# On viscosity solutions and the normalized $p$-Laplace operator

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## Abstract

In this doctoral thesis we consider a special type of degenerate elliptic partial differential equations of second order for a rather general right-hand side. We introduce the suitable notion of viscosity solutions. We first characterize these with a maximum principle and then derive useful properties. This includes a weak comparison principle, a Hopf-type Lemma, local regularity estimates and existence of viscosity solutions. Finally we consider constant right-hand sides and show $\frac{1}{2}$-power concavity of viscosity solutions. The results are applied to the normalized $p$-Laplacian.

**Zusammenfassung**

In dieser Doktorarbeit untersuchen wir einen speziellen Typ von degeneriert elliptischen partiellen Differentialgleichungen zweiter Ordnung mit allgemeiner rechten Seite. Dazu führen wir den Begriff der Viskositätslösung ein. Zunächst charakterisieren wir diese über ein Maximumprinzip und leiten dann nützliche Eigenschaften her. Dies beinhaltet ein schwaches Vergleichsprinzip, ein Hopf-artiges Lemma, lokale Regularitätsabschätzungen und Existenz von Viskositätslösungen. Zuletzt betrachten wir konstante rechte Seiten und zeigen Konkavität von Quadratwurzeln von Viskositätslösungen. Die Resultate werden auf den normalisierten $p$-Laplace angewandt.

# Contents

# Introduction

## 1.1 Overview of the problem

We consider partial differential equations of the type

$$F(\nabla u, \mathrm{D}^2 u) = f \qquad \text{in } \Omega \tag{1.1}$$

where $F : \mathbb{R}^n \setminus \{0\} \times \mathbb{R}^{n\times n} \to \mathbb{R}$ is continuous, $f \in C(\Omega)$, and where $\Omega \subset \mathbb{R}^n$ is a bounded domain. The operator $F$ may be discontinuous in points $(0, X)$. Furthermore we consider the Dirichlet boundary value problem

$$\begin{cases} F(\nabla u, \mathrm{D}^2 u) = f & \text{in } \Omega, \\ \qquad\qquad\quad u = g & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

for a continuous function $g \in C(\partial\Omega)$. Our investigation is motivated by the recently growing interest in the normalized infinity Laplacian

$$-\Delta_\infty^N u = - |\nabla u|^{-2} \left\langle \nabla u, \mathrm{D}^2 u \nabla u \right\rangle . \tag{1.3}$$

This operator can be considered as the limiting case of the normalized $p$-Laplacian

$$-\Delta_p^N u = -\tfrac{p-2}{p} |\nabla u|^{-2} \left\langle \nabla u, \mathrm{D}^2 u \nabla u \right\rangle - \tfrac{1}{p}\Delta u, \tag{1.4}$$

in which $p$ is sent to $\infty$. Questions concerning existence, regularity and properties of solutions to both problems (1.1) and (1.2) arose for the normalized infinity Laplacian (1.3) and were partially answered in the past.
We are mainly interested in the so called $\frac{1}{2}$-*power concavity* to solutions of (1.2) for constant right-hand side and vanishing boundary values in convex domains. This property is well-known for the classical Laplacian, obtained

by putting $p = 2$ in (1.4), and was recently shown for the infinity Laplacian (1.3) in [11]. It is natural to ask whether the same power concavity holds for other $p \in (2, \infty)$. To be more precise, this means to investigate whether solutions of

$$\begin{cases} -\Delta_p^N u = 1 & \text{in convex } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.5}$$

are $\frac{1}{2}$-power concave, i.e. if $u^{\frac{1}{2}}$ is a concave function. In order to adapt the methods presented in [11] it is again necessary to generalize the crucial method of *comparison with cones*, presented in [8]. Then again, after generalizing this property, it is also possible to extend some of the local regularity results, shown in [8]. These in turn can be used to answer the naturally arising question concerning existence of solutions by adapting the proof presented in [32].

Motivated by these connections, our objective is to investigate which structural properties of the normalized infinity Laplacian (1.3) are actually necessary to obtain the comparison with cones property and what additional assumptions have to be made to show the desired results.
Using this approach we are able to extend these results, not only to the normalized $p$-Laplacian (1.4), but also to more general 1-homogeneous operators like

$$F(\nabla u, \mathrm{D}^2 u) = -\alpha \, |\nabla u|^{-2} \left\langle \nabla u, \mathrm{D}^2 u \nabla u \right\rangle - \beta \operatorname{tr} \mathrm{D}^2 u, \tag{1.6}$$

with suitable constants $\alpha$ and $\beta$.

## 1.2 Related work

The notion of viscosity solutions we use was introduced for first order equations by Crandall and Lions in [10] and was shortly afterwards extended to second order equations by Lions in [31]. The boundary value problem was then introduced by Ishii and Lions in [16]. A mild introduction to this topic was given by Koike in [28]. A remarkable overview on this topic is the User's guide [9] by Crandall, Ishii and Lions. The methods presented there were used to show a wide variety of results. Barles and Busca used these methods to develop existence and comparison results in [5] for more general operators than the ones we are interested in but for vanishing right-hand side. Lu and Wang extended the comparison result for strictly positive or strictly negative right-hand sides in [33]. Comparison results for more general equations and vanishing right-hand side were discussed by Kawohl and Kutev in [24], [25], and [26]

The study of the infinity Laplacian was stimulated by the work of Aronsson [3]. He was interested in extending Lipschitz continuous functions, found the operator and the equation to describe the functions he calls *absolute minimals* and investigated them in two dimensions. He continued his research on this equation in [4]. However, since the equation is highly degenerate and the concept of viscosity solutions was not yet introduced, there was not much hope to proceed. Decades later, using the notion of viscosity solutions, Jensen was able to show that Aronsson's Lipschitz extensions are unique and minimize the sup norm of the gradient in [17]. In analogy to the operator corresponding to minimizers of the $p$-norm, Aronsson's operator is called infinity Laplacian. From then on the operator received more attention.
For the homogeneous problem Crandall, Evans and Gariepy showed a defining property of the infinity Laplacian in [8], the so called *comparison with cones*. Using this, they were able to simplify the work of Jensen and to show local regularity. It also led to an approach of showing regularity using blow-up limits and flatness decay. It was shown by Evans and Yu in [15] that the flatness decay approach happens to fail for the infinity Laplacian. In the same year Savin was able to show differentiability of solutions in two dimensions in [36], not using flatness decay but topological arguments. This result was improved in a collaboration with Evans in [13] to obtain Hölder regularity of the first derivative, also in two dimensions. Finally Evans and

Smart proved everywhere differentiability in higher dimensions in [14].
The inhomogeneous problem we are considering was motivated by the work of Peres, Schramm and Sheffield [34], showing that the normalized infinity Laplacian can be used to describe the value of a mathematical tug of war game. They showed that solutions may not exist if the right-hand side changes its sign, while a unique solution exists if the right-hand side is everywhere strictly greater or everywhere strictly less than zero. Their work was extended by Armstrong and Smart in [2]. Lu and Wang took a PDE-based approach in [32] to show that solutions exist if the right-hand side becomes zero but may fail to be unique. In the same work the authors generalized the concept of comparison with cones to non-vanishing right-hand side. Regularity of solutions can be obtained if the domain under consideration is a ball, as it was shown by Lindgren in [30]. Existence and nonuniqueness of solutions for more general right-hand sides was discussed by Bhattacharya and Mohammed in [6].

Motivated by the great interest in the infinity Laplacian and its normalized counterpart, it is reasonable to also consider the normalized $p$-Laplacian. Then one can heuristically consider the infinity Laplacian as some limiting case. This was done rigorously by Kawohl in [22], showing that for vanishing right-hand side, viscosity solutions for the normalized $p$-Laplacian converge uniformly to viscosity solutions for the infinity Laplacian. A survey on the normalized $p$-Laplacian was recently given by Kawohl and Horák in [23].

An interesting property of solutions of the type of equations we are interested in is the so called power concavity. The investigation of this property was stimulated by Korevaar in [29] who proved a concavity maximum principle which can be used to show the power concavity of solutions of a variety of equations. This problem was also discussed by Kawohl in [19]. Korevaar's result was improved by Kennington in [27] and Kawohl in [20] and [21]. Since these results apply to classical solutions, Sakaguchi used regularization techniques in [35] to obtain results for problems that might only have weak solutions in the Sobolev sense. This motivated Juutinen to generalize Korevaar's concavity maximum principle to viscosity solutions in [18]. Finally Crasta and Fragalà showed power concavity of solutions to $-\Delta_\infty^N u = 1$ in [11].

## 1.3 Summary of results

In Chapter 2 we will briefly introduce our notation and terminology used. Most of the notation surely is standard convention, therefore the Section 2.1 is intended to make our presentation self-contained. Section 2.2 is devoted to introduce the notion of viscosity solutions.

In Chapter 3 we will first prove the equivalence of $u$ being a viscosity solution to our problem (1.1) and satisfying the comparison principle in [33]. This property is well-known for linear elliptic equations of second order. The strategy to prove the equivalence is to first show another comparison principle for a smaller class of functions to compare with and then to show that all functions satisfying this weakened comparison principle are viscosity solutions. In Section 3.2 we will, inspired the by the preceding chapter, derive a comparison principle for radial functions to generalize the comparison with cones property, introduced in [8] and use it to show a Hopf-type Lemma. The derived comparison property for radial functions will then, in Section 3.3, be used to obtain local Hölder continuity of viscosity solutions, where the Hölder exponent will be explicitly calculated.

In Chapter 4 we will consider the Dirichlet boundary value problem (1.2). To this end we first introduce the notion of viscosity solutions for boundary value problems. In Section 4.2 we will use Perron's method and the results of the preceding chapter to show existence of continuous solutions. Finally, in Section 4.3, we will show power concavity of viscosity solutions for a special type of domains using the methods provided in [1] and [11].

We will present most of our results only for viscosity subsolutions. Whenever we do this, an analogous statement holds for viscosity supersolutions and the corresponding proofs work the same way.

# Preliminaries

## 2.1 Notation

The set $\Omega \subset \mathbb{R}^n$ always denotes a bounded domain, that is a bounded, open and connected subset of $\mathbb{R}^n$.
Whenever necessary, we will assume that $\Omega$ satisfies the *interior sphere condition*. That is

$$\forall x \in \partial\Omega \, \exists R > 0 : B_R(y) \subset \Omega \wedge \partial B_R(y) \cap \partial\Omega = \{x\}, \tag{2.1}$$

with $y := x - R\nu(x)$ and $\nu(x)$ denoting the outer normal vector at $x$.
For $x, y \in \mathbb{R}^n$ we denote the $i$-th entry of $x$ by $[x]_i$ or $x_i$ and use the standard scalar product

$$\langle x, y \rangle := \sum_{i=1}^{n} x_i y_i$$

and the induced Euclidean norm $|x| := \sqrt{\langle x, x \rangle}$.

Analogously for $X \in \mathbb{R}^{n\times n}$ we denote the $i$-th row of $X$ by $[X]_i$ or $X_i$ and the $j$-th entry in the $i$-th row by $[X]_{i,j}$ or $X_{i,j}$. We use

$$\operatorname{tr} X = \sum_{i=1}^{n} X_{i,i}$$

to denote the trace of $X$, the sum of all diagonal entries and

$$x \otimes y = [x_i y_j]_{i,j=1}^{n}$$

for the tensor product $\otimes : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ of two vectors $x, y \in \mathbb{R}^n$.

Furthermore we use the notation $\mathcal{O}(n) \subset \mathbb{R}^{n\times n}$ for the set of orthogonal matrices

$$\mathcal{O}(n) = \{Q \in \mathbb{R}^{n\times n} \mid Q^{\intercal} Q = \mathrm{Id}\},$$

where Id denotes the identity matrix.
For matrices $X, Y \in \mathbb{R}^{n\times n}$ we use the partial ordering

$$X \geq Y \qquad :\Longleftrightarrow \qquad X - Y \text{ is positive semidefinite.}$$

Denoting the subset of symmetric matrices with $\mathcal{S} \subset \mathbb{R}^{n\times n}$, we know that every $X \in \mathcal{S}$ has only real valued eigenvalues. We enumerate them in increasing order by

$$\lambda_{\min}(X) := \lambda_1(X) \leq \ldots \leq \lambda_n(X) =: \lambda_{\max}(X).$$

In the special case of $X, Y \in \mathcal{S}$ we can diagonalize $X$ and $Y$ using matrices in $\mathcal{O}(n)$. The partial ordering then becomes

$$X \geq Y \qquad \Longleftrightarrow \qquad \lambda_i(X - Y) \geq 0 \quad \text{for all } i \in \{1, \ldots, n\}$$

and the trace of $X$ becomes $\operatorname{tr} X = \sum_{i=1}^{n} \lambda_i(X)$.

Furthermore we denote the set of symmetric positive semidefinite matrices by $\mathcal{S}^+ \subset \mathcal{S}$ and the set of symmetric positive definite matrices by $\mathcal{S}^{++} \subset \mathcal{S}^+$.

In order to describe the asymptotic behaviour of a function $\phi : \mathbb{R} \to \mathbb{R}$ compared to a function $\psi : \mathbb{R} \to \mathbb{R}$ in a point $a \in \mathbb{R} \cup \{-\infty, \infty\}$ the Landau notation $o$ is used. We write

$$\phi \in o(\psi) :\Longleftrightarrow \lim_{x\to a} \tfrac{\phi(x)}{\psi(x)} = 0$$

or simply, by abuse of notation, $\phi = o(\psi)$.

The set of Hölder continuous functions on $\Omega$ with respect to an exponent $0 < \gamma < 1$ is denoted by $C^{0,\gamma}(\Omega)$. Finally the sets of semi-continuous functions are denoted by

$$\mathrm{USC}(\Omega) := \{u : \Omega \to \mathbb{R} \mid u \text{ is upper semicontinuous}\}$$

and

$$\mathrm{LSC}(\Omega) := \{u : \Omega \to \mathbb{R} \mid u \text{ is lower semicontinuous}\}.$$

## 2.2 Viscosity solutions

Due to the possible non-divergence structure and nonlinearity of the operator $F$ the traditional notion of weak solutions using Sobolev spaces is not sufficient. Therefore a different concept of weak solutions, the so called viscosity solutions, was introduced in [31] for degenerate elliptic operators. Degenerate elliptic means that for $X, Y \in \mathcal{S}$, it holds $F(q, X) \geq F(q, Y)$ for all $q \in \mathbb{R}^n$ whenever $X \leq Y$.

**Remark 2.1.** *There is no standard terminology for degenerate ellipticity. Depending on the field of research the definitions differ. In partial differential equations it is common to consider $-\Delta$, the Laplacian, as an elliptic operator, while in harmonic analysis it is common to consider $\Delta$ without sign instead. Therefore the authors of [33] define degenerate ellipticity with opposite sign from ours.* □

Since we will later have to consider a more general problem and do not want to repeat ourselves, we will formulate the following definitions for a more general right-hand side. That is

$$F(\nabla u(x), \mathrm{D}^2 u(x)) = f(x, u(x), \nabla u(x)) \qquad \text{in } \Omega, \tag{2.2}$$

where $-f$ is assumed to be continuous and *proper*, i.e.

$$-f(x, r, q) \leq -f(x, s, q) \tag{2.3}$$

for all $r, s \in \mathbb{R}$ with $r \leq s$.

Since we explicitly want to include operators that are singular in $(0, X)$ for any $X \in \mathbb{R}^{n \times n}$, we use the lower and upper semicontinuous envelopes of $F$. Therefore we define

$$F_*(q, X) = \begin{cases} F(q, X) & \text{if } q \neq 0, \\ \inf\limits_{a \in \mathbb{R}^n \setminus \{0\}} F(a, X) & \text{if } q = 0, \end{cases}$$

and

$$F^*(q, X) = \begin{cases} F(q, X) & \text{if } q \neq 0, \\ \sup\limits_{a \in \mathbb{R}^n \setminus \{0\}} F(a, X) & \text{if } q = 0, \end{cases}$$

as it was proposed in [9]. If one is only interested in the case of continuous $F$, one may just take $F_* = F^* = F$ in the following.

According to [9], there are two equivalent ways to define viscosity solutions. Both rely on the idea to generalize the involved derivatives. The first one uses a concept of *test functions*, that is replacing the derivatives in the equation with the derivatives of smooth functions ‘touching’ a solution from above or below. Then, if $F_*$ and $F^*$ coincide and are degenerate elliptic, viscosity solutions generalize classical solutions in the sense that smooth viscosity solutions are classical solutions and vice versa. To be more precisely we state the following definition.

**Definition 2.2.** We call $u \in \mathrm{USC}(\Omega)$ a *viscosity subsolution* of (2.2) if for every $x \in \Omega$ and $\phi \in C^2(\Omega)$

$$F_*(\nabla\phi(x), \mathrm{D}^2\phi(x)) \leq f(x, u(x), \nabla\phi(x)),$$

whenever $u - \phi$ attains a local maximum in $x$.
Likewise we call $u \in \mathrm{LSC}(\Omega)$ a *viscosity supersolution* of (2.2) if for every $x \in \Omega$ and $\phi \in C^2(\Omega)$

$$F^*(\nabla\phi(x), \mathrm{D}^2\phi(x)) \geq f(x, u(x), \nabla\phi(x)),$$

whenever $u - \phi$ attains a local minimum in $x$.
We call $u \in C(\Omega)$ a *viscosity solution* if it is both, a viscosity sub- and supersolution. □

The defining smooth functions are often referred to as *test functions*. Because of (2.3) we may always assume that for subsolutions $u \in \mathrm{USC}(\Omega)$ and test functions $\phi \in C^2(\Omega)$, whenever $u - \phi$ attains a local maximum in some $x_0 \in \Omega$, it also holds $(u - \phi)(x_0) = 0$. The analogous assumption will be made for supersolutions.

An equivalent approach to generalize the appearing derivatives is by taking all the vectors and symmetric matrices that satisfy a Taylor approximation property. Those are collected in two sets called *semijets*.

**Definition 2.3.** For a function $u : \Omega \to \mathbb{R}$ and a point $x \in \Omega$ we define

$$\begin{aligned}\mathcal{J}^{2,+}u(x) = \{(q, X) \in \mathbb{R}^n \times \mathcal{S} \mid u(y) \leq u(x) + \langle q, y - x\rangle \\ + \tfrac{1}{2}\langle y - x, X(y - x)\rangle + o(|y - x|^2) \text{ as } y \to x\},\end{aligned}$$

the *second order superjet* of $u$ at $x$ and

$$\mathcal{J}^{2,-}u(x) = \{(q, X) \in \mathbb{R}^n \times \mathcal{S} \mid u(y) \geq u(x) + \langle q, y - x \rangle \\ + \tfrac{1}{2} \langle y - x, X(y - x) \rangle + o(|y - x|^2) \text{ as } y \to x\},$$

the *second order subjet* of $u$ at $x$. □

Now we may replace the derivatives of test functions in Definition 2.2 with elements of the semijets.

**Definition 2.4.** We call $u \in \mathrm{USC}(\Omega)$ a *viscosity subsolution* of (2.2) if

$$F_*(q, X) \leq f(x, u(x), q)$$

for all $x \in \Omega$ and $(q, X) \in \mathcal{J}^{2,+}u(x)$.
Likewise we call $u \in \mathrm{LSC}(\Omega)$ a *viscosity supersolution* of (2.2) if

$$F^*(q, X) \geq f(x, u(x), q)$$

for all $x \in \Omega$ and $(q, X) \in \mathcal{J}^{2,-}u(x)$.
Again we call $u \in C(\Omega)$ a *viscosity solution* if it is both, a viscosity sub- and supersolution. □

It is not difficult to see that both definitions are equivalent. In order to do so one has to show that semijets are just exactly the sets of derivatives of valid test functions. We will adapt the common practice to drop the term *viscosity* and just speak of sub- and supersolutions from now on.

# General results on viscosity solutions

In this chapter we will consider equation (1.1) without taking into account any boundary condition. We will at first characterize subsolutions of this equation as functions satisfying a comparison principle. Then we will consider the special case of radial solutions to obtain a comparison principle with radial functions, which can be explicitly calculated. Finally we will use this to show a local regularity result for subsolutions of (1.1).

## 3.1 An equivalent condition

The main objective of this section is to show the equivalence of $u$ being a subsolution of (1.1) and satisfying the comparison principle presented in [33] for a rather general left-hand side and right-hand sides $f < 0$ or $f > 0$.

So at first we first only assume that $F$ is linear in the second argument and bounded. That means for all $\mu, \nu \in \mathbb{R}$, $q \in \mathbb{R}^n \setminus \{0\}$ and $X, Y \in \mathcal{S}$ it holds

$$F(q, \mu X + \nu Y) = \mu F(q, X) + \nu F(q, Y) \tag{3.1}$$

and there are constants $0 < c_{\min} \leq c_{\max}$ such that

$$c_{\min} \lambda_{\min}(X) \leq -F(q, X) \leq c_{\max} \lambda_{\max}(X) \,. \tag{3.2}$$

Clearly these assumptions imply superlinearity of $F_*$, sublinearity of $F^*$ and boundedness of both, that is for $\mu \in \mathbb{R}$, $\mu > 0$ and $X, Y \in \mathcal{S}$ it holds

$$F_*(q, \mu(X + Y)) = \mu F_*(q, X + Y) \geq \mu(F_*(q, X) + F_*(q, Y)), \tag{3.3}$$

$$F^*(q, \mu(X + Y)) = \mu F^*(q, X + Y) \leq \mu(F^*(q, X) + F^*(q, Y)) \tag{3.4}$$

and

$$c_{\min}\,\lambda_{\min}\,(X) \leq -F^*(q,X) \leq -F_*(q,X) \leq c_{\max}\,\lambda_{\max}\,(X)\,. \tag{3.5}$$

Before showing the announced equivalence we first want to point out that the commonly made assumption of degenerate ellipticity of the left-hand side is already implied by the assumptions we made.

**Proposition 3.1.** *The operators $F$, $F_*$ and $F^*$ are degenerate elliptic.*

*Proof.* We only show the property for $F_*$. For this let $X, Y \in \mathcal{S}$ with $Y \leq X$. Rewriting $Y = X + (Y - X)$ and using (3.3) together with (3.5) yields

$$\begin{aligned} F_*(q,Y) &\geq F_*(q,X) + F_*(q, Y-X) \\ &\geq F_*(q,X) - c_{\max}\lambda_{\max}\,(Y-X) \\ &\geq F_*(q,X) \end{aligned}$$

for any $q \in \mathbb{R}^n$. □

Due to (3.1), the non-dependency of zero order terms and the preceding Proposition 3.1 the structure assumptions made in [33] are fulfilled. Therefore we may reformulate their results to fit in our context.

**Theorem 3.2.** *Let $u \in C(\overline{\Omega})$ be a subsolution of* (1.1) *with $f < 0$ or $f > 0$. Then the strong maximum principle*

$$\sup_U (u - v) \leq \sup_{\partial U} (u - v) \tag{3.6}$$

*holds for every bounded domain $U \subset \Omega$ and every supersolution $v \in C(\overline{U})$ of* (1.1).

*Proof.* Let $U \subset \Omega$ be a bounded domain. Clearly $u$ is also a subsolution in $\overline{U}$. Let $v \in C(\overline{U})$ be a supersolution of (1.1). Invoking the maximum principle [33, Theorem 1.3], or [33, Theorem 2.4] in the singular case, we obtain the assertion. □

Another way to formulate this maximum principle is the following comparison principle. Both formulations are equivalent.

**Corollary 3.3.** *Let $u \in C(\overline{\Omega})$ satisfy the strong maximum principle* (3.6)*. Then the strong comparison principle*

$$u \leq v \quad \text{on } \partial U \quad \Longrightarrow \quad u \leq v \quad \text{in } U \tag{3.7}$$

*holds for every bounded domain $U \subset \Omega$ and every supersolution $v \in C(\overline{U})$ of* (1.1)*.*

*Proof.* Let $U \subset \Omega$ be a bounded domain and $v \in C(\overline{U})$ a supersolution of (1.1). We assume $u \leq v$ on $\partial U$ but $u(x_0) > v(x_0)$ for some $x_0$ in $U$. Then

$$\sup_U (u - v) \geq (u - v)(x_0) > 0 \geq \sup_{\partial U} (u - v),$$

contradicting the strong maximum principle. □

We will proceed showing that, under our assumptions, the strong comparison principle and the strong maximum principle are equivalent to weaker versions. The idea is that it sufficies to consider smooth functions $v$ and constant right-hand sides greater or equal to the supremum of $f$.

**Lemma 3.4.** *Let $u \in C(\overline{\Omega})$ satisfy the strong comparison principle* (3.7)*. Then the weak comparison principle*

$$u \leq \psi \quad \text{on } \partial U \quad \Longrightarrow \quad u \leq \psi \quad \text{in } U \tag{3.8}$$

*holds for all bounded domains $U \subset \Omega$ and $\psi \in C^2(U) \cap C(\overline{U})$ with $F_*(\nabla \psi, \mathrm{D}^2 \psi) \geq \sup_U f$ in $U$.*

*Proof.* Let $U \subset \Omega$ be a bounded domain and $\psi \in C^2(U) \cap C(\overline{U})$ with $F_*(\nabla \psi, \mathrm{D}^2 \psi) \geq \sup_U f$. Then it holds

$$f \leq \sup_U f \leq F_*(\nabla \psi, \mathrm{D}^2 \psi) \leq F^*(\nabla \psi, \mathrm{D}^2 \psi)$$

in $U$. So $\phi$ is a supersolution of (1.1) in $U$ while $u \in C(\overline{U})$. Then, by the strong comparison principle (3.7), it holds

$$u \leq \psi \quad \text{on } \partial U \quad \Longrightarrow \quad u \leq \psi \quad \text{in } U.$$

□

Again we may also use an equivalent formulation for this statement. This can be done because of the non-dependency of $F$ on lower order terms.

**Corollary 3.5.** *Let $u \in C(\overline{\Omega})$ satisfy the weak comparison principle* (3.8). *Then the weak maximum principle*

$$\sup_U (u - \psi) \le \sup_{\partial U} (u - \psi) \tag{3.9}$$

*holds for all bounded domains $U \subset \Omega$ and $\psi \in C^2(U) \cap C(\overline{U})$ with $F_*(\nabla\psi, \mathrm{D}^2\psi) \ge \sup_U f$ in $U$.*

*Proof.* Let $U \subset \Omega$ be a bounded subset and $\psi \in C^2(U) \cap C(\overline{U})$ with $F_*(\nabla\psi, \mathrm{D}^2\psi) \ge \sup_U f$. We define $\tilde{\psi} := \psi + \sup_{\partial U}(u - \psi) \in C^2(U) \cap C(\overline{U})$. Then

$$F_*(\nabla\tilde{\psi}, \mathrm{D}^2\tilde{\psi}) = F_*(\nabla\psi, \mathrm{D}^2\psi) \ge \sup_U f$$

and $u \le \tilde{\psi}$ on $\partial U$. Using the weak comparison principle (3.8), this implies $u \le \tilde{\psi}$ in $U$. So we have

$$\sup_U (u - \psi) \le \sup_{\partial U} (u - \psi).$$

□

In order to conclude the desired equivalence it is left to show that all upper semicontinuous functions satisfying the weak maximum principle (3.9) are subsolutions of (1.1).

**Lemma 3.6.** *Let $u \in C(\overline{\Omega})$ satisfy the weak maximum principle* (3.9). *Then $u$ is a subsolution of* (1.1).

*Proof.* We assume that $u$ is not a subsolution of (1.1). Then there is some $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ such that $u - \phi$ attains local maximum in $x_0$ but

$$F_*(\nabla\phi(x_0), \mathrm{D}^2\phi(x_0)) > f(x_0).$$

So there is some $\delta > 0$ such that

$$F_*(\nabla\phi(x_0), \mathrm{D}^2\phi(x_0)) - \delta > f(x_0).$$

We define

$$\begin{aligned} \psi(x) := {} & \phi(x_0) + \langle \nabla\phi(x_0), x - x_0 \rangle \\ & + \tfrac{1}{2} \left\langle x - x_0, \mathrm{D}^2\phi(x_0)(x - x_0) \right\rangle + \tfrac{\delta}{2\, c_{\max}} \left| x - x_0 \right|^2 . \end{aligned}$$

Then we have

$$\psi(x_0) = \phi(x_0), \quad \nabla\psi(x_0) = \nabla\phi(x_0), \quad \mathrm{D}^2\psi(x_0) = \mathrm{D}^2\phi(x_0) + \tfrac{\delta}{c_{\max}}\,\mathrm{Id}, \tag{3.10}$$

and by (3.3) and (3.5)

$$F_*(\nabla\psi(x_0), \mathrm{D}^2\psi(x_0)) \geq F_*(\nabla\phi(x_0), \mathrm{D}^2\phi(x_0)) - \delta > f(x_0).$$

By continuity of $f$ and semicontinuity of the left-hand side, there is some bounded domain $U \subset \Omega$ with $x_0 \in U$ such that

$$F_*(\nabla\psi, \mathrm{D}^2\psi) \geq \sup_V f$$

in $V$ for all $V \subset U$ with $x_0 \in V$. Also, by (3.10), $\phi - \psi$ attains a strict local maximum in $x_0$. So there is a bounded domain $V \subset U$ with $x_0$ in $V$ and

$$(u - \psi)(x_0) > \sup_{\partial V}(u - \psi),$$

while

$$F_*(\nabla\psi, \mathrm{D}^2\psi) \geq \sup_V f$$

in $V$, so the weak maximum principle (3.9) does not hold. □

We summarize the shown results in a theorem.

**Theorem 3.7.** *Let $u \in C(\overline{\Omega})$, $f < 0$ or $f > 0$, and assume that* (3.1) *and* (3.2) *hold. Then the following statements are equivalent*

*(i)* *$u$ is a subsolution of* (1.1),

*(ii)* *$u$ satisfies the strong maximum principle* (3.6),

*(iii)* *$u$ satisfies the strong comparison principle* (3.7),

*(iv)* *$u$ satisfies the weak comparison principle* (3.8),

*(v)* *$u$ satisfies the weak maximum principle* (3.9).

□

To finish this section we discuss two examples of operators satisfying the assumptions we made.

**Example 3.8.** Let $\alpha, \beta : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be continuous and bounded from above, with $\beta \geq 0$ and $\inf_{\mathbb{R}^n \setminus \{0\}}(\alpha + \beta) > 0$. We take

$$F(q, X) = -\alpha(q)\,|q|^{-2}\,\langle q, Xq\rangle - \beta(q)\,\mathrm{tr}\,X,$$

which is clearly linear in $X$, so (3.1) is satisfied. Moreover, in points where $\alpha$ is nonnegative, we have

$$\begin{aligned} -F(q, X) &\leq \alpha(q)\lambda_{\max}(X) + \beta(q)\,n\,\lambda_{\max}(X) \\ &= (\alpha(q) + \beta(q)\,n)\ \lambda_{\max}(X) \end{aligned}$$

and

$$\begin{aligned} -F(q, X) &\geq \alpha(q)\lambda_{\min}(X) + \beta(q)\,n\,\lambda_{\min}(X) \\ &= (\alpha(q) + \beta(q)\,n)\ \lambda_{\min}(X)\,. \end{aligned}$$

In all other points, where $\alpha$ is negative, we have

$$\begin{aligned} -F(q, X) &\leq \alpha(q)\lambda_{\min}(X) + \beta(q)\ \textstyle\sum_{k=1}^{n} \lambda_k(X) \\ &= (\alpha(q) + \beta(q))\lambda_{\min}(X) + \beta(q)\ \textstyle\sum_{k=2}^{n} \lambda_k(X) \\ &\leq (\alpha(q) + \beta(q))\lambda_{\max}(X) + \beta(q)\,(n-1)\lambda_{\max}(X) \\ &= (\alpha(q) + \beta(q)\,n)\lambda_{\max}(X) \end{aligned}$$

and

$$\begin{aligned} -F(q, X) &\geq \alpha(q)\lambda_{\max}(X) + \beta(q)\ \textstyle\sum_{k=1}^{n} \lambda_k(X) \\ &= (\alpha(q) + \beta(q))\lambda_{\max}(X) + \beta(q)\ \textstyle\sum_{k=1}^{n-1} \lambda_k(X) \\ &\geq (\alpha(q) + \beta(q))\lambda_{\min}(X) + \beta(q)\,(n-1)\lambda_{\min}(X) \\ &= (\alpha(q) + \beta(q)\,n)\lambda_{\min}(X)\,. \end{aligned}$$

So we may take

$$c_{\min} = \inf_{q \in \mathbb{R}^n \setminus \{0\}} (\alpha(q) + \beta(q)\,n) > 0$$

and

$$c_{\max} = \sup_{q \in \mathbb{R}^n \setminus \{0\}} (\alpha(q) + \beta(q)\,n) < \infty,$$

to see that (3.2) is also satisfied. $\square$

A special case of this example is the following.

**Example 3.9.** For the normalized $p$-Laplacian we have

$$F(q, X) = -\tfrac{p-2}{p} |q|^{-2} \langle q, Xq \rangle - \tfrac{1}{p} \operatorname{tr} X$$

for $1 < p < \infty$. Taking

$$\alpha = \tfrac{p-2}{p} \qquad \text{and} \qquad \beta = \tfrac{1}{p}$$

in the preceding example, we see $\beta \geq 0$ and $\alpha + \beta = \tfrac{p-1}{p} > 0$ to find

$$c_{\min} = c_{\max} = \tfrac{p+n-2}{p}.$$

So (3.2) is satisfied if $p + n > 2$. This also includes the limiting case of the normalized infinity Laplacian by sending $p \to \infty$ but not the normalized 1-Laplacian since the condition $\tfrac{p-1}{p} > 0$ is violated in the limit $p \to 1$. $\square$

## 3.2 Comparison with cusps

In this section we assume, in addition to the previous assumptions (3.1) and (3.2), that $F$ is normalized in the first argument and invariant under rotations. Inspired by Lemma 3.4 we derive radial functions to generalize the comparison with cones in [8] and comparison with polar quadratic functions in [32] for negative right-hand side $f < 0$.

So we assume that for all $\mu \in \mathbb{R}$, $q \in \mathbb{R}^n \setminus \{0\}$, $X \in \mathcal{S}$ and $Q \in \mathcal{O}(n)$ it holds

$$F(\mu q, X) = F(q, X) \tag{3.11}$$

and

$$F(q, X) = F(Q^\intercal q, Q^\intercal X Q). \tag{3.12}$$

In the derivation of the radial functions the values $F(q, \mathrm{Id})$ and $F(q, q \otimes q)$ will appear. As a preparation we observe that, by the assumptions we made, both of these values can be calculated.

**Proposition 3.10.** *There is a constant $c_F \in \mathbb{R}$ with $c_{\min} \leq c_F \leq c_{\max}$ such that*

$$F(q, \mathrm{Id}) = -c_F$$

*for all $q \in \mathbb{R}^n \setminus \{0\}$.*

*Proof.* Let $q_1, q_2 \in \mathbb{R}^n \setminus \{0\}$. There is a $Q \in \mathcal{O}(n)$ with

$$\tfrac{q_1}{|q_1|} = Q^{-1} \tfrac{q_2}{|q_2|} = Q^T \tfrac{q_2}{|q_2|}.$$

Then we have

$$F\left(\tfrac{q_1}{|q_1|}, \mathrm{Id}\right) = F\left(Q^T \tfrac{q_2}{|q_2|}, \mathrm{Id}\right) = F\left(Q^T \tfrac{q_2}{|q_2|}, Q^\intercal Q\right) = F\left(\tfrac{q_2}{|q_2|}, \mathrm{Id}\right),$$

so, by (3.11), there is a constant $c_F \in \mathbb{R}$ such that

$$F(q, \mathrm{Id}) = -c_F.$$

for all $q \in \mathbb{R}^n \setminus \{0\}$. Using (3.2) we obtain $c_{\min} \leq c_F \leq c_{\max}$, which completes the proof. $\square$

Furthermore we can calculate the other value that will occur.

**Proposition 3.11.** *There is a constant $c_Q \in \mathbb{R}$ with $0 \leq c_Q \leq c_F$ such that*

$$F(q, q \otimes q) = -c_Q \, |q|^2$$

*for all $q \in \mathbb{R}^n \setminus \{0\}$.*

*Proof.* Let $q = (q_1, \ldots, q_n)^\intercal \in \mathbb{R}^n \setminus \{0\}$. First we note that

$$q \otimes q = [q_1 \, q, \ldots q_n \, q],$$

so all $n$ columns are linear dependent. Thus the matrix has $n-1$ times the eigenvalue 0. Also we note that $q$ itself is an eigenvector with corresponding eigenvalue $|q|^2$. So let $Q \in \mathcal{O}(n)$ be a diagonalizing matrix in a way that $[Q^\intercal (q \otimes q) Q]_{1,1} = |q|^2$ and $[Q^\intercal (q \otimes q) Q]_{i,j} = 0$ for $2 \leq i, j \leq n$. We compute

$$[Q^\intercal (q \otimes q) Q]_{i,j} = \sum_{k=1}^{n} Q^\intercal_{i,k} \sum_{l=1}^{n} q_k q_l \, Q_{l,j} = \sum_{k=1}^{n} Q^\intercal_{i,k} \, q_k \sum_{l=1}^{n} Q^\intercal_{j,l} \, q_l = [Q^\intercal \, q]_i [Q^\intercal \, q]_j$$

to find that

$$[Q^\intercal \, q]_1^2 = [Q^\intercal (q \otimes q) Q]_{1,1} = |q|^2$$

and

$$[Q^\intercal \, q]_i^2 = [Q^\intercal (q \otimes q) Q]_{i,i} = 0,$$

so $F(q, q \otimes q) = F(Q^\intercal q, Q^\intercal (q \otimes q) Q)$ only depends on the value of $|q|$. We may deduce that there is a function $h : \mathbb{R} \to \mathbb{R}$ such that

$$F(q, q \otimes q) = h(|q|).$$

Then, by (3.1) and (3.11), for every $\mu \in \mathbb{R}$, we have

$$h(|\mu \, q|) = F(\mu q, (\mu q) \otimes (\mu q)) = \mu^2 \, F(q, q \otimes q) = \mu^2 \, h(|q|),$$

so there must be a constant $c_Q \in \mathbb{R}$ such that

$$h(|q|) = -c_Q \, |q|^2 \, .$$

Finally, by (3.2), we have

$$0 = c_{\min} \lambda_{\min} (q \otimes q) \leq -F(q, q \otimes q) = c_Q \, |q|^2$$

and

$$c_Q \, |q|^2 = -F(q, q \otimes q) \leq - |q|^2 \, F(q, \mathrm{Id}) = c_F \, |q|^2 \, ,$$

by the degenerate ellipticity of $F$ and Proposition 3.10. So $0 \leq c_Q \leq c_F$, completing the proof. $\square$

After these preparations and since $F$ is assumed to be invariant under rotations, let us now assume that $(x \mapsto \phi(|x|)) \in C^2(U \setminus \{0\})$ is a radial solution of (1.1) in $U \setminus \{0\}$ for some bounded subset $U \subset \Omega$.

We may compute for $x \neq 0$

$$\begin{aligned}
\nabla \phi(|x|) &= \phi'(|x|) \tfrac{x}{|x|}, \\
\mathrm{D}^2 \phi(|x|) &= \phi''(|x|) \tfrac{x \otimes x}{|x|^2} + \phi'(|x|) \left( \tfrac{\mathrm{Id}}{|x|} - \tfrac{x \otimes x}{|x|^3} \right) \\
&= \left( \phi''(|x|) - \tfrac{1}{|x|} \phi'(|x|) \right) \tfrac{x \otimes x}{|x|^2} + \tfrac{1}{|x|} \phi'(|x|) \, \mathrm{Id} \, .
\end{aligned}$$

Then, if $\phi'(|x|) \neq 0$, we obtain

$$\begin{aligned}
&F_*(\nabla \phi(|x|), \mathrm{D}^2 \phi(|x|)) \\
&\quad = F(\nabla \phi(|x|), \mathrm{D}^2 \phi(|x|)) \\
&\quad = \left( \phi''(|x|) - \tfrac{1}{|x|} \phi'(|x|) \right) F \left( \tfrac{x}{|x|}, \tfrac{x \otimes x}{|x|^2} \right) + \tfrac{1}{|x|} \phi'(|x|) F \left( \tfrac{x}{|x|}, \mathrm{Id} \right) \\
&\quad = - \left( \phi''(|x|) - \tfrac{1}{|x|} \phi'(|x|) \right) c_Q - \tfrac{1}{|x|} \phi'(|x|) \, c_F \\
&\quad = - \phi''(|x|) \, c_Q - \tfrac{1}{|x|} \phi'(|x|) \, (c_F - c_Q),
\end{aligned}$$

using the identities provided by Proposition 3.10 and 3.11.
Solving the ordinary differential equation

$$-\phi''(|x|) \, c_Q - \tfrac{1}{|x|} \phi'(|x|) \, (c_F - c_Q) = K \sup_U f \geq \sup_U f, \tag{3.13}$$

for any constant $K \leq 1$, yields

$$\phi(|x|) = \begin{cases} a - \frac{K}{2 \, c_F} (\sup_U f) \, |x|^2 & \text{if } c_Q = 0, \\ a + b \, \frac{c_Q}{2 \, c_Q - c_F} \, |x|^{2 - \frac{c_F}{c_Q}} - \frac{K}{2 \, c_F} (\sup_U f) \, |x|^2 & \text{if } 2 \, c_Q \neq c_F, \\ a + b \, \log |x| - \frac{K}{2 \, c_F} (\sup_U f) \, |x|^2 & \text{if } 2 \, c_Q = c_F, \end{cases}$$

with arbitrary $a, b \in \mathbb{R}$, as a candidate for the radial solution we are looking for. The constant $K$ will be chosen afterwards.

For this purpose we have to consider the case of $\phi'(|x|) = 0$.
If $c_Q = 0$ we simply have

$$\mathrm{D}^2\phi(|x|) = -\tfrac{K}{c_F}(\sup_U f)\tfrac{x\otimes x}{|x|^2}$$

and so

$$F_*(\nabla\phi(|x|), \mathrm{D}^2\phi(|x|)) \geq -c_{\max}\,\lambda_{\max}\left(\mathrm{D}^2\phi(|x|)\right) = \tfrac{c_{\max}}{c_F}\,K\,\sup_U f. \tag{3.14}$$

Taking $K := \frac{c_F}{c_{\max}} \leq 1$ then assures $F_*(\nabla\phi(|x|), \mathrm{D}^2\phi(|x|)) \geq \sup_U f$.

If $c_Q \neq 0$ then $\phi'(|x|) = 0$ implies

$$b\,|x|^{-\frac{c_F}{c_Q}} = \tfrac{K}{c_F}\sup_U f$$

and so

$$\begin{aligned}
\mathrm{D}^2\phi(|x|) &= \phi''(|x|)\tfrac{x\otimes x}{|x|^2}\\
&= \left(b\left(1-\tfrac{c_F}{c_Q}\right)|x|^{-\frac{c_F}{c_Q}} - \tfrac{K}{c_F}\sup_U f\right)\tfrac{x\otimes x}{|x|^2}\\
&= -\tfrac{K}{c_Q}(\sup_U f)\tfrac{x\otimes x}{|x|^2}.
\end{aligned}$$

Thus we have

$$F_*(\nabla\phi(|x|), \mathrm{D}^2\phi(|x|)) \geq -c_{\max}\,\lambda_{\max}\left(\mathrm{D}^2\phi(|x|)\right) = \tfrac{c_{\max}}{c_Q}\,K\,\sup_U f. \tag{3.15}$$

Taking $K := \frac{c_Q}{c_{\max}} \leq 1$ again assures $F_*(\nabla\phi(|x|), \mathrm{D}^2\phi(|x|)) \geq \sup_U f$.

We may use the radial symmetric function we just derived together with the idea of Lemma 3.4 to obtain an important lemma. Because of the shape of the function appearing in it, we call this property *comparison with cusps from above*. The shape of these cusps is fully determined by the right-hand side $f$, the values $c_F, c_Q$ and $c_{\max}$, and the values of $a$ and $b$.

**Lemma 3.12.** *Let $f < 0$, $u \in \mathrm{USC}(\overline{\Omega})$ be a subsolution of* (1.1) *and $U \subset \Omega$ be a bounded domain. Furthermore assume that* (3.1), (3.2), (3.11) *and* (3.12) *hold. Then for every $x_0 \in U$ it holds*

$$u(x) \leq C(x - x_0) \qquad \text{on } \partial\,(U \setminus \{x_0\}) \quad \Longrightarrow \quad u(x) \leq C(x - x_0) \qquad \text{in } U$$

*with*

$$C(x) := \begin{cases} a - \frac{1}{2\,c_{\max}}(\sup_U f)\,|x|^2 & \text{if } c_Q = 0, \\ a + b\,\frac{c_Q}{2\,c_Q - c_F}\;|x|^{2-\frac{c_F}{c_Q}} - \frac{c_Q}{2\,c_F\,c_{\max}}(\sup_U f)\,|x|^2 & \text{if } 2\,c_Q \neq c_F, \\ a + b\,\log|x| - \frac{c_Q}{2\,c_F\,c_{\max}}(\sup_U f)\,|x|^2 & \text{if } 2\,c_Q = c_F, \end{cases}$$

*and $a, b \in \mathbb{R}$.*

*Proof.* We only show the case of $c_Q \neq 0$, since it is obvious to transfer the proof.
So let us assume the assertion is not true. Then there exists some bounded domain $U \subset \Omega$ and a point $\hat{x} \in U \setminus \{x_0\}$ such that

$$u(x) \leq C(x - x_0) \text{ on } \partial\,(U \setminus \{x_0\}) \qquad \text{but} \qquad u(\hat{x}) > C(\hat{x} - x_0) \text{ in } U.$$

Since $U$ is bounded, we can find some ball of radius $R$ covering $U$. Therefore it holds

$$u(x) \leq C_\varepsilon(x - x_0) := C(x - x_0) + \varepsilon \tfrac{c_Q}{2\,c_F\,c_{\max}}(R^2 - |x - x_0|^2)$$

in $\partial\,(U \setminus \{x_0\})$ while still keeping $u(\hat{x}) > C_\varepsilon(\hat{x} - x_0)$ and $\sup_U f + \varepsilon \leq 0$ for $\varepsilon > 0$ sufficiently small. We may assume that $x \mapsto u(x) - C_\varepsilon(x - x_0)$ attains a local maximum in $\hat{x}$ and use the preceding calculation to obtain

$$F_*(\nabla C_\varepsilon(\hat{x} - x_0), \mathrm{D}^2 C_\varepsilon(\hat{x} - x_0)) \geq \sup_U f + \varepsilon > f(\hat{x}),$$

contradicting $u$ being a subsolution of (1.1). □

**Remark 3.13.** *We want to mention that Lemma 3.12 can be obtained for nonnegative $f \geq 0$ with slightly different cusps. Indeed we may choose $K = 1$ in* (3.13), (3.14), *and* (3.15) *to obtain the cusp*

$$C(x) := \begin{cases} a - \frac{1}{2\,c_F}(\sup_U f)\,|x|^2 & \text{if } c_Q = 0, \\ a + b\,\frac{c_Q}{2\,c_Q - c_F}\;|x|^{2-\frac{c_F}{c_Q}} - \frac{1}{2\,c_F}(\sup_U f)\,|x|^2 & \text{if } 2\,c_Q \neq c_F, \\ a + b\,\log|x| - \frac{1}{2\,c_F}(\sup_U f)\,|x|^2 & \text{if } 2\,c_Q = c_F, \end{cases}$$

*for suitable $a, b \in \mathbb{R}$.* □

This useful property of subsolutions allows us to prove a Hopf-type Lemma.

**Lemma 3.14.** *Let $f < 0$, $u \in \mathrm{USC}(\overline{\Omega})$ be a subsolution to* (1.1) *and $\Omega$ satisfy the interior sphere condition* (2.1)*. Furthermore assume that* (3.1), (3.2), (3.11) *and* (3.12) *hold, and let $x_0 \in \partial\Omega$ be a boundary point with*

$$u(x_0) > u(x) \tag{3.16}$$

*for all $x \in \Omega$. Then for every $\mu \in \partial B_1(0)$ with $\langle \mu, \nu(x_0)\rangle > 0$ holds*

$$\liminf_{r \downarrow 0} \frac{u(x_0) - u(x_0 - r\mu)}{r} > 0$$

*and therefore*

$$\frac{\partial u}{\partial \mu}(x_0) > 0,$$

*provided the derivative $\frac{\partial u}{\partial \mu}(x_0)$ exists.*

*Proof.* Again we only show the case of $c_Q \neq 0$. According to the interior sphere condition (2.1) there is some $R > 0$ such that $B_R(y_0) \subset \Omega$ and $\partial B_R(y_0) \cap \partial\Omega = \{x_0\}$ with $y_0 := x_0 - R\nu(x_0)$. We also note that $u$ satisfies

$$F_*(\nabla u, \mathrm{D}^2 u) \leq \varepsilon \sup_{\Omega} f \tag{3.17}$$

in $\Omega$ for all $\varepsilon \in (0,1)$ in the sense of viscosity solutions. We may take $\varepsilon > 0$ so small that

$$u(y_0) \leq u(x_0) + \varepsilon \tfrac{c_Q}{2c_F c_{\max}} \big( \sup_{B_R(y_0)} f \big) R^2 \tag{3.18}$$

Then we define

$$\psi(x) := u(x_0) + \varepsilon \tfrac{c_Q}{2c_F c_{\max}} \big( \sup_{B_R(y_0)} f \big) \left( R^2 - |x - y_0|^2 \right).$$

Now we have

$$u(y_0) \leq \psi(y_0),$$

by (3.18) and

$$u(x) < u(x_0) = \psi(x)$$

on $\partial B_R(y_0)$ by (3.16). Therefore we have $u \le \psi$ on $\partial(B_R(y_0) \setminus \{y_0\})$. Then Lemma 3.12 implies $u \le \psi$ in $B_R(y_0)$ by considering $\varepsilon f$ instead of $f$, which is justified by (3.17).
For every $\mu \in \partial B_1(0)$ with $\langle \mu, \nu(x_0)\rangle > 0$ holds $x_0 - r\mu \in B_R(y_0)$ for all $r > 0$ sufficiently small. Using $u(x_0) = \psi(x_0)$ we conclude

$$\frac{u(x_0) - u(x_0 - r\mu)}{r} \ge \frac{\psi(x_0) - \psi(x_0 - r\mu)}{r}$$

for all $r > 0$ sufficiently small. Sending $r \downarrow 0$ we find

$$\liminf_{r\downarrow 0} \frac{u(x_0) - u(x_0 - r\mu)}{r} \ge \frac{\partial \psi}{\partial \mu}(x_0) = -\varepsilon^{\frac{c_Q}{c_F c_{\max}}}\Big(\sup_{B_R(y_0)} f\Big) R \left\langle \nu(x_0), \mu \right\rangle > 0.$$

□

It was shown in [8] that, in case of the infinity Laplacian and vanishing right-hand side, the equivalence shown in Section 3.1 can be preserved when reducing the weak comparison principle (3.8) to these radial functions. The proof uses Lipschitz continuity of $C$, which is the case if and only if $c_Q = c_F$ or $c_Q = 0$. So, in that case, comparison with cusps from above is a defining property of subsolutions.

**Definition 3.15.** We say $u \in \mathrm{USC}(\overline{\Omega})$ *enjoys comparison with cusps from above* if for every bounded subset $U \subset \Omega$, every point $x_0 \in U$

$$u(x) \le C(x - x_0) \qquad \text{on } \partial\left(U \setminus \{x_0\}\right) \quad \Longrightarrow \quad u(x) \le C(x - x_0) \qquad \text{in } U$$

with $C(x)$ as in Lemma 3.12. Note that $u(x) \le C(x - x_0)$ on $\partial(U \setminus \{x_0\})$ only for suitable values of $a$ and $b$. □

To finish the section we want to mention that, though the function appearing in Lemma 3.12 looks complicated, the constants $c_F$ and $c_Q$ are easily computed with

$$c_F = -F(q, \mathrm{Id}),$$

according to Proposition 3.10 and

$$c_Q = -\tfrac{1}{|q|^2} F(q, q\otimes q),$$

by Proposition 3.11 for arbitrary $q \in \mathbb{R}^n \setminus \{0\}$. We discuss this in two examples.

**Example 3.16.** Due to the assumptions we made we have to take $\alpha, \beta \in \mathbb{R}$ constant in Example 3.8 with $\beta \geq 0$ and $\alpha + \beta > 0$. Then we have

$$F(q, X) = -\alpha\, |q|^{-2} \langle q, Xq\rangle - \beta \,\operatorname{tr} X.$$

By Example 3.8 we already know $c_{\max} = (\alpha + \beta\, n)$. Additionally we easily compute

$$c_F = \alpha + \beta\, n \qquad\qquad \text{and} \qquad\qquad c_Q = \alpha + \beta > 0,$$

so $2c_Q - c_F = \alpha - \beta\,(n-2)$. Then, in Lemma 3.12, we have

$$C(x) = \begin{cases} a + b\,\frac{\alpha+\beta}{\alpha-\beta\,(n-2)}\ |x|^{2-\frac{\alpha+\beta\, n}{\alpha+\beta}} - \frac{\alpha+\beta}{2\,(\alpha+\beta\, n)^2}(\sup_U f)\,|x|^2 & \text{if } \alpha \neq \beta\,(n-2),\\ a + b\,\log|x| - \frac{\alpha+\beta}{2\,(\alpha+\beta\, n)^2}(\sup_U f)\,|x|^2 & \text{if } \alpha = \beta\,(n-2). \end{cases}$$

We observe that for the lower order exponent we have $2 - \frac{\alpha+\beta\, n}{\alpha+\beta} \leq 1$, with equality if and only if $\beta = 0$ or $n = 1$. □

**Example 3.17.** Again for $1 < p < \infty$ taking $\alpha = \frac{p-2}{p}$ and $\beta = \frac{1}{p}$ in the preceding example brings us to the normalized $p$-Laplacian

$$F(q, X) = -\tfrac{p-2}{p}\, |q|^{-2} \langle q, Xq\rangle - \tfrac{1}{p} \operatorname{tr} X.$$

Then we have $c_{\max} = \frac{p+n-2}{p}$, $c_F = \frac{p+n-2}{p}$ and $c_Q = \frac{p-1}{p}$, so $2c_Q - c_F = \frac{p-n}{p}$. We obtain

$$C(x) = \begin{cases} a + b\,\frac{p-1}{p-n}\ |x|^{\frac{p-n}{p-1}} - \frac{p(p-1)}{2\,(p+n-2)^2}(\sup_U f)\,|x|^2 & \text{if } p \neq n,\\ a + b\,\log|x| - \frac{p(p-1)}{2\,(p+n-2)^2}(\sup_U f)\,|x|^2 & \text{if } p = n. \end{cases}$$

In the limiting case $p \to 1$ we have $c_Q = 0$, for which we would know how to calculate the cusp function, but in the derivation we demanded $\frac{p-2}{p} + \frac{1}{p} > 0$. So the result is not applicable in this case.
In the other limiting case $p \to \infty$ we obtain

$$C(x) = a + b\,|x| - \tfrac{1}{2}(\sup_U f)\,|x|^2 .$$

Then Lemma 3.12 becomes the so called 'Comparison with Polar Quadratic Polynomials from above' as in [32]. This property is also introduced as 'Comparison with Cones from above' for vanishing right-hand side in [8]. □

## 3.3 Local regularity

In this section we want to use comparison with cusps property to obtain local regularity of subsolutions. We use the methods provided in [8]. These apply whenever the cusps in Lemma 3.12 are bounded, which is the case if the lower order exponent is positive. So we assume

$$c_F < 2c_Q \tag{3.19}$$

or, by recalling Proposition 3.10 and 3.11, that $F(q, \mathrm{Id}) > 2\,|q|^{-2}\,F(q, q \otimes q)$ for all $q \in \mathbb{R}^n \setminus \{0\}$. In order to improve the readability we introduce

$$\gamma := 2 - \tfrac{c_F}{c_Q} \qquad \text{and} \qquad \theta := \tfrac{c_Q}{2\,c_F\,c_{\max}}$$

to replace the constants appearing in Lemma 3.12. So we are hereafter considering the case of $\gamma > 0$.

In order to obtain local estimates on subsolutions $u \in \mathrm{USC}(\Omega)$ of (1.1), we want the left-hand side of the implication in Lemma 3.12 to always hold. Therefore, for $u \in \mathrm{USC}(\Omega)$, $y \in \Omega$, $r < \operatorname{dist}(y, \partial\Omega)$, we define

$$L_r(y) := \max_{z \in \partial B_r(y)} \tfrac{u(z)-u(y)}{r^\gamma}.$$

If $u$ happens to be Hölder continuous, this value is less or equal to its Hölder norm. On the other hand, if this value is locally bounded, $u$ must be locally Hölder continuous with exponent $\gamma$. To verify that $L_r$ is locally bounded near $r = 0$, we estimate the growth of the mapping $r \mapsto L_r(y)$.

**Proposition 3.18.** *Let $u \in \mathrm{USC}(\Omega)$ enjoy comparison with cusps from above. Then it holds*

$$L_s(y) \leq L_r(y) + \big(\sup_{B_r(y)} f\big)\,\theta\,(r^{2-\gamma} - s^{2-\gamma}).$$

*for all $s \leq r$.*

*Proof.* By definition we have

$$u(z) \leq u(y) + L_r(y)\,r^\gamma = u(y) + L_r(y)\,\,|x-y|^\gamma$$

for all $z \in \partial B_r(y)$. Adding

$$0 = \big(\sup_{B_r(y)} f\big)\theta r^{2-\gamma} |z-y|^\gamma - \big(\sup_{B_r(y)} f\big)\,\theta\,|z-y|^2,$$

on the right-hand side, we find

$$u(z) \leq u(y) + \gamma\big(L_r(y) + \big(\sup_{B_r(y)} f\big)\theta r^{2-\gamma}\big)\tfrac{1}{\gamma}\,|z-y|^\gamma - \big(\sup_{B_r(y)} f\big)\,\theta\,|z-y|^2$$

for all $z \in \partial B_r(y)$. Then, by comparison with cusps from above, the same inequality holds for all $z \in B_r(y)$ and so for all $0 < s < r$ and $z \in \partial B_s(y)$. We find by replacing $|z-y| = s$ and rearranging

$$\tfrac{u(z)-u(y)}{s^\gamma} \leq L_r(y) + \big(\sup_{B_r(y)} f\big)\,\theta\,(r^{2-\gamma} - s^{2-\gamma})$$

for all $z \in \partial B_s(y)$. Taking the maximum on the left-hand side yields the assertion. □

Using this proposition we can generalize the results shown in [8, Lemma 2.4 and Lemma 2.5], carefully repeating the steps of the proof. In the special case of $\gamma = 1$ the results coincide.

**Lemma 3.19.** *Let $f \leq 0$ and $u \in \mathrm{USC}(\Omega)$ enjoy comparison with cusps from above. Then the mapping $r \mapsto L_r(y)$ is nondecreasing in $r$ and nonnegative.*

*Proof.* From Proposition 3.18 it immediately follows that the mapping is nondecreasing.
So it is left to show that $0 \leq L_r(y)$. Since $L_r(y)$ decreases as $r$ decreases, we may find some

$$M > \lim_{r\downarrow 0} L_r(y).$$

We want to show $-M < \lim_{r\downarrow 0} L_r(y)$ to conclude that $M$ can not be chosen negative, which then implies the assertion.
The monotonicity and the preceding inequality imply that there is some $r_0 > 0$ such that

$$u(x) \leq u(y) + M\,|x-y|^\gamma \tag{3.20}$$

for all $x \in B_{r_0}(y)$. Also by monotonicity we have

$$u(y) \leq u(x) + \max_{z \in \partial B_r(x)} \frac{u(z) - u(x)}{r^\gamma} |x - y|^\gamma \tag{3.21}$$

for all $r > |x - y|$. Taking $r < \frac{r_0}{2}$, we have

$$|z - y| \leq |z - x| + |x - y| < r_0,$$

so we may use (3.20) for $x = z$ in (3.21) with $|z - y|^\gamma \leq |x - y|^\gamma + r^\gamma$, which holds for $\gamma \leq 1$, to obtain

$$u(y) \leq u(x) + \frac{u(y) + M\left(|x - y|^\gamma + r^\gamma\right) - u(x)}{r^\gamma} |x - y|^\gamma .$$

This can be rearranged to

$$-M \frac{r^\gamma + |x - y|^\gamma}{r^\gamma - |x - y|^\gamma} |x - y|^\gamma \leq u(x) - u(y)$$

and so

$$-M \frac{r^\gamma + \varepsilon^\gamma}{r^\gamma - \varepsilon^\gamma} \leq \max_{x \in \partial B_\varepsilon(y)} \frac{u(x) - u(y)}{\varepsilon^\gamma} = L_\varepsilon(y).$$

Sending $\varepsilon \downarrow 0$ we conclude $-M \leq \lim_{r \downarrow 0} L_r(y)$, implying $\lim_{r \downarrow 0} L_r(y) \geq 0$ and proving the assertion. $\square$

**Theorem 3.20.** *Let $f \leq 0$ and $u \in \mathrm{USC}(\Omega)$ enjoy comparison with cusps from above. Then $u \in C^{0,\gamma}_{\mathrm{loc}}(\Omega)$.*

*Proof.* By definition of $L_r(y)$ and comparison with cusps from above we have

$$u(x) \leq u(y) + \max_{z \in \partial B_r(y)} \frac{u(z) - u(y)}{r^\gamma} |x - y|^\gamma \tag{3.22}$$

for $|x - y| \leq r$ and $r < \operatorname{dist}(y, \partial\Omega)$, since the inequality holds for $|x - y| = r$. If $|x - y| < r$, we have

$$\left(\frac{r^\gamma}{r^\gamma - |x - y|^\gamma}\right) u(x) - \max_{z \in \partial B_r(y)} u(z) \left(\frac{|x - y|^\gamma}{r^\gamma - |x - y|^\gamma}\right) \leq u(y).$$

For fixed $r$ the left-hand side is lower semicontinuous in $y \in B_r(x)$, so $u(y)$ is locally bounded from below. Since $u$ is upper semicontinuous it is also locally bounded from above, so $L_r(y)$ is locally bounded for fixed $r$.
Then we know that $0 \le L_r(y)$ is bounded for fixed $r$ in compact subsets of $\{z \in \Omega \mid \operatorname{dist}(z, \partial\Omega) < r\}$ and decreases whenever $r$ decreases, both according to Lemma 3.19.
We may now rewrite (3.22) to see

$$u(x) - u(y) \le L_r(y)\, |x - y|^\gamma$$

and by interchanging $x$ and $y$ we see

$$u(y) - u(x) \le L_r(x)\, |x - y|^\gamma .$$

Finally we put these estimates together to obtain

$$|u(x) - u(y)| \le \max\left(L_r(x), L_r(y)\right) |x - y|^\gamma$$

for $|x - y| < r$ and $\max(\operatorname{dist}(x, \partial\Omega), \operatorname{dist}(y, \partial\Omega)) < r$. This implies the assertion. $\square$

We continue developing the previous examples.

**Example 3.21.** As in Example 3.16, let $\alpha, \beta \in \mathbb{R}$ with $\beta > 0$, $\alpha + \beta > 0$ and

$$F(q, X) = -\alpha\, |q|^{-2} \langle q, Xq\rangle - \beta \operatorname{tr} X.$$

Then we have $\gamma = \frac{\alpha - \beta\,(n-2)}{\alpha+\beta} > 0$ if and only if $\alpha > \beta(n-2)$. So $\alpha > \beta(n-2)$ implies that all upper semicontinuous subsolutions of (1.1) with nonpositive right-hand side $f \le 0$ are locally Hölder continuous with exponent $\gamma$ and locally Lipschitz continuous if $\beta = 0$ or $n = 1$. $\square$

**Example 3.22.** In the case of the normalized $p$-Laplacian

$$F(q, X) = -\tfrac{p-2}{p}\, |q|^{-2} \langle q, Xq\rangle - \tfrac{1}{p} \operatorname{tr} X,$$

we find $\gamma = \frac{p-n}{p-1} > 0$ if and only if $p > n$. So $p > n$ implies local Hölder continuity with exponent $\frac{p-n}{p-1}$ for all upper semicontinuous subsolutions of (1.1) with nonpositive right-hand side $f \le 0$.
Making the assumption that $u$ is a nonnegative viscosity solution of (1.1) and the right-hand side $f$ is bounded, it was shown in [7] that $u$ is even locally Lipschitz continuous, independent of the dimension.
In the limiting case $p \to \infty$, we also obtain local Lipschitz continuity in all dimensions, as it was shown in [8]. $\square$

# The Dirichlet boundary value problem

In this chapter we will discuss the Dirichlet boundary value problem (1.2). We introduce the suitable definition of viscosity solutions to this type of problem. Then we will show existence of solutions for continuous boundary data under the assumptions made in Chapter 3. Finally we will show power concavity for constant right-hand side and vanishing boundary data under assumptions on the operator $F$ and an additional structural assumption on the domain $\Omega$.

## 4.1 Viscosity solutions of the Dirichlet boundary value problem

First we have to extend the concept of viscosity solutions to boundary value problems. The notion we introduced, does not suit for boundary points. Therefore a suitable notion was introduced in [16]. Again, there are two ways to formulate the definition. The one we will use requires the so called *closed semijets*.

**Definition 4.1.** For a function $u : \Omega \to \mathbb{R}$ and a point $x \in \overline{\Omega}$ we define

$$\begin{aligned}\overline{\mathcal{J}}^{2,+}u(x) = \{(q, X) \in \mathbb{R}^n \times \mathcal{S} \mid\ & \exists (x_n, q_n, X_n) \in \Omega \times \mathbb{R}^n \times \mathcal{S},\\ & (q_n, X_n) \in \mathcal{J}^{2,+}u(x_n) \text{ and}\\ & (x_n, u(x_n), q_n, X_n) \to (x, u(x), q, X)\},\end{aligned}$$

the *closed second order superjet* of $u$ at $x$, and

$$\begin{aligned}\overline{\mathcal{J}}^{2,-}u(x) = \{(q, X) \in \mathbb{R}^n \times \mathcal{S} \mid\ & \exists (x_n, q_n, X_n) \in \Omega \times \mathbb{R}^n \times \mathcal{S},\\ & (q_n, X_n) \in \mathcal{J}^{2,-}u(x_n) \text{ and}\\ & (x_n, u(x_n), q_n, X_n) \to (x, u(x), q, X)\},\end{aligned}$$

the *closed second order subjet* of $u$ at $x$. $\square$

The closed semijets are not what is commonly understood as the closure of set-valued functions, since they depend on the values of the function $u$ but their graphs do not record its values.
Now we may state the definition of viscosity solutions. Again we consider a more general right-hand side $f : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ that satisfies (2.3).

**Definition 4.2.** We call $u \in \mathrm{USC}(\overline{\Omega})$ a viscosity subsolution of (1.2) if it is a viscosity subsolution of (1.1) in $\Omega$ and for every $x \in \partial\Omega$ it holds

$$\min\{F_*(q,X) - f(x,u(x),q),\ u(x) - g(x)\} \leq 0,$$

for all $(q,X) \in \overline{\mathcal{J}}^{2,+}u(x)$.
Likewise we call $u \in \mathrm{LSC}(\overline{\Omega})$ a viscosity supersolution of (1.2) if it is a viscosity supersolution of (1.1) in $\Omega$ and for every $x \in \partial\Omega$ it holds

$$\max\{F^*(q,X) - f(x,u(x),q),\ u(x) - g(x)\} \geq 0,$$

for all $(q,X) \in \overline{\mathcal{J}}^{2,-}u(x)$. $\square$

So a viscosity solution of the Dirichlet boundary value problem is a viscosity solution of the (1.1) that additionally solves (1.1) in boundary points or satisfies the boundary condition, but not necessarily both. Again we will drop the term *viscosity* and just speak of sub- and supersolutions.

## 4.2 Existence of solutions

In this section we want to show, that viscosity solutions of the Dirichlet boundary value problem (1.2) actually exist. For showing existence of viscosity solutions a key tool is Perron's method [10, Chapter 4]. The idea is to consider a set of subsolutions, take the pointwise supremum of these functions, and to hopefully obtain a solution to the problem. This motivates the following lemma.

**Lemma 4.3.** *Let $\Sigma \subset C(\Omega)$. If every $v \in \Sigma$ is a subsolution of* (1.1), *then the pointwise defined function $u(x) := \sup_{v\in\Sigma} v(x)$ is also a continuous subsolution of* (1.1) *provided $u \in \mathrm{USC}(\Omega)$.*

*Proof.* First we note that $u \in \mathrm{LSC}(\Omega)$, since it is the pointwise supremum of continuous functions. By assumption we also have $u \in \mathrm{USC}(\Omega)$. Therefore $u \in C(\Omega)$.
We now assume $u$ is not a subsolution of (1.1). Then there must be some point $x_0$ and a function $\phi \in C^2(\Omega)$ such that $u - \phi$ attains a local maximum in $x_0$ with $u(x_0) = \phi(x_0)$ but $F_*(\nabla\phi(x_0), \mathrm{D}^2\phi(x_0)) > f(x_0)$. We then define

$$\phi_\delta(x) := \phi(x) + \tfrac{\delta}{2\,c_{\max}}\left|x - x_0\right|^2 .$$

Then $u - \phi_\delta$ attains a strict local maximum in $x_0$ and by (3.3)

$$F_*(\nabla\phi_\delta(x_0), \mathrm{D}^2\phi_\delta(x_0)) \geq F_*(\nabla\phi(x_0), \mathrm{D}^2\phi(x_0)) - \delta > f(x_0)$$

for $\delta > 0$ sufficiently small. By semicontinuity of the left-hand side and continuity of $f$, there is a ball $B_R(x_0)$ such that

$$F_*(\nabla\phi_\delta(x), \mathrm{D}^2\phi_\delta(x)) > f(x) \tag{4.1}$$

for all $x \in B_R(x_0)$. Then by definition, for any $\varepsilon > 0$, there is a $v_\varepsilon \in \Sigma$ such that $v_\varepsilon(x_0) > u(x_0) - \varepsilon$. Now we define $\psi_{\delta,\varepsilon}(x) := \phi_\delta(x) - \varepsilon$, so that $\psi_{\delta,\varepsilon}(x_0) = u(x_0) - \varepsilon < v_\varepsilon(x_0)$ and

$$\begin{aligned}
\psi_{\delta,\varepsilon}(x) &= \phi(x) + \tfrac{\delta}{2c_{\max}}\left|x - x_0\right|^2 - \varepsilon \\
&\geq u(x) + \tfrac{\delta}{2c_{\max}} r^2 - \varepsilon \\
&\geq u(x) \\
&\geq v_\varepsilon(x)
\end{aligned}$$

on $\partial B_r(x_0)$ for some $r < R$ and $\varepsilon > 0$ sufficiently small. Such an $r$ exists because $u - \phi_\delta$ attains a strict local maximum in $x_0$. We conclude that $\psi_{\delta,\varepsilon}(x_0) < v_\varepsilon(x_0)$ while $\psi_{\delta,\varepsilon} \geq v_\varepsilon$ on $\partial B_r(x_0)$, so there is some $\hat{x} \in B_r(x_0)$ such that $v_\varepsilon - \psi_{\delta,\varepsilon}$ attains a local maximum in $\hat{x}$. Then, since $v_\varepsilon$ is a subsolution, this implies

$$F_*(\nabla\psi_{\delta,\varepsilon}(\hat{x}), \mathrm{D}^2\psi_{\delta,\varepsilon}(\hat{x}) = F_*(\nabla\phi_\delta(\hat{x}), \mathrm{D}^2\phi_\delta(\hat{x})) \leq f(\hat{x}),$$

by construction of $\psi_{\delta,\varepsilon}$, contradicting (4.1). □

So the supremum of continuous subsolutions of (1.1) is again a continuous subsolution of the same problem if it is upper semicontinuous, which is by far not obvious and the first difficulty that arises. The second crucial part is to show that the obtained function is also a supersolution. Finally we have to make sure, that the obtained function agrees with the prescribed boundary condition. For the latter we have to restrict the set of considered subsolutions of (1.1) to a subset of those which are also subsolutions of the Dirichlet boundary value problem (1.2). The following theorem solves these problems. It is a generalization of the existence result [32, Theorem 4.1] for the normalized infinity Laplacian. Further difficulties arise in the proof whenever an explicit function is needed, since we do not demand the operator $F$ to have a specific shape which would allow us to explicitly compute $F(\nabla u, \mathrm{D}^2 u)$.

**Theorem 4.4.** *If $f > 0$, $g \in C(\partial\Omega)$, and* (3.1), (3.2), (3.11), (3.12) *and* (3.19) *hold, then there exists a viscosity solution $u \in C(\overline{\Omega})$ of* (1.2).

*Proof.* First we define a subset of all continuous subsolutions

$$\mathcal{A}_{f,g} := \{v \in C(\overline{\Omega}) \mid F_*(\nabla v, \mathrm{D}^2 v) \leq f \text{ in } \Omega \text{ and } v \leq g \text{ on } \partial\Omega\}.$$

We note that $A_{f,g}$ is nonempty since it contains all constant functions smaller than the infimum of $g$ on $\partial\Omega$. So we may define a candidate for a solution by taking the pointwise supremum

$$u(x) := \sup_{v \in \mathcal{A}_{f,g}} v(x)$$

for each $x \in \overline{\Omega}$.
We also see that $u = g$ on $\partial\Omega$. Indeed, for each $x_0 \in \partial\Omega$ and each $\varepsilon > 0$ there

is some ball $B_\delta(x_0)$ such that $|g(x)-g(x_0)|<\varepsilon$ for each $x\in B_\delta(x_0)\cap\partial\Omega$. Then, by the derivation of Lemma 3.12, the function

$$v_\varepsilon(x):=g(x_0)-\varepsilon+b\,\tfrac{1}{\gamma}\;|x-x_0|^\gamma-\tfrac{1}{2\,c_F}\,(\inf_\Omega f)\,|x-x_0|^2$$

satisfies $\nabla v_\varepsilon(x)\neq 0$ and so $F_*(\nabla v_\varepsilon,\mathrm{D}^2 v_\varepsilon)=F(\nabla v_\varepsilon,\mathrm{D}^2 v_\varepsilon)=\inf_\Omega f$ for all $x\in\Omega$ and $b\in\mathbb{R}$. When instead taking $x\in\partial\Omega$ and

$$b<-2\,\gamma\,\tfrac{\max_{z\in\partial\Omega}|g(z)|}{r^\gamma}\leq 0,$$

we see that for $|x-x_0|<r$ we have $v_\varepsilon(x)\leq g(x_0)-\varepsilon\leq g(x)$ and for $|x-x_0|\geq r$ we have $v_\varepsilon(x)\leq g(x_0)-\varepsilon+b\,\tfrac{1}{\gamma}\,r^\gamma\leq-\max_{z\in\partial\Omega}|g(z)|\leq g(x)$. So $v_\varepsilon\in\mathcal{A}_{f,g}$ and

$$g(x_0)-\varepsilon\leq v_\varepsilon(x_0)\leq u(x_0)\leq g(x_0)$$

for every $\varepsilon>0$. We conclude $u=g$ on $\partial\Omega$.

The remaining proof is done in three steps. We show that

1. $u$ is a supersolution of (1.2),
2. $u$ is continuous,
3. $u$ is a subsolution of (1.2).

We start with the first step:
For this we first note that $u$ is lower semicontinuous, as it is the pointwise supremum of continuous functions. Let us now assume $u$ is not a supersolution of (1.1). Then there must be a point $\hat{x}\in\Omega$ and a function $\phi\in C^2(\Omega)$ so that $u-\phi$ attains a local minimum in $\hat{x}$, but

$$F^*\left(\nabla\phi(\hat{x}),\mathrm{D}^2\phi(\hat{x})\right)<f(\hat{x}).$$

Defining the function

$$\phi_\delta(x):=\phi(x)-\tfrac{\delta}{c_{\min}}\;|x-\hat{x}|^2,$$

we see

$$F^*\left(\nabla\phi_\delta(\hat{x}),\mathrm{D}^2\phi_\delta(\hat{x})\right)\leq F^*\left(\nabla\phi(\hat{x}),\mathrm{D}^2\phi(\hat{x})\right)+\delta<f(\hat{x})$$

for a sufficiently small $\delta > 0$ while $u - \phi_\delta$ attains a strict local minimum in $\hat{x}$. By upper semicontinuity of the left-hand side and continuity of $f$, there must be some ball $B_r(\hat{x})$ such that

$$F^* \left(\nabla \phi_\delta(x), \mathrm{D}^2 \phi_\delta(x)\right) < f(x) \tag{4.2}$$

for all $x \in B_r(\hat{x})$.
For all $\varepsilon > 0$ there is some $v_\varepsilon \in \mathcal{A}_{f,g}$ such that $u(\hat{x}) - \varepsilon < v_\varepsilon(\hat{x}) < u(\hat{x}) + \varepsilon$. We define

$$\hat{\phi}(x) := \phi_\delta(x) + \varepsilon$$

Then we have $\hat{\phi} > v_\varepsilon$ in some domain $U_\varepsilon \subset B_r(\hat{x})$ containing $\hat{x}$ and $\hat{\phi} \leq v_\varepsilon$ outside of this domain, for $\varepsilon$ sufficiently small. We fix such a $\varepsilon > 0$.
Now we define $\hat{v} := \max\{\hat{\phi}, v_\varepsilon\} \in C(\Omega)$. Clearly we have $\hat{v} \leq g$ on $\partial\Omega$, since $\hat{v} = v_\varepsilon$ outside of $U_\varepsilon \subset \Omega$ and $v_\varepsilon \in \mathcal{A}_{f,g}$. Inside of $U_\varepsilon$ we have $\hat{v} = \hat{\phi}$, which is by (4.2) a subsolution in $U_\varepsilon$. Outside of $U_\varepsilon$ we have $\hat{v} = v_\varepsilon$, which is a subsolution by assumption.
So $\hat{v}$ satisfies $F_*(\nabla v, \mathrm{D}^2 v) \leq f$ in $\Omega$. We may conclude $\hat{v} \in \mathcal{A}_{f,g}$. On the other hand we have $\hat{v}(\hat{x}) = \hat{\phi}(\hat{x}) > \phi_\delta(\hat{x}) = u(\hat{x})$, contradicting the construction of $u$. So, together with $u = g$ on $\partial\Omega$, we conclude that $u$ is a supersolution of (1.2).

In the second step we will show continuity:
With Theorem 3.20 for supersolutions we already know that $u \in C^\gamma_{\mathrm{loc}}(\Omega)$. Therefore we only need to show

$$\lim_{x \in \Omega \to z} u(x) = g(z).$$

We denote the set of continuous supersolutions by

$$\mathcal{B}_{f,g} := \{v \in C(\overline{\Omega}) \mid F^*(\nabla v, \mathrm{D}^2 v) \geq f \text{ in } \Omega \text{ and } v \geq g \text{ on } \partial\Omega\}.$$

For an arbitrary $z \in \Omega$ we define the function

$$\psi(x) := A + \tfrac{B}{2\, c_{\max}} \left|x - z\right|^2$$

to see with (3.2) that $F^*(\nabla\psi(x), \mathrm{D}^2\psi(x)) \geq -B$. So we take $B \leq -\sup_\Omega f$ and $A$ so large that $\psi \geq g$ on $\partial\Omega$. Therefore $\mathcal{B}_{f,g}$ is not empty and we may define

$$v := \inf_{w \in \mathcal{B}_{f,g}} w(x)$$

for every $x \in \overline{\Omega}$. By construction we have $v \in \mathrm{USC}(\overline{\Omega})$ and clearly $v \leq g$.

We fix a $z \in \partial\Omega$. Then for all $\varepsilon > 0$ there is a $r > 0$ such that

$$x \in \partial\Omega \cap B_r(z) \implies |g(x) - g(z)| < \varepsilon, \tag{4.3}$$

by continuity of $g$.
Since $\Omega$ is bounded, there is some $R$ with

$$R > \sup_{x\in\Omega} |x - z| .$$

Then we may take a large number $b$ such that the mapping

$$s \mapsto b\tfrac{1}{\gamma}s^{\gamma} - \tfrac{1}{2c_F} \sup_{\Omega} f s^2 \tag{4.4}$$

is monotone increasing on $[0, R)$ and

$$b\tfrac{1}{\gamma}r^{\gamma} - \tfrac{1}{2c_F} \sup_{\Omega} f r^2 \geq 2 \sup_{\partial\Omega} |g| . \tag{4.5}$$

Then we define the function

$$w(x) := g(z) + \varepsilon + b\,\tfrac{1}{\gamma}\;|x - z|^{\gamma} - \tfrac{1}{2\,c_F}(\sup_{\Omega} f)\,|x - z|^2 .$$

Again, by the derivation of Lemma 3.12, we know $\nabla w(x) \neq 0$ and

$$F^*(\nabla w(x), \mathrm{D}^2 w(x)) = F(\nabla w(x), \mathrm{D}^2 w(x)) = \sup_{\Omega} f.$$

Moreover, by (4.4) and (4.3), we have

$$w(x) \geq g(z) + \varepsilon \geq g(x)$$

for $x \in \partial\Omega \cap B_r(z)$. On the other hand, by (4.4) and (4.5), we have

$$w(x) \geq g(z) + \varepsilon + b\,\tfrac{1}{\gamma}\,r^{\gamma} - \tfrac{1}{2\,c_F}(\sup_{\Omega} f)\,r^2 \geq g(z) + 2 \sup_{\partial\Omega} |g| \geq g(x)$$

for $x \in \partial\Omega \setminus B_r(z)$. So $w \in \mathcal{B}_{f,g}$ and by construction

$$v(z) \leq w(z) = g(z) + \varepsilon$$

for all $\varepsilon > 0$, so $v(z) \leq g(z)$.
Also $v$ must be upper semicontinuous in $\overline{\Omega}$, so we have

$$g(z) \geq v(z) \geq \limsup_{x \in \Omega \to z} v(x).$$

Using the comparison principle [33, Theorem 2.4] we know $w_1 \leq w_2$ for all $w_1 \in \mathcal{A}_{f,g}$ and $w_2 \in \mathcal{B}_{f,g}$ in $\overline{\Omega}$, so $u(x) \leq v(x)$ pointwise for all $x \in \Omega$. This implies

$$\limsup_{x \in \Omega \to z} u(x) \leq \limsup_{x \in \Omega \to z} v(x) \leq g(z).$$

On the other hand, by lower semicontinuity of $u$ in $\overline{\Omega}$, we have

$$g(z) = u(z) \leq \liminf_{x \in \Omega \to z} u(x). \tag{4.6}$$

Therefore $\lim_{x \in \Omega \to z} u(x) = g(z)$ for all $z \in \partial\Omega$, so $u \in C(\overline{\Omega})$.

We turn to the third and last step:
Knowing from the second step that $u$ is continuous, we may invoke Lemma 4.3 to see that $u$ is a subsolution of (1.1). Together with $u = g$ on $\partial\Omega$ we conclude that $u$ is subsolution of (1.2). □

If the right-hand side is $f < 0$, we may consider problem (1.2) with right-hand side $\tilde{f} := -f$ and boundary data $\tilde{g} := -g$ to obtain a continuous viscosity solution $\tilde{u}$. Then, by the structure of $F$, the continuous function $u := -\tilde{u}$ is a viscosity solution of problem (1.2). We conclude the following corollary.

**Corollary 4.5.** *If $f > 0$ or $f < 0$, $g \in C(\partial\Omega)$, and* (3.1), (3.2), (3.11), (3.12) *and* (3.19) *hold, then there exists a viscosity solution $u \in C(\overline{\Omega})$ of* (1.2). □

## 4.3 Power concavity

In this section we investigate a very special type of problem, that is showing so called *power concavity* of viscosity solutions of

$$\begin{cases} F(\nabla u, \mathrm{D}^2 u) = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.7}$$

in a convex and bounded domain $\Omega \subset \mathbb{R}^n$ that the interior sphere condition (2.1). To be more precise, we want to show that if $u$ is a positive solution to (4.7) then $u^{\frac{1}{2}}$ is a concave or equivalently $-u^{\frac{1}{2}}$ is a convex function.

Therefore we may drop the assumption (3.19) but have to assume that the boundary condition in (4.7) holds exactly and not only in the sense of viscosity solutions. We also make another assumption on the structure of $F$. We assume that for every $q \in \mathbb{R}^n$ the mapping $A \mapsto F^*(q, A^{-1})$ is concave in $\mathcal{S}^{++}$. That means

$$F^*(q, (\mu\, A_1 + (1-\mu)\, A_2)^{-1}) \geq \mu\, F^*(q, A_1^{-1}) + (1-\mu)\, F^*(q, A_2^{-1}) \tag{4.8}$$

for all $q \in \mathbb{R}^n$, $A_1, A_2 \in \mathcal{S}^{++}$ and all $\mu \in [0,1]$.

The route we take in this section was prescribed in [11]. We show that the boundary value problem $w = -u^{\frac{1}{2}}$ solves

$$\begin{cases} F(\nabla w, \mathrm{D}^2 w) = \frac{1}{w}\left(c_Q \left|\nabla w\right|^2 + \frac{1}{2}\right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.9}$$

whenever $u$ is a positive viscosity solution of (4.7). In the second step we show that the convex envelope $w_{**}$ of $-u^{\frac{1}{2}}$, that is the largest convex function below $-u^{\frac{1}{2}}$, is a viscosity supersolution of the derived equation. In the third and last step we apply the comparison principle [33, Theorem 2.4] to see that subsolutions are smaller than supersolutions and conclude $w_{**} \leq -u^{\frac{1}{2}} \leq w_{**}$, so the convex envelope is actually the function itself.

First of all we want to make sure that $u$ is positive, so $u^{\frac{1}{2}}$ is welldefined. It is easy to prove that our assumptions are sufficient to guarantee this.

**Proposition 4.6.** *Let $u \in \mathrm{LSC}(\Omega)$ be supersolution of* (4.7). *Then we have $u > 0$ in $\Omega$.*

*Proof.* We assume there is a point $x_0 \in \Omega$ such that $u(x_0) \leq 0$. Since $u$ is lower semicontinuous and $u = 0$ on $\partial\Omega$, we may assume that $u$ attains a local minimum in $x_0$. Then the constant function $\phi(x) := u(x_0)$ satisfies $\phi \in C^2(\Omega)$ and $u - \phi$ attains a local minimum in $x_0$. Therefore we have

$$F^*(\nabla\phi(x_0), \mathrm{D}^2\phi(x_0)) = F^*(0,0) \leq -c_{\min}\lambda_{\min}(0) = 0 < 1.$$

Then $u$ cannot be a supersolution of (4.7). $\square$

Knowing that $u^{\frac{1}{2}}$ is welldefined and nonzero in $\Omega$, we can now show that if $u$ is a viscosity subsolution of (1.2) then $-u^{\frac{1}{2}}$ is indeed a viscosity supersolution of (4.9). More precisely, we formulate the following proposition. Its proof is a straightforward computation.

**Proposition 4.7.** *A function $u \in \mathrm{USC}(\Omega)$ is a positive viscosity subsolution of* (4.7) *if and only if $v := -u^{\frac{1}{2}} \in \mathrm{LSC}(\Omega)$ is a negative viscosity supersolution of* (4.9).

*Proof.* We will only show one implication, as the other only involves similar calculations. So let $u \in \mathrm{USC}(\Omega)$ be a positive viscosity subsolution of (4.7). Clearly $v := -u^{\frac{1}{2}} \in \mathrm{LSC}(\Omega)$ is negative. Let $\phi \in C^2(\Omega)$ and $x_0$ be such that $v - \phi$ attains a local minimum in $x_0$ with $(v-\phi)(x_0) = 0$, so $\phi(x_0) < 0$. Then $u - \phi^2$ attains a local maximum in $x_0$. By computing

$$\nabla(\phi^2)(x_0) = 2\,\phi(x_0)\nabla\phi(x_0)$$

and

$$\mathrm{D}^2(\phi^2)(x_0) = 2\,\nabla\phi(x_0) \otimes \nabla\phi(x_0) + 2\,\phi(x_0)\mathrm{D}^2\phi(x_0)$$

we see that $\nabla(\phi^2)(x_0) = 0$ if and only if $\nabla\phi(x_0) = 0$. Then in this case we have in $x_0$

$$1 \geq F_*(\nabla(\phi^2), \mathrm{D}^2(\phi^2)) = F_*(0, 2\,\phi\mathrm{D}^2\phi) = 2\,\phi\, F^*(0, \mathrm{D}^2\phi),$$

and so

$$F^*(0, \mathrm{D}^2\phi) \geq \tfrac{1}{2\phi}$$

as desired. In the case of $\nabla(\phi^2)(x_0) \neq 0$ we have in $x_0$

$$\begin{aligned}
1 &\geq F_*(\nabla(\phi^2), \mathrm{D}^2(\phi^2)) \\
&= F(\nabla(\phi^2), \mathrm{D}^2(\phi^2)) \\
&= F(2\,\phi\nabla\phi, 2\,\nabla\phi\otimes\nabla\phi + 2\,\phi\mathrm{D}^2\phi) \\
&= 2\,F(\nabla\phi, \nabla\phi\otimes\nabla\phi) + 2\,\phi\,F(\nabla\phi, \mathrm{D}^2\phi) \\
&= 2\,(-c_Q\,|\nabla\phi|^2 + \phi\,F(\nabla\phi, \mathrm{D}^2\phi)
\end{aligned}$$

and finally

$$F(\nabla\phi, \mathrm{D}^2\phi) \leq \tfrac{1}{\phi}(c_Q\,|\nabla\phi|^2 + \tfrac{1}{2}).$$

$\square$

By Carathéodory's theorem on convex hulls, which can be found in [12], each point of the convex set $\Omega$ can be decomposed into convex combinations of at most $n+1$ other points of $\overline{\Omega}$. By then taking the convex combination of the corresponding function values, we obtain a convex function. Taking the supremum over all of these decompositions we obtain the convex envelope, the largest convex function below the function of consideration.

**Definition 4.8.** For a function $w : \overline{\Omega} \to \mathbb{R}$, we call the function

$$w_{**}(x) = \inf\left\{ \sum_{i=1}^{k} \mu_i w(x_i) \;\middle|\; x = \sum_{i=1}^{k} \mu_i x_i,\; x_i \in \overline{\Omega},\ \mu_i > 0, \sum_{i=1}^{k} \mu_i = 1,\ k \leq n+1 \right\}$$

the *convex envelope* of $w$. $\square$

The following lemma shows that, for viscosity solutions of (4.7), the defining points of their convex envelope never lie on the boundary of $\Omega$. This is a very useful property, since the boundary data itself does not contain any information about the behavior of solutions inside the domain. The proof we give requires the interior sphere condition (2.1). However, this is the only part where we actually use this condition, raising the question whether the main result of this section can be improved at this point.

**Lemma 4.9.** *Let $u \in C(\Omega)$ be a viscosity solution to* (4.7) *and $w = -u^{\frac{1}{2}}$. For every $x \in \Omega$, let $x_1, \ldots, x_k \in \overline{\Omega}$, $\mu_1, \ldots, \mu_k > 0$ with $\sum_{i=1}^k \mu_i = 1$ be such that*

$$x = \sum_{i=1}^{k} \mu_i x_i \qquad \text{and} \qquad w_{**}(x) = \sum_{i=1}^{k} \mu_i w(x_i).$$

*Then $x_1, \ldots, x_k \in \Omega$.*

*Proof.* Let $x_0 \in \partial\Omega$ be any boundary point and $\mu \in \partial B_1(0)$ be any direction with $\langle \mu, \nu(x_0)\rangle > 0$. Using the definition of $w < 0$ it is easy to see that

$$\frac{w(x_0) - w(x_0 - t\mu)}{t} \geq \frac{1}{2} \frac{1}{w(x_0 - t\mu)} \frac{u(x_0) - u(x_0 - t\mu)}{t} \tag{4.10}$$

for all $t > 0$. Using $u = 0$ on $\partial\Omega$ and $u > 0$ in $\Omega$, we may invoke Hopf's Lemma (3.14) for supersolutions and positive right-hand sides to see

$$\limsup_{t \downarrow 0} \frac{u(x_0) - u(x_0 - t\mu)}{t} < 0.$$

This together with $w(x_0 - t\mu) \uparrow 0$ for $t \downarrow 0$ and (4.10) implies

$$\frac{w(x_0) - w(x_0 - t\mu)}{t} \to \infty$$

for $t \downarrow 0$. Therefore $w$ must be convex close to the boundary and the convex envelope of $w$ can not be spanned by any boundary points. □

**Remark 4.10.** *Another proof of Lemma* (4.9) *was essentially provided for the infinity Laplacian in [11] and can be adapted using the methods derived in Chapter 3 and assuming* (3.19). □

In order to show that the convex envelope of a viscosity supersolution of (4.9) is again a viscosity supersolution of the same problem, we first need the following technical lemma. The necessity of this Lemma in turn motivates the structural assumption (4.8) made at the beginning of this section.

**Lemma 4.11.** *Let $q \in \mathbb{R}^n$, $A_1, A_2 \in \mathcal{S}^{++}$, $\mu \in [0,1]$. Then*

$$\frac{1}{-F^*(q, (\mu A_1 + (1-\mu)A_2)^{-1})} \geq \mu \frac{1}{-F^*(q, A_1^{-1})} + (1-\mu) \frac{1}{-F^*(q, A_2^{-1})},$$

*so the mapping $A \mapsto \frac{1}{-F^*(q, A^{-1})}$ is concave in $\mathcal{S}^{++}$.*

*Proof.* First let $B_1, B_2 \in \mathcal{S}^{++}$ and $\nu \in [0,1]$. Then by (4.8)

$$F^*(q, (\nu B_1 + (1-\nu)B_2)^{-1}) \geq \nu\, F^*(q, B_1^{-1}) + (1-\nu)\, F^*(q, B_2^{-1})$$

or equivalently

$$\frac{1}{-F^*(q, (\nu B_1 + (1-\nu)B_2)^{-1})} \geq \frac{1}{-\nu\, F^*(q, B_1^{-1}) - (1-\nu)\, F^*(q, B_2^{-1})}$$

for all $q \in \mathbb{R}^n$. Now let $A_1, A_2 \in \mathcal{S}^{++}$ and $\mu \in [0,1]$. To keep the readability we define

$$c_1 := -F^*(q, A_1^{-1}) \qquad \text{and} \qquad c_2 := -F^*(q, A_2^{-1}).$$

We note that $c_1, c_2 > 0$ by (3.2). Then we take

$$\nu := \frac{\mu\, c_2}{\mu\, c_2 + (1-\mu)\, c_1}, \qquad B_1 := \frac{A_1}{c_2}, \qquad B_2 := \frac{A_2}{c_1}$$

and see

$$\begin{aligned}
\frac{1}{-F^*(q, (\mu\, A_1 + (1-\mu\, A_2))^{-1})} &= \frac{\mu\, c_2 + (1-\mu)\, c_1}{-F^*(q, (\nu\, B_1 + (1-\nu)\, B_2)^{-1})} \\
&\geq \frac{\mu\, c_2 + (1-\mu)\, c_1}{-\nu\, F^*(q, B_1^{-1}) - (1-\nu)\, F^*(q, B_2^{-1})} \\
&= \frac{\mu\, c_2 + (1-\mu)\, c_1}{\nu\, c_1\, c_2 + (1-\nu)\, c_1\, c_2} \\
&= \frac{\mu\, c_2 + (1-\mu)\, c_1}{c_1\, c_2} \\
&= \mu \frac{1}{c_1} + (1-\mu) \frac{1}{c_2},
\end{aligned}$$

using the positive homogeneity of $F^*$ in the second argument, to obtain the desired. $\square$

Now we are able to prove the claimed property of the convex envelope of viscosity supersolutions to (4.9).

**Lemma 4.12.** *Let $w$ be a viscosity supersolution to* (4.9). *Then $w_{**}$ is a viscosity supersolution to* (4.9).

*Proof.* First we note that $w_{**} = w = 0$ on $\partial\Omega$, according to [1, Lemma 4]. So we only need to check that $w_{**}$ satisfies the definition of a viscosity supersolution 2.4. For this we need to check that

$$F^*(q,X) - \tfrac{1}{w_{**}(x)}\left(c_Q |q|^2 + \tfrac{1}{2}\right) \geq 0$$

holds for all $x \in \Omega$ and all $(q,X) \in \mathcal{J}^{2,-}w_{**}(x)$. So let $x \in \Omega$ and $(q,X) \in \mathcal{J}^{2,-}w_{**}(x)$. Furthermore we can reduce to the case of positive semidefinite $X$, according to [1, Lemma 3]. By Proposition 4.9 we can decompose $x$ in a convex combination of points $x_1, \dots, x_k \in \Omega$ such that

$$\sum_{i=1}^{k} \mu_i x_i = x \qquad \text{and} \qquad \sum_{i=1}^{k} \mu_i w(x_i) = w_{**}(x)$$

for some $\mu_1, \dots, \mu_k > 0$ with $\sum_{i=1}^{k} \mu_i = 1$. Then by [1, Proposition 1] for every $\varepsilon > 0$ small enough, there are positive semidefinite matrices with $X_1, \dots X_k \in \mathcal{S}$ such that $(q, X_i) \in \overline{\mathcal{J}}^{2,-} w(x_i)$ and

$$X - \varepsilon X^2 \leq \left(\sum_{i=1}^{k} \mu_i X_i^{-1}\right)^{-1} =: Y.$$

We may also assume that the matrices $X_1, \dots, X_k$ are positive definite, since the degenerate case can be handled as in [1, p. 273].
Using that by assumption $w$ is a supersolution of (4.9), we find

$$\tfrac{1}{w(x_i)}\left(c_Q |q|^2 + \tfrac{1}{2}\right) \leq F^*(q, X_i).$$

Since $X$ is positive definite we have $F^*(q,X) \leq -c_{\min}\lambda_{\min}(X) < 0$, which allows us together with $w < 0$ to rearrange the preceding inequality to

$$-w(x_i) \leq -\frac{1}{F^*(q,X_i)}\left(c_Q |q|^2 + \frac{1}{2}\right).$$

By summation and rearranging the terms again, we obtain

$$-\frac{1}{\sum_{i=1}^{k} \mu_i w(x_i)}\left(c_Q |q|^2 + \frac{1}{2}\right) \geq \left(\sum_{i=1}^{k} \mu_i \frac{1}{-F^*(q,X_i)}\right)^{-1}.$$

Using that $F^*$ is degenerate elliptic we find by plugging in $X - \varepsilon X^2 \le Y$

$$\begin{aligned}
&F^*(q, X - \varepsilon X^2) - \tfrac{1}{w_{**}(x)} \left(c_Q \left|q\right|^2 + \tfrac{1}{2}\right) \\
&\quad \ge F^*(q, Y) - \tfrac{1}{w_{**}(x)} \left(c_Q \left|q\right|^2 + \tfrac{1}{2}\right) \\
&\quad = F^*(q, Y) - \tfrac{1}{\sum_{i=1}^k \mu_i w(x_i)} \left(c_Q \left|q\right|^2 + \tfrac{1}{2}\right) \\
&\quad \ge F^*(q, Y) + \left( \sum_{i=1}^k \mu_i \, \frac{1}{-F^*(q, X_i)} \right)^{-1} \\
&\quad \ge 0,
\end{aligned}$$

by Lemma 4.11. Sending $\varepsilon \downarrow 0$ concludes the assertion. $\square$

The following theorem is the main result utilizing all the previous results of this section and the comparison principle provided in [33, Theorem 2.4].

**Theorem 4.13.** *Let $u \in C(\Omega)$ be a viscosity solution of* (4.7) *in convex $\Omega$ and assume that* (3.1), (3.2), (3.11), (3.12) *and* (4.8) *hold. Then $u^{\frac{1}{2}}$ is concave.*

*Proof.* By definition $u$ is both, a sub- and a supersolution of (4.7). Being a supersolution we know by Proposition 4.6 that $u$ is positive in $\Omega$. So $u$ is a positive subsolution of (4.7). Then, by Proposition 4.7, we find that $w := -u^{\frac{1}{2}}$ is a negative supersolution of (4.9). Using Lemma 4.12, we obtain that $w_{**} \le w$ is also a negative supersolution of (4.9). Then, again by Proposition 4.7, $(w_{**})^2$ is a positive subsolution of (4.7). Invoking the comparison principle of [33, Theorem 2.4] we find $(w_{**})^2 \le w^2$. On the other hand we have $w_{**} \le w \le 0$, so $(w_{**})^2 \ge w^2$ and finally $w_{**}^2 = w^2$. We may conclude that $w_{**} = w$, making $w$ a convex and $u^{\frac{1}{2}}$ a concave function. $\square$

We apply our main result to the two examples of the preceding chapters. For this we only have to check if (4.8) holds.

**Example 4.14.** As in Example 3.21, let $\alpha, \beta \in \mathbb{R}$ with $\beta \ge 0$, $\alpha + \beta > 0$ and

$$F(q, X) = -\alpha \left|q\right|^{-2} \langle q, Xq \rangle - \beta \operatorname{tr} X.$$

We want to check that (4.8) is satisfied for this operator.
In [1, p.286] it is shown that the mapping $(q, A) \mapsto \langle q, A^{-1} q \rangle$ is convex in $\mathbb{R}^n \times \mathcal{S}^{++}$. By that we mean

$$\begin{aligned}
&\left\langle \mu\, q_1 + (1-\mu)\, q_2, (\mu\, A_1 + (1-\mu)\, A_2)^{-1} (q_1 + (1-\mu)\, q_2) \right\rangle \\
&\qquad \ge \mu \left\langle q_1, A_1^{-1}, q_1 \right\rangle + (1-\mu) \left\langle q_2, A_2^{-1} q_2 \right\rangle
\end{aligned} \tag{4.11}$$

for all $q_1, q_2 \in \mathbb{R}^n \setminus \{0\}$, $A_1, A_2 \in \mathcal{S}^{++}$ and all $\mu \in [0,1]$.
In the case of $q \neq 0$ we may take $q_1 = q_2 = q$ in (4.11) to see that the mapping $A \mapsto F(q, A^{-1})$ is concave in $\mathcal{S}^{++}$ for $\alpha \geq 0$.
In the case of $q = 0$ we take $q_1, q_2 \in \mathbb{R}^n \setminus \{0\}$ to maximize the right-hand side of (4.11) and afterwards maximize the left-hand side to obtain that $A \mapsto F^*(0, A^{-1})$ is also concave in $\mathcal{S}^{++}$ for $\alpha \geq 0$.
We conclude that (4.8) holds, provided $\alpha \geq 0$, and that we obtain power concavity in this case. □

**Example 4.15.** For the normalized $p$-Laplacian

$$F(q, X) = -\tfrac{p-2}{p} |q|^{-2} \langle q, Xq \rangle - \tfrac{1}{p} \operatorname{tr} X,$$

we use the preceding example and have to require $p \geq 2$, so that (4.8) holds, to apply our results and to obtain power concavity. The limiting case $p \to \infty$ was discussed in [11]. □

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Michael Kühn