Quantum Superspin Systems from Conformal Field Theory

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To my family.

To W.W.
My star, my perfect silence.

How happy is the little stone
That rambles in the road alone,
And doesn't care about careers
And exigencies never fears–
Whose coat of elemental brown
A passing universe put on,
And independent as the sun
Associates or glows alone,
Fulfilling absolute decree
In casual simplicity–

– Emily Dickinson

Kurzzusammenfassung

Phasenübergänge zweiter Ordnung spielen in der Festkörperphysik eine besondere Rolle. In ihrer Natur liegt es, dass sie ausschließlich in Systemen auftreten, die sich aus unendlich vielen Einzelteilen zusammensetzen, da nur so die Voraussetzungen (bspw. Skaleninvarianz) hierfür gewährleistet werden können. Ein solches System an einem Phasenübergang zweiter Ordnung bezeichnet man als *kritisch*. Die unendlich große Teilchenzahl stellt schon für die Beschreibung klassischer Systeme eine besondere Herausforderung dar. Quantenmechanische Systeme hingegen sind in diesem Zusammenhang ungleich viel schwieriger zu behandeln, da die Dimension ihres Zustandsraums exponentiell mit der Anzahl ihrer Bestandteile wächst, ganz abgesehen davon, dass nur wenige – und meist auch nur eindimensionale – Quantensysteme exakt lösbar sind. Eines dieser exakt lösbaren eindimensionalen Quantensysteme ist die $\mathrm{SU}(N)$ Haldane-Shastry-Spinkette, die man wohl als Archetyp langreichweitiger Spinketten bezeichnen darf. Darüberhinaus ist sie kritisch im Kontinuumslimes und ihre effektive Niedrigenergie-Theorie ist das sog. $\mathrm{SU}(N)_1$ Wess-Zumino-Witten-Modell. Dabei handelt es sich um eine Quantenfeldtheorie, die nicht nur konform invariant, also insbesondere skaleninvariant, ist, sondern darüberhinaus noch zusätzliche Symmetrie in Form einer unendlichen Erweiterung der mit $\mathrm{SU}(N)$ assoziierten Lie Algebra $\mathfrak{su}(N)$ aufweist. Jüngste Untersuchungen zeigen, dass sich aus ebenjenen Strukturen des $\mathrm{SU}(N)_1$ WZW-Modells wiederum Spinsysteme ableiten lassen, deren Anordnung nicht zwangsläufig der einer Spinkette entsprechen muss; auch zweidimensionale Verteilungen der Spins in der Ebene sind möglich. Diese Systeme zeichnen sich wiederum durch langreichweitige Wechselwirkungen aus, vergleichbar mit denen der Haldane-Shastry-Spinkette, die man im Übrigen bei entsprechender Wahl der Anordnung wiederum auch erhält.

In dieser Arbeit erweitern wir die schon für den klassischen $\mathrm{SU}(N)$-Fall bekannte Konstruktion auf den supersymmetrischen Fall von $\mathrm{GL}(m|n)$. Hierbei konstruieren wir explizit sowohl einen speziellen Quantenzustand als auch einen Hamiltonoperator, der diesen Quantenzustand auf Null projiziert. Desweiteren diskutieren wir den Hamiltonoperator im Spezialfall der $\mathrm{GL}(1|1)$ Spinkette und vergleichen diese mit der entsprechenden $\mathrm{GL}(1|1)$ Haldane-Shastry-Spinkette auf einem bipartiten Zustandsraum. Beide sind kritisch und wir identifizieren die entsprechenden konformen Feldtheorien. Im Anschluss beschreiben wir eine Verallgemeinerung dieses Systems in Abhängigkeit von zwei Parametern und erklären, wie das Spektrum hierfür gefunden wurde. Dieses wird analysiert und sein Kontinuumslimes wird bestimmt. Hierbei zeigt sich, dass dieses System nur dann kritisch ist, wenn ausschließlich einer der beiden Parameter variiert werden darf.

Abstract

Phase transitions of second order play an important role in solid state physics. It is in their nature that they occur only in systems that are composed from an infinite number of components since, only that way, the necessary conditions for this (for example, scale invariance) are granted. Such a system at a second order phase transition is called *critical*. The infinite number of particles poses a particular challenge, even for the description of classical systems. Quantum mechanical systems, however, are distinctly more difficult to treat in this context, since the dimension of their state space grows exponentially with the number of their particles, not to mention the fact that only a few – and usually only one-dimensional – quantum systems are exactly solvable. One of these exactly solvable one-dimensional quantum systems is the $\mathrm{SU}(N)$ Haldane-Shastry spin chain that may be regarded as the archetype of long-range spin chains. Moreover, it is critical in the continuum limit and its effective low-energy theory is the so-called $\mathrm{SU}(N)_1$ Wess-Zumino-Witten model. It is a quantum field theory which is not only conformally invariant, so, in particular, scale invariant, but, furthermore, exhibits additional symmetry in the shape of an infinite extension of the Lie algebra $\mathfrak{su}(N)$ associated with $\mathrm{SU}(N)$. Recent studies show that, from these very structures of the $\mathrm{SU}(N)_1$ WZW model, one can, in turn, derive spin systems, whose arrangement is not necessarily the one of a spin chain, but even two-dimensional distributions of the spins in the plane are possible. These systems are again characterized by long-range interactions, comparable to those of the Haldane-Shastry spin chain, which is also obtained as a result of an appropriate choice of parameters.

In this thesis, we extend the construction already known for the $\mathrm{SU}(N)$ case to the supersymmetric case of $\mathrm{GL}(m|n)$. Here, we construct explicitly both, a special quantum state as well as a Hamiltonian that projects this quantum state to zero. We also discuss the Hamiltonian in the special case of the $\mathrm{GL}(1|1)$ spin chain and compare it to the respective $\mathrm{GL}(1|1)$ Haldane-Shastry spin chain on a bipartite state space. Both are critical and we identify the corresponding conformal field theories. Subsequently, we describe a generalization of this system in terms of two parameters and explain how its spectrum was found. It is then analyzed and its continuum limit is determined. In doing so, it shows that the system displays criticality only for generic values of one of the two parameters.

Contents

1

Setting the Stage

Not how the world is, but that it is, is the mystery.

– Ludwig Wittgenstein

Even though the foundations of quantum mechanics date back about 100 years and much progress concerning theoretical developments has been made in the field, actual calculations even in low-dimensional quantum mechanical systems are exceedingly costly and difficult when increasing the system size L. The reason for this is that the Hilbert space of such a system grows exponentially with L.

In order to tackle this problem, many useful tools have been developed. Some more general ones aim at implementing efficient algorithms for exact diagonalization of the Hamiltonian describing the system at hand.

Restricting the number of dimensions to just one, very special techniques can be employed if the system is fully integrable and can, thus, be solved analytically. Even though this may sound simple, it involves the use of a huge machinery, e.g. Bethe ansatz [1] and more refined version of the same.

In some more general one-dimensional cases, the part of the states whose entropy obeys an area law can be approximated through a *matrix product state* (MPS) [2]. Let the Hilbert space $\mathcal{H}$ be given by $\mathcal{H} = \mathcal{V}^{\otimes L}$ with an *orthonormal basis* (ONB) $\{|i\rangle\}_{i=1,\dots,d}$ on $\mathcal{V}$. An ONB on $\mathcal{H}$ is then given by the set of all combinations of the form $|i_1 \dots i_L\rangle := \prod_{j=1}^{L} |i_j\rangle$. The MPS interpolates between a product state without entanglement

$$|\Psi_{\text{classical}}\rangle = \sum_{i_1,\dots,i_L=1}^{d} A_{i_1}^{[1]} \cdots A_{i_L}^{[L]} |i_1 \dots i_L\rangle \,, \tag{1.1}$$

described by $d \cdot L$ coefficients, and a possibly fully entangled quantum state

$$|\Psi_{\text{fully entangled}}\rangle = \sum_{i_1,\dots,i_L=1}^{d} C_{i_1 \dots i_L} |i_1 \dots i_L\rangle \,, \tag{1.2}$$

described by d^L coefficients, where d is the dimensionality of the on-site Hilbert spaces $\mathcal{V}$. An MPS is fully described by $D^2 \cdot d \cdot L$ coefficients where D is an integer quantifying the

dimension of certain matrices that belong to these on-site Hilbert spaces:

$$|\Psi_{\mathrm{MPS}}\rangle = \sum_{i_1,\ldots,i_L}^{d} \sum_{\alpha,\ldots,\delta}^{D} A^{[1]}_{i_1,(\alpha\beta)} \cdot A^{[2]}_{i_2,(\beta\gamma)} \cdots A^{[L]}_{i_L,(\delta\alpha)} \, |i_1 \ldots i_L\rangle \,. \tag{1.3}$$

The bond dimension D can be tuned in order to reach the desired compromise between precision and convenience. The important observation here is that the necessary amount of information is linear in system size L.

Nevertheless, systems at criticality, in principle, become impossible to handle in this fashion as the bond dimension D of the MPS scales exponentially in L [3].

A possible way to solve this problem is the use of an *infinite-dimensional matrix product state* (IMPS) which is a generalization of the MPS through the use of chiral vertex operators from a two-dimensional *conformal field theory* (CFT). As it makes use of operator valued field insertions $\phi(z_j)$ instead of matrices $A^{[j]}_{i_j,(\alpha\beta)}$, it evades the limitation of a finite bond dimension D. Inspired by Moore and Read [4], one such construction was proposed by Cirac and Sierra in [5] in order to approximate ground state wave functions of spin chains that, in particular, can become critical for certain values of their parameters, e.g. the *XXZ spin chain*. This idea will be explained in the next Section.

1.1 The General Idea

As just mentioned, an IMPS will take the form

$$|\Psi_{\mathrm{IMPS}}(z_1, \ldots, z_L)\rangle = \sum_{i_1,\ldots,i_L=1}^{d} \langle \phi_{i_1}(z_1) \cdots \phi_{i_L}(z_L)\rangle \, |i_1 \ldots i_L\rangle \,, \tag{1.4}$$

by analogy with eq. (1.3), where z_j simply parametrizes the location of sites j in the complex plane. However, now the coefficients of the expansion in the usual basis $|i_1 \ldots i_L\rangle$ are computed by finding the vacuum expectation value (instead of taking the trace) of the product of L chiral fields $\phi(z_j)$ (instead of matrices) that belong to representations of certain algebras of two-dimensional CFTs and that fulfill model specific *operator product expansions* (OPE). In CFT, the object $\psi(\{z_m\}) := \langle \phi(z_1) \cdots \phi(z_L)\rangle$ is called a chiral correlator and it can, in principle, be calculated analytically from the OPE.

In [6], Nielsen, Cirac and Sierra have proposed an interesting construction for finding a parent Hamiltonian whose ground state is given by eq. (1.4).

1.1.1 Constructing the Hamiltonian

Their key observation was the existence of *null states* $\chi(z)$ in representations of CFTs (e.g. the $\mathrm{SU}(2)_k$ WZW model in their paper [6]). Since $\chi(z)$ decouples from the rest of the module

$$0 \equiv \langle \phi(z_1) \cdots \chi(z_i) \cdots \phi(z_L)\rangle, \tag{1.5}$$

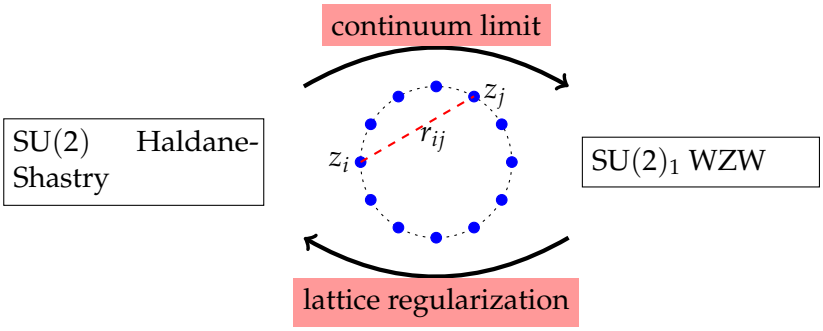


Figure 1.1: The correspondence between the SU(2) Haldane-Shastry spin chain and the SU(2) WZW model at level $k = 1$.

they are called null states. However, they can be traced back to the primary states $\phi(z)$ and thus the identity eq. (1.5) can be rewritten as

$$0 = \mathcal{P}_i(\{z_j\})\langle\phi(z_1)\cdots\phi(z_i)\cdots\phi(z_L)\rangle \tag{1.6}$$

where $\mathcal{P}_i(\{z_j\})$ is an operator carrying one adjoint index[1] (for simplicity of notation hidden at this point) of the global symmetry, acting on the full chiral correlator and, obviously, annihilating it. Repeating the same procedure with all of the field insertion at all points z_j, we find L such operators P_j, all of them annihilating the chiral correlator. To this end, we compute

$$H := \sum_{i=1}^{L} H_i := \sum_{i=1}^{L} (\mathcal{P}_i)^\dagger \mathcal{P}_i \tag{1.7}$$

which is hermitian, positive semi-definite, commutes with the global symmetry and annihilates the chiral correlator. Therefore, H must be a parent Hamiltonian[2] of the exact ground state given by eq. (1.4), which is analytically computable.

1.1.2 The $SU(2)_1$ WZW Model and the Haldane-Shastry Spin Chain

Having proposed the previously sketched construction, the authors of [6] exemplarily calculated the ground state and parent Hamiltonian for the SU(2)$_1$ WZW model with field insertions $\phi(z_j)$ distributed on the unit circle. What they found, is depicted in Figure 1.1. Making use of the symmetries of the chiral part of the SU(2)$_1$ WZW model via this construction, they obtained the Haldane-Shastry spin chain with spin-$\frac{1}{2}$ spins [7, 8], a generalization of the Heisenberg spin chain [9, 10], but with long-range interaction $\sim \mathbf{S}_i \cdot \mathbf{S}_j / r_{ij}^2$. Vice versa,

[1] The construction is not limit to only one adjoint index. Depending, e.g. on the level of the WZW model, there can be more adjoint indices.

[2] The word *parent* in this context indicates that one starts with the construction of a state and then derives a Hamiltonian associated with this state.

taking this Haldane-Shastry spin chain to the thermodynamic limit, it becomes critical and its low-energy spectrum can be analyzed using techniques called finite-size scaling [11, 12]: This way, one can extract its conformal anomaly $c = 1$ and conformal dimensions h_i, and these match the data of the $\mathrm{SU}(2)_1$ WZW model exactly. In this sense, one can refer to this construction as a discretization of the $\mathrm{SU}(2)_1$ WZW model. In the meantime it has become clear that this correspondence also carries over to $\mathrm{SU}(N)$ [13, 14].

An immediate question is whether this holds for other symmetries, e.g. $\mathrm{GL}(m|n)$. Also, and more generally, it seems worthwhile to treat the $\mathrm{GL}(m|n)$ WZW model in the same way as, e.g. $\mathfrak{gl}(1|1)$ is related to supersymmetric quantum mechanics. Additionally, supersymmetry plays an important role for the description of disordered systems [15]. Furthermore, WZW models based on supergroups describe the physics of dilute polymers [16], critical systems with quenched disorder [17] and percolation [18]. Moreover, as mentioned in [19], most of the minimal unitary CFTs that have been solved so far are actually not suitable as an ad-hoc realistic description of critical systems in, e.g. condensed matter physics because only a few of them come with no more than the necessary operator content to describe system at hand making any fine-tuning superfluous. On the other hand, it has become clear that many systems of statistical physics are, in a way, nonlocal in character with their observables, such as critical exponents, rather belonging to non-unitary CFTs with conformal anomaly $c = 0$ and, hence, described by a Hamiltonian which is not hermitian. Further, it is also expected that the two-dimensional sigma model on a super-coset space associated with the plateaux transition in the *integer quantum Hall effect* [15, 20–22] flows to a strongly interacting CFT with central charge zero [23]. Therein, it is also mentioned that this system can be mapped to a Heisenberg-like superspin chain of specific alternating infinite-dimensional $\mathfrak{gl}(2|2)$ representations that were first identified by Read (cf. reference [47] in [23], private communication between the authors). Until now, however, there exists no solution for these spin chains and, hence, no idea about their thermodynamic limit and little is known about the governing CFT. However, also therein, the hope was articulated that a deformation of this spin chain to a Haldane-Shastry-like spin chain would enhance the symmetry of the spin chain in order to be able to diagonalize it and constrain its critical behavior.

The general idea for the construction of spin chain Hamiltonians from CFT put forward by Nielsen, Cirac and Sierra might prove powerful enough to shed some light on this matter because it allows for the construction of various spin systems (on two-dimensional or one-dimensional lattices, regular or irregular, and, in particular, Haldane-Shastry-like spin chains) from any CFT that features null states. One can then hope for such a spin system of Haldane-Shastry-type to be critical and to disclose some information about the CFT it is described by. It will be shown in this thesis that this is the case for a system of alternating fundamental and antifundamental $\mathfrak{gl}(1|1)$-spins. Additionally as this case shows, these Haldane-Shastry-like spin chains seem to capture the properties of the CFT particularly well since their dispersion relation are given by the simplest relation capable of exhibiting relativistic behavior in the low-energy regime (i.e. $\varepsilon_p \sim p(\pi - p)$ with p the momentum of an excitation) while the dispersion relation of its Heisenberg version,

covered in [24], is given by a transcendental function, $\varepsilon_p \sim \sin p$. Also, as explained in [25] and going in the same vein, due to its specific long-range interaction, the Haldane-Shastry model exactly follows the scaling of the CFT, even at finite system size whereas spin chains like the Heisenberg chain come with logarithmic corrections as a result of the long-wavelength corrections depending on the system size. The reason for this particularly nice property of the Haldane-Shastry model is its Yangian symmetry [26]. Thus, it appears to be not too farfetched to hope to encounter a similarly well-behaved structure in the Haldane-Shastry-like spin chains that can be constructed from non-unitary CFTs following the idea of Nielsen, Cirac and Sierra.

1.2 The Structure of this Thesis

The construction of the superspin systems will require some preparation. Thus, in the next Chapter, some necessary concepts from super linear algebra and Lie superalgebras and some of their representations will be introduced. However, this will be kept to a minimum, mainly fixing notation. Chapter 2 concludes with a basic discussion of the GL$(1|1)$ Haldane-Shastry model since we will encounter the GL$(1|1)$ Haldane-Shastry model again in later Chapters. In [27], it was shown to be a model of non-interacting fermions and, hence, exactly solvable. In the continuum limit, it turns out to be of the same structure as the disorder sector of the Ising CFT [28].

Systems of $\mathfrak{gl}(m|n)$-superspins with long-range interactions, arbitrarily distributable in the complex plane, will be constructed in Chapter 3. The necessary ingredients from CFT will be introduced along the way. In this affair, an important ingredient will be the concept of a null state. The null states will allow us to construct a singlet wave function from particular CFT correlation functions (i.e. chiral correlators) and projection operators which project this singlet to zero. Further, the projection operators can be combined to form an operator, which will be used as the Hamiltonian of the system. The way it is constructed, the system will possess global GL$(m|n)$-invariance and long-range interactions between up to three individual spins.[3] Then, we will comment on some special cases for the choice of m and n as well as for a Haldane-Shastry-like distribution of the spins for $m = n$. This Chapter concludes with the analytic calculation of the singlet wave function from the CFT correlators for arbitrary m and n and distribution of the spins in the complex plane.

As the Hamiltonian becomes particularly simple for $m = n$ for spins uniformly distributed on the unit circle, Chapter 4 is dedicated to the discussion of these kinds of models. However, for its similarity with the alternating Haldane-Shastry model of $\mathfrak{gl}(1|1)$-spins, we will first solve this model before fleshing out our Hamiltonian. In both cases, we will shed light on their thermodynamic limit in order to eventually see some surprising interrelation emerge: Both have $c = -2$, contrary to the central charge $c = 0$ of the GL$(1|1)$ WZW model which has been discussed and solved in [29]. However, they are not described by

[3] As a manifestation of the monogamy of entanglement, this can also be expressed as the product of two spin-spin interactions with one common spin.

the same CFT as we will see. Furthermore, in the analysis of these systems, it appears that naïve finite-size scaling of the ground state energy leads to inconsistent results. The root of this seems to be that the range of summation in the definition of the long-range Haldane-Shastry spin chains must be slightly altered in order to be in perfect analogy with the energy operator L_0 stemming from the Sugawara construction of the WZW model.

In Chapter 5, we will first describe a $\mathfrak{gl}(m|n)$-spin ladder setup on a bipartite lattice built from the fundamental representation $\mathcal{V}$ and its dual $\bar{\mathcal{V}}$. It has two parameters, one angular (α), the other radial (r). We will then restrict ourselves to the case $m = n$ and give the GL$(m|m)$ Hamiltonian from our construction in terms of r and α. Then, we will propose an exact analytic expression for the full spectrum of the $\mathfrak{gl}(1|1)$-spin ladder, including its full r- and α-dependence and explain how this was found. With these solutions at hand, we discuss the continuum limit of the generic $\mathfrak{gl}(1|1)$-spin ladder at any admissible value of r and α and infer that the $\mathfrak{gl}(1|1)$-spin ladder is only critical for one particular value of r.

The final Chapter will present a summary of all the results and close with some concluding remarks and an outlook.

2

Prerequisites

If you are lost, find where you're at.
– Róisín Murphy, *Where Is The What If The What Is In Why?*

In this preliminary Chapter, we will introduce the super structures that we will need for the following construction of the superspin systems and recap the discussion of the Haldane-Shastry model of $\mathfrak{gl}(1|1)$-superspins.

2.1 Some Super Structure

In this thesis, quantum *super*spin systems are constructed from CFT. These superspins are associated with operators that act on a $\mathbb{Z}_2$-graded vector space which we will denote by $\mathcal{V}$ or $\bar{\mathcal{V}}$, depending on its transformation properties under some symmetry described in Section 2.1.3.

In the following Sections, we will recite some basics of the structures that will be necessary for the construction of the superspin systems in Chapter 3. A very nice introduction to the topic can be found in [30], for specific information on Lie superalgebras, one can consult [31] and a broader introduction to the topic of supersymmetry from a mathematical perspective is presented in [32].

2.1.1 Basic Notions in Super Linear Algebra

We start from a complex $\mathbb{Z}_2$-graded vector space

$$\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1. \tag{2.1}$$

The first direct summand $\mathcal{V}_0$ contains all elements that are called *even* (or *bosonic*) and the second summand $\mathcal{V}_1$ contains all *odd* (or *fermionic*) elements. Here and in what follows, we concentrate on the particular case of $\mathcal{V} = \mathbb{C}^{m|n}$. Its basis elements shall be denoted by e^a with $a = 1, 2, \ldots, m+n$. The grading is a map $|\bullet|$ sending a homogeneous (meaning either strictly even or odd) element to the elements 0 (even) or 1 (odd) of $\mathbb{Z}_2$ understood as a field so that they can be added and multiplied. We will always assume a homogeneous

basis $\{e^a\}_a$, so that we have

$$|e^a| = \begin{cases} 0 & \text{if } 1 \leq a \leq m, \\ 1 & \text{if } m+1 \leq a \leq m+n. \end{cases} \tag{2.2}$$

By abuse of notation, we will treat $|e^a|$ and $|a|$ as meaning the same. Also, we will often write a in exponents of (-1) instead of $|a|$. This should not lead to any confusion, though.

Next, we define the dual vector space of $\mathcal{V}$, $\bar{\mathcal{V}}$, that consists of all linear functionals mapping any element of $\mathcal{V}$ to the complex numbers. We define a homogeneous basis $\{e_b\}_b$ on $\bar{\mathcal{V}}$ by each element's action on any basis element e^c:

$$e_b(e^c) := \delta_b^c. \tag{2.3}$$

By linearity, this extends to all elements of $\mathcal{V}$ and $\bar{\mathcal{V}}$. With this definition and due to the in supermathematics omnipresent Koszul sign rule, we have on the other hand

$$e^c(e_b) = \delta_b^c(-1)^{cb} = \delta_b^c(-1)^b, \tag{2.4}$$

leading to a sign change if and only if we pass an odd element by another odd element. Also, by the last equality, it is evident that we silently used the same grading for e_b as given in eq. (2.2).

2.1.2 The Superalgebra $\mathbf{End}(\mathcal{V})$

Armed with this much, we may define the space of linear maps from $\mathcal{V}$ to itself, $\mathrm{End}(\mathcal{V})$: The basic result from usual linear algebra carries over to the super case, i.e.

$$\mathrm{End}(\mathcal{V}) := \mathcal{V} \otimes \bar{\mathcal{V}}. \tag{2.5}$$

As such, it is a graded vector space itself and the canonical (homogeneous) basis is given by the set $\{E^a{}_b := e^a \otimes e_b\}_{a,b=1,\ldots,m+n}$. Their degree is defined in terms of the grading on $\mathcal{V}$ and $\bar{\mathcal{V}}$:

$$|e^a \otimes e_b| = |e^a| + |e_b|. \tag{2.6}$$

From the choice of $\bar{\mathcal{V}}$, it is also clear how such a basis element $E^a{}_b$ acts on a basis element $e^c \in \mathcal{V}$:

$$E^a{}_b\, e^c = e^a\, \delta_b^c. \tag{2.7}$$

Again, this extends to all elements in $\mathrm{End}(\mathcal{V})$ and $\mathcal{V}$ by linearity. Furthermore, we can concatenate two maps,

$$E^d{}_f E^a{}_b\, e^c = E^d{}_f\, e^a\, \delta_b^c = e^d\, \delta_f^a \delta_b^c, \tag{2.8}$$

which defines a product in $\mathrm{End}(\mathcal{V})$:

$$E^d{}_f E^a{}_b = E^d{}_b \, \delta^a_f. \tag{2.9}$$

This turns $\mathrm{End}(\mathcal{V})$ into an algebra with consistent grading, i.e.

$$|E^d{}_f E^a{}_b| = \left(|E^d{}_f| + |E^a{}_b|\right) \delta^a_f = (d + f + a + b)\delta^a_f = (d + b)\delta^a_f \tag{2.10}$$

$$= |E^d{}_b| \, \delta^a_f \tag{2.11}$$

and, hence, into what is called a superalgebra. Not to our surprise, $\mathrm{End}(\mathcal{V})$ is also associative:

$$E^g{}_h(E^d{}_f E^a{}_b) = E^g{}_h E^d{}_b \, \delta^a_f = E^g{}_b \, \delta^d_h \delta^a_f = E^g{}_f E^a{}_b \, \delta^d_h = (E^g{}_h E^d{}_f) E^a{}_b. \tag{2.12}$$

Finally, in analogy with the classical case, $\mathrm{End}(\mathcal{V})$ is isomorphic to the associative algebra of all $(m|n) \times (m|n)$ matrices with entries in $\mathbb{C}$. Their basic structure is such that they have two even blocks, one of them $m \times m$-dimensional and the other $n \times n$-dimensional, along their diagonal and two odd blocks, $m \times n$-dimensional and $n \times m$-dimensional, on the off-diagonal. A homogeneous matrix of this form has only entries either on the even or the odd blocks in $\mathbb{C}$. It is then even or, respectively, odd. Furthermore, the supertrace of such a matrix is defined as the trace of the first even block ($m \times m$-dimensional) minus the trace of the second even block ($n \times n$-dimensional). Thus, the definition of the supertrace also carries over to elements of $\mathrm{End}(\mathcal{V})$. In particular, we find for the supertrace of the identity element of $\mathrm{End}(\mathcal{V})$

$$\mathrm{str}\, \mathbb{I} = \mathrm{str}\, E^a{}_a = \sum_{a=1}^{m} 1 - \sum_{a=m+1}^{m+n} 1 = m - n =: N. \tag{2.13}$$

The quantity N is also referred to as the superdimension of $\mathcal{V}$, $\mathrm{sdim}\, \mathcal{V}$. These are the necessary ingredients from super linear algebra for what follows.

2.1.3 The Lie Superalgebra $\mathfrak{gl}(m|n)$ and Its Defining Representation

Just as in the classical case, any associative superalgebra can be turned into a Lie superalgebra. We will only deal with the Lie superalgebra $\mathfrak{gl}(m|n)$ constructed from $\mathrm{End}(\mathcal{V})$, however, there is a variety of other complex finite-dimensional Lie superalgebras. They have been constructed and classified by Kac in [33, 34].

We start by defining a Lie bracket for the basis elements of $\mathrm{End}(\mathcal{V})$ consistent with the Koszul sign rule,

$$[E^a{}_b, E^c{}_d] := E^a{}_b E^c{}_d - (-1)^{|E^a{}_b||E^c{}_d|} E^c{}_d E^a{}_b, \tag{2.14}$$

and compatible with the grading. This extends to all elements of $\mathrm{End}(\mathcal{V})$ and defines the

product of the Lie superalgebra associated with $\mathrm{End}(\mathcal{V})$:

$$\mathfrak{gl}(m|n) := \{\mathrm{End}(\mathcal{V}), [\bullet, \bullet]\}. \tag{2.15}$$

The bracket of the Lie superalgebra $\mathfrak{gl}(m|n)$ is *anti-supersymmetric*,

$$[X, Y] = -(-1)^{|X||Y|}[Y, X], \tag{2.16}$$

and respects the *graded Jacobi-identity*,

$$[X, [Y, Z]](-1)^{|X||Z|} + [Y, [Z, X]](-1)^{|Y||X|} + [Z, [X, Y]](-1)^{|Z||Y|} = 0, \tag{2.17}$$

for any choice of homogeneous elements X, Y, and Z in $\mathfrak{gl}(m|n)$.

Next, we would like to define an invariant bilinear form. The natural choice in the classical case is the *Killing form*, which, in the super case, turns out to be nondegenerate only for $m \neq n$, and, zero when $m = n$. Since this case is of particular interest for our discussion in Chapters 4 and 5, instead of taking the supertrace in the adjoint representation, we, thus, resort to taking the supertrace in the fundamental representation,

$$\kappa(E^a{}_b, E^c{}_d) := \mathrm{str}(E^a{}_b E^c{}_d) = \delta^c_b \,\mathrm{str}(E^a{}_d) = \delta^c_b \delta^a_d (-1)^a, \tag{2.18}$$

again, extended to $\mathfrak{gl}(m|n)$ by bilinearity. For later purposes, we will also write

$$\kappa^a{}_{b,}{}^c{}_d := \kappa(E^a{}_b, E^c{}_d) = \delta^a_d \, \delta^c_b \, (-1)^a. \tag{2.19}$$

It turns out to be *consistent*,

$$\kappa(X, Y) = 0 \quad \text{for all homogeneous } X \text{ and } Y \text{ with } |X| \neq |Y|, \tag{2.20}$$

supersymmetric,

$$\kappa(X, Y) = (-1)^{|X||Y|}\kappa(Y, X), \tag{2.21}$$

and *invariant*,

$$\kappa(X, [Y, Z]) = \kappa([X, Y], Z), \tag{2.22}$$

and, in addition, *nondegenerate*,

$$(\forall Y \in \mathfrak{gl}(m|n) : \kappa(X, Y) = 0) \quad \Rightarrow \quad X = 0. \tag{2.23}$$

With this invariant bilinear form, we can now define the quadratic Casimir for some representation ρ of $\mathfrak{gl}(m|n)$,

$$C := \rho(E^a{}_b)\rho(E^b{}_a))(-1)^b. \tag{2.24}$$

For the fundamental representation, $\rho_{\mathcal{V}}(E^a{}_b) = E^a{}_b$, with representation space $V = \mathcal{V}$, we

find

$$C_{\mathcal{V}} := E^a{}_b E^b{}_a \, (-1)^b = (m-n)\, E^a{}_a = N\, \mathbb{I}. \tag{2.25}$$

The antifundamental representation is defined by the action of the $E^a{}_b$ on $\bar{\mathcal{V}}$ which is defined via the relation to $\mathcal{V}$. Therefore we calculate

$$E^a{}_b \cdot e_c(e^d) = -(-1)^{c(a+b)} e_c(E^a{}_b\, e^d) = -(-1)^{c(a+b)} e_c(e^a)\delta^d_b \tag{2.26}$$

$$= -(-1)^{a+ab}\delta^a_c e_b(e^d) \tag{2.27}$$

which implies $\rho_{\bar{\mathcal{V}}}(E^a{}_b) = -(-1)^{ab} E_b{}^a$ with $E_b{}^a := e_b \otimes e^a$ and keeping in mind the consequences of the Koszul sign rule as given in eq. (2.4), $e^a(e_c) = \delta^a_c(-1)^a$ while the overall sign change stems from turning the right module into a left module. With this, we compute the quadratic Casimir in the antifundamental representation and find

$$C_{\bar{\mathcal{V}}} \cdot e_c := (-1)^b \rho_{\bar{\mathcal{V}}}(E^a{}_b)\rho_{\bar{\mathcal{V}}}(E^b{}_a) e_c = -(-1)^{b+ba} \rho_{\bar{\mathcal{V}}}(E^a{}_b)\, E_a{}^b\, e_c \tag{2.28}$$

$$= -(-1)^{b+ba} \rho_{\bar{\mathcal{V}}}(E^a{}_b)\, e_a \otimes e^b(e_c) = -(-1)^{ba} \rho_{\bar{\mathcal{V}}}(E^a{}_b)\, e_a\, \delta^b_c \tag{2.29}$$

$$= E_b{}^a e_a\, \delta^b_c = e_b \otimes e^a(e_a)\, \delta^b_c = e_b\, \delta^a_a (-1)^a\, \delta^b_c \tag{2.30}$$

$$= e_c\, N \tag{2.31}$$

which is equivalent to the result for the fundamental representation $\mathcal{V}$.

2.1.4 Some $\mathfrak{gl}(m|n)$ Representations and Their Tensor Products

In the preceding Section, we have already learned about the fundamental and antifundamental representation, $\rho_{\mathcal{V}}$ and $\rho_{\bar{\mathcal{V}}}$, both of which are irreducible $\mathfrak{gl}(m|n)$ representations. Due to the way $\mathfrak{gl}(m|n)$ is constructed from $\mathcal{V}$, $\rho_{\mathcal{V}}$ is also referred to as the *defining* representation. However, making the distinction between $\rho_{\mathcal{V}}$ and $\rho_{\bar{\mathcal{V}}}$ is simply a matter of convention. Moreover, the systems to be constructed in Chapter 3 will be invariant under the exchange of these, i.e. under parity. In order to describe interactions between different superspins[1] meaning different representations, we will also need tensor products consisting of a choice of up to three representations. Generalizing the quadratic Casimir to those tensor products helps to find their decomposition. Nevertheless, we will see that there is a caveat: While $\mathcal{V}$ and $\bar{\mathcal{V}}$ are irreducible, we will also encounter *reducible* but *indecomposable* representations. This means that they have a proper $\mathfrak{gl}(m|n)$-submodule but it is not possible to recast the indecomposable representation as a direct sum of such proper $\mathfrak{gl}(m|n)$-submodules. For a general discussion of tensor products for $\mathfrak{gl}(m|n)$, [35] can be consulted.

[1] We will also refer to them simply as *spins*.

Twofold Tensor Products from $\mathcal{V}$ and $\bar{\mathcal{V}}$

For computing the decomposition of $\mathcal{V} \otimes \mathcal{V}$, we have to define how the quadratic Casimir acts on it. This boils down to defining a *coproduct* Δ and fixing

$$\Delta(C)\, e^a \otimes e^b = (C_{\mathcal{V}} \cdot e^a) \otimes e^b + \sum_{c,d} \Big((-1)^{d+a(d+c)}\, E^c{}_d\, e^a \otimes E^d{}_c\, e^b \tag{2.32}$$

$$+ (-1)^{d+(c+d)+a(c+d)}\, E^d{}_c\, e^a \otimes E^c{}_d\, e^b \Big) + e^a \otimes (C_{\mathcal{V}} \cdot e^b) \tag{2.33}$$

which simplifies to

$$\Delta(C)\, e^a \otimes e^b = 2 \left(N\, e^a \otimes e^b + (-1)^{ab} e^b \otimes e^a \right). \tag{2.34}$$

From this, one can infer that $\mathcal{V} \otimes \mathcal{V}$ decomposes into a symmetric and an antisymmetric representation,

$$\mathcal{V} \otimes \mathcal{V} = \mathcal{S} + \mathcal{A}, \tag{2.35}$$

which are spanned by $e^a \otimes e^b + (-1)^{ab} e^b \otimes e^a$ and $e^a \otimes e^b - (-1)^{ab} e^b \otimes e^a$, respectively, with Casimir eigenvalue $2(N+1)$ and $2(N-1)$, respectively. An analogous result holds for $\bar{\mathcal{V}} \otimes \bar{\mathcal{V}}$.

Next, we find the action of the Casimir on the elements $e^a \otimes e_b$ in the mixed tensor product $\mathcal{V} \otimes \bar{\mathcal{V}}$:

$$\Delta(C)\, e^a \otimes e_b = (C_{\mathcal{V}} \cdot e^a) \otimes e_b + \sum_{c,d} \Big(-(-1)^{d+a(d+c)}\, E^c{}_d\, e^a \otimes E^d{}_c \cdot e_b \tag{2.36}$$

$$- (-1)^{d+(c+d)+a(c+d)}\, E^d{}_c\, e^a \otimes E^c{}_d \cdot e_b \Big) + e^a \otimes (C_{\bar{\mathcal{V}}} \cdot e_b)$$

$$= 2 \left(N\, e^a \otimes e_b - \delta^a_b (-1)^a \sum_c e^c \otimes e_c \right) \tag{2.37}$$

$$= \begin{cases} 2N\, e^a \otimes e_b & \text{if } a \neq b, \\ 2\,(N\, e^a \otimes e_a - \sum_c e^c \otimes e_c) & \text{if } a = b \leq m, \\ 2\,(N\, e^a \otimes e_a + \sum_c e^c \otimes e_c) & \text{if } a = b > m. \end{cases} \tag{2.38}$$

For $N \neq 0$ (implying $m \neq n$), we, obviously, have one submodule spanned, e.g., by $\{e^a \otimes e_b\}_{a \neq b}$, $\{e^a \otimes e_a - e^{a+1} \otimes e_{a+1}\}_{a=1,\ldots,m-1}$, $\{e^a \otimes e_a - e^{a+1} \otimes e_{a+1}\}_{a=m+1,\ldots,m+n-1}$ and $e^1 \otimes e_1 + e^{m+1} \otimes e_{m+1}$, with Casimir eigenvalue $2N$ and another one-dimensional submodule spanned by $\sum_a e^a \otimes e_a$ with Casimir zero. However, this decomposition breaks down if we

choose $N = 0$: In this case, the Casimir is non-diagonalizable,

$$\Delta(C)\, e^a \otimes e_b = \begin{cases} 0 & \text{if } a \neq b, \\ -2\sum_c e^c \otimes e_c & \text{if } a = b \leq m, \\ 2\sum_c e^c \otimes e_c & \text{if } a = b > m. \end{cases} \tag{2.39}$$

While we still find the one-dimensional invariant submodule spanned by $\sum_a e^a \otimes e_a$ with eigenvalue zero, it does not decouple from the rest of the tensor product as any state $e^a \otimes e_a$ is mapped onto it. Therefore, the mixed tensor product $\mathcal{V} \otimes \bar{\mathcal{V}}$ is reducible but indecomposable if the superdimension N vanishes.

Threefold Tensor Products from $\mathcal{V}$ and $\bar{\mathcal{V}}$

The threefold tensor products are relevant for us because we will need the tensor product of the adjoint representation with representation space $\mathcal{J} = \mathcal{V} \otimes \bar{\mathcal{V}}$ with the fundamental (and antifundamental, respectively):

$$\mathcal{J} \otimes \mathcal{V} = \mathcal{V} \otimes \bar{\mathcal{V}} \otimes \mathcal{V}. \tag{2.40}$$

For the action of the quadratic Casimir on the product basis $\{e^a \otimes e_b \otimes e^c\}_{a,b,c\in\{1,\ldots,p\}}$, we have[2]

$$\Delta^2(C)\, e^a \otimes e_b \otimes e^c = \sum_{d,f} (-1)^f \Delta(E^d{}_f)\Delta(E^f{}_d)\, e^a \otimes e_b \otimes e^c \tag{2.41}$$

which, after a lengthy calculation, yields

$$\begin{aligned} \Delta^2(C)\, e^a \otimes e_b \otimes e^c = {} & 3N\, e^a \otimes e_b \otimes e^c + 2(-1)^{ab+bc+ca} e^c \otimes e_b \otimes e^a \\ & - 2(-1)^a \delta^a_b \sum_d e^d \otimes e_d \otimes e^c - 2\delta^c_b \sum_d (-1)^d e^a \otimes e_d \otimes e^d. \end{aligned} \tag{2.42}$$

We will see that the eigenvalues of the quadratic Casimir are N, $3N - 2$ and $3N + 2$.

Let us first define the $2p$ vectors $u^c := \sum_a e^a \otimes e_a \otimes e^c$ and $v^a := \sum_b (-1)^b e^a \otimes e_b \otimes e^b$ each of them spanning a submodule isomorphic to $\mathcal{V}$ (unless $N = \pm 1$ in which case the Casimir will, obviously, not resolve its eigenvectors with eigenvalue N and $3N \mp 2$):

$$\Delta^2(C)\, u^c = N\, u^c, \tag{2.43}$$

$$\Delta^2(C)\, v^c = N\, v^c. \tag{2.44}$$

Both sets of these vectors are, of course, in the span of states spanning the whole tensor

[2] For clarity, we will, in this Section, indicate summation over double indices explicitly through the summation sign $\sum$ in order to prevent confusion.

product. These states can be recast into new basis vectors,

$$w_{\pm\ b}^{\ a\ \ c} := e^a \otimes e_b \otimes e^c \pm (-1)^{ab+bc+ca} e^c \otimes e_b \otimes e^a, \tag{2.45}$$

and they are useful for forming linear combinations together with the $w_{\pm\ b}^{\ a\ \ c}$ that will feature an eigenvalue of $3N-2$ or $3N+2$. In the simplest case, a arbitrary, $b \notin \{a\}$, $c \notin \{a,b\}$, this is not even necessary:

$$\Delta^2(C)\, w_{\pm\ b}^{\ a\ \ c} = (3N \pm 2)\, w_{\pm\ b}^{\ a\ \ c}. \tag{2.46}$$

As a matter of fact, the u^c and v^c are obviously linearly independent from these $p(p-1)(p-2)$ vectors. The same is true for the $p(p-1)$ *non-zero*[3] cases $a = c$, $b \notin \{a\}$:

$$\Delta^2(C)\, w_{(-1)^a\ b}^{\ a\ \ a} = \left(3N + (-1)^a 2\right)\, w_{(-1)^a\ b}^{\ a\ \ a}. \tag{2.47}$$

In the remaining $2p(p-1)+p$ cases, $w_{\pm\ a}^{\ a\ \ c}$ ($c \neq a$) and $w_{(-1)^a\ a}^{\ a\ \ a}$[4] on the one hand and u^c and v^c on the other are not linearly independent and the ws are not in general eigenvectors of the Quadratic Casimir. In the latter cases, we find

$$\Delta^2(C)\, w_{(-1)^a\ a}^{\ a\ \ a} = \left(3N + (-1)^a 2\right)\, w_{(-1)^a\ a}^{\ a\ \ a} - \left(1 + (-1)^a\right) 2(u^a + v^a), \tag{2.48}$$

which are eigenvectors only for odd a, so with eigenvalue $3N-2$. For even a, it can be calculated that, in order to decouple the $(3N+2)$-part from $2(u^a+v^a)$, we have to form a new linear combination,

$$\Delta^2(C)\, \left(w_{+\ a}^{\ a\ \ a} - \frac{1}{N+1}(u^a+v^a)\right) = (3N+2)\, \left(w_{+\ a}^{\ a\ \ a} - \frac{1}{N+1}(u^a+v^a)\right), \tag{2.49}$$

which is possible unless $N=-1$: In that case, $w_{+\ a}^{\ a\ \ a}$ and $2(u^a+v^a)$ form an indecomposable Jordan block. An analogous procedure yields for the $2p(p-1)$ cases $w_{\pm\ a}^{\ a\ \ c}$ with $c \neq a$,

$$\Delta^2(C)\, \left(w_{\pm\ a}^{\ a\ \ c} - \frac{(-1)^a}{N \pm 1}(u^c \pm v^c)\right) = (3N \pm 2)\, \left(w_{\pm\ a}^{\ a\ \ c} - \frac{(-1)^a}{N \pm 1}(u^c \pm v^c)\right), \tag{2.50}$$

which again fails for those states with eigenvalue $3N \pm 2$ unless $N \neq \mp 1$. This failure is an important hint for the construction of the projection operator $\mathcal{P}_i$ that will be used in order to construct the GL$(m|n)$-symmetric Hamiltonian: $\mathcal{P}_i$ will be constructed such that it certainly projects out all parts in the threefold tensor product $\mathcal{V} \otimes \bar{\mathcal{V}} \otimes \mathcal{V}$ associated to the fundamental representation $\mathcal{V}$ (the antifundamental representation $\bar{\mathcal{V}}$, respectively)

[3] Obviously, we have $w_{-(-1)^a\ b}^{\ a\ \ a} \equiv 0$.

[4] Again, we have $w_{-(-1)^a\ a}^{\ a\ \ a} \equiv 0$.

as the desired null states to project onto for sure are not associated to the fundamental representation.

As a consequence, for positive N, $\mathcal{P}_i$ will project onto the submodule of eigenvalue $3N+2$ so that the indecomposability between the subspace with eigenvalue $3N-2$ and the subspaces with eigenvalue N in case $N=1$ will play no role as they are collectively projected out. For the analogous reason, for negative N, $\mathcal{P}_i$ will project onto the submodule of eigenvalue $3N-2$, so that the indecomposability between states with eigenvalue N and $3N+2$ for $N=-1$ does no harm.

To sum up, the threefold tensor product we are interested in yields

$$\mathcal{J}\otimes\mathcal{V}=\mathcal{N}\oplus\mathcal{X} \tag{2.51}$$

with $\mathcal{N}$ being defined as the submodule with

$$\Delta^2(C)\,\mathcal{N}=\big(3N+\mathrm{sgn}(N)2\big)\,\mathcal{N} \tag{2.52}$$

and $\mathcal{X}$ the span of all other states (with eigenvalues N or $3N-\mathrm{sgn}(N)2$ within $\mathcal{J}\otimes\mathcal{V}$) whose decomposition is irrelevant to us as $\mathcal{X}$ will be projected out by $\mathcal{P}_i$.

2.2 The Pure Haldane-Shastry Chain for $\mathfrak{gl}(1|1)$-Spins

In his monograph [36], Greiter gives the Hamiltonian for the Haldane-Shastry model as

$$H_{\mathrm{HS}}=\left(\frac{2\pi}{L}\right)^2\sum_{k>j=0}^{L-1}\frac{\mathbf{S}_j\cdot\mathbf{S}_k}{|\eta_j-\eta_k|^2}\quad\text{with }\eta_j:=\exp\left(j\frac{2\pi i}{L}\right). \tag{2.53}$$

Using $|\exp(j\frac{2\pi i}{L})-\exp(k\frac{2\pi i}{L})|^2=4\sin^2\left(\frac{(j-k)\pi}{L}\right)$, we arrive at

$$H_{\mathrm{HS}}=\frac{\pi^2}{L^2}\sum_{j<k}^{L-1}\frac{\mathbf{S}_j\cdot\mathbf{S}_k}{\sin^2((j-k)\pi/L)}, \tag{2.54}$$

so that, in the large L limit, the nearest-neighbor interaction becomes Heisenberg-like, which is one reason for the scale factor in front of the sum in eq. (2.53). Another reason is that, with this normalization, we can safely take the thermodynamic limit $L\to\infty$. Now using the identity $\mathbf{S}_j\cdot\mathbf{S}_k=\frac{1}{2}(P_{jk}-\frac{1}{2})$ for the $\mathfrak{su}(2)$-generators $\mathbf{S}_j$ in order to generalize this expression, we receive

$$H_{\mathrm{HS}}=\frac{\pi^2}{2L^2}\sum_{j<k}^{L-1}\frac{P_{jk}-\frac{1}{2}}{\sin^2\big((j-k)\pi/L\big)} \tag{2.55}$$

where we have introduced the permutation operator P_{jk} that permutes the spins at sites j and k.

2.2.1 Diagonalization of the Hamiltonian

For simplicity, we rewrite the $\mathfrak{gl}(1|1)$ Hamiltonian on a pure chain of fundamental representations $\mathcal{V}$ as

$$\begin{aligned} H_{\mathrm{HS}}^{\mathfrak{gl}(1|1)} &= \frac{\pi^2}{2L^2}\left(\sum_{j<k}^{L-1}\frac{P_{jk}^{\mathfrak{gl}(1|1)}-1}{\sin^2((j-k)\pi/L)}+\sum_{j<k}^{L-1}\frac{\frac{1}{2}}{\sin^2((j-k)\pi/L)}\right) \\ &= \frac{\pi^2}{2L^2}\left(H+\frac{C(L)}{2}\right) \end{aligned} \tag{2.56}$$

where we have introduced the *graded* permutation operator $P_{jk}^{\mathfrak{gl}(1|1)}$ that permutes the superspins at sites j and k. In the following, we treat the two summands separately.

In order to diagonalize the first part of the Haldane-Shastry Hamiltonian, we reformulate the permutation operators in terms of fermionic creation and annihilation operators for each site, leading to

$$H=\sum_{j<k}^{L-1}\frac{P_{jk}^{\mathfrak{gl}(1|1)}-1}{\sin^2((j-k)\pi/L)}=\sum_{j<k}^{L-1}\frac{c_j^\dagger c_k+c_k^\dagger c_j-c_j^\dagger c_j-c_k^\dagger c_k}{\sin^2((j-k)\pi/L)}, \tag{2.57}$$

where we use $P_{jk}^{\mathfrak{gl}(1|1)}=1-(c_j^\dagger-c_k^\dagger)(c_j-c_k)$ for the Fermi operators c_j and $c_j^\dagger=(c_j)^\dagger$ which fulfill the usual anticommutation relations $\{c_j^\dagger,c_k\}=\delta_{jk}$ and $\{c_j^\dagger,c_k^\dagger\}=\{c_j,c_k\}=0$.

Next, we observe that translational invariance invokes conservation of momentum suggesting the Hamiltonian to be diagonal upon Fourier transformation to momentum space. To this end, just as explained in [37], it is useful to rewrite the denominator,

$$\frac{1}{\sin^2(x)}=\left(1+i\cot(x)\right)\left(1-i\cot(x)\right), \tag{2.58}$$

and use the Fourier transform of

$$1-i\cot\left(\frac{(j-k)\pi}{L}\right)=\frac{1}{L}\sum_{u=0}^{L-1}\exp\left(-i2\pi\frac{u(j-k)}{L}\right)(L-2u) \tag{2.59}$$

and its complex conjugate. From these expressions we can already tell that, without any harm, we are allowed to sum over all values of $j,k=0,1,\ldots,L-1$ in eq. (2.57), even $j=k$, since in that case the coefficient of the Fermi operators is still finite as long as L is finite whereas the Fermi operators add up to zero for any value of L.

Likewise, the Fermi operators c_j and $c_j^\dagger$ are transformed to

$$\tilde{c}_w=\frac{1}{\sqrt{L}}\sum_{j=0}^{L-1}\exp\left(-i2\pi\frac{wj}{L}\right)c_j,\quad \tilde{c}_x^\dagger=\frac{1}{\sqrt{L}}\sum_{j=0}^{L-1}\exp\left(i2\pi\frac{xj}{L}\right)c_j^\dagger. \tag{2.60}$$

Inserting everything, we receive

$$H = \frac{1}{L} \sum_{u,v,w,x=0}^{L-1} \tilde{c}_x^\dagger \tilde{c}_w (L-2u)(L-2v)\, S(u,v,w,x) \tag{2.61}$$

with

$$\begin{aligned} S(u,v,w,x) &:= \frac{1}{2L^2} \sum_{j,k=0}^{L-1} \exp\left(-i2\pi(u-v)\frac{j-k}{L}\right) \cdot \left[\exp\left(-i2\pi\frac{jx-kw}{L}\right)\right. \\ &\quad \left. + \exp\left(-i2\pi\frac{kx-jw}{L}\right) - \exp\left(-i2\pi j\frac{x-w}{L}\right) - \exp\left(-i2\pi k\frac{x-w}{L}\right)\right] \\ &= \frac{1}{2L^2} \sum_{j,k=0}^{L-1} \left[\exp\left(-i2\pi j\frac{(u+x)-v}{L}\right)\exp\left(i2\pi k\frac{(u+w)-v}{L}\right)\right. \\ &\quad + \exp\left(-i2\pi j\frac{u-(v+w)}{L}\right)\exp\left(i2\pi k\frac{u-(v+x)}{L}\right) \\ &\quad - \exp\left(-i2\pi j\frac{(u+x)-(v+w)}{L}\right)\exp\left(i2\pi k\frac{u-v}{L}\right) \\ &\quad \left. - \exp\left(i2\pi k\frac{(u+w)-(v+x)}{L}\right)\exp\left(-i2\pi j\frac{u-v}{L}\right)\right] \end{aligned} \tag{2.62}$$

$$= \frac{1}{2}\left(\delta_{[u+x],[v]}\delta_{[u+w],[v]} + \delta_{[u],[v+w]}\delta_{[u],[v+x]} - \delta_{[u+x],[v+w]}\delta_{[u],[v]} - \delta_{[u+w],[v+x]}\delta_{[u],[v]}\right) \tag{2.63}$$

where all the indices of the Kronecker deltas must be understood as equivalence classes modulo L. Reinserting this into our expression for H, we find

$$\begin{aligned} H &= \frac{1}{2L} \sum_{u,v,w,x=0}^{L-1} \tilde{c}_x^\dagger \tilde{c}_w (L-2u)(L-2v) \left[\delta_{[u+x],[v]}\delta_{[u+w],[v]} + \delta_{[u],[v+w]}\delta_{[u],[v+x]}\right. \\ &\qquad \left. - \delta_{[u+x],[v+w]}\delta_{[u],[v]} - \delta_{[u+w],[v+x]}\delta_{[u],[v]}\right] \\ &= \frac{1}{2L} \sum_{u,v,w,x=0}^{L-1} \tilde{c}_x^\dagger \tilde{c}_w (L-2u)(L-2v) \cdot \left[(\delta_{u+x,v} + \delta_{u+x,v+L})(\delta_{u+w,v} + \delta_{u+w,v+L})\right. \\ &\quad + (\delta_{u,v+w} + \delta_{u+L,v+w})(\delta_{u,v+x} + \delta_{u+L,v+x}) - (\delta_{u+x,v+w} + \delta_{u+x,v+w-L} + \delta_{u+x-L,v+w})\delta_{u,v} \\ &\quad \left. - (\delta_{u+w,v+x} + \delta_{u+w,v+x-L} + \delta_{u+w-L,v+x})\delta_{u,v}\right] \end{aligned} \tag{2.64}$$

where, in the second step, we used the restriction $u, v, w, x \in 0, 1, \ldots, L-1$ in order to get rid of the equivalence classes. The first two products of Kronecker deltas are dual to each other under exchange of u and v, so are the last two, and so is their prefactor. This

simplifies matters to

$$H = \frac{1}{L}\sum_{u,v,w,x=0}^{L-1} \tilde{c}_x^\dagger \tilde{c}_w (L-2u)(L-2v)\left[(\delta_{u,v+w}+\delta_{u+L,v+w})(\delta_{u,v+x}+\delta_{u+L,v+x}) \right.$$
$$\left. -(\delta_{u+x,v+w}+\delta_{u+x,v+w-L}+\delta_{u+x-L,v+w})\delta_{u,v}\right]. \tag{2.65}$$

For clarity, let us treat the two parts of this expression separately. The second one works out as follows:

$$\begin{aligned}
H_2 &= -\frac{1}{L}\sum_{u,v,w,x=0}^{L-1} \tilde{c}_x^\dagger \tilde{c}_w (L-2u)(L-2v)(\delta_{u+x,v+w}+\delta_{u+x,v+w-L}+\delta_{u+x-L,v+w})\delta_{u,v} \\
&= -\frac{1}{L}\sum_{v,w,x=0}^{L-1} \tilde{c}_x^\dagger \tilde{c}_w (L-2v)^2(\delta_{v+x,v+w}+\delta_{v+x,v+w-L}+\delta_{v+x-L,v+w}) \\
&= -\frac{1}{L}\sum_{v,w,x=0}^{L-1} \tilde{c}_x^\dagger \tilde{c}_w (L-2v)^2(\delta_{x,w}+0+0) \\
&= -\frac{1}{3}\sum_{w=0}^{L-1} \tilde{c}_w^\dagger \tilde{c}_w (L^2+2).
\end{aligned} \tag{2.66}$$

The first part yields

$$\begin{aligned}
L\,H_1 &= \sum_{u,v,w,x=0}^{L-1} \tilde{c}_x^\dagger \tilde{c}_w (L-2u)(L-2v)(\delta_{u,v+w}\delta_{u,v+x}+0+0+\delta_{u+L,v+w}\delta_{u+L,v+x}) \\
&= \sum_{u,v,w=0}^{L-1} \tilde{c}_w^\dagger \tilde{c}_w (L-2u)(L-2v)(\delta_{u,v+w}\delta_{u,v+w}+\delta_{u+L,v+w}\delta_{u+L,v+w}) \\
&= \sum_{u,v,w=0}^{L-1} \tilde{c}_w^\dagger \tilde{c}_w (L-2u)(L-2v)(\delta_{u,v+w}+\delta_{u,v-(L-w)}) \\
&= \sum_{w=0}^{L-1} \tilde{c}_w^\dagger \tilde{c}_w \sum_{v=0}^{L-1-w} (L-2(v+w))(L-2v) \\
&\quad + \sum_{w=1}^{L-1} \tilde{c}_w^\dagger \tilde{c}_w \sum_{v=L-w}^{L-1} (L-2(v+w-L))(L-2v) \\
&= \tilde{c}_0^\dagger \tilde{c}_0 \sum_{v=0}^{L-1}(L-2v)^2 + \sum_{w=1}^{L-1} \tilde{c}_w^\dagger \tilde{c}_w \sum_{v=0}^{L-1-w} (L-2(v+w))(L-2v) \\
&\quad + \sum_{w=1}^{L-1} \tilde{c}_w^\dagger \tilde{c}_w \sum_{v=L-w}^{L-1} (L-2(v+w))(L-2v) + \sum_{w=1}^{L-1} \tilde{c}_w^\dagger \tilde{c}_w \sum_{v=L-w}^{L-1} 2L(L-2v) \\
&= \tilde{c}_0^\dagger \tilde{c}_0 \sum_{v=0}^{L-1}(L-2v)^2 + \sum_{w=1}^{L-1} \tilde{c}_w^\dagger \tilde{c}_w \sum_{v=0}^{L-1} (L-2(v+w))(L-2v)
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{w=1}^{L-1} \tilde{c}_w^\dagger \tilde{c}_w \sum_{v=L-w}^{L-1} 2L(L-2v) \\
&= \tilde{c}_0^\dagger \tilde{c}_0 \frac{L}{3}(L^2+2) + \sum_{w=1}^{L-1} \tilde{c}_w^\dagger \tilde{c}_w \sum_{v=0}^{L-1} (L-2v)^2 + \sum_{w=1}^{L-1} \tilde{c}_w^\dagger \tilde{c}_w (-2w) \sum_{v=0}^{L-1} (L-2v) \\
&\quad + \sum_{w=1}^{L-1} \tilde{c}_w^\dagger \tilde{c}_w (2L) \sum_{v=0}^{L-1} (L-2v) - \sum_{w=1}^{L-1} \tilde{c}_w^\dagger \tilde{c}_w (2L) \sum_{v=0}^{L-w-1} (L-2v) \\
&= \sum_{w=0}^{L-1} \tilde{c}_w^\dagger \tilde{c}_w \frac{L}{3}(L^2+2) + \sum_{w=1}^{L-1} \tilde{c}_w^\dagger \tilde{c}_w 2(L-w) \sum_{v=0}^{L-1} (L-2v) \\
&\quad - \sum_{w=1}^{L-1} \tilde{c}_w^\dagger \tilde{c}_w 2L \sum_{v=0}^{L-w-1} (L-2v) \\
&= -L\, H_2 + \sum_{w=1}^{L-1} \tilde{c}_w^\dagger \tilde{c}_w 2(L-w) \left(L^2 - 2\frac{1}{2} L(L-1) \right) \\
&\quad - \sum_{w=1}^{L-1} \tilde{c}_w^\dagger \tilde{c}_w 2L \left(L(L-w) - 2\frac{1}{2}(L-w)(L-w-1) \right). \qquad (2.67)
\end{aligned}
$$

Returning to our initial problem, we find

$$
\begin{aligned}
H &= H_1 + H_2 \\
&= \frac{1}{L} \left[\sum_{w=1}^{L-1} \tilde{c}_w^\dagger \tilde{c}_w 2(L-w)L - \sum_{w=1}^{L-1} \tilde{c}_w^\dagger \tilde{c}_w 2(L-w)L(w+1) \right] \\
&= -2 \sum_{w=1}^{L-1} \tilde{c}_w^\dagger \tilde{c}_w (L-w)w = -2 \sum_{w=0}^{L-1} \tilde{c}_w^\dagger \tilde{c}_w (L-w)w. \qquad (2.68)
\end{aligned}
$$

This concludes the diagonalization of the nontrivial part of the Hamiltonian given in eq. (2.56). Note the existence of the zero mode $\tilde{c}_0^\dagger$ which leads to a ubiquitous factor of two in the degeneracy pattern of the system.

The second part of a Haldane-Shastry-like Hamiltonian, which is proportional to the identity, plays the role of a constant that one can use to adjust the respective energy spectrum to be positive semi-definitive. In order to calculate its contribution, again, we use equations eq. (2.58) and eq. (2.59):

$$
\begin{aligned}
C(L) &= \sum_{j<k}^{L-1} \frac{1}{\sin^2((j-k)\pi/L)} \\
&= \frac{1}{2} \sum_{j \neq k}^{L-1} \sum_{u,v=0}^{L-1} \frac{(L-2u)(L-2v)}{L^2} \exp\left(-i2\pi(u-v)\frac{j-k}{L} \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{u,v=0}^{L-1} \frac{(L-2u)(L-2v)}{2L^2} \left[\sum_{j,k=0}^{L-1} \exp\left(-i2\pi(u-v)\frac{j-k}{L}\right) - L \right] \\
&= \sum_{u,v=0}^{L-1} \frac{(L-2u)(L-2v)}{2} \left[\delta_{u,v}\delta_{u,v} - \frac{1}{L} \right] \\
&= \frac{1}{2} \sum_{u=0}^{L-1} \left[(L-2u)^2 - \frac{(L-2u)(L^2 - 2\frac{1}{2}L(L-1))}{L} \right] \\
&= \frac{1}{2} \sum_{u=0}^{L-1} \left(L^2 - 4Lu + 4u^2 - L + 2u \right) \\
&= \frac{1}{2} \sum_{u=0}^{L-1} \left(L(L-1) + 2u(1-2L) + 4u^2 \right) \\
&= \frac{1}{2} \left(L(L-1)L + 2\frac{1}{2}L(L-1)(1-2L) + 4\frac{(L-1)L(2L-1)}{6} \right) \\
&= \frac{1}{2}L(L-1) \left(L + 1 - 2L + \frac{2}{3}(2L-1) \right) \\
&= \frac{1}{6}L(L-1)(L+1). \qquad (2.69)
\end{aligned}
$$

Thus, $C(L)$ is exactly such that it compensates the contribution of $-\sum_{w=0}^{L-1} \tilde{c}_w^\dagger \tilde{c}_w (L-w)w$ when acting on the completely occupied state, i.e.

$$
\sum_{w=0}^{L-1} (L-w)w = \frac{1}{6}L(L-1)(L+1) = C(L). \qquad (2.70)
$$

Now we can insert H from eq. (2.68) and $C(L)$ from eq. (2.69) into eq. (2.56) in order to fully diagonalize the Hamiltonian. This yields

$$
\begin{aligned}
H_{\mathrm{HS}}^{\mathfrak{gl}(1|1)} &= \frac{\pi^2}{2L^2} \left(H + \frac{C(L)}{2} \right) \\
&= \frac{\pi^2}{2L^2} \left(-2 \sum_{w=0}^{L-1} \tilde{c}_w^\dagger \tilde{c}_w (L-w)w + \frac{1}{12}L(L-1)(L+1) \right) \\
&= \frac{\pi^2 (L-1)(L+1)}{24L} - \frac{\pi^2}{L^2} \sum_{w=0}^{L-1} \tilde{c}_w^\dagger \tilde{c}_w (L-w)w, \qquad (2.71)
\end{aligned}
$$

which is also compatible with what Basu-Mallick et al. give in [28]:

$$
H_{\mathrm{HS,BM}}^{\mathfrak{gl}(1|1)} = \frac{1}{2} \sum_{j<k}^{L} \frac{P_{jk} + 1}{\sin^2((j-k)\pi/L)} = \frac{1}{2} \sum_{j<k}^{L} \frac{P_{jk} - 1 + 2}{\sin^2((j-k)\pi/L)}
$$

$$= \frac{1}{2}(H + 2C(L))$$
$$= -\sum_{w=0}^{L-1} \tilde{c}_w^\dagger \tilde{c}_w (L-w)w + \frac{1}{6}L(L-1)(L+1). \tag{2.72}$$

By virtue of eq. (2.70), their Hamiltonian is even positive semi-definitive. However, apart from a distinct normalization and offset, they are the same and, hence, can be discussed as one. The system has a doubly degenerate zero energy ground state which is given by the completely filled state modulo the zero mode $\tilde{c}_0^\dagger$, i.e. corresponding to the momentum quantum numbers $\{0, 1, 2, \ldots, L-1\}$ and $\{1, 2, \ldots, L-1\}$ or both corresponding to the motif $(1 \ldots 1)$ of L ones and $L-1$ ones, respectively. We dub the completely filled ground state

$$|\omega\rangle := \prod_{w=0}^{L-1} \tilde{c}_w^\dagger |0\rangle \tag{2.73}$$

with $|0\rangle$ the vacuum of the momentum modes $\tilde{c}_w$. Performing a particle-hole transformation, we introduce the operators

$$A_w := \tilde{c}_w^\dagger \qquad \text{and} \qquad A_w^\dagger := \tilde{c}_w \tag{2.74}$$

in order to rewrite the Hamiltonian into

$$H_{\text{HS,BM}}^{\mathfrak{gl}(1|1)} = -\sum_{w=0}^{L-1} A_w A_w^\dagger (L-w)w + \frac{1}{6}L(L-1)(L+1) \tag{2.75}$$
$$= -\sum_{w=0}^{L-1} (1 - A_w^\dagger A_w)(L-w)w + \frac{1}{6}L(L-1)(L+1) \tag{2.76}$$
$$= \sum_{w=0}^{L-1} A_w^\dagger A_w (L-w)w \tag{2.77}$$

where we made use of eq. (2.70). From this expression it is obvious that $|\omega\rangle$ and $A_0^\dagger |\omega\rangle = \tilde{c}_0 |\omega\rangle$ are the zero energy ground states of the system while all others come with positive energy. The first excited state is then fourfold degenerate, namely given by the states specified by $A_1^\dagger |\omega\rangle$, $A_{L-1}^\dagger |\omega\rangle$, $A_1^\dagger A_0^\dagger |\omega\rangle$ and $A_{L-1}^\dagger A_0^\dagger |\omega\rangle$.

In order to analyze the low-energy spectrum of this Hamiltonian in the thermodynamic limit, we, at first, expand the dispersion relation around its zero (separately for right- and left-moving modes, $w = 1, 2, \ldots$ and $w = L-1, L-2, \ldots$, respectively)

$$H_{\text{HS,BM}}^{\mathfrak{gl}(1|1)} = \sum_{w=1,2,\ldots} A_w^\dagger A_w \, L\, w + \sum_{w=L-1,L-2,\ldots} A_w^\dagger A_w \, L(L-w) + \mathcal{O}(w^2). \tag{2.78}$$

Next, we introduce creation and annihilation operators for the left moving modes, i.e. $B_{L-w}^\dagger = A_w^\dagger$ and $B_{L-w} = A_w$ for $w = L-1, L-2, \ldots$, in order to recast the Hamiltonian

into

$$H^{\mathfrak{gl}(1|1)}_{\text{HS,BM}} = \sum_{w=1,2,\ldots} A^\dagger_w A_w \, L\, w + \sum_{w=L-1,L-2,\ldots} B^\dagger_{L-w} B_{L-w} \, L(L-w) + \mathcal{O}(w^2) \tag{2.79}$$

$$= \sum_{w=1,2,\ldots} \left(A^\dagger_w A_w + B^\dagger_w B_w \right) L\, w + \mathcal{O}(w^2), \tag{2.80}$$

featuring a relativistic dispersion relation in the low-energy regime aside from the part high in the spectrum which is irrelevant for the identification of the underlying CFT.

Next, note that the total fermion number

$$\mathcal{N} := \sum_{w=0}^{L-1} A^\dagger_w A_w \tag{2.81}$$

of the system is conserved and we may, therefore, divide the state of spaces $\mathcal{H}$ into one part $\mathcal{H}_0$ with even fermion number and the other $\mathcal{H}_1$ with odd fermion number. As the zero mode can always be added or removed at no cost of energy, these two parts of the full state space have the same structure. Focusing on $\mathcal{H}_0$, the low-lying spectrum can be investigated in the thermodynamic limit by looking at the degeneracies $d(n)$ of states at (energy) level n, consisting exclusively of right (left) moving modes $A^\dagger_w$ ($B^\dagger_w$) and with or without the zero mode so as to meet the requirement of even fermion number. This series $\{d(n)\}_{n=1,2,\ldots}$ can then be compared to the Virasoro characters of potential primary operators. Let us go through the first few levels n (presupposing L sufficiently large):

- At $n = 0$, we have either $|\omega\rangle$ or $A^\dagger_0 |\omega\rangle$ in $\mathcal{H}_0$ depending on whether L is even or odd. Thus, $d(0) = 1$.
- At $n = 1$, focusing on right movers, we either have $A^\dagger_1 |\omega\rangle$ or $A^\dagger_1 A^\dagger_0 |\omega\rangle$ in $\mathcal{H}_0$ depending on whether L is even or odd. Thus, $d(1) = 1$.
- At $n = 2$, focusing on right movers, we have either $A^\dagger_2 |\omega\rangle$ or $A^\dagger_2 A^\dagger_0 |\omega\rangle$ in $\mathcal{H}_0$ depending on whether L is even or odd. The combination $A^\dagger_1 A^\dagger_1 |\omega\rangle$ leads to zero and not to the second excited state. Thus, $d(1) = 1$.
- At $n = 3$, focusing on right movers, we have either $A^\dagger_3 |\omega\rangle$ or $A^\dagger_3 A^\dagger_0 |\omega\rangle$ and either $A^\dagger_2 A^\dagger_1 |\omega\rangle$ or $A^\dagger_2 A^\dagger_1 A^\dagger_0 |\omega\rangle$ in $\mathcal{H}_0$ depending on whether L is even or odd. Other combinations lead to something proportional to the aforementioned states or to zero. Thus, $d(1) = 2$.

Carrying on with this procedure, we are able to find that this matches the Virasoro characters $d(\Delta, n)$ of the primary operator with conformal weight $\Delta = \frac{1}{16}$ of the Ising quantum chain with anti-periodic boundary conditions and fermionic charge 0. Further observations of the full low-lying spectrum prove that there is no other operator content so that we find that it is given by the block built from the disorder operator $\mu = (\frac{1}{16}, \frac{1}{16})$. Henkel [38] calls

this block $H_0^{(1)}$ with the superscript indicating the anti-periodic boundary conditions of the Ising quantum chain, and the subscript indicating the fermionic charge.

The same analysis can be done for $\mathcal{H}_1$ which is then identified with what Henkel calls $H_1^{(0)}$, the sector of the order operator $\sigma = (\frac{1}{16}, \frac{1}{16})$.

To summarize, we find that the pure GL$(1|1)$-invariant Haldane-Shastry spin chain is critical and described by the order and disorder sector of the Ising quantum chain. While the Virasoro algebra underlying the full Ising quantum chain features a conformal charge of $c = \frac{1}{2}$, the restriction to these two sectors has an *effective* central charge [39] of

$$\tilde{c} := c - 24\Delta = \frac{1}{2} - 24\frac{1}{16} = -1 \tag{2.82}$$

as the smallest conformal dimension, in this case, is $\Delta = \frac{1}{16}$.

3

Construction of a $\mathrm{GL}(m|n)$-invariant Quantum Superspin System

"Mario, what do you get when you cross an insomniac, an unwilling agnostic and a dyslexic?" "I give." "You get someone who stays up all night torturing himself mentally over the question of whether or not there's a dog."

– David Foster Wallace, *Infinite Jest*

In this Chapter, a globally $\mathrm{GL}(m|n)$-invariant Hamiltonian of quantum superspins will be derived from the holomorphic part of the WZW model based on the Lie supergroup $\mathrm{GL}(m|n)$. It will describe the quantum mechanics of particular superspins arbitrarily distributed in the complex plane and the interaction among them given by second and third order Casimir operators, scaled in strength as their (more or less)[1] inverse-squared distance.

This would be of little use without some information about the eigenstates of the system. However, this Hamiltonian is engineered as a parent Hamiltonian of a particular state that, in fact, is to be considered the seed of the whole construction, originally inspired by Moore and Read [4] and explored further by Cirac, Nielsen and Sierra in [5, 6].

In this context, it may be expected that this state should be of Gutzwiller-Jastrow type and thus related to the physics of the fractional quantum Hall effect. More specifically, the state will be provided by a chiral correlator of WZW primary fields that can be exactly computed from the CFT data, no matter how the positions of the superspins in the complex plane are chosen. Moreover, in general, the construction is not limited to the use of particular representations individual superspins transform under (even though, just as in [13, 14] for simplicity, the actual construction will be carried out for the fundamental representation $\mathcal{V}$ or antifundamental representation $\bar{\mathcal{V}}$ at some site z_i). Nevertheless, it should not be left unmentioned that the chiral correlator will be trivial unless the kind of WZW primary fields (and thus representations) chosen fulfill a charge neutrality condition [40] so that the global $\mathrm{GL}(m|n)$-invariance of the correlator can be met.

[1] The coefficients of the second order Casimir operators will reduce to the exact inverse-squared distance when restricting the model to a Haldane-Shastry-like setup, i.e. a setup of spins uniformly distributed on the unit circle. For more general setups, their expressions will become more involved.

Furthermore, by construction, the eigenvalue of this state will always be zero so that, in the cases covered in the literature so far [6, 13, 14] yielding positive semi-definite parent Hamiltonians, it turns out to be the singlet ground state of the system under consideration.

The rough idea of the construction to be pursued in the following Sections is sketched in Section 1.1.1 and, instead of paraphrasing [13, Section 2.1] which could hardly be put in better words for the general case of a WZW model based on a classical Lie group, we will try to stress the differences, when discussing the same structures in relation with a WZW model based on the Lie supergroups GL$(m|n)$, in the next Section synoptic in character.

3.1 Modus Operandi

The goal of the construction to be presented in the following Sections is to map the chiral part of the GL$(m|n)$ WZW model to a quantum system of superspins with long-range interaction, and, thereby, to provide a discretization scheme of the field theory on an arbitrary lattice. The GL$(m|n)$ WZW model is a CFT which is, as usually, defined on the complex plane $\mathbb{C}$ augmented with the point $\{\infty\}$ and, this way, enhanced to the Riemann sphere $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. The associated superspin system can be pictured as spins (here and subsequently, superspins will be referred to as *spins*) distributed at will on the Riemann sphere, their coordinates specified by the set $\{z_i\}_{i=1,\dots,L}$ of L points $z_i \in \overline{\mathbb{C}}$.

In this affair, it is reasonable to point out that the spins could transform under any $\mathfrak{gl}(m|n)$ representation. Nevertheless, the focus of attention will be on spins transforming under the fundamental $\mathfrak{gl}(m|n)$ representation $\mathcal{V}$ and the antifundamental $\mathfrak{gl}(m|n)$ representation $\bar{\mathcal{V}}$, both of which are $(m|n)$-dimensional, and which shall later be procured as *dual* to each other. To be precise, the two cases of interest will be

- the *pure* case with the state space given by $\mathcal{H} := \mathcal{V}^{\otimes L}$ and
- the *alternating* case with the state space given by $\mathcal{H} := (\bar{\mathcal{V}} \otimes \mathcal{V})^{\otimes L/2}$.

At this point, we would like to stress that, as the quantum spin system to be constructed is long-range in nature, the prospect of the alternating case actually requiring to place $\mathcal{V}$ and $\bar{\mathcal{V}}$ in turns onto the Riemann sphere is a fallacy, contrary to the suggestive notation $(\bar{\mathcal{V}} \otimes \mathcal{V})^{\otimes L/2}$. In fact, all that is required for the alternating case is using as many fundamental as antifundamental representations so that $L \in 2\mathbb{Z}$. Thus, in the alternating case, we can generally set

$$\mathcal{V}_i := \begin{cases} \mathcal{V} & \text{for even } i \text{ and} \\ \bar{\mathcal{V}} & \text{for odd } i \end{cases} \tag{3.1}$$

without restriction. In either case, the quantum objects to be constructed will be a singlet wave function and, based on this wave function, a parent Hamiltonian H.

To this end, let us start on the CFT-side by considering the chiral L-point correlator of WZW primary fields, $\Psi_{1\dots L}$, which can, if the fusion rules are monosemous, be considered

as a map from L copies of the Riemann sphere to the space $\bigotimes_i^L \mathcal{V}_i$,

$$\begin{array}{rcccl} \Psi_{1\ldots L} & : & \overline{\mathbb{C}}^L \otimes \bigotimes_i^L \bar{\mathcal{V}}_i & \longrightarrow & \mathbb{C}, \\ & & (z_1, \ldots, z_L) & \longmapsto & \langle \phi_1(z_1) \cdots \phi_L(z_L) \rangle, \end{array} \tag{3.2}$$

where we would like to stress that, at this point, we regard the bona fide coordinates $\{z_i\}_i$ of the L spins as *variables* of the L primary fields of the correlator $\Psi_{1\ldots L}$ while its indices signal that we have already made a specific choice of the representations at each site i (which is why we will suppress them for ease of notation).[2] The ratio behind this idea is inspired by the fact that, for the derivation of the Hamiltonian from the CFT, we need their properties as variables whereas, on the quantum spin system side, they will be treated as parameters. In this sense, the actual step of *discretizing*, illustrated in Figure 1.1, consists of actually *fixing* the $\{z_i\}_i$.

Invariance of the chiral correlator Ψ under $\mathrm{GL}(m|n)$, the global symmetry of the CFT, has the important consequence that, once $(z_1, \ldots, z_L)$ is fixed, the image will behave like a singlet wave function of a quantum spin system with respect to the global $\mathrm{GL}(m|n)$ symmetry. However, this singlet state might be embedded as an irreducible subspace in some larger representation which is reducible but indecomposable, e.g. in the cases, when we are dealing with a logarithmic CFT where the fusion product of two irreducible representations, namely in our case the (anti)-fundamental representation, might turn out to be reducible but indecomposable so that $\Psi(z_1, \ldots, z_L)$ is the socle of some larger representation. We refer the interested reader to [41] for a first taste of the subtleties involved.

Finally, non-vanishing of the chiral correlator requires some sort of charge cancellation condition to be met, the discussion of which will be postponed to Section 3.8.

With all this said, it stands to reason to view any particular image $\Psi(z_1, \ldots, z_L)$ of the chiral correlator Ψ as a candidate wave function, residing in the state space $\mathcal{H}$, and which can be assigned to a quantum system of spin-like degrees of freedom distributed on the Riemann sphere and featuring certain transformation properties under the global $\mathrm{GL}(m|n)$ symmetry.

Nonetheless, all of this would not lead very far if it was not for the existence of special WZW singular fields $\chi(z)$ contained in the Verma modules of the $\mathrm{GL}(m|n)$ WZW model, the so-called *null fields*. As these null fields are descendant *and* primary at the same time, they span submodules $\{\chi(z)\}$ of their own, nested within the Verma modules built on top of the various true primary fields $\phi(z)$. The property of being primary is analogous the well-known highest weight property from basic representation theory. In the same way as a highest weight state of, e.g. $\mathfrak{su}(2)$, spans an $\mathfrak{su}(2)$ representation, a primary field spans a (however, infinite-dimensional) representation space $\{\phi(z)\}$ of the conformal symmetry. This module consists of all the descendants of $\phi(z)$. Some of these descendants span submodules of their own which means that those are also primary. These are exactly the

[2] Namely, we have decided on whether to inspect either the pure or the alternating case.

null states $\chi(z)$. In fact, each module $\{\phi(z)\}$ may contain infinitely many of these null fields and, in the case of our interest where the primary field $\phi(z)$ will transform in $\mathcal{V}$ or $\bar{\mathcal{V}}$ under the horizontal subalgebra $\mathfrak{gl}(m|n)$, the first ones arise just at the first[3] Virasoro level. Moreover, where (global) conformal invariance strongly constrains two- and three-point correlation functions of (quasi)-primary fields in two dimensions, it offers not even nearly as much information on the form of L-point correlation functions for L greater than three. But the existence of null fields within some module $\{\phi(z)\}$ leads to a system of complex partial differential equations, the Knizhnik-Zamolodchikov equations [42], satisfied by any correlation function involving fields of the respective representation $\{\phi(z)\}$ and that way allowing for the actual computation of these correlation functions. The logarithmic case, again, needs special care [43] and we will come back to this issue in Section 3.8 where we make an attempt at evaluating the correlator.

The crucial point for the construction of the Hamiltonian, however, is that, in order to realize an indecomposable (or potentially even stronger, an irreducible) representation of the CFT, the minimum requirement is that the null fields be put to zero which can safely be done without spoiling the theory [44, Sec. 4.3]. The reason for this is that the module $\{\chi(z)\}$ decouples from the rest of the module $\{\phi(z)\}$ because every state in $\{\chi(z)\}$ is orthogonal to *every* state of $\{\phi(z)\}$ it is submodule to. Additionally, having put $\{\chi(z)\}$ to zero can be considered as a step towards unitarity of the theory as their norm is zero to begin with [38, Sec. 4.3]. And modifying Henkel's "In a sense, null-vectors may be considered as *the ingredients which make it possible to solve a model*," [ibid.] we may also consider them as the ingredients which make it possible to build our model – and this is how: By what has been explained, it is clear that any chiral correlator including an element of the null field's Verma module $\{\chi(z)\}$ will be identical to zero, in particular, replacing any primary field $\phi_i(z_i)$ in eq. (3.2) by the null field $\chi(z_i)$ yields

$$0 \equiv \langle \phi_1(z_1) \cdots \chi(z_i) \cdots \phi_L(z_L) \rangle. \tag{3.3}$$

If the null field $\chi(z_i)$ belongs to the Verma module that is built on top of some primary field, we can rewrite it in terms of some operator acting on that primary field $\phi_i(z_i)$. A part of the action of this operator on $\phi_i(z_i)$ can be shifted to all the other primary fields of the chiral correlator by means of the affine Ward identities – identities stemming from the additional affine symmetry algebra of the WZW model, analogously to the conformal Ward identities resting on the Virasoro algebra. Note, though, that the conformal Ward identities might be spoiled for particular correlators including logarithmic partners of some primary field by the fact that the logarithmic partner in some Jordan cell is not quasi-primary as is explained in [41]. Further investigation of that matter concerning generic correlation functions including logarithmic partners with respect to the affine Ward identities have, so far, not come to the author's attention but, in any case, they will not affect the general validity of the affine Ward identities for correlators involving only primary fields or the

[3] This will be shown in Section 3.2

null field at the first Virasoro level so that eq. (3.3) can be rephrased as

$$0 \equiv (\mathcal{P}_i)^{\alpha}(z_1, \ldots, z_L) \, \langle \phi_1(z_1) \cdots \phi_i(z_i) \cdots \phi_L(z_L) \rangle, \tag{3.4}$$

where we have made the index α explicit as compared to the components of all the other fields because it indicates that $\chi(z_i)$ does not transform in the (anti)-fundamental representation but in a subrepresentation of some representation tensored with the (anti)-fundamental representation.[4] The operator $(\mathcal{P}_i)^{\alpha}(z_1, \ldots, z_L)$ obviously annihilates the chiral correlator and, once the coordinates $\{z_i\}_i$ are fixed, eq. (3.4) can be interpreted as an eigenstate-equation for the linear operator $(\mathcal{P}_i)^{\alpha}$ (with parameters, i.e. $\{z_i\}_i$ suppressed) for eigenvalue zero to which Ψ is the eigenstate.

However, $(\mathcal{P}_i)^{\alpha}$ is not invariant under the symmetry of interest, i.e. the global $\mathrm{GL}(m|n)$ symmetry, since it still carries the index α. In the classical case $n = 0$, the obvious remedy lies in multiplication with its hermitian conjugate and contracting the additional index, leading to[5]

$$H_i := (\mathcal{P}_i)^{\dagger}_{\alpha} \, (\mathcal{P}_i)^{\alpha}. \tag{3.5}$$

The additional benefit of this is that H_i is not just a $\mathrm{GL}(m|n)$ invariant but also a positive semi-definite operator, and, therefore, bounded from below as expected of some physically reasonable Hamiltonian. In the generic case of $\mathrm{GL}(m|n)$, the operation of assigning an adjoint, denoted by the familiar dagger symbol †, gets promoted to the superadjoint, indicated by ‡. This involves – apart from taking the complex conjugate[6]– the definition of the dual of some representation. However, in the super case, the Killing form on $\mathfrak{gl}(m|n)$ is defined via the supertrace in the adjoint representation, and turns out to be non-degenerate for simple Lie superalgebras, so for $\mathfrak{gl}(m|n)$ with $m \neq n$ but zero for $m = n$ [31]. Nonetheless, as explained in Section 2.1.1, the most suitable substitute one can find is a supersymmetric bilinear form defined by taking the supertrace in the fundamental representation. However, for $m = n$, even this bilinear form suffers from indefiniteness and degeneracy bringing about non-unitarity. This, however, is expected for $\mathrm{GL}(m|m)$ structures that exhibit reducibility paired with indecomposability.

The result of this generalization to the fully supersymmetric expression is

$$H_i := (\mathcal{P}_i)^{\ddagger}_{\alpha} \, (\mathcal{P}_i)^{\alpha}, \tag{3.6}$$

which correctly reduces to eq. (3.5) for $n = 0$ but, as explained, comes with the drawback of lacking the guarantee for positive semi-definiteness or at least boundedness from below.

Finally, to ensure that a potential translational invariance of the quantum system manifests itself in the Hamiltonian, all sites should contribute in the same way, so that the full

[4] For a null field at the first Virasoro level, α will be an index of the adjoint representation as will be shown at the end of Section 3.2.

[5] Einstein's summation convention is understood.

[6] The definition of $(\mathcal{P}_i)^{\alpha}$ takes place on the Lie superalgebra level, so there are no Grassmann variables involved which would require special attention.

Hamiltonian is given by summation over the site index i,

$$H := \sum_{i=1}^{L} H_i, \tag{3.7}$$

endowed with global GL$(m|n)$ symmetry and, of course, still annihilating the chiral correlator:

$$H \langle \phi_1(z_1) \cdots \phi_L(z_L) \rangle = 0. \tag{3.8}$$

If H turns out to be positive semi-definite, the chiral correlator Ψ will be a ground state.

Summing up, it should be clear that the null field $\chi(z)$ allows for the construction of – as long as the coordinates $\{z_i\}_i$ have not been fixed – a differential operator H that maps the space of chiral L-point functions to itself with – at least – the chiral correlator Ψ, defined in eq. (3.2), in its kernel. The actual discretization of the chiral CFT is the mapping that takes the differential operator H acting on chiral L-point functions and the chiral correlator Ψ to some image $H(z_1, \ldots, z_L)$ acting on $\mathcal{H}$ and the wave function $\Psi(z_1, \ldots, z_L)$ by fixing the set of coordinates $\{z_i\}_i$.

3.2 The GL$(m|n)$ WZW Model, Free Fields and Null Fields

The starting point of the construction is the chiral symmetry of the GL$(m|n)$ WZW model which is provided by the current algebra $\widehat{\mathfrak{gl}}(m|n)$ built from chiral currents $J^a{}_b(z)$. These obey the mutual OPE

$$J^a{}_b(z)\, J^c{}_d(w) \sim \delta^a_d\, \delta^c_b\, \frac{(-1)^a}{(z-w)^2} + \delta^c_b\, \frac{J^a{}_d(w)}{z-w} - \delta^a_d\, \frac{(-1)^{(a+b)(c+d)}\, J^c{}_b(w)}{z-w} \tag{3.9}$$

where, here and subsequently, we sloppily write $(-1)^a$ instead of $(-1)^{|a|}$ for simplicity. However, this should not lead to any confusion. They also reflect the supercommutation relations their modes $(J^a{}_b)_n$ fulfill. The modes $(J^a{}_b)_n$ are defined through their *Laurent expansion*,

$$J^a{}_b(z) = \sum_{n \in \mathbb{Z}} z^{-n-1} (J^a{}_b)_n, \tag{3.10}$$

yielding

$$[(J^a{}_b)_n, (J^c{}_d)_m] = n\, \delta_{n+m,0}\, \delta^a_d\, \delta^c_b\, (-1)^a + \delta^c_b\, (J^a{}_d)_{n+m} - \delta^a_d\, (-1)^{(a+b)(c+d)}\, (J^c{}_b)_{n+m}. \tag{3.11}$$

In particular, we retrieve the supercommutation relations fulfilled by the modes $(J^a{}_b)_0$, i.e.

$$[(J^a{}_b)_0, (J^c{}_d)_0] = \delta^c_b\, (J^a{}_d)_0 - \delta^a_d\, (-1)^{(a+b)(c+d)}\, (J^c{}_b)_0, \tag{3.12}$$

implying that these make up the horizontal subsuperalgebra $\mathfrak{gl}(m|n)$. Note also that the first factor in eq. (3.9) is actually given by the invariant bilinear form $\kappa^a{}_{b,}{}^c{}_d$ of $\mathfrak{gl}(m|n)$

calculated in Section 2.1.3:

$$\kappa^{a}{}_{b,}{}^{c}{}_{d} = \delta^a_d\, \delta^c_b\, (-1)^a. \tag{3.13}$$

By rewriting the other two summands of eq. (3.9) and comparing these to the general form for current algebras at level $k = 1$, i.e.

$$J^{a}{}_{b}(z)\, J^{c}{}_{d}(w) \sim \frac{\kappa^{a}{}_{b,}{}^{c}{}_{d}}{(z-w)^2} + \sum_{e,f} i\, f^{a}{}_{b,}{}^{c}{}_{d,e}{}^{f}\, \frac{J^{e}{}_{f}(w)}{z-w}, \tag{3.14}$$

we can infer the matrix representation of the adjoint representation

$$(T^{a}{}_{b})^{c}{}_{d,e}{}^{f} = -i\, f^{a}{}_{b,}{}^{c}{}_{d,e}{}^{f} = -\delta^c_b\, \delta^a_e\, \delta^f_d + \delta^a_d\, \delta^c_e\, \delta^f_b\, (-1)^{(a+b)(c+d)}. \tag{3.15}$$

Next, let us give the action of the chiral currents on the primary field $\phi(w)$ which transforms in some irreducible representation of $\mathfrak{gl}(m|n)$:

$$J^{a}{}_{b}(z)\, \phi(w) \sim \frac{S^{a}{}_{b}\, \phi(w)}{z-w}. \tag{3.16}$$

The $S^{a}{}_{b}$ are the matrix representatives of the respective representation the primary field $\phi(w)$ transforms under. As our interest in this work is restricted to primary fields transforming in the fundamental or antifundamental representation (referring to these as $\psi(w)$ and $\bar{\psi}(w)$[7]), we may concretize eq. (3.16) to

$$J^{a}{}_{b}(z)\, \psi(w) \sim \frac{E^{a}{}_{b}\, \psi(w)}{z-w} \tag{3.17}$$

and

$$J^{a}{}_{b}(z)\, \bar{\psi}(w) \sim \frac{E^{a}{}_{b} \cdot \bar{\psi}(w)}{z-w} \tag{3.18}$$

with the $\mathfrak{gl}(m|n)$ generator $E^{a}{}_{b}$ canonically acting on the fundamental representation space $\mathcal{V}$ (cf. Section 2.1), and thus on $\psi(w)$, and its action on $\bar{\mathcal{V}}$ indicated by the dot in eq. (3.18).

In order to concretize the structures introduced so far, let us impose that the components of the fundamental and antifundamental primary fields $\psi(w)$ and $\bar{\psi}(z)$ are furnished by m complex free fermions and n complex free bosons. Their mutual OPE shall be given by

$$\bar{\psi}_a(z)\, \psi^b(w) \sim \frac{\delta^b_a}{z-w}. \tag{3.19}$$

As both representations are treated on equal footing, one can read off from eq. (3.19) that

[7] As we only consider the chiral part of the CFT, barred expressions always refer to anti-*fundamental* instead of anti-*chiral* objects, contrary to the common use in the physics literature. However, this should not lead to any confusion.

their anomalous dimension is constrained to $h = \frac{1}{2}$. Further, we have

$$\begin{aligned} \bar{\psi}_a(z)\,\psi^b(w) &= (-1)^{(|a|-1)(|b|-1)}\psi^b(w)\,\bar{\psi}_a(z) \\ &= (-1)^{ab-a-b+1}\psi^b(w)\,\bar{\psi}_a(z), \end{aligned} \tag{3.20}$$

reflecting that the fields have opposite parity $(|a|-1)$ as compared to the parity $|a|$ of the basis elements e^a and e_a of the fundamental and antifundamental $\mathfrak{gl}(m|n)$ representations $\mathcal{V}$ and $\bar{\mathcal{V}}$. Analogous equations hold for pure versions of eq. (3.20). By defining the chiral currents

$$J^a{}_b(z) =: \psi^a \bar{\psi}_b : (z) \tag{3.21}$$

(with $:\ldots:$ denoting normal ordering), we find that they fulfill the current algebra $\widehat{\mathfrak{gl}}(m|n)$ with OPE given in eq. (3.9) and, additionally, have the desired OPEs with $\psi(z)$ and $\bar{\psi}(z)$, namely eq. (3.17) becomes

$$J^a{}_b(z)\,\psi^c(w) \sim \frac{\delta^c_b\,\psi^a(w)}{z-w} \tag{3.22}$$

and eq. (3.18) turns into

$$J^a{}_b(z)\,\bar{\psi}_c(w) \sim -(-1)^{(a+b)c}\,\frac{\delta^a_c\,\bar{\psi}_b(w)}{z-w}. \tag{3.23}$$

Armed with the free field representation, it is not very difficult to work out that there are null fields at the first Virasoro level in the module $\{\psi\}$, i.e. that there are null fields of the form

$$(J^a{}_b)_{-1}\,\psi^c(w) = \frac{1}{2\pi i}\oint_w \frac{\mathrm{d}z}{z-w}\,J^a{}_b(z)\,\psi^c(w) = \delta^c_b\,\psi^a(w), \tag{3.24}$$

namely those for which $b \neq c$. Of course, not every field of the form given in eq. (3.24) will be a null field. Rather, the null fields at the first Virasoro level span a horizontal submodule $\mathcal{N}$ within the horizontal module spanned by fields of this form. Upon comparison of the double-index of $(J^a{}_b)_{-1}$ in eq. (3.24) with the index structure of the matrix given in eq. (3.15), we identify $(J^a{}_b)_{-1}$ as transforming under the adjoint representation $\mathcal{J}$. Thus, the null fields of interest form a submodule embedded in the (graded)[8] tensor product of the adjoint representation with the fundamental representation, $\mathcal{J} \otimes \mathcal{V}$.

Considerations along the same line lead to the insight that the analogous null fields derived from the antifundamental primary field $\bar{\psi}(z)$ reside in the analogous tensor product $\mathcal{J} \otimes \bar{\mathcal{V}}$.

[8] Since this work particularly deals with the Lie *super*algebra $\mathfrak{gl}(m|n)$, even if not explicitly mentioned in places, all structures are graded

3.3 Projection Operators onto Null Fields

Having identified $\mathcal{J} \otimes \mathcal{V}$ as the relevant horizontal submodule of $\{\psi\}$ the submodule of null fields at the first Virasoro level is submerged in, we will derive operators projecting them onto the representations $\mathcal{N}$ of the null fields. The discussion for the case $\mathcal{J} \otimes \bar{\mathcal{V}}$ runs in exactly the same vein so that we will consider the case $\mathcal{J} \otimes \mathcal{V}_i$ with,

- for the pure case, $\mathcal{V}_i = \mathcal{V}$ for all values of i and,
- for the alternating case, $\mathcal{V}_i = \begin{cases} \mathcal{V} & \text{for even } i \text{ and} \\ \bar{\mathcal{V}} & \text{for odd } i. \end{cases}$

To make it precise, the projections operators

$$\mathcal{P}_i : \mathcal{J} \otimes \mathcal{V}_i \longrightarrow \mathcal{J} \otimes \mathcal{V}_i, \tag{3.25}$$

to be constructed for each site i evaluate to the identity when restricted to $\mathcal{N}_i \subset \mathcal{J} \otimes \mathcal{V}_i$,

$$\mathcal{P}_i\Big|_{\mathcal{N}_i} \equiv \mathrm{id}, \tag{3.26}$$

and to zero when restricted to the orthogonal complement $\mathcal{N}_i^{\perp}$ of $\mathcal{N}_i$ in $\mathcal{J} \otimes \mathcal{V}_i$,

$$\mathcal{P}_i\Big|_{\mathcal{N}_i^{\perp}} \equiv 0. \tag{3.27}$$

In order to find the correct expression for $\mathcal{P}_i$, let us note that a good starting point for finding such an operator is the quadratic Casimir C_i on $\mathcal{J} \otimes \mathcal{V}_i$ since it acts proportionally to the identity on each summand[9] of the tensor product decomposition of $\mathcal{J} \otimes \mathcal{V}_i$ as we will shortly see. Thus, given the tensor product decomposition $\mathcal{J} \otimes \mathcal{V}_i = \mathcal{N}_i \oplus \mathcal{X} \oplus \ldots$ where $\mathcal{X}$ specifies one of the summands to be projected out, consider the operator

$$\big(C_i - C_i(\mathcal{X})\,\mathrm{id}\big) \tag{3.28}$$

(with $C_i(\mathcal{X})$ the actual eigenvalue of the quadratic Casimir C_i on $\mathcal{X}$), which obviously acts as zero on the unsolicited part $\mathcal{X}$. However, instead of yielding just the identity on $\mathcal{N}$, it evaluates to $\big(C_i(\mathcal{N}) - C_i(\mathcal{X})\big)\mathrm{id}$ on $\mathcal{N}$. Therefore, for every summand de trop in the tensor product decomposition of $\mathcal{J} \otimes \mathcal{V}_i$, the projection operator $\mathcal{P}_i$ must contain such a factor $\big(C_i - C_i(\mathcal{X})\,\mathrm{id}\big)$ properly normalized by $1/\big(C_i(\mathcal{N}) - C_i(\mathcal{X})\big)$, i.e.

$$\mathcal{P}_i \propto \frac{\big(C_i - C_i(\mathcal{X})\,\mathrm{id}\big)}{\big(C_i(\mathcal{N}) - C_i(\mathcal{X})\big)}. \tag{3.29}$$

[9] Of course, this only holds for a decomposition into irreducible representations which is not granted but assumed at this point nonetheless.

Thus, given the quadratic Casimir on $\mathcal{J} \otimes \mathcal{V}_i$, we can compute the projection operator $\mathcal{P}_i$.

Note that, as explained in Section 2.1.4, the adjoint representation can be written as a tensor product of two auxiliary spaces, the fundamental and antifundamental representation:

$$\mathcal{J} = \mathcal{V} \otimes \bar{\mathcal{V}}. \tag{3.30}$$

Thus, the tensor product we are seeking to decompose is truly $\mathcal{V} \otimes \bar{\mathcal{V}} \otimes \mathcal{V}_i$. The quadratic Casimir on $\mathcal{V} \otimes \bar{\mathcal{V}} \otimes \mathcal{V}_i$ is given by

$$C_i = 3(m-n)\mathrm{id} + 2(\mathbf{S}_1 \cdot \mathbf{S}_2 + \mathbf{S}_1 \cdot \mathbf{S}_i + \mathbf{S}_2 \cdot \mathbf{S}_i) \tag{3.31}$$

with the vector of spin operators $E^a{}_b$ acting on the first auxiliary space $\mathcal{V}$,

$$\mathbf{S}_1 = \left(E^a{}_b \otimes \mathbb{I} \otimes \mathbb{I}\right)_{a,b=1,\ldots,m+n}, \tag{3.32}$$

and the vector of spin operators $-(-1)^{ab}E_b{}^a$ acting on the second auxiliary space $\bar{\mathcal{V}}$ (as was explained in Section 2.1.3)

$$\mathbf{S}_2 = \left(\mathbb{I} \otimes (-(-1)^{ab}E_b{}^a) \otimes \mathbb{I}\right)_{a,b=1,\ldots,m+n}. \tag{3.33}$$

The components of $\mathbf{S}_i$ act on $\mathcal{V}_i$ either analogously to $\mathbf{S}_1$ or $\mathbf{S}_2$ depending on whether $\mathcal{V}_i = \mathcal{V}$ or $\mathcal{V}_i = \bar{\mathcal{V}}$. The dot product of two such vectors of spin operators represents either of the two mixed parts of the coproduct on these two spaces by which the quadratic Casimir operator was introduced in Section 2.1.4. For later purposes it will be useful to also introduce the map $d_\bullet : \{1, \ldots, L\} \to \mathbb{Z}_2$ which helps to keep track of the kind of space at site i (whether fundamental or antifundamental) and which is defined as

- $i \mapsto d_i := 0$ in the pure case and
- $i \mapsto d_i := i \pmod 2$ in the alternating case.

After a lengthy calculation which is sketched in Section 2.1.4 and, again, defining the superdimension $N := m - n$, one finds the eigenvalues N, $3N-2$ and $3N+2$ and the respective decomposition

$$\mathcal{V} \otimes \bar{\mathcal{V}} \otimes \mathcal{V}_i = \mathcal{N}_i \oplus \mathcal{X}_i \tag{3.34}$$

under the quadratic Casimir. The space $\mathcal{N}_i$ containing the null states is spanned by all vectors in $\mathcal{V} \otimes \bar{\mathcal{V}} \otimes \mathcal{V}_i$ that are eigenvectors of the quadratic Casimir with eigenvalue $3N + \mathrm{sgn}(N)2$ while $\mathcal{X}_i$ is a sum of eigenspaces with eigenvalues N and $3N - \mathrm{sgn}(N)2$. As a consequence, we find for positive N

$$\mathcal{P}_i^+ = \frac{(C_i - N\,\mathrm{id})}{\big((3N+2) - N\big)} \frac{\big(C_i - (3N-2)\mathrm{id}\big)}{\big((3N+2) - (3N-2)\big)} = \frac{\Big(C_i^2 - (4N-2)C_i + N(3N-2)\mathrm{id}\Big)}{8(N+1)} \tag{3.35}$$

while, for negative N, it works out to be

$$\mathcal{P}_i^- = \frac{(C_i - N\,\mathrm{id})}{((3N-2)-N)} \frac{\left(C_i - (3N+2)\mathrm{id}\right)}{((3N-2)-(3N+2))} = \frac{\left(C_i^2 - (4N+2)C_i + N(3N+2)\mathrm{id}\right)}{-8(N-1)}. \tag{3.36}$$

For $N = 0, \pm 1$, one could be led to use only one of the two factors for the construction of the proper projection operator. However, that operator turns out to be not idempotent and, thus, no projection operator. Instead, the square of such an operator is idempotent and, therefore, a projection. But only in the case $N = 0$, will the square of that operator be different from $\mathcal{P}_i^{\pm}$. This particular projection ($\frac{C_i^2}{4}$) will not be taken into consideration in the following.

After insertion of the expression for the quadratic Casimir on $\mathcal{V} \otimes \bar{\mathcal{V}} \otimes \mathcal{V}_i$ given in eq. (3.31) and either a tedious algebraic calculation or, alternatively, a more aesthetic diagrammatic calculation in terms of a graphical representation of the walled Brauer algebra, cf. [13], the operator projecting onto the null fields simplifies to

$$\begin{aligned}\mathcal{P}_i^+ = \frac{1}{2(N+1)} \big((N+1)\mathrm{id} + (\mathbf{S}_1 \cdot \mathbf{S}_i)(\mathbf{S}_2 \cdot \mathbf{S}_i) + (\mathbf{S}_2 \cdot \mathbf{S}_i)(\mathbf{S}_1 \cdot \mathbf{S}_i) + \mathbf{S}_1 \cdot \mathbf{S}_2 \\ + ((1-d_i)N+1)\mathbf{S}_1 \cdot \mathbf{S}_i + (d_i N + 1)\mathbf{S}_2 \cdot \mathbf{S}_i\big)\,.\end{aligned} \tag{3.37}$$

and

$$\begin{aligned}\mathcal{P}_i^- &= \frac{1}{2(1-N)} \big((1-N)\mathrm{id} + (\mathbf{S}_1 \cdot \mathbf{S}_i)(\mathbf{S}_2 \cdot \mathbf{S}_i) + (\mathbf{S}_2 \cdot \mathbf{S}_i)(\mathbf{S}_1 \cdot \mathbf{S}_i) - \mathbf{S}_1 \cdot \mathbf{S}_2 \\ &\qquad -(1-(1-d_i)N)\mathbf{S}_1 \cdot \mathbf{S}_i - (1-d_i N)\mathbf{S}_2 \cdot \mathbf{S}_i\big) \\ &= \frac{1}{2(1+|N|)} \big((1+|N|)\mathrm{id} + (\mathbf{S}_1 \cdot \mathbf{S}_i)(\mathbf{S}_2 \cdot \mathbf{S}_i) + (\mathbf{S}_2 \cdot \mathbf{S}_i)(\mathbf{S}_1 \cdot \mathbf{S}_i) - \mathbf{S}_1 \cdot \mathbf{S}_2 \\ &\qquad -(1+(1-d_i)|N|)\mathbf{S}_1 \cdot \mathbf{S}_i - (1+d_i|N|)\mathbf{S}_2 \cdot \mathbf{S}_i\big)\,.\end{aligned} \tag{3.38}$$

leading to the generalized form[10]

$$\begin{aligned}\mathcal{P}_i = \frac{1}{2(|N|+1)} \big((|N|+1)\mathrm{id} + (\mathbf{S}_1 \cdot \mathbf{S}_i)(\mathbf{S}_2 \cdot \mathbf{S}_i) + (\mathbf{S}_2 \cdot \mathbf{S}_i)(\mathbf{S}_1 \cdot \mathbf{S}_i) + \mathrm{sgn}(N)[\mathbf{S}_1 \cdot \mathbf{S}_2 \\ + ((1-d_i)|N|+1)\mathbf{S}_1 \cdot \mathbf{S}_i + (d_i|N|+1)\mathbf{S}_2 \cdot \mathbf{S}_i]\big)\,.\end{aligned} \tag{3.39}$$

The second and third term in the numerator are dual to each other under supertransposition while all other summands in the numerator are invariant under supertransposition. And as we have defined the representations $\mathcal{V}$ and $\bar{\mathcal{V}}$ with real-valued representation matrices (namely, somehow defined through the unit matrices $E^a{}_b$) of the generators of $\mathfrak{gl}(m|n)$, the

[10] Note that our convention will be $\mathrm{sgn}(0) := +1$ so that, for $N = 0$, we use $\mathcal{P}_i^+$. However, one could just as well use the opposite convention leading to the use of $\mathcal{P}_i^-$

operator $\mathcal{P}_i$ is not just invariant under supertransposition but also under superhermitian conjugation, $\mathcal{P}_i = \mathcal{P}_i^\ddagger$. Further, it can be shown to be idempotent, $\mathcal{P}_i^2 = \mathcal{P}_i$, and, thus, it really is the desired *projection* operator.

Another remark regards the observation that this projector is, of course, an even operator in terms of the $\mathbb{Z}_2$-grading of $\mathfrak{gl}(m|n)$ because it is built up from the identity and the quadratic Casimir – both of which preserve the parity of an element they act on.[11] Therefore, without penalty of a sign, it can be passed by any type of element, be it even or odd. This property will be relevant for the construction of the Hamiltonian in Section 3.4.

It should be mentioned that even though the terms $(\mathbf{S}_1 \cdot \mathbf{S}_i)(\mathbf{S}_2 \cdot \mathbf{S}_i)$ and $(\mathbf{S}_2 \cdot \mathbf{S}_i)(\mathbf{S}_1 \cdot \mathbf{S}_i)$ may look like they are four-spin terms, in fact, they can, of course, be reduced to third order Casimir operators. This fact implies that third order Casimir operators are, of course, even.

3.4 The Derivation of the Hamiltonian

With the projectors constructed in the previous Section, we can now explicitly express the components of the null fields at the first Virasoro level at site i as

$$\chi^{a}{}_{b,}{}^{e}(z_i) = (\mathcal{P}_i)^{a}{}_{a',b}{}^{b',e}{}_{f}\,((J^{a'}{}_{b'})_{-1}\,\psi^{f})(z_i) \tag{3.40}$$

which, as an element in $\mathcal{J} \otimes \mathcal{V}_i$, carries the adjoint bi-index ${}^{a}{}_{b}$ and the fundamental index ^{e} while the projector features the indices of the explicit matrices for $(S_1)^{a}{}_{a'}$ and $(S_2)_{b}{}^{b'}$ acting on $\mathcal{J} = \mathcal{V} \otimes \bar{\mathcal{V}}$ and for $(S_i)^{e}{}_{f}$ acting, in this case, on the fundamental representation $\mathcal{V}$ at site i. It goes without saying that an analogous expression holds for the case $\mathcal{V}_i = \bar{\mathcal{V}}$ with the index e downstairs and f switching its place. Now we can rewrite eq. (3.3) for any site i as

$$\begin{aligned} 0 \;&\equiv\; \langle \phi_1(z_1) \cdots \chi(z_i) \cdots \phi_L(z_L) \rangle && (3.41)\\ &=\; \langle \phi_1(z_1) \cdots [(\mathcal{P}_i)_{a',}{}^{b'}\,(J^{a'}{}_{b'})_{-1}\,\phi_i(z_i)] \cdots \phi_L(z_L) \rangle, && (3.42) \end{aligned}$$

suppressing any index that is not summed over, however, keeping in mind that this equation is understood componentwise. Next, we make use of the mode expansion of the current $J^{a}{}_{b}(z)$ and the affine Ward identities in order to shift the action of $(J^{a'}{}_{b'})_{-1}$ from $\phi(z_i)$ over to all the other constituents of the chiral correlator,

$$\begin{aligned} 0 \;&=\; \langle \phi_1(z_1) \cdots [(\mathcal{P}_i)_{a',}{}^{b'}\,\frac{1}{2\pi i}\oint_{z_i} \frac{\mathrm{d}z}{z - z_i}\, J^{a'}{}_{b'}(z)\,\phi_i(z_i)] \cdots \phi_L(z_L) \rangle && (3.43)\\ &=\; \sigma_i \sum_{j \notin \{i\}} \sigma_j \langle \phi_1(z_1) \cdots [(\mathcal{P}_i)_{a',}{}^{b'}\,\frac{1}{2\pi i}\oint_{z_j} \frac{-\mathrm{d}z}{z - z_i}\, J^{a'}{}_{b'}(z)\,\phi_j(z_j)] \cdots \phi_L(z_L) \rangle, && (3.44) \end{aligned}$$

[11] For the identity, this is clear. Nevertheless, as the quadratic Casimir is *quadratic* in operators of the same degree, it, indeed, is even by construction.

with the sign σ_k being defined as resulting from moving $(\mathcal{P}_i)_{a'}{}^{b'} \frac{1}{2\pi i} \oint_{z_j} \frac{-\mathrm{d}z}{z-z_i} J^{a'}{}_{b'}(z)$ from the first position of the correlator to the position just in front of the field $\phi_k(z_k)$:

$$\sigma_k := \frac{\langle (\mathcal{P}_i)_{a'}{}^{b'} \frac{1}{2\pi i} \oint_{z_i} \frac{\mathrm{d}z}{z-z_i} J^{a'}{}_{b'}(z)\, \phi_1(z_1) \cdots \phi_L(z_L) \rangle}{\langle \phi_1(z_1) \cdots [(\mathcal{P}_i)_{a'}{}^{b'} \frac{1}{2\pi i} \oint_{z_i} \frac{\mathrm{d}z}{z-z_i} J^{a'}{}_{b'}(z)\, \phi_k(z_k)] \cdots \phi_L(z_L) \rangle}. \tag{3.45}$$

Again, be reminded that, although implicit in this notation, the sign σ_k depends also on the actual components of the fields involved in the latter equation. In the next step, the OPE given in eq. (3.16) is employed, cf. [45], and focusing on the part in the square brackets in eq. (3.44), we find

$$\begin{aligned}
0 &= \ldots \cdots [(\mathcal{P}_i)_{a'}{}^{b'} \frac{1}{2\pi i} \oint_{z_j} \frac{-\mathrm{d}z}{z-z_i} J^{a'}{}_{b'}(z)\, \phi_j(z_j)] \cdots \ldots && (3.46)\\
&= \ldots \cdots [(\mathcal{P}_i)_{a'}{}^{b'} \frac{1}{2\pi i} \oint_{z_j} \frac{-\mathrm{d}z}{z-z_i} \frac{S^{a'}{}_{b'}\, \phi_j(z_j)}{z-z_j}] \cdots \ldots && (3.47)\\
&= \ldots \cdots [(\mathcal{P}_i)_{a'}{}^{b'} \frac{-1}{z_j-z_i} S^{a'}{}_{b'}\, \phi_j(z_j)] \cdots \ldots && (3.48)\\
&= \sigma_i \sum_{j \notin \{i\}} \sigma_j \langle \phi_1(z_1) \cdots [\frac{(\mathcal{P}_i)_{a'}{}^{b'} S^{a'}{}_{b'}}{z_i-z_j} \phi_j(z_j)] \cdots \phi_L(z_L) \rangle && (3.49)
\end{aligned}$$

using Cauchy's integral formula on the function $f(z) = \frac{1}{z-z_i}$ (as the correlator is independent of the variable z of integration). Note also that $S^{a'}{}_{b'}$ is not to be understood as a matrix element but, instead, refers to the respective representation matrix of the $\mathfrak{gl}(m|n)$ generator specified by the bi-index ${}^{a'}{}_{b'}$. For clarity, we will label the spin operator acting on site j by the subscript ${}_j$ which is to be understood in the form

$$S_j{}^{a'}{}_{b'} := \mathbb{I}^{\otimes j-1} \otimes S^{a'}{}_{b'} \otimes \mathbb{I}^{\otimes L-j}, \tag{3.50}$$

enhancing it to an operator acting on the full space of states $\mathcal{H} = \bigotimes_i \mathcal{V}_i$.

Now, by pulling all operators acting on some site j out in front of the chiral correlator, of course, we pick up another sign factor σ_j (again, depending on the degree of these operators and all fields they have to be passed by – however, those have not changed during the former calculation). In effect, this cancels all signs factors under the summation:

$$\begin{aligned}
0 &= \sigma_i \sum_{j \notin \{i\}} \sigma_j \langle \phi_1(z_1) \cdots [\frac{(\mathcal{P}_i)_{a'}{}^{b'} S^{a'}{}_{b'}}{z_i-z_j} \phi_j(z_j)] \cdots \phi_L(z_L) \rangle && (3.51)\\
&= \sigma_i \sum_{j \notin \{i\}} \frac{(\mathcal{P}_i)_{a'}{}^{b'} S_j{}^{a'}{}_{b'}}{z_i-z_j} \langle \phi_1(z_1) \cdots \phi_L(z_L) \rangle. && (3.52)
\end{aligned}$$

Disposing of the overall sign σ_i does not suspend the validity of the former equation, either,

$$0 = \sum_{j \notin \{i\}} \frac{(\mathcal{P}_i)_{a',}{}^{b'} S_j{}^{a'}{}_{b'}}{z_i - z_j} \langle \phi_1(z_1) \cdots \phi_L(z_L) \rangle. \tag{3.53}$$

and so we define the operator

$$(\mathcal{P}_i)^a{}_b(z_1, \ldots, z_L) := \sum_{j \notin \{i\}} \frac{(\mathcal{P}_i)^a{}_{a',b}{}^{b'} S_j{}^{a'}{}_{b'}}{z_i - z_j}, \tag{3.54}$$

at this point again additionally displaying at least the two auxiliary indices a and b.

This operator features a genuine two-spin interaction of the physical spin at site i with every physical spin at site $j \notin \{i\}$ even though the action on site i is only implicit in this notation but clear from comparing with the result for $\mathcal{P}_i$ in eq. (3.39). Again, it should be stressed that $(\mathcal{P}_i)^a{}_{a',b}{}^{b'}$ refers to matrix elements of $\mathcal{P}_i$ whereas the bi-index of $S_j{}^{a'}{}_{b'}$ specifies a particular spin operator. Furthermore, $(\mathcal{P}_i)^a{}_b(z_1, \ldots, z_L)$ explicitly depends on the coordinate z_i of every site i and annihilates the chiral correlator. Nevertheless, this still falls short of the expectations of the Hamiltonian advertised because it carries an adjoint index preventing it from being globally GL$(m|n)$-invariant.

To this end, we use the superhermitian conjugate of $(\mathcal{P}_i)^a{}_b(z_1, \ldots, z_L)$ to multiply $(\mathcal{P}_i)^a{}_b(z_1, \ldots, z_L)$ with itself and contract the spurious adjoint index. This defines, for each site i, an object

$$H_i(z_1, \ldots, z_L) := (\mathcal{P}_i^\ddagger)^b{}_a(z_1, \ldots, z_L)\ (\mathcal{P}_i)^a{}_b(z_1, \ldots, z_L) \tag{3.55}$$

which, using the properties $\mathcal{P}_i^\ddagger = \mathcal{P}_i$ and $\mathcal{P}_i^2 = \mathcal{P}_i$, can be simplified to

$$H_i(z_1, \ldots, z_L) = \sum_{j,k \notin \{i\}} \frac{\left(S_k{}^b{}_a\right)^\ddagger (\mathcal{P}_i)^a{}_{a',b}{}^{b'} S_j{}^{a'}{}_{b'}}{(\overline{z}_i - \overline{z}_k)(z_i - z_j)}. \tag{3.56}$$

Obviously, this object is GL$(m|n)$-invariant and generically exhibits a three-spin interaction on the physical spin at site i and both physical spins at sites $j, k \notin \{i\}$. Further, it annihilates the chiral correlator. However, it does not treat the site i on equal footing with all other sites. In order to cure this last flaw, we simply sum over all sites so as to arrive at the generic Hamiltonian

$$H(z_1, \ldots, z_L) = \sum_{i=1}^{L} \sum_{j,k \notin \{i\}} \frac{\left(S_k{}^b{}_a\right)^\ddagger (\mathcal{P}_i)^a{}_{a',b}{}^{b'} S_j{}^{a'}{}_{b'}}{(\overline{z}_i - \overline{z}_k)(z_i - z_j)} \tag{3.57}$$

which, after performing the summation over the matrix elements $(\mathcal{P}_i)^a{}_{a',b}{}^{b'}$, reduces to the

more instructive form[12]

$$H = \frac{1}{2(|N|+1)} \sum_{i=1}^{L} \sum_{j,k\notin\{i\}}^{L} \frac{1}{\overline{z}_{ik} z_{ij}} \cdot \Bigg((|N|+1)\, \mathbf{S}_k \cdot \mathbf{S}_j - (-1)^{d_i+d_j} \mathbf{S}_k \cdot \mathbf{S}_i - (-1)^{d_i+d_k} \mathbf{S}_j \cdot \mathbf{S}_i - \mathrm{sgn}(N) \Big[(-1)^{d_j+d_k} \mathbb{I} - \big((1-d_i)|N|+1\big)\, \mathrm{Cas}\left(\mathbf{S}_k, \mathbf{S}_j, \mathbf{S}_i\right) + (d_i|N|+1)\, \mathrm{Cas}\left(\mathbf{S}_k, \mathbf{S}_i, \mathbf{S}_j\right) \Big] \Bigg) \tag{3.58}$$

with the somewhat unusual convention

$$\mathrm{Cas}\left(\mathbf{S}_k, \mathbf{S}_j, \mathbf{S}_i\right) := S_k{}^a{}_b\, S_j{}^b{}_c\, S_i{}^c{}_a (-1)^{b+c} \tag{3.59}$$

for the third order Casimir operator of the representations at sites k, j and i, the definition of $z_{ij} := z_i - z_j$. The way it is built, namely globally invariant under $\mathrm{GL}(m|n)$, it leads to an object which, indeed, is not *hermitian* but rather *superhermitian* in general, i.e. for all cases involving fermionic degrees of freedom ($n > 0$). Thus, diagonalizability is not granted. However, certain setups may produce a hermitian and, hence, diagonalizable Hamiltonian. In particular, a certain one-dimensional, alternating setup[13] with global $\mathrm{GL}(1|1)$ symmetry features a diagonalizable (and positive semi-definite) Hamiltonian[14] even though the chiral symmetry it is based on stems from the $\mathrm{GL}(1|1)$ WZW model, a quintessential example of logarithmic CFT, well-known for indecomposable structures among its representations. In fact, the Haldane-Shastry version of the same setup[15] exhibits exactly this type of indecomposability expected from the conception about this logarithmic CFT.

Aside from this, the generic Hamiltonian ceases to be positive semi-definite as seen in numerous numerical simulations: While for $\mathrm{GL}(m|0)$ (with $m > 0$) the Hamiltonian stays positive semi-definite with exactly one ground state, we find the spectrum of the respective $\mathrm{GL}(0|m)$ Hamiltonian reflected about the zero. Adding any positive number of pairs consisting of a boson and a fermion,thus keeping the superdimension fixed, their spectra are enhanced only in degeneracy close to the zero energy state whereas far away from that state, also additional energy levels arise, some of which consist of complex-conjugate pairs as a hallmark of non-diagonalizable blocks of dimension two. The same is true for $\mathrm{GL}(m|m)$ apart from the overall degeneracy: The zero energy ground state will be fourfold degenerate and all other degeneracies will be multiples of four.

[12] Again, note that our convention will be $\mathrm{sgn}(0) := +1$ so that, for $N = 0$, we use $\mathcal{P}_i^+$. However, one could just as well use the opposite convention leading to the use of $\mathcal{P}_i^-$

[13] This is after a slight redefinition of the coordinate-dependence of the Hamiltonian was implemented, i.e. $w_{ij} := \frac{z_i+z_j}{z_i-z_j}$.

[14] This case will be treated explicitly in Section 4.2.

[15] This case will be treated explicitly in Section 4.1.

Finally, since it is built from the identity and second and third order Casimir operators, the Hamiltonian itself is an even operator, mapping even states to even states and odd states to odd ones. Anything else would raise concerns about the physical sensibility of the model as it is a well-known and well-established observational fact that the elementary excitations in nature are either bosonic or fermionic.[16]

The globally SU(2)-invariant version of this Hamiltonian was first introduced in [6] and it was shown to incorporate the SU(2) Haldane-Shastry spin chain as a special case. Furthermore, it was argued in [13] that – in the non-graded case – every singlet wave function satisfies

$$0 = \sum_{j \notin \{i\}} \langle \phi_1(z_1) \cdots [(\mathcal{P}_i)^a{}_{a',b}{}^{b'} S^{a'}{}_{b'} \phi_j(z_j)] \cdots \phi_L(z_L) \rangle \tag{3.60}$$

with the coordinate-dependence of the operator $(\mathcal{P}_i)^a{}_b(z_1, \ldots, z_L)$ entirely removed. Consequently, the supersymmetric version of this statement reads

$$\begin{aligned} 0 &= \sum_{j \notin \{i\}} \sigma_j \langle \phi_1(z_1) \cdots [(\mathcal{P}_i)^a{}_{a',b}{}^{b'} S^{a'}{}_{b'} \phi_j(z_j)] \cdots \phi_L(z_L) \rangle & (3.61) \\ &= \sum_{j \notin \{i\}} (\mathcal{P}_i)^a{}_{a',b}{}^{b'} S_j{}^{a'}{}_{b'} \langle \phi_1(z_1) \cdots \phi_L(z_L) \rangle & (3.62) \end{aligned}$$

with the additional sign factor σ_j, cf. eq. (3.44), dropping out by pulling the operator out of the correlator. The sum of this equation multiplied by some function $g_i(z_i)$ and eq. (3.53) multiplied by some other function $f_i(z_i)$ leads to

$$0 = \sum_{j \notin \{i\}} \left(\frac{f_i(z_i)}{z_i - z_j} + g_i(z_i) \right) (\mathcal{P}_i)^a{}_{a',b}{}^{b'} S_j{}^{a'}{}_{b'} \langle \phi_1(z_1) \cdots \phi_L(z_L) \rangle \tag{3.63}$$

which shows that there is some freedom in choosing the coordinate-dependence of the projection operator $(\mathcal{P}_i)^a{}_b(z_1, \ldots, z_L)$ without spoiling its main property of annihilating the chiral correlator. Thus, we will analogously redefine $(\mathcal{P}_i)^a{}_b(z_1, \ldots, z_L)$ to

$$(\mathcal{P}_i)^a{}_b(z_1, \ldots, z_L) := \sum_{j \notin \{i\}} w_{ij} (\mathcal{P}_i)^a{}_{a',b}{}^{b'} S_j{}^{a'}{}_{b'} \tag{3.64}$$

with

$$w_{ij} := \frac{f_i(z_i)}{z_i - z_j} + g_i(z_i) \tag{3.65}$$

purely for generality and without dwelling on the physical consequences an arbitrary choice of f_i and g_i could have for the system under consideration.

In the following, we will restrict ourselves to the case $w_{ij} = -w_{ji}$ which includes the

[16] Wen argues in [46] that Fermi statistics, just like fractional statistics, might be an emergent phenomenon itself, related to defects of long-range quantum entanglements.

original coordinate-dependence (with $f_i(z_i) = 1$ and $g_i(z_i) = 0$) and the admissible choice $f(z_i) = 2z_i$ and $g(z_i) = -1$ implying

$$w_{ij} := \frac{z_i + z_j}{z_i - z_j}. \tag{3.66}$$

This last choice of w_{ij} leads to the right coordinate-dependence in order to connect this Hamiltonian to the Haldane-Shastry spin chain in the case that all spins are *uniformly* distributed on the unit circle, i.e. in the case $z_j = \exp\left(i\frac{2\pi}{L}j\right)$.

Hence, finally we obtain the family of Hamiltonians with global GL$(m|n)$ invariance (with the superdimension $N = m - n$):

$$\begin{aligned} H = \frac{1}{2(|N|+1)} \sum_{i=1}^{L} \sum_{j,k\notin\{i\}}^{L} \overline{w}_{ik} w_{ij} \\ \cdot \Bigg((|N|+1)\, \mathbf{S}_k \cdot \mathbf{S}_j - (-1)^{d_i+d_j} \mathbf{S}_k \cdot \mathbf{S}_i - (-1)^{d_i+d_k} \mathbf{S}_j \cdot \mathbf{S}_i - \operatorname{sgn}(N) \Big[(-1)^{d_j+d_k} \mathbb{I} \\ - \big((1-d_i)|N|+1\big) \operatorname{Cas}\left(\mathbf{S}_k, \mathbf{S}_j, \mathbf{S}_i\right) + (d_i|N|+1) \operatorname{Cas}\left(\mathbf{S}_k, \mathbf{S}_i, \mathbf{S}_j\right) \Big] \Bigg). \end{aligned} \tag{3.67}$$

3.5 Simplification of the General Hamiltonian

After relabeling and ordering the indices of sites in the summation – which, from a computational point of view, should be a more economic representation – we arrive at

$$\begin{aligned} H = g\, \mathbb{I} + \sum_{i<j}^{L} g_{ij} \mathbb{Q}_{ij} \\ + \sum_{i<j<k}^{L} \Big(\mathbb{Q}_{ij}(g_{ijk}\mathbb{Q}_{jk} + g_{jik}\mathbb{Q}_{ik}) + \mathbb{Q}_{jk}(g_{jki}\mathbb{Q}_{ik} + g_{kji}\mathbb{Q}_{ij}) + \mathbb{Q}_{ik}(g_{kij}\mathbb{Q}_{ij} + g_{ikj}\mathbb{Q}_{jk}) \Big) \end{aligned} \tag{3.68}$$

with

$$g := \frac{\operatorname{sgn}(N)}{|N|+1} \left((|N|+2) \sum_{i<j} (|N| - \delta_{d_i,d_j}) \left| w_{ij} \right|^2 - \Re \left[\sum_{i<j} (-1)^{d_i+d_j} \sum_{k\notin\{i,j\}} \overline{w}_{ik} w_{jk} \right] \right), \tag{3.69}$$

$$g_{ij} := \frac{|N|+2}{|N|+1} \left(\delta_{d_i,d_j} |N| - 1 \right) \left| w_{ij} \right|^2 + (-1)^{d_i+d_j} \Re \left[\sum_{k\notin\{i,j\}} \overline{w}_{ik} w_{jk} \right]$$

$$+\frac{1}{|N|+1}\Re\left[w_{ij}\sum_{k\notin\{i,j\}}(-1)^{d_k}\left((-1)^{d_i}\overline{w}_{jk}-(-1)^{d_j}\overline{w}_{ik}\right)\right], \tag{3.70}$$

$$g_{ijk}:=\frac{\mathrm{sgn}(N)(-1)^{d_i+d_k}}{2(|N|+1)}\left(w_{jk}\left(\delta_{d_j,d_k}|N|+1\right)-w_{ij}\left((1-\delta_{d_i,d_j})|N|+1\right)\right)\overline{w}_{ik}, \tag{3.71}$$

and where we have introduced the invariant two-site operator $\mathbb{Q}_{ij}:=(-1)^{d_i+d_j}\mathbf{S}_i\cdot\mathbf{S}_j\equiv\mathbb{Q}_{ji}$. In this form, the globally GL$(m|n)$-invariant Hamiltonian constructed here *formally* coincides with the expression of the globally SU(N)-invariant Hamiltonian given in [13, eq. (100)] which is somewhat puzzling at first sight. Nevertheless, *true* coincidence is only given for $m=N$ and $n=0$. Therefore, at least in this case, the globally GL$(N|0)$-invariant chiral correlator Ψ given in eq. (3.2) is also annihilated by the SU(N)-invariant Hamiltonian which naturally annihilates the globally SU(N)-invariant chiral correlator $\Psi_{\mathrm{SU}(N)}$ it is constructed for. Having chosen a set of L points $\{z_i\}_i$ on the Riemann sphere, either of these two correlators maps $(z_1,\ldots,z_L)$ to some state $\Psi(z_1,\ldots,z_L)$, $\Psi_{\mathrm{SU}(N)}(z_1,\ldots,z_L)$ respectively, in the respective space of states. As both states are annihilated by either Hamiltonian, they must be proportional to each other which implies that the globally GL$(N|0)$-invariant chiral correlator Ψ must be expressible in terms of a product of the globally SU(N)-invariant chiral correlator $\Psi_{\mathrm{SU}(N)}$ and some overall factor

$$\begin{array}{rccc}\varphi_{1\ldots L} & : & \overline{\mathbb{C}}^L & \longrightarrow & \mathbb{C},\\ & & (z_1,\ldots,z_L) & \longmapsto & \varphi(z_1,\ldots,z_L).\end{array} \tag{3.72}$$

We will postpone the detailed discussion of the chiral correlator Ψ to Section 3.8 and resume the simplification of our Hamiltonian.

Next, we split the coefficients g_{ijk} into their symmetric and antisymmetric part under exchange of i and k, i.e. $\frac{1}{2}\left(g_{ijk}+g_{kji}\right)$ and $\frac{1}{2}\left(g_{ijk}-g_{kji}\right)$. Additionally, we make use of the helpful identities

$$[\mathbb{Q}_{ij},\mathbb{Q}_{jk}]=(-1)^{d_j+d_k}[\mathbb{Q}_{jk},\mathbb{Q}_{ik}]=(-1)^{d_i+d_j}[\mathbb{Q}_{ik},\mathbb{Q}_{ij}] \tag{3.73}$$

and

$$\{\mathbb{Q}_{ij},\mathbb{Q}_{jk}\}=\{\mathbb{Q}_{jk},\mathbb{Q}_{ik}\}=\{\mathbb{Q}_{ik},\mathbb{Q}_{ij}\}. \tag{3.74}$$

This streamlines the expression in eq. (3.68) to

$$H=g\,\mathbb{I}+\sum_{i<j}g_{ij}\mathbb{Q}_{ij}+\sum_{i<j<k}\left(g^A_{ijk}\left[\mathbb{Q}_{ij},\mathbb{Q}_{jk}\right]+g^S_{ijk}\left\{\mathbb{Q}_{ij},\mathbb{Q}_{jk}\right\}\right) \tag{3.75}$$

with the symmetric and antisymmetric parts of g_{ijk} given by

$$g^A_{ijk}:=\frac{1}{2}\left(g_{ijk}-g_{kji}+(-1)^{d_j+d_k}(g_{jki}-g_{ikj})+(-1)^{d_i+d_j}(g_{kij}-g_{jik})\right) \tag{3.76}$$

$$= i\,\mathrm{sgn}(N)(-1)^{d_i+d_k}\frac{(|N|+2)}{2(|N|+1)}\Im\left[\overline{w}_{ij}\left(w_{jk}+w_{ik}\right)+\overline{w}_{ik}w_{jk}\right], \tag{3.77}$$

$$g^S_{ijk} := \frac{1}{2}\left(g_{ijk}+g_{kji}+g_{jki}+g_{ikj}+g_{kij}+g_{jik}\right) \tag{3.78}$$

$$= -\,\mathrm{sgn}(N)(-1)^{d_i+d_k}\frac{|N|}{2(|N|+1)}\Re\left[\overline{w}_{ij}\left(w_{jk}-(-1)^{d_i+d_j}w_{ik}\right)-(-1)^{d_j+d_k}\overline{w}_{ik}w_{jk}\right]. \tag{3.79}$$

This last expression is obviously computationally well suited for the case $N = 0$. It also reflects the superhermiticity of the Hamiltonian in eq. (3.75) as g, g_{ij} and g^S_{ijk} are purely real, g^A_{ijk} is purely imaginary and by virtue of the invariance of $\mathbb{Q}_{ij}$ under superhermitian conjugation, i.e. $\mathbb{Q}^\ddagger_{ij} \equiv \mathbb{Q}_{ij}$, and the identity $(\mathbb{Q}_{ij}\mathbb{Q}_{jk})^\ddagger \equiv \mathbb{Q}_{jk}\mathbb{Q}_{ij}$. This insight leads to the intermediate version of the generic Hamiltonian

$$H = g\,\mathbb{I} + \sum_{i<j} g_{ij}\mathbb{Q}_{ij} + \sum_{i<j<k}\left(\overline{G}_{ijk}\mathbb{Q}_{jk}\mathbb{Q}_{ij} + G_{ijk}\mathbb{Q}_{ij}\mathbb{Q}_{jk}\right) \tag{3.80}$$

with

$$G_{ijk} := g^S_{ijk} + g^A_{ijk}. \tag{3.81}$$

In order to further reduce computational effort, we transform the former expression into one that reduces the amount of matrix multiplications to a minimum, namely

$$H = g\,\mathbb{I} + \sum_{i<j} g_{ij}\mathbb{Q}_{ij} + \sum_{i<j}\mathbb{Q}_{ij}\left(\sum_{k=1}^{i-1}\overline{G}_{kij}\mathbb{Q}_{ik} + \sum_{k=j+1}^{L} G_{ijk}\mathbb{Q}_{jk}\right) \tag{3.82}$$

$$= g\,\mathbb{I} + \sum_{i<j}\mathbb{Q}_{ij}\left(g_{ij} + \sum_{k=1}^{i-1} G_{jik}\mathbb{Q}_{ik} + \sum_{k=j+1}^{L} G_{ijk}\mathbb{Q}_{jk}\right) \tag{3.83}$$

$$= g\,\mathbb{I} + \sum_{i<j}\mathbb{Q}_{ij}\left(g_{ij} + \sum_{k=1}^{i-1}(g^S_{ijk}+g^A_{jik})\mathbb{Q}_{ik} + \sum_{k=j+1}^{L}(g^S_{ijk}+g^A_{ijk})\mathbb{Q}_{jk}\right). \tag{3.84}$$

This last version of the general Hamiltonian in eq. (3.84) seems to be the computationally most efficient expression. (Note that the antisymmetric part of the coefficient of $\mathbb{Q}_{ik}$ is, indeed, given by g^A_{jik}.) Further, it exhibits a form which demonstrates that this construction leads to a Hamiltonian whose interaction beyond the long-range two-spin interaction is rather peculiar, i.e. unless a special choice of positions $\{z_i\}$ of the representations $\{\mathcal{V}_i\}$ and possibly the representations themselves is made so as to have the sum within the parenthesis reduce to something proportional to the identity matrix $\mathbb{I}$ or a total spin term acting on the full state space $\mathcal{H}$, the two-spin interaction between every pair of sites gets enhanced by terms coupling every spin outside the range $[i, j]$ to either i or j. But what is

more, even though it might seem that the Hamiltonian would depend on the particular scheme of labeling the sites from 1 through L as in the summation over k the sites between i and j are exempt, it is, of course, independent of that pattern (as mentioned in Section 3.1) since it features – instead of a short-range interaction where this obviously would make a difference – a long-range interaction. A similar observation was made in [47] for ground state wave function derived from CFT correlators of $\mathrm{SU}(2)_1$ WZW primary fields.

3.6 Special Choices of m and n for $\mathrm{GL}(m|n)$

Let us next comment on selected choices of the parameters $M := n$ and $N := m - n$, i.e. the number of fermionic generators n and the superdimension $m - n$ of $\mathfrak{gl}(m|n)$. Some of them lead to obvious simplifications of the generic Hamiltonian while others entail rather subtle consequences.

3.6.1 Excluding fermions: $M = n = 0$

Right at first glance, the most notable simplification arises for $n = 0$: The dimension and the superdimension of GL$(m|n)$ coincide and the global symmetry reduces to the classical Lie group GL(N) as the fermionic degrees of freedom are effectively excluded from our model. Furthermore, since it is easy to see that the formal coincidence of eq. (3.68) with the respective expression for the SU(N) invariant Hamiltonian in [13] for general M and N becomes a true equivalence of both Hamiltonians whenever we restrict our considerations to the case $n = 0$, it is, at least in this case, possible to describe our system in terms of $\mathfrak{su}(N)$-spins. These systems have been analyzed in great generality in [13, 14] where, in 1D for a pure (i.e. $\mathcal{H} = \mathcal{V}^L$), uniform distribution of the z_i on the unit circle, a connection to the Haldane-Shastry model based on $\mathfrak{su}(N)$-spins was established. Thus, in this case, the spin chain derived is critical and in the universality class of the integrable *Calogero-Sutherland chain* [48–52] which is in turn given by the SU(N) WZW model at level $k = 1$. However, in the alternating case, i.e. $\mathcal{H} = \left(\bar{\mathcal{V}} \otimes \mathcal{V}\right)^{L/2}$, such a connection could not be established analytically even though the numerical treatment also indicates criticality of the alternating spin chain whereas their non-integrable Heisenberg-like analogs were shown to be gapped [53]. Additionally, while in [13] a reformulation of these alternating spin systems in terms of a loop model (that will also play a role what follows) was introduced, in [14], both 2D setups, pure as well as alternating, that can be considered lattice discretizations of fractional quantum Hall systems, were shown to be chiral spin liquids which is in accordance with what one would expect for the SU(N) WZW model at level $k = 1$.

Even more advanced insights into this family of Hamiltonians with classical global symmetry have been obtained for the simplest nontrivial choice of parameters: For $m = 2$, the formal Lie supergroup GL$(2|0)$ is represented by GL(2) which is closely related to SU(2). Paralleling arguments originally given in [4], the system based on the SU(2) WZW model at level $k = 1$ was first constructed in [5] while a rather exhaustive analysis was

presented in [6]. Here, it was shown that the procedure of discretization proposed leads to spin systems that include but are not limited to the Haldane-Shastry spin chain based on $\mathfrak{su}(2)$ which is remarkable as this spin chain sits at a fixed point of the renormalization group flow and its continuum limit effective low-energy description comes without higher order corrections for the energies. To put it differently: The scaling properties of the spin chain do not suffer from any logarithmic corrections. In this sense, the $\mathrm{SU}(2)$ Haldane-Shastry spin chain and the $\mathrm{SU}(2)$ WZW model at level $k = 1$ form a dichotomy in which the exact correspondence (cf. fig. 1.1) going from the spin chain to the CFT is provided by scaling considerations and the way back is provided by the construction presented in [5] and used here in order to arrive at eq. (3.84). Similar results were obtained in [25, 54].

In [47, 54, 55], the lattice version of the bosonic Laughlin state at half filling, i.e. the Kalmeyer-Laughlin state, was shown to be generalized on an arbitrary two-dimensional lattice in the form of the ground state derived from CFT correlators of $\mathrm{SU}(2)_1$ WZW primary fields. As such, the authors were able to prove that the topological properties of the generalized Kalmeyer-Laughlin state are stable under the changes of the lattice geometries investigated, suggesting that this holds true for any geometry.

Finally, it was shown in [56] that the low-lying spectrum of the $\mathrm{SU}(2)$ Haldane-Shastry spin chain can be obtained on the CFT side by insertion of the current operators of the underlying current algebra which proves an even broader resemblance. Alongside, supporting numerical evidence was given that this equivalence of spin chain and CFT extends to the full spectrum.

Note that, even though the coefficients in eq. (3.84) are insensitive to changes of $M = n$, i.e. they only depend on the superdimension N and the parametrization of the coordinates $\{z_i\}_i$ (cf. eqs. (3.69), (3.70), (3.77) and (3.79)). The state space and the operators involved in the definition of the Hamiltonian are subject to both, a particular choice of N *and* M, of course.

3.6.2 Superdimension $N = 1$

Within the framework provided here, a treatment of the case $N = 1$ can be achieved which would be trivial in the classical $\mathrm{SU}(N)$ case. The physical interest in this particular symmetry lies in the relevance for the description of percolation, dilute polymers [16, 57] and the discussion of the spin quantum Hall effect [58]. The authors of [59] analyze the continuum limit of the integrable $\mathfrak{sl}(2|1)$ **3**-$\bar{\mathbf{3}}$-superspin chain on the state space of alternating representations, i.e. $\mathcal{H} = (\bar{\mathcal{V}} \otimes \mathcal{V})^{L/2}$, in order to shed some light on the connection to the non-integrable Heisenberg-like $\mathfrak{sl}(2|1)$ spin chain on the same state space which the network model of the spin quantum Hall effect can be mapped to by taking the proper anisotropic limit [60–62]. It turns out that these two spin chains do not belong to the same universality class – in particular, the integrable version is captured by the $\mathrm{SU}(2|1)$ WZW model at level $k = 1$ while the non-integrable Heisenberg spin chain associated with the spin quantum Hall effect is described by some logarithmic CFT without Kac-Moody symmetry and thus not of WZW type [63].

However, we have restricted ourselves to numerical inspections of our Hamiltonian for $N = 1$: It turns out that, while the whole spectrum is positive for $N > 1$, the low-lying spectrum for $N = 1$ stays positive but high in the spectrum we observe pairs of complex-conjugate eigenvalues most likely stemming from the non-diagonalizable nature of the quadratic Casimir on e.g. $\mathcal{V} \otimes \bar{\mathcal{V}} \otimes \mathcal{V}$ in this case, cf. section 2.1.4.

3.6.3 Superdimension $N < 0$

For negative superdimension, we are able to numerically observe the analogous correspondence that was already mentioned by Haldane in the context of the SU$(m|n)$ Haldane-Shastry spin chain: The spectrum of the Haldane-Shastry Hamiltonian with SU$(m|n)$ symmetry is identical to the spectrum of the negative Haldane-Shastry Hamiltonian with SU$(n|m)$ symmetry [27]. Moreover, in the context of our constructed Hamiltonian, this correspondence is numerically observed to even extend to arbitrary positions of the spins. This being said, it is clear, that any result concerning our Hamiltonian for positive superdimension should be easily translated into the case of negative superdimension.

3.6.4 Superdimension $N = 0$

The just mentioned correspondence also extends to the case $N = 0$ if, additionally, one performs the construction of the Hamiltonian with the projection operator $\mathcal{P}_i^-$ instead of $\mathcal{P}_i^+$ and vice versa. However, for the rest of this thesis, we will restrict our attention to the case where $\mathcal{P}_i^+$ is used.

The primary physical motivation for this case stems from the fact that dense polymers are related to self-avoiding random walks which in turn are described by a certain type of loop models with loop fugacity N in the limit $N \to 0$. In order to correctly describe this limit, the authors of [16] were the first to introduce anticommuting variables in this context. There is a connection between the loop model of fugacity N constructed in [13] on the one hand and the family of Hamiltonians with global GL$(M + N|M)$ symmetry constructed in this Chapter on the other hand.

Furthermore, the GL$(M|M)$ case has been suggested to be of particular interest for the understanding of the integer quantum Hall plateau transition, by the same connection [60–62] as mentioned in Section 3.6.2. However, the spin chains defined by our general Hamiltonian in eq. (3.84) are restricted only to the fundamental representation $\mathcal{V}$ and its dual $\bar{\mathcal{V}}$ of $\mathfrak{gl}(M|M)$ as its building blocks whereas the spin chains introduced in the context of the integer quantum Hall plateau transition are defined on an alternating tensor product of some infinite-dimensional representation space $\mathcal{V}_\infty$ with its dual $\bar{\mathcal{V}}_\infty$ and the relevant Hamiltonian is given by the sum of nearest-neighbor Casimir operators which makes solving the system extremely difficult. Nonetheless, a comment in [23] points to considering a Haldane-Shastry-like deformation of the original spin chain considered. It should, in principle, be feasible to furnish such a Haldane-Shastry-like spin chain defined on $\mathcal{H} = (\bar{\mathcal{V}}_\infty \otimes \mathcal{V}_\infty)^{L/2}$ as the construction carried out in the beginning of this Chapter

allows for any $\mathfrak{gl}(m|n)$ representation to be used. Still, for the time being, we will content ourselves with the simpler version of our Hamiltonian defined on $\mathcal{H} = (\bar{\mathcal{V}} \otimes \mathcal{V})^{L/2}$ in the hope to gain some insights anyhow.

Another comment in [23] suggests that a connection similar to the one mentioned in Section 3.6.1 for $M = 0$ might be found for systems defined on an alternating lattice of some representation space $\mathcal{V}$ and its dual representation space $\bar{\mathcal{V}}$ of $\mathfrak{gl}(M|M)$ spins on the one hand and $\mathfrak{sl}(M|M)$ spins on the other.

At superdimension $N = 0$, the symmetric part of the three-spin interaction of the now globally $\mathrm{GL}(M|M)$ invariant Hamiltonian drops out as its coefficient g^S_{ijk} is identical to zero. The Hamiltonian given in eq. (3.75) thus reduces to

$$H = g^0\,\mathbb{I} + \sum_{i<j} g^0_{ij}\mathbb{Q}_{ij} + \sum_{i<j<k} g^A_{ijk}\left[\mathbb{Q}_{ij}, \mathbb{Q}_{jk}\right] \tag{3.85}$$

where the coefficients simplify to

$$g^0 = -\sum_{i\neq j}\left(\delta_{d_i,d_j}\left|w_{ij}\right|^2 + \frac{1}{2}\sum_{k\notin\{i,j\}}\overline{w}_{ki}w_{kj}\right)(-1)^{d_i+d_j}, \tag{3.86}$$

$$g^0_{ij} = \Re\left[\sum_{k\notin\{i,j\}}\overline{w}_{ik}w_{jk} + w_{ij}\sum_{k\notin\{i,j\}}(-1)^{d_k}\left((-1)^{d_j}\overline{w}_{jk} - (-1)^{d_i}\overline{w}_{ik}\right)\right](-1)^{d_i+d_j} - 2\left|w_{ij}\right|^2, \tag{3.87}$$

$$g^A_{ijk} = i\,\Im\left[\overline{w}_{ij}(w_{jk} + w_{ik}) + \overline{w}_{ik}w_{jk}\right](-1)^{d_i+d_k}. \tag{3.88}$$

This expression will only simplify further if we restrict the range of the w_{ij}, e.g. in order to get rid of the three-spin term, the most obvious restriction would be $\forall i, j \in \mathbb{N}_L : w_{ij} \in \mathbb{R}$. For either choice of coordinate dependence, $w_{ij} := \frac{1}{z_i - z_j}$ or $w_{ij} := \frac{z_i + z_j}{z_i - z_j}$, placing all spins on the real line only ($\forall i \in \mathbb{N}_L : z_i \in \mathbb{R}$) certainly implies this.

3.7 A Special Setup

As was mentioned in the beginning of this Chapter, the locations of the spins on the Riemann sphere are, *a priori*, not constrained in any way. In fact, one may consider setups of randomly distributed spins. However, an analytic treatment of anything but the computation of the chiral correlator, e.g. the analytic diagonalization of the Hamiltonian in these cases, seems hopeless. The literature concerning this construction for classical symmetries has, thus, concentrated on setups of spins forming regular lattices, be it two-dimensional or one-dimensional. Nonetheless, even the regular two-dimensional cases, related to the physics of the fractional quantum Hall effect, have only been treated numerically by means of Monte Carlo simulations. In fact, in all of these cases for classical symmetries, the only

setup where a full analytic treatment was feasible is the fully uniform one-dimensional setup, i.e. the on-site state spaces $\mathcal{V}_j$ are uniformly distributed on the unit circle, so located at $z_j = \exp\left(i\frac{2\pi j}{L}\right)$.

In the case $N = m - n = 0$ the coefficients of the generic Hamiltonian given in eq. (3.84) considerably simplify (cf. eqs. (3.86) to (3.88)). However, these expressions can be reduced even further by making the choice $w_{ij} := \frac{z_i+z_j}{z_i-z_j}$, so that eq. (3.88) yields

$$\begin{aligned} g^A_{ijk} = \Bigg(& \frac{\left(|z_i|^2 - |z_j|^2\right)\left(\overline{z}_j z_k - z_j \overline{z}_k\right) + \left(|z_j|^2 - |z_k|^2\right)\left(z_i \overline{z}_j - \overline{z}_i z_j\right)}{|z_i - z_j|^2 |z_j - z_k|^2} \\ & + \frac{\left(|z_i|^2 - |z_k|^2\right)\left(z_i \overline{z}_j - \overline{z}_i z_j\right) + \left(|z_i|^2 - |z_j|^2\right)\left(z_k \overline{z}_i - \overline{z}_k z_i\right)}{|z_i - z_j|^2 |z_i - z_k|^2} \\ & + \frac{\left(|z_i|^2 - |z_k|^2\right)\left(\overline{z}_j z_k - z_j \overline{z}_k\right) + \left(|z_j|^2 - |z_k|^2\right)\left(z_k \overline{z}_i - \overline{z}_k z_i\right)}{|z_i - z_k|^2 |z_j - z_k|^2} \Bigg) i(-1)^{d_i + d_k} \end{aligned} \tag{3.89}$$

which obviously vanishes for placing the spins somewhere on a circle of radius r centered about the origin, so $\forall j \in \mathbb{N}_L : z_j = r\exp(i\,\theta_j)$. In this case, we have $w_{ij} = -i\cot\left(\frac{\theta_{ij}}{2}\right)$ with $\theta_{ij} := \theta_i - \theta_j$, independently of r, and

$$\begin{aligned} g^{0,\text{circle}}_{ij} = & -2\cot^2\left(\frac{\theta_{ij}}{2}\right) + (-1)^{d_i+d_j} \sum_{k \notin \{i,j\}} \cot\left(\frac{\theta_{ik}}{2}\right)\cot\left(\frac{\theta_{jk}}{2}\right) \\ & + \cot\left(\frac{\theta_{ij}}{2}\right) \sum_{k \notin \{i,j\}} (-1)^{d_k} \left((-1)^{d_i} \cot\left(\frac{\theta_{jk}}{2}\right) - (-1)^{d_j} \cot\left(\frac{\theta_{ik}}{2}\right)\right). \end{aligned} \tag{3.90}$$

Using the identity

$$\cot(x)\cot(y) = \cot(x-y)\left(\cot(y) - \cot(x)\right) - 1, \tag{3.91}$$

this simplifies to

$$\begin{aligned} g^{0,\text{circle}}_{ij} = & (-1)^{d_i+d_j} \left(2 - L + 2\cot\left(\frac{\theta_{ij}}{2}\right) \sum_{k \notin \{i,j\}} \left(\delta_{d_j,d_k} \cot\left(\frac{\theta_{jk}}{2}\right) - \delta_{d_i,d_k} \cot\left(\frac{\theta_{ik}}{2}\right)\right)\right) \\ & - 2\cot^2\left(\frac{\theta_{ij}}{2}\right) \end{aligned} \tag{3.92}$$

from which we deduce the Hamiltonian for a *generic* one-dimensional setup, i.e. for the

setup of spins randomly distributed on the unit circle

$$H = g^0 \mathbb{I} - (L-2) \sum_{i<j}^{L} \mathbf{S}_i \cdot \mathbf{S}_j + 2 \sum_{i<j}^{L} \gamma_{ij}\, \mathbf{S}_i \cdot \mathbf{S}_j. \tag{3.93}$$

Here we have introduced the coefficients

$$\begin{aligned}
\gamma_{ij} &= \cot\left(\frac{\theta_{ij}}{2}\right)\left(\sum_{k\notin\{i,j\}}\left(\delta_{d_j,d_k}\cot\left(\frac{\theta_{jk}}{2}\right) - \delta_{d_i,d_k}\cot\left(\frac{\theta_{ik}}{2}\right)\right) - (-1)^{d_i+d_j}\cot\left(\frac{\theta_{ij}}{2}\right)\right) \\
&= \cot\left(\frac{\theta_{ij}}{2}\right)\left(\sum_{\substack{k\notin\{j\}\\ d_k\equiv d_j}}\cot\left(\frac{\theta_{jk}}{2}\right) - \delta_{d_i,d_j}\cot\left(\frac{\theta_{ji}}{2}\right) - \sum_{\substack{k\notin\{i\}\\ d_k\equiv d_i}}\cot\left(\frac{\theta_{ik}}{2}\right) + \delta_{d_i,d_j}\cot\left(\frac{\theta_{ij}}{2}\right)\right. \\
&\qquad\qquad \left. - (-1)^{d_i+d_j}\cot\left(\frac{\theta_{ij}}{2}\right)\right) \\
&= \cot\left(\frac{\theta_{ij}}{2}\right)\left(\sum_{\substack{k\notin\{j\}\\ d_k\equiv d_j}}\cot\left(\frac{\theta_{jk}}{2}\right) - \sum_{\substack{k\notin\{i\}\\ d_k\equiv d_i}}\cot\left(\frac{\theta_{ik}}{2}\right) + \cot\left(\frac{\theta_{ij}}{2}\right)\right).
\end{aligned} \tag{3.94}$$

From this expression, we can deduce a drastic simplification to

$$\gamma_{ij}^{\text{uniform}} = \cot^2\left(\frac{\theta_{ij}}{2}\right) = \frac{1}{\sin^2\left(\theta_{ij}/2\right)} - 1 \tag{3.95}$$

if we choose the locations of the spins such that

$$\sum_{\substack{k\notin\{j\}\\ d_k\equiv d_j}}\cot\left(\frac{\theta_{jk}}{2}\right) - \sum_{\substack{k\notin\{i\}\\ d_k\equiv d_i}}\cot\left(\frac{\theta_{ik}}{2}\right) = 0 \tag{3.96}$$

for all sites i and j. This is only the case if each set of a certain kind of spin forms a uniform sublattice on a circle centered about the origin. However, these sublattices may well be rotated against each other, e.g we may think of the fundamental representation $\mathcal{V}$ on the even sites j placed at

$$z_j = r\exp\left(i\frac{2\pi j}{L}\right) \tag{3.97}$$

and the antifundamental representation $\bar{\mathcal{V}}$ on the odd sites j placed at

$$z_j = r \exp\left(i\left(\alpha + \frac{2\pi j}{L}\right)\right) \tag{3.98}$$

with some arbitrary angle $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$ such that $\forall i,j : \theta_{ij} \neq 0$. In this case, we finally arrive at

$$\begin{aligned} H &= g\,\mathbb{I} - L\sum_{i<j}^{L} \mathbf{S}_i \cdot \mathbf{S}_j + 2\sum_{i<j}^{L} \frac{\mathbf{S}_i \cdot \mathbf{S}_j}{\sin^2\left(\theta_{ij}/2\right)} \\ &= g\,\mathbb{I} - \frac{L}{2}\left(\sum_{i,j}^{L} \mathbf{S}_i \cdot \mathbf{S}_j - \sum_{i=1}^{L} \mathbf{S}_i^2\right) + 2\sum_{i<j}^{L} \frac{\mathbf{S}_i \cdot \mathbf{S}_j}{\sin^2\left(\theta_{ij}/2\right)} \\ &= g\,\mathbb{I} - \frac{L}{2}\mathbf{S}^2 + 2\sum_{i<j}^{L} \frac{\mathbf{S}_i \cdot \mathbf{S}_j}{\sin^2\left(\theta_{ij}/2\right)} \end{aligned} \tag{3.99}$$

where we introduced the total spin operator $\mathbf{S} := \sum_i^L \mathbf{S}_i$ and its square $\mathbf{S}^2 := \mathbf{S} \cdot \mathbf{S}$, and made use of $\mathbf{S}_i^2 = N \equiv 0$. As the angular distance θ_{ij} of two spins within each sublattice is an integer multiple of $2\frac{2\pi}{L}$, this expression can be transformed even further to a form more similar to the Haldane-Shastry spin chain:

$$H = g\,\mathbb{I} - \frac{L}{2}\mathbf{S}^2 + 2\sum_{\substack{i<j \\ d_i \equiv d_j}}^{L} \frac{\mathbf{S}_i \cdot \mathbf{S}_j}{\sin^2\left((i-j)\pi/L\right)} + 2\sum_{\substack{i<j \\ d_i \neq d_j}}^{L} \frac{\mathbf{S}_i \cdot \mathbf{S}_j}{\sin^2\left((i-j)\pi/L + (-1)^{d_j}\alpha\right)}. \tag{3.100}$$

We will be able to convince ourselves in Chapter 5 that the eigenvalues of the Hamiltonian given in eq. (3.100) are – at least for $\mathfrak{gl}(1|1)$-spins – independent of the angular twist α between the two sublattices. There is numerical evidence that this behavior is inherited by such systems with global GL$(2|2)$ or GL$(3|3)$ symmetry and, thus, it may be conjectured that the α-dependency of this Hamiltonian is trivial for these systems with global GL$(m|m)$ symmetry.[17]

Of course, it is also clear from this expression that, for $\alpha = 0$, it reduces further to something very similar to the Haldane-Shastry model,

$$H = g\,\mathbb{I} - \frac{L}{2}\mathbf{S}^2 + 2\sum_{i<j}^{L} \frac{\mathbf{S}_i \cdot \mathbf{S}_j}{\sin^2\left((i-j)\pi/L\right)}, \tag{3.101}$$

apart from the – as will be shown in Section 4.2 – crucial summand $-\frac{L}{2}\mathbf{S}^2$.

[17] However, by inspection of numerical results, it can be stated that this is not the case for systems with this kind of setup and generic GL$(m|n)$-invariance with $m \neq n$.

3.8 The Seed State

This Section is dedicated to the zero energy state given by the chiral correlator of WZW primary fields as in eq. (3.2), or more explicitly, as explained in [6], the state of the spin system is given by

$$|\Psi(z_1,\ldots,z_L)\rangle := \sum_{a_1,\ldots,a_L=1}^{m+n} \Psi_{a_1,\ldots,a_L}(z_1,\ldots,z_L)\ |a_1,\ldots,a_L\rangle \tag{3.102}$$

with

$$\Psi_{a_1,\ldots,a_L}(z_1,\ldots,z_L) := \langle \phi_{1,a_1}(z_1)\cdots\phi_{L,a_L}(z_L)\rangle \tag{3.103}$$

and $|a_1,\ldots,a_L\rangle$ the product basis on $\mathcal{H} = \bigotimes_{i=1}^{L}\mathcal{V}_i$. Here $\phi_{i,a_i}(z_i)$ is the a_ith component of the WZW primary field $\phi_i(z_i)$ transforming either in the fundamental or antifundamental representation.

We start with a quick analysis of the general properties of this state in the next Subsection before presenting the actual computation of eq. (3.103) in Section 3.8.2.

3.8.1 General Properties

If the fusion rules of the fields involved are monosemous, these coefficients are unique and thus lead to only one state with the desired properties. However, if the number of fusion channels available to the fields involved is larger than one, this will lead to as many wave functions as there are fusion channels.

Further, the chiral correlator is invariant under the global $\mathrm{GL}(m|n)$ symmetry,

$$0 = \sum_{i=1}^{L} S_i{}^a{}_b \langle \phi_1(z_1)\cdots\phi_L(z_L)\rangle. \tag{3.104}$$

This was observed by Gepner and Witten [45] to be a consequence of the Ward identities applied to the correlator of the current $J^a{}_b(z)$ and a number of primary fields $\phi_i(z_i)$ on the one hand and the asymptotic behavior as $1/z^2$ of the current $J^a{}_b(z)$ as $z\to\infty$ on the other hand:

$$0 \equiv \lim_{z\to\infty} z\,\langle J^a{}_b(z)\phi_1(z_1)\cdots\phi_L(z_L)\rangle \tag{3.105}$$

$$= \lim_{z\to\infty}\sum_{i=1}^{L} z\,\sigma_i\langle\phi_1(z_1)\cdots[\frac{S^a{}_b\,\phi_i(z_i)}{(z-z_i)}]\cdots\phi_L(z_L)\rangle \tag{3.106}$$

$$\overset{|z_i|<\infty}{=} \lim_{z\to\infty}\sum_{i=1}^{L}\frac{z}{z}(\sigma_i)^2 S_i{}^a{}_b\,\langle\phi_1(z_1)\cdots\phi_L(z_L)\rangle \tag{3.107}$$

$$= \lim_{z\to\infty}\sum_{i=1}^{L} S_i{}^a{}_b\,\langle\phi_1(z_1)\cdots\phi_L(z_L)\rangle. \tag{3.108}$$

Evidently, the last line is independent of z and, hence, defining $S_{\text{tot}}{}^{a}{}_{b} := \sum_i S_i{}^{a}{}_{b}$, leads to the desired statement of global GL$(m|n)$ symmetry:

$$S_{\text{tot}}{}^{a}{}_{b} \langle \phi_1(z_1) \cdots \phi_L(z_L) \rangle = 0. \tag{3.109}$$

Before diving right into the calculation of the coefficients given in eq. (3.103), we first look at the charge neutrality condition for a correlator such as the one we seek to render nontrivial. In our case, the charges are represented by the weights μ of the representations the respective WZW primary field transforms under. In the pure case $\mathcal{H} = \mathcal{V}^{\otimes L}$, it is not clear how to tailor the correlator in order to have the cancellation of the sum of the weights. In fact, it is not clear at the moment how this can be achieved for the fundamental (or, alternatively, the antifundamental) representation. It is legitimate to numerically analyze the Hamiltonian for the pure case and various values of m and n. Therefore, we will restrict ourselves to the computation of the chiral correlator in the alternating case which – as argued in [13] – presents itself as the more natural system since it only requires an equal number of fundamental and antifundamental representation spaces for the $\mathcal{V}_i$ in order to cancel their charges.

3.8.2 Free Field Correlator

In the alternating case, the coefficients of the zero energy state we want to construct are given by

$$\Psi^{a_2 \ldots a_L}_{a_1 \ldots a_{L-1}}(z_1, \ldots, z_L) = \left\langle \bar{\psi}_{a_1}(z_1) \psi^{a_2}(z_2) \cdots \bar{\psi}_{a_{L-1}}(z_{L-1}) \psi^{a_L}(z_L) \right\rangle \tag{3.110}$$

with the free fields $\psi(z)$ and $\bar{\psi}(z)$ transforming in the fundamental and antifundamental representation as introduced in Section 3.2. It can readily be calculated by employing the OPE of the free fields given therein. The correlator in eq. (3.110) vanishes unless all fields and their duals come in pairs, i.e. $\bar{\psi}_\alpha(z_i)\psi^\alpha(z_j)$, cf. eq. (3.19). Keeping this in mind, eq. (3.110) naturally factorizes into correlators of n_α factors of pairs of fields with the same index α. In particular, it factorizes into the product of two parts: The first part is the correlator containing k fermionic components and the second part is the correlator of $L-k$ bosonic components:[18]

$$\begin{aligned} \Psi^{a_2 \ldots a_L}_{a_1 \ldots a_{L-1}}(z_1, \ldots, z_L) = &\left\langle \bar{\psi}_{a_1^-}(z_1^-) \psi^{a_2^-}(z_2^-) \cdots \bar{\psi}_{a_{k-1}^-}(z_{k-1}^-) \psi^{a_k^-}(z_k^-) \right\rangle \\ &\left\langle \bar{\psi}_{a_1^+}(z_1^+) \psi^{a_2^+}(z_2^+) \cdots \bar{\psi}_{a_{L-k-1}^+}(z_{L-k-1}^+) \psi^{a_{L-k}^+}(z_{L-k}^+) \right\rangle. \end{aligned} \tag{3.111}$$

Here we have relabeled the indices and coordinates according to whether they belong to a fermionic ($^-$) or bosonic ($^+$) component, however, especially crucial for the fermionic objects, without changing the order within each subset, i.e. $z_i \mapsto z_{i^-}^-$ such that, if we have $i < j$ for any given pair z_i and z_j they are mapped to $z_{i^-}^-$ and $z_{j^-}^-$, respectively, such that

[18] Obviously, k is an even positive integer if the correlator is to be nontrivial.

$i^- < j^-$. Explicitly, the first z_i, going from left to right in the chiral correlator, for which the corresponding field features $a_i \in \{1, \dots, m\}$ is relabeled as z_1^- and, consequently, its index as a_1^-. For the next fermionic field, we assign z_2^- and a_2^- and so on. For simplicity, we may assume the same for all indices related to the bosonic objects but this is not of importance as their rearrangement will not cause any changes of sign.

Using *Wick's theorem*, this readily reduces to

$$\Psi^{a_2 \dots a_L}_{a_1 \dots a_{L-1}}(\{z_i\}) = \det\left(\frac{\delta^{a_{2j}^-}_{a_{2i-1}^-}}{z_{2i-1}^- - z_{2j}^-}\right)_{i,j\in\{1,\dots,\frac{k}{2}\}} \operatorname{perm}\left(\frac{\delta^{a_{2j}^+}_{a_{2i-1}^+}}{z_{2i-1}^+ - z_{2j}^+}\right)_{i,j\in\{1,\dots,\frac{L-k}{2}\}} . \tag{3.112}$$

An alternative expression may be derived by noting that, instead of effectively pulling all bosonic fields in the correlator out into a separate correlator to the right and, thus, factorizing the whole object into two factors, we might as well sort the fields in the full correlator by each value $1, \dots, m+n$ an index α of any field can have. As the bosonic fields can be pulled past any other field without worrying about potential sign changes, an overall sign $\operatorname{sgn}(\pi)$ only arises from reordering the fermionic fields ($1 \leq \alpha \leq m$). Upon the rearrangement of fields by their index α, we find that the full correlator factorizes into $m+n$ correlators of n_α fields with index α:[19]

$$\Psi^{a_2 \dots a_L}_{a_1 \dots a_{L-1}}(\{z_i\}) = \operatorname{sgn}(\pi) \prod_{\alpha=1}^{m+n} \left\langle \bar{\psi}_\alpha(z_1^{(\alpha)}) \psi^\alpha(z_2^{(\alpha)}) \cdots \bar{\psi}_\alpha(z_{2n_\alpha-1}^{(\alpha)}) \psi^\alpha(z_{2n_\alpha}^{(\alpha)}) \right\rangle. \tag{3.113}$$

Note that the coordinates z_i of the fields have also been relabeled to $z_{n_\alpha}^{(\alpha)}$ in order to fit the scheme. This however belongs to the considerations concerning the sign of the permutation π which we will return to after evaluating the blocks of the correlator for particular α:[20]

$$\left\langle \prod_{i=1}^{n_\alpha} \bar{\psi}_\alpha(z_{2i-1}^{(\alpha)}) \psi^\alpha(z_{2i}^{(\alpha)}) \right\rangle = \prod_{\sigma\in S_{n_\alpha}} (\operatorname{sgn}(\sigma))^{|\alpha|-1} \prod_{i=1}^{n_\alpha} \left\langle \bar{\psi}_\alpha(z_{2i-1}^{(\alpha)}) \psi^\alpha(z_{2\sigma(i)}^{(\alpha)}) \right\rangle. \tag{3.114}$$

Note the – at first sight – peculiar use of the permutation σ and the corresponding sign arising from permuting the fundamental fields in the expression. However, when, e.g., exchanging two specific fundamental fields, it is easy to see that they are effectively blind to the antifundamental fields as both fundamental fields have to be passed by every antifundamental field in between, effectively canceling each other's sign. Evaluating this last expression, using Wick's theorem and the OPE in eq. (3.19), for $|\alpha| - 1 = -1$ for $\alpha \in \{1, \dots, m\}$ yields the determinant and for $|\alpha| - 1 = 0$ for $\alpha \in \{m+1, \dots, m+n\}$ yields

[19] If for some value of $1, \dots, m+n$ there is no such field component, we trivially include a factor of 1.

[20] The parity inversion as mentioned in Section 3.2 should be kept in mind, in particular, with respect to the exponent $^{|\alpha|-1}$.

the permanent:

$$\left\langle \prod_{i=1}^{n_\alpha} \bar{\psi}_\alpha(z_{2i-1}^{(\alpha)}) \psi^\alpha(z_{2i}^{(\alpha)}) \right\rangle = \begin{cases} \det\left(\frac{1}{z_{2i-1}^{(\alpha)} - z_{2j}^{(\alpha)}} \right)_{i,j\in\{1,\dots,n_\alpha\}} & \text{if } 1 \leq \alpha \leq m, \\ \operatorname{perm}\left(\frac{1}{z_{2i-1}^{(\alpha)} - z_{2j}^{(\alpha)}} \right)_{i,j\in\{1,\dots,n_\alpha\}} & \text{if } m+1 \leq \alpha \leq m+n. \end{cases} \tag{3.115}$$

In total we find

$$\begin{aligned} &\Psi^{a_2\dots a_L}_{a_1\dots a_{L-1}}(\{z_i\}) \\ &= \operatorname{sgn}(\pi) \prod_{\alpha=1}^{m} \det\left(\frac{1}{z_{2i-1}^{(\alpha)} - z_{2j}^{(\alpha)}} \right)_{i,j\in\{1,\dots,n_\alpha\}} \prod_{\alpha=m+1}^{m+n} \operatorname{perm}\left(\frac{1}{z_{2i-1}^{(\alpha)} - z_{2j}^{(\alpha)}} \right)_{i,j\in\{1,\dots,n_\alpha\}} \end{aligned} \tag{3.116}$$

This expression is even simpler to calculate than the one given in eq. (3.113) as the matrices involved, of which one has to compute the determinant or permanent, are, in general, a lot smaller. Nevertheless, there is a sign to be taken into account. This clearly stems from the permutation π that is necessary to reorder the $(z_1^-, \dots, z_k^-)$ so that they match the order of the $(z_1^{(1)}, z_2^{(1)}, \dots, z_{n_1}^{(1)}, z_1^{(2)}, \dots, z_{n_{m+n}}^{(m+n)})$. It also allows us to refine eq. (3.113), in case one wants to bypass the calculation of $\operatorname{sgn}(\pi)$, to

$$\Psi^{a_2\dots a_L}_{a_1\dots a_{L-1}}(z_1, \dots, z_L) = \det\left(\frac{\delta^{a_{2j}^-}_{a_{2i-1}^-}}{z_{2i-1}^- - z_{2j}^-} \right)_{i,j\in\{1,\dots,\frac{k}{2}\}} \prod_{\alpha=m+1}^{m+n} \operatorname{perm}\left(\frac{1}{z_{2i-1}^{(\alpha)} - z_{2j}^{(\alpha)}} \right)_{i,j\in\{1,\dots,n_\alpha\}} \tag{3.117}$$

as $\operatorname{sgn}(\pi)$ is purely associated to the odd degrees of freedom and, hence, irrelevant for the bosonic part of both expressions. Of course, this could also have been inferred from eq. (3.113) by noting that the permanent – as a special case of an *immanent* for the trivial character of the symmetric group – is insensitive to permutations of rows and columns of a matrix and, therefore, insensitive to permutations of rows and columns that block-diagonalize the matrix $\left(\frac{\delta^{a_{2j}^+}_{a_{2i-1}^+}}{z_{2i-1}^+ - z_{2j}^+} \right)_{i,j\in\{1,\dots,\frac{L-k}{2}\}}$.[21] Finally, using the formula for the permanent that appears in a renowned paper of Moore and Read [4, eq. (4.11)] (from now on, the range

[21] This observation leads to an alternative way of computing $\operatorname{sgn}(\pi)$: First note that the determinant is a special case of an immanent for the sign function as the defining character, leading to a change of its sign whenever two adjacent rows or columns are exchanged. Block-diagonalizing the matrix $\left(\frac{\delta^{a_{2j}^-}_{a_{2i-1}^-}}{z_{2i-1}^- - z_{2j}^-} \right)_{i,j\in\{1,\dots,\frac{k}{2}\}}$ such that the blocks are labeled by increasing indices α yields a change of sign of the determinant corresponding to $\operatorname{sgn}(\pi)$.

for the indices i and j of the matrix elements will be implicit),

$$\det\left(\frac{1}{z_i - w_j}\right)\operatorname{perm}\left(\frac{1}{z_i - w_j}\right) \;=\; \det\left(\frac{1}{(z_i - w_j)^2}\right), \tag{3.118}$$

which can be proven using bosonization of the $\beta\gamma$ system given in [64, below eq. (84)], it is only a small computation that is left for us to do. We arrive at[22]

$$\begin{aligned}
&\Psi^{a_2\ldots a_L}_{a_1\ldots a_{L-1}}(\{z_i\}) \\
&= \operatorname{sgn}(\pi)\prod_{\alpha=1}^{m}\det\left(\frac{1}{z^{(\alpha)}_{2i-1} - z^{(\alpha)}_{2j}}\right)\prod_{\beta=m+1}^{m+n}\det\left(\frac{1}{\left(z^{(\beta)}_{2i-1} - z^{(\beta)}_{2j}\right)^2}\right)\Big/\det\left(\frac{1}{z^{(\beta)}_{2i-1} - z^{(\beta)}_{2j}}\right) \\
&= \operatorname{sgn}(\pi)\prod_{\beta=1}^{m+n}\left(\det\left(\frac{1}{z^{(\beta)}_{2i-1} - z^{(\beta)}_{2j}}\right)\right)^{1-2|\beta|}\left(\det\left(\frac{1}{\left(z^{(\beta)}_{2i-1} - z^{(\beta)}_{2j}\right)^2}\right)\right)^{|\beta|}.
\end{aligned} \tag{3.119}$$

This concludes our general discussion of the GL$(m|n)$-invariant spin systems. In the following Chapters, we will analyze particular systems derived from these.

[22] Again, it should be stressed that the parity of the fields is inversed as compared to the original definition.

4

The Uniform One-Dimensional Setup of Alternating $\mathfrak{gl}(1|1)$-Spins – Simple, Yet Surprising

There is no joy in the finite; there is joy only in the Infinite.

– Anonymous, *The Upanishads*

In this Chapter, we will analyze the Hamiltonian derived in the previous Chapter for the uniform one-dimensional setup of alternating $\mathfrak{gl}(1|1)$-spins placed on the unit circle. In this case, the Hamiltonian takes the form given in eq. (3.101). But before presenting this, we will digress and inscribe to the analytic (almost) diagonalization of the analogous setup for the GL$(1|1)$ Haldane-Shastry model. This model's Hamiltonian is basically represented by the same expression apart from the summand proportional to the square of the total spin, i.e. $-\frac{L}{2}\mathbf{S}^2$, which will turn out to be the crucial ingredient only affecting the property whether or not the Hamiltonian includes a nilpotent part.

Note that, for the **remainder of this thesis**, the focus is exclusively on alternating systems with

$$\mathcal{H} := (\bar{\mathcal{V}} \otimes \mathcal{V})^{\otimes L}. \tag{4.1}$$

The reason for this is the lack of knowledge about a nontrivial seed state in the pure case as explained at the end of Section 3.8.1. Therefore, the total number of entities of the system will be given by $2L$ instead of L which leads to a slightly changed form of the Hamiltonian of our actual interest given in eq. (3.101), i.e.

$$H = g\,\mathbb{I} - L\mathbf{S}^2 + 2\sum_{i<j}^{2L} \frac{\mathbf{S}_i \cdot \mathbf{S}_j}{\sin^2\left((i-j)\pi/2L\right)}. \tag{4.2}$$

The constant offset proportional to the (L-dependent) factor g ensures that the seed state derived from the chiral correlator of primary fields in Section 3.8.2 has zero energy.

We rescale this expression by $\frac{1}{2}\left(\frac{\pi}{2L}\right)^2$,

$$H = \frac{\pi^2}{8L^2} g \mathbb{I} - \frac{\pi^2}{8L}\mathbf{S}^2 + \frac{\pi^2}{4L^2}\sum_{i<j}^{2L} \frac{\mathbf{S}_i \cdot \mathbf{S}_j}{\sin^2\left((i-j)\pi/2L\right)}, \tag{4.3}$$

in order to make contact with the alternating Haldane-Shastry spin chain,

$$H_{\text{HS,alt}} = \frac{\pi^2}{(2L)^2}\sum_{i<j}^{2L} \frac{\mathbf{S}_i \cdot \mathbf{S}_j}{\sin^2\left((i-j)\pi/2L\right)}, \tag{4.4}$$

which only consists of the usual spin-spin interaction terms that are scaled by the inverse-square of their chord-distance. With this normalization, its nearest-neighbor interaction becomes Heisenberg-like in the thermodynamic limit, i.e.

$$\left(\frac{\pi}{2L}\right)^2 \frac{1}{\sin^2\left(\pi\frac{i-(i+1)}{2L}\right)} = 1 + \mathcal{O}\left(\frac{\pi^2}{L^2}\right), \tag{4.5}$$

and the analysis of the system's behavior in the thermodynamic limit becomes directly accessible to scaling arguments.

4.1 The Alternating Haldane-Shastry Model for $\mathfrak{gl}(1|1)$-Spins

Considering the GL(1|1) Haldane-Shastry Hamiltonian on $\mathcal{H} = (\bar{\mathcal{V}} \otimes \mathcal{V})^{\otimes L}$ given in eq. (4.4), we break up the lattice into even sites $I_A = \{0, 2, \ldots, 2L-2\}$ for $\mathcal{V}$ and odd sites $I_B = \{1, 3, \ldots, 2L-1\}$ for $\bar{\mathcal{V}}$. Furthermore, as was the case for the pure chain recited in Section 2.2, if we interpret the fermionic $\mathfrak{gl}(1|1)$ generator ψ^- at an even site j as the Fermi annihilation operator c_j and likewise ψ^+ at an even site j as the Fermi creation operator $c_j^\dagger$, they fulfill the usual fermionic anticommutation relations $\{c_j^\dagger, c_k\} = \delta_{jk}$ and $\{c_j^\dagger, c_k^\dagger\} = \{c_j, c_k\} = 0$.

The dual representation space $\bar{\mathcal{V}}$ of $\mathfrak{gl}(1|1)$ located at the odd lattice sites I_B is given by $\langle -e, 1-n\rangle = \langle -1, 0\rangle$. In this case, the even $\mathfrak{gl}(1|1)$-generators take the form

$$E = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad N = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \tag{4.6}$$

while the odd ones become

$$\psi^- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \psi^+ = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \tag{4.7}$$

with the particularly interesting anticommutation relation

$$\{\psi^+, \psi^-\} = E = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.8}$$

If we now used Fermi operators c'_j for ψ^- and $c'^{\dagger 1}_j$ for ψ^+ acting on site j for all $j \in I_B$ in the very same way as is done for the fundamental representation, this would lead to $\{c'^{\dagger}_j, c'^{\dagger}_k\} = \{c'_j, c'_k\} = 0$ and $\{c'^{\dagger}_j, c'_k\} = -\delta_{jk}$, the latter based on eq. (4.8). Thus, they would not fulfill the usual anticommutation relations of the free fermion. However, in order to cure this and to have $\{c^\dagger_j, c_k\} = \delta_{jk}$ for all j and k, we use $c_j = c'_j$ and $c^\dagger_j = -c'^{\dagger}_j$ at the cost of introducing a sign at the respective places. For generality, we may also introduce the coupling γ that determines the relative strength between interactions based on the permutation operator ($\mathbf{S}_j \cdot \mathbf{S}_k$ with $d_j = d_k$) on the one hand and the Temperly-Lieb operator ($\mathbf{S}_j \cdot \mathbf{S}_k$ with $d_j \neq d_k$) on the other. Defining

$$d_{jk} := \left(\frac{2L}{\pi} \sin\left(\frac{\pi}{2L}(j-k)\right)\right)^2 = d_{kj} \tag{4.9}$$

for shorthand notation, the Haldane-Shastry Hamiltonian reads

$$H_{\mathrm{HS,alt}} = \gamma \sum_{\substack{j<k \\ d_j = d_k}}^{2L} \frac{\mathbf{S}_j \cdot \mathbf{S}_k}{d_{jk}} + \sum_{\substack{j<k \\ d_j \neq d_k}}^{2L} \frac{\mathbf{S}_j \cdot \mathbf{S}_k}{d_{jk}}. \tag{4.10}$$

After introducing

$$H_{jk} := \begin{cases} (c^\dagger_j - c^\dagger_k)(c_j + c_k) & j \in I_A, k \in I_B \\ -(c^\dagger_j - c^\dagger_k)(c_j + c_k) & j \in I_B, k \in I_A \\ \gamma(1 - (c^\dagger_j - c^\dagger_k)(c_j - c_k)) & j,k \in I_A \\ \gamma(-1 + (c^\dagger_j - c^\dagger_k)(c_j - c_k)) & j,k \in I_B \end{cases}, \tag{4.11}$$

for the part of the Hamiltonian that is sensitive to the information about the respective sublattice, we can rephrase

$$H_{\mathrm{HS,alt}} = \sum_{j<k} \frac{1}{d_{jk}} H_{jk} \tag{4.12}$$

by rearranging all terms and compensating for an adjusted range of summation by an

[1] Note that, here, the dagger clearly cannot be meant to imply hermitian conjugation of c'_j. Instead, $c'^{\dagger}_j$ is understood as a purely formal symbol.

overall factor of $\frac{1}{2}$:

$$H_{\text{HS,alt}} = \sum_{j\neq k} \frac{1}{2d_{jk}} \Bigg[\gamma \frac{(-1)^j + (-1)^k}{2} + \tag{4.13}$$
$$(c_j^\dagger - c_k^\dagger) \left(\frac{(1-\gamma)(-1)^j - (1+\gamma)(-1)^k}{2} c_j + \frac{(1+\gamma)(-1)^j - (1-\gamma)(-1)^k}{2} c_k \right) \Bigg].$$

Performing the summation for the first summand yields zero for $j \in I_A, k \in I_B$ or vice versa. Furthermore, summing the first term sorted by parity, i.e. $j,k \in I_A$ and then $j,k \in I_B$, one sees right away that the outcome is also identically zero. After further reshuffling of the second summand, we find

$$H_{\text{HS,alt}} = \sum_{j\neq k} \frac{1}{2d_{jk}} \Big[c_j^\dagger c_j \left((1-\gamma)(-1)^j - (1+\gamma)(-1)^k \right) + c_j^\dagger c_k \left((1+\gamma)(-1)^j - (1-\gamma)(-1)^k \right) \Big]. \tag{4.14}$$

Analogously to Section 2.2, we perform a Fourier transform using

$$c_j^\dagger = \frac{1}{\sqrt{2L}} \sum_{p=0}^{2L-1} \exp\left(-i2\pi \frac{pj}{2L} \right) d_p^\dagger, \qquad c_j = \frac{1}{\sqrt{2L}} \sum_{q=0}^{2L-1} \exp\left(i2\pi \frac{qj}{2L} \right) d_q \tag{4.15}$$

and

$$\frac{\pi^2}{4L^2} \frac{1}{\sin^2\left(\pi \frac{j-k}{2L}\right)} = \frac{\pi^2}{4L^4} \sum_{u,v=0}^{2L-1} \exp\left(i2\pi \frac{(u-v)(k-j)}{2L} \right) (L-u)(L-v) \tag{4.16}$$

upon which the case $j = k$ can be included in the summation in eq. (4.14) in order to get

$$H_{\text{HS,alt}} = \frac{\pi^2}{16L^5} \sum_{p,q=0}^{2L-1} d_p^\dagger d_q \sum_{u,v=0}^{2L-1} (L-u)(L-v) \cdot$$
$$\Bigg[(1-\gamma) \sum_{j=0}^{2L-1} (-1)^j \exp\left(i\pi \frac{j(v-u-p+q)}{L} \right) \sum_{k=0}^{2L-1} \exp\left(i\pi \frac{k(u-v)}{L} \right) \tag{4.17a}$$
$$- (1+\gamma) \sum_{j=0}^{2L-1} \exp\left(i\pi \frac{j(v-u-p+q)}{L} \right) \sum_{k=0}^{2L-1} (-1)^k \exp\left(i\pi \frac{k(u-v)}{L} \right) \tag{4.17b}$$
$$+ (1+\gamma) \sum_{j=0}^{2L-1} (-1)^j \exp\left(i\pi \frac{j(v-u-p)}{L} \right) \sum_{k=0}^{2L-1} \exp\left(i\pi \frac{k(u-v+q)}{L} \right) \tag{4.17c}$$

$$-(1-\gamma)\sum_{j=0}^{2L-1}\exp\left(i\pi\frac{j(v-u-p)}{L}\right)\sum_{k=0}^{2L-1}(-1)^k\exp\left(i\pi\frac{k(u-v+q)}{L}\right)\Bigg]. \quad (4.17d)$$

Omitting the prefactor $(1-\gamma)$ of eq. (4.17) for a moment, we work out the first part of the full expression:

$$\begin{aligned}
(4.17a) &= \sum_{j=0}^{2L-1}(-1)^j\exp\left(i\pi\frac{j(v-u-p+q)}{L}\right)2L\delta_{uv} && (4.18)\\
&= \sum_{j=0}^{2L-1}(-1)^j\exp\left(i\pi\frac{j(q-p)}{L}\right)2L\delta_{uv} && (4.19)\\
&= \left[\sum_{J=0}^{L-1}\exp\left(i\pi\frac{2J(q-p)}{L}\right)-\sum_{J=0}^{L-1}\exp\left(i\pi\frac{(2J+1)(q-p)}{L}\right)\right]2L\delta_{uv} && (4.20)\\
&= \sum_{J=0}^{L-1}\exp\left(i2\pi\frac{J(q-p)}{L}\right)\left[1-\exp\left(i\pi\frac{(q-p)}{L}\right)\right]2L\delta_{uv} && (4.21)\\
&= L\left(\delta_{pq}+\delta_{p,q+L}+\delta_{p+L,q}\right)\left[1-\exp\left(i\pi\frac{(q-p)}{L}\right)\right]2L\delta_{uv} && (4.22)\\
&= L\left(0+2\delta_{p,q+L}+2\delta_{p+L,q}\right)2L\delta_{uv} && (4.23)\\
&= 4L^2\left(\delta_{p,q+L}+\delta_{p+L,q}\right)\delta_{uv}. && (4.24)
\end{aligned}$$

Analogously, we proceed with the other three terms. The term in line (4.17b) is equivalent to the first term in eq. (4.19):

$$\begin{aligned}
(4.17b) &= \sum_{j=0}^{2L-1}\exp\left(i\pi\frac{j(v-u-p+q)}{L}\right)2L(\delta_{u,v+L}+\delta_{u+L,v})\\
&= 2L\left[\sum_{j=0}^{2L-1}\exp\left(i\pi\frac{j(q-p-L)}{L}\right)\delta_{u,v+L}+\sum_{j=0}^{2L-1}\exp\left(i\pi\frac{j(q-p+L)}{L}\right)\delta_{u+L,v})\right]\\
&= 2L\left[2L(\delta_{p,q+L}+\delta_{p+L,q})\delta_{u,v+L}+2L(\delta_{p,q+L}+\delta_{p+L,q})\delta_{u+L,v})\right]\\
&= 4L^2(\delta_{p,q+L}+\delta_{p+L,q})(\delta_{u,v+L}+\delta_{u+L,v}).
\end{aligned}$$

Again using eq. (4.19) along the calculation, the third term becomes

$$(4.17c) = \sum_{j=0}^{2L-1}(-1)^j\exp\left(i\pi\frac{j(v-u-p)}{L}\right)2L(\delta_{v,u+q}+\delta_{v+2L,u+q})$$

$$
\begin{aligned}
&= 2L \left[\sum_{j=0}^{2L-1} (-1)^j \exp\left(i\pi \frac{j(q-p)}{L} \right) \delta_{v,u+q} \right. \\
&\qquad \left. + \sum_{j=0}^{2L-1} (-1)^j \exp\left(i\pi \frac{j(q-2L-p)}{L} \right) \delta_{v+2L,u+q}) \right] \\
&= 2L \left[2L(\delta_{p,q+L} + \delta_{p+L,q}) \delta_{v,u+q} \right. \\
&\qquad \left. + L(\delta_{pq} + \delta_{p,q+L} + \delta_{p+L,q}) \left(1 - \exp\left(i\pi \frac{(q-2L-p)}{L} \right) \right) \delta_{v+2L,u+q} \right] \\
&= 4L^2 (\delta_{p,q+L} + \delta_{p+L,q})(\delta_{v,u+q} + \delta_{v+2L,u+q}).
\end{aligned}
$$

Completely analogously to the computation for the summand in line (4.17c), the last term yields

$$
\begin{aligned}
(4.17d) &= 2L(\delta_{v,u+p} + \delta_{v+2L,u+p}) \sum_{k=0}^{2L-1} (-1)^k \exp\left(i\pi \frac{k(u-v+q)}{L} \right) \\
&= 2L \left[\delta_{v,u+p} \sum_{k=0}^{2L-1} (-1)^k \exp\left(i\pi \frac{k(q-p)}{L} \right) \right. \\
&\qquad \left. + \delta_{v+2L,u+p} \sum_{k=0}^{2L-1} (-1)^k \exp\left(i\pi \frac{k(q-p+2L)}{L} \right) \right] \\
&= 4L^2 (\delta_{v,u+p} + \delta_{v+2L,u+p})(\delta_{p,q+L} + \delta_{p+L,q}).
\end{aligned}
$$

Putting everything back into eq. (4.17), we find

$$
\begin{aligned}
H_{\mathrm{HS,alt}} = \frac{\pi^2}{16L^5} 4L^2 \sum_{p,q=0}^{2L-1} d_p^\dagger d_q \left(\delta_{p,q+L} + \delta_{p+L,q} \right) \cdot \\
\sum_{u,v=0}^{2L-1} (L-u)(L-v) \left[(1-\gamma) \left(\delta_{uv} - (\delta_{v,u+p} + \delta_{v+2L,u+p}) \right) \right. && \text{(4.25a)} \\
\left. + (1+\gamma) \left(\delta_{v,u+q} + \delta_{v+2L,u+q} - (\delta_{u,v+L} + \delta_{u+L,v}) \right) \right]. && \text{(4.25b)}
\end{aligned}
$$

Omitting the first part of this expression and factors $(1-\gamma)$ and $(1+\gamma)$ for now, and paying close attention to the way the Kronecker deltas may change the range of summation

over u and v, we find

$$
\begin{aligned}
(4.25a) &= \sum_{u=0}^{2L-1}(L-u)^2 - \sum_{u=0}^{2L-1-p}(L-u)(L-u-p) - \sum_{u=2L-p}^{2L-1}(L-u)(L-u-p+2L) \\
&= \sum_{u=0}^{2L-1}(L-u)^2 - \sum_{u=0}^{2L-1}(L-u)(L-u-p) - \sum_{u=2L-p}^{2L-1}(L-u)2L && (4.26) \\
&= p\sum_{u=0}^{2L-1}(L-u) - 2L\sum_{u=2L-p}^{2L-1}(L-u) && (4.27) \\
&= p\left(2L^2-(2L-1)L\right) - 2L\left(pL - \left((2L-1)L-(2L-p-1)(2L-p)/2\right)\right) \\
&= (2L-p)(2L-1)L - 2L(2L-p-1)(2L-p)/2 && (4.28) \\
&= L(2L-p)p && (4.29)
\end{aligned}
$$

and

$$
\begin{aligned}
(4.25b) = &\sum_{u=0}^{2L-1-q}(L-u)(L-u-q) + \sum_{u=2L-q}^{2L-1}(L-u)(L-u-q+2L) && (4.30) \\
&- \sum_{u=0}^{L-1}(L-u)(L-u-L) - \sum_{u=L}^{2L-1}(L-u)(L-u+L) && (4.31)
\end{aligned}
$$

where, for the first two summands, we can use the result of the computation of the summand in line (4.25a) in order to eventually obtain

$$
\begin{aligned}
(4.25b) &= \sum_{u=0}^{2L-1}(L-u)^2 - L(2L-q)q + L\sum_{u=0}^{L-1}(L-u) - \sum_{u=0}^{2L-1}(L-u)(L-u) - L\sum_{u=L}^{2L-1}(L-u) \\
&= -L(2L-q)q + L(\sum_{u=0}^{L-1}(L-u) - \sum_{u=L}^{2L-1}(L-u)) && (4.32) \\
&= L(q-2L)q + L(\sum_{u=L}^{2L-1}u - \sum_{u=0}^{L-1}u) && (4.33) \\
&= L\left(q^2-2qL+L^2\right) && (4.34) \\
&= L\left(q-L\right)^2. && (4.35)
\end{aligned}
$$

Reinserting the outcome in lines (4.29) and (4.35) into eq. (4.25) yields

$$
\begin{aligned}
H_{\text{HS,alt}} &= \frac{\pi^2}{4L^3}\sum_{p,q=0}^{2L-1} d_p^\dagger d_q(\delta_{p,q+L}+\delta_{p+L,q})\left[(1-\gamma)L(2L-p)p + (1+\gamma)L\left(q-L\right)^2\right] && (4.36) \\
&= \frac{\pi^2}{4L^2}\sum_{p,q=0}^{2L-1} d_p^\dagger d_q(\delta_{p,q+L}+\delta_{p+L,q})\left[(1-\gamma)(2L-p)p + (1+\gamma)(q-L)^2\right] && (4.37)
\end{aligned}
$$

$$= \frac{\pi^2}{4L^2}\left[\sum_{q=0}^{L-1} d_{q+L}^\dagger d_q \left((1-\gamma)(L-q)(q+L)+(1+\gamma)(L-q)^2\right)\right.$$
$$\left. + \sum_{p=0}^{L-1} d_p^\dagger d_{p+L}\left((1-\gamma)(2L-p)p+(1+\gamma)p^2\right)\right] \tag{4.38}$$

$$= \frac{\pi^2}{4L^2}\left[\sum_{q=0}^{L-1} d_{q+L}^\dagger d_q (L-q)(q+L-\gamma q-\gamma L+L-q+\gamma L-\gamma q)\right.$$
$$\left. + \sum_{p=0}^{L-1} d_p^\dagger d_{p+L} p(2L-p-2\gamma L+\gamma p+p+\gamma p)\right] \tag{4.39}$$

$$= \frac{\pi^2}{4L^2}\left[\sum_{q=0}^{L-1} d_{q+L}^\dagger d_q 2(L-q)(L-\gamma q)+\sum_{p=0}^{L-1} d_p^\dagger d_{p+L} 2p(L-\gamma L+\gamma p)\right] \tag{4.40}$$

$$= \frac{\pi^2}{2L^2}\sum_{p=0}^{L-1}\left[d_{p+L}^\dagger d_p (L-p)(L-\gamma p)+d_p^\dagger d_{p+L} p(L-\gamma L+\gamma p)\right] \tag{4.41}$$

$$= \frac{\pi^2}{2} d_L^\dagger d_0 + \frac{\pi^2}{2L^2}\sum_{p=1}^{L-1}\left[d_{p+L}^\dagger d_p (L-p)(L-\gamma p)+d_p^\dagger d_{p+L} p(L-\gamma L+\gamma p)\right] \tag{4.42}$$

The first summand is non-diagonalizable. In order to diagonalize the remainder of this expression, we could use a canonical transformation. To this end, we observe that each summand of eq. (4.42) is of the form

$$\alpha\, d_{p+L}^\dagger d_p + \beta\, d_p^\dagger d_{p+L} \tag{4.43}$$

with $\alpha^* \neq \beta$, in general, so that the second part is *not* the hermitian conjugate of the first. Note that, along with the other usual fermionic anticommutation relations, $\{d_p^\dagger, d_{p+L}\} \equiv 0$, which are automatically fulfilled by making the ansatz ($p = 1, \ldots, L-1$)

$$d_p := \xi(a_p + b_p), \qquad d_{p+L} := \delta(a_p - b_p), \tag{4.44}$$
$$d_p^\dagger := \epsilon(a_p^\dagger + b_p^\dagger), \qquad d_{p+L}^\dagger := \zeta(a_p^\dagger - b_p^\dagger) \tag{4.45}$$

for fermionic modes a_p and b_p that mutually anticommute, $\{a_p, b_p\} = 0$, and arbitrary values of the parameters ϵ, δ, ξ and ζ. By invoking the given anticommutation relations

$$1 = \{d_p^\dagger, d_p\} = \xi\epsilon(\{a_p^\dagger, a_p\} + \{b_p^\dagger, b_p\}) = 2\xi\epsilon, \tag{4.46}$$
$$1 = \{d_{p+L}^\dagger, d_{p+L}\} = \delta\zeta(\{a_p^\dagger, a_p\} + \{b_p^\dagger, b_p\}) = 2\delta\zeta \tag{4.47}$$

we find $\epsilon = \frac{1}{2\xi}$ and $\zeta = \frac{1}{2\delta}$. Returning to eq. (4.43), we get

$$(4.43) = \frac{\alpha\xi}{2\delta}(a_p^\dagger - b_p^\dagger)(a_p + b_p) + \frac{\beta\delta}{2\xi}(a_p^\dagger + b_p^\dagger)(a_p - b_p) \tag{4.48}$$

$$= \frac{\alpha\xi}{2\delta}(a_p^\dagger a_p - b_p^\dagger a_p + a_p^\dagger b_p - b_p^\dagger b_p) + \frac{\beta\delta}{2\xi}(a_p^\dagger a_p + b_p^\dagger a_p - a_p^\dagger b_p - b_p^\dagger b_p) \tag{4.49}$$

$$= \left(\frac{\alpha\xi}{2\delta} + \frac{\beta\delta}{2\xi}\right)(a_p^\dagger a_p - b_p^\dagger b_p) + \left(\frac{\alpha\xi}{2\delta} - \frac{\beta\delta}{2\xi}\right)(a_p^\dagger b_p - b_p^\dagger a_p). \tag{4.50}$$

If we choose $\delta = \xi\sqrt{\alpha/\beta}$, the modes a_p and b_p decouple,

$$\frac{\alpha\xi}{2\delta} - \frac{\beta\delta}{2\xi} = 0, \tag{4.51}$$

for us to find

$$\alpha\, d_{p+L}^\dagger d_p + \beta\, d_p^\dagger d_{p+L} = \sqrt{\alpha\beta}\,(a_p^\dagger a_p - b_p^\dagger b_p). \tag{4.52}$$

It is worth pointing out that this equation always holds for anticommuting modes c and d coupled in that way:

$$\alpha\, d^\dagger c + \beta\, c^\dagger d = \sqrt{\alpha\beta}\,(a^\dagger a - b^\dagger b) \tag{4.53}$$

with

$$c := \xi(a+b), \qquad d := \delta(a-b) \quad \text{and} \tag{4.54}$$

$$c^\dagger := \epsilon(a^\dagger + b^\dagger), \qquad d^\dagger := \zeta(a^\dagger - b^\dagger) \tag{4.55}$$

Thus, our final result is acquired by using the last identity given in eq. (4.43) and the redefinition $a_0 := d_L$ and $b_0 := d_0$ on the expression for the Haldane-Shastry-like Hamiltonian in eq. (4.42),

$$H_{\text{HS,alt}} = \frac{\pi^2}{2} a_0^\dagger b_0 + \frac{\pi^2}{2L^2}\sum_{p=1}^{L-1}\sqrt{(L-p)(L-\gamma p)p(L-\gamma L+\gamma p)}(a_p^\dagger a_p - b_p^\dagger b_p). \tag{4.56}$$

Analogously to Section 2.2.1, we define the state which is completely filled by the modes $b_p^\dagger$,

$$|\omega\rangle := \prod_{p=0}^{L-1} b_p^\dagger\,|0\rangle \tag{4.57}$$

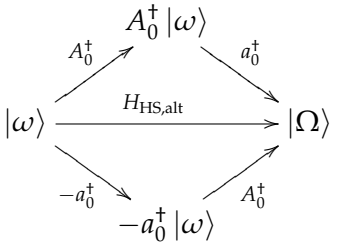
with $|0\rangle$ the vacuum of the momentum modes a_p and b_p, and introduce the operators

$$A_p := b_p^\dagger \qquad \text{and} \qquad A_p^\dagger := b_p \tag{4.58}$$

in order to rewrite the Hamiltonian into

$$H_{\text{HS,alt}} = \frac{\pi^2}{2} a_0^\dagger A_0^\dagger + \frac{\pi^2}{2L^2} \sum_{p=1}^{L-1} \sqrt{(L-p)(L-\gamma p)p(L-\gamma L+\gamma p)}(a_p^\dagger a_p + A_p^\dagger A_p) - \frac{\pi^2}{2L^2} \sum_{p=1}^{L-1} \sqrt{(L-p)(L-\gamma p)p(L-\gamma L+\gamma p)}. \tag{4.59}$$

The following analysis of the space of states goes in the same vein as the one presented in [24] for the Heisenberg version of this chain: From the expression for $H_{\text{HS,alt}}$ in eq. (4.59), it is obvious that it is not fully diagonalizable but instead it features Jordan blocks of rank two due to the term $\frac{\pi^2}{2} a_0^\dagger A_0^\dagger$. The states $|\omega\rangle$, $A_0^\dagger |\omega\rangle = b_0 |\omega\rangle$, $-a_0^\dagger |\omega\rangle$ and $a_0^\dagger A_0^\dagger |\omega\rangle =: |\Omega\rangle$ form a submodule in the space of states $\mathcal{H}$ which is isomorphic to the projective representation $\mathcal{P}_0$ of $\mathfrak{gl}(1|1)$



(4.60)

and it belongs to the Jordan block with the lowest diagonal elements of $H_{\text{HS,alt}}$. In fact, the full space of states $\mathcal{H}$ can be constructed from the cyclic state $|\omega\rangle$ by applying any combination of creation operators $\{a_p^\dagger\}$ and $\{A_q^\dagger\}$ for $p, q \in \{0, \ldots, L-1\}$. However, due to the structure depicted in eq. (4.60), we see that $|\omega\rangle$ is not an eigenstate but is mapped to $|\Omega\rangle$ by the Hamiltonian because of its Jordan block structure. $|\Omega\rangle$, on the other hand, may be considered the true vacuum of the system as it is, evidently, annihilated by both sets of operators $\{a_p\}$ and $\{A_q\}$ and is eigenstate of the Hamiltonian for its lowest eigenvalue.

The diagram in eq. (4.60) and the structure of $H_{\text{HS,alt}}$ teach us even more as the space of states $\mathcal{H}$ (at least its low-energy regime) can be divided into states forming multiplets of this type: The quadratic Casimir of $\mathfrak{gl}(1|1)$ on $\mathcal{P}_0$ is given by $2\psi^-\psi^+$ and, just like $H_{\text{HS,alt}}$, it maps $|\omega\rangle$ to $|\Omega\rangle$. Therefore, we may identify the global Casimir of $\mathfrak{gl}(1|1)$ on $\mathcal{H}$ with $\frac{\pi^2}{2} a_0^\dagger A_0^\dagger$ and, due to their role depicted in eq. (4.60), we can identify $A_0^\dagger$ with the representation of ψ^+ on $\mathcal{H}$ and $a_0^\dagger$ with the representation of ψ^- on $\mathcal{H}$. Furthermore, this is how eq. (4.60) is to be understood – as describing the action of the symmetry and of the Hamiltonian. In particular, their action only follows the arrows and, e.g. even though using A_0 on $A_0^\dagger |\omega\rangle$ we may, in principle, return to $|\omega\rangle$, the operator A_0 is not representating any of the generators of the global $\mathfrak{gl}(1|1)$ symmetry.

Independently of the particular value of γ, the Hamiltonian becomes gapless which we

demonstrate by expanding the dispersion relation around its zeros p and L,

$$\frac{2L^2}{\pi^2}\varepsilon_\gamma(p)\Big|_{p=0} = L\sqrt{L-\gamma}\sqrt{p} + \mathcal{O}(p^{3/2}) \tag{4.61}$$

and

$$\frac{2L^2}{\pi^2}\varepsilon_\gamma(p)\Big|_{p=L} = L\sqrt{L-\gamma}\sqrt{L-p} + \mathcal{O}((L-p)^{3/2}). \tag{4.62}$$

As these dispersion relations have no chance of being linear around their zeros, the spin chains for generic values of γ cannot be critical. Nonetheless, in the special case of our actual interest, i.e. for the GL(1|1) Haldane-Shastry chain of alternating spins with $\gamma = 1$, the system is critical. This becomes evident when looking at the single-particle energy of the mode generated by $a_p^\dagger$ or $A_p^\dagger$ given by the dispersion relation $\varepsilon(p)$ for $\gamma = 1$,

$$\varepsilon(p) = \frac{\pi^2}{2L^2}\sqrt{(L-p)(L-p)p(L-L+p)} = \frac{\pi^2}{2L^2}(L-p)p, \tag{4.63}$$

and expanding it around $p = 0$

$$\frac{2L^2}{\pi^2}\varepsilon(p)\Big|_{p=0} = Lp - p^2 \tag{4.64}$$

and $p = L$ (with system size given by $2L$)

$$\frac{2L^2}{\pi^2}\varepsilon(p)\Big|_{p=L} = -L(p-L) - (p-L)^2. \tag{4.65}$$

Since the admissible values of the momentum *numbers* p are $1, 2, \ldots, L-1$, we introduce $p_\mathrm{l} := -(p-L)$, rewrite eq. (4.65) as

$$\frac{2L^2}{\pi^2}\varepsilon(p_\mathrm{l})\Big|_{p_\mathrm{l}=0} = Lp_\mathrm{l} - p_\mathrm{l}^2. \tag{4.66}$$

and, hence, identify the respective modes as left movers whose dispersion relation is analogous to the one of the right movers given in eq. (4.64). Therefore, the dispersion relation of the right movers with momentum $p_\mathrm{r} := p$ and of the left moving modes with momentum p_l becomes linear in the momentum numbers p_l and p_r when they are small compared to the total number $2L$ of spins in the system.

Finally, and most importantly, the underlying conformal field theory of the system can now be identified. To this end, let us first state the actual Hamiltonian of the Haldane-Shastry chain of $2L$ alternating $\mathfrak{gl}(1|1)$-spins given by eq. (4.59) with $\gamma = 1$,

$$H_\mathrm{HS,alt} = \frac{\pi^2}{2}a_0^\dagger A_0^\dagger + \frac{\pi^2}{2L^2}\sum_{p=1}^{L-1}(L-p)p(a_p^\dagger a_p + A_p^\dagger A_p) - \frac{\pi^2}{2L^2}\sum_{p=1}^{L-1}(L-p)p \tag{4.67}$$

$$= \frac{\pi^2}{2} a_0^\dagger A_0^\dagger + \frac{\pi^2}{2L^2} \sum_{p=1}^{L-1} (L-p)p(a_p^\dagger a_p + A_p^\dagger A_p) - \frac{\pi^2}{2L^2} C(L), \tag{4.68}$$

where we used eq. (2.70) in order to introduce $C(L)$.

Finally, we find the effective low-energy version of the Hamiltonian given in eq. (4.59) by confining the sum over the momentum numbers p to momentum numbers belonging to low excitations of the system, i.e. to p_l and p_r much smaller than $2L$, and dropping terms quadratic in the respective momentum numbers:

$$H_{\mathrm{HS,alt}}^{\mathrm{low}} = \frac{\pi^2}{2} \left(a_0^\dagger A_0^\dagger + \sum_{p_\mathrm{l}>0} \frac{p_\mathrm{l}}{L} (a_{p_\mathrm{l}}^\dagger a_{p_\mathrm{l}} + A_{p_\mathrm{l}}^\dagger A_{p_\mathrm{l}}) + \sum_{p_\mathrm{r}>0} \frac{p_\mathrm{r}}{L} (a_{p_\mathrm{r}}^\dagger a_{p_\mathrm{r}} + A_{p_\mathrm{r}}^\dagger A_{p_\mathrm{r}}) \right) - \frac{\pi^2}{2L^2} C(L). \tag{4.69}$$

One may now compare eq. (4.69) to the expression given in [24, eq. (4.16)]. There, Gainutdinov, Read and Saleur investigated the Heisenberg chain of $\mathfrak{gl}(1|1)$-spins in the alternating case and deduced that its low-energy effective theory in the continuum limit is described by the symplectic fermion CFT with $c = -2$ [65]. Obviously, the structure imprinted on the space of states by both Hamiltonians coincides and we may, therefore, conclude our analysis of the alternating Haldane-Shastry chain of $\mathfrak{gl}(1|1)$-spins with the finding that this model belongs to the same universality class, with the great advantage that its dispersion relation $\varepsilon(p)$ as compared to the Heisenberg chain with $\varepsilon_{\mathrm{Heisenberg}}(p) = \sin\left(\frac{\pi p}{L}\right)$ is much simpler. In fact, it is unique in the sense that its dispersion relation is the simplest dispersion relation compatible with the existence of left and right movers as its Taylor expansion terminates after the quadratic term.

4.2 The One-Dimensional Alternating Setup for $\mathfrak{gl}(1|1)$

After having dealt with the GL(1|1) Haldane-Shastry spin chain in the alternating case, we will analyze the Hamiltonian given at the beginning of this Chapter, i.e. in eq. (4.3). Note, that apart from the term proportional to the square of the total spin of the system, it coincides with the GL(1|1)-invariant Haldane-Shastry spin chain in the alternating case. Therefore, the only task left to solving this Hamiltonian is bringing the square of the total spin into a form compatible with the almost diagonal Hamiltonian in eq. (4.59). So, neglecting the factor $\frac{\pi^2}{8L}$ for now, we have

$$\mathbf{S}^2 = \left(\sum_{i=1}^{2L} \mathbf{S}_i \right) \cdot \left(\sum_{j=1}^{2L} \mathbf{S}_j \right) = \sum_{i,j} \mathbf{S}_i \cdot \mathbf{S}_j. \tag{4.70}$$

Thus, the total spin coincides with the expression given in eq. (4.10) for $\gamma = 1$ apart from the distance-dependent part and the cases $i = j$, i.e. we may rewrite it in terms of the free

fermions introduced in Section 4.1 analogously to eq. (4.14) as

$$\mathbf{S}^2 = \sum_{j,k} \left(c_j^\dagger c_j (-2(-1)^k) + c_j^\dagger c_k (2(-1)^j) \right) \tag{4.71}$$

$$= 2 \sum_{j,k} \left(c_j^\dagger c_k (-1)^j - c_j^\dagger c_j (-1)^k \right). \tag{4.72}$$

Performing the sum over k for the second summand in the parentheses, we see that it cancels due to the alternating sign factor $(-1)^k$:

$$\mathbf{S}^2 = 2 \sum_{j,k} c_j^\dagger c_k (-1)^j. \tag{4.73}$$

Next, and just as in Section 4.1, we perform a Fourier transformation:

$$\mathbf{S}^2 = \frac{2}{2L} \sum_{p,q=0}^{2L-1} d_p^\dagger d_q \sum_{j,k} \exp\left(i2\pi \frac{qk - pj}{2L} \right) \exp\left(i\pi j \right) \tag{4.74}$$

$$= \frac{1}{L} \sum_{p,q=0}^{2L-1} d_p^\dagger d_q \sum_{j,k=1}^{2L} \exp\left(i2\pi \frac{qk - (p-L)j}{2L} \right). \tag{4.75}$$

Summation over k and then over j leads to

$$\mathbf{S}^2 = \frac{1}{L} \sum_{p,q=0}^{2L-1} d_p^\dagger d_q \sum_{j=1}^{2L} 2L\delta_{q,0} \exp\left(-i2\pi \frac{(p-L)j}{2L} \right) \tag{4.76}$$

$$= \frac{1}{L} \sum_{p,q=0}^{2L-1} d_p^\dagger d_q \, 2L\delta_{q,0} \, 2L\delta_{p,L}. \tag{4.77}$$

Finally, after summation of the remaining momentum numbers p and q, we find

$$\mathbf{S}^2 = 4L \, d_L^\dagger d_0. \tag{4.78}$$

This rather surprising outcome tells us two things about the total spin of (any) GL(1|1)-invariant spin system defined on the space of states given by an alternating tensor product, i.e. $\mathcal{H} := (\bar{\mathcal{V}} \otimes \mathcal{V})^{\otimes L/2}$: Firstly, it is non-diagonalizable as it couples the orthogonal modes generated by $d_0^\dagger$ and $d_L^\dagger$. Secondly, the diagonal elements of all of its Jordan blocks are zero and, hence, it is nilpotent.

Even more remarkable is the fact that the contribution of the summand proportional to $\mathbf{S}^2$ in the expression for the Hamiltonian of our current interest given in eq. (4.3) is such that it exactly cancels the zero mode part $\frac{\pi^2}{2} d_L^\dagger d_0$, or, expressed in the modes of eq. (4.68), the zero mode part $\frac{\pi^2}{2} a_0^\dagger A_0^\dagger$, stemming from the Haldane-Shastry-like part (cf. eq. (4.59)) of

the full Hamiltonian:

$$H = \frac{\pi^2}{8L^2} g \, \mathbb{I} - \frac{\pi^2}{8L} \mathbf{S}^2 + H_{\text{HS,alt}} \tag{4.79}$$

$$= \frac{\pi^2}{8L^2} g \, \mathbb{I} - \frac{\pi^2}{8L} 4L \, d_L^\dagger d_0 + \frac{\pi^2}{2} d_L^\dagger d_0 + \frac{\pi^2}{2L^2} \sum_{p=1}^{L-1} (L-p) p \, (a_p^\dagger a_p - b_p^\dagger b_p) \tag{4.80}$$

$$= \frac{\pi^2}{8L^2} g \, \mathbb{I} + \frac{\pi^2}{2L^2} \sum_{p=1}^{L-1} (L-p) p (a_p^\dagger a_p + A_p^\dagger A_p) - \frac{\pi^2}{2L^2} C(L) \tag{4.81}$$

$$= \frac{\pi^2}{2L^2} \sum_{p=1}^{L-1} (L-p) p (a_p^\dagger a_p + A_p^\dagger A_p). \tag{4.82}$$

Here, we made use of eq. (4.68) and, again, in the last step, we used the fact that the constant g reduces to $4C(L)$ when the sites i are uniformly distributed on the unit circle.

Hence, we find that the Hamiltonian constructed from conformal field theory in Chapter 3 becomes fully diagonalizable in the case of a uniform one-dimensional setup of alternating representation spaces $\bar{\mathcal{V}}$ and $\mathcal{V}$, antifundamental and fundamental for $\mathfrak{gl}(1|1)$, in contrast to the analogous Haldane-Shastry spin chain covered in Section 4.1 which fails to be diagonalizable due to the appearance of the zero mode part.

Furthermore, the CFT seed state, computed in Section 3.8.2, is found in the submodule of the state space $\mathcal{H}$ sketched in eq. (4.60) which, ostensibly, has zero energy by judging from eq. (4.82).

Finally, comparing this result to the Hamiltonian of the GL(1|1) Haldane-Shastry spin chain in the pure case given in eq. (2.71), we find that, for a total number of $2L$ *alternating* spins uniformly distributed on the unit circle, our system described by H is equivalent to two copies of the *pure* Haldane-Shastry model of L $\mathfrak{gl}(1|1)$ spins discussed in Section 2.2 (built on the same state $|\omega\rangle$ instead of two different states $|\omega\rangle$ and $|\omega\rangle'$), each of which becomes critical in the continuum limit and is represented by the order and disorder sector of the Ising quantum chain. Therefore, the effective central charge of this system is given by the sum of the effective central charge (cf. [39]) $\tilde{c}_{\text{pure}} = -1$ of each pure chain, i.e.

$$\tilde{c}_{\text{alt}} = -2 \tag{4.83}$$

which coincides with the central charge of the symplectic fermion CFT. This is not so surprising anymore after having understood that their only nontrivial difference lies in the zero mode part $\frac{\pi^2}{2} a_0^\dagger A_0^\dagger$ which is responsible for the fact that $|\omega\rangle$ (and all its excitations stemming from all non-zero combinations of non-zero-mode creation operators $a_p^\dagger$ $A_q^\dagger$) is an eigenstate of eq. (4.82) but not of the Haldane-Shastry version in eq. (4.68) by which it is mapped to $|\Omega\rangle$ – and not to zero.

4.3 Connecting the Ising CFT to a Logarithmic CFT

It is a good place to summarize what we have found: In Section 2.2 the GL(1|1) Haldane-Shastry spin chain on $\mathcal{H}_{\text{pure}} = \mathcal{V}^L$ was given in eq. (2.72) and solved in eq. (2.77) (here given with normalization analogously to the alternating systems just discussed and neglecting a potential constant offset):

$$H_{\text{HS,pure}} = \frac{\pi^2}{L^2} \sum_{p=0}^{L-1} A_p^\dagger A_p (L-p) p. \tag{4.84}$$

Due to its zero mode $A_0^\dagger$, the vacuum $|\omega_{\text{pure}}\rangle$ for the annihilation operators A_p and the state $A_0^\dagger |\omega_{\text{pure}}\rangle$ are both ground states of the system. The full state space $\mathcal{H}$ is built on top of $|\omega_{\text{pure}}\rangle$ by applying to it all non-zero combinations of creation operators $A_p^\dagger$ ($p = 0, 1, \ldots, L-1$). Due to the zero mode, there is a bi-partition of the states which can be pictured as

$$|\omega_{\text{pure}}\rangle \underset{A_0}{\overset{A_0^\dagger}{\rightleftarrows}} A_0^\dagger |\omega_{\text{pure}}\rangle \tag{4.85}$$

and which resembles the structure of the fundamental representation $\mathcal{V}$ of $\mathfrak{gl}(1|1)$ with $A_0^\dagger$ representing the action of ψ^+, and A_0 representing the action of ψ^- on $\mathcal{H}$. In the continuum limit, it is described by the order and disorder sector of the Ising CFT with effective central charge given by $\tilde{c} = -1$. Hence, given the character $d(I)$ of the order or disorder operator as described in Section 2.2, the decomposition of the low-energy sector of the state space is given by

$$\mathcal{H}_{\text{pure,low}} = \bigoplus_{n=0,1,2,\ldots} \mathcal{H}_{\text{pure},n} \tag{4.86}$$

with

$$\mathcal{H}_{\text{pure},n} := \bigoplus_{I,\bar{I}=0}^{n} \delta_{I+\bar{I},n}\, d(I) d(\bar{I})\, \langle n, I-\bar{I}\rangle \tag{4.87}$$

due to the existence of right moving modes I and left moving ones $\bar{I}$. With this representation, we have

$$|n, I-\bar{I}\rangle \propto A_{q_1}^\dagger \ldots A_{q_l}^\dagger A_{p_1}^\dagger \ldots A_{p_k}^\dagger |\omega\rangle \tag{4.88}$$

with $p_i, q_j = 1, 2, \ldots$ indicating left and right movers, respectively, so that $I = \sum p_i$ and $\bar{I} = \sum q_j$ while the other state $|n, I-\bar{I}-1\rangle$ also includes the zero mode[2].

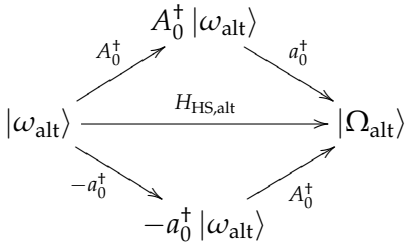
Its alternating version on $\mathcal{H}_{\text{alt}} = (\bar{\mathcal{V}} \otimes \mathcal{V})^L$ is formally given by the same expression, was

[2] Note that its spin is not $I - \bar{I} - 1$ but also $I - \bar{I}$.

solved in eq. (4.68) and, neglecting the constant, reads

$$H_{\mathrm{HS,alt}} = \frac{\pi^2}{2} a_0^\dagger A_0^\dagger + \frac{\pi^2}{2L^2} \sum_{p=1}^{L-1} (L-p)p(a_p^\dagger a_p + A_p^\dagger A_p). \tag{4.89}$$

We find the vacuum $|\omega_{\mathrm{alt}}\rangle$ for the annihilation operators a_p and A_p upon which the full state space $\mathcal{H}_{\mathrm{alt}}$ is built by applying to it all non-zero combinations of creation operators $a_p^\dagger$ and $A_q^\dagger$ ($p, q = 0, 1, \ldots, L-1$). Due to the two zero modes $a_0^\dagger$ and $A_0^\dagger$, there is a fourfold structure of the states which can be pictured as



(4.90)

with the action of the Hamiltonian, i.e. of the zero mode contribution to the Hamiltonian, mapping $|\omega_{\mathrm{alt}}\rangle$ to $|\Omega_{\mathrm{alt}}\rangle$, leading to an indecomposable structure reminiscent of the kind one expects for a logarithmic CFT. Evidently, this diagram is equivalent to the structure of the projective representation of $\mathfrak{gl}(1|1)$ and found analogously for any true excitation ($p \neq 0, L$) of $|\omega\rangle$ with its degeneracy given in terms the corresponding characters $d(I)$ of the order or disorder operator of the Ising CFT which leads to the decomposition of the low-energy sector of the state space

$$\mathcal{H}_{\mathrm{alt,low}} = \bigoplus_{n=0,1,2,\ldots} \mathcal{H}_{\mathrm{alt},n} \tag{4.91}$$

with

$$\mathcal{H}_{\mathrm{alt},n} := \bigoplus_{I,\bar{I}=0}^{n} \delta_{I+\bar{I},n}\, d(I) d(\bar{I})\, \mathcal{P}(I-\bar{I}), \tag{4.92}$$

again, due to the existence of right moving modes I and left moving ones $\bar{I}$. The integer $I - \bar{I}$ in the projective $\mathfrak{gl}(1|1)$ representation $\mathcal{P}(I-\bar{I})$ (cf. [66]), on which $H_{\mathrm{HS,alt}}$ acts like the quadratic Casimir plus n times the identity, helps to keep track of the spin of the states in that particular representation (while their scaling dimension is given by the sum $I + \bar{I}$ rather than their difference) which is the same for all four states in $\mathcal{P}(I-\bar{I})$[3]. Furthermore, in [24], the equivalence between the alternating GL(1|1)-invariant Heisenberg spin chain and a twisted XX spin chain given in terms of the Pauli matrices σ_i^x, σ_i^y and σ_i^z for each site i was shown which led to the insight that their periodic $\mathfrak{gl}(1|1)$ spin chain is equivalent to a

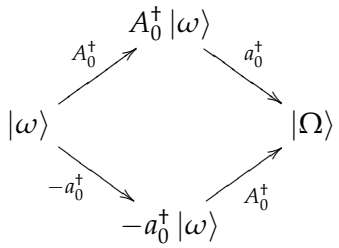
[3] Again, this is contrary to the individual state's eigenvalue with respect to the generator N of $\mathfrak{gl}(1|1)$ which would be $I - \bar{I}$ for $|\omega\rangle$ and $|\Omega\rangle$, $I - \bar{I} + 1$ for $A_0^\dagger |\omega\rangle$ and $I - \bar{I} - 1$ for $a_0^\dagger |\omega\rangle$.

periodic XX spin chain for odd values of the spin $S^z := \frac{1}{2}\sum_{i=1}^{2L}\sigma_i^z$ or an anti-periodic XX spin chain for even values of the spin S^z. Additionally, it was stated by the authors that the bosonic states $|\omega_{\text{alt}}\rangle$ and $|\Omega_{\text{alt}}\rangle$ belong to the sector with $S^z = 0$ while the fermionic states $a_0^\dagger|\omega_{\text{alt}}\rangle$ and $A_0^\dagger|\omega_{\text{alt}}\rangle$ belong to $S^z = -1$ and $S^z = +1$, respectively. The system is critical in the continuum limit and its CFT was identified as the symplectic fermion with $c = -2$.

Finally, the alternating Haldane-Shastry-like $\mathfrak{gl}(1|1)$-spin chain based on the Hamiltonian derived in Chapter 3 is represented by

$$H = \frac{\pi^2}{2L^2}\sum_{p=1}^{L-1}(L-p)p(a_p^\dagger a_p + A_p^\dagger A_p), \tag{4.93}$$

but without the non-diagonalizable part $\pi^2 a_0^\dagger A_0^\dagger$ as compared to the actual alternating Haldane-Shastry $\mathfrak{gl}(1|1)$-spin chain just discussed. Its state space structure



(4.94)

is very similar to the Haldane-Shastry version of the same which is to say that, with respect to the global symmetry, their structure is identical, i.e. we also have

$$\mathcal{H}_{\text{low}} = \bigoplus_{n=0,1,2,\ldots}\mathcal{H}_n \tag{4.95}$$

with

$$\mathcal{H}_n := \bigoplus_{I,\bar{I}=0}^{n}\delta_{I+\bar{I},n}\,d(I)d(\bar{I})\,\mathcal{P}(I-\bar{I}) \tag{4.96}$$

with the very same conventions as for the alternating Haldane-Shastry model given in the previous paragraph. However, the Hamiltonian is fully diagonalizable and, in particular, $|\omega\rangle$ is not mapped to $|\Omega\rangle$ by H but, instead, it and all its excitations are eigenstates of the Hamiltonian. For this reason, we might say that its global symmetry is that of the symplectic fermion with $c = -2$, however, with a Hamiltonian that is regularized in such a way as to cut out its indecomposable part, rendering it effectively as two coupled copies of the Ising CFT in the sector of the order and disorder operator with $\tilde{c} = -1$ each.

We would like to close the discussion of our Hamiltonian for $\mathfrak{gl}(1|1)$ in this very special uniform, one-dimensional setup by returning to the remark on numerical insights, made close to the end of Section 3.7. There, it was stated, that, twisting the two sublattices $\bar{\mathcal{V}}^L$ and $\mathcal{V}^L$ uniformly by an angle α, the spectrum does not change. However, doing so will spoil the perfect cancellation of the zero mode term. Even the tiniest contribution

from this term will render our Hamiltonian non-diagonalizable in the very same sense as the alternating Haldane-Shastry chain of $\mathfrak{gl}(1|1)$-spins is – it will create Jordan blocks of rank two. Put differently, our one-dimensional $\mathrm{GL}(1|1)$-invariant Hamiltonian on two uniform sublattices twisted against the fully uniform version by an angle α is equivalent to the alternating $\mathrm{GL}(1|1)$-invariant Haldane-Shastry spin chain[4] except for the singular point $\alpha = 0$[5] where, all of the sudden, our Hamiltonian loses its indecomposable structure making it fully diagonalizable.

4.4 Proposing a Regularization Scheme for Haldane-Shastry Spin Chains – or How To Get The Scaling Right

At this point, we would like to take a little detour since pinning down the central charge of the spin chains presented in the previous Section by means of finite-size scaling arguments led to inconclusive if not even contradictory statements.

In general, Cardy's finite-size scaling is an important tool for extracting the information relevant for indentifying the specific conformal field theory from a system that becomes critical in the thermodynamic limit. One of these data, the central charge c, of a critical one-dimensional system can be inferred once the running of the ground state energy $E_0(L)$ with the system size L is known (with v_s the speed of sound):

$$E_0(L) = \epsilon_\infty L - c\frac{\pi v_s}{6L} + \mathcal{O}(\frac{1}{L^2}). \tag{4.97}$$

This formula works very well for spin chains of, e.g., Heisenberg type but fails with respect to the conformal charge c if applied naïvely to Haldane-Shastry spin chains. Purely based on arguments of plausibility and without proof, we would like to put forward a regularization of the Haldane-Shastry models to cure this discrepancy and to make it directly comparable to CFT. To this end, we start with an analysis of the situation for the Haldane-Shastry model of $\mathfrak{su}(2)$-spins.

4.4.1 The Central Charge of the $\mathrm{SU}(2)$ Haldane-Shastry Spin Chain

Since the constructed Hamiltonians are related to Haldane-Shastry spin chains, our considerations pointed to an obvious discrepancy in the finite-size scaling analysis of the aforementioned model for $\mathfrak{su}(2)$-spins whose continuum limit effective low-energy theory is given by the free boson with $c = 1$.

As we will see, this is in contradiction with the formula for the ground state energy of

[4] Note that this chain is generically only defined on a fully uniform lattice on the unit circle.
[5] At the value $\alpha = 0$ the lattices of the two different models coincide.

this model, i.e. the lowest eigenvalue

$$E_0(L) = -\frac{\pi^2}{24}L - \frac{5\pi^2}{24L} \tag{4.98}$$

of the Hamiltonian

$$\begin{aligned} H_{\mathrm{HS}} &= \left(\frac{2\pi}{L}\right)^2 \sum_{j<k}^{L} \frac{\mathbf{S}_j \cdot \mathbf{S}_k}{\left|\exp(i2\pi j/L) - \exp(i2\pi k/L)\right|^2} \\ &= \frac{\pi^2}{L^2} \sum_{j<k}^{L} \frac{\mathbf{S}_j \cdot \mathbf{S}_k}{\sin^2(\pi(j-k)/L)} \end{aligned} \tag{4.99}$$

as given in [36]. Here, $\mathbf{S}_j$ is an operator-valued 3-vector of spin-$\frac{1}{2}$ operators acting on site j and $\mathbf{S}_j \cdot \mathbf{S}_k$ is the quadratic Casimir acting on the tensor product of the fundamental representations at sites j and k. In order to derive c from eq. (4.98) we first have to determine v_s. To this end, we compare eq. (4.99) to the corresponding Hamiltonian stated in terms of the permutation operator $P_{jk} = 2\mathbf{S}_j \cdot \mathbf{S}_k + \frac{1}{2}$ in Haldane's review [27]:

$$H_{\mathrm{HS,Haldane}} = \frac{\pi v_s}{L^2} \sum_{j<k}^{L} \frac{P_{jk}}{\sin^2(\pi(j-k)/L)} = \frac{\pi v_s}{L^2} \sum_{j<k}^{L} \frac{2\mathbf{S}_j \cdot \mathbf{S}_k + \frac{1}{2}}{\sin^2(\pi(j-k)/L)} \tag{4.100}$$

$$= \frac{\pi v_s}{6L^2} \left(12 \sum_{j<k}^{L} \frac{\mathbf{S}_j \cdot \mathbf{S}_k}{\sin^2(\pi(j-k)/L)} + 3 \sum_{j<k}^{L} \frac{1}{\sin^2(\pi(j-k)/L)} \right). \tag{4.101}$$

The comparison of eq. (4.99) and eq. (4.101) yields $v_s = \frac{\pi}{2}$, and, consequently, we can recast eq. (4.98) into

$$E_0(L) = -\frac{\pi^2}{24}L - \frac{5}{2}\frac{\pi v_s}{6L}, \tag{4.102}$$

which, evidently by looking at eq. (4.97), implies $c = \frac{5}{2}$.[6]

However, a hint to the solution of this discrepancy is the fact that the normalization of the Hamiltonian H_{HS} is such that the nearest-neighbor interaction coincides with that of the SU(2) Heisenberg spin chain which can be seen by expanding the inverse sine-squared

[6] The second summand within the parentheses of eq. (4.101) does alter this puzzling result as it contributes $\frac{\pi v_s}{L^2}\frac{L^3-L}{12}$ to the ground state energy for Haldane's original expression in eq. (4.101), effectively cancelling the energy density per site, i.e. leading to $\epsilon_\infty = 0$. However, eq. (4.102) becomes even worse, now reading

$$E_{0,\mathrm{Haldane}}(L) = -3\frac{\pi v_s}{6L} \tag{4.103}$$

for the Hamiltonian given in [36]. However, simply subtracting a multiple of this term seems to be poorly motivated as this term is absolutely meaningless in terms of the dynamics of the system.

factor in eq. (4.99):

$$H_{\mathrm{HS}} = \frac{\pi^2}{L^2} \sum_{j<k}^{L} \mathbf{S}_j \cdot \mathbf{S}_k \left(\frac{L^2}{\pi^2 (j-k)^2} + \frac{1}{3} + \mathcal{O}\left(\frac{\pi^2 (j-k)^2}{L^2} \right) \right) \tag{4.104}$$

$$= \sum_{j<k}^{L} \mathbf{S}_j \cdot \mathbf{S}_k \left(\frac{1}{(j-k)^2} + \frac{\pi^2}{L^2} \mathcal{O}\left(\frac{\pi^2 (j-k)^2}{L^2} \right) \right) + \frac{\pi^2}{3L^2} \sum_{j<k}^{L} \mathbf{S}_j \cdot \mathbf{S}_k. \tag{4.105}$$

The first sum features the aforementioned Heisenberg interaction for nearest-neighbors, i.e. for $|j-k| = 1$, in the continuum limit. The second sum can be split up into two parts,

$$\frac{\pi^2}{3L^2} \sum_{j<k}^{L} \mathbf{S}_j \cdot \mathbf{S}_k = \frac{\pi^2}{6L^2} \sum_{j \neq k}^{L} \mathbf{S}_j \cdot \mathbf{S}_k \tag{4.106}$$

$$= \frac{\pi^2}{6L^2} \sum_{j,k}^{L} \mathbf{S}_j \cdot \mathbf{S}_k - \frac{\pi^2}{6L^2} \sum_{j}^{L} \mathbf{S}_j^2, \tag{4.107}$$

of which the first part is proportional to the total spin of a state and does not enter our discussion at this point as it is identical to zero on the singlet ground state and, as its prefactor $1/L^2$ suggests, only contributes to the scaling dimension of the respective excited state.

The second part, however, does contribute the very same energy to every state, be it the ground state or any excited state,

$$-\frac{\pi^2}{6L^2} \sum_{j}^{L} \mathbf{S}_j^2 = -\frac{\pi}{2} \frac{\pi}{3L^2} L\, \mathbf{S}^2 \tag{4.108}$$

$$= -\frac{v_s \pi}{6L} 2\mathbf{S}^2, \tag{4.109}$$

with $\mathbf{S}^2$ the value of the quadratic Casimir on the fundamental representation $\mathcal{V}$. Judging by its scaling behavior, this term contributes $2\mathbf{S}^2$ to the central charge, namely $2\mathbf{S}^2 = 2\frac{N^2-1}{2N} = \frac{3}{2}$. Having reinstated the speed of sound $v_s = \frac{\pi}{2}$ in eq. (4.109), we are now able to rewrite eq. (4.102) in a more reasonable form,

$$E_0(L) = \epsilon_\infty L - (1 + \frac{3}{2}) \frac{\pi v_s}{6L} = \epsilon_\infty L - c' \frac{\pi v_s}{6L}, \tag{4.110}$$

from which we indeed derive a central charge of $c = c' - \frac{3}{2} = 1$ for the effective low-energy part of the Hamiltonian in eq. (4.105), the part governing the dynamics by creating differences between different states, and leaving out the part given in eq. (4.109) that treats all states in the same way by simply shifting any state by the same constant in the spectrum. Indeed, one could say that this term simply captures the energy from self-interaction of every spin. Hence, as an *ad-hoc* regularization procedure, one may want to consider

subtracting this self-energy from H_{HS} which leaves us with

$$H_{\mathrm{HS,reg}} = \frac{\pi^2}{L^2}\sum_{j<k}^{L}\frac{\mathbf{S}_j\cdot\mathbf{S}_k}{\sin^2(\pi(j-k)/L)} - \left(-\frac{\pi^2}{6L^2}\sum_{j}^{L}\mathbf{S}_j^2\right) \tag{4.111}$$

$$= \sum_{j<k}^{L}\mathbf{S}_j\cdot\mathbf{S}_k\left(\frac{1}{(j-k)^2}+\frac{\pi^2}{L^2}\mathcal{O}\left(\frac{\pi^2(j-k)^2}{L^2}\right)\right)+\frac{\pi^2}{3L^2}\sum_{j<k}^{L}\mathbf{S}_j\cdot\mathbf{S}_k+\frac{\pi^2}{6L^2}\sum_{j}^{L}\mathbf{S}_j^2 \tag{4.112}$$

$$= \sum_{j<k}^{L}\mathbf{S}_j\cdot\mathbf{S}_k\left(\frac{1}{(j-k)^2}+\frac{\pi^2}{L^2}\mathcal{O}\left(\frac{\pi^2(j-k)^2}{L^2}\right)\right)+\frac{\pi^2}{6L^2}\sum_{j,k}^{L}\mathbf{S}_j\cdot\mathbf{S}_k, \tag{4.113}$$

where we made use of eq. (4.107) in the last line. This expression leads to the expected central charge of $c = 1$ for the free boson, nonetheless, it looks poorly motivated. However, we can do better,

$$H_{\mathrm{HS,reg}} = \frac{\pi^2}{2L^2}\sum_{j\neq k}^{L}\mathbf{S}_j\cdot\mathbf{S}_k\left(\frac{L^2}{\pi^2(j-k)^2}+\mathcal{O}\left(\frac{\pi^2(j-k)^2}{L^2}\right)\right)+\frac{\pi^2}{2L^2}\sum_{j,k}^{L}\frac{1}{3}\mathbf{S}_j\cdot\mathbf{S}_k \tag{4.114}$$

$$= \frac{\pi^2}{2L^2}\sum_{j,k}^{L}\mathbf{S}_j\cdot\mathbf{S}_k\left((1-\delta_{jk})\frac{L^2}{\pi^2(j-k)^2}+\mathcal{O}\left(\frac{\pi^2(j-k)^2}{L^2}\right)\right)+\frac{\pi^2}{2L^2}\sum_{j,k}^{L}\frac{1}{3}\mathbf{S}_j\cdot\mathbf{S}_k, \tag{4.115}$$

where, in order to adjust the range of summation for both sums, all we had to do is to regularize the first summand by $(1-\delta_{j,k})$ while all summands of higher order in the first sum are not affected. Now we can consolidate the Hamiltonian to

$$H_{\mathrm{HS,reg}} = \frac{\pi^2}{2L^2}\sum_{j,k}^{L}\mathbf{S}_j\cdot\mathbf{S}_k\left((1-\delta_{jk})\frac{L^2}{\pi^2(j-k)^2}+\frac{1}{3}+\mathcal{O}\left(\frac{\pi^2(j-k)^2}{L^2}\right)\right), \tag{4.116}$$

which, apart from the regularization $(1-\delta_{jk})$ for $j = k$, it contains the series expansion of $\frac{1}{\sin^2(\pi(j-k)/L)}$ which we might dub $\frac{1}{\sin^2_{\mathrm{reg}}(\pi(j-k)/L)}$. With this definition, we write the Hamiltonian of the Haldane-Shastry model as

$$H_{\mathrm{HS,reg}} = \frac{\pi^2}{2L^2}\sum_{j,k}^{L}\frac{\mathbf{S}_j\cdot\mathbf{S}_k}{\sin^2_{\mathrm{reg}}(\pi(j-k)/L)} \tag{4.117}$$

which has the expected scaling behavior of its ground state energy indicating a central charge of $c = 1$. Furthermore, in the sense explained earlier, it is the bare version of the Hamiltonian dynamics of the system, writ of any trivial contribution that would only shift all states in the spectrum by the same constant.

In the following Sections, we shall put the redefinition of $H_{\text{HS,reg}}$ to the test for the cases $\mathrm{SU}(N)$ and also the pure and alternating case for $\mathrm{GL}(1|1)$.

4.4.2 The Central Charge of the $\mathrm{SU}(N)$ Haldane-Shastry Spin Chain

We want to find out whether this analysis still holds in the general $\mathrm{SU}(N)$ case. To this end, first note that, in eq. (4.110), the central charge c is given by the sum of the central charge of the $SU(N)_1$ WZW model for $N = 2$, i.e. $c = N - 1 = 1$, and twice the quadratic Casimir eigenvalue on the fundamental representation of $\mathfrak{su}(N)$ for $N = 2$, i.e. $2\frac{N^2-1}{2N} = \frac{3}{2}$. Following our arguments above, the generalization of the ground state energy scaling in eq. (4.110) to the generic case of the unregularized $\mathrm{SU}(N)$ Haldane-Shastry spin chain should be of the form

$$E_0(N, L) = \epsilon_\infty(N)L - c_{\text{unreg}}(N)\frac{\pi v_s}{6L} \tag{4.118}$$

$$= \epsilon_\infty(N)L - (N - 1 + 2\frac{N^2-1}{2N})\frac{\pi v_s}{6L} \tag{4.119}$$

$$= \epsilon_\infty(N)L - \frac{(2N+1)(N-1)}{N}\frac{\pi v_s}{6L}. \tag{4.120}$$

In particular, for $N = 3$ this yields

$$E_0(N, L) = \epsilon_\infty(N)L - \frac{14\pi v_s}{18L} \tag{4.121}$$

which, upon substitution of $v_s = \pi/2$, agrees with the corresponding formula in [67] for the Hamiltonian given by

$$H_{\text{HS}} = \left(\frac{2\pi}{L}\right)^2 \sum_{j<k}^{L} \frac{\mathbf{J}_j \cdot \mathbf{J}_k}{\left|\exp(i2\pi j/L) - \exp(i2\pi k/L)\right|^2}$$

$$= \frac{\pi^2}{L^2} \sum_{j<k}^{L} \frac{\mathbf{J}_j \cdot \mathbf{J}_k}{\sin^2(\pi(j-k)/L)} \tag{4.122}$$

with the eight-dimensional $\mathfrak{su}(3)$-spin vector $\mathbf{J}_j$. Turning our argument around, we see that the scaling of the SU(3) Haldane-Shastry spin chain as given in eq. (4.121) is, indeed, due to a dynamical part that contributes $c = 3 - 1 = 2$ for the $SU(3)_1$ WZW model and a spin self-interaction term that contributes $c = \frac{3^2-1}{3} = \frac{8}{3}$ to the central charge $c_{\text{unreg}} = \frac{14}{3}$ and that can safely be subtracted by regularizing the respective system.

In order to verify our prediction of the scaling of the ground state energy given in eq. (4.120) in the case of generic N treated by Kawakami in [68, 69], we rephrase the Hamiltonian in terms of flip operators $e_j^{\sigma\tau} := c_{j,\sigma}^\dagger c_{j,\tau}$ (as done for SU(3) in [67]). To this end

note that, for $\mathfrak{su}(N)$-spins $\mathbf{S}_i$, we have

$$2(\mathbf{S}_j \cdot \mathbf{S}_k) = \sum_{\sigma,\tau=1}^{N} e_j^{\sigma\tau} e_k^{\tau\sigma} - \frac{1}{N} \tag{4.123}$$

where the first term on the right-hand side corresponds to the permutation operator acting on sites j and k:

$$P_{jk} := \sum_{\sigma,\tau=1}^{N} e_j^{\sigma\tau} e_k^{\tau\sigma}. \tag{4.124}$$

With this, we rewrite the Hamiltonian

$$H_{\mathrm{HS}} = \frac{\pi^2}{L^2} \sum_{j<k}^{L} \frac{\mathbf{S}_j \cdot \mathbf{S}_k}{\sin^2(\pi(j-k)/L)} \tag{4.125}$$

$$= \frac{\pi^2}{2L^2} \sum_{j<k}^{L} \sum_{\sigma,\tau=1}^{N} \frac{e_j^{\sigma\tau} e_k^{\tau\sigma}}{\sin^2(\pi(j-k)/L)} - \frac{\pi^2}{2L^2} \sum_{j<k}^{L} \frac{1/N}{\sin^2(\pi(j-k)/L)} \tag{4.126}$$

$$= \frac{\pi v_s}{L^2} \sum_{j<k}^{L} \sum_{\sigma,\tau=1}^{N} \frac{e_j^{\sigma\tau} e_k^{\tau\sigma}}{\sin^2(\pi(j-k)/L)} - \frac{\pi v_s}{NL^2} C(L) \tag{4.127}$$

where we introduced $v_s = \frac{\pi}{2}$ in the last line. As Kawakami only regards the first term in his analysis, we have to add the contribution of $\frac{\pi v_s}{NL^2} C(L)$ from eq. (4.120):

$$E_0'(N,L) = \epsilon_\infty(N) L - \frac{(2N+1)(N-1)}{N} \frac{\pi v_s}{6L} + \frac{\pi v_s}{NL^2} C(L) \tag{4.128}$$

$$= \epsilon_\infty(N) L - \frac{2N^2 - N - 1}{N} \frac{\pi v_s}{6L} + \frac{\pi v_s}{NL^2} \frac{L^3 - L}{6} \tag{4.129}$$

$$= \epsilon_\infty'(N) L - \frac{2N^2 - N}{N} \frac{\pi v_s}{6L} \tag{4.130}$$

$$= \epsilon_\infty'(N) L - (2N-1) \frac{\pi v_s}{6L}. \tag{4.131}$$

With $\epsilon_\infty'(N) = \frac{\pi^2}{12} \frac{2-N}{N}$ and $v_s = \frac{\pi}{2}$, this coincides with the ground state energy given by Kawakami in [69],

$$E_0'(N,L) = \frac{\pi^2}{12} \frac{2-N}{N} L - (2N-1) \frac{\pi^2}{12L}, \tag{4.132}$$

for generic N which was first found by Kawakami by means of asymptotic Bethe ansatz in [68] and stated explicitly in [69]. Of course, our regularization procedure works analogously on Kawakami's Hamiltonian

$$H_{\mathrm{HS}}' = \frac{\pi v_s}{L^2} \sum_{j<k}^{L} \sum_{\sigma,\tau=1}^{N} \frac{e_j^{\sigma\tau} e_k^{\tau\sigma}}{\sin^2(\pi(j-k)/L)} \tag{4.133}$$

$$= \frac{\pi v_s}{2L^2} \sum_{j,k}^{L} \sum_{\sigma,\tau=1}^{N} \frac{e_j^{\sigma\tau} e_k^{\tau\sigma}}{\sin^2_{\text{reg}}(\pi(j-k)/L)} - \frac{\pi v_s}{2L^2} \frac{1}{3} \sum_{j=1}^{L} \sum_{\sigma,\tau=1}^{N} e_j^{\sigma\tau} e_j^{\tau\sigma}. \tag{4.134}$$

To see this, let us first calculate the unknown part of the correction term (omitting the site index j):

$$\sum_{\sigma,\tau=1}^{N} e^{\sigma\tau} e^{\tau\sigma} = \sum_{\sigma\neq\tau}^{N} c_\sigma^\dagger c_\tau c_\tau^\dagger c_\sigma + \sum_{\sigma=1}^{N} c_\sigma^\dagger c_\sigma c_\sigma^\dagger c_\sigma \tag{4.135}$$

$$= \sum_{\sigma\neq\tau}^{N} (1 - n_\tau) n_\sigma + \sum_{\sigma=1}^{N} n_\sigma^2 \tag{4.136}$$

$$= \sum_{\sigma=1}^{N} \big((N-1) n_\sigma + n_\sigma\big) \tag{4.137}$$

$$= N. \tag{4.138}$$

Here we defined the particle number operator $n_\sigma := c_\sigma^\dagger c_\sigma$ and made use of $n_\sigma^2 = n_\sigma$, $\sum_\tau n_\tau = 1$ and the usual anticommutation relations for the fermionic operators c_σ and $c_\sigma^\dagger$. Hence, we find

$$H'_{\text{HS}} = \frac{\pi v_s}{2L^2} \sum_{j,k}^{L} \sum_{\sigma,\tau=1}^{N} \frac{e_j^{\sigma\tau} e_k^{\tau\sigma}}{\sin^2_{\text{reg}}(\pi(j-k)/L)} - \frac{\pi v_s}{6L} N \tag{4.139}$$

and so the scaling of the regularized Hamiltonian

$$H'_{\text{HS,reg}} = \frac{\pi v_s}{2L^2} \sum_{j,k}^{L} \sum_{\sigma,\tau=1}^{N} \frac{e_j^{\sigma\tau} e_k^{\tau\sigma}}{\sin^2_{\text{reg}}(\pi(j-k)/L)} \tag{4.140}$$

is given by

$$E'_{0,\text{reg}}(N, L) = \frac{\pi v_s}{6} \frac{2-N}{N} L - (N-1) \frac{\pi v_s}{6L} \tag{4.141}$$

leading to the correct prediction of the central charge $c = N - 1$.

Finally, note that this regularized Haldane-Shastry Hamiltonian can be written more compactly and independent of the actual spins involved by introducing the permutation operator $P_{jk} = \sum_{\sigma,\tau=1}^{N} e_j^{\sigma\tau} e_k^{\tau\sigma}$,

$$H'_{\text{HS,reg}} = \frac{\pi v_s}{2L^2} \sum_{j,k}^{L} \frac{P_{jk}}{\sin^2_{\text{reg}}(\pi(j-k)/L)}, \tag{4.142}$$

and it has the desired scaling behavior as given in eq. (4.141).

4.4.3 The Central Charge of the $\mathrm{GL}(1|1)$ Haldane-Shastry Spin Chains

Next, let us put the pure and alternating GL(1|1)-invariant Haldane-Shastry spin chain to the test. Since we have $N = 0$, the regularized Hamiltonian and its precursor both yield the same scaling. The pure case was discussed in Section 2.2 and its Hamiltonian in terms of the graded permutation operator P_{jk} is given by

$$H_{\mathrm{HS,pure}} = \frac{\pi v_s}{L^2} \sum_{j<k}^{L-1} \frac{P_{jk}}{\sin^2((j-k)\pi/L)} \tag{4.143}$$

$$= \frac{\pi v_s}{L^2} \left(\sum_{j<k}^{L-1} \frac{P_{jk}-1}{\sin^2((j-k)\pi/L)} + \sum_{j<k}^{L-1} \frac{1}{\sin^2((j-k)\pi/L)} \right) \tag{4.144}$$

$$= \frac{\pi v_s}{L^2} \left(-2 \sum_{w=0}^{L-1} \tilde{c}_w^\dagger \tilde{c}_w (L-w)w + C(L) \right) \tag{4.145}$$

with the same definition of the (length-dependent but otherwise) constant offset

$$C(L) := \sum_{j<k}^{L-1} \frac{1}{\sin^2((j-k)\pi/L)} = \frac{L(L+1)(L-1)}{6}. \tag{4.146}$$

As the ground state is given by the completely filled state (in terms of the fermionic momentum creation operators $\tilde{c}_w^\dagger$) and due to

$$\sum_{w=0}^{L-1} (L-w)w = \frac{L^3-L}{6} = C(L), \tag{4.147}$$

we find the scaling of the ground state energy to be

$$E_0(L) = \frac{\pi v_s}{L^2} \left(-2C(L) + C(L)\right) = -\frac{\pi v_s}{L^2} C(L) \tag{4.148}$$

$$= -\frac{\pi v_s}{6} L - (-1)\frac{\pi v_s}{6L} \tag{4.149}$$

and therefore a central charge of -1. As explained in Section 2.2, this result is understood as an effective central charge of the Ising CFT ($c = \frac{1}{2}$) restricted to the block of conformal dimension $(\frac{1}{16}, \frac{1}{16})$, i.e. $\tilde{c} = \frac{1}{2} - 24\frac{1}{16} = -1$.

The alternating Haldane-Shastry model of $\mathfrak{gl}(1|1)$-spins was discussed in Section 4.1. Its Hamiltonian could be given in terms of the graded permutation operator P_{ij} for $d_i = d_j$ and the Temperly-Lieb operator T_{ij} for $d_i \neq d_j$. However, for $\mathfrak{gl}(1|1)$-spins $\mathbf{S}_i$, there is no difference to the quadratic Casimir $\mathbf{S}_i \cdot \mathbf{S}_j$. Therefore, we start with the expression for a

system of $2L$ spins given in eq. (4.4),

$$H_{\text{HS,alt}} = \frac{\pi^2}{2(2L)^2} \sum_{i \neq j}^{2L} \frac{\mathbf{S}_i \cdot \mathbf{S}_j}{\sin^2\left((i-j)\pi/2L\right)} = \frac{\pi^2}{(2L)^2} \sum_{i<j}^{2L} \frac{\mathbf{S}_i \cdot \mathbf{S}_j}{\sin^2\left((i-j)\pi/2L\right)}, \tag{4.150}$$

in order to make it comparable to the Hamiltonian in eq. (4.99) from which we also derive that we also have $v_s = \frac{\pi}{2}$. The scaling of this Hamiltonian can be induced from what is known for $H_{\text{HS,alt}}$ given in eq. (4.68), i.e.

$$H_{\text{HS,alt}} = \frac{\pi^2}{2L^2} \sum_{p=1}^{L-1} (L-p)p(a_p^\dagger a_p + A_p^\dagger A_p) - \frac{\pi^2}{2L^2} C(L) \tag{4.151}$$

which leads to a ground state scaling of

$$E_0(L) = -\frac{\pi^2}{2L^2} C(L) = -\frac{\pi v_s}{L^2} \frac{L^3 - L}{6} \tag{4.152}$$

$$= -\frac{\pi v_s}{6} L + \frac{\pi v_s}{6L} \tag{4.153}$$

$$= -\frac{\pi v_s}{12}(2L) - (-2)\frac{\pi v_s}{6(2L)}. \tag{4.154}$$

This is consistent with what was expected for the symplectic fermion CFT, i.e. $c = -2$.

5

The Two-Parameter Spin Ladder

And as to me, I know nothing else but miracles.

– Walt Whitman, *Leaves of Grass*

Numerical simulations of variations of the alternating one-dimensional setup for the case of $\mathfrak{gl}(1|1)$-spins, which has been treated in Chapter 4, suggested the investigation of a two-parameter spin ladder setup as its spectrum appeared to be rather well-behaved. Indeed, it became clear that the full spectrum can be represented in terms of analytic expressions, i.e. we can – by observation and all the numerical support available – state all generalized eigenvalues and their degeneracies of the Hamiltonian defined on a bipartite lattice parametrized not only by its total number of sites $2L \in 2\mathbb{N}$ but, additionally, by the quotient $r \in \mathbb{R}_{\geq 0}$ of two radii and an angular parameter α. This is remarkable as, in the case of generic values for r and α, the respective Hamiltonian features long-range three-spin interactions which render it extremely hard to diagonalize analytically – if not impossible altogether.

We will see that such a system built from $\mathfrak{gl}(1|1)$-spins will be described in terms of two mutually anticommuting sets of fermions, represented by their creation operators $a_p^\dagger(r,\alpha)$ and $A_p^\dagger(r,\alpha)$ with p running from 0 to $L-1$. Each of these fermionic modes contributes the one-particle energy

$$\varepsilon_{p,L}(r,\alpha) := \frac{\pi^2}{2L^2}\left(p(L-p) - L\frac{pr^{-L} - Lr^{L-2p} + (L-p)r^L}{r^{-L} + 2\cos(L\alpha) + r^L}\right). \tag{5.1}$$

Whether the two zero modes $a_0^\dagger(r,\alpha)$ and $A_0^\dagger(r,\alpha)$ introduce a nilpotent part in the Hamiltonian – as one might expect for a globally $\mathrm{GL}(1|1)$-invariant alternating system – depends on the actual choice of r and α.

In the following Sections, we will first describe this special setup (for the generic Hamiltonian constructed in Chapter 3) which, by itself, is not bound to the exclusive use of $\mathfrak{gl}(1|1)$-spins. Next, it will be explained how the – for $\mathfrak{gl}(1|1)$-spins particularly simple – spectrum was found to be explained in terms of the one-particle energies given by the analytic expressions in eq. (5.1). Then, we will analyze the structure of the spectrum and discuss its behavior in the continuum limit.

5.1 The Setup

The two-parameter spin ladder setup is constructed by starting with a state space $\mathcal{H}_{\text{alt}}$ of L spins transforming in the fundamental representation (i.e. $\mathcal{V}_j = \mathcal{V}$ for $j \in I_A = \{2, 4, \ldots, 2L\}$) and L spins transforming in the antifundamental representation (i.e. $\mathcal{V}_j = \bar{\mathcal{V}}$ for $j \in I_B = \{1, 3, \ldots, 2L-1\}$):

$$\mathcal{H}_{\text{alt}} = (\bar{\mathcal{V}} \otimes \mathcal{V})^{\otimes L}. \tag{5.2}$$

As illustrated in Figure 5.1 for a total system size of $2L = 8$, it deviates from the alternating setup of spins distributed fully uniformly on the unit circle ($z_j = \exp\left(i\frac{2\pi j}{2L}\right)$ for all $j \in I_A \cup I_B$)[1] by extending the range of values for the positions of the spins in the following way:

$$z_j = r_A \exp\left(i\left(\frac{\pi j}{L} + \alpha_A\right)\right) \quad \text{for all } j \in I_A, \tag{5.3}$$

$$z_j = r_B \exp\left(i\left(\frac{\pi j}{L} + \alpha_B\right)\right) \quad \text{for all } j \in I_B. \tag{5.4}$$

The parameters r_A, r_B, α_A and α_B may in principle take any real value. However, special attention is in order for those values at which the sublattices I_A and I_B coincide. We will return to this issue in the discussion of the spectrum for $\mathfrak{gl}(1|1)$-spins in Section 5.3.2.

With this choice, the sublattice I_A (I_B) makes up a system of L spins of one kind distributed uniformly on a circle centered about the origin of radius r_A (r_B). It is easy to see that, due to the particular symmetry of this special setup, it is equivalent to one with only two degrees of freedom, e.g.

$$z_j = r \exp\left(i\left(\alpha_0 j + \alpha\right)\right) \quad \text{for all } j \in I_A, \tag{5.5}$$

$$z_j = \exp\left(i\alpha_0 j\right) \quad \text{for all } j \in I_B \tag{5.6}$$

with the definitions

$$r := \frac{r_A}{r_B} \quad \in \quad \mathbb{R}_{\geq 0}, \tag{5.7}$$

$$\alpha_0 := \frac{\pi}{L}, \tag{5.8}$$

$$\alpha := \alpha_A - \alpha_B \quad \in \quad]-\alpha_0, \alpha_0]\,.^2 \tag{5.9}$$

This becomes particularly apparent when mapping the complex plane to the infinitely long cylinder of circumference $2L$ via $z \mapsto w = \ln z$: The two sublattices of the different

[1] This is the setup for the discussion in case of $\mathfrak{gl}(1|1)$-spins presented in Chapter 4.

[2] Restricting the domain of α simplifies the discussion as the spin ladder setup is translationally invariant under shifting α to $\alpha + 2\alpha_0\mathbb{Z}$.

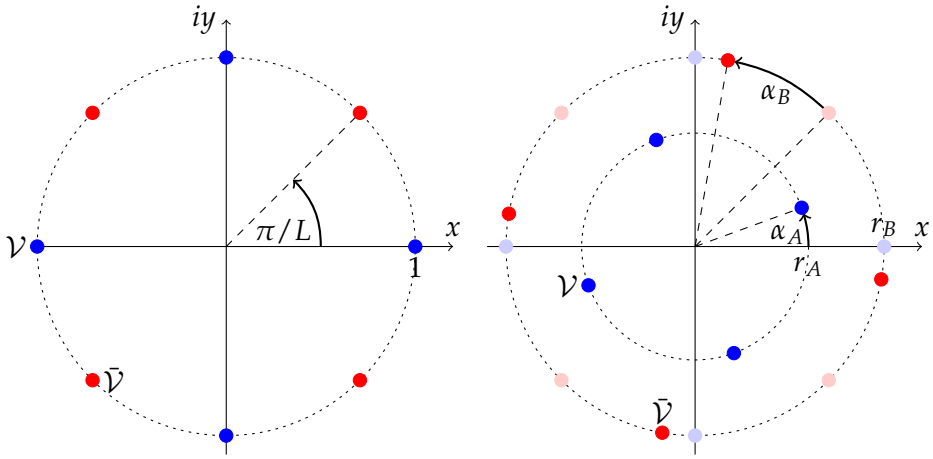


Figure 5.1: This figure shows an exemplary picture of the spin ladder setup for a system consisting of $L = 4$ spins transforming in the fundamental representation (blue points) and $L = 4$ spins transforming in the antifundamental representation (red points). The left-hand side features the Haldane-Shastry-like setup of fully uniformly distributed sites on the unit circle. This can be considered as a special case of the more general spin ladder setup illustrated on the right-hand side: The sites of each sublattice of one kind of spin are uniformly distributed on concentric circles of radii r_A or r_B, respectively. Furthermore, each sublattice might be shifted with respect to its original position (fainter points) by some angle α_A or α_B, respectively.

radii r_A and r_B are then two parallel circular sublattices along the circumference of the cylinder separated by the longitudinal distance $\ln r_A - \ln r_B = \ln \frac{r_A}{r_B} = \ln r$. On an infinitely long cylinder, the quantities $\ln r_A$ and $\ln r_B$ are defined with respect to some discretionarily chosen zero which itself has no physical meaning. Therefore, only their difference $\ln r$ can be ascribed physical relevance. Alternatively, we might say that for a lack of length scale (the w_{ij} are dimensionless) of the systems constructed, we may choose to measure all distances of the system at hand in units of r_B which leads to the definition $r_B := 1$ and in effect to $r = r_A$. In the picture of the infinitely long cylinder, this corresponds to not arbitrarily choosing the position of the origin along the cylinder but making the choice $\ln r_B := \ln 1 = 0$. A similar argument can be brought into place for the consolidation of the two angular parameter α_A and α_B to α.

By the same token and keeping in mind that fundamental and antifundamental representations are exchangeable[3] the spin ladder is invariant under the mapping $z \mapsto 1/z$. Thus, the system should be invariant under this transformation. In particular by this argument, it is expected that the spectrum (as a whole) of the spin ladder be invariant under such transformations. E.g. mapping z to its inverse z^{-1}, maps I_B to itself whereas the points of

[3] The structures we consider here are inherently invariant under exchange of particles and antiparticles unlike, e.g. what physicists know from actual particle physics where the weak force breaks this symmetry.

I_A are mapped to

$$z_j = \frac{1}{r}\exp\Big(i\left(-\alpha_0 j - \alpha\right)\Big) \quad \text{for all } j \in I_A. \tag{5.10}$$

For the sign change in α brought about by this transformation, left- and right-moving modes will be exchanged while the full spectrum should be invariant. This will be explicitly shown for the GL(1|1)-invariant system in Section 5.3.2. However, based on the generality of this argument, it should hold for spin ladders built from $\mathfrak{su}(N)$- or $\mathfrak{gl}(m|n)$-spins for arbitrary values of N or m and n, respectively.

Finally, while the coupling between spins of the same kind (so within each sublattice I_A or I_B) is fixed (as their relative location is), the coupling between spins of distinct type will be mediated by variations in r and α. It is in this sense that this kind of system may be interpreted as a spin ladder.

5.2 The $\mathfrak{gl}(m|m)$-Spin Ladder

For the systems constructed in Chapter 3 with global GL($m|n$)-invariance, it was explicitly demonstrated in Section 3.7, that for $m = n$ the part of the Hamiltonian restricted to a circular sublattice becomes independent of the radius of this circle. By virtue of this, the GL($m|m$)-invariant Hamiltonian $H(r,\alpha)$ splits into three parts, two parts H_A and H_B independent of r and α stemming from the restriction to the sublattices I_A and I_B, respectively, and one part $H_{\text{mix}}(r,\alpha)$ depending on r and α stemming from all mixed terms:

$$H(r,\alpha) = H_A + H_B + H_{\text{mix}}(r,\alpha). \tag{5.11}$$

More precisely, H_A will be of a form analogous to the one derived in eq. (3.101), i.e.

$$\frac{2(2L)^2}{\pi^2}H_A = g_A\mathbb{I} - L\,\mathbf{S}_A^2 + 2\sum_{\substack{i<j\\ i,j\in I_A}}\frac{\mathbf{S}_i\cdot\mathbf{S}_j}{\sin^2\left((i-j)\pi/(2L)\right)}, \tag{5.12}$$

with the index A indicating the restriction to the sublattice A (as in Chapter 4, we renormalize the Hamiltonian by $\frac{\pi^2}{2(2L)^2}$ in order to match the usual conventions of the Haldane-Shastry model). H_B is of corresponding form. Hence, such a system can be interpreted as two pure Haldane-Shastry spin chains of opposite type (one consisting of fundamental the other of antifundamental spins) that are coupled to each other via the interaction mediated by $H_{\text{mix}}(r,\alpha)$ which is of the more complicated form given in eq. (3.85), namely

$$\frac{2(2L)^2}{\pi^2}H_{\text{mix}}(r,\alpha) = g^0_{\text{mix}}\,\mathbb{I} + \sum_{i<j} g^0_{ij,\text{mix}}\mathbb{Q}_{ij} + \sum_{i<j<k} g^A_{ijk,\text{mix}}\Big[\mathbb{Q}_{ij},\mathbb{Q}_{jk}\Big] \tag{5.13}$$

with the subscript "mix" indicating that pure terms with all site indices from a single sublattice are set to zero. Also, we recall that we set $\mathbb{Q}_{ij} := (-1)^{d_i+d_j}\mathbf{S}_i\cdot\mathbf{S}_j$. This part of

the full Hamiltonian drastically simplifies in three cases of which two can be considered equivalent (by invariance under inversion $z \mapsto 1/z$). These two cases are realized for the limiting values of r, i.e. $r \to 0$ on the one hand and $r \to \infty$ on the other because in either case the coordinate dependence w_{ij} for i and j from the two different sublattices becomes quasi-trivial in either limit,

$$w_{ij} = \frac{z_i + z_j}{z_i - z_j} \longrightarrow \pm 1 \quad \text{for} \quad d_i \neq d_j, \tag{5.14}$$

and, therefore, insensitive to any change in α, i.e.

$$\lim_{r\to 0} H(r, \alpha) = \lim_{r\to 0} H(r, \beta) \tag{5.15}$$

for any choice of angular parameters α and β. The same holds true for the respective expression in the limit of $r \to \infty$.

The third case is given by $r = 1$ with α chosen in such a way that the two sublattices do not coalesce. As was mentioned in Section 3.7, this choice particularly simplifies the Hamiltonian as it leads to the vanishing of any contribution of the coefficients g^A_{ijk} (and thus of $g^A_{ijk,\mathrm{mix}}$). As a consequence, in this case, the Hamiltonian is limited to spin-spin interactions instead of also exhibiting the otherwise ubiquitous three-spin interaction terms for more generically distributed lattice sites. However, now it is not the Hamiltonian that is invariant under changes of α but only its spectrum. Exact diagonalization studies indicate that, e.g. for $\mathfrak{gl}(1|1)$-spins and at $r = 1$, it incorporates a nilpotent part which renders it only almost diagonalizable. This is due to the existence of two zero modes that are also present in the alternating Haldane-Shastry chain of $\mathfrak{gl}(1|1)$-spins. These are only perfectly canceled in $H(r, \alpha)$ by the term proportional to the total spin of the system if we also choose uniform angular distance of the sites in our model (cf. the analysis in Section 4.2 and the discussion in Section 4.3).

5.3 The $\mathfrak{gl}(1|1)$-Spin Ladder

Now we concretize our discussion of the aforementioned spin ladder setup to the case of $\mathfrak{gl}(1|1)$-spins which will turn out to be special for mainly two reasons. The first reason is furnished simply by the fact that the dependence of the energy levels of this model with r and α is so well-behaved that, after a slightly lengthy analysis of the numerical data of the full spectrum for system sizes $2L = 4, 6, 8$, it was possible to make a conjecture about the general analytic expressions for all energy levels $E_{j,L}(r, \alpha)$. Furthermore, these conjectured expressions for $E_{j,L}(r, \alpha)$ were found to perfectly match exact diagonalization accessible to contemporary workstations (up to $2L = 12$).

The second reason is that the analysis of the spectrum puts the fully uniform setup $H(1, 0)$ (discussed in Section 4.2) into the right perspective: Only in the fully uniform setup will the Hamiltonian be fully diagonalizable. Numerical observations show that

the slightest deviation from $\alpha = 0$ introduces a nilpotent part that can only be attributed to a term proportional to the product of the two zero modes that are also responsible for the omnipresent factor of four in the degeneracy pattern of the spectrum. This will switch the universality class of the system from two decoupled copies of the Ising CFT in the disorder sector (without the nilpotent part) to the symplectic fermion theory (with the nilpotent part). Further, in the thermodynamic limit $2L \to \infty$, we will see that any deviation from $r = 1$ will, in principle, allow for the release of an arbitrarily high amount of energy and the system will immediately switch from the antiferromagnetic phase at $r = 1$ to a ferromagnetic phase at $r \neq 1$. The reason for this is that, as we will learn in the following, when we compare the low-energy regime of the system at $r = 1$ and $r \neq 1$, adding single-particle excitation to the system in the former case increases the energy while it decreases the energy in the latter case. Thus, while breaking up the singlet state $|\Psi\rangle$, constructed in Section 3.8, bit by bit by adding single-particle excitations, the system for $r = 1$ behaves like one expects from an antiferromagnetic system, whereas, at $r \neq 1$, it exhibits ferromagnetic behavior. We will come back to this point in the discussion of the continuum limit in Section 5.3.4.

5.3.1 Establishing Analytic Expressions for the Full Spectrum

As previously mentioned, the $\mathfrak{gl}(1|1)$-spin ladder setup first moved into the spotlight when exact diagonalization indicated that the Hamiltonian for L fundamental and L antifundamental $\mathfrak{gl}(1|1)$-spins randomly distributed in the complex plane could acquire negative eigenvalues leading to the situation that the seed state (with eigenvalue zero by construction) was no longer the ground state. In fact, it was observed that there are setups were the whole spectrum can become non-positive.

Focusing on the r-Dependence

In order to investigate this situation, we started with the spin ladder setup described in the previous Section varying only r and keeping $\alpha = 0$ fixed. It was found that the first excited state of $H(1,0)$ crosses zero to become negative when lowering r to the values tabulated in Table 5.1. Due to fortunate circumstances,[4] these were identified as the positive roots of

$2L$	4	6	8	10	12
r_0	$\sqrt{2}-1$	0.5960716	0.6925048	0.7519264	0.7921520

Table 5.1: Zeros smaller than $r = 1$ of the first excited state of $H(1,0)$.

[4] At this point, Dominik Ostermayr, Daniel Wieczorek and particularly Wolfgang Palzer and *Wolfram | Alpha* shall be mentioned for assisting me in achieving this result. Therefore, the series of zeros of the first excited state of $H(1,0)$ shall be dubbed *POW | Alpha*.

$2L$	$E_{j,L}(r,0)/(\frac{\pi^2}{2L^2})$		
4	0	$-\frac{1-6r^2+r^4}{(1+r^2)^2}$	$-2\frac{1-6r^2+r^4}{(1+r^2)^2}$
6	0	$-\frac{4-8r+3r^2-2r^3+r^4}{(1-r+r^2)^2}$	$-2\frac{4-8r+3r^2-2r^3+r^4}{(1-r+r^2)^2}$
	$-\frac{1-2r+3r^2-8r^3+4r^4}{(1-r+r^2)^2}$	$-\frac{5-10r+6r^2-10r^3+5r^4}{(1-r+r^2)^2}$	$-\frac{9-18r+9r^2-12r^3+6r^4}{(1-r+r^2)^2}$
	$-2\frac{1-2r+3r^2-8r^3+4r^4}{(1-r+r^2)^2}$	$-\frac{6-12r+9r^2-18r^3+9r^4}{(1-r+r^2)^2}$	$-2\frac{5-10r+6r^2-10r^3+5r^4}{(1-r+r^2)^2}$

Table 5.2: Tabulated are the analytic expressions of all eigenvalues of $H(r,0)$ for system sizes $2L = 4, 6$ as calculated by *Mathematica*. While, for $2L = 4$, they are simply multiples of the expression in the middle of the row (i.e. contributing zero, once or twice), for $2L = 6$, they are linear combinations of the expression in the middle of the first row and the first column with coefficients 0, 1 or 2.

the polynomial $x^L + Lx + 1 - L$. This already seemed to be too much structure to be left unexplored.

The next step consisted of plotting the full spectrum of $H(r,0)$ for various system sizes. Figure 5.2 shows some exemplary plots of the four smallest non-trivial system sizes. These revealed an impressive amount of structure paired with the apparent regularity of the dependence of the single energy levels with respect to r. After some improvements of the *Mathematica*-implementation of the model, it was even possible to recover the analytic expressions of the energy levels for system sizes $2L = 4$ and 6 in terms of r with $\alpha = 0$. They are given in Table 5.2. By inspection it was clear that some of them come in pairs related by a factor of two whereas others do not. Further, for $2L = 6$ it can be seen that the energy levels in the center of the first row and column play the role of one-particle energies $\varepsilon_{p,L}(r,0)$ while the remaining levels are built up from integer linear combinations of these one-particle energies, each one of the one-particle energies contributing once, twice or not at all. This is in perfect accordance with the nature of $H(1,0)$ which, in 4.2, was shown to be given by two mutually decoupled sets of L fermionic operators $a_p^\dagger$ and $A_p^\dagger$. With this insight, it was possible to see that this behavior is inherited by all energy levels $E_{j,L}(r,0)$ throughout the whole domain of r.

However, in order to constrain the general form of these one-particle energies $\varepsilon_{p,L}(r,0)$ on the basis of Table 5.2, one thing was crucial: The factor $(x^L + Lx + 1 - L)$ should be present in exactly one of them at every system size. At $2L = 4$, this is easily seen:

$$-\frac{\pi^2}{2L^2}\frac{1-6r^2+r^4}{(1+r^2)^2} = -\frac{\pi^2}{2L^2}\frac{(r^2-2r-1)(r^2+2r-1)}{(1+r^2)^2}. \tag{5.16}$$

Moreover, this factorization reveals the polynomial factor responsible for the second zero of the one-particle energy which lies above $r = 1$ at $\frac{1}{r_0} = \frac{1}{\sqrt{2}-1} = \sqrt{2}+1$ (cf. the first plot in Figure 5.2).

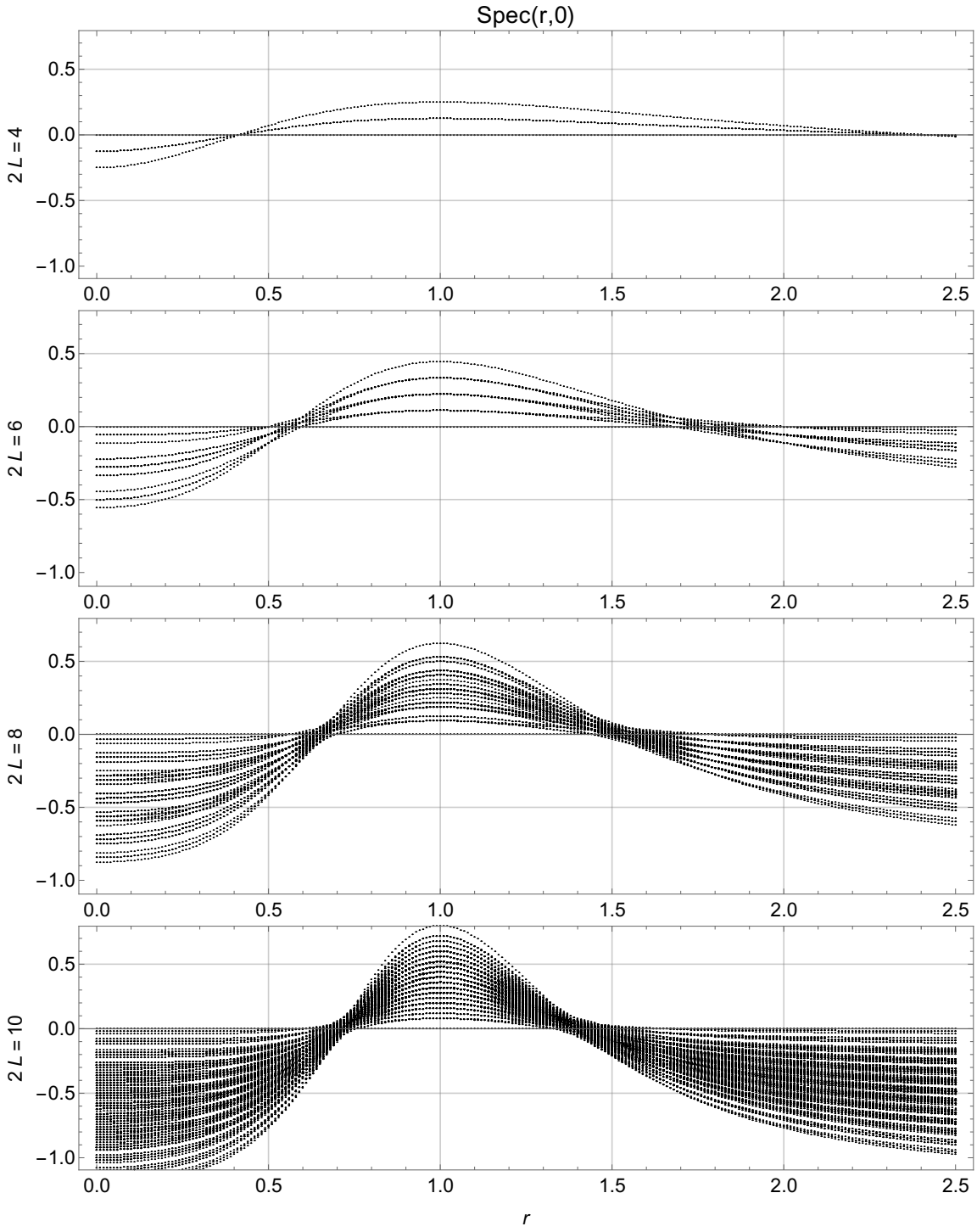


Figure 5.2: The plots show the data points from the exact diagonalization of the Hamiltonian $H(r,0)$ (in units of π^2) at the four distinct system sizes $2L = 4, 6, 8, 10$ as indicated to the left of each plot. The axes are chosen such that the plots are directly comparable.

The analogous factorization of the first one-particle energy at length $2L = 6$ leads to

$$-\frac{\pi^2}{2L^2}\frac{4-8r+3r^2-2r^3+r^4}{(1-r+r^2)^2} = -\frac{\pi^2}{2L^2}\frac{(r-2)(r^3+3r-2)}{(1-r+r^2)^2} \tag{5.17}$$

which does come with the expected factor (r^3+3r-2) but breaks the duality between r_0 and $1/r_0$ encountered in eq. (5.16). This symmetry, however, should anyway exchange left- and right-movers. Therefore, when looking at the factorization of the other one-particle energy at length $2L = 6$ we find

$$-\frac{\pi^2}{2L^2}\frac{1-2r+3r^2-8r^3+4r^4}{(1-r+r^2)^2} = -\frac{\pi^2}{2L^2}\frac{(2r-1)(2r^3-3r^2-1)}{(1-r+r^2)^2} \tag{5.18}$$

which resolves the question of the duality of the zeros ($2 \mapsto 1/2$ and $(1+\sqrt{2})^{1/3} - (1+\sqrt{2})^{-1/3} \mapsto (1+(3-2\sqrt{2})^{1/3}+(3+2\sqrt{2})^{1/3})/2$, i.e. $0.5960716 \mapsto 1.6776507$). Finally, in order to lift the veil on the general r-dependence of the one-particle energies, one last ingredient was necessary: enhancing the expressions of the one-particle energies in Table 5.2 for system size $2L = 6$ by $(1+r)^2$ (in order to have $(1-r+r^2)^2(1+r)^2 = (1+r^3)^2$) to the expressions given in Table 5.3. By the uniformity of expressions, this suggests a general rule for building the analytic expressions of the one-particle energies of any system size $2L$. These take the form

$$\varepsilon_{p,L}(r,0) := -\frac{\pi^2}{2L^2}\frac{((L-p)r^L - Lr^{L-p} - p)((L-p)r^L + Lr^{L-p} - p)}{(1+r^L)^2} \tag{5.19}$$

for all values of $p = 1, \ldots, L-1$. The actual energy levels are then given by the formal scalar product

$$E_{j,L}(r,0) := \mathbf{j} \cdot \boldsymbol{\varepsilon}_L(r,0) \tag{5.20}$$

with

$$\mathbf{j} \in \mathbb{J}_L := \{0,1,2\}^{L-1} \tag{5.21}$$

and the components of the vector $\boldsymbol{\varepsilon}_L(r,0)$ given by the one-particle energies $\varepsilon_{p,L}(r,0)$. The form of the vector $\mathbf{j}$ simply reflects the fact that the system is apparently described by two sets of $L-1$ fermionic creation operators, $a_p^\dagger(r,0)$ and $A_p^\dagger(r,0)$ for $p = 1, \ldots, L-1$

$2L$	$\varepsilon_{p,L}(r,0)/(\frac{\pi^2}{2L^2})$	
4	$-\frac{(r^2-2r-1)(r^2+2r-1)}{(1+r^2)^2}$	
6	$-\frac{(r^3-3r-2)(r^3+3r-2)}{(1+r^3)^2}$	$-\frac{(2r^3-3r^2-1)(2r^3+3r^2-1)}{(1+r^3)^2}$

Table 5.3: The one-particle energies of $H(r,0)$ for system sizes $2L = 4, 6$ are arranged in a form particularly suitable for guessing their structure.

associated with these one-particle energies at generic values of r, and not only at $r = 1$. Furthermore, due to the degeneracies of the spectrum which are always integer multiples of four, there must be two zero modes $a_0^\dagger(r,0)$ and $A_0^\dagger(r,0)$ in addition. Of course, when there is the freedom to choose either the operator $a_p^\dagger(r,0)$ *or* $A_p^\dagger(r,0)$ for the construction of some excited state, there is an additional factor of two in the degeneracy. Also, due to the r-dependence, wherever excitations may cross each other, the degeneracy will be the sum of their degeneracies. Finally it should be mentioned that, at first sight, the dimensionality of $\mathbb{J}_L$ seems to contradict the fact that the full system must be 2^{2L}-dimensional as the state space is given by $\mathcal{H}_{\text{alt}} = (\bar{\mathcal{V}} \otimes \mathcal{V})^{\otimes L}$. However, there are the two zero modes $a_0^\dagger(r,0)$ and $A_0^\dagger(r,0)$ that are not necessary to take into account in order to build up the energy levels $E_{j,L}(r,0)$. This reduces the number of necessary states to $2^{L-1}2^{L-1}$ and would rather suggest a form of $\{0,1\}^{L-1} \times \{0,1\}^{L-1}$ for $\mathbb{J}_L$. But for building up some energy level, we do not have to discern between, e.g. $(1,0) \times (0,0)$, $(0,0) \times (1,0)$ and $(1,0)$, or likewise $(1,0) \times (1,0)$ and $(2,0)$, as they lead to the same level. Hence, the form of $\mathbb{J}_L$ as given in eq. (5.21) is best suited for simply computing the energy levels while the form $\{0,1\}^{L-1} \times \{0,1\}^{L-1}$ would also keep track of the degeneracies.

In order to illustrate the agreement between these conjectured expressions and the numerical implementation of the system, the claimed dependence of the full spectrum is plotted in Figure 5.3 together with some but fewer (so as not to obscure the graphs of the analytic expressions) data points in analogy with Figure 5.2. Surely, this does not prove the validity of the analytical expressions. However, the actual agreement is perfect to any level of detail within the limits of accuracy of the data points.

Including the α-Dependence

After the question concerning the r-dependence of the spectrum was settled, the α-dependence came to the fore since numerical simulations suggested that it was similarly well-behaved. The plots in Figure 5.4 are in place to illustrate this for a system of $2L = 8$ spins. Note that, in the last plot for $\alpha = \alpha_0 := \frac{\pi}{L}$ (α_0 is the angular distance between neighboring sites in the fully uniform setup), there is a number of aberrant data points in the vicinity of $r = 1$ where the sublattices coalesce. These were identified as such by their relatively large imaginary part and we will return to the issue of how to deal with these below.

To cut a long story short, the angular dependence was found by singling out one of the energy levels in the numerical data and fitting the coefficients of eq. (5.19) adapted to this particular energy level. Again, there was a little caveat as the form of the one-particle energies as given in eq. (5.19) is not suitable in this affair. Instead, starting with the numerator and denominator expanded

$$\varepsilon_{p,L}(r,0) = -\frac{\pi^2}{2L^2}\frac{(L-p)^2 r^{2L} - L^2 r^{2(L-p)} - 2p(L-p)r^L + p^2}{1 + 2r^L + r^{2L}}, \tag{5.22}$$

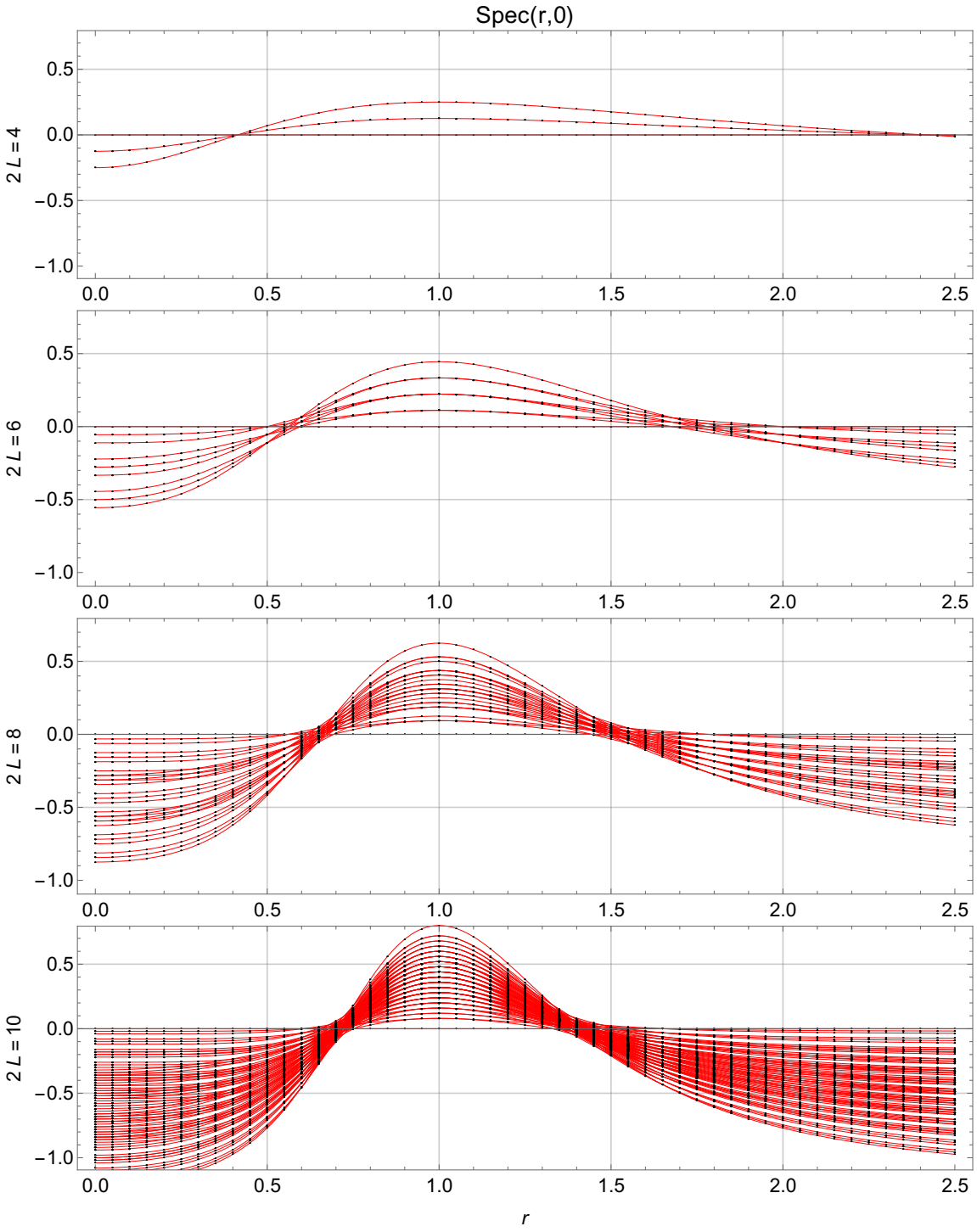


Figure 5.3: These charts show the conjectured analytic dependence of the spectrum of the Hamiltonian $H(r,0)$ (in units of π^2) at the four distinct system sizes $2L = 4, 6, 8, 10$ as indicated to the left of each plot. Some but fewer data points are included for reference.

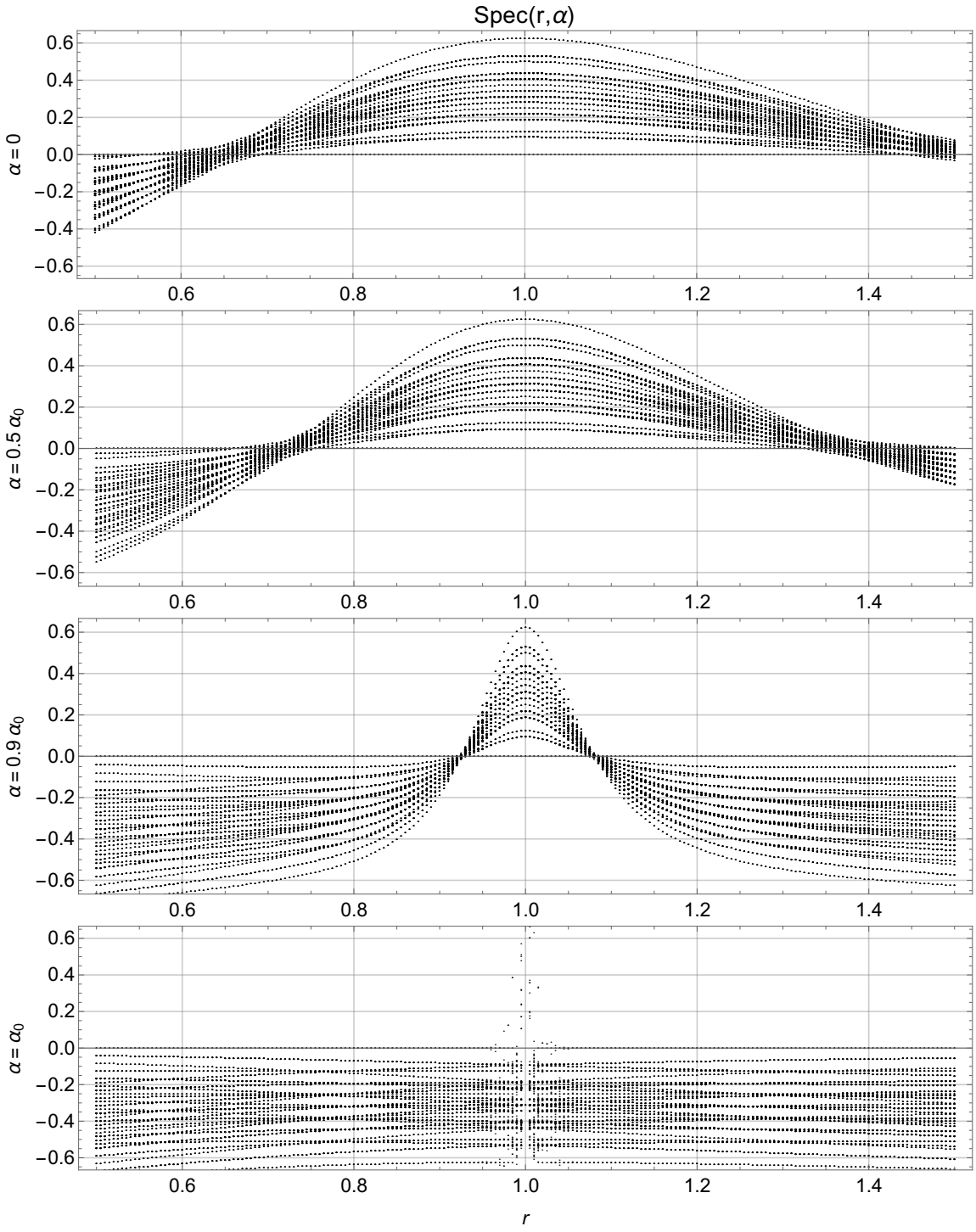


Figure 5.4: These plots show the data points from the exact diagonalization of the Hamiltonian $H(r,\alpha)$ (in units of π^2) for a system size of $2L = 8$ at the four distinct values of angular shift $\alpha/\alpha_0 = 0, 0.5, 0.9, 1$ as indicated to the left of each plot. The angular distance between neighboring sites at $\alpha = 0$ and $2L = 8$ is $\alpha_0 = \frac{\pi}{4}$.

it was only a stone's throw to arrive at the full glory of

$$\varepsilon_{p,L}(r,\alpha) := -\frac{\pi^2}{2L^2}\frac{(L-p)^2r^{2L} - L^2r^{2(L-p)} - 2p(L-p)\cos(L\alpha)r^L + p^2}{1+2\cos(L\alpha)r^L + r^{2L}}. \tag{5.23}$$

The plots documenting the legitimacy of these expressions are shown in Figure 5.5. Apart from some aberrant data points in the plot for $\alpha = \alpha_0$ in the vicinity of $r = 1$[5] and, less obvious, for $\alpha = 0.9\alpha_0$ around $r = 0.925$[6], the spectrum

$$\{E_{j,L}\}_j(r,\alpha) := \{\mathbf{j}\cdot\boldsymbol{\varepsilon}_L\}_j(r,\alpha) \quad \text{with} \quad j = 1,\ldots,|\mathbb{J}_L| \tag{5.24}$$

evidently captures any feature of the data set (with $|\mathbb{J}_L| = 3^{L-1}$). In order to demonstrate this to full extent, the diagrams exhibiting aberrant data points were recalculated using *Mathematica*'s ability to either compute exactly or control the precision of the output instead of simply using the *double-precision floating-point format* for all input values which may, depending on the operations performed during the algorithm, lead to arbitrarily high deterioration of the output's accuracy. However, as this can drastically slow down the routine, Figure 5.6 only shows the critical parts of these diagrams. Evidently, the agreement between data and analytic expressions is perfect.

5.3.2 Analytic Discussion of the One-Particle Energies

After the presentation of an account of how these results were established, this Section is dedicated to the discussion of those findings. To this end, the expression for the one-particle energies is first put into its final form by reducing the fraction to

$$\varepsilon_{p,L}(r,\alpha) := \frac{\pi^2}{2L^2}\left(p(L-p) - L\frac{pr^{-L} - Lr^{L-2p} + (L-p)r^L}{r^{-L} + 2\cos(L\alpha) + r^L}\right). \tag{5.25}$$

The first observation is that for $r = 1$, it correctly reduces to the dispersion relation of $H(1,0)$ analytically calculated in Chapter 4. Moreover, it reflects the invariance of the full spectrum with respect to $\alpha \in \,]-\alpha_0,\alpha_0[$ at $r = 1$,

$$\varepsilon_{p,L}(1,\alpha) = \frac{\pi^2}{2L^2}\left(p(L-p) - L\frac{p - L + (L-p)}{1+2\cos(L\alpha)+1}\right) \tag{5.26}$$

$$= \frac{\pi^2}{2L^2}p(L-p), \tag{5.27}$$

[5] The explanation for the existence of these aberrant data points is that the finite numerical precision is not able to appropriately consolidate the very small distances and comparably large distances between different sites in the calculations when the sublattices get too close to each other.

[6] At this value of r, all eigenvalues of the Hamiltonian become relatively small as compared to the input values and, in particular, their absolute precision. However, the two data points slightly deviating from the analytic result are only properly visible in the electronic version of this document at sufficient magnification.

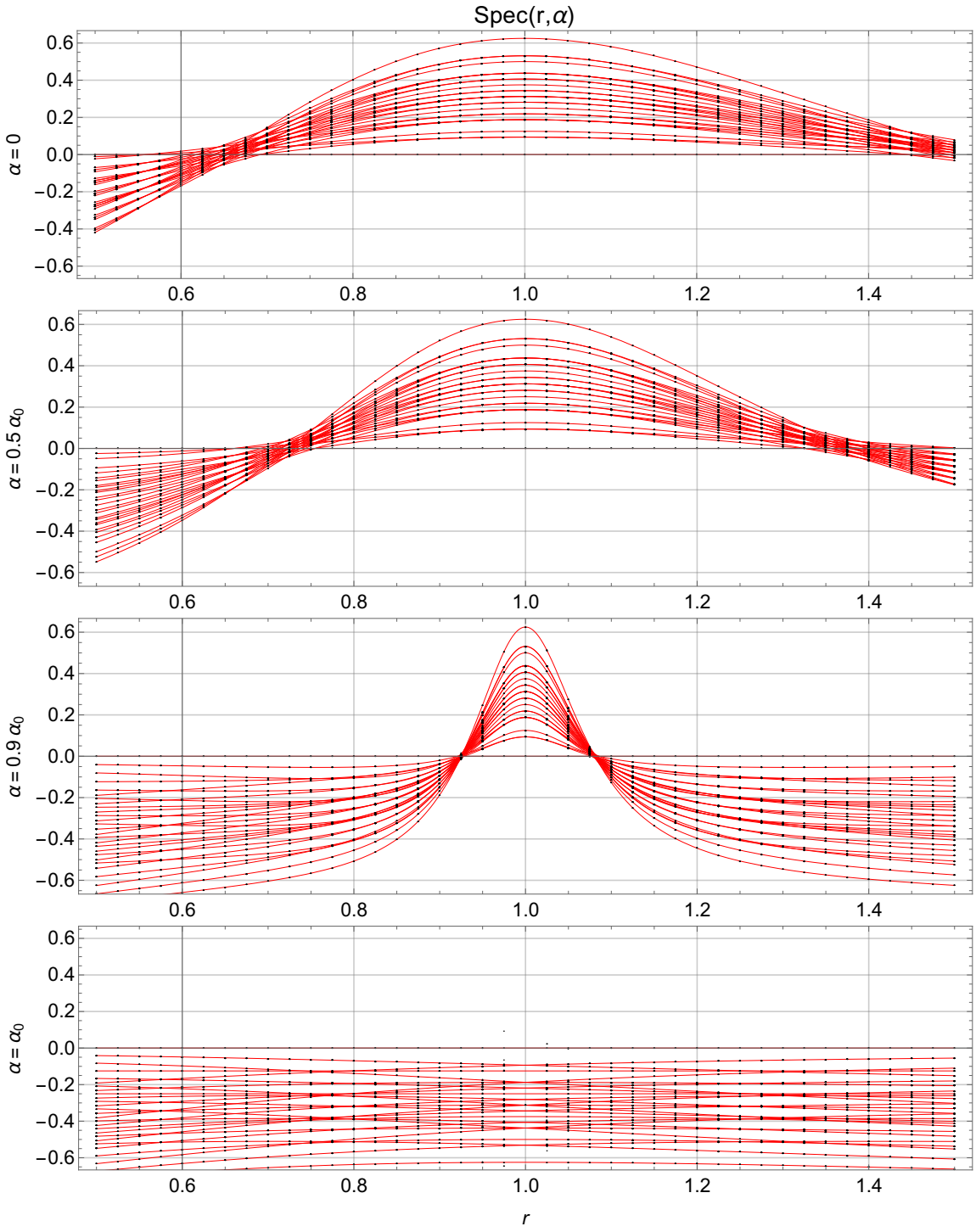


Figure 5.5: The charts show the conjectured analytic dependence of the spectrum of the Hamiltonian $H(r,\alpha)$ (in units of π^2) for a system size of $2L = 8$ at the four distinct values of angular shift $\frac{\alpha}{\alpha_0} = 0, 0.5, 0.9, 1$ as indicated to the left of each plot. Some but fewer data points are included for reference.

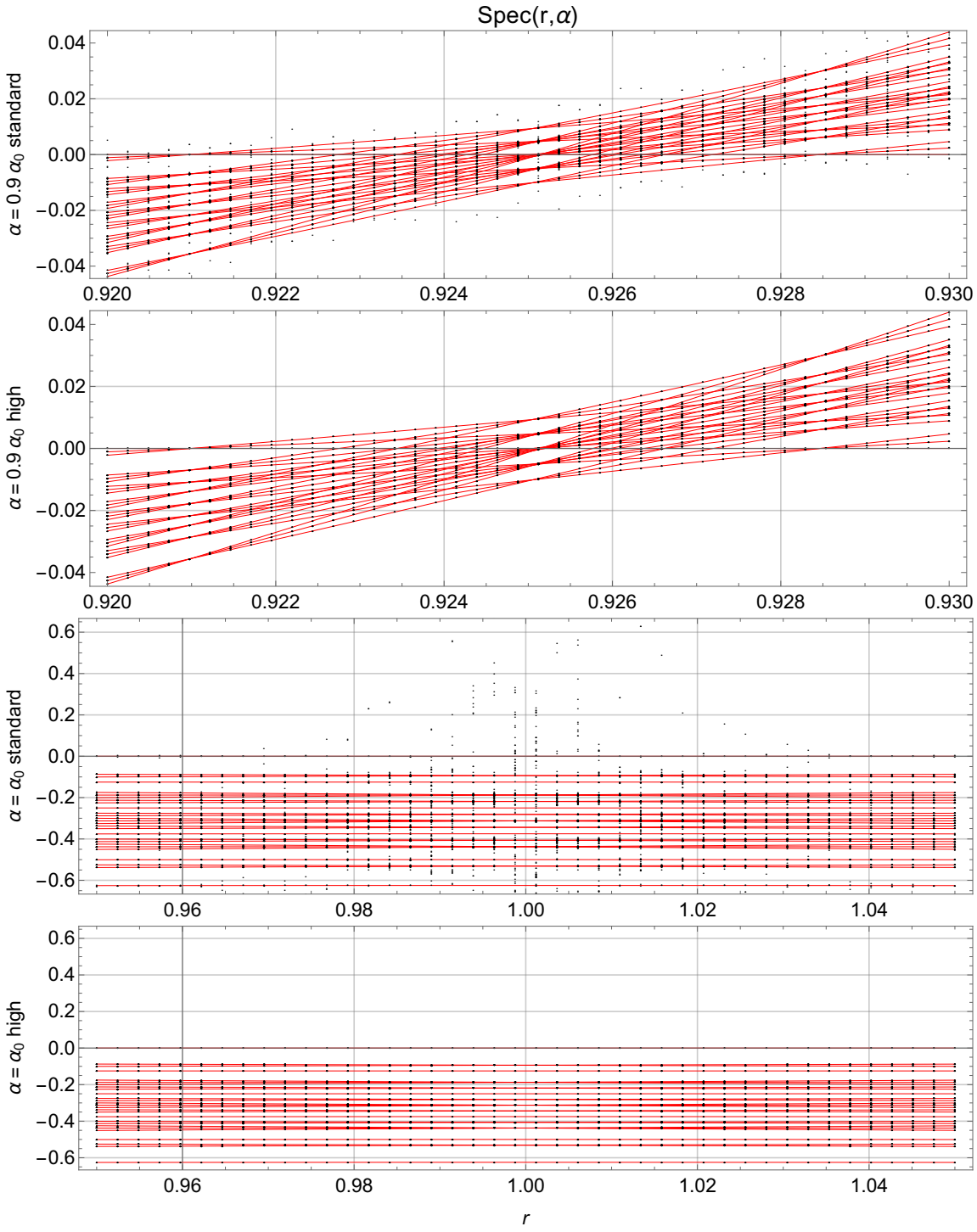


Figure 5.6: These charts show data points from calculations with *Mathematica*'s standard precision compared to data from calculations with *Mathematica*'s highest available precision for $\alpha = 0.9\alpha_0$ and from exact calculations for $\alpha = \alpha_0$ at $2L = 8$ together with the conjectured analytic dependence of the spectrum of the Hamiltonian $H(r, \alpha)$ (in units of π^2) in the domain of r where the respective plots in Figure 5.4 and Figure 5.5 are stained by aberrant data points.

a property of $H(1,\alpha)$ that was observed to be true from exact diagonalization at all accessible system sizes ($2L = 4, 6, 8, 10, 12$) and a wide range of admissible values of α (cf. Figure 5.5, demonstrating numerical invariance within at least 90% of the parameter range for $r = 1$,[7] and Figure 5.7, for the generic α-dependence of eq. (5.25) at various values of r for a system of size $2L = 14$).

Additionally, as reasoned at the end of Section 5.2, the one-particle energies become independent of α in the limit of $r \to 0$,

$$\lim_{r\to 0} \varepsilon_{p,L}(r,\alpha) = \lim_{r\to 0} \frac{\pi^2}{2L^2}\left(p(L-p) - L\frac{pr^{-L} - Lr^{L-2p} + (L-p)r^L}{r^{-L} + 2\cos(L\alpha) + r^L}\right) \tag{5.28}$$

$$= \frac{\pi^2}{2L^2}\left(p(L-p) - Lp\right) \tag{5.29}$$

$$= -\frac{1}{2}\left(\frac{\pi p}{L}\right)^2, \tag{5.30}$$

or, by the same token, in the limit of $r \to \infty$,

$$\lim_{r\to\infty} \varepsilon_{p,L}(r,\alpha) = -\frac{1}{2}\left(\frac{\pi(L-p)}{L}\right)^2. \tag{5.31}$$

This result is in unison with the next insight about the claim made in Section 5.1: It regards the fact that the full spectrum should be invariant under the mapping $z \mapsto 1/z$. The relevant parameters of the spectrum are r and α which are mapped to $1/r$ and $-\alpha$ under inversion. Consequently, we find

$$\varepsilon_{p,L}(r^{-1},-\alpha)\,\frac{2L^2}{\pi^2} = p(L-p) - L\frac{pr^L - Lr^{-L+2p} + (L-p)r^{-L}}{r^L + 2\cos(-L\alpha) + r^{-L}} \tag{5.32}$$

$$= (L-p)p - L\frac{(L-p)r^{-L} - Lr^{L-2(L-p)} + pr^L}{r^{-L} + 2\cos(L\alpha) + r^L} \tag{5.33}$$

$$= \varepsilon_{(L-p),L}(r,\alpha)\,\frac{2L^2}{\pi^2} \tag{5.34}$$

for all $p = 1, \ldots, L-1$. Since the action of the inversion on the one-particle energies closes, $\varepsilon_{p,L}(r,\alpha) \mapsto \varepsilon_{(L-p),L}(r,\alpha)$, the full spectrum of the system proves to be invariant.[8]

Another property of the solutions $\varepsilon_{p,L}(r,\alpha)$ concerns the question about their regularity at $r = 1$ and $\alpha = \alpha_0$ where both sublattices coincide. As of eq. (5.27), we know that the spectrum at $r = 1$ does not change along the α-direction, i.e. $\varepsilon_{p,L}(1,\alpha) = \frac{\pi^2}{2L^2}p(L-p)$, but is not defined at $\alpha = \alpha_0$ due to the pole. However, approaching the point in question along

[7] This seems to also hold for the $GL(2|2)$-invariant spin ladder but fails for its $SU(2)$-invariant counterpart. It is most likely a consequence of the special form of the Hamiltonian given in eq. (5.11). If this is the case, it should hold for the $\mathfrak{gl}(m|m)$-spin ladder for any positive integer m.

[8] In fact, this also holds for the separate mapping $r \mapsto 1/r$ as the one-particle energies are even functions in α.

the radial direction at $\alpha_0 = \frac{\pi}{L}$, we have to compute

$$\lim_{r\to 1} \varepsilon_{p,L}(r,\alpha_0) = \lim_{r\to 1} \frac{\pi^2}{2L^2}\left(p(L-p) - L\frac{pr^{-L} - Lr^{L-2p} + (L-p)r^L}{r^{-L} + 2\cos(\pi) + r^L}\right). \tag{5.35}$$

Focusing on the non-trivial term and using the rule of de L'Hospital twice, we find

$$\begin{aligned}
&\lim_{r\to 1} \frac{pr^{-L} - Lr^{L-2p} + (L-p)r^L}{r^{-L} - 2 + r^L} \\
&\qquad = \frac{pL(L+1) - L(L-2p)(L-2p-1) + (L-p)L(L-1)}{L(L+1) + L(L-1)} && (5.36)\\
&\qquad = \frac{p(L+1) - (L-2p)(L-2p-1) + (L-p)(L-1)}{2L} && (5.37)\\
&\qquad = \frac{4pL - 4p^2}{2L} && (5.38)\\
&\qquad = \frac{2p(L-p)}{L}. && (5.39)
\end{aligned}$$

Reinserting this result into eq. (5.35) yields the desired limit of

$$\begin{aligned}
\lim_{r\to 1} \varepsilon_{p,L}(r,\alpha_0) &= \frac{\pi^2}{2L^2}\left(p(L-p) - L\frac{2p(L-p)}{L}\right) && (5.40)\\
&= -\frac{\pi^2}{2L^2}p(L-p). && (5.41)
\end{aligned}$$

This differs from the limit along the α-direction at $r = 1$, given by $\frac{\pi^2}{2L^2}p(L-p)$. Further, the growth of values of $\varepsilon_{p,L}(r,\alpha)$, changing α from 0 to α_0 (or, equally from 0 to $-\alpha_0$), sets in at values of r ever closer to 1 and, in effect, its slope becomes ever steeper (cf. Figure 5.5). Note also that the limit calculated in eq. (5.41) coincides with that for $p \mapsto L - p$,

$$\lim_{r\to 1} \varepsilon_{p,L}(r,\alpha_0) = -\frac{\pi^2}{2L^2}p(L-p) = \lim_{r\to 1} \varepsilon_{L-p,L}(r,\alpha_0). \tag{5.42}$$

Hence, also for $\alpha = \alpha_0$ do the one-particle energies for p and $L - p$ converge when approaching $r = 1$. However, in contrast to $\alpha \neq \alpha_0$, they converge to the negative value.

And returning to the question of how to deal with the values $(1,\alpha_0)$ for (r,α), it is clear now that the limits of $\varepsilon_{p,L}(r,\alpha)$ along the radial and azimuthal direction lead to different results. This situation cannot – neither from a mathematical nor from a physical perspective – be cured in a sensible way. Evidently, these points have to be excluded from the complex plane in order to reasonably define the domain of the one-particle energies $\varepsilon_{p,L}(r,\alpha)$. Figure 5.7 illustrates the results of the analysis of the one-particle energies so far for a particular system of size $2L = 14$.

Next, we would like to mention a surprising property of the function $\varepsilon_{L/2,L}(r,\alpha)$ for

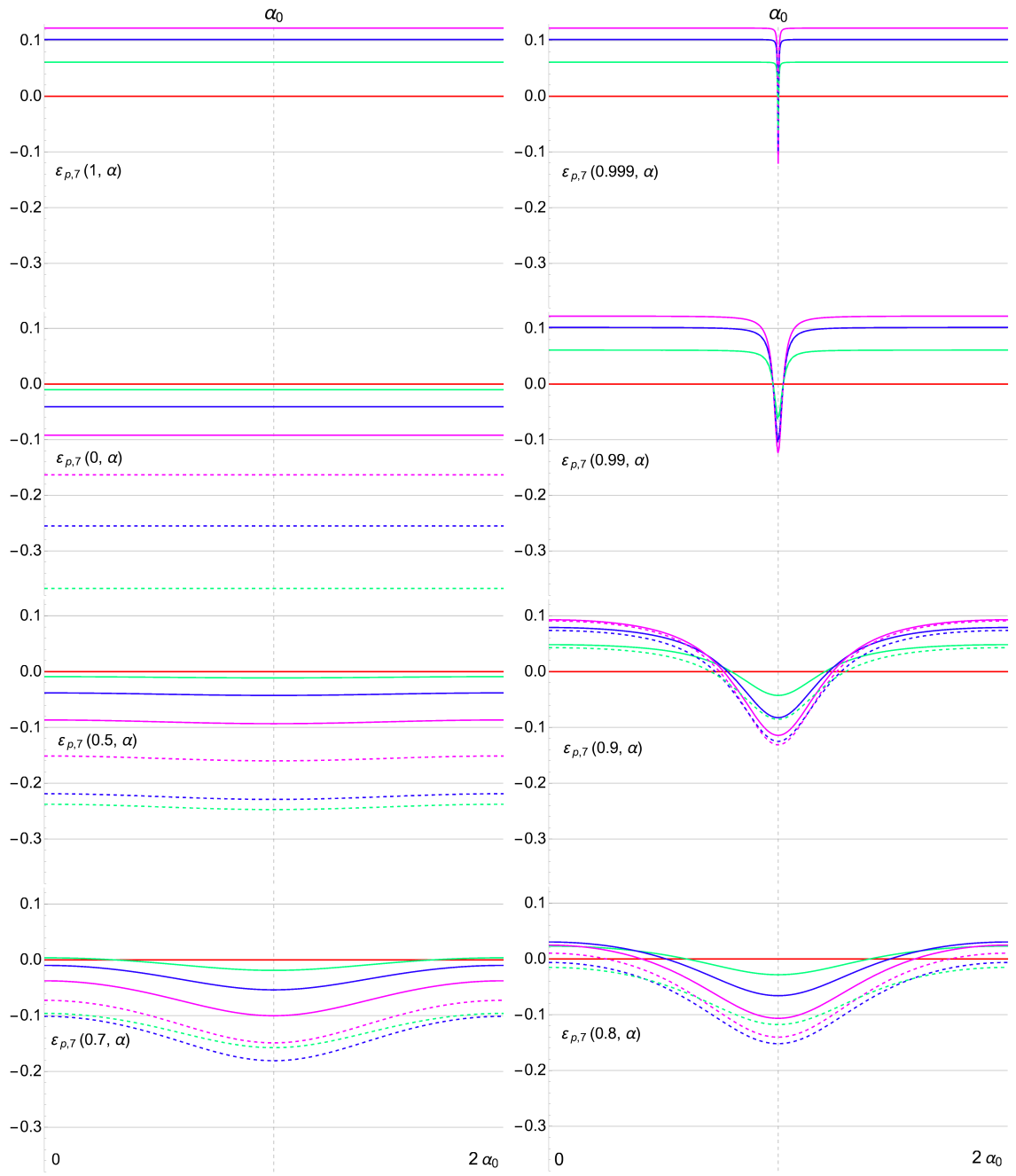


Figure 5.7: This figure is to be viewed clockwise starting at the top left. Each chart shows one period of the α-dependence of the one-particle energies $\varepsilon_{p,L}(r, \alpha)$ for $2L = 14$ at eight distinct values of r. The two dashed grey lines indicate the critical values α_0 where, as mentioned in the text, $\varepsilon_{p,L}(1, \alpha_0)$, in general, is not defined. The color coding is such that, for $p = 1, 2, 3$, $\varepsilon_{p,7}(r, \alpha)$ (solid lines) and $\varepsilon_{7-p,7}(r, \alpha)$ (dashed lines) have the same color. The solid red line belongs to the zero energy state constructed in Section 3.8. While $\varepsilon_{p,L}(1, \alpha)$ is proportional to $p(L - p)$, $\varepsilon_{p,L}(0, \alpha)$ is proportional to $-p^2$. Note that this is not the full spectrum.

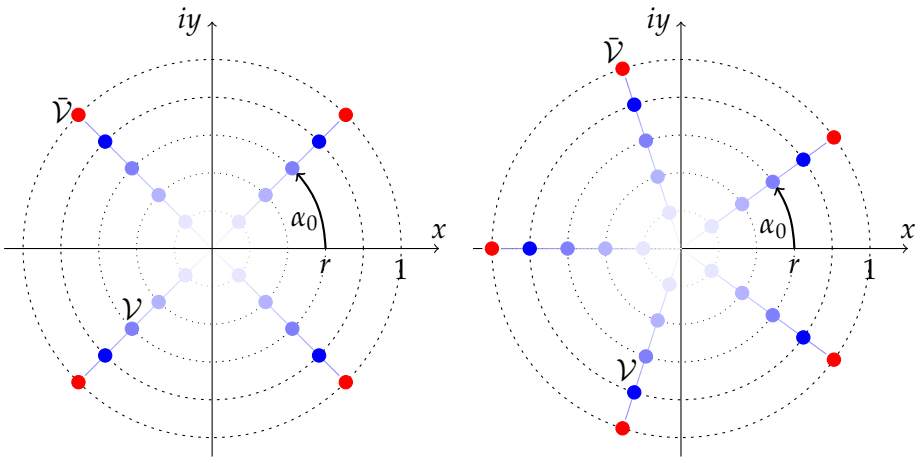


Figure 5.8: This figure illustrates the geometric difference for the spin ladder of system size $2L$ between L an even (left-hand side) and odd (right-hand side) integer larger than one. Particularly, it motivates that, for L even, every site has a partner exactly across from the origin within the same sublattice. Furthermore, for $\alpha = \alpha_0$, every such pair of one sublattice lines up with such a pair of the other sublattice. On the other hand, for odd L, the situation is different as there are no pairs of exactly opposing sites within a sublattice. This fundamental difference might explain that, for even L and $\alpha = \alpha_0$, there is a one-particle energy constant in r, i.e. $\varepsilon_{L/2,L}(r, \alpha_0) = -\frac{\pi^2}{8}$, whereas, for any odd L, there is no such contribution.

$\alpha = \alpha_0$,

$$\varepsilon_{L/2,L}(r, \alpha_0) = \frac{\pi^2}{2L^2}\left(\frac{L^2}{4} - L\frac{\frac{L}{2}r^{-L} - Lr^{L-2\frac{L}{2}} + (L-\frac{L}{2})r^L}{r^{-L}-2+r^L}\right) \tag{5.43}$$

$$= \frac{\pi^2}{2L^2}\left(\frac{L^2}{4} - L\frac{L}{2}\frac{r^{-L}-2+r^L}{r^{-L}-2+r^L}\right) \tag{5.44}$$

$$= -\frac{\pi^2}{8}, \tag{5.45}$$

i.e. it is constant for any value of r (of course, consistent with eqs. (5.30), (5.30) and (5.41)) and even independent of the total system size $2L$. However, it only plays the role of a one-particle energy in systems with $L \in 2\mathbb{Z}$, so in systems with the geometric property that, for any site of one of the two sublattices, there is a partner within that sublattice which geometrically lies exactly opposite. Further, in this setup with $\alpha = \alpha_0$, for every such pair of geometrically opposite sites, there is a pair of such sites in the other sublattice exactly in line with the former pair. The situation is depicted in Figure 5.8 which shows the two different cases of L, either an even or an odd positive integer. The dots of the same shade of blue represent the sublattice built from the fundamental representation at a certain value of r while the red dots stay for the sublattice built from the antifundamental representation.

The effect of this special one-particle energy is that any actual energy level $E_{j,L}(r,\alpha_0)$ of such a system is associated with two other energy levels $E_{k,L}(r,\alpha_0)$ and $E_{l,L}(r,\alpha_0)$ which run absolutely in parallel with each other, parted by $\frac{\pi^2}{8}$ or $\frac{\pi^2}{4}$ in energy as they only differ from each other by once or twice the contribution of $\varepsilon_{L/2,L}(r,\alpha_0)$ (cf. eq. (5.20)). In particular, since the system is designed to always yield energy zero for the seed state, in this case, there will be two other constant energy levels, one at $-\frac{\pi^2}{8}$ and the other at $-\frac{\pi^2}{4}$, independently of the actual system size as long as $L \in 2\mathbb{Z}$. Under the assumption that p plays the role of a momentum number even for generic r and α,[9] then, for even L, $\varepsilon_{L/2,L}(r,\alpha_0)$ is the one-particle energy of the state that is as much right-mover as it is left-mover. In that case, and with the different geometric pictures for even and odd L in mind, it seems most likely that it is some sort of a vibrational mode, also because it is the most energetic one-particle excitation.[10]

5.3.3 A Guess on the Hamiltonian

Finally, as mentioned at the end of Section 5.2, in the limit $r \to 1$ and $\alpha \to 0$, the model approaches the special system

$$H(1,0) = \sum_{p=1}^{L-1} \frac{\pi^2}{2L^2}(L-p)p(a_p^\dagger a_p + A_p^\dagger A_p) \tag{5.46}$$

which is, in the continuum limit, described by two copies of the Ising CFT in the disorder sector. On the other hand, for $r = 1$ and $\alpha \neq 0$, the $\mathfrak{gl}(1|1)$-spin ladder Hamiltonian $H(1,\alpha)$ exhibits a nilpotent part analogous to the Haldane-Shastry Hamiltonian

$$H_{\text{HS,alt}} = \frac{\pi^2}{2} a_0^\dagger A_0^\dagger + \sum_{p=1}^{L-1} \frac{\pi^2}{2L^2}(L-p)p(a_p^\dagger a_p + A_p^\dagger A_p), \tag{5.47}$$

there attributed to the zero-mode term $\frac{\pi^2}{2} a_0^\dagger A_0^\dagger$. Additionally, just like the Haldane-Shastry Hamiltonian and also $H(1,0)$, the spin ladder setup generically leads to conservation of momentum as translational invariance by integer multiples of $\frac{2\pi}{L}$ (the angular distance between neighboring sites within one sublattice) is preserved. With all this in mind, we conjecture the energy eigenstates to be momentum eigenstates so that the actual momentum of any given one-particle energy eigenstate would be related to the index p of its energy

[9] This is also suggested by the fact mentioned earlier that the generic spin ladder setup preserves the very same translational invariance under shifts of α to $\alpha + 2\alpha_0$ as the fully uniform alternating setup which leads to momentum conservation and, particularly, quantization of momentum given by $p\frac{2\pi}{L}$ with $p = 0, 1, \ldots, L-1$.

[10] The lowest one-particle energies should lead to local disturbances propagating clockwise or counterclockwise with some proportionality to their momentum. Increasing the momentum of the mode excited in some system will eventually lead to the transition of a right-moving mode to a left-moving mode (or vice versa) at the maximal one-particle energy available. This maximal transition energy is always given by $\varepsilon_{L/2,L}(r,\alpha_0)$ which happens to be a one-particle energy exactly if L is an even positive integer. For odd L, $\varepsilon_{(L\pm1)/2,L}(r,\alpha_0)$ will be the highest one-particle energies with their momenta oriented either clockwise or counterclockwise.

$\varepsilon_{p,L}(r,\alpha)$. However, while at $r=1$ we have

$$\varepsilon_{p,L}(1,\alpha) = \frac{\pi^2}{2L^2}p(L-p) = \varepsilon_{L-p,L}(1,\alpha), \tag{5.48}$$

this equality breaks down for $r \neq 1$.[11] Therefore, with a naïve interpretation for the index p of $\varepsilon_{p,L}(r,\alpha)$ as the straight-forward momentum number with right-movers given by $0 < p < \frac{L}{2}$ and left-movers given by $\frac{L}{2} < p < L$, this would imply that left- and right-moving modes of the same momentum (only differing in orientation) have distinct energies.

Putting together all the puzzle pieces culminates in a conjecture about the (almost) diagonal form of the Hamiltonian $H(r,\alpha)$ acting on the state space $\mathcal{H}_{\text{alt}} = (\bar{\mathcal{V}} \otimes \mathcal{V})^{\otimes L}$. It is given by

$$H(r,\alpha) = f(r,\alpha)a_0^\dagger(r,\alpha)A_0^\dagger(r,\alpha) \tag{5.49}$$

$$+ \sum_{p=1}^{L-1} \varepsilon_{p,L}(r,\alpha)\left(a_p^\dagger(r,\alpha)a_p(r,\alpha) + A_p^\dagger(r,\alpha)A_p(r,\alpha)\right) \tag{5.50}$$

which generalizes $H(1,0)$ in the following way: The only known fact that can be inferred about the function $f(r,\alpha)$ is that it must be zero at $r=1$ and $\alpha=0$ and, apparently by what is known from the numerical results, non-zero in general.[12] Furthermore, it seems likely that the general fermionic creation operators have a r- and α-dependence since they are most likely not only the Fourier transform of the particle creation operators but rather, in analogy with eq. (4.53), the pure Fourier transforms have to be decoupled by taking linear combinations with coefficients from which the dispersion relation is computed. Thus, as the dispersion relation depends on r and α in the spin ladder case, it should also be expected that the modes will depend on them. However, their mutual anticommutation relations at fixed values of r and α must, indeed, stay unaffected by this because the energy levels are built in the same manner independent of r and α (cf. eq. (5.24)). The dispersion relation is generalized in the way explained in the previous parts of this Section. While this is, of course, compatible with the form of $H(1,0)$, it is not decidable whether the index of the A-modes or the a-modes have to be adjusted in some way. Only a more rigid analysis will be able to settle this issue.

5.3.4 The Continuum Limit

In order to get a grip on the thermodynamic limit of the $\mathfrak{gl}(1|1)$-spin ladder, let us first analyze the properties of the set of one-particle energies $\{\varepsilon_{p,L}(r,\alpha)\}_p$ for $L \to \infty$. So

[11] In fact, it can be shown that, for $r \in [0,1[$ and $0 < p < \frac{L}{2}$, we have $\varepsilon_{p,L}(r,\alpha) > \varepsilon_{L-p,L}(r,\alpha)$, independently of α. The reverse is true for $r > 1$.

[12] However, the possibility that there may be other special points or domains in the parameter space of (r,α) where $f(r,\alpha)$ is zero cannot be excluded.

effectively, we will discuss what one single excitation created by one creation operator of one of the two sets of mutually anticommuting fermions, say $\{a^\dagger(r,\alpha)\}_p$, will look like in the continuum limit. Again, we will assume $r \in \mathbb{R}_{\geq 0}$ and $\alpha \in]-\alpha_0, \alpha_0]$.

Starting at $r = 0$, we draw on the result of eq. (5.30) (actually valid for all values of α):

$$\lim_{r\to 0} \varepsilon_{p,L}(r,\alpha) = -\frac{\pi^2}{2L^2} p^2. \tag{5.51}$$

As p ranges from 0 (thereby, including the zero energy level here) to $L-1$, the one-particle energies (belonging to one of the two sets of creation operators) at $r = 0$ will densely cover the real interval between 0 and $-\frac{\pi^2}{2}$ in the continuum limit.[13]

Next let us figure out how the one-particle energies vary with r as we send L to infinity. We treat the cases $r < 1$, $r = 1$ and $r > 1$ separately, starting with the first (cf. eq. (5.23)):

$$\varepsilon_{p,L}(r,\alpha) = -\frac{\pi^2}{2L^2} \frac{(L-p)^2 r^{2L} - L^2 r^{2(L-p)} - 2p(L-p)\cos(L\alpha) r^L + p^2}{1 + 2\cos(L\alpha) r^L + r^{2L}} \tag{5.52}$$

$$\overset{\substack{L\to\infty \\ r\leq 1}}{=} -\frac{\pi^2}{2L^2} \frac{0 - L^2 r^{2(L-p)} - 0 + p^2}{1 + 0 + 0}. \tag{5.53}$$

At first sight, the term proportional to $r^{2(L-p)}$ seems to be somewhat problematic, e.g. for some $p = L-1, L-2, \ldots$. However, we may limit $p < \gamma L$ for some fixed $\gamma \in [0,1[$ so as to have

$$\ln r^{2(L-p)} = 2(L-p)\ln r \overset{\ln r<0}{<} 2L(1-\gamma)\ln r \overset{L\to\infty}{\longrightarrow} -\infty \tag{5.54}$$

so that

$$r^{2(L-p)} \overset{L\to\infty}{\longrightarrow} 0. \tag{5.55}$$

Indeed, for this to hold, we may choose γ arbitrarily close to 1. Thus, in practice starting for $r = 0$ with any value $e \in \mathbb{R}^{\leq 0}_{>-\pi^2/2}$ and choosing L appropriately (large), we are able to approximate e with some $\varepsilon_{p,L}(0,\alpha)$ to any extent:

$$(\forall e \in \mathbb{R}^{\leq 0}_{>-\pi^2/2}) \wedge (\forall \epsilon > 0), (\exists L_{e,\epsilon} \in \mathbb{N}) \wedge (\forall L \geq L_{e,\epsilon} : \exists p(L) \in \mathbb{N}_{L-1}) :$$
$$\left| e - \varepsilon_{p(L),L}(0,\alpha) \right| = \left| e - \frac{\pi^2}{2} \frac{(p(L))^2}{L^2} \right| < \epsilon. \tag{5.56}$$

[13] This can be seen as follows: First strip of the factor π^2 for a moment and consider the set $\{-\frac{p^2}{2L^2}\}$ for $p = 0, 1, \ldots, L-1$ and $L \to \infty$; it is dense in the rational numbers smaller or equal to zero and larger than $-\frac{1}{2}$, $\mathbb{Q}^{\leq 0}_{>-1/2}$. This is because any rational number $-\frac{a}{2b}$ in $\mathbb{Q}^{\leq 0}_{>-1/2}$ lies, for any given L, between some $-\frac{p^2}{2L^2}$ and $-\frac{(p+1)^2}{2L^2}$. By doubling L, this interval will be divided into two, i.e. $[\frac{2p)^2}{2(2L)^2}, \frac{(2p+1)^2}{2(2L)^2}]$ and $[\frac{(2p+1)^2}{2(2L)^2}, \frac{(2p+2)^2}{2(2L)^2}]$. In order to approximate $-\frac{a}{2b}$ to arbitrary precision, one only has to choose the one interval $-\frac{a}{2b}$ lies in and iterate the procedure, leading to convergence. Further, since $\mathbb{Q}^{\leq 0}_{>-1/2}$ is dense in $\mathbb{R}^{\leq 0}_{>-1/2}$, the set $\lim_{L\to\infty}\{-\frac{p^2}{2L^2}\}_{p=0,1,\ldots,L-1}$ is also dense in the real interval $\mathbb{R}^{\leq 0}_{>-1/2}$. Evidently, the set $\lim_{L\to\infty}\{-\frac{\pi^2 p^2}{2L^2}\}_{p=0,1,\ldots,L-1}$ is, thus, dense in the real interval $\mathbb{R}^{\leq 0}_{\geq -\pi^2/2}$.

Hence, we find, with for every e and L appropriately chosen $p(L)$:

$$\lim_{L\to\infty} \varepsilon_{p(L),L}(0,\alpha) = -\frac{\pi^2}{2} \lim_{L\to\infty} \frac{(p(L))^2}{L^2} = e. \tag{5.57}$$

Additionally, by virtue of eqs. (5.53) and (5.55), even

$$\lim_{L\to\infty} \varepsilon_{p(L),L}(r,\alpha) \overset{r\leq 1}{=} -\frac{\pi^2}{2} \lim_{L\to\infty} \frac{(p(L))^2}{L^2} = e \tag{5.58}$$

holds, independently of $r \in [0,1[$.

Let us briefly return to the seemingly problematic values of $p = L-1, L-2, \ldots$ etc. Their one-particle energies $\varepsilon_{p(L),L}(r,\alpha)$ will, in the thermodynamic limit and for $r < 1$, indeed run like

$$\lim_{L\to\infty} \varepsilon_{p(L),L}(r,\alpha) = \frac{\pi^2}{2} r^{2(L-p)} - \frac{\pi^2}{2L^2} p^2 \tag{5.59}$$

with the monomial in r playing a relevant role so that they, indeed, cannot be considered constant in r. However, any finitely chosen set $\{L-1, L-2, \ldots, L-k\}$ for values of p leading to this non-constant dependence of $\varepsilon_{p(L),L}(r,\alpha)$ on r for $r < 1$ will, in the limit of $L \to \infty$, turn into a null set, i.e. for any finitely chosen k, $\frac{(L-j)^2}{L^2}$ for $j = 1, \ldots, k$ will converge to one point. In other words, all the one-particle energies $\varepsilon_{p(L),L}(r,\alpha)$ that cannot be considered constant in the thermodynamic limit for $r < 1$ run like $\frac{\pi^2}{2} r^{2(L-k)}$ for some *finite* k. Thus, in the continuum limit, they form a null set within all the one-particle energies. Furthermore, as we are mainly interested in the low-energy behavior, we should also mention that, for any finite value of $r < 1$, the states associated with this null set of one-particle energies are, with respect to the seed state $|\Psi\rangle$ high[14] in the spectrum.

Summing up, in the thermodynamic limit, for any given energy $e \in]-\frac{\pi^2}{2}, 0]$, there will be a one-particle state

$$|\psi_e\rangle := \lim_{L\to\infty} a^\dagger_{p(L)}(r,\alpha)\,|\Psi\rangle \tag{5.60}$$

which we will label by its energy e at $r = 0$:

$$\lim_{L\to\infty} H(0,\alpha)\, a^\dagger_{p(L)}(0,\alpha)\,|\Psi\rangle = e \lim_{L\to\infty} a^\dagger_{p(L)}(0,\alpha)\,|\Psi\rangle \tag{5.61}$$

Moreover, this state's energy will be constant for $r \in [0,1[$, independently of α, whereas a dependence of $|\psi_e\rangle$ on r and α cannot be excluded since the same statement applies to $a^\dagger_{p(L)}(r,\alpha)$ and, furthermore, the seed state $|\Psi\rangle$ constructed in Section 3.8 generically depends on the positions of all the sites $\{z_j\}_j$. However, for simpler notation, this dependence will be suppressed for quantum states like $|\Psi\rangle$ and $|\psi_e\rangle$. As a consequence of the existence of the two zero modes $a^\dagger_0(r,\alpha)$ and $A^\dagger_0(r,\alpha)$, there are exactly three other one-particle states of the same energy e at $r = 1$.

[14] It is understood that, when speaking about criticality, this is supposed to mean far from the vacuum which, in our considerations, is represented by the state $|\Psi\rangle$ at zero energy.

Further, due to the properties of the one-particle energies with respect to the mapping $r \mapsto 1/r$, cf. eq. (5.34), we see right away that, while the energy of $|\psi_e\rangle$ is $e = -\frac{\pi^2}{2} \lim_{L\to\infty} \frac{(p(L))^2}{L^2}$ for $r < 1$, its energy $e_>$ for $r > 1$ will be

$$e_> = -\frac{\pi^2}{2} \lim_{L\to\infty} \frac{(L - p(L))^2}{L^2} \tag{5.62}$$

$$= -\frac{\pi^2}{2} \lim_{L\to\infty} \left(1 + \frac{(p(L))^2}{L^2} - 2\frac{p(L)}{L}\right) \tag{5.63}$$

$$= -\frac{\pi^2}{2} + e + \pi\sqrt{-2\left(-\frac{\pi^2(p(L))^2}{2L^2}\right)} \tag{5.64}$$

$$= -\frac{\pi^2}{2} + e + \pi\sqrt{-2e} \tag{5.65}$$

$$= -\frac{1}{2}\left(\pi - \sqrt{-2e}\right)^2. \tag{5.66}$$

Obviously, this is also constant in r, independent of α, and lies in the interval from 0 to $-\frac{\pi^2}{2}$. Interestingly though, any one-particle state with energy close to zero for $r < 1$ will acquire an energy close to $-\frac{\pi^2}{2}$ as soon as $r > 1$, and vice versa.[15] In particular, the states which one could consider "the most constant" for $r < 1$, i.e. the ones with $p = 1, 2, \ldots, k$ for some finite k, become the ones "most rapidly changing" for $r > 1$, namely proportional to r^{-2p}, and vice versa.

Finally, as of the discussion about the one-particle energies at $r = 1$, we remind ourselves that, here, we have to make the distinction between $\alpha \in]-\alpha_0, \alpha_0[$ and $\alpha = \alpha_0$. Therefore, the energy e_1 of the state $|\psi_e\rangle$ (which is one of the four states with energy e at $r = 0$) at $r = 1$ and $\alpha \neq \alpha_0$ is given by

$$e_1 = \frac{\pi^2}{2} \lim_{L\to\infty} \frac{p(L)(L - p(L))}{L^2} \tag{5.67}$$

$$= \lim_{L\to\infty} \left(\frac{p(L)\pi^2}{2L} - \frac{(p(L))^2\pi^2}{2L^2}\right) \tag{5.68}$$

$$= \pi\sqrt{\frac{-e}{2}} + e \tag{5.69}$$

$$= \frac{1}{2}\sqrt{-2e}\left(\pi - \sqrt{-2e}\right). \tag{5.70}$$

[15] This is not really surprising: Even though the one-particle energies $\varepsilon_{p(L),L}(r, \alpha)$ are continuous functions for any finite L, their convergence towards their limit function $\varepsilon_{p(\infty),\infty}(r, \alpha)$ is uniform everywhere but at $r = 1$ which is why the limit function is continuous (in particular constant) everywhere but at $r = 1$. In the vicinity of $r = 1$, their convergence is only pointwise leading to their discontinuity at this value of r. Nevertheless, actual physical system will always be finite. Therefore, it can be expected that, e.g. a one-particle state close to zero for $r < 1$, still evolves continuously into a state close to $-\frac{\pi^2}{2}$ for $r > 1$.

This, in contrast to the other results, is strictly larger than zero for $e < 0$. As derived in eq. (5.41), the limit of $\varepsilon_{p,L}(r,\alpha)$ for $\alpha = \alpha_0$ towards $r = 1$, where both sublattices merge, is exactly the opposite:

$$\lim_{\alpha\to\alpha_0} \varepsilon_{p,L}(1,\alpha) = -\lim_{r\to 1} \varepsilon_{p,L}(r,\alpha_0). \tag{5.71}$$

Thus, the energy of $|\psi_e\rangle$ approaching $r = 1$ along α_0 is given by

$$e_0 = -\frac{1}{2}\sqrt{-2e}\left(\pi - \sqrt{-2e}\right). \tag{5.72}$$

This is also consistent with what was found in Section 5.3.2 for systems with $L \in 2\mathbb{Z}$ and $\alpha = \alpha_0$, i.e. $\varepsilon_{L/2,L}(r,\alpha_0) = -\frac{\pi^2}{8}$, as the solution to the equations $e_0 = e$ and $e_0 = e_>$ is $e = -\frac{\pi^2}{8}$.

Let us briefly summarize what we have found out, so far, about the single-particle excitations created by a single creation operator of one of the two sets of fermionic creation operators, either $\{a_p^\dagger(r,\alpha)\}_p$ or $\{A_{L-p}^\dagger(r,\alpha)\}_p$: In the thermodynamic limit $L \to \infty$, for every $e \in [0, -\frac{\pi^2}{2}[$, there is a creation operator, either $a_{p(L)}^\dagger(r,\alpha)$ or $A_{L-p(L)}^\dagger(r,\alpha)$[16], which creates the state by applying it to the seed state $|\Psi\rangle$ constructed in Section 3.8. Equation (5.58) tells us how we have to choose $p(L)$ for every L in the continuum limit so that we have

$$\lim_{L\to\infty} H(r,\alpha)\, a_{p(L)}^\dagger(r,\alpha)\,|\Psi\rangle = e \lim_{L\to\infty} a_{p(L)}^\dagger(r,\alpha)\,|\Psi\rangle \text{ [17]} \tag{5.73}$$

and the same holds for $A_{L-p(L)}^\dagger(r,\alpha)$. This means that, in the thermodynamic limit, the chart featuring all available one-particle energies is densely covered by constant functions of any value e between 0 and $-\frac{\pi^2}{2}$ for $r \in [0,1[$ and any α. Further, for $r > 1$ and any α, these constant functions have switched their value to $e_> = -\frac{(\pi-\sqrt{-2e})^2}{2}$, again, densely covering the range between 0 and $-\frac{\pi^2}{2}$. In between, at $r = 1$, we have to distinguish between the two cases $\alpha \neq \alpha_0$ and $\alpha = \alpha_0$: The former leads to positive values of e_1 densely covering the range between 0 and $\frac{\pi^2(\sqrt{2}-1)}{4}$ (cf. eq. (5.70)), while choosing $\alpha = \alpha_0$ leads to negative values $e_0 = -e_1$ densely covering the range between 0 and $\frac{\pi^2(1-\sqrt{2})}{4}$.

As was mentioned before, the zero modes $a_0^\dagger(r,\alpha)$ and $A_0^\dagger(r,\alpha)$ are ubiquitous leading to a likewise ubiquitous degeneracy of integer multiples of four in the spectrum, e.g. the seed state $|\Psi\rangle$ shares the zero energy level with three other states, i.e. $a_0^\dagger(r,\alpha)\,|\Psi\rangle$, $A_0^\dagger(r,\alpha)\,|\Psi\rangle$, and $a_0^\dagger(r,\alpha)A_0^\dagger(r,\alpha)\,|\Psi\rangle$, all of which form one of all the projective representations the state space $\mathcal{H}$ splits into. Furthermore, the nilpotent part in the Hamiltonian $H(r,\alpha)$ due to the term proportional to $a_0^\dagger(r,\alpha)A_0^\dagger(r,\alpha)$ seems to only be canceled exactly at $r = 1$ and $\alpha = 0$. From the discussion in Chapter 4, we already know that this choice of r and α leads to the description of the system in terms of two copies of the Ising CFT in the disorder sector

[16] The index is shifted to $L - p(L)$ according to the assumption made in the conjecture about the general form of the Hamiltonian in Section 5.3.3 due to equality of energies of left- and right-movers.

[17] Note that the seed state $|\Psi\rangle$, of course, depends on L, as well as r and α

while, for $r = 1$ and $\alpha \notin \{0, \alpha_0\}$ it resembles the symplectic fermion. In particular, both systems are antiferromagnetic in nature with the singlet $|\Psi\rangle$ as their ground state.

Now, deviating $r = 1$ with some arbitrary value of $\alpha \in]-\alpha_0, \alpha_0[$ immediately changes the dispersion relation in the thermodynamic limit from a relativistic version in the low-energy regime, cf. eq. (5.70), to a non-relativistic dispersion relation, cf. eqs. (5.58) and (5.66), signaling the abrupt loss of criticality. Moreover, the system becomes ferromagnetic as the singlet $|\Psi\rangle$, suddenly, is the highest excited state since any application of the creation operators $a_p^\dagger(r, \alpha)$ or $A_p^\dagger(r, \alpha)$ (for $p = 1, \ldots, L-1$, $r \neq 1$ and arbitrary α) lead to a lower eigenvalue of the resulting state. However, the global symmetry of the system is, of course, preserved and, additionally, the numerical observations indicate that the two zero modes induce an omnipresent nilpotent part of the Hamiltonian. One could, hence, speak of a non-relativistic symplectic fermion theory in this regime.

Finally, moving from somewhere with $r \neq 1$ at $\alpha = \alpha_0$ towards $r = 1$ leads right to the coalescence of both sublattices. In this limit, the non-relativistic symplectic fermion theory will abruptly switch to a theory with a relativistic dispersion relation which, compared to the symplectic fermion dispersion relation, seems to have the wrong sign (eq. (5.72)). Again, the singlet state $|\Psi\rangle$ turns out to be the highest excited state implying that the system is ferromagnetic. Numerically, it also appears as if there is a nilpotent part in the Hamiltonian, however, this is not decidable since the algorithm runs into trouble when approaching this point. In any case, these values of r and α are mathematically problematic due to their inconsistent limits along the r- or α-direction as mentioned in Section 5.3.2. If one were, nevertheless, to ascribe any validity to this choice then the system in the limit $r \to 1$ for $\alpha = \alpha_0$ appears like a ferromagnetic version of the symplectic fermion with its ground state given by the completely filled state

$$|X\rangle := \lim_{L\to\infty} a_L^\dagger(1, \alpha_0) A_L^\dagger(1, \alpha_0) \cdots a_1^\dagger(1, \alpha_0) A_1^\dagger(1, \alpha_0) |\Psi\rangle , \tag{5.74}$$

an expression of which it is far from evident if it could be well-defined as it is not clear whether this converges to some definite entity. Additionally, its energy $e_0(X)$ is not bounded from below:

$$\begin{aligned} e_0(X) &= \lim_{L\to\infty} 2 \sum_{p=1}^{L-1} \frac{\pi^2}{2} \frac{p(p-L)}{L^2} && (5.75)\\ &= \pi^2 \lim_{L\to\infty} \frac{L(L-1)(L+1)}{6L^2} && (5.76)\\ &= \frac{\pi^2}{6} \lim_{L\to\infty} L. && (5.77) \end{aligned}$$

The same is true for the completely filled state $|X\rangle$ for $r \neq 1$:

$$e_{\lessgtr}(X) = -\lim_{L\to\infty} 2 \sum_{p=1}^{L-1} \frac{\pi^2}{2} \frac{p^2}{L^2} \tag{5.78}$$

$$= -\pi^2 \lim_{L\to\infty} \frac{L(L-1)(2L-1)}{6L^2} \tag{5.79}$$

$$= \frac{\pi^2}{3} \lim_{L\to\infty} L. \tag{5.80}$$

Therefore, the conclusion of this whole discussion about the continuum limit of the $\mathfrak{gl}(1|1)$-spin ladder setup is the following: It is critical if and only if $r = 1$. For $\alpha = 0$, it yields two copies of the Ising CFT in the disorder sector. For $\alpha \in]-\alpha_0, \alpha_0[$, it is described by the symplectic fermion theory. Both systems feature a lowest energy at zero which is four-fold degenerate and they are antiferromagnetic as one of the four ground states, $|\Psi\rangle$, is a singlet. The case $\alpha = \alpha_0$ and $r \to 1$ should, if at all, be discussed in the context of $r \neq 1$. For $r \neq 1$, the dispersion relation of the one-particle energies become non-relativistic and, thus, the system cannot be critical. Further, the spectrum is now bounded from above by zero but not bounded from below. The singlet $|\Psi\rangle$ is one of the four highest states at zero energy. These two systems are, hence, ferromagnetic. Finally, approaching $r = 1$ along $\alpha = \alpha_0$ where the two sublattices coalesce does not change the ferromagnetic nature of the system but leads to a relativistic dispersion relation. If we were to allow for this to happen – although at least mathematically questionable – the system would look like a ferromagnetic version of the symplectic fermion (or Ising CFT in the disorder sector, depending on the existence of the nilpotent part in the Hamiltonian).

6

Conclusion and Outlook

You have begun to find your answers. Although it will seem difficult, the rewards will be great. Exercise your human mind as fully as possible, knowing that it is only an exercise. Build beautiful artifacts, solve problems, explore the secrets of the physical universe, savor the input from all the senses, filled with joy and sorrow and laughter, empathy, compassion, and tote the emotional memory in your travel bag. I remember where I came from, and how I became human, why I hung around – and now my final departure's scheduled. This way out, escaping velocity.

– Ryan Power, *Waking Life*

In this thesis, we tried to elucidate further the dichotomy between CFT on the one hand and long-range spin systems, in particular, spin chains of Haldane-Shastry-type, on the other hand. This intimate connection was inspired by an article by Moore and Read [4] in the context of quantum Hall physics and its evolution was reignited by Sierra and Cirac [5], and carried on by many more, to cite just a few [6, 13, 14, 47, 54–56, 70]. However, these works were always concerned with WZW models in the case of a classical symmetry, such as SU(N). Here, we presented the construction of the analogous spin systems for the GL($m|n$) WZW model. We explicitly computed the associated chiral correlators that serve as the coefficients of a quantum state $|\Psi\rangle$ in a product basis on the bipartite state space $\mathcal{H} = (\mathcal{V} \otimes \bar{\mathcal{V}})^{\otimes L}$ built from the fundamental and antifundamental $\mathfrak{gl}(m|n)$-representation. Furthermore, a parent Hamiltonian $H(\{z_i\}_i)$ was derived that always projects $|\Psi\rangle$ $(\{z_i\}_i)$ to zero. Due to failing charge cancellation of the chiral correlator in the pure case $\mathcal{H} = \mathcal{V}^{\otimes L}$, the chiral correlator used for the construction of the Hamiltonian is zero. The structure of the constructed parent Hamiltonian, though, still incorporates this *pure* case, i.e. we have a nontrivial Hamiltonian but no sensible correlator that would be projected to zero by this Hamiltonian in this case.

In general, the GL($m|n$)-invariant Hamiltonians that we constructed feature long-range spin-spin interactions, and, additionally, long-range three-spin interactions unless a special arrangement of the spins in the complex plane is imposed. For $N = m - n = 0$, for example, we found that the three-spin terms drop out when the spins are placed somewhere on the real line for the choice $w_{ij} := \frac{1}{z_i - z_j}$ or, alternatively, when the spins are placed somewhere on a circle centered about the origin for the choice $w_{ij} := \frac{z_i + z_j}{z_i - z_j}$. We also commented on

the relevance of the case $N = 1$ and the observation leading to the conjecture about a general correspondence between all such spin systems with $\mathrm{GL}(m|n)$ symmetry and their counterparts with $\mathrm{GL}(n|m)$ symmetry. Moreover, it was shown that, in the $\mathrm{GL}(N|0)$-case, our Hamiltonian is a straight forward generalization of the $\mathrm{SU}(N)$ Hamiltonian constructed in [13, 14]. Finally, again for $N = 0$, we showed explicitly how the Hamiltonian simplifies (for $w_{ij} = \frac{z_i+z_j}{z_i-z_j}$) when the spins in the bipartite case $\mathcal{H} = (\mathcal{V} \otimes \bar{\mathcal{V}})^{\otimes L}$ are all placed on the unit circle, and the $\mathcal{V}$s and $\bar{\mathcal{V}}$s each form two uniform sublattices that may be twisted against each other. In that case, the Hamiltonian constitutes two Haldane-Shastry-like Hamiltonians, each one restricted to a single sublattice $\mathcal{V}^{\otimes L}$ or $\bar{\mathcal{V}}^{\otimes L}$, and a part that captures all the mixed spin-spin term, which is also Haldane-Shastry-like but includes the angular twist in the inverse sine-squared.

After having presented the rather generic construction, we analyzed the alternating Haldane-Shastry model of $\mathfrak{gl}(1|1)$-spins to get an idea of what to expect of our Hamiltonian in an analogous setup. We found that the alternating Haldane-Shastry $\mathfrak{gl}(1|1)$-spin chain can be almost diagonalized leading to two sets of mutually anticommuting free fermions with two zero modes that are coupled in the Hamiltonian, $a_0^\dagger A_0^\dagger$, so as to yield a nilpotent part leading to the fact that the best one can do with the Hamiltonian is bring it to a Jordan normal form. This also reflects the indecomposable structure of the state space $\mathcal{H} = (\mathcal{V} \otimes \bar{\mathcal{V}})^{\otimes L}$ as this tensor product decomposes into reducible but indecomposable projective covers of one-dimensional atypical representations of $\mathfrak{gl}(1|1)$, and leads to an ubiquitous factor of four in the degeneracy pattern of the spectrum. That structure and the global symmetry of the system led to the identification of the symplectic fermion CFT as the effective low-energy description of the spin chain in the continuum limit. Hence, it was identified to be critical. On the other hand, our Hamiltonian in an analogous Haldane-Shastry-like setup turned out to look almost the same, apart from an additional term proportional to the total spin of the system. Surprisingly, it emerged that this term exactly cancels the zero mode term $a_0^\dagger A_0^\dagger$ in our $\mathrm{GL}(1|1)$-invariant Hamiltonian. With this cancellation, our alternating $\mathrm{GL}(1|1)$ Hamiltonian resembles the sum of two Hamiltonians for the *pure* Haldane-Shastry model of $\mathfrak{gl}(1|1)$-spins, which was introduced in the preliminary Chapter, with its continuum limit given by the Ising CFT in the disorder sector. Therefore, our alternating $\mathrm{GL}(1|1)$ Hamiltonian with a Haldane-Shastry-like setup, in a way, relates these with CFTs: It looks like two copies of the Ising CFT in the disorder sector that have the same global $\mathrm{GL}(1|1)$ symmetry as the symplectic fermion theory but its two copies are not coupled to each other in the way it would be expected for the symplectic fermion CFT. In other words, its state space decomposes in the same way as the symplectic fermion's state space, i.e. in summands of four-dimensional projective $\mathfrak{gl}(1|1)$-representations, but its Hamiltonian is fully diagonalizable which is in contrast to the alternating Haldane-Shastry $\mathfrak{gl}(1|1)$-spin chain.

During that analysis, we encountered one, at first, puzzling contradiction: While the Ising CFT has a central charge of $\frac{1}{2}$ and the product of two of these would thus have to have a central charge of 1, the symplectic fermion has central charge -2. This issue

could be resolved because it was realized that one has to instead consider the *effective* central charge of the Ising CFT in the disorder sector $(\frac{1}{16}, \frac{1}{16})$, $\tilde{c} = \frac{1}{2} - 24\frac{1}{16} = -1$. It also proved that the duality between the *pure* Haldane-Shastry model and the WZW model at level $k = 1$ sketched in Figure 1.1, which breaks down for the alternating Haldane-Shastry model for $\mathfrak{su}(N)$-spins with $N > 2$, also breaks down for GL(1|1) in the alternating case since the central charge of the GL(1|1) WZW model is $c = 0$. However, it also led to the critical revision of finite-size scaling results for the ground state energy scaling of the Haldane-Shastry model for SU(N) and also for GL(1|1) which somehow never produced the expected results. On the other hand, this works fine for, e.g. spin chain of Heisenberg-type. But then, critical Heisenberg spin chains are always approximated worse than Haldane-Shastry-like spin chains by the respective WZW model, the reason being that they lack long-range interactions. These seem to be necessary to, e.g. not suffer from logarithmic corrections in the energy scaling, and is also at the heart of why Haldane-Shastry spin chains also share more structure with the respective WZW model than Heisenberg spin chains: The former's symmetry is enhanced to a Yangian which makes solving it, in a way, even simpler than solving the Heisenberg spin chain. The Yangian itself is an immanent structure of $\mathrm{SU}(N)_1$ WZW models. Moreover, the stress energy tensor of this CFT is given by the Sugawara construction involving bilinears in the currents. Extracting L_0 from this involves a double contour integration which leads to an expression that is reminiscent of the Hamiltonian of the respective Haldane-Shastry model, with normal ordering of the operators regularizing the expression when both integration variables coincide. For all these similarities, it seems to be not too farfetched to define the Haldane-Shastry Hamiltonian in an analogous way, allowing summation about coincident indices but regularizing these terms by simply dropping singular terms in the Taylor expansion of the inverse sine-squared. With this prescription, finite-size scaling of the ground state energy matches perfectly the results that one expects for the Haldane-Shastry model.

In Chapter 5, we introduced a two-parameter spin ladder setup that interpolates between the fully uniform alternating setup and a setup where the two sublattices formed by $\mathcal{V}$ and $\bar{\mathcal{V}}$, respectively, form uniform sublattices on two different concentric circles of distinct radii and twisted against the fully uniform setup by an angle α. Further, we gave the GL($m|m$) Hamiltonian for this setup which decomposes into three parts: The first part looks like the GL($m|m$) Hamiltonian in the Haldane-Shastry-like setup, however, restricted to the fundamental sublattice. The second part looks the same, however, restricted to the antifundamental sublattice. The third part is constructed from all mixed terms of spin-spin and three-spin interactions and depends on the two parameters of the spin ladder setup. Additionally, we proposed the analytic expressions for the full spectrum of the generic $\mathfrak{gl}(1|1)$-spin ladder, so in terms of its system size, and the two spin ladder parameters. With this knowledge and the information we had from the fully uniform alternating setup discussed in the preceding Chapter, we took a reasonable guess at the general (almost) diagonal form of the $\mathfrak{gl}(1|1)$-spin ladder Hamiltonian.

Finally we discussed these solutions in the thermodynamic limit and saw that these

systems are only critical when both sublattices have the same radius. While the fully uniform case was already identified as given by two copies of the Ising CFT in the disorder sector with the global $\mathrm{GL}(1|1)$ symmetry of the symplectic fermion, exact diagonalization of the system of sizes up to $2L = 12$ implied that the slightest angular twist of both sublattices against each other immediately introduced the nilpotent part in the Hamiltonian known from the symplectic fermion. This shows that our $\mathrm{GL}(1|1)$-invariant Hamiltonian is capable of featuring either critical behavior, two copies of the Ising CFT in the disorder sector for a fully uniform distribution of sites on a circle, or the symplectic fermion CFT when both sublattices are twisted against each other by even the smallest angle.

Having presented this analysis various paths for further investigations can be suggested. First exact diagonalization of the $\mathfrak{gl}(2|2)$-spin ladder indicate that it seems not possible to give analytic expressions for the full spectrum in dependence of the two parameters even though they also show some sort of regularity apart from the energy levels that stem from the $\mathfrak{gl}(1|1)$ representations that are, of course, also present. For $\mathrm{SU}(2)$, there seems to be more hope, even though, the partial results for the energy levels depending on the radial parameter become much more complicated when increasing the system size from $2L = 4$ to $2L = 6$. Furthermore, it would be interesting to investigate the Haldane-Shastry-like spin chains from our Hamiltonian for other cases of $\mathfrak{gl}(m|n)$-spins for aspects of criticality. However, how to build a nonzero chiral correlator in the pure case is still an open question and may require more advanced techniques than using free fields. Nonetheless, even the seed state $|\Psi\rangle$ that we calculated for the alternating case may prove to be an interesting object to study. For the $\mathfrak{gl}(1|1)$-spin ladder setup, it would be interesting particularly to see how it behaves under braiding, i.e. under passing one of the sublattices by the other and returning it to its original position since we have full knowledge about how the energy levels would change. Finally, it could be worthwhile investigating its topological properties on two-dimensional lattices.

Lastly, the form of the Hamiltonian depends on the representations used as the basic building blocks because the projection operators that are needed change their form according to the tensor product decomposition. It would be interesting to try to carry out the construction for the infinite-dimensional $\mathfrak{gl}(2n|2n)$-representations that play an important role in the theory of the integer quantum Hall plateaux transition.

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– Walt Whitman, *Leaves of Grass*

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Erklärung

Ich versichere, dass ich die von mir vorgelegte Dissertation selbständig angefertigt, die benutzten Quellen und Hilfsmittel vollständig angegeben und die Stellen der Arbeit – einschließlich Tabellen, Karten und Abbildungen –, die anderen Werken im Wortlaut oder dem Sinn nach entnommen sind, in jedem Einzelfall als Entlehnung kenntlich gemacht habe; dass diese Dissertation noch keiner anderen Fakultät oder Universität zur Prüfung vorgelegen hat; dass sie noch nicht veröffentlicht worden ist, sowie, dass ich eine solche Veröffentlichung vor Abschluss des Promotionsverfahrens nicht vornehmen werde. Die Bestimmungen der Promotionsordnung sind mir bekannt. Die von mir vorgelegte Dissertation ist von Herrn PD Dr. Thomas Quella betreut worden.

Köln, den 1. April 2017

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Jochen C. Peschutter