Free probability approach to microscopic statistics of random matrix eigenvalues

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Abstract

We consider general ensembles of $N \times N$ random matrices in the limit of large matrix size ($N \to \infty$). Our goal is to establish a new approach for studying local eigenvalue statistics, allowing one to push boundaries of known universality classes without strong assumptions on the probability measure of matrix ensembles in question. The problem of computing many-point correlation functions is approached by means of a supersymmetric generalization of Laplace transform. The large N limit of said transform for partition functions is in many cases governed by the R-transform known from free probability theory.

We prove the existence and uniqueness of supersymmetric Laplace transform and its inverse in interesting cases of ratios of products of determinants. Our starting point is the appropriately regularized Fourier transform over the space of Hermitian matrices. A detailed derivation is given in the case of unitary symmetry, while formulas for real symmetric and quaternion self-dual matrices follow from the analytic structure of Harish-Chandra-Itzyskon-Zuber integral over orthogonal and symplectic groups respectively.

The region of applicability of our method is derived in a simple form without putting any assumptions on the form of the probability density, therefore developed formalism covers both standard cases of Wigner and invariant random matrix ensembles. We derive $N \to \infty$ applicability conditions in a way that allows us to control the order of the error term for large N.

Qualitative analysis of the region of invertibility of Green's function is performed in the case of eigenvalue densities supported on a finite number of intervals and further refined by considering invariant random matrix ensembles. We provide conditions for the appearance of singularities during a continuous deformation of matrix models in question.

Kurzzussamenfassung

Wir betrachten allgemeine Ensembles von $N \times N$ Zufallsmatrizen im Limes von großer Matrixgröße ($N \to \infty$). Unser Ziel ist es eine neue Vorgehensweise zu etablieren, um lokalen Eigenwertstatistiken zu analysieren und dadurch die Grenzen der bekannten Universalitätsklassen, ohne strenge Annahmen über die Wahrscheinlichkeitsmaße von Matrix- Ensemble, zu erweitern. Das Problem der Berechnung von n-Punktkorrelationsfunktion wird mittels der supersymmetrischen Verallgemeinerung der Laplace-Transformation gelöst. Der Limes von großem N der Transformation von Partitionsfunktionen wird in vielen Fällen durch die aus der freien Wahrscheinlichkeitstheorie bekannte R-Transformation bestimmt.

Wir beweisen die Existenz und Eindeutigkeit von supersymmetrischen Laplace-Transformation und ihre Inverse am Beispiel von interessanten Fällen, in denen ein Verhältnis von Determinantenprodukten erwägt wird. Unser Ausgangspunkt ist die in bestimmter Weise regularisierte Fourier Transformation im Raum Hermitescher Matrizen. Eine detaillierte Herleitung wird am Beispiel einer unitären Symmetrie gegeben, während Formeln für reelle symmetrische und quaternione selbstduale Matrizen jeweils aus der analytischen Struktur von Harish-Chandra-Itzyskon- Zuber Integral über orthogonal und symplektische Gruppen folgen.

Die Reichweite von Anwendbarkeit unserer Methode wird durch eine einfache Formel gezeigt, ohne auf irgendwelche Annahmen über die Form der Wahrscheinlichkeitsdichte zu beruhen. Daraus wurde Formalismus entwickelt, der sowohl Standardfälle von Wigner als auch invariante Zufallsmatrix-Ensembles umfasst. Wir entwickeln $N \to \infty$ Anwendbarkeitsbedingungen in einer Weise, die uns ermöglicht die Größenordnung von Fehlerterm für großen N zu kontrollieren.

Contents

1 Introduction

1.1 Outline

This thesis is organized in the following way. In chapter 1 we provide the motivation for our work and a short historical background of the research done in the area of random matrix theory. We explain the basic ideas of two main theories we use and combine throughout the thesis. The free probability theory is introduced in section 1.2.2 and the second one, the supersymmetry, is briefly described in section 1.3. The combination of those two formalisms is the main topic of this thesis. We close the chapter by reviewing previous results in the area of correlations between random matrix eigenvalues in section 1.4.

Chapter 2 covers the construction of the main object of interest in this thesis, the Laplace transform in space of supermatrices. We start by introducing relevant objects and motivating the need for said transform in the random matrix theory. Next, in section 2.1, we show in a heuristic way a connection between the Laplace transform of partition function for the correlation functions of random matrix eigenvalues with the supersymmetric extension of the R-transform known from the free probability theory. We proceed with the explanation of how our formalism applies in simplest cases of 1-point function in section 2.2 and continue by showing how one can extend it to many-point correlations in sections 2.3 and 2.4. We use techniques of complex analysis, i.e. contour integrations and analytic continuations to prove existence and invertibility of Laplace transform of partition functions describing correlations between random matrix eigenvalues. The derivation is based on well established Fourier transform in space of matrices. In the beginning, we restrict ourselves to the case of unitary symmetry, or in other words to the transforms over space of Hermitian matrices. Last section, 2.5, is devoted to the extension of our formalism to other symmetry classes, i.e. orthogonal and symplectic, related to real symmetric and quaternion self-dual matrices respectively.

The goal of chapter 3 is to determine the region of applicability of our approach. We give a simple requirement that is necessary for all approximations in our derivation to be exact in the large matrix size limit (often referred to as $N \to \infty$ limit). In chapter 4 special attention is placed on the regions of invertibility of Green's function and as a consequence of the inverse function theorem, analyticity of the R-transform. We start with

general considerations, but to obtain more quantitative results, we restrict ourselves first to the case of the eigenvalue distribution supported on a few intervals in section 4.1 and later to invariant random matrix ensembles in section 4.2.

Chapter 5 summarizes the thesis and describes consequences of our results. We give an outlook on possible extensions and further developments that may be accessible thanks to our Laplace transform formula.

1.2 Random matrices

1.2.1 Overview

Matrices play many roles in mathematics, physics, data analysis, telecommunication, and other numerous topics.The first work where considered matrix was taken to have random elements, was done by Wishart [1], where the correlation coefficients of multivariate data samples were computed, though his work did not get deserved recognition at the time. The real pioneering work in the field is attributed to Wigner [2]. In nuclear physics context Wigner devised a model for Hamiltonians of heavy nuclei - too complicated to write down and compute explicitly, therefore assumed to be represented by large matrices with independent random Gaussian entries with appropriate symmetries. Model turned out to describe spacings between energy levels (eigenvalues) quite well, but what is more important, is the fact that many different nuclei displayed similar level spacings, exhibiting a property called 'level repulsion'. The statistics of eigenvalues of random Hamiltonians were far from Poisson that would be expected from uncorrelated variables, showing that even though elements of the matrix are independent, the eigenvalues become highly correlated. The universality of the result suggests additionally, that the local statistics are independent of details of the system but depend only on general properties like symmetries or band structure. Eigenvalues statistics are so far the most studied property of random matrices, but there has been some interest in other quantities like e.g. eigenvectors. For a more detailed historical introduction see [3].

One of the most common ways of constructing random matrix models is to consider matrices with, up to symmetry, independent entries. A special class of those, called Wigner matrices, is constructed by requiring all elements above diagonal to have zero mean and identical second moment and requiring elements below the diagonal to reflect matrix symmetry (e.g. invariance under transposition or hermitian conjugation). A prime example from this class is a random matrix with all elements above diagonal being independent identically distributed standardized complex Gaussian random numbers, while diagonal ones are real.

The second convenient way of description is by a probability measure on space of matrices that is invariant with respect to transformation by some symmetry group. Standard example being random measures on space of Hermitian matrices invariant w.r.t. unitary transformation. Each such measure can be written in the following form:

$$\mu(H) \propto e^{-\mathrm{Tr}V(H)} , \tag{1.1}$$

where $V(x)$ is a real-valued function (ensuring positivity of the probability measure), called a potential - often considered to be a polynomial of small degree.

Three classical random matrix models are the Gaussian Orthogonal Ensemble, Gaussian Unitary Ensemble, and Gaussian Symplectic Ensemble. They all belong to both classes of Wigner and invariant random matrices and differ only by symmetry group. One can construct them by taking elements to be independent Gaussian random variables with mean zero and appropriate variance - ensuring invariance property. In the first case the matrix is real and symmetric, in second situation it is complex and Hermitian and in the third case, one considers a self-dual quaternion matrix.

It has been shown that there is a total of 10 symmetry classes [4]. In this thesis, we will restrict ourselves to 3 mentioned before, called classical symmetry classes. In a typical way, we will start by considering the computationally simplest unitary symmetry and afterward show how one can extend the results to the orthogonal and symplectic symmetries.

1.2.2 Free probability

Many techniques were used in the study of random matrix models, including enumerative combinatorics, Fredholm determinants, diffusion processes or integrable systems just to name a few (see [5, 6] for reviews).

In this section we will focus on one of them, the theory of free probability, invented by Voiculescu [7] in the context of free group factors isomorphism problem in the theory of operator algebras. Free probability describes behavior and properties of so-called 'free' non-commutative random variables with respect to addition and multiplication of said variables. The 'freeness' property is defined in the following way: two random variables A and B are free with respect to a unital linear functional ϕ if for all $n_1, m_1, n_2, \ldots \geq 1$ we have:

$$\phi\left(\left(A^{n_1} - \phi\left(A^{n_1}\right)\mathbf{1}\right)\left(B^{m_1} - \phi\left(B^{m_1}\right)\mathbf{1}\right)\left(A^{n_2} - \phi\left(A^{n_2}\right)\mathbf{1}\right)\ldots\right) = 0\ , \tag{1.2a}$$

$$\phi\left(\left(B^{n_1} - \phi\left(B^{n_1}\right)\mathbf{1}\right)\left(A^{m_1} - \phi\left(A^{m_1}\right)\mathbf{1}\right)\left(B^{n_2} - \phi\left(B^{n_2}\right)\mathbf{1}\right)\ldots\right) = 0\ . \tag{1.2b}$$

It basically allows one to compute mixed moments from moments of individual random variables, e.g. freeness ensures that:

$$\phi\left(A^n B^m\right) = \phi\left(A^n\right)\phi\left(B^m\right). \tag{1.3}$$

Soon after it was realized [8] that large independent random matrices with uncorrelated eigenvectors are mutually free with respect to the $\lim_{N\to\infty} \frac{1}{N}\mathbb{E}\left\{\mathrm{Tr}\left(\bullet\right)\right\}$ functional.

Let us introduce main objects needed when dealing with random matrices in the free probability setting. Firstly, for a $N \times N$ Hermitian matrix H one has the empirical eigenvalue distribution:

$$\rho_H(\lambda) = \frac{1}{N}\sum_{i=1}^{N}\delta\left(\lambda - \lambda_i\right)\ , \tag{1.4}$$

where $\{\lambda_i\}$ are eigenvalues of H. Now instead of considering a single deterministic matrix, we can move on to an ensemble of random matrices defined by some probability measure $\mu_N(H)$ and define the average eigenvalue density by:

$$\rho_N(\lambda) = \mathbb{E}\left\{\frac{1}{N}\sum_{i=1}^{N}\delta(\lambda-\lambda_i)\right\} = \int \frac{1}{N}\sum_{i=1}^{N}\delta(\lambda-\lambda_i)\,d\mu_N(H)\ . \quad (1.5)$$

We are also going to assume that the limit $\lim_{N\to\infty}\rho_N(\lambda)$ converges to a probability measure $\rho(\lambda)$. The form of eq. (1.5) is not very convenient for applications, because a measure on the space of matrices expressed in terms of its eigenvalues will either be very complicated to integrate or, e.g. in the case of uncorrelated eigenvalues, not interesting. Therefore one often rewrites it using following representation of real Dirac delta:

$$\delta(\lambda) = -\frac{1}{\pi}\lim_{\epsilon\to 0^+}\mathrm{Im}\frac{1}{\lambda+i\epsilon}\ . \quad (1.6)$$

Next, we define a Green's function as the Stieltjes transform of eigenvalue distribution and we can recover said distribution by taking the imaginary part of Green's function and approaching the eigenvalue support on the real line from the complex plane:

$$g(z) = \int_{\mathbb{R}}\frac{\rho(\lambda)}{z-\lambda}d\lambda = \lim_{N\to\infty}N^{-1}\mathbb{E}\left\{\mathrm{Tr}\,(z\mathbf{1}-H)^{-1}\right\}\ , \quad (1.7)$$

$$\rho(\lambda) = -\frac{1}{\pi}\lim_{\epsilon\to 0^+}\mathrm{Im}g(\lambda+i\epsilon)\ . \quad (1.8)$$

The Green's function is analytic in the complex plane away from the eigenvalue distribution, therefore it can be expanded into a series around $z=\infty$ and presented as a moment generating function:

$$g(z) = \lim_{N\to\infty}N^{-1}\mathbb{E}\left\{\mathrm{Tr}\left(z^{-1}+z^{-1}Hz^{-1}+\ldots\right)\right\} = \sum_{k=0}^{\infty}z^{-k-1}m_k\ , \quad (1.9)$$

$$m_k = \lim_{N\to\infty}\frac{1}{N}\mathbb{E}\left\{\mathrm{Tr}H^k\right\}\ . \quad (1.10)$$

This is an analog of the moment generating function known from standard commutative probability theory. We can see that knowledge of all moments allows one to determine the average eigenvalue distribution, therefore all result of the free probability apply to average eigenvalue spectra of large random matrices.

Recall that in regular commutative probability theory one can obtain a distribution of a random variable constructed as a sum of independent random variables via sum of cumulant generating functions of the summands, therefore in the setting of free probability we want to have an analogue of cumulant generating function for average eigenvalue distribution that would be additive w.r.t. addition of random matrices. For non-commutative variables this object is defined via its relation to Green's function and called the

R-transform. Not going into details that were derived in [7], the R-transform and free cumulants κ_n are defined as follows:

$$g(z) = \frac{1}{z - R(g(z))} , \tag{1.11}$$

$$R(w) = \sum_{n=1} \kappa_n w^{n-1} , \tag{1.12}$$

with the inverse relation

$$R(w) = g^{-1}(w) - w^{-1} . \tag{1.13}$$

Having defined all necessary objects, the addition law for free random matrices A and B reads:

$$R_{A+B}(z) = R_A(z) + R_B(z) . \tag{1.14}$$

Another question that arises is: can free probability theory provide information about eigenvalue spectra of products of random matrices? Even though the product of two Hermitian matrices is not Hermitian and in principle the formalism breaks down because the eigenvalue density of a product is not supported on the real axis anymore, in some cases, one can also devise the multiplication law in terms of R-transform [9]. We can consider a product of two free Hermitian matrices A, B assuming A - positive semi-definite and instead consider an equivalent problem of computing eigenvalues of the product $A^{1/2}BA^{1/2}$. In those cases one has a closed set of equations:

$$R_{AB}(z) = R_A(w) R_B(v) , \tag{1.15a}$$

$$v = zR_A(w) , \tag{1.15b}$$

$$w = zR_B(v) . \tag{1.15c}$$

The R-transform will be of great importance for us for reasons explained later, but it's worth mentioning now that if one is interested only in the average eigenvalue distributions one doesn't need to calculate R-transforms. The recently developed theory of subordination [10, 11] allows one to efficiently linearise and compute Green's function for polynomials and rational expressions in random matrices. This method doesn't reference the R-transform, which in principle might not be well defined in parts of the complex plane, therefore has to be handled with care and might not be a convenient object to manipulate numerically. I.e. R-transform is properly defined on circular sectors around the origin of the complex plane. We focus on the analysis of the analytic structure of the R-transform in chapter 4.

1.3 Supersymmetry

1.3.1 Grassmann variables

The first step in the introduction of the supersymmetry method is to recall basic information about the Grassmann variables, denoted throughout

this section by Greek letters χ_i with $i = 1, \ldots, n$. They are elements of the Grassmann algebra and obey anticommutation relations:

$$\{\chi_i, \chi_j\} = \chi_i\chi_j + \chi_j\chi_i = 0 \text{ for any } 1 \leq i, j \leq n \ . \tag{1.16}$$

Anticommutation rules imply in particular, by taking $i = j$, that

$$\chi_i^2 = 0 \ . \tag{1.17}$$

The usage of Grassmann variables in physics was significantly expanded by the introduction of the Berezin integral [12] over anticommuting variables. This integral is formally defined by two simple rules

$$\int d\chi_i = 0 \ , \tag{1.18}$$

$$\int \chi_i d\chi_i = 1 \ , \tag{1.19}$$

sufficient to integrate arbitrary function due to eq. (1.17). Any function of a single Grassmann variable must be linear in this variable and integrals of sums are taken to be equal to sums of integrals.

For the physical application, the most important Berezin integrals are the Gaussian integrals. As a further consequence of eq. (1.17), any series expansion of an analytic function of a Grassmann variable ends with the second term. Knowing that, it is easy to check by direct computation the following identity

$$\int \exp\left(-\bar{\chi}^T A\chi\right) \prod_{i=1}^{n} d\bar{\chi}_i d\chi_i = \mathrm{Det}A \ , \tag{1.20}$$

where $\{\chi_i\}$ and $\{\bar{\chi}_i\}$ are two sets of independent Grassmann variables and χ and $\bar{\chi}$ represent vectors of χ_i and $\bar{\chi}_i$ respectively. A standard counterpart of this formula is the Gaussian integral over complex variables ψ_i:

$$\int \exp\left(-\psi^\dagger A\psi\right) \prod_{i=1}^{n} d\bar{\psi}_i d\psi_i = \pi^n \mathrm{Det}^{-1}A \ . \tag{1.21}$$

Because of this analogy, some texts refer to $\bar{\chi}_i$ as a complex conjugate of χ_i but in principle there is no need to try to add this structure because ψ_i and $\bar{\psi}_i$ are independent variables in the same sense as χ_i and $\bar{\chi}_i$. The difference amounts to a change of basis, with respect to which determinants are invariant.

1.3.2 Supervectors and supermatrices

One can extend standard linear algebra by introducing an additional anticommutative structure. An $(n|m)$ supervector Θ is defined as a vector with block structure

$$\Theta = \begin{pmatrix} \chi \\ b \end{pmatrix} \ , \tag{1.22}$$

where χ is an n component vector of Grassmann variables χ_i and b is an m component vector of complex numbers b_j. A product of a complex and Grassmann numbers results in an anticommuting variable, while a product of two Grassmann numbers is, in turn, a commuting object. Therefore if we want to define linear transformations preserving the block structure of Θ, we need to represent them by matrices with a matching block structure:

$$A = \begin{pmatrix} A_{00} & \sigma \\ \rho & A_{11} \end{pmatrix} , \tag{1.23}$$

where A_{00} and A_{11} are matrices of size $n \times n$ and $m \times m$ respectively, consisting of commuting variables, while σ and ρ are $n \times m$ and $m \times n$ matrices with anticommuting elements. Such extensions of vectors and matrices are called supervectors and supermatrices.

Lastly, one needs the extension of basic operations on supermatrices. Using notation of eq. (1.23), the generalization of the trace of a matrix, preserving invariance w.r.t. cyclic permutations, called a supertrace is defined as

$$\mathrm{STr} A = \mathrm{Tr} A_{00} - \mathrm{Tr} A_{11} , \tag{1.24}$$

while equation defining a superdeterminant (also called Berezinian) is in turn determined through

$$\ln \mathrm{SDet} A = \mathrm{STr} \ln A . \tag{1.25}$$

If A_{00} and A_{11} are invertible, one has another way of expressing the superdeterminant:

$$\mathrm{SDet} A = \mathrm{Det}\left(A_{00}\right) \mathrm{Det}^{-1}\left(A_{11} - \rho A_{00}^{-1} \sigma\right) \tag{1.26a}$$

$$= \mathrm{Det}\left(A_{00} - \sigma A_{11}^{-1} \rho\right) \mathrm{Det}^{-1}\left(A_{11}\right) . \tag{1.26b}$$

Combining both commuting and anticommuting variables into one formalism significantly simplifies the notation, e.g. the Gaussian integral over complex and Grassmann variables gives

$$\int \exp\left(-\bar{\Theta}^T A \Theta\right) \prod_i^n d\bar{\chi}_i d\chi_i \prod_{j=1}^m d\bar{b}_j db_j = \pi^m \mathrm{SDet} A . \tag{1.27}$$

1.4 Previous results

As mentioned before, it is believed that many of the local eigenvalue statistics, like e.g. distribution of spacings between neighboring levels, are universal, that is independent of the details of random matrix model. There are many quantities of interest one can inspect on the local level, the simplest being aforementioned level spacing or distribution of say k'th largest eigenvalue. The object that possesses the most information about eigenvalue statistics is their joint probability distribution function (jpdf) denoted by $\rho_N(\lambda_1, \ldots, \lambda_N)$ (for a Hermitian matrix of size $N \times N$) where we take $\lambda_1 \leq \ldots \lambda_N$. For simplicity of notation, we use $\bar{\rho}_N$ for jpdf symmetrized

w.r.t. eigenvalue permutations. From the jpdf one can recover any eigenvalue statistics, the simplest being average eigenvalue density:

$$\rho(\lambda) = \int_{\mathbb{R}^{n-1}} \bar{\rho}(\lambda, \lambda_2, \ldots, \lambda_N)\, d\lambda_2 \ldots d\lambda_N \ , \tag{1.28}$$

or a single k'th eigenvalue distribution

$$\rho(\lambda_k) = \int_{\mathbb{R}^{n-1}} \rho(\lambda_1, \ldots, \lambda_N)\, d\lambda_1 \ldots d\lambda_{k-1} d\lambda_{k+1} \ldots d\lambda_N \ . \tag{1.29}$$

In a similar manner, one defines k-point correlation functions

$$\rho^k(\lambda_1, \ldots, \lambda_k) = \int_{\mathbb{R}^{n-k}} \bar{\rho}_N(\lambda_1, \ldots, \lambda_N)\, d\lambda_{k+1} \ldots d\lambda_N \ . \tag{1.30}$$

The universality of statistics tells us, that we need to be able to compute them only in one simple case to know the result in any more complicated situation falling into the same universality class. Obvious choice for the specific model for computation is the most symmetric one - GUE, defined by the probability measure

$$d\mu(H) \propto e^{-\mathrm{Tr}H^2/2} dH \ . \tag{1.31}$$

In this case one has explicit formulas for jpdf and all other correlation functions, having an exceptionally simple form

$$\rho^k(\lambda_1, \ldots, \lambda_k) = \det\left(K(\lambda_i, \lambda_j)\right)_{1 \le i,j \le k} \ , \tag{1.32}$$

$$K\left(\lambda\sqrt{N} + \frac{x}{\sqrt{N}\rho_{\mathrm{sc}}(\lambda)}, \lambda\sqrt{N} + \frac{y}{\sqrt{N}\rho_{\mathrm{sc}}(\lambda)}\right) \tag{1.33}$$

$$\xrightarrow[N\to\infty]{} K_{\mathrm{Sine}}(x, y) = \frac{\sin(\pi(x-y))}{\pi(x-y)} \ , \tag{1.34}$$

where $\rho_{\mathrm{sc}}(\lambda) = \frac{1}{2\pi}\sqrt{4-\lambda^2}\mathbf{1}_{|\lambda|\le 2}$ is the average eigenvalue density for properly rescaled GUE, called the Wigner semicircle distribution, and $|\lambda| < 2$.

The universality of eigenvalue statistics is known as "Sine kernel universality", thanks to the interpretation of eigenvalues of random matrices as particles in a determinantal point process with kernel $K_{\mathrm{Sine}}(x, y)$. Many works were devoted to proving and expanding regions of validity of the universality conjecture, in both realms of Wigner and invariant random matrices. We will shortly describe previous results and refer the reader to some of the extensive literature on the subject.

1.4.1 Wigner matrices

First of the methods to analyze the local statistics of Wigner matrices are the heat flow techniques. One starts with a Wigner matrix, say M_N^0, and considers a stochastic diffusion process defined by the equation

$$dM_N^t = d\beta_t - \frac{1}{2} M_N^t dt \ , \tag{1.35}$$

with a starting point at $M_N^t|_{t=0} = M_N^0$. β_t is a Hermitian matrix process with entries being independent Brownian motions, real on the diagonal, complex off the diagonal. This process describes a continuous flow from M_N^0 towards GUE as $t \to \infty$. Roughly speaking, one can use the dynamics of the flow of eigenvalues, established by Dyson [13], to extend the Sine kernel universality [14, 15].

Another way of dealing with Wigner matrices is the so-called 'Four Moment Theorem'. The theorem asserts that statistics of the eigenvalues on the local scale of $N^{-1/2}$ depend only on the first four moments of the matrix entries [16]. Details of that approach are beyond the scope of this thesis, but by using the Four Moment Theorem the universality has been proven for a broad class of Wigner hermitian matrices [17] and further extended to properties of eigenvectors [18] and eigenvalues of non-hermitian random matrices [19]. For a detailed review on the topic of universality in the class of Wigner random matrices see [20].

1.4.2 Invariant ensembles

Some work has been done in the case of invariant ensembles, starting with [21], where Sine kernel universality was shown for invariant unitary random matrices with potential function having sufficiently fast growing tails. The method of proof was relying strongly on the orthogonal polynomial technique. This method proved to be very effective in the case of analytic potentials with some additional requirements [22, 23, 24] and a lot of progress was made (see [25] for a review), though often restricted to the unitary symmetry class only.

More recent results proving universality hypothesis for a broader class of random matrix ensembles came from flow equation approach [26, 27] applied to so-called β-ensembles, an interpolation between the 3 classical symmetry classes. Similar to the case of Wigner matrices, one can investigate a flow in space of invariant matrices ending with the Gaussian ensemble, in this way matching the statistics of complicated models with the Gaussian ones.

Lastly, first results combining formalisms of supersymmetry and free probability came in [28], while the first attempts to relate the Laplace transform of partition function with free probabilistic R-transform arose in [29]. The relation was proven for different regimes for a low-rank argument of the transform in several papers. [30] showed the result in the case of eigenvalue distribution restricted to an interval, while [31] proved the matching in the case of analytic uniformly convex potentials. We expand and generalize those results.

2 Laplace transform

In the standard setting, the Laplace transform of a function $f(p)$ is defined for $q \geq 0$ by:

$$\tilde{f}(q) = \int_0^\infty f(p)\, e^{-pq} dp \ . \tag{2.1}$$

Formally, to invert the transform one performs an integral in the complex plane, parallel to the imaginary axis:

$$f(p) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \tilde{f}(q)\, e^{pq} dq \ , \tag{2.2}$$

where $\gamma \in \mathbb{R}$ is greater than the real part of singularities of $\tilde{f}(q)$. In practice, one can close the contour of integration to the left of the complex plane and have it encircling all singularities of $\tilde{f}(q)$.

Our goal is the description of correlation functions for eigenvalues of random matrix models. As explained in section 1.2.2, the average eigenvalue density (one-point correlation function) may be obtained by considering the Green's function, i.e.:

$$g(z) = \lim_{N \to \infty} N^{-1} \mathbb{E} \left\{ \mathrm{Tr} \left(z\mathbf{1} - H \right)^{-1} \right\} \ , \tag{2.3}$$

where H is a $N \times N$ random matrix. Trace may be expressed as a ratio of determinants, giving us an alternate expression for the resolvent:

$$g(z) = \lim_{N \to \infty} N^{-1} \mathbb{E} \left\{ \frac{d}{dz'} \frac{\mathrm{Det}\left(z'\mathbf{1} - H \right)}{\mathrm{Det}\left(z\mathbf{1} - H \right)} \bigg|_{z'=z} \right\} \ . \tag{2.4}$$

In a similar manner, the many-point correlation functions are governed by the expected value of a ratio of products of determinants. We define a general $(n|m)$ partition function by:

$$Z_{n|m}\left(\{p_0\}, \{p_1\} \right) = \mathbb{E} \left\{ \frac{\prod_{j=1}^m \mathrm{Det}\left(p_{1,j}\mathbf{1} - H \right)}{\prod_{k=1}^n \mathrm{Det}\left(p_{0,k}\mathbf{1} - H \right)} \right\} \ . \tag{2.5}$$

This object extends to a radial function of a supermatrix P of rank $(n|m)$:

$$Z(P) = \mathbb{E} \left\{ \mathrm{SDet}^{-1} \left(P \otimes \mathbf{1}_N - \mathbf{1}_{n|m} \otimes H \right) \right\} \ . \tag{2.6}$$

The goal of this chapter is to establish and prove the existence of the Laplace transform and its inverse in the context of the supersymmetric generalization of the aforementioned partition function (2.6), first in the case of one-point function as a proof of concept, and later for arbitrary integer values of n and m.

It remains to motivate the need for Laplace transform in the random matrix theory. What is the advantage of taking Laplace transform of our partition function? The $n = 1$,$m = 0$ example is enough to see the idea. We use the notation

$$G(p) = \lim_{N\to\infty} G_N(p) = \lim_{N\to\infty} N^{-1}\mathbb{E}\left\{\mathrm{Tr}\ln(p\mathbf{1} - H)\right\} , \tag{2.7}$$

for so-called integrated Green's function, related to the standard Green's function by:

$$g(p) = \frac{\partial}{\partial p} G(p) . \tag{2.8}$$

One can reduce the expected value of a determinant into a simpler form using the following approximation:

$$\begin{aligned} Z(p) &= \mathbb{E}\left\{\mathrm{Det}^{-1}(p\mathbf{1} - H)\right\} && \text{(2.9a)}\\ &= \mathbb{E}\left\{e^{-\mathrm{Tr}\ln(p\mathbf{1}-H)}\right\} && \text{(2.9b)}\\ &\approx e^{-\mathbb{E}\{\mathrm{Tr}\ln(p\mathbf{1}-H)\}} && \text{(2.9c)}\\ &= e^{-NG_N(p)} . && \text{(2.9d)} \end{aligned}$$

Unfortunately, when p is near the spectrum of H it is a bad approximation. The way to avoid it would be to keep p away from support of eigenvalue distribution of H by e.g. performing the following transformation:

$$\tilde{Z}(q) = \oint \mathbb{E}\left\{\mathrm{Det}^{-1}(p\mathbf{1} - H)\right\} e^{pq} dp , \tag{2.10}$$

with the contour of integration encircling the support of the spectrum. After performing this approximation, all calculations are reduced to one-point functions, therefore it cannot be true for any probability density. One can construct many random matrix models with same average eigenvalue density and very different correlation functions. E.g. one can choose each eigenvalue as an independent random variable distributed identically to the GUE average eigenvalue spectrum, in the first case one has Poisson statistics, vastly different from ones observed in the latter case. Validity of this approximation is discussed in detail in chapter 3. After making the approximation, one way of evaluating this integral (or in more complicated cases its supersymmetric extension) in $N \to \infty$ limit is to perform a saddle-point analysis. This form resembles the inverse Laplace transform, making it a topic worth further investigation.

2.1 R-transform as a result of the saddle-point approximation

In this section, we heuristically explain the connection between supersymmetry, free probability and local statistics of eigenvalues. We leave out many details, like e.g. normalization constants or integration domains, that are derived and explained in later parts of the thesis. For simplicity, the notation used in this section is schematic and not mathematically precise.

As discussed in the previous section, we want to calculate the Laplace transform of a supersymmetric partition function (2.6) that is schematically expressed as

$$\tilde{Z}(Q) = \int dP \exp(\mathrm{STr} PQ)\, Z(P) \tag{2.11a}$$

$$= \int dP \exp(\mathrm{STr} PQ)\, \mathbb{E}\left\{\mathrm{SDet}^{-1}(P \otimes \mathbf{1} - \mathbf{1} \otimes H)\right\} , \tag{2.11b}$$

where Q and P are supermatrices of appropriate sizes and symmetries. Now we assume that in $N \to \infty$ limit we can move the expected value under the exponential. Denoting a supersymmetric lift of the integrated Green's function by $G(P)$ we arrive at

$$= \int dP \exp(\mathrm{STr} PQ) \exp(-\mathrm{STr} \log \mathbb{E}\{P \otimes \mathbf{1} - \mathbf{1} \otimes H\}) \tag{2.11c}$$

$$= \int dP \exp(\mathrm{STr} PQ - N \mathrm{STr} G(P)) \ . \tag{2.11d}$$

Now taking $\tilde{Z}(NQ)$ for large N, we can perform a saddle-point approximation of the integral. We make a variation of P in a direction of some δP and calculate a directional derivative of the exponent w.r.t. a parameter t. The matrices P_0 for which the derivative vanishes form a critical subspace

$$\mathrm{STr}(\delta P Q) - \lim_{t \to 0} \mathrm{STr} \frac{G(P + t\delta P) - G(P)}{t}\bigg|_{P=P_0} = 0 \ . \tag{2.12}$$

Writing the lift of Green's function as

$$g(P) = \lim_{N \to \infty} N^{-1} \mathbb{E}\left\{\left(P \otimes \mathbf{1} - \mathbf{1}_{n|m} \otimes H\right)^{-1}\right\} ,$$

and requiring condition (2.12) to be true for any δP, we have:

$$Q - g(P_0) = 0 \ , \tag{2.13}$$

resulting in

$$\begin{aligned} \Gamma(Q) :&= \log N^{-1} \tilde{Z}(NQ) \propto \mathrm{STr}(P_0 Q - G(P_0)) \\ &= \mathrm{STr}\left(g^{-1}(Q)\, Q - G\left(g^{-1}(Q)\right)\right) \ . \end{aligned} \tag{2.14}$$

The last step is to again take a derivative (this time simply denoted by prime symbol "$'$") and remove the singularity at $Q = 0$, resulting in

$$(\Gamma(Q) - \mathrm{STr} \log Q)' = \mathrm{STr}\left(g^{-1}(Q) - Q^{-1}\right) , \tag{2.15}$$

which has exactly the form of eq. (1.13), defining the R-transform, or in this case a supersymmetric extension thereof.

2.2 One-point function

We use the one-point function as an example allowing us to present the reasoning behind the more complicated analysis of the Laplace transform for many-point correlation functions. The main idea is to start with the well-established Fourier transform and to use techniques of complex analysis, i.e. contour integration and analytic continuations in order to establish relations between the transforms. The standard Fourier transform and its inverse are given by:

$$\hat{f}(q) = \int_{-\infty}^{\infty} f(p)\, e^{-ipq} dp \ , \tag{2.16}$$

$$f(p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(q)\, e^{ipq} dq \ . \tag{2.17}$$

2.2.1 Fermion-fermion sector

In this section we will consider the relation between Fourier and Laplace transforms of the function:

$$f(p) = \mathrm{Det}\,(p\mathbf{1} - H)\,. \tag{2.18}$$

which is a polynomial in variable p, and therefore doesn't have any singularities. The Laplace transform, defined for $\mathrm{Re}\,(q) > 0$, extends to a holomorphic function on $\mathbb{C} \setminus \{0\}$ by following an analytic continuation procedure. We start by drawing a contour of integration consisting of real positive semi-axis, another ray starting at the origin and lying in the right side of the complex plane, denoted by γ, and an arc connecting those two in the infinity as presented in fig. 2.1. We perform $\oint f(p)\, e^{-pq} dp$ along this contour in the counter-clockwise direction and using Cauchy integral theorem we know that such integral is equal to zero. Therefore, as the integral on the arc in infinity vanishes, we know that

$$\int_0^{\infty} f(p)\, e^{-pq} dp = -\int_{\gamma} f(p)\, e^{-pq} dp = \int_{-\gamma} f(p)\, e^{-pq} dp \ , \tag{2.19}$$

where the two integrals are properly defined. If we denote by α the angle between real positive semi-axis and γ, the integral over γ is well defined in the region described by inequality

$$\mathrm{Re}\,(q) \cos\alpha - \mathrm{Im}\,(q) \sin\alpha > 0 \ . \tag{2.20}$$

In fact, we can define the analytical continuation of Laplace transform for any $q \in \mathbb{C}/\{0\}$ by repeating this procedure. In particular, we have

$$\tilde{f}(q) = \int_0^{-\infty} f(p)\, e^{-pq} dp \ , \tag{2.21}$$

for $\mathrm{Re}\,(q) < 0$.

Let us turn our attention to the Fourier transform of $f(p)$. It's not well defined unless we perform some regularization procedure. We will regularize

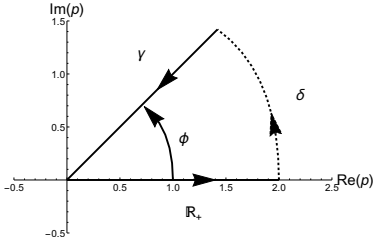

Figure 2.1: Sketch of the contour of integration used for analytical continuation of the Laplace transform in the fermion-fermion sector. Integral over the real positive semi-axis coincides with integral over $(-\gamma)$ as there are no singularities inside of the contour and integral over δ vanishes.

the transform by using an exponential cutoff and relate it to the Laplace transform:

$$\hat{f}(q) = \lim_{\epsilon\to 0^+} \int_{-\infty}^{\infty} f(p)\, e^{-ipq} e^{-\epsilon|p|} dp \tag{2.22a}$$

$$= \lim_{\epsilon\to 0^+} \left(\int_0^{\infty} f(p)\, e^{-ipq} e^{-\epsilon p} dp - \int_0^{-\infty} f(p)\, e^{-ipq} e^{\epsilon p} dp \right) \tag{2.22b}$$

$$= \lim_{\epsilon\to 0^+} \left(\tilde{f}(ip+\epsilon) - \tilde{f}(ip-\epsilon) \right) . \tag{2.22c}$$

Having this relation, we can use the inverse Fourier transform to construct the inverse of the Laplace transform in turn showing it's existence and form. The inverse relation goes as follows,

$$f(p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(q)\, e^{ipq} dq \tag{2.23a}$$

$$= \lim_{\epsilon\to 0^+} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\tilde{f}(iq+\epsilon) - \tilde{f}(iq-\epsilon) \right) e^{ipq} dq \tag{2.23b}$$

$$= \lim_{\epsilon\to 0^+} \frac{1}{2\pi} \left(\int_{-\infty-i\epsilon}^{\infty-i\epsilon} \tilde{f}(iq)\, e^{ipq} dq - \int_{-\infty+i\epsilon}^{\infty+i\epsilon} \tilde{f}(iq)\, e^{ipq} dq \right) . \tag{2.23c}$$

After a change of the integration variable to $z := iq$, the two integrals can be collapsed to a contour integral running counter-clockwise around the imaginary axis (see fig. 2.2). Any contour deformation is allowed, as long as it doesn't pass through the possible singularity at 0. Therefore we end with the following inverse Laplace transform:

$$f(p) = \frac{1}{2\pi i} \oint \tilde{f}(z)\, e^{pz} dz , \tag{2.24}$$

where integration contour encircles the origin of the complex plane.

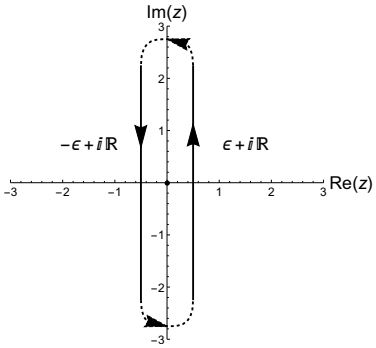

Figure 2.2: Sketch of the contour of integration used for calculation of the inverse Laplace transform in the fermion-fermion sector. We join two integrals over lines parallel to the imaginary axis, one slightly to the right, and one slightly to the left, in $\pm i\infty$, in turn replacing them by one contour integral around the imaginary axis.

2.2.2 Boson-boson sector

Regularization of the denominator in the Green's function can be done in one of two ways, advanced or retarded, denoted in this section by

$$f(p) = \mathrm{Det}^{-1}(p\mathbf{1} \pm i\epsilon - H) \ . \tag{2.25}$$

For the simplicity we will consider only the advanced case, the other one is analogical. Let us again start with the Fourier transform, adjust contours and arrive at Laplace transform and its inverse in the boson-boson sector. Starting with the integral over the real axis, we close the contour of integration in either upper or lower complex plane, depending on the sign of the transform argument (see fig. 2.3). Therefore we have:

$$\hat{f}(q) = \int_{-\infty}^{\infty} f(p)\, e^{-ipq} dp = \oint_{\kappa} f(p)\, e^{-ipq} dp \ , \tag{2.26}$$

where κ goes along the real axis to the right and closes in the upper complex plane for $q < 0$ or lower for $q > 0$. $f(p)$ is holomorphic in the upper complex plane, so again using Cauchy theorem we obtain that $\hat{f}(q) = 0$ for $q < 0$. In case of $q > 0$, we may deform the contour into any shape encircling all singularities of $f(p)$, that is positions of the eigenvalues shifted by $i\epsilon$ into the lower complex plane. After making the $p \to ip$ variable change, we end up with the following reduced formulas for the Fourier transform and its

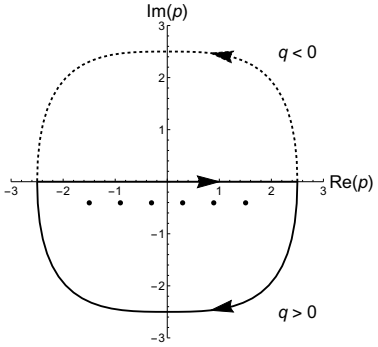

Figure 2.3: Sketch of contour of integration used for reduction of Fourier transform in boson-boson sector. Contour is closed in upper or lower complex plane, depending on sgn (q) s.t. contour integral coincides with integral over the real axis. As a result of Cauchy theorem it is equal zero for $q < 0$, while for $q > 0$ the contour encircles all singularities (represented by black dots) of integrated function.

inverse:

$$\hat{f}(q) = \begin{cases} i \oint_\kappa f(ip)\, e^{pq} dp & \text{for } q < 0 \\ 0 & \text{for } q > 0 \end{cases} , \tag{2.27}$$

$$f(ip) = \frac{1}{2\pi} \int_0^\infty \hat{f}(q)\, e^{-pq} dq \, . \tag{2.28}$$

The first expression (up to a factor of i and orientation of the contour κ) is the inverse Laplace transform of the function $F(p) := f(ip)$ we were looking for, and the second one is the forward transform. To obtain the standard form of the transform, we may change the orientation of the contour κ and include the resulting $(-i)$ factor in the inverse transform instead of the forward one. Explicitly, the final results reads:

$$\tilde{f}(q) = \frac{1}{2\pi i} \oint F(p)\, e^{pq} dp \, , \tag{2.29}$$

$$F(p) = \int_0^\infty \tilde{f}(q)\, e^{-pq} dq \, , \tag{2.30}$$

where $F(p) = \mathrm{Det}^{-1}(i(p+\epsilon)\mathbf{1} - H)$. In the boson-boson sector, the forward transform is an integral along a closed curve encircling positions of all the eigenvalues, while the inverse one goes along the real positive semi-axis.

In the following sections, we will generalize this approach to deal with the ratio of products of determinants.

2.3 Many-point functions

To gain access to local scales one has to go beyond one-point function, as it gives information only about the average eigenvalue distribution. We need to consider cases with $n, m > 1$, where the Fourier transform is not a simple integral over the real axis anymore, and the resulting Laplace transform has to be a multidimensional integral too. Our starting points will be the Fourier transform over the space of Hermitian matrices and its inverse, that will be expressed in terms of the Laplace transform over the Hermitian positive-definite matrices and unitary matrices respectively. In later sections, we will show how our construction generalizes to real symmetric and quaternion self-dual matrices.

Before we move to specific calculations, let us recall a very important result, the Harish-Chandra–Itzykson-Zuber (HCIZ) integral formula [32, 33]. Let A, B be $n \times n$ Hermitian matrices with eigenvalues (by convention in increasing order) denoted by $\lambda_i(A)$ and $\lambda_i(B)$. The formula states that if eigenvalues are non-degenerate then

$$\int_{U(n)} \exp\left(x \mathrm{Tr}\left(AUBU^\dagger\right)\right) dU = \tag{2.31a}$$

$$= \prod_k^{n-1} k! \frac{\mathrm{Det}\left(\exp\left(x\lambda_i(A)\lambda_j(B)\right)\right)_{1\le i,j\le n}}{x^{(n^2-n)/2}\Delta(\lambda(A))\Delta(\lambda(B))} , \tag{2.31b}$$

where dU is the Haar probability measure over the group of unitary matrices of size $n \times n$. $\Delta(\lambda(A))$ represents the Vandermonde determinant:

$$\Delta(\lambda(A)) = \prod_{1\le i<j\le n} (\lambda_j(A) - \lambda_i(A)) . \tag{2.32}$$

HCIZ formula is especially useful when dealing with Fourier type transforms of class functions of matrices, i.e. ones that depend only on the set of matrix eigenvalues. One can first perform the eigenvalue reduction and then integrate out angular degrees of freedom using eq. (2.31). E.g. for an integral over the unitary group with invariant measure:

$$\int_{U(n)} dP e^{x\mathrm{Tr}PQ} F(\lambda(P)) = \tag{2.33a}$$

$$= \int_{(S^1)^{\times n}} \prod_{j=1}^n dp_j \Delta(\lambda(P))^2 \int_{U(n)} dU e^{x\mathrm{Tr}U\Lambda_P U^\dagger Q} . \tag{2.33b}$$

2.3.1 Fermion-fermion sector

Let us start, in a similar manner to the one variable case, by analyzing the Fourier transform of the product of determinants:

$$F(P) = \prod_{j=0}^m \mathrm{Det}(p_j \mathbf{1} - H) , \tag{2.34}$$

over the space of Hermitian matrices of size $m \times m$, denoted by $H(m)$, where p_j denote eigenvalues of said matrices P. We know that properly regularized Fourier transform and its inverse exist for such functions. By relating Fourier and Laplace transforms we will show that the latter is well defined and invertible in an analogous way to the one-dimensional case presented previously. Again using the exponential cutoff, the regularized Fourier transform reads:

$$\hat{F}_\epsilon(Q) = \int_{H(m)} dP e^{-i\mathrm{Tr}PQ - \epsilon \mathrm{Tr}|P|} F(P) \ , \tag{2.35}$$

where dP denotes a flat measure. The Fourier transform is recovered in the $\epsilon \to 0^+$ limit. Using the fact that $F(P)$ is a radial function, we start by performing eigenvalue reduction and use the HCIZ formula to evaluate the resulting integral over the unitary group:

$$\hat{F}_\epsilon(Q) = \int_{\mathbb{R}^m} \prod_{j=1}^{m} dp_j \Delta(\lambda(P))^2 e^{-\epsilon \mathrm{Tr}|\Lambda_P|} \int_{U(m)} e^{-i\mathrm{Tr}U\Lambda_P U^\dagger Q} F(\Lambda_P)\, dU \tag{2.36a}$$

$$= \int_{\mathbb{R}^m} \prod_{j=1}^{m} dp_j \Delta(\lambda(P))^2 e^{-\epsilon \mathrm{Tr}|\Lambda_P|} \prod_{k}^{m-1} k! \tag{2.36b}$$

$$\times \frac{\mathrm{Det}\left(\exp(-ip_i q_j)\right)_{1\le i,j\le n}}{(-i)^{(m^2-m)/2} \Delta(\lambda(P)) \Delta(\lambda(Q))} F(\Lambda_P) \ . \tag{2.36c}$$

We will use the combinatorial definition of the determinant, as a sum over all permutations of products of matrix elements with the sign of the permutation. One can move the sum in front of the integral and, noting the transformation law for Vandermonde determinant under permutation of matrix eigenvalues:

$$p_i \to p_{\sigma(i)} \ , \tag{2.37a}$$

$$\Delta(\lambda(P)) \to \mathrm{sgn}(\sigma) \Delta(\lambda(P)) \ , \tag{2.37b}$$

one can perform such change of variables in each integral separately and end up with the following expression ($F(\Lambda_P)$ and $\mathrm{Tr}|\Lambda_P|$ are invariant w.r.t. permutation of p_i's):

$$\hat{F}_\epsilon(Q) = \sum_\sigma \int_{\mathbb{R}^m} \prod_{j=1}^{m} dp_j \frac{\Delta(\lambda(P))}{\Delta(\lambda(Q))} e^{-\epsilon \mathrm{Tr}|\Lambda_P|} \left(\prod_{k}^{m-1} k! \right) i^{(m^2-m)/2} \tag{2.38a}$$

$$\times \exp\left(-i \sum_l p_l q_l \right) F(\Lambda_P) \ . \tag{2.38b}$$

The sum amounts to an adjustment in a constant factor while the exponents can be combined in the following way:

$$\exp(-\epsilon \mathrm{Tr}|\Lambda_P|) \exp\left(-i \sum_l p_l q_l \right) = \exp\left(-\sum_l p_l (i q_l + \epsilon\, \mathrm{sgn}(p_l)) \right) \tag{2.39}$$

yielding the Fourier transform expressed in a way convenient for comparison with the Laplace transform:

$$\hat{F}(Q) = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^m} \prod_{j=1}^{m} dp_j \frac{\Delta(\lambda(P))}{\Delta(\lambda(Q))} \left(\prod_k k! \right) i^{(m^2-m)/2} \tag{2.40a}$$

$$\times \exp\left(-\sum_l p_l \left(iq_l + \epsilon \operatorname{sgn}(p_l) \right) \right) F(\Lambda_P) \ . \tag{2.40b}$$

Now let us turn our attention to the Laplace transform of functions $F(P)$. The same approach as in the case of the Fourier transform may be employed mutatis mutandis to arrive at the following form:

$$\tilde{F}(Q) = \int_{H^+(m)} dP e^{-\mathrm{Tr} PQ} F(P) \tag{2.41a}$$

$$= \int_{\mathbb{R}_+^m} \prod_{j=1}^{m} dp_j \frac{\Delta(\lambda(P))}{\Delta(\lambda(Q))} \left(\prod_k k! \right) (-1)^{(m^2-m)/2} \tag{2.41b}$$

$$\times \exp\left(-\sum_l p_l q_l \right) F(\Lambda_P) \ , \tag{2.41c}$$

properly defined when $\mathrm{Re}(q_l) > 0$ for all l. A procedure identical to the one for one-point function (eq. (2.21)) can be performed for any number of the integrals over p_j's:

$$\int_{\mathbb{R}_+} dp_j \Delta(\lambda(P)) \exp(-p_j q_j) F(\Lambda_P) \ . \tag{2.42}$$

in order to obtain an analytical continuation of $\tilde{F}(Q)$ valid for matrices with any possible signature. Now we can split regions of integration over p_j's in eq. (2.41a) to positive and negative real semi-axes and match those with Laplace transform up to a constant and slightly different dependence on q_l's. Doing so allows one to evaluate sgn functions explicitly. Additionally each integral over negative real semi-axis only introduces (-1) factor. In the end, we have a relation between Fourier and Laplace transforms analogous to the one in the one-point function case (2.22c):

$$\hat{F}(Q) = \lim_{\epsilon \to 0^+} \sum_{S = \mathrm{diag}(\pm 1, \ldots, \pm 1)} (-1)^{\mathrm{Tr} S} \tilde{F}\left(i\Lambda_Q + \epsilon S \right) \ . \tag{2.43}$$

Now we will obtain an inverse Laplace transform formula starting with inverse Fourier transform, inserting formula (2.43) into it and performing

eigenvalue reduction:

$$F(P) = \frac{1}{(2\pi)^{m^2}} \int_{H(m)} dQ e^{i\mathrm{Tr}PQ} \hat{F}(Y) \tag{2.44a}$$

$$= \lim_{\epsilon\to 0^+} \frac{1}{(2\pi)^{m^2}} \int_{\mathbb{R}^m} \prod_{j=1}^{m} dq_j \Delta(\lambda(Q))^2 \tag{2.44b}$$

$$\times \int_{U(m)} dU e^{i\mathrm{Tr}PU^\dagger \Lambda_Q U} \sum_{S=\mathrm{diag}(\pm 1,\ldots,\pm 1)} (-1)^{\mathrm{Tr}S} \tilde{F}\left(i\Lambda_Q + \epsilon S\right) . \tag{2.44c}$$

Again, as $\tilde{F}$ depends only on the set of eigenvalues of Q, we can follow the same reduction as in the case of the forward transform. We perform HCIZ integral, write determinant explicitly, perform necessary permutations of q_j's as before, and continue the previous series of equations in the following way:

$$= \lim_{\epsilon\to 0^+} \frac{i^{-(m^2-m)/2}}{(2\pi)^{m^2}} \int_{\mathbb{R}^m} \prod_{j=1}^{m} dq_j \frac{\Delta(\lambda(Q))}{\Delta(\lambda(P))} \left(\prod_{k=1}^{m} k!\right) e^{\sum_l ip_l q_l} \tag{2.44d}$$

$$\times \sum_{S=\mathrm{diag}(\pm 1,\ldots,\pm 1)} (-1)^{\mathrm{Tr}S} \tilde{F}\left(i\Lambda_Q + \epsilon S\right) . \tag{2.44e}$$

Now we can take the sum in front of integration and for each term in the sum perform $\Lambda_Z := i\Lambda_Q + \epsilon S$ change of variables. Then the integrated function will be the same in each of the terms after taking $\epsilon \to 0^+$ limit, but integration domains will be different. Each of integrals over eigenvalues z_j of Z will be performed parallel to the imaginary axis, either on the left or right depending on the element of the sum. Keeping track of proper prefactor coming from the change of variables we continue:

$$= \lim_{\epsilon\to 0^+} \frac{1}{(2\pi i)^{m^2}} \prod_{j=1}^{m} \left(\int_{i\mathbb{R}+\epsilon} - \int_{i\mathbb{R}-\epsilon} dz_j\right) \frac{\Delta(\lambda(\Lambda_Z))}{\Delta(\lambda(P))} \tag{2.44f}$$

$$\times \left(\prod_{k=1}^{m} k!\right) e^{\sum_l p_l z_l} \tilde{F}(\Lambda_Z) . \tag{2.44g}$$

Each of the integrals over dz_j can be replaced by a contour integral going counter-clockwise around the imaginary axis, see fig. 2.3. As the function under integral doesn't have any singularities apart from possible one at $z_j = 0$, each contour can be deformed to a circle. Finally, comparing this result with HCIZ integral in eq. (2.33), we see that

$$F(P) = \frac{1}{(2\pi i)^{m^2}} \int_{U(m)} e^{\mathrm{Tr}PQ} \tilde{F}(Q)\, dQ \tag{2.45}$$

is a formula for inverse Laplace transform in question.

2.3.2 Boson-boson sector

In this section, we will focus on denominator part of the partition function in eq. (2.5). In the beginning, we will construct the Laplace transform and its inverse of functions of the type:

$$F(P) = \lim_{\eta \to 0^+} \prod_{j=0}^{n} \mathrm{Det}^{-1} \left((p_j + i\eta) \mathbf{1} - H \right) \ , \tag{2.46}$$

which has poles in variables p_j in lower complex half-plane. In appendix A.1 we show how our approach generalizes in order to include conjugate situation of poles in the upper complex half-plane (which amounts to change $\eta \to -\eta$), as well as mixed cases, where some of the factors contain $(+i\eta)$ and some of them have $(-i\eta)$.

Our starting point again is the Fourier transform over the space of Hermitian matrices. We can perform exactly the same reduction as in the fermion-fermion case up to the eq. (2.40), the only difference being that we don't need any regularization, so we can straight ahead put $\epsilon = 0$. We have therefore for each j:

$$\hat{F}(Q) \propto \int_{\mathbb{R}} dp_j \frac{\Delta(\lambda(P))}{\Delta(\lambda(Q))} \exp(-ip_j q_j) F(\Lambda_P) \ . \tag{2.47}$$

We may rephrase the integral over the real axis as a contour integral by closing it in the upper (lower) complex half-plane for $q_j < 0$ ($q_j > 0$), so that the contribution from the closure vanishes. In the first case, the integrated function doesn't have any singularities inside the contour, therefore it is equal to zero. In the latter, we have enclosed all the singularities in a clockwise direction and we can deform the contour to a circle around the origin of the complex plane with a radius bigger than the absolute value of largest singularity. The same procedure may be performed for each and every j, resulting in the following formula for Fourier transform:

$$\hat{F}(Q) = \begin{cases} \oint \prod_j dp_j \frac{\Delta(\lambda(P))}{\Delta(\lambda(Q))} \left(\prod_{k=1}^{n} k! \right) i^{(n^2-n)/2} & \text{if } \forall_j q_j > 0 \Leftrightarrow \\ \times \exp\left(-i \sum_l p_l q_l \right) F(\Lambda_P) & Q \in H_+(n) \\ 0 & \text{otherwise} \end{cases} . \tag{2.48}$$

Lastly, we just need to reverse the orientation of integration contours (introduces $(-1)^n$ factor) and change variables $z_l = -ip_l$ (introduces $i^{(n^2+n)/2}$ factor). Similar to the case of the inverse transform in the previous section, for positive definite matrices Q we are left with an integral over the unitary group:

$$\hat{F}(Q) = (-1)^n i^{n^2} \int_{U(n)} dZ \exp(\mathrm{Tr} ZQ) F(iZ) \ . \tag{2.49}$$

Defining $G(Z) := F(iZ)$ we arrive at the Laplace transform:

$$\tilde{G}(Q) = \int_{U(n)} dZ \exp(\mathrm{Tr} ZQ)\, G(Z) = (-i)^{n^2}\, \hat{F}(Q) \ . \tag{2.50}$$

For consistency, we will move constant factor to the inverse transform.

Calculation of the inverse Laplace transform follows straightforward from the inverse Fourier transform and second case in eq. (2.48), by considering $F(iP)$. We immediately obtain a reduction in the integration space of Fourier transform and, as a result, the inverse Laplace transform formula:

$$G(P) = F(iP) = \frac{1}{(2\pi)^{n^2}} \int_{H(n)} dQ \exp(i\mathrm{Tr}(iPQ))\, \hat{F}(Q) \tag{2.51a}$$

$$= \frac{1}{(2\pi i)^{n^2}} \int_{H_+(n)} dQ \exp(-\mathrm{Tr} PQ)\, \tilde{G}(Q) \ . \tag{2.51b}$$

The same formula can be derived for a product of an arbitrary number of 'advanced' and 'retarded' determinants. One has to carefully handle subspaces of $H(n)$ with different signatures and combinations thereof, but the main idea remains the same. A detailed derivation is given in the appendix (A.1).

Concluding, we have forward and inverse Laplace transform formulas for the product of determinants or the product of the inverse of determinants. In the first case, the forward transform is an integral over the space of positive-definite Hermitian matrices and its inverse is an integral over the unitary group. For a product of inverse of determinants, the situation is opposite, Laplace transform is performed by integration over the unitary group and the inverse one is computed by integrating over Hermitian positive-definite matrices.

2.4 Supersymmetric Laplace transform

Having developed formulas in cases of products of determinants and inverse of thereof separately, now let us combine both approaches by usage of the supersymmetry. We will consider functions of the type:

$$F\left(\{p_{0,j}\}, \{p_{1,k}\}\right) = \frac{\prod_{k=1}^{m} \mathrm{Det}\left(p_{1,k}\mathbf{1} - H\right)}{\prod_{j=1}^{n} \mathrm{Det}\left(p_{0,j}\mathbf{1} - H\right)} \ . \tag{2.52}$$

We choose to interpret p variables as the eigenvalues of a supermatrix:

$$P = \begin{pmatrix} P_{00} & P_{01} \\ P_{10} & iP_{11} \end{pmatrix} \ . \tag{2.53}$$

The function F can be lifted to a function of a supermatrix P:

$$F(P) = \mathrm{SDet}^{-1}\left(P \otimes \mathbf{1} - \mathbf{1}_{n|m} \otimes H\right) \ , \tag{2.54}$$

that has the form of our partition function. As in previous sections, we can apply the Fourier transform for $P_{00} \in H(n)$ and $P_{11} \in iH(m)$:

$$\hat{F}(Q) = \int_{H(n)\times iH(m)} dP \exp\left(-i\mathrm{STr}PQ\right) F(P) \tag{2.55a}$$

$$= \int_{H(n)} dP_{00} \int_{iH(m)} dP_{11} \prod_{i,j}^{n,m} \frac{\partial^2}{\partial P_{01_{i,j}} \partial P_{10_{j,i}}} F(P) \tag{2.55b}$$

$$\times \exp\left(-i\mathrm{Tr}P_{00}Q_{00} - i\mathrm{Tr}P_{01}Q_{10} + i\mathrm{Tr}P_{10}Q_{01} - i\mathrm{Tr}P_{11}Q_{11}\right) , \tag{2.55c}$$

with parametrization of supermatrix Q analogous to eq. (2.53). We can transform any such supermatrix P by a similarity transformation, that brings P_{00} and P_{11} to diagonal forms, denoted by Λ_{00} and Λ_{11} respectively, in the following way:

$$\begin{pmatrix} P_{00} & P_{01} \\ P_{10} & iP_{11} \end{pmatrix} = \begin{pmatrix} U^\dagger & 0 \\ 0 & V^\dagger \end{pmatrix} \begin{pmatrix} \Lambda_{00} & UP_{01}V^\dagger \\ VP_{10}U^\dagger & i\Lambda_{11} \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} , \tag{2.56}$$

where $U \in U(n)$ and $V \in U(m)$. This allows us to perform eigenvalue reduction in both boson-boson and fermion-fermion sectors of our supersymmetric Fourier transform. Additionally, derivatives over anticommuting variables are invariant w.r.t. following simultaneous change of variables:

$$P'_{01} = UP_{01}V^\dagger , \tag{2.57a}$$

$$P'_{10} = VP_{10}U^\dagger . \tag{2.57b}$$

The function of interest in our new variables reads:

$$F\left(\begin{pmatrix} \Lambda_{00} & P'_{01} \\ P'_{10} & i\Lambda_{11} \end{pmatrix}\right) = \mathrm{SDet}^{-1}\left(\begin{pmatrix} \Lambda_{00} - H & P'_{01} \\ P'_{10} & i\Lambda_{11} - H \end{pmatrix}\right) \tag{2.58a}$$

$$= \mathrm{Det}^{-1}\left(\Lambda_{00} - H\right) \mathrm{Det}\left((i\Lambda_{11} - H) - P'_{10}\left(\Lambda_{00} - H\right)^{-1} P'_{01}\right) . \tag{2.58b}$$

It is clear now, that $F(P)$ is a polynomial in eigenvalues of P_{11} and is meromorphic in eigenvalues of P_{00}. Derivatives over anticommuting variables P'_{01} and P'_{10} will not change the type of this dependence, therefore we can proceed with computations for P_{00} as in boson-boson case by shifting eigenvalues of H by some small $i\eta$ and for P_{11} as in fermion-fermion sector, by regularizing the Fourier transform with an exponential cutoff. Since this regularization is done in an invariant way, the Fourier transform $\hat{F}(Q)$ will be a radial function of Q. For simplicity we can straight ahead consider only

diagonal Q's by writing:

$$\hat{F}_\epsilon\left(\Lambda_Q\right) = \int_{R^n} d\Lambda_{00}\Delta\left(\lambda\left(\Lambda_{00}\right)\right)^2 \int_{U(n)} dU \tag{2.59a}$$

$$\times \int_{iR^m} d\Lambda_{11}\Delta\left(\lambda\left(\Lambda_{11}\right)\right)^2 \int_{U(m)} dV \tag{2.59b}$$

$$\times \prod_{i,j}^{n,m} \frac{\partial^2}{\partial P'_{01_{i,j}} \partial P'_{10_{i,j}}} \exp\left(-\epsilon \mathrm{Tr}\left|P\right|\right) F\left(\begin{pmatrix} \Lambda_{00} & P'_{01} \\ P'_{10} & i\Lambda_{11} \end{pmatrix}\right) \tag{2.59c}$$

$$\times \exp\left(-i\mathrm{Tr}U^\dagger \Lambda_{00} U \Lambda_{Q,00} - i\mathrm{Tr}V^\dagger \Lambda_{11} V \Lambda_{Q,11}\right) . \tag{2.59d}$$

The integral factorizes and each sector can be treated separately, exactly like in previous sections. Therefore we immediately obtain following final formulas for supersymmetric Laplace transform and its inverse for the inverse of a superdeterminant:

$$\tilde{F}\left(Q\right) = \int dP \exp\left(-\mathrm{STr}PQ\right) F\left(P\right) \ , \tag{2.60}$$

$$F\left(P\right) = c_{n,m} \int dQ \exp\left(\mathrm{STr}PQ\right) \tilde{F}\left(Q\right) \tag{2.61}$$

where integration is done over $H_+\left(n\right) \times U\left(m\right)$ for the forward transform and over $U\left(n\right) \times H_+\left(m\right)$ for its inverse with the constant $c_{n,m} = 1/\left(2\pi i\right)^{n^2+m^2}$

2.5 Non-unitary symmetry classes

Formalism we developed applies to Hermitian matrices, or in other words matrices diagonalizable by a unitary similarity transformation. The calculation we performed relies heavily on the usage of HCIZ integral formula (2.31), which has been derived explicitly only in the case of an integral over the unitary group. Although analogs of HCIZ formula are not known in closed form, it turns out that our results are not limited to the unitary symmetry class only. The crucial observation is, that our derivation doesn't require exact HCIZ formula, but exploits only the symmetry of the result and analytic properties of integrands. In fact, it has been conjectured in [34] and proven in [35], that the HCIZ type integral over some other symmetry groups, in particular, orthogonal group $O\left(N\right)$ and symplectic group $Sp\left(2N\right)$, possess properties required by our formalism. We will shortly summarize those results here and explain how to apply them to obtain Laplace transform formulas for real symmetric matrices (diagonalized by orthogonal transformations) and quaternion self-dual matrices (diagonalized by symplectic transformations).

Following (non-standard) notation of [35], we denote compact Lie groups

$G_{\beta,N}$ as follows:

$$G_{1/2,N} = O(N) \ , \tag{2.62a}$$

$$G_{1,N} = U(N) \ , \tag{2.62b}$$

$$G_{2,N} = Sp(2N) \ , \tag{2.62c}$$

and are interested in integrals of the form:

$$I_{\beta,N}(P,Q) = \int_{G_{\beta,N}} dU e^{\mathrm{Tr}PUQU^{-1}} \ , \tag{2.63}$$

where dU is the Haar measure on the Lie group $G_{\beta,N}$, while P and Q are matrices diagonalizable by similarity transformation by an element of $G_{\beta,N}$. Because of the invariance of dU, without loss of generality, one can consider P and Q to be diagonal matrices with eigenvalues $\{p_j\}$ and $\{q_j\}$ respectively. It has been shown in [35] that one can write such integrals in the following form:

$$I_{\beta,N}(P,Q) = \sum_{\sigma} \frac{\mathrm{Det}\left(\exp\left(p_i q_j\right)\right)_{1\leq i,j\leq N}}{\Delta(\lambda(P))^{2\beta}\,\Delta(\lambda(Q_\sigma))^{2\beta}} \hat{\mathcal{I}}_{\beta,N}(P,Q_\sigma) \ , \tag{2.64}$$

where $\hat{\mathcal{I}}_{\beta,N}(P,Q)$ are so-called principal terms that can be derived via certain recursion relation, the sum is performed over permutations and matrix subscript σ means, that the order of eigenvalues is changed by according permutation. In particular, for β integers, the principal terms are symmetric polynomials of degree β in

$$\tau_{i,j} = -\frac{(p_i - p_j)(q_i - q_j)}{2} \tag{2.65}$$

variables. In the case of $\beta = 1/2$ (orthogonal symmetry group), it can be expressed as a series in $\tau_{i,j}$.

We will sketch the derivation of Laplace transform formulas for matrices with other than unitary symmetries. Evaluation of the Fourier transform over real symmetric or quaternion self-dual matrices (denoted here by $H^\beta(n)$) begins with, as in previous sections, eigenvalue reduction, which for general β reads:

$$\int_{H^\beta(n)} dP \exp\left(-i\mathrm{Tr}PQ\right) F(P) = \int_{R^n} \prod_{j=1}^{n} dp_j \Delta(\lambda(P))^{2\beta} \tag{2.66a}$$

$$\times \int_{G_{\beta,n}} dU \exp\left(-i\mathrm{Tr}U\Lambda_P U^{-1}Q\right) F(\Lambda_P) \ . \tag{2.66b}$$

The $\Delta(\lambda(P))^{2\beta}$ term cancels with the same term in the denominator in eq. (2.64). As a result, even though the integrand differs from the one in the case of unitary symmetry, it possesses exactly the same analytic structure. This fact, together with the symmetry of $\tau_{i,j}$ variables, is all

we needed to proceed with derivation analogous to the one in the case of transform over Hermitian matrices presented in previous sections. We use $H_{+}^{\beta}(n)$ notation for the positive definite subspace of $H^{\beta}(n)$. The resulting Laplace transform formulas have exactly the same form as eq. (2.60,2.61) with following changes:

- Integration is performed over $H_{+}^{\beta}(n) \times U(m) / G_{\beta,m}$ for forward transform and $U(n) / G_{\beta,n} \times H_{+}^{\beta}(m)$ for the inverse.
- Constant factor is equal $c_{n,m} = 1/(2\pi i)^{n_{\mathrm{dof}}+m_{\mathrm{dof}}}$.

n_{dof} and m_{dof} denote the number of degrees of freedom in $H^{\beta}(n)$ and $H^{\beta}(m)$ respectively. In the case of orthogonal symmetry $n_{\mathrm{dof}} = \left(n^2+n\right)/2$ while for $\beta = 2$ we have $n_{\mathrm{dof}} = n(2n+1)$

3 Region of applicability

The partition functions, and as a result determination of the universality class of a random matrix ensemble, are determined within our formalism via the supersymmetric extension of the R-transform. However, the form of the R-transform depends on the average eigenvalue spectrum only. One can easily construct a random matrix ensemble having the same eigenvalue distribution while possessing different higher correlation functions. As an example in the realm of complex Hermitian matrices, take the GUE, belonging to the Sine-kernel universality class. One can consider an ensemble of $N \times N$ diagonal matrices, where each element is drawn independently according to the Wigner semicircle distribution as its counterpart. Eigenvalues of matrices constructed in such a way are independent and therefore experience Poissonian statistics. Obviously, in the $N \to \infty$, both ensembles have the same average eigenvalue distribution, therefore their R-transforms are identical, while they belong to different universality classes. Clearly, independent eigenvalues do not experience the level repulsion property that is a key feature of invariant ensembles like e.g. the GUE.

At which point our approach fails in the case of independent eigenvalues? One crucial step of our derivation is the approximation of the type taken in the eq. (2.9). Shortly speaking, we would want to be able to replace the expected value of a determinant by the exponent of the expected value of trace of a logarithm. Without loss of generality, we will consider the simplest case of $Z_{0|1}(p)$ partition function and for the convenience of notation we will take its logarithm:

$$\log \mathbb{E}\left\{\exp\left(\operatorname{Tr}\log\left(p\mathbf{1}-H\right)\right)\right\} \approx \log\exp\left(\mathbb{E}\left\{\operatorname{Tr}\log\left(p\mathbf{1}-H\right)\right\}\right) \tag{3.1a}$$

$$= \mathbb{E}\left\{\operatorname{Tr}\log\left(p\mathbf{1}-H\right)\right\} \tag{3.1b}$$

$$= NG(p) \ . \tag{3.1c}$$

In the aforementioned example of independent eigenvalues distributed according to the Wigner semicircle distribution, the left-hand side of eq. (3.1a) is equal to

$$\log \mathbb{E}\left\{\exp\left(\operatorname{Tr}\log\left(p\mathbf{1}-H\right)\right)\right\} = N\log p \ , \tag{3.2}$$

while the integrated Green's function, in this case, is given by

$$NG(p) = \frac{N}{\pi}\int_{-2}^{2}\log(p-\lambda)\sqrt{1-\frac{\lambda^2}{4}} \tag{3.3}$$

$$= N\left(\frac{p\left(p-\sqrt{p^2-4}\right)}{4}+\log\left(p+\sqrt{p^2-4}\right)-\frac{1}{2}-\log 2\right)\ . \tag{3.4}$$

We clearly obtain different results. In this case, the error of our approximation scales proportionally to N, which makes it unusable. In next sections, we will check under what conditions this approximation is justified in the $N\to\infty$ limit and explore examples satisfying our requirements.

3.1 Moment generating function and variance of integrated Green's function

To further shorten the notation, we will denote any term of the form $\operatorname{Tr}\log(p\mathbb{1}-H)$ by a random variable x. It is obvious from eq. (3.1c) that the result of our approximation behaves as $\mathcal{O}(N)$, therefore we want to keep the error term,

$$\log\mathbb{E}\left\{\exp\left(x-\mathbb{E}\left\{x\right\}\right)\right\}\ , \tag{3.5}$$

behaving at most as $\mathcal{O}\left(N^{1-\epsilon}\right)$ for some $\epsilon>0$. This can be replaced by a slightly more conservative requirement for the central moment generating function:

$$\left|\mathbb{E}\left\{\exp\left(x-\mathbb{E}\left\{x\right\}\right)\right\}\right| < \mathbb{E}\left\{\left|\exp\left(x-\mathbb{E}\left\{x\right\}\right)\right|\right\} \sim \mathcal{O}\left(\exp\left(aN^{1-\epsilon}\right)\right) \tag{3.6}$$

for some fixed constant a.

First, we employ the Chebyshev inequality [36], for the random variable x, that for any $k>0$ reads:

$$P\left(\left|x-\mathbb{E}\left\{x\right\}\right|\geq k\sigma\right)\leq\frac{1}{k^2}\ , \tag{3.7}$$

where σ is the square root of the variance of x. If we take $k=N^{\delta}$ for some small $\delta>0$, we can see that the probability of x to fluctuate more than $N^{\delta}\sigma$ from its mean goes to zero as $N\to\infty$:

$$P\left(\left|x-\mathbb{E}\left\{x\right\}\right|\geq N^{\delta}\sigma\right)\leq N^{-2\delta}\to 0\ . \tag{3.8}$$

As a result, we will treat $x-\mathbb{E}\left\{x\right\}$ as a centered random variable bounded to the interval $\left[-N^{\delta}\sigma, N^{\delta}\sigma\right]$. The size of the effective support of x depends on N in a way determined by the behavior of its variance. Having that, we will apply the Hoeffding's lemma [37], to move the requirement from the central moment generating function to the endpoints of the effective domain, and as a consequence onto the variance of x. The lemma states that for a centered

random variable y bounded to an interval, its moment generating function is bounded in the following way:

$$\mathbb{E}\left\{\exp\left(ty\right)\right\} \leq \exp\left(\frac{t^2\left(b-a\right)^2}{8}\right), \tag{3.9}$$

where $a < b$ are endpoints of the support of random variable y.

One should note, that Hoeffding's lemma holds for random variables bound almost surely, though one can extend this result to cases of random variables with tails decaying sufficiently fast. Random variables constructed in a similar way to the (integrated) Green's function generically have tails decaying fast, i.e. faster than Gaussian, see e.g. [38] for a recent review. Nevertheless, we do not give a proof of required tail bounds here.

Applying the lemma (3.9) to our case we obtain the following bound:

$$\mathbb{E}\left\{\left|\exp\left(x-\mathbb{E}\left\{x\right\}\right)\right|\right\} \leq \exp\left(\frac{N^{2\delta}\sigma^2}{2}\right) \sim \mathcal{O}\left(\exp\left(N^{2\delta}\sigma^2\right)\right), \tag{3.10}$$

Comparing this bound with our previous considerations (3.6), we can move the requirement from moment generating function onto the variance of x:

$$\sigma^2 \sim \mathcal{O}\left(N^{1-\epsilon-2\delta}\right), \tag{3.11}$$

or equivalently, denoting $\kappa = \epsilon + 2\delta$, to deviations of the integrated Green's function

$$\mathbb{E}\left\{\left|G\left(p\right)-\mathbb{E}\left\{G\left(p\right)\right|\right\}\right\} \sim \mathcal{O}\left(N^{-(1+\kappa)/2}\right). \tag{3.12}$$

In most of the cases, the object analyzed in literature is not the integrated, but regular Green's function. We need one more step to relate results on the deviations of Green's function to ones required by our method. I.e. usual form of bounds proven in the literature is

$$\left|g\left(p\right)-\mathbb{E}\left\{g\left(p\right)\right\}\right| < \mathcal{O}\left(f\left(N\right)\right), \tag{3.13}$$

for some function f. The behavior of integrated Green's function is bounded in the same way, through the following reasoning:

$$\left|G\left(p\right)-\mathbb{E}\left\{G\left(p\right)\right\}\right| = \left|\int_{\gamma(p)} g\left(q\right)dq - \mathbb{E}\left\{\int_{\gamma(p)} g\left(q\right)dq\right\} + C\right| \tag{3.14}$$

$$\leq \int_{\gamma(p)} \left|g\left(q\right)-\mathbb{E}\left\{g\left(q\right)\right\}\right|\left|dq\right| + \left|C\right|, \tag{3.15}$$

for some constant C, where $\gamma\left(p\right)$ denotes a path in the upper (lower) complex plane ending at p for $\mathrm{Im}p > 0 (\mathrm{Im}p < 0)$. As a consequence, all results about deviations of $g\left(p\right)$ for large N apply when considering $G\left(p\right)$ as well.

3.2 Known variance estimates

Much attention in random matrix theory was given to the question of self-averaging of Green's function. Namely, if Green's function variance goes to zero as matrix size grows to infinity, one converges to a deterministic limit for the average eigenvalue spectrum. Results of the type:

$$\mathbb{E}\left\{\left|g\left(p\right)-\mathbb{E}\left\{g\left(p\right)\right\}\right|\right\}\leq f\left(N\right)\xrightarrow[N\to\infty]{}0 \tag{3.16}$$

have been proven for the first time for some Wigner random matrices in [2] and for a broad class of invariant random matrices in [39]. To no surprise, the bounds obtained in those papers are not good enough to conform to our requirements. Better control of the rate of convergence is required for many methods, including one developed in this thesis, in order to access many-point correlation functions.

In the case of matrices with independent entries, one can standardize them by a simple linear scaling to zero mean and normalized variance:

$$\sum_{j=1}^{N}\sigma_{ij}^{2}=1,\quad i=1,2,\ldots,N. \tag{3.17}$$

One calls such ensembles "generalized Wigner" if all of the variances for individual entries are of the same order, $\sigma_{ij}^{2}\sim\mathcal{O}\left(1/N\right)$. In those cases, not only the existence of a limit can be proven, but it is universally given by the Stieltjes transform of the semicircle density:

$$g_{sc}\left(p\right)=\int_{-2}^{2}\frac{\rho_{sc}\left(\lambda\right)d\lambda}{p-\lambda},\quad \rho_{sc}\left(\lambda\right)=\frac{1}{2\pi}\sqrt{4-\lambda^{2}}, \tag{3.18}$$

as in the purely Gaussian case. The best bound of this type is given in [40]:

$$\mathbb{E}\left\{\left|g\left(p\right)-g_{sc}\left(p\right)\right|\right\}\leq C\frac{\left(\log N\right)^{L}}{N} \tag{3.19}$$

with some constants C,L for sufficiently large N. This result is strong enough for our method to apply.

In the realm of invariant random matrices a similar result, yielding a sufficient self-averaging of Green's function, was proven and studied in great detail in [27, 41, 42]. In all three classical cases of $\beta=1,2,4$ and for a broad class of potential functions, including eigenvalue densities supported on single or multiple intervals, one has:

$$\mathbb{E}\left\{\left|g\left(p\right)-\mathbb{E}\left\{g\left(p\right)\right\}\right|\right\}\leq C\frac{\log N}{N}, \tag{3.20}$$

for some constant C.

The next case to discuss here, are random band matrices, constructed in a similar way to (generalized) Wigner matrices, but with entries set to zero beyond a diagonal band. This type of models are probably the most interesting ones, as correlations universality has been conjectured, but not

rigorously proven yet, if one lets the band width M scale at least as fast as $\sqrt{N}$. Numerical evidence confirming the conjecture is extensive, showing the Sine-kernel universality for $M >> \sqrt{N}$ and Poisson statistics for eigenvalues if $M << \sqrt{N}$. In this case authors of [43] have shown that

$$\mathbb{E}\left\{\left|g\left(p\right) - g_{sc}\left(p\right)\right|\right\} \leq C\frac{N^{\epsilon}}{M} \tag{3.21}$$

away from edges of the eigenvalue distribution for any $\epsilon > 0$. Comparing this bound with our requirement translates to

$$M \sim \mathcal{O}\left(N^{1/2+\delta}\right) \tag{3.22}$$

for any $\delta > 0$, therefore only slightly exceeding the conjectured $1/2$ exponent.

Finally, in the realm of non-Hermitian random matrices, proven self-averaging bounds aren't tight enough yet to comply with our requirements. As an example and not going into details of extended formalism, the best estimate in the case of matrices with independent entries for the variance of the quaternionic generalization of Green's function is [44]:

$$\mathbb{E}\left\{\left|\mathbf{g}\left(q\right) - \mathbb{E}\left\{\mathbf{g}_{sc}\left(q\right)\right\}\right|\right\} \leq CN^{-1/2}. \tag{3.23}$$

Nevertheless, the construction of a quaternionic Green's function and R-transform is a relatively new concept, requiring further work. Many objects are not well defined or studied in detail to date, therefore almost surely there is room for improvement on this front.

4 Singularities of the R-transform

The Green's function, as defined in (1.7), is holomorphic on $\mathbb{C}/\mathrm{supp}\,(\rho)$. Its non-analyticity near the eigenvalue spectrum is exactly what allows us to relate it to the eigenvalue density. In the most extreme case of $\mathrm{supp}\,(\rho) = \mathbb{R}$ one has two equivalent disjoint domains of definition for Green's function: $\mathbb{C}^+$ and $\mathbb{C}^-$, denoting upper and lower complex half-plane respectively. Those regions are equivalent because of the symmetry of the Stieltjes transform of a real-valued function $\rho(\lambda)$ w.r.t. complex conjugation $\overline{g(z)} = g(\bar{z})$. Therefore without loss of generality, we can treat $g(z)$ as a function on $\mathbb{C}^+$ or in fact, as a map from upper to lower half of the complex plane

$$\mathbb{C}^+ \ni z \mapsto g(z) \in \mathbb{C}^- \ . \tag{4.1}$$

In our approach to local eigenvalue statistics, we expect two models to belong to the same universality class if their $\Gamma(Q)$ functions, as defined in (2.14), have the same analytic structure. Most notably, we expect the Sine kernel universality to hold if $\Gamma(Q)$ is holomorphic. Therefore we need to have good control over the analyticity of the R-transform.

Invertibility of the Green's function was already considered in the case of a compactly supported probability measures in one of the first works introducing the free probability theory [7] and the results were further extended to non-compact cases in [45, 46]. It was shown, that

$$g(z) = \frac{1}{z}\left(1 + o(1)\right) \tag{4.2}$$

as $|z| \to \infty$ with $|\arg(z)| < \pi/2 - \theta$ for some $\theta > 0$. This result implies, that Green's function is invertible in some cone-shaped neighborhood of infinity. Formally, denoting:

$$\Gamma_{\theta,\beta} = \left\{ z \in \mathbb{C}^+ : \arg(z) \in (\theta, \pi - \theta)\,; |z| > \beta \right\} \ , \tag{4.3}$$

$$D_{\theta,\beta} = \left\{ z \in \mathbb{C}^- : \arg(z) \in (-\pi + \theta, -\theta)\,; |z| < \beta \right\} \ , \tag{4.4}$$

$$F(z) = 1/g(z) = z\left(1 + o(1)\right) \ , \tag{4.5}$$

it was shown that for any probability measure and $\alpha \in \mathbb{R}$ there exists $\beta > 0$, such that $F(z)$ is invertible in the truncated cone $\Gamma_{\alpha,\beta}$. Additionally $F\left(\Gamma_{\alpha,\beta}\right) \supset \Gamma_{\alpha-\epsilon,\beta(1+\epsilon)}$ for any $0 < \epsilon < \alpha$. As a result, the inverse of Green's function, and by extension the R-transform, is properly defined

in the circular sectors $D_{\alpha,\beta}$. The addition law in eq. (1.14) is true in the region of the complex plane where both of summed R-transforms are properly defined, and the resulting R-transform for the sum is well behaved in the intersection of two regions.

Those considerations provide us with some grasp of the possible non-analyticity of the R-transform, but results are rather qualitative. In next sections, we will provide a more quantitative approach to this problem in the case of eigenvalue density $\rho(\lambda)$ with a compact, but possibly disjoint, support as well as consider the birth of singularities during a continuous deformation of the potential function for invariant ensembles.

4.1 Densities with a compact support

Very often in $N \to \infty$ limit, with appropriate scaling, the resulting average spectrum of a random matrix ensemble has a compact support. Examples include Gaussian ensembles, other invariant ensembles with polynomial potential $NV(H)$ or Wishart ensemble. In this section we will put a restriction on positions of singularities of the R-transform for matrix models with eigenvalue spectra supported on a finite number of intervals on the real axis.

Let us start by recalling the inverse function theorem for holomorphic functions. If

$$g'(z_0) := \frac{\partial}{\partial z} g(z)\Big|_{z=z_0} \neq 0 \ , \tag{4.6}$$

then g is invertible in the neighborhood of z_0. We will explicitly look for constraints that eq. (4.6) provides without further restrictions on random matrix model.

Conversely, we can ask the question, what are the solutions of the equation

$$g'(z) = -\int_{-\infty}^{\infty} \frac{\rho(\lambda)}{(z-\lambda)^2} d\lambda = 0 \ . \tag{4.7}$$

In fact, one can express it not as one, but a set of two linearly independent real equations for real and imaginary parts of $g'(z)$ separately. In other words, writing $z = x + iy$, singularities can appear if for all $\alpha \in \mathbb{R}$

$$\mathrm{Re}g'(z) + \alpha \mathrm{Im}g'(z) = -\int_{-\infty}^{\infty} \frac{(x-\lambda)^2 - y^2 + 2\alpha(x-\lambda)y}{|x+iy-\lambda|^4} \rho(\lambda)\, d\lambda = 0 \ . \tag{4.8}$$

After a $\lambda \to -\lambda + x$ change of variables, denoting $f_{y,\alpha}(\lambda) = \frac{\lambda^2 - 2\alpha\lambda y - y^2}{\left(\lambda^2+y^2\right)^2}$ we can write our requirement in a concise way:

$$\forall_{\alpha\in\mathbb{R}} \int_{-\infty}^{\infty} f_{y,\alpha}(\lambda)\, \rho(x-\lambda)\, d\lambda = 0 \ . \tag{4.9}$$

The shape of function $f_{y,\alpha}$ for some values of y and α is presented in fig. 4.1. It has zeroes at $\lambda_\pm = y\left(\alpha \pm \sqrt{1+\alpha^2}\right)$ and is negative in between those zeroes, therefore if the support of $\rho(x-\lambda)$ lies in the region where $f_{y,\alpha}$ is negative, the integral (4.9) cannot be equal to zero. If the eigenvalue

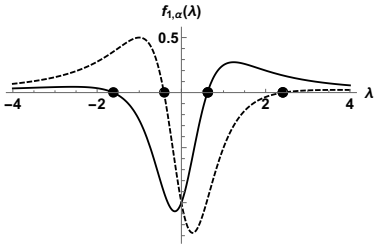

Figure 4.1: Two examples presenting shape of function $f_{y,\alpha}(\lambda)$. Solid line corresponds to parameters $y = 1, \alpha = -1/2$ and dashed line is drawn for $y = 1, \alpha = 1$. Zeroes of $f_{y,\alpha}$ are marked by dots on the real axis, the function is negative in between those points.

distribution is positive only on some interval (a, b), then we have the allowed region for zeroes of $g'(z)$ given by:

$$\forall_{\alpha\in\mathbb{R}} \left(x + a < y\left(\alpha - \sqrt{1+\alpha^2}\right) \vee x + b > y\left(\alpha + \sqrt{1+\alpha^2}\right)\right) . \quad (4.10)$$

Boundary of this region is given by saturation of the inequality, which solved for x and y gives the circle equation around the center of the interval (a, b) with a radius being half of its width

$$\left(x - \frac{a+b}{2}\right)^2 + y^2 = \left(\frac{b-a}{2}\right)^2 . \quad (4.11)$$

Lastly, if the average eigenvalue distribution is supported on two disjoint intervals, say $(a, a') \cup (b', b)$, with $a' < b'$, the same analysis as before applies to the hole (a', b') in the spectrum. Only difference being, that if the hole lies in between zeroes of $f_{y,\alpha}$ for some α, the resulting region is forbidden for zeroes of $g'(z)$.

To summarize, in the case of the eigenvalue density supported on multiple intervals, the domain in which Green's function may not be invertible, is given by a circle in the complex plane around the whole eigenvalue density, while each hole in the spectrum excludes a smaller circle from this domain. An example of such configuration is given in fig. 4.2.

4.2 Birth of singular values

It is easy to construct examples of eigenvalue densities, that make the Green's function non-invertible arbitrarily close to the boundaries of the region described in the previous section. One can do this by considering random variable given by two-point distributions and small deformations of thereof. To proceed further with our analysis we have to assume something more than just the compact support of the spectrum.

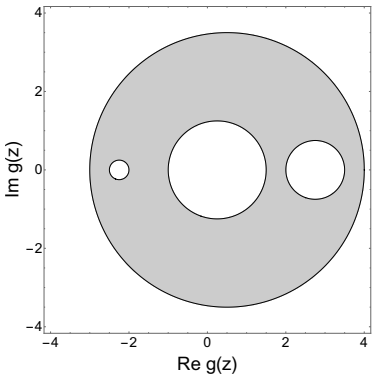

Figure 4.2: Example of a region, where singularities of R-transform may appear for eigenvalue support consisting of a few disjoint intervals. Particular example shows this domain in case of 4 disjoint intervals $\mathrm{supp}\,(\rho) = (-3.0, -2.5) \cup (-2.0, -1.0) \cup (1.5, 2.0) \cup (3.5, 4.0)$.

Firstly we consider $\rho\,(\lambda) \in \mathcal{C}^1$ and integrate by parts in (4.9) with $\alpha = 0$. The resulting condition

$$\int_{-\infty}^{\infty} \frac{y}{\lambda^2 + y^2} \rho'\,(x - \lambda)\, d\lambda = 0\ , \tag{4.12}$$

simplifies significantly in the vicinity of the real axis, i.e. in the limit $y \to 0$, as the first factor under the integral converges to the Dirac delta at $\lambda = 0$. This is where we expect the non-analytic structure to appear if the probability measure is continuously deformed from the case without any singularities. As a result, we are simply left with a requirement that the eigenvalue density has a critical point at x

$$\rho'\,(x) = 0. \tag{4.13}$$

Let us now further specify to the description of invariant random matrix ensembles. It is known [47, 48], that in the case of a probability measure of the type

$$d\mu\,(H) \propto e^{-N \mathrm{Tr} V(H)} dH\ , \tag{4.14}$$

on the space of Hermitian $N \times N$ matrices, where $V\,(H)$ is a polynomial of degree $d \geq 2$ and dH is a flat measure, the eigenvalue density is supported on a finite number of intervals. Without a loss of generality, we assume that $V'\,(x)$ is a monic polynomial. A way of computing arbitrary partition functions for this particular type of matrices was derived in [49] via the method of topological expansion, but for our purposes, we only need the first one, the Green's function. It was shown in multiple ways that $g\,(x)$ is algebraic, i.e.

$$g\,(x) = \frac{1}{N} \mathbb{E}\left\{ \mathrm{Tr}\,(x - H)^{-1} \right\} = \frac{1}{2}\left(V'\,(x) - M\,(x) \sqrt{\sigma\,(x)} \right)\ , \tag{4.15}$$

where M is a monic polynomial and if we denote by $a_1, \ldots, a_{2s}$ boundary points of the s intervals forming the support of eigenvalue distribution, we have

$$\sigma(x) = \prod_{i=1}^{2s} (x - a_i) \ . \tag{4.16}$$

From previous considerations we know, that singularities may appear only near the eigenvalue support, where $\sigma(x)$ is purely imaginary. This time looking at the real part of the eq. (4.7) we have

$$\lim_{y \to 0} \mathrm{Re}\left(g'(z = x + iy)\right) = \frac{1}{2} V''(x) = 0 \ . \tag{4.17}$$

as a second independent condition complementing the eq. (4.13).

To summarize, the necessary condition for non-invertibility of Green's function of invariant random matrix ensemble with potential V in the neighborhood of a point x on the real axis is

$$V''(x) = \rho'(x) = 0. \tag{4.18}$$

4.3 Example of singularities evolution

Let us consider a simple example illustrating a possible behavior of singularities of R-transform. We take

$$\rho(\lambda) = \alpha\delta(\lambda) + \frac{(1-\alpha)}{2} \left(\delta(\lambda - 1) + \delta(\lambda + 1)\right) \ , \tag{4.19}$$

as an average eigenvalue distribution of some invariant random matrix ensemble in the limit of the potential function being symmetric and having 3 very deep minima at $\lambda = -1, 0, 1$. Parameter α controls relative depth of potential wells. In this case, we have

$$g'(z) = -\frac{\alpha}{z^2} - \frac{1-\alpha}{2} \left(\frac{1}{(z-1)^2} + \frac{1}{(z+1)^2} \right) = -\frac{z^4 + z^2(1 - 3\alpha) + \alpha}{z^2(-1 + z^2)^2} \ , \tag{4.20}$$

for which 4 distinct zeroes can be easily computed. A plot presenting the evolution of singularities is presented in fig. 4.3 with the full allowed region.

This example behaves according to predictions from previous sections, singularities do not appear beyond the region calculated for a general density supported on 3 points $\mathrm{supp}(\rho) = \{-1, 0, 1\}$.

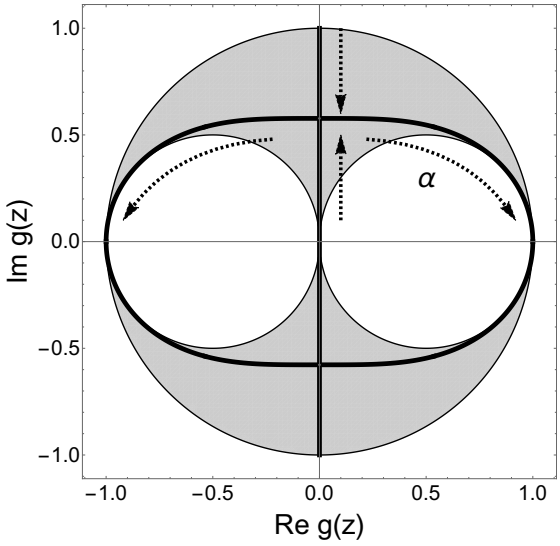

Figure 4.3: An example of the behaviour of singularities of R-transform for 3-point eigenvalue density (4.19). There are a total of 4 singularities in this model, starting for α close to zero at $z = 0, -i, i$ and evolving along the imaginary axis as α increases. After the points meet, they evolve in direction of $z = -1, 1$ symmetrically. This evolution is indicated by arrows. Shaded region indicates possible location of singularities for most general eigenvalue density supported on 3 points $-1, 0, 1$.

5 Summary and outlook

We have generalized the concept of the Laplace transform to the space of supermatrices with the transformed function being a ratio of characteristic polynomials. The existence of the transform and its inverse were proven and exact formulas were given in the cases of partition functions of Hermitian, real symmetric and quaternion self-dual ensembles. In the limit of large matrix size ($N \to \infty$) the transform is often governed by the free probabilistic R-transform. In this way, we have extended the results of [30] and [31] by considering a full supersymmetric extension of angular integrals. Our analysis is general in the sense that it does not rely on a particular form of a probability measure. It applies to both classical cases of Wigner and invariant random matrix ensembles, as well as other matrix models. One way of obtaining the R-transform equation is the saddle point method for integral of an approximation of a partition function. We provide a simple condition under which this approximation becomes exact in the $N \to \infty$ limit, while keeping track of the order of magnitude of the error term. Lastly, we qualitatively analyze the analyticity of the R-transform in more concrete examples of eigenvalue densities supported on multiple intervals and show, that the region of non-invertibility of Green's function is restricted to a domain not far from the support of eigenvalue density.

Further work should include using our method to calculate the correlation functions for some particular physical models. One can further refine applicability region of approximation provided in chapter 3, by considering certain classes of random matrix ensembles. In particular, it may be possible to, first of all, recreate universality results in the realms of Wigner or invariant random matrices. Having done that, many results will follow from the additivity of R-transforms or the multiplication law for free random matrix ensembles. In this way, one can extend the boundaries of known universality classes.

Another direction of research may include analysis of circular ensembles, that is unitary, orthogonal or symplectic random matrices. It has been shown in many cases that correlation functions have a very similar form to those for Hermitian, real symmetric or quaternion self-dual matrices respectively. The significant difference is, that analytic structure of partition functions in the boson-boson sector is very different in those cases. The singularities lie on a unit circle, therefore a good contour of integration cannot simply involve the real axis and a proper regularization method must be

employed.

Lastly, the extension of the free probability theory to the realm of non-Hermitian matrices (i.e. with complex eigenvalues) was preliminary developed in recent years [50, 9, 51]. Instead of being a complex function, the R-transform in this cases is considered as a 2×2 matrix valued function on the space of 2×2 matrices, or equivalently as a quaternionic map. It is a very recent topic of research, but it seems like both our and quaternionic formalisms may be combined in order to give insight into correlations of eigenvalues of non-Hermitian random matrices.

A Appendix

A.1 Shifting poles into different parts of the complex plane

In section 2.3.2 we discussed the construction of the Laplace transform of a product of inverse determinants. To avoid contour integration going through the singularities, one has to add a small imaginary number to the argument of the transformed function. One can do that in two different ways, by shifting poles into upper or lower complex plane, for each determinant in the product. We already discussed the simplest case of adding $i\eta$ with $\eta > 0$ to each variable p_j in eq. (2.46). We will now present the detailed derivation of Laplace transform formulas in the case of arbitrary shifts of singularities, by considering the (non-radial) function:

$$F(P) = \lim_{\eta \to 0^+} \prod_{j=0}^{n-n_0} \mathrm{Det}^{-1}\left((p_j + i\eta)\,\mathbf{1} - H \right) \tag{A.1a}$$

$$\times \prod_{j=n-n_0+1}^{n} \mathrm{Det}^{-1}\left((p_j - i\eta)\,\mathbf{1} - H \right) \ . \tag{A.1b}$$

As $F(P)$ does depend on first n_0 of p_j's in a different way than on the last $(n - n_0)$ ones, we cannot simply perform a variable change in (2.37). To circumvent this problem, let us start with observation, that we can still express the Fourier transform as:

$$\hat{F}(Q) = \int_{\mathbb{R}^n} \prod_{j=1}^{n} dp_j \frac{\Delta(\lambda(P))}{\Delta(\lambda(Q))} \prod_{k}^{n-1} k! i^{(n^2-n)/2} \tag{A.2a}$$

$$\times \sum_{\sigma} \mathrm{sgn}(\sigma) \exp\left(-i \sum_{l} p_l q_{\sigma(l)} \right) F(\Lambda_P) \ . \tag{A.2b}$$

We can interchange the sum over permutations and integral and close contours in upper/lower complex half-plane depending on signs of $q_{\sigma(l)}$ in each term of the sum and each integral respectively. By the same argument as in section 2.3.2, the transform will be equal to zero if Q does not have a particular signature. For the Fourier transform not to vanish, Q has to have

exactly n_0 positive and $(n-n_0)$ negative eigenvalues. Due to the sum over σ, the ordering of q_j's is not important. For simplicity take the case of $q_j > 0$ for $j \leq n_0$ (small indices) and $q_j < 0$ for $j > n_0$ (large indices). We can still freely permute variables p_j in each of the subsets $j \leq n_0$ and $j > n_0$, but if a permutation σ mixes a small index with a large one the resulting integral will vanish. As a result, we have the analog of Eq. (2.48):

$$\hat{F}(Q) = \begin{cases} n_0!\,(n-n_0)! \oint \prod_j dp_j \dfrac{\Delta(\lambda(P))}{\Delta(\lambda(Q))} \left(\prod_{k=1}^{n-1} k!\right) & \text{if } \forall_{j\leq n_0} q_j > 0 \\ \times\, i^{(n^2-n)/2} \exp\left(-i \sum_l p_l q_l\right) F(\Lambda_P) & \quad \forall_{j>n_0} q_j < 0 \\ 0 & \text{otherwise} \end{cases} . \tag{A.3}$$

Finally, in the same manner as in section 2.3.2, we can reverse contours, change variables et cetera to arrive at an integral over the unitary group, with an additional factor $\frac{n!}{n_0!(n-n_0)!}$. This factor is absorbed during the calculation of inverse Laplace transform. Analogous to the reduction in the integration region in eq. (2.51) the inverse transform is now performed not over $H_+(n)$ but over the space of matrices with $(n_0, n-n_0)$ signature. We perform analytic continuations for different eigenvalues, from $\mathbb{R}_-$ to $\mathbb{R}_+$ when needed, exactly like in section 2.3.1. Because of different possible orderings of positive and negative eigenvalues one arrives at $\frac{n!}{n_0!(n-n_0)!}$ copies of $H_+(n)$. The resulting Laplace transform formula is the same as in the case of shifting all singularities into lower complex half-plane.

A.2 Non-unitary symmetry classes

In this appendix, we will expand on the topic of Laplace transform in cases of matrix ensembles with orthogonal or symplectic symmetries. The computation follows essentially the same steps and formulas resemble ones seen in the case of the unitary symmetry. We will point out all the adjustments needed in order to apply our reasoning to transform over real symmetric or quaternion self-dual matrices. We keep the same notation as in section 2.5

Firstly, after performing eigenvalue reduction in eq. (2.36), we obtain

$$\hat{F}_{\epsilon}(Q) = \int_{\mathbb{R}^m} \prod_{j=1}^{m} dp_j \Delta(\lambda(P))^{2\beta} e^{-\epsilon \mathrm{Tr}|\Lambda_P|} \tag{A.4a}$$

$$\times \frac{\mathrm{Det}\left(\exp\left(-ip_i q_j\right)\right)_{1\leq i,j\leq n}}{\Delta(\lambda(-iP))^{2\beta}\Delta(\lambda(Q_\sigma))^{2\beta}} \hat{\mathcal{I}}_{\beta,N}(-iP, Q_\sigma) F(\Lambda_P) \tag{A.4b}$$

$$= \int_{\mathbb{R}^m} \prod_{j=1}^{m} dp_j e^{-\epsilon \mathrm{Tr}|\Lambda_P|} \tag{A.4c}$$

$$\times \sum_{\sigma} \frac{\mathrm{Det}\left(\exp\left(-ip_i q_j\right)\right)_{1\leq i,j\leq n}}{\Delta(\lambda(-iQ_\sigma))^{2\beta}} \hat{\mathcal{I}}_{\beta,N}(-iP, Q_\sigma) F(\Lambda_P) \ . \tag{A.4d}$$

The additional factor $\hat{\mathcal{I}}_{\beta,N}(-iP, Q_\sigma)$ doesn't change the analytic structure of the integrand, therefore all changes of contour and analytic continuations of chapter 2 remain the same.

Secondly, instead of integral over the unitary group in eq. (2.45), one is left with integral over a circle for each eigenvalue and integral over $O(n)$ or $Sp(2n)$ for the similarity transformations. As a result, we get symmetric or self-dual matrices, with eigenvalues described only by the phase, which is $U(n)/O(n)$ and $U(2n)/Sp(2n)$ respectively.

The last point to address, is the value of the constant $c_{n,m}$. The prefactor for inverse Fourier transform over $n \times n$ real symmetric or quaternion self-dual matrices is $1/(2\pi)^{n_{\mathrm{dof}}}$ as defined in the end of section 2.5. The additional i factor comes first of all from $1/\Delta(\lambda(-iQ_\sigma))^{2\beta}$ in eq. (A.4c) and change of variables $z_j = -ip_j$ as in sections 2.3.1 and 2.3.2. This concludes all the differences occurring in the derivation of Laplace transform for non-unitary symmetry classes, i.e. the orthogonal and symplectic symmetries.

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