Effective equations for a cloud of ultracold atoms in an optical lattice

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Marcel Hendrik Braukhoff

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Berichterstatter:	Prof. Dr. Markus Kunze
	Prof. Dr. Bernd Kawohl
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Kurzzusammenfassung

Bereits seit einigen Jahrzehnten blüht die Forschung in der angewandten Mathematik zu Ladungstransport in Halbleitern auf. Da ein Halbleiter aus einigen 10^{23} Atomen besteht, sind vor allem effektive Gleichungen von großem Interesse, um einzelne Phänomene erklären zu können. Vor ein paar Jahren wurden die zahlreichen Modelle um ein weiteres ergänzt, dass im Gegensatz zu den bisherigen Modellen selbst ein Experiment ist. In diesem Modell werden die geladenen Elektronen durch ultrakalte Atome dargestellt und das periodische Potential des Halbleiterkristalls wird mithilfe eines optischen Gitters realisiert.

Das Ziel dieser Dissertation ist die mathematische Behandlung von effektiven Gleichungen zur Beschreibung einer Wolke aus ultrakalten Atomen in einem optischen Gitter. Der Hauptunterschied von dem Experiment mit ultrakalten Atomen zu dem Ladungstransport in Halbleitern liegt in der unterschiedlichen Wechselwirkung. Bei ultrakalten, nicht geladenen Atomen tritt eine sehr singuläre Wechselwirkung auf. Da diese wesentlich irregulärer ist als die Coulombwechselwirkung zwischen Elektronen, erschwert dies erheblich die Analysis der Gleichungen.

Eine Halbleiter-Boltzmann Gleichung mit einem BGK-Stoßoperator und einem singulären Potential ist ein geeignetes mikroskopisches Modell für eine Wolke aus ultrakalten Atomen in einem optischen Gitter. Es wird gezeigt, dass diese Gleichung für kurze Zeit eine analytische Lösung besitzt. Dafür wird allerdings vorausgesetzt, dass geeignete und analytische Anfangswerte vorliegen, deren Energiedichten klein genug sind. Ersetzt man den BGK-Stoßoperator durch eine lineare Relaxationszeit-Approximation mit konstantem Gleichgewicht, so wird für diese Halbleiter-Boltzmann Gleichung eine globale Lösung gefunden. Dafür müssen analytische Anfangswerte vorliegen, von denen jegliche Ableitungen genügend klein sind.

Aus den mikroskopischen Gleichungen kann man mittels eines diffusiven Limes mikroskopische Gleichungen erhalten. In dieser Dissertation werden aus der Halbleiter-Boltzmann Gleichung mit einem BGK-Stoßoperator sowohl eine Driftdiffusionsgleichung als auch zwei Energietransport Gleichungen for-

mal hergeleitet. Die Driftdiffusionsgleichung ist bei kleiner Dichte der Atomwolke die logarithmische Diffusionsgleichung. Diese Gleichung wird auf einem beschränktem Gebiet betrachtet und mit Randbedingungen versehen, die aus der mikroskopischen Gleichung motiviert werden. Mit diesen Randwerten besitzt die logarithmische Diffusionsgleichung eine globale Lösung die exponentiell abfällt.

Die erste der beiden Energietransport Gleichungen wird direkt über einen diffusiven Limes der Halbleiter-Boltzmann Gleichung mit einem BGK-Stoßoperator erhalten und besteht aus einer Kreuzdiffusiongleichung. Bei solchen Kreuzdiffsusionsgleichungen wird häufig die Wohlgestelltheit mittels Entropieabschätzungen bewiesen. In dieser Dissertation wird allerdings gezeigt, dass diese Entropieabschätzungen wesentlich schwächer sind als bei gewöhnlichen Kreuzdiffusionsgleichungen für Halbleiter. Der Grund dafür ist das singuläre Wechselwirkungspotential, das zu starken Degeneriertheiten in der Entropiedissipation führt. Um dieses System an Gleichungen zu vereinfachen, kann man formal eine Entwicklung nach hohen Temperaturen durchführen und erhält ein zweites Energietransport Gleichungssystem. Für die Hochtemperatur-Energietransport Gleichungen wird in dieser Dissertation eine schwache untere Lösung, sowie eine numerische Lösung ermittelt.

Abstract

In the last decades, the theory of charge transport in semiconductors has become a thriving field in applied mathematics. Due to the complexity of semiconductors consisting of some 10^{23} atoms, there are several effective equations describing different phenomenological properties of semiconductors. Recently, the description of charge transport in semiconductors was extended by an experimental model: a cloud of ultracold atoms in an optical lattice. In this model, the ultracold atoms stand for the charged electrons and the optical lattice describes the periodic potential of the crystal formed by the ions of the semiconductor.

This thesis is dedicated to effective equations for this experimental model of the charge transport in semiconductors. The main difference between a cloud of ultracold atoms and a system of electrons is the interaction. Assuming that the atoms are uncharged, the interaction potential is significantly more singular than the Coulomb potential of the electrons causing major structural difficulties in the analysis.

In the microscopic description, this thesis investigates a semiconductor Boltzmann equation with BGK-type collision operator and a singular interaction potential. It is shown that for adequate analytic initial data, this equation possesses a local, analytic solution. Moreover, replacing the collision operator by a linear relaxation time approximation with constant equilibrium, this thesis provides a proof of the global existence of an analytic solution if all derivatives of the analytic initial data are sufficiently small.

Using a diffusive limit, some macroscopic models are formally derived from the semiconductor Boltzmann equation with BGK-type collision operator: a drift diffusion equation being equal to the logarithmic diffusion equation and two systems of energy transport equations.

The logarithmic diffusion equation is treated on a bounded domain with non-standard boundary conditions motivated by the microscopic picture of ultracold atoms. It is shown that the logarithmic diffusion equation admits a global classical solution which decays exponentially in time. The first energy transport model is a cross-diffusion equation which formally admits an entropy

function. However, the singular interaction potential leads to degeneracies in the entropy dissipation, undermining a rigorous solution so far. Approximating the system formally, this thesis simplifies the first energy transport model to a second energy transport model, namely its high temperature expansion. This high temperature energy transport model is solved numerically and possesses a non-trivial weak lower solution.

Contents

Part I

Introduction

Computer, mobile phones and other devices based on semiconductors have developed a growing influence on our daily life. Therefore, it is not surprising that the mathematical modeling of charge transport in semiconductors has become more and more popular. The simplest picture of a solid state material is a crystal formed by a lattice of ions and electrons moving inside this crystal. Noting that the ions are heavier than electrons by a magnitude of $10^3 - 10^5$, it is reasonable to assume that the electrons are much faster than the ions so that the movement of the ions can be neglected in the phenomenological way. This means that the charge transport is given by the movement of the electrons in the semi-classical picture. Since a semiconductor consists of the order of 10^{23} ions and electrons, further simplified models are required to understand the effective behavior of the electrons. So far, mathematical literature [27, 26, 37] basically considered the interactions between ions and electrons and the electron-electron interaction to be of long range, e.g., given by the electrostatic force. In this context, several effective equations have been derived. These equations can be divided into quantum mechanical models or semi-classical models. Moreover, one can distinguish between microscopic and macroscopic models (see [27]).

In [49], Schneider et al. use a cloud of ultracold fermions in an optical lattice as an experimental model of some semiconductor materials. In their experiment, the potential of the optical lattice represents the periodic potential of the ions in a semiconductor. The electrons are modeled by ultracold fermions. From the experimental point of view, an optical lattice can easily be adjusted, i.e. its lattice constants and intensity, in contrast to solid state materials. Changing the lattice constant of a crystal for an experiment results in growing a new crystal and dismissing the previous one.

The aim of this thesis is to consider semi-classical models for this experiment. This includes equations in the microscopic as well as in the macroscopic picture describing a short range interactions between the fermions. The thesis is based on the thesis of Mandt [36], who derived some effective equations for an ultracold cloud of fermions in an optical lattice and solved them numerically. A mathematical rigorous theory has been missing so far. The basic difference of the models from [36] to the standard models in [27, 26, 37] is the interaction potential. In semiconductors, the interaction between the electrons is given by the Coulomb force

$$F(x,t) = -\nabla_x (e\Phi * n)(x,t) \quad \text{for} \quad \Delta\Phi(x) = 4\pi\delta_x,$$

where $e > 0$ denotes the absolute value of the charge of the electrons and n the electron particle density. However, for the model of ultracold fermions in an optical lattice, one assumes that the fermions are not charged. Mandt [36] introduces an short ranged interaction between the fermions with

$$F(x,t) = -\nabla_x (\delta_x * n)(x,t) = -\nabla_x n(x,t).$$

Chapter 1

The microscopic picture

As a starting point, we choose a semi-classical approach to describe the behavior of an ultracold cloud of fermions in an optical lattice. In general, the density distribution $f = f(x, p, t)$ of a cloud of indistinguishable particles in a force field $F = F(x, t)$ can be modeled by a semiconductor Boltzmann equation (cf. [37, 26, 15]), namely

$$\partial_t f(x,p,t) + v(p) \cdot \nabla_x f(x,p,t) + F(x,t) \cdot \nabla_p f(x,p,t) = Q(f(x,\cdot,t))(p). \quad (1.1)$$

The value $f(x, p, t)$ equals the density of particles at point x with momentum p and at time t. Let ϵ denote the dispersive relation, i.e. the function connecting the momentum p to the energy. For free particles the dispersion relation is given by $\epsilon(p) = \frac{1}{2m}p^2$, where m is the mass of each particle. The velocity in (1.1) is defined by $v(p) := \nabla\epsilon(p)$ for all p. This implies that the velocity is proportional to momentum with $p = mv(p)$ for free particles. In addition, the scattering operator $Q(\cdot)$, being in general non-local in p, models short ranged collisions of the particles.

1.1 Vlasov equation

If the scattering operator vanishes, Eq. (1.1) transforms into the Vlasov equation

$$\partial_t f(x,p,t) + v(p) \cdot \nabla_x f(x,p,t) + F(x,t) \cdot \nabla_p f(x,p,t) = 0. \quad (1.2)$$

The Vlasov equation can directly be motivated by the Newton law for each particle

$$\partial_t x = v \quad \text{and} \quad \partial_t p = F. \quad (1.3)$$

According to Newton's law, a test particle in the force field F moves along the trajectory (1.3) through the phase space.

With the assumption that f is constant along the trajectories from (1.3), we obtain that

$$0 = \frac{d}{dt} f = \partial_t f + \partial_t x \cdot \nabla_x f + \partial_t p \cdot \nabla_p f = \partial_t f + v \cdot \nabla_x f + F \cdot \nabla_p f.$$

Note that for now, this equation is only justified on the trajectories from above. However, one can also extend this result for all x, p and t and obtains Eq. (1.2).

Therefore, the Vlasov equation is an easy way to model a large cloud of indistinguishable particles driven by the force F in the microscopic picture. Modeling free particles, the momentum is proportional to the velocity as seen above. By choosing the right coordinates, one can assume without loss or generality that the mass equals one implying that $v = v(p) = p$. We thus can rewrite the Vlasov equation by

$$\begin{cases} \partial_t f(x,v,t) + v \cdot \nabla_x f(x,v,t) + F(x,t) \cdot \nabla_v f(x,v,t) = 0, \\ f(x,v,0) = f_0(x,v), \end{cases} \tag{1.4}$$

where $x \in \mathbb{R}^d$ represents the position in space, $v \in \mathbb{R}^d$ the velocity and $t > 0$ the time. Note that in (1.4), the velocity v is used as a coordinate. In plasma physics, assuming that the influence of magnetic fields as well as the movement of the ions are small and therefore neglectable, dilute plasma can be modeled by a self-consistent version of (1.4), where

$$F(x,t) = -\nabla_x (e\Phi * n_f)(x,t) \text{ for } \Delta\Phi = 4\pi\delta_0 \text{ and } n_f(x,t) = \int_{\mathbb{R}^d} f(x,v,t)dv. \tag{1.5}$$

Here, $e > 0$ denotes the absolute value of the electron charge. This system, known as the Vlasov-Poisson equation, can be solved globally in time in spatial dimension three [34, 42]. Moreover, there are several articles devoted to the decay properties of the solution. In [40], Mouhot and Villani prove the global existence of classical solutions of a non-linear Vlasov equation of the form

$$\begin{cases} \partial_t f(x,v,t) + v \cdot \nabla_x f(x,v,t) + \nabla_x \Phi * n_f(x,t) \cdot \nabla_v f(x,v,t) = 0, \\ f(x,v,0) = f_0(x,v) \end{cases} \tag{1.6}$$

for $x \in \mathbb{T}^d, v \in \mathbb{R}^d$ and $t > 0$, where the Fourier transform $\hat{\Phi}$ of the periodic interaction potentials Φ satisfies

$$\left|\hat{\Phi}(l)\right| \leq \frac{C_\Phi}{|l|^\gamma} \quad \text{for all } l \in \mathbb{Z}^d$$

and for some constants $C_\Phi > 0$, $\gamma \geq 2$. Mouhot and Villani [40] work on the d-dimensional torus instead of the whole $\mathbb{R}^d$. If the initial data of (1.4) is sufficiently close to an appropriate velocity profile $f^0 = f^0(v)$, then the unique classical solution of (1.6) converges exponentially fast in the weak topology of $L^2_x(\mathbb{T}^d; L^1_v(\mathbb{R}^d))$ to a spatially homogeneous equilibrium $f_{\pm\infty}$ as $t \to \pm\infty$ [40]. In addition, the particle density as well as the force converge strongly in $L^2(\mathbb{T}^d)$ to a constant, again exponentially fast. The key ingredient for their proof is the concept of analytic norms and the smoothing effect of the interaction potential W.

Inserting the delta distribution δ_0 for $e\Phi$ in (1.5), we obtain

$$F(x,t) = -\nabla_x(\delta_0 * n_f)(x,t) = -\nabla_x n_f(x,t) \quad \text{for } n_f(x,t) = \int_{\mathbb{R}^d} f(x,v,t)dv. \tag{1.7}$$

Combining this with the Vlasov equation yields the Vlasov-Dirac-Benny equation

$$\partial_t f(x,v,t) + v \cdot \nabla_x f(x,v,t) - \nabla n_f(x,t) \cdot \nabla_v f(x,v,t) = 0 \tag{1.8}$$

for $x \in \mathbb{R}^d, v \in \mathbb{R}^d$ and $t > 0$. In spatial dimension one, this equation can be used to describe the density of fusion plasma in a strong magnetic field in direction of the field [8]. Comparing the Vlasov-Poisson equation ((1.4) in conjunction with (1.5)) to the Equation (1.8), we see that the interaction potential Φ is long ranged (i.e., the support is the whole space) in contrast to the delta distribution with $\text{supp}(\delta_0) = \{0\}$. Therefore, we can understand (1.8) as a version of the classical Vlasov-Poisson system with a short-ranged Dirac potential, which motivated the "Dirac" in the name of the Vlasov-Dirac-Benny equation. The name Benny is due to its relation to the Benny equation in dimension one (for details see [5]).

However, the analysis of a Vlasov-Dirac-Benny equation is more delicate as in [25] only local in time solvability was shown for analytic initial data in spatial dimension one. Moreover, it is shown in [5] that this system is not locally weakly ($H^m - H^1$) well-posed in the sense of Hadamard. Very recently, [17] show that the Vlasov-Dirac-Benny is ill-posed in $d = 3$, requiring that the spatial domain is restricted to the 3-dimensional torus $\mathbb{T}^3$. More precisely, they show that the flow of solutions does not belong to $C^\alpha(H^{s,m}(\mathbb{R}^3 \times \mathbb{T}^3), L^2(\mathbb{R}^3 \times \mathbb{T}^3))$ for any $s \geq 0, \alpha \in (0,1]$ and $m \in \mathbb{N}_0$. Here, $H^{s,m}(\mathbb{R}^3 \times \mathbb{T}^3)$ denotes the weighted Sobolev space of order s with weight $(x,v) \mapsto \langle v\rangle^m (1+|v|^2)^{m/2}$: they prove that there exists a stationary solution $\mu = \mu(v)$ of (1.8) and a family of solutions $(f_\varepsilon)_{\varepsilon>0}$, times $t_\varepsilon = O(\varepsilon\,|\log\varepsilon|)$ and $(x_0, v_0) \in \mathbb{T}^3 \times \mathbb{R}^3$ such that

$$\lim_{\varepsilon\to 0} \frac{\|f_\varepsilon - \mu\|_{L^2([0,t_\varepsilon]\times B_\varepsilon(x_0)\times B_\epsilon(v_0))}}{\|\langle v\rangle^m (f_\varepsilon|_{t=0} - \mu)\|^\alpha_{H^s(\mathbb{T}^3_x\times\mathbb{R}^3_v)}} = \infty,$$

where $B_\varepsilon(x_0)$ denotes the ball with radius ε centered at x_0. In this thesis, however, we face another difficulty by introducing (nonlinear) collision operators to the Vlasov equation. A Vlasov equation with collisions is in general called a semiconductor Boltzmann equation.

1.2 Semiconductor Boltzmann equation

In contrast to plasma physics, the kinetic transport of electrons in semiconductors is in general modeled by a semiconductor Boltzmann equation consisting of the left-hand side of the Vlasov equation in combination with a scattering operator modeling short range interactions in form of “collisions” of the particles [37].

Supposing that the density distribution f can change on the Newtonian trajectories due to a scattering process, we write

$$\frac{d}{dt} f(x(t), p(t), t) = Q(f(x(t), \cdot, t))(p(t))$$

for characteristics $(x(t), p(t))$ solving $\partial_t x = v$ and $\partial_t p = F$. Similarly to the Vlasov case, one can motivate the semiconductor Boltzmann equation for all x, p and t from Eq. (1.1), namely

$$\partial_t f(x,p,t) + v(p) \cdot \nabla_x f(x,p,t) + F(x,t) \cdot \nabla_p f(x,p,t) = Q(f(x,\cdot,t))(p). \quad (1.9)$$

Note that a scattering event taking place at a certain position x at a time t may change the momentum p. Therefore, the scattering operator $Q(f)$ is local in time and space but not necessarily local in p. In addition, the dispersive relation, i.e. the relation between momentum and energy, may differ in semiconductors from the standard case, where the free energy ϵ is given by $\epsilon(p) = \frac{1}{2m} |p|^2$. This means that the velocity $v(p) = \nabla \epsilon(p)$ does not necessarily depend linearly on the momentum p. For example, in the lowest band approximation for semiconductors, the energy dispersion relation is given by

$$\epsilon : p = (p_1, \dots, p_d) \mapsto -2J \sum_{i=1}^{d} \cos(p_i)$$

for some $J > 0$. This implies that the velocity is no longer equivalent to the momentum, i.e. $p \mapsto v(p)$ is not bijective. Therefore, the density function f will be considered as a function of the position x, the momentum p and the time t. A large cloud of charged particles with short ranged collisions can be described by the semiconductor Boltzmann-Poisson equation (1.9) in conjunction with

$$F(x,t) = -\nabla_x (e\Phi * n_f)(x,t) \text{ for } \Delta\Phi(x) = 4\pi\delta_x$$

and $n_f(x,t) = \int_{\mathbb{R}^d} f(x,p,t)dp$ as well as

$$(Q(g))(p) = \int_B (s(p,p')g(p')(1-g(p)) - s(p',p)g(p)(1-g(p')))dp' \quad (1.10)$$

for $g = g(p)$, where $s(p,p')$ is called the transition rate. Here, B denotes the momentum space, which - depending on the context - is $\mathbb{R}^d$ or the first Brillouin zone being a bounded domain in $\mathbb{R}^d$. The presented collision operator in (1.10) is one of the numerous physical relevant choices of the collision operators [27, 26, 37]. The semiconductor Boltzmann-Poisson system from (1.9) was weakly solved by Poupaud [43]. The existence of a global smooth solution is due to Andréasson [3].

However, in the description of ultracold fermions in an optical lattice, Schneider et al. [49] consider a semiconductor Boltzmann-type equation with the singular potential as in the Vlasov-Dirac-Benny equation (1.8). They use the dispersive relation $\epsilon : \mathbb{T}^d \to \mathbb{R}$ defined by $(p_1, \dots, p_d) \mapsto -2J\sum_i \cos(p_i)$. The semiconductor Boltzmann-type equation for ultracold fermions is then given by

$$\begin{cases} \partial_t f(x,p,t) + \nabla_p \epsilon(p) \cdot \nabla_x f(x,p,t) - \nabla_x n_f(x,t) \cdot \nabla_p f = Q_{ee}(f(x,\cdot,t))(p), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad f(x,p,0) = f_0(x,p), \end{cases}$$

where $n_f(x,t) = \int_{\mathbb{T}^d} f(x,p,t)dp$ (see [49]). In contrast to (1.10), the ultracold cloud of fermions permits only two or more particle scattering between the fermions. Therefore, another collision operator is required. Mandt [36] neglects the three or more particle scattering and states the two particle scattering operator for ultracold fermions in an optical lattice as

$$\begin{aligned} Q_{ee}(g)(p) := \sum_{G \in 2\pi\mathbb{Z}^d} \int_B \int_{\substack{p_{\text{tot}}(\mathbf{p})=G \\ \epsilon_{\text{tot}}(\mathbf{p})=0}} Z(\mathbf{p}) \Big(& g(p)g(p')(1-\eta g(p''))(1-\eta g(p''')) \\ & - g(p'')g(p''')(1-\eta g(p))(1-\eta g(p')) \Big) \frac{d\mathcal{H}^{d-1}_{p''}}{|\nabla_{p''}\epsilon_{\text{tot}}(\mathbf{p})|} dp'. \end{aligned}$$

for some $\eta \geq 0$, where $\mathbf{p} = (p,p',p'',p''')$ and $\mathcal{H}^{d-1}_{p''}$ denotes the $d-1$ dimensional Hausdorff measure w.r.t. p''. The function $Z(\mathbf{p})$, modeling the probability of a scattering event from state (p,p_1) to the state (p_2,p_3), shall be sufficiently regular, positive and satisfy

$$Z(\mathbf{p}) := Z(p,p',p'',p''') = Z(p'',p''',p,p').$$

Moreover, the total change of momentum and energy are denoted by

$$p_{\text{tot}}(\mathbf{p}) := p + p' - p'' - p''' \quad \text{and} \quad \epsilon_{\text{tot}}(\mathbf{p}) = \epsilon(p) + \epsilon(p') - \epsilon(p'') - \epsilon(p'''),$$

respectively. The sum over G runs over all reciprocal lattice vectors $G \in 2\pi\mathbb{Z}^d$. Note that in fact only finite summands contribute to the sum since p_{tot} is bounded. This scattering operator is also well-known as the electron-electron scattering operator [9]. We can formally rewrite this collision operator with the aid of the delta distribution by

$$Q_{ee}(g)(p) := \sum_G \int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \delta_{p_{\text{tot}}(\mathbf{p})-G}\delta_{\epsilon_{\text{tot}}(\mathbf{p})} Z(\mathbf{p})\times \\ \times \Big(g(p)g(p')(1-\eta g(p''))(1-\eta g(p''')) \\ - g(p'')g(p''')(1-\eta g(p))(1-\eta g(p'))\Big) dp'dp''dp'''.$$

As the formal definition may suggest, the collision operator conserves the local particle due to its symmetry as well as the local energy due to the $\delta_{\epsilon_{\text{tot}}}$. This is rigorously proved in [27]. However, the local momentum is not conserved because of umklapp processes, i.e. scatter events with $G \neq 0$.

1.3 Relaxation time approximation

Due to the complexity of the two particle scattering operator, the analysis as well as the numerics of Eq. (1.1) with $Q = Q_{ee}$ are very difficult. Therefore, we search for a less complicated physical approximation of Q_{ee}. In [27], Jüngel proves in Proposition 4.6 that the zero set of Q_{ee} consists of Fermi-Dirac distribution functions, i.e. it holds formally that $Q_{ee}(g) = 0$ if and only if there exists a $\lambda = (\lambda_0, \lambda_1) \in \mathbb{R}^2$ with

$$g(p) = \mathcal{F}(\lambda, p) := \frac{1}{\eta + e^{-\lambda_0 - \lambda_1 \epsilon(p)}}.$$

Hence, $\mathcal{F}$ annihilates the collision operator and can be seen as an equilibrium distribution. The idea of the relaxation time approximation is to assume that the collision operator drives the solution into the equilibrium. We define

$$Q_\tau(g)(p) := -\frac{g(p) - \mathcal{F}(\lambda, p)}{\tau}$$

for some $\lambda \in \mathbb{R}^2$, $\tau > 0$ and $g = g(p)$ (see [4]). The parameter τ is called the relaxation time and represents the average time between two scattering events. Since $\mathcal{F}(\lambda, \cdot)$ is a fixed function, the relaxation time approximation collision operator neither conserves the local particle nor the local energy. The simplest version of the relaxation time approximation is to assume that λ_1 vanishes. Then, $\mathcal{F}(\lambda_0, 0)$ equals a constant $\overline{n} \in [0, 1/\eta]$. However, there are also more complicated versions of the relaxation time approximation like the BGK-collision operator.

BGK collision operator

The idea of a Bhatnagar-Gross-Krook-type (BGK-type) collision operator is to combine the simplicity of the relaxation time approximation while keeping the conservation of the local particle and energy (see [13]). Fortunately, the Fermi-Dirac equilibrium distribution offers two free parameter. Thus, we can use those parameter to make the collision operator particle and energy conservative. Doing this, we interpret $\lambda = (\lambda_0, \lambda_1)$ as a function of the densities defined by

$$\begin{pmatrix} n \\ E \end{pmatrix} = \int_{\mathbb{T}^d} \begin{pmatrix} 1 \\ \epsilon(p) \end{pmatrix} \frac{dp}{\eta + e^{-\lambda_0(n,E) - \lambda_1(n,E)\epsilon(p)}} \tag{1.11}$$

and write

$$\mathcal{F}^0(n, E, p) := \mathcal{F}((\lambda_0(n, E), \lambda_1(n, E)), p).$$

Note that this is well-defined according to chapter 5. In addition to this, we may assume that the relaxation time also depends on the densities. In the description of ultracold fermions, Mandt [36] motivates numerically that the inverse of τ is proportional to $n(1-n)$ at high temperatures with $\eta = 1$. Note that for $\eta = 1$, the particle density $n = \int_{\mathbb{T}^d} \mathcal{F}(\lambda, p)dp$ is bounded by 1. We thus generalize the relaxation time for any $\eta \geq 0$ to

$$\tau(n) = \frac{1}{\gamma n(1 - \eta n)}$$

for some $\gamma \geq 0$ since the density $n = \int_{\mathbb{T}^d} \mathcal{F}(\lambda, p)dp$ is bounded by η^{-1}. This leads to the BGK-type collision operator

$$Q_{\mathrm{BGK}}(g)(p) := \gamma n_g(1 - \eta n_g)(\mathcal{F}^0(n_g, E_g, p) - g(p)), \tag{1.12}$$

using $n_g = \int_{\mathbb{T}^d} g(p)dp$ and $E_g = \int_{\mathbb{T}^d} \epsilon(p)g(p)dp$. Now, this collision operator is again non-linear. In [36], the author is also interested in the high temperature limit of (1.12). For the Fermi-Dirac equilibrium distribution $\mathcal{F}(\lambda, \cdot)$, the second variable λ_1 coincides with the negative inverse temperature. Therefore, high temperatures are attained at $\lambda_1 \approx 0$. It is shown in chapter 5 with (1.11) that $\lambda_1 \approx 0$ corresponds to $E \approx 0$ and that at $E \approx 0$ the Fermi-Dirac distribution function is approximately given by

$$\mathcal{F}^0(n, E, p) = n + \frac{\epsilon(p)}{2J^2 d} E + o(E^2). \tag{1.13}$$

Thus, for high temperatures, the BGK-type collision operator is heuristically, approximately given by its Taylor expansion w.r.t. E, namely

$$Q^{\mathrm{hT}}_{\mathrm{BGK}}(g)(p) := \gamma n_g(1 - \eta n_g) \left(n_g + \frac{\epsilon(p)}{2J^2 d} E_g - g(p) \right). \tag{1.14}$$

This collision operator is called the (first order) high temperature expansion of (1.12). An even more drastic approximation can be realized by setting $E = 0$ in (1.13). This leads to the zeroth order high temperature expansion, namely

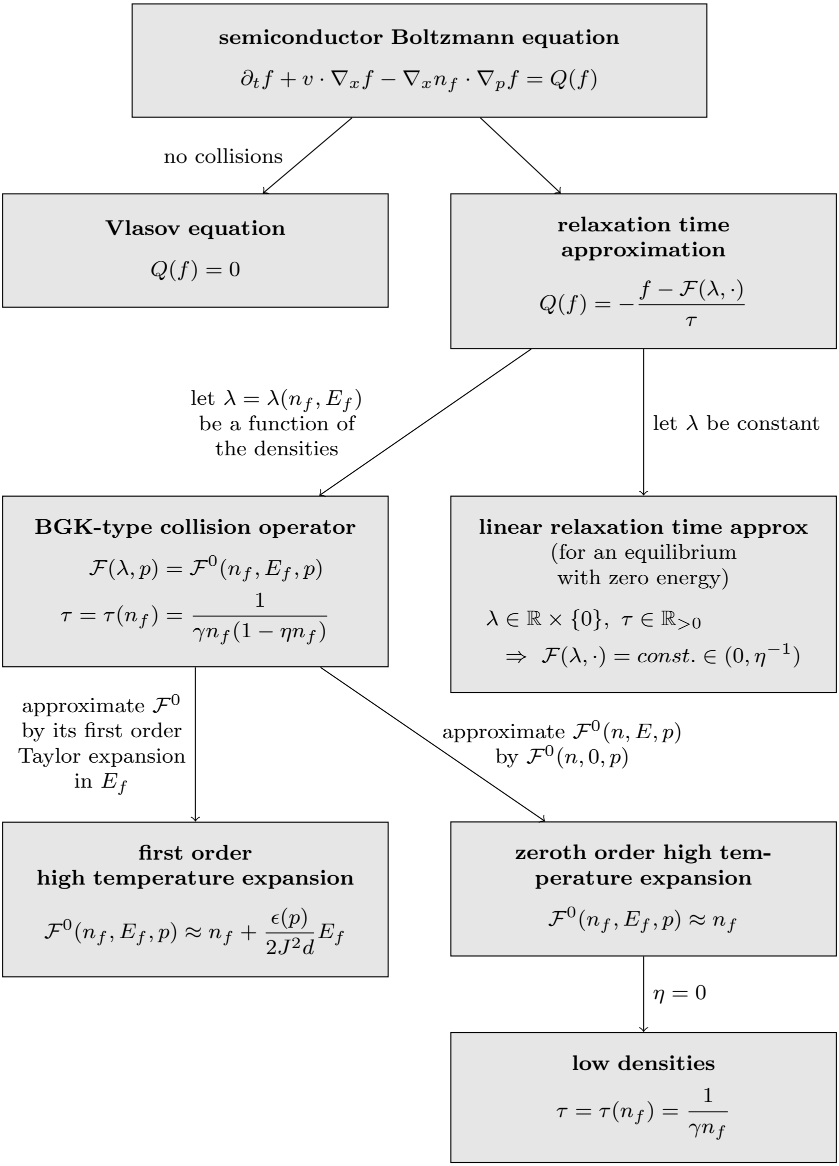
$$Q_{\mathrm{BGK}}^{\mathrm{hT},0}(g)(p) := \gamma n_g (1 - \eta n_g) \left(n_g - g(p) \right) . \tag{1.15}$$

The particle density is very low for dilute gases, implying that $n_g(1 - \eta n_g) \approx n_g$. This motivates the following BGK-type collision operator for dilute gases at high temperatures

$$Q_{\mathrm{dilute}}^{\mathrm{hT},0}(g)(p) := \gamma n_g \left(n_g - g(p) \right) , \tag{1.16}$$

corresponding to $Q_{\mathrm{BGK}}^{\mathrm{hT},0}(g)(p)$ for $\eta = 0$.

1.4 Diagram of the models



1.5 Main Results

In this thesis, we consider the semiconductor Boltzmann-type equation

$$\begin{cases} \partial_t f + \nabla_p \epsilon(p) \cdot \nabla_x f - \nabla n_f \cdot \nabla_p f = Q(f(x,\cdot,t))(p), \\ \qquad\qquad\qquad\qquad\qquad f(x,p,0) = f_0(x,p), \end{cases} \tag{1.17}$$

for several collision operators $Q(\cdot)$, where $n_f(x,t) = \int_{\mathbb{T}^d} f(x,p,t)dp$ and $d \in \mathbb{N}$ (see [49]). Here, $x \in \mathbb{R}^d$ denotes the position, $p \in \mathbb{T}^d$ the momentum and $t > 0$ the time. The dispersion relation is given by $\epsilon(p_1,\dots,p_d) = -2J\sum_{i=1}^d \cos(p_i)$.

Let $\tau_0 \in (0, \frac{1}{12Je})$. For a linear relaxation time approximation with

$$Q(g)(p) = -\frac{g(p)-\overline{n}}{\tau_0} \quad \text{for } g = g(p) \text{ and some fixed } \overline{n} \in [0,1],$$

it is shown that (1.17) admits a global analytic solution f requiring that the initial data is close to the equilibrium $\overline{n}$ in the sense of analytic norms, i.e.,

$$\sum_{i,j=0}^{\infty} \frac{\nu^{i+j}}{i!j!} \left\| \partial_x^i \partial_p^j (f_0 - \bar{n}) \right\|_{L^\infty(\mathbb{R}^d\times\mathbb{T}^d)} \le \varepsilon\nu^2$$

for some $\nu \in (0,1)$ and sufficiently small $\varepsilon > 0$ (for the size of ε see Theorem 10.3.1). Moreover, it holds

$$\|f(\cdot,\cdot,t) - \overline{n}\|_{L^\infty(\mathbb{R}^d\times\mathbb{T}^d)} \le \|f_0 - \overline{n}\|_{L^\infty(\mathbb{R}^d\times\mathbb{T}^d)} \exp(-\frac{t}{\tau_0}).$$

This is the first time that a global solution was found for an equation similarly to the Vlasov-Dirac-Benny equation.

From the physical point of view, the presence of collisions should facilitate the problem since collisions are assumed to drive the solution into an equilibrium given by the Fermi-Dirac distribution. However, if the collision operator is given by the BGK-type collision operator

$$Q(g)(p) = \gamma n_g (1-\eta n_g)(\mathcal{F}^0(n_g, E_g, p) - g(p)) \quad \text{for some } \gamma \ge 0,$$

only the existence of an analytic local in time solution is proved if the initial data f_0 is analytic and satisfies

$$|E_{f_0}(x)| \le \frac{n_{f_0}(x)(1-\eta n_{f_0}(x))}{19200(2J+1)^3} \quad \text{for } E_{f_0}(x) = \int_{\mathbb{T}^d} \epsilon(p) f_0(x,p)dp$$

and

$$\sum_{i,j=0}^{\infty} \frac{\nu^{i+j}}{i!j!} \left\| \partial_x^i \partial_p^j f_0(x,p) \right\|_{L_p^\infty(\mathbb{T}^d)} \le C n_{f_0}(x)(1-\eta n_{f_0}(x))0$$

for some $C, \nu > 0$ and for all $x \in \mathbb{R}^d$ (for more details see Theorem 10.2.7 and the preceding remark). The restriction of this result is due to the fact that the BGK-collision operator is rather difficult to cope with in analytic norms. The reason for this is that it is implicitly defined and involves a composition of functions.

In the high temperature expansion

$$Q(g)(p) = \gamma n_g (1 - \eta n_g) \left(n_g + \frac{\epsilon(p)}{2J^2 d} E_g - g(p) \right) \quad \text{for some } \gamma \geq 0,$$

some of the technical issues can be omitted such that an analytic solution $f : \mathbb{R}^d \times \mathbb{T}^d \times [0, T)$ exists for small $T > 0$ if the initial data fulfills

$$\sup_{x \in \mathbb{R}^d} \sum_{i,j=0}^{\infty} \frac{\nu^{i+j}}{i!j!} \left\| \partial_x^i \partial_p^j f_0(x,p) \right\|_{L_p^\infty(\mathbb{T}^d)} < \infty$$

for some $\nu > 0$ (for the size of T see Theorem 10.1.6).

Chapter 2

The macroscopic picture

2.1 Drift diffusion equation

The semiconductor Boltzmann equations as in (1.9) or in (1.17) are integro-differential equations and therefore, rather complicated from the analytical and numerical point of view. Thus, one is interested in effective equations for the macroscopic particle density simplifying (1.9) or (1.17). This can be implemented by choosing the right scaling for the equation. In the diffusive scaling, it is supposed that for a large time scale, the collisions dominate the kinetics of the equation. This means that the mean free path between collisions is relatively small in comparison to the reference length. Therefore, the time variable and position variable transform to

$$t' = \alpha^2 t \quad \text{and} \quad x' := \alpha x$$

for $\alpha > 0$, respectively (see e.g. [27] or [9]).

Defining $f_\alpha(x, p, t) := f\left(\frac{x'}{\alpha}, p, \frac{t'}{\alpha^2}\right)$, it holds

$$\partial_t f\left(\frac{x'}{\alpha}, p, \frac{t'}{\alpha^2}\right) = \alpha^2 \partial_{t'} f\left(\frac{x'}{\alpha}, p, \frac{t'}{\alpha^2}\right) = \alpha^2 \partial_{t'} f_\alpha(x', p, t')$$

and

$$\nabla_x f\left(\frac{x'}{\alpha}, p, \frac{t'}{\alpha^2}\right) = \alpha \nabla_{x'} f\left(\frac{x'}{\alpha}, p, \frac{t'}{\alpha^2}\right) = \alpha \nabla_{x'} f_\alpha(x', p, t')$$

as well as

$$\nabla_x n_f\left(\frac{x'}{\alpha}, \frac{t'}{\alpha^2}\right) = \alpha \nabla_{x'} n_{f_\alpha}(x', t'),$$

where $n_{f_\alpha}(x', t') = \int_B f_\alpha(x', p, t') dp$. The diffusive scaling of the semiconductor Boltzmann equation for ultracold fermions in an optical lattice (1.17) reads

$$\alpha^2 \partial_t f_\alpha + \alpha \nabla_p \epsilon(p) \cdot \nabla_x f_\alpha - \alpha \nabla_x n_{f_\alpha} \cdot \nabla_p f_\alpha = Q(f_\alpha(x,\cdot,t))(p). \tag{2.1}$$

Finally, we are interested in the limit $\alpha \to 0$ in (2.1) which is called the diffusive limit of (2.1). This procedure is already well-studied in the semiconductor Boltzmann case with Poisson potential. In [39], Masmoudi and Tayeb consider the limit $\alpha \to 0$ of the scaled Boltzmann-Poisson equation

$$\begin{cases} \alpha^2 \partial_t f_\alpha + \alpha \nabla \epsilon(p) \cdot \nabla_x f_\alpha + \alpha \nabla V_\alpha(x,t) \cdot \nabla_p f_\alpha = Q(f_\alpha(x,\cdot,t))(p), \\ \Delta V_\alpha(x,t) - \int_{\mathbb{T}^d} f_\alpha(x,p,t)dp + D(x) = 0, \\ f_\alpha(x,p,0) = f_0(x,p), \end{cases} \tag{2.2}$$

with

$$(Q(g))(p) = \int_{\mathbb{T}^d} (s(p,p')g(p')(1-g(p)) - s(p',p)g(p)(1-g(p')))dp'$$

for $x, p \in \mathbb{T}^d, t > 0$, where $f_0 \in L^\infty(\mathbb{T}^d \times \mathbb{T}^d)$. The function $D \in C^\infty(\mathbb{T}^d)$ denotes the doping profile fulfilling $\int_{\mathbb{T}^d} D dx = \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} f_0 dp dx$. Requiring that $s \in L^\infty(\mathbb{T}^d \times \mathbb{T}^d)$ and that s is bounded from below by a positive constant, it is shown that the solution of (2.2) converges to a unique equilibrium $F = F(n(x,t),p)$ such that

$$0 \leq F \leq 1, \quad \int_{\mathbb{T}^d} F(n(x,t),p)dp = n(x,t) \quad \text{and} \quad Q(F) = 0$$

for $x \in \mathbb{T}^d, t > 0$ and a unique $n \in L^\infty(\mathbb{T}^d \times \mathbb{R}_{>0})$. Moreover, Goudon and Mellet [20] prove that this density $n = n(x,t)$ solves a diffusion equation of the form

$$\begin{cases} \partial_t n - \nabla_x \cdot (\Pi(n,\cdot)\nabla_x n + \Theta(n,\cdot)\nabla_x V + \chi(n,\cdot)) = 0, \\ \Delta_x V = n + D, \\ n\big|_{t=0} = \int_{\mathbb{T}^d} f_0(\cdot,p)dp, \end{cases} \tag{2.3}$$

for some matrix valued functions Π, Θ and vector valued function χ. This extends previous results from Goudon and Mellet [20] as well as Masmoudi and Tayeb [38], who require the symmetry condition $s(p,p')e^{\epsilon(p)} = s(p',p)e^{\epsilon(p')}$. This condition is for example fulfilled if $s(p,p') = \theta(p)e^{-\epsilon(p)}\theta(p')$ for some $\theta = \theta(p)$.

In case the ultracold atoms in an optical lattice, the drift diffusion limit can only be formally derived since it is unknown whether the underlying semiconductor Boltzmann-type equation can be solved globally. So far, whether one can prove the convergence rigorously remains an open problem. The key difficulty is the fact that the time of existence of the scaled equation proved in chapter 10 is of order α^2.

For the drift diffusion limit, we need to choose a collision operator, for which the equilibrium function is independent of the energy density. Otherwise, as we will see in the next section, we would derive an energy transport model. Therefore, we consider the following scaled equations:

$$\begin{cases} \alpha^2 \partial_t f_\alpha + \alpha \nabla_p \epsilon \cdot \nabla_x f_\alpha - \alpha \nabla_x n_{f_\alpha} \cdot \nabla_p f_\alpha = -\gamma n_{f_\alpha}(1 - \eta n_{f_\alpha})(f_\alpha - n_{f_\alpha}), \\ \qquad f_\alpha\big|_{t=0} = f_0 > 0 \end{cases} \tag{2.4}$$

and

$$\begin{cases} \alpha^2 \partial_t f_\alpha + \alpha \nabla_p \epsilon \cdot \nabla_x f_\alpha - \alpha \nabla_x n_{f_\alpha} \cdot \nabla_p f_\alpha = -\gamma n_{f_\alpha}(f_\alpha - n_{f_\alpha}), \\ \qquad f_\alpha\big|_{t=0} = f_0 > 0. \end{cases} \tag{2.5}$$

In chapter 7, it is shown that solutions f_α of (2.4) and (2.5) tend formally to an equilibrium $n = n(x,t)$ fulfilling

$$\partial_t n = \Delta \log\left(\frac{n}{1 - \eta n}\right) \tag{2.6}$$

and

$$\partial_t n = \Delta \log n \tag{2.7}$$

with $n(\cdot, 0) = n_0 = \int_B f_0(\cdot, p)dp$, respectively. This macroscopic description for ultracold fermions has already been established by [48]. Equation (2.7) is a particular case of the porous-medium equation and belongs to the type of super fast diffusion equation which is well studied in [51]. It is called the logarithmic diffusion equation. One of the main properties of the logarithmic diffusion equation is the fact that the total mass, i.e. the total number of particles, is not conserved in spatial dimension greater than one. More precisely, in dimension two there is a loss rate of the total mass which is greater or equal than 4π [52], e.g. for every $n_0 \in L^1(\mathbb{R}^2)$, with $n_0 \geq 0$ there exists a unique function $n \in C([0,T), L^1(\mathbb{R}^2))$, which is a classical ($C^\infty$ and positive) solution of (2.7) and satisfies the mass constraint

$$\int_{\mathbb{R}^2} n(x,t)dx = \int_{\mathbb{R}^2} n_0(x)dx - 4\pi t.$$

Therefore, after a finite time, there is no particle left. In dimension three and above, this behavior is even more drastic: there exists no solution with finite mass [50]. From the physical point of view, these properties are undesired since they undermine the conservation of mass. Since the logarithmic diffusion equation was derived by a (formal) drift diffusion limit of (1.17), one would physically expect that the Equation (2.7) would conserve the mass just as the semiconductor Boltzmann-type equation (1.17). This is the reason for Schneider et al. [48] to call this phenomenon the "breakdown of diffusion".

However, on a bounded domain with Neumann-boundary conditions, the logarithmic diffusion equation can be solved globally for any dimension [22, 23, 24] requiring that the initial data belongs to some L^p space. Therefore the "breakdown of diffusion" is caused by the tail of the particle cloud, i.e., by the region outside a bounded domain. This may be explained by the diffusive limit procedure which was used in [48] in order to derive (2.7). It was assumed that the collisions have the main influence on the dynamics. Considering a cloud of fermions, there are only two or more particle scattering events. Thus, the collisions become less important the fewer particles are considered.

For low densities, it is not reasonable to assume that the collisions are dominant. Therefore, Schneider et al. [48] divide the whole space into two regions, the diffusive regime and the ballistic regime. They argue that inside the diffusive regime, a drift diffusion limit shall be considered. Outside this regime, the particles can be assumed to move on straight lines since interactions with other particles can be neglected. In this thesis, we give an example of a rigorous setting to these ideas: we artificially fix the diffusive regime to be a bounded domain and set reasonable boundary conditions, being in the simplest case

$$\partial_\nu \log(n) = -\beta n, \quad (x,t) \in \partial\Omega \times \mathbb{R}_+ \tag{2.8}$$

for some $\beta > 0$, where ∂_ν denotes to outer normal derivative. Unfortunately, the optimal size of the diffusive regime in this model is unknown.

For this type of system, the local solvability in combination with a criterion for a global solution was extensively studied in [1]. The aim of this thesis is to extend the local classical solution of (2.7) with (2.8) globally.

One can argue that the diffusive regime shall also change in time. This leads to a model with moving boundary conditions. Such a model is stated in the comments of chapter 11. However, if this model admits a classical solution or not is still an open problem.

2.2 Energy transport models

In addition to the drift diffusion limit, there are several other relevant scalings of semiconductor Boltzmann-type equations [27]. Adding to the semiconductor Boltzmann-Poisson system (1.9) further collision operators modeling the electron-electron interaction and the impurity scattering events, Ben Abdallah et al. [12] show that the scaling $t' = \alpha^2 t$ and $x' = \alpha x$ leads formally to a cross-diffusion system for the energy density E and the particle density n.

Using the same scaling ($t' = \alpha^2 t'$ and $x' = \alpha x$) for the semiconductor Boltzmann-type equation with the BGK-type collision operator from (1.12), modeling ultracold fermions in an optical lattice, entails

$$\begin{cases} \alpha^2\partial_t f_\alpha + \alpha\nabla_p\epsilon\cdot\nabla_x f_\alpha - \alpha\nabla_x n_{f_\alpha}\cdot\nabla_p f_\alpha \\ \qquad\qquad = -\gamma n_{f_\alpha}(1-\eta n_{f_\alpha})(f_\alpha - \mathcal{F}^0(n_{f_\alpha}, E_{f_\alpha}, \cdot)), \\ f_\alpha\big|_{t=0} = f_0, \end{cases} \tag{2.9}$$

where $n_{f_\alpha}(x,t) = \int f_\alpha(x,p,t)dp$, $E_{f_\alpha}(x,t) = \int \epsilon(p) f_\alpha(x,p,t)dp$ and $d \in \mathbb{N}$. Again, $\mathcal{F}^0$ is given by

$$\mathcal{F}^0(n,E,p) := \mathcal{F}(\lambda_0(n,E), \lambda_1(n,E), p),$$

where λ_0, λ_1 are implicitly defined by

$$\int_{\mathbb{T}^d} \begin{pmatrix} 1 \\ \epsilon(p) \end{pmatrix} \mathcal{F}(\lambda_0(n,E), \lambda_1(n,E), p)dp = \begin{pmatrix} n \\ E \end{pmatrix}. \tag{2.10}$$

Here, $\mathcal{F}(\lambda,p) := 1/(1+e^{-\lambda_0-\lambda_1\epsilon(p)})$ denotes the Fermi-Dirac distribution function for the entropy parameter $\lambda = (\lambda_0, \lambda_1)$. In this thesis it is shown that f_α converges formally to $\mathcal{F} = \mathcal{F}(\lambda,p)$, where $\lambda = \lambda(x,t)$ is considered as a function depending on space x and time t. In accord with (2.10), we define the particle and energy densities as functions of λ by $n(\lambda) = \int_B \mathcal{F}(\lambda,p)dp$, $E(\lambda) = \int_B \epsilon(p)\mathcal{F}(\lambda,p)dp$, respectively. Then the entropy parameter $\lambda = \lambda(x,t)$ formally fulfills the energy transport model for ultracold fermions in an optical lattice

$$\begin{aligned} n'(\lambda)\partial_t\lambda + \nabla\cdot\big(J_0(\lambda)\nabla\lambda\big) &= 0, \\ E'(\lambda)\partial_t\lambda + \nabla\cdot\big(J_1(\lambda)\nabla\lambda\big) &= -J_0(\lambda)\nabla\lambda\cdot n'(\lambda)\nabla\lambda, \end{aligned} \tag{2.11}$$

where the currents are

$$J_j(\lambda)\nabla\lambda = -\sum_{i=0}^{1} D_{ji}(\lambda)\nabla\lambda_i + \lambda_1 D_{j0}(\lambda)n'(\lambda)\nabla\lambda,$$

being defined with the aid of the diffusion matrix

$$D_{ij}^{kl}(\lambda) = \int_{\mathbb{T}^d} \epsilon^{i+j}(p)\partial_k\epsilon(p)\partial_l\epsilon(p)\frac{\mathcal{F}(\lambda,p)(1-\mathcal{F}(\lambda,p))}{n(\lambda)(1-n(\lambda))}dp$$

for $k,l,i,j = 1,\dots,d$.

Noting that the energy transport model is a macroscopic model describing a cloud of ultracold fermions in an optical lattice, it is no surprise that these equations admit an entropy $H(t) = \int_{\mathbb{R}^d} h(\lambda(t,x))dx$ which fulfills

$$\partial_t H(t) + \int_{\mathbb{R}^d} S(\lambda(x,t), \nabla\lambda(x,t))dx = 0$$

for some $S \geq 0$, where λ is a solution of the energy transport model. In [28], making use of such entropy inequalities is the key ingredient to solve cross-diffusion problems. In this thesis, however, it is shown that there are degeneracies of the entropy dissipation. More precisely, $S(\lambda(x,t), \nabla\lambda(x,t))$ vanishes in all points (x,t), where

$$\lambda_1(x,t) \int_{\mathbb{T}^d} \mathcal{F}(\lambda(x,t),p)(1 - \mathcal{F}(\lambda(x,t),p))dp = 1. \tag{2.12}$$

Hence, these degeneracies depend on the explicit values of λ_0 and λ_1 in contrast to the degeneracies treated in [28]. In particular, it is shown in this thesis that one cannot extract a non-degenerate estimate for the gradients of λ in the points, where (2.12) is fulfilled in contrast to the standard systems (e.g. [28]). Therefore, the solvability of (2.11) remains an open problem. In order to understand the system, one can heuristically approximate it similarly as in (1.13) and [36] using

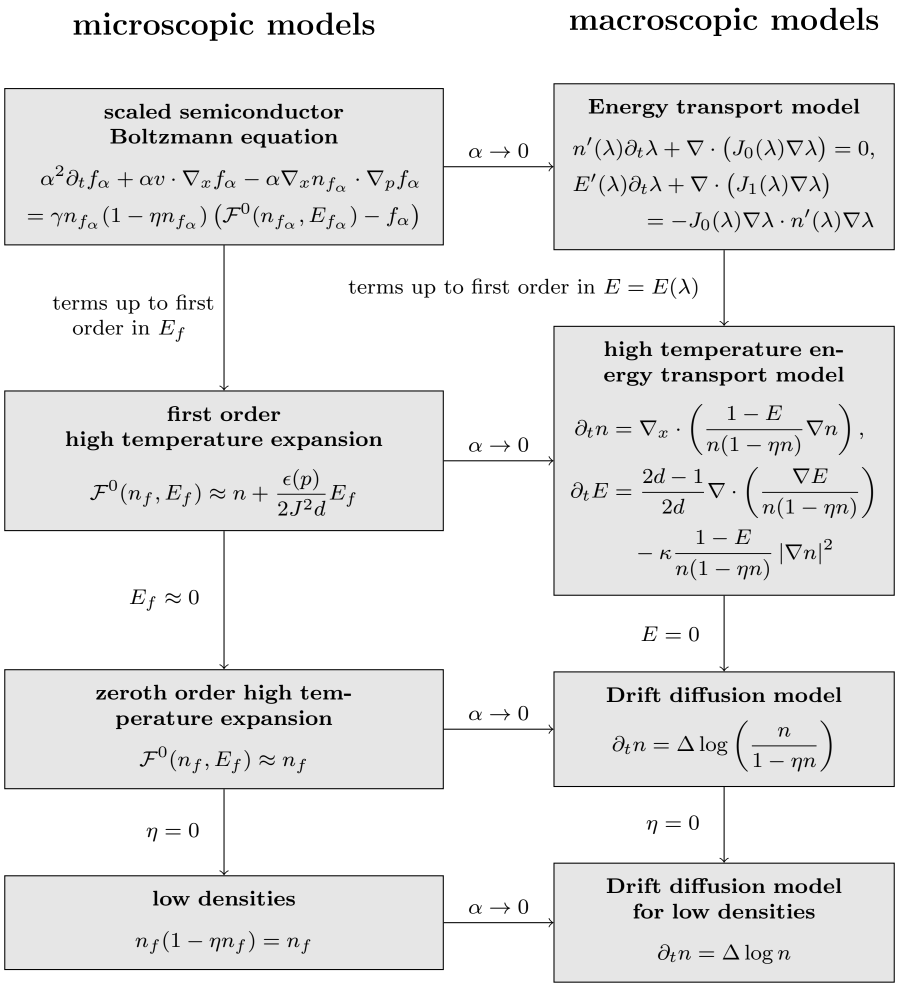
$$\mathcal{F}(\lambda,p) = n(\lambda) + \frac{\epsilon(p)}{2J^2 d} E(\lambda) + o(E(\lambda)^2).$$

This approximation is called the high-temperature expansion since high temperatures correspond to small absolute values of $E(\lambda)$ [36]. Therefore, replacing $\mathcal{F}(\lambda,p)$ by $n(\lambda) + \frac{\epsilon(p)}{2J^2 d}E(\lambda)$ in the definition of the diffusion matrix, it is possible to write the high temperature expansion of (2.11) in a closed form in $n = n(x,t)$ and $E = E(x,t)$ by

$$\begin{aligned} \partial_t n &= \nabla_x \cdot \left(\frac{1-E}{n(1-\eta n)} \nabla_x n \right), \\ \partial_t E &= \frac{2d-1}{2d} \nabla_x \cdot \left(\frac{\nabla E}{n(1-\eta n)} \right) - \kappa \frac{1-E}{n(1-\eta n)} \left|\nabla_x n\right|^2 \end{aligned} \tag{2.13}$$

for some $\kappa > 0$. Although this high temperature approximation model is more explicit than the full energy transport model, it still admits degeneracies (at $E = 1$).

2.3 Diagram of the models



2.4 Main Results

This thesis shows that the logarithmic diffusion equation

$$\begin{cases} \partial_t n(x,t) = \Delta \log\big(n(x,t)\big), & (x,t) \in \Omega \times \mathbb{R}_+, \\ \partial_\nu \log\big(n(x,t)\big) = -\beta n(x,t), & (x,t) \in \partial\Omega \times \mathbb{R}_+, \\ n(x,0) = n_0 > 0, & x \in \overline{\Omega} \end{cases}$$

on a bounded and regular domain Ω admits a unique classical, global solution, where ∂_ν denotes the normal derivative on $\partial\Omega$. Moreover, it is shown that the solution decays exponentially in time, i.e. for all $p > 1$ there exists a constant $C_p > 0$ such that

$$\sqrt[p]{|\Omega|} \exp\left(-\frac{1}{|\Omega|}\int_\Omega |\log(n_0)|\, dx\right) e^{-\beta t} \leq \|n(t)\|_{L^p(\Omega)} \leq \|n_0\|_{L^p(\Omega)}\, e^{-C_p t}$$

if $n_0 \leq 1$.

We say that the high temperature limit of the energy transport equation (2.13) has a weak lower solution if

$$\begin{aligned} &\int_0^\infty \langle \partial_t n, \phi_0 \rangle_{H^1(\Omega)', H^1(\Omega)}\, dt + \int_0^\infty \int_\Omega \frac{(1-E)\nabla n}{n(1-\eta n)} \cdot \nabla \phi_0 dx dt = 0, \\ &\int_\Omega \left(E^0 + \frac{1}{2}(n^0)^2\right) \phi_1(0) dx + \int_0^\infty \int_\Omega \left(E + \frac{1}{2}n^2\right) \partial_t \phi_1 dx dt \\ &\quad \leq \frac{2d-1}{2d} \int_0^\infty \int_\Omega \frac{\nabla E \cdot \nabla \phi_1}{n(1-\eta n)} dx dt + \int_0^\infty \int_\Omega \frac{(1-E)\nabla n}{1-\eta n} \cdot \nabla \phi_1 dx dt \end{aligned} \tag{2.14}$$

for all $\phi_0 \in L^2(0,\infty; H^1(\Omega))$ and $\phi_1 \in L^2(0,\infty; H^1(\Omega)) \cap W^{1,1}(0,\infty; L^1(\Omega))$, being compactly supported in $[0,\infty)$ with $\partial_t \phi_1 \geq 0$. In this thesis we are able to derive a non-trivial weak lower solution for suitable initial data of (2.14). So far it is not possible to show that (2.13) admits a weak solution since the degeneracy in $E = 1$ undermines an a priori estimate of ∇n in $L^2(0,\infty, L^2(\Omega))$. The lower solution was obtained by a weak limit of approximating solutions. However, an argument whether the particle density converges strongly on $\{E = 1\}$ in the weak limit is still missing. Moreover, we are not able to apply a proper maximum principle for the approximating energy density due to the critical exponent in the second equation. Therefore, it remains an open problem, whether (2.14) possesses a weak solution if we replace the inequality by an equality.

In addition, the high temperature limit of the energy transport model is solved numerically on the one dimensional torus. It can be seen that the solution decays exponentially to a stationary solution. Due to the degeneracies, the rate of convergence to the final stationary solution highly depends on the initial data.

Chapter 3

Overview

This thesis is divided into four parts with this introduction as part one. The second part provides more or less well-known results for analytic norms and the Fermi-Dirac distribution. Moreover, in chapter 4, tailor-made analytic norms for the analysis of the semiconductor Boltzmann equations modeling ultracold atoms in an optical lattice are presented. They thus differ from the typical definitions according to Mouhot and Villani [40]. In addition, the analytic norm of the Fermi-Dirac distribution as a function of the particle and energy density is estimated in chapter 5.

The third part concentrates on the modeling. At first, the ill-posedness of a simplified version of the semiconductor Boltzmann equation for ultracold atoms in an optical lattice is discussed, showing the structural difficulties of the microscopic equations. In chapter 7, the diffusion limit of the semiconductor Boltzmann equation is formally proved, linking the microscopic equations to the macroscopic equations. This chapter is followed by a chapter on the energy transport model exploring entropy methods. In this chapter, degeneracies of the entropy dissipation are discovered. In the last chapter of part two, a numerical solution of the high-temperature energy transport model on the one dimensional torus is presented, emphasizing the importance of the degeneracies in the energy transport models.

The content of the fourth part is to exploit mathematical tools in order to solve the presented equation rigorously. In chapter 10, the semiconductor Boltzmann equation is considered with three different collision operators: the standard BGK-type operator, its high-temperature approximation and the linear relaxation time approximation. Chapter 11 deals with the logarithmic diffusion equation on a bounded domain with nonlinear boundary conditions and in the last chapter the existence of a weak lower solution of the high temperature energy transport model is shown.

Part II

Fundamentals

Chapter 4

Analytic norms

4.1 One variable analytic norms

In contrast to the typical approach to weaken the definition of differentiable functions in order to solve partial differential equations, we work on a Banach space consisting of analytic functions. The easiest way to write down such a Banach space can be seen in the following definitions which are motivated by Mouhot and Villani [40].

Definition 4.1.1. Let $d \in \mathbb{N}$ and X be a Banach space with norm $|\cdot|_X$. For $l \in \mathbb{N}$, we define $X_0 := X$ and $X_{l+1} := L(\mathbb{R}^d, X_l)$ as the linear continuous functions $\mathbb{R}^d \mapsto X_l$, which is isomorph to $(X_l)^d$ equipped with the operator norm induced by X_l, i.e. $|T|_{X_{l+1}} := \sup_{x\in\mathbb{R}^d} |Tx|_{X_l}$ for $T \in X_{l+1}$. Now let $T \in \dot{\bigcup}_l X_l$. We write

$$|T|_{\mathrm{Op}} := |T|_{X_l}\,, \quad \text{where } T \in X_l.$$

Definition 4.1.2. Let X be a Banach space, $d \in \mathbb{N}$ and $U \subset \mathbb{R}^d$ be open. We denote $\mathcal{O}(U,X)$ as the set of all functions $U \mapsto V$ and write $\mathcal{O}(U) := \mathcal{O}(U,\mathbb{R})$.

Definition 4.1.3. Let U be an open subset of $\mathbb{R}^d$, X a Banach space. Given $\lambda \in \mathbb{R}_{\geq 0}$, we define

$$|f|_{\dot{C}^\lambda_x} := \sum_{a=1}^{\infty} \frac{\lambda^a}{a!} \left|f^{(a)}(x)\right|_{\mathrm{Op}} \quad \text{and} \quad |f|_{\dot{C}^\lambda(U,X)} := \sup_{x\in U} |f|_{\dot{C}^\lambda_x}$$

as well as

$$|f|_{C^\lambda_x} := |f(x)| + |f|_{\dot{C}^\lambda_x} \quad \text{and} \quad |f|_{C^\lambda(U,X)} := \sup_{x\in U} |f|_{C^\lambda_x}$$

for $f : U \to \mathcal{O}(U,X)$ and $x \in U$.

The following statement is due to Mouhot and Villani [40] and will be stated without a proof:

Lemma 4.1.4. *If X is a Banach algebra and hence its norm submultiplicative, then $|\cdot|_{\dot{C}_x^\lambda}$, $|\cdot|_{C^\lambda(U)}$ are also submultiplicative.*

Example 4.1.5. Let $U := \mathbb{R} \setminus \{0\}$ and $f(x) = \frac{1}{x}$ and $\lambda > 0$. We have for $|x| > \lambda$

$$|f|_{\dot{C}_x^\lambda} = \sum_{a=1}^{\infty} \frac{\lambda^a}{a!} \frac{a!}{|x|^{a+1}} = \frac{1}{|x|} \frac{\lambda}{|x| - \lambda}.$$

4.2 Composition of functions

In this subsection we are analyze the composition of two functions f and g in the norms given above. There is also a statement concerning compositions in Mouhot and Villani [40]. However, the proof of it is rather a sketch of a proof. Therefore, we elaborate the proof with some additional information.

Let X be a Banach space and let $f : U_2 \to X$, $g : U_1 \to U_2$ be two analytic functions on open sets $U_i \subset \mathbb{R}^{d_i}$ with $d_i \in \mathbb{N}$.

Lemma 4.2.1 (Faà di Bruno Formula)**.** *Let $n \in \mathbb{N}$ and let Π_n be the set of all partitions of $\{1, \ldots, n\}$. For each $1 \leq i \leq n$, let ∂_i be a partial derivative operator, i.e. $\partial_i = \partial_{x_j}$ for some $j = 1, \ldots, l$. Then for every $\pi \in \Pi_n$ there exists an $|\pi| + 1$ linear mapping M_π of norm equal to 1 such that*

$$\partial_1 \ldots \partial_n f \circ g(x) = \sum_{\pi \in \Pi_n} M_\pi \left(f^{(|\pi|)} \circ g(x), \left(\frac{\partial^{|B|}}{\prod_{i \in B} \partial_i} g(x) \right)_{B \in \pi} \right)$$

Proof. The proof can directly be obtained by induction. □

Corollary 4.2.2 (Faà di Bruno Formula)**.** *For $l = 1$, this facilitates to*

$$(f \circ g)^{(n)}(x) = \sum_{\pi \in \Pi_n} M_\pi \left(f^{(|\pi|)} \circ g(x), \left(g^{(|B|)}(x) \right)_{B \in \pi} \right). \tag{4.1}$$

In the following, assume that $l = 1$ and hence $g : U_1 \to U_2$. In dimension one the $|\pi| + 1$ linear mappings are given by multiplication. Thus, we can write a more explicit and well-known formula:

Lemma 4.2.3 (Faà di Bruno Formula)**.** *Let $n \in \mathbb{N}$ and $d_i = 1$. We have*

$$(f \circ g)^{(n)}(x) = \sum_{\sum_{j=1}^n j m_j = n} \frac{n!}{\prod_j m_j!} f^{(m_1 + \ldots + m_n)} \circ g(x) \prod_{j=1}^n \left(\frac{g^{(j)}(x)}{j!} \right)^{m_j}.$$

Corollary 4.2.4. *Let $n \in \mathbb{N}$ and d_i be arbitrary. We have*

$$\left|(f \circ g)^{(n)}(x)\right|_{\mathrm{Op}} \le \sum_{\substack{\sum_{j=1}^n j m_j = n \\ m = \sum_j m_j}} \frac{n!}{\prod_j m_j!} \left|f^{(m)} \circ g(x)\right|_{\mathrm{Op}} \prod_{j=1}^{n} \left(\frac{\left|g^{(j)}(x)\right|_{\mathrm{Op}}}{j!}\right)^{m_j}.$$

Proof. In dimension $d_i = 1$ in conjunction with $X = \mathbb{R}$, we have two formulas for $(f \circ g)^{(n)}$. Introducing $f : y \mapsto \sum_{k=1}^n \frac{a_k}{k!} y^k$ and $g : x \mapsto \sum_{l=1}^n \frac{b_l}{l!} x^l$ for $k, l \in \mathbb{N}$, we infer setting $x = 0$

$$\sum_{\pi \in \Pi_n} a_{|\pi|} \prod_{B \in \pi} b_{|B|} = \sum_{\sum_{j=1}^n j m_j = n} \frac{n!}{\prod_j m_j!} a_{(m_1 + \ldots + m_n)} \prod_{j=1}^{n} \left(\frac{b_j}{j!}\right)^{m_j}$$

for all $a_k, b_l \in \mathbb{R}$. Finally, we obtain the assertion using the preceding equality and

$$\left|(f \circ g)^{(n)}(x)\right|_{\mathrm{Op}} \le \sum_{\pi \in \Pi_n} \left|f^{(|\pi|)} \circ g(x)\right|_{\mathrm{Op}} \prod_{B \in \pi} \left|g^{(|B|)}(x)\right|_{\mathrm{Op}},$$

since the operator norm is by definition submultiplicative. □

Lemma 4.2.5. *Let f, g and U_i, X be as above. We have*

$$|f \circ g|_{\dot{C}^{\lambda}_x} \le |f|_{\dot{C}^{\mu}_y} \qquad \text{with } \mu = |g|_{\dot{C}^{\lambda}_x} \text{ and } y = g(x).$$

Proof. Let $N \in \mathbb{N}$ be arbitrary. The computation

$$\begin{aligned}
&\sum_{n=1}^{N} \frac{\lambda^n}{n!} \left|(f \circ g)^{(n)}(x)\right|_{\mathrm{Op}} \\
&\le \sum_{n=1}^{N} \sum_{m=1}^{N} \sum_{(m_j)_j \in \mathbb{N}^N} \delta_{\sum_j m_j, m} \delta_{\sum_j j m_j, n} \frac{\lambda^n}{\prod_j m_j!} \left|f^{(m)} \circ g(x)\right|_{\mathrm{Op}} \prod_{j=1}^{n} \frac{\left|g^{(j)}(x)\right|_{\mathrm{Op}}^{m_j}}{j!^{m_j}} \\
&\le \sum_{m=1}^{N} \frac{1}{m!} \left|f^{(m)} \circ g(x)\right|_{\mathrm{Op}} \sum_{\sum_j m_j = m} \binom{m}{m_1, \ldots, m_N} \prod_{j=1}^{N} \left(\frac{\lambda^j}{j!} \left|g^{(j)}(x)\right|_{\mathrm{Op}}\right)^{m_j} \\
&= \sum_{m=1}^{N} \frac{1}{m!} \left|f^{(m)} \circ g(x)\right|_{\mathrm{Op}} \left(\sum_{j=1}^{N} \frac{\lambda^j}{j!} \left|g^{(j)}(x)\right|_{\mathrm{Op}}\right)^{m}
\end{aligned}$$

ensures the assertion in the limit $N \to \infty$. □

Example 4.2.6. Let $U := \mathbb{R}^d$ be equipped with the euclidean norm and $f(x) := \frac{1}{a+b|x|^2}$ for $a, b > 0$. Then for $\lambda \in (0, \sqrt{\frac{a}{5b}})$ and $x \in \mathbb{R}^d$,

$$|f|_{\dot{C}^\lambda_x} \leq \left|\frac{1}{(\cdot)}\right|_{\dot{C}^\mu_y} = \frac{b\lambda(\lambda + 2|x|)}{a + b|x|^2 - b\lambda(\lambda + 2|x|)} |f(x)| \leq \sqrt{\frac{20b}{a}} \lambda |f(x)| .$$

where $y := g(x) := a + b|x|^2$ and

$$\mu := |g|_{\dot{C}^\lambda_x} = b\lambda(\lambda + 2|x|).$$

Here we used that for $\lambda \leq \sqrt{\frac{a}{5b}}$

$$\begin{aligned} \frac{a}{2} + \frac{b}{2}|x|^2 - b\lambda(\lambda + 2|x|) &= \frac{a}{2} + \frac{b}{2}\left(x^2 - 2\lambda|x| - 4\lambda^2\right) \\ &= \frac{a}{2} - \frac{5}{2}b\lambda^2 + \frac{b}{2}(|x| - \lambda)^2 \geq 0, \end{aligned}$$

which implies that

$$\frac{b\lambda(\lambda + 2|x|)}{a + b|x|^2 - b\lambda(\lambda + 2|x|)} \leq \frac{\lambda}{\sqrt{\frac{a}{5b}}} \frac{b\sqrt{\frac{a}{5b}}(\sqrt{\frac{a}{5b}} + 2|x|)}{a + b|x|^2 - b\lambda(\lambda + 2|x|)} \leq 2\sqrt{\frac{5b}{a}}\lambda.$$

4.3 Inverse function

Let U, V be two open subsets of $\mathbb{R}^d$. We denote $\mathcal{O}(U, V)$ as the set of all analytic functions $U \to V$.

The idea of estimating the analytic norm of an inverse function was already established in [40]. However, in the third part of this thesis, we require a different version being based on Theorem 13 in [35]:

Proposition 4.3.1 (Inverse function theorem, [35])**.** *Let $x \in U$ and $f \in \mathcal{O}(U, V)$ such that $\left|f^{(n)}\right| \leq CL^n n!$ for $n \geq 2$ as well as $\left|f'(x)^{-1}\right|_{\mathrm{Op}} \leq H$ for some $C, L, H > 0$. Then there exist an open set $V' \ni f(x)$ and an $g \in \mathcal{O}(V', U)$ which is inverse to f such that*

$$\left|g^{(m)}(f(x))\right|_{\mathrm{Op}} \leq 2CH(HL)^m(3 + 4CHL)^{m-2}m^{-\Delta-1}m!$$

for all $m \geq 3$, where $\frac{1}{2} < \Delta < 1$ is depending on CHL. Especially,

$$\left|g^{(m)}(f(x))\right|_{\mathrm{Op}} \leq \frac{CH}{8(1 + CHL)^2}(4HL(1 + CHL))^m m^{-\frac{3}{2}} m!.$$

Corollary 4.3.2. *Let $\lambda > 0$, $x \in U$ and let $f \in \mathcal{O}(U,V)$ be bijective and fulfill $|f|_{\dot{C}^\lambda_x} < \infty$ as well as $\left|(f')^{-1}(y)\right|_{L^\infty_{Op}} < \infty$ with $y = f(x)$. Then its inverse function $g \in \mathcal{O}(V,U)$ satisfies*

$$|g|_{\dot{C}^\mu_y} \le \left|(f')^{-1}(y)\right|_{\mathrm{Op}} \times \\ \times \left(\mu + |f|_{\dot{C}^\lambda_x}\left(\left|(f')^{-1}(y)\right|^2_{\mathrm{Op}}\frac{\mu^2}{\lambda} - \frac{(\mu C_{\lambda,f})^2}{12}\log(1-\mu C_{\lambda,f})\right)\right)$$

for all $\mu \ge 0$ fulfilling

$$C_{\lambda,f} := \frac{4}{\lambda}\left|(f')^{-1}(y)\right|_{\mathrm{Op}}\left(1 + \left|(f')^{-1}(y)\right|_{\mathrm{Op}}|f|_{\dot{C}^\lambda_x}\frac{1}{\lambda}\right) < \frac{1}{\mu}.$$

Proof. A direct calculation ensures

$$\sum_{m=1,2}\frac{\mu^m}{m!}\left|g^{(m)}(y)\right|_{\mathrm{Op}} = \mu\left|(f')^{-1}(y)\right|_{\mathrm{Op}} + \left(\frac{\mu}{\lambda}\right)^2\left|(f')^{-1}(y)\right|^3_{\mathrm{Op}}|f|_{\dot{C}^\lambda_x}.$$

The rest of the sum can be estimated using Proposition 4.3.1, leading to

$$\sum_{m\ge3}\frac{\mu^m}{m!}\left|g^{(m)}(y)\right|_{\mathrm{Op}} \le \frac{1}{12}\left|(f')^{-1}(y)\right|_{\mathrm{Op}}|f|_{\dot{C}^\lambda_x}\sum_{m\ge3}\frac{a^m}{m},$$

where

$$a := 4\frac{\mu}{\lambda}\left|(f')^{-1}(y)\right|_{\mathrm{Op}}\left(1 + \left|(f')^{-1}(y)\right|_{\mathrm{Op}}|f|_{\dot{C}^\lambda_x}\frac{1}{\lambda}\right).$$

By assumption $a < 1$ and hence

$$\sum_{m\ge3}\frac{a^m}{m} \le a^2\sum_{m\ge1}\frac{a^m}{m} = -a^2\log(1-a). \qquad \square$$

4.4 Two variable analytic norms

So far we treated functions with one (d-dimensional) variable. The idea of this section is to extend these definitions and properties to functions with two variables. Of course, one can treat two d-dimensional variables as one $2d$-dimensional variable. However, those two variables may have a different physical interpretation so that we would like to treat them separately. Again, the idea of a two variable analytic norm was first introduced by [40]. Nevertheless, we will require a modified version in the application in part three.

Definition 4.4.1. Let $U \subset \mathbb{R}^d$ be open. The set $\mathcal{O}(U \times \mathbb{T}^d)$ denotes space of all real valued analytic functions $f : U \times \mathbb{T}^d \to \mathbb{R}$. For $\lambda_1, \lambda_2 \geq 0$ and $x \in U$, we define

$$|f|_{\dot{C}_x^{\lambda_1,\lambda_2}} := \sum_{a+b>0} \frac{\lambda_1^a \lambda_2^b}{a!b!} \left\|\partial_x^a \partial_p^b f(x,p)\right\|_{L_p^\infty(\mathbb{T}^d)},$$
$$|f|_{C_x^{\lambda_1,\lambda_2}} := \|f(x,\cdot)\|_{L_p^\infty(\mathbb{T}^d)} + |f|_{\dot{C}_x^{\lambda_1,\lambda_2}}$$

for $f \in \mathcal{O}(U \times \mathbb{T}^d)$. Here, $\partial_x^a \partial_p^b f(x,p)$ is assumed to be a multilinear operator and

$$\|M(x,p)\|_{L_p^\infty(\mathbb{T}^d)} = \sup_p |M(x,p)|_{\mathrm{Op}}$$

for every multilinear valued function $(x,p) \mapsto M(x,p)$, where $|\cdot|_{\mathrm{Op}}$ denotes an adequate operator norm. Moreover, we set as before:

$$|f|_{\dot{C}^{\lambda_1,\lambda_2}(U)} := \sup_{x\in U} |f|_{\dot{C}_x^{\lambda_1,\lambda_2}}, \quad |f|_{C^{\lambda_1,\lambda_2}(U)} = \sup_{x\in U} |f|_{C_x^{\lambda_1,\lambda_2}}.$$

We usually write $|f|_{C_x^\lambda} = |f|_{C_x^{\lambda_1,\lambda_2}}$ and $|f|_{C^\lambda(U)} = |f|_{C^{\lambda_1,\lambda_2}(U)}$ for $\lambda = \lambda_1 = \lambda_2$. Furthermore, if $U = \mathbb{R}^d$, we neglect the U in the notation.

Remark 4.4.2. Note that this notation coincides with the one used for the one variable analytic norms. However, the space $\mathcal{O}(U)$ can be embedded into $\mathcal{O}(U \times \mathbb{T}^d)$ by

$$\iota : \mathcal{O}(U) \to \mathcal{O}(U \times \mathbb{T}^d);\ \big(x \mapsto f(x)\big) \mapsto \big((x,p) \mapsto f(x)\big).$$

This embedding is in accord with the norm $|\cdot|_{C_x^\lambda}$ for $U \subset \mathbb{R}^d$ open and $x \in U$, i.e. $|\iota(f)|_{C_x^\lambda} = |f|_{C_x^\lambda}$ for all $f \in \mathcal{O}(U)$. The same remains true for $|\cdot|_{\dot{C}_x^\lambda}$.

Lemma 4.4.3. *The norm $|\cdot|_{C_x^\lambda}$ is submultiplicative, i.e.,*

$$|fg|_{C_x^\lambda} \leq |f|_{C_x^\lambda} |g|_{C_x^\lambda}.$$

Proof. The proof can be done by a straightforward calculation as in [40]. □

Lemma 4.4.4. *Let $g : U_1 \mapsto U_2$, $h : V_1 \mapsto V_2$ and $f : U_2 \times V_2 \mapsto X$ be analytic, where $U_1, V_1 \subseteq \mathbb{R}^d$, $U_2, V_2 \subseteq \mathbb{R}^N$ are open and X is a Banach space. For every $\lambda_1, \lambda_2 > 0$, we have*

$$|f \circ (g,h)|_{\mathcal{C}^{\lambda_1,\lambda_2}} \leq |f|_{\mathcal{C}^{\mu_1,\mu_2}},$$

where $\mu_1 := |g|_{\dot{\mathcal{C}}^{\lambda_1}(U_1,\mathbb{R}^N)}$ and $\mu_2 := |h|_{\dot{\mathcal{C}}^{\lambda_2}(V_1,\mathbb{R}^N)}$.

Proof. We may write the two variable norm $\|\cdot\|_{\mathcal{C}^{\lambda_1,\lambda_2}}$ as a one variable Banach space valued norm and apply Lemma 4.2.5. We therefore define $\tilde{f} : U_2 \mapsto \mathcal{C}^{\lambda_2}(V_1, X)$ mapping $x \mapsto f(x, h(\cdot))$. This enables the calculation

$$|f \circ (g,h)|_{\dot{\mathcal{C}}^{\lambda_1,\lambda_2}} = \left|\tilde{f} \circ g\right|_{\dot{\mathcal{C}}^{\lambda_1}(U_1,\dot{\mathcal{C}}^{\lambda_2}(V_1))} \leq \left|\tilde{f}\right|_{\dot{\mathcal{C}}^{\mu_1}(U_2,\dot{\mathcal{C}}^{\lambda_2}(V_1))},$$

where $\mu_1 := |g|_{\dot{\mathcal{C}}^{\lambda_1}(U_1,\mathbb{R}^N)}$. In order to profit a second time from Lemma 4.2.5, we utilize the trick above again. Setting $\hat{f} : V_2 \mapsto \mathcal{C}^{\mu_1}(U_2, X)$, $x \mapsto f(\cdot, x)$, we derive

$$\begin{aligned}
\left|\tilde{f}\right|_{\dot{\mathcal{C}}^{\mu_1}(U_2,\dot{\mathcal{C}}^{\lambda_2}(V_1))} &= |f \circ (\cdot, h)|_{\dot{\mathcal{C}}^{\mu_1,\lambda_2}} \\
&= \left|\hat{f} \circ h\right|_{\dot{\mathcal{C}}^{\lambda_2}(V_1,\dot{\mathcal{C}}^{\mu_1}(U_2))} \\
&\leq \left|\hat{f}\right|_{\dot{\mathcal{C}}^{\mu_2}(V_1,\dot{\mathcal{C}}^{\mu_1}(U_2))} = |f|_{\dot{\mathcal{C}}^{\mu_1,\mu_2}}
\end{aligned}$$

for $\mu_2 := |h|_{\dot{\mathcal{C}}^{\lambda_2}(V_1,\mathbb{R}^N)}$, completing the proof. □

Definition 4.4.5. Let $U \subset \mathbb{R}^d$ be open and $x \in U$. For $\lambda > 0$ and $f \in \mathcal{O}(U \times \mathbb{T}^d)$, we put

$$\|f\|_{C_x^\lambda} := |f|_{C_x^\lambda} + |\partial_x f|_{C_x^\lambda} + |\partial_p f|_{C_x^\lambda}, \quad \|f\|_{C^\lambda(U)} := \sup_{y\in U} \|f\|_{C_y^\lambda}$$

and $\|Df\|_{C^\lambda(U)} := \sup_{y\in U} \|Df\|_{C_y^\lambda}$, where

$$\|Df\|_{C_x^\lambda} := |\partial_x f|_{C_x^\lambda} + |\partial_p f|_{C_x^\lambda} + \left|\partial_x^2 f\right|_{C_x^\lambda} + \left|\partial_p^2 f\right|_{C_x^\lambda} + 2\,|\partial_x \partial_p f|_{C_x^\lambda}.$$

The main advantage of these modified norms can be seen in the following remark:

Remark 4.4.6. Let $\partial_1, \partial_2 \in \{\partial_x, \partial_p\}$. It holds $\|fg\|_{C_x^\lambda} \leq \|f\|_{C_x^\lambda} \|g\|_{C_x^\lambda}$ and

$$\begin{aligned}
\|f\partial_1 g\|_{C_x^\lambda} &\leq |f|_{C_x^\lambda} |\partial_1 g|_{C_x^\lambda} + |\partial_x f|_{C_x^\lambda} |\partial_1 g|_{C_x^\lambda} + |\partial_p f|_{C_x^\lambda} |\partial_1 g|_{C_x^\lambda} \\
&\quad + |f|_{C_x^\lambda} |\partial_x \partial_1 g|_{C_x^\lambda} + |f|_{C_x^\lambda} |\partial_p \partial_1 g|_{C_x^\lambda} \\
&\leq \|f\|_{C_x^\lambda} \|Dg\|_{C_x^\lambda}
\end{aligned}$$

as well as

$$\begin{aligned}
\|\partial_1 f \partial_2 g\|_{C_x^\lambda} &\leq |\partial_1 f|_{C_x^\lambda} |\partial_2 g|_{C_x^\lambda} + |\partial_x \partial_1 f|_{C_x^\lambda} |\partial_2 g|_{C_x^\lambda} + |\partial_p \partial_1 f|_{C_x^\lambda} |\partial_2 g|_{C_x^\lambda} \\
&\quad + |\partial_1 f|_{C_x^\lambda} |\partial_x \partial_2 g|_{C_x^\lambda} + |\partial_1 f|_{C_x^\lambda} |\partial_p \partial_2 g|_{C_x^\lambda} \\
&\leq \|f\|_{C_x^\lambda} \|Dg\|_{C_x^\lambda} + \|Df\|_{C_x^\lambda} \|g\|_{C_x^\lambda}
\end{aligned}$$

for $f, g \in C^\infty(\mathbb{R}^d \times \mathbb{T}^d)$ and $x \in \mathbb{R}^d$.

Lemma 4.4.7. *Let $\nu > \lambda > 0$ and $U \subseteq \mathbb{R}^d$ be open. Then there exists a $C_{\lambda,\nu} > 0$ such that for all $f \in \mathcal{O}(\mathbb{R}^d \times \mathbb{T}^d)$ and all $x \in \mathbb{R}^d$, it holds*

$$\|f\|_{C_x^\lambda} \leq C_{\lambda,\nu} \, |f|_{C_x^\nu} .$$

Proof. It suffices to show that we have $|\partial f|_{C_x^\lambda} \leq C \, |f|_{C_x^\nu}$ for $\partial \in \{\partial_x, \partial_p\}$ for some $C > 0$. Let $\partial = \partial_x$ and compute

$$\begin{aligned}
|\partial_x f|_{C_x^\lambda} &\leq \sum_{i,j \in \mathbb{N}_0} \frac{\lambda^{i+j}}{i!j!} \left\| \partial_x^{i+1} \partial_p^j f(x,\cdot) \right\|_{L^\infty(\mathbb{T}^d)} \\
&= \frac{1}{\lambda} \sum_{i,j \in \mathbb{N}_0} i \frac{\lambda^{i+j}}{i!j!} \left\| \partial_x^{i} \partial_p^j f(x,\cdot) \right\|_{L^\infty(\mathbb{T}^d)} \\
&\leq \frac{1}{\lambda} \sup_{a \in \mathbb{N}} a \frac{\lambda^a}{\nu^a} \sum_{i,j \in \mathbb{N}_0} \frac{\nu^{i+j}}{i!j!} \left\| \partial_x^{i} \partial_p^j f(x,\cdot) \right\|_{L^\infty(\mathbb{T}^d)} = C \, |\partial_x f|_{C_x^\nu}
\end{aligned}$$

for $C = \sup_{a \in \mathbb{N}} a \frac{\lambda^{a-1}}{\nu^a} < \infty$. The estimate for $\partial = \partial_p$ can be proved similarly. □

Lemma 4.4.8. *Let $g, h : U_1 \to U_2$ and $f : U_2 \times V \to \mathbb{R}$ are analytic, where $U_1, V \in \mathbb{R}^d$ and $U_2 \subset \mathbb{R}^N$ are open sets. Then for every $\lambda_1, \lambda_2 \geq 0$ it holds*

$$\|f(g,\cdot)h\|_{C_x^\lambda} \leq |\partial_1 f|_{C_y^{\mu,\lambda}} \, \|g\|_{C_x^\lambda} \, |h|_{C_x^\lambda} + \left(|f|_{C_y^{\mu,\lambda}} + |\partial_2 f|_{C_y^{\mu,\lambda}} \right) \|h\|_{C_y^{\mu,\lambda}}$$

for $x \in U_1$, $y = g(x)$ and $\mu = |g|_{\dot{C}_x^\lambda}$.

Proof. Let us compute

$$\begin{aligned}
|\partial_x (f(g,\cdot)h)|_{C_x^\lambda} &\leq |\partial_x f(g,\cdot)|_{C_x^\lambda} \, |h|_{C_x^\lambda} + |f(g,\cdot)|_{C_x^\lambda} \, |\partial_x h|_{C_x^\lambda} \\
&\leq |(\partial_1 f)(g,\cdot)|_{C_x^\lambda} \, |\partial_x g|_{C_x^\lambda} \, |h|_{C_x^\lambda} + |f(g,\cdot)|_{C_x^\lambda} \, |\partial_x h|_{C_x^\lambda} \\
&\leq |\partial_1 f|_{C_y^{\mu,\lambda}} \, |\partial_x g|_{C_x^\lambda} \, |h|_{C_x^\lambda} + |f|_{C_y^{\mu,\lambda}} \, |\partial_x h|_{C_x^\lambda} ,
\end{aligned}$$

where $y = g(x)$ and $\mu = |g|_{\dot{C}_x^\lambda}$. □

The main advantage of the analytic norms with parameter λ is the fact that we can estimate the derivative of a function in the analytic norms by the analytic norms of the function with a larger parameter λ'. This phenomenon is the key ingredient in [40] for the concept of nonlinear Landau damping.

Lemma 4.4.9. *Let $\lambda > 0, 0 \leq \mu < \lambda$, $U \subset \mathbb{R}^d$ be open and $f \in \mathcal{O}(U \times \mathbb{T}^d)$. For $x \in U$, we have the Lipschitz estimate*

$$\|f\|_{C_x^{\lambda-\mu}} + \mu \, \|Df\|_{C_x^{\lambda-\mu}} \leq \|f\|_{C_x^\lambda} \leq \|f\|_{C_x^{\lambda-\mu}} + \mu \, \|Df\|_{C_x^\lambda} .$$

In particular, if $\|Df\|_{C_x^\lambda} < \infty$, it holds

$$\partial_\lambda^- \|f\|_{C_x^\lambda} := \lim_{\mu \searrow 0} \frac{\|f\|_{C_x^\lambda} - \|f\|_{C_x^{\lambda-\mu}}}{\mu} = \|Df\|_{C_x^\lambda}.$$

Proof. The proof is straightforward. The crucial part is the estimate

$$\frac{(\lambda-\mu+\mu)^a - (\lambda-\mu)^a}{a!} = \mu \sum_{j=0}^{a-1} \frac{(\lambda-\mu)^j \mu^{a-1-j}}{j!(a-1-j)!(a-j)} \begin{cases} \le \mu \frac{\lambda^{a-1}}{(a-1)!} \\ \ge \mu \frac{(\lambda-\mu)^{a-1}}{(a-1)!}. \end{cases}$$

The remaining part follows by summation and the definition of $\|D\cdot\|_{C_x^\lambda}$. □

Lemma 4.4.10. *Let $\lambda : \mathbb{R} \to \mathbb{R}_{\ge 0}$ be differentiable and monotonically decreasing, $U \subset \mathbb{R}^d$ be open and $f \in C^{\infty,1}((U \times \mathbb{T}^d) \times (0,T))$ for $T > 0$. We write $f(t) := f(\cdot, t)$. Then*

$$\|f(t)\|_{C_x^{\lambda(t)}} - \int_s^t \dot{\lambda}(\tau) \|Df(s)\|_{C_x^{\lambda(\tau)}} \, d\tau \le \|f(s)\|_{C_x^{\lambda(s)}} + \int_s^t \|\partial_t f(\tau)\|_{C_x^{\lambda(\tau)}} \, d\tau$$

for $0 < s < t < T$.

Proof. As a consequence of Lemma 4.4.9 and the monotone convergence theorem, we have

$$\|f(s)\|_{C_x^{\lambda(s)}} = \|f(s)\|_{C_x^{\lambda(t)}} - \int_s^t \dot{\lambda}(\tau) \|Df(s)\|_{C_x^{\lambda(\tau)}} \, d\tau$$

for $0 < s < t < T$ and $\dot{\lambda} \le 0$. Using the monotone convergence theorem,

$$\begin{aligned} \|f(t)\|_{C_x^{\lambda(t)}} &\le \|f(s)\|_{C_x^{\lambda(t)}} + \left\| \int_s^t \partial_t f(\tau) d\tau \right\|_{C_x^{\lambda(t)}} \\ &\le \|f(s)\|_{C_x^{\lambda(t)}} + \int_s^t \|\partial_t f(\tau)\|_{C_x^{\lambda(\tau)}} \, d\tau \end{aligned}$$

since $\|\cdot\|_{C_x^\lambda}$ is monotonically increasing in λ and $\dot{\lambda} \le 0$. □

Lemma 4.4.11. *Let $\lambda \in C^1(\mathbb{R}, \mathbb{R}_{\ge 0})$ with $\dot{\lambda} \le 0$, $U \subset \mathbb{R}^d$ open, $x \in U$ and $f \in C^{\infty,1}((U \times \mathbb{T}^d) \times (0,T))$ for $T > 0$ such that*

$$t \mapsto \|Df(t)\|_{C_x^{\lambda(t)}} \in L^\infty_{\mathrm{loc}}((0,T)) \quad \textit{and} \quad t \mapsto \|\partial_t f(t)\|_{C_x^{\lambda(t)}} \in L^\infty_{\mathrm{loc}}((0,T)).$$

Then $t \mapsto \|f(t)\|_{C_x^{\lambda(t)}} \in W^{1,\infty}_{\mathrm{loc}}((0,T))$ with

$$\frac{d}{dt} \|f(t)\|_{C_x^{\lambda(t)}} \le \dot{\lambda}(t) \|Df(t)\|_{C_x^{\lambda(t)}} + \|\partial_t f(t)\|_{C_x^{\lambda(t)}}.$$

Proof. Similar to the proof of Lemma 4.4.10, we see that

$$\left| \|f(t)\|_{C_x^{\lambda(t)}} - \|f(s)\|_{C_x^{\lambda(s)}} + \int_s^t \dot{\lambda}(\tau) \|Df(s)\|_{C_x^{\lambda(\tau)}} \, d\tau \right| \leq \int_s^t \|\partial_t f(\tau)\|_{C_x^{\lambda(\tau)}} \, d\tau.$$

for $0 < s < t < T$ such that $\|Df(s)\|_{C_x^{\lambda(s)}} < \infty$. Thus, $\|f(t)\|_{C_x^{\lambda(t)}}$ admits a local Lipschitz continuous representant and in particular, $t \mapsto \|f(t)\|_{C_x^{\lambda(t)}} \in W^{1,\infty}_{\mathrm{loc}}((0,T))$. Moreover, let $s > 0$ be a Lebesgue point of $\frac{d}{dt} \|f(s)\|_{C_x^{\lambda(s)}}$ and $\|Df(s)\|_{C_x^{\lambda(s)}} < \infty$. Then,

$$\frac{d}{dt} \|f(s)\|_{C_x^{\lambda(s)}} \leq \|\partial_t f(s)\|_{C_x^{\lambda(s)}} + \dot{\lambda}(s) \|Df(s)\|_{C_x^{\lambda(t)}}$$

for all $t > s$. Finally, the assertion follows by taking the limit $t \to s$. □

Lemma 4.4.12. *Let $\lambda : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be continuously differentiable and monotonely decreasing, $U \subset \mathbb{R}^d$ be open and $f \in C^{\infty,1}((U \times \mathbb{T}^d) \times (0,T))$ for $T > 0$. Moreover, we assume that*

$$\textit{either} \quad \|Df(t)\|_{C_x^{\lambda(t)}} \quad \textit{or} \quad \|\partial_t f(t)\|_{C_x^{\lambda(t)}} \quad \textit{is bounded in } t \in (0,T)$$

writing $f(t) := f(\cdot, t)$. Then

$$\|f(t)\|_{C_x^{\lambda(t)}} - \int_s^t \dot{\lambda}(\tau) \|Df(\tau)\|_{C_x^{\lambda(\tau)}} \, d\tau \leq \|f(s)\|_{C_x^{\lambda(s)}} + \int_s^t \|\partial_t f(\tau)\|_{C_x^{\lambda(\tau)}} \, d\tau$$

for all $0 < s < t < T$.

Proof. Setting

$$P_{f,N}(\lambda, t) := \sum_{0 \leq i+j \leq 1} \sum_{a,b \leq N} \frac{\lambda^{a+b}}{a!b!} \left\| \partial_x^{i+a} \partial_p^{j+b} f(x, \cdot, t) \right\|_{L^\infty(\mathbb{T}^d)}$$

and

$$Q_N(\lambda, t) := \sum_{1 \leq i+j \leq 2} \sum_{\substack{a,b \leq N \\ a+b < 2N}} \frac{\lambda^{a+b}}{a!b!} \left\| \partial_x^{i+a} \partial_p^{j+b} f(x, \cdot, t) \right\|_{L^\infty(\mathbb{T}^d)},$$

we have $P_{f,N}(\lambda, t) \to \|f(t)\|_{C_x^\lambda}$ and $Q_N(\lambda, t) \to \|Df(t)\|_{C_x^\lambda}$ as $N \to \infty$. Let $i, j, a, b \in \mathbb{N}_0$ and $0 < s < t$. Then

$$\begin{aligned}
&\left| \left\| \partial_x^{i+a} \partial_p^{j+b} f(x, \cdot, t) \right\|_{L^\infty(\mathbb{T}^d)} - \left\| \partial_x^{i+a} \partial_p^{j+b} f(x, \cdot, s) \right\|_{L^\infty(\mathbb{T}^d)} \right| \\
&\leq \left\| \partial_x^{i+a} \partial_p^{j+b} f(x, \cdot, t) - \partial_x^{i+a} \partial_p^{j+b} f(x, \cdot, s) \right\|_{L^\infty(\mathbb{T}^d)} \\
&\leq \sup_{s \leq \tau \leq t} \left\| \partial_x^{i+a} \partial_p^{j+b} \partial_t f(x, \cdot, \tau) \right\|_{L^\infty(\mathbb{T}^d)} (t - s)
\end{aligned}$$

implies

$$|P_{f,N}(\lambda, t) - P_{f,N}(\lambda, s)| \le \sup_{s \le \tau \le t} P_{\partial_t f,N}(\lambda, \tau)(t - s).$$

Next, let $\lambda : \mathbb{R} \to \mathbb{R}_{\ge 0}$ be continuously differentiable. Proceeding similarly as in the proof of Lemma 4.4.9, we have

$$\begin{aligned} |P_{f,N}(\lambda(t), t) - P_{f,N}(\lambda(s), s)| \le \sup_{s \le \tau \le t} & P_{\partial_t f,N}(\lambda(t), \tau)(t - s) \\ & + \sup_{s \le \tau \le t} \left|\dot{\lambda}(\tau)\right| Q_N(\lambda(\tau), s)(t - s). \end{aligned}$$

Therefore, $P_{f,N}(\lambda(t), t)$ is Lipschitz continuous w.r.t. t and in addition, it belongs to $W^{1,\infty}((0, T))$ with

$$\frac{d}{dt} P_{f,N}(\lambda(t), t) \le P_{\partial_t f,N}(\lambda(t), t) + \dot{\lambda}(t) Q_N(\lambda(t), t),$$

since $P_{f,N}$, $P_{\partial_t f,N}$ and Q_N are continuous. Using the monotone convergence theorem, we obtain

$$\begin{aligned} & \|f(t)\|_{C_x^{\lambda(t)}} + \int_s^t \left(-\dot{\lambda}(\tau) \|Df(\tau)\|_{C_x^{\lambda(\tau)}} - \|\partial_t f(\tau)\|_{C_x^{\lambda(\tau)}}\right) d\tau \\ & \quad \overset{N\to\infty}{\longleftarrow} P_{f,N}(\lambda(t), t) + \int_s^t \left(-\dot{\lambda}(\tau) Q_N(\lambda(\tau), \tau) + P_{\partial_t f,N}(\lambda(\tau), \tau)\right) d\tau \\ & \quad \le P_{f,N}(\lambda(s), s) \le \|f(s)\|_{C_x^{\lambda(s)}}, \end{aligned}$$

because either $\|Df(\tau)\|_{C_x^{\lambda(\tau)}}$ or $\|\partial_t f(\tau)\|_{C_x^{\lambda(\tau)}}$ is bounded. □

4.5 Time dependent analytic norms

In this section we introduce different analytic norms depending on another parameter t, which we call time. In the previous section, we already proved some estimate for the analytic norm $\|f(t)\|_{C_x^\lambda}$ for some time depending λ. So far we assumed that $\lambda(t) = \lambda_0 - \mu t$. However, this definition has the side effect that the norm $\|f(t)\|_{C_x^{\lambda(t)}}$ is only well-defined on a small time interval. In the following, we replace λ by an exponentially decaying function and adjust the definition of the analytic norm in order to obtain stronger estimates.

Definition 4.5.1. Let $\epsilon : \mathbb{T}^d \to \mathbb{R}$ be defined by

$$\epsilon : p = (p_1, \dots, p_d) \mapsto -2J \sum_{i=1}^d \cos(p_i)$$

for fixed $J > 0$ and let $v := \nabla\epsilon$. For $\psi : \mathbb{R}^d \times \mathbb{T}^d \to \mathbb{R}$, we define

$$\tilde{\partial}_{vt}\psi := \tilde{\partial} := \partial_p \psi - tv'(p)\partial_x \psi$$

as well as

$$T_{\pm vt}\psi(x,k) := \psi(x \pm tv(p), p) \quad \text{for } x \in \mathbb{R}^d, p \in \mathbb{T}^d.$$

The notation $\tilde{\partial}_{vt}$ is motivated by the property

$$\begin{aligned} \tilde{\partial}_{vt} T_{vt}\psi(x,p) &= (\partial_p - tv'(p)\partial_x)\psi(x + tv(p), p) \\ &= (\partial_p \psi - t(v'(p) - v'(p))\partial_x \psi)(x + tv(p), p) = T_{vt}\partial_p \psi(x,p). \end{aligned}$$

Definition 4.5.2. For $t \geq 0$, $J, \lambda > 0$ and $\psi : \mathbb{R}^d \times \mathbb{T}^d \to \mathbb{R}$ analytic, let

$$|\psi|_{\dot{\mathcal{O}}_t^\lambda} := \sum_{(a,b)\in\mathbb{N}^2\setminus\{0\}} \frac{\lambda^{a+b}}{a!b!} \left\| \partial_x^a \tilde{\partial}_{vt}^b \psi \right\|_{L^\infty}$$

and

$$|\psi|_{\mathcal{O}_t^\lambda} = \|\psi\|_{L^\infty} + |\psi|_{\dot{\mathcal{O}}_t^\lambda}.$$

Moreover, we set

$$\|\psi\|_{\dot{\mathcal{O}}_t^\lambda} := |\psi|_{\dot{\mathcal{O}}_t^\lambda} + |\partial_x \psi|_{\dot{\mathcal{O}}_t^\lambda} + \left|\tilde{\partial}_{vt}\psi\right|_{\dot{\mathcal{O}}_t^\lambda},$$

$$\|D\psi\|_{\mathcal{O}_t^\lambda} := \sum_{i+j=1}^{2} \left|\partial_x^i \tilde{\partial}_{vt}^j \psi\right|_{\mathcal{O}_t^\lambda}.$$

Lemma 4.5.3.

$$|\psi\phi|_{\dot{\mathcal{O}}_t^\lambda} \leq |\psi|_{\dot{\mathcal{O}}_t^\lambda} |\phi|_{\mathcal{O}_t^\lambda} + \|\psi\|_{L^\infty} |\phi|_{\dot{\mathcal{O}}_t^\lambda}.$$

Proof. At first, we compute the $|\cdot|_{\mathcal{O}_t^\lambda}$ norm of the product by

$$\begin{aligned} |\psi\phi|_{\mathcal{O}_t^\lambda} &\leq \sum_{a,b\in\mathbb{N}} \frac{\lambda^{a+b}}{a!b!} \left\|\partial_x^a \tilde{\partial}_{vt}^b(\psi\phi)\right\|_{L^\infty} \\ &\leq \sum_{a,b\in\mathbb{N}} \sum_{\substack{a_1+a_2=a \\ b_1+b_2=b}} \frac{\lambda^{a+b}}{a!b!} \binom{a}{a_1}\binom{b}{b_1} \left\|\partial_x^{a_1}\tilde{\partial}_{vt}^{b_1}\psi\right\|_{L^\infty} \left\|\partial_x^{a_2}\tilde{\partial}_{vt}^{b_2}\phi\right\|_{L^\infty} \\ &= |\psi|_{\mathcal{O}_t^\lambda} |\phi|_{\mathcal{O}_t^\lambda}. \end{aligned}$$

Thus, the assertion follows by making use of $|\cdot|_{\dot{\mathcal{O}}_t^\lambda} = |\cdot|_{\mathcal{O}_t^\lambda} - \|\cdot\|_{L^\infty}$. □

Lemma 4.5.4. *For $T > 0$ let $\lambda \in C^1([0,T], \mathbb{R}_{>0})$ be decreasing and $f \in C^{\infty,1-}((\mathbb{R}^d \times \mathbb{T}^d) \times [0,T])$ such that*

$$\sup_{0\le t\le T} \left(\|f\|_{\dot{\mathcal{O}}_t^{\lambda(t)}} + \|\partial_t f\|_{\dot{\mathcal{O}}_t^{\lambda(t)}} + \|Df\|_{\mathcal{O}_t^{\lambda(t)}} \right) \le const.$$

Writing $f(\cdot,\cdot,t) = f(t)$, the function $t \mapsto \|f(t)\|_{\mathcal{O}_t^{\lambda(t)}}$ belongs to $W^{1,\infty}([0,T])$ and fulfills

$$\frac{d}{dt}\|f(t)\|_{\dot{\mathcal{O}}_t^{\lambda(t)}} \le \|\partial_t f\|_{\dot{\mathcal{O}}_t^{\lambda(t)}} + \left(4J(e^{\lambda(t)}-1) + \frac{\lambda'(t)}{p}\right)\|Df(t)\|_{\mathcal{O}_t^{\lambda(t)}} + \left(2Je^{\lambda(t)} + \frac{\lambda'(t)}{q\lambda(t)}\right)\|f(t)\|_{\dot{\mathcal{O}}_t^{\lambda(t)}} \tag{4.2}$$

for all $p, q \ge 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. At first, we fix $\tau \in [0,T]$ and show that $t \mapsto \|f(t)\|_{\mathcal{O}_\tau^{\lambda(t)}}$ is Lipschitz continuous and therefore an element of $W^{1,\infty}([0,T])$: Let $a, b \in \mathbb{N}_0$. Since $g = \partial_x^a \tilde{\partial}_{vt}^b f$ is differentiable w.r.t. t, we have

$$\begin{aligned} |\|g(t)\|_{L^\infty} - \|g(s)\|_{L^\infty}| &\le \|g(t) - g(s)\|_{L^\infty} \\ &\le \|\partial_t g\|_{L_t^\infty([s,t];L^\infty)} |t-s| \end{aligned} \tag{4.3}$$

for all $0 \le t \le T$. In order to proceed, we claim

$$[\partial_t, \tilde{\partial}_{vt}^b] = -\sum_{i=1}^b \binom{b}{i} v^{(i)} \tilde{\partial}_{vt}^{b-i} \partial_x, \tag{4.4}$$

where $[\cdot,\cdot]$ denotes the commutator, i.e. $[A,B] = AB - BA$. This can be proved by induction, since

$$\partial_t \tilde{\partial}_{vt} = \partial_t(\partial_p - tv'(p)\partial_x) = (\partial_p - tv'(p)\partial_x)\partial_t - v'(p)\partial_x = \tilde{\partial}_{vt}\partial_t - v^{(1)}(p)\partial_x$$

and

$$\begin{aligned} [\partial_t, \tilde{\partial}_{vt}^{b+1}] &= \tilde{\partial}_{vt}[\partial_t, \tilde{\partial}_{vt}^b] + [\partial_t, \tilde{\partial}_{vt}]\tilde{\partial}_{vt}^b \\ &\overset{(4.4)}{=} -\sum_{i=1}^b \binom{b}{i} \partial_{vt} v^{(i)} \tilde{\partial}_{vt}^{b-i}\partial_x - v^{(1)}\partial_x \partial_{vt} \\ &= -\sum_{i=1}^b \binom{b}{i} \left(v^{(i+1)}\tilde{\partial}_{vt}^{b-i} + v^{(i)}\tilde{\partial}_{vt}^{b+1-i} + \delta_{i1} v^{(1)}\tilde{\partial}_{vt}^b \right)\partial_x \\ &= -v^{(b+1)}\partial_x - \sum_{i=1}^b \left(\binom{b}{i} + \binom{b}{i-1} \right) \partial_{vt} v^{(i)} \tilde{\partial}_{vt}^{b+1-i}\partial_x \\ &= -\sum_{i=1}^{b+1} \binom{b+1}{i} v^{(i)} \tilde{\partial}_{vt}^{b+1-i}\partial_x \end{aligned}$$

hold due to $[\partial_x, \tilde{\partial}_{vt}^b] = 0$ and $[\tilde{\partial}_{vt}^b, v^{(i)}] = v^{(i+1)}$. Fix $s \in (0,T), \delta \in (0,1)$ and let $\lambda_0 = \delta\lambda(s)$. Thus, we can estimate

$$\begin{aligned}
\sum_{b=0}^{\infty} \frac{\lambda_0^b}{b!} \left\| \partial_t \partial_x^a \tilde{\partial}_{vt}^b f \right\|_{L^\infty} &- \sum_{b=0}^{\infty} \frac{\lambda_0^b}{b!} \left\| \partial_x^a \tilde{\partial}_{vt}^b \partial_t f \right\|_{L^\infty} \\
&\leq \sum_{b=1}^{\infty} \frac{\lambda_0^b}{b!} \sum_{i=1}^{b} \binom{b}{i} \left\| v^{(i)} \right\|_{L^\infty} \left\| \partial_x^a \tilde{\partial}_{vt}^{b-i} \partial_x f \right\|_{L^\infty} \\
&\leq \underbrace{\sum_{i=1}^{\infty} \frac{\lambda_0^i}{i!} \left\| v^{(i)} \right\|_{L^\infty}}_{\leq 2J(e^{\lambda_0}-1) \leq C_{\lambda_0}\lambda_0} \sum_{b=0}^{\infty} \frac{\lambda_0^b}{b!} \left\| \partial_x^a \tilde{\partial}_{vt}^b \partial_x f \right\|_{L^\infty}
\end{aligned}$$

with $C_{\lambda_0} := 2J\frac{e^{\lambda_0}-1}{\lambda_0} \leq 2J\frac{e^{\max\lambda}-1}{\max\lambda} = C_{\max\lambda} < \infty$. With this calculation and Inequality (4.3), we obtain by summing over all a that $t \mapsto \|f(t)\|_{\mathcal{O}_t^{\lambda_0}}$ is Lipschitz continuous in a neigborhood of s and satisfies

$$\left.\frac{d}{dt}\right|_{t=s} |f(t)|_{\mathcal{O}_t^{\lambda_0}} \leq |\partial_t f(s)|_{\mathcal{O}_s^{\lambda_0}} + C_M \lambda_0 \, |\partial_x f(s)|_{\mathcal{O}_s^{\lambda_0}} .$$

Derivation w.r.t. λ of $|f|_{\mathcal{O}_t^\lambda}$ yields on the one hand

$$\begin{aligned}
\partial_{\lambda_0} |f|_{\dot{\mathcal{O}}_t^{\lambda_0}} &= \sum_{0 \neq (a,b)} \left(\frac{\lambda^{a-1+b}}{(a-1)!b!} + \frac{\lambda^{a+b-1}}{a!(b-1)!} \right) \left\| \partial_x^a \tilde{\partial}_{vt}^b f \right\|_{L^\infty} \\
&= |\partial_x f|_{\mathcal{O}_t^{\lambda_0}} + \left| \tilde{\partial}_{vt} f \right|_{\mathcal{O}_t^{\lambda_0}} .
\end{aligned}$$

On the other hand, we may estimate the same series by

$$\partial_{\lambda_0} |f|_{\dot{\mathcal{O}}_t^{\lambda_0}} = \sum_{0 \neq (a,b) \in \mathbb{N}_0^2} (a+b) \frac{\lambda^{a+b-1}}{a!b!} \left\| \partial_x^a \tilde{\partial}_{vt}^b f \right\|_{L^\infty} \geq \frac{1}{\lambda_0} |f|_{\dot{\mathcal{O}}_t^{\lambda_0}} .$$

Combining the foregoing calculations, we have for $\tilde{\lambda} = \delta\lambda$

$$\begin{aligned}
\frac{d}{dt} |f(t)|_{\mathcal{O}_t^{\tilde{\lambda}(t)}} \leq |\partial_t f|_{\mathcal{O}_t^{\tilde{\lambda}(t)}} &+ (C_M \tilde{\lambda}(t) + \tilde{\lambda}'(t)) \, |\partial_x f(t)|_{\mathcal{O}_t^{\tilde{\lambda}(t)}} \\
&+ \tilde{\lambda}'(t) \left| \tilde{\partial}_{vt} f(t) \right|_{\mathcal{O}_t^{\tilde{\lambda}(t)}} \qquad (4.5)
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dt} |f(t)|_{\dot{\mathcal{O}}_t^{\tilde{\lambda}(t)}} \leq |\partial_t f|_{\dot{\mathcal{O}}_t^{\tilde{\lambda}(t)}} &+ \left(C_M \tilde{\lambda}(t) + \frac{\tilde{\lambda}'(t)}{p} \right) |\partial_x f(t)|_{\mathcal{O}_t^{\tilde{\lambda}(t)}} \\
&+ \frac{\tilde{\lambda}'(t)}{p} \left| \tilde{\partial}_{vt} f(t) \right|_{\mathcal{O}_t^{\tilde{\lambda}(t)}} + \frac{\tilde{\lambda}'(t)}{q\tilde{\lambda}(t)} |f(t)|_{\dot{\mathcal{O}}_t^{\tilde{\lambda}(t)}}
\end{aligned}$$

for all $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ if $\tilde{\lambda}' \leq 0$. In order to conclude the proof, we must translate the estimate above for the norm $\|\cdot\|_{\mathcal{O}_t^{\tilde{\lambda}(t)}}$. This can directly be done by the definition of $\|\cdot\|_{\mathcal{O}_t^{\tilde{\lambda}(t)}}$ and the calculation

$$\begin{aligned} \left|\partial_t \tilde{\partial}_{vt} f\right|_{\mathcal{O}_t^{\tilde{\lambda}(t)}} &\leq \left|\tilde{\partial}_{vt} \partial_t f\right|_{\mathcal{O}_t^{\tilde{\lambda}(t)}} + |v' \partial_x f|_{\mathcal{O}_t^{\tilde{\lambda}(t)}} \\ &\leq \left|\tilde{\partial}_{vt} \partial_t f\right|_{\mathcal{O}_t^{\tilde{\lambda}(t)}} + 2J e^{\tilde{\lambda}(t)} \|f\|_{\mathcal{O}_t^{\tilde{\lambda}(t)}} . \end{aligned}$$

The estimate for $\dot{\mathcal{O}}_t^{\tilde{\lambda}(t)}$ is slightly different and reads

$$\begin{aligned} \left|\partial_t \tilde{\partial}_{vt} f\right|_{\dot{\mathcal{O}}_t^{\tilde{\lambda}(t)}} - \left|\tilde{\partial}_{vt} \partial_t f\right|_{\dot{\mathcal{O}}_t^{\tilde{\lambda}(t)}} &= |v' \partial_x f|_{\dot{\mathcal{O}}_t^{\tilde{\lambda}(t)}} \\ &\leq 2J e^{\tilde{\lambda}(t)} |\partial_x f|_{\dot{\mathcal{O}}_t^{\tilde{\lambda}(t)}} + 2J(e^{\tilde{\lambda}(t)} - 1) \|\partial_x f\|_{L^\infty} . \end{aligned}$$

Using $|\partial_x f|_{\dot{\mathcal{O}}_t^{\tilde{\lambda}(t)}} \leq \|f\|_{\dot{\mathcal{O}}_t^{\tilde{\lambda}(t)}}$, we estimate either

$$(e^{\tilde{\lambda}(t)} - 1) \|\partial_x f\|_{L^\infty} \leq (e^{\tilde{\lambda}(t)} - 1) \|Df\|_{\mathcal{O}_t^{\tilde{\lambda}(t)}}$$

or

$$(e^{\tilde{\lambda}(t)} - 1) \|\partial_x f\|_{L^\infty} \leq \frac{e^{\tilde{\lambda}(t)} - 1}{\tilde{\lambda}} \|f\|_{\dot{\mathcal{O}}_t^{\tilde{\lambda}(t)}} . \qquad \square$$

Finally, we have proved the assertion for $\tilde{\lambda} := \delta\lambda$. Since the assertion holds uniformly in δ, we can take the limit $\delta \to 1$.

Corollary 4.5.5. *Let the assumptions of the previous lemma be fulfilled. Let $\lambda_0, \alpha, \beta \geq 0$ and $\mu := 2J \frac{(2+\lambda_0)e^{\lambda_0} - 2}{\lambda_0} + \alpha + \beta \leq 6Je^{\lambda_0} + \alpha + \beta$. Then it holds*

$$\frac{d}{dt} \|f(t)\|_{\dot{\mathcal{O}}_t^{\lambda(t)}} + \alpha\lambda(t) \|Df(t)\|_{\mathcal{O}_t^{\lambda(t)}} + \beta \|f(t)\|_{\dot{\mathcal{O}}_t^{\lambda(t)}} \leq \|\partial_t f\|_{\dot{\mathcal{O}}_t^{\lambda(t)}} \tag{4.6}$$

for $\lambda(t) = \lambda_0 \exp(-\mu t)$. Note that $\mu \geq 6J$ is true for all possible combinations of λ_0, α, β.

Proof. Recall

$$\begin{aligned} \frac{d}{dt} \|f(t)\|_{\dot{\mathcal{O}}_t^{\lambda(t)}} \leq \|\partial_t f\|_{\dot{\mathcal{O}}_t^{\lambda(t)}} &+ \left(4J(e^{\lambda(t)} - 1) + \frac{\lambda'(t)}{p}\right) \|Df(t)\|_{\dot{\mathcal{O}}_t^{\lambda(t)}} \\ &+ \left(2Je^{\lambda(t)} + \frac{\lambda'(t)}{q\lambda(t)}\right) \|f(t)\|_{\dot{\mathcal{O}}_t^{\lambda(t)}} . \end{aligned}$$

We choose $\frac{1}{p} = \frac{4J(e^\lambda - 1) + \alpha\lambda}{\lambda\mu}$, $\frac{1}{q} = \frac{2Je^\lambda + \beta}{\mu}$ and $\mu = \frac{2J(2+\lambda)e^\lambda - 4J}{\lambda} + \alpha + \beta$ and obtain the corollary using $2e^{2J\lambda} - 2 \leq 4J\lambda e^{2J\lambda}$ for $\lambda \geq 0$. $\square$

Chapter 5

Fermi Function

5.1 Definition of the Fermi-Dirac distribution

In statistical physics, a distribution function provides the density of particles at a certain point for a given momentum. The momentum space for free particles is the whole $\mathbb{R}^d$, see [32]. However, in the case of a periodic potential, a connected bounded subset $B \subset \mathbb{R}^d$, the so-called first Brillouin zone, is enough to describe the momentum space due to the present periodic potential. This well-known fact in the theory of semiconductors is a consequence of the Bloch Theorem [14].

Let dx denote the Lebesgue measure. We define

$$dp = \frac{dx}{\int_B 1 dx}$$

as the normalized Lebesgue measure, which satisfies $|B| := dp(B) = 1$.

Definition 5.1.1. The dispersive relation, relating the momentum p to the kinetic energy, is a continuous function $\epsilon : B \to \mathbb{R}$. Furthermore, we assume

$$|\{\epsilon = c\}| = 0 \qquad \text{for all } c \in \mathbb{R} \tag{5.1}$$

and that ϵ is symmetric by means of

$$\int\limits_B \phi(\epsilon(p))dp = \int\limits_B \phi(-\epsilon(p))dp \qquad \text{for all measurable } \phi. \tag{5.2}$$

Example 5.1.2. In the prototype case, we assume that the potential forms a simple cubic lattice. Then we can identify the first Brillouin zone $B :=$

$[0, 2\pi)^d \subset \mathbb{R}^d$ with the torus $\mathbb{T}^d$. The function

$$\epsilon : \mathbb{T}^d \simeq [0, 2\pi)^d \to \mathbb{R}, \quad p = (p_1, \dots, p_d) \mapsto -2J \sum_{i=1}^{d} \cos(p_i)$$

for $J > 0$ fulfills the requirements of Definition 5.1.1 and is an approximation for the lowest energy band (see [4]).

Remark 5.1.3. Since B is a connected set and ϵ is continuous, the image $\epsilon(B)$ equals an interval I. The function

$$h : \overline{I} \to \mathbb{R}; \; c \mapsto |\{\epsilon \geq c\}|$$

is continuous, because

$$h(s) = |\{\epsilon \geq c\}| = \int_B \chi_{\{\epsilon \geq s\}}(p)dp \to \int_B \chi_{\{\epsilon \geq c\}}(p)dp \qquad \text{as } s \to c$$

according to Lebesgue's theorem. Moreover, the condition $|\{\epsilon = c\}| = 0$ for all $c \in \mathbb{R}$ implies that h decreases monotonically. Now, let $s, c \in \overline{I}$ with $c \leq s$. Assuming $h(c) = h(s)$ yields $|\{c < \epsilon < s\}| = 0$, which thereby entails that $\{c < \epsilon < d\}$ is empty by being open due to the continuity of ϵ. Thus, $c = s$.

Remark 5.1.4. Writing $\phi = \max(\phi, 0) + \min(\phi, 0)$, we infer from the symmetry of ϵ that

$$\int_B \phi(\epsilon(p))dp = \int_{\{\epsilon>0\}} \phi(\epsilon(p))dp + \int_{\{\epsilon>0\}} \phi(-\epsilon(p))dp \tag{5.3}$$

holds for all measurable ϕ and in particular, $|\{\epsilon > 0\}| = 1/2$ if we choose $\phi \equiv 1$ in the equation (5.3).

Definition 5.1.5. Every element of

$$L^1(B; [0, \eta^{-1}]) := \left\{ f \in L^1(B) : 0 \leq f \leq \frac{1}{\eta} \right\}$$

being the equivalence class of a

$$p \mapsto \mathcal{F}(\lambda, p) := \frac{1}{\eta + e^{-\lambda_0 - \lambda_1 \epsilon(p)}} \tag{5.4}$$

for some $\lambda = (\lambda_0, \lambda_1) \in \mathbb{R}^2$ is called a (generalized) Fermi-Dirac distribution. The parameter λ_0, λ_1 are sometimes called entropy parameters, where physically $-\lambda_1$ equals the inverse temperature.

The main objective in this section is to associate a given particle density and a given energy with a Fermi-Dirac distribution. Before we are able to achieve this, we derive a relation between (n, E) and the entropy parameters λ_0, λ_1.

Definition 5.1.6. We define the particle and energy density for given $\lambda \in \mathbb{R}^2$ as

$$\tilde{n}(\lambda) := \int_B \mathcal{F}(\lambda, p)dp \quad \text{and} \quad \tilde{E}(\lambda, p) := \int_B \epsilon(p)\mathcal{F}(\lambda, p)dp,$$

respectively.

Remark 5.1.7. The functions $\tilde{n}$ and $\tilde{E}$ are analytic.

Proposition 5.1.8. *The mapping*

$$\begin{aligned}\mathbb{R}^2 &\to \left\{\int_B (1, \epsilon(p))f(p)dp : f \in L^1(B; [0, \eta^{-1}])\right\} \setminus \left\{(0,0), \left(\frac{1}{\eta}, 0\right)\right\},\\ \lambda &\mapsto \int_B (1, \epsilon(p))\mathcal{F}(\lambda, p)dp\end{aligned}$$

is bijective and smooth. Moreover, its inverse is smooth as well.

Before we prove this proposition, we need some results on $\tilde{n}$ and $\tilde{E}$ first.

Lemma 5.1.9. *The functions $\tilde{n}$ and $\tilde{E}$ from Definition* (5.1.6) *are analytic and fulfill*

$$\partial_{\lambda_0}\tilde{n}(\lambda), \partial_{\lambda_1}\tilde{E}(\lambda) > 0 \tag{5.5}$$

as well as

$$\partial_{\lambda_1}\tilde{n}(\lambda), \partial_{\lambda_0}\tilde{E}(\lambda) \lesseqqgtr 0 \quad \text{if and only if} \quad \left(\lambda_0 - \log\frac{1}{\eta}\right)\lambda_1 \lesseqqgtr 0 \tag{5.6}$$

for $\lambda = (\lambda_0, \lambda_1)$.

Proof. To begin with, we observe that

$$\partial_{\lambda_i}\int_B \epsilon(p)^j\mathcal{F}(\lambda, p)dp = \int_B \epsilon(p)^{i+j}\mathcal{F}(\lambda, p)(1 - \eta\mathcal{F}(\lambda, p))dp$$

for $i, j = 0, 1$. The basic idea of this proof is to split this integral into two parts, one with $\epsilon > 0$ and the other with $\epsilon < 0$. Note that changing the sign of ϵ in the definition of $\mathcal{F}$ behaves like a change of the sign of λ_1 since $\mathcal{F}(\lambda, p) = 1/(\eta + e^{-\lambda_0 - \lambda_1\epsilon(p)})$. Thus, the symmetry of ϵ entails

$$\begin{aligned}\partial_{\lambda_i}\int_B \epsilon(p)^j\mathcal{F}(\lambda, p)dp &= \int_{B/2} \epsilon(p)^{i+j}\mathcal{F}(\lambda, p)(1 - \eta\mathcal{F}(\lambda, p))dp\\ &\quad + (-1)^{i+j}\int_{\{\epsilon>0\}} \epsilon(p)^{i+j}\mathcal{F}((\lambda_0, -\lambda_1)p)(1 - \eta\mathcal{F}((\lambda_0, -\lambda_1), p))dp.\end{aligned}$$

Now putting $g(\lambda,p) := \mathcal{F}(\lambda,p)(1-\eta\mathcal{F}(\lambda,p)) = \frac{\exp(-\lambda_0-\lambda_1\epsilon(p))}{(\eta+\exp(-\lambda_0-\lambda_1\epsilon(p)))^2}$, we directly see that $g((\lambda_0,-\lambda_1),p) = \frac{\exp(-\lambda_0-\lambda_1\epsilon(p))}{(\exp(-\lambda_0)+\eta\exp(-\lambda_1\epsilon(p)))^2}$. Then g is positive and

$$\partial_{\lambda_i}\int_B \epsilon(p)^i\mathcal{F}(\lambda,p)dp = \int_{\{\epsilon>0\}}\epsilon(p)(g(\lambda,p)+g((\lambda_0,-\lambda_1),p))dp > 0$$

for $i=0,1$. Comparing the denominator of $g(\lambda,p)$ with the denominator of $g((\lambda_0,-\lambda_1),p)$ yields that

$$g((\lambda_0,\lambda_1),p) \lesseqqgtr g((\lambda_0,-\lambda_1),p) \quad \text{whenever } (e^{-\lambda_0}-\eta)(1-e^{-\lambda_1\epsilon(p)}) \lesseqqgtr 0.$$

We finally deduce the assertion from

$$\partial_{\lambda_i}\int_B \epsilon(p)^j\mathcal{F}(\lambda,p)dp = \int_{\{\epsilon>0\}}\epsilon(p)(g(\lambda,p)-g((\lambda_0,-\lambda_1),p))dp > 0$$

for $(i,j) = (0,1),(1,0)$ and the fact that

$$(1-e^{-\lambda_1\epsilon(p)}) \lesseqqgtr 0 \quad \text{whenever } \lambda_1 \gtreqqless 0$$

for $p\in\{\epsilon>0\}$. ☐

Lemma 5.1.10. *The Jacobian determinant* $\det\partial_\lambda(\tilde{n},\tilde{E})$ *is positive.*

Proof. Starting similarly to the proof of the foregoing lemma, we deduce

$$\det\partial_\lambda(\tilde{n},\tilde{E}) = \int_B\int_B \epsilon(\epsilon-\epsilon')d\mu d\mu'$$
$$= \int_B \epsilon^2 d\mu\int_B 1d\mu - \left(\int_B \epsilon\cdot 1d\mu\right)^2,$$

where $d\mu := d\mu_p := \mathcal{F}(\lambda,p)(1-\eta\mathcal{F}(\lambda,p))dp$ is a positive measure. Thus, an application of the Cauchy Schwarz theorem yields the assertion. ☐

The previous Lemma entails in particular that $\lambda\to(\tilde{n}(\lambda),\tilde{E}(\lambda))$ is a local isomorphism. The goal of the following Lemmata is to show that there is indeed a global isomorphism.

Lemma 5.1.11. *Let* $n\in(0,1/\eta)$. *There exists a unique function* $\phi:\mathbb{R}\to\mathbb{R}$ *such that*

$$n = \int_B \mathcal{F}((\phi(\lambda_1),\lambda_1),p)dp.$$

In particular, it holds $\phi(0) = \log\frac{n}{1-\eta n}$. *Moreover, defining* $\epsilon_{\mathrm{F}}\in\overline{\epsilon(B)}$ *as the unique solution of*

$$|\{\epsilon<\epsilon_{\mathrm{F}}\}| = \eta n,$$

we have

$$\phi(\lambda_1)\mp\epsilon_{\mathrm{F}}\lambda_1 = o(\lambda_1) \text{ as } \lambda_1\to\pm\infty.$$

Remark 5.1.12. Note that ϵ_F is well-defined due to Remark 5.1.3 in conjunction with the symmetry of ϵ. In particular, observe that $-\epsilon_F$ fulfills

$$|\{\epsilon > -\epsilon_F\}| = \eta n.$$

Proof of Lemma 5.1.11. To begin with, we fix λ_1 and compute

$$\int_B \mathcal{F}(\lambda, p)dp = \int_B \frac{dp}{\eta + e^{-\lambda_0 - \lambda_1 \epsilon(p)}} \to \begin{cases} \frac{1}{\eta} & \text{as } \lambda_0 \to \infty, \\ 0 & \text{as } \lambda_0 \to -\infty \end{cases}$$

in use of Lebesgue's theorem. According to Lemma 5.1.9, $\int_B \mathcal{F}(\lambda, p)dp$ is strictly monotone and continuous in λ_0. Therefore, ϕ is well-defined and unique due to the mean-value theorem. Finally, the function ϕ is smooth as a consequence of the implicit function theorem and the positivity of $\partial_{\lambda_0}\tilde{n}$.

Let $r_\pm(\lambda_1) := \phi(\lambda_1) \mp \epsilon_F \lambda_1$. We want to show that $r_\pm(\lambda_1) = o(\lambda_1)$ as $\lambda_1 \to \pm\infty$.

Case 1: Let $\lambda_{1,i}^\pm \to \pm\infty$ as $i \to \infty$. Assume that $r(\lambda_{1,i}^\pm)/\lambda_{1,i}^\pm \to c^\pm \in \mathbb{R}$. Thus, for every $\delta > 0$ there exists an $i_0 \in \mathbb{N}$ such that $\lambda_1(c - \delta) < r_\pm(\lambda_1) < \lambda_1(c + \delta)$ for all $i \geq i_0$ abbreviating $\lambda_1 = \lambda_{1,i}^\pm$ and $c = c^\pm$. We have

$$\begin{aligned} n = \int_B \frac{dp}{\eta + e^{-\lambda_1(\epsilon(p) \pm \epsilon_F) - r_\pm(\lambda_1)}} &\leq \lim_{\lambda_1 \to \pm\infty} \int_B \frac{dp}{\eta + e^{-\lambda_1(\epsilon(p) \pm \epsilon_F + c + \delta)}} \\ &= \frac{1}{\eta} \int_B \chi_{\{\pm(\epsilon \pm \epsilon_F + c + \delta) < 0\}}(p)dp \\ &\to \frac{1}{\eta} \left|\{\pm\epsilon > -\epsilon_F \mp c\}\right| \end{aligned}$$

as $\delta \to 0$. The lower bound can be found analogously and we conclude $\eta n = \left|\{\pm\epsilon > -\epsilon_F \mp c)\}\right|$. According to Remark 5.1.3 and the symmetry of ϵ, this is only true if $c = c^\pm = 0$.

Case 2: Suppose that $r_\pm(\lambda_1)/\lambda_1$ are unbounded at $\pm$ infinity such that there exist sequences $\lambda_{1,i}^\pm \to \pm\infty$ fulfilling $r_\pm(\lambda_{1,i}^\pm)/\lambda_{1,i}^\pm \to \pm\infty$. Similarly to the proof of case 1 and using the boundedness of ϵ, we derive $n = 0$, which undermines the assumption $n \neq 0$. □

Now we are able to define the basic properties of distribution functions. In general, a distribution function depends on the spatial position, the momentum and time. Nevertheless, in the present chapter only the dependency on the momentum is of interest. Thus, we neglect the spatial and time variable for the moment to simplify the notations.

Definition 5.1.13. Let $\eta \in [0, \infty)$. For $f \in L^1(B; [0, \eta^{-1}])$, we define its particle density and energy density as

$$n_f := \int_B f(p)dp \qquad \text{and} \qquad E_f := \int_B \epsilon(p) f(p)dp, \tag{5.7}$$

respectively.

Lemma 5.1.14. *The set of all admissible particle and energy densities is given by*

$$\left\{\int_B (1,\epsilon(p))f(p)dp : f \in L^1(B;[0,\eta^{-1}])\right\}$$
$$= \left\{(n,E) \in \mathbb{R}^2 : 0 \le n \le \frac{1}{\eta} \text{ and } |E| \le e_{\max}(n)\right\}, \quad (5.8)$$

where $e_{\max} = \infty$ *for* $\eta = 0$ *and*

$$e_{\max} : \left[0, \frac{1}{\eta}\right] \to \mathbb{R};\ n \mapsto \frac{1}{\eta}\int_{\{\epsilon \ge c\}} \epsilon(p)dp \text{ with } |\{\epsilon \ge c\}| = \eta n, \quad (5.9)$$

otherwise bounded by $e_{\max}(\eta^{-1}) = \eta^{-1}\int_{\{\epsilon>0\}} \epsilon(p)dp$.

Proof. The assertion is obvious for $\eta = 0$ and hence we can assume that η is positive. The symmetry of ϵ, see condition (5.2), in conjunction with $0 \le f \le \eta^{-1}$ implies the inclusion "$\subseteq$".

Therefore, let $(n,E) \in \mathbb{R}^2$ such that $0 \le n \le 1/\eta$ and $|E| \le e_{\max}(n)$. We define

$$\lambda_s(p) := \frac{s}{\eta}\chi_{\{\epsilon \ge c\}}(p) + \frac{1-s}{\eta}\chi_{\{\epsilon \le -c\}}(p) \text{ with } |\{\epsilon \ge c\}| = \eta n \quad (5.10)$$

for $s \in [0,1]$ and $p \in B$, where χ denotes the characteristic function. Clearly, $\lambda_s \in L^1(B;[0,\eta^{-1}])$. The definition of λ_s and the symmetry of ϵ ensue that $n = \int_B \lambda_s(p)dp$ for each s. Moreover, we infer $-\int_B \lambda_0(p)\epsilon(p)dp = \int_B \lambda_1(p)\epsilon(p)dp = e_{\max}(n)$ once again from the symmetry of ϵ. Since $s \mapsto E(\lambda_s) := \int_B \epsilon(p)\lambda_s(p)dp$ is continuous, the mean value theorem implies the existence of a $t \in [0,1]$ such that $\int_B \lambda_t(p)\epsilon(p)dp = E$, finalizing the proof. □

Lemma 5.1.15. *Let* $0 < n < 1/\eta < \infty$ *and* ϕ *be given by Lemma 5.1.11. Then for every* $-e_{\max} < E < e_{\max}$*, there exists a unique* $\lambda_1 \in \mathbb{R}$ *such that*

$$E = \int_B \epsilon(p)\mathcal{F}((\phi(\lambda_1),\lambda_1),p)dp$$

Proof. The idea of the proof is to make use of the mean value theorem once again to find λ_1. Recalling

$$0 = \frac{d}{d\lambda_1}\tilde{n}(\phi(\lambda_1),\lambda_1) = \phi'(\lambda_1)\partial_{\lambda_0}n(\phi(\lambda_1),\lambda_1) + \partial_{\lambda_1}\tilde{n}(\phi(\lambda_1),\lambda_1)$$

from Lemma 5.1.11, we observe that

$$\begin{aligned}\frac{d}{d\lambda_1}\tilde{E}(\phi(\lambda_1),\lambda_1) &= \phi'(\lambda_1)\partial_{\lambda_0}\tilde{E}(\phi(\lambda_1),\lambda_1) + \partial_{\lambda_1}\tilde{E}(\phi(\lambda_1),\lambda_1)\\ &= \frac{1}{\partial_{\lambda_0} n(\phi(\lambda_1),\lambda_1)}\det \mathcal{J}_{(\tilde{n},\tilde{E})}(\phi(\lambda_1),\lambda_1)\end{aligned}$$

is positive due to Lemmata 5.1.9 and 5.1.10. According to the mean value theorem, it remains to show that

$$\lim_{\lambda\to\pm\infty}\tilde{E}(\phi(\lambda_1),\lambda_1) = \pm\eta e_{\max}(n). \tag{5.11}$$

In order to prove this, recall from Lemma 5.1.11 that

$$r_\pm(\lambda_1) := \mp\epsilon_{\mathrm{F}}\lambda_1 + \phi(\lambda_1) = o(\lambda_1) \text{ as } \lambda_1 \to \pm\infty.$$

Turning to Equation (5.11), we observe

$$\begin{aligned}\tilde{E}((\phi(\lambda_1),\lambda_1),p) &= \int_B \frac{\epsilon(p)dp}{\eta + e^{-\lambda_1(\epsilon(p)\pm\epsilon_{\mathrm{F}})-r_\pm(\lambda_1)}}\\ &\to \frac{1}{\eta}\int_B \epsilon(p)\chi_{\{\mp\epsilon<\epsilon_{\mathrm{F}}\}}(p)dp\end{aligned}$$

as $\lambda_1 = \lambda_1^\pm \to \pm\infty$. By means of the symmetry of ϵ, we obtain

$$\int_B \epsilon(p)\chi_{\{-\epsilon<\epsilon_{\mathrm{F}}\}}(p)dp = \int_B -\epsilon(p)\chi_{\{\epsilon>\epsilon_{\mathrm{F}}\}}(p)dp = \eta e_{\max}$$

and

$$\int_B \epsilon(p)\chi_{\{\epsilon<\epsilon_{\mathrm{F}}\}}(p)dp = \int_B \epsilon(p)\chi_{\{\epsilon>-\epsilon_{\mathrm{F}}\}}(p)dp = -\eta e_{\max}. \qquad \square$$

Remark 5.1.16. The parameters λ_0, λ_1 are sometimes called the entropy parameters. Note that λ_1 has the same sign as $\tilde{E}(\phi(\lambda_1),\lambda_1)$, since the energy $\tilde{E}(\phi(\lambda_1),\lambda_1)$ increases in λ_1 and we may observe that $\tilde{E}(\phi(0),0)$ vanishes.

Proof of Proposition 5.1.8. Recall that $(\lambda,p)\mapsto \mathcal{F}(\lambda,p) = 1/(\eta+e^{-\lambda_0-\lambda_1\epsilon(p)})$ is an analytic in λ and continuous in p. From this, one can easily check the analyticity of $(\tilde{n},\tilde{E})$. Since Lemma 5.1.10 provides the invertibility of the Jacobian $\mathcal{J}(\tilde{n},\tilde{E})$, the implicit function theorem ensures the analyticity and hence the smoothness of the inverse mapping. $\square$

As we have seen in Proposition 5.1.8, we can either use (n,E) or λ to describe the Fermi-Dirac distribution. Sometimes its more useful to describe the Fermi function with the help of the particle density and λ_1. This is also possible as shown by the next lemma:

Lemma 5.1.17. *The mapping* $(0, 1/\eta) \times \mathbb{R} \to \mathbb{R}^2$

$$(n, \lambda_1) \mapsto (\lambda_0, E) \quad \text{with} \quad (n, E) = \int_B (1, \epsilon(p))\mathcal{F}(\lambda, p)dp \tag{5.12}$$

is injective and smooth.

Proof. Given $\lambda_1 \in \mathbb{R}$, we define $g : \mathbb{R} \to (0, 1/\eta)$ by

$$g : \lambda_0 \mapsto \int_B \mathcal{F}((\lambda_0, \lambda_1), p)dp,$$

which has the properties $\lim_{\lambda_0 \to -\infty} g(\lambda_0) = 0$, $\lim_{\lambda_0 \to \infty} g(\lambda_0) = 1/\eta$ and $g'(\lambda) > 0$ according to Lemma 5.1.9. Thus, there exists a unique $\lambda_0 \in \mathbb{R}$ fulfilling $g(\lambda_0) = n$. Therefore, the mapping is well-defined and injective. The smoothness is a direct consequence of the smoothness of the mapping $\lambda \to (\tilde{n}(\lambda), \tilde{E}(\lambda))$. □

5.2 The Fermi energy and chemical potential

Throughout this section, we assume in addition to the hypothesis on ϵ that it is Lipschitz continuous and fulfills $\|\epsilon\|_\infty = 2J$. Moreover, we fix $2J > 0$ and fix $\eta \geq 0$. As we have seen in Section 5.1, there exists a C^∞ diffeomorphism connecting the entropy parameters λ_0, λ_1 to the particle and energy densities n, E. This motivates the following definition:

Definition 5.2.1. Let $\eta \geq 0$ and $L^1(B; [0, \eta^{-1}]) := \{f \in L^1(B) \text{ with } 0 \leq f \leq 1/\eta\}$. We define the $\mathcal{F}^0 : \left\{\int_B (1, \epsilon(p))f(p)dp : f \in L^1(B; [0, \eta^{-1}])\right\} \times B \to [0, \eta]$ by

$$(n, E, p) \mapsto \begin{cases} n & \text{if } n \in \{0, 1/\eta\}, \\ \chi_{\{\pm\epsilon < \epsilon_{\mathrm{F}}(n)\}}(p) & \text{if } E = \mp e_{\max} \text{ and } 0 < n < 1, \\ \frac{1}{\eta + e^{-\lambda_0(n,E) - \lambda_1(n,E)\epsilon(p)}} & \text{else,} \end{cases} \tag{5.13}$$

where $\epsilon_{\mathrm{F}}(n)$ and $\lambda(n, E) = (\lambda_0(n, E), \lambda_1(n, E))$ are implicitly given by

$$|\{\epsilon < \epsilon_{\mathrm{F}}(n)\}| = \eta n, \quad (n, E) = \int_B (1, \epsilon(p))\mathcal{F}(\lambda(n, E), p)dp$$

for $n \in (0, \eta^{-1})$. Thus, $\mathcal{F}^0$ maps the particle density and energy density to its corresponding Fermi-Dirac distribution. The parameter $\epsilon_{\mathrm{F}} = \epsilon_{\mathrm{F}}(n)$ is called the Fermi Energy and describes the energy level below which every state is occupied at zero temperature (i.e. at $E = -e_{\max}$). Moreover, we can extend the definition of ϵ_{F} as a function of the density by

$$\epsilon_{\mathrm{F}}(\eta^{-1}) = 2J \quad \text{and} \quad \epsilon_{\mathrm{F}}(0) = -2J.$$

Since the definition is rather implicit, we seek for a more direct method to compute ϵ_{F}. On the one hand, we can use the notion of the chemical potential in order to compute the Fermi energy.

Definition 5.2.2. For $\lambda \in \mathbb{R} \times \mathbb{R} \setminus \{0\}$ the chemical potential is defined as $\mu := \lambda_0/\lambda_1$. We may rewrite any Fermi-Dirac function $\mathcal{F}(\lambda, \cdot)$ using the chemical potential in combination with λ_1 as variables by

$$\bar{\mathcal{F}}(\mu, \lambda_1, p) := \frac{1}{\eta + e^{-\lambda_1(\epsilon(p)-\mu)}} \quad \text{for } p \in B.$$

Moreover, we denote the corresponding particle density as

$$\bar{n}(\mu, \lambda_1) := \int_B \bar{\mathcal{F}}(\mu, \lambda_1, p) dp.$$

Corollary 5.2.3. $\lim\limits_{\substack{\lambda_1 \to \pm\infty \\ \bar{n}(\mu,\lambda_1)=const.}} \mu = \mp\epsilon_{\mathrm{F}}.$

Proof. The assertion is a direct consequence of the proof of Lemma 5.1.15 writing

$$\mu := \mp\epsilon_{\mathrm{F}} - \frac{r_{\pm}(\lambda_1)}{\lambda_1}.$$ □

On the other hand, we may compute ϵ_{F} using moments of ϵ with respect to the measure $\mathcal{F}(\lambda, p)(1 - \eta\mathcal{F}(\lambda, p))dp$, which we have already required for the proof of Lemma 5.1.10. For this, we state some preliminary definitions first:

Definition 5.2.4. We set $\omega_i := \omega_i(\lambda) := \int_B \epsilon(p)^i \mathcal{F}(\lambda, p)(1 - \eta\mathcal{F}(\lambda, p))dp$ for $i \in \mathbb{N}_0$.

Definition 5.2.5. Let $N(e)$ denote the density of states at energy level e given by

$$N(e) := \int_{\epsilon(p)=e} \frac{d\mathcal{H}_p^{d-1}}{|\nabla\epsilon(p)|}.$$

Here, $\mathcal{H}_p^{d-1}$ denotes the $d-1$ dimensional Hausdorff measure on B.

Lemma 5.2.6. *Let $\eta > 0$ and*

$$\tilde{\mathcal{F}}(\lambda, a) := \frac{1}{\eta + e^{-\lambda_0 - \lambda_1 a}}.$$

With this definition, we have $\tilde{\mathcal{F}}(\lambda, \epsilon(p)) = \mathcal{F}(\lambda, p)$. Considering λ_0 as a function of n and λ_1, we have

$$n^2(1-\eta n)^2 N(\cdot) \frac{\tilde{\mathcal{F}}(\lambda, \cdot)(1 - \eta\tilde{\mathcal{F}}(\lambda, \cdot))}{\omega_0(\lambda)} \rightrightarrows n^2(1-\eta n)^2 \delta_{\mp\epsilon_{\mathrm{F}}(n)} \quad \text{as } \lambda_1 \to \pm\infty$$

in $C^0(\mathbb{R})'$ as well as

$$n^2(1-\eta n)^2\frac{\omega_i(\lambda)}{\omega_0(\lambda)} \rightrightarrows n^2(1-\eta n)^2(\mp\epsilon_{\mathrm{F}}(n))^i \quad \text{as } \lambda_1 \to \pm\infty$$

for $i \in \mathbb{N}$. Here $\rightrightarrows$ denotes uniform convergence w.r.t. $n \in (0,\eta^{-1})$.

In order to prove this, we require an Arzelà-Ascoli type lemma, where the equicontiuity is replaced by monotonicity:

Lemma 5.2.7. *Let $I \subset \mathbb{R}$ be an compact interval and let $\phi : I \to \mathbb{R}$ be continuous. Furthermore, let $\psi : I \times \mathbb{R} \to \mathbb{R}$ be monotonically increasing in its first variable such that*

$$\psi(x,y) \to \phi(x) \quad \text{pointwise as } y \to \infty$$

for every $x \in I$. Then the convergence is uniform, i.e.

$$\sup_x |\psi(x,y) - \phi(x)| \quad \text{as } y \to \infty.$$

Proof. We encounter from the hypothesis that for every $\epsilon > 0$ and every x, there exists a $r_{x,\epsilon} > 0$ such that

$$|\psi(x,y) - \phi(x)| \leq \frac{\epsilon}{2} \quad \text{for all } y \geq r_{x,\epsilon}. \tag{5.14}$$

In order to show that the convergence is uniform, we need some preliminary considerations. First, for every $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ such that

$$|\phi(x) - \phi(z)| \leq \frac{\epsilon}{2} \quad \text{if } |x-z| \leq \delta_\epsilon \tag{5.15}$$

due to the uniform continuity of ϕ. Combining (5.14) with (5.15) and using the monotonicity of $\psi(x,y)$ yields

$$\psi(x,y) - \phi(x) \leq \psi(z,y) - \phi(z) + \frac{\epsilon}{2} \leq \epsilon$$

for $y \geq r_{z,\epsilon}$ and $0 \leq z - x \leq \delta_\epsilon$. Likewise, we obtain

$$\psi(x,y) - \phi(x) \geq \psi(\tilde{z},y) - \phi(\tilde{z}) - \frac{\epsilon}{2} \geq -\epsilon$$

for $y \geq r_{\tilde{z},\epsilon}$ and $0 \leq x - \tilde{z} \leq \delta_\epsilon$. Since I is compact, for $\epsilon > 0$ there exists an $N_\epsilon \in \mathbb{N}$ and $z_{0,\epsilon} \in \mathbb{R}$, such that $\bigcup_{i=0}^{N_\epsilon - 1}[z_{i,\epsilon}, z_{i+1,\epsilon}] \supset I$, where $z_{i,\epsilon} = z_{0,\epsilon} + i\delta_\epsilon$ for $i = 1,\ldots,N_\epsilon$. Thus, for every $x \in I$, we can find and i such that

$$0 \leq x - z_{i,\epsilon} \leq \delta_\epsilon \quad \text{and} \quad 0 \leq z_{i+1,\epsilon} - x \leq \delta_\epsilon.$$

Let $\epsilon > 0$. Defining $r_\epsilon = \max_i r_{z_{i,\epsilon},\epsilon}$, we have

$$\sup_x |\psi(x,y) - \phi(x)| \leq \epsilon$$

for $y \geq r_\epsilon$, which proves the claim. □

Proof of Lemma 5.2.6. At first, we fix $n \in (0,1)$, let us define

$$\phi(n,\lambda_1,a) := N(\epsilon)\frac{\tilde{\mathcal{F}}(\lambda,a)(1-\eta\tilde{\mathcal{F}}(\lambda,a))}{\omega_0(\lambda)} \quad \text{with} \quad \lambda = (\lambda_0(n,\lambda_1),\lambda_1).$$

Note that $\tilde{\mathcal{F}}(\lambda,\epsilon(\lambda)) = \mathcal{F}(\lambda,p)$. Here, λ_0 may be computed via

$$\lambda_0(n,\lambda_1) = \log\frac{n}{1-\eta n} - \int_0^{\lambda_1}\frac{\omega_1(\lambda_0(n,\lambda_1'),\lambda_1'))}{\omega_0(\lambda_0(n,\lambda_1'),\lambda_1'))}d\lambda_1'$$

by integrating the ODE

$$0 = \partial_{\lambda_1} n = \omega_1 + \omega_0\partial_{\lambda_1}\lambda_0 \quad \text{and} \quad \lambda_0(n,0) = \log\frac{n}{1-\eta n}.$$

Recall that ϵ_{F} is the solution of $|\{\epsilon+\epsilon_{\mathrm{F}} > 0\}| = \eta n$. Now taking the limit in $\lambda_1 \to \infty$ with $\lambda = (\lambda_0(n,\lambda_1),\lambda_1)$ yields

$$\begin{aligned} e^{\nu\lambda_1}\tilde{\mathcal{F}}(\lambda,a)(1-\eta\tilde{\mathcal{F}}(\lambda,a)) &= \frac{e^{-\lambda_1(a+\epsilon_{\mathrm{F}}-\nu)-\lambda_0(n,\lambda_1)+\epsilon_{\mathrm{F}}\lambda_1}}{(\eta + e^{-\lambda_1(a+\epsilon_{\mathrm{F}})-\lambda_0(n,\lambda_1)+\epsilon_{\mathrm{F}}\lambda_1})^2} \\ &\to \begin{cases} \infty, & \text{if } |a+\epsilon_{\mathrm{F}}| < \nu \\ 0, & \text{else} \end{cases} \end{aligned}$$

for every positive ν as $\lambda_1 \to \infty$, because $\lambda_0(n,\lambda_1) + \epsilon_{\mathrm{F}}\lambda_1 = o(\lambda_1)$ according to Lemma 5.1.11. Hence, we obtain

$$\phi(n,\lambda_1,a) = \frac{e^{\nu\lambda_1}\mathcal{F}(\lambda,a)(1-\eta\mathcal{F}(\lambda,a))}{e^{\nu\lambda_1}\omega_0(\lambda)} \to 0 \quad \text{with} \quad \lambda = (\lambda_0(n,\lambda_1),\lambda_1)$$

as $\lambda_1 \to \infty$ uniformly in a for $|a+\epsilon_{\mathrm{F}}| > 2\nu$. Since $\nu > 0$ was arbitrary, we infer by means of the co-area formula

$$\int_{|a+\epsilon_{\mathrm{F}}|>2\nu}\phi(n,\lambda_1,a)N(a)da = \int_{|\epsilon(p)+\epsilon_{\mathrm{F}}|>2\nu}\phi(n,\lambda_1,\epsilon(p))dp \to 0 \text{ as } \lambda_1 \to \infty.$$

This and the fact that $\int_{\mathbb{R}}\phi(n,\lambda_1,a)da = 1$ (due to the co-area formula) as well as the positivity of ϕ show that

$$N(\cdot)\frac{\mathcal{F}(\lambda,\cdot)(1-\eta\mathcal{F}(\lambda,\cdot))}{\omega_0(\lambda)} \to \delta_{-\epsilon_{\mathrm{F}}} \quad \text{with } \lambda = (\lambda_0(n,\lambda_1),\lambda_1) \text{ in } C^0(\mathbb{R})' \tag{5.16}$$

as $\lambda_1 \to \infty$. In particular, for $i \in \mathbb{N}_0$ we have

$$\frac{\omega_i(\lambda_0(n,\lambda_1),\lambda_1)}{\omega_0(\lambda_0(n,\lambda_1),\lambda_1)} \to (-\epsilon_{\mathrm{F}})^i \quad \text{as } \lambda_1 \to \infty.$$

Until now, every convergence has been point-wise w.r.t. n. In order to prove uniform convergence, we define

$$\psi(n,\lambda_1) := \frac{1}{\lambda_1}\int_0^{\lambda_1} \frac{\omega_1(\lambda_0(n,\lambda_1'),\lambda_1'))}{\omega_0(\lambda_0(n,\lambda_1'),\lambda_1'))} d\lambda_1' \quad \text{for } 0 < n < 1$$

and $\psi(1,\lambda_1) = -2J = -\epsilon_F(1)$, $\psi(0,\lambda_1) = 2J = -\epsilon_F(0)$ such that $\psi(1,\lambda_1) \leq \psi(n,\lambda_1) \leq \psi(0,\lambda_1)$. For $n \in (0,1)$, we can calculation ψ with the aid of λ_0 by

$$\lambda_1\psi(n,\lambda_1) = \log\frac{n}{1-\eta n} - \lambda_0(n,\lambda_1)$$

We want to apply Lemma 5.2.7 and therefore we need that ψ is monotone. For this we take the derivative

$$\lambda_1\partial_n\psi(n,\lambda_1) = \frac{1}{n(1-\eta n)} - \partial_n\lambda_0(n,\lambda_1) = \frac{1}{n(1-\eta n)} - \frac{1}{\omega_0},$$

where we have used

$$\partial_n\lambda_0 = \frac{1}{\omega_0} \quad \Leftarrow \quad 1 = \partial_n n = \partial_n\lambda_0\omega_0.$$

The derivative of $\psi(n,\lambda_1)$ w.r.t. n is non-positive and hence ψ monotone since $\omega_0 \leq n(1-\eta n)$. From the first part of the proof, we deduct that $\psi(n,\lambda_1) \to -\epsilon_F(n)$ as $\lambda_1 \to \infty$ for every $n \in (0,1)$. Now, we have prepared everything to apply Lemma 5.2.7 and obtain that $\psi(n,\lambda_1) \rightrightarrows -\epsilon_F(n)$ as $\lambda_1 \to \infty$ since ϵ_F is continuous. With this additional knowledge we can reuse the ideas of the first part. However, we have to pay attention that $\frac{\lambda_0}{\lambda_1} + \epsilon_F\lambda_1$ does not converge uniformly in contrast to ψ. Therefore, the convergence of $e^{\nu\lambda_1}\tilde{\mathcal{F}}(\lambda,a)(1-\eta\tilde{\mathcal{F}}(\lambda,a))$ is not uniform w.r.t. a; we have to replace it on the one hand with $\frac{1}{n(1-\eta n)}e^{\nu\lambda_1}\tilde{\mathcal{F}}(\lambda,a)(1-\eta\tilde{\mathcal{F}}(\lambda,a)) \rightrightarrows \infty$ if $|a+\epsilon_F| < \nu$ and on the other hand with $n(1-\eta n)e^{\nu\lambda_1}\tilde{\mathcal{F}}(\lambda,a)(1-\eta\tilde{\mathcal{F}}(\lambda,a)) \rightrightarrows 0$ if $|a+\epsilon_F| > \nu$ as $\lambda_1 \to \infty$. This implies the assertion for $\lambda_1 \to \infty$. Finally, we note that we can do exactly the same for the case $\lambda_1 \to -\infty$ replacing $-\epsilon_F$ by ϵ_F. □

Remark 5.2.8. The foregoing proof entails in addition

$$\frac{1}{\lambda_1}\int_0^{\lambda_1} \frac{\omega_1(\lambda_0(n,\lambda_1'),\lambda_1'))}{\omega_0(\lambda_0(n,\lambda_1'),\lambda_1'))} d\lambda_1' \rightrightarrows \mp\epsilon_F(n)$$

as $\lambda_1 \to \pm\infty$.

Remark 5.2.9. Let $\eta = 1$. Due to the definition of λ_0, we see that it is monotonically increasing in λ_1 for $\lambda_0\lambda_1 \geq 0$ and monotonically decreasing for $\lambda_0\lambda_1 \leq 0$ since

$$\omega_1(\lambda_0,\lambda_1) = \int_B \frac{\epsilon(p)e^{-\lambda_0}dp}{(1+e^{-\lambda_0}e^{-\lambda_1\epsilon(p)})(e^{-\lambda_0}+e^{\lambda_1\epsilon(p)})} \begin{cases} \leq 0, & \text{if } \lambda_0\lambda_1 \geq 0 \\ \geq 0, & \text{else.} \end{cases}$$

Therefore, we infer that λ_0 is monotone in λ_1. Note that the proof of Lemma 5.2.6 provides that λ_0 is also monotonically increasing in n.

5.3 The capacity

In semiconductor physics, the capacity is given by

$$\kappa(\lambda) := \partial_\mu \tilde{n}(\lambda) = \lambda_1 \omega_0(\lambda),$$

where $\mu = \lambda_1/\lambda_0$ is the chemical potential, $\tilde{n}(\lambda)$ the particle density and $\lambda = (\lambda_0, \lambda_1)$. As we will see in the chapter of the hydrodynamic description of our model, the ultracold cloud of fermions in an optical lattice, certain values of the capacity lead to degeneracies in our equation and produce difficulties. In this section we prove that the capacity is unbounded and in particular that $\kappa : \mathbb{R}^2 \to \mathbb{R}$ is surjective.

Lemma 5.3.1. $\sup_{\lambda_0 \in \mathbb{R}, \lambda_1 > 0} \lambda_1 \omega_0(\lambda_0, \lambda_1) = \infty$.

Proof. Recall that $\lambda_1 \omega_0(\lambda_0, \lambda_1)$ is defined as the integral $\lambda_1 \int_B \mathcal{F}(\lambda, p)(1 - \eta \mathcal{F}(\lambda, p))dp$ with $\lambda = (\lambda_0, \lambda_1)$. Thus, showing that the integral over a smaller set is already unbounded ensures the assertion. We calculate the value of the integral using the co-area formula and the definition

$$\tilde{\mathcal{F}}(\lambda, a) = \frac{1}{\eta + e^{-\lambda_0 - \lambda_1 a}}, \quad \text{where } \lambda = (\lambda_0, \lambda_1).$$

For $\nu > 0$, we compute

$$\begin{aligned}
&\sup_{\lambda_1 > 0} \int_{-\nu - \epsilon_F}^{\nu - \epsilon_F} \lambda_1 \tilde{\mathcal{F}}(\lambda, a)(1 - \eta \tilde{\mathcal{F}}(\lambda, a)) N(a) da \\
&\quad \geq \inf_{|e + \epsilon_F| \leq \nu} N(e) \sup_{\lambda_1 > 0} \int_{-\nu - \epsilon_F}^{\nu - \epsilon_F} \frac{\lambda_1 e^{-\lambda_0 - \lambda_1 a}}{(\eta + e^{-\lambda_1 a - \lambda_0})^2} da \\
&\quad = \inf_{|e + \epsilon_F| \leq \nu} N(e) \sup_{\lambda_1 > 0} \frac{e^{\lambda_1 \nu} - e^{-\lambda_1 \nu}}{(\eta e^{\lambda_0 - \lambda_1 \epsilon_F} + e^{-\lambda_1 \nu})(\eta + e^{-\lambda_0 + \lambda_1 \epsilon_F} e^{\lambda_1 \nu})} \\
&\quad = \inf_{|e + \epsilon_F| \leq \nu} N(e) \sup_{\lambda_1 > 0} \frac{\sinh(\lambda_1 \nu)}{e^{\lambda_0 - \lambda_1 \epsilon_F}(1 + \eta^2) + \eta \cosh(\lambda_1 \nu)} = \eta^{-1} \inf_{|e + \epsilon_F| \leq \nu} N(e)
\end{aligned}$$

since $\lambda_0 - \lambda_1 \epsilon_F = o(\lambda_1)$. Now, let $\nu = \frac{2J + \epsilon_F}{2}$. The unboundedness of $N(e)$ and, being more precise, the fact that $\lim_{\epsilon_F \to \pm 2J} N(\epsilon_F) = \infty$ imply

$$\begin{aligned}
&\sup_{n \in (0,1)} \sup_{\lambda_1 > 0} \int_{-J - \frac{\epsilon_F}{2}}^{-J - \frac{3}{2}\epsilon_F} \lambda_1 \tilde{\mathcal{F}}(\lambda, a)(1 - \eta \tilde{\mathcal{F}}(\lambda, a)) N(a) da \\
&\qquad \geq \eta^{-1} \lim_{\epsilon_F \to -2J} N\Big(-J - \frac{3}{2}\epsilon_F \Big) = \infty.
\end{aligned}$$

This finishes the proof. □

In the case $\eta \neq 0$, the statement of Lemma 5.3.1 changes completely assuming that the density $\tilde{n}(\lambda_0, \lambda_1) = \int_B \mathcal{F}((\lambda_0, \lambda_1), p)dp$ does not approach 0 and η^{-1} as λ_1 tends to ∞.

Lemma 5.3.2. *Let $\eta > 0$ and I be a compact subset of $(0, \eta^{-1})$; then*

$$\sup_{\tilde{n}(\lambda_0,\lambda_1)\in I, \lambda_1>0} \lambda_1 \omega_0(\lambda_0, \lambda_1) < \infty.$$

Proof. As in the proof of Lemma 5.3.1, we treat the case $\lambda_1 \to \infty$ first. Note that $\lambda_1\omega_0(\lambda_0, \lambda_1)$ is defined as the integral $\lambda_1 \int_B \mathcal{F}(\lambda, p)(1-\eta\mathcal{F}(\lambda, p))dp$ with $\lambda = (\lambda_0, \lambda_1)$. Let $\lambda_1 > 0$ and $n \in I$. We start as in the proof of Lemma 5.3.1 and define

$$\tilde{\mathcal{F}}(\lambda, a) = \frac{1}{\eta + e^{-\lambda_0 - \lambda_1 a}}, \quad \text{where } \lambda = (\lambda_0, \lambda_1).$$

Likewise to the proof of Lemma 5.3.1,

$$\int_{\underline{\nu}-\epsilon_{\mathrm{F}}}^{\bar{\nu}-\epsilon_{\mathrm{F}}} \lambda_1 \tilde{\mathcal{F}}(\lambda, a)(1 - \eta\tilde{\mathcal{F}}(\lambda, a))N(a)da \leq \eta^{-1} \sup_{\underline{\nu}\leq e \leq \bar{\nu}} N(e).$$

for $\underline{\nu} < 0 < \bar{\nu}$. Since $N(e)$ may be singular for $e = \pm 2J$, we need to estimate the remaining part of the integral separately. To begin with, we assume $a + \epsilon_{\mathrm{F}} \geq \bar{\nu}$ and derive

$$\begin{aligned} e^{\frac{\bar{\nu}}{2}\lambda_1} \tilde{\mathcal{F}}(\lambda, a)(1 - \eta\tilde{\mathcal{F}}(\lambda, a)) &\leq e^{-\lambda_1(a+\epsilon_{\mathrm{F}}-\bar{\nu})} e^{-\frac{\lambda_1}{2}\bar{\nu} + |\lambda_0 - \epsilon_{\mathrm{F}}\lambda_1|} \\ &\leq n(1-\eta n) e^{-\frac{\lambda_1}{2}(\bar{\nu}-2|\theta|)}, \end{aligned}$$

where we know from the proof of Lemma 5.2.6 that $\theta(n, \lambda_1) := \frac{1}{\lambda_1}\lambda_0(n, \lambda_1) - \epsilon_{\mathrm{F}}(n) - \frac{1}{\lambda_1}\log\frac{n}{1-\eta n} \rightrightarrows 0$ as $\lambda_1 \to \infty$. In fact, the same estimate holds for $\epsilon + \epsilon_{\mathrm{F}} \leq \underline{\nu}$ replacing $\bar{\nu}$ by $-\underline{\nu}$. Thus, using $\lambda_1 e^{-\lambda_1\nu/2} \leq 1/\nu$, the inequalities for $\bar{\nu}$ and $\underline{\nu}$ entail

$$\begin{aligned} &\left(\int_{\epsilon+\epsilon_{\mathrm{F}}\geq\bar{\nu}} + \int_{\epsilon+\epsilon_{\mathrm{F}}\leq\underline{\nu}}\right) \lambda_1 \mathcal{F}(\lambda, p)(1 - \eta\mathcal{F}(\lambda, p))dp \\ &\qquad\leq n(1-\eta n)\left(\frac{|\{p : \epsilon + \epsilon_{\mathrm{F}} \geq \bar{\nu}\}|}{\bar{\nu}e^{\frac{\lambda_1}{2}(\bar{\nu}-2|\theta|)}} + \frac{|\{p : \epsilon + \epsilon_{\mathrm{F}} \leq \underline{\nu}\}|}{-\underline{\nu}e^{\frac{\lambda_1}{2}(-\underline{\nu}-2|\theta|)}}\right), \end{aligned}$$

which is bounded uniformly in n as $2|\theta| \leq \bar{\nu}, -\underline{\nu}$ for sufficiently large λ_1 for fixed $\underline{\nu}, \bar{\nu}$. However, since $n \in I \subset\subset (0, 1)$, we can choose $\bar{\nu}, \underline{\nu}$ independently from n. Combining both estimates shows the assertion. □

5.4 The Fermi-Dirac distribution as a function of the densities

This section is devoted to estimates on the derivatives of $\mathcal{F}^0$. Since $\mathcal{F}^0$ is not directly given, we may consider it as composition of two functions. The natural choice would be

$$\mathcal{F}^0(n,E,p) = \left(\mathcal{F}(\cdot,p)\circ(\tilde{n},\tilde{E})^{-1}\right)(n,E)$$

for $\tilde{n}(\lambda) = \int_B \mathcal{F}(\lambda,p)dp$ and $\tilde{E}(\lambda) = \int_B \epsilon(p)\mathcal{F}(\lambda,p)dp$. Thus, we can combine estimates on $\mathcal{F}$ and $\tilde{n},\tilde{E}$ in order to find estimates on the derivatives of $\mathcal{F}^0$.

Remark 5.4.1. The Fermi-Dirac distribution functions are related to each other for different $\eta > 0$. In order to see this, we denote $\mathcal{F}^0_\eta(n,E,p) = (\mathcal{F}_\eta(\cdot,p)\circ(\tilde{n}_\eta,\tilde{E}_\eta)^{-1})(n,E)$ for $\eta > 0$ with

$$\mathcal{F}_\eta(\lambda,p) := \frac{1}{\eta + e^{-\lambda_0-\lambda_1\epsilon(p)}} \quad \text{and} \quad (\tilde{n}_\eta(\lambda),\tilde{E}_\eta(\lambda)) := \int_B (1,\epsilon(p))\mathcal{F}_\eta(\lambda,p)dp.$$

Defining the transformation

$$\psi_\eta : \mathbb{R}^2 \to \mathbb{R}^2,\ (x,y) \mapsto (x - \log\eta, y),$$

we observe that

$$\mathcal{F}_\eta(\cdot,\cdot,p) = \frac{1}{\eta}\mathcal{F}_1(\cdot,\cdot,p)\circ\psi_\eta \quad \text{and} \quad (\tilde{n}_\eta,\tilde{E}_\eta) = \frac{1}{\eta}(\tilde{n}_1,\tilde{E}_1)\circ\psi_\eta.$$

as well as

$$\begin{aligned}\mathcal{F}^0_\eta(\cdot,\cdot,p) = \mathcal{F}_\eta(\cdot,\cdot,p)\circ(\tilde{n}_\eta,\tilde{E}_\eta)^{-1} &= \frac{1}{\eta}\left(\mathcal{F}_1(\cdot,\cdot,p)\circ(\tilde{n}_1,\tilde{E}_1)^{-1}\right)(\eta\cdot)\\ &= \frac{1}{\eta}\mathcal{F}^0_1(\eta\cdot,\eta\cdot,p).\end{aligned}$$

Therefore, if we can prove estimates for the derivatives of $\mathcal{F}^0$ with respect to n and E for $\eta = 1$, we can easily extend these estimates to all $\eta > 0$ and also for $\eta = 0$ by taking the limit. Throughout this section, we assume that $\eta = 1$.

Since we are also interested in the case of small densities, $n \ll 1$. The desired estimates shall not be singular in n. Nevertheless, it turns out that we need different coordinates as λ.

Definition 5.4.2 ($\eta = 1$)**.** We define the transformation $\phi : \lambda \mapsto (\delta,\gamma)$ by setting

$$\delta := \frac{1}{1+e^{-\lambda_0}} \quad \text{and} \quad \gamma := \frac{1}{1+e^{-\lambda_0-\lambda_1}}.$$

Using $\tilde{\delta} := \frac{1-\delta}{\delta} = e^{-\lambda_0}$ and $\tilde{\gamma} := \frac{1-\gamma}{\gamma} = e^{-\lambda_0-\lambda_1}$ as well as $\hat{\delta} = 1/\tilde{\delta}$ and $\hat{\gamma} = 1/\tilde{\gamma}$, we define

$$\hat{F}(\gamma,\delta,\beta) := \frac{1}{1+\tilde{\delta}\left(\frac{\tilde{\gamma}}{\tilde{\delta}}\right)^{\beta}} = \frac{1}{1+\frac{1}{\hat{\delta}}\left(\frac{\hat{\gamma}}{\hat{\delta}}\right)^{-\beta}}. \tag{5.17}$$

We set

$$\hat{\mathbf{n}}(\gamma,\delta) := \int_B \begin{pmatrix} 1 \\ \epsilon(p) \end{pmatrix} \hat{F}(\gamma,\delta,\epsilon(p))dp, \tag{5.18}$$

which entails $\mathcal{F}^0(\cdot,\cdot,p) = \hat{F}(\cdot,\cdot,\epsilon(p)) \circ \hat{\mathbf{n}}^{-1}$ by requiring that $\hat{\mathbf{n}}$ is invertible.

Lemma 5.4.3 ($\eta = 1$)**.** *Let* $(n,E) = \hat{\mathbf{n}}(\gamma,\delta)$*. Let*

$$\partial_{(n,E)} = \frac{\partial}{\partial(n,E)}$$

be the derivative w.r.t. (n,E)*. We have*

$$\partial_{(n,E)}\hat{\mathbf{n}}^{(-1)}(n,E) = \frac{1}{\langle \epsilon \perp 1\rangle_\mu^2} \int_B \begin{pmatrix} \frac{\gamma^2}{\delta^2}\frac{(\epsilon-1)}{X} \\ \epsilon \end{pmatrix} \cdot \big(-\epsilon, 1\big) d\mu \tag{5.19}$$

with $d\mu = \frac{X^{\epsilon(p)}}{(\delta+(1-\delta)X^{\epsilon(p)})^2}dp$ *and* $X = \frac{\tilde{\gamma}}{\tilde{\delta}}$*, where*

$$\langle \epsilon \perp 1\rangle_\mu^2 := \int_B \int_B (\epsilon'\epsilon - \epsilon^2) d\mu d\mu'$$

fulfilling

$$\langle \epsilon \perp 1\rangle_\mu^2 \geq \|\epsilon\|_2^2 \exp(-2\,\|\epsilon \log X\|_\infty).$$

In particular, it holds

$$\partial_{(n,E)}\hat{\mathbf{n}}^{(-1)}(n,0) = -\begin{pmatrix} 1 & \frac{1}{\|\epsilon\|_2^2} \\ 1 & 0 \end{pmatrix}.$$

Proof. We compute

$$\partial_{(\gamma,\delta)}\hat{F}(\gamma,\delta,\beta) = \frac{X^\beta}{(1+\tilde{\delta}X^\beta)^2}\left(\frac{\epsilon X}{\gamma^2}, \frac{1-\beta}{\delta^2}\right),$$

where $\partial_{(\gamma,\delta)} = \frac{\partial}{\partial(\gamma,\delta)}$. This leads to

$$\partial_{(\gamma,\delta)}\hat{\mathbf{n}}(\gamma,\delta) = \int_B \begin{pmatrix} \frac{\delta^2}{\gamma^2}\epsilon X & 1-\epsilon \\ \frac{\delta^2}{\gamma^2}\epsilon^2 X & \epsilon(1-\epsilon) \end{pmatrix} d\mu$$

having set $d\mu = \frac{X^{\epsilon(p)}}{\delta^2(1+\tilde{\delta}X^{\epsilon(p)})^2}dp = \frac{X^{\epsilon(p)}}{(\delta+(1-\delta)X^{\epsilon(p)})^2}dp$. Its determinant, given by

$$\det\left(\partial_{(\gamma,\delta)}\hat{\mathbf{n}}(\gamma,\delta)\right) = \frac{\delta^2}{\gamma^2}X\int_B\int_B(\epsilon'\epsilon-\epsilon^2)d\mu d\mu' =: \frac{\delta^2}{\gamma^2}X\langle\epsilon\perp 1\rangle_\mu^2,$$

can be estimated by

$$\begin{aligned}\left|\det\left(\partial_{(\gamma,\delta)}\hat{\mathbf{n}}(\gamma,\delta)\right)\right| &\leq \frac{\gamma^2\tilde{\delta}}{\delta\tilde{\gamma}}\frac{\exp(2\left\|\epsilon\log(X)\right\|)}{\|\epsilon\|_2^2}\\ &= \left(\frac{1+e^{-\lambda_0}}{1+e^{-\lambda_0-\lambda_1}}\right)^2\frac{1}{\|\epsilon\|_2^2}\exp(2\left\|\lambda_1\epsilon\right\|).\end{aligned}$$

Hence,

$$\begin{aligned}\partial_{(n,E)}\hat{\mathbf{n}}^{(-1)}(n,E) &= \frac{1}{\langle\epsilon\perp 1\rangle_\mu^2}\int_B\begin{pmatrix}\frac{\gamma^2}{\delta^2}\frac{\epsilon(1-\epsilon)}{X} & \frac{\gamma^2}{\delta^2}\frac{(\epsilon-1)}{X}\\ -\epsilon^2 & \epsilon\end{pmatrix}d\mu\\ &= \frac{1}{\langle\epsilon\perp 1\rangle_\mu^2}\int_B\begin{pmatrix}\frac{\gamma^2}{\delta^2}\frac{(\epsilon-1)}{X}\\ \epsilon\end{pmatrix}\cdot\big(-\epsilon,1\big)d\mu\end{aligned}$$

can be estimated using

$$\langle\epsilon\perp 1\rangle_\mu^2 \geq \|\epsilon\|_2^2\exp(-2\left\|\lambda_1\epsilon\right\|_\infty).$$

We conclude the assertion, noting that $E=0$ corresponds to $\gamma=\delta$ as well as $X=1$. □

Remark 5.4.4. Let $U\subset\mathbb{R}^2$ and $f:U\to\mathbb{R}$ be smooth than for $v^1,\ldots,v^j\in\mathbb{R}^2$ it holds

$$f^{(j)}(x)(v^1,\ldots,v^j) = \sum_{i\in\{1,2\}^j}\partial_{x_{i(j)}}\cdots\partial_{x_{i(j)}}f(x)v^1_{i(1)}\cdots v^j_{i(j)}$$

for $x\in U$. Thus,

$$\left|f^{(j)}(x)\right|_{\mathrm{Op}} \leq \sup_{|\alpha|=j}\left|\partial^\alpha f(x)\right|2^j.$$

Lemma 5.4.5 ($\eta=1$)**.** *Let $j\in\mathbb{N}$ and let*

$$\partial_{(\tilde{\gamma},\tilde{\delta})} = \frac{\partial}{\partial(\tilde{\gamma},\tilde{\delta})}$$

the derivative w.r.t. $(\tilde{\gamma},\tilde{\delta})$. *Then there exist tensor valued polynomials* A^j_{il} *of degree at most* j *for* $i,l = 1,\dots,j$ *such that*

$$\partial^j_{(\tilde{\gamma},\tilde{\delta})}\hat{F}(\gamma,\delta,\beta) = \sum_{l=0}^{j}\sum_{i=1}^{j} A^j_{il}(\beta)\left(\frac{\delta}{1-\delta}\right)^l\left(\frac{\gamma}{1-\gamma}\right)^{j-l}(1-\hat{F}(\gamma,\delta,\beta))^i\hat{F}(\gamma,\delta,\beta)^{j-i+1},$$

using $\delta = \delta(\tilde{\delta}) = \frac{1}{1+\tilde{\delta}}$ *and* $\gamma = \gamma(\tilde{\gamma}) = \frac{1}{1+\tilde{\gamma}}$ *(cf. Definition 5.4.2). These tensors may be estimate via*

$$\left|A^j_{il}(\beta)\right| \le 2^i(1+|\beta|)^j j!\binom{j-1}{i-1}.$$

In particular, it holds

$$\left|\partial^j_{(\tilde{\gamma},\tilde{\delta})}\hat{F}(\gamma,\delta,\beta)\right| \le j!(2+2\,|\beta|)^j\sum_{l=0}^{j}\left(\frac{\delta}{1-\delta}\right)^l\left(\frac{\gamma}{1-\gamma}\right)^{j-l}\hat{F}(\gamma,\delta,\beta)(1-\hat{F}(\gamma,\delta,\beta)).$$

Proof. Fix $\beta\in\mathbb{R}$ and define

$$g_{l,m}(\gamma,\delta) := (-1)^{l+m}\frac{\sum_i a^{lm}_i\left(\frac{\gamma}{\delta}\right)^{i\beta-m}\frac{1}{\delta^{m+l-i}}}{\left(1+\delta\left(\frac{\gamma}{\delta}\right)^\beta\right)^{l+m+1}}$$

with $a^{00}_i = 1$ if $i=0$ and $a^{00}_i = 0$ else. We demand $g_{l+1,m} = \partial_\delta g_{l,m}$ by setting

$$a^{l+1,m}_i := \big(l+(l+m+2-i)(1-\beta)\big)a^{lm}_{i-1} + (l+i(\beta-1))a^{lm}_i$$

for $0\le i\le l+m$ and $a^{l+1,m}_i = 0$ otherwise; this can be seen by the following calculation:

$$\begin{aligned}\partial_\delta g_{lm} &= (-1)^{l+m+1}(l+m+1)(1-\beta)\frac{\sum_i a^{lm}_i\left(\frac{\gamma}{\delta}\right)^{(i+1)\beta-m}\frac{1}{\delta^{l+m-i}}}{\left(1+\delta\left(\frac{\gamma}{\delta}\right)^\beta\right)^{l+m+2}}\\ &\quad+(-1)^{l+m+1}(l-i+i\beta)\frac{\sum_i a^{lm}_i\left(\frac{\gamma}{\delta}\right)^{i\beta-m}\frac{1}{\delta^{l+m+1-i}}}{\left(1+\delta\left(\frac{\gamma}{\delta}\right)^\beta\right)^{l+m+1}}\\ &= \frac{(-1)^{l+m+1}}{\left(1+\delta\left(\frac{\gamma}{\delta}\right)^\beta\right)^{l+m+2}}\sum_i\left(\frac{\gamma}{\delta}\right)^{i\beta-m}\frac{1}{\delta^{l+m+1-i}}\big((l+m+1)(1-\beta)a^{lm}_{i-1}\\ &\qquad+(l+i(\beta-1))a^{lm}_i+(l+(1-i)(1-\beta))a^{lm}_{i-1}\big).\end{aligned}$$

In addition, we obtain $g_{l,m+1} = \partial_\gamma g_{l,m}$ by setting

$$a_i^{l,m+1} := (l+m+1)\beta a_{i-1}^{lm} + (m - i\beta)a_i^{lm} + (m-(i-1)\beta)a_{i-1}^{lm}$$

for $0 \le i \le l+m$ and $a_i^{l,m+1} = 0$ otherwise, since

$$\begin{aligned}
\partial_\gamma g_{lm} &= (-1)^{l+m+1}(l+m+1)\beta \frac{\sum_{ij} a_i^{lm} \left(\frac{\gamma}{\delta}\right)^{(i+1)\beta-m-1} \frac{1}{\delta^{l+m-i}}}{\left(1+\delta\left(\frac{\gamma}{\delta}\right)^\beta\right)^{l+m+2}} \\
&\quad + (-1)^{l+m+1}(m-i\beta)\frac{\sum_i a_i^{lm}\left(\frac{\gamma}{\delta}\right)^{i\beta-m-1}\frac{1}{\delta^{l+m+1-i}}}{\left(1+\delta\left(\frac{\gamma}{\delta}\right)^\beta\right)^{l+m+1}} \\
&= \frac{(-1)^{l+m+1}}{\left(1+\delta\left(\frac{\gamma}{\delta}\right)^\beta\right)^{l+m+2}} \sum_i \left(\frac{\gamma}{\delta}\right)^{i\beta-(m+1)} \frac{1}{\delta^{l+m+1-i}} \big((l+m+1)\beta a_{i-1}^{lm} \\
&\qquad + (m-i\beta)a_i^{lm} + (m+(i-1)\beta)a_{i-1}^{lm}\big) .
\end{aligned}$$

Note that

$$a_0^{lm} = \begin{cases} 1 & \text{if } l = m = 0, \\ 0 & \text{else.} \end{cases}$$

Thus, we have found the derivatives of $\hat{F}$ by

$$\begin{aligned}
\partial_{\tilde{\delta}}^l \partial_{\tilde{\gamma}}^m \hat{F}(\gamma,\delta,\beta) &= g_{lm}(\tilde{\gamma},\tilde{\delta}) \\
&= (-1)^{l+m} \frac{\left(\frac{\tilde{\gamma}}{\tilde{\delta}}\right)^{-m} \tilde{\delta}^{-m-l}}{\left(1+\tilde{\delta}\left(\frac{\tilde{\gamma}}{\tilde{\delta}}\right)^\beta\right)^{l+m+1}} \sum_{i=1}^{l+m} a_i^{lm} \tilde{\delta}^i \left(\frac{\tilde{\gamma}}{\tilde{\delta}}\right)^{i\beta} \\
&= \frac{(-1)^{l+m}}{\tilde{\delta}^l \tilde{\gamma}^m} \sum_{i=1}^{l+m} a_i^{lm} (1-\hat{F}(\delta,\gamma))^i \hat{F}(\delta,\gamma)^{l+m-i+1}.
\end{aligned}$$

This proves the first assertion with

$$A_{il}^j(\beta) = \left((-1)^j a_i^{l,j-l} \delta_{l,|\kappa|}\right)_{\kappa_1,\ldots,\kappa_j=0,1},$$

where $\delta_{l,m}$ denotes the Kronaecker δ and $|\kappa| = \sum_\iota \kappa_\iota$. In order to derive the estimate, we observe that

$$\left|a_i^{l+1,m}\right|, \left|a_i^{lm+1}\right| \le (l+m+1)(1+|\beta|)(2\left|a_{i-1}^{lm}\right| + \left|a_i^{lm}\right|)$$

and especially

$$\left|a_i^{lm}\right| \leq 2^i(1+|\beta|)^{l+m}(l+m)!\binom{l+m-1}{i-1}$$

for $l+m \geq 1$. This entails

$$\begin{aligned}
|g_{lm}(\gamma,\delta)| &\leq \frac{\sum_{i=1}^{l+m}\left|a_i^{lm}\right|\left(\frac{\gamma}{\delta}\right)^{i\beta-m}\frac{1}{\delta^{l+m-i}}}{\left(1+\delta\left(\frac{\gamma}{\delta}\right)^{\beta}\right)^{l+m+1}} \\
&= \frac{\left(\frac{\gamma}{\delta}\right)^{-m}}{\left(1+\delta\left(\frac{\gamma}{\delta}\right)^{\beta}\right)^{l+m+1}}\left(\frac{\gamma}{\delta}\right)^{\beta}\sum_{i=0}^{l+m-1}\left|a_{i+1}^{lm}\right|\left(\frac{\gamma}{\delta}\right)^{i\beta}\frac{1}{\delta^{l+m-1-i}} \\
&\leq \frac{\left(\frac{\gamma}{\delta}\right)^{-m}}{\left(1+\delta\left(\frac{\gamma}{\delta}\right)^{\beta}\right)^{l+m+1}}(2+2|\beta|)^{l+m}(l+m)!\times \\
&\qquad\qquad \times\left(\frac{\gamma}{\delta}\right)^{\beta}\left(\left(\frac{\gamma}{\delta}\right)^{\beta}+\frac{1}{\delta}\right)^{l+m-1} \\
&\leq \frac{\left(\frac{\gamma}{\delta}\right)^{\beta-m}\frac{1}{\delta^{l+m-1}}}{\left(1+\delta\left(\frac{\gamma}{\delta}\right)^{\beta}\right)^{2}}(2+2|\beta|)^{l+m}(l+m)!
\end{aligned}$$

for $l+m \geq 1$. □

Corollary 5.4.6 ($\eta = 1$)**.** *Let $j \in \mathbb{N}$. Then there exist tensors B_{il}^j for $i,l = 1,\ldots,j$ such that*

$$\partial^j_{(\tilde{\gamma}\tilde{\delta})}\hat{\mathbf{n}}(\gamma,\delta) = \sum_{l=0}^{j}\sum_{i=1}^{j}B_{il}^j\left(\frac{\delta}{1-\delta}\right)^{l}\left(\frac{\gamma}{1-\gamma}\right)^{j-l}\times$$
$$\times\int_B(1-\hat{F}(\gamma,\delta,\epsilon(p)))^i\hat{F}(\gamma,\delta,\epsilon(p))^{j-i+1}dp$$

with $\gamma = \frac{1}{1+\tilde{\gamma}}$, $\delta = \frac{1}{1+\tilde{\delta}}$ and

$$\left|B_{il}^j\right| \leq 2^i\left(1+\|\epsilon\|_\infty\right)^{j+1}j!\binom{j-1}{i-1}.$$

In particular, we have

$$\left|\partial^j_{(\tilde{\gamma},\tilde{\delta})}\hat{\mathbf{n}}(\delta,\gamma)\right| \leq j!3^j\left(1+\|\epsilon\|_\infty\right)^{j+1}\max\left\{\frac{\delta}{1-\delta},\frac{\gamma}{1-\gamma}\right\}^j\cdot$$
$$\cdot\int_B\hat{F}(\gamma,\delta,\epsilon(p))(1-\hat{F}(\gamma,\delta,\epsilon(p)))dp.$$

Lemma 5.4.7 ($\eta = 1$). *Let $\gamma, \delta \in (0,1)$ and $\lambda \geq 0$ such that*

$$3(2+2\|\epsilon\|_\infty)C_1\lambda \leq \min\{\delta,\gamma\}(1-\max\{\delta,\gamma\}),$$

where $C_1 := \frac{\max\{\delta,\gamma\}}{\min\{\delta,\gamma\}}$. Then we have

$$\sum_l \frac{\lambda^l}{l!}\left|\partial^l \hat{\mathbf{n}}(\gamma,\delta)\right| \leq \frac{3(2+2\|\epsilon\|_\infty)^2 C_1\lambda}{\min\{\delta,\gamma\}(1-\max\{\delta,\gamma\})}\int_B \hat{F}(\delta,\gamma)(1-\hat{F}(\delta,\gamma))dp$$

and in particular

$$\left|\partial^l_{(\gamma,\delta)}\hat{\mathbf{n}}(\gamma,\delta)\right| \leq l!C\left(\frac{L}{\min\{\delta,\gamma\}(1-\max\{\delta,\gamma\})}\right)^l \times$$
$$\times \int_B \hat{F}(\delta,\gamma,\epsilon(p))(1-\hat{F}(\delta,\gamma,\epsilon(p)))dp,$$

where

$$C := 2+2\|\epsilon\|_\infty \quad \text{and} \quad L := 3(2+2\|\epsilon\|_\infty)\frac{\max\{\delta,\gamma\}}{\min\{\delta,\gamma\}}.$$

Proof. We define $g(\gamma,\delta) = (\tilde{\gamma},\tilde{\delta})$. According to Faà di Bruno's formula, we have

$$\sum_l \frac{\lambda^l}{l!}\left|\partial^l \hat{\mathbf{n}}(\gamma,\delta)\right| \leq \sum_j \frac{\mu^j}{j!}\left|\partial^j_{\tilde{\gamma}\tilde{\delta}}\hat{\mathbf{n}}(\gamma,\delta)\right| \quad \text{with} \quad \mu = \sum_j \frac{\lambda^m}{m!}\left|\partial^j g(\gamma,\delta)\right|.$$

We can directly estimate μ via

$$\mu \leq \sum_{m=1}^\infty \lambda^m \max\left\{\frac{1}{\delta^{m+1}},\frac{1}{\gamma^{m+1}}\right\} \leq \frac{2\lambda}{\min\{\delta,\gamma\}^2}$$

by assuming the $2\lambda \leq \min\{\delta,\gamma\}$. Let $\omega_0 = \int_B \hat{F}(\gamma,\delta,\epsilon(p))(1-\hat{F}(\gamma,\delta,\epsilon(p)))dp$. Corollary 5.4.6 entails

$$\begin{aligned}\sum_{j=1}^\infty \frac{\mu^j}{j!}\left|\partial^j_{\tilde{\gamma}\tilde{\delta}}\hat{\mathbf{n}}(\gamma,\delta)\right| &\leq \sum_{j=1}^\infty \mu^j 3^j(1+\|\epsilon\|_\infty)^{j+1}\max\left\{\frac{\delta}{1-\delta},\frac{\gamma}{1-\gamma}\right\}^j\omega_0\\ &\leq \sum_{j=1}^\infty \frac{2^j\lambda^j}{\min\{\delta,\gamma\}^{2j}}3^j(1+\|\epsilon\|_\infty)^{j+1}\max\left\{\frac{\delta}{1-\delta},\frac{\gamma}{1-\gamma}\right\}^j\omega_0\\ &\leq 3(2+2\|\epsilon\|_\infty)^2\frac{C_1\omega_0\lambda}{\min\{\delta,\gamma\}(1-\max\{\delta,\gamma\})}\end{aligned}$$

where $C_1 := \frac{\max\{\delta,\gamma\}}{\min\{\delta,\gamma\}}$ if

$$3(2+2\|\epsilon\|_\infty)C_1\lambda \leq \min\{\delta,\gamma\}(1-\max\{\delta,\gamma\}).$$

□

Corollary 5.4.8. *Lemma 5.4.7 remains true, replacing C_1 and L by*

$$\tilde{C}_1 = \min\left\{\frac{\max\{\delta,\gamma\}}{\min\{\delta,\gamma\}}, \frac{1-\min\{\delta,\gamma\}}{1-\max\{\delta,\gamma\}}\right\} \quad \textit{and} \quad \tilde{L} := 3(2+2\left\|\epsilon\right\|_\infty)\tilde{C}_1,$$

respectively. Moreover, let $I = [a, 1/a]$ for $a > 0$. Then we have

$$\left|\partial^l \hat{\mathbf{n}}(\gamma,\delta)\right| \leq l! C L' \left(\frac{L'}{\hat{\mathbf{n}}_1(\gamma,\delta)(1-\hat{\mathbf{n}}_1(\gamma,\delta))}\right)^{l-1}$$

with

$$C = 2 + 2\left\|\epsilon\right\|_\infty \quad \textit{and} \quad L' := 3(2+2\left\|\epsilon\right\|_\infty)a^{\|\epsilon\|_\infty + 1}$$

for all $\gamma, \delta \in (0,1)$ such that $\tilde{\gamma}/\tilde{\delta} \in I$. Especially, it holds

$$\left|\partial^l \hat{\mathbf{n}}(\gamma,\gamma)\right| \leq l! \cdot 2(\left\|\epsilon\right\|_\infty + 1)\gamma(1-\gamma)\left(\frac{6(\left\|\epsilon\right\|_\infty + 1)}{\gamma(1-\gamma)}\right)^l$$

for $\gamma \in (0,1)$ and $\gamma = \hat{\mathbf{n}}_1(\gamma,\gamma)$.

Proof. Recalling the definition of $\hat{F}$, we observe that

$$\hat{F}(1-\gamma, 1-\delta, \beta) := \frac{\tilde{\delta}\left(\frac{\tilde{\gamma}}{\tilde{\delta}}\right)^\beta}{1+\tilde{\delta}\left(\frac{\tilde{\gamma}}{\tilde{\delta}}\right)^\beta} = 1 - \frac{1}{1+\tilde{\delta}\left(\frac{\tilde{\gamma}}{\tilde{\delta}}\right)^\beta} = 1 - \hat{F}(\gamma,\delta,\beta)$$

is valid, where $\tilde{\delta} := \frac{1-\delta}{\delta}$ and $\tilde{\gamma} := \frac{1-\gamma}{\gamma}$. Hence, we can exchange (δ,γ) by $(1-\delta, 1-\gamma)$ without changing the previous lemma. Moreover, we have

$$C_1 = \frac{\max\{\delta,\gamma\}}{\min\{\delta,\gamma\}} = \max\left\{\frac{\delta}{\gamma}, \frac{\gamma}{\delta}\right\} \leq \max\left\{\frac{\tilde{\delta}}{\tilde{\gamma}}, \frac{\tilde{\gamma}}{\tilde{\delta}}\right\} \leq a^2$$

for all $\tilde{\delta}/\tilde{\gamma} \in I = [1/a, a]$. It remains to show that we can find an estimate as in Lemma 5.4.7 using $\hat{\mathbf{n}}_1(\gamma,\delta)$ and not the variables γ,δ themselves. In order to prove this, we compare δ and $1-\gamma$ with $\hat{F}(\gamma,\delta,\epsilon(p))$ and $1-\hat{F}(\gamma,\delta,\epsilon(p))$ and see

$$\begin{aligned}\delta(1-\gamma) = \frac{1}{1+\tilde{\delta}}\frac{\tilde{\gamma}}{1+\tilde{\gamma}} &\geq \frac{1}{1+\tilde{\delta}\left(\frac{\tilde{\gamma}}{\tilde{\delta}}\right)^{\epsilon(p)}} \frac{\tilde{\gamma}}{\left(\frac{\tilde{\gamma}}{\tilde{\delta}}\right)^{1-\epsilon(p')} + \tilde{\gamma}} a^{-\|\epsilon\|_\infty} \\ &= \hat{F}(\gamma,\delta,\epsilon(p))(1-\hat{F}(\gamma,\delta,\epsilon(p')))a^{-1-\|\epsilon\|_\infty}\end{aligned}$$

for $p, p' \in B$. Similarly, we obtain a lower bound for $(1-\delta)\gamma$. Finally integration w.r.t p and p' entails

$$\min\{\gamma,\delta\}(1-\max\{\gamma,\delta\}) \geq \hat{\mathbf{n}}_1(\gamma,\delta)(1-\hat{\mathbf{n}}_1(\gamma,\delta))a^{-1-\|\epsilon\|_\infty}. \qquad \square$$

Lemma 5.4.9. *Let $I = [1/a, a] \subset \mathbb{R}_{>0}$. Then there exist constants $A, B > 0$ depending only on a, $\|\epsilon\|_\infty$ such that*

$$\left|(\hat{\mathbf{n}}^{-1})^{(m)}(n,E)\right| \leq m! m^{-\frac{3}{2}} \frac{AB^m}{\big(n(1-n)\big)^{m-1}} \tag{5.20}$$

and $(n,E) = \hat{\mathbf{n}}(\delta,\gamma) \in \hat{\mathbf{n}}((0,1)^2)$ with

$$\frac{\tilde{\delta}}{\tilde{\gamma}} = \frac{1-\delta}{1-\gamma}\frac{\gamma}{\delta} \in I.$$

Furthermore, it holds

$$\left|(\hat{\mathbf{n}}^{-1})^{(m)}(n,0)\right| \leq m! m^{-\frac{3}{2}} (\|\epsilon\|_\infty + 1)^4 \left(\frac{1200(\|\epsilon\|_\infty+1)^3}{n(1-n)}\right)^{m-1}. \tag{5.21}$$

Proof. We start with recalling that Lemma 5.4.3 which entails that

$$H := \sup_{0\leq\delta\leq 1} \sup_{\frac{\tilde{\delta}}{\tilde{\gamma}}\in I} \left|(\partial\hat{\mathbf{n}}(\gamma,\delta))^{-1}\right|_{Op} < \infty.$$

Moreover, due to Corollary 5.4.8, there exist $C, L > 0$ such that

$$\left|\partial^l \hat{\mathbf{n}}_i(\gamma,\delta)\right| \leq l! \tilde{C}\tilde{L}^l.$$

with $\tilde{C} := n(1-n)C$ and $\tilde{L} = \frac{L'}{n(1-n)}$ for all $\gamma, \delta \in (0,1)$ with $\frac{\tilde{\delta}}{\tilde{\gamma}} \in I$ and $l \in \mathbb{N}$, where $n = \hat{\mathbf{n}}_1(\gamma,\delta)$. Thus, we can apply Proposition 4.3.1 to obtain for $m \geq 3$

$$\begin{aligned}\left|\partial^m(\hat{\mathbf{n}}^{-1})(\hat{\mathbf{n}}(\gamma,\delta))\right| &\leq \frac{\tilde{C}H}{8(1+\tilde{C}H\tilde{L})^2}(4H\tilde{L}(1+\tilde{C}H\tilde{L}))^m m^{-\frac{3}{2}} m! \\ &\leq m! m^{-\frac{3}{2}} \tilde{A} n(1-n)\left(\frac{B}{n(1-n)}\right)^m\end{aligned}$$

with

$$\tilde{A} := \frac{CH}{8(1+CHL)^2} \quad \text{and} \quad B := 4LH(1+CHL).$$

Now, we turn to $m = 2$ and have

$$\left|\partial^2(\hat{\mathbf{n}}^{-1})(n,E)\right| \leq 2! C(L')^2 H^3 \frac{1}{n(1-n)} = \tilde{A}B^2 \frac{1}{n(1-n)}.$$

For the first assertion, we define $A = \tilde{A}B$. The second assertion treats the case where $I = \{1\}$ implying $\delta = \gamma$ or equivalently $E = 0$. We obtain from Lemma 5.4.3 that

$$H = \sup_{0\leq\delta\leq 1} \left|(\partial\hat{\mathbf{n}}(\delta,\delta))^{-1}\right|_{Op} = \left|\begin{pmatrix} 1 & \frac{1}{\|\epsilon\|_2^2} \\ 1 & 0 \end{pmatrix}\right|_{Op} \leq 2.$$

Moreover, Corollary 5.4.8 entails that

$$C = 2(\|\epsilon\|_\infty + 1) \quad \text{and} \quad L = 6(\|\epsilon\|_\infty + 1).$$

Therefore $\tilde{A}$ and $\tilde{B}$ can be estimated from above as

$$\tilde{A} \leq \frac{4(\|\epsilon\|_\infty + 1)}{8(1 + 2\cdot 2\cdot 6)^2} = \frac{\|\epsilon\|_\infty + 1}{1250}$$

and

$$B \leq 4\cdot 6(\|\epsilon\|_\infty + 1)\cdot 2\cdot(1 + 2\cdot 2\cdot 6)(\|\epsilon\|_\infty + 1)^2 = 1200(\|\epsilon\|_\infty + 1)^3. \quad \square$$

Lemma 5.4.10. *Let $i + j \geq 1$.*

$$\left|\partial^i_{\tilde{\gamma}\tilde{\delta}}\partial^j_\beta \hat{F}(\gamma,\delta,\beta)\right| \leq 3^{i+j}(1 + |\beta|)^i(i+j)! \sum_{l+m=i} \frac{1}{\tilde{\gamma}^m\tilde{\delta}^l}\left(1 + \left|\log\frac{\tilde{\gamma}}{\tilde{\delta}}\right|\right)^j \times$$
$$\times(1 - \hat{F}(\gamma,\delta,\beta))\hat{F}(\gamma,\delta,\beta).$$

Proof. Define

$$g_{l,m,j}(\gamma,\delta,\beta) := (-1)^{l+m}\sum_{i,\kappa}\left(\log\frac{\gamma}{\delta}\right)^\kappa \frac{a^{lmj}_{i\kappa}\left(\frac{\gamma}{\delta}\right)^{i\beta - m}\frac{1}{\delta^{l+m-i}}}{\left(1 + \delta\left(\frac{\gamma}{\delta}\right)^\beta\right)^{l+m+\kappa+1}}.$$

Here the sum runs over all $0 \leq i \leq l + m + j$. We compute

$$\begin{aligned}
&\partial_\beta g_{l,m,j}(\gamma,\delta,\beta) \\
&= (-1)^{l+m}\sum_{i,\kappa}\left(\log\frac{\gamma}{\delta}\right)^\kappa \frac{\partial_\beta a^{lmj}_{i\kappa}\left(\frac{\gamma}{\delta}\right)^{i\beta-m}\frac{1}{\delta^{l+m-i}}}{\left(1 + \delta\left(\frac{\gamma}{\delta}\right)^\beta\right)^{l+m+\kappa+1}} \\
&\quad - (-1)^{l+m}\sum_{i\kappa}(l+m+\kappa+1)\left(\log\frac{\gamma}{\delta}\right)^{\kappa+1}\frac{a^{lmj}_{i\kappa}\left(\frac{\gamma}{\delta}\right)^{(i+1)\beta-m}\frac{1}{\delta^{l+m-i-1}}}{\left(1 + \delta\left(\frac{\gamma}{\delta}\right)^\beta\right)^{l+m+\kappa+2}} \\
&\quad + (-1)^{l+m}\sum_{i\kappa}\left(\log\frac{\gamma}{\delta}\right)^{\kappa+1}\frac{i a^{lmj}_{i\kappa}\left(\frac{\gamma}{\delta}\right)^{i\beta-m}\frac{1}{\delta^{l+m-i}}}{\left(1 + \delta\left(\frac{\gamma}{\delta}\right)^\beta\right)^{l+m+\kappa+1}}.
\end{aligned}$$

We assume that $a_{i\kappa}^{lm0} := a_i^{lm}$ if $\kappa = 0$ and $a_{i\kappa}^{lm0} = 0$ otherwise, where a_i^{lm} is defined in the proof of Lemma 5.4.5. In addition, we put $a_{i,0}^{lm,j+1} := \partial_\beta a_{i,0}^{lmj}$ and

$$a_{i\kappa}^{lm,j+1} := \partial_\beta a_{i\kappa}^{lmj} + \partial_\beta a_{i-1,\kappa}^{lmj} + (i-1-l-m-\kappa-1)a_{i-1,\kappa-1}^{lmj} + i a_{i,\kappa-1}^{lmj}$$

for $\kappa \geq 0$ and $a_{i,-1}^{lmj} = a_{i-1,-1}^{lmj} = 0$. This entails that $\partial_\beta^j g_{lm} = g_{lmj}$, where g_{lm} is defined in the proof of Lemma 5.4.5. Note that this implies $a_{0,\kappa}^{lmj} = 0$. As in the proof of Lemma 5.4.5, we see that

$$\begin{aligned}
\partial_{\tilde{\delta}}^l \partial_{\tilde{\gamma}}^m \partial_\beta^j \hat{F}(\gamma,\delta) &= g_{l,m,j}(\tilde{\gamma},\tilde{\delta},\beta) \\
&= \frac{(-1)^{l+m}}{\tilde{\gamma}^m \tilde{\delta}^l} \sum_{\kappa=0}^{j} \left(\log \frac{\tilde{\gamma}}{\tilde{\delta}}\right)^{\kappa} \sum_{i=1}^{l+m+\kappa} a_{i\kappa}^{lmj} (1-\hat{F}(\gamma,\delta))^i \hat{F}(\gamma,\delta)^{l+m+\kappa+1-i}.
\end{aligned}$$

We recall from Lemma 5.4.5 that

$$a_i^{lm} = \begin{cases} 1 & \text{if } i = l = m = 0, \\ 0 & \text{if } l+m > i = 0 \text{ or } i > l+m \end{cases}$$

as well as

$$a_i^{l,m+1} := (l+m+1)\beta a_{i-1}^{lm} + (m-i\beta)a_i^{lm} + (m+(i-1)\beta)a_{i-1}^{lm}$$

and

$$a_i^{l+1,m} := \big(l+(l+m+2-i)(1-\beta)\big)a_{i-1}^{lm} + (l+i(\beta-1))a_i^{lm}.$$

Let $\delta_{\alpha,\beta}$ denote the Kronaecker δ. Thus, $\partial_\beta^\alpha a_{i\kappa}^{lmj}$ can be computed iteratively by $\partial_\beta^\alpha a_{i\kappa}^{000} = \delta_{\alpha,0}\delta_{i,0}\delta_{\kappa,0}$ and

$$\begin{aligned}
\partial_\beta^\alpha a_i^{l+1,0} &= -\alpha(l+2-i)\partial_\beta^{\alpha-1} a_{i-1}^{l0} + \alpha i \partial_\beta^{\alpha-1} a_i^{l0} \\
&\quad + \big(l+(l+2-i)(1-\beta)\big)\partial_\beta^\alpha a_{i-1}^{l0} + (l+i(\beta-1))\partial_\beta^\alpha a_i^{l0}, \\
\partial_\beta^\alpha a_i^{l,m+1} &= \alpha(l+m+1)\partial_\beta^{\alpha-1} a_{i-1}^{lm} - \alpha i \partial_\beta^{\alpha-1} a_i^{lm} - \alpha(i-1)\partial^{\alpha-1} a_{i-1}^{lm} \\
&\quad + (l+m+1)\beta\partial_\beta^\alpha a_{i-1}^{lm} + (m-i\beta)\partial_\beta^\alpha a_i^{lm} \\
&\quad + (m-(i-1)\beta)\partial_\beta^\alpha a_{i-1}^{lm}, \\
\partial_\beta^\alpha a_{i\kappa}^{lm,j+1} &= \partial_\beta^{\alpha+1} a_{i\kappa}^{lmj} + \partial_\beta^{\alpha+1} a_{i-1,\kappa}^{lmj} \\
&\quad + (i-1-l-m-\kappa-1)\partial_\beta^\alpha a_{i-1,\kappa-1}^{lmj} + i\partial_\beta^\alpha a_{i,\kappa-1}^{lmj}
\end{aligned}$$

for all $\alpha \in \mathbb{N}$ with $a_{i,0}^{lm,0} = a_i^{lm}$. Here, we have used the convention that $\partial_\beta^\alpha a_{i\kappa}^{lmj} = 0$ for $i \notin \{0,\dots,l+m+j\}$ or $\kappa \notin \{0,\dots,j\}$. From this we directly derive the estimates

$$\begin{aligned}
\left|\partial_\beta^\alpha a_i^{l+1,0}\right| &\le \alpha(l+1)\left(\left|\partial_\beta^{\alpha-1} a_{i-1}^{l0}\right| + \left|\partial_\beta^{\alpha-1} a_i^{l0}\right|\right) \\
&\quad + (l+1)(1+|\beta|)\left(2\left|\partial_\beta^\alpha a_{i-1}^{l0}\right| + \left|\partial_\beta^\alpha a_i^{l0}\right|\right), \\
\left|\partial_\beta^\alpha a_i^{l,m+1}\right| &\le \alpha(l+m+1)\left(\left|\partial_\beta^{\alpha-1} a_{i-1}^{lm}\right| + \left|\partial_\beta^{\alpha-1} a_i^{lm}\right|\right) \\
&\quad + (l+m+1)(1+|\beta|)\left(2\left|\partial_\beta^\alpha a_{i-1}^{lm}\right| + \left|\partial_\beta^\alpha a_i^{lm}\right|\right), \\
\left|\partial_\beta^\alpha a_{i\kappa}^{lm,j+1}\right| &\le \left|\partial_\beta^{\alpha+1} a_{i\kappa}^{lmj}\right| + \left|\partial_\beta^{\alpha+1} a_{i-1,\kappa}^{lmj}\right| \\
&\quad + (l+m+\kappa+1)\left(\left|\partial_\beta^\alpha a_{i-1,\kappa-1}^{lmj}\right| + \left|\partial_\beta^\alpha a_{i,\kappa-1}^{lmj}\right|\right).
\end{aligned}$$

Now by induction, we deduce successively that

$$\begin{aligned}
\left|\partial_\beta^\alpha a_i^{l,m}\right| &\le \alpha!(l+m)!(1+|\beta|)^{l+m}3^{l+m}\binom{l+m-1}{i-1}, \\
\left|\partial_\beta^\alpha a_{i\kappa}^{lmj}\right| &\le (\alpha+l+m+j)!(1+|\beta|)^{l+m}3^{l+m}2^j\binom{l+m+\kappa-1}{i-1}.
\end{aligned}$$

This yields

$$\begin{aligned}
\left|\partial_{\tilde\delta}^l \partial_{\tilde\gamma}^m \partial_\beta^j \hat F(\gamma,\delta)\right| &\le 3^{l+m+j}(1+|\beta|)^{l+m}(l+m+j)!\times \\
&\quad \times \frac{1}{\tilde\gamma^m\tilde\delta^l}\left(1+\left|\log\frac{\tilde\gamma}{\tilde\delta}\right|\right)^j (1-\hat F(\gamma,\delta))\hat F(\gamma,\delta).
\end{aligned}$$

From this, we easily derive the assertion by connecting the derivative with respect to $(\tilde\gamma,\tilde\delta)$ to the partial derivatives $\partial_{\tilde\gamma}$ and $\partial_{\tilde\delta}$. □

Lemma 5.4.11. *Let $\gamma,\delta \in (0,1)$ and define $\tilde\delta := \frac{1-\delta}{\delta}$ as well as $\tilde\gamma := \frac{1-\gamma}{\gamma}$. Moreover, let λ be non-negative such that*

$$\lambda \le \frac{\min\{\gamma,\delta\}(1-\max\{\gamma,\delta\})}{36(1+|\beta|)C_1} \quad \textit{with} \quad C_1 := \frac{\max\{\gamma,\delta\}}{\min\{\gamma,\delta\}}.$$

Then, we have

$$\begin{aligned}
\sum_{l=1}^\infty \frac{\lambda^l}{l!}\left|\partial_{\gamma\delta}^l \partial_\beta^j \hat F(\gamma,\delta,\beta)\right| &\le (1-\hat F(\gamma,\delta,\beta))\hat F(\gamma,\delta,\beta)\times \\
&\quad \times j!6^j\left(1+\left|\log\frac{\tilde\gamma}{\tilde\delta}\right|\right)^j \frac{36(1+|\beta|)C_1}{\min\{\gamma,\delta\}(1-\max\{\gamma,\delta\})}\lambda
\end{aligned}$$

and in particular it holds

$$\left|\partial^l_{\gamma\delta}\partial^j_\beta \hat{F}(\gamma,\delta,\beta)\right| \leq \hat{F}(\gamma,\delta,\beta)(1-\hat{F}(\gamma,\delta,\beta))\times$$
$$\times\, j!6^j\left(1+\left|\log\frac{\tilde{\gamma}}{\tilde{\delta}}\right|\right)^j l!\left(\frac{36(1+|\beta|)C_1}{\min\{\gamma,\delta\}(1-\max\{\gamma,\delta\})}\right)^l.$$

as well as

$$\left|\partial^l_{\gamma\delta}\partial^j_\beta \hat{F}(\gamma,\delta,\beta)\big|_{\delta=\gamma}\right| \leq \hat{F}(\gamma,\gamma,\beta)(1-\hat{F}(\gamma,\gamma,\beta))j!6^j l!\left(\frac{36(1+|\beta|)}{\gamma(1-\gamma)}\right)^l.$$

Proof. Likewise to the proof of Lemma 5.4.7, we make use of Lemma 5.4.10 and estimate

$$\sum_l \frac{\lambda^l}{l!}\left|\partial^l_{\gamma\delta}\partial^j_\beta \hat{F}(\gamma,\delta,\beta)\right| \leq \sum_i \frac{\mu^i}{i!}\left|\left(\frac{d}{d(\tilde{\gamma},\tilde{\delta})}\right)^i \partial^j_\beta \hat{F}(\gamma,\delta,\beta)\right|.$$

for $\mu = \frac{2\lambda}{\min\{\delta,\gamma\}^2}$. Hence,

$$\begin{aligned}
&\sum_{l=1}^\infty \frac{\lambda^l}{l!}\left|\partial^l_{\gamma\delta}\partial^j_\beta \hat{F}(\gamma,\delta,\beta)\right| \\
&\quad\leq \sum_{i=1}^\infty \mu^i 6^{i+j}(1+|\beta|)^i j! \sum_{l+m=i}\frac{1}{\tilde{\gamma}^m\tilde{\delta}^l}\left(1+\left|\log\frac{\tilde{\gamma}}{\tilde{\delta}}\right|\right)^j \times \\
&\qquad\qquad \times (1-\hat{F}(\gamma,\delta,\beta))\hat{F}(\gamma,\delta,\beta) \\
&\quad\leq (1-\hat{F}(\gamma,\delta,\beta))\hat{F}(\gamma,\delta,\beta)j!6^j\left(1+\left|\log\frac{\tilde{\gamma}}{\tilde{\delta}}\right|\right)^j \frac{18\mu(1+|\beta|)}{\min\{\tilde{\gamma},\tilde{\delta}\}}
\end{aligned}$$

since

$$\frac{18\mu(1+|\beta|)}{\min\{\tilde{\gamma},\tilde{\delta}\}} = \frac{18(1+|\beta|)}{\min\{\tilde{\gamma},\tilde{\delta}\}}\frac{2\lambda}{\min\{\delta,\gamma\}^2} \leq \frac{36(1+|\beta|)C_1}{\min\{\gamma,\delta\}(1-\max\{\gamma,\delta\})}\lambda \leq 1,$$

where $C_1 := \frac{\max\{\gamma,\delta\}}{\min\{\gamma,\delta\}}$. Finally, the same trick as in the proof of Corollary 5.4.8 ensures the assertion. □

In the following, we only want to specify the energy dispersion being given by Example 5.1.2 and use its concrete form in the sequel.

Definition 5.4.12. Identifying the first Brillouin zone $B := [0,2\pi)^d \subset \mathbb{R}^d$ with the torus $\mathbb{T}^d$, we define

$$\epsilon : \mathbb{T}^d \to \mathbb{R},\ p = (p_1,\dots,p_d) \mapsto -2J\sum_{i=1}^d \cos(p_i)$$

for a given $J > 0$.

Definition 5.4.13. For $a \geq 1$ and $n \in [0,1]$, let $\mathcal{E}_a(n)$ be the set of all possible energy densities for a particle cloud in equilibrium with density n, where the inverse temperature $-\lambda_1$ is restricted to the interval $[-\log a, \log a]$. More precisely, we define

$$\mathcal{E}_a(n) := \left\{ \int_B \frac{\epsilon(p)dp}{1+e^{-\lambda_0-\lambda_1\epsilon(p)}} : |\lambda_1| \leq \log a \text{ and } \int_B \frac{dp}{1+e^{-\lambda_0-\lambda_1\epsilon(p)}} = n \right\}$$

being equivalent to

$$\mathcal{E}_a(n) = \left\{ \hat{\mathbf{n}}_2(\gamma,\delta) : \ \gamma,\delta \in (0,1),\ \frac{1-\gamma}{1-\delta}\frac{\delta}{\gamma} \in \left[\frac{1}{a}, a\right] \text{ and } \hat{\mathbf{n}}_1(\gamma,\delta) = n \right\}.$$

Note that $\mathcal{E}_1(n) = \{0\}$.

Lemma 5.4.14. *Let $a > 0$. Then there exist constants $A_a, B_a > 0$ only depending on a and $J = \frac{1}{2}\|\epsilon\|_\infty$ such that for*

$$\mathcal{F}^0(\cdot,\cdot,p) := \hat{F}(\cdot,\cdot,\epsilon(p)) \circ \hat{\mathbf{n}}^{-1}$$

it holds

$$\left|\partial^i_{(n,E)}\partial^j_p \mathcal{F}^0(n,E,p)\right| \leq i!j!A_a^j \left(\frac{B_a}{n(1-n)}\right)^i \mathcal{F}^0(n,E,p)(1-\mathcal{F}^0(n,E,p))$$

for all $p \in \mathbb{T}^d$, $i+j \geq 1$ and $(n,E) = (0,1)\times\mathbb{R}$ with $E \in \mathcal{E}_a(n)$. In addition, we have

$$\left|\partial^i_{(n,E)}\partial^j_p \mathcal{F}^0(n,E,p)\big|_{E=0}\right| \leq 3i!j! \left(\frac{1}{\log\left(1+\frac{1}{24J}\right)}\right)^j \times \\ \times \left(\frac{2400(2J+1)^3}{n(1-n)}\right)^i n(1-n)$$

for all $n \in (0,1)$ and $p \in \mathbb{T}^d$.

Proof. At first, we see that we can prove a version of Corollary 5.4.8 for Lemma 5.4.11 following the same method as above. Thus, replace $\min\{\gamma,\delta\}\cdot(1-\max\{\gamma,\delta\})$ by $n(1-n)a^{-1-|\epsilon|}$ in the estimate of Lemma 5.4.11. In particular, there exist $A', B' > 0$ only depending on a and $|\epsilon|_\infty$ such that

$$\left|\partial^l_{(\gamma,\delta)}\partial^j_\beta \hat{F}(\gamma,\delta,\beta)\big|_{\beta=\epsilon(p)}\right| \leq j!(A')^j l! \left(\frac{B'}{n(1-n)}\right)^l \times \\ \times \hat{F}(\gamma,\delta,\epsilon(p))(1-\hat{F}(\gamma,\delta,\epsilon(p))).$$

If $a = 1$ implying that $\gamma = \delta$, it holds

$$A' = 6 \quad \text{and} \quad B' \leq 36(\|\epsilon\|_\infty + 1).$$

According to Lemma 5.4.9, we have for $m \geq 1$

$$\left|\partial^m(\hat{\mathbf{n}}^{-1})(n, E)\right| \leq m! m^{-\frac{3}{2}} A \left(\frac{B}{n(1-n)}\right)^{m-1} \tag{5.22}$$

for some $A, B > 0$ depending only on a and $\|\epsilon\|_\infty$, where

$$A = (\|\epsilon\|_\infty + 1)^3 \quad \text{and} \quad B = 1200(\|\epsilon\|_\infty + 1)^3 \quad \text{if } a = 1.$$

Let $\lambda > 0$ be sufficiently small, we have

$$\begin{aligned}
&\sum_{i=1}^{\infty} \frac{\lambda^i}{i!} \left|\partial_{nE}^i \partial_\beta^j \hat{F}(\cdot,\cdot,\beta) \circ (\hat{\mathbf{n}}^{-1})(n, E)\big|_{\beta=\epsilon(p)}\right| \\
&\qquad \leq \sum_{l=1}^{\infty} \frac{\mu^l}{l!} \left|\partial_{\gamma\delta}^l \partial_\beta^j \hat{F}(\gamma,\delta,\beta)\big|_{\beta=\epsilon(p)}\right| \\
&\qquad \leq j!(A')^j \frac{2B'\mu}{n(1-n)} \hat{F}(\gamma,\delta,\epsilon(p))(1 - \hat{F}(\gamma,\delta,\epsilon(p)))
\end{aligned}$$

for $n(1-n) \geq 2B'\mu$, where $\hat{\mathbf{n}}(\gamma, \delta) = (n, E)$ and

$$\mu = \sum_{m=1}^{\infty} \frac{\lambda^m}{m!} \left|\partial^m(\hat{\mathbf{n}}^{-1})(n, E)\right| \leq 2A\lambda$$

for $n(1-n) \geq 2B\lambda$. The next step is to cope with the derivation w.r.t. p. For this purpose we abbreviate

$$g_i(\beta) := \frac{\lambda^i}{i!} \partial_{nE}^i \hat{F}(\cdot,\cdot,\beta) \circ (\hat{\mathbf{n}}^{-1})(n, E) \quad \text{and} \quad C := \frac{4AB'\lambda}{n(1-n)}.$$

We have

$$\begin{aligned}
\sum_{i+j=1}^{\infty} \frac{\nu^j}{j!} \left|\partial_p^j g_i(\epsilon(p))\right| &\leq \sum_{i=1}^{\infty} |g_i(\epsilon(p))| + \sum_{i=0,l=1}^{\infty} \frac{(2J(e^\nu - 1))^l}{l!} \left|\partial_\beta^l g_i(\beta)\big|_{\beta=\epsilon(p)}\right| \\
&\leq (C + 8J(e^\nu - 1)A') \, \hat{F}(\gamma,\delta,\epsilon(p))(1 - \hat{F}(\gamma,\delta,\epsilon(p)))
\end{aligned}$$

for $4J(e^\nu - 1)A' \leq 1$ since

$$\sum_{j \geq 1} \frac{\nu^j}{j!} \left|\partial^j \epsilon(p)\right| \leq 2J(e^\nu - 1).$$

We can summarize this by

$$\sum_{i+j=1}^{\infty} \frac{\lambda^i \nu^j}{i!j!} \left|\partial^i_{(n,E)} \partial^j_p \mathcal{F}^0(n,E,p)\right| \\ \leq \left(\frac{4AB'\lambda}{n(1-n)} + 8J(e^\nu - 1)A'\right) \mathcal{F}^0(n,E,p)(1-\mathcal{F}^0(n,E,p)) \tag{5.23}$$

for

$$\nu \leq \log\left(\frac{1}{4JA'} + 1\right) \quad \text{and} \quad \lambda \leq \frac{n(1-n)}{\max\{4AB', 2B\}}.$$

As before, we deduce the assertion from this by taking the largest possible values for λ and ν in the estimate. In the case of $a = 1$ or equivalently $E = 0$, we put

$$\lambda = \frac{n(1-n)}{\max\{4AB', 2B\}} = \frac{n(1-n)}{2400(\|\epsilon\|_\infty + 1)^3} = \frac{n(1-n)}{2400(2J+1)^3}$$

and

$$\nu = \log\left(\frac{1}{4JA'} + 1\right) = \log\left(\frac{1}{24J} + 1\right).$$

Finally, we use estimate (5.23) and see

$$\frac{\lambda^{i_0}\nu^{j_0}}{i_0!j_0!} \left|\partial^{i_0}_{(n,E)} \partial^{j_0}_p \mathcal{F}^0(n,E,p)\right| \leq \sum_{i+j=1}^{\infty} \frac{\lambda^i \nu^j}{i!j!} \left|\partial^i_{(n,E)} \partial^j_p \mathcal{F}^0(n,E,p)\right| \\ \leq 3\mathcal{F}^0(n,E,p)(1-\mathcal{F}^0(n,E,p))$$

for $i_0 + j_0 \geq 1$. □

Corollary 5.4.15. *For $a \geq 1$ let $A_a, B_a > 0$ be given by Lemma 5.4.14. Moreover, we define*

$$\mathcal{F}^0_\eta(n,E,p) := \left(\mathcal{F}_\eta(\cdot,\cdot,p) \circ (\tilde{n}_\eta, \tilde{E}_\eta)^{-1}\right)(n,E)$$

with $\mathcal{F}_\eta(\lambda,p) := \frac{1}{\eta + e^{-\lambda_0 - \lambda_1 \epsilon(p)}}$ and $(\tilde{n}_\eta(\lambda), \tilde{E}_\eta(\lambda)) := \int_B (1,\epsilon(p))\mathcal{F}_\eta(\lambda,p)dp$. Then for all $i + j \geq 1$ and all $(n,E) \in (0,\eta^{-1}) \times \mathbb{R}$ satisfying $\eta E \in \mathcal{E}_a(\eta n)$, we have

$$\left|\partial^i_{(n,E)} \partial^j_p \mathcal{F}^0_\eta(n,E,p)\right| \leq i!j!A^j_a \left(\frac{B_a}{n(1-\eta n)}\right)^i \mathcal{F}^0_\eta(n,E,p)(1-\eta\mathcal{F}^0_\eta(n,E,p)).$$

In particular, for $\eta = 0$ it holds

$$\left|\partial^i_{n,E}\partial^j_p \mathcal{F}^0_0(n,E,p)\right| \leq i!j!A_1^j \frac{B_1^i}{n^i}\mathcal{F}^0_0(n,E,p)$$

for any $i+j \geq 1$ and all $(n,E) \in [0,\infty) \times \mathbb{R}$.

Remark 5.4.16. For $a \geq 1$, $\eta > 0$ and $n \in [0,\eta^{-1}]$. It holds that $\eta E \in \mathcal{E}_a(\eta n)$ if and only if

$$E \in \left\{ \int_B \frac{\epsilon(p)dp}{\eta + e^{-\lambda_0-\lambda_1\epsilon(p)}} : |\lambda_1| \leq \log a \text{ and } \int_B \frac{dp}{\eta + e^{-\lambda_0-\lambda_1\epsilon(p)}} = n \right\}.$$

This can easily be proved by the identities

$$\eta\tilde{n}_\eta(\lambda_0,\lambda_1) = \tilde{n}_1(\lambda_0 - \log\eta, \lambda_1) \quad \text{and} \quad \eta\tilde{E}_\eta(\lambda_0,\lambda_1) = \tilde{E}_1(\lambda_0 - \log\eta, \lambda_1).$$

Proof of Corollary 5.4.15. The proof is a direct consequence of the fact that we can rewrite $\mathcal{F}^0_\eta(\cdot,\cdot,p)$ as $\frac{1}{\eta}\mathcal{F}^0(\eta\cdot,\eta\cdot,p)$ for $\eta > 0$ and that we have that $\mathcal{F}^0_0(n,E,p) = \lim_{\eta\to 0}\mathcal{F}^0_\eta(n,E,p)$. □

With the same proof, we can derive an estimate for the derivatives of $\mathcal{F}^0_\eta$ at all points, where $E = 0$ is satisfied:

Corollary 5.4.17. *For $\eta \geq 0$ let again*

$$\mathcal{F}^0_\eta(n,E,p) := \left(\mathcal{F}_\eta(\cdot,\cdot,p) \circ (\tilde{n}_\eta,\tilde{E}_\eta)^{-1}\right)(n,E)$$

with $\mathcal{F}_\eta(\lambda,p) := \frac{1}{\eta+e^{-\lambda_0-\lambda_1\epsilon(p)}}$ and $(\tilde{n}_\eta(\lambda),\tilde{E}_\eta(\lambda)) := \int_B (1,\epsilon(p))\mathcal{F}_\eta(\lambda,p)dp$. Then for all $n \in (0,\eta^{-1})$, it holds

$$\left|\partial^i_{(n,E)}\partial^j_p\mathcal{F}^0_\eta(n,E,p)\big|_{E=0}\right| \leq 3i!j!\left(\frac{1}{\log\left(1+\frac{1}{24J}\right)}\right)^j \times$$
$$\times \left(\frac{2400(2J+1)^3}{n(1-\eta n)}\right)^i n(1-\eta n).$$

Proposition 5.4.18. *For $\eta \geq 0$ let $\mathcal{F}^0_\eta$, $a \geq 1$ and A_a, B_a be given by Corollary 5.4.15 and let $A'_a := a^{\|\epsilon\|_\infty}A_a$. Then for all $n \in (0,\eta^{-1})$ and $E \in \mathbb{R}$ such that*

$$\inf_{\eta E_0 \in \mathcal{E}_a(\eta n)} 2B_a|E - E_0| \leq n(1-\eta n)$$

it holds

$$\sum_{i+j=1}^{\infty} \frac{\lambda^i \nu^j}{i!j!} \left| \partial^i_{(n,E)} \partial^j_p \mathcal{F}^0_\eta(n,E,p) \right|$$
$$\leq 2 \left(B_a \left(\lambda + \inf_{\eta E_0 \in \mathcal{E}_a(\eta n)} |E - E_0| \right) + A'_a \nu n(1-\eta n) \right)$$
$$\leq (1 + a^{\|\epsilon\|_\infty}) n(1-\eta n)$$

for non-negative λ, ν such that

$$2A_a \nu \leq 1 \quad \textit{and} \quad 2B_a \lambda \leq n(1-\eta n) - \inf_{\eta E_0 \in \mathcal{E}_a(\eta n)} 2B_a |E - E_0| .$$

Moreover, if

$$\lambda \leq \frac{n(1-\eta n)}{4800(2J+1)^3} \quad \textit{and} \quad \nu \leq \frac{1}{2} \log\left(\frac{1}{24J} + 1 \right),$$

we have

$$\sum_{i+j=1}^{\infty} \frac{\lambda^i \nu^j}{i!j!} \left| \partial^i_{(n,E)} \partial^j_p \mathcal{F}^0_\eta(n,E,p) \right|$$
$$\leq 12 \left(2400(2J+1)^3(\lambda + |E|) + \frac{n(1-\eta n)}{\log(1+\frac{1}{24J})} \nu \right) \leq 12$$

for all $n \in (0, \eta^{-1})$ and $E \in \mathbb{R}$ such that

$$|E| \leq \frac{n(1-\eta n)}{4800(2J+1)^3} - \lambda.$$

Proof. This proposition is a direct consequence of Corollaries 5.4.15 and 5.4.17 as well as Taylor's formula. □

Remark 5.4.19. Let $\eta > 0$. We have $\mathcal{E}_1(n) = \{0\}$ and thus

$$\inf_{E_0 \in \mathcal{E}_1(n)} 2B_1 |E - E_0| = 2B_1 |E| .$$

In addition, $\mathcal{E}_a(n)$ is symmetric in n for all a in such a way that

$$\mathcal{E}_a(n) = \mathcal{E}_a(1-n) .$$

Lemma 5.4.20. *For $n_2 \geq n_1 \geq \frac{1}{2}$ it holds*

$$\mathcal{E}_a(n_2) \subseteq \mathcal{E}_a(n_1)$$

for all $a > 1$.

Proof. Let $\frac{1}{2\eta} \leq n \leq \frac{1}{\eta}$, $a > 1$ and $E \in \mathcal{E}_a(n)$ as well as $\lambda = (\lambda_0, \lambda_1)$ being defined by

$$(n, E) = \left(\tilde{n}(\lambda), \tilde{E}(\lambda)\right) := \int_B \frac{(1, \epsilon(p))}{1 + e^{-\lambda_0 - \lambda_1 \epsilon(p)}} dp.$$

As a direct consequence of Lemma 5.1.9, we see that

$$\lambda_0 = 0 \quad \Leftrightarrow \quad n = \frac{1}{2}.$$

From $E \in \mathcal{E}_a(n)$ we know that λ_1 fulfills $|\lambda_1| \leq \log a$. As the statement suggests, we are interested in what happens if we change n. Therefore we define the auxiliary function

$$\phi : [0,1] \times \mathbb{R} \to \mathbb{R}, \; (n, \lambda_1) \mapsto \tilde{E}(\lambda_0, \lambda_1), \quad \text{where } \tilde{n}(\lambda_0, \lambda_1) = n.$$

The change in n can qualitatively be treated with the aid of Lemmata 5.1.9 and 5.1.10 by

$$\partial_n \phi(n, \lambda_1) = \partial_{\lambda_0} \tilde{E}(\lambda) \partial_n \lambda_0 = \partial_{\lambda_0} \tilde{E}(\lambda) \frac{\partial_{\lambda_1} \tilde{E}(\lambda)}{\det \mathcal{J}_{(\tilde{n}, \tilde{E})}(\lambda)} \lesseqqgtr 0 \;\Leftrightarrow\; \lambda_0 \lambda_1 \lesseqqgtr 0,$$

where $\tilde{n}(\lambda_0, \lambda_1) = n$. Thus, if λ_1 is non-negative (non-positive), then $\phi(\cdot, \lambda_1)$ has a global maximum (minimum) in $n = \frac{1}{2}$. Moreover, beyond this maximum (minimum), $\phi(\cdot, \lambda_1)$ is monotone. Finally, the assertion follows the fact that ϕ is strictly monotone w.r.t. λ_1 due to Lemma 5.1.9. □

Proposition 5.4.21. *Let $\eta \geq 0$, $x \in \mathbb{R}^d$, U be a neighborhood of x and let $n : U \to (0, \eta^{-1})$, $E : U \to \mathbb{R}$ be analytic. Moreover, for $a \geq 1$, let A_a, B_a be as in Lemma 5.4.14. We either assume that there exists an $a \geq 1$ and a $\lambda_0 \in (0, \frac{1}{2A_a})$ such that*

$$|n|_{\dot{C}_x^{\lambda_0}} + |E|_{\dot{C}_x^{\lambda_0}} + \inf_{\eta E_0 \in \mathcal{E}_a(\eta n(x))} |E(x) - E_0| \leq \frac{n(x)(1 - \eta n(x))}{2B_a} \tag{5.24}$$

is fulfilled or we assume that

$$|n|_{\dot{C}_x^{\lambda_0}} + |E|_{\dot{C}_x^{\lambda_0}} + |E(x)| \leq \frac{n(x)(1 - \eta n(x))}{4800(2J+1)^3}.$$

is satisfied for some $\lambda_0 \in (0, \frac{1}{2} \log(1 + \frac{1}{24J}))$.

1. *Then, for all positive $\lambda \leq \lambda_0$, we have*

$$\left|\mathcal{F}_\eta^0(n, E, \cdot)\right|_{\dot{C}_x^{\lambda}} \leq C\lambda \left(n(x)(1 - \eta n(x))\right),$$

for some $C > 0$ depending on a and λ_0.

2. *Moreover, it holds*

$$\left\|n(1-\eta n)\mathcal{F}^0_\eta(n,E,\cdot)\right\|_{C_x^{\lambda_0}} \leq \tilde{C}\left(n(x)(1-\eta n(x))\right),$$

for some $\tilde{C} > 0$ *only depending on* a *and* λ_0.

Proof. We only consider the case $\eta = 1$ since the the other cases are similar. To prove the first assertion, we directly derive from Lemma 4.4.4 that

$$\left|\mathcal{F}^0(n,E,\cdot)\right|_{\dot{C}_x^{\lambda_0}} \leq \left|\mathcal{F}^0\right|_{\dot{C}_y^{\mu,\lambda_0}}$$

with $y = (n(x), E(x))$ and $\mu := |(n,E)|_{\dot{C}_x^{\lambda_0}} \leq |n|_{\dot{C}_x^{\lambda_0}} + |E|_{\dot{C}_x^{\lambda_0}}$. We consider only the case, where $a > 1$ since the remaining cas can be proved similarly. Let A_a, B_a be given by Lemma 5.4.14. Then Proposition 5.4.18 implies

$$\left|\mathcal{F}^0(n,E,\cdot)\right|_{\dot{C}_x^{\lambda_0}} \leq \left(1 + a^{\|\epsilon\|_\infty}\right) n(x)(1-n(x))$$

assuming $2A_a\lambda_0 \leq 1$ and

$$2B_a\mu \leq n(x)(1-n(x)) - \inf_{\eta E_0 \in \mathcal{E}_a(\eta n)} 2B_a\left|E(x) - E_0\right|,$$

which is a consequence of Eq. (5.24). We can prove the second assertion similarly. This time, we only have to combine Lemma 5.4.14 with Lemma 4.4.8 instead of Lemma 4.4.4. □

Proposition 5.4.22. *For* $a \geq 1$, *let* A_a, B_a *be as in Corollary 5.4.15 and let* $\lambda_0 \leq \frac{1}{2A_a}$ *be positive and* $U \subset \mathbb{R}^d$ *be open and* $x \in U$. *Moreover, let* $n_0, n_1 : U \to (0,1)$, $E_0, E_1 : U \to \mathbb{R}$ *be analytic. We assume that* $n_\theta := \theta n_1 + (1-\theta)n_2$ *and* $E_\theta := \theta E_1 + (1-\theta)E_2$ *satisfy*

$$|n_\theta|_{\dot{C}_x^{\lambda_0}} + |E_\theta|_{\dot{C}_x^{\lambda_0}} + \inf_{\eta E_0 \in \mathcal{E}_a(\eta n_\theta(x))} |E_\theta(x) - E_0| \leq \frac{1}{2B_a} n_\theta(x)(1-\eta n_\theta(x)) \tag{5.25}$$

for all $\theta \in [0,1]$. *Then there exists a* $C > 0$ *only depending on* a *and* λ_0 *such that*

$$\begin{aligned}\left\|n_0(1-\eta n_0)\mathcal{F}^0(n_0,E_0,\cdot) - n_1(1-\eta n_1)\mathcal{F}^0(n_1,E_1,\cdot)\right\|_{C_x^{\lambda_0}} \\ \leq C\left\|(n_0,E_0) - (n_1,E_1)\right\|_{C_x^{\lambda_0}}.\end{aligned}$$

In addition, this estimate holds true for some C *depending on* λ_0 *if*

$$\lambda_0 \leq \frac{1}{2}\log\left(1 + \frac{1}{24J}\right)$$

and $n_\theta := \theta n_1 + (1-\theta)n_2$, $E_\theta := \theta E_1 + (1-\theta)E_2$ *fulfill*

$$|n_\theta|_{\dot{C}_x^{\lambda_0}} + |E_\theta(x)|_{\dot{C}_x^{\lambda_0}} + |E_\theta| \leq \frac{n_\theta(x)(1-\eta n_\theta(x))}{4800(2J+1)^3}.$$

Proof. The proof can be done exactly as the proof of Proposition 5.4.21 using the identity

$$G(n_0, E_0) - G(n_1, E_1) = \int_0^1 G'(n_\theta, E_\theta) d\theta \left((n_0, E_0) - (n_1, E_1)\right)$$

for $G(n, E) := n(1 - \eta n)\mathcal{F}^0(n, E, \cdot)$. □

5.5 High temperature expansion

The high temperature expansion of $\mathcal{F}^0_\eta$ is defined as its Taylor expansion w.r.t. E at $E = 0$. It was first computed in [36] for $\eta = 1$. Let us write

$$\mathcal{F}^0_\eta(n, E, p) = \left(\mathcal{F}_\eta(\cdot, \cdot, p) \circ \mathbf{n}_\eta^{-1}\right)(n, E)$$

with

$$\mathcal{F}_\eta(\lambda, p) = \frac{1}{\eta + \exp(-\lambda_0 - \lambda_1 \epsilon(p))} \quad \text{and} \quad \mathbf{n}(\lambda) := \int_B \begin{pmatrix} 1 \\ \epsilon(p) \end{pmatrix} \mathcal{F}_\eta(\lambda, p) dp.$$

Note that $\mathbf{n}$ is invertible due to Proposition 5.1.8 and $E = 0$ corresponds to $\lambda_1 = 0$. The second order Taylor expansion w.r.t. E at zero is given by

$$\mathcal{F}^0_\eta(n, E, p) = \mathcal{F}^0_\eta(n, 0, p) + \partial_\lambda \mathcal{F}_\eta(\lambda_0, 0, p) \left(\partial_\lambda \mathbf{n}_\eta(\lambda_0, 0)\right)^{-1} \begin{pmatrix} 0 \\ E \end{pmatrix} + \frac{1}{2} \partial_E^2 \mathcal{F}^0_\eta(n, 0, p) E^2 + o(E^3).$$

The second derivative of $\mathcal{F}^0_\eta$ can be computed via the formula

$$\partial_E^2 \mathcal{F}^0_\eta = \partial_E \mathbf{n}_\eta^{-1} \cdot \partial_\lambda^2 \mathcal{F}_\eta \circ \mathbf{n}_\eta^{-1} \partial_E \mathbf{n}_\eta^{-1} - \partial_\lambda \mathcal{F}_\eta \circ \mathbf{n}_\eta^{-1} \left(\partial_\lambda \mathbf{n}_\eta\right)^{-1} \begin{pmatrix} \partial_E \mathbf{n}_\eta^{-1} \cdot \partial_\lambda^2 \mathbf{n}_{\eta,1} \circ \mathbf{n}_\eta^{-1} \partial_E \mathbf{n}_\eta^{-1} \\ \partial_E \mathbf{n}_\eta^{-1} \cdot \partial_\lambda^2 \mathbf{n}_{\eta,2} \circ \mathbf{n}_\eta^{-1} \partial_E \mathbf{n}_\eta^{-1} \end{pmatrix},$$

Thus, we compute

$$\partial_\lambda \mathcal{F}_\eta(\lambda, p) = \mathcal{F}_\eta(\lambda, p)(1 - \eta \mathcal{F}_\eta(\lambda, p))(1, \epsilon(p))$$
$$\partial_\lambda^2 \mathcal{F}_\eta(\lambda, p) = \mathcal{F}_\eta(\lambda, p)(1 - \eta \mathcal{F}_\eta(\lambda, p))(1 - 2\eta \mathcal{F}_\eta(\lambda, p)) \begin{pmatrix} 1 & \epsilon(p) \\ \epsilon(p) & \epsilon(p)^2 \end{pmatrix}.$$

Note that we are only interested in the case $\lambda_1 = 0$ which facilitates the computations since $\mathcal{F}_\eta(\lambda_0, 0, p) = \mathbf{n}_{\eta,1}(\lambda_0, 0) =: n$ for all $p \in B$. We have

$$\partial_\lambda \mathcal{F}_\eta(\lambda_0, 0) = n(1 - \eta n)(1, \epsilon(p)),$$
$$\partial_\lambda^2 \mathcal{F}_\eta(\lambda_0, 0) = n(1 - \eta n)(1 - 2\eta n) \begin{pmatrix} 1 & \epsilon(p) \\ \epsilon(p) & \epsilon(p)^2 \end{pmatrix}$$

and

$$\partial_\lambda \mathbf{n}_\eta(\lambda_0, 0) = n(1-\eta n)\begin{pmatrix} 1 & 0 \\ 0 & 2J^2 d \end{pmatrix},$$

$$\partial_E \mathbf{n}^{-1}(n, 0) = \frac{1}{2J^2 dn(1-\eta n)}\begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$\partial_\lambda^2 \mathbf{n}_{\eta,1}(\lambda_0, 0) = n(1-\eta n)(1-2\eta n)\begin{pmatrix} 1 & 0 \\ 0 & 2J^2 d \end{pmatrix},$$

$$\partial_\lambda^2 \mathbf{n}_{\eta,2}(\lambda_0, 0) = n(1-\eta n)(1-2\eta n)\begin{pmatrix} 0 & 2J^2 d \\ 2J^2 d & 0 \end{pmatrix}$$

using $\int_B \epsilon(p)^2 dp = 2J^2 d$. Finally, we obtain that

$$\partial_E^2 \mathcal{F}^0(n, 0, p) = \frac{n(1-\eta n)(1-2\eta n)}{(2J^2 dn(1-\eta n))^2}\epsilon(p)^2 - \frac{n(1-\eta n)}{2J^2 dn(1-\eta n)}(2J^2 d, \epsilon(p))\frac{n(1-\eta n)(1-2\eta n)}{(2J^2 dn(1-\eta n))^2}\begin{pmatrix} 2J^2 d \\ 0 \end{pmatrix}.$$

Definition 5.5.1. We define the zeroth, first and second order high temperature expansion as

$$\mathcal{F}_0^{\mathrm{hT}}(n, E, p) = n, \tag{5.26}$$

$$\mathcal{F}_1^{\mathrm{hT}}(n, E, p) = n + \frac{\epsilon(p)}{2J^2 d}E \tag{5.27}$$

$$\mathcal{F}_2^{\mathrm{hT}}(n, E) = n + \frac{\epsilon(p)}{2J^2 d}E + \frac{1-2\eta n}{8J^4 d^2 n(1-\eta n)}(\epsilon(p)^2 - 2J^2 d)E^2 \tag{5.28}$$

for $p \in B = \mathbb{T}^d$, respectively. They formally fulfill

$$\mathcal{F}_\eta^0(n, E, p) = \mathcal{F}_i^{\mathrm{hT}}(n, E, p) + O(E^{i+1}).$$

5.6 Comments

Recall that n and E describe the particle and energy density, respectively. If we think of an experimental realization, we may assume that most of the particles are located near to the origin. A common model for this assumption is given by

$$n(x) = n_0 e^{-\frac{1}{2}x^2}$$

with $|E(x)| \ll n(x) < 1$.

Lemma 5.6.1. *Let $\eta = 1$ and $n(x) = n_0 e^{-\frac{1}{2}x^2}$ be as above. We have*

$$\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \left\| \partial_x^l \partial_p^j \partial_E^2 \mathcal{F}^0(n(x), 0, p) \right\|_{L_x^\infty(r,\infty))} = \infty.$$

for all $r, \lambda > 0$ and almost all $p \in \mathbb{T}^d$.

Proof. The assertion is a direct consequence of the observation

$$\partial_p^j \partial_E^2 \mathcal{F}^0(n(x), 0) = \frac{\partial_p^j \left(\epsilon^2 - 2J^2 d\right)}{8J^4 d^2 n_0} \left(e^{\frac{1}{2}x^2} - \frac{2n_0}{2 - n_0 e^{-\frac{1}{2}x^2}} \right) \to \pm\infty$$

as $|x| \to \infty$ for almost all $p \in \mathbb{T}^d$. □

Part III

Modeling

Chapter 6

Ill-posedness of Semiconductor Boltzmann-type equation

This chapter is devoted to the ill-posedness of Eq. (1.1) in combination with the potential proportional to the particle density. For the sake of simplicity, we focus in this chapter on a relaxation time approximation with the fixed equilibrium $\overline{n} + \frac{\overline{E}}{2J^2 d}\epsilon$. Thus, we consider

$$\begin{cases} \partial_t f + v(p) \cdot \nabla_x f - U \nabla n_f \cdot \nabla_p f = \dfrac{\overline{n} + \frac{\overline{E}}{2J^2 d}\epsilon(p) - f}{\tau_0}, \\ f(x,p,0) = f_0(x,p) \end{cases} \tag{6.1}$$

for $x \in \mathbb{R}^d, p \in \mathbb{T}^d$ and $t > 0$, where $\tau_0, \overline{n} \in \mathbb{R}_{>0}$ and $v(p) = \nabla\epsilon(p) := (2J\sin(p_i))_i$ as well as $n_f(x,t) = \int_{\mathbb{T}^d} f(x,p,t)dp$. Note that this equation is closely related to the Vlasov-Dirac-Benny equation [6, 7, 17], where $v(p) = p$ with $p \in \mathbb{R}^d$ and where the right-hand side vanishes.

Everything in this chapter is based on [8] and [5]. We only adapt their methods and calculations to the present setting in order to show the difficulties of the analysis of the semiconductor Boltzmann equation for ultracold atoms in an optical lattice.

6.1 Linearized equation

Similar to [8], we formally linearize Eq. (6.1) around $G = G(p) = \overline{n} + \frac{\overline{E}}{2J^2 d}\epsilon(p)$ by assuming that $g(x,p,t) = f(x,p,t) - G(p)$ is relatively small. Inserting this into Eq. (6.1) and dropping the quadratic terms, we obtain

$$\partial_t g + v(p) \cdot \nabla_x g(x,p,t) - U \nabla_x n_g(x,p) \cdot \nabla_p G(p) = -\frac{g(x,p,t)}{\tau_0}. \tag{6.2}$$

In oder to derive a explicit solution, we need the following auxiliary lemma first.

Lemma 6.1.1. *Let $U\overline{E} > 2J^2 d$. We denote $v_j := v \cdot \hat{e}_j$ for $j = 1, \dots, d$. Then*

$$1 = U \frac{\overline{E}}{2J^2 d} \int_{\mathbb{T}^d} \frac{v_j(p)}{v_j(p) - ic} dp = U \frac{\overline{E}}{2J^2 d} \int_{\mathbb{T}^d} \frac{v_j(p)^2}{v_j(p)^2 + c^2} dp, \tag{6.3}$$

admits a unique positive solution c independent from j.

Proof. The r.h.s. of (6.3) attains its minimum at $c = 0$ and tends to zero as $c \to \infty$. Using that $v_j = 2J \sin(p_j)$, we obtain the assertion. □

With c as in the previous lemma, we can find special solutions of (6.2):

Proposition 6.1.2. *Let $j = 1, \dots, d$, $U\overline{E} > 2J^2 d$ and let c be a solution of (6.3) for $\xi = \hat{e}_j$. For $T > 0$, we assume that $n_0 : \{z \in \mathbb{C} : |c\Im z| < T\} \to \mathbb{C}$ is holomorph. Then*

$$g(x,p,t) := \frac{v_j(p)}{v_j(p) - ic} n_0(x_j - ict) e^{-\frac{t}{\tau_0}} \tag{6.4}$$

is a classical solution of (6.2) on the time interval $(-T, T)$ for $j = 1, \dots, d$, where $x_j = x \cdot \hat{e}_j$.

Proof. According to (6.3), we have

$$\begin{aligned} n_g(x,t) :=& \int_{\mathbb{T}^d} g(x,p,t) dp = \int_{\mathbb{T}^d} \frac{v_j(p)}{v_j(p) - ic} dp n_0(x_j - ict) e^{-\frac{t}{\tau_0}} \\ =& \frac{2J^2 d}{U\overline{E}} n_0(x_j - ict) e^{-\frac{t}{\tau_0}}. \end{aligned}$$

Since n_0 is complex differentiable, we see that

$$\begin{aligned} &(\partial_t + v \cdot \nabla_x) g(x,p,t) + \frac{g(x,p,t)}{\tau_0} \\ &= \frac{v_j(p)}{v_j(p) - ic} n_0'(x_j - ict) e^{-\frac{t}{\tau_0}} \cdot (-ic + v_j(p)) \\ &= v_j(p) n_0'(x_j - ict) e^{-\frac{t}{\tau_0}} = U \frac{\overline{E}}{2J^2 d} v(p) \cdot \nabla_x n_g(x,t) \end{aligned}$$

This finishes the proof, because $\nabla_p G(p) := \frac{\overline{E}}{2J^2 d} v(p)$. □

Remark 6.1.3. Note that (6.2) is a linear PDE with real coefficients. Thus, we obtain real valued solutions by taking the real part or the imaginary part of g. Assuming that $n_0(\mathbb{R}^d) \subseteq \mathbb{R}^d$, we have found solutions to the initial values

$$g_0(x,p) := (av(p)+b)\,\frac{v(p)}{v(p)^2+c^2}n_0(x)$$

for $a, b \in \mathbb{R}$.

Indeed, we have found the only solutions to these initial values due to the following proposition, which corresponds to Theorem 3.1 in [25].

Definition 6.1.4. Let $T > 0$. We say $g \in (L^1(0,T;H^1(\mathbb{R}^d;C^0(\mathbb{T}^d))))'$ is a weak solution of (6.2) to the initial guess g_0 if

$$\begin{aligned}
-\int_0^T \int_{\mathbb{R}^d} \langle g, \partial_t \phi\rangle dxdt - \int_{\mathbb{R}^d}\int_{\mathbb{T}^d} g_0\phi(0)dpdx + \int_0^T\int_{\mathbb{R}^d} \langle v\cdot\nabla_x g, \phi(t)\rangle\, dpdt \\
= -U\int_0^T\int_{\mathbb{R}^d} \langle \nabla n_g, G\nabla_p\phi(t)\rangle\, dxdt + -\frac{1}{\tau_0}\int_0^T\int_{\mathbb{R}^d}\langle g,\phi(t)\rangle\, dxdt \quad (6.5)
\end{aligned}$$

holds for all $\phi \in C_c^1([0,T);L^2(\mathbb{R}^d;C^1(\mathbb{T}^d)))$, where $\langle \mu,\phi\rangle := \mu(\phi) = \int_{\mathbb{T}^d}\phi d\mu$ for $\mu \in \mathcal{M} = C^0(\mathbb{T}^d)'$.

Remark 6.1.5. The classical solutions of Eq. (6.2) found in Proposition 6.1.2 belong to $(L^1(0,T;H^1(\mathbb{R}^d;C^0(\mathbb{T}^d))))'$ and are also weak solution w.r.t. Definition 6.1.4 if $n_0 \in H^1(\mathbb{R})$ is additionally fulfilled. This is a consequence of the fact that we can interpret $f \in L^\infty(0,T;H^1(\mathbb{R}^d;L^1(\mathbb{T}^d)))$ as an element of $(L^1(0,T;H^1(\mathbb{R}^d;C^0(\mathbb{T}^d))))'$ with

$$\begin{aligned}
&\|f\|_{(L^1(0,T;H^1(\mathbb{R}^d;C^0(\mathbb{T}^d))))'} \\
&= \sup_{\|\phi\|_{L_t^1(H_x^1(C_p^0))}=1}\left|\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{T}^d} f(x,p,t)\phi(x,p,t)dpdxdt\right| \\
&\leq \|f\|_{L^\infty(0,T;H^1(\mathbb{R}^d;L^1(\mathbb{T}^d)))}\,.
\end{aligned}$$

Proposition 6.1.6. *Any weak solution $g \in (L^1(0,T;H^1(\mathbb{R}^d;C^0(\mathbb{T}^d))))'$ to the initial guess $g_0 = 0$ is identical to zero.*

Proof. Note that Eq. (6.2) can directly be solved in Fourier space. By an argument using Gronwall, one easily obtains that the solution must vanish. For more details, we refer to Theorem 3.1 in [8]. □

6.2 Ill-posedness

In this section, we show that the nonlinear semiconductor Boltzmann type equation in (6.1) is not well-posed by means of the following definition. This definition is motivated by [8]. However, it differs slightly to its corresponding version in [8] since we corrected some minor mistakes.

Definition 6.2.1. Let $m \in \mathbb{N}$, $T' > 0$ and let $f \in L^\infty(0,T'; H^1_{\text{loc}}(\mathbb{R}^d; L^1(\mathbb{T}^d)))$ be a weak solution of (6.1) with $f(0) = f_0 \in H^m_{\text{loc}}(\mathbb{R}^d; L^1(\mathbb{T}^d))$. We call the Cauchy problem (6.1) locally $(H^m - H^1)$ well-posed at f_0 if there exists a constant $c_m > 0$ such that for all $C \in (0, 1/c_m)$, there exist a time $T \in (0, T')$ with the following property:

For any $\delta f \in H^m(\mathbb{R}^d; L^1(\mathbb{T}^d))$ with

$$\|\delta f_0\|_{H^m(\mathbb{R}^d;L^1(\mathbb{T}^d))} \leq C,$$

there exists a weak solution

$$f + \delta f \in L^\infty(0,T; H^1_{\text{loc}}(\mathbb{R}^d; L^1(\mathbb{T}^d)))$$

of (6.1) for $(f + \delta f)(0) = f_0 + \delta f_0$ with

$$\operatorname{esssup}_{0<t<T} \|\delta f(t)\|_{H^1(L^1)} \leq c_m \|\delta f_0\|_{H^m(L^1)}.$$

Definition 6.2.2. Let $T > 0$ and $f_0 \in L^2_{\text{loc}}(\mathbb{R}^d; L^1(\mathbb{T}^d))$. We call a function

$$f \in L^\infty(0,T; H^1_{\text{loc}}(\mathbb{R}^d; L^1(\mathbb{T}^d)))$$

a weak solution of (6.1) if

$$\begin{aligned}
&- \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} f \partial_t \phi dp dx dt - \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} f_0 \phi(0) dp dx \\
&+ \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} v \cdot \nabla_x f \phi dp dx dt + U \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} \nabla n_f f \cdot \nabla_p \phi dp dx dt \\
&= \frac{1}{\tau_0} \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} \left(\overline{n} + \frac{\overline{E}}{2J^2 d} \epsilon - f \right) \phi dp dx dt
\end{aligned}$$

for all $\phi \in C^1_c([0,T); C^0_c(\mathbb{R}^d; C^1(\mathbb{T}^d)))$.

Proposition 6.2.3. *Let $U\overline{E} > 2J^2 d$. The Cauchy problem (6.1) is not locally $(H^m - H^1)$ well-posed at $G = G(p) := \overline{n} + \frac{\overline{E}}{2J^2 d}\epsilon(p)$ for any $m \in \mathbb{N}$.*

Proof. The proof is due to [8] which itself was inspired by [21]. Therefore, we only sketch the proof. Let $G := \overline{n} + \frac{\overline{E}}{2J^2 d}\epsilon$. First, a straightforward computation that G is indeed a stationary classical solution of (6.1). We now assume to the contrary that the Cauchy problem is $(H^m - H^1)$ well-posed for some m at G and fix $c_m, C, T > 0$ from the definition. Now let

$$g_0(x,p) := \frac{v_j(p)}{v_j(p) - ic} e^{-x_j^2}$$

for some $j = 1, \ldots, d$ with c being a solution of (6.3). Rescaling this function, we obtain with

$$\delta f_0^a(x,p) := \frac{C}{\|g_0\|_{H^1(L^1)}} g_0(x,p) e^{iax_j} \quad \text{for } a \in \mathbb{N}$$

a sequence of functions with $\|\delta f_0^a\|_{H^1(L^1)} = C$. By assumption, for any $G + \frac{1}{n}\delta f_0^a$ with $n, a \in \mathbb{N}$, there exists a weak solution $G + \delta f_n^a$ of (6.1) which belongs to $L^\infty(0,T; H^1_{\text{loc}}(\mathbb{R}^d; L^1(\mathbb{T}^d)))$ such that

$$\operatorname{esssup}_{0<t<T} \|\delta f_n^a(t)\|_{H^1(L^1)} \leq \frac{1}{n} c_m \|\delta f_0^a\|_{H^m(L^1)}.$$

Then $g_n := n\delta f_n^a$ solves

$$\begin{aligned}
&- \int_0^T \int_{\mathbb{R}^d} \langle g_n, \partial_t \phi \rangle dx dt - \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} \delta f_0^a \phi(0) dp dx \\
&+ \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} v \cdot \nabla_x g_n \phi(t) dp dx dt \\
&+ U \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} \nabla n_{g_n} \left(G(p) + \frac{1}{n} g_n \right) \cdot \nabla_p \phi(t) dp dx dt \\
&\qquad = - \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} \frac{g_n}{\tau_0} \phi(t) dp dx dt
\end{aligned}$$

for all $\phi \in C_c^1([0,T); C_c^0(\mathbb{R}^d; C^1(\mathbb{T}^d)))$. According to the hypothesis, the sequence $(g_n)_n$ admits a weakly* convergent subsequence $(g_{n_m})_m$ in the dual space of $L^1(0,T; H^1(\mathbb{R}^d; \mathcal{C}^0(\mathbb{T}^d)))$. Moreover, we see that the second part of the fourth term on the left-hand side converges to 0 as $n \to \infty$ since

$$\begin{aligned}
&\left| \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} \nabla n_{g_n} \frac{1}{n} g_n \cdot \nabla_p \phi(t) dp dx dt \right| \\
&\qquad \leq \frac{1}{n} \|g_n\|_{L^\infty(H^1(L^1))} \|g_n\|_{L^\infty(L^2(L^1))} \|\nabla_p \phi\|_{L^\infty(L^\infty(L^\infty))}.
\end{aligned}$$

Thus, the limit g of the subsequence g_{n_m} solves the linearized equation

$$\begin{aligned}
&- \int_0^T \int_{\mathbb{R}^d} \langle g, \partial_t \phi \rangle dx dt - \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} \delta f_0^a \phi(0) dp dx + \int_0^T \int_{\mathbb{R}^d} \langle v \cdot \nabla_x g, \phi(t) \rangle \, dp dt \\
&\quad = -U \int_0^T \int_{\mathbb{R}^d} \langle \nabla n_g, G \nabla_p \phi(t) \rangle \, dx dt - \frac{1}{\tau_0} \int_0^T \int_{\mathbb{R}^d} \langle g, \phi(t) \rangle \, dx dt
\end{aligned}$$

for all $\phi \in C_c^1([0,T); C_c^0(\mathbb{R}^d; C^1(\mathbb{T}^d)))$. Moreover, we can estimate the norm of g by

$$\|g\|_{L^1(0,T;H^1(\mathbb{R}^d;\mathcal{C}^0(\mathbb{T}^d))))'} \leq c_m \|\delta f_0^a\|_{H^m(L^1)} \leq 1. \tag{6.6}$$

By Propositions 6.1.2 and 6.1.6, we can compute this solution by

$$g(x,p,t) = \frac{C}{\|g_0\|_{H^m(L^1)}} \left(\frac{v(p)}{v(p) - ic} e^{-x_j^2 + 2ictx_j + c^2t^2} e^{iax_j} e^{act} \right) e^{-\frac{t}{\tau_0}}.$$

Defining

$$K_m := \frac{C}{\|g_0\|_{H^m(L^1)}} \left\| \frac{v(p)}{v(p) - ic} e^{-x_j^2} \right\|_{(H^1(\mathbb{R}^d;\mathcal{C}^0(\mathbb{T}^d)))'} > 0,$$

it holds

$$\|g(t)\|_{H^1(\mathbb{R}^d;\mathcal{C}^0(\mathbb{T}^d)))'} \geq K_m e^{c^2t^2 + act - \frac{t}{\tau_0}}.$$

However, this tends to infinity for fixed $t > 0$ as $a \to \infty$, contradicting (6.6). $\square$

Chapter 7

Diffusion limit

The thesis of Mandt [36] has shown that there are different models to describe the ultracold atoms in an optical lattice. These models can be divided into the microscopic picture and the macroscopic picture. This chapter provides a link between the two descriptions which is called the diffusion limit. For this, we rescale the semiconductor Boltzmann equation for ultracold atoms in an optical lattice by a diffusive scaling with parameter α. The next step is to take the limit $\alpha \to 0$ to obtain a macroscopic description of the model.

7.1 The diffusive scaling

Let us introduce the scaled semiconductor Boltzmann equation for ultracold atoms in an optical lattice

$$\alpha \partial_t f_\alpha + v(p) \cdot \nabla_x f_\alpha + \nabla_x V(x,t,n_{f_\alpha}) \cdot \nabla_p f_\alpha = \frac{1}{\alpha} Q(f_\alpha(x,\cdot,t))(p), \qquad (7.1)$$

where $\alpha > 0$ is the Knudsen number, v the velocity, V the potential and Q the collision operator. The velocity v is defined by the energy dispersion $\epsilon(p) = -2J\sum_{i=1}^{d}\cos(p_i)$ via $v(p) = \nabla_p \epsilon(p)$. Moreover, the potential V is defined by

$$V(x,t,n_{f_\alpha}) = -Un_{f_\alpha}(x,t) + V_{\text{ext}}(x,t).$$

using the particle density $n_{f_\alpha}(x,t) = \int_{\mathbb{T}^d} f_\alpha(x,p,t)dp$. Similarly to the particle density, we can define the energy density $E_{f_\alpha}(x,t) = \int_{\mathbb{T}^d} \epsilon(p) f_\alpha(x,p,t)dp$.

There are several choices of collision operators. Let $F = F(n_{f_\alpha}, E_{f_\alpha}, \epsilon)$ be a function depending smoothly on the particle density n, the energy density E and the internal energy e. In the following, we consider a BGK-type collision operator similarly as in (1.11) by

$$Q(g)(p) = Q_F(g)(p) := \gamma n_g(1-\eta n_g)(F(n_g, E_g, \epsilon(p)) - g(p)) \quad \text{for } g = g(p) \tag{7.2}$$

for some $\gamma > 0$ and $\eta \geq 0$, where $n_g = \int_{\mathbb{T}^d} f_g(p)dp$ and $E_g = \int_{\mathbb{T}^d} \epsilon(p)g(p)dp$. For this collision operator, we say that G is an equilibrium since

$$p \mapsto F(n, E, \epsilon(p))$$

annihilates Q_F for all possible n, E.

Note that the scaling is in accord with [9, 12]. In the diffusion limit, we assumes that the Knudsen number is small (see, e.g., [29]). Therefore, we are interested in the limit $\alpha \to 0$ of the semiconductor Boltzmann equation (7.1). These types of limits have been widely studied for similar semiconductor Boltzmann equations [10, 11, 19, 39, 44].

Proposition 7.1.1. *Let f_α be a formal solution of* (7.1)*. Assume that the formal limits* $\lim_{\alpha\to 0} f_\alpha = f_0$ *exists such that $f_\alpha = f_0 + \mathcal{O}(\alpha)$.*

Then $f_0(x,p,t) = F(n_{f_0}(x,t), E_{f_0}(x,t), p)$. Moreover, $n_0 := n_{f_0}$ and $E_0 := E_{f_0}$ formally solve

$$\begin{aligned} &\partial_t n_0 + \nabla_x \cdot \int_{\mathbb{T}^d} v(p) \frac{G_F(n_0, E_0, p)}{\gamma n_0(1-\eta n_0)} dp = 0, \\ &\partial_t E_0 + \nabla_x \cdot \int_{\mathbb{T}^d} v(p)\epsilon(p) \frac{G_F(n_0, E_0, p)}{\gamma n_0(1-\eta n_0)} dp \\ &\qquad = \nabla_x V(\cdot,\cdot,n_0) \cdot \int_{\mathbb{T}^d} v(p) \frac{G_F(n_0, E_0, \cdot)}{\gamma n_0(1-\eta n_0)} dp, \end{aligned} \tag{7.3}$$

where

$$G_F(n_0, E_0, p) := -v(p) \cdot \nabla_x F(n_0, E_0, \epsilon(p)) + \nabla_x V(n_0) \cdot \nabla_p F(n_0, E_0, \epsilon(p)).$$

Proof. Let us introduce the Chapman-Enskog expansion (see, e.g., [16]) by

$$f_\alpha = f_0 + \alpha f^1_\alpha. \tag{7.4}$$

and define

$$n_0(x,t) := \int_{\mathbb{T}^d} f_0(x,p,t)dp \quad \text{and} \quad n^1_\alpha(x,t) := \int_{\mathbb{T}^d} f^1_\alpha(x,p,t)dp$$

as well as

$$E_0(x,t) := \int_{\mathbb{T}^d} \epsilon(p) f_0(x,p,t)dp \quad \text{and} \quad E^1_\alpha(x,t) := \int_{\mathbb{T}^d} \epsilon(p) f^1_\alpha(x,p,t)dp.$$

Now, we insert the Chapman-Enskog expansion (7.4) in Equation (7.1) and identify equal powers of α. This yields $f_0(x,p,t) = F(n_0(x,t), E_0(x,t), p)$ for terms of order α^{-1} and

$$n_\alpha^1 \partial_1 F(n_0, E_0, \epsilon) + E_\alpha^1 \partial_2 F(n_0, E_0, \epsilon) - f_\alpha^1 = \frac{v \cdot \nabla_x f_0 + \nabla_x V(\cdot,\cdot,n_0) \cdot \nabla_p f_0}{\gamma n_0 (1-\eta n_0)}. \tag{7.5}$$

for terms of order α^0. We define the functions $G_0 = G_0(n_0, E_0, p)$ as well as $G_1 = G_1(n_0, E_0, n_\alpha^1, E_\alpha^1, p)$ by

$$G_0(n_0, E_0, p) := -v(p) \cdot \nabla_x F(n_0, E_0, \epsilon(p)) + \nabla_x V(\cdot,\cdot,n_0) \cdot \nabla_p F(n_0, E_0, \epsilon(p))$$

and

$$G_1(n_0, E_0, n_\alpha^1, E_\alpha^1, p) := n_\alpha^1 \partial_1 F(n_0, E_0, \epsilon(p)) + E_\alpha^1 \partial_2 F(n_0, E_0, \epsilon(p)).$$

Thus, we have

$$f_\alpha^1 = G_1(n_0, E_0, n_\alpha^1, E_\alpha^1, \cdot) + \frac{G_0(n_0, E_0, \cdot)}{\gamma n_0 (1-\eta n_0)}. \tag{7.6}$$

On the one hand, Gauß' theorem implies that

$$\begin{aligned}\int_{\mathbb{T}^d} & v(p)\epsilon(p)^i G_1(n_0, E_0, n_\alpha^1, E_\alpha^1, p) dp \\ &= \int_{\mathbb{T}^d} \nabla_p \epsilon(p) \epsilon(p)^i \left(n_\alpha^1 \partial_1 F(n_0, E_0, \epsilon(p)) + E_\alpha^1 \partial_2 F(n_0, E_0, \epsilon(p)) \right) dp\end{aligned}$$

vanishes for all $i = 0, 1$ using that $\mathbb{T}^d$ has no boundary. On the other hand, by the same reason, the function $G_0(n_0, E_0, \cdot)$ has neither mass nor energy, i.e. it holds

$$\begin{aligned}-\int_{\mathbb{T}^d} G_0(n_0, E_0, p)\epsilon(p)^i dp = &\int_{\mathbb{T}^d} \nabla_p \epsilon(p) \cdot \nabla_x F(n_0, E_0, \epsilon(p))\epsilon(p)^i dp \\ &+ \int_{\mathbb{T}^d} \nabla_x V(\cdot,\cdot,n_0) \cdot \nabla_p F(n_0, E_0, \epsilon(p))\epsilon(p)^i dp = 0\end{aligned}$$

for $i = 0, 1$.

However, we still need an equation determining n_0 and E_0. For this, we insert the Chapman-Enskog expansion (7.4) satisfying (7.6) into (7.1) and compute the first two moments in ϵ using the properties of G_1 and G_2 from above. We have

$$\partial_t n_0 + \int_{\mathbb{T}^d} v(p) \cdot \nabla_x \frac{G_0(n_0, E_0, p)}{\gamma n_0 (1-\eta n_0)} dp = 0$$

and

$$\partial_t E_0 + \int_{\mathbb{T}^d} v(p)\epsilon(p) \cdot \nabla_x \frac{G_0(n_0, E_0, p)}{\gamma n_0(1-\eta n_0)} dp$$
$$= \nabla_x V(\cdot, \cdot, n_0) \cdot \int_{\mathbb{T}^d} v(p) \frac{G_0(n_0, E_0, \cdot)}{\gamma n_0(1-\eta n_0)} dp$$

in the limits $\alpha \to 0$, which implies the assertion. □

Definition 7.1.2. Let $\eta \geq 0$. For $(n, E) \in \{\int_{\mathbb{T}^d}(1, \epsilon(p))g(p)dp : g \in L^1(\mathbb{T}^d)$ with $0 < g < \frac{1}{\eta}\}$, let $\tilde{\lambda} = \tilde{\lambda}(n, E)$ be the unique solution of

$$\begin{pmatrix} n \\ E \end{pmatrix} = \int_B \begin{pmatrix} 1 \\ \epsilon \end{pmatrix} \frac{dp}{\eta + e^{-\tilde{\lambda}_0(n,E) - \tilde{\lambda}_1(n,E)\epsilon(p)}} \tag{7.7}$$

for $\tilde{\lambda} = (\tilde{\lambda}_0, \tilde{\lambda}_1)$ and let

$$\mathcal{F}^0(n, E, p) := \mathcal{F}(\tilde{\lambda}(n, E), p) := \frac{1}{\eta + e^{-\tilde{\lambda}_0(n,E) - \tilde{\lambda}_1(n,E)\epsilon(p)}}$$

(see chapter 5, Definition 5.2.1).

Corollary 7.1.3. *Let $\gamma > 0$ and $\eta \geq 0$. Assume that for all $\alpha > 0$, there exists a formal solution f_α of (7.1) with*

$$Q(g)(p) = Q_{\mathrm{BGK}}(g)(p) := \gamma n_g(1-\eta n_g)(\mathcal{F}^0(n_g, E_g, \epsilon(p)) - g(p)) \quad \textit{for } g = g(p).$$

If this solution admits the formal limit $f_\alpha = f_0 + \mathcal{O}(\alpha)$, there exists a function $\lambda = (\lambda_0, \lambda_1) : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}^2$ fulfilling $f_0(x, p, t) = \mathcal{F}(\lambda(x, t), p)$. Moreover, λ is a formal solution of

$$\begin{aligned} \partial_t n(\lambda) + \nabla \cdot J_n(\lambda, \nabla\lambda) = 0, \\ \partial_t E(\lambda) + \nabla \cdot J_E(\lambda, \nabla\lambda) - J_n(\lambda, \nabla\lambda) \cdot \nabla V(\lambda) = 0, \end{aligned} \tag{7.8}$$

where $n(\lambda) = \int_{\mathbb{T}^d} \mathcal{F}(\lambda, p)dp$, $E(\lambda) = \int_{\mathbb{T}^d} \epsilon(p)\mathcal{F}(\lambda, p)dp$ and $V(\lambda) = -Un(\lambda) + V_{\mathrm{ext}}$. Here, the particle current $J_n(\lambda, \nabla\lambda) = \int_{\mathbb{T}^d} v(p)G(\lambda, \nabla\lambda, p)dp$ and the energy current $J_E(\lambda, \nabla\lambda) = \int_{\mathbb{T}^d} v(p)\epsilon(p)G(\lambda, \nabla\lambda, p)dp$ are defined using

$$G(\lambda, \nabla_x\lambda, p) := -\left(\sum_{i=0}^{1} \epsilon(p)^i v(p) \cdot \nabla_x \lambda_i + \nabla_x V(\lambda) \cdot v(p)\lambda_1\right) \times$$
$$\times \frac{\mathcal{F}(\lambda, p)(1-\eta\mathcal{F}(\lambda, p))}{\gamma n(\lambda)(1-\eta n(\lambda))}.$$

Proof. This Corollary is a consequence of Proposition 7.1.1 for $F(n, E, \epsilon(p)) = \mathcal{F}^0(n, E, p)$, which provides two equations for the particle density $n_0(x,t) = \int_{\mathbb{T}^d} f_0(x,p,t)dp$ and the energy density $E_0(x,t) = \int_{\mathbb{T}^d} \epsilon(p) f_0(x,p,t)dp$. Let us define $\lambda = \lambda(x,t)$ implicitly by the relation

$$\begin{pmatrix} n_0(x,t) \\ E_0(x,t) \end{pmatrix} = \begin{pmatrix} n(\lambda(x,t)) \\ E(\lambda(x,t)) \end{pmatrix} := \int_{\mathbb{T}^d} \mathcal{F}(\lambda(x,t),p).$$

Note that this λ is well-defined and unique (see chapter 5). We thus only need to verify the formula for G. By Proposition 7.1.1, we have

$$\begin{aligned}
&\gamma n(\lambda)(1-\eta n(\lambda))G(\lambda, \nabla\lambda, p) \\
&\quad = G_{\mathcal{F}^0}(n(\lambda), E(\lambda), p) \\
&\quad = v(p)\cdot\nabla_x \mathcal{F}^0(n(\lambda), E(\lambda), p) + \nabla_x V(\cdot,\cdot,n(\lambda))\cdot\nabla_p \mathcal{F}^0(n(\lambda), E(\lambda), p) \\
&\quad = v(p)\cdot\nabla_x \mathcal{F}(\lambda,p) + \nabla_x V(\lambda))\cdot\nabla_p \mathcal{F}(\lambda,p) \\
&\quad = \left(\sum_{i=0}^{1} \epsilon(p)^i v(p)\cdot\nabla_x\lambda_i + \nabla_x V(\lambda)\cdot v(p)\lambda_1\right)\mathcal{F}(\lambda,p)(1-\eta\mathcal{F}(\lambda,p))
\end{aligned}$$

using

$$\nabla_x \mathcal{F}(\lambda,p) = \nabla_x \frac{1}{\eta + e^{-\lambda_1\epsilon-\lambda_0}} = \sum_{i=0}^{1} \nabla_x\lambda_i\epsilon^i\mathcal{F}(\lambda,p))(1-\eta\mathcal{F}(\lambda,p))$$

and $\nabla_p\mathcal{F}(\lambda,p) = v(p)\lambda_1\mathcal{F}(\lambda,p)(1-\eta\mathcal{F}(\lambda,p))$. □

7.2 High temperature expansion

In this thesis the prototype (scaled) semiconductor Boltzmann equation for a cloud of ultracold atoms in an optical lattice is Eq. (7.1) with

$$Q(g)(p) = Q_{\mathrm{BGK}}(g)(p) := \gamma n_g(1-\eta n_g)(\mathcal{F}^0(n_g, E_g, p) - g(p)) \quad \text{for } g = g(p).$$

We already have seen by chapter 5 that $\mathcal{F}^0(n,E,p)$ is the generalized Fermi-Dirac distribution for a fixed particle and energy density. In Definition 7.1.2, we need to define $\tilde\lambda = (\tilde\lambda_0, \tilde\lambda_1)$ implicitly by

$$\begin{pmatrix} n \\ E \end{pmatrix} = \int_B \begin{pmatrix} 1 \\ \epsilon \end{pmatrix} \frac{dp}{\eta + e^{-\tilde\lambda_0(n,E)-\tilde\lambda_1(n,E)\epsilon(p)}}$$

in order to provide the formula

$$\mathcal{F}^0(n,E,p) := \mathcal{F}(\tilde\lambda(n,E),p) := \frac{1}{\eta + e^{-\tilde\lambda_0(n,E)-\tilde\lambda_1(n,E)\epsilon(p)}}.$$

Assuming that the temperature of the cloud is large meaning that $|\lambda_1(n,E)| \approx 0$, Mandt [36] approximates $\mathcal{F}^0$ by its high temperature expansion. In the high temperature expansion we replace $\mathcal{F}^0$ by a Taylor polynomial of $\mathcal{F}^0$ w.r.t. E at $E=0$. Note that high temperatures ($|\lambda_1(n,E)| \approx 0$) correspond to small absolute values of the energy density E. We recall from (5.27) that

$$\mathcal{F}^0(n,E,p) = n + \frac{\epsilon(p)}{2J^2d}E + \frac{1-2n}{8J^4d^2n(1-\eta n)}(\epsilon(p)^2 - 2J^2d)E^2 + \mathcal{O}(E^3).$$

From this we can define a high temperature expansion collision operator for the microscopic picture by

$$Q^{\mathrm{hT},1}_{\mathrm{BGK}}(g) := \gamma n_g(1-\eta n_g)(n_g - g) \quad \text{for } g = g(p). \tag{7.9}$$

Similarly, we obtain the first order high temperature expansion by

$$Q^{\mathrm{hT},1}_{\mathrm{BGK}}(g) := \gamma n_g(1-\eta n_g)\left(n_g + \frac{\epsilon}{2J^2d}E_g - g\right) \quad \text{for } g = g(p) \tag{7.10}$$

as well as the second order high temperature expansion by

$$Q^{\mathrm{hT},2}_{\mathrm{BGK}}(g) := Q^{\mathrm{hT},1}_{\mathrm{BGK}}(g) + \gamma\frac{1-2\eta n_g}{8J^4d^2}(\epsilon^2 - 2J^2d)E_g^2 \quad \text{for } g = g(p). \tag{7.11}$$

In the macroscopic picture, i.e. for (7.3), we can also define a high temperature expansion. Using the expansion of $\mathcal{F}^0$, we can formally approximate the System (7.3) for $F(n,E,\epsilon(p)) = \mathcal{F}^0(n_0,E_0,p)$ in different orders.

In the zeroth order high temperature expansion, we apply Proposition 7.1.1 to

$$F(n,E,\epsilon(p)) = \mathcal{F}^{\mathrm{hT}}_0(n,E,p) = n.$$

and obtain

$$\begin{aligned}
&\partial_t n_0 - \int_{\mathbb{T}^d} v(p)\cdot\nabla_x \frac{v(p)\cdot\nabla n_0}{\gamma n_0(1-\eta n_0)} = 0,\\
&\partial_t E_0 - \int_{\mathbb{T}^d} v(p)\epsilon(p)\cdot\nabla_x \frac{v(p)\cdot\nabla n_0}{\gamma n_0(1-\eta n_0)}\\
&\qquad = -\nabla_x V(\cdot,\cdot,n_0)\cdot\int_{\mathbb{T}^d} v(p)\frac{v(p)\cdot\nabla n_0}{\gamma n_0(1-\eta n_0)}dp,
\end{aligned} \tag{7.12}$$

since $G_{\mathcal{F}^{\mathrm{hT}}_0}(n_0,E_0,p) = -v(p)\cdot\nabla n_0$. Note that these system can be iteratively solved since the first equation is independent from E_0.

In the first order high temperature expansion, we need to use

$$F(n,E,\epsilon(p)) = \mathcal{F}^{\mathrm{hT}}_1(n,E,p) = n + \frac{\epsilon(p)}{2J^2d}E$$

in Proposition 7.1.1. We thus obtain

$$G_{\mathcal{F}_1^{\mathrm{hT}}}(n_0, E_0, p) = -\left(v(p) \cdot \nabla n_0 + \frac{\epsilon(p)}{2J^2 d} v(p) \cdot \nabla E_0 + \nabla V(n_0) \cdot \frac{v(p)}{2J^2 d} E_0\right).$$

Similarly, one can also derive a formula for the second order high temperature expansion of (7.3). However, we leave this to the reader since the formula is rather long. In the zeroth order high temperature, we call the first equation of (7.12) the drift diffusion equation for ultracold atoms in an optical lattice. Moreover, in the first order high temperature expansion of (7.12), we call the system in (7.8) with $G_F = G_{\mathcal{F}_1^{\mathrm{hT}}}$ the high temperature energy transport equations. Note that Corollary 7.1.3 provides the diffusive limit for the prototype case with $Q = Q_{\mathrm{BGK}}$. However, the equations for the diffusive limit involve $\mathcal{F}(\lambda, p)$. We thus can also formally approximate $\mathcal{F}(\lambda, p)$ by its high temperature expansion and derive formal approximations for the diffusive equation (7.8).

Therefore, if we use a high temperature expansion for the scaled Boltzmann equation and then perform the formal limit will give us the same system as if we approximate G in the formal limit of the standard scaled Boltzmann-equation (7.1). In particular, the diagram in the introduction in section 2.3 commutes w.r.t. the formal limits.

7.3 Drift diffusion equation

The drift diffusion equation for ultracold atoms in an optical lattice is given by the first equation of (7.12). Hence it involves only an equation for the particle density n. With the aid of the symmetry of v, the matrix $\int v(p) \otimes v(p) dp$ can be identified with the number $2J^2$ due to the calculation

$$\int_B v_i v_j dp = \int v_i^2 dp \delta_{i,j} = \int 4J^2 \sin^2(p) dp \delta_{i,j} = 2J^2.$$

Thus, we can rewrite the Eq. (7.12) for $n = n_0$ by

$$\partial_t n = 2J^2 \nabla \cdot \left(\frac{\nabla n}{\gamma n(1 - \eta n)}\right). \tag{7.13}$$

Let us transform the time variable t to $t' := 2J^2 t/\gamma$ and write again t instead of t'. Then Equation (7.13) transforms to

$$\partial_t n = \Delta \log\left(\frac{n}{1 - \eta n}\right). \tag{7.14}$$

By defining the fugacity $\tilde{n} := \frac{n}{1-\eta n}$, this equation is equivalent to

$$\partial_t \tilde{n} = (1 + \eta\tilde{n})^2 \Delta \log \tilde{n}, \tag{7.15}$$

which is a super fast diffusion type equation and very similar to the logarithmic diffusion equation, in which the prefactor of $\Delta \log \tilde{n}$ equals a constant. Note that for $\eta = 1$, Equation (7.14) is invariant to the transformation $n \mapsto 1 - n$ and therefore, Equation (7.15) remains the same after replacing $\tilde{n}$ by $\frac{1}{\tilde{n}}$.

A new set of the boundary conditions

The drift diffusion limit is a super fast diffusion equation which does not conserve the mass in dimension $d \geq 2$ (see [51] for $\eta = 0$). In [48], this property was called "breakdown of diffusion". In order to understand the lack of particle conservation, we need to investigate the formal limit of Proposition 7.1.1. In the diffusive limit $\alpha \to 0$, it was assumed that the collision term dominates the other terms. However, in the region where the particle density is small, i.e. $n \approx 0$, the relaxation time is small as well. This again entails that there are only little collisions and hence the collision operator is rather neglectable, contradicting the assumption that the collisions dominate the kinetics.

With this observation, [48] distinguishes between a diffusive region and a ballistic regime. In the diffusive regime, the dynamics of the particle cloud can be described by the diffusive limit according to Proposition 7.1.1. In contrast to that, Schneider et al. [48] argue that in the ballistic regime, the particles move almost along straight lines with constant velocity. Since the particle density in the ballistic regime is assumed to be very small, the main interest lies on understanding the diffusive regime.

In order to derive a complete model, we suppose that $\mathbb{R}^d$ can be divided in the diffusive regime $\Omega \subset \mathbb{R}^d$ being a bounded domain with smooth boundary and the ballistic regime. In the diffusive regime, we consider the system

$$\begin{cases} n_t = \Delta \log\left(\dfrac{n}{1-\eta n}\right), & (x,t) \in \Omega \times \mathbb{R}_+, \\ n(\cdot, 0) = n_0, & x \in \overline{\Omega}, \end{cases} \tag{7.16}$$

where u_0 is the sufficiently regular initial guess with values in $(0, \eta^{-1})$. However, we require boundary conditions to guaranty that the solution is unique. So far we may suppose that the number of particles leaving the diffusive regime at a point $x \in \partial\Omega$ depend on the particle density in x. This leads to generalized mixed boundary condition, namely

$$\partial_\nu g_1(n) + g_2(n) = 0 \qquad (x,t) \in \partial\Omega \times \mathbb{R}_+$$

for some g_i to be determined.

In the following, we try to motivate a suitable choice for g_i by estimating the number of particles entering the ballistic regime: we assume that the unscaled semiconductor Boltzmann equation

$$\partial_t f + v(p) \cdot \nabla_x f - \nabla_x V(x,t) \cdot \nabla_p f = Q(f)$$

is fulfilled for all $t > 0$ in the ballistic regime $\mathbb{R}^d \setminus \Omega \times \mathbb{T}^d$. Thus, we obtain

$$\partial_t \int_{\mathbb{R}^d \setminus \Omega} n dx = -\int_{\mathbb{R}^d \setminus \Omega} \nabla \cdot \int_{\mathbb{T}^d} v(p) f dp dx = \int_{\partial\Omega} \int_{\mathbb{T}^d} \nu \cdot v f dp dx,$$

where ν denotes the outer normal vector. Using the physical assumption that the total mass $\int_{\mathbb{R}^d} n dx$ is conserved, we derive

$$\left| \partial_t \int_{\mathbb{R}^d} n dx \right| = \left| \int_{\partial\Omega} \int_{\mathbb{T}^d} \nu \cdot v f dp dx \right| \leq \|v\|_\infty \int_{\partial\Omega} n dx \tag{7.17}$$

and likewise

$$\left| \partial_t \int_{\mathbb{R}^d} n dx \right| \leq \|v\|_\infty \int_{\partial\Omega} (1 - \eta n) dx. \tag{7.18}$$

Coming back to the generalized mixed boundary conditions, we see that $g_0(n) := \log(\frac{n}{1-\eta n})$ and $g_1(n) = b_0(n) n (1 - \eta n)$, where $\|b_0\|_\infty \leq \|v\|_\infty$ fulfills (7.17) since

$$\partial_t \int_\Omega n dx = \int_{\partial\Omega} \partial_\nu \log\left(\frac{n}{1 - \eta n}\right) dx = -\int_{\partial\Omega} b_0(n) n (1 - \eta n) dx.$$

Moreover, if we assume that Ω is convex and that every particle in the ballistic regime had its origin in the diffusive regime and escaped, b_0 must be non-negative. In order to prove decay estimates, we may assume in addition that

$$b_0(n) \geq \beta > 0.$$

7.4 High temperature energy transport model

In this approximation, we apply Proposition 7.1.1 for

$$F(n, E, \epsilon(p)) = \mathcal{F}_1^{\mathrm{hT}}(n, E, p) = n + \frac{\epsilon(p)}{2J^2 d} E$$

and derive the System (7.3) with

$$G_{\mathcal{F}_1^{\mathrm{hT}}}(n_0, E_0, p) = -\left(v(p) \cdot \nabla n_0 + \frac{\epsilon(p)}{2J^2 d} v(p) \cdot \nabla E_0 + \nabla V(n_0) \cdot \frac{v(p)}{2J^2 d} E_0 \right).$$

This is called the first order high temperature expansion. In [36], the second order high temperature expansion of $\mathcal{F}^0$ is used to derive the high temperature energy transport model. In addition to that, Mandt [36] simplifies this system by neglecting some of the quadratic terms in E. This thesis presents a more direct method, where all second order terms in E are neglected in the approximation of $\mathcal{F}^0(n, E, p)$.

Let $n = n_0$ and $E = E_0$. We recall the equations of (7.3) by

$$\begin{aligned}
\partial_t n &= \nabla \cdot \int_{\mathbb{T}^d} v(p) \left(-\frac{G_{\mathcal{F}_1^{\mathrm{hT}}}(n, E, p)}{\gamma n(1 - \eta n)} \right) dp \\
\partial_t E &= \nabla \cdot \int_{\mathbb{T}^d} v(p)\epsilon(p) \left(-\frac{G_{\mathcal{F}_1^{\mathrm{hT}}}(n, E, p)}{\gamma n(1 - \eta n)} \right) dp \\
&\quad - \nabla V(\cdot, \cdot, n) \cdot \int_{\mathbb{T}^d} v(p) \left(-\frac{G_{\mathcal{F}_1^{\mathrm{hT}}}(n, E, p)}{\gamma n(1 - \eta n)} \right) dp.
\end{aligned}$$

In order to write it in a closed form, we need to compute the integrals involving $G_{\mathcal{F}_1^{\mathrm{hT}}}$. We have

$$\begin{aligned}
&\int \epsilon(p)^a v(p)(-G_{\mathcal{F}_1^{\mathrm{hT}}}(n, E, p))dp = \int_{\mathbb{T}^d} \epsilon(p)^a v(p) \otimes v(p) dp \cdot \nabla n \\
&+ \int_{\mathbb{T}^d} \frac{\epsilon(p)^{a+1}}{2J^2 d} v(p) \otimes v(p) dp \nabla E + \int_{\mathbb{T}^d} \epsilon(p)^a v(p) \otimes v(p) dp \nabla V(\cdot, \cdot, n) \frac{E}{2J^2 d}.
\end{aligned}$$

As in the zeroth order case, we can identify $\int_{\mathbb{T}^d} v(p) \otimes v(p) dp$ with $2J^2$. Furthermore, we have $\int_B \epsilon(p)^a v_i(p) v_j(p) dp = \delta_{ij} \int_{\mathbb{T}^d} \epsilon(p)^a (\partial_1 \epsilon(p))^2 dp$ by the symmetry of $\epsilon(p)$. We define

$$\kappa_a := \frac{1}{2J^2 d} \int_{\mathbb{T}^d} \epsilon(p)^a (\partial_1 \epsilon(p))^2 dp = \begin{cases} 0, & a = 1, \\ J^2 \frac{2d-1}{d}, & a = 2. \end{cases}$$

Hence, we conclude

$$\begin{aligned}
\int_{\mathbb{T}^d} v(p)(-G_{\mathcal{F}_1^{\mathrm{hT}}}(n, E, p))dp &= 2J^2 \nabla n + \frac{1}{d} \nabla V(\cdot, \cdot, n) E, \\
\int_{\mathbb{T}^d} \epsilon(p) v(p)(-G_{\mathcal{F}_1^{\mathrm{hT}}}(n, E, p))dp &= \kappa_2 \nabla E = J^2 \frac{2d-1}{d} \nabla E.
\end{aligned}$$

Therefore, the energy transport model in the first order high temperature approximation is given by

$$\partial_t n = \nabla \cdot \left(\frac{2J^2 \nabla n + \frac{1}{d} E \nabla V(\cdot,\cdot,n)}{\gamma(n(1-\eta n))} \right)$$
$$\partial_t E = J^2 \frac{2d-1}{d} \nabla \cdot \frac{\nabla E}{\gamma(n(1-\eta n))} - \frac{\nabla V(\cdot,\cdot,n) \cdot 2J^2 d \nabla n + E \left|\nabla V(\cdot,\cdot,n)\right|^2}{d\gamma(n(1-\eta n))}. \tag{7.19}$$

As in the zeroth order approximation, we suppose that

$$V(x,t,n) := -Un(x,t) + V_{\text{ext}}(x,t) \tag{7.20}$$

for some $U \in \mathbb{R}$ and all $x \in \mathbb{R}^d, t > 0$. With these assumptions we can rewrite (7.19) as

$$\partial_t n = \frac{1}{\gamma d} \nabla \cdot \left(\frac{2J^2 d - UE}{n(1-\eta n)} \nabla n + \frac{E}{n(1-\eta n)} \nabla V_{\text{ext}} \right),$$
$$\partial_t E = \frac{J^2(2d-1)}{\gamma d} \nabla \cdot \frac{\nabla E}{n(1-\eta n)} + \frac{1}{\gamma d} \frac{2J^2 d - UE}{n(1-\eta n)} \left(U \left|\nabla n\right|^2 - \nabla V_{\text{ext}} \cdot \nabla n \right)$$
$$+ \frac{1}{\gamma d} \frac{UE}{n(1-\eta n)} \left(\nabla V_{\text{ext}} \cdot \nabla n - \frac{1}{U} \left|\nabla V_{\text{ext}}\right|^2 \right).$$

Remark 7.4.1 (Dimensionless parameters). Since $2J \neq 0$, we can rescale the energy density E to $E' := \frac{U}{2J^2 d} E$. In addition, we introduce the time scaling $t \mapsto t' := 2J^2 t/\gamma$ as well as the external potential $UV'_{\text{ext}} := V_{\text{ext}}$. Finally, we write again t for t', E for E' and V_{ext} for V'_{ext} in order to facilitate the notation. The parameters $\kappa := \frac{U^2}{2J^2 d}$ describes the intensity of the interactions. Altogether, we end up with

$$\partial_t n = \nabla \cdot \left(\frac{1-E}{n(1-\eta n)} \nabla n + \frac{E}{n(1-\eta n)} \nabla V_{\text{ext}} \right),$$
$$\partial_t E = \frac{2d-1}{2d} \nabla \cdot \frac{\nabla E}{n(1-\eta n)} + \kappa \frac{1-E}{n(1-\eta n)} \left(\left|\nabla n\right|^2 - \nabla V_{\text{ext}} \cdot \nabla n \right) \tag{7.21}$$
$$+ \kappa \frac{E}{n(1-\eta n)} \left(\nabla V_{\text{ext}} \cdot \nabla n - \left|\nabla V_{\text{ext}}\right|^2 \right)$$

and in particular with

$$\partial_t n = \nabla \cdot \left(\frac{1-E}{n(1-\eta n)} \nabla n \right),$$
$$\partial_t E = \frac{2d-1}{2d} \nabla \cdot \frac{\nabla E}{n(1-\eta n)} + \kappa \frac{1-E}{n(1-\eta n)} \left|\nabla n\right|^2 \tag{7.22}$$

if $V_{\text{ext}} = const.$. Note that we recall Equation (7.14) by setting $E \equiv 0$ in the first equation.

Chapter 8

Energy transport model

8.1 The model and its structure

Let us consider a system of indistinguishable particles in the potential V. We assume that the momentum space is $\mathbb{T}^d$ with the energy dispersion defined by $\epsilon : \mathbb{T}^d \to \mathbb{R}$,

$$p = (p_1, \ldots, p_d) \mapsto -2J \sum_{i=1}^{d} \cos(p_i) \tag{8.1}$$

for a given, positive J. Thus, the dispersion relation is an approximation for the lowest band (see. [4]). The velocity can be computed by $v(p) := \nabla\epsilon(p) = 2J \sum_i \sin(p_i)\hat{e}_i$. Throughout this thesis, we normalize the Lebesgue measure on $\mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d \simeq [0, 2\pi)^d$ such that $\left|\mathbb{T}^d\right| = 1$, i.e., we define

$$dp = \frac{1}{(2\pi)^d} dx.$$

An energy-transport model for a particle distribution in the generalized Fermi-Dirac equilibrium $\mathcal{F}(\lambda, p) := 1/(\eta + e^{-\lambda_0 - \lambda_1 \epsilon(p)})$ for fixed $\nu \geq 0$ is given by

$$\begin{aligned} \partial_t \tilde{n}(\lambda) + \nabla \cdot J_n(\lambda, \nabla\lambda) = 0, \\ \partial_t \tilde{E}(\lambda) + \nabla \cdot J_E(\lambda, \nabla\lambda) - J_n(\lambda, \nabla\lambda) \cdot \nabla V = 0, \end{aligned} \tag{8.2}$$

with the densities $\tilde{n}(\lambda) = \int_{\mathbb{T}^d} \mathcal{F}(\lambda, p)dp$ and $\tilde{E}(\lambda) = \int_{\mathbb{T}^d} \epsilon(p)\mathcal{F}(\lambda, p)dp$ as well as the currents $J_n(\lambda, \nabla\lambda) = \tau \int_{\mathbb{T}^d} v(p)G(\lambda, \nabla\lambda, p)dp$ and $J_E(\lambda, \nabla\lambda) = \tau \int_{\mathbb{T}^d} v(p)\epsilon(p)G(\lambda, \nabla\lambda, p)dp$ for $\tau > 0$, where

$$G(\lambda, \nabla\lambda, p) := -\left(\sum_{i=0}^{1} \epsilon(p)^i v(p) \cdot \nabla\lambda_i + \nabla V \cdot v(p)\lambda_1\right) \mathcal{F}(\lambda, p)(1 - \eta\mathcal{F}(\lambda, p)).$$

Here n and E denote the particle density and the energy density, restrictively. Moreover, we call J_n the particle current and J_E the energy current.

Remark 8.1.1. If we assume that the parameter τ is a function of λ with

$$\tau = \tau(\lambda) = \frac{1}{\gamma\tilde{n}(\lambda)(1-\eta\tilde{n}(\lambda))}$$

for $\gamma > 0$, we obtain with (8.2) the system from (7.8). However, the structural analysis remains the same. Therefore, for the sake of simplicity, we only treat $\tau \in \mathbb{R}_{>0}$ in this chapter.

Definition 8.1.2. The matrix $D_{ij}(\lambda) = (D(\lambda)_{ij}^{kl})_{kl}$ with

$$D(\lambda)_{ij}^{kl} = \tau \int_{\mathbb{T}^d} \epsilon(p)^{i+j} v(p)_k v(p)_l \mathcal{F}(\lambda,p)(1-\eta\mathcal{F}(\lambda,p))dp, \tag{8.3}$$

is called the diffusion matrix for the entropy parameters λ_0, λ_1. It can be used to rewrite the currents to

$$\begin{aligned} J_n(\lambda,\nabla\lambda) &= -\sum_{i=0}^{1} D(\lambda)_{0i}\nabla\lambda_i - \lambda_1 D(\lambda)_{00}\nabla V, \\ J_E(\lambda,\nabla\lambda) &= -\sum_{i=0}^{1} D(\lambda)_{1i}\nabla\lambda_i - \lambda_1 D(\lambda)_{10}\nabla V. \end{aligned} \tag{8.4}$$

The diffusion matrix and its properties for similar systems are already introduced in [30]. Similarly as in [30], we see that it admits the following properties.

Lemma 8.1.3. *The $2d\times 2d$ diffusion matrix $D(\lambda) = (D(\lambda)_{ij})_{ij}$ is symmetric, positive definite if $\mathcal{F}(\lambda,p)(1-\eta\mathcal{F}(\lambda,p))$ is positive a.e..*

Proof. The property $D(\lambda)_{01} = D(\lambda)_{10}$ is a direct consequence of the definition of D. We easily verify $D(\lambda)_{ij}^{lm} = D(\lambda)_{ij}^{ml}$ for $1 \le l,m \le d$ from the definition and see that D is symmetric. Let $z = (\xi,\zeta)^T \neq 0$ with $\xi,\zeta \in \mathbb{R}^d$; we have

$$\frac{1}{\tau} z^T D(\lambda) z = \int_{\mathbb{T}^d} (\xi + \epsilon(p)\zeta)^T v(p) \otimes v(p)(\xi + \epsilon(p)\zeta)\mathcal{F}(\lambda,p)(1-\eta\mathcal{F}(\lambda,p))dp.$$

Note that the off-diagonal elements of $v(p)\otimes v(p)$ have zero contributions, since the summand of the right-hand side, involving the i,j-th entry of $v(p)\otimes v(p)$, can be written in the form of $\int_{\mathbb{T}^d} v(p)_i g_{ij}(\epsilon(p))dp$ for some differentiable $g : \mathbb{R} \to \mathbb{R}$. Therefore, it vanishes due to $v(p)_i = \partial_i \epsilon(p)$. Thus, we have

$$\begin{aligned} \frac{1}{\tau} z^T D(\lambda) z &= \sum_i \int_{\mathbb{T}^d} (\xi_i + \epsilon(p)\zeta_i)^2 \left|v(p)_i\right|^2 \mathcal{F}(\lambda,p)(1-\eta\mathcal{F}(\lambda,p))dp \\ &= 4J^2 \sum_i \int_{\mathbb{T}^d} (\xi_i + \epsilon(p)\zeta_i)^2 \left|\sin(p_i)\right|^2 \mathcal{F}(\lambda,p)(1-\eta\mathcal{F}(\lambda,p))dp > 0 \end{aligned}$$

since the integrand is positive almost everywhere in $\mathbb{T}^d$. □

In the following lemma, we give a condition for D to be coercive.

Lemma 8.1.4. *Let* $M := \int_{\epsilon(p)>0} \epsilon(p)^2 |\sin(p_i)|^2 \, dp \cdot \min\{1, \frac{1}{4J^2}\}$. *Then we have*

$$z^T D(\lambda) z \geq \frac{\tau M}{e^{8J|\lambda_1|}} \tilde{n}(\lambda)(1 - \eta \tilde{n}(\lambda)) |z|^2 \tag{8.5}$$

for all $z = (\xi, \zeta)^T$ *with* $\xi, \zeta \in \mathbb{R}^d$.

Proof. Let $i \in \{1, \dots, d\}$ and $z_i = (\xi_i/(2J), \zeta_i) \neq 0$ be fixed. According to Lemma 8.1.3, we require an estimate for

$$A := \frac{1}{|z|^2} \int_{\mathbb{T}^d} (\xi_i + \epsilon(p)\zeta_i)^2 |\sin(p_i)|^2 \mathcal{F}(\lambda, p)(1 - \eta \mathcal{F}(\lambda, p)) dp > 0.$$

We start with the case: $\xi_i \zeta_i \geq 0$. This implies

$$\begin{aligned} A &\geq \int_{\epsilon(p)>0} \frac{\xi_i^2 + (\epsilon(p)\zeta_i)^2}{|z|^2} |\sin(p_i)|^2 \mathcal{F}(\lambda, p)(1 - \eta \mathcal{F}(\lambda, p)) dp \\ &\geq \int_{\epsilon(p)>0} \epsilon(p)^2 |\sin(p_i)|^2 \mathcal{F}(\lambda, p)(1 - \eta \mathcal{F}(\lambda, p)) dp. \end{aligned}$$

We have

$$\mathcal{F}(\lambda, p) = \frac{1}{\eta + e^{-\lambda_0 - \lambda_1 \epsilon(p)}} \geq \int_{\mathbb{T}^d} \frac{e^{-4J\lambda_1} d\tilde{p}}{\eta + e^{-\lambda_0 - \lambda_1 \epsilon(\tilde{p})}} = e^{-4J\lambda_1} \tilde{n}(\lambda).$$

Likewise, we can estimate $1 - \eta \mathcal{F}(\lambda, p)$ and combine

$$\mathcal{F}(\lambda, p)(1 - \eta \mathcal{F}(\lambda, p)) \geq e^{-8J\lambda_1} \tilde{n}(\lambda)(1 - \eta \tilde{n}(\lambda)).$$

Defining $M := \int_{\epsilon(p)>0} \epsilon(p)^2 |\sin(p_i)|^2 \, dp$, we obtain

$$A \geq M e^{-8J\lambda_1} \tilde{n}(\lambda)(1 - \eta \tilde{n}(\lambda))$$

and conclude the assertion for $\xi_i \zeta_i \geq 0$. The remaining case can be treated similarly by integration over $\{\epsilon(p) < 0\}$ and utilizing the symmetry of $\epsilon(p)$. □

8.2 Entropy structure and dual entropy parameters

The entropy structure of the system (8.2) was already described in [29] and [30]. However, for the convenience of the reader, we state the main ideas and sketch the proofs whenever it helps to understand the structure.

Definition 8.2.1. Let $\lambda = (\lambda_0, \lambda_1)$ be as above. The entropy of the system (8.2) is defined as

$$H(t) = -\int_{\mathbb{R}^d}\int_{\mathbb{T}^d} \big(\mathcal{F}(\lambda,p)\log\mathcal{F}(\lambda,p) + \eta^{-1}(1-\eta\mathcal{F}(\lambda,p))\log(1-\eta\mathcal{F}(\lambda,p))\big)dpdx \quad (8.6)$$

with $\mathcal{F}(\lambda,p) = 1/(\eta + e^{-\lambda_1\epsilon(p)-\lambda_0})$.

Lemma 8.2.2. *The entropy can alternatively be written as* $H(t) = \int_{\mathbb{R}^d} h(\lambda)dx$ *with*

$$h(\lambda) := -m(\lambda)\cdot\lambda + \eta^{-1}\int_{\mathbb{T}^d}\log(1+\eta e^{\lambda_0+\lambda_1\epsilon(p)})dp, \quad (8.7)$$

where $m(\lambda) := \int_{\mathbb{T}^d}\binom{1}{\epsilon(p)}\mathcal{F}(\lambda,p)dp$.

Proof. The assertion can be verified directly by

$$\begin{aligned} h(\lambda) &= \int_{\mathbb{T}^d}\mathcal{F}(\lambda,p)(-\lambda_1\epsilon(p)-\lambda_0) + \eta^{-1}\log(1/(1-\eta\mathcal{F}(\lambda,p)))dp \\ &= -\int_{\mathbb{T}^d}\mathcal{F}(\lambda,p)\log\frac{\mathcal{F}(\lambda,p)}{(1-\eta\mathcal{F}(\lambda,p))} + \eta^{-1}\log(1-\eta\mathcal{F}(\lambda,p))dp \\ &= -\int_{\mathbb{T}^d}\mathcal{F}(\lambda,p)\log\mathcal{F}(\lambda,p) + \eta^{-1}(1-\eta\mathcal{F}(\lambda,p))\log(1-\eta\mathcal{F}(\lambda,p))dp. \quad \square \end{aligned}$$

Remark 8.2.3. Let $\eta = 1$. The fact that $\mathcal{F}(\lambda,p) = 1-\mathcal{F}(-\lambda,p)$ and the transformation $p \to p+(\pi,\dots,\pi)$ in the integration of h leads to the identities

$$h(\lambda) = h(-\lambda_0,\lambda_1) = h(\lambda_0,-\lambda_1) = h(-\lambda).$$

In order to show that the entropy is monotone in time, we need to define the dual-entropy parameters as in [29].

Definition 8.2.4. Die dual-entropy parameter $\bar\lambda = (\bar\lambda_0, \bar\lambda_1)$ is given by

$$\bar\lambda_0 = \lambda_0 + V\lambda_1 \quad \text{and} \quad \bar\lambda_1 = \lambda_1.$$

The diffusion matrix C for the dual-entropy parameters $\bar\lambda_0, \bar\lambda_1$ is defined as

$$C^{kl}(\bar\lambda) = P^T D(\lambda)^{kl} P, \quad \text{where} \quad P = \begin{pmatrix} 1 & -V \\ 0 & 1 \end{pmatrix}.$$

Corollary 8.2.5. *The Matrix* $C(\bar\lambda)$ *is symmetric and positive semi-definite for any* $\bar\lambda \in \mathbb{R}^2$.

The following proposition is due to [29] and can be obtained by a direct computation. Therefore, the proof is omitted.

Proposition 8.2.6. *The System* (8.2) *in conjunction with* (8.4) *is equivalent to*

$$\begin{aligned} \partial_t \bar{n}(\bar{\lambda}) + \nabla \cdot \bar{J}_n(\bar{\lambda}, \nabla\bar{\lambda}) = 0, \\ \partial_t \bar{E}(\bar{\lambda}) - V\partial_t \bar{n}(\bar{\lambda}) + \nabla \cdot \bar{J}_E(\bar{\lambda}, \nabla\bar{\lambda}) = 0 \end{aligned} \tag{8.8}$$

with $\bar{J}_n(\bar{\lambda}, \nabla\bar{\lambda}) = -C_{00}(\bar{\lambda})\nabla\bar{\lambda}_0 - C_{01}(\bar{\lambda})\nabla\bar{\lambda}_1$ *and* $\bar{J}_E(\bar{\lambda}, \nabla\bar{\lambda}) = -C_{10}(\bar{\lambda})\nabla\bar{\lambda}_0 - C_{11}(\bar{\lambda})\nabla\bar{\lambda}_1$ *as well as* $\bar{n}(\bar{\lambda}) = n(\lambda)$ *and* $\bar{E}(\bar{\lambda}) = \tilde{E}(\lambda)$, *where*

$$\bar{\lambda}_0 = \lambda_0 + V\lambda_1 \quad \textit{and} \quad \bar{\lambda}_1 = \lambda_1.$$

Proposition 8.2.7.

$$-\frac{d}{dt}H(\lambda) + \sum_{i,j=0}^{1} \int_{\mathbb{R}^d} \nabla\bar{\lambda}_i \cdot C_{ij}(\bar{\lambda})\nabla\bar{\lambda}_j = 0. \tag{8.9}$$

Proof. The proof is exactly the same as in [29], Proposition 4.9. However, we sketch the calculation and abbreviate $m := (\tilde{n}(\lambda), \tilde{E}(\lambda))$ as well as $\bar{J} = (J_n(\lambda, \nabla\lambda), J_E(\lambda, \nabla\lambda))$. It holds

$$-\partial_t h(\lambda) = -\sum_{i=0}^{1} \frac{\partial h}{\partial \lambda_i}\partial_t \lambda_i = \sum_{i=0}^{1} \frac{\partial m_i}{\partial \lambda_i} \cdot \lambda \partial_t \lambda_i = \partial_t m \cdot \lambda.$$

As in [29], we use $\lambda = P\bar{\lambda}$ in order to transform this equation to

$$\partial_t h(\lambda) = -P^T \partial_t m \cdot P^{-1}\lambda = -P^T \partial_t m \cdot \bar{\lambda} = \nabla \cdot \bar{J} \cdot \bar{\lambda}.$$

by making use of the equations in (8.8). Finally, an integration by part and the definition of C yields the assertion with

$$-\int_{\mathbb{R}^d} \partial_t h(\lambda) dx + \sum_{i,j=0}^{1} \int_{\mathbb{R}^d} \nabla\bar{\lambda}_i \cdot C_{ij}(\bar{\lambda})\nabla\bar{\lambda}_j = 0. \qquad \square$$

8.3 Entropy dissipation estimates

As we have seen so far, the dual entropy variables play an important role analyzing the entropy. Throughout this section, we formally derive some estimates for the entropy dissipation. Lemma 8.3.2 provides an formal estimate for the transformation $(n, E) \leftrightarrow \bar{\lambda}$ for the special case $V = -Un$. However, we do not prove that the mapping $(n, E) \mapsto \bar{\lambda}$ is bijective. The idea of this section is to argue that this mapping has to be treated with caution (see, e.g., Eq. (8.10)).

Definition 8.3.1. Let $\bar\lambda = (\bar\lambda_0, \bar\lambda_1) \in \mathbb{R}^2$. In accord with the definitions of the dual-entropy variables from Definition 8.2.4, we obtain the particle and energy density by

$$\bar n(\bar\lambda) := \int_{\mathbb{T}^d} \frac{dp}{\eta + e^{-\bar\lambda_0 + U\bar n(\bar\lambda) - \bar\lambda_1 \epsilon(p)}} \text{ and } \bar E(\bar\lambda) := \int_{\mathbb{T}^d} \frac{\epsilon(p)dp}{\eta + e^{-\bar\lambda_0 + U\bar n(\bar\lambda) - \bar\lambda_1 \epsilon(p)}}.$$

Using this (implicit) definitions, we can define the generalized Fermi-Dirac distribution for the dual-entropy variables as

$$\bar{\mathcal{F}}(\bar\lambda, p) = \frac{1}{\eta + e^{-\bar\lambda_0 + U\bar n(\bar\lambda) - \bar\lambda_1 \epsilon(p)}}$$

and set $\bar\omega_i := \bar\omega_i(\bar\lambda) := \int_{\mathbb{T}^d} \epsilon(p)^i \bar{\mathcal{F}}(\bar\lambda, p)(1 - \eta\bar{\mathcal{F}}(\bar\lambda, p))dp$ for $i \in \mathbb{N}_0$.

Lemma 8.3.2. *There exist constants C_1, C_2 depending on the used norms such that*

$$\left\|\partial_{\bar\lambda}(\bar n(\bar\lambda), \bar E(\bar\lambda))\right\| \le C_2 \frac{\bar\omega_0(1 + |U|\,\bar n(\bar\lambda)) + \left(\bar\omega_2\bar\omega_0 - \bar\omega_1^2\right)\left|U\bar\lambda_1\right|}{\left|1 - U\bar\lambda_1\bar\omega_0\right|},$$

$$\left\|\partial_{\bar\lambda}(\bar n(\bar\lambda), \bar E(\bar\lambda))\right\| \ge C_1 \frac{\bar\omega_2\bar\omega_0 - \bar\omega_1^2}{\bar\omega_0(1 + |U|\,\bar n(\bar\lambda)) + (\bar\omega_2\bar\omega_0 - \bar\omega_1^2)\left|U\bar\lambda_1\right|}.$$

In addition, it holds

$$\det\left(\partial_{\bar\lambda}(\bar n(\bar\lambda), \bar E(\bar\lambda))\right) = \frac{\bar\omega_2\bar\omega_0 - \bar\omega_1^2}{1 - U\bar\lambda_1\bar\omega_0}. \tag{8.10}$$

Proof. Let $m = (\bar n(\bar\lambda), \bar E(\bar\lambda))^T$ with $m_i = \int_{\mathbb{T}^d} \frac{\epsilon(p)^i dp}{\eta + e^{-\bar\lambda_1(\epsilon(p) + Un) - \bar\lambda_0}}$, we infer

$$\partial_{\bar\lambda_0} m_i = \bar\omega_i\left(1 + U\bar\lambda_1\partial_{\bar\lambda_0} m_0\right) \quad \text{and} \quad \partial_{\bar\lambda_1} m_i = \bar\omega_{i+1} + U\bar\omega_i\left(m_0 + \bar\lambda_1\partial_{\bar\lambda_1} m_0\right).$$

Rearranging the equations leads to

$$\partial_{\bar\lambda_0}\bar n = \frac{\bar\omega_0}{1 - \bar\omega_0 U\bar\lambda_1} \quad \text{and} \quad \partial_{\bar\lambda_1}\bar n = \frac{\bar\omega_1 + U\bar\omega_0\bar n}{1 - \bar\omega_0 U\bar\lambda_1}$$

as well as

$$\partial_{\bar\lambda_0}\bar E = \bar\omega_1\left(1 + U\bar\lambda_1\frac{\bar\omega_0}{1 - \bar\omega_0 U\bar\lambda_1}\right) = \frac{\bar\omega_1}{1 - \bar\omega_0 U\bar\lambda_1}$$

and

$$\begin{aligned}\partial_{\bar\lambda_1}\bar E &= \bar\omega_2 + U\bar\omega_1\left(\bar n + \bar\lambda_1\frac{\bar\omega_1 + U\bar\omega_0\bar n}{1 - \bar\omega_0 U\bar\lambda_1}\right) = \bar\omega_2 + U\bar\omega_1\frac{\bar\lambda_1\bar\omega_1 + \bar n}{1 - \bar\omega_0 U\bar\lambda_1} \\ &= \frac{\bar\omega_2 + U\bar n\bar\omega_1 - U\bar\lambda_1(\bar\omega_0\bar\omega_2 - \bar\omega_1^2)}{1 - \bar\omega_0 U\bar\lambda_1}.\end{aligned}$$

In order to obtain a lower bound, we require the inverse of $\bar{\lambda} \mapsto m$ and its Jacobian. Therefore, we calculate its determinant first by

$$\begin{aligned}
&\det \begin{pmatrix} \partial_{\bar{\lambda}_0}\bar{n} & \partial_{\bar{\lambda}_1}\bar{n} \\ \partial_{\bar{\lambda}_0}E & \partial_{\bar{\lambda}_1}\bar{E} \end{pmatrix} (1-\bar{\omega}_0 U\bar{\lambda}_1)^2 \\
&\quad = \bar{\omega}_0 \left(\bar{\omega}_2 + U\bar{n}\bar{\omega}_1 - U\bar{\lambda}_1(\bar{\omega}_0\bar{\omega}_2 - \bar{\omega}_1^2)\right) - \bar{\omega}_1 \left(\bar{\omega}_1 + U\bar{\omega}_0\bar{n}\right) \\
&\quad = (1-\bar{\omega}_0 U\bar{\lambda}_1)(\bar{\omega}_0\bar{\omega}_2 - \bar{\omega}_1^2).
\end{aligned}$$

This entails

$$\begin{pmatrix} \partial_n\bar{\lambda}_0 & \partial_E\bar{\lambda}_0 \\ \partial_n\bar{\lambda}_1 & \partial_E\bar{\lambda}_1 \end{pmatrix} = \frac{1}{\bar{\omega}_0\bar{\omega}_2 - \bar{\omega}_1^2} \begin{pmatrix} \bar{\omega}_2 + Un\bar{\omega}_1 & -Un\bar{\omega}_0 - \bar{\omega}_1 \\ -\bar{\omega}_1 & \bar{\omega}_0 \end{pmatrix} + \begin{pmatrix} 0 & -U\bar{\lambda}_1 \\ 0 & 0 \end{pmatrix}$$

as well as the estimate stated in the assertion using that $\|\cdot\|$ is equivalent to a matrix norm. □

Remark 8.3.3. Lemma 8.3.2 provides useful estimates for the mapping $\bar{\lambda} \to m$. However, these estimates are either singular or degenerated for

$$1 = U\bar{\omega}_0(\bar{\lambda})\bar{\lambda}_1 \quad \text{and} \quad \frac{\bar{\omega}_2(\bar{\lambda})}{\bar{\omega}_0(\bar{\lambda})} = \left(\frac{\bar{\omega}_1(\bar{\lambda})}{\bar{\omega}_0(\bar{\lambda})}\right)^2. \tag{8.11}$$

Lemma 8.3.4. *Let $\eta = 0$. We have $\bar{\omega}_0 = \bar{n}$, $\bar{\omega}_1 = \bar{E}$ and*

$$\bar{\omega}_2(\bar{\lambda}) = 4J^2\bar{n}(\bar{\lambda})d - \frac{\bar{E}(\bar{\lambda})}{\bar{\lambda}_1} + 2(d-1)\frac{\bar{E}(\bar{\lambda})^2}{\bar{n}(\bar{\lambda})}.$$

In particular, it holds $U\bar{\omega}_0(\bar{\lambda})\bar{\lambda}_1 = U\bar{n}(\bar{\lambda})\bar{\lambda}_1$ and

$$\bar{\omega}_2(\bar{\lambda})\bar{\omega}_0(\bar{\lambda}) - \bar{\omega}_1(\bar{\lambda})^2 = 4J^2\bar{n}(\bar{\lambda})^2 d - \bar{n}(\bar{\lambda})\frac{\bar{E}(\bar{\lambda})}{\bar{\lambda}_1} + (2d-3)\bar{E}(\bar{\lambda})^2.$$

Note that for $\eta > 0$, we cannot calculate the values of $\bar{\omega}_i$ as explicitly as in the Maxwell-Boltzmann case ($\eta = 0$).

Proof. The the fact that $\bar{\omega}_0 = \bar{n}$ and $\bar{\omega}_1 = \bar{E}$ is clear by definition. For $\bar{\omega}_2$, we observe that

$$\epsilon(p)^2 = 4J^2 \sum_i \left(\cos^2(p_i) + 2\sum_{j\neq i} \cos(p_i)\cos(p_j) \right).$$

Using $\cos^2 + \sin^2 = 1$ and integration by parts, we compute

$$\begin{aligned}
4J^2 \sum_i \int_{\mathbb{T}^d} \cos^2(p_i)\bar{\mathcal{F}}(\bar{\lambda},p)dp &= 4J^2\bar{n}(\bar{\lambda})d - \frac{2J}{\bar{\lambda}_1}\sum_i \int_{\mathbb{T}^d} \sin(p_i)\partial_{p_i}\bar{\mathcal{F}}(\bar{\lambda},p)dp \\
&= 4J^2\bar{n}(\bar{\lambda})d - \frac{\bar{E}(\bar{\lambda})}{\bar{\lambda}_1}.
\end{aligned}$$

Since $\bar{\mathcal{F}}(\bar{\lambda}, p)$ can be rewritten as

$$\bar{\mathcal{F}}(\bar{\lambda}, p) = e^{\bar{\lambda}_0 + U\bar{n}(\bar{\lambda})\bar{\lambda}_1 - 2J\bar{\lambda}_1 \cos(p_i)} \prod_{j \neq i} e^{-2J\bar{\lambda}_1 \cos(p_i)},$$

we have

$$4J^2 \int_{\mathbb{T}^d} \cos(p_i) \cos(p_j) \bar{\mathcal{F}}(\bar{\lambda}, p) dp = \frac{\bar{E}(\bar{\lambda})^2}{\bar{n}(\bar{\lambda})}.$$

Thus, we conclude the assertion. □

Corollary 8.3.5. *let $\lambda = (\lambda_0, \lambda_1) : \mathbb{R}^d \to \mathbb{R}^2$ and let*

$$\bar{\lambda}_0 = \lambda_0 - \lambda_1 U \int_{\mathbb{T}^d} \frac{dp}{\eta + e^{-\lambda_0 - \lambda_1 \epsilon(p)}} \quad \textit{and} \quad \bar{\lambda}_1 = \lambda_1.$$

Moreover, let $n, E, \omega_i : \mathbb{R}^d \to \mathbb{R}$ be defined by $n := \bar{n} \circ \bar{\lambda}$ and $E := \bar{E} \circ \bar{\lambda}$ as well as $\omega_i := \bar{\omega}_i \circ \bar{\lambda}$ for $i \in \mathbb{N}_0$. There exists constants $C_1, C_2 > 0$ depending on U such that

$$|\nabla n|^2 + |\nabla E|^2 \leq C_1 \frac{\omega_0^2 + \left(\omega_2\omega_0 - \omega_1^2\right)^2 \bar{\lambda}_1^2}{(1 - U\bar{\lambda}_1\omega_0)^2} \left|\nabla\bar{\lambda}\right|^2$$

and

$$\left|\nabla\bar{\lambda}\right|^2 \leq C_2 \frac{\omega_0^2 + (\omega_2\omega_0 - \omega_1^2)^2 \bar{\lambda}_1^2}{(\omega_2\omega_0 - \omega_1^2)^2} \left(|\nabla n|^2 + |\nabla E|^2\right).$$

Proof. Using Lemma 8.3.2, we obtain

$$|\nabla n| \leq \sum_i \left|\partial_{\bar{\lambda}_i} n\right| \left|\nabla\bar{\lambda}_i\right| \leq \varpi \left(\left|\nabla\bar{\lambda}_0\right| + (2J + |U|\, n) \left|\nabla\bar{\lambda}_1\right|\right)$$

with $\varpi := \omega_0 / \left|1 - \omega_0 U \bar{\lambda}_1\right|$ and

$$|\nabla E| \leq 2J\varpi \left(\left|\nabla\bar{\lambda}_0\right| + (2J + |U|\, n) \left|\nabla\bar{\lambda}_1\right|\right) + \varpi \left|U\lambda_1\right| \left(\omega_2 - \frac{\omega_1^2}{\omega_0}\right) \left|\nabla\bar{\lambda}_1\right|.$$

The second assertion can be derived similarly. □

Recalling the statements of section 5.2, we see that the conditions in Remark 8.3.3 are indeed critical. Therefore, we see that the inequalities in Corollary 8.3.5 connecting $\nabla n, \nabla E$ and $\nabla\bar{\lambda}$ have to be used with caution. Nevertheless, the next lemma and its corollaries try to obtain an estimate for ∇n and ∇E directly from the entropy dissipation:

Lemma 8.3.6. *Let $\bar\lambda : \mathbb{R}^d \to \mathbb{R}^2$ and let $n, E, \omega_i : \mathbb{R}^d \to \mathbb{R}$ be defined by $n := \bar n \circ \bar\lambda$ and $E := \bar E \circ \bar\lambda$ as well as $\omega_i := \bar\omega_i \circ \bar\lambda$ for $i \in \mathbb{N}_0$. Moreover, we define the diffusion matrix C^{kl} as a matrix valued function $\mathbb{R}^d \to \mathbb{R}^{2\times 2}$ by*

$$C^{kl}(x) = P(x)^T D(x)^{kl} P(x), \quad \textit{where} \quad P(x) = \begin{pmatrix} 1 & Un(x) \\ 0 & 1 \end{pmatrix}$$

and $D_{ij}(x) = (D(x)^{kl}_{ij})_{kl}$ with

$$D(x)^{kl}_{ij} = \tau \int_{\mathbb{T}^d} \epsilon(p)^{i+j} v(p)_k v(p)_l \bar{\mathcal{F}}(\bar\lambda(x), p)(1 - \eta \bar{\mathcal{F}}(\bar\lambda(x), p))dp \tag{8.12}$$

for $x \in \mathbb{R}^d$ and $k, l = 1, \dots, d$. Let $C := (C^{kl})_{k,l=1,\dots,d}$. Then for every positive δ exists an $R_\delta > 0$ such that

$$\nabla\bar\lambda : C\nabla\bar\lambda + \delta \frac{\tau\omega_0}{n^2(1-\eta n)^2} \geq \frac{\tau N_{-2}(\mu)}{\omega_0 d}(1 - U\bar\lambda_1\omega_0)^2 |\nabla n|^2$$

and

$$\nabla\bar\lambda : C\nabla\bar\lambda + \delta \frac{\tau\omega_0}{n^2(1-\eta n)^2} \geq \frac{\tau N_{-2}(\mu)}{4J^2\omega_0 d} \left|\nabla E - U\bar\lambda_1\omega_1\nabla n\right|^2$$

for $\left|\bar\lambda_1\right| \geq R_\delta$ setting

$$N_{-2}(e) = \frac{1}{N(e)} \int_{\epsilon(p)=e} |\nabla\epsilon(p)| d\mathcal{H}^{d-1}_p.$$

Definition 8.3.7. Let $z_0 = (z_{01}, \dots, z_{0d})^T, z_1 = (z_{10}, \dots, z_{1d})^T \in \mathbb{R}^d$ and $z = (z_0, z_1)$. Then we define

$$z : Cz := \sum_{i,j} z_i \cdot C_{ij} z_j = \sum_{i,j}\sum_{k,l} z_{ik} C^{kl}_{ij} z_{jl}.$$

Proof. Let $z_0, z_1 \in \mathbb{R}^d$, $z = (z_0, z_1)$ and $C_{ij} := (C^{kl}_{ij})_{k,l=1,\dots,d}$ and let $V := -Un : \mathbb{R}^d \to \mathbb{R}$. We have

$$\begin{aligned} z : Cz &:= \sum_{i,j} z_i \cdot C_{ij} z_j = \sum_{i,j,k,l} P_{ij} z_j \cdot D_{ik} P_{kl} z_l \\ &= \tau \sum_k \int_{\mathbb{T}^d} (z_{0k} - Vz_{1k} + \epsilon(p) z_{1k})^2 \left|v(p)_k\right|^2 \bar{\mathcal{F}}(\bar\lambda, p)(1 - \eta\bar{\mathcal{F}}(\bar\lambda, p))dp. \end{aligned} \tag{8.13}$$

$$= \tau \sum_k \int_{\mathbb{T}^d} (z_{0k} - Vz_{1k} + ez_{1k})^2 \psi_k(e) \tilde{\mathcal{F}}(\lambda, e)(1 - \eta\tilde{\mathcal{F}}(\lambda, e)) N(e) de.$$

for $\tilde{\mathcal{F}}(\lambda, a) := \frac{1}{\eta+e^{-\lambda_0-\lambda_1 a}}$ with λ depending on $\bar{\lambda}$, where we have used the co-area formula in the last step as well as

$$\psi_k(e) = \frac{1}{N(e)} \int_{\epsilon(p)=e} \frac{|\partial_k \epsilon(p)|^2}{|\nabla \epsilon(p)|} d\mathcal{H}_p^{d-1}.$$

We infer from Lemma 5.2.6 that for all $\delta > 0$, there exists an $R_\delta > 0$ such that

$$\frac{n^2(1-\eta n)^2}{\omega_0} \bigg| \int_{\mathbb{T}^d} (z_{0k} - V z_{1k} + \epsilon(p) z_{1k})^2 \times \\ \times (|v(p)_k|^2 - \psi_k(\mp\mu)) \bar{\mathcal{F}}(\bar{\lambda}, p)(1 - \eta \bar{\mathcal{F}}(\bar{\lambda}, p)) dp \bigg| < \delta$$

if $\pm\lambda_1 \geq R_\delta$. The Cauchy-Schwartz Inequality for the measure $\bar{\mathcal{F}}(\bar{\lambda}, p)(1 - \eta \bar{\mathcal{F}}(\bar{\lambda}, p))dp$ ensures

$$\int_{\mathbb{T}^d} (z_{0k} - V z_{1k} + \epsilon(p) z_{1k}) \bar{\mathcal{F}}(\bar{\lambda}, p)(1 - \eta \bar{\mathcal{F}}(\bar{\lambda}, p)) dp \\ \leq \sqrt{\omega_0 \int_{\mathbb{T}^d} (z_{0k} - V z_{1k} + \epsilon(p) z_{1k})^2 \bar{\mathcal{F}}(\bar{\lambda}, p)(1 - \eta \bar{\mathcal{F}}(\bar{\lambda}, p)) dp}.$$

Therefore, in conjunction with the estimate from above, we infer

$$\begin{aligned} z : Cz \\ &\geq \frac{\tau}{\omega_0} \sum_k \psi_k(\mp\mu) \left| \int_{\mathbb{T}^d} (z_{0k} - V z_{1k} + \epsilon(p) z_{1k}) \bar{\mathcal{F}}(\bar{\lambda}, p)(1 - \eta \bar{\mathcal{F}}(\bar{\lambda}, p)) dp \right|^2 \\ &\quad - \delta \frac{\tau \omega_0}{n^2(1-\eta n)^2} \\ &= \frac{\tau \psi_1(\mu)}{\omega_0} |z_0 \omega_0 - V z_1 \omega_0 + z_1 \omega_1|^2 - \delta \frac{\tau \omega_0}{n^2(1-\eta n)^2}. \end{aligned} \tag{8.14}$$

Here, we have applied that $\psi_k(\pm e)$ and $\psi_1(e)$ coincide due to the symmetry of $\epsilon(p)$. For the assertion, we note that $N_{-2} = \psi d$. Making use of $|\epsilon(p)| \leq 2J$, we can analogously prove

$$4J^2 z : Cz \geq \frac{\tau \psi_1(\mu)}{\omega_0} |z_0 \omega_1 - V z_1 \omega_1 + z_1 \omega_2|^2 - 4J^2 \delta \frac{\tau \omega_0}{n^2(1-\eta n)^2}. \tag{8.15}$$

For the next step we need the precise definition of $V = -Un$. The main idea is to utilize $\lambda_0 = \bar{\lambda}_0 + U \bar{\lambda}_1 n$ to obtain an estimate for the gradient of

n. The definition of λ and the fact $\bar{\lambda}_1 = \lambda_1$ yield $\nabla n = \nabla\bar{\lambda}_1\omega_1 + \nabla\lambda_0\omega_0$. Thereby, the relation between λ_0 und $\bar{\lambda}$ implies

$$\nabla n = \nabla\bar{\lambda}_1\omega_1 + \left(\nabla\bar{\lambda}_0 + U\nabla\bar{\lambda}_1 n + U\bar{\lambda}_1\nabla n\right)\omega_0,$$

which is equivalent to

$$\nabla n(1 - U\bar{\lambda}_1\omega_0) = \nabla\bar{\lambda}_1(\omega_1 - V\omega_0) + \nabla\bar{\lambda}_0\omega_0.$$

Combining this with the estimate on $z : Cz$ in (8.14) entails the assertion concerning ∇n. Finally, we insert

$$\nabla E - U\bar{\lambda}_1\omega_1\nabla n = \nabla\bar{\lambda}_1(\omega_2 - V\omega_1) + \nabla\bar{\lambda}_0\omega_1$$

in (8.15) in order to obtain the desired estimate involving ∇E. □

Corollary 8.3.8. *Let $\eta \neq 0$ and let n, E as well as C be defined as above. Assume that $\bar{\lambda}_1\omega_0$ is bounded uniformly in $\bar{\lambda}$ such that* $\sup U\bar{\lambda}_1\omega_0 < 1$. *Then there exists a $\delta > 0$ such that*

$$N_{-2}(\mu)\left(|\nabla n|^2 + |\nabla E|^2\right) \leq \delta\sum_{i,j}\nabla\bar{\lambda}_i\cdot C_{ij}\nabla\bar{\lambda}_j + 1.$$

Proof. For large $\left|\bar{\lambda}_1\right|$, this corollary is a direct consequence of Lemma 8.3.6. Now assuming $\left|\bar{\lambda}_1\right|$ is bounded by some R, we need to apply Lemma 8.1.4 and the definition of C in conjunction with Corollary 8.3.5. □

Remark 8.3.9. For $\eta = 1$ and in dimension $d = 1$, we have

$$N(e) = \frac{2}{\sqrt{4J^2 - e^2}} \quad \text{and} \quad N_{-2}(e) = 4J^2 - e^2.$$

Combining this with $\mu = 2J\cos(\pi n)$ yields

$$N_{-2}(\mu) = 4J^2\sin^2(\pi n) \geq 4J^2\pi^2 n^2(1-n)^2.$$

For the Maxwell-Boltzmann case, we have a slightly different result:

Lemma 8.3.10. *Let $\eta = 0$ and let $\bar{\lambda}, n, E$ as well as C be defined as above. Then*

$$\sum_{i,j}\nabla\bar{\lambda}_i\cdot C_{ij}\nabla\bar{\lambda}_j \geq \frac{\tau\bar{\lambda}_1}{Ed}\left|\nabla\frac{E}{\bar{\lambda}_1} - EU\nabla n\right|^2. \tag{8.16}$$

Proof. For $a, b \in \mathbb{R}$, we have

$$\begin{aligned}\int_{\mathbb{T}^d}(a\epsilon(p) + b)\left|\nabla\epsilon(p)\right|^2\bar{\mathcal{F}}(\bar{\lambda}, p)dp &= \frac{1}{\bar{\lambda}_1}\int_{\mathbb{T}^d}(a\epsilon(p) + b)\epsilon(p)\bar{\mathcal{F}}(\bar{\lambda}, p)dp - a\frac{E}{\bar{\lambda}_1^2} \\ &= \frac{a\omega_2 + b\omega_1}{\bar{\lambda}_1} - a\frac{E}{\bar{\lambda}_1^2}\end{aligned}$$

and especially $\int_{\mathbb{T}^d} |\nabla\epsilon(p)|^2 \bar{\mathcal{F}}(\bar\lambda,p)dp = \frac{E}{\bar\lambda_1}$ (for more details see Lemma 8.4.2). Let $z_0, z_1 \in \mathbb{R}^d$. Using (8.13) and the symmetry of $\epsilon(p)$ ensures

$$\begin{aligned} z : Cz &:= \sum_{i,j} z_i \cdot C_{ij} z_j \\ &= \frac{\tau}{d} \sum_k \int_{\mathbb{T}^d} (z_{0k} - V z_{1k} + \epsilon(p) z_{1k})^2 \, |\nabla\epsilon(p)|^2 \, \bar{\mathcal{F}}(\bar\lambda, p) dp, \end{aligned} \tag{8.17}$$

where we can apply Cauchy-Schwartz' inequality in $L^2(|\nabla\epsilon(p)|^2 \bar{\mathcal{F}}(\bar\lambda,p)dp)$ in order to estimate the integral and obtain

$$\begin{aligned} z : Cz &\geq \frac{\tau}{d} \sum_k \left(\int_{\mathbb{T}^d} (z_{0k} - V z_{1k} + \epsilon(p) z_{1k}) \, |\nabla\epsilon(p)|^2 \, \bar{\mathcal{F}}(\bar\lambda, p) dp \right)^2 \frac{\bar\lambda_1}{E} \\ &= \frac{\tau}{d} \left| \frac{z_0 \omega_1 - V z_1 \omega_1 + z_1 \omega_2}{\bar\lambda_1} - \frac{E}{\bar\lambda_1^2} z_1 \right|^2 \frac{\bar\lambda_1}{E}. \end{aligned}$$

Likewise to the proof of Lemma 8.3.6, we have

$$\nabla E - U \bar\lambda_1 \omega_1 \nabla n = \nabla \bar\lambda_1 (\omega_2 - V\omega_1) + \nabla \bar\lambda_0 \omega_1$$

implying

$$\frac{d}{\tau} \nabla\bar\lambda : C \nabla\bar\lambda \geq \left| \frac{\nabla E - U \bar\lambda_1 \omega_1 \nabla n}{\bar\lambda_1} + E \nabla \frac{1}{\bar\lambda_1} \right|^2 \frac{\bar\lambda_1}{E} = \left| \nabla \frac{E}{\bar\lambda_1} - EU \nabla n \right|^2 \frac{\bar\lambda_1}{E}$$

since $\omega_0 = n$. □

8.4 Degeneracies of the entropy dissipation

Throughout this section, let $\lambda = (\lambda_0, \lambda_1) : \mathbb{R}^d \to \mathbb{R}^2$ and let

$$\bar\lambda_0 = \lambda_0 - \lambda_1 U \int_{\mathbb{T}^d} \frac{dp}{\eta + e^{-\lambda_0 - \lambda_1 \epsilon(p)}} \quad \text{and} \quad \bar\lambda_1 = \lambda_1.$$

Moreover, we define $n, E, \omega_i : \mathbb{R}^d \to \mathbb{R}$ by $n := \int_{\mathbb{T}^d} \frac{dp}{\eta + e^{-\lambda_0 - \lambda_1 \epsilon(p)}}$, $E := \int_{\mathbb{T}^d} \frac{\epsilon(p) dp}{\eta + e^{-\lambda_0 - \lambda_1 \epsilon(p)}}$ as well as

$$\omega_i := \int_{\mathbb{T}^d} \epsilon(p)^i \mathcal{F}(\lambda, p)(1 - \eta \mathcal{F}(\lambda, p)) dp$$

for $i \in \mathbb{N}_0$. In addition, the diffusion matrix C^{kl} is a matrix valued function $\mathbb{R}^d \to \mathbb{R}^{2\times 2}$ given by

$$C^{kl}(x) = P(x)^T D(x)^{kl} P(x), \quad \text{where} \quad P(x) = \begin{pmatrix} 1 & Un(x) \\ 0 & 1 \end{pmatrix}$$

and $D_{ij}(x) = (D(x)_{ij}^{kl})_{kl}$ with

$$D(x)_{ij}^{kl} = \tau \int_{\mathbb{T}^d} \epsilon(p)^{i+j} v(p)_k v(p)_l \mathcal{F}(\lambda(x), p)(1 - \eta \mathcal{F}(\lambda(x), p))dp \tag{8.18}$$

for $x \in \mathbb{R}^d$ and $k, l = 1, \ldots, d$. Let $C := (C^{kl})_{k,l=1,\ldots,d}$. Moreover, we define $V := -Un$.

Definition 8.4.1. Put $\Gamma_i := \int_{\mathbb{T}^d} \epsilon(p)^i \left|\nabla \epsilon(p)\right|^2 \mathcal{F}(\lambda, p)(1 - \eta \mathcal{F}(\lambda, p))dp$ and

$$\gamma := \int_0^\infty \frac{s}{\lambda_1} E(\lambda_0 - s, \lambda_1)ds = \eta^{-1} \int_{\mathbb{T}^d} \left|\nabla \epsilon(p)\right|^2 \log\left(1 + \eta e^{\lambda_0 + \lambda_1 \epsilon(p)}\right) dp.$$

Lemma 8.4.2 ($\eta = 0$)**.** *$\lambda_1 \Gamma_0 = \omega_1 = E$, $\lambda_1 \Gamma_1 = \omega_2 - \frac{E}{\lambda_1} = \omega_2 - \Gamma_0$ and*

$$\lambda_1 \Gamma_2 = \omega_3 - \Gamma_1 = \omega_3 - \frac{2\omega_2}{\lambda_1} + \frac{2E}{\lambda_1^2}.$$

Proof. The functions $G_0(\epsilon(p)) := \frac{1}{\lambda_1} e^{\lambda_0 + \lambda_1 \epsilon(p)}$ and $G_1(\epsilon(p)) := \frac{1}{\lambda_1^2}(\lambda_1 \epsilon(p) - 1)e^{\lambda_0 + \lambda_1 \epsilon(p)}$ as well as $G_2(\epsilon(p)) := \frac{1}{\lambda_1^3}(\lambda_1^2 \epsilon(p)^2 - 2\lambda_1 \epsilon(p) + 2)e^{\lambda_0 + \lambda_1 \epsilon(p)}$ fulfill $G_i'(\epsilon(p)) = \epsilon(p)^i \mathcal{F}(\lambda, p)$. Thus, integration by parts as well as the property $\Delta \epsilon(p) = -\epsilon(p)$ entail

$$\Gamma_i = \int_{\mathbb{T}^d} \nabla \epsilon(p) \cdot \nabla G_i(\epsilon(p))dp = -\int_{\mathbb{T}^d} \Delta \epsilon(p) G_i(\epsilon(p))dp = \int_{\mathbb{T}^d} \epsilon(p) G_i(\epsilon(p))dp.$$

□

Lemma 8.4.3. *γ is well-defined and fulfills*

$$\frac{\Gamma_0 d}{\tau} \sum_{i,j} \nabla \bar{\lambda}_i \cdot C_{ij} \nabla \bar{\lambda}_j = \left|\nabla \gamma + \lambda_1 \Gamma_0 \nabla V\right|^2 + (\Gamma_2 \Gamma_0 - \Gamma_1^2) \left|\nabla \lambda_1\right|^2.$$

Proof. We can rewrite (8.13) by using the symmetry of $\epsilon(p)$ to

$$\begin{aligned} z : Cz &:= \sum_{i,j} z_i \cdot C_{ij} z_j \\ &= \frac{\tau}{d} \int_{\mathbb{T}^d} \left|z_0 - V z_1 + \epsilon(p) z_1\right|^2 \left|\nabla \epsilon(p)\right|^2 \mathcal{F}(\lambda, p)(1 - \eta \mathcal{F}(\lambda, p))dp. \end{aligned} \tag{8.19}$$

Inserting the definition of Γ_i yields

$$\begin{aligned} \frac{d}{\tau} z : Cz &= \Gamma_0 \left|z_0\right|^2 + 2(\Gamma_1 - V\Gamma_0) z_0 \cdot z_1 + (V^2 \Gamma_0 - 2V\Gamma_1 + \Gamma_2) \left|z_1\right|^2 \\ &= \frac{1}{\Gamma_0} \left(\left|\Gamma_0 z_0 + (\Gamma_1 - V\Gamma_0) z_1\right|^2 + (\Gamma_2 \Gamma_0 - \Gamma_1^2) \left|z_1\right|^2 \right). \end{aligned}$$

Here, $\Gamma_2\Gamma_0 - \Gamma_1^2$ is positive for $n \in (0, \eta^{-1})$ according to the Cauchy-Schwartz Inequality. Choosing $z = \nabla\bar{\lambda}$, we want to simplify $\Gamma_0\nabla\bar{\lambda}_0 + (\Gamma_1 - V\Gamma_0)\nabla\bar{\lambda}_1$. For this, we consider ψ fulfilling $\psi'(s) = \eta^{-1}\log(1+\eta e^s)$ and observe $\psi''(\lambda_0 + \lambda_1\epsilon(p)) = \mathcal{F}(\lambda, p)$. Note that ψ is given by a complete Fermi-Dirac integral, namely

$$\psi(\lambda_0 + \lambda_1\epsilon(p)) = \int_0^\infty \frac{tdt}{\eta + e^{t-\lambda_0-\lambda_1\epsilon(p)}} + const.$$

Using $\Delta\epsilon(p) = -\epsilon(p)$, we can compute the gradient of

$$\gamma := \gamma(\lambda) := \frac{1}{\lambda_1}\int_{\mathbb{T}^d} \epsilon(p)\psi(\lambda_0 + \epsilon(p)\lambda_1)dp = \int_{\mathbb{T}^d} |\nabla\epsilon(p)|^2\,\psi'(\lambda_0 + \lambda_1\epsilon(p))dp$$

w.r.t. x by

$$\begin{aligned}\nabla\gamma &= \int_{\mathbb{T}^d} (\nabla\bar{\lambda}_0 + (\epsilon(p) - V)\nabla\bar{\lambda}_1 - \bar{\lambda}_1\nabla V)\psi''(\lambda_0 + \lambda_1\epsilon(p))dp \\ &= \Gamma_0\nabla\bar{\lambda}_0 + (\Gamma_1 - V\Gamma_0)\nabla\bar{\lambda}_1 - \Gamma_0\bar{\lambda}_1\nabla V\end{aligned}$$

recalling the relations $\bar{\lambda}_0 = \lambda_0 - V\lambda_1$ and $\bar{\lambda}_1 = \lambda_1$. □

Lemma 8.4.4. *Let $V = -Un$ and define $\upsilon_{n,1} = \omega_1$, $\upsilon_{E,1} = \omega_0$ and*

$$\begin{pmatrix}\upsilon_{n,0}\\ \upsilon_{E,0}\end{pmatrix} := \frac{1}{\sqrt{\Gamma_2\Gamma_0 - \Gamma_1^2}}\begin{pmatrix}\Gamma_0\omega_2 - \omega_1\Gamma_1 - \lambda_1 U\Gamma_0(\omega_0\omega_2 - \omega_1^2)\\ \Gamma_0\omega_1 - \omega_0\Gamma_1\end{pmatrix}.$$

Then, we have

$$\frac{\Gamma_0 d}{\tau}\sum_{i,j}\nabla\bar{\lambda}_i \cdot C_{ij}\nabla\bar{\lambda}_j = \frac{\Gamma_2\Gamma_0 - \Gamma_1^2}{\omega_2\omega_0 - \omega_1^2}\sum_i |\upsilon_{n,i}\nabla n - \upsilon_{E,i}\nabla E|^2.$$

Proof. Similarly as in the proof of Lemma 8.4.3, we have $\nabla\gamma = \Gamma_0\nabla\lambda_0 + \Gamma_1\nabla\lambda_1$. With this relation we can compute the gradient of n in terms of γ and λ_1 via

$$\Gamma_0\nabla n = \Gamma_0\omega_0\nabla\lambda_0 + \Gamma_0\omega_1\nabla\lambda_1 = \omega_0\nabla\gamma + (\Gamma_0\omega_1 - \omega_0\Gamma_1)\nabla\lambda_1$$

and in particular for $V = -Un$

$$\Gamma_0(1 - \lambda_1 U\omega_0)\nabla n = \omega_0(\nabla\gamma + \lambda_1\Gamma_0\nabla V) + (\Gamma_0\omega_1 - \omega_0\Gamma_1)\nabla\lambda_1.$$

Likewise, we obtain

$$\Gamma_0(\nabla E - \lambda_1 U\omega_1\nabla n) = \omega_1(\nabla\gamma + \lambda_1\Gamma_0\nabla V) + (\Gamma_0\omega_2 - \omega_1\Gamma_1)\nabla\lambda_1$$

and therefore

$$\begin{aligned}
&\Gamma_0(1-\lambda_1 U\omega_0)\nabla E \\
&\quad = (1-\lambda_1 U\omega_0)\left(\omega_1(\nabla\gamma+\lambda_1\Gamma_0\nabla V)+(\Gamma_0\omega_2-\omega_1\Gamma_1)\nabla\lambda_1\right) \\
&\qquad + \lambda_1 U\omega_1\Gamma_0(1-\lambda_1 U\omega_0)\nabla n \\
&\quad = \omega_1(1-\lambda_1 U\omega_0+\lambda_1 U\omega_0)(\nabla\gamma+\lambda_1\Gamma_0\nabla V) \\
&\qquad + \big((1-\lambda_1 U\omega_0)(\Gamma_0\omega_2-\omega_1\Gamma_1)+\lambda_1 U\omega_1(\Gamma_0\omega_1-\omega_0\Gamma_1)\big)\nabla\lambda_1 \\
&\quad = \omega_1(\nabla\gamma+\lambda_1\Gamma_0\nabla V)+\big((\Gamma_0\omega_2-\omega_1\Gamma_1)-\lambda_1 U\Gamma_0(\omega_2\omega_0-\omega_1^2)\big)\nabla\lambda_1.
\end{aligned}$$

We define the Matrix

$$A := \begin{pmatrix} \omega_0 & \Gamma_0\omega_1-\omega_0\Gamma_1 \\ \omega_1 & \Gamma_0\omega_2-\omega_1\Gamma_1-\lambda_1 U\Gamma_0(\omega_2\omega_0-\omega_1^2) \end{pmatrix}$$

and compute its determinant by

$$\begin{aligned}
\det A &= \omega_0\big(\Gamma_0\omega_2-\omega_1\Gamma_1-\lambda_1 U\Gamma_0(\omega_2\omega_0-\omega_1^2)\big)-\omega_1(\Gamma_0\omega_1-\omega_0\Gamma_1) \\
&= \Gamma_0(1-\lambda_1 U\omega_0)(\omega_0\omega_2-\omega_1^2),
\end{aligned}$$

which leads to

$$\begin{aligned}
&\Gamma_0(1-\lambda_1 U\omega_0)A^{-1} \\
&\qquad = \frac{1}{\omega_0\omega_2-\omega_1^2}\begin{pmatrix} \Gamma_0\omega_2-\omega_1\Gamma_1 & \omega_0\Gamma_1-\Gamma_0\omega_1 \\ -\omega_1 & \omega_0 \end{pmatrix} - \begin{pmatrix} \lambda_1 U\Gamma_0 & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

and concludes the assertion by using Lemma 8.4.3. □

Lemma 8.4.5. *Let $V=-Un$. Setting*

$$\begin{aligned}
\psi_1(\lambda) &:= \frac{\Gamma_0(\omega_0\omega_2-\omega_1^2)(\Gamma_2\Gamma_0-\Gamma_1^2)}{(\Gamma_0\omega_1-\omega_0\Gamma_1)^2+(\Gamma_2\Gamma_0-\Gamma_1^2)\omega_0^2}, \\
\psi_2(\lambda) &:= \frac{\Gamma_2\Gamma_0-\Gamma_1^2}{\psi_1(\lambda)} = \frac{(\Gamma_0\omega_1-\omega_0\Gamma_1)^2+(\Gamma_2\Gamma_0-\Gamma_1^2)\omega_0^2}{\Gamma_0(\omega_2\omega_0-\omega_1^2)}
\end{aligned}$$

and

$$\Upsilon_n := \frac{\Gamma_0(\Gamma_0\omega_1-\omega_0\Gamma_1)\Big((1-\lambda_1 U\omega_0)(\omega_2-\frac{\omega_1^2}{\omega_0})+\frac{\omega_1^2}{\omega_0}\Big)+\Gamma_0\omega_1(\Gamma_2\omega_0-\Gamma_1\omega_1)}{(\Gamma_0\omega_1-\omega_0\Gamma_1)^2+(\Gamma_2\Gamma_0-\Gamma_1^2)\omega_0^2}.$$

Then we have

$$\frac{d}{\tau}\sum_{i,j}\nabla\bar\lambda_i\cdot C_{ij}\nabla\bar\lambda_j = \psi_1(\lambda)(1-\lambda_1 U\omega_0)^2\left|\nabla n\right|^2+\psi_2(\lambda)\left|\Upsilon_n\nabla n-\nabla E\right|^2. \tag{8.20}$$

Proof. We define $z_i := \upsilon_{n,i}\nabla n - \upsilon_{E,i}\nabla E$ and the rotation

$$R(s)z := \begin{pmatrix} sz_0 + \sqrt{1-s^2}z_1 \\ -\sqrt{1-s^2}z_0 + sz_1 \end{pmatrix}$$

and compute

$$R(s)z = \begin{pmatrix} (s\upsilon_{n,0} + \sqrt{1-s^2}\upsilon_{n,1})\nabla n - (s\upsilon_{E,0} + \sqrt{1-s^2}\upsilon_{E,1})\nabla E \\ (s\upsilon_{n,1} - \sqrt{1-s^2}\upsilon_{n,0})\nabla n + (\sqrt{1-s^2}\upsilon_{E,0} - s\upsilon_{E,1})\nabla E \end{pmatrix}$$

Choose s such that $0 = s\upsilon_{E,0} + \sqrt{1-s^2}\upsilon_{E,1}$. Then s satisfies

$$s^2\upsilon_{E,0}^2 = (1-s^2)\upsilon_{E,1}^2 \quad \Leftrightarrow \quad s^2 = \frac{\upsilon_{E,1}^2}{\upsilon_{E,0}^2 + \upsilon_{E,1}^2}.$$

Now $(R(s)z)_0 := (s\upsilon_{n,0} + \sqrt{1-s^2}\upsilon_{n,1})\nabla n = s(\upsilon_{n,0} - \frac{\upsilon_{E,0}}{\upsilon_{E,1}}\upsilon_{n,1})\nabla n$. Inserting s^2 calculated above yields

$$(R(s)z)_0^2 = \frac{(\upsilon_{n,0}\upsilon_{E,1} - \upsilon_{E,0}\upsilon_{n,1})^2}{\upsilon_{E,0}^2 + \upsilon_{E,1}^2} |\nabla n|^2 .$$

We recall $\upsilon_{n,1} = \omega_1$, $\upsilon_{E,1} = \omega_0$ and

$$\begin{pmatrix} \upsilon_{n,0} \\ \upsilon_{E,0} \end{pmatrix} = \frac{1}{\sqrt{\Gamma_2\Gamma_0 - \Gamma_1^2}} \begin{pmatrix} \Gamma_0\omega_2 - \omega_1\Gamma_1 - \lambda_1 U\Gamma_0(\omega_0\omega_2 - \omega_1^2) \\ \Gamma_0\omega_1 - \omega_0\Gamma_1 \end{pmatrix}$$

from Lemma 8.4.4. We compute

$$\begin{aligned} &\sqrt{\Gamma_2\Gamma_0 - \Gamma_1^2}(\upsilon_{n,0}\upsilon_{E,1} - \upsilon_{E,0}\upsilon_{n,1}) \\ &\qquad = \omega_0(\Gamma_0\omega_2 - \omega_1\Gamma_1 - \lambda_1 U\Gamma_0(\omega_0\omega_2 - \omega_1^2)) - \omega_1(\Gamma_0\omega_1 - \omega_0\Gamma_1) \\ &\qquad = \Gamma_0(1 - \lambda_1 U\omega_0)(\omega_0\omega_2 - \omega_1^2) \end{aligned}$$

and

$$\begin{aligned} (\Gamma_2\Gamma_0 - \Gamma_1^2)(\upsilon_{E,0}^2 + \upsilon_{E,1}^2) &= (\Gamma_0\omega_1 - \omega_0\Gamma_1)^2 + (\Gamma_2\Gamma_0 - \Gamma_1^2)\omega_0^2 \\ &= \omega_1^2\Gamma_0^2 - 2\Gamma_0\omega_1\omega_0\Gamma_1 + \Gamma_2\Gamma_0\omega_0^2. \end{aligned}$$

Thus, we conclude

$$(R(s)z)_0^2 = \frac{\Gamma_0^2(1-\lambda_1 U\omega_0)^2(\omega_0\omega_2 - \omega_1^2)^2}{(\Gamma_0\omega_1 - \omega_0\Gamma_1)^2 + (\Gamma_2\Gamma_0 - \Gamma_1^2)\omega_0^2} |\nabla n|^2 .$$

Lemma 8.4.4 implies

$$\sum_{i,j} \nabla\bar{\lambda}_i \cdot C_{ij}\nabla\bar{\lambda}_j \geq \frac{\tau}{d}\psi_1(\lambda)(1 - \lambda_1 U\omega_0)^2 |\nabla n|^2 .$$

Moreover, we are able to find an equality instead of an estimate. We therefore need to investigate $(R(s)z)_1$ with the help of

$$\begin{aligned}
(R(s)z)_1 &= (s\upsilon_{n,1} - \sqrt{1-s^2}\upsilon_{n,0})\nabla n + (\sqrt{1-s^2}\upsilon_{E,0} - s\upsilon_{E,1})\nabla E \\
&= s\left(\left(\upsilon_{n,1} + \frac{\upsilon_{E,0}}{\upsilon_{E,1}}\upsilon_{n,0}\right)\nabla n - \left(\frac{\upsilon_{E,0}}{\upsilon_{E,1}}\upsilon_{E,0} + \upsilon_{E,1}\right)\nabla E\right), \\
(R(s)z)_1^2 &= \frac{\left((\upsilon_{n,1}\upsilon_{E,1} + \upsilon_{E,0}\upsilon_{n,0})\nabla n - \left(\upsilon_{E,0}^2 + \upsilon_{E,1}^2\right)\nabla E\right)^2}{\upsilon_{E,0}^2 + \upsilon_{E,1}^2} \\
&= (\upsilon_{E,0}^2 + \upsilon_{E,1}^2)\left(\frac{\upsilon_{n,1}\upsilon_{E,1} + \upsilon_{E,0}\upsilon_{n,0}}{\upsilon_{E,0}^2 + \upsilon_{E,1}^2}\nabla n - \nabla E\right)^2 .
\end{aligned}$$

In order to compute the prefactor for ∇n, we observe

$$\begin{aligned}
(\Gamma_2\Gamma_0 - \Gamma_1^2)\upsilon_{n,0}\upsilon_{E,0} &= (\Gamma_0\omega_2 - \omega_1\Gamma_1 - \lambda_1 U\Gamma_0(\omega_0\omega_2 - \omega_1^2))(\Gamma_0\omega_1 - \omega_0\Gamma_1) \\
&= \Gamma_0\omega_1\Gamma_0\omega_2 - \Gamma_0\omega_1\omega_1\Gamma_1 - \lambda_1 U\Gamma_0\Gamma_0\omega_1(\omega_0\omega_2 - \omega_1^2) \\
&\quad - \omega_0\Gamma_1\Gamma_0\omega_2 + \omega_0\Gamma_1\omega_1\Gamma_1 + \lambda_1 U\Gamma_0\omega_0\Gamma_1(\omega_0\omega_2 - \omega_1^2)
\end{aligned}$$

and

$$\begin{aligned}
&(\Gamma_2\Gamma_0 - \Gamma_1^2)\upsilon_{n,0}\upsilon_{E,0} + (\Gamma_2\Gamma_0 - \Gamma_1^2)\omega_0\omega_1 \\
&\quad = \Gamma_0\left(\Gamma_0\omega_1\omega_2 - \omega_1\omega_1\Gamma_1 - \lambda_1 U(\Gamma_0\omega_1 - \omega_0\Gamma_1)(\omega_0\omega_2 - \omega_1^2)\right. \\
&\qquad \left. - \omega_0\Gamma_1\omega_2 + \omega_0\Gamma_2\omega_1\right) \\
&\quad = \Gamma_0\left(\Gamma_0\omega_1\omega_2 - \omega_1\omega_1\Gamma_1 + (1 - \lambda_1 U\omega_0)(\Gamma_0\omega_1 - \omega_0\Gamma_1)(\omega_2 - \frac{\omega_1^2}{\omega_0})\right. \\
&\qquad \left. - \omega_0\Gamma_1\omega_2 + \omega_0\Gamma_2\omega_1 - \Gamma_0\omega_1\omega_2 + \Gamma_0\omega_1\frac{\omega_1^2}{\omega_0} + \omega_0\Gamma_1\omega_2 - \omega_0\Gamma_1\frac{\omega_1^2}{\omega_0}\right) \\
&\quad = \Gamma_0\left(-2\omega_1^2\Gamma_1 + (1 - \lambda_1 U\omega_0)(\Gamma_0\omega_1 - \omega_0\Gamma_1)(\omega_2 - \frac{\omega_1^2}{\omega_0})\right. \\
&\qquad \left. + \omega_0\Gamma_2\omega_1 + \Gamma_0\omega_1\frac{\omega_1^2}{\omega_0}\right) \\
&\quad = \Gamma_0(1 - \lambda_1 U\omega_0)(\Gamma_0\omega_1 - \omega_0\Gamma_1)(\omega_2 - \frac{\omega_1^2}{\omega_0}) \\
&\qquad + \Gamma_0\frac{\omega_1}{\omega_0}\left(\omega_1(\Gamma_0\omega_1 - \Gamma_1\omega_0) + \omega_0(\Gamma_2\omega_0 - \Gamma_1\omega_1)\right).
\end{aligned}$$

Hence,

$$\Upsilon_n := \frac{\upsilon_{n,1}\upsilon_{E,1} + \upsilon_{E,0}\upsilon_{n,0}}{\upsilon_{E,0}^2 + \upsilon_{E,1}^2}$$
$$= \frac{\Gamma_0(\Gamma_0\omega_1 - \omega_0\Gamma_1)\left((1-\lambda_1 U\omega_0)(\omega_2 - \frac{\omega_1^2}{\omega_0}) + \frac{\omega_1^2}{\omega_0}\right) + \Gamma_0\omega_1(\Gamma_2\omega_0 - \Gamma_1\omega_1)}{(\Gamma_0\omega_1 - \omega_0\Gamma_1)^2 + (\Gamma_2\Gamma_0 - \Gamma_1^2)\omega_0^2}.$$

Then, by means of Lemma 8.4.4 we have

$$\frac{d}{\tau}\sum_{i,j}\nabla\bar{\lambda}_i \cdot C_{ij}\nabla\bar{\lambda}_j = \psi_1(\lambda)(1-\lambda_1 U\omega_0)^2\left|\nabla n\right|^2 + \psi_2(\lambda)\left|\Upsilon_n\nabla n - \nabla E\right|^2,$$

where $\psi_2(\lambda) := \frac{(\Gamma_0\omega_1-\omega_0\Gamma_1)^2+(\Gamma_2\Gamma_0-\Gamma_1^2)\omega_0^2}{\Gamma_0(\omega_2\omega_0-\omega_1^2)}$. □

Lemma 8.4.6. *Let $V = -Un$ and define*

$$\varsigma(\lambda) := \frac{(\Gamma_0\omega_1 - \omega_0\Gamma_1)^2 + (\Gamma_2\Gamma_0 - \Gamma_1^2)\omega_0^2}{\left(\frac{\omega_1}{\omega_0}(\Gamma_0\omega_1 - \omega_0\Gamma_1) + (1-\lambda_1 U\omega_0)\Gamma_0\left(\omega_2 - \frac{\omega_1^2}{\omega_0}\right)\right)^2 + (\Gamma_2\Gamma_0 - \Gamma_1^2)\omega_1^2}$$

as well as

$$\psi_1(\lambda) := \frac{\Gamma_0(\omega_0\omega_2 - \omega_1^2)(\Gamma_2\Gamma_0 - \Gamma_1^2)}{(\Gamma_0\omega_1 - \omega_0\Gamma_1)^2 + (\Gamma_2\Gamma_0 - \Gamma_1^2)\omega_0^2}.$$

Then we have for any $\alpha \in [0,1]$

$$\sum_{i,j}\nabla\bar{\lambda}_i \cdot C_{ij}\nabla\bar{\lambda}_j \geq \frac{\tau}{d}\psi_1(\lambda)(1-\lambda_1 U\omega_0)^2\left(\alpha\left|\nabla n\right|^2 + \varsigma(\lambda)(1-\alpha)\left|\nabla E\right|^2\right). \tag{8.21}$$

Proof. We can similarly find an estimate for ∇E by using another rotation angle such that the zeroth component from $R(\tilde{s})z$ is independent from ∇n. We choose $\tilde{s}$, satisfying $0 = \tilde{s}\upsilon_{n,0} + \sqrt{1-\tilde{s}^2}\upsilon_{n,1}$ and especially $\tilde{s}^2 = \upsilon_{n,1}^2/(\upsilon_{n,0}^2 + \upsilon_{n,1}^2)$. Likewise before, we observe

$$(R(\tilde{s})z)_0^2 = \frac{(\upsilon_{n,0}\upsilon_{E,1} - \upsilon_{E,0}\upsilon_{n,1})^2}{\upsilon_{n,0}^2 + \upsilon_{n,1}^2}\left|\nabla E\right|^2.$$

Fortunately, compared to $(R(s)z)_0^2$, only the denominator has changed to

$$(\Gamma_2\Gamma_0 - \Gamma_1^2)(\upsilon_{n,0}^2 + \upsilon_{n,1}^2)$$
$$= (\Gamma_0\omega_2 - \omega_1\Gamma_1 - \lambda_1 U\Gamma_0(\omega_0\omega_2 - \omega_1^2))^2 + (\Gamma_2\Gamma_0 - \Gamma_1^2)\omega_1^2$$
$$= \left(\frac{\omega_1}{\omega_0}(\Gamma_0\omega_1 - \omega_0\Gamma_1) + (1-\lambda_1 U\omega_0)\Gamma_0\left(\omega_2 - \frac{\omega_1^2}{\omega_0}\right)\right)^2 + (\Gamma_2\Gamma_0 - \Gamma_1^2)\omega_1^2.$$

Define $\varsigma = \frac{\upsilon_{E,0}^2+\upsilon_{E,1}^2}{\upsilon_{n,0}^2+\upsilon_{n,1}^2}$. □

Chapter 9

High temperature energy transport model

Assume (n, E) is a solution of (7.22). For $\mathcal{E} = 1 - E$ holds

$$\begin{aligned} \partial_t n &= \nabla \cdot \left(\frac{\mathcal{E}}{n(1-\eta n)} \nabla n \right), \\ \partial_t \mathcal{E} &= \frac{2d-1}{2d} \nabla \cdot \frac{\nabla \mathcal{E}}{n(1-\eta n)} - \frac{\kappa \mathcal{E}}{n(1-\eta n)} \left| \nabla n \right|^2 . \end{aligned} \tag{9.1}$$

9.1 Numeric realization

This section is devoted to the visualization of the solution and its long term behavior. For this, we consider $\Omega := \mathbb{R}/\mathbb{Z}$ as the one dimensional torus. As we can easily see, the system (9.1) admits two conserved quantities: the total number of particles $\int_\Omega n dx$ and the total amount of energy $\int_\Omega \mathcal{E}_{\text{tot}} dx$, where $\mathcal{E}_{\text{tot}} = \mathcal{E} - \frac{\kappa}{2} n^2$. Therefore, we discretize the time variable by an semi-implicit Euler-scheme conserving these quantities. Let $\tau > 0$ and $k \in \mathbb{N}$. Given $n^{k-1}, \mathcal{E}^{k-1}$, we compute n^k and $\mathcal{E}^k$ by

$$\frac{1}{\tau}(n^k - n^{k-1}) = \partial_x \left(\frac{\mathcal{E}^{k-1}}{n^{k-1}(1-\eta n^{k-1})} \partial_x n^k \right), \tag{9.2}$$

$$\frac{1}{\tau}(\mathcal{E}_{\text{tot}}^k - \mathcal{E}_{\text{tot}}^{k-1}) = \partial_x \left(\frac{\partial_x \mathcal{E}^k}{2n^k(1-\eta n^k)} + \frac{\kappa \mathcal{E}^k}{1-\eta n^k} \partial_x n^k \right), \tag{9.3}$$

where $\mathcal{E}_{\text{tot}}^k = \mathcal{E}^k - \frac{\kappa}{2}(n^k)^2$ for $k \in \mathbb{N}_0$. Here, we use the method of centered finite differences to calculate the spatial derivatives. Note that we can solve the two equations separately since Eq. (9.2) is independent from $\mathcal{E}^k$. After

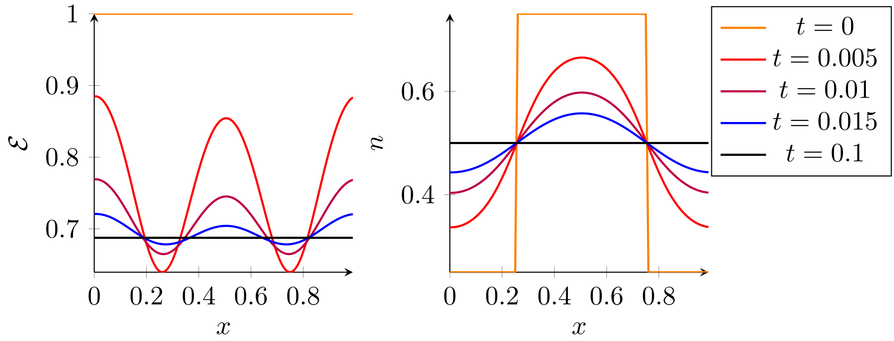


Figure 9.1: $h = 1/100$, $\tau = 1/10000$, $\kappa = 10$, $\eta = 1$.

having solved Eq. (9.2) for some k, we can insert n^k in Eq. (9.3). Thus, $\mathcal{E}^k$ can also be computed using a solver for linear systems.

9.2 Interpretation

In figure 9.1 and figure 9.2, we see the numeric realization of the high temperature energy transport model for $\kappa = 10$ and $\eta = 1$ on the torus. The torus was discretized in 100 uniformly distributed points and the time steps are all of size 1/10000. The initial data was chosen in such a way that the initial kinetic energy $\mathcal{E}_0$ is constant with $\mathcal{E}_0 = 1$ in figure 9.1 and $\mathcal{E}_0 = 1/4$ in figure 9.2. Moreover, the initial particle density distribution is in both figures given by

$$n_0(x) = \begin{cases} \frac{1}{4}, & m - \frac{1}{4} \leq x < m + \frac{1}{4},\ m \in \mathbb{Z} \\ \frac{3}{4}, & \text{else.} \end{cases}$$

Therefore, the only difference between figure 9.1 and 9.2 is the size of the initial kinetic energy $\mathcal{E}_0$. Analyzing figures 9.1 and 9.2, we firstly see the smoothing effects due to the diffusive type of the equations for n and $\mathcal{E}$. Hence, it is no surprise that the solution in figure 9.1 tends numerically to a constant. Note that by construction, the numeric realization conserves the total number of particles $\int_0^1 n_0(x)dx = 0$ as well as the total energy $\int_0^1 \left(\mathcal{E}_0(x) - 5n_0(x)^2\right) dx = 0$. From this, we can easily compute the final particle density by

$$n^\infty = \int_0^1 n_0(x)dx = \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{3}{4} = \frac{1}{2}$$

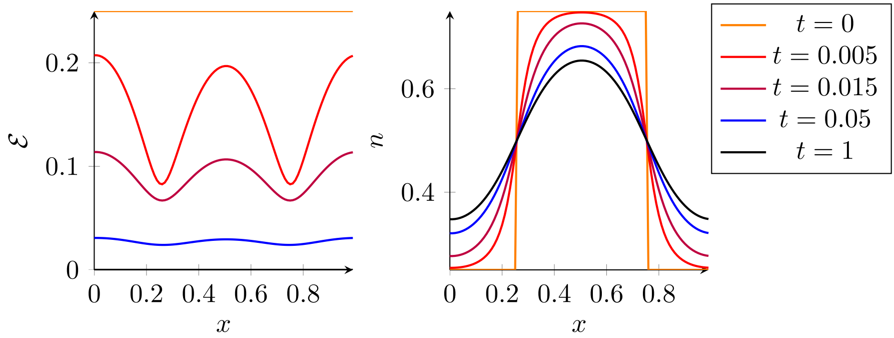


Figure 9.2: $h = 1/100$, $\tau = 1/10000$, $\kappa = 10$, $\eta = 1$.

and the final kinetic energy by

$$\mathcal{E}^\infty = \int_0^1 \left(\mathcal{E}_0(x) - 5n_0(x)^2\right) dx + 5(n^\infty)^2 = 1 - 5\left(\frac{1}{2}\cdot\frac{3^2}{4^2} + \frac{1}{2}\cdot\frac{1^2}{4^2}\right) + \cdot\frac{5}{2^2}$$
$$= 0.6875.$$

We can see that these final distributions are already almost attained after $t = 0.1$. However, the situation in figure 9.2 is quite different. The reason for this can easily be deduced by making the false assumption that the kinetic energy as well as the particle density again numerically converge to constants. Implementing the same computation as above, these constants would be $n^\infty = \frac{1}{2}$ and $\mathcal{E}^\infty = -0.0625$. However, this contradicts the fact that $\mathcal{E}$ always remains non-negative due to the maximum principle. Therefore, either n or $\mathcal{E}$ cannot converge to a constant. Since the diffusion equation of n is degenerated at $\mathcal{E} = 0$ in contrast to the diffusion equation for $\mathcal{E}$, it is reasonable that $\mathcal{E}$ numerically converges to 0 and that n does not converge to a constant. Due to the degeneracy in $\mathcal{E} = 0$, all solutions of the form $\mathcal{E} = 0 = const.$ with n being any bounded function being constant in time with $0 < n < 1$ are stationary solutions of the high temperature energy transport model. Unfortunately, this means that we cannot find a concrete formula for the final density distribution $n^\infty = n^\infty(x)$. The easiest interpretation for n in figure 9.2 is that there is not enough kinetic energy in order to level the distribution density.

Another difference between figure 9.2 and figure 9.1 is the time scale. This can be seen with the help of figure 9.3. The rate of convergence for n and $\mathcal{E}$, with the initial data from figure 9.1, is faster than for n and $\mathcal{E}$, satisfying the initial conditions from figure 9.2. Moreover, figure 9.3 shows that n and $\mathcal{E}$ numerically converge exponentially fast to the final distribution n^∞ and $\mathcal{E}^\infty$, respectively. However, it is remarkable that the convergence rate highly depends on the initial kinetic energy. Smaller initial kinetic energy leads to

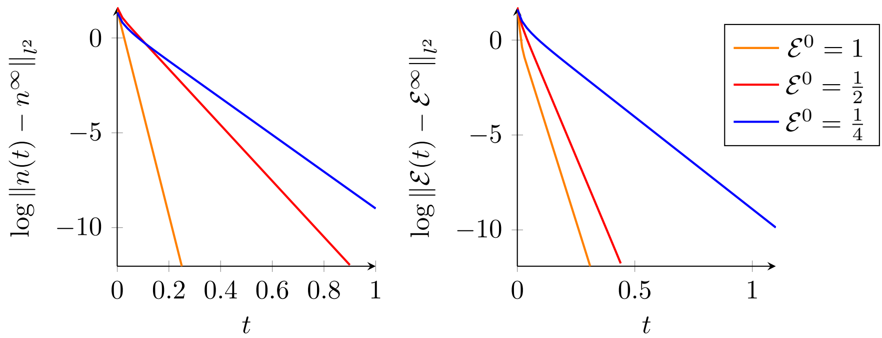


Figure 9.3: $h = 1/100$, $\tau = 1/5000$, $\kappa = 10$, $\eta = 1$. The initial particle density is given by $n^0(x) = \frac{1}{4}$ for $0 \leq x < \frac{1}{2}$, $n^0(x) = \frac{3}{4}$ for $\frac{1}{2} \leq x < 1$. For $\mathcal{E}^0 = 1, \frac{1}{2}$, we have $n^\infty = \frac{1}{2}$ and $\mathcal{E}^\infty = \mathcal{E}^0 - \frac{\kappa}{2} \int_0^1 (n^0 - \frac{1}{2})^2 dx$. For $\mathcal{E} = \frac{1}{4}$, it holds $\mathcal{E}^\infty = 0$ and we have set $n^\infty = n(2)$.

a slower rate of convergence. As usual for energy transport models [28], it is possible to use entropy estimates in order to derive an exponential convergence of the solution to an equilibrium. However, these convergence rates do not depend on the initial data. This leads to the conjecture that it is not possible to derive an exponential decay of the solution of the high temperature energy transport models using entropy methods. It seems that it is necessary to find another tool in order to prove analytically that a solution of (9.1) converges exponentially fast to a stationary solution.

9.3 Numerical convergence

In order to show numerical convergence, we compare different step sizes of Δx and Δx to a reference solution. Since there is no analytic formula for the solution of Eq. (12.6), we use a numerically computed solution of (9.2) and (9.3) with comparably small step size. For the l_x^2 error in the spatial coordinate (see Fig. 9.4) we choose $\Delta x_{\text{ref}} = \frac{1}{1680}$. On the other hand, we take $\Delta t_{\text{ref}} = \frac{1}{5040}$ for the $l_t^2 l_x^2$ error as the reference time step size (see Fig. 9.5). We see that the error in t converges linearly and the error in x converges quadratically.

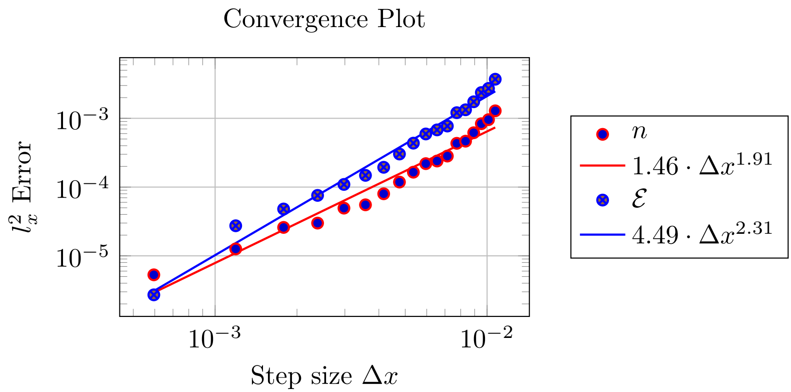


Figure 9.4: $T = 0.01$ with $\tau = \frac{1}{5000}$, $x \in \{\frac{N}{100} : 0 \leq N \leq 1\}$.

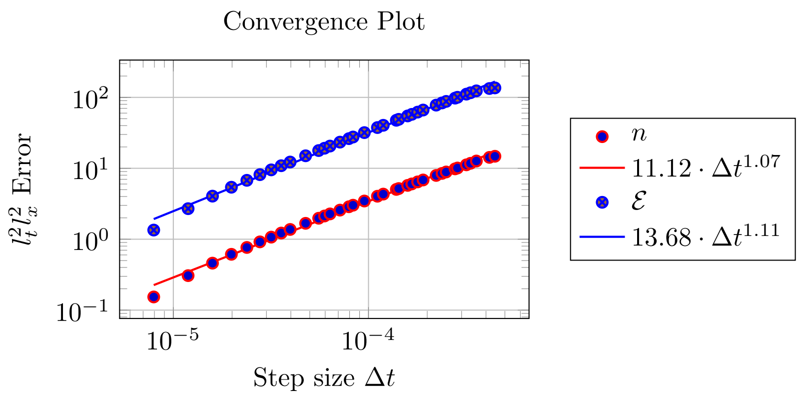


Figure 9.5: $(t, x) \in \{(\frac{M}{5040}, \frac{N}{100}) : 0 \leq M \leq 50, 0 \leq N \leq 100\}$.

Part IV

Analysis of the models

Chapter 10

Semiconductor Boltzmann-type equations

Let $V_{\text{ext}} : \mathbb{R}^d \to \mathbb{R}$ be an analytic potential. Moreover, we consider dispersive relation $\epsilon : \mathbb{T}^d \to \mathbb{R}$ defined by $p = (p_1, \ldots, p_d) \mapsto -2J \sum_i \cos(p_i)$ and the velocity $v(p) = \nabla_p \epsilon(p)$. The semiconductor Boltzmann equation for ultracold atoms in an optical lattice and in an the external potential V_{ext} is given by

$$\begin{cases} \partial_t f + v(p) \cdot \nabla_x f - \nabla_x (U n_f + V_{\text{ext}}(x)) \cdot \nabla_p f \\ \qquad\qquad = -\gamma n(1 - \eta n)(f - \mathcal{F}^0(n_f, E_f, p)), \\ \qquad f\big|_{t=0} = f_0, \end{cases} \tag{10.1}$$

where $n_f(x,t) = \int_{\mathbb{T}^d} f(x,p,t)dp$, $E_f(x,p,t) = \int_{\mathbb{T}^d} \epsilon(p) f(x,p,t)dp$ and $d \in \mathbb{N}$, $\gamma \geq 0$, $\eta \geq 0$, $U \neq 0$. Here, $\mathcal{F}^0$ denotes the generalized Fermi-Dirac distribution given by

$$\mathcal{F}^0(n,E,p) := \frac{1}{\eta + e^{-\lambda_0(n,E) - \lambda_1(n,E)\epsilon(p)}},$$

where λ_0 and λ_1 are functions of n, E being implicitly defined by

$$\int_{\mathbb{T}^d} \begin{pmatrix} 1 \\ \epsilon(p) \end{pmatrix} \frac{dp}{\eta + e^{-\lambda_0(n,E) - \lambda_1(n,E)\epsilon(p)}} = \begin{pmatrix} n \\ E \end{pmatrix}.$$

For more detail on the generalized Fermi-Dirac distribution see chapter 5. A similar equation in one dimension without collisions, i.e. $\gamma = 0$, and $v(p) = p$ for $p \in \mathbb{R}$ was analytically solved for short times in [25]. Jabin and Nouri [25] used analytic norms motivated by [40]. However, it is not clear to the author why theses norms consisting of infinite series converge. In this chapter we establish a solution of 10.1 for adequate initial data with a similar technique making use of the norms defined in chapter 4. We will do this in two steps: first we replace the collision operator by its high-temperature approximation. Second, we solve the original Problem 10.1. In a third section, we will exchange the collision operator by a linear relaxation time approximation and prove the global existence of a classical solution for sufficiently small and regular initial data.

10.1 High temperature expansion

In the first order high temperature expansion of the semiconductor Boltzmann equation for ultracold fermions in an optical lattice, we replace $\mathcal{F}^0(n_f, E_f, p)$ in Equation (10.1) by

$$\mathcal{F}_1^{\mathrm{hT}}(n_f, E_f, p) = n_f + E_f \frac{\epsilon(p)}{2J^2 d}.$$

Moreover, we may add an additional source term $G \in C^{\infty,0}((\mathbb{R}^d \times \mathbb{T}^d) \times (0,T)) \to \mathbb{R}$ and consider the following equation

$$\begin{aligned} &\partial_t f + v(p) \cdot \nabla_x f - \nabla_x (n_f + V_{\mathrm{ext}}(x,t)) \cdot \nabla_p f \\ &\qquad = -\gamma n_f (1 - \eta n_f)(f - \mathcal{F}_1^{\mathrm{hT}}(n_f, E_f, p)) + G(x,p,t) \end{aligned} \tag{10.2}$$

with $f(x,p,0) = f_0(x,p)$. Our strategy to solve (10.2) lies on an iterative explicit scheme: Given f_j and $n_j = \int f_j dp$, $E_j = \int_\epsilon f_j dp$, we define f_{j+1} as the solution of

$$\partial_t f_{j+1} + v \cdot \nabla_x f_j - \nabla_x (U n_j + V_{\mathrm{ext}}) \cdot \nabla_p f_j = -\gamma n_j (1 - \eta n_j)(f_j - \mathcal{F}_1^{\mathrm{hT}}(n_j, E_j)) + G, \tag{10.3}$$

with $f_{j+1}(x,p,0) = f_0(x,p)$. In order to proof that f_j converges, we need to work with the analytic norms from chapter 4:

Definition 10.1.1. For λ_0, T and $\mu \in [0, \lambda_0/T)$, we define $\lambda_t := \lambda_0 - \mu t$ and the norm

$$\|f\|_{\lambda_0,\mu,T,x} := \sup_{0 \le t < T} \|f(\cdot,t)\|_{C_x^{\lambda_t}} + \mu \int_0^T \|Df(\cdot,t)\|_{C_x^{\lambda_t}} \, dt$$

for $f \in C^{\infty,0}((\mathbb{R}^d \times \mathbb{T}^d) \times [0,T))$. We put $\|f\|_{\lambda_0,\mu,T} = \sup_x \|f\|_{\lambda_0,\mu,T,x}$.

Lemma 10.1.2. *Given $f_0 \in C^\infty(\mathbb{R}^d \times \mathbb{T}^d)$, $f_j, G \in C^{\infty,0}((\mathbb{R}^d \times \mathbb{T}^d) \times [0,T))$ for some $j \in \mathbb{N}$, we assume that there exist $\lambda_0, T > 0$ and $\mu \in (0, \lambda/T)$. Then,* *it holds*

$$\begin{aligned}&\frac{1}{2}\left\|f_{j+1}\right\|_{\lambda_0,\mu,T,x}\\&\quad\le\left\|f_0\right\|_{C_x^{\lambda_0}}+\frac{1}{\mu}\left(\|v\|_{C_x^{\lambda_0}}+\|\nabla V_{\mathrm{ext}}\|_{C_x^{\lambda_0}}+|U|\left\|f_j\right\|_{\lambda_0,\mu,T,x}\right)\left\|f_j\right\|_{\lambda_0,\mu,T,x}\\&+2\gamma T\left\|f_j\right\|^2_{\lambda_0,\mu,T,x}\left(1+\eta\left\|f_j\right\|_{\lambda_0,\mu,T,x}\right)\left(1+\frac{1}{2J}\|\epsilon\|_{C_x^{\lambda_0}}\right)+T\sup_{t<T}\|G\|_{C_x^{\lambda_t}}\end{aligned}$$

for f_{j+1} being defined via the iterative scheme and $x\in\mathbb{R}^d$.

Proof. Due to the iterative scheme, f_{j+1} is defined as $f_{j+1}\big|_{t=0}=f_0$ and

$$\begin{aligned}\partial_t f_{j+1}&=-v\cdot\nabla_x f_j\\&\qquad+\nabla_x(Un_j+V_{\mathrm{ext}})\cdot\nabla_p f_j-\gamma n_j(1-\eta n_j)(f_j-\mathcal{F}^{\mathrm{hT}}(n_j,E_j))+G.\end{aligned}$$

For $x\in\mathbb{R}^d$, we have

$$\begin{aligned}\left\|\partial_t f_{j+1}\right\|_{C_x^{\lambda_t}}&\le\left\|v\cdot\nabla_x f_j\right\|_{C_x^{\lambda_t}}+\left\|\nabla V_{\mathrm{ext}}\cdot\nabla_p f_j\right\|_{C_x^{\lambda_t}}+|U|\left\|\nabla_x n_j\cdot\nabla_p f_j\right\|_{C_x^{\lambda_t}}\\&\quad+\gamma\left\|n_j(1-\eta n_j)(f_j-\mathcal{F}^{\mathrm{hT}}(n_j,E_j))\right\|_{C_x^{\lambda_t}}+\sup_t\|G\|_{C_x^{\lambda_t}}\\&=:I+II+III+IV+V.\end{aligned}$$

We can estimate I, II and III using the submultiplicativity obtained by Remark 4.4.6 and see

$$\begin{aligned}I+II+III&\le\left(\|v\|_{C_x^{\lambda_t}}+\|\nabla V_{\mathrm{ext}}\|_{C_x^{\lambda_t}}+|U|\left\|n_j\right\|_{C_x^{\lambda_t}}\right)\left\|Df_j\right\|_{C_x^{\lambda_t}}\\&\quad+|U|\left\|Dn_j\right\|_{C_x^{\lambda_t}}\left\|f_j\right\|_{C_x^{\lambda_t}}\\&\le\left(\|v\|_{C_x^{\lambda_t}}+\|\nabla V_{\mathrm{ext}}\|_{C_x^{\lambda_t}}+2\left|U\right|\left\|f_j\right\|_{C_x^{\lambda_t}}\right)\left\|Df_j\right\|_{C_x^{\lambda_t}}\end{aligned}$$

since $\|n_j\|_{C_x^{\lambda_t}}\le\|f_j\|_{C_x^{\lambda_t}}$ and $\|Dn_j\|_{C_x^{\lambda_t}}\le\|Df_j\|_{C_x^{\lambda_t}}$. Likewise,

$$\begin{aligned}IV&\le\left\|n_j\right\|_{C_x^{\lambda_t}}\left(1+\eta\left\|n_j\right\|_{C_x^{\lambda_t}}\right)\left(\left\|f_j\right\|_{C_x^{\lambda_t}}+\left\|n_j\right\|_{C_x^{\lambda_t}}+\frac{1}{2J^2d}\left\|E_j\right\|_{C_x^{\lambda_t}}\|\epsilon\|_{C_x^{\lambda_t}}\right)\\&\le2\left\|f_j\right\|^2_{C_x^{\lambda_t}}\left(1+\eta\left\|f_j\right\|_{C_x^{\lambda_t}}\right)\left(1+\frac{1}{2J}\|\epsilon\|_{C_x^{\lambda_t}}\right).\end{aligned}$$

We conclude that

$$\begin{aligned}
&\int_0^T \|\partial_t f_{j+1}\|_{C_x^{\lambda_\tau}}\, d\tau \\
&\le \frac{1}{\mu}\left(\sup_{0\le\tau<T}\left(\|v\|_{C_x^{\lambda_\tau}} + \|\nabla V_{\mathrm{ext}}\|_{C_x^{\lambda_\tau}}\right) + |U|\,\|f_j\|_{\lambda_0,\mu,T,x}\right)\|f_j\|_{\lambda_0,\mu,T,x} \\
&\quad + 2\gamma T\,\|f_j\|^2_{\lambda_0,\mu,T,x}\left(1+\eta\,\|f_j\|_{\lambda_0,\mu,T,x}\right)\left(1+\frac{1}{2J}\sup_{0\le\tau<T}\|\epsilon\|_{C_x^{\lambda_\tau}}\right) \\
&\qquad + \sup_t \|G\|_{C_x^{\lambda_t}}
\end{aligned}$$

and in particular $t \mapsto \|\partial_t f_{j+1}\|_{C_x^{\lambda_t}} \in L^1((0,T))$. Finally, Lemma 4.4.12 entails

$$\begin{aligned}
&\sup_{0\le t<T}\|f_{j+1}\|_{C_x^{\lambda_t}} + \mu\int_0^T \|Df_{j+1}\|_{C_x^{\lambda_t}}\, ds \\
&\qquad\le 2\,\|f_{j+1}(0)\|_{C_x^{\lambda_0}} + 2\int_0^T \|\partial_t f_{j+1}\|_{C_x^{\lambda_t}}\, dt
\end{aligned}$$

and thus, the assertion is proved. □

Lemma 10.1.3. *For $x\in\mathbb{R}^d$ let $T,\lambda_0>0$. Then for every $\mu\in(0,\lambda_0/T)$*

$$\|f_0\|_{\lambda_0,\mu,T,x} = \|f_0\|_{C_x^{\lambda_0}} + \mu\int_0^T \|Df_0\|_{C_x^{\lambda_t}}\, dt \le 2\,\|f_0\|_{C_x^{\lambda_0}},$$

where $\lambda_t = \lambda_0 - \mu T$.

Proof. The assertion is a direct consequence of Lemma 4.4.12 since

$$\|f_0\|_{C_x^{\lambda_T}} + \mu\int_0^T \|Df_0\|_{C_x^{\lambda_t}}\, dt \le \|f_0\|_{C_x^{\lambda_0}}.$$ □

Proposition 10.1.4. *Let $x\in\mathbb{R}^d$, $\lambda_0>0$ such that $\|f_0\|_{C_x^{\lambda_0}} < \frac{R}{2}$ for some $R>0$. Moreover, let V be a neighborhood of x and*

$$\mu := \frac{2\left(\|v\|_{C_x^{\lambda_0}} + \|\nabla V_{\mathrm{ext}}\|_{C_x^{\lambda_0}} + |U|\,R\right)R + 4\lambda_0\gamma R^2\left(1+\eta R\right)\left(1+\frac{1}{2J}\|\epsilon\|_{C_x^{\lambda_0}}\right)}{R - 2\,\|f_0\|_{C_x^{\lambda_0}}}$$

assume that there exists a $\mu>\tilde\mu$ and a $T'\in(0,\lambda_0/\mu)$ such that $G: V\times\mathbb{T}^d\times[0,T')\to\mathbb{R}$ is analytic in its first two variables and continuous in $t\in[0,T')$ with $\sup_{0\le t\le T'}\|G\|_{C_x^{\lambda_0-\mu t}}<\infty$. Then there exists a time $T\in(0,T']$ such that

$$\|f_j\|_{\lambda_0,\mu,T,x} \le R \quad \text{for all } j\in\mathbb{N}. \tag{10.4}$$

A sufficient condition for the smallness of $T \in (0, T')$ is given by

$$T \leq \frac{(\mu - \tilde{\mu}) \left(R - 2 \|f_0\|_{C_x^{\lambda_0}} \right)}{2\mu \sup_{0 \leq t < \frac{\lambda_0}{\mu_0}} \|G\|_{C_x^{\lambda_0 - \mu t}}}.$$

Proof. By the hypothesis, we have

$$\begin{aligned} 2 \left(\|v\|_{C_x^{\lambda_0}} + \|\nabla V_{\text{ext}}\|_{C_x^{\lambda_0}} + |U| R \right) R + 4\lambda_0 \gamma R^2 (1 + \eta R) \left(1 + \frac{1}{2J} \|\epsilon\|_{C_x^{\lambda_0}} \right) \\ = \alpha\mu \left(R - 2 \|f_0\|_{C_x^{\lambda_0}} \right) \quad (10.5) \end{aligned}$$

for some $\alpha \in (0, 1)$. Now let $T \in (0, \lambda_0/\mu)$ satisfy

$$2T \sup_{0 \leq t < T} \|G\|_{C_x^{\lambda_0 - \mu t}} \leq (1 - \alpha) \left(R - 2 \|f_0\|_{C_x^{\lambda_0}} \right). \quad (10.6)$$

According to Lemma 10.1.2, we have

$$\begin{aligned} \|f_{j+1}\|_{\lambda_0, \mu, T, x} \leq 2 \|f_0\|_{C_x^{\lambda_0}} + \frac{2}{\mu} \left(\|v\|_{C_x^{\lambda_0}} + \|\nabla V_{\text{ext}}\|_{C_x^{\lambda_0}} + |U| R \right) R \\ + 4 \frac{\lambda_0}{\mu} \gamma R^2 (1 + \eta R) \left(1 + \frac{1}{2J} \|\epsilon\|_{C_x^{\lambda_0}} \right) + 2T \sup_{0 \leq t < \frac{\lambda_0}{\mu}} \|G\|_{C_x^{\lambda_0 - \mu t}} \end{aligned}$$

Thus, using (10.5) and (10.6),

$$\begin{aligned} \|f_{j+1}\|_{\lambda_0, \mu, T, x} &\leq 2 \|f_0\|_{C_x^{\lambda_0}} + \alpha \left(R - 2 \|f_0\|_{C_x^{\lambda_0}} \right) + (1 - \alpha) \left(R - 2 \|f_0\|_{C_x^{\lambda_0}} \right) \\ &= R. \end{aligned}$$

This finishes the proof. □

In order to use the Banach fixed point theorem, we need the following estimate on $f_{j+1} - f_j$.

Proposition 10.1.5. *Let $\lambda_0, \mu, R > 0$ and $T \in (0, \lambda_0/\mu)$ and define*

$$\begin{aligned} C_{\lambda_0, \mu, R} := \frac{2}{\mu} \Bigg(\|v\|_{C_x^{\lambda_0}} + \|\nabla V_{\text{ext}}\|_{C_x^{\lambda_0}} \\ + 4 |U| R \Bigg) + 2 \frac{\lambda_0}{\mu} \gamma R (2 + 3\eta R) \left(2 + \frac{1}{J} \|\epsilon\|_{C_x^{\lambda_0}} \right). \end{aligned}$$

If $\|f_j\|_{\lambda_0, \mu, T, x}, \|f_{j-1}\|_{\lambda_0, \mu, T, x} \leq R$, then

$$\|f_{j+1} - f_j\|_{\lambda_0, \mu, T, x} \leq C_{\lambda_0, \mu, R} \|f_j - f_{j-1}\|_{\lambda_0, \mu, T, x}.$$

Proof. The difference $g_{j+1} := f_{j+1} - f_j$ is given by

$$g_{j+1}(t) = \int_0^t \big(-v\cdot\nabla_x g_j + \nabla_x(Un_j + V_{\text{ext}})\cdot\nabla_p g_j - U\nabla_x(n_j - n_{j-1})\cdot\nabla_p f_{j-1} + Q^{\text{hT},1}_{\text{BGK}}(f_j) - Q^{\text{hT},1}_{\text{BGK}}(f_{j-1})\big) ds$$

where

$$Q^{\text{hT},1}_{\text{BGK}}(f_j) := -\gamma n_{f_j}(1-\eta n_{f_j})\left(f_j - n_{f_j} + E_{f_j}\frac{\epsilon}{2J^2 d}\right).$$

Since $Q^{\text{hT},1}_{\text{BGK}}(f_j)$ is cubic in f_j, we use the submultiplicativity of the norm $\|\cdot\|_{C^\lambda_x}$ to ensure that

$$\left\|Q^{\text{hT},1}_{\text{BGK}}(f_j) - Q^{\text{hT},1}_{\text{BGK}}(f_{j-1})\right\|_{C^{\lambda_t}_x} \le \gamma R(2+3\eta R)\left(2 + \frac{1}{J}\|\epsilon\|_{C^{\lambda_t}_x}\right)\|g_j\|_{C^\lambda_x}$$

since $\|f_{j-1}\|_{C^{\lambda_t}_x}, \|f_j\|_{C^{\lambda_t}_x} \le R$. We derive similarly to the proof of Lemma 10.1.2 that

$$\begin{aligned}
&\|\partial_t g_{j+1}\|_{C^{\lambda_t}_x} \\
&\le \left(\|v\|_{C^{\lambda_0}_x} + \|\nabla V_{\text{ext}}\|_{C^{\lambda_0}_x} + |U|\,\|f_j\|_{C^{\lambda_t}_x} + |U|\,\|f_{j-1}\|_{C^{\lambda_t}_x}\right)\|Dg_j\|_{C^{\lambda_t}_x} \\
&\quad + \Bigg(|U|\,\|Df_j\|_{C^{\lambda_t}_x} + |U|\,\|Df_{j-1}\|_{C^{\lambda_t}_x} \\
&\qquad + \gamma R^2(2+3\eta R)\left(2 + \frac{1}{J}\|\epsilon\|_{C^{\lambda_t}_x}\right)\Bigg)\|g_j\|_{C^\lambda_x}.
\end{aligned}$$

Using the hypothesis $\|f_j\|_{\lambda_0,\mu,T} \le R$ entails

$$\begin{aligned}
\int_0^T \|\partial_t g_{j+1}\|_{C^{\lambda_t}_x} dt \le{}& \frac{1}{\mu}\left(\|v\|_{C^{\lambda_0}_x} + \|\nabla V_{\text{ext}}\|_{C^{\lambda_t}_x} + 4|U|R\right)\|g_j\|_{\lambda_0,\mu,T} \\
&+ \frac{\lambda_0}{\mu}\gamma R(2+3\eta R)\left(2 + \frac{1}{J}\|\epsilon\|_{C^{\lambda_t}_x}\right)\|g_j\|_{\lambda_0,\mu,T}
\end{aligned}$$

using $T \le \frac{\lambda_0}{\mu}$. Hence, we conclude the assertion with the aid of Lemma 4.4.12 by

$$\|g_{j+1}\|_{\lambda_0,\mu,T,x} \le 2\int_0^T \|\partial_t g_{j+1}\|_{C^{\lambda_t}_x} dt < C_{\lambda_0,\mu,R}\|g_j\|_{\lambda_0,\mu,T}. \qquad \square$$

Theorem 10.1.6. *For $\lambda_0 > 0$, let $\|f_0\|_{C^{\lambda_0}(\mathbb{R}^d)} < \infty$. We choose any $R \ge 4\|f_0\|_{C^{\lambda_0}(\mathbb{R}^d)}$ and*

$$\mu > \bar{\mu} = 2\left(\|v\|_{C^{\lambda_0}} + \|\nabla V_{\text{ext}}\|_{C^{\lambda_0}(\mathbb{R}^d)} + 4|U|R\right) + 2\lambda_0\gamma R(2+3\eta R)\left(2 + \frac{1}{J}\|\epsilon\|_{C^{\lambda_0}}\right).$$

and a $T' \in (0, \lambda_0/\mu)$ such that $G : V \times \mathbb{T}^d \times [0, T') \to \mathbb{R}$ is analytic in its first two variables and continuous in $t \in [0, T')$ with $\sup_{0 \le t \le T'} \|G\|_{C^{\lambda_0 - \mu t}(\mathbb{R}^d)} < \infty$. Then there exists an analytic solution f of (10.2) on $\mathbb{R}^d \times \mathbb{T}^d \times [0, T)$ for $T \in (0, T']$ satisfying

$$T \le \frac{(\mu - \bar{\mu})R}{4\mu \sup_{0 \le t < T} \|G\|_{C^{\lambda_0 - \mu t}(\mathbb{R}^d)}}.$$

The solution is unique in the space of all $g \in C^{\infty,0}((\mathbb{R}^d \times \mathbb{T}^d) \times [0,T)])$ such that $\|g\|_{\lambda_0,\mu,T} \le R$. Moreover, f fulfills

$$\|f(\cdot, t)\|_{C_x^{\lambda_0 - \mu t}} \le 4 \|f_0\|_{C_x^{\lambda_0}} \quad \text{for } x \in \mathbb{R}^d,\ t \le \frac{(\mu - \bar{\mu}) \|f_0\|_{C_x^{\lambda_0}}}{\mu \sup_{0 \le t < T} \|G\|_{C_x^{\lambda_0 - \mu t}}}.$$

Corollary 10.1.7. *Let $\nu > 0$. Assume that*

$$|V_{\text{ext}}|_{C^\nu(\mathbb{R}^d)} + |f_0|_{C^\nu(\mathbb{R}^d)} < \infty.$$

Then there exists a $T > 0$ such that (10.2) admits an analytic solution $f : \mathbb{R}^d \times \mathbb{T}^d \times [0, T)$.

Proof of Theorem 10.1.6. Let X denote the Banach space consisting of all functions $g \in C^{\infty,0}((\mathbb{R}^d \times \mathbb{T}^d) \times [0,T)])$ such at that $\|g\|_{\lambda_0,\mu,T} < \infty$. We use the Banach scheme $(f_j)_j$ defined above. Since $R \ge 4 \|f_0\|_{C^{\lambda_0}(\mathbb{R}^d)}$, we have $R - 2\|f_0\|_{C^{\lambda_0}} \ge R/2$. Thus, $\bar{\mu} \ge \tilde{\mu}$ for

$$\tilde{\mu} := \frac{2\left(\|v\|_{C^{\lambda_0}} + \|\nabla V_{\text{ext}}\|_{C^{\lambda_0}(\mathbb{R}^d)} + |U|\, R\right) R}{R - 2\|f_0\|_{C^{\lambda_0}(\mathbb{R}^d)}} + \frac{4\lambda_0 \gamma R^2 (1 + \eta R) \left(1 + \frac{1}{2J} \|\epsilon\|_{C^{\lambda_0}}\right)}{R - 2\|f_0\|_{C^{\lambda_0}(\mathbb{R}^d)}}.$$

Proposition 10.1.4 guarantees that $f_j \in B_R := \{g \in X : \|g\|_{\lambda_0,\mu,T} \le R\}$. Moreover, according to Proposition 10.1.5, we have $\|f_{j+1} - f_j\|_{\lambda_0,\mu,T} \le C_{\lambda_0,\mu,R} \|f_j - f_{j-1}\|_{\lambda_0,\mu,T}$ with $C_{\lambda_0,\mu,R} < 1$. Thus, the series f_j converges to the (unique) fixed point of the mapping $\Phi : B_R \to B_R$ with

$$f \mapsto f_0 - \int_0^t \big(v \cdot \nabla_x f - \nabla_x (U n_f + V_{\text{ext}}) \cdot \nabla_p f + \gamma n (1 - \eta n)(f - \mathcal{F}^{\text{hT}}(n_f, E_f, \cdot)) + G\big) dt,$$

where $(n_f, E_f) := \int_{\mathbb{T}^d} (1, \epsilon(p)) f(\cdot, p, \cdot) dp$. For more details to the proof of uniqueness, see Proposition 10.1.8. Note that the fixed point f belongs to

$C^{\infty,0}((\mathbb{R}^d \times \mathbb{T}^d) \times [0,T)])$ by the definition of X. Making use of the analyticity of f_0 and the fixed point equation, a bootstrap argument shows that f is smooth (also in the time variable) and hence, f is a classical solution of (10.2). The remaining part of the assertion is finally a direct consequence of Proposition 10.1.4 if we set $R = R_x = 4\,\|f_0\|_{C_x^{\lambda_0}}$ and use Proposition 10.1.4 for fixed $x \in \mathbb{R}^d$. □

Proof of Corollary 10.1.7. By Lemma 4.4.7, we infer that there exists a $\lambda_0 \in (0,\nu)$ such that $\|\nabla V_{\text{ext}}\|_{C^{\lambda_0}(\mathbb{R}^d)}$ and $\|f_0\|_{C^{\lambda_0}(\mathbb{R}^d)}$ are finite. We thus can apply Theorem 10.1.6. □

Proposition 10.1.8. *Let f_0, λ_0, R, V_{ext}, $\bar{\mu}$ and $\mu > \bar{\mu}$ be as in the previous theorem. Moreover, let $G^i \in C^{\infty,0}((\mathbb{R}^d \times \mathbb{T}^d) \times (0, \lambda_0/\mu))$ for $= 1,2$ such that*

$$\sup_{0 \le t < \frac{\lambda_0}{\mu}} \left\|G^i\right\|_{C^{\lambda_0 - \mu t}(\mathbb{R}^d)} < \infty$$

for $i = 1,2$. Let f^i denote the solution of (10.2) for $G = G^i$, respectively. Then there exists a constant $C > 0$ such that

$$\left\|f^2 - f^1\right\|_{\lambda_0,\mu,T,x} \le CT \sup_{0 \le t < T} \left\|G^2 - G^1\right\|_{C_x^{\lambda_0 - \mu t}}$$

for $x \in \mathbb{R}^d$ if $T \in (0, \frac{\lambda_0}{\mu}]$ satisfies

$$T \le \min_{i=1,2} \frac{(\mu - \bar{\mu})\,\|f_0\|_{C_x^{\lambda_0}}}{\mu \sup_{0 \le t < T} \|G_i\|_{C_x^{\lambda_0 - \mu t}}}.$$

Proof. Let $g := f^2 - f^1$. Proceeding similarly to the proof of Proposition 10.1.5, we can find a constant $C' < \frac{1}{2}$ such that

$$\int_0^T \|\partial_t g\|_{C_x^{\lambda_t}}\, dt \le C\,\|g\|_{\lambda_0,\mu,T,x} + T \sup_{0 \le t < T} \left\|G^2 - G^1\right\|_{C_x^{\lambda_t}}.$$

Thus, the definition of $\|g\|_{\lambda_0,\mu,T,x}$ directly entails that

$$\|g\|_{\lambda_0,\mu,T,x} \le \frac{T}{1 - 2C} \sup_{0 \le t < T} \left\|G^2 - G^1\right\|_{C_x^{\lambda_t}}.$$ □

10.2 BGK-type collision operator

In this section we solve Equation (10.1) without replacing the Fermi-Dirac distribution function by a high temperature expansion. The basic idea of the proof is to use the results of the high temperature expansion in combination with the Banach fixed point theorem. However, in this case controlling $\mathcal{F}^0$ in the norm $\|\cdot\|_{C^\lambda(\mathbb{R}^d)}$ is more cumbersome since this norm involves every derivative of $\mathcal{F}^0$. Fortunately, Chapter 5 provides us with the required estimates for the Fermi-Dirac distribution function.

We define the mapping

$$\Phi : (n_g, E_g) \mapsto (n_f, E_f) \tag{10.7}$$

with

$$n_f = \int_{\mathbb{T}^d} f dp \quad \text{as well as} \quad E_f = \int_{\mathbb{T}^d} \epsilon f dp,$$

where f is the solution of (5.27) for

$$G = \gamma n_g(1 - \eta n_g)(\mathcal{F}^0(n_g, E_g, \cdot) - \mathcal{F}^{\mathrm{hT}}(n_g, E_g, \cdot)).$$

Clearly a fixed point of this mapping provides an f solving the desired equation. However, we still have to concretize the domain and range of Φ. On the one hand, this is crucial for the well-definedness of Φ due to the lack of analyticity of $\mathcal{F}^0$ in 0. On the other hand, we want Φ to be a contraction in order to apply the Banach fixed point theorem.

Definition 10.2.1. Let λ_0, μ and $T \leq \frac{\lambda_0}{\mu}$ be positive. Let $\mathcal{Y}$ be the set of all $(n, E) : \mathbb{R}^d \times (0, T) \to \mathbb{R}^2$ such that there exists a $f : \mathbb{R}^d \times \mathbb{T}^d \times (0, T) \to [0, \eta^{-1}]$ being smooth in $x \in \mathbb{R}^d$ with

$$(n, E) = \int_{\mathbb{T}^d} (1, \epsilon(p)) f(\cdot, p, \cdot) dp.$$

Moreover, let

$$d_{\lambda_0,\mu,T}((n_0, E_0), (n_1, E_1)) := \sup_{0<t<T} \|(n_0, E_0) - (n_1, E_1)\|_{C^{\lambda_0 - \mu t}(\mathbb{R}^d)} .$$

Finally, we can define the metric space

$$Y_{\lambda_0,\mu,T} := \overline{\{(n, E) \in \mathcal{Y} : d_{\lambda_0,\mu,T}((n, E), 0) < \infty\}}^{d_{\lambda_0,\mu,T}}$$

with metric $d_{\lambda_0,\mu,T}$.

Definition 10.2.2. Let $\eta \geq 0$ and $a \geq 1$. Choose $A_a, B_a > 0$ such that for all $(n, E) \in (0, \eta^{-1}) \times \mathbb{R}$ satisfying

$$\eta E \in \mathcal{E}_a(\eta n) = \left\{ \int_{\mathbb{T}^d} \frac{\epsilon(p) dp}{1 + e^{-\lambda_0 - \lambda_1 \epsilon(p)}} : |\lambda_1| \leq \log a \right.$$
$$\left. \text{and } \int_{\mathbb{T}^d} \frac{dp}{1 + e^{-\lambda_0 - \lambda_1 \epsilon(p)}} = \eta n \right\}$$

it holds

$$\left|D^i_{(n,E)} D^j_p \mathcal{F}^0_\eta(n,E,p)\right| \leq i!j!A^j_a \left(\frac{B_a}{n(1-\eta n)}\right)^i \mathcal{F}^0_\eta(n,E,p)(1-\eta\mathcal{F}^0_\eta(n,E,p)).$$

These numbers $A_a, B_a \in \mathbb{R}_{>0}$ exist and can be chosen independently from η according to Corollary 5.4.15. Moreover, for $a = 1$, Corollary 5.4.17 states that we can take

$$A_1 = \frac{3}{\log\left(1 + \frac{1}{24J}\right)} \quad \text{and} \quad B_1 = 2400(2J+1)^3$$

recalling that $2J = \|\epsilon\|_\infty$.

Definition 10.2.3. Let $a \geq 1$, $\lambda_0 \in (0, \frac{1}{2A_a})$, $\mu > 0$, $T \in (0, \frac{\lambda_0}{\mu})$. Moreover, let M_a be the set of functions $(n_g, E_g) \in Y_{\lambda_0,\mu,T}$ such that

$$|n_g(\cdot,t)|_{\dot{C}_x^{\lambda_0-\mu t}} + |E_g(\cdot,t)|_{\dot{C}_x^{\lambda_0-\mu t}} \leq \frac{1}{4B_a} n_g(x)(1-\eta n_g(x)) \tag{10.8}$$

and

$$\inf_{\eta E_0 \in \mathcal{E}_a(\eta n_g(x,t))} |E_g(x,t) - E_0| \leq \frac{1}{4B_a} n_g(x)(1-\eta n_g(x)). \tag{10.9}$$

for all $x \in \mathbb{R}^d$ and $t \in (0,T)$. Note that M_a is a set of functions to which we can apply Proposition 5.4.21 at each point (x,t). Finally, for $f_0 : \mathbb{R}^d \times \mathbb{T}^d \to (0, \eta^{-1})$ analytic, let $M(f_0, \lambda_0, \mu, T, a)$ be the set of all $(n_g, E_g) \in M_a$ such that

$$n_g(x,t)(1-\eta n_g(x,t)) \leq 2n_{f_0}(x)(1-\eta n_{f_0}(x)) \tag{10.10}$$

for all $x \in \mathbb{R}^d$.

Example 10.2.4. The set $M(f_0, \lambda_0, \mu, T, a)$ is not empty for appropriate $f_0 \in C^\infty(\mathbb{R}^d \times \mathbb{T}^d)$. For $\alpha > \eta$ and $\beta > 0$, let

$$f_0(x,p) := \frac{1}{\alpha + \beta|x|^2} \quad \text{for } x \in \mathbb{R}^d, p \in \mathbb{T}^d.$$

According to Example 4.2.6, it holds

$$\begin{aligned} |f_0|_{\dot{C}_x^{\lambda_0}} &\leq \sqrt{\frac{20\beta}{\alpha}}\lambda_0 |f_0(x,p)| = \sqrt{\frac{20\beta}{\alpha}}\lambda_0 n_{f_0}(x) \\ &\leq \frac{\sqrt{20\alpha\beta}}{\alpha-\eta}\lambda_0 n_{f_0}(x)(1-\eta n_{f_0}(x)) \end{aligned}$$

for $\lambda_0 \in (0, \sqrt{\frac{\alpha}{5\beta}})$ and for all $x \in \mathbb{R}^d, p \in \mathbb{T}^d$. Then $\big((x,p,t) \mapsto f_0(x,p)\big) \in M(f_0, \lambda_0, \mu, T, a)$ if $\sqrt{20\alpha\beta}\lambda_0 \leq \frac{\alpha-\eta}{4B_a}$.

Proposition 10.2.5. *Let $a \geq 1$ and $f_0 : \mathbb{R}^d \times \mathbb{T}^d \to (0, \eta^{-1})$ analytic such that*

$$\inf_{\eta E_0 \in \mathcal{E}_a(\eta\bar{n}(x))} |E_{f_0}(x) - E_0| \leq \frac{n_{f_0}(x)(1 - \eta n_{f_0}(x))}{8B_a}.$$

for $\eta\bar{n}(x) := \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{\eta}{2} n_{f_0}(x)(1 - \eta n_{f_0}(x))}$ and

$$\|f_0\|_{C_x^\nu} \leq C_0 n_{f_0}(x)(1 - \eta n_{f_0}(x))$$

for some $C_0 > 0$ and some $\nu \in (0, \frac{1}{2A_a})$. Thus, there exists a $\lambda_0 < \nu$ such that

$$\|f_0\|_{\dot{C}_x^{\lambda_0}} \leq \frac{n_{f_0}(x)(1 - \eta n_{f_0}(x))}{20B_a(1 + 2J)}.$$

For this λ_0, we define $\mu > 0$ as in Theorem 10.1.6. Then there exists a $T' > 0$ such that $\Phi : M(f_0, \lambda_0, \mu, T, a) \to M(f_0, \lambda_0, \mu, T, a)$ with $\Phi : (n_g, E_g) \mapsto (n_f, E_f)$ as in (10.7) is well-defined for all $0 < T \leq T'$.

Proof. Let $g \in M = M(f_0, \lambda_0, \mu, T, a)$ and $G = \gamma n_g(1 - \eta n_g)(\mathcal{F}^0(n_g, E_g, \cdot) - \mathcal{F}^{\mathrm{hT}}(n_g, E_g, \cdot))$. Similar to the proof of Lemma 10.1.2 (setting $f_{j+1} = f_j = f$), we have

$$\begin{aligned}
\|\partial_t f\|_{\dot{C}_x^{\lambda_0 - \mu t}} &\leq \left(\|v\|_{C_x^{\lambda_0 - \mu t}} + \|\nabla V_{\mathrm{ext}}\|_{C_x^{\lambda_t}} + 2\,|U|\, \|f\|_{C_x^{\lambda_0 - \mu t}} \right) \|Df\|_{C_x^{\lambda_0 - \mu t}} \\
&\quad + \gamma \left\| n_f(1 - \eta n_f)(f - \mathcal{F}^{\mathrm{hT}}(n_f, E_f)) \right\|_{\dot{C}_x^{\lambda_0 - \mu t}} + \|G\|_{\dot{C}_x^{\lambda_0 - \mu t}} \\
&\leq \mu \|Df\|_{C_x^{\lambda_0 - \mu t}} + \|G\|_{\dot{C}_x^{\lambda_0 - \mu t}}
\end{aligned}$$

since $\|f\|_{\dot{C}_x^{\lambda_0 - \mu t}} \leq \lambda_0 \|Df\|_{C_x^{\lambda_0 - \mu t}}$. Thus, we have

$$\begin{aligned}
\|f\|_{\dot{C}_x^{\lambda_0 - \mu t}} - \|f_0\|_{\dot{C}_x^{\lambda_0}} &\leq \int_0^t \partial_t \|f\|_{\dot{C}_x^{\lambda_0 - \mu s}} \, ds \\
&\leq \int_0^t \|\partial_t f\|_{\dot{C}_x^{\lambda_0 - \mu s}} \, ds - \mu \int_0^t \|Df\|_{C_x^{\lambda_0 - \mu s}} \, ds \\
&\leq t \sup_{0 \leq s < t} \|G\|_{\dot{C}_x^{\lambda_0 - \mu s}} . \qquad (10.11)
\end{aligned}$$

Combining this with Proposition 5.4.21 and Eq. (10.10), we obtain

$$\sup_{0 \leq t \leq T} \|f\|_{\dot{C}_x^{\lambda_0 - \mu t}} \leq \|f_0\|_{\dot{C}_x^{\lambda_0}} + C_1 T n_{f_0}(x)(1 - \eta n_{f_0}(x))$$

for some $C_1 > 0$. This directly implies

$$\begin{aligned}
|n_f|_{\dot{C}_x^{\lambda_0 - \mu t}} + |E_f|_{\dot{C}_x^{\lambda_0 - \mu t}} &\leq (1 + 2J) \|f_0\|_{\dot{C}_x^{\lambda_0}} + C_1 T n_{f_0}(x)(1 - \eta n_{f_0}(x)) \\
&\leq \frac{1}{6B_a} n_{f_0}(x)(1 - \eta n_{f_0}(x))
\end{aligned}$$

using the hypothesis on f_0 if $T > 0$ is sufficiently small. Note that E_f and E_{f_0} are close to each other since

$$\begin{aligned}\partial_t E_f &+ \int_{\mathbb{T}^d} v(p)\epsilon(p)\cdot\nabla_x f dp + U\nabla_x n_f \cdot \int_{\mathbb{T}^d} v(p) f dp \\ &= \partial_t E_f + \int_{\mathbb{T}^d} v(p)\epsilon(p)\cdot\nabla_x f dp - U\nabla_x n_f\cdot\int_{\mathbb{T}^d}\epsilon(p)\nabla_p f dp = 0\end{aligned}$$

yields that

$$|\partial_t E_f| \le \bar{C}\int_{\mathbb{T}^d} |\nabla_x f|\, dp$$

with $\bar{C} = 2J(|U|\,\eta^{-1} + 2J)$. Similar to (10.11), we see that

$$\begin{aligned}|E_f(x,t) - E_{f_0}(x)| &\le \bar{C}\int_0^t \|Df\|_{C_x^{\lambda_0-(1+\bar{C})\mu s}}\, ds \\ &\le \|f_0\|_{\dot{C}_x^{\lambda_0}} + t\sup_{0\le s\le t}\|G\|_{\dot{C}_x^{\lambda_0-(1+\bar{C})\mu s}} \\ &\le \frac{n_{f_0}(x)(1-\eta n_{f_0}(x))}{20B_a(1+2J)} + C_1 T n_{f_0}(x)(1-\eta n_{f_0}(x))\end{aligned}$$

We conclude

$$\begin{aligned}&\inf_{\eta E_0\in\mathcal{E}_a(\eta\bar{n}(x))} |E_f(x,t) - E_0| \\ &\qquad\le \inf_{\eta E_0\in\mathcal{E}_a(\eta\bar{n}(x))} |E_{f_0}(x) - E_0| + |E_f(x,t) - E_{f_0}(x)| \\ &\qquad\le \frac{1}{6B_a} n_{f_0}(x)(1-\eta n_{f_0}(x))\end{aligned}$$

for $\eta\bar{n}(x) := \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{\eta}{2} n_{f_0}(x)(1-\eta n_{f_0}(x))}$ if T is sufficiently small. Note that these estimates are already close to the asserted ones. However, we still need to "replace" f_0 and $\eta\bar{n}$ by the solution f and ηn_f in the estimate, respectively. At first, we show as above, that n_f and n_{f_0} are closely related, i.e. we can show that

$$\begin{aligned}&|(n_f(x,t)(1-\eta n_f(x,t))) - (n_{f_0}(x)(1-\eta n_{f_0}(x)))| \\ &\quad\le \|v\|_\infty \int_0^t\int_{\mathbb{T}^d} |\nabla f(x,p,s)|\, dpds \\ &\quad\le \|f_0\|_{\dot{C}_x^{\lambda_0}} + t\sup_{0\le s\le t}\|G\|_{\dot{C}_x^{\lambda_0-(1+2J)\mu s}}\end{aligned}$$

Therefore, if $T > 0$ is sufficiently small we have

$$\frac{1}{2} n_{f_0}(x)(1-\eta n_{f_0}(x)) \le n_f(x,t)(1-\eta n_f(x,t)) \le 2n_{f_0}(x)(1-\eta n_{f_0}(x))$$

for all $x \in \mathbb{R}^d$ and all $0 \le t \le T$. and

$$|n_f|_{\dot{C}_x^{\lambda_0-\mu t}} + |E_f|_{\dot{C}_x^{\lambda_0-\mu t}} \le \frac{1}{4B_a} n_f(x)(1-\eta n_f(x))$$

as well as

$$\inf_{\eta E_0 \in \mathcal{E}_a(\eta \bar{n}(x))} |E_f(x,t) - E_0| \le \frac{1}{4B_a} n_f(x)(1-\eta n_f(x))$$

In the next and final step, we need to replace $\mathcal{E}_a(\eta \bar{n}(x,t))$ by $\mathcal{E}_a(\eta n_f(x,t))$ in the estimate. Due to the symmetry $\mathcal{E}_a(\eta n) = \mathcal{E}_a(1-\eta n)$, we can assume w.l.o.g. that ηn_{f_0} and ηn_f are greater or equal to $\frac{1}{2}$. Therefore,

$$n_f(x,t)(1-\eta n_f(x,t)) \ge \frac{1}{2} n_{f_0}(x)(1-\eta n_{f_0}(x)) \ge \bar{n}(x)(1-\eta \bar{n}(x)),$$

which directly implies that $n_f \le \bar{n}$. Finally, Lemma 5.4.20 ensures

$$\inf_{\eta E_0 \in \mathcal{E}_a(\eta n_f(x,t))} |E_f(x,t) - E_0| \le \frac{1}{4B_a} n_f(x)(1-\eta n_f(x)),$$

implying the assertion. □

Proposition 10.2.6. *Assuming the hypothesis of Proposition 10.2.5, let* λ_0*,* μ*,* T' *and* a *be as in Proposition 10.2.5. Then there exists a* $T'' \in (0, T')$ *such that* $\phi : M(f_0, \lambda_0, \mu, T, a) \to M(f_0, \lambda_0, \mu, T, a)$ *is a contraction for all* $0 \le T \le T''$*.*

Proof. Let $(n_{g_0}, E_{g_0}), (n_{g_1}, E_{g_1}) \in M(f_0, \lambda_0, \mu, T, a)$. The idea of the proof is to use Proposition 5.4.22. Therefore, we would like to show that the set $M(f_0, \lambda_0, \mu, T, a)$ is a convex set and define $g_\theta := \theta g_1 + (1-\theta) g_0$ for $\theta \in (0,1)$. Unfortunately, we cannot prove that $M(f_0, \lambda_0, \mu, T, a)$ is convex due to the restriction in (10.10). In contrast to this, the condition (10.8) can be verified for g_θ by

$$\begin{aligned}
|n_{g_\theta}|_{\dot{C}_x^{\lambda_0-\mu t}} + |E_{g_\theta}|_{\dot{C}_x^{\lambda_0-\mu t}} &\le \theta \left(|n_{g_0}|_{\dot{C}_x^{\lambda_0-\mu t}} + |E_{g_0}|_{\dot{C}_x^{\lambda_0-\mu t}} \right) \\
&\quad + (1-\theta) \left(|n_{g_1}|_{\dot{C}_x^{\lambda_0-\mu t}} + |E_{g_1}|_{\dot{C}_x^{\lambda_0-\mu t}} \right) \\
&\le \theta \frac{1}{4B_a} n_{g_0}(x)(1-\eta n_{g_0}(x)) \\
&\quad + (1-\theta) \frac{1}{4B_a} n_{g_1}(x)(1-\eta n_{g_1}(x)) \\
&\le \frac{1}{4B_a} n_{g_\theta}(x)(1-\eta n_{g_\theta}(x))
\end{aligned}$$

since $x \mapsto x(1-x)$ is concave. However, in order to bypass the lack of convexity of $M(f_0, \lambda_0, \mu, T, a)$, we may split $M(f_0, \lambda_0, \mu, T, a)$ into its connected parts. Then we connect two points in the connected parts of $M(f_0, \lambda_0, \mu, T, a)$ by a curve consisting of two or three straight lines. For this we fix $x \in \mathbb{R}^d, t > 0$ and proceed with different cases:

At first we assume w.l.o.g. that

$$n_{g_1}(x,t)(1 - \eta n_{g_1}(x,t)) \leq n_{g_0}(x,t)(1 - \eta n_{g_0}(x,t)) \tag{10.12}$$

implying

$$n_{g_1}(x,t)(1 - \eta n_{g_1}(x,t)) \leq n_{g_\theta}(x,t)(1 - \eta n_{g_\theta}(x,t)).$$

Case 1: We suppose in addition that $E_{g_0} = E_{g_1}$. Then clearly

$$\begin{aligned} \inf_{\eta E_0 \in \mathcal{E}_a(\eta n_{g_\theta}(x,t))} |E_{g_\theta}(x,t) - E_0| &\leq \inf_{\eta E_0 \in \mathcal{E}_a(\eta n_{g_1}(x,t))} |E_{g_1}(x,t) - E_0| \\ &\leq \frac{1}{4B_a} n_{g_1}(x)(1 - \eta n_{g_1}(x)) \\ &\leq \frac{1}{4B_a} n_{g_\theta}(x)(1 - \eta n_{g_\theta}(x)). \end{aligned}$$

and hence $(n_{g_\theta}, E_{g_\theta}) \in M^{x,t}$.
Case 2: We assume that $n_{g_0} = n_{g_1} = n_{g_\theta}$ and $E_{g_0} \leq E_{g_1}$. Thus,

$$E_{g_0} - E_0 \leq E_{g_\theta} - E_0 \leq E_{g_1} - E_0 \quad \text{for all } E_0 \in \mathbb{R},$$

which entails

$$\begin{aligned} \inf_{\eta E_0 \in \mathcal{E}_a(\eta n_{g_\theta}(x,t))} |E_{g_\theta}(x,t) - E_0| &\leq \max_{i=0,1} \inf_{\eta E_0 \in \mathcal{E}_a(\eta n_{g_i}(x,t))} |E_{g_i}(x,t) - E_0| \\ &\leq \frac{1}{4B_a} n_{g_\theta}(x)(1 - \eta n_{g_\theta}(x)). \end{aligned}$$

Again, we have $(n_{g_\theta}, E_{g_\theta}) \in M_a$.

Finally, we can come back to the general case, where we only have the restriction from (10.12). We see that the curve

$$\gamma : [0,2] \to M^{x,t}, \ t \mapsto \begin{cases} (n_{g_0}, E_{g_t}), & t \in [0,1], \\ (n_{g_{t-1}}, E_{g_1}), & t \in [1,2]. \end{cases}$$

is well-defined.

Therefore, we have found a curve γ consisting of two straight lines parallel to the coordinate axis n and E that connect (n_{g_0}, E_{g_0}) with (n_{g_1}, E_{g_1}) in M_a.

This means that we can use Proposition 5.4.22: there exists a $C_0 > 0$ only depending on ε_0, ε_1, a and λ_0 such that

$$\left\| n_{g_0}(1-n_{g_0})\mathcal{F}^0(n_{g_0}, E_{g_0}) - n_{g_1}(1-n_{g_1})\mathcal{F}^0(n_{g_1}, E_{g_1}) \right\|_{C_x^{\lambda_0}} \leq C_0 \left(\|n_{g_0} - n_{g_1}\|_{C_x^{\lambda_0}} + \|E_{g_0} - E_{g_1}\|_{C_x^{\lambda_0}} \right).$$

Since $\mathcal{F}_1^{\mathrm{hT}}$ is linear in n and E, this directly entails

$$\|G(n_0, E_0) - G(n_1, E_1)\|_{C_x^{\lambda_0}} \leq C_1 \|(n_0, E_0) - (n_1, E_1)\|_{C_x^{\lambda_0}},$$

for some $C_1 > 0$ only depending on ε_0, ε_1, a and λ_0, where

$$G(n_g, E_g) = \gamma n_g(1-\eta n_g)(\mathcal{F}^0(n_g, E_g) - \mathcal{F}^{\mathrm{hT}}(n_g, E_g)).$$

Finally, the assertion is a consequence of this and Proposition 10.1.8, providing

$$\|\phi(n_{g_0}, E_{g_0}) - \phi(n_{g_1}, E_{g_1})\|_{\lambda_0,\mu,T} \leq C_3 T \sup_{0\leq t<T} \|G(n_{g_0}, E_{g_0}) - G(n_{g_1}, E_{g_1})\|_{C_x^{\lambda_t}}$$

for some $C_3 > 0$ and some small $T > 0$. □

Theorem 10.2.7. *Let $f_0 : \mathbb{R}^d \times \mathbb{T}^d \to (0, \eta^{-1})$ be analytic such that*

$$|E_{f_0}(x)| \leq \frac{n_{f_0}(x)(1-\eta n_{f_0}(x))}{19200(2J+1)^3} \quad \text{and} \quad |f_0|_{C_x^\nu} \leq C n_{f_0}(x)(1-\eta n_{f_0}(x)) \tag{10.13}$$

for some $C, \nu > 0$ and all $x \in \mathbb{R}^d$. Moreover, assume that $|V_{\mathrm{ext}}|_{C_x^\nu} < \infty$.

Then there exist a time $T > 0$ and an analytic $f : \mathbb{R}^d \times \mathbb{T}^d \times [0,T) \to [0, \eta^{-1}]$ being a classical solution of Eq. (10.1) on the time interval $[0,T)$.

Proof. The theorem can be proved by Banach's fixed point theorem. At first, we assume by shrinking ν that $\nu \in (0, \frac{1}{2A_1}) = (0, \frac{1}{2}\log(1+\frac{1}{24J}))$ and that

$$\|f_0\|_{C_x^\nu} \leq C_0 n_{f_0}(x)(1-\eta n_{f_0}(x)) \quad \text{and} \quad \|\nabla V_{\mathrm{ext}}\|_{C_x^\nu} < \infty$$

for some $C_0 > 0$ and all $x \in \mathbb{R}^d$ using Lemma 4.4.7. Recalling that $B_1 = 2400(2J+1)^3$, we choose $\lambda_0 < \nu$ sufficiently small and define

$$\mu \geq 2\left(\|v\|_{C_x^{\lambda_0}} + \|\nabla V_{\mathrm{ext}}\|_{C_x^{\lambda_0}} + 4|U|R\right) + 2\lambda_0\gamma R^2(2+3\eta R)\left(2 + \frac{1}{J}\|\epsilon\|_{C_x^{\lambda_0}}\right).$$

for some $R > 4\|f_0\|_{C^\nu(\mathbb{R}^d)}$ as in Proposition 10.2.6. Then the mapping

$$\phi : M(f_0, \lambda_0, \mu, T, 1) \to M(f_0, \lambda_0, \mu, T, 1),\ (n, E) \mapsto \int_{\mathbb{T}^d} (1, \epsilon(p)) f(\cdot, p, \cdot)\, dp,$$

where f is the solution of (5.27) for $G = \gamma n(1-\eta n)(\mathcal{F}^0(n,E) - \mathcal{F}^{\mathrm{hT}}(n,E))$, is a contraction for sufficiently small $T > 0$ according to Proposition 10.2.6. Hence, it can be continuously extended to the closure of $M(f_0, \lambda_0, \mu, T, 1)$ in $Y_{\lambda_0,\mu,T}$. The Banach fixed point theorem finally guarantees a unique fixed point of ϕ in $\overline{M(f_0, \lambda_0, \mu, T, 1)}$. Then Theorem 10.1.6 provides a unique solution of (10.1) in $\overline{M(f_0, \lambda_0, \mu, T, 1)}$. □

Remark 10.2.8. We can replace the first hypothesis in (10.13) from the previous theorem by a less restricting condition, which reads

$$\inf\left\{|E_{f_0}(x) - E_0| : \eta E_0 \in \mathcal{E}_a\left(\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{\eta}{2} n_{f_0}(x,t)(1-\eta n_{f_0}(x,t))}\right)\right\} \leq \frac{n_{f_0}(x)(1-\eta n_{f_0}(x))}{8B_a}.$$

Moreover, the densities (n_f, E_f) of the solution f found in the previous theorem are unique in the space $\overline{M(f_0, \lambda_0, \mu, T, a)}$ for $\lambda_0, \mu > 0$ as in the proof of Theorem 10.2.7. Hence, due to Theorem 10.1.6, the solution f itself is unique in the space of all smooth g with $\|g\|_{\lambda_0,\mu,T} \leq R$ fulfilling that $(n_g, E_g) \in \overline{M(f_0, \lambda_0, \mu, T, a)}$, where $\|g\|_{\lambda_0,\mu,T} \leq R$ is given by Definition 10.1.1.

10.3 Global solvability for linear relaxation time approximation

Let $\overline{n} \in [0,1]$ and $\tau_0 > 0$, we consider the equation

$$\begin{cases} \partial_t f + v(p) \cdot \nabla_x f - \nabla_x n_f \cdot \nabla_p f = -\dfrac{f - \overline{n}}{\tau_0} \\ \qquad f(x,p,0) = f_0(x,p) \end{cases} \tag{10.14}$$

for $x \in \mathbb{R}^d, p \in \mathbb{T}^d, t > 0$ with $n_f = \int f(\cdot, p, \cdot)dp$ and $f_0 \colon \mathbb{R}^d \times \mathbb{T}^d \to \mathbb{R}$ as well as $v(p) := \nabla_p \epsilon(p)$ with $\epsilon(p_1, \ldots, p_d) := -2J\sum_{i=1}^d \cos(p_i)$ and $p = (p_1, \ldots, p_d)$.

In order to find such a solution, we transform f into

$$g(x,p,t) := (f(x + tv(k), p, t) - \overline{n})\, e^{\frac{t}{\tau_0}}. \tag{10.15}$$

This entails

$$\begin{cases} \partial_t g(x,p,t) = \partial_x n(x + tv(k), t) \cdot \tilde{\partial}_{vt} g(x,p,t) \\ \qquad g\big|_{t=0} = g_0 := f_0 - \overline{n}, \end{cases} \tag{10.16}$$

where $n(x,t) = e^{-\frac{t}{\tau_0}} \int g(x - tv(p), p, t)dp = e^{-\frac{t}{\tau_0}} \int T_{-vt}g(x,p,t)dp$ and

$$\tilde{\partial}_{vt} := \tilde{\partial} := \partial_p - tv'(p)\partial_x \tag{10.17}$$

as well as

$$T_{\pm vt}f(x,k,t) := f(x \pm tv(k), k, t). \tag{10.18}$$

The notation $\tilde{\partial}_{vt}$ is motivated by the property

$$\begin{aligned} \tilde{\partial}_{vt}T_{vt}f(x,p,t) &= (\partial_p - tv'(p)\partial_x)\psi(x + tv(p), p) \\ &= (\partial_p\psi - t(v'(p) - v'(p))\partial_x\psi)(x + tv(p), p) = T_{vt}\partial_p\psi(x,p). \end{aligned}$$

Theorem 10.3.1. *Let $\alpha, \lambda_0 > 0$, $\overline{n} \in \mathbb{R}$ and $f_0 \in C^\infty(\mathbb{R}^d \times \mathbb{T}^d)$ satisfy*

$$R := \|f_0 - \overline{n}\|_\infty + 4\|f_0\|_{\dot{\mathcal{O}}_0^{\lambda_0}} \leq \frac{\alpha\lambda_0^2}{8(\lambda_0 + 1)}.$$

If $\frac{1}{\tau_0} \geq 12Je^{\lambda_0} + 2\alpha$, then Eq. (10.14) *admits a global analytic solution f. For any $T > 0$, the solution is unique in the space of $h \in C^{\infty,0}((\mathbb{R}^d \times \mathbb{T}^d) \times [0,T))$ with*

$$\sup_{0 \leq t < T} \left\|(h(t) - \overline{n})e^{\frac{t}{\tau_0}}\right\|_{\mathcal{O}_0^{\lambda(t)}} + \alpha\int_0^T \lambda(t)\left\|h(t)e^{\frac{t}{\tau_0}}\right\|_{\mathcal{O}_0^{\lambda(t)}} dt \leq R,$$

where $\lambda(t) = \lambda_0 e^{-\mu t}$ for $\mu = 6Je^{\lambda_0} + \alpha$. In particular, we have

$$\|f\|_{\dot{\mathcal{O}}_0^{\lambda(t)}} \leq 4\|f_0\|_{\dot{\mathcal{O}}_0^{\lambda_0}} e^{-\frac{t}{\tau_0}}.$$

Proposition 10.3.2. *A classical solution f of* (10.14) *fulfills*

$$\|f - \overline{n}\|_\infty \leq \|f_0 - \overline{n}\|_\infty e^{-\frac{t}{\tau_0}}.$$

Proof. The solution f can be rewritten as

$$f(x,p,t) := (f_0(X(0), K(0)) - \overline{n})e^{-\frac{t}{\tau_0}} + \overline{n},$$

where X, K are the characteristics with

$$\partial_s X(s) = v(K(s)), \quad \partial_s K(s) = -\nabla_x\sigma(X(s), s)$$

with $X(t) = x$ and $K(t) = p$. Thus, $\|f - \overline{n}\|_\infty \leq \|f_0 - \overline{n}\|_\infty e^{-\frac{t}{\tau_0}}$. □

Proof of Theorem 10.3.1: linearized equation

Let $g : \mathbb{R}^d \times \mathbb{T}^d \times (0, \infty) \to \mathbb{R}$ be analytic in $x \in \mathbb{R}^d$ and $p \in \mathbb{T}^d$ and measurable in t. We define h by

$$\begin{cases} \partial_t h(x,p,t) = \partial_x \sigma_g(x + tv(p), t) \cdot \tilde{\partial}_{vt} g(x,p,t) \\ \quad h\big|_{t=0} = g_0 := f_0 - n_0, \end{cases} \tag{10.19}$$

where $\sigma_g(x,t) := e^{-\frac{t}{\tau_0}} \int g(x - tv(p), p, t)dp = e^{-\frac{t}{\tau_0}} \int T_{-vt} g(x,p,t)dp.$

Lemma 10.3.3. *Let h be defined by* (10.19). *Then*

$$\|\partial_t h\|_{\dot{\mathcal{O}}_t^\lambda} \leq \left(\|DT_{vt}\sigma_g\|_{\mathcal{O}_t^{\lambda,\infty}} \|g\|_{\dot{\mathcal{O}}_t^\lambda} + \|T_{vt}\sigma_g\|_{\dot{\mathcal{O}}_t^{\lambda,\infty}} \|Dg\|_{\mathcal{O}_t^\lambda} \right) \frac{\lambda+1}{\lambda} \tag{10.20}$$

Proof. In use of Eq. (10.19) and the triangle inequality, we infer

$$\begin{aligned}
\left\| T_{vt}\partial_x\sigma_g \cdot \tilde{\partial}_{vt} g \right\|_{\dot{\mathcal{O}}_t^\lambda} &\leq \left| \partial_x T_{vt}\sigma_g \tilde{\partial}_{vt} g \right|_{\dot{\mathcal{O}}_t^\lambda} + \left| \partial_x^2 T_{vt}\sigma_g \tilde{\partial}_{vt} g \right|_{\dot{\mathcal{O}}_t^\lambda} \\
&\quad + \left| \partial_x T_{vt}\sigma_g \partial_x \tilde{\partial}_{vt} g \right|_{\dot{\mathcal{O}}_t^\lambda} + \left| \partial_x T_{vt}\sigma_g \tilde{\partial}_{vt}^2 g \right|_{\dot{\mathcal{O}}_t^\lambda} \\
&\leq |\partial_x T_{vt}\sigma_g|_{\dot{\mathcal{O}}_t^{\lambda,\infty}} \left| \tilde{\partial}_{vt} g \right|_{\mathcal{O}_t^\lambda} + \|\partial_x\sigma_g\|_{L_x^\infty} \left| \tilde{\partial}_{vt} g \right|_{\dot{\mathcal{O}}_t^\lambda} \\
&\quad + \left| \partial_x^2 T_{vt}\sigma_g \right|_{\mathcal{O}_t^{\lambda,\infty}} \left| \tilde{\partial}_{vt} g \right|_{\dot{\mathcal{O}}_t^\lambda} + \left| \partial_x^2 T_{vt}\sigma_g \right|_{\dot{\mathcal{O}}_t^{\lambda,\infty}} \left\| \tilde{\partial}_{vt} g \right\|_{L^\infty} \\
&\quad + |\partial_x T_{vt}\sigma_g|_{\dot{\mathcal{O}}_t^{\lambda,\infty}} \left(\left| \partial_x \tilde{\partial}_{vt} g \right|_{\mathcal{O}_t^\lambda} + \left| \tilde{\partial}_{vt}^2 g \right|_{\mathcal{O}_t^\lambda} \right) \\
&\quad + \|\partial_x\sigma_g\|_{L_x^\infty} \left(\left| \tilde{\partial}_{vt}^2 g \right|_{\dot{\mathcal{O}}_t^\lambda} + \left| \partial_x \tilde{\partial}_{vt} g \right|_{\dot{\mathcal{O}}_t^\lambda} \right) \\
&\leq \|DT_{vt}\sigma_g\|_{\mathcal{O}_t^{\lambda,\infty}} \|g\|_{\dot{\mathcal{O}}_t^\lambda} + \|T_{vt}\sigma_g\|_{\dot{\mathcal{O}}_t^{\lambda,\infty}} \|Dg\|_{\mathcal{O}_t^\lambda} \\
&\quad + \|\partial_x\sigma_g\|_{L_x^\infty} \|Dg\|_{\dot{\mathcal{O}}_t^\lambda} + \|DT_{vt}\sigma_g\|_{\dot{\mathcal{O}}_t^{\lambda,\infty}} \left\| \tilde{\partial}_{vt} g \right\|_{L^\infty} \\
&\leq \left(\|DT_{vt}\sigma_g\|_{\mathcal{O}_t^{\lambda,\infty}} \|g\|_{\dot{\mathcal{O}}_t^\lambda} + \|T_{vt}\sigma_g\|_{\dot{\mathcal{O}}_t^{\lambda,\infty}} \|Dg\|_{\mathcal{O}_t^\lambda} \right) \frac{\lambda+1}{\lambda}
\end{aligned}$$

in order to encounter the first statement. The remaining part can be obtained by straightforward computations. □

Definition 10.3.4. We define

$$\|h\|_{\dot{\mathcal{O}}} = \sup_{0 \leq t < T} \|h(t)\|_{\dot{\mathcal{O}}_t^{\lambda(t)}} + \alpha \int_0^T \lambda(t) \|Dh\|_{\mathcal{O}_t^{\lambda(t)}} dt$$

Proposition 10.3.5. *For $\lambda_0, \alpha, T > 0$, let $\lambda : t \mapsto \lambda_0 e^{-\mu t}$ with $\mu = 6Je^{\lambda_0} + \alpha$. Moreover, assume that $g_0 \in C^\infty(\mathbb{R}^d \times \mathbb{T}^d)$ and $g \in C^{\infty,0}(\mathbb{R}^d \times \mathbb{T}^d \times (0,\infty))$ fulfill*

$$\frac{1}{4}\left\|g\right\|_{\dot{\mathcal{O}}} \le \left\|g_0\right\|_{\dot{\mathcal{O}}_0^{\lambda_0}} \le \frac{\alpha\lambda_0^2}{32(\lambda_0+1)}.$$

For h being given by (10.19), *it holds*

$$\left\|h\right\|_{\dot{\mathcal{O}}} \le 3\left\|g_0\right\|_{\dot{\mathcal{O}}_0^{\lambda_0}}.$$

Proof. Throughout this proof we write $\mathcal{O}_t^{\lambda}$ instead of $\mathcal{O}_t^{\lambda(t)}$. Applying Corollary 4.5.5 and Lemma 10.3.3, we arrive at

$$\begin{aligned}\frac{d}{dt}\left\|h(t)\right\|_{\dot{\mathcal{O}}_t^{\lambda}} + \alpha\lambda\left\|Dh\right\|_{\mathcal{O}_t^{\lambda}} &\le \left\|\partial_t h\right\|_{\dot{\mathcal{O}}_t^{\lambda}}\\ &\le \left(\left\|DT_{vt}\sigma_g\right\|_{\mathcal{O}_t^{\lambda,\infty}}\left\|g\right\|_{\dot{\mathcal{O}}_t^{\lambda}} + \left\|T_{vt}\sigma_g\right\|_{\dot{\mathcal{O}}_t^{\lambda,\infty}}\left\|Dg\right\|_{\mathcal{O}_t^{\lambda}}\right)\frac{\lambda+1}{\lambda}.\end{aligned}$$

Thus, integration w.r.t. t yields

$$\left\|h\right\|_{\dot{\mathcal{O}}} \le 2\left\|g_0\right\|_{\dot{\mathcal{O}}_0^{\lambda_0}} + \frac{2}{\alpha}\left\|g\right\|_{\dot{\mathcal{O}}}\left\|\frac{\lambda+1}{\lambda^2}\sigma_g\right\|_{\dot{\mathcal{O}}}.$$

Thus, for $\sigma_g := e^{-\frac{t}{\tau_0}}\int g(x-tv(p),p,t)dp$, it holds

$$\left\|\frac{\lambda+1}{\lambda^2}\sigma_g\right\|_{\dot{\mathcal{O}}} \le \left\|\frac{\lambda+1}{\lambda^2}e^{-\frac{t}{\tau_0}}g\right\|_{\dot{\mathcal{O}}} \le \frac{\lambda_0+1}{\lambda_0^2}\left\|e^{-(\frac{1}{\tau_0}-2\mu)t}g\right\|_{\dot{\mathcal{O}}} \le \frac{\lambda_0+1}{\lambda_0^2}\left\|g\right\|_{\dot{\mathcal{O}}}$$

since $\frac{1}{\tau_0} \ge 2\mu$. Now, we can estimate the semi-norm of h by means of the initial value by

$$\left\|h\right\|_{\dot{\mathcal{O}}} \le \left(2 + 32\frac{\lambda_0+1}{\alpha\lambda_0^2}\left\|g_0\right\|_{\dot{\mathcal{O}}_0^{\lambda_0}}\right)\left\|g_0\right\|_{\dot{\mathcal{O}}_0^{\lambda_0}} \le 3\left\|g_0\right\|_{\dot{\mathcal{O}}_0^{\lambda_0}} \le \frac{3\alpha\lambda_0^2}{32(\lambda_0+1)}$$

using the hypothesis

$$\frac{1}{4}\left\|g\right\|_{\dot{\mathcal{O}}} \le \left\|g_0\right\|_{\dot{\mathcal{O}}_0^{\lambda_0}} \le \frac{\alpha\lambda_0^2}{32(\lambda_0+1)}.$$

□

Lemma 10.3.6. *Let g_0, g, h be as in Proposition 10.3.5. Then*

$$\left\|h\right\|_\infty \le \left\|g_0\right\|_\infty + \frac{16}{\alpha\lambda_0^2}\left\|g_0\right\|_{\dot{\mathcal{O}}}^2 \le \left\|g_0\right\|_\infty + \frac{1}{2(\lambda_0+1)}\left\|g_0\right\|_{\dot{\mathcal{O}}}.$$

Proof. Again, we write $\mathcal{O}_t^\lambda$ instead of $\mathcal{O}_t^{\lambda(t)}$. Using the definition of h, we have

$$\begin{aligned}\|\partial_t h(x,p,t)\|_\infty &= \|\partial_x \sigma_g(x+tv,t)\|_\infty \left\|\tilde{\partial}_{vt} g(x,p,t)\right\|_\infty \\ &\le \frac{1}{\lambda^2}\lambda \|D\sigma_g\|_{\mathcal{O}_t^\lambda} \|g\|_{\mathcal{O}_t^\lambda}.\end{aligned}$$

Integration w.r.t. t yields

$$\|h(\cdot,\cdot,t)\|_\infty \le \|g_0\|_\infty + \frac{1}{\alpha\lambda_0^2}\left\|e^{-2\mu t}\sigma_g\right\|_{\dot{\mathcal{O}}} \|g\|_{\dot{\mathcal{O}}} \le \|g_0\|_\infty + \frac{1}{\alpha\lambda_0^2}\|g\|_{\dot{\mathcal{O}}}^2$$

for all $t > 0$. □

Proposition 10.3.7. *For $\lambda_0, \alpha, T > 0$, let $\lambda : t \mapsto \lambda_0 e^{-\mu t}$ with $\mu = 6Je^{\lambda_0} + \alpha$. In addition, we assume that $g_0 \in C^\infty(\mathbb{R}^d \times \mathbb{T}^d)$ and $g_1, g_2 \in C^\infty(\mathbb{R}^d \times \mathbb{T}^d \times (0,\infty))$ satisfy*

$$\frac{1}{4}\|g_i\|_{\dot{\mathcal{O}}} \le \|g_0\|_{\dot{\mathcal{O}}_0^{\lambda_0}} \le \frac{\alpha\lambda_0^2}{32(\lambda_0+1)} \quad \textit{for } i = 1,2.$$

We define $h_i = h$ by (10.19) for $g = g_i$ and $i = 1,2$.

Then it holds

$$\|h_1 - h_2\|_{\dot{\mathcal{O}}} \le 16\frac{\lambda_0+1}{\alpha\lambda_0^2}\|g_0\|_{\dot{\mathcal{O}}_0^{\lambda_0}} \|g_1 - g_2\|_{\dot{\mathcal{O}}} \le \frac{1}{2}\|g_1 - g_2\|_{\dot{\mathcal{O}}}.$$

Proof. The function $H := h_1 - h_2$ solves

$$\partial_t H(x,p,t) = \partial_x \sigma_{g_1}(x+tv(p),t)\cdot \tilde{\partial}_{vt} G(x,p,t) - \partial_x(\sigma_G)\cdot\tilde{\partial}_{vt} g_2(x,p,t)$$

with $H_0 := H(\cdot,\cdot,0) = 0$, where $G := g_1 - g_2$. We derive similarly to the proof of Proposition 10.3.5 that

$$\begin{aligned}\|H\|_{\dot{\mathcal{O}}} &\le \frac{2}{\alpha}\left\|\frac{\lambda+1}{\lambda^2}\sigma_{g_1}\right\|_{\dot{\mathcal{O}}} \|G\|_{\dot{\mathcal{O}}} + \frac{2}{\alpha}\left\|\frac{\lambda+1}{\lambda^2}\sigma_G\right\|_{\dot{\mathcal{O}}}\|g_2\|_{\dot{\mathcal{O}}} \\ &\le 2\frac{\lambda_0+1}{\alpha\lambda_0^2}\left(\|g_1\|_{\dot{\mathcal{O}}} + \|g_2\|_{\dot{\mathcal{O}}}\right)\|G\|_{\dot{\mathcal{O}}}.\end{aligned}$$

Finally, the assertion can be concluded using Proposition 10.3.5. □

Lemma 10.3.8. *Let g_0, g_1, g_2, h_1, h_2 be as in Proposition 10.3.7. Then*

$$\|h_1 - h_2\|_\infty \le \frac{8}{\alpha\lambda_0^2}\|g_0\|_{\dot{\mathcal{O}}_0^{\lambda_0}} \|g_1 - g_2\|_{\dot{\mathcal{O}}} \le \frac{1}{4(1+\lambda_0)}\|g_1 - g_2\|_{\dot{\mathcal{O}}}.$$

Proof. Defining $G := g_1 - g_2$, then $H := h_1 - h_2$ is given by

$$\partial_t H(x,p,t) = \partial_x \sigma_{g_1}(x + tv(p), t) \cdot \tilde{\partial}_{vt} G(x,p,t) - \partial_x \sigma_G \cdot \tilde{\partial}_{vt} g_2(x,p,t)$$

with $H_0 = H(\cdot,\cdot,0) = 0$. We can now proceed similarly to the proof of Lemma 10.3.6 and obtain

$$\begin{aligned} \|H\|_\infty &\le \frac{1}{\alpha\lambda_0^2} \left\|e^{-2\mu t}\sigma_{g_1}\right\|_{\dot{\mathcal{O}}} \|G\|_{\dot{\mathcal{O}}} + \frac{1}{\alpha\lambda_0^2} \left\|e^{-2\mu t}\sigma_G\right\|_{\dot{\mathcal{O}}} \|g_2\|_{\dot{\mathcal{O}}} \\ &\le \frac{1}{\alpha\lambda_0^2} \left(\|g_1\|_{\dot{\mathcal{O}}} + \|g_2\|_{\dot{\mathcal{O}}}\right) \|G\|_{\dot{\mathcal{O}}} \end{aligned}$$

for all $t > 0$. □

Definition 10.3.9. Let $\mathcal{Y}$ be the space of all $g \in C^{\infty,0}((\mathbb{R}^d \times \mathbb{T}^d) \times (0,\infty))$ such that

$$|||g||| := \|g\|_\infty + \|g\|_{\dot{\mathcal{O}}} < \infty.$$

Clearly, the space $(\mathcal{Y}, |||\cdot|||)$ is a Banach space. We denote B_R as the closed ball of radius $R := \|g_0\|_\infty + 4\|g_0\|_{\dot{\mathcal{O}}_0^{\lambda_0}}$ in $\mathcal{Y}$.

Corollary 10.3.10. *Let $\Psi : B_R \to \mathcal{Y}$ denote the mapping $g \mapsto h$ defined by*

$$\begin{cases} \partial_t h(x,p,t) = \partial_x \sigma_g(x + tv(p), t) \cdot \tilde{\partial}_{vt} g(x,p,t) \\ \qquad h\big|_{t=0} = g_0 := f_0 - n_0. \end{cases}$$

If $g_0 \in C^\infty(\mathbb{R}^d \times \mathbb{T}^d)$ fulfills

$$R := \|g_0\|_\infty + 4\|g_0\|_{\dot{\mathcal{O}}_0^{\lambda_0}} < \frac{\alpha\lambda_0^2}{8(\lambda_0 + 1)},$$

then Ψ is a contraction with $\Psi(B_R) \subseteq B_R$.

In particular, Ψ admits a unique fixed point $g \in B_R$.

Proof. According to Proposition 10.3.5 and Lemma 10.3.6, we have

$$|||\Psi(g)||| \le \|g_0\|_\infty + 4\|g_0\|_{\dot{\mathcal{O}}_0^{\lambda_0}} \le R \quad \text{for all } g \in B_R.$$

This time, using Proposition 10.3.7 and Lemma 10.3.6 yields

$$|||\Psi(g_1) - \Psi(g_2)||| \le \frac{3}{4}\|g_1 - g_2\|_{\dot{\mathcal{O}}} \le \frac{3}{4}|||g_1 - g_2|||$$

for any $g_1, g_2 \in B_R$. Finally, the Banach fixed point theorem proves the claim. □

Proof of Theorem 10.3.1. Now, we derive the solution of (10.14) by back-transforming the fixed point g of Ψ. □

Chapter 11

Drift diffusion equation

11.1 The model

Let Ω be a bounded domain with smooth boundary and let b_0 belong to $C^2(\partial\Omega, [0,\infty), [0,\eta^{-1}])$ with $\|b_0\|_\infty \le \|v\|_\infty = 2Jd$ and let $\eta \ge 0$. We introduce the equation

$$\begin{cases} u_t = \Delta \log\left(\dfrac{u}{1-\eta u}\right), & (x,t) \in \Omega \times \mathbb{R}_+, \\ \partial_\nu \log\left(\dfrac{u}{1-\eta u}\right) = -b_0(x,t,u)u(1-\eta u), & (x,t) \in \partial\Omega \times \mathbb{R}_+, \\ u(\cdot,0) = u_0 > 0, & x \in \overline{\Omega}, \end{cases} \tag{11.1}$$

where $u_0 \in C^2(\overline{\Omega}, (0,\eta^{-1}))$.

Proposition 11.1.1 (Amann [1]). *There exists a unique maximal solution*

$$u \in C([0,T), C(\overline{\Omega}, (0,\eta^{-1}))) \cap C((0,T), C^2(\overline{\Omega})) \cap C^1((0,T), C(\overline{\Omega}))$$

of (11.1) *for* $T \in (0,\infty]$. *Moreover, u is global, i.e. $T = \infty$, if $u([0,t])$ is bounded away from* 0 *and η^{-1} for all $t \in [0,T)$ and u is bounded in $C(\overline{\Omega})$.*

Remark 11.1.2. The condition of Proposition 11.1.1 of u being bounded in $C(\overline{\Omega})$ is trivially satisfied for $\eta > 0$ since $0 < u < \eta^{-1}$.

Remark 11.1.3. Let u denote the solution of Proposition 11.1.1. Then $w := \log(\frac{u}{1-\eta u})$ solves

$$\begin{cases} w_t = e^{-w}\left(1+\eta e^w\right)^2 \Delta w, & (x,t) \in \Omega \times \mathbb{R}_+, \\ \partial_\nu w = -b_0\left(x,t,\dfrac{e^w}{1+\eta e^w}\right)\dfrac{e^w}{(1+\eta e^w)^2}, & (x,t) \in \partial\Omega \times \mathbb{R}_+, \\ u(\cdot,0) = u_0 > 0, & x \in \overline{\Omega}. \end{cases} \tag{11.2}$$

11.2 Low density approximation

In the drift diffusive picture, the difference between Fermions and Bosons relies on the Pauli exclusion principle, since states can only be occupied once the density is bounded by 1 describing occupied states. Therefore, the inverse relaxation time vanishes for $n = 1$. In the low density approximation, we assume that n is sufficiently small such that the Pauli exclusion principle is neglectable since almost no state is occupied. In this case, Fermions behave like Bosons. This section analyzes Eq. (11.1) for $\eta = 0$. Let u be the (local) positive, classical solution of

$$\begin{cases} u_t = \Delta \log u, & (x,t) \in \Omega \times \mathbb{R}_+, \\ \partial_\nu \log(u) = -b_0(x,t,u)u, & (x,t) \in \partial\Omega \times \mathbb{R}_+, \\ u(\cdot,0) = u_0 > 0, & x \in \overline{\Omega} \end{cases} \tag{11.3}$$

with $u_0 \in C^2(\overline{\Omega})$. Furthermore, we assume that

$$b_0 \geq \beta \quad \text{for some } \beta \in (0,\infty).$$

Lemma 11.2.1 (Maximum principle)**.**

$$\max_x u(x,t) = \max_x u_0(x)$$

Proof. This Lemma is a direct consequence of chapter 3, Theorem 6 from [45] since $\partial_\nu u = -b_0(x,t,u)u^2 \leq -\beta u^2 \leq 0$ on $\partial\Omega \times \mathbb{R}_+$. □

Lemma 11.2.2. *Let $p \neq -1$. It holds*

$$\frac{1}{p+1}\partial_t \int_\Omega u^{p+1} dx + \beta \int_{\partial\Omega} u^{p+1} d\mathcal{H}_x^{d-1} + p \int_\Omega u^p \left|\nabla \log u\right|^2 dx = 0. \tag{11.4}$$

Moreover, let $q \in (1,\infty)$; we have

$$\|u(t)\|_{L^q(\Omega)} \leq \|u_0\|_{L^q(\Omega)} e^{-C_q t}$$

for some $C_q > 0$ depending only on β, $\|u_0\|_{L^\infty}$ and q. In addition,

$$\|u(t)\|_{L^1(\Omega)} \leq |\Omega|^{\frac{q-1}{q}} \|u_0\|_{L^q(\Omega)} e^{-C_q t}$$

for all $q \geq 1$ putting $C_1 = 0$.

Proof. This first assertion can easily be deduced by multiplying Eq. $(11.1)_1$ by u^p and using integration by parts as well as the boundary equation. Now, we note that

$$\|\cdot\|'_{W^{1,p}(\Omega)} := \left(\|\cdot\|^p_{L^p(\partial\Omega)} + \|\nabla\cdot\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}}$$

defines an equivalent norm on $W^{1,p}(\Omega)$. Hence, there exists a constant $C_q > 0$ such that

$$\begin{aligned}\partial_t \int_\Omega u^q dx &= -q \int_{\partial\Omega} b_0(u) u^q d\mathcal{H}_x^{d-1} - q(q-1) \int_\Omega u^{q-1} \left|\nabla \log u\right|^2 dx \\ &\leq -q\beta \int_{\partial\Omega} \left|u^{\frac{q}{2}}\right|^2 d\mathcal{H}_x^{d-1} - \frac{4(q-1)}{q} \int_\Omega \left|\nabla u^{\frac{q}{2}}\right|^2 \frac{1}{u} dx \\ &\leq -C_q q \min\left\{\beta, \frac{4(q-1)}{q^2 \max_x u}\right\} \int_\Omega u^q dx.\end{aligned}$$

Finally, by Gronwall, the second assertion is proved. The estimate of the L^1 norm of u is a consequence of Hölder's inequalities. □

The following lemma is a modified version of Lemma 3.1 in [2], which itself has its counterpart Proposition 2.1 from [47]. However, we require an additional non-divergence part in the equation as we will see later on, which motivates the new version of this lemma.

Lemma 11.2.3 (Maximum principle)**.** *Define* $\Omega_T := \Omega \times (0,T)$ *and* $\Sigma_T = \partial\Omega \times (0,T)$*. Let* $a_2, b \in C^1(\Omega_T, \mathbb{R})$*,* $a_1, c \in C^0(\Omega_T \times \mathbb{R})$ *and* $g \in C^0(\Sigma_T \times \mathbb{R})$ *such that*

1. $\big(a_1(x,t,w) + \partial_w a_1(x,t,w)\big) a_2(x,t,w) \geq \theta$ *for* $(x,t) \in \Omega_T$ *and* $w > k_0$*,*
2. $|b|, |c|, |\nabla_x a_1| \leq \alpha$ *for* $(x,t) \in \Omega_T$ *and* $w > k_0$*,*
3. $|g|(x,t,w) \leq \alpha$ *for* $(x,t) \in \Sigma_T$ *and* $w > k_0$

for some constants k_0, α, β*. Then every classical solution of*

$$\begin{aligned} w_t &= a_1 \nabla \cdot (a_2 \nabla w + b) + c && \text{on } \Omega_T, \\ a_1 a_2 \partial_\nu w &= g && \text{on } \Sigma_T, \\ w &= w_0 && \text{on } \overline{\Omega} \times \{0\} \end{aligned} \tag{11.5}$$

fulfills $w(x,t) \leq \max w_0 + C(1+t)t^{\frac{2}{d+2}}$ *for all* $0 \leq t \leq T$ *and some* $C > 0$ *only depending on* α, θ, Ω *and the dimension* d*.*

Proof. The proof can be done as in [2]. Since the proof in [2] contains a false application of the Gagliardo Nirenberg inequalities leading to a different time dependency, we redo the proof. However, the main steps will coincide.

Let $\mu(k) := \int_0^t |\{x \in \Omega : w(x,s) > k\}| \, ds$. By applying integration by parts and standard theory for the positive part ($r_+ = \max\{r, 0\}$) of a function, Equation (11.5) entails

$$\begin{aligned}\frac{1}{2}\int_\Omega (w-k)_+^2 dx \\ &= -\int_0^t\int_\Omega \big((a_1+\partial_w a_1)a_2 \left|\nabla (w-k)_+\right|^2 + b\cdot\nabla(w-k)_+\big)dxds \\ &\quad + \int_0^t\int_\Omega c(w-k)_+ dxds + \int_0^t\int_{\partial\Omega}(w-k)_+(g+b\cdot\nu)d\mathcal{H}_x^{d-1}ds\end{aligned}$$

for all $0 \le t < T$ and all $k > k_0$ such that $\mu(k_0) = 0$. Here ν is defined as the outer normal vector of Ω at a point in $\partial\Omega$. As in the proof of Lemma 3.1 in [2], the trace theorem ensures

$$\begin{aligned}\frac{1}{2}\int_\Omega (w-k)_+^2 dx + \frac{\theta}{2}\int_0^t\int_\Omega \left|\nabla(w-k)_+\right|^2 dxds \\ \le C\mu(k) + C\int_0^t\int_\Omega (w-k)_+ dxds \qquad (11.6)\end{aligned}$$

for some constant C only depending on Ω, α, θ and the dimension d. Similar to [33], we infer from Gagliardo Nirenbergs inequality

$$\begin{aligned}\|f\|_{L^r(\Omega\times(0,t))}^2 &= \left(\int_0^t \|f\|_{L^r(\Omega)}^r ds\right)^{\frac{2}{r}} \\ &\le C\left(\int_0^t \|f\|_{W^{1,2}(\Omega)}^2 ds\right)^{\frac{2}{r}} \operatorname{ess\,sup} \|f\|_{L^2(\Omega)}^{1-\frac{2}{r}}\end{aligned}$$

for $r = 2(d+2)/d$ and some $C > 0$ and all suitable f. Using Young's inequality, we infer for $f = (w-k)_+^2$

$$\begin{aligned}&\|(w-k)_+\|_{L^{2(d+2)/d}(\Omega\times(0,t))}^2 \\ &\quad \le C\int_0^t\int_\Omega \|\nabla(w-k)_+\|^2 dxds + C(1+t)\operatorname{ess\,sup}\int_\Omega \|(w-k)_+\|^2 dx.\end{aligned}$$

Thus, the previous consideration in Eq. (11.6) yields

$$\|(w-k)_+\|_{L^{2(d+2)/d}(\Omega\times(0,t))}^2 \le C(1+t)\mu(k) + C(1+t)\int_0^t\int_\Omega (w-k)_+ dxds$$

for some $C > 0$ only depending on $\Omega, \alpha, \theta, d$. The previous inequality implies by interpolation of L^p spaces

$$\begin{aligned}(1+t)\int_0^t\int_\Omega (w-k)_+dxds\\ &\le \|(w-k)_+\|_{2(d+2)/d}\,\|1\|_{2(d+2)/(d+4)}\\ &\le (1+t)\frac{1}{2C}\,\|(w-k)_+\|^2_{2(d+2)/d}+\frac{C(1+t)^2}{2}\mu(k)^{\frac{d+4}{d+2}}\\ &\le \frac{1}{2C}\,\|(w-k)_+\|^2_{2(d+2)/d}+|\Omega|^{\frac{2}{d+2}}\,\frac{C(1+t)^2}{2}\mu(k)t^{\frac{2}{d+2}}\end{aligned}$$

the following estimate

$$(h-k)^2\mu(h)^{\frac{d}{d+2}}\le \|(w-k)_+\|^2_{L^{2(d+2)/d}(\Omega\times(0,t))}\le \left(C+C't^{\frac{d+3}{d+2}}\right)^2\mu(k)$$

for some $C'>0$ independent from t and for all $h>k$. In particular, we have

$$(h-k)^{\frac{2(d+2)}{d}}\mu(h)\le \left(C+C't\right)^{2\frac{d+2}{d}}t^{\frac{2}{d}}\mu(k)^{\frac{d+2}{d}}$$

for some $\tilde{C},\tilde{C}'>0$ independent from t and all $h>k$. In conjunction with [31], this entails $\mu(k_0+K')=0$ for

$$K'^{\frac{2(d+2)}{d}}=2(C+C't)^{\frac{2(d+2)}{d}}t^{\frac{2}{d}}\mu(k_0)^{\frac{2}{d}}.$$

If we roughly estimate $\mu(k_0)\le|\Omega|\,t$, we infer $\mu(k_0+K'')=0$ for

$$K''=C''(1+t)t^{\frac{2}{(d+2)}}$$

for some $C''>0$ independent from t. Since this is true for all k_0 satisfying $\mu(k_0)=0$, we end up with the assertion. □

Theorem 11.2.4. *The classical solution u of* (11.3) *is global. Moreover, there exists a constant $C>0$ such that*

$$\min_{x\in\overline{\Omega}}u(x,t)\ge\min_{x\in\overline{\Omega}}u_0(x)\exp(-C(1+t)t^{\frac{2}{d+2}}).$$

In particular, we have

$$\min_{x\in\overline{\Omega}}u(x,t)\ge e^{-C}\min_{x\in\overline{\Omega}}u_0(x)\exp(-2Ct).$$

Proof. The function $w=-\log u$ solves

$$\begin{aligned} w_t&=e^w\nabla\cdot\nabla w &&\text{on }\Omega_T,\\ e^w\partial_\nu w&=b_0 &&\text{on }\Sigma_T,\end{aligned}$$

where $e^w + \partial_w e^w \geq 2$. Hence, Lemma 11.2.3 is applicable and ensures the first part of the assertion. For the second one, we write $t = i + s$ for $i \in \mathbb{N}_0$ and $s \in [0,1)$; we observe inductively

$$\begin{aligned}\min_{x\in\overline{\Omega}} u(x,i+s) &\geq e^{-C} \min_{x\in\overline{\Omega}} u(x,i) e^{-Cs} \geq e^{-C} \min_{x\in\overline{\Omega}} u(x,i-1) e^{-2C(1+s)} \\ &\geq \ldots \geq e^{-C} \min_{x\in\overline{\Omega}} u_0(x) \exp(-2C(N+s)). \qquad \square\end{aligned}$$

In order to learn more about the decay of the solution, we analyze the L^p norms of the logarithm.

Lemma 11.2.5. *Let $p \neq -1$. We have*

$$\begin{aligned}\frac{1}{p+1}\partial_t \int_\Omega (\log u)^{p+1} dx + \int_\Omega \frac{(\log u)^{p-1}}{u}(p - \log u)\left|\nabla \log u\right|^2 dx \\ = \|b_0\|_\infty \int_{\partial\Omega} (\log u)^p d\mathcal{H}_x^{d-1}. \quad (11.7)\end{aligned}$$

In addition, there exist a constant $C > 0$ fulfilling

$$\|\log u(t)\|_{L^{p+1}(\Omega)} \leq \|\log u_0\|_{L^{p+1}(\Omega)} + \sqrt{p} C \|b_0\|_\infty \left(t + \frac{1}{4}\right)$$

for all $p \in \mathbb{N}$ and it holds

$$\|\log u(t)\|_{L^1(\Omega)} \leq \|\log u_0\|_{L^1(\Omega)} + \|b_0\|_\infty t.$$

for $u_0 \leq 1$.

Proof. We obtain the first equality by multiplying Eq. (11.1) by $\frac{1}{p}(\log u)^p$ and using integration by parts. For the next step, we assume $\max_x u_0 \leq 1$ for simplicity. In the case $p = 0$, the desired inequality can directly be obtained by choosing $C = 1$. Now, let $p > 0$; we have

$$\begin{aligned}&\frac{1}{p+1}\partial_t \int_\Omega |\log u|^{p+1} dx + \int_\Omega \frac{|\log u|^{p-1}}{u}(p - \log u)\left|\nabla \log u\right|^2 dx \\ &= \int_{\partial\Omega} b_0(u) |\log u|^p \, d\mathcal{H}_x^{d-1} \\ &\leq C\beta \left(\int_\Omega |\log u|^p \, dx + p \int_\Omega |\log u|^{p-1} |\nabla \log u| \, dx\right) \\ &\leq \|b_0\|_\infty C \left(\int_\Omega |\log u|^p \, dx + \frac{Cp}{4} \|b_0\|_\infty \int_\Omega |\log u|^{p-1} dx\right) \\ &\quad + p \int_\Omega |\log u|^{p-1} |\nabla \log u|^2 \, dx\end{aligned}$$

for some constant $C > 0$ independent from p, since we have applied the trace theorem for $|\log u|^p \in W^{1,1}(\Omega)$ and Young's inequality. Defining $y_p := |\log u|_{L^{p+1}(\Omega)}$ and $\alpha = \|b_0\|_\infty$, Hölder's inequality entails an ordinary differential inequality, namely

$$\dot{y}_p y_p^p = \frac{1}{1+p}\partial_t y_p^{p+1} \leq C\alpha y_p^p + p\frac{C^2\alpha^2}{4}y_p^{p-1}.$$

From this inequality, we obtain

$$\begin{aligned} C\alpha t = \int_0^t C\alpha ds \geq \int_0^t \frac{4\dot{y}_p y}{4y + pC\alpha} ds = y_p(t) - y_p(0) - \frac{pC\alpha}{4}\log\frac{4y_p(t) + pC\alpha}{4y_p(0) + pC\alpha} \\ = \frac{1}{4}pC\alpha\left(4\frac{y_p(t) - y_p(0)}{pC\alpha} - \log\left(4\frac{y_p(t) - y_p(0)}{pC\alpha} + 1\right)\right). \end{aligned}$$

Abbreviating $z = (y_p(t) - y_p(0))$, we can rewrite this inequality as

$$\frac{4z}{pC\alpha} - \log\left(\frac{4z}{pC\alpha} + 1\right) \leq \frac{4}{p}t.$$

In order to cope with the logarithm in the previous inequality, we observe that

$$s \geq s - \log(1+s) \geq \frac{1}{\sqrt{p}}s - \frac{1}{p}$$

holds for all non-negative s. Thus,

$$\frac{4}{p}t \geq \frac{1}{\sqrt{p}}\frac{4z}{pC\alpha} - \frac{1}{p}$$

which directly implies that

$$\|\log u(t)\|_{L^{p+1}(\Omega)} \leq \|\log u_0\|_{L^{p+1}(\Omega)} + \sqrt{p}C\alpha\left(t + \frac{1}{4}\right).$$

Finally, we can easily extend this result for $\max_x u_0 > 0$: We define $\tilde{u} = u/\max_x u_0$ and observe that $\tilde{u}$ solves Eq. (11.1) after readjusting the time $\tilde{t} = \max_x u_0 t$ and $\tilde{\alpha} = \max_x u_0 \alpha$ for the initial value $u_0/\max_x u_0$. □

Corollary 11.2.6. *Let $u_0 \leq 1$. Then it holds*

$$\|u(t)\|_{L^p(\Omega)} \geq \sqrt[p]{|\Omega|}\exp\left(-\frac{1}{|\Omega|}\int_\Omega |\log(u_0)|\,dx\right)e^{-\|b_0\|_\infty t}.$$

Proof. This can easily be obtained by Jensen's inequality and the previous Lemma using

$$\begin{aligned}\frac{1}{|\Omega|}\int_\Omega u^p dx &= \frac{1}{|\Omega|}\int_\Omega \exp(-p\,|\log(u)|)dx \\ &\geq \exp\left(-\frac{p}{|\Omega|}\int_\Omega |\log(u)|\,dx\right) \\ &\geq \exp\left(-\frac{p}{|\Omega|}\int_\Omega |\log(u_0)|\,dx\right) e^{-p\|b_0\|_\infty t}.\end{aligned}$$

□

Lemma 11.2.7.

$$\begin{aligned}\frac{1}{2}\partial_t \int_\Omega |\nabla \log u|^2\,dx + \partial_t \int_{\partial\Omega} B_0(u)d\mathcal{H}_x^{d-1} + \int_\Omega \frac{|\Delta \log u|^2}{u}dx \\ = \int_{\partial\Omega}(\partial_t B_0)(u)d\mathcal{H}_x^{d-1}, \quad (11.8)\end{aligned}$$

where $B_0(x,t,u) = \int_1^u b_0(x,t,s)ds$ *for* $x \in \overline{\Omega}$ *and* $t > 0$. *In particular, if* $\partial_t b_0$ *is bounded, then*

$$u \in L^\infty_{\text{loc}}(0,\infty; H^1(\Omega)).$$

Proof. The assertion can be obtained by the straightforward calculation

$$\begin{aligned}\frac{1}{2}\partial_t \int_\Omega |\nabla \log u|^2\,dx &= \int_\Omega \nabla \log u \cdot \nabla\left(\frac{u_t}{u}\right)dx \\ &= -\int_{\partial\Omega} b_0(u)u_t d\mathcal{H}_x^{d-1} - \int_\Omega \Delta \log u \frac{u_t}{u}dx \\ &= -\int_{\partial\Omega} b_0(u)u_t d\mathcal{H}_x^{d-1} - \int_\Omega \frac{|\Delta \log u|^2}{u}dx\end{aligned}$$

and the property that $u > 0$ for all $t > 0$. □

At most exponential decay for spherical symmetry

Lemma 11.2.8. *Let* $\Omega = B_R$ *equal the ball of radius* R *in* $\mathbb{R}^d$ *and* $0 \leq r < r_0 < R$. *Let* $t \in (0,T)$. *The condition*

$$\int_{\mathbb{S}^{d-1}} \log u_0(ry)d\mathcal{H}_y^{d-1} \geq \int_{\mathbb{S}^{d-1}} \log u_0(r_0 y)d\mathcal{H}_y^{d-1}$$

implies

$$\int_{\mathbb{S}^{d-1}} \log u(ry,t)d\mathcal{H}_y^{d-1} \geq \int_{\mathbb{S}^{d-1}} \log u(r_0 y,t)d\mathcal{H}_y^{d-1}. \quad (11.9)$$

Proof. We denote Δ_r and Δ_ϕ as the radial and spherical component of the Laplace operator, respectively. Since u is positive and continuous, there exists a constant $C > 0$ such that

$$\begin{aligned}
\partial_t \int_{\mathbb{S}^{d-1}} \log u(ry,t) d\mathcal{H}^{d-1}_y &= \int_{\mathbb{S}^{d-1}} \frac{\partial_t u(ry,t)}{u(ry,t)} d\mathcal{H}^{d-1}_y \\
&\geq \frac{1}{C} \int_{\mathbb{S}^{d-1}} \partial_t u(ry,t) d\mathcal{H}^{d-1}_y \\
&= \frac{1}{C} \int_{\mathbb{S}^{d-1}} (\Delta_r + \Delta_\phi) \log u(ry,t) d\mathcal{H}^{d-1}_y \\
&= \frac{1}{C} \Delta_r \int_{\mathbb{S}^{d-1}} \log u(ry,t) d\mathcal{H}^{d-1}_y
\end{aligned}$$

since integration of the part concerning Δ_ϕ yields zero contribution. Likewise, we obtain

$$\partial_t \int_{\mathbb{S}^{d-1}} \log u(ry,t) d\mathcal{H}^{d-1}_y \leq \frac{1}{C'} \Delta_r \int_{\mathbb{S}^{d-1}} \log u(ry,t) d\mathcal{H}^{d-1}_y$$

for some $C' > 0$ different from C. According to the boundary conditions, the function $r \mapsto \int_{\mathbb{S}^{d-1}} \log u(ry,t) d\mathcal{H}^{d-1}_y$ cannot attain its local maximum at $r = R$. Fix for a moment $t \in (0,T)$. Assuming that there exists a local minimum in $r' \in [0,R)$ implies the existence of a local maximum in at some $r'' \in (r', R)$. The derived heat inequalities for $\int_{\mathbb{S}^{d-1}} \log u(ry,t) d\mathcal{H}^{d-1}_y$ entail

$$\partial_t \int_{\mathbb{S}^{d-1}} \log u(r''y,t) d\mathcal{H}^{d-1}_y \leq 0 \leq \partial_t \int_{\mathbb{S}^{d-1}} \log u(r'y,t) d\mathcal{H}^{d-1}_y.$$

Since this is true for all t, the hypothesis of the lemma implies that a local in space minimum at $r' \in [0,R)$ is greater or equal to an arbitrary local maximum. Consequently, $\int_{\mathbb{S}^{d-1}} \log u(ry,t) d\mathcal{H}^{d-1}_y$ is non-increasing in r. □

Let $\Omega = B_R$ be the ball of radius R in $\mathbb{R}^d$ and u_0 be spherically symmetric and let u be a solution of Eq. (11.1). The symmetry of u_0 directly implies that u must also be spherically symmetric. We denote $u(r,t) = u(x,t)$ and $u_0(r) = u_0(x)$ for $r = |x|$.

Lemma 11.2.9. *Let $\Omega = B_R$ be the ball of radius R in $\mathbb{R}^d$ and u_0 be spherical symmetric and let u be a solution of Eq.* (11.1). *Assume that*

$$u_0(r_1) \geq u_0(r_2) \quad \text{for all } 0 \leq r_1 \leq r_2 \leq R, \tag{11.10}$$

which implies $u(r_1,t) \geq u(r_2,t)$ according to Lemma 11.2.8.

Then we have

$$\int_{B_R} u(t)dx \geq \int_{B_R} u_0 dx \exp\left(-\frac{\|b_0\|_\infty t}{R \max\{1, d-1\}}\right)$$

Proof. We integrate Eq. (11.1) and use Lemma 11.2.8 in conjunction with Condition (11.10) to derive

$$\begin{aligned}
-\partial_t \int_{B_R} u dx &= \int_{\partial B_R} b_0(u) u d\mathcal{H}_x^{d-1} = b_0(u(R,t)) \left|\mathbb{S}^{d-1}\right| R^{d-1} u(R,t) \\
&\leq \frac{\|b_0\|_\infty}{(d-1)R} \left|\mathbb{S}^{d-1}\right| \int_0^R r^{d-1} u(R,t) dr \\
&\leq \frac{\|b_0\|_\infty}{(d-1)R} \int_{B_R} u dx
\end{aligned}$$

for $d > 1$ and likewise, we obtain $-\partial_t \int_{B_R} u \leq \frac{1}{R} \|b_0\|_\infty \int_{B_R} u$ for $d = 1$. Finally, the assertion follows using Gronwall's inequality. □

Corollary 11.2.10.

$$\begin{aligned}
\max_{x \in \overline{\Omega}} u(x,t) &\geq \frac{1}{|B_R|} \int_{B_R} u(y,t) dy \\
&\geq \frac{1}{|B_R|} \int_{B_R} u_0(y) dy \exp\left(-\frac{\beta t}{R \max\{1, d-1\}}\right).
\end{aligned}$$

11.3 Comments

So far we have assumed that the diffusive regime is fixed to a bounded domain. However, if we assume that the cloud of ultracold atoms expands, we may suppose that the diffusive regime also expands. This leads to a diffusive equation with moving boundaries. The idea of this section is to present the problem and rewrite it into a fixed boundary problem. Whether it is possible to derive a global solution for this problem as in the previous section remains open and needs further investigation.

Let B_R the ball of radius R in $\mathbb{R}^d$ and $R(t) = v(t + t_0)$ the radius of the atom cloud. We define $\Omega_T := \{(x,t) \in \mathbb{R}^d[d+1] : t > 0,\ x \in B_{R(t)}\}$, $\partial\Omega_T : \{(x,t) \in \mathbb{R}^d[d+1] : t > 0,\ x \in \partial B_{R(t)}\}$. Given $u_0 > 0$ being a smooth function on $B_{R(0)}$, we consider the equation

$$\begin{cases} u_t = \Delta \log u, & (x,t) \in \Omega_T, \\ \partial_\nu \log u = -b_0(u) u, & (x,t) \in \partial\Omega_T, \\ u(\cdot, 0) = u_0 > 0, & x \in \overline{B_{R(0)}}. \end{cases} \tag{11.11}$$

We note that Eq. 11.11 can be transformed into a fixed boundary problem by setting

$$n(x,t) := R(t)^m u(R(t)x, t)$$

for some fixed $m \in \mathbb{N}$. We compute

$$\begin{aligned} n_t &= m\dot{R}R^{m-1}u(Rx,t) + R^m\big(\Delta \log u(Rx,t)) + \dot{R}x \cdot \nabla u(Rx,t)\big) \\ &= R^{m-2}\Delta \log n + (mn + x \cdot \nabla n)\partial_t \log R \end{aligned}$$

as well as

$$\begin{aligned} \partial_\nu \log n(1,t) &= R\partial_\nu \log u(Rx,t) = -Rb_0(u(Rx,t))u(Rx,t) \\ &= -R^{1-m}b_0(R^{-m}n(1,t))n(1,t) \end{aligned}$$

and infer

$$\begin{cases} n_t = R^{m-2}\Delta \log n + (mn + x\cdot\nabla n)f, & (x,t) \in B_1 \times \mathbb{R}_+, \\ \partial_\nu \log n = -Rg(n)n, & (x,t) \in \partial B_1 \times \mathbb{R}_+, \\ n(\cdot,0) = n_0 > 0, & x \in \overline{B_1}, \end{cases} \tag{11.12}$$

where $n_0 := R(0)^m u(R(0)\cdot)$, $f := \partial_t \log R$ and $g(n) = R^{-m}b_0(R^{-m}n(1,t))$.

Proposition 11.3.1 (Amann)**.** *There exists a unique maximal solution*

$$n \in C([0,T), C(\overline{\Omega})) \cap C((0,T), C^2(\overline{\Omega})) \cap C^1((0,T), C(\overline{\Omega}))$$

of (11.12) *for* $T \in (0,\infty]$. *Moreover, n is global, i.e. $T = \infty$ if $n([0,t])$ is bounded away from* 0 *for all $t \in [0,T)$ and u is bounded in $C(\overline{\Omega})$ for all $t < T$.*

Proof. See Theorem 1 and 3 of [1]. □

Lemma 11.3.2 (Loss rate)**.**

$$\partial_t \int_{B_R} u dx = \frac{1}{R}\int_{\partial B_R} \Big(\dot{R} - b_0(u)\Big)\, u d\mathcal{H}_x^{d-1}. \tag{11.13}$$

Proof. We can write the total mass of u in terms of n and calculate the derivative via

$$\begin{aligned}
\partial_t \int_{B_R} u dx &= \partial_t \Big(R^{d-m} \int_{B_1} n dx \Big) \\
&= (d-m) R^{d-m} f \int_{B_1} n dx \\
&\quad + R^{d-m} \int_{B_1} \big(R^{m-2} \Delta \log n + (mn + x \cdot \nabla n) f \big) dx \\
&= d R^{d-m} f \int_{B_1} n dx - R^{d-1} \int_{\partial B_1} g(n) n d\mathcal{H}^{d-1}_x \\
&\quad + f R^{d-m} \int_{\partial B_1} n d\mathcal{H}^{d-1}_x - d R^{d-m} f \int_{B_1} n dx \\
&= R^{d-m} f \int_{\partial B_1} n d\mathcal{H}^{d-1}_x - R^{d-1} \int_{\partial B_1} g(n) n d\mathcal{H}^{d-1}_x \\
&= f \int_{\partial B_R} u d\mathcal{H}^{d-1}_x - \frac{1}{R} \int_{\partial B_R} g(n) n d\mathcal{H}^{d-1}_x . \qquad \square
\end{aligned}$$

Chapter 12

High temperature energy transport model

Assume (n, E) is a solution of (7.22). For $\mathcal{E} = 1 - E$ we recall the system (9.1) by

$$\begin{aligned} \partial_t n &= \nabla \cdot \left(\frac{\mathcal{E}}{n(1-\eta n)} \nabla n \right), \\ \partial_t \mathcal{E} &= \frac{2d-1}{2d} \nabla \cdot \frac{\nabla \mathcal{E}}{n(1-\eta n)} - \frac{\kappa \mathcal{E}}{n(1-\eta n)} |\nabla n|^2 . \end{aligned} \tag{12.1}$$

In this chapter, we try to find a weak solution of (12.1) and face certain difficulties. The main two difficulties of these equations are the degeneracy of $(12.1)_1$ in $\mathcal{E} = 0$ and the last term on the right-hand side of $(12.1)_2$, which has a critical exponent in ∇n. Introducing the total energy $\mathcal{E}_{\text{tot}} = \mathcal{E} - \frac{\kappa}{2} n^2$, we can rewrite this system of equations to

$$\begin{aligned} \partial_t n &= \nabla \cdot \left(\frac{\mathcal{E}}{n(1-\eta n)} \nabla n \right), \\ \partial_t \mathcal{E}_{\text{tot}} &= \frac{2d-1}{2d} \nabla \cdot \frac{\nabla \mathcal{E}}{n(1-\eta n)} - \kappa \nabla \cdot \frac{\mathcal{E} \nabla n}{(1-\eta n)} . \end{aligned} \tag{12.2}$$

This chapter shows the existence of an almost weak solution of (12.2) by means of Definition 12.1.1. In the proof, we approximate (12.2) by a time discrete version without degeneracies. However, in the limit of the approximating solutions, we cannot prove that the approximating n converges strongly in L^2, because we cannot apply an Aubin-Lions Lemma for the approximating n due to the degeneracy in the first equation. In addition, if we work with system of equations from (12.2), we will loose the maximum principle for the lower bound of $\mathcal{E}$ in the approximation. These two issues are the main reason, why we only find an almost weak upper solution.

12.1 Results

Definition 12.1.1. We call $(n, \mathcal{E}, \mathcal{E}_{\text{tot}})$ with $n, \mathcal{E} \in L^\infty((0,\infty)\times\Omega)$, $\mathcal{E} \in L^2_{\text{loc}}(0,\infty; H^1(\Omega))$, $\mathcal{E}n \in L^2_{\text{loc}}(0,\infty; H^1(\Omega))$ and $n, \mathcal{E}_{\text{tot}} \in H^1_{\text{loc}}(0,\infty; H^1(\Omega)')$ a weak upper solution of (12.1) if $\nabla\mathcal{E}, \mathcal{E}\nabla n \in L^2(0,\infty; L^2(\Omega))$, $\partial_t n, \partial_t \mathcal{E}_{\text{tot}} \in L^2(0,\infty, H^1(\Omega)')$ and

$$\begin{aligned}
&\int_0^\infty \langle \partial_t n, \phi_0\rangle\, dt + \int_0^\infty \int_\Omega \frac{\mathcal{E}\nabla n}{n(1-\eta n)} \cdot \nabla\phi_0 dxdt = 0\\
&\int_0^\infty \langle \partial_t \mathcal{E}_{\text{tot}}, \phi_1\rangle\, dt + \frac{2d-1}{2d}\int_0^\infty \int_\Omega \frac{\nabla\mathcal{E}\cdot\nabla\phi_1}{n(1-\eta n)} dxdt \\
&\qquad - \kappa \int_0^\infty \int_\Omega \frac{\mathcal{E}\nabla n}{1-\eta n}\cdot\nabla\phi_1 dxdt = 0
\end{aligned} \tag{12.3}$$

for all $\phi_i \in L^2(0,\infty; H^1(\Omega))$, where

$$\mathcal{E}\nabla n = \nabla(\mathcal{E}n) - n\nabla\mathcal{E} \quad \text{and} \quad \mathcal{E}_{\text{tot}} \le \mathcal{E} - \frac{\kappa}{2}n^2.$$

The last inequality has to be understood in $L^2_{\text{loc}}(0,\infty; H^1(\Omega))$ as

$$\int_0^\infty \langle \mathcal{E}_{\text{tot}}, \psi\rangle\, dt \le \int_0^\infty \left\langle \mathcal{E} - \frac{\kappa}{2}n^2, \psi \right\rangle dt$$

for all $\psi \in L^2_c(0,T; H^1(\Omega))$ with $\psi \ge 0$.

Theorem 12.1.2. *Let $\mathcal{E}^0, n^0 \in L^\infty(\Omega)$ be nonnegative such that $\delta \le n^0 \le \frac{1-\delta}{\eta}$ and*

$$\int_\Omega \mathcal{E}_{\text{tot}}(0)dx := \int_\Omega \left(\mathcal{E}^0 - \frac{\kappa}{2}(n^0)^2\right) dx > -\frac{\kappa}{2\,|\Omega|}\left(\int_\Omega n^0 dx\right)^2.$$

Then there exists a non-trivial weak upper solution $(n, \mathcal{E}, \mathcal{E}_{\text{tot}})$ of the system (12.1). Moreover, $n, \mathcal{E}_{\text{tot}} \in C^0([0,\infty); H^1(\Omega)')$ fulfill the initial conditions

$$n(0) = n^0 \quad \text{and} \quad \mathcal{E}_{\text{tot}}(0) = \mathcal{E}^0 - \frac{\kappa}{2}(n^0)^2.$$

In addition, the solution satisfies $\delta \le n \le \|n^0\|_\infty \le \frac{1-\delta}{\eta}$ and $0 \le \mathcal{E} \le \|\mathcal{E}^0\|_\infty$ as well as

$$\int_\Omega \left(\mathcal{E}(t) - \frac{\kappa}{2}n(t)^2\right) dx \ge \int_\Omega \left(\mathcal{E}^0 - \frac{\kappa}{2}(n^0)^2\right) dx \quad \textit{for a.e. } t > 0. \tag{12.4}$$

Corollary 12.1.3. *The solutions n and $\mathcal{E}$ found in the proof of Theorem 12.1.2 fulfill*

$$\begin{aligned}
&\int_0^\infty \langle \partial_t n, \phi_0 \rangle \, dt + \int_0^\infty \int_\Omega \frac{\mathcal{E} \nabla n}{n(1-\eta n)} \cdot \nabla \phi_0 dx dt = 0 \\
&\Big\langle \mathcal{E}^0 - \frac{\kappa}{2}(n^0)^2, \phi_1(0) \Big\rangle + \int_0^\infty \Big\langle \mathcal{E} - \frac{\alpha}{2} n^2, \partial_t \phi_1 \Big\rangle \, dt \\
&\quad \geq \frac{2d-1}{2d} \int_0^\infty \int_\Omega \frac{\nabla \mathcal{E} \cdot \nabla \phi_1}{n(1-\eta n)} dx dt + \kappa \int_0^\infty \int_\Omega \frac{\mathcal{E} \nabla n}{1-\eta n} \cdot \nabla \phi_1 dx dt
\end{aligned} \tag{12.5}$$

for all $\phi_0 \in L^2(0, \infty; H^1(\Omega))$ and $\phi_1 \in L^2(0, \infty; H^1(\Omega)) \cap W^{1,1}(0, \infty; L^1(\Omega))$, being compactly supported in $[0, \infty)$ with $\partial_t \phi_1 \geq 0$.

12.2 Proof of Theorem 12.1.2

Existence of approximate solution of (12.2)

In order to find a solution of (12.2), we need to approximate it. We use an approximation with discrete time steps as in [28]. Let $\tau > 0$, $k \in \mathbb{N}$, $\varepsilon, \gamma > 0$ and let $\delta > 0$ be sufficiently small. Given $\mathcal{E}^{k-1}, n^{k-1}$ such that $\delta \leq n^{k-1} \leq \|n^0\|_\infty \leq \frac{1-\delta}{\eta}$ and $\mathcal{E}^{k-1} \leq \|\mathcal{E}^0\|_\infty$, we introduce

$$\begin{cases}
0 = \frac{1}{\tau} \langle n^k - n^{k-1}, \phi_0 \rangle + \int_\Omega \frac{\mathcal{E}^k_{[\gamma]}}{n^k(1-\eta n^k)} \nabla n^k \cdot \nabla \phi_0 dx, \\
\varepsilon \sum_{\substack{a \in \mathbb{N}_0^d \\ |a| \leq 2}} \langle \partial^a \mathcal{E}^k, \partial^a \phi_1 \rangle = -\frac{1}{\tau} \Big\langle \mathcal{E}^k - \mathcal{E}^{k-1} - \frac{\kappa}{2}((n^k)^2 - (n^{k-1})^2), \phi_1 \Big\rangle \\
\qquad - \gamma \int_\Omega \nabla \mathcal{E}^k \cdot \nabla \phi_1 dx - \kappa \int_\Omega \frac{\mathcal{E}^k_{[\gamma]} \nabla n^k}{1-\eta n^k} \cdot \nabla \phi_1 dx \\
\qquad - \frac{2d-1}{2d} \int_0^\infty \int_\Omega \frac{\nabla \mathcal{E}^k_{[\gamma]} \cdot \nabla \phi_1}{n^k(1-\eta n^k)} dx
\end{cases} \tag{12.6}$$

for all $\phi_0 \in H^1(\Omega)$, where $\langle f, g \rangle = \int_\Omega f(x) g(x) dx$ and

$$f_{[\gamma]} = \min\{1/\gamma, \max\{\gamma, f\}\}.$$

In contrast to [28], we need further terms in the approximation: The term involving ε and the truncation $\mathcal{E}^k_{[\gamma]}$ in the first equation are necessary in order to guarantee a global weak solution of this system. Letting $\varepsilon \to 0$, we still need an estimate for the L^2 norm of $\nabla \mathcal{E}^k$ for the limit $\tau \to 0$, which motivates the extra integral over $\nabla \mathcal{E}^k \cdot \nabla \phi$ in the second equation. This could also be achieved if we replaced the second to last integral by $\frac{2d-1}{2d} \int_0^\infty \int_\Omega \frac{\nabla \mathcal{E}^k \cdot \nabla \phi_1}{n^k(1-\eta n^k)} dx$ without truncation. However, this would result in some difficulties in the limit $\gamma \to 0$ since we are not able to show that $\mathcal{E}^k$ and n^k converge strongly as $\gamma \to 0$.

Proposition 12.2.1. *There exists a solution of* (12.6) *such that* $\delta \le n^k \le \|n^0\|_\infty \le \frac{1-\delta}{\eta}$.

Proof. We solve this equation using Leray-Schauder's fixed point theorem and thus need to define the fixed point operator $S : L^2(\Omega) \times H^1(\Omega) \times [0,1] \to L^2(\Omega) \times H^1(\Omega)$. For $\tilde{n} \in L^2(\Omega)$, $\tilde{\mathcal{E}} \in H^1(\Omega)$ and $\theta \in [0,1]$, let $S((\tilde{n}, \tilde{\mathcal{E}}), \theta) := (n, \mathcal{E})$ be the unique solution of

$$\langle n, \phi_0 \rangle_0 = F_0(\phi_0) \quad \text{and} \quad \langle \mathcal{E}, \phi_1 \rangle_1 = F_1(n, \phi_1) \quad \text{for all } \phi_0, \phi_1 \in H^1(\Omega), \tag{12.7}$$

where

$$\langle n, \phi_0 \rangle_0 := \int_\Omega \frac{\tilde{\mathcal{E}}_{[\gamma]}}{\tilde{n}^{[\delta]}(1-\eta\tilde{n}^{[\delta]})} \nabla n \cdot \nabla \phi_0 dx + \frac{1}{\tau} \langle n, \phi_0 \rangle ,$$

and $F_0(\phi_0) = \frac{\theta}{\tau} \langle n^{k-1}, \phi_0 \rangle$ and

$$\langle \mathcal{E}, \phi_1 \rangle_1 := \varepsilon \sum_{\substack{a \in \mathbb{N}_0^d \\ |a| \le 2}} \langle \partial^a \mathcal{E}, \partial^a \phi_1 \rangle + \gamma \int_\Omega \nabla \mathcal{E} \cdot \nabla \phi_1 dx$$

as well as

$$\begin{aligned} F_1(n, \phi_1) := &-\frac{\theta}{\tau} \left\langle \tilde{\mathcal{E}} - \mathcal{E}^{k-1} - \frac{\kappa}{2}((\tilde{n}^{[\delta]})^2 - (n^{k-1})^2), \phi_1 \right\rangle \\ &- \frac{2d-1}{2d} \theta \int_0^\infty \int_\Omega \frac{\nabla \tilde{\mathcal{E}}_{[\gamma]} \cdot \nabla \phi_1}{\tilde{n}^{[\delta]}(1-\eta\tilde{n}^{[\delta]})} dx - \theta\kappa \int_\Omega \frac{\tilde{\mathcal{E}}_{[\gamma]} \nabla n}{1-\eta\tilde{n}^{[\delta]}} \cdot \nabla \phi_1 dx. \end{aligned}$$

Here, we have used the notation $\tilde{n}^{[\delta]} := \min\{\frac{1-\delta}{\eta}, \max\{\delta, \tilde{n}\}\}$. The function n is well-defined due to the Lax-Milgram Lemma and the facts that $\langle \cdot, \cdot \rangle_0$, F_0 are bounded on $H^1(\Omega)$ and $\langle \cdot, \cdot \rangle_0$ is coercive. Moreover, Lax-Milgram's lemma provides that $n \in H^1(\Omega)$. As a consequence, $F_1(n, \cdot) \in H^1(\Omega)' \subseteq H^2(\Omega)'$. Thus, $\mathcal{E}$ is also well-defined by the same argumentation and it holds $\mathcal{E} \in H^2(\Omega)$. In particular, we see that S has a compact range. In the following, we will assume that $\delta \le \tilde{n} \le \frac{1-\delta}{\eta}$, motivated by the fact that $S((\tilde{n}^{[\delta]}, \tilde{\mathcal{E}}), \theta) = S((\tilde{n}, \tilde{\mathcal{E}}), \theta)$. Thus, we can drop the ${}^{[\delta]}$ in the equations.

Next, we are going to prove that S is continuous. Let $\tilde{n}_l \to \tilde{n}$ in $L^2(\Omega)$, $\tilde{\mathcal{E}}_l \to \tilde{\mathcal{E}}_l$ in $H^1(\Omega)$ and $\theta_l \to \theta$ for $l \to \infty$. Since $\langle \cdot, \cdot \rangle_0$ and $\langle \cdot, \cdot \rangle_1$ are coercive, we obtain $\|n_l\|_{H^1(\Omega)} \le C + C\|\tilde{n}_l\|_{L^2(\Omega)}$ and $\|\mathcal{E}_l\|_{H^2(\Omega)} \le C + C\|\tilde{\mathcal{E}}_l\|_{H^1(\Omega)}$ for some $C > 0$, where we have set $(n_l, \mathcal{E}_l) := S((\tilde{n}_l, \tilde{\mathcal{E}}_l), \theta_l)$. According to the compact embeddings $H^i(\Omega) \subset\subset H^{i-1}(\Omega)$ for $i = 1, 2$, we can extract a subsequence which is not relabeled such that

$$(n_l, \mathcal{E}_l) \to y \text{ in } L^2(\Omega) \times H^1(\Omega) \quad \text{and} \quad (n_l, \mathcal{E}_l) \rightharpoonup y \text{ weakly in } H^1(\Omega) \times H^2(\Omega)$$

for some $y = (y_0, y_1) \in H^1(\Omega) \times H^2(\Omega)$. For $\phi_0 \in W^{1,\infty}(\Omega)$, we can take the limit in the first equation of (12.7) since $(\mathcal{E}_l)_{[\gamma]}/(\tilde{n}_l(1-\eta\tilde{n}_l))$ converges strongly in $L^2(\Omega)$ to $\mathcal{E}_{[\gamma]}/(\tilde{n}(1-\eta\tilde{n}))$ and ∇n_l converges weakly in $L^2(\Omega)$. Since the solution of (12.7) is unique, we have $y_0 = n$ and that the whole series converges. Likewise, we can prove that the integrals in the definition of $F_2(n_l, \phi_1)$ converge setting $\tilde{n} = \tilde{n}_l$ and $\tilde{\mathcal{E}} = \tilde{\mathcal{E}}_l$ if $\phi_1 \in W^{2,\infty}(\Omega)$. Therefore, using the same arguments as before, we end up with $y_1 = \mathcal{E}$ and in particular with the result that S is continuous. Furthermore, we easily deduce that $S((\tilde{n}, \tilde{\mathcal{E}}), 0) = 0$.

For the Leray-Schauder's fixed point theorem we require additionally that all potential fixed points of $S(\cdot, \theta)$ are uniformly bounded: Inserting $\phi_0 = \tilde{n} = n$ in (12.7), we have

$$\frac{1}{2\tau}\int_\Omega n^2 dx + \frac{\gamma}{(\eta-\delta)(1-\eta\delta)}\int_\Omega |\nabla n|^2 dx \le \frac{\theta}{2\tau}\int_\Omega (n^{k-1})^2 dx$$

due to Cauchy-Schwarz' inequality for $\langle\cdot,\cdot\rangle$. Let $\Pi_1 : L^2(\Omega) \times H^1(\Omega) \to L^2(\Omega)$ be the projection on the first component. Then, we have shown that the image of $\Pi_1 \circ S$ is uniformly bounded in $H^1(\Omega)$. Now, we insert $\phi_1 = \tilde{\mathcal{E}} = \mathcal{E}$ (12.7) and estimate

$$\begin{aligned}\varepsilon \|\mathcal{E}\|^2_{H^2(\Omega)} \le \frac{\theta}{\tau}\left\|\mathcal{E}^{k-1}\right\|_{L^2(\Omega)} \|\mathcal{E}\|_{L^2(\Omega)} \\ + \frac{\theta\kappa}{\tau}(\delta^2)\sqrt{|\Omega|}\,\|\mathcal{E}\|_{L^2(\Omega)} + \frac{\theta\kappa}{\gamma(1-\delta}\|n\|_{L^2(\Omega)}\|\mathcal{E}\|_{L^2(\Omega)}\end{aligned}$$

again by Cauchy-Schwarz' inequality. Since, we already know that image of $\Pi_1 \circ S$ is uniformly bounded in $H^1(\Omega)$, we can use the embedding $L^2(\Omega) \subseteq H^2(\Omega)$ and Young's inequality to see that the whole image of S is uniformly bounded in $H^1(\Omega)$.

Finally, we have shown the hypothesis of Leray-Schauder's fixed point theorem. Beyond this, by the Stampaccia method/maximum principle, we are able to ensure that the solution $(n^k, \mathcal{E}^k)$ fulfills $\delta \le n^k \le \|n^0\|_\infty \le \frac{1-\delta}{\eta}$. Thus, we obtain a weak solution of (12.6). $\square$

The limit $\varepsilon \to 0$

In order to emphasize the dependency of ε, we write n^k_ε and $\mathcal{E}^k_\varepsilon$ for the solution of (12.6). Testing Eq. (12.6) with $\phi_0 = n^k_\varepsilon$ and $\phi_1 = \mathcal{E}^k_\varepsilon$ ensures that

$$\frac{1}{2\tau}\int_\Omega (n^k_\varepsilon)^2 dx + \frac{\gamma}{(\eta-\delta)(1-\eta\delta)}\int_\Omega \left|\nabla n^k_\varepsilon\right|^2 dx \le \frac{1}{2\tau}\int_\Omega (n^{k-1})^2 dx$$

and

$$\frac{1}{\tau}\left\|(\mathcal{E}^k)_{[\gamma]}\right\|^2_{L^2(\Omega)} + \varepsilon\left\|\mathcal{E}^k_\varepsilon\right\|^2_{H^2(\Omega)} + \gamma\left\|\nabla\mathcal{E}^k_\varepsilon\right\|^2_{L^2(\Omega)} \\ \leq \frac{1}{\tau}\left\|\mathcal{E}^{k-1}\right\|_{L^2(\Omega)}\left\|\mathcal{E}^k_\varepsilon\right\|_{L^2(\Omega)} + \frac{\kappa}{\tau}(\delta^2)\sqrt{|\Omega|}\left\|\mathcal{E}^k_\varepsilon\right\|_{L^2(\Omega)} \\ + \frac{\kappa}{\gamma(1-\delta)}\left\|n^k_\varepsilon\right\|_{L^2(\Omega)}\left\|(\mathcal{E}^k_\varepsilon)_{[\gamma]}\right\|_{L^2(\Omega)}. \tag{12.8}$$

Thus, n^k_ε is uniformly bounded in $H^1(\Omega)$. There exists a sequence $\varepsilon = \varepsilon(l) \to 0$ such that $n^k_{\varepsilon(l)} \rightharpoonup n^k$ in $H^1(\Omega)$ and $n^k_{\varepsilon(l)} \to n^k$ in $L^2(\Omega)$ for $l \to \infty$ for some $n^k \in H^1(\Omega)$. Using the uniform bound on n^k_ε in $H^1(\Omega)$ and Young's inequality, we infer from (12.8) that there exist a constant $C > 0$ independent from ε such that

$$\frac{1}{2\tau}\left\|\mathcal{E}^k_\varepsilon\right\|^2_{L^2(\Omega)} + \varepsilon\left\|\mathcal{E}^k_\varepsilon\right\|^2_{H^2(\Omega)} + \gamma\left\|\nabla\mathcal{E}^k_\varepsilon\right\|^2_{L^2(\Omega)} \leq C.$$

In particular, we can find a subsequence of $\varepsilon(l)$, which we again denote by $\varepsilon(l)$ and $\mathcal{E}, \xi \in H^1(\Omega)$ such that $\mathcal{E}^k_{\varepsilon(l)} \rightharpoonup \mathcal{E}^k$, $(\mathcal{E}_{\varepsilon(l)})^k_{[\gamma]} \rightharpoonup \xi$ in $H^1(\Omega)$ and $\mathcal{E}^k_{\varepsilon(l)} \to \mathcal{E}^k$, $(\mathcal{E}_{\varepsilon(l)})^k_{[\gamma]} \to \xi$ in $L^2(\Omega)$ for $l \to \infty$. Du to the strong convergence in $L^2(\Omega)$, it holds $\xi = (\mathcal{E}^k)_{[\gamma]}$. Moreover, we easily check that $\varepsilon(l)\mathcal{E}^k_{\varepsilon(l)} \rightharpoonup 0$ in $H^2(\Omega)$. Finally, we can take the limit $\varepsilon = \varepsilon(l) \to 0$ in (12.6) for $\phi_0, \phi_1 \in W^{2,\infty}(\Omega)$ and obtain

$$0 = \frac{1}{\tau}\left\langle n^k - n^{k-1}, \phi_0\right\rangle + \int_\Omega \frac{\mathcal{E}^k_{[\gamma]}}{n^k(1-\eta n^k)}\nabla n^k \cdot \nabla\phi_0 dx, \\ 0 = \frac{1}{\tau}\left\langle \mathcal{E}^k - \mathcal{E}^{k-1} - \frac{\kappa}{2}((n^k)^2 - (n^{k-1})^2), \phi_1\right\rangle + \gamma\int_\Omega \nabla\mathcal{E}^k \cdot \nabla\phi_1 dx \\ + \frac{2d-1}{2d}\int_0^\infty \int_\Omega \frac{\nabla\mathcal{E}^k_{[\gamma]} \cdot \nabla\phi_1}{n^k(1-\eta n^k)} dx + \kappa\int_\Omega \frac{\mathcal{E}^k_{[\gamma]}\nabla n^k}{1-\eta n^k} \cdot \nabla\phi_1 dx \tag{12.9}$$

for all $\phi_0, \phi_1 \in H^1(\Omega)$ since $W^{2,\infty}(\Omega) \subseteq H^1(\Omega)$ is dense. Again $\delta \leq n^k \leq \left\|n^0\right\|_\infty \leq \frac{1-\delta}{\eta}$ holds true, because of the strong convergence of $n^k_{\varepsilon(l)}$ and the fact that $\delta \leq n^k_{\varepsilon(l)} \leq \left\|n^0\right\|_\infty$ for all $l \in \mathbb{N}$. We can rewrite the second equation with the aid of the first one by

$$0 = \frac{1}{\tau}\left\langle \mathcal{E}^k - \mathcal{E}^{k-1} + \frac{\kappa}{2}(n^k - n^{k-1})^2, \phi_1\right\rangle + \gamma\int_\Omega \nabla\mathcal{E}^k \cdot \nabla\phi_1 dx \\ + \frac{2d-1}{2d}\int_0^\infty \int_\Omega \frac{\nabla\mathcal{E}^k_{[\gamma]} \cdot \nabla\phi_1}{n^k(1-\eta n^k)} dx + \kappa\int_\Omega \frac{\mathcal{E}^k_{[\gamma]}\left|\nabla n^k\right|^2}{n^k(1-\eta n^k)}\phi_1 dx. \tag{12.10}$$

Thus, the maximum principle ensures that $\mathcal{E}^k \leq \operatorname{ess\,sup}_x \mathcal{E}^{k-1}(x) \leq \left\|\mathcal{E}^0\right\|_\infty$.

The limit $\tau \to 0$

To start with, let $T \in \mathbb{N}$ and assume that $T/\tau \in \mathbb{N}$. We define $n_\tau(t) := n^k$ and $\mathcal{E}_\tau(t) := \mathcal{E}^k$ for $t \in ((k-1)\tau, k\tau]$. Moreover, for any function $\psi : (0,T) \to L^1(\Omega)$, we define the discrete time derivation D_τ by $D_\tau \psi(t) := \frac{1}{\tau}(\psi(t) - \psi(t-\tau))$. Summing Equations (12.9) over all $k \le T/\tau$ yields that $n_\tau, \mathcal{E}_\tau$ solve

$$\begin{aligned}
&\int_0^T \langle D_\tau n_\tau, \phi_0 \rangle \, dt + \int_0^T \int_\Omega \frac{(\mathcal{E}_\tau)_{[\gamma]}}{n_\tau(1-\eta n_\tau)} \nabla n_\tau \cdot \nabla \phi_0 dx dt = 0 \\
&\int_0^T \left\langle D_\tau \left(\mathcal{E}_\tau - \frac{\kappa}{2} n_\tau^2 \right), \phi_1 \right\rangle dt + \frac{2d-1}{2d} \int_0^T \int_\Omega \frac{\nabla (\mathcal{E}_\tau)_{[\gamma]} \cdot \nabla \phi_1}{n_\tau(1-\eta n_\tau)} dx dt \\
&\qquad + \gamma \int_0^T \int_\Omega \nabla \mathcal{E}_\tau \cdot \nabla \phi_1 dx dt + \kappa \int_0^T \int_\Omega \frac{(\mathcal{E}_\tau)_{[\gamma]}}{1-\eta n_\tau} \nabla n_\tau \cdot \nabla \phi_1 dx dt = 0
\end{aligned} \tag{12.11}$$

for all piecewise constant functions $\phi := (\phi_0, \phi_1) : (0,T) \to H^1(\Omega) \times (H^1(\Omega) \cap L^\infty(\Omega))$ which are dense in $L^2(0,T;H^1(\Omega)) \times L^2(0,T;H^1(\Omega)) \cap L^\infty((0,T) \times \Omega))$, see [46, Prop 1.36].

In order to extract a converging subsequence of n_τ and $\mathcal{E}_\tau$, we need some a priori estimates. The main ingredient will be the discrete Aubin-Lions Lemma from [18]. Let $0 < t \le T$ and let $\chi_{[0,t]}$ denote the characteristic function on $[0,t]$. Inserting $\phi_0 := n_\tau \chi_{[0,t]}$ in Eq. $(12.11)_1$ entails

$$\frac{1}{2} \int_\Omega n_\tau(t)^2 dx + \int_0^t \int_\Omega \frac{(\mathcal{E}_\tau)_{[\gamma]}}{n_\tau(1-\eta n_\tau)} |\nabla n_\tau|^2 \, dx ds \le \frac{1}{2} \int_\Omega (n^0)^2 dx \tag{12.12}$$

Thus, n_τ is bounded in $L^\infty(0,T;L^\infty(\Omega)) \cap L^2(0,T;H^1(\Omega))$. Note that we have a uniform bound for all T. Next, we derive an estimate on the discrete time derivative of n_τ by

$$\left| \int_\tau^T \langle D_\tau n_\tau, \phi_0 \rangle \, dt \right| \le \left\| \frac{(\mathcal{E}_\tau)_{[\gamma]}}{n_\tau(1-\eta n_\tau)} \right\|_{L^\infty} \|\nabla n_\tau\|_{L^2 L^2} \|\phi_0\|_{L^2 H^1},$$

which is again uniform in T. A similar estimate for the energy is not possible due to the latter two terms on the left-hand side. However, switching to the total energy $\mathcal{E}_{\mathrm{tot},\tau} := \mathcal{E}_\tau - \frac{\kappa}{2} n_\tau^2$, we insert $\phi_0 := \mathcal{E}_{\mathrm{tot},\tau} \chi_{[0,t]}$ in Eq. $(12.11)_2$

and estimate

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \mathcal{E}_{\text{tot},\tau}(t)^2 dx + \frac{2d-1}{4}\int_0^t\int_\Omega \frac{\left|\nabla(\mathcal{E}_\tau)_{[\gamma]}\right|^2}{n_\tau(1-\eta n_\tau)} dxds + \frac{\gamma}{2}\int_0^t\int_\Omega \left|\nabla\mathcal{E}_\tau\right|^2 dxds \\
&\quad \leq \frac{1}{2}\int_\Omega \mathcal{E}_{\text{tot},\tau}(0)^2 dx + \kappa\frac{2d-1}{4}\int_0^t\int_\Omega \frac{\left|\nabla n_\tau\right|^2}{1-\eta n_\tau} dxds \\
&\qquad + \frac{\kappa\gamma}{2}\int_0^t\int_\Omega n_\tau^2\left|\nabla n_\tau\right|^2 dxds - \kappa\int_0^t\int_\Omega \frac{(\mathcal{E}_\tau)_{[\gamma]}}{1-\eta n_\tau}\nabla n_\tau\cdot\nabla\mathcal{E}_\tau dxds \\
&\qquad\qquad + \kappa^2\int_0^t\int_\Omega \frac{(\mathcal{E}_\tau)_{[\gamma]} n_\tau}{1-\eta n_\tau}\left|\nabla n_\tau\right|^2 dxds \qquad (12.13)
\end{aligned}$$

using $a\cdot(a-b)\geq\frac{a^2}{2}-\frac{b^2}{2}$ for $a,b\in\mathbb{R}^d$. Again by Young's inequality, we can treat the first term of the last line by

$$\begin{aligned}
&-\kappa\int_0^t\int_\Omega \frac{(\mathcal{E}_\tau)_{[\gamma]}}{1-\eta n_\tau}\nabla n_\tau\cdot\nabla\mathcal{E}_\tau dxds \\
&\qquad \leq \frac{\gamma}{4}\int_0^t\int_\Omega \left|\nabla\mathcal{E}_\tau\right|^2 dxds + \frac{\kappa^2}{\gamma}\int_0^t\int_\Omega \frac{(\mathcal{E}_\tau)_{[\gamma]}^2}{(1-\eta n_\tau)^2}\left|\nabla n_\tau\right|^2 dxds.
\end{aligned}$$

Putting both estimates together, there exists a constant $C>0$ independent from t such that

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \mathcal{E}_{\text{tot},\tau}(t)^2 dx + \frac{2d-1}{4}\int_0^t\int_\Omega \frac{\left|\nabla(\mathcal{E}_\tau)_{[\gamma]}\right|^2}{n_\tau(1-\eta n_\tau)} dxds + \frac{\gamma}{4}\int_0^t\int_\Omega \left|\nabla\mathcal{E}_\tau\right|^2 dxds \\
&\qquad \leq \frac{1}{2}\int_\Omega \mathcal{E}_{\text{tot},\tau}(0)^2 dx + C \qquad (12.14)
\end{aligned}$$

since $n_\tau, (\mathcal{E}_\tau)_{[\gamma]}$ are uniformly bounded and ∇n_τ is uniformly bounded in $L^2(0,\infty;L^2(\Omega))$. Therefore, we can use (12.12) and (12.14) to see that $\mathcal{E}_{\text{tot},\tau}$ is also (uniformly w.r.t. τ) bounded in $L^2(0,T;H^1(\Omega))$. Moreover, we calculate

$$\begin{aligned}
&\left|\int_\tau^T \left\langle D_\tau\mathcal{E}_{\text{tot},\tau},\phi_1\right\rangle dt\right| \leq \left(\left\|\frac{n_\tau(\mathcal{E}_\tau)_{[\gamma]}+1}{n_\tau(1-\eta n_\tau)}\right\|_{L^\infty}+\gamma\right) \\
&\qquad\qquad \cdot\left(\left\|\nabla\mathcal{E}_\tau\right\|_{L^2L^2}+\left\|\nabla n_\tau\right\|_{L^2L^2}\right)\left\|\phi_1\right\|_{L^2H^1}.
\end{aligned}$$

Finally, we have shown the hypothesis for n_τ and $\mathcal{E}_{\text{tot},\tau}$ of the discrete Aubin-Lions Lemma from [18]. Hence, there exists a subsequence $\tau(l)\to 0$ and $n,\mathcal{E}_{\text{tot}}\in\mathcal{H}:=L^2(0,T;H^1(\Omega))\cap H^1(0,T;H^1(\Omega)')$ such that see that $n_\tau\to n$ and $\mathcal{E}_{\text{tot},\tau}\to\mathcal{E}_{\text{tot}}$ in $L^2(0,T;L^2(\Omega))$ as well as $n_\tau\rightharpoonup n$ and $\mathcal{E}_{\text{tot},\tau}\rightharpoonup\mathcal{E}_{\text{tot}}$ in $\mathcal{H}$. Since n_τ is uniformly bounded in $L^\infty(0,T;L^\infty(\Omega))$, n_τ convergences strongly in every $L^p(0,T;L^p(\Omega))$ with $p<\infty$. In particular, we have that

$\mathcal{E}_\tau \to \mathcal{E} := \mathcal{E}_{\text{tot}} + \frac{\kappa}{2} n^2$ in $L^2(0,T;L^2(\Omega))$ and $\mathcal{E}_\tau \rightharpoonup \mathcal{E}$ in $L^2(0,T;H^1(\Omega))$. Thus, $(\mathcal{E}_\tau)_{[\gamma]} \to \mathcal{E}_{[\gamma]}$ in every $L^p(0,T;L^p(\Omega))$ since $(\mathcal{E}_\tau)_{[\gamma]}$ is uniformly bounded in $L^\infty(0,\infty;L^\infty(\Omega))$. Moreover, we can easily check that $(\mathcal{E}_\tau)_{[\gamma]} \rightharpoonup \mathcal{E}_{[\gamma]}$ in $L^2(0,T;H^1(\Omega))$. Finally, we note that we can extract a diagonal sequence such that every convergence stated above holds simultaneously for all $T \in \mathbb{N}$ and consequently for all $T > 0$.

Let $\phi_0, \phi_1 \in L^2(0,\infty;H^1(\Omega))$ have compact support in $[0,\infty)$. We can take the limit in Eq. (12.11) along a subsequence and obtain

$$\begin{aligned}
&\int_0^\infty \langle \partial_t n, \phi_0 \rangle \, dt + \int_0^\infty \int_\Omega \frac{\mathcal{E}_{[\gamma]} \nabla n}{n(1-\eta n)} \cdot \nabla \phi_0 dx dt = 0, \\
&\int_0^\infty \left\langle \partial_t \big(\mathcal{E} - \frac{\kappa}{2} n^2\big), \phi_1 \right\rangle dt + \frac{2d-1}{2d} \int_0^\infty \int_\Omega \frac{\nabla \mathcal{E}_{[\gamma]} \cdot \nabla \phi_1}{n(1-\eta n)} dx dt \\
&\qquad + \gamma \int_0^\infty \int_\Omega \nabla \mathcal{E} \cdot \nabla \phi_1 dx dt + \kappa \int_0^\infty \int_\Omega \frac{\mathcal{E}_{[\gamma]}}{1-\eta n} \nabla n \cdot \nabla \phi_1 dx dt = 0
\end{aligned} \tag{12.15}$$

for $n, \mathcal{E} \in \mathcal{H} \cap L^\infty((0,\infty)\times\Omega))$ with $\mathcal{H} := L^2(0,\infty;H^1(\Omega)) \cap H^1(0,\infty;H^1(\Omega)')$ form above and the uniform bounds w.r.t. T. As before, n and $\mathcal{E}$ fulfill $\delta \le n \le \|n^0\|_\infty \le \frac{1-\delta}{\eta}$ and $\mathcal{E} \le \|\mathcal{E}^0\|_\infty$, respectively.

Remark 12.2.2. The fact that $n, \mathcal{E} \in \mathcal{H}$ implies that $n, \mathcal{E} \in C^0([0,\infty);L^p(\Omega))$ for all $1 \le p < \infty$. Since the space of function from $L^2(0,\infty;H^1(\Omega))$ with compact support in $[0,\infty)$ are dense in $L^2(0,\infty;H^1(\Omega))$, Equation (12.15) is fulfilled for all $\phi_0, \phi_1 \in L^2(0,\infty;H^1(\Omega))$ without further restrictions.

In order to show that $n(0) = n^0$ and $\mathcal{E}(0) = \mathcal{E}^0$, we proceed similarly as in [28] and define for $u \in \{n, \mathcal{E}_{\text{tot}}\}$ the linear interpolation $\tilde{u}_\tau$ as

$$\tilde{u}_\tau(t) = u^k - \frac{k\tau - t}{\tau}(u^k - u^{k-1}) \quad \text{for } (k-1)\tau \le t \le k\tau.$$

Let $T > 0$. Since $D_\tau n_\tau, D_\tau \mathcal{E}_{\text{tot},\tau}$ are uniformly bounded in $L^2(\tau,T;H^1(\Omega)')$, we see by

$$\|\partial_t \tilde{u}_\tau\|_{L^2(0,T-\tau,H^1(\Omega)')} \le \|D_t u_\tau\|_{L^2(0,T-\tau,H^1(\Omega)')} \le C$$

for some $C > 0$ that $\tilde{u}_\tau$ is bounded in $H^1(0,T;H^1(\Omega)')$. Hence, a subsequence converges weakly to some w in $H^1(0,T;H^1(\Omega)')$ which is continuously embedded in $C^0([0,T];H^1(\Omega)')$. The fact that $\tilde{u}_\tau$ and u_τ converge to the same limit due to

$$\|\tilde{u}_\tau - u_\tau\|_{L^2(0,T-\tau,H^1(\Omega)')} \le \tau \|D_t u_\tau\|_{L^2(0,T-\tau,H^1(\Omega)')} \le C\tau \to 0 \quad \text{as } \tau \to 0$$

implies that $u^0 = \lim_{\tau\to 0} \tilde{u}_\tau(0) = \lim_{\tau \to 0} u_\tau(0) = u(0)$. Finally, we conclude that $n(0) = n^0$ and $\mathcal{E}(0) = \mathcal{E}_{\text{tot}}(0) + \frac{\kappa}{2} n(0)^2 = \mathcal{E}^0$.

The limit $\gamma \to 0$

In the sequel, let us denote n_γ and $\mathcal{E}_\gamma$ as the solution of (12.15) in order to emphasize the dependency on γ. This shall not be confused with n_τ and $\mathcal{E}_\tau$ from above. Let $T > 0$. The limit $\gamma \to 0$ is more delicate due to the degeneracy of the equation concerning n. However, the equation for $\mathcal{E}$ does not degenerate and we are able to show that $((\mathcal{E}_\gamma)_{[\gamma]})_\gamma \subset L^2(0,T;L^2(\Omega))$ is precompact. Clearly, this will be a result of an Aubin-Lions type lemma. Let $\phi_1 \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^\infty(\Omega))$. Testing the equation for n_γ with $\kappa n_\gamma \phi_1$ and subtracting this from Eq. $(12.15)_2$, we derive

$$\int_0^T \langle \partial_t \mathcal{E}_\gamma, \phi_1 \rangle \, dt + \frac{2d-1}{2d} \int_0^T \int_\Omega \frac{\nabla (\mathcal{E}_\gamma)_{[\gamma]} \cdot \nabla \phi_1}{n_\gamma (1 - \eta n_\gamma)} dx dt \\ + \gamma \int_0^T \int_\Omega \nabla \mathcal{E}_\gamma \cdot \nabla \phi_1 dx dt + \kappa \int_0^T \left\langle \frac{(\mathcal{E}_\gamma)_{[\gamma]} \left| \nabla n_\gamma \right|^2}{n_\gamma (1 - \eta n_\gamma)}, \phi_1 \right\rangle = 0. \quad (12.16)$$

Since $\mathcal{E}_\gamma \leq \left\| \mathcal{E}^0 \right\|_\infty$, we have for sufficiently small $\gamma > 0$ that

$$(\mathcal{E}_\gamma)_{[\gamma]} = (\mathcal{E}_\gamma - \gamma)_+ + \gamma := \max\{\mathcal{E}_\gamma - \gamma, 0\} + \gamma$$

and in particular $\nabla (\mathcal{E}_\gamma - \gamma)_+ = \nabla (\mathcal{E}_\gamma)_{[\gamma]}$. Setting $\phi_1 := (\mathcal{E}_\gamma(t) - \gamma)_+$ in (12.16), we see

$$\int_\Omega (\mathcal{E}_\gamma(T) - \gamma)_+^2 dx + \frac{2d-1}{2d} \int_0^T \int_\Omega \frac{\left| \nabla (\mathcal{E}_\gamma - \gamma)_+ \right|^2}{n_\gamma (1 - \eta n_\gamma)} dx dt \\ + \gamma \int_0^T \int_\Omega \left| \nabla (\mathcal{E}_\gamma - \gamma)_+ \right|^2 dx dt + \kappa \int_0^T \int_\Omega \frac{(\mathcal{E}_\gamma)_{[\gamma]} \left| \nabla n_\gamma \right|^2}{n_\gamma (1 - \eta n_\gamma)} (\mathcal{E}_\gamma - \gamma)_+ dx dt \\ \leq \int_\Omega (\mathcal{E}^0 - \gamma)_+^2 dx, \quad (12.17)$$

implying that $\nabla (\mathcal{E}_\gamma)_{[\gamma]} = \nabla (\mathcal{E}_\gamma - \gamma)_+$ is uniformly bounded in $L^2(0,T;L^2(\Omega))$. Moreover, testing Eq. $(12.15)_1$ with n_γ, we obtain that $\sqrt{(\mathcal{E}_\gamma)_{[\gamma]}} \nabla n_\gamma$ is also bounded in $L^2(0,T;L^2(\Omega))$. In both cases, the bound can be chosen in such a way that it is uniform for all T. We estimate

$$\left| \int_0^T \langle \partial_t (\mathcal{E}_\gamma - \gamma)_+^2, \phi \rangle \, dt \right| = 2 \left| \int_0^T \langle \partial_t (\mathcal{E}_\gamma - \gamma), (\mathcal{E}_\gamma - \gamma)_+ \phi \rangle \, dt \right| \\ \leq C \left(\left\| \nabla (\mathcal{E}_\gamma)_{[\gamma]} \right\|_{L^2 L^2}^2 + \left\| \sqrt{(\mathcal{E}_\gamma)_{[\gamma]}} \nabla n_\gamma \right\|_{L^2 L^2}^2 \right) \|\phi\|_{L^\infty} \\ + C \left\| \nabla (\mathcal{E}_\gamma)_{[\gamma]} \right\|_{L^2 L^2} \|\nabla \phi\|_{L^2 L^2} .$$

Using that $L^\infty((0,T)\times\Omega)\cap L^2(0,T;H^1(\Omega)) \supset C^0([0,T];H^s(\Omega))$ for $s > \min\{1,d/2\}$, we just have proved that $\partial_t(\mathcal{E}_\gamma-\gamma)_+^2$ is uniformly bounded in the space $C^0([0,T];H^s(\Omega))'$. Note that in addition, $\nabla(\mathcal{E}_\gamma-\gamma)_+^2$ is uniformly bounded in $L^2(0,T;L^2(\Omega))$. Therefore, we can apply the version of the Aubin-Lions Lemma from [41] to infer that $(\mathcal{E}_\gamma-\gamma)_+^2$ converges strongly in $L^2(0,T;L^2(\Omega))$ and weakly in $L^2(0,T;H^1(\Omega))$ to some $\xi\in L^2(0,T;H^1(\Omega))$ along a subsequence $\gamma=\gamma(l)\to 0$. Note that ξ is non-negative such that there exists its non-negative root $\mathcal{E}=\sqrt{\xi}$. Since $(\mathcal{E}_\gamma)_{[\gamma]}$ is uniformly bounded, we obtain that

$$(\mathcal{E}_{\gamma(l)})_{[\gamma(l)]} = (\mathcal{E}_{\gamma(l)}-\gamma(l))_+ + \gamma(l) \to \mathcal{E} \quad \text{in } L^2(0,T;L^2(\Omega)) \text{ for } l\to\infty$$

using Lebesgue's theorem. Moreover, we can assume that along this subsequence, we have $(\mathcal{E}_{\gamma(l)})_{[\gamma(l)]} \rightharpoonup \mathcal{E}$ in $L^2(0,T;H^1(\Omega))$. Regarding the variable n_γ, so far, we can only derive a weak convergence of a subsequence: there exist some $n\in L^2(0,T;L^2(\Omega))$ such that $n_{\gamma(l)}\rightharpoonup n$ in $L^2(0,T;L^2(\Omega))$.

The next step is to improve the convergence of $n_{\gamma(l)}$ – at least for a subsequence. Because of the degeneracy in Eq. $(12.15)_1$, we consider $z_l := (\mathcal{E}_{\gamma(l)}-\gamma)_+^2 n_{\gamma(l)}$ instead of $n_{\gamma(l)}$. Since $\nabla(\mathcal{E}_\gamma-\gamma)_+ = \nabla(\mathcal{E}_\gamma)_{[\gamma]}$, $\sqrt{(\mathcal{E}_\gamma)_{[\gamma]}}\nabla n_\gamma$ are bounded in $L^2(0,T;L^2(\Omega))$ and $\delta\le n_\gamma\le \left\|n^0\right\|_\infty \le \frac{1-\delta}{\eta}$, we see that

$$\nabla z_l = 2(\mathcal{E}_{\gamma(l)})_+\nabla(\mathcal{E}_{\gamma(l)})_+ n_{\gamma(l)} + (\mathcal{E}_{\gamma(l)})_+^2\nabla n_{\gamma(l)}$$

is uniformly bounded in $L^2(0,T;L^2(\Omega))$, because $(\mathcal{E}_\gamma-\gamma)_+ \le (\mathcal{E}_\gamma)_{[\gamma]} \le \left\|\mathcal{E}^0\right\|_\infty$. We aim to apply Aubin-Lions's lemma once again; we thus need an estimate on the time derivative of z_l. Making use of Eq. $(12.15)_1$ and (12.16) ensures that

$$\begin{aligned}
&\left|\int_0^T \left\langle \partial_t((\mathcal{E}_\gamma-\gamma)_+^2 n_\gamma),\phi\right\rangle dt\right| \\
&\quad\le 2\left|\int_0^T \left\langle \partial_t(\mathcal{E}_\gamma-\gamma),(\mathcal{E}_\gamma-\gamma)_+ n_\gamma\phi\right\rangle dt\right| + \left|\int_0^T \left\langle \partial_t n_\gamma,(\mathcal{E}_\gamma-\gamma)_+^2\phi\right\rangle dt\right| \\
&\quad\le C(T,\Omega,\kappa,\delta)\left(\|\phi\|_{L^\infty((0,T)\times\Omega)} + \|\phi\|_{L^2H^1}\right).
\end{aligned}$$

As above, we can apply the Aubin-Lions Lemma from [41] to z_l. Hence, along a subsequence which is not relabeled, $z_l\to z$ in $L^2(0,T;L^2(\Omega))$ for some $z\in L^2(0,T;L^2(\Omega))$. Let us return to the main purpose: the convergence of $n_{\gamma(l)}$. We have

$$n_{\gamma(l)} = \frac{z_l}{(\mathcal{E}_{\gamma(l)}-\gamma)_+^2} \to \frac{z}{\mathcal{E}^2} \quad \text{a.e. in } \{\mathcal{E}>0\}.$$

Since $n_{\gamma(l)}$ is bounded, the dominated convergence theorem implies that $n_{\gamma(l)}$ converges strongly in $L^2(\{\mathcal{E} > 0\})$. Its limit coincides with its weak limit n. Note that we cannot expect to prove that n_γ converges strongly on the whole Ω because its gradient is eventually not uniformly bounded in $L^2(\{\mathcal{E} = 0\})$.

Keeping in mind that we would like to take the limit in Eq. (12.15), we turn to analyze the behavior of the gradient of n_γ in the limit. As we have seen before, it is useful to distinguish the cases $\mathcal{E} = 0$ and $\mathcal{E} \neq 0$.

First, we treat the easier case $\mathcal{E} = 0$. Let $g : \mathbb{R} \to \mathbb{R}$ be a locally bounded function. Then $g(n_{\gamma(l)})\sqrt{(\mathcal{E}_{\gamma(l)})_{[\gamma(l)]}}\nabla n_{\gamma(l)}$ is bounded in $L^2(0, T; L^2(\Omega))$ and admits a weakly converging subsequence labeled again with the same index. Therefore, we directly deduce that

$$g(n_{\gamma(l)})(\mathcal{E}_{\gamma(l)})_{[\gamma(l)]}\nabla n_{\gamma(l)} \rightharpoonup 0 \quad \text{in } L^2(\{\mathcal{E} = 0\})$$

as $\gamma(l) \to 0$. We emphasize that we would have been completely lost at this point without the multiplicator $(\mathcal{E}_{\gamma(l)})_{[\gamma(l)]}$. Fortunately, the gradient of n_γ always occurs in combination with it.

For the other case ($\mathcal{E} \neq 0$), we again require an auxiliary variable, namely $y_l = (\mathcal{E}_{\gamma(l)})_{[\gamma]} n_{\gamma(l)}$. This can be seen by the following identity:

$$\iint_{\{\mathcal{E}\neq 0\}} (\mathcal{E}_{\gamma(l)})_{[\gamma(l)]}\nabla n_{\gamma(l)} \cdot \Phi dxdt$$
$$= \iint_{\{\mathcal{E}\neq 0\}} \nabla y_l \cdot \Phi dxdt - \iint_{\{\mathcal{E}\neq 0\}} \nabla(\mathcal{E}_{\gamma(l)})_{[\gamma(l)]} n_{\gamma(l)} \cdot \Phi dxdt.$$

We want to prove the convergence of the left-hand side for some fixed and regular Φ. The second integral on the right-hand side converges for $\Phi \in L^\infty((0, T) \times \Omega)^d$, since $\nabla(\mathcal{E}_{\gamma(l)})_{[\gamma(l)]} \rightharpoonup \nabla\mathcal{E}$ and $n_{\gamma(l)} \to n$ both along a subsequence in $L^2(0, T; L^2(\Omega))$. For this, we can use that $(\nabla\mathcal{E}_{\gamma(l)})_{[\gamma(l)]}$ is uniformly bounded in $L^2(0, T; L^2(\Omega))$ and that $(\mathcal{E}_{\gamma(l)})_{[\gamma(l)]}$ converges strongly. Nevertheless, we need to get more involved to prove the convergence of the first term, because we are not able to integrate by parts. Due to the strong convergence of $(\mathcal{E}_{\gamma(l)})_{[\gamma(l)]}$ and $n_{\gamma(k)} \rightharpoonup n$ in $L^2(0, T; L^2(\Omega))$, we have $y_l \rightharpoonup \mathcal{E}n$ in $L^1(0, T; L^1(\Omega))$. Similarly as above for z_l, we derive that ∇y_l is bounded in $L^2(0, T; H^1(\Omega))$. This entails that for a subsequence, which is not relabeled, that $\nabla y_l \rightharpoonup \nabla(\mathcal{E}n)$ in $L^1(0, T; L^1(\Omega))$. According to chapter V, Theorem 3 in [53], this convergence holds in $L^2(0, T; L^2(\Omega))$, since $L^1(0, T; L^1(\Omega))$ is dense in $L^2(0, T; L^2(\Omega))$. As in the first case, we can do the same if we multiply $(\mathcal{E}_{\gamma(l)})_{[\gamma(l)]}\nabla n_{\gamma(l)}$ by a (Lipschitz) function of $n_{\gamma(l)}$.

Finally, we can summarize the convergence of $\nabla n_{\gamma(l)}$: Let $g : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function. Then

$$g(n_{\gamma(l)})(\mathcal{E}_{\gamma(l)})_{[\gamma(l)]}\nabla n_{\gamma(l)} \rightharpoonup g(n)(\nabla(\mathcal{E}n) - n\nabla\mathcal{E}) \quad \text{in } L^2(0, T; L^2(\Omega))$$

as $\gamma(l) \to 0$.

Lemma 12.2.3. *By interpreting $L^2(\Omega) \subset C_c^\infty(\Omega)'$, we have*

$$\mathcal{E}\nabla n = \nabla(\mathcal{E}n) - n\nabla\mathcal{E} \in L^2(0,T;L^2(\Omega)).$$

Proof. Since $n \in L^2(0,T;L^2(\Omega))$, we have that the distributional derivative of n w.r.t. $x \in \Omega$ belongs to $L^2(0,T;H^1(\Omega)')$. Now let $\phi \in L^2(0,T;C_c^\infty(\Omega))$. Then it holds by definition

$$\begin{aligned}\langle \mathcal{E}\nabla n, \phi\rangle_{L^2(H^1)',L^2H^1} &= \langle \nabla n, \mathcal{E}\phi\rangle_{L^2(H^1)',L^2H^1} \\ &= -\langle n, \nabla\mathcal{E}\phi\rangle_{L^2(H^1)',L^2H^1} - \langle n, \mathcal{E}\nabla\phi\rangle_{L^2(H^1)',L^2H^1} \\ &= -\langle n\nabla\mathcal{E}, \phi\rangle_{L^2(H^1)',L^2H^1} + \langle \nabla(n\mathcal{E}), \phi\rangle_{L^2(H^1)',L^2H^1}.\end{aligned}$$

Thus, we have that $\mathcal{E}\nabla n = \nabla(\mathcal{E}n) - n\nabla\mathcal{E}$ holds in $L^2(0,T;C_c^\infty(\Omega)')$. Finally the fact that $\nabla(\mathcal{E}n) - n\nabla\mathcal{E}$ belongs to $L^2(0,T;L^2(\Omega))$ yields the assertion by identifying $L^2(\Omega)$ with the right subset of regular distribution of $C_c^\infty(\Omega)'$. $\square$

We are now able to take the limit in the second term of Equation $(12.15)_1$, assuming $\phi_0 \in L^2(0,T;H^1(\Omega))$. For the first term involving the time derivative, we see by

$$\left|\int_0^T \langle \partial_t n_\gamma, \phi_0\rangle \, dt\right| \le \left\| \frac{\sqrt{(\mathcal{E}_\gamma)_{[\gamma]}}}{n_\gamma(1-\eta n_\gamma)} \right\|_{L^\infty} \left\| (\mathcal{E}_\gamma)_{[\gamma]} \left|\nabla n_\gamma\right|^2 \right\|_{L^1L^1} \|\nabla\phi_0\|_{L^2L^2}$$

that n_γ is bounded in $H^1(0,T;H^1(\Omega)')$. Thus, taking the right sequence $\gamma(l) \to 0$, we obtain

$$\int_0^\infty \langle \partial_t n, \phi_0\rangle \, dt + \int_0^\infty \int_\Omega \frac{\mathcal{E}\nabla n}{n(1-\eta n)} \cdot \nabla\phi_0 dx dt = 0.$$

Now let us turn to $(12.15)_2$. Here, we still need to argue why the first three integrals converge. For the convergence of the integral concerning the time derivative of $\mathcal{E}_{\mathrm{tot},\gamma} := \mathcal{E}_\gamma - \frac{\kappa}{2}n_\gamma^2$, we compute

$$\begin{aligned}\left|\int_0^T \left\langle \partial_t(\mathcal{E}_\gamma - \frac{\kappa}{2}n_\gamma^2), \phi_1 \right\rangle dt\right| &\le \left|\int_0^T \int_\Omega \frac{\nabla(\mathcal{E}_\gamma)_{[\gamma]} \cdot \nabla\phi_1}{n_\gamma(1-\eta n_\gamma)} dxdt\right| \\ &\quad + \gamma\left|\int_0^T \int_\Omega \nabla\mathcal{E}_\gamma \cdot \nabla\phi_1 dxdt\right| \\ &\quad + \kappa\left|\int_0^T \int_\Omega \frac{(\mathcal{E}_\gamma)_{[\gamma]}}{1-\eta n_\gamma}\nabla n_\gamma \cdot \nabla\phi_1 dxdt\right| \\ &\le C\left(\left\|\nabla(\mathcal{E}_\gamma)_{[\gamma]}\right\|_{L^2L^2} + \|\nabla n_\gamma\|_{L^2L^2}\right)\|\phi_1\|_{L^2H^1} \\ &\quad + \gamma\left|\int_0^T \int_\Omega \mathcal{E}_\gamma \Delta\phi_1 dxdt\right| \\ &\le C\left(\left\|\nabla(\mathcal{E}_\gamma)_{[\gamma]}\right\|_{L^2L^2} + \|\nabla n_\gamma\|_{L^2L^2}\right)\|\phi_1\|_{L^2H^1} \\ &\quad + \gamma\|\mathcal{E}_\gamma\|_{L^\infty L^1}\|\phi_1\|_{L^1W^{2,\infty}}\end{aligned}$$

for $\phi_1 \in C^0([0,T];C_N^2(\overline{\Omega}))$, where $C_N^2(\Omega) := \{f \in C^2(\overline{\Omega}) : \partial_\nu f = 0 \text{ on } \partial\Omega\}$. Let $H_N^s(\Omega)$ be the completion of $C_N^2(\Omega)$ in $H^s(\Omega)$ for $s > 0$. Since $\mathcal{E}_\gamma$ is bounded from above by $\left\|\mathcal{E}^0\right\|_\infty$, we obtain from the energy conservation

$$\int_\Omega \left(\mathcal{E}_\gamma(t) - \frac{\kappa}{2}n_\gamma^2\right) dx = \int_\Omega \left(\mathcal{E}^0 - \frac{\kappa}{2}(n^0)^2\right) dx$$

(put $\phi_1 = 1$ in $(12.15)_2$) that $\|\mathcal{E}_\gamma(t)\|_{L^1(\Omega)}$ is uniformly bounded w.r.t. $t > 0$ and $\gamma > 0$. Thus, $\mathcal{E}_{\mathrm{tot},\gamma}$ is uniformly bounded in $H^1(0,T;H_N^s(\Omega)')$ for $s > 2 + \frac{d}{2}$. This implies that $\mathcal{E}_{\mathrm{tot},\gamma}$ converges weakly to some $\mathcal{E}_{\mathrm{tot}} \in H^1(0,T;H^s(\Omega)')$ along a subsequence $\gamma(l) \to 0$.

The following lemma links $\mathcal{E}_{\mathrm{tot}}$ to $\mathcal{E}$ and n. Due to the lack of the strong convergence of $n_{\gamma(l)}$ and $\mathcal{E}_{\gamma(l)}$ in $L^2(0,T;L^2(\Omega))$, it presents only a partial result.

Lemma 12.2.4. *It holds $\mathcal{E}_{\mathrm{tot}} \in H^1(0,T;H^1(\Omega)')$ and*

$$\int_0^\infty \langle \mathcal{E}_{\mathrm{tot}}, \phi\rangle\, dt \le \int_0^\infty \left\langle \mathcal{E} - \frac{\kappa}{2}n^2, \phi\right\rangle dt$$

for all $\phi \in L_c^2(0,\infty;H^1(\Omega))$ with $\phi \ge 0$.

Proof. Let $T > 0$. To begin with, we test $(12.15)_2$ with $\gamma\mathcal{E}_{\mathrm{tot},\gamma} = \gamma(\mathcal{E}_\gamma - \frac{\kappa}{2}n_\gamma^2)$

and obtain similarly to (12.13) and its following estimate that

$$
\begin{aligned}
&\frac{\gamma}{2}\int_\Omega \mathcal{E}_{\mathrm{tot},\gamma}(t)^2 dx + \gamma\frac{2d-1}{4}\int_0^t\int_\Omega \frac{\left|\nabla(\mathcal{E}_\gamma)_{[\gamma]}\right|^2}{n_\gamma(1-\eta n_\gamma)} dxds + \frac{\gamma^2}{4}\int_0^t\int_\Omega \left|\nabla\mathcal{E}_\gamma\right|^2 dxds \\
&\quad\le \frac{\gamma}{2}\int_\Omega \mathcal{E}_{\mathrm{tot},\gamma}(0)^2 dx + \kappa\frac{2d-1}{4}\int_0^t\int_\Omega \frac{\gamma\left|\nabla n_\gamma\right|^2}{1-\eta n_\gamma} dxds \\
&\qquad + \frac{\kappa\gamma}{2}\int_0^t\int_\Omega n_\gamma^2\gamma\left|\nabla n_\gamma\right|^2 dxds + \kappa^2\int_0^t\int_\Omega \frac{(\mathcal{E}_\gamma)_{[\gamma]}^2}{(1-\eta n_\gamma)^2}\left|\nabla n_\gamma\right|^2 dxds \\
&\qquad\qquad + \kappa^2\gamma\int_0^t\int_\Omega \frac{(\mathcal{E}_\gamma)_{[\gamma]}n_\gamma}{1-\eta n_\gamma}\left|\nabla n_\gamma\right|^2 dxds. \qquad (12.18)
\end{aligned}
$$

Since $\gamma \le (\mathcal{E}_\gamma)_{[\gamma]}$, all integrals involving $\gamma\left|\nabla n_\gamma\right|^2$ are uniformly bounded due to the boundedness of $\sqrt{(\mathcal{E}_\gamma)_{[\gamma]}}\nabla n_\gamma$ in $L^2(0,\infty,L^2(\Omega))$. Thus, $\sqrt{\gamma}\mathcal{E}_\gamma$ and $\gamma\nabla\mathcal{E}_\gamma$ are uniformly bounded in $L^\infty(0,\infty;L^2(\Omega))$ and $L^2(0,\infty;L^2(\Omega))$, respectively. Therefore, there exists a subsequence of $\gamma(l)$, which is again denoted by $\gamma(l)$, such that

$$\gamma(l)\nabla\mathcal{E}_{\gamma(l)} \rightharpoonup 0 \quad \text{in } L^2(0,\infty;L^2(\Omega)).$$

This implies that $\phi \mapsto \int_0^\infty \left\langle \partial_t\mathcal{E}_{\mathrm{tot},\gamma},\phi\right\rangle ds$ is bounded in $H^1(0,T;H^1(\Omega'))$ for every $T>0$. As we have seen above, we can take the limit (along the right subsequence of $\gamma(l)$ which is not relabeled) such that

$$\int_0^T \left\langle \partial_t\mathcal{E}_{\mathrm{tot},\gamma(l)},\phi\right\rangle dt \to \int_0^T \left\langle \partial_t\mathcal{E}_{\mathrm{tot}},\phi\right\rangle dt$$

for all $\phi\in L^2(0,T;H^1(\Omega))$ as $\gamma(l)\to 0$. For the next step, let the test function $\phi\in L^2(0,T;H^1(\Omega))$ be non-negative. Then

$$
\begin{aligned}
\int_0^T \left\langle \mathcal{E}_{\mathrm{tot}},\phi\right\rangle dt &= \lim_{l\to\infty}\int_0^T \left\langle \mathcal{E}_{\mathrm{tot},\gamma(l)},\phi\right\rangle dt \\
&= \lim_{l\to\infty}\int_0^T \left\langle \mathcal{E}_{\gamma(l)} - \frac{\kappa}{2}n_{\gamma(l)}^2,\phi\right\rangle dt \\
&\le \limsup_{l\to\infty}\int_0^T \left\langle (\mathcal{E}_{\gamma(l)})_{[\gamma(l)]} - \frac{\kappa}{2}n_{\gamma(l)}^2,\phi\right\rangle dt \\
&\le \int_0^T \left\langle \mathcal{E} - \frac{\kappa}{2}n^2,\phi\right\rangle dt.
\end{aligned}
$$

Finally, this implies the assertion since T was arbitrary. □

Let us come back to the convergence of $(12.15)_2$. The behavior of the second integral in $(12.15)_2$ can be understood by using integration by parts, i.e.

$$\begin{aligned}
&\int_0^T \int_\Omega \frac{\nabla(\mathcal{E}_{\gamma(l)})_{[\gamma(l)]} \cdot \nabla\phi_1}{n_{\gamma(l)}(1-\eta n_{\gamma(l)})} dx dt \\
&\quad = -\int_0^T \int_\Omega \frac{(\mathcal{E}_{\gamma(l)})_{[\gamma(l)]} \Delta\phi_1}{n_{\gamma(l)}(1-\eta n_{\gamma(l)})} dx dt \\
&\qquad + \int_0^T \int_\Omega \frac{1-2\eta n_{\gamma(l)}}{n_{\gamma(l)}(1-\eta n_{\gamma(l)})} (\mathcal{E}_{\gamma(l)})_{[\gamma(l)]} \nabla n_{\gamma(l)} \cdot \nabla\phi_1 dx dt \\
&\quad \to -\int_\Omega \frac{\mathcal{E}\Delta\phi_1}{n(1-\eta n)} dx + \int_\Omega \frac{1-2\eta n}{n(1-\eta n)} \mathcal{E}\nabla n \cdot \nabla\phi_1 \\
&\quad = \int_\Omega \frac{\nabla\mathcal{E} \cdot \nabla\phi_1}{n(1-\eta n)} dx
\end{aligned}$$

for $\phi_1 \in L^2(0,T;H^2_N(\Omega))$.

For the third integral, we already have seen in the proof of Lemma 12.2.4 that

$$\gamma \int_0^T \int_\Omega \nabla\mathcal{E}_\gamma \cdot \nabla\phi_1 dx dt \to 0 \quad \text{as } \gamma \to 0 \tag{12.19}$$

if $\phi_1 \in L^2(0,T;H^1(\Omega))$. The last term of $(12.15)_2$ can be treated similarly as in the proof of the convergence of the first equation. Thus, if we take the appropriate subsequence $\gamma(l) \to 0$, we obtain that every integral in $(12.15)_2$ converges for $\phi_1 \in L^2(0,T;H^s_N(\Omega))$ for $s > 2 + \frac{d}{2}$. We finally obtain that $(n, \mathcal{E}, \mathcal{E}_{\text{tot}})$ fulfills

$$\begin{aligned}
&\int_0^\infty \langle \partial_t n, \phi_0 \rangle \, dt + \int_0^\infty \int_\Omega \frac{\mathcal{E}\nabla n}{n(1-\eta n)} \cdot \nabla\phi_0 dx dt = 0 \\
&\int_0^\infty \langle \partial_t \mathcal{E}_{\text{tot}}, \phi_1 \rangle \, dt + \frac{2d-1}{2d} \int_0^\infty \int_\Omega \frac{\nabla\mathcal{E} \cdot \nabla\phi_1}{n(1-\eta n)} dx dt \\
&\qquad + \kappa \int_0^\infty \int_\Omega \frac{\mathcal{E}\nabla n}{1-\eta n} \cdot \nabla\phi_1 dx dt = 0
\end{aligned} \tag{12.20}$$

for $\phi_0 \in L^2(0,T;H^1(\Omega))$ and $\phi_1 \in L^2(0,T;H^s_N(\Omega))$ for $s > 2 + \frac{d}{2}$. Note that $(12.20)_2$ holds true for all $\phi_1 \in L^2(0,T;H^1(\Omega))$ since $L^2(0,T;H^s_N(\Omega))$ is dense in $L^2(0,T;H^1(\Omega))$.

In addition, our solution shall fulfill the initial data. We have already seen that $\mathcal{E}_{\text{tot},\gamma}$ converges weakly to $\mathcal{E}_{\text{tot}}$ in $H^1(0,T;H^s(\Omega)')$, which is continuously embedded in $C^0([0,T];H^s(\Omega)')$. Thus, $\mathcal{E}_{\text{tot}}(0) \leftarrow \mathcal{E}_{\text{tot},\gamma(l)}(0) = \mathcal{E}^0 - \frac{\kappa}{2}(n^0)^2$. Likewise, we recall that $n_{\gamma(l)}$ is uniformly bounded in $H^1(0,T;H^1(\Omega)')$ w.r.t.

$\gamma(l)$ and T. Following the same arguments as before, $n \in C^0([0,T];H^1(\Omega)')$ and $n(0)=n^0$. For this, we may have to take another subsequence of $\gamma(l)$ in the procedure above.

Finally, we notice that the key estimates were uniform w.r.t. T. As in the case $\tau \to 0$, we use the fact that the functions with compact support in $[0,\infty)$ are dense in $L^2(0,\infty;H^1(\Omega))$ such that we have proved the first part of Theorem 12.1.2 without any restrictions on the support of the test functions.

The second part of Theorem 12.1.2 can be proved as follows. We observe that

$$\begin{aligned}\int_s^t\int_\Omega \mathcal{E}dxd\tau &= \lim_{l\to\infty}\int_s^t\int_\Omega (\mathcal{E}_{\gamma(l)})_{[\gamma(l)]}dxd\tau\\ &\geq \limsup_{l\to\infty}\int_s^t\int_\Omega \mathcal{E}_{\gamma(l)}dxd\tau\\ &= \limsup_{l\to\infty}\int_s^t\int_\Omega \mathcal{E}_{\mathrm{tot},\gamma(l)}dxd\tau + \frac{\kappa}{2}\limsup_{l\to\infty}\int_s^t\int_\Omega n_{\gamma(l)}^2dxd\tau\\ &\geq (t-s)\int_\Omega \mathcal{E}_{\mathrm{tot}}(0)dx + \frac{\kappa}{2}\int_s^t\int_\Omega n^2dxd\tau.\end{aligned}$$

Here, we used that $(\mathcal{E}_{\gamma(l)})_{[\gamma(l)]}$ converges strongly in $L^1(s,t;L^1(\Omega))$ and $n_{\gamma(l)}$ converges weakly in $L^2(s,t;L^2(\Omega))$. Finally, the energy given by

$$\int_\Omega\left(\mathcal{E}(t)-\frac{\kappa}{2}n(t)^2\right)dx$$

is alway greater or equal than the initial total energy $\int_\Omega \mathcal{E}_{\mathrm{tot}}(0)dx$ for all almost every $t>0$ due to the Lebesgue differentiation theorem. Moreover, by Hölder's inequality, we see that $\mathcal{E}\geq 0$ does not vanish, because

$$\begin{aligned}\int_s^t\int_\Omega \mathcal{E}dxd\tau &\geq (t-s)\int_\Omega \mathcal{E}_{\mathrm{tot}}(0)dx + \frac{\kappa}{2}\int_s^t\int_\Omega n^2dxd\tau\\ &\geq (t-s)\int_\Omega \mathcal{E}_{\mathrm{tot}}(0)dx + \frac{\kappa}{2}\left(\int_s^t\int_\Omega ndxd\tau\right)^2(t-s)^{-1}\left|\Omega\right|^{-1}\\ &\geq (t-s)\left(\int_\Omega \mathcal{E}_{\mathrm{tot}}(0) + \frac{\kappa}{2\left|\Omega\right|}\left(\int_\Omega n(0)dx\right)^2\right)\end{aligned}$$

is positive.

Appendix A

Notation

In this chapter we state some special abbreviations defined throughout this thesis.

Symbol	**Explication**	**Def.**	**page**
B	1) first Brillouin zone $\subset \mathbb{R}^d$	5.1.1	45
	2) $[0,2\pi)^d \cong \mathbb{T}^d$	4.5.1, 5.4.12	39, 71
$\tilde{\partial}_{vt}$	$\partial_p - tv'(p)\partial_x$	4.5.1	39
$(\tilde{\delta},\tilde{\gamma})$	$(\frac{1-\delta}{\delta},\frac{1-\gamma}{\gamma})$	5.4.2	59
$\tilde{E}(\lambda)$	$\int_B \epsilon(p)\mathcal{F}(\lambda,p)dp$	5.1.6	47
E_f	$\int_B \epsilon(p)f(p)dp$	5.1.13	49
$e_{\max}(n)$	$\frac{1}{\eta}\int_{\{\epsilon\geq c\}}\epsilon(p)dp$ with $\lvert\{\epsilon\geq c\}\rvert=\eta n$ and $\eta>0$	Lemma 5.1.14	50
$\epsilon_{\mathrm{F}}(n)$	$\lvert\{\epsilon<\epsilon_{\mathrm{F}}\}\rvert=\eta n$ with $n=\tilde{n}(\lambda)$ and $\eta>0$	5.13	52
$\mathcal{E}_a(n)$	$\mathcal{E}_a(n)=\{\int_B \frac{\epsilon(p)dp}{1+e^{-\lambda_0-\lambda_1\epsilon(p)}} : \lvert\lambda_1\rvert\leq\log a$ and $\int_B \frac{dp}{1+e^{-\lambda_0-\lambda_1\epsilon(p)}}=n\}$;	5.4.13	72
ϵ	1) energy dispersion	5.1.1	45
	2) $(p_1,\dots,p_d)\mapsto -2J\sum_{i=1}^d\cos(p_i)$	4.5.1, 5.4.12	39, 71
$\mathcal{F}(\lambda,p)$	Fermi Dirac distribution $\frac{1}{\eta+e^{-\lambda_0-\lambda_1\epsilon(p)}}$ with $\lambda=(\lambda_0,\lambda_1)$ and $\eta\geq 0$	5.1.5	46

$\mathcal{F}^0_\eta(n,E,p)$	Fermi Dirac distribution $\frac{1}{\eta+e^{-\lambda_0-\lambda_1\epsilon(p)}}$, where $\lambda_0=\lambda_0(n,E)$ and $\lambda_1=\lambda_1(n,E)$ are implicitly defined by $(\tilde{n}(\lambda_0,\lambda_1),\tilde{E}(\lambda_0,\lambda_1))=(n,E)$	5.13	52
$\mathcal{F}^0(n,E,p)$	another notation for $\mathcal{F}^0_\eta(n,E,p)$		
$\mathcal{F}^{\mathrm{hT}}_0(n,E,p)$	n	5.5.1	80
$\mathcal{F}^{\mathrm{hT}}_1(n,E,p)$	$n+\frac{\epsilon(p)}{2J^2d}E$	5.5.1	80
$\mathcal{F}^{\mathrm{hT}}_2(n,E,p)$	$n+\frac{\epsilon(p)}{2J^2d}E+\frac{1-2\eta n}{8J^4d^2n(1-\eta n)}(\epsilon(p)^2-2J^2d)E^2$	5.5.1	80
$\hat{F}(\gamma,\delta,\beta)$	$\frac{1}{1+\tilde{\delta}\left(\frac{\tilde{\gamma}}{\tilde{\delta}}\right)^\beta}$	5.4.2	59
Γ_i	$\int_{\mathbb{T}^d}\epsilon(p)^i\left\|\nabla\epsilon(p)\right\|^2\mathcal{F}(\lambda,p)dp$	8.4.1	115
$\mathcal{H}^{d-1}$	$d-1$ dimensional Hausdorff measure on B		
J	$\frac{1}{2}\left\|\epsilon\right\|_\infty$	5.4.12	71
$N(e)$	density of states, $\int_{\epsilon(p)=e}\frac{d\mathcal{H}^{d-1}_p}{\|\nabla\epsilon(p)\|}$	5.2.5	53
$\mathbf{\hat{n}}(\gamma,\delta)$	$\int_B\binom{1}{\epsilon(p)}\hat{F}(\gamma,\delta,\epsilon(p))dp$	5.4.2	59
n_f	$\int_B f(p)dp$	5.1.13	49
$\tilde{n}(\lambda)$	$\int_B\mathcal{F}(\lambda,p)dp$	5.1.6	47
$\mathcal{O}(U,X)$	analytic function from $U\subset\mathbb{R}^d$ open to an Banach space X	4.1.2	29
$\mathcal{O}(U)$	equals $\mathcal{O}(U,\mathbb{R})$	4.1.2	29
$\omega_i,\omega_i(\lambda)$	$\int_B\epsilon(p)^i\mathcal{F}(\lambda,p)(1-\eta\mathcal{F}(\lambda,p))dp$	5.2.4	53
$T_{\pm vt}f(x,k)$	$f(x\pm tv(k),k)$	4.5.1	39
v	$\nabla\epsilon$	4.5.1	39

Overview of the analytic norms

Symbol	**Explication**	**Def.**	**page**
$\|\cdot\|_{\mathrm{Op}}$	Operator norm	4.1.1	29
$\|f\|_{\dot{C}^\lambda_x}$	$\sum_{a=1}^\infty\frac{\lambda^a}{a!}\left\|f^{(a)}(x)\right\|_{\mathrm{Op}}$ for $f=f(x)$	4.1.3	29
$\|f\|_{\dot{C}^\lambda(U,X)}$	$\sup_{x\in U}\|f\|_{\dot{C}^\lambda_x}$ for $f=f(x)$	4.1.3	29

$\lvert f\rvert_{C_x^\lambda}$	$\lvert f(x)\rvert + \lvert f\rvert_{\dot{C}_x^\lambda}$ for $f = f(x)$	4.1.3	29
$\lvert f\rvert_{C^\lambda(U,X)}$	$\sup_{x\in U} \lvert f\rvert_{C_x^\lambda}$ for $f = f(x)$	4.1.3	29
$\lvert f\rvert_{\dot{C}_x^{\lambda_1,\lambda_2}}$	$\sum_{a+b>0} \frac{\lambda_1^a \lambda_2^b}{a!b!} \left\lVert \partial_x^a \partial_p^b f(x,p)\right\rVert_{L_p^\infty(\mathbb{T}^d)}$ for $f = f(x,p)$	4.4.1	34
$\lvert f\rvert_{C_x^{\lambda_1,\lambda_2}}$	$\lVert f(x,\cdot)\rVert_{L_p^\infty(\mathbb{T}^d)} + \lvert f\rvert_{\dot{C}_x^{\lambda_1,\lambda_2}}$ for $f = f(x,p)$	4.4.1	34
$\lVert f\rVert_{C_x^\lambda}$	$\lvert f\rvert_{C_x^\lambda} + \lvert\partial_x f\rvert_{C_x^\lambda} + \lvert\partial_p f\rvert_{C_x^\lambda}$ for $f = f(x,p)$	4.4.5	35
$\lVert f\rVert_{C^\lambda(U)}$	$\sup_{x\in U} \lVert f\rVert_{C_x^\lambda}$ for $f = f(x,p)$	4.4.5	35
$\lVert Df\rVert_{C_x^\lambda}$	$\lVert\partial_x f\rVert_{C_x^\lambda} + \lVert\partial_p f\rVert_{C_x^\lambda} = \lvert\partial_x f\rvert_{C_x^\lambda} + \lvert\partial_p f\rvert_{C_x^\lambda} + \left\lvert\partial_x^2 f\right\rvert_{C_x^\lambda} + \left\lvert\partial_p^2 f\right\rvert_{C_x^\lambda} + 2\lvert\partial_x\partial_p f\rvert_{C_x^\lambda}$ for $f = f(x,p)$	4.4.5	35
$\lVert Df\rVert_{C^\lambda(U)}$	$\sup_{x\in U} \lVert Df\rVert_{C_x^\lambda}$ for $f = f(x,p)$	4.4.5	35
$\tilde{\partial}_{vt}$	$\partial_p - tv'(p)\partial_x$	4.5.1	39
$\lvert f\rvert_{\dot{\mathcal{O}}_t^\lambda}$	$\sum_{(a,b)\in\mathbb{N}^2\setminus\{0\}} \frac{\lambda^{a+b}}{a!b!} \left\lVert \partial_x^a \tilde{\partial}_{vt}^b f\right\rVert_{L^\infty}$ for $f = f(x,p)$	4.5.2	40
$\lvert f\rvert_{\mathcal{O}_t^\lambda}$	$\lVert f\rVert_{L^\infty} + \lvert f\rvert_{\dot{\mathcal{O}}_t^\lambda}$ for $f = f(x,p)$	4.5.2	40
$\lVert f\rVert_{\dot{\mathcal{O}}_t^\lambda}$	$\lvert f\rvert_{\dot{\mathcal{O}}_t^\lambda} + \lvert\partial_x f\rvert_{\dot{\mathcal{O}}_t^\lambda} + \left\lvert\tilde{\partial}_{vt} f\right\rvert_{\dot{\mathcal{O}}_t^\lambda}$ for $f = f(x,p)$	4.5.2	40
$\lVert Df\rVert_{\mathcal{O}_t^\lambda}$	$\sum_{i+j=1}^2 \left\lvert\partial_x^i \tilde{\partial}_{vt}^j f\right\rvert_{\mathcal{O}_t^\lambda}$ for $f = f(x,p)$	4.5.2	40

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