrak{G}^{\mathrm{c}}(\mathbf{p}_1,\ldots,\mathbf{p}_m)\right]^k \equiv \mathrm{e}^{-\mathrm{S}_{\mathrm{eff,L}}}.$$

Or

$$\mathrm{S}_{\mathrm{eff,L}} = \sum_{m=1}^{\infty}\frac{1}{m!}\sum_{\mathbf{p}_1,\cdots\mathbf{p}_m}\mathfrak{G}^{\mathrm{c}}(\mathbf{p}_1,\ldots,\mathbf{p}_m). \tag{B.3}$$

APPENDIX C

SU(2)–INDUCED CONDITION ON COLOR-CURRENT

When the color group is SU(2), the integration of A_μ in (3.35) only takes out the components of J^μ in Lie(SU(2)). Using the fact that

$$X = -\varepsilon^{-1} X^{\mathrm{T}} \varepsilon \qquad \forall X \in \mathrm{Lie}(\mathrm{SU}(2)),$$

where $\varepsilon \equiv i\sigma^2$, the Lie(SU(2)) components of J^μ are $(J^\mu - \varepsilon^{-1}(J^\mu)^{\mathrm{T}}\varepsilon)/2$ and hence ZC becomes

$$J^\mu - \varepsilon^{-1}(J^\mu)^{\mathrm{T}}\varepsilon \stackrel{!}{=} 0 \qquad \forall \mu. \tag{C.1}$$

In the boson-boson sector, (C.1) is equivalent to $(\forall \mu)$

$$\begin{aligned} 0 &\stackrel{!}{=} \mathrm{tr}(X^\mu - \varepsilon^{-1}(X^\mu)^{\mathrm{T}}\varepsilon)(X^\mu - \varepsilon^{-1}(X^\mu)^{\mathrm{T}}\varepsilon)^\dagger \\ &\propto \mathrm{tr} X^\mu (X^\mu)^\dagger - \mathrm{tr}\varepsilon^{-1}(X^\mu)^{\mathrm{T}}\varepsilon(X^\mu)^\dagger, \end{aligned} \tag{C.2}$$

where $X^\mu \equiv \phi \Gamma^\mu \bar{\phi}$.

Introducing one *meson* matrix $Q \equiv \bar{\phi}\phi$ and two *baryon* matrices $B \equiv \bar{\phi}\epsilon^{-1}\bar{\phi}$, $C \equiv \phi\epsilon\phi$, we can rewrite (C.2) as

$$\mathrm{Tr}\Gamma^\mu Q (\Gamma^\mu)^\dagger Q - \mathrm{Tr}(\Gamma^\mu)^{\mathrm{T}} C (\Gamma^\mu)^\dagger B = 0. \tag{C.3}$$

To simplify the equations further, we can incorporate the "baryon number" into the spinor-flavor space by defining

$$\begin{aligned} &\tilde{\Gamma}^\mu \equiv \mathbb{1}_2 \otimes \Gamma^\mu; \\ &\mathscr{T} \equiv \tau \otimes \mathbb{1}_2, \quad \tau \equiv \mathrm{diag}(\varepsilon, \varepsilon); \\ &\mathscr{Q} \equiv \begin{pmatrix} Q & B\mathscr{T} \\ \mathscr{T}^{-1}C & -\mathscr{T}^{-1}Q^{\mathrm{T}}\mathscr{T} \end{pmatrix}. \end{aligned}$$

Then (C.3) is written in the following compact form

$$\tilde{\mathrm{Tr}}\tilde{\Gamma}^{\mu}\mathscr{Q}\tilde{\Gamma}^{\mu}\mathscr{Q}=0 \qquad \forall\mu. \tag{C.4}$$

By construction, $\mathscr{Q}$ has the following properties:

$$\mathscr{Q}=(\mathscr{Q})^{\dagger}, \qquad \mathscr{Q}=-\Sigma^{-1}\mathscr{Q}^{\mathrm{T}}\Sigma,$$

where $\Sigma\equiv i\sigma^2\otimes\mathscr{T}$. One can say $i\mathscr{Q}$ lives in Lie(USp(8)).

Appendix D

Single-Flavor Massless Symmetry Group

A proof regarding the homogeneity of the full symmetry group K for the case of $N_f = 1$ is presented here.

Decompose any $k \in K$ into the *even* part (commuting with γ_5) and the *odd* part (anticommuting with γ_5) $k \equiv k_{\mathrm{E}} + k_{\mathrm{O}}$. In the Euclidean-Weyl representation they look like

$$k_{\mathrm{E}} \equiv \begin{pmatrix} k_+ & 0 \\ 0 & k_- \end{pmatrix}; \quad k_{\mathrm{O}} \equiv \begin{pmatrix} 0 & \tilde{k}_+ \\ \tilde{k}_- & 0 \end{pmatrix}. \tag{D.1}$$

Lemma D.1. *When $N = N_f = 1$, any $k \in K$ is homogeneous. That is, either $k = k_E$, or $k = k_O$.*

Proof. Any $k \in K$ is *invertible* and it satisfies

$$k\gamma^\mu k^\dagger = \Lambda^\mu{}_\nu \gamma^\nu \quad \forall \mu. \tag{D.2}$$

Since the RHS of (D.2) is always *odd*, it is necessary that

$$k_{\mathrm{E}}\gamma^\mu k_{\mathrm{O}}^\dagger + k_{\mathrm{O}}\gamma^\mu k_{\mathrm{E}}^\dagger = 0 \quad \forall \mu. \tag{D.3}$$

Using (D.1), (D.3) reads

$$k_+\tilde{k}_+^\dagger + \tilde{k}_+ k_+^\dagger = 0; \quad k_+\sigma^l \tilde{k}_+^\dagger - \tilde{k}_+\sigma^l k_+^\dagger = 0 \;\forall l. \tag{D.4}$$

and likewise for $k_-, \tilde{k}_-$. Here, $k_+, \tilde{k}_+$ are 2×2 matrices, so the algebra is easy and one concludes that for (D.4) to be true, either $k_+ = 0$ or $\tilde{k}_+ = 0$. Similarly we also need either $k_- = 0$ or $\tilde{k}_- = 0$. As the result, if one insists that $k_{\mathrm{E}} \neq 0$ and $k_{\mathrm{O}} \neq 0$, then for (D.3) to hold it is necessary that we turn one (and only one) out of the two blocks in k_{E} to zero and same for k_{O}, which in turn results in

$$k_{\mathrm{E}}\gamma^\mu k_{\mathrm{E}}^\dagger = 0; \; k_{\mathrm{O}}\gamma^\mu k_{\mathrm{O}}^\dagger = 0 \quad \forall \mu. \tag{D.5}$$

This contradicts that fact that the RHS of (D.2) must be *invertible* (hence nonvanishing). ■

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