# STABILITY OF THE SOLITARY MANIFOLD OF THE SINE-GORDON EQUATION

INAUGURAL-DISSERTATION

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I dedicate this thesis to the memory of my grandparents
Nila and Wilen.

# Abstract

We consider the sine-Gordon equation in the presence of a small forcing term $F(\varepsilon, x)$:

$$\theta_{tt} - \theta_{xx} + \sin\theta = F(\varepsilon, x)\,,$$

where $t, x \in \mathbb{R}$, with $\varepsilon$ being a small parameter. The equation without the perturbation ($F(\varepsilon, x) = 0$) admits soliton solutions which define a two dimensional (classical) solitary manifold. We consider different types of forcing terms $F(\varepsilon, x)$ and establish stability results for the corresponding initial value problems with initial state close to the solitary manifold. These results are proven for the following perturbations $F(\varepsilon, x)$:

(a) $F(\varepsilon, x) = \varepsilon f(\varepsilon x)$, where $f \in H^1(\mathbb{R})$;

(b) $F(\varepsilon, x) = \varepsilon^2 f(\varepsilon x)$, where $f(x) := V'(x)$ for any $V \in H^4(\mathbb{R})$;

(c) $F : (-1, 1) \to L^2(\mathbb{R})$, $\varepsilon \mapsto F(\varepsilon, x)$, such that $F \in C^{k+1}((-1, 1), L^2(\mathbb{R}))$ and $\partial_\varepsilon^l F(0, \cdot) = 0$ for $0 \le l \le k$.

Further, we consider

$$F : (-1, 1) \to H^{1,1}(\mathbb{R}),\ \varepsilon \mapsto F(\varepsilon, x),$$

such that $F \in C^n((-1, 1), H^{1,1}(\mathbb{R}))$ and $\partial_\varepsilon^l F(0, \cdot) = 0$ for $0 \le l \le k$, where $k + 1 \le n$ and $n \ge 1$. By solving successively equations depending on $F(\varepsilon, x)$, we define implicitly a virtual solitary manifold which is adjusted on the forcing term $F(\varepsilon, x)$. This allows us to prove a stability result of higher accuracy for an initial value problem (with $F(\varepsilon, x)$) and with initial state close to the virtual solitary manifold.

The approach is based on the Lyapunov energy method, symplectic projection in Hilbert space onto virtual/classical solitary manifold, and modulation equations for the parameters of the projection.

# Zusammenfassung

Wir betrachten die sine-Gordon Gleichung mit einem kleinen Störungsterm $F(\varepsilon, x)$:

$$\theta_{tt} - \theta_{xx} + \sin\theta = F(\varepsilon, x)\,,$$

wobei $t, x \in \mathbb{R}$ und $\varepsilon$ einen kleinen Parameter darstellt. Die ungestörte sine-Gordon Gleichung ($F(\varepsilon, x) = 0$) besitzt Solitonlösungen, welche eine zweidimensionale (klassische) Solitonenmannigfaltigkeit definieren. Wir untersuchen verschiedene Störungsterme $F(\varepsilon, x)$ und beweisen Stabilitätsaussagen für die entsprechenden Anfangswertprobleme mit Anfangsdaten nahe an der Solitonenmannigfaltigkeit. Die Resultate werden für folgende Störungsterme $F(\varepsilon, x)$ formuliert:

(a) $F(\varepsilon, x) = \varepsilon f(\varepsilon x)$ mit $f \in H^1(\mathbb{R})$;

(b) $F(\varepsilon, x) = \varepsilon^2 f(\varepsilon x)$ mit $f(x) := V'(x)$, wobei $V \in H^4(\mathbb{R})$;

(c) $F : (-1, 1) \to L^2(\mathbb{R})$, $\varepsilon \mapsto F(\varepsilon, x)$ derart, dass $F \in C^{k+1}((-1, 1), L^2(\mathbb{R}))$ und $\partial_\varepsilon^l F(0, \cdot) = 0$ für $0 \leq l \leq k$.

Desweiteren betrachten wir

$$F : (-1, 1) \to H^{1,1}(\mathbb{R}),\ \varepsilon \mapsto F(\varepsilon, x)$$

derart, dass $F \in C^n((-1, 1), H^{1,1}(\mathbb{R}))$ und $\partial_\varepsilon^l F(0, \cdot) = 0$ für $0 \leq l \leq k$ für $k + 1 \leq n$ und $n \geq 1$. Durch sukzessives Lösen von Gleichungen, die von $F(\varepsilon, x)$ abhängen, definieren wir implizit eine virtuelle Solitonenmannigfaltigkeit, die an die Störung $F(\varepsilon, x)$ angepasst ist. Wir beweisen eine Stabilitätsaussage für das Anfangswertproblem (mit der Störung $F(\varepsilon, x)$) und mit Anfangsdaten nahe an der virtuellen Solitonenmannigfaltigkeit. Dieses Resultat liefert eine höhere Genauigkeit für die Beschreibung der Lösung. Die Beweise der Stabilitätsaussagen beruhen auf der Lyapunov-Energie-Methode, symplektischen Projektionen in Hilbert Räumen auf die virtuelle bzw. klassische Solitonenmannigfaltigkeit und Modulationsgleichungen für die Parameter der Projektionen.

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# Introduction

## Sine-Gordon Equation and Hamiltonian Structure

The sine-Gordon equation in the presence of a forcing term $F(\varepsilon, x)$ is

$$\theta_{tt} - \theta_{xx} + \sin\theta = F(\varepsilon, x), \tag{1}$$

where $t, x \in \mathbb{R}$ and $\varepsilon$ is a small real parameter such that $F(\varepsilon, \cdot) \in L^2(\mathbb{R})$ for every $\varepsilon$. The equation (1) is a Hamiltonian evolution equation with Hamiltonian given by

$$H^\varepsilon(\theta, \psi) = \frac{1}{2}\int \psi^2 + \theta_x^2 + 2(1-\cos\theta) - 2F(\varepsilon, x)\theta \, dx\,.$$

In first order formulation (1) can be written as a system:

$$\partial_t \begin{pmatrix} \theta \\ \psi \end{pmatrix} = \begin{pmatrix} \psi \\ \theta_{xx} - \sin\theta + F(\varepsilon, x) \end{pmatrix}.$$

We define a symplectic form on appropriate spaces to be specified later by

$$\Omega\left(\begin{pmatrix} \theta' \\ \psi' \end{pmatrix}, \begin{pmatrix} \theta \\ \psi \end{pmatrix}\right) = \left\langle \begin{pmatrix} \theta' \\ \psi' \end{pmatrix}, \mathbb{J}\begin{pmatrix} \theta \\ \psi \end{pmatrix}\right\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} = \int_{\mathbb{R}} \psi'(x)\theta(x) - \theta'(x)\psi(x)\, dx, \tag{2}$$

where

$$\mathbb{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

For the vector field

$$X_{H^\varepsilon}\begin{pmatrix} \theta \\ \psi \end{pmatrix} = \begin{pmatrix} \psi \\ \theta_{xx} - \sin\theta + F(\varepsilon, x) \end{pmatrix}$$

it holds that

$$\Omega\left(X_{H^\varepsilon}\begin{pmatrix} \theta \\ \psi \end{pmatrix}, \begin{pmatrix} \theta' \\ \psi' \end{pmatrix}\right) = -d_\theta H^\varepsilon(\theta, \psi)\theta' - d_\psi H^\varepsilon(\theta, \psi)\psi' = -dH^\varepsilon(\theta, \psi)\begin{pmatrix} \theta' \\ \psi' \end{pmatrix},$$

or in shorthand notation

$$\Omega(X_{H^\varepsilon}, \cdot) = -dH^\varepsilon\,.$$

# Soliton Solutions ($\boldsymbol{F(\varepsilon, x) = 0}$)

Let $u \in (-1,1)$, $\xi \in \mathbb{R}$ and

$$\gamma(u) = \frac{1}{\sqrt{1-u^2}}.$$

We introduce the function

$$\theta_K(x) = 4\arctan(e^x),$$

and call $\theta_K$ the kink. It holds that $\theta_K'(x) = 2\operatorname{sech}(x)$. The following general relations

$$2\operatorname{sech}^2(x)\sinh(x) + \sin(4\arctan(e^x)) = 0\,,$$

$$-2\operatorname{sech}^2(x) + 1 - \cos(4\arctan(e^x)) = 0\,,$$

imply that (see [Kun12])

$$\theta_K''(x) = \sin\theta_K(x), \qquad 1 - \cos\theta_K(x) = \frac{1}{2}(\theta_K'(x))^2\,.$$

We set

$$\begin{pmatrix} \theta_0(\xi,u,x) \\ \psi_0(\xi,u,x) \end{pmatrix} := \begin{pmatrix} \theta_K(\gamma(u)(x-\xi)) \\ -u\gamma(u)\theta_K'(\gamma(u)(x-\xi)) \end{pmatrix}\,, \; u \in (-1,1), \quad \xi, x \in \mathbb{R}\,. \tag{3}$$

The sine-Gordon equation with no perturbation ($F(\varepsilon, x) = 0$),

$$\partial_t \begin{pmatrix} \theta \\ \psi \end{pmatrix} = \begin{pmatrix} \psi \\ \theta_{xx} - \sin\theta \end{pmatrix}, \tag{4}$$

admits solutions of the form:

$$\begin{pmatrix} \theta_0(\xi(t),u(t),x) \\ \psi_0(\xi(t),u(t),x) \end{pmatrix}, \tag{5}$$

as long as the differential equations

$$\dot{\xi} = u\,,$$

$$\dot{u} = 0\,,$$

are satisfied, where we assume that $\xi(0) = a,\; u(0) = v$ for $(a, v) \in \mathbb{R} \times (-1,1)$. Namely, for $\gamma = \gamma(u)$ the chain rule gives

$$\begin{aligned}
&\partial_t \begin{pmatrix} \theta_0(\xi,u,x) \\ \psi_0(\xi,u,x) \end{pmatrix} \\
&= \partial_t \begin{pmatrix} \theta_K(\gamma(x-\xi)) \\ -u\gamma\theta_K'(\gamma(x-\xi)) \end{pmatrix} \\
&= \begin{pmatrix} \Big[\dot\gamma(x-\xi) - \gamma\dot\xi\Big]\theta_K'(\gamma(x-\xi)) \\ \\ -\dot u\gamma\theta_K'(\gamma(x-\xi)) - u\dot\gamma\theta_K'(\gamma(x-\xi)) - u\gamma\Big[\dot\gamma(x-\xi) - \gamma\dot\xi\Big]\theta_K''(\gamma(x-\xi)) \end{pmatrix} \\
&= -u\partial_x \begin{pmatrix} \theta_0(\xi,u,x) \\ \psi_0(\xi,u,x) \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
&\begin{pmatrix} \psi_0(\xi,u,x) \\ \partial_x^2\theta_0(\xi,u,x) - \sin\theta_0(\xi,u,x) \end{pmatrix} \\
&= \begin{pmatrix} -u\gamma\theta_K'(\gamma(x-\xi)) \\ \gamma^2\theta_K''(\gamma(x-\xi)) - \sin(\theta_K(\gamma(x-\xi))) \end{pmatrix} \\
&= \begin{pmatrix} -u\gamma\theta_K'(\gamma(x-\xi)) \\ \gamma^2\theta_K''(\gamma(x-\xi)) - \theta_K''(\gamma(x-\xi)) \end{pmatrix} \\
&= \begin{pmatrix} -u\gamma\theta_K'(\gamma(x-\xi)) \\ u^2\gamma^2\theta_K''(\gamma(x-\xi)) \end{pmatrix} \\
&= -u\partial_x \begin{pmatrix} \theta_0(\xi,u,x) \\ \psi_0(\xi,u,x) \end{pmatrix} .
\end{aligned}$$

These solutions are solitons and we obtain

$$-u\partial_x \begin{pmatrix} \theta_0(\xi,u,x) \\ \psi_0(\xi,u,x) \end{pmatrix} - \begin{pmatrix} \psi_0(\xi,u,x) \\ \partial_x^2\theta_0(\xi,u,x) - \sin\theta_0(\xi,u,x) \end{pmatrix} = 0 \tag{6}$$

as the equation characterizing the (classical) solitons. Notice that the energy

$$H(\theta,\psi) = \frac{1}{2}\int \psi^2 + \theta_x^2 + 2(1-\cos\theta)\,dx \tag{7}$$

and the momentum

$$\Pi(\theta,\psi) = \int \psi\theta_x\,dx \tag{8}$$

are conserved quantities of the sine-Gordon equation (4).

# Orbital Stability

For the unperturbed initial value problem

$$\partial_t \begin{pmatrix} \theta(t,x) \\ \psi(t,x) \end{pmatrix} = \begin{pmatrix} \psi(t,x) \\ \partial_x^2\theta(t,x) - \sin\theta(t,x) \end{pmatrix},$$

$$\begin{pmatrix} \theta(0,x) \\ \psi(0,x) \end{pmatrix} = \begin{pmatrix} \theta_0(\xi_0,u_0,x) \\ \psi_0(\xi_0,u_0,x) \end{pmatrix} + \begin{pmatrix} v(0,x) \\ w(0,x) \end{pmatrix},$$

where $(v(0,\cdot),w(0,\cdot)) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$, the following stability result was proven in [Stu12, Section 4]: If $|v(0)|_{H^1(\mathbb{R})} + |w(0)|_{L^2(\mathbb{R})} = \varepsilon$ is sufficiently small then the Cauchy problem has a unique solution which may be written in the form

$$\begin{pmatrix} \theta(t,x) \\ \psi(t,x) \end{pmatrix} = \begin{pmatrix} \theta_0(\xi(t),u(t),x) \\ \psi_0(\xi(t),u(t),x) \end{pmatrix} + \begin{pmatrix} v(t,x) \\ w(t,x) \end{pmatrix},$$

where $v, w, u, \xi$ have regularity

$$(\xi(t),u(t)) \in C^1(\mathbb{R}, \mathbb{R}\times(-1,1)),$$
$$(v(t),w(t)) \in C(\mathbb{R}, H^1(\mathbb{R})\times L^2(\mathbb{R})),$$

and satisfy

$$\sup_{t\in\mathbb{R}} \left( |u(t)-u_0| + |v(t)|_{H^1(\mathbb{R})} + |w(t)|_{L^2(\mathbb{R})} \right) \le c\varepsilon$$

for some $c > 0$.

# Perturbation Theory for Kinks

The following result was proven in [Stu92]: Let the perturbation $g = g(\theta)$ be a smooth function such that $g_0(Z) = g(\theta_K(Z)) \in L^2(dZ)$. Then for sufficiently small $\varepsilon$ there exists $T_* = \mathcal{O}\left(\frac{1}{\varepsilon}\right)$ such that for $T \le T_*$ there is a unique solution to the initial value problem:

$$\theta_{TT} - \theta_{XX} + \sin\theta + \varepsilon g = 0,$$

$$\theta(0,X) = \theta_K(Z(0)) + \varepsilon\tilde{\theta}(0,X),$$

$$\theta_T(0,X) = \frac{-u(0)}{\sqrt{1-u(0)^2}}\theta_K'(Z(0)) + \varepsilon\tilde{\theta}_T(0,X),$$

where $(\tilde{\theta}(0,X), \tilde{\theta}_T(0,X)) \in H^1 \oplus L^2$, of the form

$$\theta(T,X) = \theta_K(Z) + \varepsilon\tilde{\theta}(T,X), \quad Z = \frac{X - \int^T u - C(T)}{\sqrt{1-u^2}},$$

where $\tilde{\theta} \in C([0,T_*], H^1),\ \theta_T \in C([0,T_*], L^2)$ and

$$\begin{aligned} &C(T) = C_0(\varepsilon T) + \varepsilon \tilde{C}, \\ &u(T) = u_0(\varepsilon T) + \varepsilon \tilde{u}(T) \left( \Rightarrow\ p = \frac{u}{\sqrt{1-u^2}} = p_0(\varepsilon T) + \varepsilon \tilde{p}(T) \right) \end{aligned}$$

with $\tilde{p},\ \tilde{u}, \tilde{C}, \frac{d\tilde{u}}{dT}, \frac{d\tilde{C}}{dT}, |\tilde{\theta}|_{H^1(\mathbb{R})}$ bounded independent of $\varepsilon$, and $u_0, C_0$ the solutions of certain modulation equations. This theorem is also valid for perturbations of the form

$$g = g(\varepsilon T, \varepsilon X, \theta),$$

if among others the following assumption is satisfied: Let $Z$ be as above. There exists a time interval $[0, T_+] = \left[0, \frac{t_+}{\varepsilon}\right]$, where $t_+$ is independent of $\varepsilon$ such that for all $T \in [0, T_+]$ :

$$\left( \int g(\varepsilon T, \varepsilon X, \theta_K(Z))^2 \, dZ \right)^{\frac{1}{2}} \leq A, \quad dZ = \gamma dX, \quad \gamma = 1/\sqrt{1-u^2}, \tag{9}$$

where $A$ is independent of $\varepsilon$ (see [Stu92, p. 442]). The proof is based on an orthogonal decomposition of the solution into an oscillatory part and a one-dimensional "zero-mode" term.

# Main Results

For $(a,v) \in \mathbb{R} \times (-1,1)$ the states $\begin{pmatrix} \theta_0(a,v,\cdot) \\ \psi_0(a,v,\cdot) \end{pmatrix}$ form the (classical) solitary manifold

$$\mathcal{S}_0 := \left\{ \begin{pmatrix} \theta_0(a,v,\cdot) \\ \psi_0(a,v,\cdot) \end{pmatrix} \ :\ v \in (-1,1),\ a \in \mathbb{R} \right\}.$$

In Part I - III we consider for different perturbations $F(\varepsilon, x)$ initial value problems of type

$$\begin{aligned} \partial_t \begin{pmatrix} \theta(t,x) \\ \psi(t,x) \end{pmatrix} &= \begin{pmatrix} \psi(t,x) \\ \partial_x^2 \theta(t,x) - \sin\theta(t,x) + F(\varepsilon,x) \end{pmatrix}, \\ \begin{pmatrix} \theta(0,x) \\ \psi(0,x) \end{pmatrix} &= \begin{pmatrix} \theta_0(\xi_s, u_s, x) \\ \psi_0(\xi_s, u_s, x) \end{pmatrix} + \begin{pmatrix} v(0,x) \\ w(0,x) \end{pmatrix}, \end{aligned}$$

such that $(v(0,\cdot),w(0,\cdot)) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ and $(v(0,\cdot),w(0,\cdot))$ is symplectic orthogonal to the tangent space of $\mathcal{S}_0$ at the point $(\theta_0(\xi_s,u_s,\cdot),\psi_0(\xi_s,u_s,\cdot))$, where $(\xi_s,u_s) \in \mathbb{R} \times (-1,1)$. We consider the following perturbations $F(\varepsilon,x)$:

| | |
|---|---|
| Part I: | $F(\varepsilon,x) = \varepsilon f(\varepsilon x)$, where $f \in H^1(\mathbb{R})$. |
| Part II: | $F(\varepsilon,x) = \varepsilon^2 f(\varepsilon x)$, where $f(x) := V'(x)$ for any $V \in H^4(\mathbb{R})$. |
| Part III: | $F : (-1,1) \to L^2(\mathbb{R}),\ \varepsilon \mapsto F(\varepsilon,x)$,<br>such that $F \in C^{k+1}((-1,1),L^2(\mathbb{R}))$ and $\partial_\varepsilon^l F(0,\cdot) = 0$ for $0 \le l \le k$. |

We establish results of the following type. Suppose that

$$|v(0,\cdot)|^2_{H^1(\mathbb{R})} + |w(0,\cdot)|^2_{L^2(\mathbb{R})} \le \varepsilon^\eta$$

for a sufficiently small $\varepsilon > 0$. Then the solution $(\theta,\psi)$ (whose existence will be established) of the initial value problem can be split for times

$$0 \le t \le \frac{1}{\varepsilon^\mu}$$

as the sum of two components

$$\begin{pmatrix}\theta(t,x)\\ \psi(t,x)\end{pmatrix} = \begin{pmatrix}\theta_0(\xi(t),u(t),x)\\ \psi_0(\xi(t),u(t),x)\end{pmatrix} + \begin{pmatrix}v(t,x)\\ w(t,x)\end{pmatrix}, \tag{10}$$

where $(\theta_0(\xi(t),u(t),\cdot),\psi_0(\xi(t),u(t),\cdot))$ is a point on the solitary manifold and $(v(t,\cdot),w(t,\cdot))$ is a transversal component which is symplectic orthogonal to the tangent space of $\mathcal{S}_0$ in the corresponding point. We state ordinary differential equations that are exactly or up to certain errors in $\varepsilon$ satisfied by the parameters $(\xi(t),u(t))$. We show that there exists a positive constant $c$ such that

$$|v(t,\cdot)|^2_{H^1(\mathbb{R})} + |w(t,\cdot)|^2_{L^2(\mathbb{R})} \le c\varepsilon^{\tilde{\eta}}, \tag{11}$$

for times

$$0 \le t \le \frac{1}{\varepsilon^\mu}.$$

$\eta, \mu, \tilde{\eta}$ are positive numbers, that differ in the theorems. Thus we are able to control each of the components of the solution, namely $(\theta_0(\xi(t),u(t),\cdot),\psi_0(\xi(t),u(t),\cdot))$ by the ODE's and $(v(t,\cdot),w(t,\cdot))$ by the upper bound. The ODE's established in Part I and Part III are those which describe the evolution of a soliton, whereas the ODE's established in Part II contain the potential $V$ and are obtained by considering the restricted Hamilton equations.

We assume throughout the whole thesis that

$$0 < \delta < \frac{1}{32}.$$

The following table gives an overview of the results.

| | $\varepsilon^{\eta}$ | $\frac{1}{\varepsilon^{\mu}}$ | $\varepsilon^{\tilde{\eta}}$ | ODE's satisfied up to an error of order |
|---|---|---|---|---|
| Part I, Theorem 1.2 (i) | $\varepsilon$ | $\frac{1}{\varepsilon^{\rho(\delta)}}$, $\rho(\delta) = \frac{1}{2} - 2\delta$ | $\varepsilon$ | $\varepsilon$ |
| Part I, Theorem 1.2 (ii) | $\varepsilon$ | $\frac{1}{\varepsilon^{\beta(\delta)}}$, $\beta(\delta) = \frac{1-\delta}{4}$ | $\varepsilon$ | exactly |
| Part II, Theorem 13.1 | $\varepsilon^3$ | $\frac{1}{\varepsilon^{\beta(\delta)}}$, $\beta(\delta) = 1 - \delta$ | $\varepsilon^3$ | $\varepsilon^3$ |
| Part II, Theorem 9.1 | $\varepsilon^3$ | $\frac{1}{\varepsilon^{\beta(\delta)}}$, $\beta(\delta) = 1 - \delta$ | $\varepsilon^2$ | exactly |
| Part III, Theorem 14.1 (i) | $\varepsilon^{k+1}$ | $\frac{1}{\varepsilon^{\rho(k,\delta)}}$, $\rho(k,\delta) = \frac{k+1}{2} - 2\delta$ | $\varepsilon^{k+1}$ | $\varepsilon^{k+1}$ |
| Part III, Theorem 14.1 (ii) | $\varepsilon^{k+1}$ | $\frac{1}{\varepsilon^{\rho(k,\delta)}}$, $\rho(k,\delta) = \frac{k+1-\delta}{4}$ | $\varepsilon^{k+1}$ | exactly |

We obtain analogous stability statements in Part I and Part III for the case $k = 0$. But these results are established for different classes of perturbations and are not contained in each other, since there does not exist a funcion $f \neq 0, f \in L^2(\mathbb{R})$ such that the mapping

$$(-1,1) \to L^2(\mathbb{R}), \varepsilon \mapsto \varepsilon f(\varepsilon \cdot)\,,$$

is differentiable. This fact is evident from

$$\left| \frac{\varepsilon f(\varepsilon \cdot) - 0 f(0 \cdot)}{\varepsilon} \right|_{L^2(\mathbb{R})} = |f(\varepsilon \cdot)|_{L^2(\mathbb{R})} = \frac{1}{\varepsilon^{\frac{1}{2}}} \, |f(\cdot)|_{L^2(\mathbb{R})} \,.$$

Notice that in the result [Stu92] by D. M. Stuart mentioned above, there is considered a different perturbation than in our case in Part I. Since our perturbation does not depend on time and since

$$|f(\varepsilon \cdot)|_{L^2(\mathbb{R})} = \varepsilon^{-\frac{1}{2}} \, |f(\cdot)|_{L^2(\mathbb{R})}\,,$$

the condition (9) is in our case not satisfied.

Stability of solitons has been studied for a long time and for several equations. Just to mention some examples: in [Wei86] there were proven orbital stability of ground state solitary waves of the nonlinear Schrödinger equation and stability of the solitary wave for the generalized Korteweg-de Vries equation. In [IKV12] there was established a long time soliton asymptotics for a nonlinear system of wave equation coupled to a charged particle. In [HZ08] there was considered the Gross-Pitaevskii equation

$$\begin{cases} i\partial_t u + \frac{1}{2}\partial_x^2 u - V(x)u + u|u|^2 = 0 \\ u(x,0) = e^{iv_0 x} \operatorname{sech}(x - a_0)\,, \end{cases}$$

with a slowly varying smooth potential $V(x) = W(\varepsilon x)$ where $W \in C^3(\mathbb{R}, \mathbb{R})$. It was shown that up to time $\frac{\log(1/\varepsilon)}{\varepsilon}$ and errors of size $\varepsilon^2$ in $H^1$, the solution is a soliton evolving according to the classical dynamics of a natural effective Hamiltonian. This work was our starting point.

In our approach we prove first that a symplectic decomposition described above is possible close to the solitary manifold. We establish existence of a solution $(\theta, \psi)$ with initial state close to the solitary manifold, decompose the solution as in (10) and derive modulation equations for the parameters $(\xi(t), u(t))$, that describe the position on the manifold. Next, we introduce a Lyapunov functional in order to control the transversal component $(v, w)$. Therefor we consider a linear combination of $H$ and $\Pi$ at $(\theta, \psi)$ minus the same quantities at $(\theta_0(\xi, u, \cdot), \psi_0(\xi, u, \cdot))$:

$$H(\theta, \psi) + u\Pi(\theta, \psi) - H(\theta_0(\xi, u, \cdot), \psi_0(\xi, u, \cdot)) - u\Pi(\theta_0(\xi, u, \cdot), \psi_0(\xi, u, \cdot)) \,. \tag{12}$$

In stability questions of our type (in the cases of other differential equations) it is typical to decompose the solution close to the manifold and to consider a Lyapunov functional that is given by the difference of a linear combination of conserved or almost conserved quantities evaluated at the solution minus the same quantities evaluated at a point on the manifold (see [Wei86], [HZ08], [FJL07], [JFGS06]).

The linear part in (12) vanishes due to symplectic orthogonality. Motivated by [HZ08], we choose our Lyapunov functional to be the quadratic approximation of (12) and call it $L$. We bound $L$ from below in terms of $|v(t, \cdot)|^2_{H^1(\mathbb{R})} + |w(t, \cdot)|^2_{L^2(\mathbb{R})}$ by using symplectic orthogonality. Utilizing the modulation equations for the parameters $(\xi(t), u(t))$ we are able to control $L$ and as a consequence also the transversal component $(v, w)$ from above, which yields the bound (11).

In Part IV we establish a stability statement of higher accuracy. Suppose that a perturbation

$$F : (-1, 1) \to H^{1,1}(\mathbb{R}), \ \varepsilon \mapsto F(\varepsilon, x)$$

is given such that $F \in C^n((-1, 1), H^{1,1}(\mathbb{R}))$ and $\partial^l_\varepsilon F(0, \cdot) = 0$ for $0 \le l \le k$, where $n \ge 1$, $k + 1 \le n$ and $H^{1,1}(\mathbb{R})$ denotes the weighted Sobolev space of functions with finite norm

$$|\theta|_{H^{1,1}(\mathbb{R})} = |(1 + |x|^2)^{\frac{1}{2}} \theta(x)|_{H^1_x(\mathbb{R})} \,.$$

Let $\xi_s \in \mathbb{R}$ be given. We want to study an initial value problem analogous to that in Part I-III but with different initial data in order to obtain a more accurate statement. The idea of our approach is to adjust the solitary manifold to the perturbation term $F(\varepsilon, \cdot)$, i.e., to bend the classical solitary manifold $\mathcal{S}_0$ in such a way that we obtain a statement of higher accuracy compared to that with the original solitary manifold. This is done by solving successively certain equations. (6) is the equation characterizing the (classical) solitons that can be written as $\mathcal{G}_0(\theta, \psi) = 0$, where $(\theta_0, \psi_0)$ is a solution of the equation, i.e., $\mathcal{G}_0(\theta_0, \psi_0) = 0$. We add some terms involving the perturbation $F(\varepsilon, \cdot)$ to $\mathcal{G}_0(\theta, \psi)$ and consider a new equation $\mathcal{G}^\varepsilon_1(\theta, \psi, \lambda_u) = 0$, where $\lambda_u$ is an additional unknown variable. We solve this equation implicitly for $(\theta, \psi, \lambda_u)$ in terms of $\varepsilon$ and call the solution $(\theta^\varepsilon_1, \psi^\varepsilon_1, \lambda^\varepsilon_{u,1})$. We are able to define a new virtual solitary manifold by $(\theta^\varepsilon_1, \psi^\varepsilon_1)$ and formulate a result analogous to Part I-III with initial data close to the new virtual solitary manifold. This would already give us an improvement of accuracy. Due to the assumption that $F$ is of class $C^n, n \in \mathbb{N}$, it is possible to iterate our adjustment. We add some terms involving $(\theta_1^\varepsilon, \psi_1^\varepsilon)$ to $\mathcal{G}_1^\varepsilon(\theta, \psi, \lambda_u)$ and consider a new equation $\mathcal{G}_2^\varepsilon(\theta, \psi, \lambda_u) = 0$. We solve this equation implicitly for $(\theta, \psi, \lambda_u)$ in terms of $\varepsilon$ and call the solution $(\theta_2^\varepsilon, \psi_2^\varepsilon, \lambda_{u,2}^\varepsilon)$. Defining a virtual solitary manifold by $(\theta_2^\varepsilon, \psi_2^\varepsilon)$ would yield a further improvement of accuracy in the stability statement. We iterate this procedure by adding terms involving $(\theta_j^\varepsilon, \psi_j^\varepsilon)$ to $\mathcal{G}_j^\varepsilon(\theta, \psi, \lambda_u)$ and solving the new equations implicitly until we obtain the solution $(\theta_n^\varepsilon, \psi_n^\varepsilon, \lambda_{u,n}^\varepsilon)$ of the nth equation $\mathcal{G}_n^\varepsilon(\theta, \psi, \lambda_u) = 0$.

The existence of the implicit solutions $(\theta_j^\varepsilon, \psi_j^\varepsilon, \lambda_{u,j}^\varepsilon)$ to the equations $\mathcal{G}_j^\varepsilon(\theta, \psi, \lambda) = 0$ for $1 \leq j \leq n$ is ensured by the implicit function theorem and $(\theta_j^\varepsilon, \psi_j^\varepsilon, \lambda_{u,j}^\varepsilon)$ depend smoothly (of class $C^n$) on $\varepsilon$, such that $(\theta_j^0, \psi_j^0, \lambda_{u,j}^0) = (\theta_0, \psi_0, 0)$. The maps $\mathcal{G}_j$ are defined on spaces of different regularity and satisfy $\mathcal{G}_j^0(\theta_0, \psi_0, 0) = 0$.

We define the virtual solitary manifold by

$$\mathcal{S}_n^\varepsilon := \left\{ \begin{pmatrix} \theta_n^\varepsilon(a, v, \cdot) \\ \psi_n^\varepsilon(a, v, \cdot) \end{pmatrix} \ : \ v \in (-u_*, u_*), \ a \in \mathbb{R} \right\},$$

where $u_* \in (0, 1]$. The idea of deforming the classical solitary manifold by defining functions implicitly appears in [Stu12] with the purpose of rewriting the Hamiltonian in a neighbourhood of the manifold of virtual solitons (see [Stu12, Section 3]). Our virtual solitary manifold, given by $(\theta_n^\varepsilon, \psi_n^\varepsilon)$, is defined by solving successively equations that were not considered in [Stu12]. Now we are able to formulate the main result of Part IV. We consider the initial value problem

$$\partial_t \begin{pmatrix} \theta(t,x) \\ \psi(t,x) \end{pmatrix} = \begin{pmatrix} \psi(t,x) \\ \partial_x^2 \theta(t,x) - \sin \theta(t,x) + F(\varepsilon, x) \end{pmatrix},$$

$$\begin{pmatrix} \theta(0,x) \\ \psi(0,x) \end{pmatrix} = \begin{pmatrix} \theta_n^\varepsilon(\xi_s, u_s, x) \\ \psi_n^\varepsilon(\xi_s, u_s, x) \end{pmatrix} + \begin{pmatrix} v(0,x) \\ w(0,x) \end{pmatrix}.$$

such that $(v(0,\cdot), w(0,\cdot)) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ and $(v(0,\cdot), w(0,\cdot))$ is symplectic orthogonal to the tangent space of $\mathcal{S}_n^\varepsilon$ at the point $(\theta_n^\varepsilon(\xi_s, u_s, x), \psi_n^\varepsilon(\xi_s, u_s, x))$. Suppose that

$$|v(0,\cdot)|^2_{H^1(\mathbb{R})} + |w(0,\cdot)|^2_{L^2(\mathbb{R})} \leq \varepsilon^{2n}$$

and

$$|u_s| \leq \varepsilon^{\frac{k+1}{2}} \tag{13}$$

for a sufficiently small $\varepsilon > 0$. Then the solution $(\theta, \psi)$ (whose existence will be established) of the initial value problem can be split for times

$$0 \leq t \leq \left(\frac{1}{\varepsilon}\right)^{\frac{k+1-\delta}{2}}$$

as the sum of two components

$$\begin{pmatrix} \theta(t,x) \\ \psi(t,x) \end{pmatrix} = \begin{pmatrix} \theta_n^\varepsilon(\xi(t),u(t),x) \\ \psi_n^\varepsilon(\xi(t),u(t),x) \end{pmatrix} + \begin{pmatrix} v(t,x) \\ w(t,x) \end{pmatrix},$$

where $(\theta_n^\varepsilon(\xi(t),u(t),\cdot),\psi_n^\varepsilon(\xi(t),u(t),\cdot))$ is a point on the solitary manifold and $(v(t,\cdot),w(t,\cdot))$ is a transversal component which is symplectic orthogonal to the tangent space of $\mathcal{S}_n^\varepsilon$ in the corresponding point. We state ordinary differential equations that are exactly satisfied by the parameters $(\xi(t),u(t))$ and we show that

$$|v(t,\cdot)|^2_{H^1(\mathbb{R})} + |w(t,\cdot)|^2_{L^2(\mathbb{R})} \leq c\varepsilon^{2n},$$

for times

$$0 \leq t \leq \left(\frac{1}{\varepsilon}\right)^{\frac{k+1-\delta}{2}}.$$

Thus we are able to control each of the components of the solution as in Part I-III. By varying $n$ we are able to change the accuracy of the statement whereas $\eta,\tilde{\eta}$ were fixed numbers in Part I-II and dependent on $k$ only in Part III. Each iteration step gave us an improvement of the order $\varepsilon$ of the accuracy (accuracy in measuring the norm of the transversal component $(v,w)$).

We restrict the possible initial data by the smallness assumption (13) on $u_s$ whereas there are no restrictions on the coordinates $(\xi_s,u_s)$ of the classical solitary manifold in Part III.

We abstained from considering a perturbation of type $\varepsilon f(\varepsilon x)$ in Part IV. This is because in our approach we do need the assumption that the perturbation $F(\varepsilon,\cdot)$ is differentiable with respect to $\varepsilon$, but there does not exist a function $f \neq 0, f \in L^2(\mathbb{R})$ such that the mapping $\varepsilon \mapsto \varepsilon f(\varepsilon\cdot)$ is differentiable in $L^2(\mathbb{R})$, as mentioned above.

In the broadest sense, a similar approach has been used in [HL12, Section 4, Section 5] for the NLS equation. The solitary manifold has been corrected there once, which correspond to the first iteration in our case. The existence of the correction was not concluded by the implicit function theorem as in our case.

The proof of the result in Part IV is similar to those of the results in Part I-III, whereas we decompose the solution in a point on the virtual solitary manifold $\mathcal{S}_n^\varepsilon$ and a transversal component which is symplectic orthogonal to the tangent space of $\mathcal{S}_n^\varepsilon$ in the corresponding point. A further major difference is that we use the quadratic part of

$$H(\theta,\psi) + u\Pi(\theta,\psi)$$

above the virtual solitary manifold $\mathcal{S}_n^\varepsilon$ (instead of the classical solitary manifold $\mathcal{S}_0$) as our Lyapunov function in Part IV.

We formulate at the beginning of each part the main result and give a detailed chapter-wise overview of our approach afterwards.

# Notation and Convention

*Let $M$ be a closed subspace of a Hilbert space $H$. We use the following notation.*

*(a)* $M^{\perp,H}$ *denotes the orthogonal complement of $M$ in $H$.*

*(b)* $(\cdot)_M$ *denotes the orthogonal projection on $M$.*

*(c) If $v_1, \dots v_p \in H$, $\langle v_1, \dots v_p \rangle$ denotes the span of $v_1, \dots v_p$.*

*(d) $\gamma$ without an argument denotes always $\gamma(u)$.*

*(e) We denote by $\lambda_u$ functions which depend on $(\xi, u)$. One should understand the subscript $u$ in our notation just as a symbol.*

*(f) We denote by $Z$ either a variable or the function $Z = \gamma(x - \xi)$.*

*(g) We denote by $\langle \cdot, \cdot \rangle_H$ the inner product in $H$.*

*(h) For functions $\lambda$ which depend on $(\xi, u)$ we use the following notation $\lambda(\xi, u) = \lambda(u)(\xi)$.*

*(i) For functions $\theta$ which depend on $(\xi, u, x)$ we use the following notation $\theta(\xi, u, x) = \theta(u)(\xi, x)$.*

We will often use in Part I-IV Morrey's embedding theorem without further mention.

# Part I

# Classical Solitons in the Presence of a Forcing $\varepsilon f(\varepsilon x)$

# Chapter 1

# Main Result and Overview

To formulate our results precisely, we need some definitions.

**Definition 1.1.** *(a) Let us denote by* $U(l) := \frac{1-U}{l}$ *for* $0 < U < 1$.

*(b) We introduce the parameter area*

$$\Sigma(l,U) := \Big\{(\xi,u) \in \mathbb{R} \times (-1,1) : u \in \Big(-U-U(l), U+U(l)\Big)\Big\}\,.$$

*(c) Let* $\mathcal{N} : L^\infty(\mathbb{R}) \times L^2(\mathbb{R}) \times \Sigma(2,U) \to \mathbb{R}^2$ *be the map given by*

$$\begin{aligned}\mathcal{N}(\theta,\psi,\xi,u) &:= \begin{pmatrix}\mathcal{C}_1(\theta,\psi,\xi,u)\\ \mathcal{C}_2(\theta,\psi,\xi,u)\end{pmatrix}\\ &:= \begin{pmatrix}\Omega\left(\begin{pmatrix}\partial_\xi\theta_0(\xi,u,\cdot)\\ \partial_\xi\psi_0(\xi,u,\cdot)\end{pmatrix}, \begin{pmatrix}\theta(\cdot)-\theta_0(\xi,u,\cdot)\\ \psi(\cdot)-\psi_0(\xi,u,\cdot)\end{pmatrix}\right)\\ \Omega\left(\begin{pmatrix}\partial_u\theta_0(\xi,u,\cdot)\\ \partial_u\psi_0(\xi,u,\cdot)\end{pmatrix}, \begin{pmatrix}\theta(\cdot)-\theta_0(\xi,u,\cdot)\\ \psi(\cdot)-\psi_0(\xi,u,\cdot)\end{pmatrix}\right)\end{pmatrix}.\end{aligned}$$

We consider the initial value problem

$$\partial_t\begin{pmatrix}\theta(t,x)\\ \psi(t,x)\end{pmatrix} = \begin{pmatrix}\psi(t,x)\\ \partial_x^2\theta(t,x)-\sin\theta(t,x)+\varepsilon f(\varepsilon x)\end{pmatrix}, \tag{1.1}$$

$$\theta(0,x) = \theta_0(\xi_s,u_s,x)+v(0,x)\,, \tag{1.2}$$

$$\psi(0,x) = \psi_0(\xi_s,u_s,x)+w(0,x)\,, \tag{1.3}$$

where

$$(v(0,x),w(0,x)) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})\,.$$

The main result of Part I is the following theorem.

**Theorem 1.2.** *We consider the Cauchy problem defined by* (1.1)-(1.3) *and assume that*

*(a)* $\varepsilon$ *is sufficiently small,*

*(b)* $f \in H^1(\mathbb{R})$,

*(c)* $(\xi_s, u_s) \in \mathbb{R} \times (-U, U)$ , *where* $0 < U < 1$;

*(d)* $\mathcal{N}(\theta(0,x), \psi(0,x), \xi_s, u_s) = 0$ ,

*(e)* $|v(0)|^2_{H^1(\mathbb{R})} + |w(0)|^2_{L^2(\mathbb{R})} \leq \varepsilon$.

*Then*

*(i) The Cauchy problem defined by* (1.1)-(1.3) *has a unique solution on the time interval*

$$0 \leq t \leq T\,, \ \text{where } T = T(\varepsilon, \delta) := \frac{1}{\varepsilon^{\rho(\delta)}}, \quad \rho(\delta) = \frac{1}{2} - 2\delta\,.$$

*The solution may be written in the form*

$$\theta(t,x) = \theta_0(\xi(t), u(t), x) + v(t,x)\,,$$
$$\psi(t,x) = \psi_0(\xi(t), u(t), x) + w(t,x)\,,$$

*where* $v, w, u, \xi$ *have regularity*

$$(\xi(t), u(t)) \in C^1([0,T], \mathbb{R} \times (-1,1)),$$
$$(v(t), w(t)) \in C^1([0,T], H^1(\mathbb{R}) \oplus L^2(\mathbb{R})),$$

*such that the orthogonality condition*

$$\mathcal{N}(\theta(t,x), \psi(t,x), \xi(t), u(t)) = 0$$

*is satisfied. There exist positive constants* $c, C$ *such that*

$$|\dot{\xi}(t) - u(t)| \leq C\varepsilon,$$
$$|\dot{u}(t)| \leq C\varepsilon\,,$$

*and*

$$|v|^2_{L^\infty([0,T],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,T],L^2(\mathbb{R}))} \leq c\varepsilon\,.$$

*The constants* $c, C$ *depend on* $f$.

*(ii) The Cauchy problem defined by* (1.1)-(1.3) *has a unique solution on the time interval*

$$0 \le t \le T\,, \ \textit{where}\ T = T(\varepsilon,\delta) = \frac{1}{\varepsilon^{\beta(\delta)}}, \quad \beta(\delta) = \frac{1-\delta}{4}\,.$$

*The solution may be written in the form*

$$\theta(t,x) = \theta_0\left(\bar{\xi}(t), \bar{u}(t), x\right) + v(t,x)\,,$$
$$\psi(t,x) = \psi_0\left(\bar{\xi}(t), \bar{u}(t), x\right) + w(t,x)\,,$$

*where* $v, w$ *have regularity*

$$(v(t), w(t)) \in C([0,T], H^1(\mathbb{R}) \oplus L^2(\mathbb{R})),$$

$\bar{\xi}, \bar{u}$ *solve the following system of equations*

$$\bar{\xi}'(t) = \bar{u}(t)\,, \tag{1.4}$$

$$\bar{u}'(t) = 0 \tag{1.5}$$

*with initial data* $\bar{\xi}(0) = \xi_s,\ \bar{u}(0) = u_s$ *and there exists a positive constant* $c$ *such that*

$$|v|^2_{L^\infty([0,T],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,T],L^2(\mathbb{R}))} \le c\varepsilon\,.$$

*The constant* $c$ *depends on* $f$.

The following chapter-wise outline provides an overview of our approach.

**Solitary Manifold** We define the (classical) solitary manifold as the set

$$\mathcal{S}_0 := \left\{ \begin{pmatrix} \theta_0(\xi,u,\cdot) \\ \psi_0(\xi,u,\cdot) \end{pmatrix} \ :\ u \in (-1,1),\ \xi \in \mathbb{R} \right\},$$

which contains the soliton solutions (5) discussed in the introduction.

**Symplectic Orthogonal Decomposition** We show that if $(\theta,\psi) \in L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R})$ is close enough (in the $L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R})$ norm) to the region

$$\mathcal{S}_0(U) := \left\{ \begin{pmatrix} \theta_0(\xi,u,\cdot) \\ \psi_0(\xi,u,\cdot) \end{pmatrix} \ :\ (\xi,u) \in \Sigma(4,U) \right\},$$

of the solitary manifold $\mathcal{S}_0$, then there exists a unique $(\xi,u) \in \Sigma(2,U)$ such that

$$\begin{pmatrix} \theta(\cdot) \\ \psi(\cdot) \end{pmatrix} - \begin{pmatrix} \theta_0(\xi,u,\cdot) \\ \psi_0(\xi,u,\cdot) \end{pmatrix} =: \begin{pmatrix} v(\cdot) \\ w(\cdot) \end{pmatrix},$$

is symplectic orthogonal to the tangent vectors $\begin{pmatrix} \partial_\xi \theta_0(\xi, u, \cdot) \\ \partial_\xi \psi_0(\xi, u, \cdot) \end{pmatrix}$ and $\begin{pmatrix} \partial_u \theta_0(\xi, u, \cdot) \\ \partial_u \psi_0(\xi, u, \cdot) \end{pmatrix}$ of the solitary manifold $\mathcal{S}_0$, i.e.,

$$\mathcal{N}(\theta, \psi, \xi, u) = 0\,.$$

We prove that the symplectic decomposition is possible in a small uniform distance to the manifold $\mathcal{S}_0$.

**Existence of Dynamics and the Orthogonal Component** The existence theory provides that there is a local solution $(\theta, \psi)$ of (1.1)-(1.3), which might be written in the form

$$\theta(t, x) = \bar{v}(t, x) + \theta_0(\xi_s, u_s, x)\,,$$

$$\psi(t, x) = \bar{w}(t, x) + \psi_0(\xi_s, u_s, x)\,,$$

where $(\bar{v}, \bar{w}) \in C^1([0, T_{loc}], H^1(\mathbb{R}) \oplus L^2(\mathbb{R}))$ . Due to Morrey's embedding theorem it holds that $(\theta, \psi) \in C^1([0, T_{loc}], L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R}))$. In the following we assume that a solution $(\theta, \psi)$ of (1.1)-(1.3) is given on the time interval $[0, \overline{T}]$, which might be written as above where $(\bar{v}, \bar{w}) \in C^1([0, \overline{T}], H^1(\mathbb{R}) \oplus L^2(\mathbb{R}))$ .

$\varepsilon$ is chosen so small that due to assumptions (c), (e) in Theorem 1.2 the initial state $(\theta(0), \psi(0))$ is so close to the region $\mathcal{S}_0(U)$ of the solitary manifold that the symplectic orthogonal decomposition is possible in a neighbourhood of $(\theta(0), \psi(0))$.

In (1.2)-(1.3) the initial state $(\theta(0), \psi(0))$ is already written as a sum of a point on the solitary manifold $\mathcal{S}_0$ and a transversal component $(v(0), w(0))$ such that the symplectic orthogonality condition is satisfied due to assumption (d) in Theorem 1.2.

For times $t > 0$ we are able to choose the parameters $(\xi(t), u(t))$ according to the decomposition in Chapter 3 (Symplectic Orthogonal Decomposition ) as long as $(\theta(t), \psi(t))$ stays close enough to $\mathcal{S}_0(U)$. As long as $(\theta(t), \psi(t))$ stays close enough to $\mathcal{S}_0(U)$ we define $(v, w)$ by

$$v(t, x) = \theta(t, x) - \theta_0(\xi(t), u(t), x)\,, \tag{1.6}$$

$$w(t, x) = \psi(t, x) - \psi_0(\xi(t), u(t), x)\,, \tag{1.7}$$

where the parameter $(\xi(t), u(t))$ are obtained from the decomposition in Chapter 3 (Symplectic Orthogonal Decomposition ), such that

$$\mathcal{N}(\theta(t), \psi(t), \xi(t), u(t)) = 0\,. \tag{1.8}$$

Thus we decompose the dynamics in two components, namely a point on the solitary manifold $(\theta_0(\xi(t),u(t),\cdot),\psi_0(\xi(t),u(t),\cdot))$ and a transversal component $(v(t,\cdot),w(t,\cdot))$ which is symplectic orthogonal to the tangent vectors

$$\begin{pmatrix}\partial_\xi\theta_0(\xi(t),u(t),\cdot)\\ \partial_\xi\psi_0(\xi(t),u(t),\cdot)\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}\partial_u\theta_0(\xi(t),u(t),\cdot)\\ \partial_u\psi_0(\xi(t),u(t),\cdot)\end{pmatrix}$$

of $\mathcal{S}_0$. Finally we compute the time derivatives of $v$ and $w$ which will be needed in the following chapters.

In Chapter 8 (Proof of Theorem 1.2) we will obtain a bound on $|v|^2_{L^\infty([0,T],H^1(\mathbb{R}))}+|w|^2_{L^\infty([0,T],L^2(\mathbb{R}))}$ (where $T\le\overline{T}$) which will give us control over the distance of $(\theta,\psi)$ to the solitary manifold and which will imply that the local solution $(\theta,\psi)$ is indeed continuable.

**Modulation Equations** We want to consider the longitudinal dynamics on $\mathcal{S}_0$, which is described by the parameters $(\xi(t),u(t))$. In order to be able to understand the dynamics on $\mathcal{S}_0$ we derive a system of ordinary differential equations (modulation equations) for the parameters $(\xi(t),u(t))$ which is satisfied up to a certain error. We examine up to what errors the ordinary differential equations that describe the evolution of a soliton,

$$\dot\xi(t)=u(t)\,,$$

$$\dot u(t)=0\,,$$

are satisfied. For this purpose we take the time derivative of (1.8) and obtain a system of differential equations. Using Neumann's theorem we conclude that the estimates

$$|\dot\xi(t)-u(t)|\le C\varepsilon+C\,|v(t)|^2_{H^1(\mathbb{R})}\,,$$

$$|\dot u(t)|\le C\varepsilon+C\,|v(t)|^2_{H^1(\mathbb{R})}\,,$$

are satisfied if $|v(t)|_{H^1(\mathbb{R})}\,,|w(t)|_{L^2(\mathbb{R})}$ are less than a certain $\varepsilon_0>0$ and as long as the time $t$ is such as described in the introduction of $(v,w)$ above.

**Lyapunov Functional** In order to obtain control on the transversal component $(v,w)$ we introduce the Lyapunov function

$$L(t)=\int\frac{w^2(x)}{2}+\frac{(\partial_x v(x))^2}{2}+\frac{\cos(\theta_K(\gamma(u)(x-\xi)))v^2(x)}{2}+uw(x)\partial_x v(x)\,dx\,,$$

where $(v,w)$ are given by (1.6)-(1.7), $(\xi,u)$ are obtained from the decomposition in Chapter 3 (Symplectic Orthogonal Decomposition), such that the orthogonality conditions hold and $\gamma(u)=1/\sqrt{1-u^2}$. $L$ is the quadratic part of

$$H(\theta,\psi)+u\Pi(\theta,\psi)$$

above the solitary manifold $\mathcal{S}_0$, where $H$ and $\Pi$, given by (7) and (8) are conserved quantities of the sine-Gordon equation. Finally we compute the time derivative of $L(t)$ which will be needed later.

**Lower Bound** We consider for $(v,w) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$, $(\xi,u) \in \mathbb{R} \times (-1,1)$, the functional

$$\mathcal{E}(v,w,\xi,u) := \frac{1}{2} \int (w(x) + u\partial_x v(x))^2 + v_Z^2(x) + \cos(\theta_K(Z))v^2(x)\, dx\,,$$

where $Z = \gamma(x-\xi)$ and $v_Z(x) = \partial_Z v(\frac{Z}{\gamma} + \xi) = \frac{1}{\gamma}\partial_x v(x)$.

We prove that there exists a $c > 0$ such that if $(\xi,u) \in \mathbb{R} \times [-U - U(2), U + U(2)] \subset \mathbb{R} \times (-1,1)$ and $(v,w) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ satisfies the orthogonality condition

$$\check{\mathcal{C}}_2(v,w,\xi,u) := \int \partial_u \psi_0(\xi,u,x)v(x) - \partial_u \theta_0(\xi,u,x)w(x)\, dx = 0$$

(which is related to the second component in (1.8)), then the lower bound on $\mathcal{E}$,

$$\mathcal{E}(v,w,\xi,u) \geq c(|v|^2_{H^1(\mathbb{R})} + |w|^2_{L^2(\mathbb{R})})$$

holds. In the next chapter we relate this lower bound to the lyapunov function $L$, since it holds that

$$L(t) = \mathcal{E}(v(t), w(t), \xi(t), u(t))\,,$$

where $(v,w)$ are given by (1.6)-(1.7) and $(\xi,u)$ are obtained from the decomposition in Chapter 3 (Symplectic Orthogonal Decomposition).

**Proof of Theorem 1.2** First of all we prove the statement of Theorem 1.2 (i). We suppose that (1.1)-(1.3) has a solution and we make some assumptions on $(\xi,u)$ obtained from the decomposition in Chapter 3 (Symplectic Orthogonal Decomposition) and on $(v,w)$ given by (1.6)-(1.7). The modulation equations allow us to control $(\xi,u)$. The Lyapunov functions and the lower bound on $\mathcal{E}$ allow us to control $(v,w)$, since we are able to estimate

$$\begin{aligned} & c(|v(t)|^2_{H^1(\mathbb{R})} + |w(t)|^2_{L^2(\mathbb{R})}) \\ & \leq L(t) \\ & = L(0) + \int_0^t \dot{L}(s)\, ds \end{aligned}$$

and to control the right hand side (after bringing some terms on the left hand side). All in all, we obtain more accurate information about $(v,w)$ and $(\xi,u)$. Using a continuity argument this implies the bound on $(v,w)$ claimed in Theorem 1.2 (i) and approximate equations for the parameters $(\xi,u)$. The bound on $(v,w)$ implies that the local solution discussed in Chapter 4 (Existence of Dynamics and the Orthogonal Component) is continuable up to times $\frac{1}{\varepsilon^{\rho(\delta)}}$ ($\rho(\delta) = \frac{1}{2} - 2\delta$), which establishes the statement of Theorem 1.2 (i).

Using Theorem 1.2 (i) and Gronwall's lemma we show that the dynamics on the solitary manifold can be described by $(\bar{\xi},\bar{u})$ that satisfy the ODE's (1.4)-(1.5), which establishes the statement of Theorem 1.2 (ii).

# Chapter 2

# Solitary Manifold

We recall the definition of the solitary manifold presented in the introduction.

**Definition 2.1.** *The (classical) solitary manifold is the set*

$$\mathcal{S}_0 := \left\{ \begin{pmatrix} \theta_0(\xi, u, \cdot) \\ \psi_0(\xi, u, \cdot) \end{pmatrix} \ : \ u \in (-1, 1), \ \xi \in \mathbb{R} \right\} .$$

$\mathcal{S}_0$ is a two dimensional manifold.

## 2.1 Tangent Vectors

We introduce some further definitions and notation.

**Definition 2.2.** *(a)* $m := \int \left[\theta_K'(y)\right]^2 dy \,,$

*(b)* $t_\xi(\xi, u, x) := \partial_\xi \begin{pmatrix} \theta_0(\xi, u, x) \\ \psi_0(\xi, u, x) \end{pmatrix} = \begin{pmatrix} -\gamma \theta_K'(\gamma(x-\xi)) \\ u\gamma^2 \theta_K''(\gamma(x-\xi)) \end{pmatrix} \,,$

*(c)* $t_u(\xi, u, x) := \partial_u \begin{pmatrix} \theta_0(\xi, u, x) \\ \psi_0(\xi, u, x) \end{pmatrix} = \begin{pmatrix} u\gamma^3 (x-\xi) \theta_K'(\gamma(x-\xi)) \\ -\gamma^3 \theta_K'(\gamma(x-\xi)) - u^2\gamma^4 (x-\xi) \theta_K''(\gamma(x-\xi)) \end{pmatrix} \,.$

**Remark 2.3.** (a) One should understand the subscripts $\xi$ and $u$ in our notation of $t_\xi$ and $t_u$ just as symbols. $t_\xi$ and $t_u$ always really depend on $(\xi, u, x)$.

(b) The vectors $t_\xi(\xi, u, \cdot)$ and $t_u(\xi, u, \cdot)$ are tangent vectors of the manifold $\mathcal{S}_0$ at the point $(\theta_0(\xi, u, \cdot), \psi_0(\xi, u, \cdot))$ and form a basis of the tangent space at this point.

(c) Notice that

$$\mathbb{J}t_\xi(\xi,u,x) = \begin{pmatrix} -u\gamma^2\theta_K''(\gamma(x-\xi)) \\ -\gamma\theta_K'(\gamma(x-\xi)) \end{pmatrix} ,$$

$$\mathbb{J}t_u(\xi,u,x) = \begin{pmatrix} \gamma^3\theta_K'(\gamma(x-\xi)) + u^2\gamma^4(x-\xi)\theta_K''(\gamma(x-\xi)) \\ u\gamma^3(x-\xi)\theta_K'(\gamma(x-\xi)) \end{pmatrix} ,$$

where

$$\mathbb{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

# Chapter 3

# Symplectic Orthogonal Decomposition

As mentioned in Chapter 1 we show that if $(\theta, \psi) \in L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R})$ is close enough (in the $L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R})$ norm) to the region

$$\mathcal{S}_0(U) := \left\{ \begin{pmatrix} \theta_0(\xi, u, \cdot) \\ \psi_0(\xi, u, \cdot) \end{pmatrix} \ : \ (\xi, u) \in \Sigma(4, U) \right\} ,$$

of the solitary manifold $\mathcal{S}_0$, then there exists a unique $(\xi, u) \in \Sigma(2, U)$ such that we are able to decompose

$$\begin{pmatrix} \theta(\cdot) \\ \psi(\cdot) \end{pmatrix} = \begin{pmatrix} \theta_0(\xi, u, \cdot) \\ \psi_0(\xi, u, \cdot) \end{pmatrix} + \begin{pmatrix} v(\cdot) \\ w(\cdot) \end{pmatrix} ,$$

in a point on the solitary manifold $(\theta_0(\xi, u, \cdot), \psi_0(\xi, u, \cdot))$ and a transversal component $(v(\cdot), w(\cdot))$, which is symplectic orthogonal to the tangent vectors $\begin{pmatrix} \partial_\xi \theta_0(\xi, u, \cdot) \\ \partial_\xi \psi_0(\xi, u, \cdot) \end{pmatrix}$ and $\begin{pmatrix} \partial_u \theta_0(\xi, u, \cdot) \\ \partial_u \psi_0(\xi, u, \cdot) \end{pmatrix}$ of the solitary manifold $\mathcal{S}_0$, i.e., the orthogonality condition

$$\mathcal{N}(\theta, \psi, \xi, u) = 0$$

is satisfied. We prove in the following lemma that the symplectic decomposition is possible in a small uniform distance to the solitary manifold $\mathcal{S}_0$.

**Lemma 3.1.** *Let* $0 < U < 1$. *Let*

$$\mathcal{O} = \mathcal{O}_{U,p} = \left\{ (\theta, \psi) \in L^\infty(\mathbb{R}) \times L^2(\mathbb{R}) : \inf_{(\xi,u) \in \Sigma(4,U)} \left\| \begin{pmatrix} \theta(\cdot) \\ \psi(\cdot) \end{pmatrix} - \begin{pmatrix} \theta_0(\xi, u, \cdot) \\ \psi_0(\xi, u, \cdot) \end{pmatrix} \right\|_{L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R})} < p \right\} .$$

*There exists* $r > 0$ *such that if* $p \leq r$ *then for any* $(\theta, \psi) \in \mathcal{O}_{U,p}$ *there exists a unique* $(\xi, u) \in \Sigma(2, U)$ *such that*

$$\mathcal{N}(\theta, \psi, \xi, u) = 0$$

*and the map*

$$(\theta,\psi) \mapsto (\xi(\theta,\psi), u(\theta,\psi))$$

*is in* $C^1(\mathcal{O}_{U,p}, \Sigma(2,U))$.

**Proof.** Notice that

$$\begin{aligned}&U(4) \leq U(3) \leq U(2)\,,\\ &\Sigma(4,U) \subset \Sigma(3,U) \subset \Sigma(2,U)\,.\end{aligned}$$

$\mathcal{N}$ is given by

$$\mathcal{N}(\theta,\psi,\xi,u) = \begin{pmatrix} \int \partial_\xi \psi_0(\xi,u,x)\Big[\theta(x) - \theta_0(\xi,u,x)\Big] - \partial_\xi \theta_0(\xi,u,x)\Big[\psi(x) - \psi_0(\xi,u,x)\Big]\,dx \\ \int \partial_u \psi_0(\xi,u,x)\Big[\theta(x) - \theta_0(\xi,u,x)\Big] - \partial_u \theta_0(\xi,u,x)\Big[\psi(x) - \psi_0(\xi,u,x)\Big]\,dx \end{pmatrix}.$$

Consider $(\xi_0,u_0) \in \Sigma(3,U)$.
**i)** It holds that $\mathcal{N}(\theta_0(\xi_0,u_0,x), \psi_0(\xi_0,u_0,x), \xi_0, u_0) = (0,0)$. Since

$$\begin{aligned}&D_{\xi,u}\mathcal{N}(\theta_0(\xi_0,u_0,x), \psi_0(\xi_0,u_0,x), \xi_0, u_0)\\ &= \begin{pmatrix} \partial_\xi \mathcal{N}^1_{\xi_0,u_0}(\theta_0(\xi_0,u_0,x), \psi_0(\xi_0,u_0,x), \xi_0, u_0) & \partial_u \mathcal{N}^1_{\xi_0,u_0}(\theta_0(\xi_0,u_0,x), \psi_0(\xi_0,u_0,x), \xi_0, u_0) \\ \partial_\xi \mathcal{N}^2_{\xi_0,u_0}(\theta_0(\xi_0,u_0,x), \psi_0(\xi_0,u_0,x), \xi_0, u_0) & \partial_u \mathcal{N}^2_{\xi_0,u_0}(\theta_0(\xi_0,u_0,x), \psi_0(\xi_0,u_0,x), \xi_0, u_0) \end{pmatrix}\\ &= \begin{pmatrix} 0 & \gamma^3(u_0)m \\ -\gamma^3(u_0)m & 0 \end{pmatrix},\end{aligned}$$

we obtain:

$$\det D_{\xi,u}\mathcal{N}(\theta_0(\xi_0,u_0,x), \psi_0(\xi_0,u_0,x), \xi_0, u_0) \neq 0.$$

It follows by the implicit function theorem that there exist balls

$$B_r(\theta_0(\xi_0,u_0,\cdot), \psi_0(\xi_0,u_0,\cdot)) \subset L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R}), \quad B_{\bar{\delta}}(\xi_0,u_0) \subset \Sigma(2,U)$$

and exactly one map

$$T_{\xi_0,u_0} : B_r(\theta_0(\xi_0,u_0,\cdot), \psi_0(\xi_0,u_0,\cdot)) \to B_{\bar{\delta}}(\xi_0,u_0)$$

such that

$$T_{\xi_0,u_0}(\theta_0(\xi_0,u_0,x), \psi_0(\xi_0,u_0,x)) = (\xi_0,u_0)$$

and

$$\mathcal{N}(\theta,\psi,T_{\xi_0,u_0}(\theta,\psi)) = 0$$

on $B_r(\theta_0(\xi_0,u_0,\cdot),\psi_0(\xi_0,u_0,\cdot))$.
**ii)** We refer to [Dei85, Theorem 15.1] and we are going to prove:
There exist $r>0,\bar{\delta}>0$ such that $\forall(\xi_0,u_0)\in\Sigma(3,U)$ there exist balls

$$B_r(\theta_0(\xi_0,u_0,\cdot),\psi_0(\xi_0,u_0,\cdot))\subset L^\infty(\mathbb{R})\oplus L^2(\mathbb{R})\,,\quad B_{\bar{\delta}}(\xi_0,u_0)\subset\Sigma(2,U)\,,$$

and a map

$$T_{\xi_0,u_0}:B_r(\theta_0(\xi_0,u_0,\cdot),\psi_0(\xi_0,u_0,\cdot))\to B_{\bar{\delta}}(\xi_0,u_0)$$

such that

$$\mathcal{N}(\theta,\psi,T_{\xi_0,u_0}(\theta,\psi))=0$$

on $B_r(\theta_0(\xi_0,u_0,\cdot),\psi_0(\xi_0,u_0,\cdot))$. The claim follows from this statement. In order to obtain the same setting as in [Dei85, Theorem 15.1] we introduce

$$\bar{\mathcal{N}}_{\xi_0,u_0}(\theta,\psi,\xi,u)=\mathcal{N}(\theta(\cdot)+\theta_0(\xi_0,u_0,\cdot),\psi(\cdot)+\psi_0(\xi_0,u_0,\cdot),\xi+\xi_0,u+u_0).$$

Then

$$\mathcal{N}(\theta_0(\xi_0,u_0,\cdot),\psi_0(\xi_0,u_0,\cdot),\xi_0,u_0)=\bar{\mathcal{N}}_{\xi_0,u_0}(0,0,0,0)=(0,0).$$

Set

$$\begin{aligned}K_{\xi_0,u_0}&:=D_{(\xi,u)}\mathcal{N}_{\xi_0,u_0}(\theta_0(\xi_0,u_0,\cdot),\psi_0(\xi_0,u_0,\cdot),\xi_0,u_0)\\&=D_{(\xi,u)}\bar{\mathcal{N}}_{\xi_0,u_0}(0,0,0,0)\\&=\begin{pmatrix}0&\gamma(u_0)^3m\\-\gamma(u_0)^3m&0\end{pmatrix}.\end{aligned}$$

Hence

$$K^{-1}_{\xi_0,u_0}=\frac{1}{\gamma(u_0)^3m}\begin{pmatrix}0&-1\\1&0\end{pmatrix}.$$

We define

$$S_{\xi_0,u_0}(\theta,\psi,\xi,u)=K^{-1}_{\xi_0,u_0}\bar{\mathcal{N}}_{\xi_0,u_0}(\theta,\psi,\xi,u)-I(\xi,u).$$

The following norms

$$|\partial_\xi\theta_0(\xi,u,x)|_{L^2_x(\mathbb{R})}\,,|\partial_u\theta_0(\xi,u,x)|_{L^2_x(\mathbb{R})}\,,|\partial_\xi\psi_0(\xi,u,x)|_{L^2_x(\mathbb{R})}\,,|\partial_u\psi_0(\xi,u,x)|_{L^2_x(\mathbb{R})}\,,$$

$$|\partial_\xi\theta_0(\xi,u,x)|_{L^1_x(\mathbb{R})}\,,|\partial_u\theta_0(\xi,u,x)|_{L^1_x(\mathbb{R})}\,,|\partial_\xi\psi_0(\xi,u,x)|_{L^1_x(\mathbb{R})}\,,|\partial_u\psi_0(\xi,u,x)|_{L^1_x(\mathbb{R})}\,,$$

$$\left|\partial^2_\xi\theta_0(\xi,u,x)\right|_{L^2_x(\mathbb{R})}\,,\left|\partial^2_u\theta_0(\xi,u,x)\right|_{L^2_x(\mathbb{R})}\,,|\partial_\xi\partial_u\theta_0(\xi,u,x)|_{L^2_x(\mathbb{R})}\,,|\partial_\xi\partial_u\psi_0(\xi,u,x)|_{L^2_x(\mathbb{R})}\,,$$

$$\left|\partial^2_\xi\psi_0(\xi,u,x)\right|_{L^1_x(\mathbb{R})}\,,\left|\partial^2_u\psi_0(\xi,u,x)\right|_{L^1_x(\mathbb{R})}\,,|\partial_\xi\partial_u\theta_0(\xi,u,x)|_{L^1_x(\mathbb{R})}\,,|\partial_\xi\partial_u\psi_0(\xi,u,x)|_{L^1_x(\mathbb{R})}\,,$$

are bounded from above, uniformly in $(\xi, u)$ by

$$B := \max\Big\{ \sup_{u\in(-U-U(2),U+U(2))} \Big|\partial_\xi^{\beta_1}\partial_u^{\beta_2}\theta_0(\xi,u,x)\Big|_{L^p_x(\mathbb{R})}\,,$$
$$\sup_{u\in(-U-U(2),U+U(2))} \Big|\partial_\xi^{\beta_1}\partial_u^{\beta_2}\psi_0(\xi,u,x)\Big|_{L^p_x(\mathbb{R})} \;\Big|\; \beta_1+\beta_2\le 2,\ p=1,2\Big\} + 1\,.$$

Notice that

$$\forall\ (\xi,u)\in\Sigma(2,U):\quad \frac{1}{|\gamma(u)^3 m|}\le\frac{1}{c}\,.$$

In this proof we denote by $\|\cdot\|$ the maximum row sum norm of a $2\times 2$ matrix induced by the maximum norm $|\cdot|_\infty$ in $\mathbb{R}^2$.

$\underline{\text{We show } \exists k\in(0,1),\bar\delta>0\ \Big[\text{independent of } \Big(\theta_0(\xi_0,u_0,\cdot),\psi_0(\xi_0,u_0,\cdot),(\xi_0,u_0)\Big)\Big]}$
$\underline{\forall\ (\theta,\psi),(\xi,u)\in B_{\bar\delta}(0)\times B_{\bar\delta}(0)\ :\ \|D_{(\xi,u)}S_{\xi_0,u_0}(\theta,\psi,\xi,u)\|\le k<1\ :}$

We set

$$k := \frac{1}{2},\quad K := \frac{k}{30}\,.$$

Consider $|\theta_0(\xi_0,u_0,x)-\theta_0(\bar\xi,\bar u,x)|_{L^\infty_x(\mathbb{R})}$ and $|\psi_0(\xi_0,u_0,x)-\psi_0(\bar\xi,\bar u,x)|_{L^2_x(\mathbb{R})}$, where $(\xi_0,u_0),(\bar\xi,\bar u)\in\overline{\Sigma(2,U)}$:

Since

$$\theta_K(\gamma(u_0)(x-\xi_0))-\theta_K(\gamma(u_0)(x-\bar\xi)) = \theta_K'(\gamma(u_0)(x-\hat\xi[\xi_0,\bar\xi,u_0,x]))\cdot\gamma(u_0)(\xi_0-\bar\xi)\,,$$
$$\theta_K(\gamma(u_0)(x-\bar\xi))-\theta_K(\gamma(\bar u)(x-\bar\xi)) = \theta_K'(\hat\gamma[\gamma(u_0),\gamma(\bar u),\bar\xi,x](x-\bar\xi))\cdot(\gamma(u_0)-\gamma(\bar u))\,,$$

by the mean value theorem, there exist a $\eta>0$ s.t. for all $(\xi_0,u_0),(\bar\xi,\bar u)\in\overline{\Sigma(2,U)}$ with $|(\xi_0,u_0)-(\bar\xi,\bar u)|<\eta$, we obtain:

$$\Big|\theta_0(\xi_0,u_0,x)-\theta_0(\bar\xi,\bar u,x)\Big|_{L^\infty_x(\mathbb{R})}\le\frac{Kc}{B},$$

Since

$$\begin{aligned}
&-u_0\gamma(u_0)\theta_K'(\gamma(u_0)(x-\xi_0))+\bar u\gamma(\bar u)\theta_K'(\gamma(\bar u)(x-\bar\xi))\\
=\ &-u_0\gamma(u_0)\theta_K'(\gamma(u_0)(x-\xi_0))+u_0\gamma(u_0)\theta_K'(\gamma(u_0)(x-\bar\xi))\\
&-u_0\gamma(u_0)\theta_K'(\gamma(u_0)(x-\bar\xi))+u_0\gamma(u_0)\theta_K'(\gamma(\bar u)(x-\bar\xi))\\
&-u_0\gamma(u_0)\theta_K'(\gamma(\bar u)(x-\bar\xi))+\bar u\gamma(\bar u)\theta_K'(\gamma(\bar u)(x-\bar\xi))
\end{aligned}$$

and

$$\theta_K'(\gamma(u_0)(x-\xi_0)) - \theta_K'(\gamma(u_0)(x-\bar{\xi})) = \theta_K''(\gamma(u_0)(x-\hat{\xi}[\xi_0,\bar{\xi},u_0,x]))\cdot\gamma(u_0)(\xi_0-\bar{\xi})\,,$$
$$\theta_K'(\gamma(u_0)(x-\bar{\xi})) - \theta_K'(\gamma(\bar{u})(x-\bar{\xi})) = \theta_K''(\hat{\gamma}[\gamma(u_0),\gamma(\bar{u}),\bar{\xi},x](x-\bar{\xi}))\cdot(\gamma(u_0)-\gamma(\bar{u}))\,,$$

by the mean value theorem, there exist a $\eta > 0$ s.t. for all $(\xi_0, u_0), (\bar{\xi}, \bar{u}) \in \overline{\Sigma(2,U)}$ with $|(\xi_0,u_0)-(\bar{\xi},\bar{u})| < \eta$, we obtain:

$$\left|\psi_0(\xi_0,u_0,x)-\psi_0(\bar{\xi},\bar{u},x)\right|_{L_x^\infty(\mathbb{R})} \leq \frac{Kc}{B}$$

and

$$|(\gamma(u_0)^3-\gamma(\bar{u})^3)m| \leq \frac{Kc}{B}\,.$$

We set

$$\bar{\delta} := \min\left\{\eta, \frac{Kc}{B}, U(7)\right\}$$

and get back to our case where $(\xi_0, u_0) \in \Sigma(3, U)$. Notice that

$$\begin{aligned}
&D_{(\xi,u)}S_{\xi_0,u_0}(\theta,\psi,\xi,u)\\
&= \frac{1}{\gamma(u_0)^3m}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} D_{(\xi,u)}\bar{\mathcal{N}}_{\xi_0,u_0}(\theta,\psi,\xi,u) - \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\\
&= \frac{1}{\gamma(u_0)^3m}\begin{pmatrix}-\partial_\xi\bar{\mathcal{N}}^2_{\xi_0,u_0}(\theta,\psi,\xi,u) & -\partial_u\bar{\mathcal{N}}^2_{\xi_0,u_0}(\theta,\psi,\xi,u)\\ \partial_\xi\bar{\mathcal{N}}^1_{\xi_0,u_0}(\theta,\psi,\xi,u) & \partial_u\bar{\mathcal{N}}^1_{\xi_0,u_0}(\theta,\psi,\xi,u)\end{pmatrix} - \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\,.
\end{aligned}$$

For all $(\theta, \psi), (\xi, u) \in B_{\bar{\delta}}(0) \times B_{\bar{\delta}}(0)$:

$$\begin{aligned}
&|\frac{1}{\gamma(u_0)^3m}\partial_\xi\bar{\mathcal{N}}^1_{\xi_0,u_0}(\theta,\psi,\xi,u)|\\
&= |\frac{1}{\gamma(u_0)^3m}\partial_\xi\mathcal{N}^1((\theta(\cdot)+\theta_0(\xi_0,u_0,\cdot),\psi(\cdot)+\psi_0(\xi_0,u_0,\cdot)),(\underbrace{\xi+\xi_0}_{=\bar{u}},\underbrace{u+u_0}_{=\bar{\xi}}))|\\
&\leq \frac{1}{|\gamma(u_0)^3m|}\Big(|\partial_\xi^2\psi_0(\bar{\xi},\bar{u},x)|_{L_x^1(\mathbb{R})}|\theta(x)+\theta_0(\xi_0,u_0,x)-\theta_0(\bar{\xi},\bar{u},x)|_{L_x^\infty(\mathbb{R})}\\
&\quad + |\partial_\xi^2\theta_0(\bar{\xi},\bar{u},x)|_{L_x^2(\mathbb{R})}|\psi(x)|_{L_x^2(\mathbb{R})}\\
&\quad + |\partial_\xi^2\theta_0(\bar{\xi},\bar{u},x)|_{L_x^1(\mathbb{R})}|\psi_0(\xi_0,u_0,x)-\psi_0(\bar{\xi},\bar{u},x)|_{L_x^\infty(\mathbb{R})}\\
&\quad + |\int -\partial_\xi\psi_0(\bar{\xi},\bar{u},x)\partial_\xi\theta_0(\bar{\xi},\bar{u},x)+\partial_\xi\theta_0(\bar{\xi},\bar{u},x)\partial_\xi\psi_0(\bar{\xi},\bar{u},x)\,dx|\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{|\gamma(u_0)^3 m|}\Big(|\partial_\xi^2\psi_0(\bar{\xi},\bar{u},x)|_{L^1_x(\mathbb{R})}|\theta(x)+\theta_0(\xi_0,u_0,x)-\theta_0(\bar{\xi},\bar{u},x)|_{L^\infty_x(\mathbb{R})} \\
&\quad + |\partial_\xi^2\theta_0(\bar{\xi},\bar{u},x)|_{L^2_x(\mathbb{R})}|\psi(x)|_{L^2_x(\mathbb{R})} \\
&\quad + |\partial_\xi^2\theta_0(\bar{\xi},\bar{u},x)|_{L^1_x(\mathbb{R})}|\psi_0(\xi_0,u_0,x)-\psi_0(\bar{\xi},\bar{u},x)|_{L^\infty_x(\mathbb{R})}\Big) \\
&< \frac{k}{2},
\end{aligned}
$$

$$
\begin{aligned}
&|-\frac{1}{\gamma(u_0)^3 m}\partial_\xi\bar{\mathcal{N}}^2_{\xi_0,u_0}(\theta,\psi,\xi,u)-1| \\
&= |-\frac{1}{\gamma(u_0)^3 m}\partial_\xi\mathcal{N}^2((\theta(\cdot)+\theta_0(\xi_0,u_0,\cdot),\psi(\cdot)+\psi_0(\xi_0,u_0,\cdot)),(\underbrace{\xi+\xi_0}_{=\bar{u}},\underbrace{u+u_0}_{=\bar{\xi}}))-1| \\
&\le \frac{1}{|\gamma(u_0)^3 m|}\Big(|\partial_\xi\partial_u\psi_0(\bar{\xi},\bar{u},x)|_{L^1_x(\mathbb{R})}|\theta(x)+\theta_0(\xi_0,u_0,x)-\theta_0(\bar{\xi},\bar{u},x)|_{L^\infty_x(\mathbb{R})} \\
&\quad + |\partial_\xi\partial_u\theta_0(\bar{\xi},\bar{u},x)|_{L^2_x(\mathbb{R})}|\psi(x)|_{L^2_x(\mathbb{R})} \\
&\quad + |\partial_\xi\partial_u\theta_0(\bar{\xi},\bar{u},x)|_{L^1_x(\mathbb{R})}|\psi_0(\xi_0,u_0,x)-\psi_0(\bar{\xi},\bar{u},x)|_{L^\infty_x(\mathbb{R})}\Big) \\
&\quad + |-\frac{1}{\gamma(u_0)^3 m}\int -\partial_u\psi_0(\bar{\xi},\bar{u},x)\partial_\xi\theta_0(\bar{\xi},\bar{u},x)+\partial_u\theta_0(\bar{\xi},\bar{u},x)\partial_\xi\psi_0(\bar{\xi},\bar{u},x)\,dx-1| \\
&\le \frac{1}{|\gamma(u_0)^3 m|}\Big(|\partial_\xi\partial_u\psi_0(\bar{\xi},\bar{u},x)|_{L^1_x(\mathbb{R})}|\theta(x)+\theta_0(\xi_0,u_0,x)-\theta_0(\bar{\xi},\bar{u},x)|_{L^\infty_x(\mathbb{R})} \\
&\quad + |\partial_\xi\partial_u\theta_0(\bar{\xi},\bar{u},x)|_{L^2_x(\mathbb{R})}|\psi(x)|_{L^2_x(\mathbb{R})} \\
&\quad + |\partial_\xi\partial_u\theta_0(\bar{\xi},\bar{u},x)|_{L^1_x(\mathbb{R})}|\psi_0(\xi_0,u_0,x)-\psi_0(\bar{\xi},\bar{u},x)|_{L^\infty_x(\mathbb{R})} \\
&\quad + |\gamma(\bar{u})^3 m-\gamma(u_0)^3 m|\Big) \\
&< \frac{k}{2},
\end{aligned}
$$

$$
\begin{aligned}
&|\frac{1}{\gamma(u_0)^3 m}\partial_u\bar{\mathcal{N}}^1_{\xi_0,u_0}(\theta,\psi,\xi,u)-1| \\
&= |\frac{1}{\gamma(u_0)^3 m}\partial_u\mathcal{N}^1((\theta(\cdot)+\theta_0(\xi_0,u_0,\cdot),\psi(x)+\psi_0(\xi_0,u_0,x),(\underbrace{\xi+\xi_0}_{=\bar{u}},\underbrace{u+u_0}_{=\bar{\xi}}))-1|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{|\gamma(u_0)^3 m|}\Big(|\partial_u\partial_\xi\psi_0(\bar{\xi},\bar{u},x)|_{L^1_x(\mathbb{R})}|\theta(x)+\theta_0(\xi_0,u_0,x)-\theta_0(\bar{\xi},\bar{u},x)|_{L^\infty_x(\mathbb{R})}\\
&\quad+|\partial_u\partial_\xi\theta_0(\bar{\xi},\bar{u},x)|_{L^2_x(\mathbb{R})}|\psi(x)|_{L^2_x(\mathbb{R})}\\
&\quad+|\partial_u\partial_\xi\theta_0(\bar{\xi},\bar{u},x)|_{L^1_x(\mathbb{R})}|\psi_0(\xi_0,u_0,x)-\psi_0(\bar{\xi},\bar{u},x)|_{L^\infty_x(\mathbb{R})}\Big)\\
&\quad+|\frac{1}{\gamma(u_0)^3 m}\int -\partial_\xi\psi_0(\bar{\xi},\bar{u},x)\partial_u\theta_0(\bar{\xi},\bar{u},x)+\partial_\xi\theta_0(\bar{\xi},\bar{u},x)\partial_u\psi_0(\bar{\xi},\bar{u},x)\,dx-1|\\
&\leq \frac{1}{|\gamma(u_0)^3 m|}\Big(|\partial_u\partial_\xi\psi_0(\bar{\xi},\bar{u},x)|_{L^1_x(\mathbb{R})}|\theta(x)+\theta_0(\xi_0,u_0,x)-\theta_0(\bar{\xi},\bar{u},x)|_{L^\infty_x(\mathbb{R})}\\
&\quad+|\partial_u\partial_\xi\theta_0(\bar{\xi},\bar{u},x)|_{L^2_x(\mathbb{R})}|\psi(x)|_{L^2_x(\mathbb{R})}\\
&\quad+|\partial_u\partial_\xi\theta_0(\bar{\xi},\bar{u},x)|_{L^1_x(\mathbb{R})}|\psi_0(\xi_0,u_0,x)-\psi_0(\bar{\xi},\bar{u},x)|_{L^\infty_x(\mathbb{R})}\\
&\quad+|\gamma(\bar{u})^3 m-\gamma(u_0)^3 m|\Big)\\
&< \frac{k}{2}\,,
\end{aligned}
$$

$$
\begin{aligned}
&\quad|\frac{1}{\gamma(u_0)^3 m}\partial_u\bar{\mathcal{N}}^2_{\xi_0,u_0}(\theta,\psi,\xi,u)|\\
&=|\frac{1}{\gamma(u_0)^3 m}\partial_u\mathcal{N}^2((\theta(\cdot)+\theta_0(\xi_0,u_0,\cdot),\psi(\cdot)+\psi_0(\xi_0,u_0,\cdot)),(\underbrace{\xi+\xi_0}_{=\bar{u}},\underbrace{u+u_0}_{=\bar{\xi}}))|\\
&\leq \frac{1}{|\gamma(u_0)^3 m|}\Big(|\partial_u^2\psi_0(\bar{\xi},\bar{u},x)|_{L^1_x(\mathbb{R})}|\theta(x)+\theta_0(\xi_0,u_0,x)-\theta_0(\bar{\xi},\bar{u},x)|_{L^\infty_x(\mathbb{R})}\\
&\quad+|\partial_u^2\theta_0(\bar{\xi},\bar{u},x)|_{L^2_x(\mathbb{R})}|\psi(x)|_{L^2_x(\mathbb{R})}\\
&\quad+|\partial_u^2\theta_0(\bar{\xi},\bar{u},x)|_{L^1_x(\mathbb{R})}|\psi_0(\xi_0,u_0,x)-\psi_0(\bar{\xi},\bar{u},x)|_{L^\infty_x(\mathbb{R})}\\
&\quad+|\int -\partial_u\psi_0(\bar{\xi},\bar{u},x)\partial_u\theta_0(\bar{\xi},\bar{u},x)+\partial_u\theta_0(\bar{\xi},\bar{u},x)\partial_u\psi_0(\bar{\xi},\bar{u},x)\,dx|\Big)\\
&= \frac{1}{|\gamma(u_0)^3 m|}\Big(|\partial_u^2\psi_0(\bar{\xi},\bar{u},x)|_{L^1_x(\mathbb{R})}|\theta(x)+\theta_0(\xi_0,u_0,x)-\theta_0(\bar{\xi},\bar{u},x)|_{L^\infty_x(\mathbb{R})}\\
&\quad+|\partial_u^2\theta_0(\bar{\xi},\bar{u},x)|_{L^2_x(\mathbb{R})}|\psi(x)|_{L^2_x(\mathbb{R})}\\
&\quad+|\partial_u^2\theta_0(\bar{\xi},\bar{u},x)|_{L^1_x(\mathbb{R})}|\psi_0(\xi_0,u_0,x)-\psi_0(\bar{\xi},\bar{u},x)|_{L^\infty_x(\mathbb{R})}\Big)\\
&< \frac{k}{2}\,.
\end{aligned}
$$

We show $\exists\ r \leq \bar{\delta}$ $\Big[$independent of $\Big(\theta_0(\xi_0, u_0, \cdot), \psi_0(\xi_0, u_0, \cdot), (\xi_0, u_0)\Big)\Big]$
$\forall\ (\theta, \psi) \in B_r(0):\ \ |S_{\xi_0,u_0}(\theta, \psi, 0, 0)|_\infty < \bar{\delta}(1-k)$ :

Notice that

$$S_{\xi_0,u_0}(\theta, \psi, 0, 0) = \frac{1}{\gamma(u_0)^3 m} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \bar{\mathcal{N}}_{\xi_0,u_0}(\theta, \psi, 0, 0)\,.$$

We set

$$r := \min\left\{\frac{\bar{\delta}}{2}, \frac{\bar{\delta} C}{3B}\right\}\,.$$

For all $(\theta, \psi) \in B_r(0)$:

$$\begin{aligned}
&|\frac{1}{\gamma(u_0)^3 m}\bar{\mathcal{N}}^1_{\xi_0,u_0}(\theta, \psi, 0, 0)| \\
&= |\frac{1}{\gamma(u_0)^3 m}\mathcal{N}^1_{\xi_0,u_0}((\theta(\cdot) + \theta_0(\xi_0, u_0, \cdot), \psi(\cdot) + \psi_0(\xi_0, u_0, \cdot), (\xi_0, u_0))| \\
&= \frac{1}{|\gamma(u_0)^3 m|}\left|\int \partial_\xi \psi_0(\xi_0, u_0, x)\theta(x) - \partial_\xi \theta_0(\xi_0, u_0, x)\psi(x)\, dx\right| \\
&\leq \frac{1}{|\gamma(u_0)^3 m|}\Big(|\partial_\xi \psi_0(\xi_0, u_0, x)|_{L^1_x(\mathbb{R})}|\theta(x)|_{L^\infty_x(\mathbb{R})} + |\partial_\xi \theta_0(\xi_0, u_0, x)|_{L^2_x(\mathbb{R})}|\psi(x)|_{L^2_x(\mathbb{R})}\Big)\,,
\end{aligned}$$

$$\begin{aligned}
&|\frac{1}{\gamma(u_0)^3 m}\bar{\mathcal{N}}^2_{\xi_0,u_0}(\theta, \psi, 0, 0)| \\
&= |\frac{1}{\gamma(u_0)^3 m}\mathcal{N}^2((\theta(\cdot) + \theta_0(\xi_0, u_0, \cdot), \psi(\cdot) + \psi_0(\xi_0, u_0, \cdot), (\xi_0, u_0))| \\
&= \frac{1}{|\gamma(u_0)^3 m|}\left|\int \partial_u \psi_0(\xi_0, u_0, x)\theta(x) - \partial_u \theta_0(\xi_0, u_0, x)\psi(x)\, dx\right| \\
&\leq \frac{1}{|\gamma(u_0)^3 m|}\Big(|\partial_u \psi_0(\xi_0, u_0, x)|_{L^1_x(\mathbb{R})}|\theta(x)|_{L^\infty_x(\mathbb{R})} + |\partial_u \theta_0(\xi_0, u_0, x)|_{L^2_x(\mathbb{R})}|\psi(x)|_{L^2_x(\mathbb{R})}\Big)\,.
\end{aligned}$$

□

# Chapter 4

# Existence of Dynamics and the Orthogonal Component

We argue similar to [Stu98, Proof of theorem 2.1]. In order to be able to make use of existence theory we set

$$\bar{v}(t,x) = \theta(t,x) - \theta_0(\xi_s, u_s, x)\,,$$
$$\bar{w}(t,x) = \psi(t,x) - \psi_0(\xi_s, u_s, x)$$

and consider the problem

$$\begin{pmatrix} \bar{v}(0,x) \\ \bar{w}(0,x) \end{pmatrix} = \begin{pmatrix} \theta(0,x) - \theta_0(\xi_s, u_s, x) \\ \psi(0,x) - \psi_0(\xi_s, u_s, x) \end{pmatrix}, \tag{4.1}$$

$$\partial_t \begin{pmatrix} \bar{v}(t,x) \\ \bar{w}(t,x) \end{pmatrix} = \begin{pmatrix} \bar{w}(t,x) - \psi_0(\xi_s, u_s, x) \\ [\bar{v}(t,x) + \theta_0(\xi_s, u_s, x)]_{xx} - \sin(\bar{v}(t,x) + \theta_0(\xi_s, u_s, x)) + \varepsilon f(\varepsilon x) \end{pmatrix}. \tag{4.2}$$

By [Mar76, Theorem VIII 2.1, Theorem VIII 3.2 ] there exists a local solution (see also [Stu98, Proof of theorem 2.1], [Stu92, p.434 ]) with

$$(\bar{v}, \bar{w}) \in C^1([0, T_{loc}], H^1(\mathbb{R}) \oplus L^2(\mathbb{R}))\,.$$

$(\theta, \psi)$ given by $\theta(t,x) = \bar{v}(t,x) + \theta_n^\varepsilon(\xi_s, u_s, x)$ and $\psi(t,x) = \bar{w}(t,x) + \psi_n^\varepsilon(\xi_s, u_s, x)$ solves obviously locally the Cauchy problem (1.1)-(1.3) and $(\theta, \psi) \in C^1([0, T_{loc}], L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R}))$ due to Morrey's embedding theorem.

We are going to obtain a bound in Chapter 8 which will imply that the local solutions are indeed continuable.

So from now we assume that $(\bar{v}, \bar{w}) \in C^1([0, \overline{T}], H^1(\mathbb{R}) \oplus L^2(\mathbb{R}))$ is a solution of (4.1)-(4.2) and $(\theta, \psi)$ is a solution of (1.1)-(1.3) such that $(\theta, \psi) \in C^1([0, \overline{T}], L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R}))$.

Given $(\theta, \psi)$ we choose the parameters $(\xi(t), u(t))$ according to Lemma 3.1 and define $(v, w)$ as follows:

$$v(t,x) = \theta(t,x) - \theta_0(\xi(t), u(t), x)\,, \tag{4.3}$$

$$w(t,x) = \psi(t,x) - \psi_0(\xi(t), u(t), x)\,. \tag{4.4}$$

$(v(t,x), w(t,x))$ is well defined for $t \geq 0$ so small that

$$|v(t)|_{L^\infty(\mathbb{R})} + |w(t)|_{L^2(\mathbb{R})} \leq r$$

and

$$(\xi(t), u(t)) \in \Sigma(4, U)\,,$$

where $r$ and $U$ are from Lemma 3.1. We formalize this by the following definition.

**Definition 4.1.** *Let $t^*$ be the "exit time":*

$$t^* := \sup\Big\{T > 0 : |v|_{L^\infty(\mathbb{R})L^\infty([0,t])} + |w|_{L^\infty([0,t],L^2(\mathbb{R}))} \leq r,$$

$$(\xi(t), u(t)) \in \Sigma(4, U),\ 0 \leq t \leq T\Big\}\,,$$

*where $r$ and $U$ are from Lemma 3.1.*

Notice that $(\xi_s, u_s) = (\xi(0), u(0)) \in \Sigma(4, U)$. We will choose $\varepsilon$ such that, among others,

$$|v(0)|_{L^\infty(\mathbb{R})} + |w(0)|_{L^2(\mathbb{R})} \leq \frac{r}{2},$$

where $(v(0), w(0))$ is given by (1.2)-(1.3). Thus $(v(t,x), w(t,x))$ is well defined for $0 \leq t \leq t^*$. The following technical lemma will be needed later.

**Lemma 4.2.** *Let $(\xi_0, u_0), (\bar\xi, \bar u) \in \mathbb{R} \times (-1, 1)$.*
*The difference $\Big(\theta_K(\gamma(u_0)(\cdot - \xi_0)) - \theta_K(\gamma(\bar u)(\cdot - \bar\xi))\Big)$ is in $L^2(\mathbb{R})$.*

**Proof.** The mean value theorem yields:

$$\theta_K(\gamma(u_0)(x - \xi_0)) - \theta_K(\gamma(u_0)(x - \bar\xi)) = \theta_K'(\gamma(u_0)(x - \hat\xi[\xi_0, \bar\xi, u_0, x])) \cdot \gamma(u_0)(\xi_0 - \bar\xi)\,,$$

$$\theta_K(\gamma(u_0)(x - \bar\xi)) - \theta_K(\gamma(\bar u)(x - \bar\xi)) = \theta_K'(\hat\gamma[\gamma(u_0), \gamma(\bar u), \bar\xi, x](x - \bar\xi)) \cdot (\gamma(u_0) - \gamma(\bar u))\,.$$

Assume without loss of generality $\xi_0 < \bar\xi$. For $x \geq \bar\xi \geq \hat\xi[\xi_0, \bar\xi, u_0, x] \geq \xi_0$ it follws that

$$\gamma(u_0)(x - \xi_0) \geq \gamma(u_0)(x - \hat\xi[\xi_0, \bar\xi, u_0, x]) \geq \gamma(u_0)(x - \bar\xi) \geq 0$$

and

$$|\theta_K'(\gamma(u_0)(x - \hat\xi[\xi_0, \bar\xi, u_0, x]))| \leq |\theta_K'(\gamma(u_0)(x - \bar\xi))|\,.$$

For $\bar{\xi} \geq \hat{\xi}[\xi_0, \bar{\xi}, u_0, x] \geq \xi_0 \geq x$ it follws that

$$0 \geq \gamma(u_0)(x - \xi_0) \geq \gamma(u_0)(x - \hat{\xi}[\xi_0, \bar{\xi}, u_0, x]) \geq \gamma(u_0)(x - \bar{\xi})$$

and

$$|\theta'_K(\gamma(u_0)(x - \hat{\xi}[\xi_0, \bar{\xi}, u_0, x]))| \leq |\theta'_K(\gamma(u_0)(x - \xi_0))| \,.$$

Assume without loss of generality $0 < u_0 < \bar{u}$. For $x \geq \bar{\xi}$ it follows that

$$\gamma(\bar{u})(x - \bar{\xi}) \geq \hat{\gamma}[\gamma(u_0), \gamma(\bar{u}), \bar{\xi}, x](x - \bar{\xi}) \geq \gamma(u_0)(x - \bar{\xi}) \geq 0$$

and

$$|\theta'_K(\hat{\gamma}[\gamma(u_0), \gamma(\bar{u}), \bar{\xi}, x](x - \bar{\xi}))| \leq |\theta'_K(\gamma(u_0)(x - \bar{\xi}))|.$$

For $\bar{\xi} \geq x$ it follows that

$$0 \geq \gamma(\bar{u})(x - \bar{\xi}) \geq \hat{\gamma}[\gamma(u_0), \gamma(\bar{u}), \bar{\xi}, x](x - \bar{\xi}) \geq \gamma(u_0)(x - \bar{\xi})$$

and

$$|\theta'_K(\hat{\gamma}[\gamma(u_0), \gamma(\bar{u}), \bar{\xi}, x](x - \bar{\xi}))| \leq |\theta'_K(\gamma(\bar{u})(x - \bar{\xi}))|.$$

Since we are able to estimate $|\theta'_K(\gamma(u_0)(x - \hat{\xi}[\xi_0, \bar{\xi}, u_0, x]))|$ and $|\theta'_K(\hat{\gamma}[\gamma(u_0), \gamma(\bar{u}), \bar{\xi}, x](x - \bar{\xi}))|$ for large $|x|$ by integrable functions, this implies the claim:

$$\begin{aligned}
&\left|\theta_K(\gamma(u_0)(x - \xi_0)) - \theta_K(\gamma(\bar{u})(x - \bar{\xi}))\right|_{L^2(\mathbb{R})} \\
\leq\ &\left|\theta_K(\gamma(u_0)(x - \xi_0)) - \theta_K(\gamma(u_0)(x - \bar{\xi}))\right|_{L^2(\mathbb{R})} + \left|\theta_K(\gamma(u_0)(x - \bar{\xi})) - \theta_K(\gamma(\bar{u})(x - \bar{\xi}))\right|_{L^2(\mathbb{R})} \\
\leq\ &\left|\theta'_K(\gamma(u_0)(x - \hat{\xi}[\xi_0, \bar{\xi}, u_0, x]))\right|_{L^2(\mathbb{R})} |\gamma(u_0)(\xi_0 - \bar{\xi})| \\
&+ \left|\theta'_K(\hat{\gamma}[\gamma(u_0), \gamma(\bar{u}), \bar{\xi}, x](x - \bar{\xi}))\right|_{L^2(\mathbb{R})} |\gamma(u_0) - \gamma(\bar{u})| \,.
\end{aligned}$$

□

Using the previous lemma we obtain in the following lemma more information on $(v, w)$.

**Lemma 4.3.** *Let* $T = \min\{t^*, \overline{T}\}$ *and let* $(v, w)$ *be defined by* (4.3)-(4.4). *Then* $(v, w) \in C^1([0, T], H^1(\mathbb{R}) \oplus L^2(\mathbb{R}))$.

**Proof.** Using (4.3)-(4.4), Lemma 4.2 and the fact that $(\bar{v}, \bar{w}) \in C^1([0,T], H^1(\mathbb{R}) \oplus L^2(\mathbb{R}))$, we obtain

$$\begin{aligned} v(t,x) &= \theta(t,x) - \theta_0(\xi_s, u_s, x) + \theta_0(\xi_s, u_s, x) - \theta_0(\xi(t), u(t), x) \\ &= \bar{v}(t,x) + \theta_0(\xi_s, u_s, x) - \theta_0(\xi(t), u(t), x) \in H^1_x(\mathbb{R}) \,, \end{aligned}$$

$$w(t,x) = \psi(t,x) - \psi_0(\xi(t), u(t), x) \in L^2_x(\mathbb{R}) \,.$$

This implies the claim. □

We compute the time derivatives of $v$ and $w$ which will be needed in the following chapters.

**Lemma 4.4.** *The equations for* $(v, w)$ *defined by* (4.3)-(4.4)*, are*

$$\begin{aligned} \dot{v}(x) &= w(x) - \dot{\xi}\partial_\xi\theta_0(\xi, u, x) - \dot{u}\partial_u\theta_0(\xi, u, x) + u\partial_\xi\theta_0(\xi, u, x) \,, \\ \dot{w}(x) &= \partial_x^2 v(x) - \cos\theta_0(\xi, u, x)v(x) + \varepsilon f(\varepsilon x) + \frac{\sin\theta_0(\xi, u, x)v^2(x)}{2} + \tilde{R}(v)(x) \\ &\quad + u\partial_\xi\psi_0(\xi, u, x) - \dot{\xi}\partial_\xi\psi_0(\xi, u, x) - \dot{u}\partial_u\psi_0(\xi, u, x) \,, \end{aligned}$$

*for times* $t \in [0, t^*]$*, where* $\tilde{R}(v) = \mathcal{O}(|v|^3_{H^1(\mathbb{R})})$.

**Proof.** We take the time derivatives of $(v, w)$ and use (4.3)-(4.4), (1.1):

$$\begin{aligned} \dot{v}(x) &= w(x) + \psi_0(\xi, u, x) \\ &\quad - \dot{\xi}\partial_\xi\theta_0(\xi, u, x) - \dot{u}\partial_u\theta_0(\xi, u, x) \\ &= w(x) - \dot{\xi}\partial_\xi\theta_0(\xi, u, x) - \dot{u}\partial_u\theta_0(\xi, u, x) + u\partial_\xi\theta_0(\xi, u, x) \,, \end{aligned}$$

$$\begin{aligned} \dot{w}(x) &= \partial_x^2\theta(x) - \sin\theta(x) + \varepsilon f(\varepsilon x) - \dot{\xi}\partial_\xi\psi_0(\xi, u, x) - \dot{u}\partial_u\psi_0(\xi, u, x) \\ &= \partial_x^2\theta_0(\xi, u, x) + \partial_x^2 v(x) - \sin\theta_0(\xi, u, x) \\ &\quad - \cos\theta_0(\xi, u, x)v(x) + \frac{\sin\theta_0(\xi, u, x)v^2(x)}{2} \\ &\quad + \tilde{R}(v)(x) + \varepsilon f(\varepsilon x) - \dot{\xi}\partial_\xi\psi_0(\xi, u, x) - \dot{u}\partial_u\psi_0(\xi, u, x) \\ &\quad + u\partial_x\psi_0(\xi, u, x) - u\partial_x\psi_0(\xi, u, x) \end{aligned}$$

$$
\begin{aligned}
&= \partial_x^2 v(x) - \cos\theta_0(\xi,u,x)v(x) + \varepsilon f(\varepsilon x) + \frac{\sin\theta_0(\xi,u,x)v^2(x)}{2} + \tilde{R}(v)(x) \\
&\quad + u\partial_\xi\psi_0(\xi,u,x) - \dot{\xi}\partial_\xi\psi_0(\xi,u,x) - \dot{u}\partial_u\psi_0(\xi,u,x)\,,
\end{aligned}
$$

where we have expanded the term $\sin(\theta_0(\xi,u,x)+v(x))$. □

# Chapter 5

# Modulation Equations

In the following lemma we derive modulation equations for the parameters $(\xi(t), u(t))$.

**Lemma 5.1.** *There exists an* $\varepsilon_0 > 0$ *such that the following statement holds. Let* $(v, w)$ *be given by* (4.3)-(4.4), *with* $(\xi, u)$ *obtained from Lemma 3.1 and let*

$$|v|_{L^\infty([0,t^*],H^1(\mathbb{R}))}\,, |w|_{L^\infty([0,t^*],L^2(\mathbb{R}))} \leq \varepsilon_0\,,$$

*where* $t^*$ *is from Definition 4.1. Then*

$$|\dot{\xi}(t) - u(t)| \leq C\varepsilon + C\,|v(t)|^2_{H^1(\mathbb{R})}\,,$$

$$|\dot{u}(t)| \leq C\varepsilon + C\,|v(t)|^2_{H^1(\mathbb{R})}\,,$$

*for* $0 \leq t \leq t^*$, *where* $C$ *depends on* $f$.

**Proof.** The technique we use is similar to that in the proof of [IKV12, Lemma 6.2]. Using Definition 1.1 and (4.3)-(4.4) we write the orthogonality conditions as follows:

$$0 = \mathcal{C}_1(\theta, \psi, \xi, u) = \int \partial_\xi \psi_0(\xi, u, x) v(x) - \partial_\xi \theta_0(\xi, u, x) w(x)\, dx\,,$$

$$0 = \mathcal{C}_2(\theta, \psi, \xi, u) = \int \partial_u \psi_0(\xi, u, x) v(x) - \partial_u \theta_0(\xi, u, x) w(x)\, dx\,.$$

In the following we skip $(\theta, \psi, \xi, u)$ for simplicity of further notation and take the derivatives of $\mathcal{C}_1$, $\mathcal{C}_2$ with respect to $t$. Using Lemma 4.4 we obtain:

$$
\begin{aligned}
\dot{\mathcal{C}}_1 = & \int \partial_t[\partial_\xi \psi_0(\xi,u,x)]v(x) + \partial_\xi \psi_0(\xi,u,x)\partial_t v(x) \\
& - \partial_t[\partial_\xi \theta_0(\xi,u,x)]w(x) - \partial_\xi \theta_0(\xi,u,x)\partial_t w(x)\, dx \\
= & \int \left\{ \dot{\xi}\partial_\xi^2 \psi_0(\xi,u,x) + \dot{u}\partial_u \partial_\xi \psi_0(\xi,u,x) \right\} v(x) \\
& + \left\{ \partial_\xi \psi_0(\xi,u,x) \right\} \left\{ w(x) - \dot{\xi}\partial_\xi \theta_0(\xi,u,x) - \dot{u}\partial_u \theta_0(\xi,u,x) \right. \\
& \left. + u\partial_\xi \theta_0(\xi,u,x) \right\} \\
& - \left\{ \dot{\xi}\partial_\xi^2 \theta_0(\xi,u,x) + \dot{u}\partial_u \partial_\xi \theta_0(\xi,u,x) \right\} w(x) \\
& - \left\{ \partial_\xi \theta_0(\xi,u,x) \right\} \left\{ \partial_x^2 v(x) - \cos\theta_0(\xi,u,x)v(x) + \varepsilon f(\varepsilon x) + \frac{\sin\theta_0(\xi,u,x)v^2(x)}{2} \right. \\
& \left. + \tilde{R}(v)(x) + u\partial_\xi \psi_0(\xi,u,x) - \dot{\xi}\partial_\xi \psi_0(\xi,u,x) - \dot{u}\partial_u \psi_0(\xi,u,x) \right\} dx \\
= & \underbrace{\int -\partial_\xi \psi_0(\xi,u,x)\partial_u \theta_0(\xi,u,x) + \partial_\xi \theta_0(\xi,u,x)\partial_u \psi_0(\xi,u,x)\, dx}_{=\Omega\left(t_\xi(\xi,u,\cdot),\, t_u(\xi,u,\cdot)\right)} \cdot \dot{u} \\
& \underbrace{+ \int \partial_\xi \psi_0(\xi,u,x)\partial_\xi \theta_0(\xi,u,x) - \partial_\xi \psi_0(\xi,u,x)\partial_\xi \theta_0(\xi,u,x)\, dx}_{=\Omega\left(t_\xi(\xi,u,\cdot),\, t_\xi(\xi,u,\cdot)\right)} \cdot (u - \dot{\xi}) \\
& \underbrace{+ \int \partial_u \partial_\xi \psi_0(\xi,u,x)v(x) - \partial_u \partial_\xi \theta_0(\xi,u,x)w(x)\, dx}_{=[M(\xi,u,v,w)]_{12}} \cdot \dot{u} \\
& \underbrace{- \int \partial_\xi^2 \psi_0(\xi,u,x)v(x) - \partial_\xi^2 \theta_0(\xi,u,x)w(x)\, dx}_{=[M(\xi,u,v,w)]_{11}} \cdot (u - \dot{\xi}) \\
& \underbrace{+ \int \partial_\xi \psi_0(\xi,u,x)w(x) - \partial_\xi \theta_0(\xi,u,x)\left( \partial_x^2 v(x) - \cos\theta_0(\xi,u,x)v(x) \right) dx} \cdots
\end{aligned}
$$

$$\underbrace{\begin{aligned}&+\int u\partial_\xi^2\psi_0(\xi,u,x)v(x) - u\partial_\xi^2\theta_0(\xi,u,x)w(x)\,dx\\ &-\int\left\{\partial_\xi\theta_0(\xi,u,x)\right\}\left\{\varepsilon f(\varepsilon x)\right\}dx\\ &-\int\partial_\xi\theta_0(\xi,u,x)\left\{\frac{\sin\theta_0(\xi,u,x)v^2(x)}{2}+\tilde{R}(v)(x)\right\}dx\end{aligned}}_{=[P(\xi,u,v,w)]_1}\ .$$

$$\begin{aligned}\dot{\mathcal{C}}_2 = &\int \partial_t[\partial_u\psi_0(\xi,u,x)]v(x) + \partial_u\psi_0(\xi,u,x)\partial_t v(x)\\ &-\partial_t[\partial_u\theta_0(\xi,u,x)]w(x) - \partial_u\theta_0(\xi,u,x)\partial_t w(x)\,dx\\ = &\int\left\{\dot{\xi}\partial_\xi\partial_u\psi_0(\xi,u,x)+\dot{u}\partial_u^2\psi_0(\xi,u,x)\right\}v(x)\\ &+\left\{\partial_u\psi_0(\xi,u,x)\right\}\left\{w(x)-\dot{\xi}\partial_\xi\theta_0(\xi,u,x)-\dot{u}\partial_u\theta_0(\xi,u,x)\right.\\ &\left.+u\partial_\xi\theta_0(\xi,u,x)\right\}\\ &-\left\{\dot{\xi}\partial_\xi\partial_u\theta_0(\xi,u,x)+\dot{u}\partial_u^2\theta_0(\xi,u,x)\right\}w(x)\\ &-\left\{\partial_u\theta_0(\xi,u,x)\right\}\left\{\partial_x^2 v(x)-\cos\theta_0(\xi,u,x)v(x)+\varepsilon f(\varepsilon x)\right.\\ &+\frac{\sin\theta_0(\xi,u,x)v^2(x)}{2}+\tilde{R}(v)(x)\\ &\left.+u\partial_\xi\psi_0(\xi,u,x)-\dot{\xi}\partial_\xi\psi_0(\xi,u,x)-\dot{u}\partial_u\psi_0(\xi,u,x)\right\}dx\end{aligned}$$

$$
\begin{aligned}
&= \underbrace{\int \partial_u\theta_0(\xi,u,x)\partial_u\psi_0(\xi,u,x) - \partial_u\theta_0(\xi,u,x)\partial_u\psi_0(\xi,u,x)\,dx}_{=\Omega\left(t_u(\xi,u,\cdot),t_u(\xi,u,\cdot)\right)} \cdot \dot{u} \\
&\quad + \underbrace{\int \partial_\xi\theta_0(\xi,u,x)\partial_u\psi_0(\xi,u,x) - \partial_\xi\psi_0(\xi,u,x)\partial_u\theta_0(\xi,u,x)\,dx}_{=\Omega\left(t_u(\xi,u,\cdot),t_\xi(\xi,u,\cdot)\right)} \cdot (u-\dot{\xi}) \\
&\quad + \underbrace{\int \partial_u^2\partial_\xi\psi_0(\xi,u,x)v(x) - \partial_u^2\theta_0(\xi,u,x)w(x)\,dx}_{=[M(\xi,u,v,w)]_{22}} \cdot \dot{u} \\
&\quad - \underbrace{\int \partial_\xi\partial_u\psi_0(\xi,u,x)v(x) - \partial_\xi\partial_u\theta_0(\xi,u,x)w(x)\,dx}_{=[M(\xi,u,v,w)]_{21}} \cdot (u-\dot{\xi}) \\
&\quad \underbrace{\begin{aligned}
&+\int \left\{\partial_u\psi_0(\xi,u,x)\right\} w(x) - \partial_u\theta_0(\xi,u,x)\left(\partial_x^2 v(x) - \cos\theta_0(\xi,u,x)v(x)\right)dx \\
&+u\int \partial_\xi\partial_u\psi_0(\xi,u,x)v(x) - \partial_\xi\partial_u\theta_0(\xi,u,x)w(x)\,dx \\
&-\int \left\{\partial_u\theta_0(\xi,u,x)\right\}\left\{\varepsilon f(\varepsilon x)\right\} dx \\
&-\int \left\{\partial_u\theta_0(\xi,u,x)\right\}\left\{\frac{\sin\theta_0(\xi,u,x)v^2(x)}{2} + \tilde{R}(v)(x)\right\} dx
\end{aligned}}_{=[P(\xi,u,v,w)]_2} \; .
\end{aligned}
$$

We set

$$
\begin{aligned}
\Omega(u) &:= \begin{pmatrix} \Omega\left(t_\xi(\xi,u,\cdot),t_\xi(\xi,u,\cdot)\right) & \Omega\left(t_\xi(\xi,u,\cdot),t_u(\xi,u,\cdot)\right) \\ \Omega\left(t_u(\xi,u,\cdot),t_\xi(\xi,u,\cdot)\right) & \Omega\left(t_u(\xi,u,\cdot),t_u(\xi,u,\cdot)\right) \end{pmatrix} \\
&= \begin{pmatrix} 0 & \Omega\left(t_\xi(\xi,u,\cdot),t_u(\xi,u,\cdot)\right) \\ \Omega\left(t_u(\xi,u,\cdot),t_\xi(\xi,u,\cdot)\right) & 0 \end{pmatrix}
\end{aligned}
$$

$$= \begin{pmatrix} 0 & \gamma(u)^3 m \\ -\gamma(u)^3 m & 0 \end{pmatrix}$$

$$= \gamma(u)^3 m \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

Now we consider for any $(\bar{\xi}, \bar{u}) \in \mathbb{R} \times [-U - U(2), U + U(2)]$, $(\bar{v}, \bar{w}) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ the matrix:

$$M(\bar{\xi}, \bar{u}, \bar{v}, \bar{w})$$

$$= \begin{pmatrix} \left\langle \begin{pmatrix} \partial_\xi^2 \psi_0(\bar{\xi}, \bar{u}, \cdot) \\ -\partial_\xi^2 \theta_0(\bar{\xi}, \bar{u}, \cdot) \end{pmatrix}, \begin{pmatrix} \bar{v}(\cdot) \\ \bar{w}(\cdot) \end{pmatrix} \right\rangle_{L^2(\mathbb{R}) \oplus L^2(\mathbb{R})} & \left\langle \begin{pmatrix} \partial_u \partial_\xi \psi_0(\bar{\xi}, \bar{u}, \cdot) \\ -\partial_u \partial_\xi \theta_0(\bar{\xi}, \bar{u}, \cdot) \end{pmatrix}, \begin{pmatrix} \bar{v}(\cdot) \\ \bar{w}(\cdot) \end{pmatrix} \right\rangle_{L^2(\mathbb{R}) \oplus L^2(\mathbb{R})} \\ \left\langle \begin{pmatrix} \partial_\xi \partial_u \psi_0(\bar{\xi}, \bar{u}, \cdot) \\ -\partial_\xi \partial_u \theta_0(\bar{\xi}, \bar{u}, \cdot) \end{pmatrix}, \begin{pmatrix} \bar{v}(\cdot) \\ \bar{w}(\cdot) \end{pmatrix} \right\rangle_{L^2(\mathbb{R}) \oplus L^2(\mathbb{R})} & \left\langle \begin{pmatrix} \partial_u^2 \psi_0(\bar{\xi}, \bar{u}, \cdot) \\ -\partial_u^2 \theta_0(\bar{\xi}, \bar{u}, \cdot) \end{pmatrix}, \begin{pmatrix} \bar{v}(\cdot) \\ \bar{w}(\cdot) \end{pmatrix} \right\rangle_{L^2(\mathbb{R}) \oplus L^2(\mathbb{R})} \end{pmatrix} .$$

We use Hölder's inequality and obtain for all $(\bar{\xi}, \bar{u}) \in \mathbb{R} \times [-U - U(2), U + U(2)]$, $(\bar{v}, \bar{w}) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$:

$$\| \left[\Omega(\bar{u})\right]^{-1} M(\bar{\xi}, \bar{u}, \bar{v}, \bar{w}) \| \leq C(|\bar{v}|_{H^1(\mathbb{R})} + |\bar{w}|_{L^2(\mathbb{R})}) , \tag{5.1}$$

where we denote by $\| \cdot \|$ a matrix norm. Let $I = I_2$ be the identety matrix of dimension 2. Due to (16.1) we are able to find an $\varepsilon_0 > 0$ such that if $|\bar{v}|_{H^1(\mathbb{R})}, |\bar{w}|_{L^2(\mathbb{R})} \leq \varepsilon_0$ then the matrix

$$I + \left[\Omega(\bar{u})\right]^{-1} M(\bar{\xi}, \bar{u}, \bar{v}, \bar{w})$$

is invertible by Neumann's theorem. We write the time derivatives of $(\mathcal{C}_1^\varepsilon, \mathcal{C}_2^\varepsilon)$ in matrix form and use the notation $P(\xi, u, v, w) = P$, $M(\xi, u, v, w) = M$, $\Omega(u) = \Omega$:

$$0 = \begin{pmatrix} \dot{\mathcal{C}}_1 \\ \dot{\mathcal{C}}_2 \end{pmatrix}$$

$$= \Omega \begin{pmatrix} \dot{\xi} - u \\ \dot{u} \end{pmatrix} + M \begin{pmatrix} \dot{\xi} - u \\ \dot{u} \end{pmatrix} + P .$$

This implies

$$-\Omega^{-1} P = \Big( I + \Omega^{-1} M \Big) \begin{pmatrix} \dot{\xi} - u \\ \dot{u} \end{pmatrix} .$$

If $|v|_{H^1(\mathbb{R})}, |w|_{L^2(\mathbb{R})} \leq \varepsilon_0$ then we obtain as mentioned above by Neumann's theorem that

$$\begin{pmatrix} \dot{\xi} - u \\ \dot{u} \end{pmatrix} = -\Big( I + \Omega^{-1} M \Big)^{-1} [\Omega^{-1} P] .$$

We will show later in Corollary 20.6 that

$$\left\langle \begin{pmatrix} -u\partial_x v(\cdot) - w(\cdot) \\ -\partial_x^2 v(\cdot) + \cos(\theta_0(\xi, u, \cdot))v(\cdot) - u\partial_x w(\cdot) \end{pmatrix}, \begin{pmatrix} -\partial_\xi \psi_0(\xi, u, \cdot) \\ \partial_\xi \theta_0(\xi, u, \cdot) \end{pmatrix} \right\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} = 0\,,$$

$$\left\langle \begin{pmatrix} -u\partial_x v(\cdot) - w(\cdot) \\ -\partial_x^2 v(\cdot) + \cos(\theta_0(\xi, u, \cdot))v(\cdot) - u\partial_x w(\cdot) \end{pmatrix}, \begin{pmatrix} -\partial_u \psi_0(\xi, u, \cdot) \\ \partial_u \theta_0(\xi, u, \cdot) \end{pmatrix} \right\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} = 0\,.$$

These identities can also be checked by hand using integration by parts and the symplectic orthogonality. We will use them in the following computations. We consider $P_1$ and $P_2$:

$$\begin{aligned}
P_1 = &\int \partial_\xi \psi_0(\xi, u, x) w(x) - \partial_\xi \theta_0(\xi, u, x) \left( \partial_x^2 v(x) - \cos\theta_0(\xi, u, x) v(x) \right) dx \\
&+ u \int \partial_\xi^2 \psi_0(\xi, u, x) v(x) - \partial_\xi^2 \theta_0(\xi, u, x) w(x)\, dx \\
&- \int \left\{ \partial_\xi \theta_0(\xi, u, x) \right\} \left\{ \varepsilon f(\varepsilon x) \right\} dx \\
&- \int \partial_\xi \theta_0(\xi, u, x) \left\{ \frac{\sin\theta_0(\xi, u, x) v^2(x)}{2} + \tilde{R}(v)(x) \right\} dx \\
= &\left\langle \begin{pmatrix} -u\partial_x v(\cdot) - w(\cdot) \\ -\partial_x^2 v(\cdot) + \cos(\theta_0(\xi, u, \cdot))v(\cdot) - u\partial_x w(\cdot) \end{pmatrix}, \begin{pmatrix} -\partial_\xi \psi_0(\xi, u, \cdot) \\ \partial_\xi \theta_0(\xi, u, \cdot) \end{pmatrix} \right\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} \\
&- \int \left\{ \partial_\xi \theta_0(\xi, u, x) \right\} \left\{ \varepsilon f(\varepsilon x) \right\} dx \\
&- \int \partial_\xi \theta_0(\xi, u, x) \left\{ \frac{\sin\theta_0(\xi, u, x) v^2(x)}{2} + \tilde{R}(v)(x) \right\} dx
\end{aligned}$$

Consequently using Corollary 20.6 we obtain

$$|P_1| \leq C\varepsilon + C \left| v \right|_{H^1(\mathbb{R})}^2 \,.$$

$$\begin{aligned}
P_2 = &\int \left\{ \partial_u \psi_0(\xi, u, x) \right\} w(x) - \partial_u \theta_0(\xi, u, x) \left( \partial_x^2 v(x) - \cos\theta_0(\xi, u, x) v(x) \right) dx \\
&+ u \int \partial_\xi \partial_u \psi_0(\xi, u, x) v(x) - \partial_\xi \partial_u \theta_0(\xi, u, x) w(x)\, dx
\end{aligned}$$

$$
\begin{aligned}
&- \int \left\{ \partial_u \theta_0(\xi, u, x) \right\} \left\{ \varepsilon f(\varepsilon x) \right\} dx \\
&- \int \left\{ \partial_u \theta_0(\xi, u, x) \right\} \left\{ \frac{\sin \theta_0(\xi, u, x) v^2(x)}{2} + \tilde{R}(v)(x) \right\} dx \\
= &\left\langle \begin{pmatrix} -u\partial_x v(\cdot) - w(\cdot) \\ -\partial_x^2 v(\cdot) + \cos(\theta_0(\xi, u, \cdot))v(\cdot) - u\partial_x w(\cdot) \end{pmatrix}, \begin{pmatrix} -\partial_u \psi_0(\xi, u, \cdot) \\ \partial_u \theta_0(\xi, u, \cdot) \end{pmatrix} \right\rangle_{L^2(\mathbb{R}) \oplus L^2(\mathbb{R})} \\
&- \int \left\{ \partial_u \theta_0(\xi, u, x) \right\} \left\{ \varepsilon f(\varepsilon x) \right\} dx \\
&- \int \left\{ \partial_u \theta_0(\xi, u, x) \right\} \left\{ \frac{\sin \theta_0(\xi, u, x) v^2(x)}{2} + \tilde{R}(v)(x) \right\} dx
\end{aligned}
$$

Consequently using Corollary 20.6 we obtain

$$
|R_2| \leq C\varepsilon + C\,|v|^2_{H^1(\mathbb{R})} \ .
$$

□

# Chapter 6

# Lyapunov Functional

We introduce the Lyapunov function.

**Definition 6.1.** *Let* $(v, w)$ *be given by* (4.3)-(4.4), *with* $(\xi, u)$ *obtained from Lemma 3.1. We set*

$$L(t) = \int \frac{w^2(x)}{2} + \frac{(\partial_x v(x))^2}{2} + \frac{\cos(\theta_K(\gamma(x-\xi)))v^2(x)}{2} + uw(x)\partial_x v(x)\, dx\,. \tag{6.1}$$

**Remark 6.2.** $L$ is the quadratic part of

$$H(\theta, \psi) + u\Pi(\theta, \psi)$$

above the solitary manifold $\mathcal{S}_0$, since due to (7)

$$\begin{aligned}
&H\Big(\theta_0(\xi, u, \cdot) + v(\cdot), \psi_0(\xi, u, \cdot) + w(\cdot)\Big) \\
&= \frac{1}{2}\int (\psi_0(\xi, u, x) + w(x))^2 + (\partial_x\theta_0(\xi, u, x) + \partial_x v(x))^2 \\
&\quad + 2(1 - \cos(\theta_0(\xi, u, x) + v(x)))\, dx \\
&= \frac{1}{2}\int (\psi_0(\xi, u, x))^2 + 2\psi_0(\xi, u, x)w(x) + w^2(x) \\
&\quad + (\partial_x\theta_0(\xi, u, x))^2 + 2\partial_x\theta_0(\xi, u, x)\partial_x v(x) + (\partial_x v(x))^2 \\
&\quad + 2 - 2\cos\theta_0(\xi, u, x) + 2\sin(\theta_0(\xi, u, x))v(x) \\
&\quad + \cos(\theta_0(\xi, u, x))v^2(x)\, dx + \mathcal{O}(v^3) + \mathcal{O}(w^3)
\end{aligned}$$

and due to (8)

$$\begin{aligned}
&\Pi\Big(\theta_0(\xi,u,\cdot)+v(\cdot),\psi_0(\xi,u,\cdot)+w(\cdot)\Big)\\
&=\int(\psi_0(\xi,u,x)+w(x))(\partial_x\theta_0(\xi,u,x)+\partial_x v(x))\,dx\\
&=\int\psi_0(\xi,u,x)\partial_x\theta_0(\xi,u,x)+\psi_0(\xi,u,x)\partial_x v(x)\\
&\quad+\partial_x\theta_0(\xi,u,x)\;w(x)+w(x)\partial_x v(x)\,dx\,.
\end{aligned}$$

A simple computation shows that the linear part of $H(\theta,\psi)+u\Pi(\theta,\psi)$ vanishes if $(v,w)$ is symplectic orthogonal to the tangent space of $\mathcal{S}_0$ at the point $(\theta_0(\xi,u,\cdot),\psi_0(\xi,u,\cdot))$.

The time derivative of $L(t)$ is computed in the following lemma. This will be one of the main ingredients in the proof of the main result.

**Lemma 6.3.**

$$\begin{aligned}
\frac{d}{dt}L(t)=&\int w(x)\left[\frac{\sin(\theta_0(\xi,u,x))v^2(x)}{2}+\tilde{R}(v)(x)\right]\\
&+u\partial_x v(x)\left[\frac{\sin(\theta_0(\xi,u,x))v^2(x)}{2}+\tilde{R}(v)(x)\right]\,dx\\
&-\dot{u}\int\frac{\sin(\theta_0(\xi,u,x))}{2}\partial_u\theta_0(\xi,u,x)v^2(x)\,dx\\
&+(\dot{\xi}-u)\int\cos(\theta_0(\xi,u,x))v(x)\partial_x v(x)\,dx+\dot{u}\int w(x)\partial_x v(x)\,dx\\
&+\varepsilon\int\dot{v}(x)f(\varepsilon x)\,dx-u\dot{u}\gamma\varepsilon\int Z\theta_K'(Z)\left[f(\varepsilon\xi)+\frac{\varepsilon}{\gamma(u)}f'(\varepsilon\xi)Z+\dots\right]\,dZ\\
&+(u-\dot{\xi})\gamma\varepsilon\int\theta_K'(\gamma(x-\xi))f(\varepsilon x)\,dx-u\varepsilon^2\int vf'(\varepsilon x)\,dx\,.
\end{aligned}$$

**Proof.** We use a similar technique as in the proof of [KSK97, Lemma 2.1]. We can assume that the initial data of our problem has compact support. This allows us to do the following computations (integration by parts etc.). The claim for non-compactly supported initial data follows by density arguments. Firstly, we do some preliminary computations.

$$\begin{aligned}
&\int\frac{\partial_t[\cos\theta_0(\xi,u,x)]}{2}v^2(x)\,dx\\
=&\int-\frac{\sin(\theta_0(\xi,u,x))}{2}\dot{\xi}\partial_\xi\theta_0(\xi,u,x)v^2(x)-\frac{\sin(\theta_0(\xi,u,x))}{2}\dot{u}\partial_u\theta_0(\xi,u,x)v^2(x)\,dx
\end{aligned}$$

$$= \int -\dot{\xi}\frac{\partial_x(\cos(\theta_0(\xi,u,x)))}{2}v^2(x) - \frac{\sin(\theta_0(\xi,u,x))}{2}\dot{u}\partial_u\theta_0(\xi,u,x)v^2(x)\,dx$$

$$= \int \dot{\xi}\cos(\theta_0(\xi,u,x))v(x)\partial_x v(x) - \dot{u}\frac{\sin(\theta_0(\xi,u,x))}{2}\partial_u\theta_0(\xi,u,x)v^2(x)\,dx\,.$$

Notice that due to (4.3)-(4.4)

$$\begin{aligned} v(t,x) &= \theta(t,x) - \theta_0(\xi(t),u(t),x)\,,\\ w(t,x) &= \psi(t,x) - \psi_0(\xi(t),u(t),x) = \psi(t,x) + u\gamma\theta_K'(\gamma(x-\xi))\,. \end{aligned}$$

Partial integration yields $\int \partial_x v(x)\partial_x^2 v(x) + w(x)\partial_x w(x)\,dx = 0$. We differentiate the Lyapunov function (6.1) with respect to $t$ and use Lemma 4.4:

$$\begin{aligned} \dot{L}(t) = &\int w(x)\dot{w}(x) + \partial_x v(x)\partial_x\dot{v}(x) + \cos\theta_0(\xi,u,x)v(x)\dot{v}(x) + \frac{\partial_t[\cos\theta_0(\xi,u,x)]}{2}v^2(x)\\ &+ u\dot{w}(x)\partial_x v(x) + uw(x)\partial_x\dot{v}(x) + \dot{u}w(x)\partial_x v(x)\,dx\\ = &\int w(x)\Big[\partial_x^2 v(x) - \cos\theta_0(\xi,u,x)v(x) + \frac{\sin\theta_0(\xi,u,x)v^2(x)}{2}\\ &+ \tilde{R}(v)(x) + u\partial_\xi\psi_0(\xi,u,x) + \varepsilon f(\varepsilon x)\\ &- \dot{\xi}\partial_\xi\psi_0(\xi,u,x) - \dot{u}\partial_u\psi_0(\xi,u,x)\Big]\\ &+ \partial_x v(x)\partial_x\Big[w(x) - \dot{\xi}\partial_\xi\theta_0(\xi,u,x) - \dot{u}\partial_u\theta_0(\xi,u,x)\\ &+ u\partial_\xi\theta_0(\xi,u,x)\Big]\\ &+ \cos(\theta_0(\xi,u,x))v(x)\Big[w(x) - \dot{\xi}\partial_\xi\theta_0(\xi,u,x) - \dot{u}\partial_u\theta_0(\xi,u,x)\\ &+ u\partial_\xi\theta_0(\xi,u,x)\Big]\\ &+ \dot{\xi}\cos(\theta_0(\xi,u,x))v(x)\partial_x v(x) - \dot{u}\frac{\sin\theta_0(\xi,u,x)}{2}\partial_u\theta_0(\xi,u,x)v^2(x)\,dx\\ &+ u\partial_x v(x)\Big[\partial_x^2 v(x) - \cos\theta_0(\xi,u,x)v(x) + \frac{\sin\theta_0(\xi,u,x)v^2(x)}{2}\\ &+ \tilde{R}(v)(x) + u\partial_\xi\psi_0(\xi,u,x) + \varepsilon f(\varepsilon x)\\ &- \dot{\xi}\partial_\xi\psi_0(\xi,u,x) - \dot{u}\partial_u\psi_0(\xi,u,x)\Big] \end{aligned}$$

$$
\begin{aligned}
&+ uw(x)\partial_x\Big[w(x) - \dot{\xi}\partial_\xi\theta_0(\xi,u,x) - \dot{u}\partial_u\theta_0(\xi,u,x) \\
&+ u\partial_\xi\theta_0(\xi,u,x)\Big] + \dot{u}\int w(x)\partial_x v(x)\,dx \\
= (u-\dot{\xi})&\Bigg[\int -u\partial_x v(x)\Big\{-\partial_\xi\psi_0(\xi,u,x)\Big\} - w(x)\Big\{-\partial_\xi\psi_0(\xi,u,x)\Big\} \\
&+ \left[\cos(\theta_0(\xi,u,x))v(x) - \partial_x^2 v(x)\right]\partial_\xi\theta_0(\xi,u,x) + uw(x)\partial_x\partial_\xi\theta_0(\xi,u,x)\,dx\Bigg] \\
&- \dot{u}\Bigg[\int -u\partial_x v(x)\Big\{-\partial_u\psi_0(\xi,u,x)\Big\} - w(x)\Big\{-\partial_u\psi_0(\xi,u,x)\Big\} \\
&+ \left[\cos(\theta_0(\xi,u,x))v(x) - \partial_x^2 v(x)\right]\partial_u\theta_0(\xi,u,x) + uw(x)\partial_x\partial_u\theta_0(\xi,u,x)\,dx\Bigg] \\
&+ w(x)\left[\frac{\sin\theta_0(\xi,u,x)v^2(x)}{2} + \tilde{R}(v)(x)\right] + u\partial_x v(x)\left[\frac{\sin\theta_0(\xi,u,x)v^2(x)}{2} + \tilde{R}(v)(x)\right] \\
&- \dot{u}\int\frac{\sin(\theta_0(\xi,u,x))}{2}\partial_u\theta_0(\xi,u,x)v^2(x)\,dx + (\dot{\xi}-u)\int\cos(\theta_0(\xi,u,x))v(x)\partial_x v(x)\,dx \\
&+ \dot{u}\int w(x)\partial_x v(x)\,dx + \int w(x)\varepsilon f(\varepsilon x)\,dx + \int u\partial_x v(x)\varepsilon f(\varepsilon x)\,dx\,.
\end{aligned}
$$

We take a closer look at these terms and observe that some terms disappear. We will show later in Corollary 20.6 that

$$
\left\langle\begin{pmatrix} -u\partial_x v(\cdot) - w(\cdot) \\ -\partial_x^2 v(\cdot) + \cos(\theta_0(\xi,u,\cdot))v(\cdot) - u\partial_x w(\cdot)\end{pmatrix}, \begin{pmatrix} -\partial_\xi\psi_0(\xi,u,\cdot) \\ \partial_\xi\theta_0(\xi,u,\cdot)\end{pmatrix}\right\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} = 0\,,
$$

$$
\left\langle\begin{pmatrix} -u\partial_x v(\cdot) - w(\cdot) \\ -\partial_x^2 v(\cdot) + \cos(\theta_0(\xi,u,\cdot))v(\cdot) - u\partial_x w(\cdot)\end{pmatrix}, \begin{pmatrix} -\partial_u\psi_0(\xi,u,\cdot) \\ \partial_u\theta_0(\xi,u,\cdot)\end{pmatrix}\right\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} = 0\,.
$$

These identities can also be checked by hand using integration by parts and the symplectic orthogonality. We will use them in the following computations.

It holds that

$$
\begin{aligned}
&\Bigg[\int -u\partial_x v(x)\Big\{-\partial_\xi\psi_0(\xi,u,x)\Big\} - w(x)\Big\{-\partial_\xi\psi_0(\xi,u,x)\Big\}\\
&\quad + [\cos(\theta_0(\xi,u,x))v(x) - \partial_x^2 v(x)]\partial_\xi\theta_0(\xi,u,x) + uw(x)\partial_x\partial_\xi\theta_0(\xi,u,x)\,dx\Bigg]\\
&= \left\langle \begin{pmatrix} -u\partial_x v(\cdot) - w(\cdot) \\ -\partial_x^2 v(\cdot) + \cos(\theta_0(\xi,u,\cdot))v(\cdot) - u\partial_x w(\cdot)\end{pmatrix}, \begin{pmatrix} -\partial_\xi\psi_0(\xi,u,\cdot) \\ \partial_\xi\theta_0(\xi,u,\cdot)\end{pmatrix}\right\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})}\\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
&\Bigg[\int -u\partial_x v(x)\Big\{-\partial_u\psi_0(\xi,u,x)\Big\} - w(x)\Big\{-\partial_u\psi_0(\xi,u,x)\Big\}\\
&\quad + [\cos(\theta_0(\xi,u,x))v(x) - \partial_x^2 v(x)]\partial_u\theta_0(\xi,u,x) + uw(x)\partial_x\partial_u\theta_0(\xi,u,x)\,dx\Bigg]\\
&= \left\langle \begin{pmatrix} -u\partial_x v(\cdot) - w(\cdot) \\ -\partial_x^2 v(\cdot) + \cos(\theta_0(\xi,u,\cdot))v(\cdot) - u\partial_x w(\cdot)\end{pmatrix}, \begin{pmatrix} -\partial_u\psi_0(\xi,u,\cdot) \\ \partial_u\theta_0(\xi,u,\cdot)\end{pmatrix}\right\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})}\\
&= 0\,.
\end{aligned}
$$

Since

$$
\begin{aligned}
\partial_t[\theta_K(\gamma(x-\xi))] &= \Big[-u\dot{u}\gamma^3(x-\xi) - \gamma\dot{\xi}\Big]\,\theta_K'(\gamma(x-\xi))\\
&= \Big[-u\dot{u}\gamma^3(x-\xi) - \gamma\dot{\xi} + \gamma u\Big]\,\theta_K'(\gamma(x-\xi)) - \gamma u\theta_K'(\gamma(x-\xi))\,,
\end{aligned}
$$

we obtain

$$
\gamma u\theta_K'(\gamma(x-\xi)) = -\partial_t[\theta_K(\gamma(x-\xi))] + \Big[-u\dot{u}\gamma^3(x-\xi) + (u-\dot{\xi})\gamma\Big]\,\theta_K'(\gamma(x-\xi))\,.
$$

Thus using (1.1) and (4.3)-(4.4) it follows that

$$
\begin{aligned}
&\varepsilon\int wf(\varepsilon x)\,dx\\
&= \varepsilon\int [\psi(x) + u\gamma\theta_K'(\gamma(x-\xi))]\,f(\varepsilon x)\,dx
\end{aligned}
$$

$$
\begin{aligned}
&= \varepsilon \int \Big[\dot\theta(x) - \partial_t[\theta_K(\gamma(x-\xi))] + [-u\dot u\gamma^3(x-\xi) + (u-\dot\xi)\gamma]\theta_K'(\gamma(x-\xi))\Big]\, f(\varepsilon x)\, dx \\
&= \varepsilon \int \dot v(x) f(\varepsilon x)\, dx - u\dot u\gamma^3\varepsilon \int (x-\xi)\theta_K'(\gamma(x-\xi)) f(\varepsilon x)\, dx \\
&\quad + (u-\dot\xi)\gamma\varepsilon \int \theta_K'(\gamma(x-\xi)) f(\varepsilon x)\, dx \\
&= \varepsilon \int \dot v(x) f(\varepsilon x)\, dx - u\dot u\gamma\varepsilon \int Z\theta_K'(Z) f(\varepsilon(\frac{Z}{\gamma}+\xi))\, dZ \\
&\quad + (u-\dot\xi)\gamma\varepsilon \int \theta_K'(\gamma(x-\xi)) f(\varepsilon x)\, dx \\
&= \varepsilon \int \dot v(x) f(\varepsilon x)\, dx - u\dot u\gamma\varepsilon \int Z\theta_K'(Z)\Big[f(\varepsilon\xi) + \frac{\varepsilon}{\gamma(u)} f'(\varepsilon\xi) Z + \dots\Big]\, dZ \\
&\quad + (u-\dot\xi)\gamma\varepsilon \int \theta_K'(\gamma(x-\xi)) f(\varepsilon x)\, dx\,.
\end{aligned}
$$

Notice that

$$\int u\partial_x v(x)\varepsilon f(\varepsilon x)\, dx = -u\varepsilon^2 \int v(x) f'(\varepsilon x)\, dx\,.$$

Feeding the identities above in the formula for $\dot L(t)$ yields the claim.

□

# Chapter 7

# Lower Bound

In this chapter we introduce a functional $\mathcal{E}$ and prove a lower bound on $\mathcal{E}$ under the assumption that an orthogonality condition is satisfied. This will be one of the main ingredients in the proof of the main result. We will relate the functional $\mathcal{E}$ to the Lyapunov function $L$ later and obtain in this way a lower bound on $L$.

**Definition 7.1.** *For $(v, w) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$, $(\xi, u) \in \mathbb{R} \times (-1, 1)$ we set*

$$\mathcal{E}(v, w, \xi, u) := \frac{1}{2} \int (w(x) + u\partial_x v(x))^2 + v_Z^2(x) + \cos(\theta_K(Z))v^2(x)\, dx\,,$$

*where $Z = \gamma(x - \xi)$ and $v_Z(x) = \partial_Z v(\frac{Z}{\gamma} + \xi) = \frac{1}{\gamma}\partial_x v(x)$.*

In the following lemma we express $\mathcal{E}$ in a slightly different way.

**Lemma 7.2.**

$$\mathcal{E}(v, w, \xi, u) = \int \frac{w^2(x)}{2} + \frac{(\partial_x v(x))^2}{2} + \frac{\cos(\theta_K(\gamma(x - \xi)))v^2(x)}{2} + uw(x)\partial_x v(x)\, dx\,.$$

**Proof.** Using $Z = \gamma(x - \xi)$ and $v_Z(x) = \partial_Z v(\frac{Z}{\gamma} + \xi) = \frac{1}{\gamma}\partial_x v(x)$ we obtain:

$$\begin{aligned}
\mathcal{E}(v, w, \xi, u) &= \frac{1}{2} \int (w(x) + u\partial_x v(x))^2 + v_Z^2(x) + \cos(\theta_K(Z))v^2(x)\, dx \\
&= \frac{1}{2} \int w^2(x) + 2uw(x)\partial_x v(x) + u^2(\partial_x v(x))^2 \\
&\quad + \Big(\partial_Z v(\frac{Z}{\gamma} + \xi)\Big)^2 + \cos(\theta_K(Z))v^2(x)\, dx \\
&= \frac{1}{2} \int w^2(x) + 2uw(x)\partial_x v(x) + u^2(\partial_x v(x))^2 + \frac{1}{\gamma^2}(\partial_x v(x))^2 + \cos(\theta_K(Z))v^2(x)\, dx \\
&= \frac{1}{2} \int w^2(x) + 2uw(x)\partial_x v(x) + (\partial_x v(x))^2 + \cos(\theta_K(Z))v^2(x)\, dx\,.
\end{aligned}$$

□

Due to (4.3)-(4.4) $(v,w)$ is given by

$$v(t,x) = \theta(t,x) - \theta_0(\xi(t),u(t),x)\,,$$
$$w(t,x) = \psi(t,x) - \psi_0(\xi(t),u(t),x)\,.$$

Thus the orthogonality conditions

$$\mathcal{C}_1(\theta,\psi,\xi,u) = 0$$
$$\mathcal{C}_2(\theta,\psi,\xi,u) = 0$$

can be expressed in terms of the variables $(\theta,\psi,\xi,u)$ and in terms of the variables $(v,w,\xi,u)$. First we introduce a notation in order to be able to express the orthogonality conditions in terms of the variables $(v,w,\xi,u)$.

**Definition 7.3.**

$$\check{\mathcal{C}}_1(v,w,\xi,u) = \int \partial_\xi\psi_0(\xi,u,x)v(x) - \partial_\xi\theta_0(\xi,u,x)w(x)\,dx\,,$$
$$\check{\mathcal{C}}_2(v,w,\xi,u) = \int \partial_u\psi_0(\xi,u,x)v(x) - \partial_u\theta_0(\xi,u,x)w(x)\,dx\,.$$

Now we prove the lower bound on the functional $\mathcal{E}$ mentioned above.

**Lemma 7.4 (Stuart).** *There exists $c>0$ such that if $(\xi,u) \in \mathbb{R}\times[-U-U(2),U+U(2)] \subset \mathbb{R}\times(-1,1)$ and $(v,w) \in H^1(\mathbb{R})\times L^2(\mathbb{R})$ satisfies*

$$\check{\mathcal{C}}_2(v,w,\xi,u) = 0$$

*then*

$$\mathcal{E}(v,w,\xi,u) \geq c(|v|^2_{H^1(\mathbb{R})} + |w|^2_{L^2(\mathbb{R})})\,.$$

**Proof.** We follow closely [Stu12] and [Stu98]. This proof is a slight modification of the proof of [Stu12, Lemma 4.3]. Notice that the operator $-\partial_Z^2 + \cos\theta_K(Z)$ is nonnegative. It has (see [Stu92]) an one dimensional null space spanned by $\theta_K'(\cdot)$ and the essential spectrum $[1,\infty)$.

We argue by contradiction and asume that the following statement is false. For $\xi \in \mathbb{R}$ there exists $c = c(\xi) > 0$ such that if $u \in [-U-U(2),U+U(2)] \subset (-1,1)$ and $(v,w) \in H^1(\mathbb{R})\times L^2(\mathbb{R})$ satisfies $\check{\mathcal{C}}_2(v,w,\xi,u) = 0$ then

$$\mathcal{E}(v,w,\xi,u) \geq c(|v|^2_{H^1(\mathbb{R})} + |w|^2_{L^2(\mathbb{R})})\,.$$

This implies

$$\begin{aligned}&\exists\xi\in\mathbb{R}\ \ \forall j\in\mathbb{N}\ \exists u_j\in[-U-U(2),U+U(2)]\ \ \exists(\bar{v}_j,\bar{w}_j)\in H^1(\mathbb{R})\times L^2(\mathbb{R})\ :\\ &\check{\mathcal{C}}_2(\bar{v}_j,\bar{w}_j,\xi,u_j) = 0\,,\quad \mathcal{E}(\bar{v}_j,\bar{w}_j,\xi,u_j) < \frac{1}{j}(|\bar{v}_j|^2_{H^1(\mathbb{R})} + |\bar{w}_j|^2_{L^2(\mathbb{R})})\,,\end{aligned} \tag{7.1}$$

where

$$\mathcal{E}(\bar{v}_j, \bar{w}_j, \xi, u_j) = \frac{1}{2}\int (\bar{w}_j(x) + u_j\bar{v}_j'(x))^2 + [\frac{1}{\gamma(u_j)}(\bar{v}_j)'(x)]^2 + \cos(\theta_K(\gamma(u_j)(x-\xi)))\bar{v}_j^2(x)\,dx\,.$$

This statement is also true for the sequences

$$v_j := \frac{\bar{v}_j}{(|\bar{v}_j|^2_{H^1(\mathbb{R})} + |\bar{w}_j|^2_{L^2(\mathbb{R})})^{\frac{1}{2}}} \quad \text{and} \quad w_j := \frac{\bar{w}_j}{(|\bar{v}_j|^2_{H^1(\mathbb{R})} + |\bar{w}_j|^2_{L^2(\mathbb{R})})^{\frac{1}{2}}}\,.$$

Assuming that $|v_j|_{L^2(\mathbb{R})} \overset{j\to\infty}{\to} 0$ we get: $|(v_j)_x|_{L^2(\mathbb{R})} \overset{j\to\infty}{\to} 0$ and $|w_j|_{L^2(\mathbb{R})} \overset{j\to\infty}{\to} 0$. This is a contradiction to the fact that $|v_j|^2_{H^1(\mathbb{R})} + |w_j|^2_{L^2(\mathbb{R})} = 1\ \forall j \in \mathbb{N}$. By passing to a subsequence we may assume (without loss of generality) that there exists an $\bar{\delta} > 0$ such that

$$|v_j|^2_{L^2(\mathbb{R})} \geq \bar{\delta}\ \forall j \in \mathbb{N}\,. \tag{7.2}$$

Since $(v_j, w_j)$ is bounded in $H^1(\mathbb{R}) \times L^2(\mathbb{R})$ we may assume that $v_j \overset{H^1(\mathbb{R})}{\rightharpoonup} v$ and $w_j \overset{L^2(\mathbb{R})}{\rightharpoonup} w$ by taking subsequances. Due to Rellich's theorem we may assume again by passing to subsequences that $v_j \overset{L^2(\Omega)}{\to} v$ for $\Omega \subset \mathbb{R}$ bounded and open. Passing to a further subsequence we may assume almost everywhere convergence. Apart from that we may assume that $u_j \overset{\mathbb{R}}{\to} u$. The fact that

$$\exists\, r > 0 \quad \text{s.t.} \quad |\cos(\theta_K(Z))| > \frac{1}{2} \quad \text{for} \quad |Z| > r \tag{7.3}$$

and that $-\partial_Z^2 + \cos\theta_K(Z)$ is a nonnegative operator implies the following estimate if we assume that $(v_j, w_j) \in H^2(\mathbb{R}) \times L^2(\mathbb{R})$.

$$\begin{aligned}
&\mathcal{E}(v_j, w_j, \xi, u_j) \\
&= \frac{1}{2}\int (w_j(x) + u_j v_j'(x))^2 + [\frac{1}{\gamma(u_j)}(v_j)'(x)]^2 + \cos(\theta_K(\gamma(u_j)(x-\xi)))v_j^2(x)\,dx \\
&= \frac{1}{2\gamma(u_j)}\int (w_j(\frac{Z}{\gamma(u_j)} + \xi) + u_j v_j'(\frac{Z}{\gamma(u_j)} + \xi))^2 + [\frac{1}{\gamma(u_j)}(v_j)'(\frac{Z}{\gamma(u_j)} + \xi)]^2 \\
&\quad + \cos(Z)v_j^2(\frac{Z}{\gamma(u_j)} + \xi)\,dZ \\
&= \frac{1}{2\gamma(u_j)}\int (w_j(\frac{Z}{\gamma(u_j)} + \xi) + u_j v_j'(\frac{Z}{\gamma(u_j)} + \xi))^2 + [\partial_Z v_j(\frac{Z}{\gamma(u_j)} + \xi)]^2 \\
&\quad + \cos(Z)v_j^2(\frac{Z}{\gamma(u_j)} + \xi)\,dZ
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\gamma(u_j)} \int (w_j(\frac{Z}{\gamma(u_j)} + \xi) + u_j v_j'(\frac{Z}{\gamma(u_j)} + \xi))^2 - \partial_Z^2 v_j(\frac{Z}{\gamma(u_j)} + \xi) v_j(\frac{Z}{\gamma(u_j)} + \xi) \\
&\quad + \cos(Z) v_j^2(\frac{Z}{\gamma(u_j)} + \xi)\, dZ \\
&\geq \frac{1}{2\gamma(u_j)} \int_{\{Z\in\mathbb{R}:|Z|\geq r\}} (w_j(\frac{Z}{\gamma(u_j)} + \xi) + u_j v_j'(\frac{Z}{\gamma(u_j)} + \xi))^2 + [\partial_Z v_j(\frac{Z}{\gamma(u_j)} + \xi)]^2 \\
&\quad + \cos(Z) v_j^2(\frac{Z}{\gamma(u_j)} + \xi)\, dZ \\
&\geq \frac{1}{4\gamma(u_j)} \int_{\{Z\in\mathbb{R}:|Z|\geq r\}} v_j^2(\frac{Z}{\gamma(u_j)} + \xi)\, dZ \\
&= \frac{1}{4} \int_{-\infty}^{\frac{-r}{\gamma(u_j)}+\xi} v_j^2(x)\, dx + \frac{1}{4} \int_{\frac{r}{\gamma(u_j)}+\xi}^{\infty} v_j^2(x)\, dx\,.
\end{aligned}
$$

It follows by approximation

$$
\mathcal{E}(v_j, w_j, \xi, u_j) \geq \frac{1}{4} \int_{-\infty}^{\frac{-r}{\gamma(u_j)}+\xi} v_j^2(x)\, dx + \frac{1}{4} \int_{\frac{r}{\gamma(u_j)}+\xi}^{\infty} v_j^2(x)\, dx
$$

for $(v_j, w_j) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ and $(\xi, u_j)$ as above. Hence (7.1) implies that

$$
\int_{\{x\in\mathbb{R}:|x|\geq\tilde{r}\}} v_j^2(x)\, dx \overset{j\to\infty}{\to} 0
$$

for a sufficiently large $\tilde{r}$. As a consequence (7.2) and the strong convergence on bounded domains yield

$$
\int_{\{x\in\mathbb{R}:|x|\leq\tilde{r}\}} v^2(x)\, dx \geq \bar{\delta},
$$

from which it follows that $v \not\equiv 0$. Weak convergence implies using the triangle inequality:

$$
\check{\mathcal{C}}_2(v, w, \xi, u) = 0 \tag{7.4}
$$

and

$$
\frac{1}{2} \int \left(w(x) + uv'(x)\right)^2\, dx \leq \liminf_{j\to\infty} \frac{1}{2} \int \left(w_j(x) + u_j v_j'(x)\right)^2\, dx, \tag{7.5}
$$

$$
\frac{1}{2} \int \left(\frac{1}{\gamma(u)}(v)'(x)\right)^2\, dx \leq \liminf_{j\to\infty} \frac{1}{2} \int \left(\frac{1}{\gamma(u_j)}(v_j)'(x)\right)^2\, dx. \tag{7.6}
$$

Due to (7.3) we are able to apply Fatou's lemma for a sufficiently large $\tilde{r}$ and obtain:

$$\begin{aligned}&\frac{1}{2}\int_{\{x\in\mathbb{R}:|x|>\tilde{r}\}}\cos(\theta_K(\gamma(u)(x-\xi)))v^2(x)\,dx\\ &\leq \liminf_{j\to\infty}\frac{1}{2}\int_{\{x\in\mathbb{R}:|x|>\tilde{r}\}}\cos(\theta_K(\gamma(u_j)(x-\xi_j)))v_j^2(x)\,dx,\end{aligned}\tag{7.7}$$

where we have used that $(v_j)$ converges almost everywhere. Dominated convergence theorem yields:

$$\begin{aligned}&\frac{1}{2}\int_{\{x\in\mathbb{R}:|x|\leq\tilde{r}\}}\cos(\theta_K(\gamma(u)(x-\xi)))v^2(x)\,dx\\ &= \lim_{j\to\infty}\frac{1}{2}\int_{\{x\in\mathbb{R}:|x|\leq\tilde{r}\}}\cos(\theta_K(\gamma(u_j)(x-\xi_j)))v_j^2(x)\,dx\,.\end{aligned}\tag{7.8}$$

(7.1) together with (7.5)-(7.8) imply:

$$\mathcal{E}(v,w,\xi,u)=0\,.\tag{7.9}$$

Since $v\not\equiv 0$, (7.9) yields $(v(x),w(x))=\alpha(\theta_K'(\gamma(u)(x-\xi)),-u\gamma(u)\theta_K''(\gamma(u)(x-\xi)))$ for some $\alpha\neq 0$. This is a contradiction to (7.4). The constant $c$ does not depend on $\xi$, since

$$\check{\mathcal{C}}_2(v,w,\xi,u)=\check{\mathcal{C}}_2(v(\cdot+\xi),w(\cdot+\xi),0,u)=0$$

implies

$$\mathcal{E}(v,w,\xi,u)=\mathcal{E}(v(\cdot+\xi),w(\cdot+\xi),0,u)\geq c(0)(|v|^2_{H^1(\mathbb{R})}+|w|^2_{L^2(\mathbb{R})})\,.$$

□

**Remark 7.5.** Let $(v,w)$ be given by (4.3)-(4.4), with $(\xi,u)$ obtained from Lemma 3.1. It holds that

$$L(t)=\mathcal{E}(v(t),w(t),\xi(t),u(t))\,.$$

# Chapter 8

# Proof of Theorem 1.2

We prove Theorem 1.2.

## 8.1 Proof of Theorem 1.2 (i)

First we suppose that (1.1)-(1.3) has a solution and we make some assumptions on $(v, w)$ given by (4.3)-(4.4) and on $(\xi, u)$ obtained from Lemma 3.1. Then the following lemma yields us more accurate information about $(v, w)$ and $(\xi, u)$.

**Lemma 8.1.** *Let $\varepsilon$ be sufficiently small, $\rho(\delta) = \frac{1}{2} - 2\delta$. Assume that the assumptions (b),(c),(d) of Theorem 1.2 are satisfied. Assume that (1.1)-(1.3) has a solution $(\theta, \psi)$ on $[0, \overline{T}]$ such that*

$$(\theta, \psi) \in C^1([0, \overline{T}], L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R}))\,.$$

*Suppose that*

$$0 \le T \le t^* \le \overline{T}$$

*where $t^*$ is from Definition 4.1. Let $(v, w)$ be given by (4.3)-(4.4), with $(\xi, u)$ obtained from Lemma 3.1 such that*

$$|v|^2_{L^\infty([0,T],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,T],L^2(\mathbb{R}))} \le \varepsilon^{1-\delta}\,.$$

*Then , provided*

$$0 \le T \le \frac{1}{\varepsilon^{\rho(\delta)}},$$

*it holds*

- *(a)* $\forall t \in [0, T]: \ (\xi(t), u(t)) \in \Sigma(5, U)\,;$
- *(b)* $|v|^2_{L^\infty([0,T],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,T],L^2(\mathbb{R}))} \le \bar{C}(L(0) + \varepsilon)\,,$ *where $\bar{C}$ depends on $f$ (and c from Lemma 7.4).*

**Proof.** Choose $\varepsilon$ such that the following holds:

(1) $\varepsilon \in (0, \varepsilon_0)$ where $\varepsilon_0$ is from Lemma 5.1.

(2) $\varepsilon$ is so small that

$$C\varepsilon^{1-\delta}\left[\frac{1}{\varepsilon^{\rho(\delta)}}\right] + |u(0)| \leq \frac{U(5)}{2} + U\,,$$

where C is a constant that appears in (8.1) further in this proof which depends on $f$.

Lemma 5.1 yields $\forall t \in [0, T]$:

$$\begin{aligned} |\dot{\xi}(t) - u(t)| &\leq C\varepsilon + C\,|v|^2_{H^1(\mathbb{R})} \\ &\leq C\varepsilon + C\varepsilon^{1-\delta} \\ &\leq C\varepsilon^{1-\delta}, \\ |\dot{u}(t)| &\leq C\varepsilon + C\,|v|^2_{H^1(\mathbb{R})} \\ &\leq C\varepsilon + C\varepsilon^{1-\delta} \\ &\leq C\varepsilon^{1-\delta}. \end{aligned}$$

Thus we obtain $\forall t \in [0, T]$:

$$\begin{aligned} |u(t) - u(0)| &\leq \int_0^t |\dot{u}(s)|\, ds \\ &\leq C\varepsilon^{1-\delta} t \end{aligned}$$

$$\Rightarrow \quad |u(t)| \leq C\varepsilon^{1-\delta} t + |u(0)|\,. \tag{8.1}$$

This implies (a) due to assumptions (c) of Theorem 1.2 and (2). Using Lemma 6.3, Lemma 7.2 and Lemma 7.4 we obtain for times

$$0 \leq t \leq T \leq \frac{1}{\varepsilon^{\rho(\delta)}},$$

the following estimate,

$$\begin{aligned} &c(|v(t)|^2_{H^1(\mathbb{R})} + |w(t)|^2_{L^2(\mathbb{R})}) \\ &\leq L(t) = L(0) + \int_0^t \dot{L}(t)\, dt \end{aligned}$$

$$\begin{aligned}
&\leq L(0) + \int_0^t w(x)[\frac{\sin\theta_0(\xi,u,x)v^2(x)}{2} + \tilde{R}(v)(x)] + u\partial_x v(x)[\frac{\sin\theta_0(\xi,u,x)v^2(x)}{2} + \tilde{R}(v)(x)] \\
&\quad - \dot{u}\int \frac{\sin(\theta_0(\xi,u,x))}{2}\partial_u\theta_0(\xi,u,x)v^2(x)\,dx + (\dot{\xi}-u)\int \cos(\theta_0(\xi,u,x))v(x)\partial_x v(x)\,dx \\
&\quad + \dot{u}\int w(x)\partial_x v(x)\,dx \\
&\quad - u\dot{u}\gamma\varepsilon\int Z\theta_K'(Z)\Big[f(\varepsilon\xi) + \frac{\varepsilon}{\gamma(u)}f'(\varepsilon\xi)Z + \dots\Big]\,dZ \\
&\quad + (u-\dot{\xi})\gamma\varepsilon\int \theta_K'(\gamma(x-\xi))f(\varepsilon x)\,dx - u\varepsilon^2\int v(x)f'(\varepsilon x)\,dx\,dt \\
&\quad + |v(t,\cdot)|_{H^1(\mathbb{R})}\,\varepsilon^{\frac{1}{2}}\,|f(\cdot)|_{L^2(\mathbb{R})} + |v(0,\cdot)|_{H^1(\mathbb{R})}\,\varepsilon^{\frac{1}{2}}\,|f(\cdot)|_{L^2(\mathbb{R})} \\
&\leq L(0) + C\int_0^t \varepsilon^{1-\delta+\frac{1-\delta}{2}}\,dt + \frac{c}{8}|v(t,\cdot)|^2_{H^1(\mathbb{R})} + \frac{c}{8}|v(0,\cdot)|^2_{H^1(\mathbb{R})} + \frac{4}{c}|f(\cdot)|^2_{L^2(\mathbb{R})}\,\varepsilon\,,
\end{aligned}$$

since

$$-u\varepsilon^2\int v(x)f'(\varepsilon x)\,dx \leq |u|\varepsilon^{\frac{3}{2}}\,|v|_{L^2(\mathbb{R})}\,|f(\cdot)|_{L^2(\mathbb{R})}\,.$$

After bringing some terms on the left hand side we obtain

$$\tilde{c}(|v|^2_{L^\infty([0,t],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,t],L^2(\mathbb{R}))}) \leq L(0) + C\int_0^t \varepsilon^{1-\delta+\frac{1-\delta}{2}}\,dt + \frac{4}{c}|f(\cdot)|^2_{L^2(\mathbb{R})}\,\varepsilon\,.$$

□

**Theorem 8.2.** *Let $\varepsilon$ be sufficiently small, $\rho(\delta) = \frac{1}{2} - 2\delta$. Assume that the assumptions (b),(c),(d),(e) of Theorem 1.2 are satisfied. Assume that* (1.1)-(1.3) *has a a solution $(\theta,\psi)$ on $[0,\overline{T}]$ such that*

$$(\theta,\psi) \in C^1([0,\overline{T}], L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R}))\,.$$

*Suppose that*

$$0 \leq T \leq \overline{T}\,.$$

*Then, provided*

$$0 \leq T \leq \frac{1}{\varepsilon^{\rho(\delta)}},$$

*it holds that $(v,w)$ given by* (4.3)-(4.4) *is well defined for times $[0,T]$ and there exists a constant $\hat{c}$ such that* *(a)* $|v|^2_{L^\infty([0,T],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,T],L^2(\mathbb{R}))} \leq \hat{c}\varepsilon\,,$

*(b)* $\forall t \in [0,T]: \ (\xi(t), u(t)) \in \Sigma(5, U)\,.$

**Proof.** Choose $\varepsilon$ such that the following holds:

(1) $\varepsilon$ satisfies all smallness assumtions of Lemma 8.1;

(2) $2\bar{C}(L(0)+\varepsilon) < \varepsilon^{1-\delta}$, where $L(0) = \mathcal{E}(v(0), w(0), \xi_s, u_s)$ and $\bar{C}$ is from Lemma 8.1 (b);

(3) $\varepsilon$ is so small that if $(v,w) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ satisfies $|v|^2_{H^1(\mathbb{R})} + |w|^2_{L^2(\mathbb{R})} \leq \varepsilon^{1-\delta}$ then it holds that $|v|_{L^\infty(\mathbb{R})} + |w|_{L^2(\mathbb{R})} \leq \frac{r}{2}$, where $r$ is from Lemma 3.1. This can be ensured by Morrey's embedding theorem.

Notice that $\Sigma(5,U) \subset \Sigma(4,U)$. We define an exit time

$$t_* := \sup\Big\{T > 0 : |v|^2_{L^\infty([0,t],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,t],L^2(\mathbb{R}))} \leq 2\bar{C}(L(0)+\varepsilon),$$
$$(\xi(t), u(t)) \in \Sigma(5,U),\ 0 \leq t \leq T\Big\}.$$

Suppose $t_* < \frac{1}{\varepsilon^{\rho(\delta)}}$. Then there exists a time $\hat{t}$ s.t.

$$\frac{1}{\varepsilon^{\rho(\delta)}} > \hat{t} > t_*\,,$$

with

$$\forall t \in [0,\hat{t}]: \ (\xi(t), u(t)) \in \Sigma(4,U), \qquad (\xi(\hat{t}), u(\hat{t})) \notin \Sigma(5,U)$$

or

$$\bar{C}(L(0)+\varepsilon) < 2\bar{C}(L(0)+\varepsilon) < |v|^2_{L^\infty([0,\hat{t}],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,\hat{t}],L^2(\mathbb{R}))} < \varepsilon^{1-\delta}\,.$$

This leads a contradiction to the previous lemma. Thus

$$|v|^2_{L^\infty([0,T],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,T],L^2(\mathbb{R}))} \leq 2\bar{C}(L(0)+\varepsilon) \leq \hat{c}\varepsilon\,,$$

and

$$\forall t \in [0,T]: \ (\xi(t), u(t)) \in \Sigma(5,U)\,.$$

□

The previous theorem implies that the local solution of (1.1)-(1.3) discussed in Chapter 4 is indeed continuable up to times $\frac{1}{\varepsilon^{\rho(\delta)}}$, where $\rho(\delta) = \frac{1}{2} - 2\delta$. Theorem 8.2 and Lemma 5.1 yield the approximate equations for the parameters $(\xi, u)$. This verifies the claim of Theorem 1.2 (i).

## 8.2 Proof of Theorem 1.2 (ii)

### 8.2.1 ODE Analysis

In this section we lay the groundwork for passing from the approximate equations for the parameters $(\xi, u)$ in Theorem 1.2 (i) to the ODE's (1.4)-(1.5) in Theorem 1.2 (ii). We start with a preparing lemma.

**Lemma 8.3.** *Let* $\tilde{\xi} = \tilde{\xi}(s)$, $\tilde{u} = \tilde{u}(s)$, $\epsilon_1 = \epsilon_1(s)$, $\epsilon_2 = \epsilon_2(s)$ *be* $C^1$ *real-valued functions. Suppose that*

$$|\epsilon_j(s)| \leq \bar{c}\varepsilon^{\frac{3+\delta}{4}}$$

*on* $[0, T]$ *for* $j = 1, 2$. *Assume that on* $[0, T]$,

$$\frac{d}{ds}\tilde{\xi}(s) = \tilde{u}(s) + \epsilon_1(s)\,, \quad \tilde{\xi}(0) = \tilde{\xi}_0$$

$$\frac{d}{ds}\tilde{u}(s) = \epsilon_2(s)\,, \quad \tilde{u}(0) = \tilde{u}_0$$

*Let* $\hat{\xi} = \hat{\xi}(s)$ *and* $\hat{u} = \hat{u}(s)$ *be* $C^1$ *real-valued functions which satisfy the exact equations*

$$\frac{d}{ds}\hat{\xi}(s) = \hat{u}(s)\,, \quad \hat{\xi}(0) = \tilde{\xi}_0\,,$$

$$\frac{d}{ds}\hat{u}(s) = 0\,, \quad \hat{u}(0) = \tilde{u}_0\,.$$

*Then provided* $T \leq 1$, *there exists a* $c > 0$ *such that the estimates*

$$|\tilde{\xi}(s) - \hat{\xi}(s)| \leq c\varepsilon^{\frac{3+\delta}{4}}, \qquad |\tilde{u}(s) - \hat{u}(s)| \leq c\varepsilon^{\frac{3+\delta}{4}}\,.$$

*hold on* $[0, T]$.

**Proof.** In the following proof we follow very closely [HZ08, Lemma 6.1]. Let $x = x(s)$ and $y = y(s)$ be $C^1$ real-valued functions, $C \geq 1$, and $(x, y)$ satisfy the differential inequalities:

$$\begin{cases} |\dot{x}| \leq |y| & x(0) = x_0 \\ |\dot{y}| \leq 0 & y(0) = y_0 \end{cases}, \quad .$$

We are going to apply the Gronwall lemma. Let $z(s) = x^2 + y^2$. Then

$$|\dot{z}| = |2x\dot{x} + 2y\dot{y}| \leq 2|x||y| \leq (x^2 + y^2) = z\,,$$

and hence $z(s) \leq z(0)e^s$. Thus

$$\begin{aligned} |x(s)| &\leq \sqrt{2}\max(|x_0|, |y_0|)\exp(\frac{s}{2})\,, \\ |y(s)| &\leq \sqrt{2}\max(|x_0|, |y_0|)\exp(\frac{s}{2})\,. \end{aligned} \tag{8.2}$$

Now we recall the Duhamel's formula. Let $X(s) : \mathbb{R} \to \mathbb{R}^2$ be a two-vector function, $X_0 \in \mathbb{R}^2$ a two-vector, and $A(s) : \mathbb{R} \to (2 \times 2 \text{ matrices})$ a $2 \times 2$ matrix function. We consider the ODE system

$$\dot{X}(s) = A(s)X(s), \qquad X(s') = X_0$$

and denote its solution by $X(s) = S(s, s')X_0$ such that

$$\frac{d}{ds}S(s,s')X_0 = A(s)S(s,s')X_0\,, \quad S(s',s')X_0 = X_0\,.$$

Let $F(s) : \mathbb{R} \to \mathbb{R}^2$ be a two-vector function. We can express the solution to the inhomogeneous ODE system

$$\dot{X}(s) = A(s)X(s) + F(s)$$

with initial condition $X(0) = 0$ by the Duhamel's formula

$$X(s) = \int_0^s S(s,s')F(s')ds'\,.$$

Let $U = \hat{u} - \tilde{u}$ and $\Xi = \hat{\xi} - \tilde{\xi}$. These functions satisfy

$$\begin{aligned} \frac{d}{ds}\Xi(s) &= U(s) + \epsilon_1(s) \\ \frac{d}{ds}U(s) &= \epsilon_2(s) \end{aligned}$$

We set

$$A(s) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F(s) = \begin{bmatrix} \epsilon_1(s) \\ \epsilon_2(s) \end{bmatrix}, \quad X(s) = \begin{bmatrix} \Xi(s) \\ U(s) \end{bmatrix}$$

and obtain by Duhamel's formula:

$$\begin{bmatrix} \Xi(s) \\ U(s) \end{bmatrix} = \int_0^s S(s,s') \begin{bmatrix} \epsilon_1(s') \\ \epsilon_2(s') \end{bmatrix} ds' \tag{8.3}$$

We apply (8.2) with

$$\begin{bmatrix} x(s) \\ y(s) \end{bmatrix} = S(s+s', s') \begin{bmatrix} \epsilon_1(s') \\ \epsilon_2(s') \end{bmatrix}, \quad \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} \epsilon_1(s') \\ \epsilon_2(s') \end{bmatrix}.$$

It follows that

$$\left| S(s,s') \begin{bmatrix} \epsilon_1(s') \\ \epsilon_2(s') \end{bmatrix} \right| \leq \sqrt{2} \begin{bmatrix} \exp(\frac{1}{2}(s-s')) \\ \exp(\frac{1}{2}(s-s')) \end{bmatrix} \max(|\epsilon_1(s')|, |\epsilon_2(s')|)\,.$$

Using (8.3) we obtain that on $[0,T]$

$$|\Xi(s)| \leq \sqrt{2}\,T\mathrm{exp}(\frac{1}{2}T) \sup_{0\leq s\leq T} \max(|\epsilon_1(s)|, |\epsilon_2(s)|)\,,$$

$$|U(s)| \leq \sqrt{2}\,T\exp(\frac{1}{2}T) \sup_{0\leq s\leq T} \max(|\epsilon_1(s)|, |\epsilon_2(s)|)\,,$$

which yields the claim. □

In the following we show the relation between the parameters $(\xi, u)$ selected by the implicit function theorem according to Lemma 3.1 and the solutions $(\hat{\xi}, \hat{u})$ of the exact ODE's from the previous lemma.

**Lemma 8.4.** *Let $\varepsilon$ be sufficiently small, $\beta(\delta) = \frac{1-\delta}{4}$ and*

$$s = \varepsilon^{\beta(\delta)}t\,,$$

*where*

$$0 \leq s \leq 1, \quad 0 \leq t \leq \frac{1}{\varepsilon^{\beta(\delta)}}\,.$$

*Let $(\xi, u)$ be the parameters selected according to Lemma 3.1 and $(\hat{\xi}, \hat{u})$ from Lemma 8.3. Then it holds that*

$$|\xi(t) - \frac{\hat{\xi}(\varepsilon^{\beta(\delta)}t)}{\varepsilon^{\beta(\delta)}}| \leq \varepsilon^{\frac{1+\delta}{2}}\,,$$

$$|u(t) - \hat{u}(\varepsilon^{\beta(\delta)}t)| \leq \varepsilon^{\frac{3+\delta}{4}}\,.$$

**Proof.** We set

$$\tilde{\xi}(s) = \varepsilon^{\beta(\delta)}\xi(s/\varepsilon^{\beta(\delta)}), \qquad \tilde{u}(s) = u(s/\varepsilon^{\beta(\delta)})\,.$$

For times

$$0 \leq t \leq \frac{1}{\varepsilon^{\beta(\delta)}}$$

Lemma 5.1 and Theorem 8.2 yield:

$$\begin{aligned}|\dot{\xi}(t) - u(t)| &\leq C\varepsilon + C\,|v|^2_{H^1(\mathbb{R})}\\ &\leq C\varepsilon\,,\end{aligned}$$

$$\begin{aligned}|\dot{u}(t)| &\leq C\varepsilon + C\,|v|^2_{H^1(\mathbb{R})}\\ &\leq C\varepsilon\,.\end{aligned}$$

Thus $(\tilde{\xi}, \tilde{u})$ satisfy the assumptions of Lemma 8.3, since

$$\begin{aligned}
&\frac{d}{ds}\tilde{\xi}(s)\\
=&\xi'\left(\frac{s}{\varepsilon^{\beta(\delta)}}\right)\\
=&u\left(\frac{s}{\varepsilon^{\beta(\delta)}}\right) + \mathcal{O}(\varepsilon^1)\\
=&\tilde{u}(s) + \mathcal{O}(\varepsilon^1)\,,
\end{aligned}$$

$$\begin{aligned}
&\frac{d}{ds}\tilde{u}(s)\\
=&\frac{1}{\varepsilon^{\beta(\delta)}}u'\left(\frac{s}{\varepsilon^{\beta(\delta)}}\right)\\
=&\frac{1}{\varepsilon^{\beta(\delta)}}\mathcal{O}(\varepsilon^1)\\
=&\mathcal{O}(\varepsilon^{\frac{3+\delta}{4}})\,.
\end{aligned}$$

Hence Lemma 8.3 yields:

$$\begin{aligned}
&|\varepsilon^{\beta(\delta)}\xi(t) - \hat{\xi}(\varepsilon^{\beta(\delta)}t)| = |\tilde{\xi}(s) - \hat{\xi}(s)| \leq \varepsilon^{\frac{3+\delta}{4}}\\
\Rightarrow &|\xi(t) - \frac{\hat{\xi}(\varepsilon^{\beta(\delta)}t)}{\varepsilon^{\beta(\delta)}}| \leq \varepsilon^{1-2\beta(\delta)} = \varepsilon^{\frac{1+\delta}{2}}\,,
\end{aligned}$$

$$\begin{aligned}
&|u(t) - \hat{u}(\varepsilon^{\beta(\delta)}t)| = |\tilde{u}(s) - \hat{u}(s)| \leq \varepsilon^{\frac{3+\delta}{4}}\\
\Rightarrow &|u(t) - \hat{u}(\varepsilon^{\beta(\delta)}t)| \leq \varepsilon^{\frac{3+\delta}{4}}\,.
\end{aligned}$$

□

### 8.2.2 Completion of the Proof of Theorem 1.2 (ii)

Theorem 1.2 (i) yields the dynamics with the parameters $(\xi, u)$ selected by the implicit function theorem according to Lemma 3.1 on the time interval

$$0 \leq t \leq \frac{1}{\varepsilon^{\beta(\delta)}}\,.$$

Using Lemma 8.4 and the the triangle inequality we can replace

$$(\xi(t), u(t))$$

with

$$(\frac{\hat{\xi}(\varepsilon^{\beta(\delta)}t)}{\varepsilon^{\beta(\delta)}}, \hat{u}(\varepsilon^{\beta(\delta)}t))\,.$$

We set

$$(\bar{\xi}(t), \bar{u}(t)) = (\frac{\hat{\xi}(\varepsilon^{\beta(\delta)}t)}{\varepsilon^{\beta(\delta)}}, \hat{u}(\varepsilon^{\beta(\delta)}t))$$

and conclude that the equations claimed are satisfied. □

# Part II

# Classical Solitons in the Presence of a Forcing $\varepsilon^2 f(\varepsilon x)$

# Chapter 9

# Main Result and Overview

We use the notation from Definition 1.1 and we consider the initial value problem

$$\partial_t \begin{pmatrix} \theta(t,x) \\ \psi(t,x) \end{pmatrix} = \begin{pmatrix} \psi(t,x) \\ \partial_x^2 \theta(t,x) - \sin\theta(t,x) + \varepsilon^2 f(\varepsilon x) \end{pmatrix} , \tag{9.1}$$

$$\theta(0,x) = \theta_0(\xi_s, u_s, x) + v(0,x) \,, \tag{9.2}$$

$$\psi(0,x) = \psi_0(\xi_s, u_s, x) + w(0,x) \,, \tag{9.3}$$

where

$$(v(0,x), w(0,x)) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}).$$

The main result of Part II is the following theorem.

**Theorem 9.1.** *We consider the Cauchy problem defined by* (9.1)-(9.3) *and assume that*

- *(a)* $\varepsilon$ *is sufficiently small,*
- *(b)* $V \in H^4(\mathbb{R})$ *and* $f = V'$,
- *(c)* $(\xi_s, u_s) \in \mathbb{R} \times (-U, U)$ , *where* $0 < U < 1$;
- *(d)* $\mathcal{N}(\theta(0,x), \psi(0,x), \xi_s, u_s) = 0$ ,
- *(e)* $|v(0)|^2_{H^1(\mathbb{R})} + |w(0)|^2_{L^2(\mathbb{R})} \leq \varepsilon^3$.

*Then the Cauchy problem defined by* (9.1)-(9.3) *has a unique solution on the time interval*

$$0 \leq t \leq T, \ \textit{where}\ T := T(\varepsilon, \delta) = \frac{1}{\varepsilon^{\beta(\delta)}}, \quad \beta(\delta) = 1 - \delta \,.$$

*The solution may be written in the form*

$$\theta(t,x) = \theta_0\left(\bar{\xi}(t), \bar{u}(t), x\right) + v(t,x) \,,$$

$$\psi(t,x) = \psi_0\left(\bar{\xi}(t), \bar{u}(t), x\right) + w(t,x) \,,$$

*where $v, w$ have regularity*

$$(v(t), w(t)) \in C^1([0,T], H^1(\mathbb{R}) \oplus L^2(\mathbb{R})),$$

*$\bar{\xi}, \bar{u}$ solve the following system of equations*

$$\bar{\xi}'(t) = \bar{u}(t)\,, \tag{9.4}$$

$$\bar{u}'(t) = \,-\,\varepsilon^2 \frac{V'(\varepsilon\bar{\xi}(t))}{[\gamma(\bar{u}(t))]^3 m} \int \theta_K'(Z)\,dZ\,, \tag{9.5}$$

*with initial data $\bar{\xi}(0) = \xi_s,\ \bar{u}(0) = u_s$ and there exists a positive constant $c$ such that*

$$|v|^2_{L^\infty([0,T],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,T],L^2(\mathbb{R}))} \leq c\varepsilon^2\,.$$

*The constant $c$ depends on $V$.*

The following chapter-wise outline provides an overview of our approach. We use the notation and the results from Chapter 2 (Solitary Manifold ) and Chapter 3 (Symplectic Orthogonal Decomposition).

**Existence of Dynamics and the Orthogonal Component** The existence theory provides that there is a local solution $(\theta, \psi)$ of (9.1)-(9.3), which might be written in the form

$$\theta(t,x) = \bar{v}(t,x) + \theta_0(\xi_s, u_s, x)\,,$$

$$\psi(t,x) = \bar{w}(t,x) + \psi_0(\xi_s, u_s, x)\,,$$

where $(\bar{v}, \bar{w}) \in C^1([0, T_{loc}], H^1(\mathbb{R}) \oplus L^2(\mathbb{R}))$ . Due to Morrey's embedding theorem it holds that $(\theta, \psi) \in C^1([0, T_{loc}], L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R}))$. In the following we assume that a solution $(\theta, \psi)$ of (9.1)-(9.3) is given on the time interval $[0, \overline{T}]$, which might be written as above where $(\bar{v}, \bar{w}) \in C^1([0, \overline{T}], H^1(\mathbb{R}) \oplus L^2(\mathbb{R}))$ .

$\varepsilon$ is chosen so small that due to assumptions (c), (e) in Theorem 9.1 the initial state $(\theta(0), \psi(0))$ is so close to the region

$$\mathcal{S}_0(U) := \left\{ \begin{pmatrix} \theta_0(\xi, u, \cdot) \\ \psi_0(\xi, u, \cdot) \end{pmatrix} \ :\ (\xi, u) \in \Sigma(4, U) \right\}\,,$$

of the solitary manifold that the symplectic orthogonal decomposition is possible in a neighbourhood of $(\theta(0), \psi(0))$.

In (9.2)-(9.3) the initial state $(\theta(0), \psi(0))$ is already written as a sum of a point on the solitary manifold $\mathcal{S}_0$ and a transversal component $(v(0), w(0))$ such that the symplectic orthogonality condition is satisfied due to assumption (d) in Theorem 9.1.

For times $t > 0$ we are able to choose the parameters $(\xi(t), u(t))$ according to the decomposition in Chapter 3 (Symplectic Orthogonal Decomposition) as long as $(\theta(t), \psi(t))$ stays close enough to $\mathcal{S}_0(U)$. As long as $(\theta(t), \psi(t))$ stays close enough to $\mathcal{S}_0(U)$ we define $(v, w)$ by

$$v(t,x) = \theta(t,x) - \theta_0(\xi(t), u(t), x)\,, \tag{9.6}$$

$$w(t,x) = \psi(t,x) - \psi_0(\xi(t), u(t), x)\,, \tag{9.7}$$

where the parameter $(\xi(t), u(t))$ are obtained from the decomposition in Chapter 3 (Symplectic Orthogonal Decomposition), such that

$$\mathcal{N}(\theta(t), \psi(t), \xi(t), u(t)) = 0\,. \tag{9.8}$$

Thus we decompose the dynamics in two components, namely a point on the solitary manifold $(\theta_0(\xi(t), u(t), \cdot), \psi_0(\xi(t), u(t), \cdot))$ and a transversal component $(v(t, \cdot), w(t, \cdot))$ which is symplectic orthogonal to the tangent vectors

$$\begin{pmatrix} \partial_\xi \theta_0(\xi(t), u(t), \cdot) \\ \partial_\xi \psi_0(\xi(t), u(t), \cdot) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \partial_u \theta_0(\xi(t), u(t), \cdot) \\ \partial_u \psi_0(\xi(t), u(t), \cdot) \end{pmatrix}$$

of $\mathcal{S}_0$. Finally we compute the time derivatives of $v$ and $w$ which will be needed in the following chapters.

In Chapter 13 (Proof of Theorem 9.1) we will obtain a bound on $|v|^2_{L^\infty([0,T],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,T],L^2(\mathbb{R}))}$ (where $T \leq \overline{T}$) which will give us control over the distance of $(\theta, \psi)$ to the solitary manifold and which will imply that the local solution $(\theta, \psi)$ is indeed continuable.

**Modulation Equations** We want to consider the longitudinal dynamics on $\mathcal{S}_0$, which is described by the parameters $(\xi(t), u(t))$. In order to be able to understand the dynamics on $\mathcal{S}_0$ we derive a system of ordinary differential equations (modulation equations) for the parameters $(\xi(t), u(t))$ which is satisfied up to a certain error. We set

$$W(\varepsilon, \xi, u) := \frac{\varepsilon^2 V'(\varepsilon\xi) \int \theta_K'(Z)\, dZ}{[\gamma(u)]^3 m}\,.$$

for $(\xi, u) \in \mathbb{R} \times (-1, 1)$ and examine up to what errors the ordinary differential equations

$$\dot{\xi}(t) = u(t), \tag{9.9}$$

$$\dot{u}(t) + W(\varepsilon, \xi(t), u(t)) = 0, \tag{9.10}$$

are satisfied. For this purpose we take the time derivative of (9.8) and obtain a system of differential equations. Using Neumann's theorem we conclude that the estimates

$$\begin{aligned} |\dot{\xi}(t) - u(t)| &\leq C[|v|_{H^1(\mathbb{R})} + |w|_{L^2(\mathbb{R})}]\varepsilon^2 + C\varepsilon^3 + C\,|v|^2_{H^1(\mathbb{R})}\,, \\ |\dot{u}(t) + W(\varepsilon, \xi, u)| &\leq C[|v|_{H^1(\mathbb{R})} + |w|_{L^2(\mathbb{R})}]\varepsilon^2 + C\varepsilon^4 + C\,|v|^2_{H^1(\mathbb{R})}\,, \end{aligned}$$

are satisfied if $|v(t)|_{H^1(\mathbb{R})}\,, |w(t)|_{L^2(\mathbb{R})}$ are less than a certain $\varepsilon_0 > 0$ and as long as the time $t$ is such as described in the introduction of $(v, w)$ above. The reason for examining the equations (9.9)-(9.10) is that these are restricted Hamilton equations up to an error of order $\varepsilon^3$, which will be established in Section 13.2 (ODE Analysis).

**Lyapunov Functional** In order to obtain control on the transversal component $(v, w)$ we introduce the Lyapunov function

$$L(t) = \int \frac{w^2(x)}{2} + \frac{(\partial_x v(x))^2}{2} + \frac{\cos(\theta_K(\gamma(u)(x-\xi)))v^2(x)}{2} + uw(x)\partial_x v(x)\,dx\,,$$

where $(v, w)$ are given by (9.6)-(9.7), $(\xi, u)$ are obtained from the decomposition in Chapter 3 (Symplectic Orthogonal Decomposition), such that the orthogonality conditions hold and $\gamma(u) = 1/\sqrt{1-u^2}$. $L$ is the quadratic part of

$$H(\theta, \psi) + u\Pi(\theta, \psi)$$

above the solitary manifold $\mathcal{S}_0$, where $H$ and $\Pi$, given by (7) and (8) are conserved quantities of the sine-Gordon equation. Finally we compute the time derivative of $L(t)$ which will be needed later.

**Proof of Theorem 9.1** In Section 13.1 we prove the statement of Theorem 9.1 with a better bound on $(v, w)$ and with approximate equations for the parameters $(\xi, u)$ instead of the exact ODE's (9.4)-(9.5). We suppose that (9.1)-(9.3) has a solution and we make some assumptions on $(\xi, u)$ obtained from the decomposition in Chapter 3 (Symplectic Orthogonal Decomposition) and on $(v, w)$ given by (9.6)-(9.7). The modulation equations allow us to control $(\xi, u)$. The Lyapunov functions and the the lower bound on $\mathcal{E}$ allow us to control $(v, w)$, since we are able to estimate

$$\begin{aligned} &c(|v(t)|^2_{H^1(\mathbb{R})} + |w(t)|^2_{L^2(\mathbb{R})}) \\ &\leq L(t) \\ &= L(0) + \int_0^t \dot{L}(s)\,ds \end{aligned}$$

and to control the right hand side (after bringing some terms on the left hand side). All in all, we obtain more accurate information about $(v, w)$ and $(\xi, u)$. Using a continuity argument this implies the bound on $(v, w)$ claimed in Theorem 9.1 and approximate equations

for the parameters $(\xi, u)$. The bound on $(v, w)$ implies that the local solution discussed in Chapter 10 (Existence of Dynamics and the Orthogonal Component) is continuable up to times $\frac{1}{\varepsilon^{\beta(\delta)}}$, $\beta(\delta) = 1 - \delta$, which establishes the statement of Theorem 9.1 with approximate equations for $(\xi, u)$.
Motivated by [HL12, Section 3] we compute in Section 13.2 Hamiltonian's equations of motion for the restricted (to $\mathcal{S}_0$) Hamiltonian

$$(\xi, u) \mapsto H^{\varepsilon}(\theta_0(\xi, u, x), \psi_0(\xi, u, x)) \,,$$

with respect to the restricted (to $\mathcal{S}_0$) symplectic form, where

$$H^{\varepsilon}(\theta, \psi) = \frac{1}{2} \int \psi^2(x) + (\partial_x \theta(x))^2 + 2(1 - \cos\theta(x)) - 2\varepsilon^2 f(\varepsilon x)\theta(x)\, dx \,.$$

These equations coincide up to an error of the order $\varepsilon^3$ with the equations (9.9)-(9.10). Using Gronwall's lemma we show that the dynamics on the solitary manifold can be described by $(\bar{\xi}, \bar{u})$ which satisfy the ODE's (9.4)-(9.5).

# Chapter 10

# Existence of Dynamics and the Orthogonal Component

We argue similar to [Stu98, Proof of theorem 2.1]. In order to be able to make use of existence theory we set

$$\bar{v}(t,x) = \theta(t,x) - \theta_0(\xi_s, u_s, x)\,,$$
$$\bar{w}(t,x) = \psi(t,x) - \psi_0(\xi_s, u_s, x)$$

and consider the problem

$$\begin{pmatrix} \bar{v}(0,x) \\ \bar{w}(0,x) \end{pmatrix} = \begin{pmatrix} \theta(0,x) - \theta_0(\xi_s, u_s, x) \\ \psi(0,x) - \psi_0(\xi_s, u_s, x) \end{pmatrix}, \tag{10.1}$$

$$\partial_t \begin{pmatrix} \bar{v}(t,x) \\ \bar{w}(t,x) \end{pmatrix} = \begin{pmatrix} \bar{w}(t,x) - \psi_0(\xi_s, u_s, x) \\ [\bar{v}(t,x) + \theta_0(\xi_s, u_s, x)]_{xx} - \sin(\bar{v}(t,x) + \theta_0(\xi_s, u_s, x)) + \varepsilon^2 f(\varepsilon x) \end{pmatrix}. \tag{10.2}$$

By [Mar76, Theorem VIII 2.1, Theorem VIII 3.2 ] there exists a local solution (see also [Stu98, Proof of theorem 2.1], [Stu92, p.434 ]) with

$$(\bar{v}, \bar{w}) \in C^1([0, T_{loc}], H^1(\mathbb{R}) \oplus L^2(\mathbb{R}))\,.$$

$(\theta, \psi)$ given by $\theta(t,x) = \bar{v}(t,x) + \theta_0(\xi_s, u_s, x)$ and $\psi(t,x) = \bar{w}(t,x) + \psi_0(\xi_s, u_s, x)$ solves obviously locally the Cauchy problem (9.1)-(9.3) and $(\theta, \psi) \in C^1([0, T_{loc}], L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R}))$ due to Morrey's embedding theorem.

We are going to obtain some bounds in Chapter 13 which will imply that the local solutions are indeed continuable.

So from now we assume that $(\bar{v}, \bar{w}) \in C^1([0, \overline{T}], H^1(\mathbb{R}) \oplus L^2(\mathbb{R}))$ is a solution of (10.1)-(10.2) and $(\theta, \psi)$ is a solution of (9.1)-(9.3) such that $(\theta, \psi) \in C^1([0, \overline{T}], L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R}))$. Given $(\theta, \psi)$ we choose the parameters $(\xi(t), u(t))$ according to Lemma 3.1 and define $(v, w)$ as follows:

$$v(t,x) = \theta(t,x) - \theta_0(\xi(t), u(t), x)\,, \tag{10.3}$$

$$w(t,x) = \psi(t,x) - \psi_0(\xi(t), u(t), x)\,. \tag{10.4}$$

$(v(t,x), w(t,x))$ is well defined for $t \geq 0$ so small that

$$|v(t)|_{L^\infty(\mathbb{R})} + |w(t)|_{L^2(\mathbb{R})} \leq r$$

and

$$(\xi(t), u(t)) \in \Sigma(4, U)\,,$$

where $r$ and $U$ are from Lemma 3.1. We formalize this by the following definition.

**Definition 10.1.** *Let $t^*$ be the "exit time":*

$$t^* := \sup \Big\{ T > 0 : |v|_{L^\infty(\mathbb{R}) L^\infty([0,t])} + |w|_{L^\infty([0,t], L^2(\mathbb{R}))} \leq r,$$
$$(\xi(t), u(t)) \in \Sigma(4, U),\ 0 \leq t \leq T \Big\}\,,$$

*where $r$ and $U$ are from Lemma 3.1.*

Notice that $(\xi_s, u_s) = (\xi(0), u(0)) \in \Sigma(4, U)$. We will choose $\varepsilon$ such that, among others,

$$|v(0)|_{L^\infty(\mathbb{R})} + |w(0)|_{L^2(\mathbb{R})} \leq \frac{r}{2},$$

where $(v(0), w(0))$ is given by (9.2)-(9.3). Thus $(v(t,x), w(t,x))$ is well defined for $0 \leq t \leq t^*$. In the following lemma we obtain more information on $(v, w)$.

**Lemma 10.2.** *Let $T = \min\{t^*, \overline{T}\}$ and let $(v, w)$ be defined by* (10.3)-(10.4)*. Then $(v, w) \in C^1([0, T], H^1(\mathbb{R}) \oplus L^2(\mathbb{R}))$.*

**Proof.** Analogous to Lemma 4.3. □

We compute the time derivatives of $v$ and $w$ which will be needed in the following chapters.

**Lemma 10.3.** *The equations for $(v, w)$ defined by* (10.3)-(10.4)*, are*

$$\dot{v}(x) = w(x) - \dot{\xi}\partial_\xi \theta_0(\xi, u, x) - \dot{u}\partial_u \theta_0(\xi, u, x) + u\partial_\xi \theta_0(\xi, u, x)\,,$$

$$\dot{w}(x) = \partial_x^2 v(x) - \cos\theta_0(\xi, u, x) v(x) + \varepsilon^2 f(\varepsilon x) + \frac{\sin\theta_0(\xi, u, x) v^2(x)}{2} + \tilde{R}(v)(x)$$
$$+ u\partial_\xi \psi_0(\xi, u, x) - \dot{\xi}\partial_\xi \psi_0(\xi, u, x) - \dot{u}\partial_u \psi_0(\xi, u, x)\,,$$

*for times $t \in [0, t^*]$, where $\tilde{R}(v) = \mathcal{O}(|v|^3_{H^1(\mathbb{R})})$.*

**Proof.** We take the time derivatives of $(v, w)$ and use (10.3)-(10.4), (9.1):

$$\begin{aligned}
\dot{v}(x) &= w(x) + \psi_0(\xi, u, x) \\
&\quad - \dot{\xi}\partial_\xi \theta_0(\xi, u, x) - \dot{u}\partial_u \theta_0(\xi, u, x) \\
&= w(x) - \dot{\xi}\partial_\xi \theta_0(\xi, u, x) - \dot{u}\partial_u \theta_0(\xi, u, x) + u\partial_\xi \theta_0(\xi, u, x)\,,
\end{aligned}$$

$$\begin{aligned}
\dot{w}(x) &= \partial_x^2 \theta(x) - \sin\theta(x) + \varepsilon^2 f(\varepsilon x) - \dot{\xi}\partial_\xi \psi_0(\xi, u, x) - \dot{u}\partial_u \psi_0(\xi, u, x) \\
&= \partial_x^2 \theta_0(\xi, u, x) + \partial_x^2 v(x) - \sin\theta_0(\xi, u, x) \\
&\quad - \cos\theta_0(\xi, u, x) v(x) + \frac{\sin\theta_0(\xi, u, x) v^2(x)}{2} \\
&\quad + \tilde{R}(v)(x) + \varepsilon^2 f(\varepsilon x) - \dot{\xi}\partial_\xi \psi_0(\xi, u, x) - \dot{u}\partial_u \psi_0(\xi, u, x) \\
&\quad + u\partial_x \psi_0(\xi, u, x) - u\partial_x \psi_0(\xi, u, x) \\
&= \partial_x^2 v(x) - \cos\theta_0(\xi, u, x) v(x) + \varepsilon^2 f(\varepsilon x) + \frac{\sin\theta_0(\xi, u, x) v^2(x)}{2} + \tilde{R}(v)(x) \\
&\quad + u\partial_\xi \psi_0(\xi, u, x) - \dot{\xi}\partial_\xi \psi_0(\xi, u, x) - \dot{u}\partial_u \psi_0(\xi, u, x)\,,
\end{aligned}$$

where we have expanded the term $\sin(\theta_0(\xi, u, x) + v(x))$. ☐

# Chapter 11

# Modulation Equations

In this chapter we derive modulation equations for the parameters $(\xi(t), u(t))$.

**Definition 11.1.** *Let* $(\xi, u) \in \mathbb{R} \times (-1, 1)$. *We set*

$$W(\varepsilon, \xi, u) := \frac{\varepsilon^2 V'(\varepsilon\xi) \int \theta_K'(Z)\, dZ}{[\gamma(u)]^3 m} \,.$$

**Lemma 11.2.** *There exists an* $\varepsilon_0 > 0$ *such that the following statement holds. Let* $(v, w)$ *be given by* (10.3)-(10.4)*, with* $(\xi, u)$ *obtained from Lemma 3.1 and let*

$$|v|_{L^\infty([0,t^*],H^1(\mathbb{R}))} \,, |w|_{L^\infty([0,t^*],L^2(\mathbb{R}))} \leq \varepsilon_0 \,,$$

*where* $t^*$ *is from Definition 10.1. Then*

$$|\dot{\xi}(t) - u(t)| \leq C[|v|_{H^1(\mathbb{R})} + |w|_{L^2(\mathbb{R})}]\varepsilon^2 + C\varepsilon^3 + C\,|v|^2_{H^1(\mathbb{R})} \,,$$

$$|\dot{u}(t) + W(\varepsilon, \xi, u)| \leq C[|v|_{H^1(\mathbb{R})} + |w|_{L^2(\mathbb{R})}]\varepsilon^2 + C\varepsilon^4 + C\,|v|^2_{H^1(\mathbb{R})} \,,$$

*for* $0 \leq t \leq t^*$*, where* $C$ *depends on* $f$.

**Proof.** The technique we use is similar to that in the proof of [IKV12, Lemma 6.2]. Using Definition 1.1 and (10.3)-(10.4) we write the orthogonality conditions as follows:

$$0 = \mathcal{C}_1(\theta, \psi, \xi, u) = \int \partial_\xi \psi_0(\xi, u, x) v(x) - \partial_\xi \theta_0(\xi, u, x) w(x)\, dx$$

$$0 = \mathcal{C}_2(\theta, \psi, \xi, u) = \int \partial_u \psi_0(\xi, u, x) v(x) - \partial_u \theta_0(\xi, u, x) w(x)\, dx$$

In the following we skip $(\theta, \psi, \xi, u)$ for simplicity of further notation and take the derivatives of $\mathcal{C}_1$, $\mathcal{C}_2$ with respect to $t$. Using Lemma 10.3 we obtain:

$$
\begin{aligned}
\dot{\mathcal{C}}_1 = & \int \partial_t[\partial_\xi \psi_0(\xi,u,x)]v(x) + \partial_\xi \psi_0(\xi,u,x)\partial_t v(x) \\
& - \partial_t[\partial_\xi \theta_0(\xi,u,x)]w(x) - \partial_\xi \theta_0(\xi,u,x)\partial_t w(x)\,dx \\
= & \int \left\{ \dot{\xi}\partial_\xi^2 \psi_0(\xi,u,x) + \dot{u}\partial_u \partial_\xi \psi_0(\xi,u,x) \right\} v(x) \\
& + \left\{ \partial_\xi \psi_0(\xi,u,x) \right\} \left\{ w(x) - \dot{\xi}\partial_\xi \theta_0(\xi,u,x) - \dot{u}\partial_u \theta_0(\xi,u,x) \right. \\
& \left. + u\partial_\xi \theta_0(\xi,u,x) \right\} \\
& - \left\{ \dot{\xi}\partial_\xi^2 \theta_0(\xi,u,x) + \dot{u}\partial_u \partial_\xi \theta_0(\xi,u,x) \right\} w(x) \\
& - \left\{ \partial_\xi \theta_0(\xi,u,x) \right\} \left\{ \partial_x^2 v(x) - \cos\theta_0(\xi,u,x)v(x) + \varepsilon^2 f(\varepsilon x) + \frac{\sin\theta_0(\xi,u,x)v^2(x)}{2} \right. \\
& \left. + \tilde{R}(v)(x) + u\partial_\xi \psi_0(\xi,u,x) - \dot{\xi}\partial_\xi \psi_0(\xi,u,x) - \dot{u}\partial_u \psi_0(\xi,u,x) \right\} dx \\
= & \underbrace{\int -\partial_\xi \psi_0(\xi,u,x)\partial_u \theta_0(\xi,u,x) + \partial_\xi \theta_0(\xi,u,x)\partial_u \psi_0(\xi,u,x)\,dx}_{=\Omega\left(t_\xi(\xi,u,\cdot),\,t_u(\xi,u,\cdot)\right)} \cdot(\dot{u} + W(\varepsilon,\xi,u)) \\
& + \underbrace{\int \partial_\xi \psi_0(\xi,u,x)\partial_\xi \theta_0(\xi,u,x) - \partial_\xi \psi_0(\xi,u,x)\partial_\xi \theta_0(\xi,u,x)\,dx}_{=\Omega\left(t_\xi(\xi,u,\cdot),\,t_\xi(\xi,u,\cdot)\right)} \cdot(u - \dot{\xi}) \\
& + \underbrace{\int \partial_u \partial_\xi \psi_0(\xi,u,x)v(x) - \partial_u \partial_\xi \theta_0(\xi,u,x)w(x)\,dx}_{=[M(\xi,u,v,w)]_{12}} \cdot(\dot{u} + W(\varepsilon,\xi,u)) \\
& - \underbrace{\int \partial_\xi^2 \psi_0(\xi,u,x)v(x) - \partial_\xi^2 \theta_0(\xi,u,x)w(x)\,dx}_{=[M(\xi,u,v,w)]_{11}} \cdot(u - \dot{\xi})
\end{aligned}
$$

$$\underbrace{\begin{aligned}
&+\int \partial_\xi\psi_0(\xi,u,x)w(x) - \partial_\xi\theta_0(\xi,u,x)\left(\partial_x^2 v(x) - \cos\theta_0(\xi,u,x)v(x)\right) dx \\
&+\int u\partial_\xi^2\psi_0(\xi,u,x)v(x) - u\partial_\xi^2\theta_0(\xi,u,x)w(x)\,dx \\
&-\int \left\{\partial_\xi\theta_0(\xi,u,x)\right\}\left\{\varepsilon^2 f(\varepsilon x)\right\} dx - W(\varepsilon,\xi,u)\gamma(u)^3 m \\
&-\int \partial_u\partial_\xi\psi_0(\xi,u,x)v(x) - \partial_u\partial_\xi\theta_0(\xi,u,x)w(x)\,dx\cdot W(\varepsilon,\xi,u) \\
&-\int \partial_\xi\theta_0(\xi,u,x)\left\{\frac{\sin\theta_0(\xi,u,x)v^2(x)}{2} + \tilde{R}(v)(x)\right\} dx\,,
\end{aligned}}_{=[P(\xi,u,v,w)]_1}$$

and

$$\begin{aligned}
\dot{\mathcal{C}}_2 = &\int \partial_t[\partial_u\psi_0(\xi,u,x)]v(x) + \partial_u\psi_0(\xi,u,x)\partial_t v(x) \\
&- \partial_t[\partial_u\theta_0(\xi,u,x)]w(x) - \partial_u\theta_0(\xi,u,x)\partial_t w(x)\,dx \\
= &\int \left\{\dot{\xi}\partial_\xi\partial_u\psi_0(\xi,u,x) + \dot{u}\partial_u^2\psi_0(\xi,u,x)\right\}v(x) \\
&+ \left\{\partial_u\psi_0(\xi,u,x)\right\}\left\{w(x) - \dot{\xi}\partial_\xi\theta_0(\xi,u,x) - \dot{u}\partial_u\theta_0(\xi,u,x)\right. \\
&\left. + u\partial_\xi\theta_0(\xi,u,x)\right\} \\
&- \left\{\dot{\xi}\partial_\xi\partial_u\theta_0(\xi,u,x) + \dot{u}\partial_u^2\theta_0(\xi,u,x)\right\}w(x)
\end{aligned}$$

$$
\begin{aligned}
&- \left\{ \partial_u \theta_0(\xi, u, x) \right\} \left\{ \partial_x^2 v(x) - \cos\theta_0(\xi, u, x) v(x) + \varepsilon^2 f(\varepsilon x) \right. \\
&+ \frac{\sin\theta_0(\xi, u, x) v^2(x)}{2} + \tilde{R}(v)(x) \\
&\left. + u \partial_\xi \psi_0(\xi, u, x) - \dot{\xi} \partial_\xi \psi_0(\xi, u, x) - \dot{u} \partial_u \psi_0(\xi, u, x) \right\} dx \\
= {} & \underbrace{\int \partial_u \theta_0(\xi, u, x) \partial_u \psi_0(\xi, u, x) - \partial_u \theta_0(\xi, u, x) \partial_u \psi_0(\xi, u, x) \, dx}_{= \Omega\left(t_u(\xi, u, \cdot), t_u(\xi, u, \cdot)\right)} \cdot (\dot{u} + W(\varepsilon, \xi, u)) \\
&+ \underbrace{\int \partial_\xi \theta_0(\xi, u, x) \partial_u \psi_0(\xi, u, x) - \partial_\xi \psi_0(\xi, u, x) \partial_u \theta_0(\xi, u, x) \, dx}_{= \Omega\left(t_u(\xi, u, \cdot), t_\xi(\xi, u, \cdot)\right)} \cdot (u - \dot{\xi}) \\
&+ \underbrace{\int \partial_u^2 \partial_\xi \psi_0(\xi, u, x) v(x) - \partial_u^2 \theta_0(\xi, u, x) w(x) \, dx}_{= [M(\xi, u, v, w)]_{22}} \cdot (\dot{u} + W(\varepsilon, \xi, u)) \\
&- \underbrace{\int \partial_\xi \partial_u \psi_0(\xi, u, x) v(x) - \partial_\xi \partial_u \theta_0(\xi, u, x) w(x) \, dx}_{= [M(\xi, u, v, w)]_{21}} \cdot (u - \dot{\xi}) \\
& \underbrace{\begin{aligned}
&+ \int \left\{ \partial_u \psi_0(\xi, u, x) \right\} w(x) - \partial_u \theta_0(\xi, u, x) \left( \partial_x^2 v(x) - \cos\theta_0(\xi, u, x) v(x) \right) dx \\
&+ u \int \partial_\xi \partial_u \psi_0(\xi, u, x) v(x) - \partial_\xi \partial_u \theta_0(\xi, u, x) w(x) \, dx \\
&- \int \left\{ \partial_u \theta_0(\xi, u, x) \right\} \left\{ \varepsilon^2 f(\varepsilon x) \right\} dx \\
&- \int \partial_u^2 \partial_\xi \psi_0(\xi, u, x) v(x) - \partial_u^2 \theta_0(\xi, u, x) w(x) \, dx \cdot W(\varepsilon, \xi, u) \\
&- \int \left\{ \partial_u \theta_0(\xi, u, x) \right\} \left\{ \frac{\sin\theta_0(\xi, u, x) v^2(x)}{2} + \tilde{R}(v)(x) \right\} dx
\end{aligned}}_{= [P(\xi, u, v, w)]_2}
\end{aligned}
$$

We set

$$\begin{aligned}\Omega(u) &:= \begin{pmatrix} \Omega\left(t_\xi(\xi,u,\cdot),t_\xi(\xi,u,\cdot)\right) & \Omega\left(t_\xi(\xi,u,\cdot),t_u(\xi,u,\cdot)\right) \\ \Omega\left(t_u(\xi,u,\cdot),t_\xi(\xi,u,\cdot)\right) & \Omega\left(t_u(\xi,u,\cdot),t_u(\xi,u,\cdot)\right)\end{pmatrix} \\ &= \begin{pmatrix} 0 & \Omega\left(t_\xi(\xi,u,\cdot),t_u(\xi,u,\cdot)\right) \\ \Omega\left(t_u(\xi,u,\cdot),t_\xi(\xi,u,\cdot)\right) & 0\end{pmatrix} \\ &= \begin{pmatrix} 0 & \gamma(u)^3 m \\ -\gamma(u)^3 m & 0\end{pmatrix} \\ &= \gamma(u)^3 m \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}.\end{aligned}$$

Now we consider for any $(\bar{\xi},\bar{u}) \in \mathbb{R}\times[-U-U(2),U+U(2)]$, $(\bar{v},\bar{w}) \in H^1(\mathbb{R})\times L^2(\mathbb{R})$ the matrix:

$$\begin{aligned}&M(\bar{\xi},\bar{u},\bar{v},\bar{w}) \\ &= \begin{pmatrix} \left\langle \begin{pmatrix} \partial_\xi^2\psi_0(\bar{\xi},\bar{u},\cdot) \\ -\partial_\xi^2\theta_0(\bar{\xi},\bar{u},\cdot)\end{pmatrix}, \begin{pmatrix}\bar{v}(\cdot) \\ \bar{w}(\cdot)\end{pmatrix}\right\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} & \left\langle \begin{pmatrix} \partial_u\partial_\xi\psi_0(\bar{\xi},\bar{u},\cdot) \\ -\partial_u\partial_\xi\theta_0(\bar{\xi},\bar{u},\cdot)\end{pmatrix}, \begin{pmatrix}\bar{v}(\cdot) \\ \bar{w}(\cdot)\end{pmatrix}\right\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} \\ \left\langle \begin{pmatrix} \partial_\xi\partial_u\psi_0(\bar{\xi},\bar{u},\cdot) \\ -\partial_\xi\partial_u\theta_0(\bar{\xi},\bar{u},\cdot)\end{pmatrix}, \begin{pmatrix}\bar{v}(\cdot) \\ \bar{w}(\cdot)\end{pmatrix}\right\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} & \left\langle \begin{pmatrix} \partial_u^2\psi_0(\bar{\xi},\bar{u},\cdot) \\ -\partial_u^2\theta_0(\bar{\xi},\bar{u},\cdot)\end{pmatrix}, \begin{pmatrix}\bar{v}(\cdot) \\ \bar{w}(\cdot)\end{pmatrix}\right\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})}\end{pmatrix}.\end{aligned}$$

We use Hölder's inequality and obtain for all $(\bar{\xi},\bar{u}) \in \mathbb{R}\times[-U-U(2),U+U(2)]$, $(\bar{v},\bar{w}) \in H^1(\mathbb{R})\times L^2(\mathbb{R})$:

$$\| \left[\Omega(\bar{u})\right]^{-1} M(\bar{\xi},\bar{u},\bar{v},\bar{w})\| \leq C(|\bar{v}|_{H^1(\mathbb{R})} + |\bar{w}|_{L^2(\mathbb{R})}), \tag{11.1}$$

where we denote by $\|\cdot\|$ a matrix norm. Let $I = I_2$ be the identety matrix of dimension 2. Due to (11.1) we are able to find an $\varepsilon_0 > 0$ such that if $|\bar{v}|_{H^1(\mathbb{R})}, |\bar{w}|_{L^2(\mathbb{R})} \leq \varepsilon_0$ then the matrix

$$I + \left[\Omega(\bar{u})\right]^{-1} M(\bar{\xi},\bar{u},\bar{v},\bar{w})$$

is invertible by Neumann's theorem. We write the time derivatives of $(\mathcal{C}_1^\varepsilon,\mathcal{C}_2^\varepsilon)$ in matrix form and use the notation $P(\xi,u,v,w) = P$, $M(\xi,u,v,w) = M$, $\Omega(u) = \Omega$:

$$\begin{aligned}0 &= \begin{pmatrix}\dot{\mathcal{C}}_1 \\ \dot{\mathcal{C}}_2\end{pmatrix} \\ &= \Omega\begin{pmatrix}\dot{\xi}-u \\ \dot{u}+W(\varepsilon,\xi,u)\end{pmatrix} + M\begin{pmatrix}\dot{\xi}-u \\ \dot{u}+W(\varepsilon,\xi,u)\end{pmatrix} + P\end{aligned}$$

This implies

$$-\Omega^{-1}P = \Big(I + \Omega^{-1}M\Big)\begin{pmatrix}\dot{\xi} - u\\ \dot{u} + W(\varepsilon,\xi,u)\end{pmatrix}.$$

If $|v|_{H^1(\mathbb{R})}\,, |w|_{L^2(\mathbb{R})} \le \varepsilon_0$ then we obtain as mentioned above by Neumann's theorem that

$$\begin{pmatrix}\dot{\xi} - u\\ \dot{u} + W(\varepsilon,\xi,u)\end{pmatrix} = -\Big(I + \Omega^{-1}M\Big)^{-1}[\Omega^{-1}P]\,.$$

We will show later in Corollary 20.6 that

$$\left\langle \begin{pmatrix} -u\partial_x v(\cdot) - w(\cdot)\\ -\partial_x^2 v(\cdot) + \cos(\theta_0(\xi,u,\cdot))v(\cdot) - u\partial_x w(\cdot)\end{pmatrix}, \begin{pmatrix}-\partial_\xi\psi_0(\xi,u,\cdot)\\ \partial_\xi\theta_0(\xi,u,\cdot)\end{pmatrix}\right\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} = 0\,,$$

$$\left\langle \begin{pmatrix} -u\partial_x v(\cdot) - w(\cdot)\\ -\partial_x^2 v(\cdot) + \cos(\theta_0(\xi,u,\cdot))v(\cdot) - u\partial_x w(\cdot)\end{pmatrix}, \begin{pmatrix}-\partial_u\psi_0(\xi,u,\cdot)\\ \partial_u\theta_0(\xi,u,\cdot)\end{pmatrix}\right\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} = 0\,.$$

These identities can also be checked by hand using integration by parts and the symplectic orthogonality. We will use them in the following computations. We consider $P_1$ and $P_2$. Since

$$\begin{aligned}
&-\int \Big\{\partial_\xi\theta_0(\xi,u,x)\Big\}\Big\{\varepsilon^2 f(\varepsilon x)\Big\}\,dx - W(\varepsilon,\xi,u)\gamma(u)^3 m\\
&= \gamma(u)\int \theta_K'(\gamma(u)(x-\xi))\varepsilon^2 f(\varepsilon x)\,dx - \varepsilon^2 f(\varepsilon\xi)\int \theta_K'(Z)\,dZ\\
&= \int \theta_K'(Z)\varepsilon^2 f(\varepsilon(\frac{Z}{\gamma(u)} + \xi))\,dZ - \varepsilon^2 f(\varepsilon\xi)\int \theta_K'(Z)\,dZ\\
&= \int \theta_K'(Z)\varepsilon^2\Big[f(\varepsilon\xi) + \frac{\varepsilon}{\gamma(u)}f'(\varepsilon\xi)Z + \ldots\Big]\,dZ - \varepsilon^2 f(\varepsilon\xi)\int \theta_K'(Z)\,dZ\,,
\end{aligned}$$

it follows that

$$\begin{aligned}
P_1 = &\int \partial_\xi \psi_0(\xi,u,x) w(x) - \partial_\xi \theta_0(\xi,u,x)\left(\partial_x^2 v(x) - \cos\theta_0(\xi,u,x) v(x)\right) dx \\
&+ u \int \partial_\xi^2 \psi_0(\xi,u,x) v(x) - \partial_\xi^2 \theta_0(\xi,u,x) w(x)\, dx \\
&- \int \left\{\partial_\xi \theta_0(\xi,u,x)\right\}\left\{\varepsilon^2 f(\varepsilon x)\right\} dx - W(\varepsilon,\xi,u)\gamma(u)^3 m \\
&- \int \partial_u \partial_\xi \psi_0(\xi,u,x) v(x) - \partial_u \partial_\xi \theta_0(\xi,u,x) w(x)\, dx \cdot W(\varepsilon,\xi,u) \\
&- \int \partial_\xi \theta_0(\xi,u,x)\left\{\frac{\sin\theta_0(\xi,u,x) v^2(x)}{2} + \tilde{R}(v)(x)\right\} dx \\
= &\left\langle \begin{pmatrix} -u\partial_x v(\cdot) - w(\cdot) \\ -\partial_x^2 v(\cdot) + \cos(\theta_0(\xi,u,\cdot)) v(\cdot) - u\partial_x w(\cdot) \end{pmatrix}, \begin{pmatrix} -\partial_\xi \psi_0(\xi,u,\cdot) \\ \partial_\xi \theta_0(\xi,u,\cdot) \end{pmatrix} \right\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} \\
&+ \int \theta_K'(Z)\varepsilon^2 \Big[ f(\varepsilon\xi) + \frac{\varepsilon}{\gamma(u)} f'(\varepsilon\xi) Z + \dots \Big]\, dZ - \varepsilon^2 f(\varepsilon\xi) \int \theta_K'(Z)\, dZ \\
&- \int \partial_u \partial_\xi \psi_0(\xi,u,x) v(x) - \partial_u \partial_\xi \theta_0(\xi,u,x) w(x)\, dx \cdot W(\varepsilon,\xi,u) \\
&- \int \partial_\xi \theta_0(\xi,u,x)\left\{\frac{\sin\theta_0(\xi,u,x) v^2(x)}{2} + \tilde{R}(v)(x)\right\} dx \\
= &\int \theta_K'(Z)\varepsilon^2 \Big[ f(\varepsilon\xi) + \frac{\varepsilon}{\gamma(u)} f'(\varepsilon\xi) Z + \dots \Big]\, dZ - \varepsilon^2 f(\varepsilon\xi) \int \theta_K'(Z)\, dZ \\
&- \int \partial_u \partial_\xi \psi_0(\xi,u,x) v(x) - \partial_u \partial_\xi \theta_0(\xi,u,x) w(x)\, dx \cdot W(\varepsilon,\xi,u) \\
&- \int \partial_\xi \theta_0(\xi,u,x)\left\{\frac{\sin\theta_0(\xi,u,x) v^2(x)}{2} + \tilde{R}(v)(x)\right\} dx\,.
\end{aligned}$$

Since $Z\theta_K'(Z)$ is an odd function we obtain

$$|P_1| \le C[|v|_{H^1(\mathbb{R})} + |w|_{L^2(\mathbb{R})}]\varepsilon^2 + C\varepsilon^4 + C\,|v|_{H^1(\mathbb{R})}^2\;.$$

Since

$$
\begin{aligned}
&-\int \Big\{\partial_u\theta_0(\xi,u,x)\Big\}\Big\{\varepsilon^2 f(\varepsilon x)\Big\}\,dx \\
&= -\int u\gamma^3(x-\xi)\theta_K'(\gamma(u)(x-\xi))\varepsilon^2 f(\varepsilon x)\,dx \\
&= -u\gamma(u)\int Z\theta_K'(Z)\varepsilon^2 f(\varepsilon(\frac{Z}{\gamma(u)}+\xi))\,dZ \\
&= -u\gamma(u)\int Z\theta_K'(Z)\varepsilon^2\Big[f(\varepsilon\xi)+\frac{\varepsilon}{\gamma(u)}f'(\varepsilon\xi)Z+\dots\Big]\,dZ\,,
\end{aligned}
$$

it follows that

$$
\begin{aligned}
P_2 = &\int \Big\{\partial_u\psi_0(\xi,u,x)\Big\}w(x)-\partial_u\theta_0(\xi,u,x)\Big(\partial_x^2 v(x)-\cos\theta_0(\xi,u,x)v(x)\Big)\,dx \\
&+u\int \partial_\xi\partial_u\psi_0(\xi,u,x)v(x)-\partial_\xi\partial_u\theta_0(\xi,u,x)w(x)\,dx \\
&-\int \Big\{\partial_u\theta_0(\xi,u,x)\Big\}\Big\{\varepsilon^2 f(\varepsilon x)\Big\}\,dx \\
&-\int \partial_u^2\partial_\xi\psi_0(\xi,u,x)v(x)-\partial_u^2\theta_0(\xi,u,x)w(x)\,dx\cdot W(\varepsilon,\xi,u) \\
&-\int \Big\{\partial_u\theta_0(\xi,u,x)\Big\}\Big\{\frac{\sin\theta_0(\xi,u,x)v^2(x)}{2}+\tilde{R}(v)(x)\Big\}\,dx \\
= &\left\langle \begin{pmatrix} -u\partial_x v(\cdot)-w(\cdot) \\ -\partial_x^2 v(\cdot)+\cos(\theta_0(\xi,u,\cdot))v(\cdot)-u\partial_x w(\cdot)\end{pmatrix}, \begin{pmatrix}-\partial_u\psi_0(\xi,u,\cdot) \\ \partial_u\theta_0(\xi,u,\cdot)\end{pmatrix}\right\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} \\
&-u\gamma(u)\int Z\theta_K'(Z)\varepsilon^2\Big[f(\varepsilon\xi)+\frac{\varepsilon}{\gamma(u)}f'(\varepsilon\xi)Z+\dots\Big]\,dZ \\
&-\int \partial_u^2\partial_\xi\psi_0(\xi,u,x)v(x)-\partial_u^2\theta_0(\xi,u,x)w(x)\,dx\cdot W(\varepsilon,\xi,u) \\
&-\int \Big\{\partial_u\theta_0(\xi,u,x)\Big\}\Big\{\frac{\sin\theta_0(\xi,u,x)v^2(x)}{2}+\tilde{R}(v)(x)\Big\}\,dx
\end{aligned}
$$

$$
\begin{aligned}
= \; & -u\gamma(u) \int Z\theta_K'(Z)\varepsilon^2 \Big[ f(\varepsilon\xi) + \frac{\varepsilon}{\gamma(u)} f'(\varepsilon\xi) Z + \dots \Big] \, dZ \\
& - \int \partial_u^2 \partial_\xi \psi_0(\xi, u, x) v(x) - \partial_u^2 \theta_0(\xi, u, x) w(x) \, dx \cdot W(\varepsilon, \xi, u) \\
& - \int \Bigg\{ \partial_u \theta_0(\xi, u, x) \Bigg\} \Bigg\{ \frac{\sin\theta_0(\xi, u, x) v^2(x)}{2} + \tilde{R}(v)(x) \Bigg\} \, dx \, .
\end{aligned}
$$

Since $Z\theta_K'(Z)$ is an odd function we obtain

$$
|P_2| \le C[|v|_{H^1(\mathbb{R})} + |w|_{L^2(\mathbb{R})}]\varepsilon^2 + C\varepsilon^3 + C\,|v|_{H^1(\mathbb{R})}^2 \ .
$$

□

# Chapter 12

# Lyapunov Functional

We introduce the Lyapunov function.

**Definition 12.1.** *Let* $(v, w)$ *be given by* (10.3)-(10.4)*, with* $(\xi, u)$ *obtained from Lemma 3.1. We set*

$$L(t) = \int \frac{w^2(x)}{2} + \frac{\partial_x v^2(x)}{2} + \frac{\cos(\theta_K(\gamma(x-\xi)))v^2(x)}{2} + uw(x)\partial_x v(x)\, dx\,.$$

**Remark 12.2.** $L$ is the quadratic part of

$$H(\theta, \psi) + u\Pi(\theta, \psi)$$

above the solitary manifold $\mathcal{S}_0$ as in Chapter 6.

The time derivative of $L(t)$ is computed in the following lemma. This will be one of the main ingredients in the proof of the main result.

**Lemma 12.3.**

$$\begin{aligned}
\frac{d}{dt}L(t) = \int & w(x)\left[\frac{\sin(\theta_0(\xi, u, x))v^2(x)}{2} + \tilde{R}(v)(x)\right] \\
& + u\partial_x v(x)\left[\frac{\sin(\theta_0(\xi, u, x))v^2(x)}{2} + \tilde{R}(v)(x)\right] dx \\
& - \dot{u}\int \frac{\sin(\theta_0(\xi, u, x))}{2}\partial_u\theta_0(\xi, u, x)v^2(x)\, dx \\
& + (\dot{\xi} - u)\int \cos(\theta_0(\xi, u, x))v(x)\partial_x v(x)\, dx + \dot{u}\int w(x)\partial_x v(x)\, dx \\
& + \varepsilon^2\int \dot{v}(x)f(\varepsilon x)\, dx - u\dot{u}\gamma\varepsilon^2\int Z\theta_K'(Z)\Big[f(\varepsilon\xi) + \frac{\varepsilon}{\gamma(u)}f'(\varepsilon\xi)Z + \dots\Big]\, dZ \\
& + (u - \dot{\xi})\gamma\varepsilon^2\int \theta_K'(\gamma(x-\xi))f(\varepsilon x)\, dx - u\varepsilon^3\int vf'(\varepsilon x)\, dx\,.
\end{aligned}$$

**Proof.** Analogous to the proof of Lemma 6.3.

□

# Chapter 13

# Proof of Theorem 9.1

## 13.1 Dynamics with Approximate Equations for the Parameters $(\xi, u)$

The goal of this section is to prove the following theorem. We consider again the Cauchy problem defined by (9.1)-(9.3).

**Theorem 13.1.** *We consider the Cauchy problem defined by* (9.1)-(9.3) *and assume that*

*(a)* $\varepsilon$ *is sufficiently small,*

*(b)* $V \in H^4(\mathbb{R})$ *and* $f = V'$,

*(c)* $(\xi_s, u_s) \in \mathbb{R} \times (-U, U)$, *where* $0 < U < 1$;

*(d)* $\mathcal{N}(\theta(0,x), \psi(0,x), \xi_s, u_s) = 0$,

*(e)* $|v(0)|^2_{H^1(\mathbb{R})} + |w(0)|^2_{L^2(\mathbb{R})} \leq \varepsilon^3$.

*Then the Cauchy problem defined by* (9.1)-(9.3) *has a unique solution on the time interval*

$$0 \leq t \leq T \text{ where } T := T(\varepsilon, \delta) = \frac{1}{\varepsilon^{\rho(\delta)}}, \quad \rho(\delta) = 1 - \delta\,.$$

*The solution may be written in the form*

$$\theta(t,x) = \theta_0(\xi(t), u(t), x) + v(t,x)\,,$$
$$\psi(t,x) = \psi_0(\xi(t), u(t), x) + w(t,x)\,,$$

*where* $v, w, u, \xi$ *have regularity*

$$(\xi(t), u(t)) \in C^1([0,T], \mathbb{R} \times (-1,1)),$$
$$(v(t), w(t)) \in C^1([0,T], H^1(\mathbb{R}) \times L^2(\mathbb{R})),$$

*such that the orthogonality condition*

$$\mathcal{N}(\theta(t,x),\psi(t,x),\xi(t),u(t))=0$$

*is satisfied. There exist positive constants* $c, C$ *such that*

$$|\dot{\xi}(t)-u(t)|\leq C\varepsilon^3,$$
$$|\dot{u}(t)+W(\varepsilon,\xi,u)|\ \leq C\varepsilon^3\,,$$

*and*

$$|v|^2_{L^\infty([0,T],H^1(\mathbb{R}))}+|w|^2_{L^\infty([0,T],L^2(\mathbb{R}))}\leq c\varepsilon^3\,.$$

*The constants* $c, C$ *depend on* $V$*.*

Theorem 13.1 yields us only approximate equations for the parameters $(\xi, u)$ whereas Theorem 9.1 provides ODE's (9.4)-(9.5).
Notice that Theorem 13.1 yields us a better bound for the transversal component $(v, w)$ than Theorem 9.1. In Theorem 13.1 the orthogonality conditions are satisfied which do not have to hold in Theorem 9.1.

Now we suppose that (9.1)-(9.3) has a solution and we make some assumptions on $(v, w)$ given by (10.3)-(10.4) and on $(\xi, u)$ obtained from Lemma 3.1. Then the following lemma yields us more accurate information about $(v, w)$ and $(\xi, u)$.

**Lemma 13.2.** *Let* $\varepsilon$ *be sufficiently small. Assume that assumptions (b),(c),(d) of Theorem 9.1 are satisfied. Assume that* (9.1)-(9.3) *has a solution* $(\theta, \psi)$ *on* $[0, \overline{T}]$ *such that*

$$(\theta,\psi)\in C^1([0,\overline{T}],L^\infty(\mathbb{R})\oplus L^2(\mathbb{R}))\,.$$

*Suppose that*

$$0\leq T\leq t^*\leq\overline{T}$$

*where* $t^*$ *is from Definition 10.1. Let* $(v, w)$ *be given by* (10.3)-(10.4), *with* $(\xi, u)$ *obtained from Lemma 3.1 such that*

$$|v|^2_{L^\infty([0,T],H^1(\mathbb{R}))}+|w|^2_{L^\infty([0,T],L^2(\mathbb{R}))}\leq\varepsilon^{3-\delta}\,.$$

*Then , provided*

$$0\leq T\leq\frac{1}{\varepsilon^{1-\delta}},$$

*it holds that*

*(a)* $\forall t\in[0,T]:\ (\xi(t),u(t))\in\Sigma(5,U)$*;*

*(b)* $|v|^2_{L^\infty([0,T],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,T],L^2(\mathbb{R}))} \leq \bar{C}(L(0)+\varepsilon^3)$ *, where* $\bar{C}$ *depends on* $f$ *(and* $c$ *from Lemma 7.4).*

**Proof.** Choose $\varepsilon$ such that the following holds:

(1) $\varepsilon \in (0, \varepsilon_0)$ where $\varepsilon_0$ is from Lemma 11.2.

(2) $\varepsilon$ is so small that

$$C\varepsilon^2 \left[\frac{1}{\varepsilon^{1-\delta}}\right] + |u(0)| \leq \frac{U(5)}{2} + U\,,$$

where C is a constant that appears in (13.1) further in this proof which depends on $f$.

Lemma 11.2 yields $\forall t \in [0,T]$:

$$\begin{aligned} |\dot{\xi}(t) - u(t)| &\leq C[|v|_{H^1(\mathbb{R})} + |w|_{L^2(\mathbb{R})}]\varepsilon^2 + C\varepsilon^3 + C\,|v|^2_{H^1(\mathbb{R})} \\ &\leq C\varepsilon^{\frac{3-\delta}{2}+2} + C\varepsilon^3 + C\varepsilon^{3-\delta} \\ &\leq C\varepsilon^{3-\delta}, \\ |\dot{u}(t) + W(\varepsilon,\xi,u)| &\leq C[|v|_{H^1(\mathbb{R})} + |w|_{L^2(\mathbb{R})}]\varepsilon^2 + C\varepsilon^4 + C\,|v|^2_{H^1(\mathbb{R})} \\ &\leq C\varepsilon^{\frac{3-\delta}{2}+2} + C\varepsilon^4 + C\varepsilon^{3-\delta} \\ &\leq C\varepsilon^{3-\delta}\,. \end{aligned}$$

Thus we obtain $\forall t \in [0,T]$:

$$\begin{aligned} |u(t) - u(0)| &\leq \int_0^t |\dot{u}(s)|\,ds \\ &\leq C\varepsilon^2 t \end{aligned}$$

$$\Rightarrow \quad |u(t)| \leq C\varepsilon^2 t + |u(0)|\,. \tag{13.1}$$

This implies (a) due to assumption (c) of Theorem 9.1 and (2). Using Lemma 12.3, Lemma 7.2 and Lemma 7.4 we obtain for times

$$0 \leq t \leq T \leq \frac{1}{\varepsilon^{1-\delta}},$$

the following estimate,

$$
\begin{aligned}
& c(|v(t)|^2_{H^1(\mathbb{R})}+|w(t)|^2_{L^2(\mathbb{R})}) \\
& \leq L(t)=L(0)+\int_0^t \dot{L}(t)\,dt \\
& = L(0)+\int_0^t w(x)[\frac{\sin\theta_0(\xi,u,x)v^2(x)}{2}+\tilde{R}(v)(x)]+u\partial_x v(x)[\frac{\sin\theta_0(\xi,u,x)v^2(x)}{2}+\tilde{R}(v)(x)] \\
& \quad -\dot{u}\int\frac{\sin(\theta_0(\xi,u,x))}{2}\partial_u\theta_0(\xi,u,x)v^2(x)\,dx \\
& \quad +(\dot{\xi}-u)\int\cos(\theta_0(\xi,u,x))v(x)\partial_x v(x)\,dx+\dot{u}\int w(x)\partial_x v(x)\,dx \\
& \quad +\varepsilon^2\int\dot{v}(x)f(\varepsilon x)\,dx-u\dot{u}\gamma\varepsilon^2\int Z\theta_K'(Z)\Big[f(\varepsilon\xi)+\frac{\varepsilon}{\gamma(u)}f'(\varepsilon\xi)Z+\dots\Big]\,dZ \\
& \quad +(u-\dot{\xi})\gamma\varepsilon^2\int\theta_K'(\gamma(x-\xi))f(\varepsilon x)\,dx-u\varepsilon^3\int v(x)f'(\varepsilon x)\,dx\,dt \\
& \leq L(0)+\int_0^t w[\frac{\sin\theta_0(\xi,u,x)v^2(x)}{2}+\tilde{R}(v)(x)]+u\partial_x v(x)[\frac{\sin\theta_0(\xi,u,x)v^2(x)}{2}+\tilde{R}(v)(x)] \\
& \quad -\dot{u}\int\frac{\sin(\theta_0(\xi,u,x))}{2}\partial_u\theta_0(\xi,u,x)v^2(x)\,dx \\
& \quad +(\dot{\xi}-u)\int\cos(\theta_0(\xi,u,x))v(x)\partial_x v(x)\,dx+\dot{u}\int w(x)\partial_x v(x)\,dx \\
& \quad -u\dot{u}\gamma\varepsilon^2\int Z\theta_K'(Z)\Big[f(\varepsilon\xi)+\frac{\varepsilon}{\gamma(u)}f'(\varepsilon\xi)Z+\dots\Big]\,dZ \\
& \quad +(u-\dot{\xi})\gamma\varepsilon^2\int\theta_K'(\gamma(x-\xi))f(\varepsilon x)\,dx-u\varepsilon^3\int v(x)f'(\varepsilon x)\,dx\,dt \\
& \quad +|v(t,\cdot)|_{H^1(\mathbb{R})}\,\varepsilon^{\frac{3}{2}}\,|f(\cdot)|_{L^2(\mathbb{R})}+|v(0,\cdot)|_{H^1(\mathbb{R})}\,\varepsilon^{\frac{3}{2}}\,|f(\cdot)|_{L^2(\mathbb{R})} \\
& \leq L(0)+C\int_0^t\varepsilon^{4-\frac{\delta}{2}}\,dt+\frac{c}{8}\,|v(t,\cdot)|^2_{H^1(\mathbb{R})}+\frac{c}{8}\,|v(0,\cdot)|^2_{H^1(\mathbb{R})}+\frac{4}{c}\,|f(\cdot)|^2_{L^2(\mathbb{R})}\,\varepsilon^3\,,
\end{aligned}
$$

since

$$-u\varepsilon^3\int v(x)f'(\varepsilon x)\,dx\leq|u|\varepsilon^{\frac{5}{2}}\,|v|_{L^2(\mathbb{R})}\,|f(\cdot)|_{L^2(\mathbb{R})}\;.$$

After bringing some terms on the left hand side we obtain

$$\tilde{c}(|v|^2_{L^\infty([0,t],H^1(\mathbb{R}))}+|w|^2_{L^\infty([0,t],L^2(\mathbb{R}))})\leq L(0)+C\int_0^t\varepsilon^{4-\frac{\delta}{2}}\,dt+\frac{4}{c}\,|f(\cdot)|^2_{L^2(\mathbb{R})}\,\varepsilon^3\,.$$

$\square$

**Theorem 13.3.** *Let $\varepsilon$ be sufficiently small. Assume that the assumptions (b),(c),(d),(e) of Theorem 13.1 are satisfied. Assume that* (9.1)-(9.3) *has a solution $(\theta, \psi)$ on $[0, \overline{T}]$ such that*

$$(\theta, \psi) \in C^1([0, \overline{T}], L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R}))\,.$$

*Suppose that*

$$0 \leq T \leq \overline{T}\,.$$

*Then, provided*

$$0 \leq T \leq \frac{1}{\varepsilon^{1-\delta}},$$

*it holds that $(v, w)$ given by* (10.3)-(10.4) *is well defined for times $[0, T]$ and there exists a constant $\hat{c}$ such that*

*(a)* $|v|^2_{L^\infty([0,T],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,T],L^2(\mathbb{R}))} \leq \hat{c}\varepsilon^3\,,$

*(b)* $\forall t \in [0, T]:\ (\xi(t), u(t)) \in \Sigma(5, U)\,.$

**Proof.** Choose $\varepsilon$ such that the following holds:

(1) $\varepsilon$ satisfies all smallness assumtions of Lemma 13.2;

(2) $2\bar{C}(L(0) + \varepsilon^3) < \varepsilon^{3-\delta}$ , where $L(0) = \mathcal{E}(v(0), w(0), \xi_s, u_s)$ and $\bar{C}$ is from Lemma 13.2 (b);

(3) $\varepsilon$ is so small that if $(v, w) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ satisfies $|v|^2_{H^1(\mathbb{R})} + |w|^2_{L^2(\mathbb{R})} \leq \varepsilon^{1-\delta}$ then it holds that $|v|_{L^\infty(\mathbb{R})} + |w|_{L^2(\mathbb{R})} \leq \frac{r}{2}$, where $r$ is from Lemma Lemma 3.1. This can be ensured by Morrey's embedding theorem.

Notice that $\Sigma(5, U) \subset \Sigma(4, U)$. We define an exit time

$$t_* := \sup\Big\{T > 0 : |v|^2_{L^\infty([0,t],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,t],L^2(\mathbb{R}))} \leq 2\bar{C}(L(0) + \varepsilon^3),$$
$$(\xi(t), u(t)) \in \Sigma(5, U),\ 0 \leq t \leq T\Big\}\,.$$

Suppose $t_* < \frac{1}{\varepsilon^{1-\delta}}$. Then there exists a time $\hat{t}$ s.t.

$$\frac{1}{\varepsilon^{1-\delta}} > \hat{t} > t_*\,,$$

with

$$\forall t \in [0, \hat{t}]:\ (\xi(t), u(t)) \in \Sigma(4, U), \qquad (\xi(\hat{t}), u(\hat{t})) \notin \Sigma(5, U)$$

or

$$\bar{C}(L(0)+\varepsilon^3) < 2\bar{C}(L(0)+\varepsilon^3) < |v|^2_{L^\infty([0,\hat{t}],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,\hat{t}],L^2(\mathbb{R}))} < \varepsilon^{3-\delta}\,.$$

This leads a contradiction to the previous lemma. Thus

$$|v|^2_{L^\infty([0,T],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,T],L^2(\mathbb{R}))} \leq 2\bar{C}(L(0)+\varepsilon^3) \leq \hat{c}\varepsilon^3$$

and

$$\forall t \in [0,T]:\ (\xi(t),u(t)) \in \Sigma(5,U)\,.$$

$\square$

The previous theorem implies that the local solution of (9.1)-(9.3) discussed in Chapter 10 is indeed continuable up to times $\frac{1}{\varepsilon^{1-\delta}}$. Theorem 13.3 and Lemma 11.2 yield the approximate equations for the parameters $(\xi,u)$. This verifies the claim of Theorem 13.1.

## 13.2 ODE Analysis

In this section we lay the groundwork for passing from the approximate equations for the parameters $(\xi,u)$ in Theorem 13.1 to the ODE's in (9.4)-(9.5).

### 13.2.1 Restricted Hamilton Equations

In the following lemma we compute the derivatives of the restricted (to $\mathcal{S}_0$) Hamiltonian.

**Lemma 13.4.** *Let*

$$H^\varepsilon(\theta,\psi) = \frac{1}{2}\int \psi^2(x) + (\partial_x\theta(x))^2 + 2(1-\cos\theta(x)) - 2\varepsilon^2 f(\varepsilon x)\theta(x)\,dx\,.$$

*It holds that*

$$\begin{aligned}&\partial_u H^\varepsilon(\theta_0(\xi,u,x),\psi_0(\xi,u,x))\\ &= mu[\gamma(u)]^3 - u\varepsilon^3 V''(\varepsilon\xi)\int Z^2\theta_K'(Z)\,dZ + \mathcal{O}(\varepsilon^5),\end{aligned}$$

$$\begin{aligned}&\partial_\xi H^\varepsilon(\theta_0(\xi,u,x),\psi_0(\xi,u,x))\\ &= \varepsilon^2 V'(\varepsilon\xi)\int \theta_K'(Z)\,dZ + \frac{1}{2\gamma(u)^2}\varepsilon^4 V'''(\varepsilon\xi)\int Z^2\theta_K'(Z)\,dZ + \mathcal{O}(\varepsilon^6)\,.\end{aligned}$$

**Proof.** Notice that

$$\begin{pmatrix}\theta_0(\xi,u,x)\\ \psi_0(\xi,u,x)\end{pmatrix} = \begin{pmatrix}\theta_K(\gamma(x-\xi))\\ -u\gamma(u)\theta_K'(\gamma(x-\xi))\end{pmatrix}.$$

It holds that

$$\begin{aligned}
&H^\varepsilon(\theta_0(\xi,u,x),\psi_0(\xi,u,x))\\
&= m\gamma(u) - \int \varepsilon^2 f(\varepsilon x)\theta_K(\gamma(u)(x-\xi))\,dx\\
&= m\gamma(u) - \varepsilon\int \partial_x[V(\varepsilon x)]\theta_K(\gamma(u)(x-\xi))\,dx\\
&= m\gamma(u) + \varepsilon\gamma(u)\int V(\varepsilon x)\theta_K'(\gamma(u)(x-\xi))\,dx\\
&= m\gamma(u) + \varepsilon\gamma(u)\int V(\varepsilon(y+\xi))\theta_K'(\gamma(u)y)\,dy\,.
\end{aligned}$$

We expand around $y = 0$:

$$\begin{aligned}
&H^\varepsilon(\theta_0(\xi,u,x),\psi_0(\xi,u,x))\\
&= m\gamma(u) + \varepsilon\gamma(u)V(\varepsilon\xi)\int \theta_K'(\gamma(u)y)\,dy\\
&\quad + \gamma(u)\varepsilon^2 V'(\varepsilon\xi)\int y\theta_K'(\gamma(u)y)\,dy\\
&\quad + \frac{\gamma(u)\varepsilon^3 V''(\varepsilon\xi)}{2}\int y^2\theta_K'(\gamma(u)y)\,dy + \mathcal{O}(\varepsilon^5)\\
&= m\gamma(u) + \varepsilon\gamma(u)V(\varepsilon\xi)\int \theta_K'(\gamma(u)y)\,dy\\
&\quad + \frac{\gamma(u)\varepsilon^3 V''(\varepsilon\xi)}{2}\int y^2\theta_K'(\gamma(u)y)\,dy + \mathcal{O}(\varepsilon^5)\\
&= m\gamma(u) + \varepsilon V(\varepsilon\xi)\int \theta_K'(Z)\,dZ\\
&\quad + \frac{1}{2\gamma(u)^2}\varepsilon^3 V''(\varepsilon\xi)\int Z^2\theta_K'(Z)\,dZ + \mathcal{O}(\varepsilon^5)\,.
\end{aligned}$$

We take the derivatives with respect to $u$ and $\xi$.

$$\begin{aligned}&\partial_u H^\varepsilon(\theta_0(\xi,u,x),\psi_0(\xi,u,x))\\=\,&mu[\gamma(u)]^3-u\varepsilon^3V''(\varepsilon\xi)\int Z^2\theta_K'(Z)\,dZ+\mathcal{O}(\varepsilon^5),\end{aligned}$$

$$\begin{aligned}&\partial_\xi H^\varepsilon(\theta_0(\xi,u,x),\psi_0(\xi,u,x))\\=\,&\varepsilon^2V'(\varepsilon\xi)\int\theta_K'(Z)\,dZ+\frac{1}{2\gamma(u)^2}\varepsilon^4V'''(\varepsilon\xi)\int Z^2\theta_K'(Z)\,dZ+\mathcal{O}(\varepsilon^6)\,.\end{aligned}$$

□

Considering the equations

$$\begin{aligned}&[\gamma(u)]^3m\dot{\xi}\\=\,&mu[\gamma(u)]^3-u\varepsilon^3V''(\varepsilon\xi)\int Z^2\theta_K'(Z)\,dZ+\mathcal{O}(\varepsilon^5),\end{aligned}$$

$$\begin{aligned}&-[\gamma(u)]^3m\dot{u}\\=\,&\varepsilon^2V'(\varepsilon\xi)\int\theta_K'(Z)\,dZ+\frac{1}{2\gamma(u)^2}\varepsilon^4V'''(\varepsilon\xi)\int Z^2\theta_K'(Z)\,dZ+\mathcal{O}(\varepsilon^6)\,,\end{aligned}$$

which are Hamiltonian's equations of motion for the restricted (to $\mathcal{S}_0$) Hamiltonian

$$(\xi,u)\mapsto H^\varepsilon(\theta_0(\xi,u,x),\psi_0(\xi,u,x))\,,$$

with respect to the restricted (to $\mathcal{S}_0$) symplectic form, allows us to draw conclusions about the ODE's that are exactly satisfied by the parameters that describe the component of the dynamics on the manifold. This approach is motivated by [HL12, Section 3] and the restricted equations emerged out of discussions with Justin Holmer.

### 13.2.2 Refernce Trajectory

We start with a preparing lemma.

**Lemma 13.5.** *Let $\tilde{\xi}=\tilde{\xi}(s)$, $\tilde{u}=\tilde{u}(s)$, $\epsilon_1=\epsilon_1(s)$, $\epsilon_2=\epsilon_2(s)$ be $C^1$ real-valued functions. Suppose that $V\in H^4(\mathbb{R})$. and that*

$$|\epsilon_j|\le\bar{c}\varepsilon^2$$

*on* $[0,T]$ *for* $j=1,2$. *Assume that on* $[0,T]$,

$$\frac{d}{ds}\tilde{\xi}(s) = \tilde{u}(s) + \epsilon_1(s)\,, \quad \tilde{\xi}(0) = \tilde{\xi}_0$$
$$\frac{d}{ds}\tilde{u}(s) = -\varepsilon \frac{V'(\tilde{\xi}(s))}{[\gamma(\tilde{u}(s))]^3 m} \int \theta'_K(Z)\,dZ + \epsilon_2(s)\,, \quad \tilde{u}(0) = \tilde{u}_0.$$

*Let* $\hat{\xi} = \hat{\xi}(s)$ *and* $\hat{u} = \hat{u}(s)$ *be* $C^1$ *real-valued functions which satisfy the exact equations*

$$\frac{d}{ds}\hat{\xi}(s) = \hat{u}(s)\,, \quad \hat{\xi}(0) = \tilde{\xi}_0$$
$$\frac{d}{ds}\hat{u}(s) = -\varepsilon \frac{V'(\hat{\xi}(s))}{[\gamma(\hat{u}(s))]^3 m} \int \theta'_K(Z)\,dZ\,, \quad \hat{u}(0) = \tilde{u}_0.$$

*Then provided* $T \leq 1$, *there exists a* $c > 0$ *such that the estimates*

$$|\tilde{\xi} - \hat{\xi}| \leq c\varepsilon^2, \qquad |\tilde{u} - \hat{u}| \leq c\varepsilon^2\,.$$

*hold on* $[0,T]$.

**Proof.** In the following proof we follow very closely [HZ08, Lemma 6.1]. Let $x = x(t)$ and $y = y(t)$ be $C^1$ real-valued functions, $C \geq 1$, and let $(x,y)$ satisfy the differential inequalities:

$$\begin{cases} |\dot{x}| \leq |y| & x(0) = x_0 \\ |\dot{y}| \leq C|x| + C|y| \,, & y(0) = y_0 \end{cases}.$$

We are going to apply the Gronwall lemma. Let $z(t) = x^2 + y^2$. Then

$$|\dot{z}| = |2x\dot{x} + 2y\dot{y}| \leq 2|x||y| + 2C|x||y| + 2C|y||y| \leq 4C(x^2+y^2) = 4Cz$$

and hence $z(t) \leq z(0)e^{4Ct}$. Thus

$$|x(t)| \leq \sqrt{2}\max(|x_0|,|y_0|)\exp(2Ct)\,, \tag{13.2}$$
$$|y(t)| \leq \sqrt{2}\max(|x_0|,|y_0|)\exp(2Ct)\,.$$

Now we recall the Duhamel's formula. Let $X(s) : \mathbb{R} \to \mathbb{R}^2$ be a two-vector function, $X_0 \in \mathbb{R}^2$ a two-vector, and $A(s) : \mathbb{R} \to (2 \times 2 \text{ matrices})$ a $2 \times 2$ matrix function. We consider the ODE system

$$\dot{X}(s) = A(s)X(s), \qquad X(s') = X_0$$

and denote its solution by $X(s) = S(s,s')X_0$ such that

$$\frac{d}{ds}S(s,s')X_0 = A(s)S(s,s')X_0\,, \quad S(s',s')X_0 = X_0\,.$$

Let $F(s) : \mathbb{R} \to \mathbb{R}^2$ be a two-vector function. We can express the solution to the inhomogeneous ODE system

$$\dot{X}(s) = A(s)X(s) + F(s)$$

with initial condition $X(0) = 0$ by the Duhamel's formula

$$X(s) = \int_0^s S(s, s')F(s')ds' .$$

Let $U = \hat{u} - \tilde{u}$ and $\Xi = \hat{\xi} - \tilde{\xi}$. These functions satisfy

$$\begin{aligned} &\frac{d}{ds}\Xi(s) = U(s) + \epsilon_1(s) , \\ &\frac{d}{ds}U(s) = \Big[\frac{V'(\hat{\xi}(s))}{[\gamma(\hat{u}(s))]^3 m} - \frac{V'(\tilde{\xi}(s))}{[\gamma(\tilde{u}(s))]^3 m}\Big] \int \theta'_K(Z)\, dZ + \epsilon_2(s) . \end{aligned}$$

Notice that

$$\begin{aligned} &\frac{V'(\hat{\xi}(s))}{[\gamma(\hat{u}(s))]^3} - \frac{V'(\tilde{\xi}(s))}{[\gamma(\tilde{u}(s))]^3} \\ &= \frac{V'(\hat{\xi}(s)) - V'(\tilde{\xi}(s))}{[\gamma(\hat{u}(s))]^3} + \frac{V'(\tilde{\xi}(s))}{[\gamma(\hat{u}(s))]^3} - \frac{V'(\tilde{\xi}(s))}{[\gamma(\tilde{u}(s))]^3} \\ &= \frac{1}{[\gamma(\hat{u}(s))]^3} \frac{V'(\hat{\xi}(s)) - V'(\tilde{\xi}(s))}{\hat{\xi}(s) - \tilde{\xi}(s)} \Big[\hat{\xi}(s) - \tilde{\xi}(s)\Big] \\ &\quad + \frac{V'(\tilde{\xi}(s))}{[\gamma(\tilde{u}(s))]^3[\gamma(\hat{u}(s))]^3} \frac{[\gamma(\tilde{u}(s))]^3 - [\gamma(\hat{u}(s))]^3}{\tilde{u}(s) - \hat{u}(s)} \Big[\tilde{u}(s) - \hat{u}(s)\Big] . \end{aligned}$$

Let

$$g(s) = \begin{cases} \dfrac{1}{[\gamma(\hat{u}(s))]^3 m} \displaystyle\int \theta'_K(Z)\, dZ \frac{V'(\hat{\xi}(s)) - V'(\tilde{\xi}(s))}{(\hat{\xi}(s) - \tilde{\xi}(s))} & \text{if } \hat{\xi}(s) \neq \tilde{\xi}(s) \\ \dfrac{V''(\hat{\xi}(s))}{[\gamma(\hat{u}(s))]^3 m} \displaystyle\int \theta'_K(Z)\, dZ & \text{if } \hat{\xi}(s) = \tilde{\xi}(s) \end{cases} ,$$

$$h(s) = \begin{cases} \dfrac{V'(\tilde{\xi}(s))}{[\gamma(\tilde{u}(s))]^3[\gamma(\hat{u}(s))]^3 m} \displaystyle\int \theta'_K(Z)\, dZ \frac{[\gamma(\tilde{u}(s))]^3 - [\gamma(\hat{u}(s))]^3}{\tilde{u}(s) - \hat{u}(s)} & \text{if } \hat{u}(s) \neq \tilde{u}(s) \\ \dfrac{V''(\tilde{\xi}(s))}{[\gamma(\tilde{u}(s))]^3[\gamma(\hat{u}(s))]^3 m} \displaystyle\int \theta'_K(Z)\, dZ & \text{if } \tilde{u}(s) = \tilde{u}(s) \end{cases} .$$

We set

$$A(s) = \begin{bmatrix} 0 & 1 \\ g(s) & h(s) \end{bmatrix}, \quad F(s) = \begin{bmatrix} \epsilon_1(s) \\ \epsilon_2(s) \end{bmatrix}, \quad X(s) = \begin{bmatrix} \Xi(s) \\ U(s) \end{bmatrix}$$

and obtain by Duhamel's:

$$\begin{bmatrix} \Xi(s) \\ U(s) \end{bmatrix} = \int_0^s S(s,t') \begin{bmatrix} \epsilon_1(t') \\ \epsilon_2(t') \end{bmatrix} dt' \tag{13.3}$$

We apply (13.2) with

$$\begin{bmatrix} x(s) \\ y(s) \end{bmatrix} = S(s+t',t') \begin{bmatrix} \epsilon_1(t') \\ \epsilon_2(t') \end{bmatrix}, \quad \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} \epsilon_1(t') \\ \epsilon_2(t') \end{bmatrix}.$$

It follows that

$$\left| S(s,t') \begin{bmatrix} \epsilon_1(t') \\ \epsilon_2(t') \end{bmatrix} \right| \leq \sqrt{2} \begin{bmatrix} \exp(2C(s-t')) \\ \exp(2C(s-t')) \end{bmatrix} \max(|\epsilon_1(t')|, |\epsilon_2(t')|).$$

Using (13.3) we obtain that on $[0,T]$

$$|\Xi(s)| \leq \sqrt{2}\,T\exp(2CT) \sup_{0\leq t\leq T} \max(|\epsilon_1(t)|, |\epsilon_2(t)|),$$

$$|U(s)| \leq \sqrt{2}\,T\exp(2CT) \sup_{0\leq t\leq T} \max(|\epsilon_1(t)|, |\epsilon_2(t)|),$$

which yields the claim. □

In the following we show the relation between the parameters $(\xi, u)$ selected by the implicit function theorem according to Lemma 3.1 and the solutions $(\hat{\xi}, \hat{u})$ of the exact ODE's from the previous lemma.

**Lemma 13.6.** *Let $\varepsilon$ be sufficiently small, $\beta(\delta) = 1 - \delta$ ,*

$$s = \varepsilon t,$$

*where*

$$0 \leq s \leq \varepsilon^\delta, \quad 0 \leq t \leq \frac{1}{\varepsilon^{\beta(\delta)}}.$$

*Let $(\xi, u)$ be the parameters selected according to Lemma 3.1 and $(\hat{\xi}, \hat{u})$ from Lemma 13.5. Then it holds that*

$$|\xi(t) - \frac{\hat{\xi}(\varepsilon t)}{\varepsilon}| \leq c\varepsilon,$$

$$|u(t) - \hat{u}(\varepsilon t)| \leq c\varepsilon^2.$$

**Proof.** We choose $\varepsilon$ so small that $\varepsilon^\delta \le 1$. We set

$$\tilde{\xi}(s) = \varepsilon\xi(s/\varepsilon), \qquad \tilde{u}(s) = u(s/\varepsilon)\,.$$

For times

$$0 \le t \le \frac{1}{\varepsilon^{\beta(\delta)}}$$

Lemma 11.2 and Theorem 13.3 yield:

$$\begin{aligned}
|\dot{\xi}(t) - u(t)| &\le C[|v|_{H^1}(\mathbb{R}) + |w|_{L^2(\mathbb{R})}]\varepsilon^2 + C\varepsilon^3 + C\,|v|^2_{H^1(\mathbb{R})} \\
&\le C\varepsilon^{\frac{3}{2}+2} + C\varepsilon^3 + C\varepsilon^3 \\
&\le C\varepsilon^3, \\
|\dot{u}(t) + W(\varepsilon,\xi,u)| &\le C[|v|_{H^1}(\mathbb{R}) + |w|_{L^2(\mathbb{R})}]\varepsilon^2 + C\varepsilon^4 + C\,|v|^2_{H^1(\mathbb{R})} \\
&\le C\varepsilon^{\frac{3}{2}+2} + C\varepsilon^4 + C\varepsilon^3 \\
&\le C\varepsilon^3\,.
\end{aligned}$$

Thus $(\tilde{\xi}, \tilde{u})$ satisfy the assumptions of Lemma 13.5, since

$$\begin{aligned}
&\frac{d}{ds}\tilde{\xi}(s) \\
=&\xi'\left(\frac{s}{\varepsilon}\right) \\
=&u\left(\frac{s}{\varepsilon}\right) + \mathcal{O}(\varepsilon^3) \\
=&\tilde{u}(s) + \mathcal{O}(\varepsilon^3)\,,
\end{aligned}$$

$$\begin{aligned}
&\frac{d}{ds}\tilde{u}(s) + \varepsilon\frac{V'(\tilde{\xi}(s))}{[\gamma(\tilde{u}(s))]^3 m}\int \theta_K'(Z)\,dZ \\
=&\frac{1}{\varepsilon}u'\left(\frac{s}{\varepsilon}\right) + \frac{1}{\varepsilon}W(\varepsilon, \xi\left(\frac{s}{\varepsilon}\right), u\left(\frac{s}{\varepsilon}\right)) \\
=&\frac{1}{\varepsilon}\mathcal{O}(\varepsilon^3) \\
=&\mathcal{O}(\varepsilon^2)\,.
\end{aligned}$$

Hence Lemma 13.5 yields:

$$|\varepsilon\xi(t) - \hat{\xi}(\varepsilon t)| = |\tilde{\xi}(s) - \hat{\xi}(s)| \leq c\varepsilon^2$$

$$\Rightarrow |\xi(t) - \frac{\hat{\xi}(\varepsilon t)}{\varepsilon}| \leq c\varepsilon\,,$$

$$|u(t) - \hat{u}(\varepsilon t)| = |\tilde{u}(s) - \hat{u}(s)| \leq c\varepsilon^2$$

$$\Rightarrow |u(t) - \hat{u}(\varepsilon t)| \leq c\varepsilon^2\,.$$

□

## 13.3 Completion of the Proof of Theorem 9.1

We choose $\varepsilon$ such that the assumption in Lemma 13.6 is satisfied. Theorem 13.1 yields the dynamics with the parameters $(\xi, u)$ selected by the implicit function theorem according to Lemma 3.1 on the time interval

$$0 \leq t \leq \frac{1}{\varepsilon^{1-\delta}}\,.$$

Using Lemma 13.6 and the the triangle inequality we can replace

$$(\xi(t), u(t))$$

with

$$(\frac{\hat{\xi}(\varepsilon t)}{\varepsilon}, \hat{u}(\varepsilon t))\,.$$

We have to replace $\varepsilon^3$ with $\varepsilon^2$ in the estimate for the transversal component $(v, w)$ since the difference of the parameters $|\xi(t) - \frac{\hat{\xi}(\varepsilon t)}{\varepsilon}|$ in Lemma 13.6 is of order $\varepsilon$. We set

$$(\bar{\xi}(t), \bar{u}(t)) = (\frac{\hat{\xi}(\varepsilon t)}{\varepsilon}, \hat{u}(\varepsilon t))$$

and conclude that the equations claimed are satisfied.

□

# Part III

# Classical Solitons in the Presence of a Forcing $F(\varepsilon, x)$

# Chapter 14

# Main Result and Overview

We use the notation from Definition 1.1 and we consider the initial value problem

$$\partial_t \begin{pmatrix} \theta(t,x) \\ \psi(t,x) \end{pmatrix} = \begin{pmatrix} \psi(t,x) \\ \partial_x^2\theta(t,x) - \sin\theta(t,x) + F(\varepsilon,x) \end{pmatrix}, \tag{14.1}$$

$$\theta(0,x) = \theta_0(\xi_s,u_s,x) + v(0,x)\,, \tag{14.2}$$

$$\psi(0,x) = \psi_0(\xi_s,u_s,x) + w(0,x)\,, \tag{14.3}$$

where

$$(v(0,x),w(0,x)) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}).$$

The main result of Part III is the following theorem.

**Theorem 14.1.** *We consider the Cauchy problem defined by* (14.1)-(14.3) *and assume that*

*(a)* $\varepsilon$ *is sufficiently small;*

*(b)* $F \in C^{k+1}((-1,1),L^2(\mathbb{R}))$ , $\partial_\varepsilon^l F(0,\cdot) = 0$, *where* $0 \le l \le k$, $k \in \mathbb{N}$*;*

*(c)* $(\xi_s,u_s) \in \mathbb{R} \times (-U,U)$ , *where* $0 < U < 1$*;*

*(d)* $\mathcal{N}(\theta(0,x),\psi(0,x),\xi_s,u_s) = 0$ *;*

*(e)* $|v(0)|^2_{H^1(\mathbb{R})} + |w(0)|^2_{L^2(\mathbb{R})} \le \varepsilon^{k+1}$.

*Then*

*(i) The Cauchy problem defined by* (14.1)-(14.3) *has a unique solution on the time interval*

$$0 \le t \le T, \text{ where } T = T(\varepsilon,k,\delta) := \frac{1}{\varepsilon^{\rho(k,\delta)}}, \quad \rho(k,\delta) = \frac{k+1}{2} - 2\delta\,.$$

*The solution may be written in the form*

$$\theta(t,x) = \theta_0(\xi(t), u(t), x) + v(t,x)\,,$$
$$\psi(t,x) = \psi_0(\xi(t), u(t), x) + w(t,x)\,,$$

*where* $v, w, u, \xi$ *have regularity*

$$(\xi(t), u(t)) \in C^1([0,T], \mathbb{R} \times (-1,1)),$$
$$(v(t), w(t)) \in C^1([0,T], H^1(\mathbb{R}) \oplus L^2(\mathbb{R})),$$

*such that the orthogonality condition*

$$\mathcal{N}(\theta(t,x), \psi(t,x), \xi(t), u(t)) = 0$$

*is satisfied. There exist positive constants* $c, C$ *such that*

$$|\dot{\xi}(t) - u(t)| \leq C\varepsilon^{k+1},$$
$$|\dot{u}(t)| \leq C\varepsilon^{k+1}\,,$$

*and*

$$|v|^2_{L^\infty([0,T],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,T],L^2(\mathbb{R}))} \leq c\varepsilon^{k+1}\,.$$

*The constants* $c, C$ *depend on* $F$*.*

*(ii) The Cauchy problem defined by* (14.1)-(14.3) *has a unique solution on the time interval*

$$0 \leq t \leq T, \;\text{ where } T = T(\varepsilon, k, \delta) := \frac{1}{\varepsilon^{\beta(k,\delta)}}, \quad \beta(k,\delta) = \frac{k+1-\delta}{4}\,.$$

*The solution may be written in the form*

$$\theta(t,x) = \theta_0\left(\bar{\xi}(t), \bar{u}(t), x\right) + v(t,x)\,,$$
$$\psi(t,x) = \psi_0\left(\bar{\xi}(t), \bar{u}(t), x\right) + w(t,x)\,,$$

*where* $v, w$ *have regularity*

$$(v(t), w(t)) \in C([0,T], H^1(\mathbb{R}) \oplus L^2(\mathbb{R})),$$

$\bar{\xi}, \bar{u}$ *solve the following system of equations*

$$\bar{\xi}'(t) = \bar{u}(t)\,, \tag{14.4}$$
$$\bar{u}'(t) = 0 \tag{14.5}$$

*with initial data* $\bar{\xi}(0) = \xi_s,\; \bar{u}(0) = u_s$ *and there exists a positive constant* $c$ *such that*

$$|v|^2_{L^\infty([0,T],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,T],L^2(\mathbb{R}))} \leq c\varepsilon^{k+1}\,.$$

*The constant* $c$ *depends on* $F$*.*

The following chapter-wise outline provides an overview of our approach. We use the notation and the results from Chapter 2 (Solitary Manifold ) and Chapter 3 (Symplectic Orthogonal Decomposition).

**Existence of Dynamics and the Orthogonal Component** The existence theory provides that there is a local solution $(\theta, \psi)$ of (14.1)-(14.3), which might be written in the form

$$\theta(t,x) = \bar{v}(t,x) + \theta_0(\xi_s, u_s, x)\,,$$
$$\psi(t,x) = \bar{w}(t,x) + \psi_0(\xi_s, u_s, x)\,,$$

where $(\bar{v}, \bar{w}) \in C^1([0, T_{loc}], H^1(\mathbb{R}) \oplus L^2(\mathbb{R}))$ . Due to Morrey's embedding theorem it holds that $(\theta, \psi) \in C^1([0, T_{loc}], L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R}))$. In the following we assume that a solution $(\theta, \psi)$ of (14.1)-(14.3) is given on the time interval $[0, \overline{T}]$, which might be written as above where $(\bar{v}, \bar{w}) \in C^1([0, \overline{T}], H^1(\mathbb{R}) \oplus L^2(\mathbb{R}))$ .

$\varepsilon$ is chosen so small that due to assumptions (c), (e) in Theorem 14.1 the initial state $(\theta(0), \psi(0))$ is so close to the region

$$\mathcal{S}_0(U) := \left\{ \begin{pmatrix} \theta_0(\xi, u, \cdot) \\ \psi_0(\xi, u, \cdot) \end{pmatrix} \ : \ (\xi, u) \in \Sigma(4, U) \right\} ,$$

of the solitary manifold that the symplectic orthogonal decomposition is possible in a neighbourhood of $(\theta(0), \psi(0))$.

In (14.2)-(14.3) the initial state $(\theta(0), \psi(0))$ is already written as a sum of a point on the solitary manifold $\mathcal{S}_0$ and a transversal component $(v(0), w(0))$ such that the symplectic orthogonality condition is satisfied due to assumption (d) in Theorem 14.1.

For times $t > 0$ we are able to choose the parameters $(\xi(t), u(t))$ according to the decomposition in Chapter 3 (Symplectic Orthogonal Decomposition) as long as $(\theta(t), \psi(t))$ stays close enough to $\mathcal{S}_0(U)$. As long as $(\theta(t), \psi(t))$ stays close enough to $\mathcal{S}_0(U)$ we define $(v, w)$ by

$$v(t,x) = \theta(t,x) - \theta_0(\xi(t), u(t), x)\,, \tag{14.6}$$
$$w(t,x) = \psi(t,x) - \psi_0(\xi(t), u(t), x)\,, \tag{14.7}$$

where the parameter $(\xi(t), u(t))$ are obtained from the decomposition in Chapter 3 (Symplectic Orthogonal Decomposition), such that

$$\mathcal{N}(\theta(t), \psi(t), \xi(t), u(t)) = 0\,. \tag{14.8}$$

Thus we decompose the dynamics in two components, namely a point on the solitary manifold $(\theta_0(\xi(t), u(t), \cdot), \psi_0(\xi(t), u(t), \cdot))$ and a transversal component $(v(t, \cdot), w(t, \cdot))$ which is symplectic orthogonal to the tangent vectors

$$\begin{pmatrix} \partial_\xi \theta_0(\xi(t), u(t), \cdot) \\ \partial_\xi \psi_0(\xi(t), u(t), \cdot) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \partial_u \theta_0(\xi(t), u(t), \cdot) \\ \partial_u \psi_0(\xi(t), u(t), \cdot) \end{pmatrix}$$

of $\mathcal{S}_0$. Finally we compute the time derivatives of $v$ and $w$ which will be needed in the following chapters.

In Chapter 18 (Proof of Theorem 14.1) we will obtain a bound on $|v|^2_{L^\infty([0,T],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,T],L^2(\mathbb{R}))}$ (where $T \leq \overline{T}$) which will give us control over the distance of $(\theta, \psi)$ to the solitary manifold and which will imply that the local solution $(\theta, \psi)$ is indeed continuable.

**Modulation Equations** We want to consider the longitudinal dynamics on $\mathcal{S}_0$, which is described by the parameters $(\xi(t), u(t))$. In order to be able to understand the dynamics on $\mathcal{S}_0$ we derive a system of ordinary differential equations (modulation equations) for the parameters $(\xi(t), u(t))$ which is satisfied up to a certain error. We examine up to what errors the ordinary differential equations that describe the evolution of a soliton,

$$\begin{aligned} \dot{\xi}(t) &= u(t)\,, \\ \dot{u}(t) &= 0\,, \end{aligned}$$

are satisfied. For this purpose we take the time derivative of (14.8) and obtain a system of differential equations. Using Neumann's theorem we conclude that the estimates

$$\begin{aligned} |\dot{\xi}(t) - u(t)| &\leq C\varepsilon^{k+1} + C\,|v|^2_{H^1(\mathbb{R})}\,, \\ |\dot{u}(t)| &\leq C\varepsilon^{k+1} + C\,|v|^2_{H^1(\mathbb{R})}\,, \end{aligned}$$

are satisfied if $|v(t)|_{H^1(\mathbb{R})}\,, |w(t)|_{L^2(\mathbb{R})}$ are less than a certain $\varepsilon_0 > 0$ and as long as the time $t$ is such as described in the introduction of $(v, w)$ above.

**Lyapunov Functional** In order to obtain control on the transversal component $(v, w)$ we introduce the Lyapunov function

$$L(t) = \int \frac{w^2(x)}{2} + \frac{(\partial_x v(x))^2}{2} + \frac{\cos(\theta_K(\gamma(u)(x-\xi)))v^2(x)}{2} + uw(x)\partial_x v(x)\,dx\,,$$

where $(v, w)$ are given by (14.6)-(14.7), $(\xi, u)$ are obtained from the decomposition in Chapter 3 (Symplectic Orthogonal Decomposition), such that the orthogonality conditions hold and $\gamma(u) = 1/\sqrt{1-u^2}$. $L$ is the quadratic part of

$$H(\theta, \psi) + u\Pi(\theta, \psi)$$

above the solitary manifold $\mathcal{S}_0$, where $H$ and $\Pi$, given by (7) and (8) are conserved quantities of the sine-Gordon equation. Finally we compute the time derivative of $L(t)$ which will be needed later.

**Proof of Theorem 14.1** First of all we prove the statement of Theorem 14.1 (i). We suppose that (14.1)-(14.3) has a solution and we make some assumptions on $(\xi, u)$ obtained from the decomposition in Chapter 3 (Symplectic Orthogonal Decomposition) and on $(v, w)$ given by (14.6)-(14.7). The modulation equations allow us to control $(\xi, u)$. The Lyapunov functions and the lower bound on $\mathcal{E}$ allow us to control $(v, w)$, since we are able to estimate

$$\begin{aligned} & c(|v(t)|^2_{H^1(\mathbb{R})} + |w(t)|^2_{L^2(\mathbb{R})}) \\ \leq\; & L(t) \\ =\; & L(0) + \int_0^t \dot{L}(s)\, ds \end{aligned}$$

and to control the right hand side. All in all, we obtain more accurate information about $(v, w)$ and $(\xi, u)$. Using a continuity argument this implies the bound on $(v, w)$ claimed in Theorem 14.1 (i) and approximate equations for the parameters $(\xi, u)$. The bound on $(v, w)$ implies that the local solution discussed in Chapter 15 (Existence of Dynamics and the Orthogonal Component) is continuable up to times $\frac{1}{\varepsilon^{\rho(\delta)}}$ ($\rho(\delta) = \frac{1}{2} - 2\delta$), which establishes the statement of Theorem 14.1 (i).

Using Theorem 14.1 (i) and Gronwall's lemma we show that the dynamics on the solitary manifold can be described by $(\bar{\xi}, \bar{u})$ that satisfy the ODE's (14.4)-(14.5), which establishes the statement of Theorem 14.1 (ii).

# Chapter 15

# Existence of Dynamics and the Orthogonal Component

We argue similar to [Stu98, Proof of theorem 2.1]. In order to be able to make use of existence theory we set

$$\bar{v}(t,x) = \theta(t,x) - \theta_0(\xi_s, u_s, x)\,,$$
$$\bar{w}(t,x) = \psi(t,x) - \psi_0(\xi_s, u_s, x)$$

and consider the problem

$$\begin{pmatrix} \bar{v}(0,x) \\ \bar{w}(0,x) \end{pmatrix} = \begin{pmatrix} \theta(0,x) - \theta_0(\xi_s, u_s, x) \\ \psi(0,x) - \psi_0(\xi_s, u_s, x) \end{pmatrix}, \tag{15.1}$$

$$\partial_t \begin{pmatrix} \bar{v}(t,x) \\ \bar{w}(t,x) \end{pmatrix} = \begin{pmatrix} \bar{w}(t,x) - \psi_0(\xi_s, u_s, x) \\ [\bar{v}(t,x) + \theta_0(\xi_s, u_s, x)]_{xx} - \sin(\bar{v}(t,x) + \theta_0(\xi_s, u_s, x)) + F(\varepsilon, x) \end{pmatrix}. \tag{15.2}$$

By [Mar76, Theorem VIII 2.1, Theorem VIII 3.2 ] there exists a local solution (see also [Stu98, Proof of theorem 2.1], [Stu92, p.434 ]) with

$$(\bar{v}, \bar{w}) \in C^1([0, T_{loc}], H^1(\mathbb{R}) \oplus L^2(\mathbb{R}))\,.$$

$(\theta, \psi)$ given by $\theta(t,x) = \bar{v}(t,x) + \theta_0(\xi_s, u_s, x)$ and $\psi(t,x) = \bar{w}(t,x) + \psi_0(\xi_s, u_s, x)$ solves obviously locally the Cauchy problem (14.1)-(14.3) and $(\theta, \psi) \in C^1([0, T_{loc}], L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R}))$ due to Morrey's embedding theorem.

We are going to obtain some bounds in Chapter 18 which will imply that the local solutions are indeed continuable.

So from now we assume that $(\bar{v}, \bar{w}) \in C^1([0, \overline{T}], H^1(\mathbb{R}) \oplus L^2(\mathbb{R}))$ is a solution of (15.1)-(15.2) and $(\theta, \psi)$ is a solution of (14.1)-(14.3) such that $(\theta, \psi) \in C^1([0, \overline{T}], L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R}))$. Given $(\theta, \psi)$ we choose the parameters $(\xi(t), u(t))$ according to Lemma 3.1 and define $(v, w)$ as follows:

$$v(t,x) = \theta(t,x) - \theta_0(\xi(t), u(t), x)\,, \tag{15.3}$$

$$w(t,x) = \psi(t,x) - \psi_0(\xi(t), u(t), x)\,. \tag{15.4}$$

$(v(t,x), w(t,x))$ is well defined for $t \geq 0$ so small that

$$|v(t)|_{L^\infty(\mathbb{R})} + |w(t)|_{L^2(\mathbb{R})} \leq r$$

and

$$(\xi(t), u(t)) \in \Sigma(4, U)\,,$$

where $r$ and $U$ are from Lemma 3.1. We formalize this by the following definition.

**Definition 15.1.** *Let $t^*$ be the "exit time":*

$$t^* := \sup\Big\{T > 0 : |v|_{L^\infty(\mathbb{R})L^\infty([0,t])} + |w|_{L^\infty([0,t],L^2(\mathbb{R}))} \leq r,$$
$$(\xi(t), u(t)) \in \Sigma(4, U),\ 0 \leq t \leq T\Big\}\,,$$

*where $r$ and $U$ are from Lemma 3.1.*

Notice that $(\xi_s, u_s) = (\xi(0), u(0)) \in \Sigma(4, U)$. We will choose $\varepsilon$ such that, among others,

$$|v(0)|_{L^\infty(\mathbb{R})} + |w(0)|_{L^2(\mathbb{R})} \leq \frac{r}{2},$$

where $(v(0), w(0))$ is given by (14.2)-(14.3). Thus $(v(t,x), w(t,x))$ is well defined for $0 \leq t \leq t^*$. In the following lemma we obtain more information on $(v, w)$.

**Lemma 15.2.** *Let $T = \min\{t^*, \overline{T}\}$ and let $(v, w)$ be defined by* (15.3)-(15.4)*. Then $(v, w) \in C^1([0, T], H^1(\mathbb{R}) \oplus L^2(\mathbb{R}))$.*

**Proof.** Analogous to Lemma 4.3. □

We compute the time derivatives of $v$ and $w$ which will be needed in the following chapters.

**Lemma 15.3.** *The equations for $(v, w)$ defined by* (15.3)-(15.4)*, are*

$$\dot v(x) = w(x) - \dot\xi \partial_\xi \theta_0(\xi, u, x) - \dot u \partial_u \theta_0(\xi, u, x) + u \partial_\xi \theta_0(\xi, u, x)\,,$$

$$\dot w(x) = \partial_x^2 v(x) - \cos(\theta_0(\xi, u, x)) v(x) + F(\varepsilon, x) + \frac{\sin(\theta_0(\xi, u, x)) v^2(x)}{2} + \tilde R(v)(x)$$
$$+ u \partial_\xi \psi_0(\xi, u, x) - \dot\xi \partial_\xi \psi_0(\xi, u, x) - \dot u \partial_u \psi_0(\xi, u, x)\,,$$

*for times $t \in [0, t^*]$, where $\tilde R(v) = \mathcal{O}(|v|^3_{H^1(\mathbb{R})})$.*

**Proof.** Take the time derivatives of $(v, w)$ and use (15.3)-(15.4), (14.1):

$$\begin{aligned}\dot{v}(x) &= w(x) + \psi_0(\xi, u, x) \\ &\quad - \dot{\xi}\partial_\xi\theta_0(\xi, u, x) - \dot{u}\partial_u\theta_0(\xi, u, x) \\ &= w(x) - \dot{\xi}\partial_\xi\theta_0(\xi, u, x) - \dot{u}\partial_u\theta_0(\xi, u, x) + u\partial_\xi\theta_0(\xi, u, x)\,,\end{aligned}$$

$$\begin{aligned}\dot{w}(x) &= \partial_x^2\theta(x) - \sin\theta(x) + F(\varepsilon, x) - \dot{\xi}\partial_\xi\psi_0(\xi, u, x) - \dot{u}\partial_u\psi_0(\xi, u, x) \\ &= \partial_x^2\theta_0(\xi, u, x) + \partial_x^2 v(x) - \sin\theta_0(\xi, u, x) \\ &\quad - \cos\theta_0(\xi, u, x)v(x) + \frac{\sin\theta_0(\xi, u, x)v^2(x)}{2} \\ &\quad + \tilde{R}(v)(x) + F(\varepsilon, x) - \dot{\xi}\partial_\xi\psi_0(\xi, u, x) - \dot{u}\partial_u\psi_0(\xi, u, x) \\ &\quad + u\partial_x\psi_0 - u\partial_x\psi_0 \\ &= \partial_x^2 v(x) - \cos\theta_0(\xi, u, x)v(x) + F(\varepsilon, x) + \frac{\sin\theta_0(\xi, u, x)v^2(x)}{2} + \tilde{R}(v)(x) \\ &\quad + u\partial_\xi\psi_0(\xi, u, x) - \dot{\xi}\partial_\xi\psi_0(\xi, u, x) - \dot{u}\partial_u\psi_0(\xi, u, x)\,,\end{aligned}$$

where we have expanded the term $\sin(\theta_0(\xi, u, x) + v(x))$. □

# Chapter 16

# Modulation Equations

In the following lemma we derive modulation equations for the parameters $(\xi(t), u(t))$.

**Lemma 16.1.** *There exists an* $\varepsilon_0 > 0$ *such that the following statement holds. Let* $(v, w)$ *be given by* (15.3)-(15.4)*, with* $(\xi, u)$ *obtained from Lemma 3.1 and let*

$$|v|_{L^\infty([0,t^*],H^1(\mathbb{R}))}\,, |w|_{L^\infty([0,t^*],L^2(\mathbb{R}))} \leq \varepsilon_0\,,$$

*where* $t^*$ *is from Definition 15.1. Then*

$$|\dot{\xi}(t) - u(t)| \leq C\varepsilon^{k+1} + C\,|v|^2_{H^1(\mathbb{R})}\,,$$

$$|\dot{u}(t)| \leq C\varepsilon^{k+1} + C\,|v|^2_{H^1(\mathbb{R})}\,,$$

*for* $0 \leq t \leq t^*$*, where* $C$ *depends on* $F$*.*

**Proof.** The technique we use is similar to that in the proof of [IKV12, Lemma 6.2]. Using Definition 1.1 and (15.3)-(15.4) we write the orthogonality conditions as follows:

$$0 = \mathcal{C}_1(\theta, \psi, \xi, u) = \int \partial_\xi \psi_0(\xi, u, x) v(x) - \partial_\xi \theta_0(\xi, u, x) w(x)\, dx$$

$$0 = \mathcal{C}_2(\theta, \psi, \xi, u) = \int \partial_u \psi_0(\xi, u, x) v(x) - \partial_u \theta_0(\xi, u, x) w(x)\, dx$$

In the following we skip $(\theta, \psi, \xi, u)$ for simplicity of further notation and take the derivatives of $\mathcal{C}_1$, $\mathcal{C}_2$ with respect to $t$. Using Lemma 15.3 we obtain:

$$
\begin{aligned}
\dot{\mathcal{C}}_1 = &\int \partial_t[\partial_\xi \psi_0(\xi,u,x)]v(x) + \partial_\xi \psi_0(\xi,u,x)\partial_t v(x) \\
&- \partial_t[\partial_\xi \theta_0(\xi,u,x)]w(x) - \partial_\xi \theta_0(\xi,u,x)\partial_t w(x)\, dx \\
= &\int \left\{ \dot{\xi}\partial_\xi^2 \psi_0(\xi,u,x) + \dot{u}\partial_u \partial_\xi \psi_0(\xi,u,x) \right\} v(x) \\
&+ \left\{ \partial_\xi \psi_0(\xi,u,x) \right\} \left\{ w(x) - \dot{\xi}\partial_\xi \theta_0(\xi,u,x) - \dot{u}\partial_u \theta_0(\xi,u,x) \right. \\
&\left. + u\partial_\xi \theta_0(\xi,u,x) \right\} \\
&- \left\{ \dot{\xi}\partial_\xi^2 \theta_0(\xi,u,x) + \dot{u}\partial_u \partial_\xi \theta_0(\xi,u,x) \right\} w(x) \\
&- \left\{ \partial_\xi \theta_0(\xi,u,x) \right\} \left\{ \partial_x^2 v(x) - \cos\theta_0(\xi,u,x)v(x)(x) + F(\varepsilon,x) + \frac{\sin\theta_0(\xi,u,x)v^2(x)}{2} \right. \\
&\left. + \tilde{R}(v)(x) + u\partial_\xi \psi_0(\xi,u,x) - \dot{\xi}\partial_\xi \psi_0(\xi,u,x) - \dot{u}\partial_u \psi_0(\xi,u,x) \right\} dx \\
= &\underbrace{\int -\partial_\xi \psi_0(\xi,u,x)\partial_u \theta_0(\xi,u,x) + \partial_\xi \theta_0(\xi,u,x)\partial_u \psi_0(\xi,u,x)\, dx}_{=\Omega\left(t_\xi(\xi,u,\cdot),t_u(\xi,u,\cdot)\right)} \cdot \dot{u} \\
&+ \underbrace{\int \partial_\xi \psi_0(\xi,u,x)\partial_\xi \theta_0(\xi,u,x) - \partial_\xi \psi_0(\xi,u,x)\partial_\xi \theta_0(\xi,u,x)\, dx}_{=\Omega\left(t_\xi(\xi,u,\cdot),t_\xi(\xi,u,\cdot)\right)} \cdot (u-\dot{\xi}) \\
&+ \underbrace{\int \partial_u \partial_\xi \psi_0(\xi,u,x)v(x) - \partial_u \partial_\xi \theta_0(\xi,u,x)w(x)\, dx}_{=[M(\xi,u,v,w)]_{12}} \cdot \dot{u} \\
&- \underbrace{\int \partial_\xi^2 \psi_0(\xi,u,x)v(x) - \partial_\xi^2 \theta_0(\xi,u,x)w(x)\, dx}_{=[M(\xi,u,v,w)]_{11}} \cdot (u-\dot{\xi}) \\
&\underbrace{+ \int \partial_\xi \psi_0(\xi,u,x)w(x) - \partial_\xi \theta_0(\xi,u,x)\left( \partial_x^2 v(x) - \cos\theta_0(\xi,u,x)v(x) \right) dx}
\end{aligned}
$$

$$\underbrace{\begin{aligned}
&+\int u\partial_\xi^2\psi_0(\xi,u,x)v(x) - u\partial_\xi^2\theta_0(\xi,u,x)w(x)\,dx\\
&-\int \left\{\partial_\xi\theta_0(\xi,u,x)\right\}\left\{F(\varepsilon,x)\right\}dx\\
&-\int \partial_\xi\theta_0(\xi,u,x)\left\{\frac{\sin\theta_0(\xi,u,x)v^2(x)}{2}+\tilde{R}(v)(x)\right\}dx
\end{aligned}}_{=[P(\xi,u,v,w)]_1},$$

$$\begin{aligned}
\dot{\mathcal{C}}_2 = &\int \partial_t[\partial_u\psi_0(\xi,u,x)]v(x) + \partial_u\psi_0(\xi,u,x)\partial_t v(x)\\
&-\partial_t[\partial_u\theta_0(\xi,u,x)]w(x) - \partial_u\theta_0(\xi,u,x)\partial_t w(x)\,dx\\
= &\int \left\{\dot{\xi}\partial_\xi\partial_u\psi_0(\xi,u,x)+\dot{u}\partial_u^2\psi_0(\xi,u,x)\right\}v(x)\\
&+\left\{\partial_u\psi_0(\xi,u,x)\right\}\left\{w(x)-\dot{\xi}\partial_\xi\theta_0(\xi,u,x)-\dot{u}\partial_u\theta_0(\xi,u,x)\right.\\
&\left.+u\partial_\xi\theta_0(\xi,u,x)\right\}\\
&-\left\{\dot{\xi}\partial_\xi\partial_u\theta_0(\xi,u,x)+\dot{u}\partial_u^2\theta_0(\xi,u,x)\right\}w(x)\\
&-\left\{\partial_u\theta_0(\xi,u,x)\right\}\left\{\partial_x^2 v(x)-\cos\theta_0(\xi,u,x)v(x)+F(\varepsilon,x)+\frac{\sin\theta_0(\xi,u,x)v^2(x)}{2}\right.\\
&\left.+\tilde{R}(v)(x)+u\partial_\xi\psi_0(\xi,u,x)-\dot{\xi}\partial_\xi\psi_0(\xi,u,x)-\dot{u}\partial_u\psi_0(\xi,u,x)\right\}dx
\end{aligned}$$

$$
\begin{aligned}
&= \underbrace{\int \partial_u\theta_0(\xi,u,x)\partial_u\psi_0(\xi,u,x) - \partial_u\theta_0(\xi,u,x)\partial_u\psi_0(\xi,u,x)\,dx}_{=\Omega\left(t_u(\xi,u,\cdot),t_u(\xi,u,\cdot)\right)}\cdot\dot{u} \\
&\quad + \underbrace{\int \partial_\xi\theta_0(\xi,u,x)\partial_u\psi_0(\xi,u,x) - \partial_\xi\psi_0(\xi,u,x)\partial_u\theta_0(\xi,u,x)\,dx}_{=\Omega\left(t_u(\xi,u,\cdot),t_\xi(\xi,u,\cdot)\right)}\cdot(u-\dot{\xi}) \\
&\quad + \underbrace{\int \partial_u^2\partial_\xi\psi_0(\xi,u,x)v(x) - \partial_u^2\theta_0(\xi,u,x)w(x)\,dx}_{=[M(\xi,u,v,w)]_{22}}\cdot\dot{u} \\
&\quad - \underbrace{\int \partial_\xi\partial_u\psi_0(\xi,u,x)v(x) - \partial_\xi\partial_u\theta_0(\xi,u,x)w(x)\,dx}_{=[M(\xi,u,v,w)]_{21}}\cdot(u-\dot{\xi}) \\
&\quad + \int \left\{\partial_u\psi_0(\xi,u,x)\right\}w(x) - \partial_u\theta_0(\xi,u,x)\left(\partial_x^2 v(x) - \cos\theta_0(\xi,u,x)v(x)\right)dx \\
&\quad + u\int \partial_\xi\partial_u\psi_0(\xi,u,x)v(x) - \partial_\xi\partial_u\theta_0(\xi,u,x)w(x)\,dx \\
&\quad - \int \left\{\partial_u\theta_0(\xi,u,x)\right\}\left\{F(\varepsilon,x)\right\}dx \\
&\quad \underbrace{- \int \left\{\partial_u\theta_0(\xi,u,x)\right\}\left\{\frac{\sin\theta_0(\xi,u,x)v^2(x)}{2} + \tilde{R}(v)(x)\right\}dx}_{=[P(\xi,u,v,w)]_2}\,.
\end{aligned}
$$

We set

$$
\begin{aligned}
\Omega(u) &:= \begin{pmatrix} \Omega\left(t_\xi(\xi,u,\cdot),t_\xi(\xi,u,\cdot)\right) & \Omega\left(t_\xi(\xi,u,\cdot),t_u(\xi,u,\cdot)\right) \\ \Omega\left(t_u(\xi,u,\cdot),t_\xi(\xi,u,\cdot)\right) & \Omega\left(t_u(\xi,u,\cdot),t_u(\xi,u,\cdot)\right) \end{pmatrix} \\
&= \begin{pmatrix} 0 & \Omega\left(t_\xi(\xi,u,\cdot),t_u(\xi,u,\cdot)\right) \\ \Omega\left(t_u(\xi,u,\cdot),t_\xi(\xi,u,\cdot)\right) & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \gamma(u)^3 m \\ -\gamma(u)^3 m & 0 \end{pmatrix}
\end{aligned}
$$

$$= \gamma(u)^3 m \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

Now we consider for any $(\bar{\xi}, \bar{u}) \in \mathbb{R} \times [-U - U(2), U + U(2)]$, $(\bar{v}, \bar{w}) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ the matrix:

$$M(\bar{\xi}, \bar{u}, \bar{v}, \bar{w})$$

$$= \begin{pmatrix} \left\langle \begin{pmatrix} \partial_\xi^2 \psi_0(\bar{\xi}, \bar{u}, \cdot) \\ -\partial_\xi^2 \theta_0(\bar{\xi}, \bar{u}, \cdot) \end{pmatrix}, \begin{pmatrix} \bar{v}(\cdot) \\ \bar{w}(\cdot) \end{pmatrix} \right\rangle_{L^2(\mathbb{R}) \oplus L^2(\mathbb{R})} & \left\langle \begin{pmatrix} \partial_u \partial_\xi \psi_0(\bar{\xi}, \bar{u}, \cdot) \\ -\partial_u \partial_\xi \theta_0(\bar{\xi}, \bar{u}, \cdot) \end{pmatrix}, \begin{pmatrix} \bar{v}(\cdot) \\ \bar{w}(\cdot) \end{pmatrix} \right\rangle_{L^2(\mathbb{R}) \oplus L^2(\mathbb{R})} \\ \left\langle \begin{pmatrix} \partial_\xi \partial_u \psi_0(\bar{\xi}, \bar{u}, \cdot) \\ -\partial_\xi \partial_u \theta_0(\bar{\xi}, \bar{u}, \cdot) \end{pmatrix}, \begin{pmatrix} \bar{v}(\cdot) \\ \bar{w}(\cdot) \end{pmatrix} \right\rangle_{L^2(\mathbb{R}) \oplus L^2(\mathbb{R})} & \left\langle \begin{pmatrix} \partial_u^2 \psi_0(\bar{\xi}, \bar{u}, \cdot) \\ -\partial_u^2 \theta_0(\bar{\xi}, \bar{u}, \cdot) \end{pmatrix}, \begin{pmatrix} \bar{v}(\cdot) \\ \bar{w}(\cdot) \end{pmatrix} \right\rangle_{L^2(\mathbb{R}) \oplus L^2(\mathbb{R})} \end{pmatrix} .$$

We use Hölder's inequality and obtain for all $(\bar{\xi}, \bar{u}) \in \mathbb{R} \times [-U - U(2), U + U(2)]$, $(\bar{v}, \bar{w}) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$:

$$\| [\Omega(\bar{u})]^{-1} M(\bar{\xi}, \bar{u}, \bar{v}, \bar{w}) \| \leq C(|\bar{v}|_{H^1(\mathbb{R})} + |\bar{w}|_{L^2(\mathbb{R})}) , \tag{16.1}$$

where we denote by $\| \cdot \|$ a matrix norm. Let $I = I_2$ be the identety matrix of dimension 2. Due to (16.1) we are able to find an $\varepsilon_0 > 0$ such that if $|\bar{v}|_{H^1(\mathbb{R})}, |\bar{w}|_{L^2(\mathbb{R})} \leq \varepsilon_0$ then the matrix

$$I + [\Omega(\bar{u})]^{-1} M(\bar{\xi}, \bar{u}, \bar{v}, \bar{w})$$

is invertible by Neumann's theorem. We write the time derivatives of $(\mathcal{C}_1^\varepsilon, \mathcal{C}_2^\varepsilon)$ in matrix form and use the notation $P(\xi, u, v, w) = P$, $M(\xi, u, v, w) = M$, $\Omega(u) = \Omega$:

$$\begin{aligned} 0 &= \begin{pmatrix} \dot{\mathcal{C}}_1 \\ \dot{\mathcal{C}}_2 \end{pmatrix} \\ &= \Omega \begin{pmatrix} \dot{\xi} - u \\ \dot{u} \end{pmatrix} + M \begin{pmatrix} \dot{\xi} - u \\ \dot{u} \end{pmatrix} + P \end{aligned}$$

This implies

$$-\Omega^{-1} P = \left( I + \Omega^{-1} M \right) \begin{pmatrix} \dot{\xi} - u \\ \dot{u} \end{pmatrix} .$$

If $|v|_{H^1(\mathbb{R})}, |w|_{L^2(\mathbb{R})} \leq \varepsilon_0$ then we obtain as mentioned above by Neumann's theorem that

$$\begin{pmatrix} \dot{\xi} - u \\ \dot{u} \end{pmatrix} = -\left( I + \Omega^{-1} M \right)^{-1} [\Omega^{-1} P]$$

We will show later in Corollary 20.6 that

$$\left\langle \begin{pmatrix} -u\partial_x v(\cdot) - w(\cdot) \\ -\partial_x^2 v(\cdot) + \cos(\theta_0(\xi,u,\cdot))v(\cdot) - u\partial_x w(\cdot) \end{pmatrix}, \begin{pmatrix} -\partial_\xi \psi_0(\xi,u,\cdot) \\ \partial_\xi \theta_0(\xi,u,\cdot) \end{pmatrix} \right\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} = 0\,,$$

$$\left\langle \begin{pmatrix} -u\partial_x v(\cdot) - w(\cdot) \\ -\partial_x^2 v(\cdot) + \cos(\theta_0(\xi,u,\cdot))v(\cdot) - u\partial_x w(\cdot) \end{pmatrix}, \begin{pmatrix} -\partial_u \psi_0(\xi,u,\cdot) \\ \partial_u \theta_0(\xi,u,\cdot) \end{pmatrix} \right\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} = 0\,.$$

These identities can also be checked by hand using integration by parts and the symplectic orthogonality. We will use them in the following computations. We consider $P_1$ and $P_2$:

$$\begin{aligned} P_1 = & \int \partial_\xi \psi_0(\xi,u,x)w(x) - \partial_\xi \theta_0(\xi,u,x)\left(\partial_x^2 v(x) - \cos\theta_0(\xi,u,x)v(x)\right)dx \\ & + u\int \partial_\xi^2 \psi_0(\xi,u,x)v(x) - \partial_\xi^2 \theta_0(\xi,u,x)w(x)\,dx \\ & - \int \left\{\partial_\xi \theta_0(\xi,u,x)\right\}\left\{F(\varepsilon,x)\right\}dx \\ & - \int \partial_\xi \theta_0(\xi,u,x)\left\{\frac{\sin\theta_0(\xi,u,x)v^2(x)}{2} + \tilde{R}(v)(x)\right\}dx \\ = & \left\langle \begin{pmatrix} -u\partial_x v(\cdot) - w(\cdot) \\ -\partial_x^2 v(\cdot) + \cos(\theta_0(\xi,u,\cdot))v(\cdot) - u\partial_x w(\cdot) \end{pmatrix}, \begin{pmatrix} -\partial_\xi \psi^0(\xi,u,\cdot) \\ \partial_\xi \theta^0(\xi,u,\cdot) \end{pmatrix} \right\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} \\ & - \int \left\{\partial_\xi \theta_0(\xi,u,x)\right\}\left\{F(\varepsilon,x)\right\}dx \\ & - \int \partial_\xi \theta_0(\xi,u,x)\left\{\frac{\sin\theta_0(\xi,u,x)v^2(x)}{2} + \tilde{R}(v)(x)\right\}dx\,. \end{aligned}$$

Consequently using Corollary 20.6 we obtain

$$|P_1| \leq C\varepsilon^{k+1} + C\,|v|_{H^1(\mathbb{R})}^2\,.$$

$$\begin{aligned} P_2 = & \int \left\{\partial_u \psi_0(\xi,u,x)\right\}w(x) - \partial_u \theta_0(\xi,u,x)\left(\partial_x^2 v(x) - \cos\theta_0(\xi,u,x)v(x)\right)dx \\ & + u\int \partial_\xi \partial_u \psi_0(\xi,u,x)v(x) - \partial_\xi \partial_u \theta_0(\xi,u,x)w(x)\,dx \end{aligned}$$

$$
\begin{aligned}
&- \int \left\{ \partial_u \theta_0(\xi, u, x) \right\} \left\{ F(\varepsilon, x) \right\} dx \\
&- \int \left\{ \partial_u \theta_0(\xi, u, x) \right\} \left\{ \frac{\sin \theta_0(\xi, u, x) v^2(x)}{2} + \tilde{R}(v)(x) \right\} dx \\
= &\left\langle \begin{pmatrix} -u\partial_x v(\cdot) - w(\cdot) \\ -\partial_x^2 v(\cdot) + \cos(\theta_0(\xi, u, \cdot)) v(\cdot) - u\partial_x w(\cdot) \end{pmatrix}, \begin{pmatrix} -\partial_u \psi^0(\xi, u, \cdot) \\ \partial_u \theta^0(\xi, u, \cdot) \end{pmatrix} \right\rangle_{L^2(\mathbb{R}) \oplus L^2(\mathbb{R})} \\
&- \int \left\{ \partial_u \theta_0(\xi, u, x) \right\} \left\{ F(\varepsilon, x) \right\} dx \\
&- \int \left\{ \partial_u \theta_0(\xi, u, x) \right\} \left\{ \frac{\sin \theta_0(\xi, u, x) v^2(x)}{2} + \tilde{R}(v)(x) \right\} dx \, .
\end{aligned}
$$

Consequently using Corollary 20.6 we obtain

$$
|P_2| \leq C\varepsilon^{k+1} + C \left| v \right|^2_{H^1(\mathbb{R})} \, .
$$

□

# Chapter 17

# Lyapunov Functional

We introduce the Lyapunov function.

**Definition 17.1.** *Let* $(v, w)$ *be given by* (15.3)-(15.4)*, with* $(\xi, u)$ *obtained from Lemma 3.1. We set*

$$L(t) = \int \frac{w^2(x)}{2} + \frac{\partial_x v^2(x)}{2} + \frac{\cos(\theta_K(\gamma(x-\xi)))v^2(x)}{2} + uw(x)\partial_x v(x)\, dx\,. \tag{17.1}$$

**Remark 17.2.** $L$ is the quadratic part of

$$H(\theta, \psi) + u\Pi(\theta, \psi)$$

above the solitary manifold $\mathcal{S}_0$ as in Chapter 6.

The time derivative of $L(t)$ is computed in the following lemma. This will be one of the main ingredients in the proof of the main result.

**Lemma 17.3.**

$$\begin{aligned} &\frac{d}{dt}L(t) \\ &= \int w(x)\left[\frac{\sin(\theta_0(\xi,u,x))v^2(x)}{2} + \tilde{R}(v)(x)\right] + u\partial_x v(x)\left[\frac{\sin(\theta_0(\xi,u,x))v^2(x)}{2} + \tilde{R}(v)(x)\right] dx \\ &\quad - \dot{u}\int \frac{\sin(\theta_0(\xi,u,x))}{2}\partial_u\theta_0(\xi,u,x)v^2(x)\, dx + (\dot{\xi} - u)\int \cos(\theta_0(\xi,u,x))v(x)\partial_x v(x)\, dx \\ &\quad + \dot{u}\int w(x)\partial_x v(x)\, dx + \int w(x)F(\varepsilon,x)\, dx + \int u\partial_x v(x)F(\varepsilon,x)\, dx\,. \end{aligned}$$

**Proof.** We use a similar technique as in the proof of [KSK97, Lemma 2.1]. We can assume that the initial data of our problem has compact support. This allows us to do the following computations (integration by parts etc.). The claim for non-compactly supported initial data follows by density arguments.

As we have seen in the proof of Lemma 6.3 it holds that

$$\begin{aligned}
&\int \frac{\partial_t[\cos\theta_0(\xi,u,x)]}{2} v^2(x)\,dx \\
&= \int \dot{\xi}\cos(\theta_0(\xi,u,x))v(x)\partial_x v(x) - \dot{u}\frac{\sin(\theta_0(\xi,u,x))}{2}\partial_u\theta_0(\xi,u,x)v^2(x)\,dx\,.
\end{aligned}$$

Notice that due to (15.3)-(15.4)

$$\begin{aligned}
&v(t,x) = \theta(t,x) - \theta_0(\xi(t),u(t),x)\,, \\
&w(t,x) = \psi(t,x) - \psi_0(\xi(t),u(t),x) = \psi(t,x) + u\gamma(u)\theta_K'(\gamma(x-\xi))\,.
\end{aligned}$$

Partial integration yields $\int \partial_x v(x)\partial_x^2 v(x) + w(x)\partial_x w(x)\,dx = 0$. We differentiate the Lyapunov function (17.1) with respect to $t$ and use Lemma 15.3:

$$\begin{aligned}
\dot{L}(t) = &\int w(x)\dot{w}(x) + \partial_x v(x)\partial_x\dot{v}(x) + \cos\theta_0(\xi,u,x)v(x)\dot{v}(x) + \frac{\partial_t[\cos\theta_0(\xi,u,x)]}{2}v^2(x) \\
&+ u\dot{w}(x)\partial_x v(x) + uw(x)\partial_x\dot{v}(x) + \dot{u}w(x)\partial_x v(x)\,dx \\
= &\int w(x)\Big[\partial_x^2 v(x) - \cos\theta_0(\xi,u,x)v(x) + \frac{\sin\theta_0(\xi,u,x)v^2(x)}{2} \\
&+ \tilde{R}(v)(x) + u\partial_\xi\psi_0(\xi,u,x) + F(\varepsilon,x) \\
&- \dot{\xi}\partial_\xi\psi_0(\xi,u,x) - \dot{u}\partial_u\psi_0(\xi,u,x)\Big] \\
&+ \partial_x v(x)\partial_x\Big[w(x) - \dot{\xi}\partial_\xi\theta_0(\xi,u,x) - \dot{u}\partial_u\theta_0(\xi,u,x) \\
&+ u\partial_\xi\theta_0(\xi,u,x)\Big] \\
&+ \cos(\theta_0(\xi,u,x))v(x)\Big[w(x) - \dot{\xi}\partial_\xi\theta_0(\xi,u,x) - \dot{u}\partial_u\theta_0(\xi,u,x) \\
&+ u\partial_\xi\theta_0(\xi,u,x)\Big] \\
&+ \dot{\xi}\cos(\theta_0(\xi,u,x))v(x)\partial_x v(x) - \dot{u}\frac{\sin\theta_0(\xi,u,x)}{2}\partial_u\theta_0(\xi,u,x)v^2(x)\,dx \\
&+ u\partial_x v(x)\Big[\partial_x^2 v(x) - \cos\theta_0(\xi,u,x)v(x) + \frac{\sin\theta_0(\xi,u,x)v^2(x)}{2} \\
&+ \tilde{R}(v)(x) + u\partial_\xi\psi_0(\xi,u,x) + F(\varepsilon,x) \\
&- \dot{\xi}\partial_\xi\psi_0(\xi,u,x) - \dot{u}\partial_u\psi_0(\xi,u,x)\Big]
\end{aligned}$$

$$
\begin{aligned}
&+ uw(x)\partial_x\Big[w(x) - \dot{\xi}\partial_\xi\theta_0(\xi,u,x) - \dot{u}\partial_u\theta_0(\xi,u,x)\\
&+ u\partial_\xi\theta_0(\xi,u,x)\Big] + \dot{u}\int w(x)\partial_x v(x)\,dx\\
= (u-\dot{\xi})&\Bigg[\int -u\partial_x v(x)\Big\{-\partial_\xi\psi_0(\xi,u,x)\Big\} - w\Big\{-\partial_\xi\psi_0(\xi,u,x)\Big\}\\
&+ [\cos(\theta_0(\xi,u,x))v(x) - \partial_x^2 v(x)]\partial_\xi\theta_0(\xi,u,x) + uw(x)\partial_x\partial_\xi\theta_0(\xi,u,x)\,dx\Bigg]\\
&- \dot{u}\Bigg[\int -u\partial_x v(x)\Big\{-\partial_u\psi_0(\xi,u,x)\Big\} - w(x)\Big\{-\partial_u\psi_0(\xi,u,x)\Big\}\\
&[\cos(\theta_0(\xi,u,x))v(x) - \partial_x^2 v(x)]\partial_u\theta_0(\xi,u,x) + uw(x)\partial_x\partial_u\theta_0(\xi,u,x)\,dx\Bigg]\\
&+ w(x)[\frac{\sin\theta_0(\xi,u,x)v^2(x)}{2} + \tilde{R}(v)(x)] + u\partial_x v(x)[\frac{\sin\theta_0(\xi,u,x)v^2(x)}{2} + \tilde{R}(v)(x)]\\
&- \dot{u}\int\frac{\sin(\theta_0(\xi,u,x))}{2}\partial_u\theta_0(\xi,u,x)v^2(x)\,dx + (\dot{\xi}-u)\int\cos(\theta_0(\xi,u,x))v(x)\partial_x v(x)\,dx\\
&+ \dot{u}\int w(x)\partial_x v(x)\,dx + \int w(x)F(\varepsilon,x)\,dx + \int u\partial_x v(x)F(\varepsilon,x)\,dx\,.
\end{aligned}
$$

Due to Corollary 20.6

$$
\begin{aligned}
&\Bigg[\int -u\partial_x v(x)\Big\{-\partial_\xi\psi_0(\xi,u,x)\Big\} - w(x)\Big\{-\partial_\xi\psi_0(\xi,u,x)\Big\}\\
&+ [\cos(\theta_0(\xi,u,x))v(x) - \partial_x^2 v(x)]\partial_\xi\theta_0(\xi,u,x) + uw(x)\partial_x\partial_\xi\theta_0(\xi,u,x)\,dx\Bigg]\\
&= \left\langle \begin{pmatrix} -u\partial_x v(\cdot) - w(\cdot) \\ -\partial_x^2 v(\cdot) + \cos(\theta_0(\xi,u,\cdot))v(\cdot) - u\partial_x w(\cdot)\end{pmatrix}, \begin{pmatrix} -\partial_\xi\psi_0(\xi,u,\cdot) \\ \partial_\xi\theta_0(\xi,u,\cdot)\end{pmatrix}\right\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})}\\
&= 0
\end{aligned}
$$

and

$$\begin{aligned}
&\Bigg[\int -u\partial_x v(x)\Big\{-\partial_u\psi_0(\xi,u,x)\Big\} - w(x)\Big\{-\partial_u\psi_0(\xi,u,x)\Big\} \\
&[\cos(\theta_0(\xi,u,x))v(x) - \partial_x^2 v(x)]\partial_u\theta_0(\xi,u,x) + uw(x)\partial_x\partial_u\theta_0(\xi,u,x)\,dx\Bigg] \\
= &\left\langle \begin{pmatrix} -u\partial_x v(\cdot) - w(\cdot) \\ -\partial_x^2 v(\cdot) + \cos(\theta_0(\xi,u,\cdot))v(\cdot) - u\partial_x w(\cdot) \end{pmatrix}, \begin{pmatrix} -\partial_u\psi_0(\xi,u,\cdot) \\ \partial_u\theta_0(\xi,u,\cdot) \end{pmatrix} \right\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} \\
= &\,0.
\end{aligned}$$

These identities can also be checked by hand using integration by parts and the symplectic orthogonality. □

# Chapter 18

# Proof of Theorem 14.1

We prove Theorem 14.1.

## 18.1 Proof of Theorem 14.1 (i)

First we suppose that (14.1)-(14.3) has a solution and we make some assumptions on $(v, w)$ given by (15.3)-(15.4) and on $(\xi, u)$ obtained from Lemma 3.1. Then the following lemma yields us more accurate information about $(v, w)$ and $(\xi, u)$.

**Lemma 18.1.** *Let $\varepsilon$ be sufficiently small, $\rho(k, \delta) = \frac{k+1}{2} - 2\delta$. Assume that the assumptions (b),(c),(d) from Theorem 14.1 are satisfied. Assume that* (14.1)-(14.3) *has a solution $(\theta, \psi)$ on $[0, \overline{T}]$ such that*

$$(\theta, \psi) \in C^1([0, \overline{T}], L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R}))\,.$$

*Suppose that*

$$0 \leq T \leq t^* \leq \overline{T}\,,$$

*where $t^*$ is from Definition 15.1. Let $(v, w)$ be given by* (15.3)-(15.4)*, with $(\xi, u)$ obtained from Lemma 3.1 such that*

$$|v|^2_{L^\infty([0,T],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,T],L^2(\mathbb{R}))} \leq \varepsilon^{k+1-\delta}\,.$$

*Then, provided*

$$0 \leq T \leq \frac{1}{\varepsilon^{\rho(k,\delta)}},$$

*it holds*

*(a)* $\forall t \in [0, T]:\ (\xi(t), u(t)) \in \Sigma(5, U)\,,$

*(b)* $|v|^2_{L^\infty([0,T],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,T],L^2(\mathbb{R}))} \leq \frac{1}{c}(L(0) + C\varepsilon^{k+1})\,,$ *where $C$ depends on $F$ and $c$ is from Lemma 7.4.*

**Proof.** Choose $\varepsilon$ such that the following holds:

(1) $\varepsilon \in (0, \varepsilon_0)$ where $\varepsilon_0$ is from Lemma 16.1,

(2) $\varepsilon$ is so small that

$$C\varepsilon^{k+1-\delta}\left[\frac{1}{\varepsilon^{\rho(k,\delta)}}\right] + |u(0)| \leq \frac{U(5)}{2} + U\,,$$

where C is a constant that appears in (18.1) further in this proof which depends on $F$.

Lemma 16.1 yields $\forall t \in [0,T]$:

$$\begin{aligned}|\dot{\xi}(t) - u(t)| &\leq C\varepsilon^{k+1} + C\,|v|^2_{H^1(\mathbb{R})}\\ &\leq C\varepsilon^{k+1} + C\varepsilon^{k+1-\delta}\\ &\leq C\varepsilon^{k+1-\delta}\,,\end{aligned}$$

$$\begin{aligned}|\dot{u}(t)| &\leq C\varepsilon^{k+1} + C\,|v|^2_{H^1(\mathbb{R})}\\ &\leq C\varepsilon^{k+1} + C\varepsilon^{k+1-\delta}\\ &\leq C\varepsilon^{k+1-\delta}\,,\end{aligned}$$

Thus we obtain $\forall t \in [0,T]$:

$$\begin{aligned}|u(t) - u(0)| &\leq \int_0^t |\dot{u}(s)|\,ds\\ &\leq C\varepsilon^{k+1-\delta}t\end{aligned}$$

$$\Rightarrow \quad |u(t)| \leq C\varepsilon^{k+1-\delta}t + |u(0)|\,. \tag{18.1}$$

This implies (a) due to assumption (c) from Theorem 14.1 and (2). Using Lemma 17.3, Lemma 7.2 and Lemma 7.4 we obtain for times

$$0 \leq t \leq T \leq \frac{1}{\varepsilon^{\rho(k,\delta)}},$$

the following estimate,

$$\begin{aligned}&c(|v(t)|^2_{H^1(\mathbb{R})} + |w(t)|^2_{L^2(\mathbb{R})})\\ &\leq L(t) = L(0) + \int_0^t \dot{L}(t)\,dt\end{aligned}$$

$$
\begin{aligned}
&= L(0) + \int_0^t w(x) \left[\frac{\sin\theta_0(\xi,u,x)v^2(x)}{2} + \tilde{R}(v)(x)\right] \\
&\quad + u\partial_x v(x) \left[\frac{\sin\theta_0(\xi,u,x)v^2(x)}{2} + \tilde{R}(v)(x)\right] dx \\
&\quad - \dot{u} \int \frac{\sin(\theta_0(\xi,u,x))}{2} \partial_u \theta_0(\xi,u,x) v^2(x)\, dx + (\dot{\xi} - u) \int \cos(\theta_0(\xi,u,x)) v(x) \partial_x v(x)\, dx \\
&\quad + \dot{u} \int w(x)\partial_x v(x)\, dx + \int w(x) F(\varepsilon,x)\, dx + \int u\partial_x v(x) F(\varepsilon,x)\, dx\, dt \\
&\le L(0) + C\int_0^t \varepsilon^{\frac{k+1-\delta}{2}+k+1-\delta}\, dt \\
&\le L(0) + C\varepsilon^{k+1}\,.
\end{aligned}
$$

□

**Theorem 18.2.** *Let* $\varepsilon$ *be sufficiently small,* $\rho(k,\delta) = \frac{k+1}{2} - 2\delta$*. Assume that the assumptions (b),(c),(d),(e) of Theorem 14.1 are satisfied. Assume that* (14.1)-(14.3) *has a solution* $(\theta,\psi)$ *on* $[0,\overline{T}]$ *such that*

$$(\theta,\psi) \in C^1([0,\overline{T}], L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R}))\,.$$

*Suppose that*

$$0 \le T \le \overline{T}\,.$$

*Then, provided*

$$0 \le T \le \frac{1}{\varepsilon^{\rho(\delta)}}\,,$$

*it holds that* $(v,w)$ *be given by* (15.3)-(15.4) *is well defined for times* $[0,T]$ *and there exists a constant* $\hat{c}$ *such that*

*(a)* $|v|^2_{L^\infty([0,T],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,T],L^2(\mathbb{R}))} \le \hat{c}\varepsilon^{k+1}$,

*(b)* $\forall t \in [0,T]:\ (\xi(t),u(t)) \in \Sigma(5,U)$.

**Proof.** Choose $\varepsilon$ such that the following holds:

(1) $\varepsilon$ satisfies all smallness assumtions of Lemma 18.1;

(2) $\frac{2}{c}(L(0) + C\varepsilon^{k+1}) < \varepsilon^{k+1-\delta}$, where $L(0) = \mathcal{E}(v(0), w(0), \xi_s, u_s)$ and $c, C$ are from Lemma 18.1 (b);

(3) $\varepsilon$ is so small that if $(v,w) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ satisfies $|v|^2_{H^1(\mathbb{R})} + |w|^2_{L^2(\mathbb{R})} \leq \varepsilon^{1-\delta}$ then it holds that $|v|_{L^\infty(\mathbb{R})} + |w|_{L^2(\mathbb{R})} \leq \frac{r}{2}$, where $r$ is from Lemma Lemma 3.1. This can be ensured by Morrey's embedding theorem.

Notice that $\Sigma(5,U) \subset \Sigma(4,U)$. We define an exit time

$$t_* := \sup \Big\{ T > 0 : |v|^2_{L^\infty([0,t],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,t],L^2(\mathbb{R}))} \leq \frac{2}{c}(L(0) + C\varepsilon^{k+1}), \\ (\xi(t), u(t)) \in \Sigma(5,U),\ 0 \leq t \leq T \Big\} .$$

Suppose $t_* < \frac{1}{\varepsilon^{\rho(k,\delta)}}$. Then there exists a time $\hat{t}$ such that

$$\frac{1}{\varepsilon^{\rho(k,\delta)}} > \hat{t} > t_* ,$$

with

$$\forall t \in [0,\hat{t}] :\ (\xi(t), u(t)) \in \Sigma(4,U), \qquad (\xi(\hat{t}), u(\hat{t})) \notin \Sigma(5,U)$$

or

$$\frac{1}{c}(L(0) + C\varepsilon^{k+1}) < \frac{2}{c}(L(0) + C\varepsilon^{k+1}) < |v|^2_{L^\infty([0,\hat{t}],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,\hat{t}],L^2(\mathbb{R}))} < \varepsilon^{k+1-\delta} .$$

This leads a contradiction to the previous lemma. Thus

$$|v|^2_{L^\infty([0,T],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,T],L^2(\mathbb{R}))} \leq \frac{2}{c}(L(0) + C\varepsilon^{k+1}) \leq \hat{c}\varepsilon^{k+1}$$

and

$$\forall t \in [0,T] :\ (\xi(t), u(t)) \in \Sigma(5,U) .$$

□

The previous theorem implies that the local solution of (14.1)-(14.3) discussed in Chapter 15 is indeed continuable up to times $\frac{1}{\varepsilon^{\rho(k,\delta)}}$, where $\rho(k,\delta) = \frac{1}{2} - 2\delta$. Theorem 18.2 and Lemma 16.1 yield the approximate equations for the parameters $(\xi, u)$. This verifies the claim of Theorem 14.1 (i).

## 18.2 Proof of Theorem 14.1 (ii)

### 18.2.1 ODE Analysis

In this section we lay the groundwork for passing from the approximate equations for the parameters $(\xi, u)$ in Theorem 14.1 (i) to the ODE's (14.4)-(14.5) in Theorem 14.1 (ii). We start with a preparing lemma.

**Lemma 18.3.** *Let* $\tilde{\xi} = \tilde{\xi}(s)$, $\tilde{u} = \tilde{u}(s)$, $\epsilon_1 = \epsilon_1(s)$, $\epsilon_2 = \epsilon_2(s)$ *be* $C^1$ *real-valued functions. Suppose that*

$$|\epsilon_j(s)| \leq \bar{c}\varepsilon^{\frac{3(k+1)+\delta}{4}}$$

*on* $[0,T]$ *for* $j = 1, 2$. *Assume that on* $[0,T]$,

$$\begin{aligned}
&\frac{d}{ds}\tilde{\xi}(s) = \tilde{u}(s) + \epsilon_1(s)\,, \quad \tilde{\xi}(0) = \tilde{\xi}_0 \\
&\frac{d}{ds}\tilde{u}(s) = \epsilon_2(s)\,, \quad \tilde{u}(0) = \tilde{u}_0\,.
\end{aligned}$$

*Let* $\hat{\xi} = \hat{\xi}(s)$ *and* $\hat{u} = \hat{u}(s)$ *be* $C^1$ *real-valued functions which satisfy the exact equations*

$$\begin{aligned}
&\frac{d}{ds}\hat{\xi}(s) = \hat{u}(s)\,, \quad \hat{\xi}(0) = \tilde{\xi}_0\,, \\
&\frac{d}{ds}\hat{u}(s) = 0\,, \quad \hat{u}(0) = \tilde{u}_0\,.
\end{aligned}$$

*Then provided* $T \leq 1$, *there exists a* $c > 0$ *such that the estimates*

$$|\tilde{\xi}(s) - \hat{\xi}(s)| \leq c\varepsilon^{\frac{3(k+1)+\delta}{4}}, \qquad |\tilde{u}(s) - \hat{u}(s)| \leq c\varepsilon^{\frac{3(k+1)+\delta}{4}}\,,$$

*hold on* $[0,T]$.

**Proof.** In the following proof we follow very closely [HZ08, Lemma 6.1]. Let $x = x(s)$ and $y = y(s)$ be $C^1$ real-valued functions, $C \geq 1$, and $(x,y)$ satisfy the differential inequalities:

$$\begin{cases} |\dot{x}| \leq |y| & x(0) = x_0 \\ |\dot{y}| \leq 0 & y(0) = y_0 \end{cases}, \qquad .$$

We are going to apply the Gronwall lemma. Let $z(s) = x^2 + y^2$. Then

$$|\dot{z}| = |2x\dot{x} + 2y\dot{y}| \leq 2|x||y| \leq (x^2 + y^2) = z\,,$$

and hence $z(s) \leq z(0)e^s$ . Thus

$$\begin{aligned}
|x(s)| &\leq \sqrt{2}\max(|x_0|, |y_0|)\exp(\frac{s}{2})\,, \\
|y(s)| &\leq \sqrt{2}\max(|x_0|, |y_0|)\exp(\frac{s}{2})\,.
\end{aligned} \tag{18.2}$$

Now we recall the Duhamel's formula. Let $X(s) : \mathbb{R} \to \mathbb{R}^2$ be a two-vector function, $X_0 \in \mathbb{R}^2$ a two-vector, and $A(s) : \mathbb{R} \to (2 \times 2 \text{ matrices})$ a $2 \times 2$ matrix function. We consider the ODE system

$$\dot{X}(s) = A(s)X(s), \qquad X(s') = X_0$$

and denote its solution by $X(s) = S(s,s')X_0$ such that

$$\frac{d}{ds}S(s,s')X_0 = A(s)S(s,s')X_0\,, \quad S(s',s')X_0 = X_0\,.$$

Let $F(s) : \mathbb{R} \to \mathbb{R}^2$ be a two-vector function. We can express the solution to the inhomogeneous ODE system

$$\dot{X}(s) = A(s)X(s) + F(s)$$

with initial condition $X(0) = 0$ by the Duhamel's formula

$$X(s) = \int_0^s S(s,s')F(s')ds'\,.$$

Let $U = \hat{u} - \tilde{u}$ and $\Xi = \hat{\xi} - \tilde{\xi}$. These functions satisfy

$$\begin{aligned} \frac{d}{ds}\Xi(s) &= U(s) + \epsilon_1(s)\,, \\ \frac{d}{ds}U(s) &= \epsilon_2(s)\,. \end{aligned}$$

We set

$$A(s) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F(s) = \begin{bmatrix} \epsilon_1(s) \\ \epsilon_2(s) \end{bmatrix}, \quad X(s) = \begin{bmatrix} \Xi(s) \\ U(s) \end{bmatrix}$$

and obtain by Duhamel's formula:

$$\begin{bmatrix} \Xi(s) \\ U(s) \end{bmatrix} = \int_0^s S(s,s') \begin{bmatrix} \epsilon_1(s') \\ \epsilon_2(s') \end{bmatrix} \, ds'\,. \tag{18.3}$$

We apply (18.2) with

$$\begin{bmatrix} x(s) \\ y(s) \end{bmatrix} = S(s+s',s') \begin{bmatrix} \epsilon_1(s') \\ \epsilon_2(s') \end{bmatrix}, \quad \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} \epsilon_1(s') \\ \epsilon_2(s') \end{bmatrix}.$$

It follows that

$$\left| S(s,s') \begin{bmatrix} \epsilon_1(s') \\ \epsilon_2(s') \end{bmatrix} \right| \leq \sqrt{2} \begin{bmatrix} \exp(\frac{1}{2}(s-s')) \\ \exp(\frac{1}{2}(s-s')) \end{bmatrix} \max(|\epsilon_1(s')|, |\epsilon_2(s')|)\,.$$

Using (18.3) we obtain that on $[0,T]$

$$\begin{aligned} |\Xi(s)| &\leq \sqrt{2}\,T\exp(\frac{1}{2}T) \sup_{0\leq s\leq T} \max(|\epsilon_1(s)|, |\epsilon_2(s)|)\,, \\ |U(s)| &\leq \sqrt{2}\,T\exp(\frac{1}{2}T) \sup_{0\leq s\leq T} \max(|\epsilon_1(s)|, |\epsilon_2(s)|)\,, \end{aligned}$$

which yields the claim. □

In the following we show the relation between the parameters $(\xi, u)$ selected by the implicit function theorem according to Lemma 3.1 and the solutions $(\hat{\xi}, \hat{u})$ of the exact ODE's from the previous lemma.

**Lemma 18.4.** *Let* $\varepsilon$ *be sufficiently small,* $\beta(k,\delta) = \frac{k+1-\delta}{4}$ *and*

$$s = \varepsilon^{\beta(k,\delta)} t,$$

*where*

$$0 \le s \le 1, \quad 0 \le t \le \frac{1}{\varepsilon^{\beta(k,\delta)}}\,.$$

*Let* $(\xi, u)$ *be the parameters selected according to Lemma 3.1 and* $(\hat{\xi}, \hat{u})$ *from Lemma 18.3. Then it holds that*

$$|\xi(t) - \frac{\hat{\xi}(\varepsilon^{\beta(k,\delta)} t)}{\varepsilon^{\beta(k,\delta)}}| \le \varepsilon^{\frac{(k+1)+\delta}{2}}\,,$$

$$|u(t) - \hat{u}(\varepsilon^{\beta(k,\delta)} t)| \le \varepsilon^{\frac{3(k+1)+\delta}{4}}\,.$$

**Proof.** We set

$$\tilde{\xi}(s) = \varepsilon^{\beta(k,\delta)} \xi(s/\varepsilon^{\beta(k,\delta)}), \qquad \tilde{u}(s) = u(s/\varepsilon^{\beta(k,\delta)})\,.$$

For times

$$0 \le t \le \frac{1}{\varepsilon^{\beta(k,\delta)}}$$

Lemma 16.1 and Theorem 18.2 yield:

$$\begin{aligned} |\dot{\xi}(t) - u(t)| &\le C\varepsilon^{k+1} + C\,|v|^2_{H^1(\mathbb{R})} \\ &\le C\varepsilon^{k+1}\,, \end{aligned}$$

$$\begin{aligned} |\dot{u}(t)| &\le C\varepsilon^{k+1} + C\,|v|^2_{H^1(\mathbb{R})} \\ &\le C\varepsilon^{k+1}\,. \end{aligned}$$

Thus $(\tilde{\xi}, \tilde{u})$ satisfy the assumptions of Lemma 18.3, since

$$\begin{aligned} &\frac{d}{ds}\tilde{\xi}(s) \\ =&\xi'\left(\frac{s}{\varepsilon^{\beta(k,\delta)}}\right) \end{aligned}$$

$$
\begin{aligned}
&= u\left(\frac{s}{\varepsilon^{\beta(k,\delta)}}\right) + \mathcal{O}(\varepsilon^{k+1}) \\
&= \tilde{u}(s) + \mathcal{O}(\varepsilon^{k+1}) ,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{d}{ds}\tilde{u}(s) \\
&= \frac{1}{\varepsilon^{\beta(k,\delta)}} u'\left(\frac{s}{\varepsilon^{\beta(k,\delta)}}\right) \\
&= \frac{1}{\varepsilon^{\beta(k,\delta)}} \mathcal{O}(\varepsilon^{k+1}) \\
&= \mathcal{O}(\varepsilon^{\frac{3(k+1)+\delta}{4}}) .
\end{aligned}
$$

Hence Lemma 18.3 yields:

$$
\begin{aligned}
&|\tilde{\xi}(s) - \hat{\xi}(s)| = |\varepsilon^{\beta(k,\delta)}\xi(t) - \hat{\xi}(\varepsilon^{\beta(k,\delta)}t)| \leq c\varepsilon^{\frac{3(k+1)+\delta}{4}} \\
\Rightarrow &|\xi(t) - \frac{\hat{\xi}(\varepsilon^{\beta(k,\delta)}t)}{\varepsilon^{\beta(k,\delta)}}| \leq c\varepsilon^{\frac{(k+1)+\delta}{2}} ,
\end{aligned}
$$

$$
\begin{aligned}
&|\tilde{u}(s) - \hat{u}(s)| = |u(t) - \hat{u}(\varepsilon^{\beta(k,\delta)}t)| \leq c\varepsilon^{\frac{3(k+1)+\delta}{4}} , \\
\Rightarrow &|u(t) - \hat{u}(\varepsilon^{\beta(k,\delta)}t)| \leq c\varepsilon^{\frac{3(k+1)+\delta}{4}} .
\end{aligned}
$$

□

### 18.2.2 Completion of the Proof of Theorem 14.1 (ii)

Theorem 14.1 (i) yields the dynamics with the parameters $(\xi, u)$ selected by the implicit function theorem according to Lemma 3.1 on the time interval

$$0 \leq t \leq \frac{1}{\varepsilon^{\beta(k,\delta)}} .$$

Using Lemma 18.4 and the the triangle inequality we can replace

$$(\xi(t), u(t))$$

with

$$(\frac{\hat{\xi}(\varepsilon^{\beta(k,\delta)}t)}{\varepsilon^{\beta(k,\delta)}}, \hat{u}(\varepsilon^{\beta(k,\delta)}t)) .$$

We set

$$(\bar{\xi}(t), \bar{u}(t)) = (\frac{\hat{\xi}(\varepsilon^{\beta(k,\delta)}t)}{\varepsilon^{\beta(k,\delta)}}, \hat{u}(\varepsilon^{\beta(k,\delta)}t))$$

and conclude that the equations claimed are satisfied.

□

# Part IV

# Virtual Solitons in the Presence of a Forcing $F(\varepsilon, x)$

# Chapter 19

# Main Result and Overview

To formulate our results precisely, we need some definitions.

**Definition 19.1.** *Let $\alpha, n \in \mathbb{N}$ and $u_* > 0$.*

*(a) We set $I(u_*) = [-u_*, u_*]$.*

*(b) Let $H^{k,\alpha}(\mathbb{R})$ denote the weighted Sobolev space of functions with finite norm*

$$|\theta|_{H^{k,\alpha}(\mathbb{R})} = |(1+|x|^2)^{\frac{\alpha}{2}}\theta(x)|_{H^k_x(\mathbb{R})} .$$

*(c) Let $H^{k,\alpha}(\mathbb{R}^2)$ denote the weighted Sobolev space of functions with finite norm*

$$|\theta|_{H^{k,\alpha}(\mathbb{R}^2)} = |(1+|\xi|^2+|x|^2)^{\frac{\alpha}{2}}\theta(\xi,x)|_{H^k_{\xi,x}(\mathbb{R}^2)} .$$

*(d) $(\bar{Y}^\alpha)'$ is the space $H^{3,\alpha}(\mathbb{R}^2) \oplus H^{2,\alpha}(\mathbb{R}^2) \oplus H^{2,\alpha}(\mathbb{R})$ with the finite norm*

$$|y|_{(\bar{Y}^\alpha)'} = |\theta|_{H^{3,\alpha}(\mathbb{R}^2)} + |\psi|_{H^{2,\alpha}(\mathbb{R}^2)} + |\lambda|_{H^{2,\alpha}(\mathbb{R})} .$$

*(e) $Y^\alpha_n(u_*)$ is the space*

$$\Bigg\{ y = (\theta, \psi, \lambda_u) \in C^n(I(u_*), (\bar{Y}^\alpha)') : \|y\|_{Y^\alpha_n(u_*)} < \infty;\ \forall\ u \in I(u_*),\ \forall\ \lambda \in H^{2,\alpha}(\mathbb{R}) :$$
$$\left\langle \begin{pmatrix} \theta(u)(\xi,x) \\ \psi(u)(\xi,x) \end{pmatrix}, \lambda(\xi) \begin{pmatrix} \theta_K'(\gamma(x-\xi)) \\ -u\gamma\theta_K''(\gamma(x-\xi)) \end{pmatrix} \right\rangle_{L^{2,\alpha}_{\xi,x}(\mathbb{R}^2)\oplus L^{2,\alpha}_{\xi,x}(\mathbb{R}^2)} = 0 \Bigg\}$$

*with the finite norm*

$$\|y\|_{Y^\alpha_n(u_*)} = \sup_{u\in I(u_*)} \left( \sum_{i=0}^{n} |\partial_u^i y(u)|_{(\bar{Y}^\alpha)'} \right) .$$

*(f) For* $0 < U < u_*$ *we introduce the parameter area*

$$\Sigma(l, U, u_*) := \Big\{(\xi, u) \in \mathbb{R} \times (-1,1) : u \in \Big(-U - U(l,u_*), U + U(l,u_*)\Big)\Big\}\,,$$

*where* $U(l, u_*) := \frac{u_* - U}{l}$.

**Remark 19.2.** (a) The weighted Sobolev spaces are defined as in [Kop15].

The main result of Part IV is the following theorem.

**Theorem 19.3.** *Let* $n \geq 1$, $k+1 \leq n$. *Assume that*

*(a)* $F \in C^n((-1,1), H^{1,1}(\mathbb{R}))$, $\partial_\varepsilon^l F(0,\cdot) = 0$, *where* $0 \leq l \leq k$*;*

*(b)* $\xi_s \in \mathbb{R}$.

*There exist* $\varepsilon_0 > 0$, $u_* > 0$ *and a map*

$$(-\varepsilon_0, \varepsilon_0) \to Y_2^1(u_*),\ \varepsilon \mapsto (\hat{\theta}_n^\varepsilon, \hat{\psi}_n^\varepsilon, \lambda_{u,n}^\varepsilon) \tag{19.1}$$

*of class* $C^n$ *such that the following holds. Let*

*(c)* $\varepsilon \in (0, \varepsilon_0)$.

*Let* $0 < U < u_*$. *We consider the initial value problem*

$$\partial_t \begin{pmatrix} \theta(t,x) \\ \psi(t,x) \end{pmatrix} = \begin{pmatrix} \psi(t,x) \\ \partial_x^2 \theta(t,x) - \sin\theta(t,x) + F(\varepsilon, x) \end{pmatrix}\,, \tag{19.2}$$

$$\theta(0,x) = \theta_n^\varepsilon(\xi_s, u_s, x) + v(0,x)\,, \tag{19.3}$$

$$\psi(0,x) = \psi_n^\varepsilon(\xi_s, u_s, x) + w(0,x)\,, \tag{19.4}$$

*where* $(\theta_n^\varepsilon, \psi_n^\varepsilon, \lambda_{u,n}^\varepsilon) = (\theta_0 + \hat{\theta}_n^\varepsilon, \psi_0 + \hat{\psi}_n^\varepsilon, \lambda_{u,n}^\varepsilon)$ *with* $(\theta_0, \psi_0)$ *given by* (3), $(v(0,x), w(0,x)) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ *such that the following assumptions are satisfied:*

*(d)* $|u_s| \leq \varepsilon^{\frac{k+1}{2}}$,

*(e)* $\mathcal{N}^\varepsilon(\theta(0,\cdot), \psi(0,\cdot), \xi_s, u_s) = 0$, *where* $\mathcal{N}^\varepsilon : L^\infty(\mathbb{R}) \times L^2(\mathbb{R}) \times \Sigma(2, U, u_*) \to \mathbb{R}^2$ *is given by*

$$\mathcal{N}^\varepsilon(\theta, \psi, \xi, u) := \begin{pmatrix} \Omega\left(\begin{pmatrix} \partial_\xi \theta_n^\varepsilon(\xi, u, \cdot) \\ \partial_\xi \psi_n^\varepsilon(\xi, u, \cdot) \end{pmatrix}, \begin{pmatrix} \theta(\cdot) - \theta_n^\varepsilon(\xi, u, \cdot) \\ \psi(\cdot) - \psi_n^\varepsilon(\xi, u, \cdot) \end{pmatrix}\right) \\ \Omega\left(\begin{pmatrix} \partial_u \theta_n^\varepsilon(\xi, u, \cdot) \\ \partial_u \psi_n^\varepsilon(\xi, u, \cdot) \end{pmatrix}, \begin{pmatrix} \theta(\cdot) - \theta_n^\varepsilon(\xi, u, \cdot) \\ \psi(\cdot) - \psi_n^\varepsilon(\xi, u, \cdot) \end{pmatrix}\right) \end{pmatrix}$$

*and the symplectic form* $\Omega$ *is given by* (2).

*(f)* $|v(0,\cdot)|^2_{H^1(\mathbb{R})} + |w(0,\cdot)|^2_{L^2(\mathbb{R})} \leq \varepsilon^{2n}$, *where* $(v(0,\cdot), w(0,\cdot))$ *is given by* (19.3)-(19.4).

*Then the Cauchy problem defined by* (19.2)-(19.4) *has a unique solution on the time interval*

$$0 \leq t \leq T, \text{ where } T = T(\varepsilon, k, \delta) = \frac{1}{\varepsilon^{\beta(k,\delta)}}, \quad \beta(k,\delta) = \frac{k+1-\delta}{2}\,.$$

*The solution may be written in the form*

$$\theta(t,x) = \theta^\varepsilon_n(\bar\xi(t), \bar u(t), x) + v(t,x)\,,$$
$$\psi(t,x) = \psi^\varepsilon_n(\bar\xi(t), \bar u(t), x) + w(t,x)\,,$$

*where* $v, w$, *have regularity*

$$(v(t), w(t)) \in C^1([0,T], H^1(\mathbb{R}) \oplus L^2(\mathbb{R})),$$

$\bar\xi, \bar u$ *solve the following system of equations*

$$\dot{\bar\xi}(t) = \bar u(t)\,, \tag{19.5}$$

$$\dot{\bar u}(t) = \lambda^\varepsilon_{u,n}\left(\bar\xi(t), \bar u(t)\right)\,, \tag{19.6}$$

*with initial data* $\bar\xi(0) = \xi_s$, $\bar u(0) = u_s$ *and there exists a positive constant c such that*

$$|v|^2_{L^\infty([0,T],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,T],L^2(\mathbb{R}))} \leq c\varepsilon^{2n}\,.$$

*The constant c depends on* $F$ *and* $\xi_s$.

The following chapter-wise outline provides an overview of our approach.

**Virtual Solitary Manifold** The classical soliton is defined by functions $(\theta_0, \psi_0)$ depending on variables $\xi, u, x$ such that

$$\begin{pmatrix}\theta_0(\xi(t), u(t), x)\\ \psi_0(\xi(t), u(t), x)\end{pmatrix} := \begin{pmatrix}\theta_K(\gamma(u(t))(x-\xi(t)))\\ -u(t)\gamma(u(t))\theta_K'(\gamma(u(t))(x-\xi(t)))\end{pmatrix}$$

solves

$$\partial_t\begin{pmatrix}\theta\\ \psi\end{pmatrix} - \begin{pmatrix}\psi\\ \theta_{xx} - \sin\theta\end{pmatrix} = 0,$$

as long as $(\xi, u)$ satisfy the ODE system $\dot\xi(t) = u(t)$ and $\dot u(t) = 0$. The idea of virtual solitons is to find functions $(\theta^\varepsilon, \psi^\varepsilon)$ depending on variables $\xi, u, x$ and functions $(\mu^\varepsilon, \lambda^\varepsilon)$ depending on variables $\xi, u$ such that

$$\begin{pmatrix}\theta^\varepsilon(\xi(t), u(t), x)\\ \psi^\varepsilon(\xi(t), u(t), x)\end{pmatrix}$$

solves

$$\partial_t \begin{pmatrix} \theta \\ \psi \end{pmatrix} - \begin{pmatrix} \psi \\ \theta_{xx} - \sin\theta + F(\varepsilon, x) \end{pmatrix} = 0$$

up to an error of the order $\varepsilon^{n+1}$ as long as $(\xi, u)$ satisfy an ODE system $\dot{\xi}(t) = \mu^\varepsilon(\xi(t), u(t))$ and $\dot{u}(t) = \lambda^\varepsilon(\xi(t), u(t))$ up to an error of a certain order (in $\varepsilon$). This can be done by solving certain equations successively.
Due to technical reasons we consider first a general $C^n$ function

$$\tilde{F} : (-1, 1) \to H^{1,\alpha}(\mathbb{R}^2)\,, \varepsilon \mapsto \tilde{F}(\varepsilon)$$

with $\tilde{F}(0) = 0$ and the equation

$$\underbrace{u\partial_\xi \begin{pmatrix} \theta \\ \psi \end{pmatrix} + \lambda_u \partial_u \begin{pmatrix} \theta \\ \psi \end{pmatrix} - \begin{pmatrix} \psi \\ \theta_{xx} - \sin\theta + \tilde{F}(\varepsilon) \end{pmatrix}}_{=: \mathcal{G}^\varepsilon(\theta, \psi, \lambda_u)} = 0\,, \tag{19.7}$$

where $\theta, \psi$ depend on $\xi, u, x$ and $\lambda_u$ depends on $\xi, u$ and $\tilde{F}(\varepsilon)$ depends on $\xi, x$. By making an assumption on the parameter area of $\xi$ later we will be able to replace $\tilde{F}(\varepsilon)$ with $F(\varepsilon)$ in (19.7). One should understand the subscript $u$ of $\lambda_u$ just as a symbol (not as variable) referring to the fact that $\lambda_u$ is a coefficient in front of the derivative with respect to $u$. $(\theta_0, \psi_0, 0)$ solves (19.7) up to an error of the order $\varepsilon$. Using $(\theta_0, \psi_0)$ we define a map $\mathcal{G}_1$ and we solve the equation

$$\underbrace{u\partial_\xi \begin{pmatrix} \theta \\ \psi \end{pmatrix} + \lambda_u \partial_u \begin{pmatrix} \theta_0 \\ \psi_0 \end{pmatrix} - \begin{pmatrix} \psi \\ \theta_{xx} - \sin\theta + \tilde{F}(\varepsilon) \end{pmatrix}}_{=: \mathcal{G}_1^\varepsilon(\theta, \psi, \lambda_u)} = 0\,,$$

implicitly for $(\theta, \psi, \lambda_u)$ in terms of $\varepsilon$. We call the implicitly defined map $(\theta_1^\varepsilon, \psi_1^\varepsilon, \lambda_{u,1}^\varepsilon)$. $(\theta_1^\varepsilon, \psi_1^\varepsilon, \lambda_{u,1}^\varepsilon)$ solves (19.7) up to an error of the order $\varepsilon^2$. Using $(\theta_1^\varepsilon, \psi_1^\varepsilon)$ we define a map $\mathcal{G}_2$, we solve the equation

$$\underbrace{u\partial_\xi \begin{pmatrix} \theta \\ \psi \end{pmatrix} + \lambda_u \partial_u \begin{pmatrix} \theta_1^0 + \partial_\varepsilon \theta_1^0 \varepsilon \\ \psi_1^0 + \partial_\varepsilon \psi_1^0 \varepsilon \end{pmatrix} - \begin{pmatrix} \psi \\ \theta_{xx} - \sin\theta + \tilde{F}(\varepsilon) \end{pmatrix}}_{=: \mathcal{G}_2^\varepsilon(\theta, \psi, \lambda_u)} = 0\,,$$

implicitly for $(\theta, \psi, \lambda_u)$ in terms of $\varepsilon$. We call the implicitly defined map $(\theta_2^\varepsilon, \psi_2^\varepsilon, \lambda_{u,2}^\varepsilon)$. $(\theta_2^\varepsilon, \psi_2^\varepsilon, \lambda_{u,1}^\varepsilon)$ solves (19.7) up to an error of the order $\varepsilon^3$. We continue the procedure until we obtain an equation

$$\underbrace{u\partial_\xi \begin{pmatrix} \theta \\ \psi \end{pmatrix} + \lambda_u \partial_u \begin{pmatrix} \sum_{i=0}^{n-1} \frac{\partial_\varepsilon^i \theta_{n-1}^0}{i!} \varepsilon^i \\ \sum_{i=0}^{n-1} \frac{\partial_\varepsilon^i \psi_{n-1}^0}{i!} \varepsilon^i \end{pmatrix} - \begin{pmatrix} \psi \\ \theta_{xx} - \sin\theta + \tilde{F}(\varepsilon) \end{pmatrix}}_{=: \mathcal{G}_n^\varepsilon(\theta, \psi, \lambda_u)} = 0\,.$$

and its implicitly defined map $(\theta_n^\varepsilon, \psi_n^\varepsilon, \lambda_{u,n}^\varepsilon)$ which solves (19.7) up to an error of the order $\varepsilon^{n+1}$. The existence of the implicit solutions $(\theta_k^\varepsilon, \psi_k^\varepsilon, \lambda_{u,k}^\varepsilon)$ to the equations $\mathcal{G}_k^\varepsilon(\theta, \psi, \lambda) = 0$ for $1 \leq k \leq n$ is ensured by the implicit function theorem and $(\theta_k^\varepsilon, \psi_k^\varepsilon, \lambda_{u,k}^\varepsilon)$ depend smoothly (of class $C^n$) on $\varepsilon$, such that $(\theta_k^0, \psi_k^0, \lambda_{u,k}^0) = (\theta_0, \psi_0, 0)$. The maps $\mathcal{G}_k^\varepsilon$ are defined on spaces of different regularity and satisfy $\mathcal{G}_k^0(\theta_0, \psi_0, 0) = 0$. We apply the implicit function theorem on the equations $\mathcal{G}_k^\varepsilon(\theta, \psi, \lambda) = 0$ for $1 \leq k \leq n$ and show that the implicit solutions satisfy (19.7) up to the errors mentioned above in Section 20.3.

Actually we would like to solve the equations with the $F$ from the Cauchy problem (19.2)-(19.4) (instead of the $\tilde{F}$), but this is not possible, since the equations are defined on Sobolev spaces on $\mathbb{R}^2$ for technical reasons, but $F(\varepsilon) \in H^{1,1}(\mathbb{R})$. Thus we choose the special $\tilde{F}$ given by

$$\tilde{F}(\varepsilon, \xi, x) := F(\varepsilon, x)\chi(\xi)\,,$$

where $\chi$ is a smooth cutoff function that is equal to 1 in $[-\Xi, \Xi]$ and vanishes in $\mathbb{R} \setminus (-(\Xi + 1), \Xi + 1)$ for a fixed $\Xi > |\xi_s| + 2$. From then we denote by $(\theta_n^\varepsilon, \psi_n^\varepsilon, \lambda_n^\varepsilon)$ always the nth solution obtained from $\mathcal{G}_n^\varepsilon(\theta, \psi, \lambda) = 0$ with the special $\tilde{F}$. We define the virtual solitary manifold by

$$\mathcal{S}_n^\varepsilon := \left\{ \begin{pmatrix} \theta_n^\varepsilon(\xi, u, \cdot) \\ \psi_n^\varepsilon(\xi, u, \cdot) \end{pmatrix} \ : \ u \in (-u_*, u_*),\ \xi \in \mathbb{R} \right\}\,,$$

where $u_* \in (0, 1]$. In order to be able to apply the implicit function theorem we need to show that the linearizations of $\mathcal{G}_k$ are invertible. This will be proved in Section 20.1 and in Section 20.2. We will show in Section 20.2 that certain operators

$$\mathfrak{M}_n^\alpha : Y_n^\alpha(u_*) \to Z_n^\alpha(u_*)$$

are invertible, which will ensure the applicability of the implicit function theorem. $Y_n^\alpha(u_*), Z_n^\alpha(u_*)$ are Banach spaces such that

$$Y_n^\alpha(u_*) \subset C^n(I(u_*), (\bar{Y}^\alpha)')\,,$$

$$Z_n^\alpha(u_*) \subset C^n(I(u_*), (\bar{Z}^\alpha)')\,,$$

where $(\bar{Y}^\alpha)', (\bar{Z}^\alpha)'$ are weighted Sobolev spaces. But first we will show in Section 20.1 the invertibility of analogous operators

$$\mathfrak{M}_n : Y_n(u_*) \to Z_n(u_*)\,,$$

where

$$Y_n(u_*) \subset C^n(I(u_*), (\bar{Y})')\,,$$

$$Z_n(u_*) \subset C^n(I(u_*), (\bar{Z})')\,.$$

and $(\bar{Y})', (\bar{Z})'$ are Sobolev spaces without weights. The reason for starting with the operators $\mathfrak{M}_n$ is that the treatment is technically easier.

**Symplectic Orthogonal Decomposition** We choose a sufficiently small $\varepsilon_0$ and consider $\varepsilon \in (0, \varepsilon_0]$. We show that if $(\theta, \psi) \in L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R})$ is close enough (in the $L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R})$ norm) to the region

$$\mathcal{S}_n^\varepsilon(U) := \left\{ \begin{pmatrix} \theta_n^\varepsilon(\xi, u, \cdot) \\ \psi_n^\varepsilon(\xi, u, \cdot) \end{pmatrix} \ : \ (\xi, u) \in \Sigma(4, U) \right\} ,$$

of the virtual solitary manifold $\mathcal{S}_n^\varepsilon$, then there exists a unique $(\xi, u) \in \Sigma(2, U)$ such that

$$\begin{pmatrix} \theta(\cdot) \\ \psi(\cdot) \end{pmatrix} - \begin{pmatrix} \theta_n^\varepsilon(\xi, u, \cdot) \\ \psi_n^\varepsilon(\xi, u, \cdot) \end{pmatrix} =: \begin{pmatrix} v(\cdot) \\ w(\cdot) \end{pmatrix}$$

is symplectic orthogonal to the tangent vectors $\begin{pmatrix} \partial_\xi \theta_n^\varepsilon(\xi, u, \cdot) \\ \partial_\xi \psi_n^\varepsilon(\xi, u, \cdot) \end{pmatrix}$ and $\begin{pmatrix} \partial_u \theta_n^\varepsilon(\xi, u, \cdot) \\ \partial_u \psi_n^\varepsilon(\xi, u, \cdot) \end{pmatrix}$ of the virtual solitary manifold $\mathcal{S}_n^\varepsilon$, i.e.,

$$\mathcal{N}^\varepsilon(\theta, \psi, \xi, u) = 0 \,.$$

We prove that the symplectic decomposition is possible in a small uniform distance to the manifold $\mathcal{S}_n^\varepsilon$, where the distance might depend on $\varepsilon_0$ but does not depend on $\varepsilon$.

**Existence of Dynamics and the Orthogonal Component** The existence theory provides that there is a local solution $(\theta, \psi)$ of (19.2)-(19.4), which might be written in the form

$$\theta(t, x) = \bar{v}(t, x) + \theta_n^\varepsilon(\xi_0, u_0, x) \,,$$

$$\psi(t, x) = \bar{w}(t, x) + \psi_n^\varepsilon(\xi_0, u_0, x) \,,$$

where $(\bar{v}, \bar{w}) \in C^1([0, T_{loc}], H^1(\mathbb{R}) \oplus L^2(\mathbb{R}))$ . Due to Morrey's embedding theorem it holds that $(\theta, \psi) \in C^1([0, T_{loc}], L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R}))$. In the following we assume that a solution $(\theta, \psi)$ of (19.2)-(19.4) is given on the time interval $[0, \overline{T}]$, which might be written as above where $(\bar{v}, \bar{w}) \in C^1([0, \overline{T}], H^1(\mathbb{R}) \oplus L^2(\mathbb{R}))$ .

$\varepsilon$ is chosen so small that due to assumptions (d), (f) in Theorem 19.3 the initial data $(\theta(0), \psi(0))$ is so close to the region $\mathcal{S}_n^\varepsilon(U)$ of the virtual solitary manifold that the symplectic orthogonal decomposition is possible in a neighbourhood of $(\theta(0), \psi(0))$.

In (19.3)-(19.4) the initial data $(\theta(0), \psi(0))$ is already written as a sum of a point on the virtual solitary manifold $\mathcal{S}_n^\varepsilon$ and a transversal component $(v(0), w(0))$ such that the symplectic orthogonality condition is satisfied due to assumption (e) in Theorem 19.3.

For times $t > 0$ we are able to choose the parameters $(\xi(t), u(t))$ according to the decomposition in Chapter 21 (Symplectic Orthogonal Decomposition) as long as $(\theta(t), \psi(t))$ stays close enough to $\mathcal{S}_n^\varepsilon(U)$. We introduce the parameter area

$$\Sigma(l, U, \Xi) := \Big\{(\xi, u) \in (-\Xi + 1 - U(l), \Xi - 1 + U(l)) \times (-U - U(l), U + U(l))\Big\}$$

with $u_* \in (0, 1]$ and $\Xi, U, U(l)$ as above. The solution $(\theta_n^\varepsilon, \psi_n^\varepsilon, \lambda_n^\varepsilon)$ of the equation $\mathcal{G}_n^\varepsilon(\theta, \psi, \lambda) = 0$ with the special $\tilde{F}$ satisfies

$$\begin{aligned} &u\partial_\xi \begin{pmatrix} \theta_n^\varepsilon(\xi, u, x) \\ \psi_n^\varepsilon(\xi, u, x) \end{pmatrix} - \begin{pmatrix} \psi_n^\varepsilon(\xi, u, x) \\ \partial_x^2 \theta_n^\varepsilon(\xi, u, x) - \sin \theta_n^\varepsilon(\xi, u, x) + F(\varepsilon, x) \end{pmatrix} \\ &\quad + \lambda_{u,n}^\varepsilon(\xi, u)\partial_u \begin{pmatrix} \theta_n^\varepsilon(\xi, u, x) \\ \psi_n^\varepsilon(\xi, u, x) \end{pmatrix} + \mathcal{R}_n^\varepsilon(\xi, u, x) = 0 \end{aligned}$$

for $(\xi, u, x) \in \Sigma(4, U, \Xi) \times \mathbb{R}$, where $\mathcal{R}_n^\varepsilon$ is a term of the order $\varepsilon^{n+1}$. Thus $(\theta_n^\varepsilon, \psi_n^\varepsilon, \lambda_{u,n}^\varepsilon)$ solves (19.7) up to an error of the order $\varepsilon^{n+1}$ and we are able to replace $\tilde{F}(\varepsilon, \xi, x)$ with $F(\varepsilon, x)$ as long as $(\xi(t), u(t), x) \in \Sigma(4, U, \Xi) \times \mathbb{R}$ (notice that $(\xi_s, u_s) = (\xi(0), u(0)) \in \Sigma(4, U, \Xi)$).

As long as $(\theta(t), \psi(t))$ stays close enough to $\mathcal{S}_n^\varepsilon(U)$ and $(\xi(t), u(t)) \in \Sigma(4, U, \Xi)$ we define $(v, w)$ by

$$v(t, x) = \theta(t, x) - \theta_n^\varepsilon(\xi(t), u(t), x)\,, \tag{19.8}$$

$$w(t, x) = \psi(t, x) - \psi_n^\varepsilon(\xi(t), u(t), x)\,, \tag{19.9}$$

where the parameter $(\xi(t), u(t))$ are obtained from the decomposition in Chapter 21 (Symplectic Orthogonal Decomposition), such that

$$\mathcal{N}^\varepsilon(\theta(t), \psi(t), \xi(t), u(t)) = 0\,. \tag{19.10}$$

Thus we decompose the dynamics in two components, namely a point on the virtual solitary manifold $(\theta_n^\varepsilon(\xi(t), u(t), \cdot), \psi_n^\varepsilon(\xi(t), u(t), \cdot))$ and a transversal component $(v(t, \cdot), w(t, \cdot))$ which is symplectic orthogonal to the tangent vectors

$$\begin{pmatrix} \partial_\xi \theta_n^\varepsilon(\xi(t), u(t), \cdot) \\ \partial_\xi \psi_n^\varepsilon(\xi(t), u(t), \cdot) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \partial_u \theta_n^\varepsilon(\xi(t), u(t), \cdot) \\ \partial_u \psi_n^\varepsilon(\xi(t), u(t), \cdot) \end{pmatrix}$$

of $\mathcal{S}_n^\varepsilon$. Finally, we compute the time derivatives of $v$ and $w$ which will be needed in the following chapters.

In Chapter 26 (Proof of Theorem 19.3) we will obtain a bound on $|v|^2_{L^\infty([0,T],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,T],L^2(\mathbb{R}))}$ (where $T \leq \overline{T}$) which will give us control over the distance of $(\theta, \psi)$ to the virtual solitary manifold and which will imply that the local solution $(\theta, \psi)$ is indeed continuable.

**Modulation Equations** We want to consider the longitudinal dynamics on $\mathcal{S}_n^\varepsilon$, which is described by the parameters $(\xi(t), u(t))$. In order to be able to understand the dynamics on $\mathcal{S}_n^\varepsilon$ we derive a system of ordinary differential equations (modulation equations) for the parameters $(\xi(t), u(t))$ which is satisfied up to a certain error. Considering (19.7) it makes sense to examine up to what errors the ordinary differential equations

$$\begin{aligned}\dot{\xi}(t) &= u(t)\,,\\ \dot{u}(t) &= \lambda_{u,n}^\varepsilon(\xi(t), u(t))\,,\end{aligned}$$

are satisfied. For this purpose we take the time derivative of (19.10) and obtain a system of differential equations. Using Neumann's theorem we conclude that the estimates

$$\begin{aligned}|\dot{\xi}(t) - u(t)| &\leq C[|v(t)|_{H^1(\mathbb{R})} + |w(t)|_{L^2(\mathbb{R})}]\varepsilon^{k+1} + C\,|v(t)|_{H^1(\mathbb{R})}^2 + C\varepsilon^{n+k+1},\\ |\dot{u}(t) - \lambda_{u,n}^\varepsilon(\xi(t), u(t))| &\leq C[|v(t)|_{H^1(\mathbb{R})} + |w(t)|_{L^2(\mathbb{R})}]\varepsilon^{k+1} + C\,|v(t)|_{H^1(\mathbb{R})}^2 + C\varepsilon^{n+k+1}\,,\end{aligned}$$

are satisfied if $\varepsilon, |v(t)|_{H^1(\mathbb{R})}, |w(t)|_{L^2(\mathbb{R})}$ are less than a certain $\varepsilon_0 > 0$ and as long as the time $t$ is such as described in the introduction of $(v, w)$ above.

**Lyapunov Functional** In order to obtain control on the transversal component $(v, w)$ we introduce Lyapunov functions

$$\begin{aligned}L(t) &= \int \frac{w^2(x)}{2} + \frac{(\partial_x v(x))^2}{2} + \frac{\cos(\theta_K(\gamma(u)(x-\xi)))v^2(x)}{2} + uw(x)\partial_x v(x)\,dx\,,\\ L^\varepsilon(t) &= \int \frac{w^2(x)}{2} + \frac{(\partial_x v(x))^2}{2} + \frac{\cos(\theta_n^\varepsilon(\xi, u, x))v^2(x)}{2} + uw(x)\partial_x v(x)\,dx\,,\end{aligned}$$

where $(v, w)$ are given by (19.8)-(19.9), $(\xi, u)$ are obtained from the decomposition in Chapter 21 (Symplectic Orthogonal Decomposition), such that the orthogonality conditions hold and $\gamma(u) = 1/\sqrt{1-u^2}$. $L^\varepsilon$ is the quadratic part of

$$H(\theta, \psi) + u\Pi(\theta, \psi)$$

above the virtual solitary manifold $\mathcal{S}_n^\varepsilon$, where $H$ and $\Pi$, given by (7) and (8) are conserved quantities of the sine-Gordon equation. Finally we compute the time derivative of $L^\varepsilon(t)$ which will be needed later.

**Lower Bound** We consider for $(v, w) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$, $(\xi, u) \in \mathbb{R} \times (-1, 1)$, the functional

$$\mathcal{E}(v, w, \xi, u) := \frac{1}{2}\int (w(x) + u\partial_x v(x))^2 + v_Z^2(x) + \cos(\theta_K(Z))v^2(x)\,dx\,,$$

where $Z = \gamma(x - \xi)$ and $v_Z(x) = \partial_Z v(\frac{Z}{\gamma} + \xi) = \frac{1}{\gamma}\partial_x v(x)$.

We choose an $\varepsilon_0 > 0$ and prove that there exists a $c > 0$ such that if $0 < \varepsilon \leq \varepsilon_0$, $(\xi, u) \in \mathbb{R} \times [-U - U(2), U + U(2)] \subset \mathbb{R} \times (-1, 1)$ and $(v, w) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ satisfies the orthogonality condition

$$\check{\mathcal{C}}_2^{\varepsilon}(v, w, \xi, u) := \int \partial_u \psi_n^{\varepsilon}(\xi, u, x) v(x) - \partial_u \theta_n^{\varepsilon}(\xi, u, x) w(x)\, dx = 0$$

(which is related to the second component in (19.10)), then the lower bound on $\mathcal{E}$,

$$\mathcal{E}(v, w, \xi, u) \geq c(\|v\|_{H^1}^2 + \|w\|_{L^2}^2)$$

holds. In the next chapter we relate this lower bound to the Lyapunov function $L$, since it holds that

$$L(t) = \mathcal{E}(v(t), w(t), \xi(t), u(t))\,,$$

where $(v, w)$ are given by (19.8)-(19.9) and $(\xi, u)$ are obtained from the decomposition in Chapter 21 (Symplectic Orthogonal Decomposition).

**Proof of Theorem 19.3** First of all we prove the statement of Theorem 19.3 with approximate equations for the parameters $(\xi, u)$ instead of the exact ODE's (19.5)-(19.6). The existence of the map (19.1) is ensured by the implicit function theorem in Chapter 20 (Virtual Solitary Manifold). We suppose that (19.2)-(19.4) has a solution and we make some assumptions on $(\xi, u)$ obtained from the decomposition in Chapter 21 (Symplectic Orthogonal Decomposition) and on $(v, w)$ given by (19.8)-(19.9). The modulation equations allow us to control $(\xi, u)$. The Lyapunov functions and the lower bound on $\mathcal{E}$ allow us to control $(v, w)$, since we are able to estimate

$$\begin{aligned} & c(|v(t)|_{H^1(\mathbb{R})}^2 + |w(t)|_{L^2(\mathbb{R})}^2) \\ \leq\ & L(t) \\ =\ & L^{\varepsilon}(t) + C\varepsilon^{k+1}\, |v|_{L^\infty([0,t],H^1(\mathbb{R}))}^2 \\ =\ & L^{\varepsilon}(0) + \int_0^t \frac{d}{dt} L^{\varepsilon}(t)\, dt + C\varepsilon\, |v|_{L^\infty([0,t],H^1(\mathbb{R}))}^2 \end{aligned}$$

and to control the right hand side. All in all, we obtain more accurate information about $(v, w)$ and $(\xi, u)$. Using a continuity argument this implies the bound on $(v, w)$ claimed in Theorem 19.3 and approximate equations for the parameters $(\xi, u)$. The bound on $(v, w)$ implies that the local solution discussed in Chapter 22 (Existence of Dynamics and the Orthogonal Component) is continuable up to times $\frac{1}{\varepsilon^{\beta(k,\delta)}}$, which establishes the statement of Theorem 19.3 with approximate equations for $(\xi, u)$.
By using Gronwall's lemma we show that the dynamics on the virtual solitary manifold can be described by $(\bar{\xi}, \bar{u})$ which satisfy the ODE's (19.5)-(19.6).

# Chapter 20

# Virtual Solitary Manifold

Let $(\xi, u) \in \mathbb{R} \times (-1, 1)$. The aim of this chapter is to construct the virtual solitary manifold. The virtual solitary manifold is going to be defined by an implicit solution of an equation

$$\mathcal{G}_n^{\varepsilon}(\theta, \psi, \lambda_u) = 0\,,$$

in some Banach spaces, where $\mathcal{G}_n$ will be defined in Section 20.3. We solve the equation implicitly for $(\theta, \psi, \lambda_u)$ in terms of $\varepsilon$ by using the implicit function theorem. In order to be able to apply the implicit function theorem we need to show that certain operators are invertible. This will be done in Section 20.1 and in Section 20.2.

In this chapter we use the notation from Definition 2.2 (see also Remark 2.3).

## 20.1 Inverse Operator in Sobolev Spaces

In this section we will show that certain operators

$$\mathfrak{M}_n : Y_n(u_*) \to Z_n(u_*)\,, \quad n \in \mathbb{N}$$

are invertible, where

$$\begin{aligned} Y_n(u_*) &\subset C^n(I(u_*), (\bar{Y})')\,, \\ Z_n(u_*) &\subset C^n(I(u_*), (\bar{Z})')\,, \end{aligned}$$

and $(\bar{Y})'$, $(\bar{Z})'$ are Sobolev spaces (without weights).

### 20.1.1 Preliminary Decomposition

Our first goal is to prove some decompositions for some Lebesgue and Sobolev spaces on $\mathbb{R}$ and on $\mathbb{R}^2$. We start with Lebesgue and Sobolev spaces on $\mathbb{R}$ and define some spaces and operators.

**Definition 20.1.** *We define the following spaces.*

*(a)* $H^2_{\xi,u,\perp}(\mathbb{R}) := \{\theta \in H^2(\mathbb{R}) \ : \ \langle\theta(\cdot), \theta'_K(\gamma(\cdot - \xi))\rangle_{L^2(\mathbb{R})} = 0\}$.

*(b)* $L^2_{\xi,u,\perp}(\mathbb{R}) := \{\theta \in L^2(\mathbb{R}) \ : \ \langle\theta(\cdot), \theta'_K(\gamma(\cdot - \xi))\rangle_{L^2(\mathbb{R})} = 0\}$.

*We define the following operators.*

*(a)* $L_{\xi,u} : H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$, *where*

$$(L_{\xi,u}\theta)(x) = -(1-u^2)\partial_x^2\theta(x) + \cos(\theta_K(\gamma(x-\xi)))\theta(x).$$

*(b)* $\hat{L}_{\xi,u} : H^2_{\xi,u,\perp}(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$, *where*

$$(L_{\xi,u}\theta)(x) = -(1-u^2)\partial_x^2\theta(x) + \cos(\theta_K(\gamma(x-\xi)))\theta(x).$$

*(c)* $M_{\xi,u} : H^2(\mathbb{R}) \oplus \mathbb{R} \to L^2(\mathbb{R})$, *where*

$$\left(M_{\xi,u}\begin{pmatrix}\theta\\ \lambda\end{pmatrix}\right)(x) = (L_{\xi,u}\theta)(x) + \lambda\theta'_K(\gamma(x-\xi)).$$

*(d)* $\hat{M}_{\xi,u} : H^2_{\xi,u,\perp}(\mathbb{R}) \oplus \mathbb{R} \to L^2(\mathbb{R})$, *where*

$$\left(\hat{M}_{\xi,u}\begin{pmatrix}\theta\\ \lambda\end{pmatrix}\right)(x) = \left(\hat{L}_{\xi,u}\theta\right)(x) + \lambda\theta'_K(\gamma(x-\xi)).$$

First of all, we would like to prove the following decomposition

$$L^2(\mathbb{R}) = \operatorname{ran}\hat{L}_{\xi,u} \overset{L^2}{\oplus} \langle\theta'_K(\gamma(\cdot - \xi))\rangle.$$

**Lemma 20.2.** $\operatorname{ran} L_{\xi,u}$ *is closed with respect to* $L^2(\mathbb{R})$.

**Proof.** We consider the case $(\xi, u) = (0, 0)$. The proof for a general $(\xi, u) \in \mathbb{R} \times (-1, 1)$ works in the same way.
$L_{0,0}$ is self-adjoint and 0 is an isopated point of $\sigma(L_{0,0})$. $l := L_{0,0}|_{H^2(\mathbb{R})\cap\langle\theta'_K\rangle^\perp}$ is self-adjoint and has a bounded inverse (see [HS96, Proposition 6.6]). Notice $\operatorname{ran} L_{0,0} = \operatorname{ran} l$. Let $y_n = Mx_n \overset{L^2}{\to} y$. Boundness yields $x_n = l^{-1}y_n \overset{L^2}{\to} \overline{l^{-1}}y$, where $\overline{l^{-1}}$ denotes the bounded extension of $l^{-1}$ on the closure $\overline{\operatorname{ran} l}$. Since $l^* = l$ is a closed operator (see [HS96, Proposition 4.9]), we obtain $l(\overline{l^{-1}}y) = y$. □

We use the previous lemma to show some properties of $L_{\xi,u}$ and $\hat{L}_{\xi,u}$.

**Lemma 20.3.** *(a)* $\ker L_{\xi,u} = \langle\theta'_K(\gamma(\cdot - \xi))\rangle$, $L^2(\mathbb{R}) = \operatorname{ran} L_{\xi,u} \overset{L^2}{\oplus} \langle\theta'_K(\gamma(\cdot - \xi))\rangle$.

*(b)* $L^2(\mathbb{R}) = \operatorname{ran}\hat{L}_{\xi,u} \overset{L^2}{\oplus} \langle\theta'_K(\gamma(\cdot - \xi))\rangle$.

*(c)* $\hat{L}_{\xi,u} \in L(H^2_{\xi,u,\perp}(\mathbb{R}), L^2_{\xi,u,\perp}(\mathbb{R}))$.

*(d)* $\hat{L}^{-1}_{\xi,u} \in L(L^2_{\xi,u,\perp}(\mathbb{R}), H^2_{\xi,u,\perp}(\mathbb{R}))$.

*(e)* $\hat{M}_{\xi,u}$ *is one-to-one and onto.*

**Proof.** Notice that $\operatorname{ran} L_{\xi,u}$ is closed due to Lemma 20.2. (a) Since $\ker L = \langle \theta'_K(\cdot)\rangle$ (see: [Stu92]), we get:

$$\begin{aligned}
&\tilde{\theta} \in \ker L_{\xi,u}\\
\Rightarrow\ &\text{for a.e. } x \in \mathbb{R}: \ -\partial_x^2\tilde{\theta}(x) + u^2\partial_x^2\tilde{\theta}(x) + \cos(\theta_K(\gamma(x-\xi)))\tilde{\theta}(x) = 0\\
\Rightarrow\ &\text{for a.e. } Z \in \mathbb{R}: \ -\partial_Z^2\tilde{\theta}(\frac{Z}{\gamma}+\xi) + \cos(\theta_K(Z))\tilde{\theta}(\frac{Z}{\gamma}+\xi) = 0\\
\Rightarrow\ &\text{for a.e. } Z \in \mathbb{R}: \ \tilde{\theta}(\frac{Z}{\gamma}+\xi) = \theta'_K(Z)\\
\Rightarrow\ &\text{for a.e. } x \in \mathbb{R}: \ \tilde{\theta}(x) = \theta'_K(\gamma(x-\xi))\,.
\end{aligned}$$

(b) The claim follows by orthogonal projecion.
(c) clear
(d) Thus the claim follows from the inverse mapping theorem since $L_{u,\xi} : H^2_{\xi,u,\perp}(\mathbb{R}) \to L^2_{\xi,u,\perp}$ is one-to-one, onto, and bounded.
(e) clear. □

Next, we define some more complicated operators.

**Definition 20.4.** *(a)* $\mathcal{L}_{\xi,u} : H^2(\mathbb{R}) \oplus H^1(\mathbb{R}) \to H^1(\mathbb{R}) \oplus L^2(\mathbb{R})$*, where*

$$\left(\mathcal{L}_{\xi,u}\begin{pmatrix}\theta\\ \psi\end{pmatrix}\right)(x) = \begin{pmatrix} -u\partial_x\theta(x) - \psi(x)\\ -\partial_x^2\theta(x) + \cos(\theta_K(\gamma(x-\xi)))\theta(x) - u\partial_x\psi(x)\end{pmatrix}.$$

*(b)* $\hat{\mathcal{L}}_{\xi,u} : \Big[H^2(\mathbb{R}) \oplus H^1(\mathbb{R})\Big] \cap (\ker \mathcal{L}_{\xi,u})^{\perp, L^2\oplus L^2} \to H^1(\mathbb{R}) \oplus L^2(\mathbb{R})$*, where*

$$\left(\hat{\mathcal{L}}_{\xi,u}\begin{pmatrix}\theta\\ \psi\end{pmatrix}\right)(x) = \begin{pmatrix} -u\partial_x\theta(x) - \psi(x)\\ -\partial_x^2\theta(x) + \cos(\theta_K(\gamma(x-\xi)))\theta(x) - u\partial_x\psi(x)\end{pmatrix}.$$

Our next goal is to show the following decomposition

$$H^1(\mathbb{R}) \oplus L^2(\mathbb{R}) = \mathcal{L}_{\xi,u}\Big(\Big[H^2(\mathbb{R}) \oplus H^1(\mathbb{R})\Big] \cap \langle \mathbb{J}t_\xi(\xi,u,\cdot)\rangle^{\perp, L^2\oplus L^2}\Big) \overset{L^2\oplus L^2}{\oplus} \langle \mathbb{J}t_\xi(\xi,u,\cdot), \mathbb{J}t_u(\xi,u,\cdot)\rangle\,.$$

In the following lemma we obtain an orthogonal decomposition of $H^1(\mathbb{R}) \oplus L^2(\mathbb{R})$.

**Lemma 20.5 (orthogonal sum).**

$$H^1(\mathbb{R}) \oplus L^2(\mathbb{R}) = \hat{\mathcal{L}}_{\xi,u}\left(\Big[H^2(\mathbb{R}) \oplus H^1(\mathbb{R})\Big] \cap \langle t_\xi(\xi,u,\cdot)\rangle^{\perp,L^2\oplus L^2}\right) \overset{L^2\oplus L^2}{\oplus} \langle \mathbb{J}t_\xi(\xi,u,\cdot)\rangle\,.$$

**Proof.** Notice $\ker \mathcal{L}^*_{\xi,u} = \langle \mathbb{J}t_\xi(\xi,u,\cdot)\rangle = \langle \begin{pmatrix} -u\gamma^2\theta''_K(\gamma(\cdot-\xi)) \\ -\gamma\theta'_K(\gamma(\cdot-\xi)) \end{pmatrix}\rangle$.

"$\supset$": clear.

"$\subset$": Let $\begin{pmatrix}\bar{v}\\ \bar{w}\end{pmatrix} \in H^1(\mathbb{R}) \oplus L^2(\mathbb{R})$. Since $L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) = \overline{\mathcal{L}_{\xi,u}(H^2(\mathbb{R}) \oplus H^1(\mathbb{R}))} \overset{L^2\oplus L^2}{\oplus} \ker \mathcal{L}^*_{\xi,u}$, there exists $\mu(\xi,u) \in \mathbb{R}$, s.t.

$$\begin{pmatrix}\bar{v}\\ \bar{w}\end{pmatrix} = {}_{L^2\oplus L^2-}\lim_{n\to\infty} \mathcal{L}_{\xi,u}\begin{pmatrix}\theta_n\\ \psi_n\end{pmatrix} + \mu(\xi,u)\mathbb{J}t_\xi(\xi,u,\cdot).$$

Hence $\begin{pmatrix}v\\ w\end{pmatrix} := {}_{L^2\oplus L^2-}\lim_{n\to\infty} \mathcal{L}_{\xi,u}\begin{pmatrix}\theta_n\\ \psi_n\end{pmatrix} \in \langle \mathbb{J}t_\xi(\xi,u,\cdot)\rangle^{\perp,L^2\oplus L^2} \cap \Big[H^1(\mathbb{R}) \oplus L^2(\mathbb{R})\Big]$.

$$\begin{aligned}
&\begin{pmatrix}v\\ w\end{pmatrix} \in \langle \mathbb{J}t_\xi(\xi,u,\cdot)\rangle^{\perp,L^2\oplus L^2} \\
\Rightarrow\quad & 0 = \int u\gamma^2\theta''_K(\gamma(x-\xi))v(x) + \gamma\theta'_K(\gamma(x-\xi))w(x)\,dx \\
&\quad = \int \Big[-u\gamma\partial_x v(x) + \gamma w(x)\Big]\theta'_K(\gamma(x-\xi))\,dx \\
\Rightarrow\quad & \Big[-u\gamma v'(\cdot) + \gamma w(\cdot)\Big] \in \langle \theta'_K(\gamma(\cdot-\xi))\rangle^{\perp,L^2} = \operatorname{ran}\hat{L}_{\xi,u} \quad \text{(due to Lemma 20.3)}
\end{aligned}$$

We set $\begin{cases} \tilde{\theta}(x) := [\hat{L}_{\xi,u}]^{-1}\Big(-u\gamma\partial_x v(x) + \gamma w(x)\Big) \\ \tilde{\psi}(x) := -u\partial_x\tilde{\theta}(x) - v(x) \end{cases}$. Let $Z = \gamma(x-\xi)$.

Due to Lemma 20.3 $\begin{pmatrix}\tilde{\theta}\\ \tilde{\psi}\end{pmatrix} \in H^2(\mathbb{R}) \oplus H^1(\mathbb{R})$ and

$$\begin{cases} -\tilde{\psi} - u\partial_x\tilde{\theta} = v \\ \hat{L}_{\xi,u}\tilde{\theta}(x) = -u\gamma\partial_x v(x) + \gamma w(x) \end{cases}$$

$$\begin{cases} -\tilde{\psi} - u\partial_x\tilde{\theta} = v \\ -(1-u^2)\partial_x^2\tilde{\theta} + \cos(\theta_K(Z))\tilde{\theta} + u\partial_x v = w \end{cases}$$

$$\begin{cases} -\tilde{\psi} - u\partial_x\tilde{\theta} = v \\ -(1-u^2)\partial_x^2\tilde{\theta} + \cos(\theta_K(Z))\tilde{\theta} - u\partial_x\tilde{\psi} - u^2\partial_x^2\tilde{\theta} = w \end{cases}$$

$$\mathcal{L}_{\xi,u}\begin{pmatrix}\tilde{\theta}\\ \tilde{\psi}\end{pmatrix} = \begin{pmatrix} -u\partial_x\tilde{\theta} - \tilde{\psi} \\ -\partial_x^2\tilde{\theta} + \cos(\theta_K(Z))\tilde{\theta} - u\partial_x\tilde{\psi}\end{pmatrix} = \begin{pmatrix} v\\ w\end{pmatrix}.$$

Set $\begin{pmatrix}\theta\\ \psi\end{pmatrix} := \begin{pmatrix}\tilde{\theta}\\ \tilde{\psi}\end{pmatrix}_{(\ker\mathcal{L}_{\xi,u})^{\perp,L^2\times L^2}}$, so $\hat{\mathcal{L}}_{\xi,u}\begin{pmatrix}\theta\\ \psi\end{pmatrix} = \begin{pmatrix} v\\ w\end{pmatrix}$. □

The following corollary will be needed in the proof of Lemma 23.1 , where we will derive the modulation equation for the parameters $(\xi(t), u(t))$.

**Corollary 20.6.**

$$H^1(\mathbb{R}) \oplus L^2(\mathbb{R}) = \mathcal{L}_{\xi,u}\Big(\Big[H^2(\mathbb{R}) \oplus H^1(\mathbb{R})\Big] \cap \langle \mathbb{J}t_\xi(\xi,u,\cdot)\rangle^{\perp,L^2\oplus L^2}\Big) \overset{L^2\oplus L^2}{\oplus} \langle \mathbb{J}t_\xi(\xi,u,\cdot), \mathbb{J}t_u(\xi,u,\cdot)\rangle .$$

**Proof.**

$$\begin{aligned} & H^1(\mathbb{R}) \oplus L^2(\mathbb{R}) \\ &= [\mathcal{L}_{\xi,u}]^2\left(\Big[H^3(\mathbb{R}) \oplus H^2(\mathbb{R})\Big] \cap \langle t_\xi(\xi,u,\cdot)\rangle^{\perp,L^2\oplus L^2}\right) \overset{L^2\oplus L^2}{\oplus} \ker\left([\mathcal{L}_{\xi,u}]^2\right)^* \\ &= \mathcal{L}_{\xi,u}\Big[\mathcal{L}_{\xi,u}\left(\Big[H^3(\mathbb{R}) \oplus H^2(\mathbb{R})\Big] \cap \langle t_\xi(\xi,u,\cdot)\rangle^{\perp,L^2\oplus L^2}\right)\Big] \overset{L^2\oplus L^2}{\oplus} \langle \mathbb{J}t_\xi(\xi,u,\cdot), \mathbb{J}t_u(\xi,u,\cdot)\rangle \\ &= \mathcal{L}_{\xi,u}\Big(\Big[H^2(\mathbb{R}) \oplus H^1(\mathbb{R})\Big] \cap \langle \mathbb{J}t_\xi(\xi,u,\cdot)\rangle^{\perp,L^2\oplus L^2}\Big) \overset{L^2\oplus L^2}{\oplus} \langle \mathbb{J}t_\xi(\xi,u,\cdot), \mathbb{J}t_u(\xi,u,\cdot)\rangle . \end{aligned}$$

The first and last identities follow from the proof of Lemma 20.5 and elliptic regularity. □

Now we consider Lebesgue and Sobolev spaces on $\mathbb{R}^2$ and define again some spaces and operators. Unlike the onedimensional case we consider $\xi$ not as a fixed parameter anymore, but as a new variable.

**Definition 20.7.** *We define the following spaces.*

*(a)* $H^2_\perp(\mathbb{R}^2) := \{\theta \in H^2(\mathbb{R}^2) \mid \forall\lambda \in H^2(\mathbb{R}):\ \langle \theta(\xi,Z), \lambda(\xi)\theta_K'(Z)\rangle_{L^2_{\xi,Z}(\mathbb{R}^2)} = 0\}$.

*(b)* $H^2_{u,\perp}(\mathbb{R}^2) := \{\theta \in H^2(\mathbb{R}^2) \mid \forall\lambda \in H^2(\mathbb{R}):\ \langle \theta(\xi,x), \lambda(\xi)\theta_K'(\gamma(x-\xi))\rangle_{L^2_{\xi,x}(\mathbb{R}^2)} = 0\}$.

*(c)* $H^3_{u,\perp}(\mathbb{R}^2) := \{\theta \in H^3(\mathbb{R}^2) \mid \forall\lambda \in H^2(\mathbb{R}):\ \langle \theta(\xi,x), \lambda(\xi)\theta_K'(\gamma(x-\xi))\rangle_{L^2_{\xi,x}(\mathbb{R}^2)} = 0\}$.

*(d)* $L^2_{u,\perp}(\mathbb{R}^2) := \{\theta \in L^2(\mathbb{R}^2) \mid \forall\lambda \in H^2(\mathbb{R}):\ \langle \theta(\xi,x), \lambda(\xi)\theta_K'(\gamma(x-\xi))\rangle_{L^2_{\xi,x}(\mathbb{R}^2)} = 0\}$.

*(e)* $I := I(u_*) = [-u_*, u_*]$ *for* $u_* > 0$.

*(f)* $\bar{\mathcal{Y}} = H^2(\mathbb{R}^2) \oplus H^2(\mathbb{R})$ *with the finite norm*

$$|y|_{\bar{\mathcal{Y}}} = |\theta|_{H^2(\mathbb{R}^2)} + |\lambda|_{H^2(\mathbb{R})}.$$

*(g)* $\bar{\mathcal{Z}} = L^2(\mathbb{R}^2)$ *with the finite norm*

$$|z|_{\bar{\mathcal{Z}}} = |z|_{L^2(\mathbb{R}^2)}.$$

*(h)* $$\mathcal{Y} = \mathcal{Y}(u^*)$$
$$= \left\{ y = (\theta, \lambda_u) \in C(I(u_*), \bar{\mathcal{Y}}) : \|y\|_{\mathcal{Y}(u_*)} < \infty,\ \forall u \in I(u_*) : \theta(u) \in H^2_{u,\perp}(\mathbb{R}^2) \right\}$$

*with the finite norm*

$$\|y\|_{\mathcal{Y}(u_*)} = \sup_{u \in I(u_*)} |y|_{\bar{\mathcal{Y}}}\,.$$

*(i)* $\mathcal{Z} = \mathcal{Z}(u_*) = \left\{ z \in C(I(u_*), \bar{\mathcal{Z}}) : \|z\|_{\mathcal{Z}(u_*)} < \infty \right\}$ *with the finite norm*

$$\|z\|_{\mathcal{Z}(u_*)} = \sup_{u \in I(u_*)} |z|_{\bar{\mathcal{Z}}}\,.$$

*We define the following operators.*

*(a)* $L : H^2(\mathbb{R}^2) \subset L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$, *where*

$$(L\theta)(\xi, Z) = -\partial_Z^2 \theta(\xi, Z) + \cos(\theta_K(Z))\theta(\xi, Z).$$

*(b)* $\hat{L} : H^2_\perp(\mathbb{R}^2) \subset L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$, *where*

$$(\hat{L}\theta)(\xi, Z) = -\partial_Z^2 \theta(\xi, Z) + \cos(\theta_K(Z))\theta(\xi, Z).$$

*(c)* $L_u : H^2(\mathbb{R}^2) \subset L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$, *where*

$$(L_u\theta)(\xi, x) = -(1 - u^2)\partial_x^2 \theta(\xi, x) + \cos(\theta_K(\gamma(x - \xi)))\theta(\xi, x).$$

*(d)* $\hat{L}_u : H^2_{u,\perp}(\mathbb{R}^2) \subset L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$, *where*

$$(\hat{L}_u\theta)(\xi, x) = -(1 - u^2)\partial_x^2 \theta(\xi, x) + \cos(\theta_K(\gamma(x - \xi)))\theta(\xi, x).$$

*(e)* $M_u : H^2(\mathbb{R}^2) \oplus H^2(\mathbb{R}) \to L^2(\mathbb{R})$, *where*

$$\left( M_u \begin{pmatrix} \theta \\ \lambda \end{pmatrix} \right)(\xi, x) = (L_u\theta)(\xi, x) + \lambda(\xi)\theta_K'(\gamma(x - \xi)).$$

*(f)* $\hat{M}_u : H^2_{u,\perp}(\mathbb{R}^2) \oplus H^2(\mathbb{R}) \to L^2(\mathbb{R}^2)$, *where*

$$\left(\hat{M}_u \begin{pmatrix}\theta\\ \lambda\end{pmatrix}\right)(\xi,x) = (\hat{L}_u\theta)(\xi,x) + \lambda(\xi)\theta_K'(\gamma(x-\xi)).$$

*(g)* $M : H^2(\mathbb{R}^2) \oplus H^2(\mathbb{R}) \to L^2(\mathbb{R})$, *where*

$$\left(M \begin{pmatrix}\theta\\ \lambda\end{pmatrix}\right)(\xi,Z) = (L\theta)(\xi,Z) + \lambda(\xi)\theta_K'(Z).$$

*(h)* $\hat{M} : H^2_{\perp}(\mathbb{R}^2) \oplus H^2(\mathbb{R}) \to L^2(\mathbb{R}^2)$, *where*

$$\left(\hat{M} \begin{pmatrix}\theta\\ \lambda\end{pmatrix}\right)(\xi,Z) = (\hat{L}\theta)(\xi,Z) + \lambda(\xi)\theta_K'(Z).$$

Now we start to examine the operator $L_u$. Further we want to show the following decomposition

$$L^2(\mathbb{R}^2) = \operatorname{ran}\hat{L}_u \overset{L^2}{\oplus} \ker L_u\,.$$

**Lemma 20.8.** *(a)* $\ker L_u = \{\ \theta \in H^2(\mathbb{R}^2) \mid \theta(\xi,x) = \lambda(\xi)\theta_K'(\gamma(x-\xi)),\ \lambda \in H^2(\mathbb{R})\}$.

*(b)* $0$ *is an isolated eigenvalue of* $L_u$.

**Proof.** Let $u \in (-1,1)$.
(a) Kernel of $L_u$: Let $w \in H^2(\mathbb{R}^2)$ and $L_u w = 0$.

$$\begin{aligned}
&L_u w = 0\\
\Rightarrow{}& \text{for a.e. } (\xi,x) \in \mathbb{R}^2 :\ \Big[u^2\partial_x^2 - \partial_x^2 + \cos(\theta_K(\gamma(x-\xi)))\Big]w(\xi,x) = 0\\
\overset{\text{Lemma 20.3}}{\Rightarrow}{}& \text{for a.e. } \xi \in \mathbb{R} :\ w(\xi,\cdot) = \lambda(\xi)\theta_K'(\gamma(\cdot-\xi))\\
\Rightarrow{}& \ker L_u = \{\ \theta \in H^2(\mathbb{R}^2) \mid \theta(\xi,x) = \lambda(\xi)\theta_K'(\gamma(x-\xi)),\ \lambda \in H^2(\mathbb{R})\}\,.
\end{aligned}$$

This implies that 0 is an eigenvalue of $L_u$.

(b) 0 is an isolated eigenvalue of $L_u$:
Let $w \in H^2(\mathbb{R}^2)$, $0 < |\lambda| < 1$ and $(L_u - \lambda)w = 0$.

$$\begin{aligned}
&(L_u - \lambda)w = 0\\
\Rightarrow{}& \text{for a.e. } (\xi,x) \in \mathbb{R}^2 :\ \Big(\Big[u^2\partial_x^2 - \partial_x^2 + \cos(\theta_K(\gamma(x-\xi)))\Big] - \lambda\Big)w(\xi,x) = 0\\
\Rightarrow{}& \text{for a.e. } \xi \in \mathbb{R} :\ w(\xi,\cdot) = 0\\
\Rightarrow{}& \ker(L_u - \lambda) = \{0\}
\end{aligned}$$

The second implication holds because of the spectral gap of $L_{\xi,u}$ (see: [Stu92]). Therefore

$$L^2(\mathbb{R}^2) = \overline{\operatorname{ran}(L_u - \lambda)} \overset{L^2}{\oplus} \ker(L_u - \lambda) = \overline{\operatorname{ran}(L_u - \lambda)}\,.$$

$(L_u - \lambda)^{-1} : \operatorname{ran} L_u \cap L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is bounded:
Let $v \in \operatorname{ran} L_u$, $|v(\xi, x)|_{L^2(\mathbb{R}^2)} \leq 1$. Let $\theta \in H^2(\mathbb{R}^2)$ be such that

$$\begin{aligned} & (L_u - \lambda)\theta = v \\ \Rightarrow \text{for a.e. } \xi \in \mathbb{R} : \quad & (L_{\xi,u} - \lambda)\theta(\xi, \cdot) = v(\xi, \cdot) \\ \Rightarrow \text{for a.e. } \xi \in \mathbb{R} : \quad & (L_{0,0} - \lambda)\theta(\xi, \frac{\cdot}{\gamma} + \xi) = v(\xi, \frac{\cdot}{\gamma} + \xi)\,. \end{aligned}$$

This implies the following identities

$$\begin{aligned} & \theta = (L_u - \lambda)^{-1} v\,, \\ \text{for a.e. } \xi \in \mathbb{R} : \quad & \theta(\xi, \cdot) = (L_{\xi,u} - \lambda)^{-1} v(\xi, \cdot)\,, \\ \text{for a.e. } \xi \in \mathbb{R} : \quad & \theta(\xi, \frac{\cdot}{\gamma} + \xi) = (L_{0,0} - \lambda)^{-1} v(\xi, \frac{\cdot}{\gamma} + \xi)\,. \end{aligned} \tag{20.1}$$

Since $(L_{0,0} - \lambda)^{-1}$ is bounded, we obtain

$$\begin{aligned} \text{for a.e. } \xi \in \mathbb{R} : \quad & \gamma\, |\theta(\xi, \cdot)|^2_{L^2(\mathbb{R})} \\ & = \left|\theta(\xi, \frac{\cdot}{\gamma} + \xi)\right|^2_{L^2(\mathbb{R})} \\ & = \left|(L_{0,0} - \lambda)^{-1} v(\xi, \frac{\cdot}{\gamma} + \xi)\right|^2_{L^2(\mathbb{R})} \\ & = \int ((L_{0,0} - \lambda)^{-1} v(\xi, \frac{Z}{\gamma} + \xi))^2\, dZ \\ & \leq \|(L_{0,0} - \lambda)^{-1}\|^2 \int v(\xi, \frac{Z}{\gamma} + \xi))^2\, dZ \\ & \leq \gamma \|(L_{0,0} - \lambda)^{-1}\|^2\, |v(\xi, \cdot)|^2_{L^2(\mathbb{R})}\,. \end{aligned}$$

Integration with respect to $\xi$ yields due to (20.1):

$$\begin{aligned} & \left|(L_u - \lambda)^{-1} v\right|^2_{L^2(\mathbb{R}^2)} \\ = \ & |\theta|^2_{L^2(\mathbb{R}^2)} \\ \leq \ & \|(L_{0,0} - \lambda)^{-1}\|^2\, |v|^2_{L^2(\mathbb{R}^2)}\,. \end{aligned}$$

$\operatorname{ran}(L_u - \lambda)$ is closed:
Let $y_n = (L_u - \lambda)x_n \overset{L^2}{\to} y$. Boundness of $(L_u - \lambda)^{-1}$ yields $x_n = (L_u - \lambda)^{-1}y_n \overset{L^2}{\to} \overline{(L_u - \lambda)^{-1}}y$, where $\overline{(L_u - \lambda)^{-1}}$ denotes the bounded extension of $(L_u - \lambda)^{-1}$ on the closure $\overline{\operatorname{ran}(L_u - \lambda)}$. Since $(L_u - \lambda)^* = (L_u - \lambda)$ is a closed operator we obtain $(L_u - \lambda)(\overline{(L_u - \lambda)^{-1}}y) = y$.

□

Using the previous lemma we show the following lemma.

**Lemma 20.9.** $\operatorname{ran} L_u$ *is closed with respect to* $L^2(\mathbb{R}^2)$.

**Proof.** $L_u$ is self-adjoint and 0 is an isolated point of $\sigma(L_u)$. $l_u := L_u|_{H^2(\mathbb{R})\cap \ker L_u^{\perp}}$ is self-adjoint and has a bounded inverse (see [HS96, Proposition 6.6]). $\operatorname{ran} L_u = \operatorname{ran} l_u$. Let $y_n = l_u x_n \overset{L^2}{\to} y$. Boundness yields $x_n = l_u^{-1} y_n \overset{L^2}{\to} \overline{l_u^{-1}}y$, where $\overline{l_u^{-1}}$ denotes the bounded extension of $l_u^{-1}$ on the closure $\overline{\operatorname{ran} l_u}$. Since $l_u^* = l_u$ is a closed operator (see [HS96, Proposition 4.9]), we obtain $l_u(\overline{l_u^{-1}}y) = y$. □

**Lemma 20.10.** *(a)* $L^2(\mathbb{R}^2) = \operatorname{ran} L_u \overset{L^2}{\oplus} \ker L_u$.

*(b)* $L^2(\mathbb{R}^2) = \operatorname{ran} \hat{L}_u \overset{L^2}{\oplus} \ker L_u$.

*(c)* $\hat{L}_u \in L(H^2_{u,\perp}(\mathbb{R}^2), L^2_{u,\perp}(\mathbb{R}^2))$.

*(d)* $\hat{L}_u^{-1} \in L(L^2_{u,\perp}(\mathbb{R}^2), H^2_{u,\perp}(\mathbb{R}^2))$.

*(e)* $\hat{M}_u, \hat{M}$ *are one-to-one, onto, bounded and the inverse mappings are also bounded.*

**Proof.** (a) Follows from Lemma 20.9.
(b) The claim follows by orthogonal projecion.
(c) clear.
(d) Follows from the inverse mapping theorem since $L_u : H^2_{u,\perp}(\mathbb{R}^2) \to L^2_{u,\perp}(\mathbb{R}^2)$ is one-to-one, onto, and bounded.
(e) It follows from (b),(c) and the inverse mapping theorem. □

**Remark 20.11.** It follows from elliptic regularity of $\hat{L}_u$ that

$$H^1(\mathbb{R}^2) = \hat{L}_u(H^3_{u,\perp}(\mathbb{R}^2)) \overset{L^2}{\oplus} \ker L_u \,.$$

The following lemma will be one ingredient in the proof of the invertibility of $\mathfrak{M}_n$.

**Lemma 20.12.** *Let* $\mathfrak{m}$ *be the linear operator, defined by*

$$\mathfrak{m} : \mathcal{Y} \to \mathcal{Z},$$

$$(\theta, \lambda) \mapsto \mathfrak{m}(\theta, \lambda)\,, \quad s.t. \;\; \mathfrak{m}(\theta, \lambda)(u) = \hat{M}_u(\theta(u), \lambda(u)).$$

$\mathfrak{m}$ *is is one-to-one, onto and bounded, i.e.,* $\mathfrak{m}^{-1}$ *is bounded.*

**Proof.** $\mathfrak{m}$ is well defined: clear.
$\mathfrak{m}$ is one-to-one: Let $(\theta,\lambda)\in\mathcal{Y}$ with $\mathfrak{m}(\theta,\lambda)=0$. It follows

$$\forall u\in I:\mathfrak{m}(\theta,\lambda)(u)=\hat{M}_u(\theta(u),\lambda(u))=\hat{L}_u\theta(u)+\lambda(u)\theta_K'(\gamma(x-\xi))=0\,.$$

Lemma 20.10 yields

$$\forall u\in I:\theta(u)=0,\lambda(u)=0\Rightarrow\theta=0,\lambda=0\,.$$

$\mathfrak{m}$ is onto: Let $v\in\mathcal{Z}$. Due to Lemma 20.10 there exists for all $u\in I$ a $(\theta(u),\lambda(u))\in H^2_{u,\perp}(\mathbb{R}^2)\oplus H^2(\mathbb{R})$, s.t.

$$v(u)(\xi,x)=\hat{M}_u\begin{pmatrix}\theta(u)\\ \lambda(u)\end{pmatrix}(\xi,x)=(\hat{L}_u\theta(u))(\xi,x)+\lambda(u)(\xi)\theta_K'(\gamma(x-\xi))\,.$$

Thus

$$v(u)(\xi,\frac{Z}{\gamma}+\xi)=\hat{M}\begin{pmatrix}\bar{\theta}(u)\\ \lambda(u)\end{pmatrix}(\xi,Z)=\hat{L}\,\bar{\theta}(u)(\xi,Z)+\lambda(u)(\xi)\theta_K'(Z)\,,$$

where $\bar{\theta}(u)(\xi,Z)=\theta(u)(\xi,\frac{Z}{\gamma}+\xi)$. Abusing notation, we write

$$\begin{aligned}&v(u)(\xi,\frac{Z}{\gamma}+\xi)\\ &=\hat{M}\begin{pmatrix}\theta(u)(\xi,\frac{Z}{\gamma}+\xi)\\ \lambda(u)(\xi)\end{pmatrix}(\xi,Z)=\hat{L}\left(\theta(u)(\xi,\frac{Z}{\gamma}+\xi)\right)(\xi,Z)+\lambda(u)(\xi)\theta_K'(Z)\,.\end{aligned}\tag{20.2}$$

For $f\in H^2(\mathbb{R}^2)$ it holds

$$\begin{aligned}&|f(\xi,\gamma(x-\xi))|_{H^2_{\xi,x}(\mathbb{R}^2)}\\ &=\Big(|f(\xi,\gamma(x-\xi))|^2_{L^2_{\xi,x}(\mathbb{R}^2)}+|\partial_x f(\xi,\gamma(x-\xi))|^2_{L^2_{\xi,x}(\mathbb{R}^2)}+\left|\partial_x^2 f(\xi,\gamma(x-\xi))\right|^2_{L^2_{\xi,x}(\mathbb{R}^2)}\\ &\quad+|\partial_\xi\partial_x f(\xi,\gamma(x-\xi))|^2_{L^2_{\xi,x}(\mathbb{R}^2)}+|\partial_x\partial_\xi f(\xi,\gamma(x-\xi))|^2_{L^2_{\xi,x}(\mathbb{R}^2)}\\ &\quad+|\partial_\xi f(\xi,\gamma(x-\xi))|^2_{L^2_{\xi,x}(\mathbb{R}^2)}+\left|\partial_\xi^2 f(\xi,\gamma(x-\xi))\right|^2_{L^2_{\xi,x}(\mathbb{R}^2)}\Big)^{\frac{1}{2}}\end{aligned}$$

$$
\begin{aligned}
&= \Big( |f(\xi,\gamma(x-\xi))|^2_{L^2_{\xi,x}(\mathbb{R}^2)} + \gamma^2 |\partial_2 f(\xi,\gamma(x-\xi))|^2_{L^2_{\xi,x}(\mathbb{R}^2)} + \gamma^4 \left|\partial_2^2 f(\xi,\gamma(x-\xi))\right|^2_{L^2_{\xi,x}(\mathbb{R}^2)} \\
&\quad + \gamma^2 |\partial_1\partial_2 f(\xi,\gamma(x-\xi))|^2_{L^2_{\xi,x}(\mathbb{R}^2)} + \gamma^2 |\partial_2\partial_1 f(\xi,\gamma(x-\xi))|^2_{L^2_{\xi,x}(\mathbb{R}^2)} \\
&\quad + |\partial_1 f(\xi,\gamma(x-\xi)) - \gamma\partial_2 f(\xi,\gamma(x-\xi))|^2_{L^2_{\xi,x}(\mathbb{R}^2)} \\
&\quad + \left|\partial_1^2 f(\xi,\gamma(x-\xi)) - \gamma\partial_2\partial_1 f(\xi,\gamma(x-\xi))\right. \\
&\quad \left. -\gamma\partial_1\partial_2 f(\xi,\gamma(x-\xi)) + \gamma^2\partial_2^2 f(\xi,\gamma(x-\xi))\right|^2_{L^{2,\xi,x}(\mathbb{R}^2)} \Big)^{\frac{1}{2}} \\
&= \Big(\frac{1}{\gamma} |f(\xi,Z)|^2_{L^2_{\xi,Z}(\mathbb{R}^2)} + \gamma |\partial_2 f(\xi,Z)|^2_{L^2_{\xi,Z}(\mathbb{R}^2)} + \gamma^3 \left|\partial_2^2 f(\xi,Z)\right|^2_{L^2_{\xi,Z}(\mathbb{R}^2)} \\
&\quad + \gamma |\partial_1\partial_2 f(\xi,Z)|^2_{L^2_{\xi,Z}(\mathbb{R}^2)} + \gamma |\partial_2\partial_1 f(\xi,Z)|^2_{L^2_{\xi,Z}(\mathbb{R}^2)} \\
&\quad + \frac{1}{\gamma} |\partial_1 f(\xi,Z) - \gamma\partial_2 f(\xi,Z)|^2_{L^2_{\xi,Z}(\mathbb{R}^2)} \\
&\quad + \frac{1}{\gamma} \left|\partial_1^2 f(\xi,Z) - \gamma\partial_2\partial_1 f(\xi,Z) - \gamma\partial_1\partial_2 f(\xi,Z) + \gamma^2\partial_2^2 f(\xi,Z)\right|^2_{L^2_{\xi,Z}(\mathbb{R}^2)} \Big)^{\frac{1}{2}} \\
&\le \Big(\frac{1}{\gamma} |f(\xi,Z)|^2_{L^2_{\xi,Z}(\mathbb{R}^2)} + \gamma |\partial_2 f(\xi,Z)|^2_{L^2_{\xi,Z}(\mathbb{R}^2)} + \gamma^3 \left|\partial_2^2 f(\xi,Z)\right|^2_{L^2_{\xi,Z}(\mathbb{R}^2)} \\
&\quad + \gamma |\partial_1\partial_2 f(\xi,Z)|^2_{L^2_{\xi,Z}(\mathbb{R}^2)} + \gamma |\partial_2\partial_1 f(\xi,Z)|^2_{L^2_{\xi,Z}(\mathbb{R}^2)} \\
&\quad + \frac{2}{\gamma} |\partial_1 f(\xi,Z)|_{L^2_{\xi,Z}(\mathbb{R}^2)} + 2\gamma |\partial_2 f(\xi,Z)|^2_{L^2_{\xi,Z}(\mathbb{R}^2)} \\
&\quad + \frac{4}{\gamma} \left|\partial_1^2 f(\xi,Z)\right|^2_{L^2_{\xi,Z}(\mathbb{R}^2)} + 4\gamma |\partial_2\partial_1 f(\xi,Z)|^2_{L^2_{\xi,Z}(\mathbb{R}^2)} \\
&\quad + 4\gamma |\partial_1\partial_2 f(\xi,Z)|^2_{L^2_{\xi,Z}(\mathbb{R}^2)} + 4\gamma^3 \left|\partial_2^2 f(\xi,Z)\right|^2_{L^2_{\xi,Z}(\mathbb{R}^2)} \Big)^{\frac{1}{2}} \\
&\le \sqrt{5}\gamma^{\frac{3}{2}} |f(\xi,Z)|_{H^2_{\xi,Z}(\mathbb{R}^2)} .
\end{aligned}
$$

For $h \in H^2(\mathbb{R}^2)$ and $f(\xi,Z) = h(\xi,\frac{Z}{\gamma}+\xi)$ we obtain the inequality

$$
|h(\xi,x)|_{H^2_{\xi,x}(\mathbb{R}^2)} \le \sqrt{5}\gamma(u)^{\frac{3}{2}} \left|h(\xi,\frac{Z}{\gamma(u)}+\xi)\right|_{H^2_{\xi,Z}(\mathbb{R}^2)} . \tag{20.3}
$$

Similarly, we obtain

$$|h(\xi,x)|_{H^3_{\xi,x}(\mathbb{R}^2)} \leq \Gamma(u)\left|h(\xi,\frac{Z}{\gamma(u)}+\xi)\right|_{H^3_{\xi,Z}(\mathbb{R}^2)}, \tag{20.4}$$

where $\Gamma:(-1,1)\to\mathbb{R}$ is a continuous function.

Using (20.3) for $h=\theta(u)-\theta(\bar{u})$, (20.2) and Lemma 20.10 we obtain:

$$\begin{aligned}
&\left|\begin{pmatrix}\theta(u)-\theta(\bar{u})\\ \lambda(u)-\lambda(\bar{u})\end{pmatrix}\right|_{\bar{y}} \\
=\ &|\theta(u)(\xi,x)-\theta(\bar{u})(\xi,x)|_{H^2_{\xi,x}(\mathbb{R}^2)}+|\lambda(u)(\xi)-\lambda(\bar{u})(\xi)|_{H^2_{\xi}(\mathbb{R})} \\
\leq\ &c\gamma(u)^{\frac{3}{2}}\left|\theta(u)(\xi,\frac{Z}{\gamma(u)}+\xi)-\theta(\bar{u})(\xi,\frac{Z}{\gamma(u)}+\xi)\right|_{H^2_{\xi,Z}(\mathbb{R}^2)} \\
&+|\lambda(u)(\xi)-\lambda(\bar{u})(\xi)|_{H^2_{\xi}(\mathbb{R})} \\
\leq\ &c\gamma(u)^{\frac{3}{2}}\left|\theta(u)(\xi,\frac{Z}{\gamma(u)}+\xi)-\theta(\bar{u})(\xi,\frac{Z}{\gamma(\bar{u})}+\xi)\right|_{H^2_{\xi,Z}(\mathbb{R}^2)} \\
&+c\gamma(u)^{\frac{3}{2}}\left|\theta(\bar{u})(\xi,\frac{Z}{\gamma(\bar{u})}+\xi)-\theta(\bar{u})(\xi,\frac{Z}{\gamma(u)}+\xi)\right|_{H^2_{\xi,Z}(\mathbb{R}^2)} \\
&+|\lambda(u)(\xi)-\lambda(\bar{u})(\xi)|_{H^2_{\xi}(\mathbb{R})} \\
\leq\ &c\gamma(u)^{\frac{3}{2}}\left|\hat{M}^{-1}\left(v(u)(\xi,\frac{Z}{\gamma(u)}+\xi)\right)-\hat{M}^{-1}\left(v(\bar{u})(\xi,\frac{Z}{\gamma(\bar{u})}+\xi)\right)\right|_{H^2_{\xi,Z}(\mathbb{R}^2)\oplus H^2_{\xi}(\mathbb{R})} \\
&+c\gamma(u)^{\frac{3}{2}}\left|\theta(\bar{u})(\xi,\frac{Z}{\gamma(\bar{u})}+\xi)-\theta(\bar{u})(\xi,\frac{Z}{\gamma(u)}+\xi)\right|_{H^2_{\xi,Z}(\mathbb{R}^2)} \\
\leq\ &c\gamma(u)^{\frac{3}{2}}\|\hat{M}^{-1}\|\left|v(u)(\xi,\frac{Z}{\gamma(u)}+\xi)-v(\bar{u})(\xi,\frac{Z}{\gamma(\bar{u})}+\xi)\right|_{L^2_{\xi,Z}(\mathbb{R}^2)} \\
&+c\gamma(u)^{\frac{3}{2}}\left|\theta(\bar{u})(\xi,\frac{Z}{\gamma(\bar{u})}+\xi)-\theta(\bar{u})(\xi,\frac{Z}{\gamma(u)}+\xi)\right|_{H^2_{\xi,Z}(\mathbb{R}^2)}
\end{aligned}$$

$$\leq c\gamma(u)^{\frac{3}{2}}\|\hat{M}^{-1}\| \left| v(u)(\xi, \frac{Z}{\gamma(u)} + \xi) - v(u)(\xi, \frac{Z}{\gamma(\bar{u})} + \xi) \right|_{L^2_{\xi,Z}(\mathbb{R}^2)}$$

$$+ c\gamma(u)^{\frac{3}{2}}\|\hat{M}^{-1}\| \left| v(u)(\xi, \frac{Z}{\gamma(\bar{u})} + \xi) - v(\bar{u})(\xi, \frac{Z}{\gamma(\bar{u})} + \xi) \right|_{L^2_{\xi,Z}(\mathbb{R}^2)}$$

$$+ c\gamma(u)^{\frac{3}{2}} \left| \theta(\bar{u})(\xi, \frac{Z}{\gamma(\bar{u})} + \xi) - \theta(\bar{u})(\xi, \frac{Z}{\gamma(u)} + \xi) \right|_{H^2_{\xi,Z}(\mathbb{R}^2)} .$$

This implies that $(\theta, \lambda) \in \mathcal{Y}$, since $v \in \mathcal{Z}$. The inverse mapping theorem yields that $\mathfrak{m}^{-1}$ is bounded, since $\mathfrak{m}$ is bounded.

□

We define some more operators.

**Definition 20.13.** *(a)* $\mathcal{L}_u : H^2(\mathbb{R}^2) \oplus H^1(\mathbb{R}^2) \to H^1(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)$, *where*

$$\left(\mathcal{L}_u \begin{pmatrix} \theta \\ \psi \end{pmatrix}\right)(\xi, x) = \begin{pmatrix} -u\partial_x\theta(\xi, x) - \psi(\xi, x) \\ -\partial_x^2\theta(\xi, x) + \cos(\theta_K(\gamma(x-\xi)))\theta(\xi, x) - u\partial_x\psi(\xi, x) \end{pmatrix} .$$

*(b)* $\hat{\mathcal{L}}_u : \left[H^2(\mathbb{R}^2) \oplus H^1(\mathbb{R}^2)\right] \cap \ker \mathcal{L}_u^{\perp, L^2 \oplus L^2} \to H^1(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)$, *where*

$$\left(\hat{\mathcal{L}}_u \begin{pmatrix} \theta \\ \psi \end{pmatrix}\right)(\xi, x) = \begin{pmatrix} -u\partial_x\theta(\xi, x) - \psi(\xi, x) \\ -\partial_x^2\theta(\xi, x) + \cos(\theta_K(\gamma(x-\xi)))\theta(\xi, x) - u\partial_x\psi(\xi, x) \end{pmatrix} .$$

*(c)* $\mathcal{M}_u : H^2(\mathbb{R}^2) \oplus H^1(\mathbb{R}^2) \oplus H^2(\mathbb{R}) \to H^1(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)$, *where*

$$\left(\mathcal{M}_u \begin{pmatrix} \theta \\ \psi \\ \lambda \end{pmatrix}\right)(\xi, x) = \left(\mathcal{L}_u \begin{pmatrix} \theta \\ \psi \end{pmatrix}\right)(\xi, x) + \lambda(\xi) t_u(\xi, u, x) .$$

*(d)* $\hat{\mathcal{M}}_u : \left[H^2(\mathbb{R}^2) \oplus H^1(\mathbb{R}^2)\right] \cap \ker \mathcal{L}_u^{\perp, L^2 \oplus L^2} \oplus H^2(\mathbb{R}) \to H^1(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)$, *where*

$$\left(\hat{\mathcal{M}}_u \begin{pmatrix} \theta \\ \psi \\ \lambda \end{pmatrix}\right)(\xi, x) = \left(\hat{\mathcal{L}}_u \begin{pmatrix} \theta \\ \psi \end{pmatrix}\right)(\xi, x) + \lambda(\xi) t_u(\xi, u, x) .$$

*(e)* $\mathcal{K}_u : H^2(\mathbb{R}^2) \oplus H^1(\mathbb{R}^2) \to H^1(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)$, *where*

$$\left(\mathcal{K}_u \begin{pmatrix} \theta \\ \psi \end{pmatrix}\right)(\xi, x) = \begin{pmatrix} u\partial_\xi\theta(\xi, x) - \psi(\xi, x) \\ -\partial_x^2\theta(\xi, x) + \cos(\theta_K(\gamma(x-\xi)))\theta(\xi, x) + u\partial_\xi\psi(\xi, x) \end{pmatrix} .$$

*(f)* $\hat{\mathcal{K}}_u : \Big[H^2(\mathbb{R}^2) \oplus H^1(\mathbb{R}^2)\Big] \cap \ker \mathcal{L}_u^{\perp, L^2 \oplus L^2} \to H^1(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)$, *where*

$$\left(\hat{\mathcal{K}}_u \begin{pmatrix} \theta \\ \psi \end{pmatrix}\right)(\xi, x) = \begin{pmatrix} u\partial_\xi \theta(\xi, x) - \psi(\xi, x) \\ -\partial_x^2 \theta(\xi, x) + \cos(\theta_K(\gamma(x-\xi)))\theta(\xi, x) + u\partial_\xi \psi(\xi, x) \end{pmatrix}.$$

*(g)* $\mathcal{N}_u : H^2(\mathbb{R}^2) \oplus H^1(\mathbb{R}^2) \oplus H^2(\mathbb{R}) \to H^1(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)$, *where*

$$\left(\mathcal{N}_u \begin{pmatrix} \theta \\ \psi \\ \lambda \end{pmatrix}\right)(\xi, x) = \left(\mathcal{K}_u \begin{pmatrix} \theta \\ \psi \end{pmatrix}\right)(\xi, x) + \lambda(\xi) t_u(\xi, u, x).$$

*(h)* $\hat{\mathcal{N}}_u : \Big[H^2(\mathbb{R}^2) \oplus H^1(\mathbb{R}^2)\Big] \cap \ker \mathcal{L}_u^{\perp, L^2 \oplus L^2} \oplus H^2(\mathbb{R}) \to H^1(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)$, *where*

$$\left(\hat{\mathcal{N}}_u \begin{pmatrix} \theta \\ \psi \\ \lambda \end{pmatrix}\right)(\xi, x) = \left(\hat{\mathcal{K}}_u \begin{pmatrix} \theta \\ \psi \end{pmatrix}\right)(\xi, x) + \lambda(\xi) t_u(\xi, u, x).$$

We examine the operator $\mathcal{L}_u$. Our next goal is to prove the following direct sum decomposition of the space $H^1(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)$:

$$H^1(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2) = \hat{\mathcal{L}}_u\left(\Big[H^2(\mathbb{R}^2) \oplus H^1(\mathbb{R}^2)\Big] \cap (\ker \mathcal{L}_u)^{\perp, L^2 \oplus L^2}\right) \oplus \{\lambda t_u(u), \lambda \in H^2(\mathbb{R})\}.$$

**Lemma 20.14.**

*(a)* $\ker \mathcal{L}_u$

$$= \left\{ \begin{pmatrix} \theta \\ \psi \end{pmatrix} \in H^2(\mathbb{R}^2) \oplus H^1(\mathbb{R}^2) \;\middle|\; \begin{pmatrix} \theta(\xi, x) \\ \psi(\xi, x) \end{pmatrix} = \lambda(\xi) \begin{pmatrix} -\theta_K'(\gamma(x-\xi)) \\ u\gamma\theta_K''(\gamma(x-\xi)) \end{pmatrix},\ \lambda \in H^2(\mathbb{R}) \right\}.$$

*(b)* $\ker \mathcal{L}_u^*$

$$= \left\{ \begin{pmatrix} v \\ w \end{pmatrix} \in H^1(\mathbb{R}^2) \oplus H^2(\mathbb{R}^2) \;\middle|\; \begin{pmatrix} v(\xi, x) \\ w(\xi, x) \end{pmatrix} = \lambda(\xi) \begin{pmatrix} u\gamma\theta_K''(\gamma(x-\xi)) \\ \theta_K'(\gamma(x-\xi)) \end{pmatrix},\ \lambda \in H^2(\mathbb{R}) \right\}.$$

**Proof.** Notice that $\mathcal{L}_u : H^2(\mathbb{R}^2) \oplus H^1(\mathbb{R}^2) \to H^1(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)$ is given by

$$\left(\mathcal{L}_u \begin{pmatrix} \theta \\ \psi \end{pmatrix}\right)(\xi, x) = \begin{pmatrix} -u\partial_x \theta(\xi, x) - \psi(\xi, x) \\ -\partial_x^2 \theta(\xi, x) + \cos(\theta_K(\gamma(x-\xi)))\theta(\xi, x) - u\partial_x \psi(\xi, x) \end{pmatrix}$$

and the adjoint $\mathcal{L}_u^* : H^1(\mathbb{R}^2) \oplus H^2(\mathbb{R}^2) \to H^1(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)$ is given by

$$\left(\mathcal{L}_u^* \begin{pmatrix} v \\ w \end{pmatrix}\right)(\xi, x) = \begin{pmatrix} u\partial_x v(\xi, x) - \partial_x^2 w(\xi, x) + \cos(\theta_K(\gamma(x-\xi)))w(\xi, x) \\ u\partial_x w(\xi, x) - v(\xi, x) \end{pmatrix},$$

since partial integration yields

$$\begin{aligned}
&\left\langle \mathcal{L}_u \begin{pmatrix} \theta \\ \psi \end{pmatrix}, \begin{pmatrix} v \\ w \end{pmatrix} \right\rangle_{L^2(\mathbb{R}^2)\oplus L^2(\mathbb{R}^2)} \\
&= \left\langle \begin{pmatrix} -u\partial_x\theta - \psi \\ -\partial_x^2\theta + \cos(\theta_K(\gamma(x-\xi)))\theta - u\partial_x\psi \end{pmatrix}, \begin{pmatrix} v \\ w \end{pmatrix} \right\rangle_{L^2(\mathbb{R}^2)\oplus L^2(\mathbb{R}^2)} \\
&= \int_{\mathbb{R}^2} [-u\partial_x\theta - \psi]v + [-\partial_x^2\theta + \cos(\theta_K(\gamma(x-\xi)))\theta - u\partial_x\psi]w\,dx \\
&= \int_{\mathbb{R}^2} u\theta\partial_x v - \psi v - \theta\partial_x^2 w + \cos(\theta_K(\gamma(x-\xi)))\theta w + u\psi\partial_x w\,dx \\
&= \int_{\mathbb{R}^2} [u\partial_x v - \partial_x^2 w + \cos(\theta_K(\gamma(x-\xi)))w]\theta + [u\partial_x w - v]\psi\,dx \\
&= \left\langle \begin{pmatrix} \theta \\ \psi \end{pmatrix}, \mathcal{L}_u^* \begin{pmatrix} v \\ w \end{pmatrix} \right\rangle_{L^2(\mathbb{R}^2)\oplus L^2(\mathbb{R}^2)} .
\end{aligned}$$

Let $Z = \gamma(x-\xi)$.

(a) Using Lemma 20.8 we obtain:

$$\begin{aligned}
&\mathcal{L}_u \begin{pmatrix} \tilde\theta \\ \tilde\psi \end{pmatrix} = \begin{pmatrix} -u\partial_x\tilde\theta - \tilde\psi \\ -u\partial_x\tilde\psi - \partial_x^2\tilde\theta + \cos(\theta_K(Z))\tilde\theta \end{pmatrix} = 0 \\
&\Rightarrow \begin{cases} -u\partial_x\tilde\theta = \tilde\psi \\ -u\partial_x\tilde\psi - \partial_x^2\tilde\theta + \cos(\theta_K(Z))\tilde\theta = 0 \end{cases} \\
&\Rightarrow \begin{cases} -u\partial_x\tilde\theta = \tilde\psi \\ u^2\partial_x^2\tilde\theta - \partial_x^2\tilde\theta + \cos(\theta_K(Z))\tilde\theta = 0 \end{cases} \\
&\Rightarrow \begin{cases} \tilde\theta(\xi,x) = \lambda(\xi)\theta_K'(\gamma(x-\xi)) \\ \tilde\psi(\xi,x) = -u\gamma\lambda(\xi)\theta_K''(\gamma(x-\xi)) \end{cases} .
\end{aligned}$$

(b) Using Lemma 20.8 we obtain:

$$\begin{aligned}
&\mathcal{L}_u^* \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} u\partial_x v - \partial_x^2 w + \cos(\theta_K(Z))w \\ u\partial_x w - v \end{pmatrix} = 0 \\
&\Rightarrow \begin{cases} u\partial_x v - \partial_x^2 w + \cos(\theta_K(Z))w = 0 \\ u\partial_x w = v \end{cases}
\end{aligned}$$

$$\Rightarrow \begin{cases} u^2\partial_x^2 w - \partial_x^2 w + \cos(\theta_K(Z))w = 0 \\ u\partial_x w = v \end{cases}$$

$$\Rightarrow \begin{cases} w(\xi,x) = \lambda(\xi)\theta_K'(\gamma(x-\xi)) \\ v(\xi,x) = u\gamma\lambda(\xi)\theta_K''(\gamma(x-\xi)) \end{cases} .$$

□

**Remark 20.15.** In the following we denote by $t_\xi(u)$ and $t_u(u)$ the functions $t_\xi(\cdot_\xi, u, \cdot_x)$ and $t_u(\cdot_\xi, u, \cdot_x)$ which depend on the variables $\xi$ and $x$.

**Remark 20.16.** Notice that

$$\ker \mathcal{L}_u = \{\lambda t_\xi(u), \lambda \in H^2(\mathbb{R})\},$$
$$\ker \mathcal{L}_u^* = \{\lambda \mathbb{J} t_\xi(u), \lambda \in H^2(\mathbb{R})\}.$$

We prove an orthogonal decomposition of $H^1(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)$.

**Lemma 20.17 (orthogonal sum).**

$$H^1(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2) = \hat{\mathcal{L}}_u\left(\Big[H^2(\mathbb{R}^2) \oplus H^1(\mathbb{R}^2)\Big] \cap \ker \mathcal{L}_u^{\perp, L^2\oplus L^2}\right) \overset{L^2\oplus L^2}{\oplus} \ker \mathcal{L}_u^*.$$

**Proof.** "$\supset$": clear.
"$\subset$":
Let $\begin{pmatrix} \bar{v} \\ \bar{w} \end{pmatrix} \in H^1(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)$. Since $L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2) = \overline{\mathcal{L}_u(H^2(\mathbb{R}^2) \oplus H^1(\mathbb{R}^2))} \overset{L^2\oplus L^2}{\oplus} \ker \mathcal{L}_u^*$, using Lemma 20.14 there exists $\mu(u) \in H^2(\mathbb{R})$, s.t.

$$\begin{pmatrix} \bar{v} \\ \bar{w} \end{pmatrix} = {}_{L^2\oplus L^2}\text{-} \lim_{n\to\infty} \mathcal{L}_u \begin{pmatrix} \theta_n \\ \psi_n \end{pmatrix} + \mu(u)\mathbb{J} t_\xi(u).$$

Hence $\begin{pmatrix} v \\ w \end{pmatrix} := {}_{L^2\oplus L^2}\text{-} \lim_{n\to\infty} \mathcal{L}_u \begin{pmatrix} \theta_n \\ \psi_n \end{pmatrix} \in \ker \mathcal{L}_u^{\perp, L^2\oplus L^2} \cap \Big[H^1(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)\Big]$.
$\begin{pmatrix} v \\ w \end{pmatrix} \in \ker \mathcal{L}_u^{\perp, L^2\oplus L^2}$ yields

$$\begin{aligned} 0 &= \int \lambda(\xi)\Big(u\gamma^2\theta_K''(\gamma(x-\xi))v(\xi,x) + \gamma\theta_K'(\gamma(x-\xi))w(\xi,x)\Big)\, d\xi\, dx \\ &= \int \Big[ -u\gamma\partial_x v(\xi,x) + \gamma w(\xi,x)\Big]\lambda(\xi)\theta_K'(\gamma(x-\xi))\, d\xi\, dx, \end{aligned}$$

which implies

$$\Big[-u\gamma\partial_x v(\xi,x)+\gamma w(\xi,x)\Big]\in\ker L_u^{\perp,L^2}=\operatorname{ran}\hat{L}_u\quad\text{(due to Lemma 20.10).}$$

We set

$$\begin{cases}\tilde{\theta}(\xi,x):=[\hat{L}_u]^{-1}\Big(-u\partial_x\gamma v(\xi,x)+w(\xi,x)\Big)\\ \tilde{\psi}(\xi,x):=-u\partial_x\tilde{\theta}(\xi,x)-v(\xi,x)\end{cases}.$$

Let $Z=\gamma(x-\xi)$. Due to Lemma 20.10 $\begin{pmatrix}\tilde{\theta}\\ \tilde{\psi}\end{pmatrix}\in H^2(\mathbb{R}^2)\oplus H^1(\mathbb{R}^2)$ and

$$\begin{cases}-\tilde{\psi}-u\partial_x\tilde{\theta}=v\\ \hat{L}_u\tilde{\theta}(x)=-u\gamma\partial_x v(x)+\gamma w(x)\end{cases}$$

$$\begin{cases}-\tilde{\psi}-u\partial_x\tilde{\theta}=v\\ -(1-u^2)\partial_x^2\tilde{\theta}+\cos(\theta_K(Z))\tilde{\theta}+u\partial_x v=w\end{cases}$$

$$\begin{cases}-\tilde{\psi}-u\partial_x\tilde{\theta}=v\\ -(1-u^2)\partial_x^2\tilde{\theta}+\cos(\theta_K(Z))\tilde{\theta}-u\partial_x\tilde{\psi}-u^2\partial_x^2\tilde{\theta}=w\end{cases}$$

$$\mathcal{L}_u\begin{pmatrix}\tilde{\theta}\\ \tilde{\psi}\end{pmatrix}=\begin{pmatrix}-u\partial_x\tilde{\theta}-\tilde{\psi}\\ -\partial_x^2\tilde{\theta}+\cos(\theta_K(Z))\tilde{\theta}-u\partial_x\tilde{\psi}\end{pmatrix}=\begin{pmatrix}v\\ w\end{pmatrix}.$$

Set $\begin{pmatrix}\theta\\ \psi\end{pmatrix}:=\begin{pmatrix}\tilde{\theta}\\ \tilde{\psi}\end{pmatrix}_{\langle t_\xi(u)\rangle^{\perp,L^2\oplus L^2}}$, so $\hat{\mathcal{L}}_u\begin{pmatrix}\theta\\ \psi\end{pmatrix}=\begin{pmatrix}v\\ w\end{pmatrix}$. □

The following corollaries will be another ingredient in the proof of the invertibility of $\mathfrak{M}_n$.

**Corollary 20.18 (direct sum).**

$$H^1(\mathbb{R}^2)\oplus L^2(\mathbb{R}^2)=\hat{\mathcal{L}}_u\left(\Big[H^2(\mathbb{R}^2)\oplus H^1(\mathbb{R}^2)\Big]\cap(\ker\mathcal{L}_u)^{\perp,L^2\oplus L^2}\right)\oplus\{\lambda t_u(u),\lambda\in H^2(\mathbb{R})\}.$$

**Proof.** "$\supset$": clear.
"$\subset$": Let $(v,w)\in H^1(\mathbb{R}^2)\oplus L^2(\mathbb{R}^2)$ then there exists due to Lemma 20.17

$$\begin{pmatrix}\theta\\ \psi\end{pmatrix}=\begin{pmatrix}\theta(u)\\ \psi(u)\end{pmatrix}\in H^2(\mathbb{R}^2)\oplus H^1(\mathbb{R}^2)\cap(\ker\mathcal{L}_u)^{\perp,L^2\oplus L^2}$$

and $\lambda=\lambda(u)\in H^2(\mathbb{R})$ s.t.

$$\begin{pmatrix}v\\ w\end{pmatrix}=\hat{\mathcal{L}}_u\begin{pmatrix}\theta\\ \psi\end{pmatrix}+\lambda\mathbb{J}t_\xi(u).$$

Let $Z = \gamma(x-\xi)$. Assume without loss of generality $|\lambda|_{H^2(\mathbb{R})} \neq 0$, then

$$\begin{aligned}
&\langle \lambda t_\xi(u), \lambda \mathbb{J} t_u(u)\rangle_{L^2(\mathbb{R}^2)\oplus L^2(\mathbb{R}^2)}\\
&= \int \lambda(\xi)^2 \Big( -\gamma^4 \Big[\theta_K'(Z)\Big]^2 - u^2\gamma^5 (x-\xi)\theta_K''(Z)\theta_K'(Z) + u^2\gamma^5\theta_K''(Z)(x-\xi)\theta_K'(Z)\Big)\, d\xi\, dx\\
&= \int -\gamma^4 \lambda(\xi)^2 \Big[\theta_K'(Z)\Big]^2 d\xi\, dx\\
&= -\gamma^3 \int \lambda(\xi)^2 \Big[\theta_K'(y)\Big]^2 d\xi\, dy\\
&= -\gamma^3 m \int \lambda(\xi)^2\, d\xi\,.
\end{aligned}$$

Since

$$\langle \lambda t_u(u), \lambda \mathbb{J} t_\xi(u)\rangle_{L^2(\mathbb{R}^2)\oplus L^2(\mathbb{R}^2)} \neq 0,$$

due to Lemma 20.17 there exist

$$\begin{aligned}
&\begin{pmatrix}\bar\theta\\ \bar\psi\end{pmatrix} = \begin{pmatrix}\bar\theta(u)\\ \bar\psi(u)\end{pmatrix} \in H^2(\mathbb{R}^2)\oplus H^1(\mathbb{R}^2) \cap \langle t_\xi(u)\rangle^{\perp, L^2\times L^2} \quad \text{and}\\
&0 \neq \mu = \mu(u) \in H^2(\mathbb{R}) \quad \text{s.t.} \qquad (20.5)\\
&\lambda(u) t_u(u) = \mathcal{L}_u \begin{pmatrix}\bar\theta(u)\\ \bar\psi(u)\end{pmatrix} + \mu(u)\mathbb{J} t_\xi(u)\,.
\end{aligned}$$

This is an identity in $H^1(\mathbb{R}^2)\oplus L^2(\mathbb{R}^2)$. We fix $\xi$ and pair this identity with $\mathbb{J}t_\xi(\xi, u, \cdot)$ in $L^2_x(\mathbb{R})\oplus L^2_x(\mathbb{R})$. It follows due to Lemma 20.5 for a.e. $\xi \in \mathbb{R}$:

$$\lambda(\xi,u)\gamma(u)^3 m = \mu(\xi,u)\|\mathbb{J}t_\xi(\xi,u,\cdot)\|^2_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} = \mu(\xi,u)(u^2\gamma^3\, |\theta_K''|^2_{L^2(\mathbb{R})} + \gamma\, |\theta_K'|^2_{L^2(\mathbb{R})}),$$

since

$$\begin{aligned}
&\|\mathbb{J}t_\xi(\xi,u,\cdot)\|^2_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})}\\
&= \| \begin{pmatrix} -u\gamma^2\theta_K''(\gamma(\cdot-\xi))\\ -\gamma\theta_K'(\gamma(\cdot-\xi))\end{pmatrix} \|^2_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})}\\
&= u^2\gamma^3\, |\theta_K''|^2_{L^2(\mathbb{R})} + \gamma\, |\theta_K'|^2_{L^2(\mathbb{R})}\ .
\end{aligned}$$

Consequently

$$\lambda(u) = \eta(u)\mu(u)\,,$$

where

$$\eta(u) := \frac{u^2\gamma^3 \left|\theta''_K\right|^2_{L^2(\mathbb{R})} + \gamma \left|\theta'_K\right|^2_{L^2(\mathbb{R})}}{\gamma(u)^3 m} \in \mathbb{R}\,.$$

Thus

$$\begin{aligned} &\lambda(u)\mathbb{J}t_\xi(u) \\ &\eta(u)\mu(u)\mathbb{J}t_\xi(u) \\ &= \eta(u)\lambda(u)t_u(u) - \eta(u)\mathcal{L}_u \begin{pmatrix} \bar{\theta}(u) \\ \bar{\psi}(u) \end{pmatrix} \end{aligned}$$

and

$$\begin{pmatrix} v \\ w \end{pmatrix} = \hat{\mathcal{L}}_u \begin{pmatrix} \theta(u) \\ \psi(u) \end{pmatrix} + \lambda(u)\mathbb{J}t_\xi(u) = \hat{\mathcal{L}}_u \left( \begin{pmatrix} \theta(u) \\ \psi(u) \end{pmatrix} - \eta(u) \begin{pmatrix} \bar{\theta}(u) \\ \bar{\psi}(u) \end{pmatrix} \right) + \eta(u)\lambda(u)t_u(u)\,.$$

The sum is direct, i.e.,

$$\{\lambda t_u(u); \lambda \in H^2(\mathbb{R})\} \cap \mathcal{L}_u(H^2(\mathbb{R}^2) \oplus H^1(\mathbb{R}^2) \cap \langle t_\xi(u)\rangle^{\perp, L^2 \oplus L^2}) = 0$$

due to (20.5). □

**Remark 20.19.** It follows from elliptic regularity of $\hat{L}_u$ that

$$H^2(\mathbb{R}^2) \oplus H^1(\mathbb{R}^2) = \hat{\mathcal{L}}_u \left( \left[ H^3(\mathbb{R}^2) \oplus H^2(\mathbb{R}^2) \right] \cap (\ker \mathcal{L}_u)^{\perp, L^2 \oplus L^2} \right) \oplus \{\lambda t_u(u), \lambda \in H^2(\mathbb{R})\}\,.$$

**Corollary 20.20.**

*(a)* $\ker \mathcal{M}_u$

$$= \left\{ \begin{pmatrix} \theta \\ \psi \\ \lambda \end{pmatrix} \in H^2(\mathbb{R}^2) \oplus H^1(\mathbb{R}^2) \oplus H^2(\mathbb{R}) \;\middle|\; \begin{pmatrix} \theta(\xi, x) \\ \psi(\xi, x) \\ \lambda(\xi) \end{pmatrix} = \mu(\xi) \begin{pmatrix} -\theta'_K(\gamma(x-\xi)) \\ u\gamma\theta''_K(\gamma(x-\xi)) \\ 0 \end{pmatrix},\ \mu \in H^2(\mathbb{R}) \right\}.$$

*(b) $\hat{\mathcal{M}}_u$ is one-to-one, onto, bounded and the inverse mapping is also bounded.*

**Proof.** Follows from Lemma 20.14, Corollary 20.18 and the inverse mapping theorem. □

### 20.1.2 Inverse Operator

We start with a definition.

**Definition 20.21.** *(a)* $X = \mathbb{R}$ *with the absolute value on* $\mathbb{R}$ *as a norm.*

*(a)* $\bar{Y} = H^2(\mathbb{R}^2) \oplus H^1(\mathbb{R}^2) \oplus H^2(\mathbb{R})$ *with the finite norm*

$$|y|_{\bar{Y}} = |\theta|_{H^2(\mathbb{R}^2)} + |\psi|_{H^1(\mathbb{R}^2)} + |\lambda|_{H^2(\mathbb{R})} \,.$$

*(b)* $\bar{Z} = H^1(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)$ *with the finite norm*

$$|z|_{\bar{Z}} = |v|_{H^1(\mathbb{R}^2)} + |w|_{L^2(\mathbb{R}^2)} \,.$$

*(c)*
$$\begin{aligned} Y &= Y(u_*) \\ &= \Bigg\{ y = (\theta, \psi, \lambda_u) \in C(I(u_*), \bar{Y}) : \|y\|_{Y(u_*)} < \infty;\ \forall\ u \in I(u_*),\ \forall\ \lambda \in H^2(\mathbb{R}) : \\ &\quad \left\langle \begin{pmatrix} \theta(u)(\xi, x) \\ \psi(u)(\xi, x) \end{pmatrix}, \lambda(\xi) \begin{pmatrix} \theta_K'(\gamma(x-\xi)) \\ -u\gamma\theta_K''(\gamma(x-\xi)) \end{pmatrix} \right\rangle_{L^2_{\xi,x}(\mathbb{R}^2) \oplus L^2_{\xi,x}(\mathbb{R}^2)} = 0 \Bigg\} \end{aligned}$$

*with the finite norm*

$$\|y\|_{Y(u_*)} = \sup_{u \in I(u_*)} |y|_{\bar{Y}} \,.$$

*(d)* $Z = Z(u_*) = \Big\{ z = (v, w) \in C(I(u_*), \bar{Z}) : \|z\|_{Z(u_*)} < \infty \Big\}$ *with the finite norm*

$$\|z\|_{Z(u_*)} = \sup_{u \in I(u_*)} |z|_{\bar{Z}} \,.$$

We want to show that the linear operator

$$\mathfrak{M} : Y(u_*) \to Z(u_*),$$

given by

$$\mathfrak{M} \begin{pmatrix} \tilde{\theta} \\ \tilde{\psi} \\ \tilde{\lambda}_u \end{pmatrix} (u) = \hat{\mathcal{N}}_u \begin{pmatrix} \tilde{\theta}(u) \\ \tilde{\psi}(u) \\ \tilde{\lambda}_u(u) \end{pmatrix} ,$$

is invertible if $u_*$ is small. The operator $\hat{\mathcal{N}}_u$ contains derivatives with respect to $\xi$ and with respect to $x$. This fact makes it difficult to analyze the operator $\mathfrak{M}$. Therefore we consider first the operator

$$\tilde{\mathfrak{M}} : Y \to Z,$$

given by

$$\tilde{\mathfrak{M}}\begin{pmatrix}\tilde{\theta}\\ \tilde{\psi}\\ \tilde{\lambda}_u\end{pmatrix}(u)=\hat{\mathcal{M}}_u\begin{pmatrix}\tilde{\theta}(u)\\ \tilde{\psi}(u)\\ \tilde{\lambda}_u(u)\end{pmatrix}.$$

The operator $\hat{\mathcal{M}}_u$ contains only derivatives with respect to $x$. This allows us to prove invertibility of $\tilde{\mathfrak{M}}$ by using the statements from Section 20.1.1.

**Lemma 20.22.** *The linear operator*

$$\tilde{\mathfrak{M}}:Y\to Z,$$
$$\begin{pmatrix}\tilde{\theta}\\ \tilde{\psi}\\ \tilde{\lambda}_u\end{pmatrix}\mapsto\tilde{\mathfrak{M}}\begin{pmatrix}\tilde{\theta}\\ \tilde{\psi}\\ \tilde{\lambda}_u\end{pmatrix},$$

*given by*

$$\tilde{\mathfrak{M}}\begin{pmatrix}\tilde{\theta}\\ \tilde{\psi}\\ \tilde{\lambda}_u\end{pmatrix}(u)=\hat{\mathcal{M}}_u\begin{pmatrix}\tilde{\theta}(u)\\ \tilde{\psi}(u)\\ \tilde{\lambda}_u(u)\end{pmatrix},$$

*is invertible.*

**Proof.** $\tilde{\mathfrak{M}}$ is onto: Let $\begin{pmatrix}\theta\\ \psi\end{pmatrix}\in Z$. Due to Corollary 20.18 for all $u\in I$ there exists

$$\begin{pmatrix}\tilde{\theta}(u)\\ \tilde{\psi}(u)\end{pmatrix}\in\left[H^2(\mathbb{R}^2)\oplus H^1(\mathbb{R}^2)\right]\cap\ker\mathcal{L}_u^{\perp,L^2\oplus L^2}$$

and $\tilde{\lambda}_u(u)\in H^2(\mathbb{R})$ s.t.

$$\begin{pmatrix}\theta(u)\\ \psi(u)\end{pmatrix}=\hat{\mathcal{L}}_u\begin{pmatrix}\tilde{\theta}(u)\\ \tilde{\psi}(u)\end{pmatrix}+\tilde{\lambda}_u(u)t_u(u).\tag{20.6}$$

First of all we would like to prove that $\tilde{\lambda}_u\in C(I,L^2(\mathbb{R}))$. (20.6) is an identity in $H^1(\mathbb{R}^2)\oplus L^2(\mathbb{R}^2)$. We fix $\xi$ and pair this identity with

$$\mathbb{J}t_\xi(\xi,u,\cdot)=\begin{pmatrix}-u\gamma^2\theta_K''(\gamma(\cdot-\xi))\\ -\gamma\theta_K'(\gamma(\cdot-\xi))\end{pmatrix}=:\begin{pmatrix}t_1(u)(\xi,\cdot)\\ t_2(u)(\xi,\cdot)\end{pmatrix}$$

in $L^2_x(\mathbb{R}) \oplus L^2_x(\mathbb{R})$. It follows due to Lemma 20.5 for a.e. $\xi \in \mathbb{R}$:

$$\left\langle \begin{pmatrix} \theta(u)(\xi,x) \\ \psi(u)(\xi,x) \end{pmatrix}, \begin{pmatrix} t_1(u)(\xi,x) \\ t_2(u)(\xi,x) \end{pmatrix} \right\rangle_{L^2_x(\mathbb{R})\oplus L^2_x(\mathbb{R})} = \tilde{\lambda_u}(\xi,u)\gamma(u)^3 m\,.$$

We have to prove that the map

$$\begin{aligned} I &\to L^2_\xi(\mathbb{R}) \\ u &\mapsto f(u)\,, \end{aligned}$$

where

$$f(u)(\xi) = \left\langle \begin{pmatrix} \theta(u)(\xi,x) \\ \psi(u)(\xi,x) \end{pmatrix}, \begin{pmatrix} t_1(u)(\xi,x) \\ t_2(u)(\xi,x) \end{pmatrix} \right\rangle_{L^2_x(\mathbb{R})\oplus L^2_x(\mathbb{R})}\,,$$

is continuous. Therefore we deduce an estimate for the difference

$$\begin{aligned} &\int \Big(f(u)(\xi) - f(\bar{u})(\xi)\Big)^2 d\xi \\ &\le 2\int \Big( \langle \theta(u)(\xi,x), t_1(u)(\xi,x)\rangle_{L^2_x(\mathbb{R})} - \langle \theta(\bar{u})(\xi,x), t_1(\bar{u})(\xi,x)\rangle_{L^2_x(\mathbb{R})} \Big)^2 d\xi \\ &\quad + 2\int \Big( \langle \psi(u)(\xi,x), t_2(u)(\xi,x)\rangle_{L^2_x(\mathbb{R})} - \langle \psi(\bar{u})(\xi,x), t_2(\bar{u})(\xi,x)\rangle_{L^2_x(\mathbb{R})} \Big)^2 d\xi\,. \end{aligned}$$

We consider the first integral, the second one can be treated in the same way:

$$\begin{aligned} &2\int \Big( \langle \theta(u)(\xi,x), t_1(u)(\xi,x)\rangle_{L^2_x(\mathbb{R})} - \langle \theta(\bar{u})(\xi,x), t_1(\bar{u})(\xi,x)\rangle_{L^2_x(\mathbb{R})} \Big)^2 d\xi \\ &= 2\int \Big( \langle \theta(u)(\xi,x), t_1(u)(\xi,x)\rangle_{L^2_x(\mathbb{R})} - \langle \theta(\bar{u})(\xi,x), t_1(u)(\xi,x)\rangle_{L^2_x(\mathbb{R})} \\ &\quad + \langle \theta(\bar{u})(\xi,x), t_1(u)(\xi,x)\rangle_{L^2_x(\mathbb{R})} - \langle \theta(\bar{u})(\xi,x), t_1(\bar{u})(\xi,x)\rangle_{L^2_x(\mathbb{R})} \Big)^2 d\xi \\ &\le 4\left|\theta(u)(\xi,x) - \theta(\bar{u})(\xi,x)\right|^2_{L^2(\mathbb{R}^2)} \left|t_1(\bar{u})(\xi,x)\right|^2_{L^2_x(\mathbb{R})} \\ &\quad + 4\left|t_1(u)(\xi,x) - t_1(\bar{u})(\xi,x)\right|^2_{L^2_x(\mathbb{R})} \left|\theta(\bar{u})(\xi,x)\right|^2_{L^2(\mathbb{R}^2)}\,. \end{aligned}$$

The terms on the right hand side converge to 0 as $\bar{u} \to u$, since $\begin{pmatrix} \theta \\ 0 \end{pmatrix} \in Z$. Notice that the norms of $t_1$ do not depend on $\xi$. It follows that $\tilde{\lambda_u} \in C(I, L^2(\mathbb{R}))$. Now we set

$$\begin{pmatrix} v(u) \\ w(u) \end{pmatrix} := \begin{pmatrix} \theta(u) \\ \psi(u) \end{pmatrix} - \tilde{\lambda_u}(u)t_u(u)\,. \tag{20.7}$$

Let $Z = \gamma(x-\xi)$. We determine $\begin{pmatrix}\tilde{\theta}(u)\\ \tilde{\psi}(u)\end{pmatrix}$:

$$\mathcal{L}_u \begin{pmatrix}\tilde{\theta}\\ \tilde{\psi}\end{pmatrix} = \begin{pmatrix} -u\partial_x\tilde{\theta} - \tilde{\psi} \\ -\partial_x^2\tilde{\theta} + \cos(\theta_K(Z))\tilde{\theta} - u\partial_x\tilde{\psi}\end{pmatrix} = \begin{pmatrix} v\\ w\end{pmatrix}$$

$$\begin{cases} \tilde{\psi} = -u\partial_x\tilde{\theta} - v \\ -\partial_x^2\tilde{\theta} + u^2\partial_x^2\tilde{\theta} + \cos(\theta_K(Z))\tilde{\theta} + u\partial_x v = w \end{cases}$$

$$\begin{cases} \tilde{\psi} = -u\partial_x\tilde{\theta} - v \\ -(1-u^2)\partial_x^2\tilde{\theta} + \cos(\theta_K(Z))\tilde{\theta} = w - u\partial_x v \end{cases}$$

$$\begin{cases} \tilde{\psi} = -u\partial_x\tilde{\theta} - v \\ L_u\tilde{\theta} = w - u\partial_x v \end{cases} .$$

Using (20.7) we compute

$$\begin{aligned} &w - u\partial_x v \\ &= \psi - \tilde{\lambda}_u[-\gamma^3\theta_K'(Z) - u^2\gamma^4(x-\xi)\theta_K''(Z)] - u\partial_x\theta + u\tilde{\lambda}_u\partial_x[u\gamma^3(x-\xi)\theta_K'(Z)] \\ &= \psi - u\partial_x\theta - \tilde{\lambda}_u[-\gamma^3\theta_K'(Z) - u^2\gamma^4(x-\xi)\theta_K''(Z)] + \tilde{\lambda}_u[u^2\gamma^3\theta_K'(Z) + u^2\gamma^4(x-\xi)\theta_K''(Z)] \\ &= \psi - u\partial_x\theta + \tilde{\lambda}_u[(1+u^2)\gamma^3\theta_K'(Z)] + 2\tilde{\lambda}_u[u^2\gamma^4(x-\xi)\theta_K''(Z)] . \end{aligned}$$

This yields

$$\begin{cases} \tilde{\psi} = -u\partial_x\tilde{\theta} - v \\ L_u\tilde{\theta} + \tilde{\lambda}_u[(1+u^2)\gamma^3\theta_K'(\gamma(x-\xi))] = \psi - u\partial_x\theta - 2\tilde{\lambda}_u[u^2\gamma^4(x-\xi)\theta_K''(Z)] \end{cases} ,$$

$$\begin{cases} \tilde{\psi} = -u\partial_x\tilde{\theta} - v \\ L_u\left[\frac{1}{(1+u^2)\gamma^3}\tilde{\theta}\right] + \tilde{\lambda}_u[\theta_K'(\gamma(x-\xi))] = \frac{1}{(1+u^2)\gamma^3}\left(\psi - u\partial_x\theta - 2\tilde{\lambda}_u[u^2\gamma^4(x-\xi)\theta_K''(Z)]\right) \end{cases} ,$$

$$\begin{cases} \begin{pmatrix} \left(\frac{1}{(1+u^2)\gamma^3}\tilde{\theta}\right)_{\ker L_u} \\ \tilde{\lambda}_u \end{pmatrix} = \hat{M}_u^{-1}\left[\frac{1}{(1+u^2)\gamma^3}\left(\psi - u\partial_x\theta - 2\tilde{\lambda}_u[u^2\gamma^4(x-\xi)\theta_K''(Z)]\right)\right] \\ \tilde{\psi} = -u\partial_x\tilde{\theta} - \theta + \tilde{\lambda}_u[u\gamma^3(x-\xi)\theta_K'(Z)] \end{cases} .$$

Hence $\left(\left(\frac{1}{(1+u^2)\gamma^3}\tilde{\theta}\right)_{\ker L_u}, \tilde{\lambda}_u\right) \in \mathcal{Y}$ due to Lemma 20.12, since $\frac{1}{(1+u^2)\gamma^3}(\psi - u\partial_x\theta) \in \mathcal{Z}$ and $\tilde{\lambda}_u \in C(I, L^2(\mathbb{R}))$. Thus $(\tilde{\theta}, \tilde{\psi}, \tilde{\lambda}_u) \in Y$.

$\tilde{\mathfrak{M}}$ is one-to-one: Let $(\tilde{\theta}, \tilde{\psi}, \tilde{\lambda}_u) \in Y$ with

$$\tilde{\mathfrak{M}} \begin{pmatrix} \tilde{\theta} \\ \tilde{\psi} \\ \tilde{\lambda}_u \end{pmatrix} = 0 .$$

This implies $\forall u \in I$:

$$\hat{\mathcal{L}}_u \begin{pmatrix} \tilde{\theta}(u) \\ \tilde{\psi}(u) \end{pmatrix} + \tilde{\lambda}_u(u) t_u(u) = 0 .$$

It follows from Corollary 20.18 that

$$\forall u \in I : \quad \tilde{\theta}(u) = 0, \tilde{\psi}(u) = 0, \tilde{\lambda}_u(u) = 0 \qquad \Rightarrow \qquad \tilde{\theta} = 0, \tilde{\psi} = 0, \tilde{\lambda}_u = 0 .$$

The inverse mapping theorem yields the claim. □

Next, we want to show that the operator norm $\|\hat{\mathcal{M}}_u^{-1}\|$ is bounded by a continuous function in $u$. We start with a preparing Lemma.

**Lemma 20.23 (Norm of $\hat{M}_u^{-1}$).** *There exists a constant $c > 0$ such that*

$$\|\hat{M}_u^{-1}\|_{L(L^2(\mathbb{R}^2), H^2(\mathbb{R}^2) \times H^2(\mathbb{R}))} \leq c\gamma(u) \|\hat{M}^{-1}\|_{L(L^2(\mathbb{R}^2), H^2(\mathbb{R}^2) \times H^2(\mathbb{R}))} .$$

**Proof.** Let

$$|v|_{L^2(\mathbb{R}^2)} \leq 1 .$$

Due to Lemma 20.10 there exists

$$\begin{pmatrix} \theta \\ \lambda \end{pmatrix} \in H^2_{u,\perp}(\mathbb{R}^2) \oplus H^2(\mathbb{R}) ,$$

such that

$$v(\xi, x) = \hat{M}_u \begin{pmatrix} \theta \\ \lambda \end{pmatrix} (\xi, x) = (\hat{L}_u \theta)(\xi, x) + \lambda(\xi) \theta_K'(\gamma(x - \xi)) .$$

Thus abusing notation in the same sense as in (20.2), we write

$$\begin{aligned} & v(\xi, \frac{Z}{\gamma} + \xi) \\ = \hat{M} & \begin{pmatrix} \theta(\xi, \frac{Z}{\gamma} + \xi) \\ \lambda(\xi) \end{pmatrix} (\xi, Z) = \hat{L} \left( \theta(\xi, \frac{Z}{\gamma} + \xi) \right) (\xi, Z) + \lambda(\xi) \theta_K'(Z) . \end{aligned}$$

Using the inequality

$$|h(\xi,x)|_{H^2_{\xi,x}(\mathbb{R}^2)} \leq \sqrt{5}\gamma(u)^{\frac{3}{2}} \left| h(\xi, \frac{Z}{\gamma(u)} + \xi) \right|_{H^2_{\xi,Z}(\mathbb{R}^2)}$$

from (20.3) we obtain

$$\begin{aligned}
&\left| \left( \hat{M}_u^{-1} v \right)(\xi,x) \right|_{H^2_{\xi,x}(\mathbb{R}^2)\oplus H^2_\xi(\mathbb{R})} \\
&= \left| \begin{pmatrix} \theta(\xi,x) \\ \lambda(\xi) \end{pmatrix} \right|_{H^2_{\xi,x}(\mathbb{R}^2)\oplus H^2_\xi(\mathbb{R})} \\
&= \ |\theta(\xi,x)|_{H^2_{\xi,x}(\mathbb{R}^2)} + |\lambda(\xi)|_{H^2_\xi(\mathbb{R})} \\
&\leq c\gamma(u)^{\frac{3}{2}} \left| \theta(\xi, \frac{Z}{\gamma(u)} + \xi) \right|_{H^2_{\xi,Z}(\mathbb{R}^2)} + c\gamma(u)^{\frac{3}{2}} \, |\lambda(\xi)|_{H^2_\xi(\mathbb{R})} \\
&= c\gamma(u)^{\frac{3}{2}} \left| \begin{pmatrix} \theta(\xi, \frac{Z}{\gamma} + \xi) \\ \lambda(\xi) \end{pmatrix} \right|_{H^2_{\xi,Z}(\mathbb{R}^2)\oplus H^2_\xi(\mathbb{R})} \\
&= c\gamma(u)^{\frac{3}{2}} \left| \hat{M}^{-1} \left( v(\xi, \frac{Z}{\gamma(u)} + \xi) \right) \right|_{H^2_{\xi,Z}(\mathbb{R}^2)\oplus H^2_\xi(\mathbb{R})} \\
&\leq c\gamma(u)^{\frac{3}{2}} \|\hat{M}^{-1}\| \left| v(\xi, \frac{Z}{\gamma(u)} + \xi) \right|_{L^2_{\xi,Z}(\mathbb{R}^2)} \\
&\leq c\gamma(u) \|\hat{M}^{-1}\| \, |v(\xi,Z)|_{L^2_{\xi,Z}(\mathbb{R}^2)} \ ,
\end{aligned}$$

where $c = \sqrt{5}$ □

**Lemma 20.24 (Norm of $\hat{\mathcal{M}}_u^{-1}$).** *There exists a continuous function $C : (-1,1) \to \mathbb{R}$ such that*

$$\|\hat{\mathcal{M}}_u^{-1}\| \leq C(u)\,.$$

**Proof.** Let

$$\left| \begin{pmatrix} \theta \\ \psi \end{pmatrix} \right|_{H^1(\mathbb{R}^2)\oplus L^2(\mathbb{R}^2)} \leq 1\,.$$

Due to Corollary 20.18 there exists

$$\begin{pmatrix} \tilde{\theta} \\ \tilde{\psi} \\ \tilde{\lambda}_u \end{pmatrix} \in \left[ H^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \right] \cap \ker \mathcal{L}_u^{\perp, L^2 \times L^2} \times H^2(\mathbb{R})\,,$$

such that

$$\begin{pmatrix}\theta\\ \psi\end{pmatrix} = \hat{\mathcal{M}}_u \begin{pmatrix}\tilde{\theta}\\ \tilde{\psi}\\ \tilde{\lambda}_u\end{pmatrix} = \hat{\mathcal{L}}_u \begin{pmatrix}\tilde{\theta}\\ \tilde{\psi}\end{pmatrix} + \tilde{\lambda}_u t_u(u)\,.$$

Now we set

$$\begin{pmatrix}v\\ w\end{pmatrix} := \begin{pmatrix}\theta\\ \psi\end{pmatrix} - \tilde{\lambda}_u t_u(u)\,. \tag{20.8}$$

Let $Z = \gamma(x-\xi)$. We are going to find an expression for $\begin{pmatrix}\tilde{\theta}\\ \tilde{\psi}\\ \tilde{\lambda}_u\end{pmatrix}$.

$$\mathcal{L}_u \begin{pmatrix}\tilde{\theta}\\ \tilde{\psi}\end{pmatrix} = \begin{pmatrix}-u\partial_x\tilde{\theta} - \tilde{\psi}\\ -\partial_x^2\tilde{\theta} + \cos(\theta_K(Z))\tilde{\theta} - u\partial_x\tilde{\psi}\end{pmatrix} = \begin{pmatrix}v\\ w\end{pmatrix}$$

$$\begin{cases}\tilde{\psi} = -u\partial_x\tilde{\theta} - v\\ -\partial_x^2\tilde{\theta} + u^2\partial_x^2\tilde{\theta} + \cos(\theta_K(Z))\tilde{\theta} + u\partial_x v = w\end{cases}$$

$$\begin{cases}\tilde{\psi} = -u\partial_x\tilde{\theta} - v\\ -(1-u^2)\partial_x^2\tilde{\theta} + \cos(\theta_K(Z))\tilde{\theta} = w - u\partial_x v\end{cases}$$

$$\begin{cases}\tilde{\psi} = -u\partial_x\tilde{\theta} - v\\ L_u\tilde{\theta} = w - u\partial_x v\end{cases}.$$

Using (20.8) we compute

$$\begin{aligned}
&w - u\partial_x v\\
&= \psi - \tilde{\lambda}_u[-\gamma^3\theta_K'(Z) - u^2\gamma^4(x-\xi)\theta_K''(Z)] - u\partial_x\theta + u\tilde{\lambda}_u\partial_x[u\gamma^3(x-\xi)\theta_K'(Z)]\\
&= \psi - u\partial_x\theta - \tilde{\lambda}_u[-\gamma^3\theta_K'(Z) - u^2\gamma^4(x-\xi)\theta_K''(Z)] + \tilde{\lambda}_u[u^2\gamma^3\theta_K'(Z) + u^2\gamma^4(x-\xi)\theta_K''(Z)]\\
&= \psi - u\partial_x\theta + \tilde{\lambda}_u[(1+u^2)\gamma^3\theta_K'(Z)] + 2\tilde{\lambda}_u[u^2\gamma^4(x-\xi)\theta_K''(Z)]\,.
\end{aligned}$$

This yields

$$\begin{cases} \tilde{\psi} = -u\partial_x\tilde{\theta} - v \\ L_u\tilde{\theta} + \tilde{\lambda_u}[(1+u^2)\gamma^3\theta'_K(\gamma(x-\xi))] = \psi - u\partial_x\theta - 2\tilde{\lambda_u}[u^2\gamma^4(x-\xi)\theta''_K(Z)] \end{cases} ,$$

$$\begin{cases} \tilde{\psi} = -u\partial_x\tilde{\theta} - v \\ L_u\left[\frac{1}{(1+u^2)\gamma^3}\tilde{\theta}\right] + \tilde{\lambda_u}[\theta'_K(\gamma(x-\xi))] = \frac{1}{(1+u^2)\gamma^3}\left(\psi - u\partial_x\theta - 2\tilde{\lambda_u}[u^2\gamma^4(x-\xi)\theta''_K(Z)]\right) \end{cases} ,$$

$$\begin{cases} \begin{pmatrix} \left(\frac{1}{(1+u^2)\gamma^3}\tilde{\theta}\right)_{\ker L_u} \\ \tilde{\lambda_u} \end{pmatrix} = \hat{M}_u^{-1}\left[\frac{1}{(1+u^2)\gamma^3}\left(\psi - u\partial_x\theta - 2\tilde{\lambda_u}[u^2\gamma^4(x-\xi)\theta''_K(Z)]\right)\right] \\ \tilde{\psi} = -u\partial_x\tilde{\theta} - \theta + \tilde{\lambda_u}[u\gamma^3(x-\xi)\theta'_K(Z)] \end{cases} .$$

Lemma 20.23 yields the claim.

□

Now we are able to prove that $\mathfrak{M} : Y(u_*) \to Z(u_*)$ is invertible for small $u_*$ by using the Neumann Theorem. In order to specify $u_*$ we introduce the following definition.

**Definition 20.25.** *Let $C$ be a specific fixed function from Lemma 20.24. Set*

$$u^* = u^*(\|\hat{M}^{-1}\|) = \sup\{u \in (-1,1) \mid \forall s,t \in \mathbb{R} : \ |s|,|t| \leq |u| : |s|C(t) < 1\} . \tag{20.9}$$

**Corollary 20.26.** *The linear operator*

$$\mathfrak{M} : Y(u_*) \to Z(u_*),$$

$$\begin{pmatrix} \tilde{\theta} \\ \tilde{\psi} \\ \tilde{\lambda}_u \end{pmatrix} \mapsto \mathfrak{M}\begin{pmatrix} \tilde{\theta} \\ \tilde{\psi} \\ \tilde{\lambda}_u \end{pmatrix} ,$$

*given by*

$$\mathfrak{M}\begin{pmatrix} \tilde{\theta} \\ \tilde{\psi} \\ \tilde{\lambda}_u \end{pmatrix}(u) = \hat{\mathcal{N}_u}\begin{pmatrix} \tilde{\theta}(u) \\ \tilde{\psi}(u) \\ \tilde{\lambda}_u(u) \end{pmatrix} ,$$

*is invertible if $u_* < u^*$.*

**Proof.**

$$\tilde{\mathfrak{M}} : Y(u_*) \to Z(u_*),$$

$$\begin{pmatrix} \tilde{\theta} \\ \tilde{\psi} \\ \tilde{\lambda}_u \end{pmatrix} \mapsto \tilde{\mathfrak{M}} \begin{pmatrix} \tilde{\theta} \\ \tilde{\psi} \\ \tilde{\lambda}_u \end{pmatrix}$$

is invertible by Lemma 20.22. Let $C$ be from Definition 20.25. For

$$\begin{pmatrix} \theta \\ \psi \end{pmatrix} \in Z(u_*), \quad \left\| \begin{pmatrix} \theta \\ \psi \end{pmatrix} \right\|_{Z(u_*)} \leq 1,$$

we obtain

$$\begin{aligned} & \|\tilde{\mathfrak{M}}^{-1} \begin{pmatrix} \theta \\ \psi \end{pmatrix} \|_{Y(u_*)} \\ = & \sup_{|u| \leq u_*} |\mathcal{M}_u^{-1} \begin{pmatrix} \theta(u) \\ \psi(u) \end{pmatrix} |_{\bar{Y}} \\ \leq & \sup_{|u| \leq u_*} \|\mathcal{M}_u^{-1}\| | \begin{pmatrix} \theta(u) \\ \psi(u) \end{pmatrix} |_{\bar{Z}} \\ \leq & \sup_{|u| \leq u_*} \|\mathcal{M}_u^{-1}\| \\ \leq & \sup_{|u| \leq u_*} C(u)\,, \end{aligned}$$

i.e.,

$$\|\tilde{\mathfrak{M}}^{-1}\| \leq \sup_{|u| \leq u_*} C(u)\,.$$

Let $\mathfrak{P}$ be given by

$$\mathfrak{P} : Y(u_*) \to Z(u_*),$$

$$\begin{pmatrix} \tilde{\theta} \\ \tilde{\psi} \\ \tilde{\lambda}_u \end{pmatrix} \mapsto \mathfrak{P} \begin{pmatrix} \tilde{\theta} \\ \tilde{\psi} \\ \tilde{\lambda}_u \end{pmatrix}$$

where

$$\mathfrak{P} \begin{pmatrix} \tilde{\theta} \\ \tilde{\psi} \\ \tilde{\lambda}_u \end{pmatrix} (u) = u \begin{pmatrix} \partial_\xi \tilde{\theta}(u) \\ \partial_\xi \tilde{\psi}(u) \end{pmatrix} + u \begin{pmatrix} \partial_x \tilde{\theta}(u) \\ \partial_x \tilde{\psi}(u) \end{pmatrix} .$$

For

$$\begin{pmatrix}\tilde{\theta}\\ \tilde{\psi}\\ \tilde{\lambda}_u\end{pmatrix} \in Y(u_*), \quad \left\| \begin{pmatrix}\tilde{\theta}\\ \tilde{\psi}\\ \tilde{\lambda}_u\end{pmatrix} \right\|_{Y(u_*)} \leq 1,$$

we obtain

$$\begin{aligned} &\|\mathfrak{P} \begin{pmatrix}\tilde{\theta}\\ \tilde{\psi}\\ \tilde{\lambda}_u\end{pmatrix} \|_{Z(u_*)} \\ &= \sup_{|u|\leq u_*} \left| u \begin{pmatrix} \partial_\xi \tilde{\theta}(u) \\ \partial_\xi \tilde{\psi}(u) \end{pmatrix} + u \begin{pmatrix} \partial_x \tilde{\theta}(u) \\ \partial_x \tilde{\psi}(u) \end{pmatrix} \right|_{\bar{Z}} \\ &\leq \sup_{|u|\leq u_*} |u|, \end{aligned}$$

i.e.,

$$\|\mathfrak{P}\| \leq \sup_{|u|\leq u_*} |u| \, .$$

Thus

$$\begin{aligned} &\|\mathfrak{P}\| \|\tilde{\mathfrak{M}}^{-1}\| \\ &\leq \sup_{|u|\leq u_*} |u| \sup_{|u|\leq u_*} C(u) \\ &< 1 \, , \end{aligned}$$

due to (20.9), since $u_* < u^*$. Hence $\mathfrak{P} + \tilde{\mathfrak{M}} = \mathfrak{M}$ is invertible by Neumann Theorem. □

### 20.1.3 Inverse Operator in Spaces of Higher Regularity

Let $n \in \mathbb{N}$. In this section we want to prove invertibility of an analogous operator to $\mathfrak{M}$ in spaces of higher regularity. We define analogous spaces to $Y(u_*), Z(u_*)$ from Definition 20.21 but with higher regularity in $u$.

**Definition 20.27.**

*(a)* $\widetilde{Y}_n = \widetilde{Y}_n(u_*)$

$$= \Bigg\{ y = (\theta, \psi, \lambda_u) \in C^n(I(u_*), \bar{Y}) : \|y\|_{\widetilde{Y}_n(u_*)} < \infty; \ \forall\ u \in I(u_*), \ \forall\ \lambda \in H^2(\mathbb{R}) :$$

$$\left\langle \begin{pmatrix} \theta(u)(\xi, x) \\ \psi(u)(\xi, x) \end{pmatrix}, \lambda(\xi) \begin{pmatrix} \theta'_K(\gamma(x-\xi)) \\ -u\gamma\theta''_K(\gamma(x-\xi)) \end{pmatrix} \right\rangle_{L^2_{\xi,x}(\mathbb{R}^2) \oplus L^2_{\xi,x}(\mathbb{R}^2)} = 0 \Bigg\},$$

*with the finite norm*

$$\|y\|_{\widetilde{Y}_n(u_*)} = \sup_{u \in I(u_*)} \left( \sum_{i=0}^{n} |\partial_u^i y(u)|_{\bar{Y}} \right).$$

*(b)* $\widetilde{Z}_n = \widetilde{Z}_n(u_*) = \left\{ z = (v, w) \in C^n(I(u_*), \bar{Z}) : \|z\|_{\widetilde{Z}_n(u_*)} < \infty \right\}$ *with the finite norm*

$$\|z\|_{\widetilde{Z}_n(u_*)} = \sup_{u \in I(u_*)} \left( \sum_{i=0}^{n} |\partial_u^i z(u)|_{\bar{Z}} \right).$$

The following lemma provides invertibility of an analogous operator to $\mathfrak{M}$ in spaces of higher regularity in $u$.

**Lemma 20.28.** *The linear operator*

$$\widetilde{\mathfrak{M}}_n : \widetilde{Y}_n(u_*) \to \widetilde{Z}_n(u_*),$$
$$\begin{pmatrix} \tilde{\theta} \\ \tilde{\psi} \\ \tilde{\lambda}_u \end{pmatrix} \mapsto \widetilde{\mathfrak{M}}_n \begin{pmatrix} \tilde{\theta} \\ \tilde{\psi} \\ \tilde{\lambda}_u \end{pmatrix},$$

*given by*

$$\widetilde{\mathfrak{M}}_n \begin{pmatrix} \tilde{\theta} \\ \tilde{\psi} \\ \tilde{\lambda}_u \end{pmatrix} (u) = \hat{\mathcal{N}}_u \begin{pmatrix} \tilde{\theta}(u) \\ \tilde{\psi}(u) \\ \tilde{\lambda}_u(u) \end{pmatrix},$$

*is invertible if $u_* < u^*$.*

**Proof.** We skip $u_*$ in the notation. $\widetilde{\mathfrak{M}}_n$ is well defined: Let $\begin{pmatrix} \theta \\ \psi \\ \lambda \end{pmatrix} \in \widetilde{Y}_n$. $\widetilde{\mathfrak{M}}_n \begin{pmatrix} \theta \\ \psi \\ \lambda \end{pmatrix}$ is differentiable, since

$$\frac{1}{h} \left| \widetilde{\mathfrak{M}}_n \begin{pmatrix} \theta \\ \psi \\ \lambda \end{pmatrix} (u+h) - \widetilde{\mathfrak{M}}_n \begin{pmatrix} \theta \\ \psi \\ \lambda \end{pmatrix} (u) - \partial_u \left[ \mathcal{N}_u \begin{pmatrix} \theta(u) \\ \psi(u) \\ \lambda(u) \end{pmatrix} \right] h \right|_{H^1(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)}$$

$$
\begin{aligned}
&= \frac{1}{h} \left| \widetilde{\mathfrak{M}}_n \begin{pmatrix} \theta \\ \psi \\ \lambda \end{pmatrix} (u+h) - \widetilde{\mathfrak{M}}_n \begin{pmatrix} \theta \\ \psi \\ \lambda \end{pmatrix} (u) - \partial_u[\mathcal{K}_u] \begin{pmatrix} \theta(u) \\ \psi(u) \end{pmatrix} h - \mathcal{K}_u \partial_u \begin{pmatrix} \theta(u) \\ \psi(u) \end{pmatrix} h \right. \\
&\qquad \left. - \partial_u[\lambda(u) t_u(u)] h \right|_{H^1(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)} \\
&\leq \frac{1}{h} \left| \hat{\mathcal{K}}_{u+h} \begin{pmatrix} \theta(u+h) \\ \psi(u+h) \end{pmatrix} - \hat{\mathcal{K}}_u \begin{pmatrix} \theta(u) \\ \psi(u) \end{pmatrix} - \partial_u[\mathcal{K}_u] \begin{pmatrix} \theta(u) \\ \psi(u) \end{pmatrix} h - \mathcal{K}_u \partial_u \begin{pmatrix} \theta(u) \\ \psi(u) \end{pmatrix} h \right|_{H^1(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)} \\
&\quad + \frac{1}{h} \left| \lambda(u+h) t_u(u+h) - \lambda(u) t_u(u) - \partial_u[\lambda(u) t_u(u)] h \right|_{H^1(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)} \\
&\leq \frac{1}{h} \left| \hat{\mathcal{K}}_{u+h} \begin{pmatrix} \theta(u+h) \\ \psi(u+h) \end{pmatrix} - \mathcal{K}_u \begin{pmatrix} \theta(u+h) \\ \psi(u+h) \end{pmatrix} - \partial_u[\mathcal{K}_u] \begin{pmatrix} \theta(u) \\ \psi(u) \end{pmatrix} h + \mathcal{K}_u \begin{pmatrix} \theta(u+h) \\ \psi(u+h) \end{pmatrix} \right. \\
&\quad \left. - \hat{\mathcal{K}}_u \begin{pmatrix} \theta(u) \\ \psi(u) \end{pmatrix} - \mathcal{K}_u \partial_u \begin{pmatrix} \theta(u) \\ \psi(u) \end{pmatrix} h \right|_{H^1(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)} \\
&\quad + \frac{1}{h} \left| \lambda(u+h) t_u(u+h) - \lambda(u) t_u(u) - \partial_u[\lambda(u) t_u(u)] h \right|_{H^1(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)} \\
&\leq \frac{1}{h} \left| [\mathcal{K}_{u+h} - \mathcal{K}_u] \left( \begin{pmatrix} \theta(u+h) \\ \psi(u+h) \end{pmatrix} - \begin{pmatrix} \theta(u) \\ \psi(u) \end{pmatrix} \right) \right|_{H^1(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)} \\
&\quad + \frac{1}{h} \left| [\mathcal{K}_{u+h} - \mathcal{K}_u] \begin{pmatrix} \theta(u) \\ \psi(u) \end{pmatrix} - \partial_u[\mathcal{K}_u] \begin{pmatrix} \theta(u) \\ \psi(u) \end{pmatrix} h \right|_{H^1(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)} \\
&\quad + \frac{1}{h} \left| \mathcal{K}_u \begin{pmatrix} \theta(u+h) \\ \psi(u+h) \end{pmatrix} - \mathcal{K}_u \begin{pmatrix} \theta(u) \\ \psi(u) \end{pmatrix} - \mathcal{K}_u \partial_u \begin{pmatrix} \theta(u) \\ \psi(u) \end{pmatrix} h \right|_{H^1(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)} \\
&\quad + \frac{1}{h} \left| \lambda(u+h) t_u(u+h) - \lambda(u) t_u(u) - \partial_u[\lambda(u) t_u(u)] h \right|_{H^1(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)} \\
&\overset{h \to 0}{\to} 0 \,.
\end{aligned}
$$

Higher regularity and the continuity of the derivatives can be shown in a similar way. $\widetilde{\mathfrak{M}}_n$ is one-to-one due to Corollary 20.26.

$\widetilde{\mathfrak{M}}_n$ is onto: Let $\begin{pmatrix} v \\ w \end{pmatrix} \in \widetilde{Z}_n$. Due to Corollary 20.26 there exists $\begin{pmatrix} \theta \\ \psi \\ \lambda \end{pmatrix} \in Y$ such that

$$\begin{pmatrix} \theta(u) \\ \psi(u) \\ \lambda(u) \end{pmatrix} = \hat{\mathcal{N}}_u^{-1} \begin{pmatrix} v(u) \\ w(u) \end{pmatrix} .$$

We are going to show that $\begin{pmatrix} \theta \\ \psi \\ \lambda \end{pmatrix} \in \widetilde{Y}_n$. $\ker \mathcal{M}_u$ is a closed subspace of the Hilbert space $L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times L^2(\mathbb{R})$ with the standard inner product. We denote by

$$P_u : L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times L^2(\mathbb{R}) \to \ker \mathcal{M}_u ,$$

$$\begin{pmatrix} \theta \\ \psi \\ \lambda \end{pmatrix} \mapsto P_u \begin{pmatrix} \theta \\ \psi \\ \lambda \end{pmatrix} = \begin{bmatrix} \theta \\ \psi \\ \lambda \end{bmatrix}_{\ker \mathcal{M}_u}$$

the orthogonal projection on $\ker \mathcal{M}_u$. Let

$$R_u : L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times L^2(\mathbb{R}) \to (\ker \mathcal{M}_u)^{\perp} ,$$

$$\begin{pmatrix} \theta \\ \psi \\ \lambda \end{pmatrix} \mapsto (Id - P_u) \begin{pmatrix} \theta \\ \psi \\ \lambda \end{pmatrix} = \begin{bmatrix} \theta \\ \psi \\ \lambda \end{bmatrix}_{(\ker \mathcal{M}_u)^{\perp}} .$$

Recall that

$$\ker \mathcal{M}_u$$
$$= \left\{ \begin{pmatrix} \theta \\ \psi \\ \lambda \end{pmatrix} \in H^2(\mathbb{R}^2) \oplus H^1(\mathbb{R}^2) \oplus H^2(\mathbb{R}) \;\middle|\; \begin{pmatrix} \theta(\xi,x) \\ \psi(\xi,x) \\ \lambda(\xi) \end{pmatrix} = \mu(\xi) \begin{pmatrix} t_\xi(\xi,u,x) \\ 0 \end{pmatrix} ,\ \mu \in H^2(\mathbb{R}) \right\} ,$$

due to Corollary 20.20. An orthonormal basis of $\ker \mathcal{M}_u$ can be constructed by choosing an orthonormal basis of $L^2(\mathbb{R})$ and normalizing the vectors. Thus we are able to write an orthonormal basis of

$$\ker \mathcal{M}_u \subset L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}) ,$$

in the form

$$S_u$$
$$= \left\{ \begin{pmatrix} \theta \\ \psi \\ \lambda \end{pmatrix} \in L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}) \;\middle|\; \begin{pmatrix} \theta(\xi,x) \\ \psi(\xi,x) \\ \lambda(\xi) \end{pmatrix} = \begin{pmatrix} e_k(\xi,u) t_\xi(\xi,u,x) \\ 0 \end{pmatrix} ,\ k \in \mathbb{N} \right\}$$
$$\subset L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}) ,$$

where $e_k(\cdot, u)$ depends smoothly on $u$. Therefore it holds that

$$\overline{\operatorname{span}(S_u)} = \ker \mathcal{M}_u \,.$$

Notice that $P_u$ is given by

$$P_u \begin{pmatrix} \theta \\ \psi \\ \lambda_u \end{pmatrix} (\xi, x) = \sum_{k=1}^{\infty} \left\langle \begin{pmatrix} \theta(\xi, x) \\ \psi(\xi, x) \end{pmatrix}, e_k(\xi, u) t_\xi(\xi, u, x) \right\rangle_{L^2_{\xi,x}(\mathbb{R}^2) \oplus L^2_{\xi,x}(\mathbb{R}^2)} \begin{pmatrix} e_k(\xi, u) t_\xi(\xi, u, x) \\ 0 \end{pmatrix} .$$

and

$$\begin{aligned}
&\frac{1}{h} \Bigg| \begin{pmatrix} \theta(u+h) \\ \psi(u+h) \\ \lambda(u+h) \end{pmatrix} - \begin{pmatrix} \theta(u) \\ \psi(u) \\ \lambda(u) \end{pmatrix} - \hat{\mathcal{N}}_u^{-1} \Big[ \partial_u \begin{pmatrix} v(u) \\ w(u) \end{pmatrix} - (\partial_u \mathcal{N}_u) \begin{pmatrix} \theta(u) \\ \psi(u) \\ \lambda(u) \end{pmatrix} \Big] h \\
&\quad + \sum_{k=1}^{\infty} \begin{pmatrix} e_k(u) t_\xi(u) \left\langle \begin{pmatrix} \theta(u) \\ \psi(u) \end{pmatrix}, \partial_u [e_k(u) t_\xi(u)] \right\rangle_{L^2(\mathbb{R}^2)} \\ 0 \end{pmatrix} h \Bigg|_{H^2(\mathbb{R}^2) \oplus H^1(\mathbb{R}^2) \oplus H^2(\mathbb{R})} \\
&\leq \frac{1}{h} \Bigg| \hat{\mathcal{N}}_u^{-1} \hat{\mathcal{N}}_u \left[ \begin{pmatrix} \theta(u+h) \\ \psi(u+h) \\ \lambda(u+h) \end{pmatrix} - \begin{pmatrix} \theta(u) \\ \psi(u) \\ \lambda(u) \end{pmatrix} \right]_{(\ker \mathcal{M}_u)^\perp} \\
&\quad - \left[ \hat{\mathcal{N}}_u^{-1} [\partial_u \begin{pmatrix} v(u) \\ w(u) \end{pmatrix} - (\partial_u \mathcal{N}_u) \begin{pmatrix} \theta(u) \\ \psi(u) \\ \lambda(u) \end{pmatrix} ] \right]_{(\ker \mathcal{M}_u)^\perp} h \Bigg|_{H^2(\mathbb{R}^2) \oplus H^1(\mathbb{R}^2) \oplus H^2(\mathbb{R})} \\
&\quad + \frac{1}{h} \Bigg|\Bigg| \left[ \begin{pmatrix} \theta(u+h) \\ \psi(u+h) \\ \lambda(u+h) \end{pmatrix} - \begin{pmatrix} \theta(u) \\ \psi(u) \\ \lambda(u) \end{pmatrix} - \hat{\mathcal{N}}_u^{-1} [\partial_u \begin{pmatrix} v(u) \\ w(u) \end{pmatrix} - (\partial_u \mathcal{N}_u) \begin{pmatrix} \theta(u) \\ \psi(u) \\ \lambda(u) \end{pmatrix} ] h \right]_{\ker \mathcal{M}_u} \\
&\quad + \sum_{k=1}^{\infty} \begin{pmatrix} e_k(u) t_\xi(u) \left\langle \begin{pmatrix} \theta(u) \\ \psi(u) \end{pmatrix}, \partial_u [e_k(u) t_\xi(u)] \right\rangle_{L^2(\mathbb{R}^2)} \\ 0 \end{pmatrix} h \Bigg|_{H^2(\mathbb{R}^2) \oplus H^1(\mathbb{R}^2) \oplus H^2(\mathbb{R})}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{h}\|\hat{\mathcal{N}}_u^{-1}\| \Bigg| \hat{\mathcal{N}}_u \left[ \begin{pmatrix} \theta(u+h) \\ \psi(u+h) \\ \lambda(u+h) \end{pmatrix} - \begin{pmatrix} \theta(u) \\ \psi(u) \\ \lambda(u) \end{pmatrix} \right]_{(\ker \mathcal{M}_u)^{\perp}} \\
&\quad - \left[ \partial_u \begin{pmatrix} v(u) \\ w(u) \end{pmatrix} - (\partial_u \mathcal{N}_u) \begin{pmatrix} \theta(u) \\ \psi(u) \\ \lambda(u) \end{pmatrix} \right]_{(\ker \mathcal{M}_u)^{\perp}} h \Bigg|_{H^1(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)} \\
&\quad + \frac{1}{h} \Bigg\| \left[ \begin{pmatrix} \theta(u+h) \\ \psi(u+h) \\ \lambda(u+h) \end{pmatrix} - \begin{pmatrix} \theta(u) \\ \psi(u) \\ \lambda(u) \end{pmatrix} - \hat{\mathcal{N}}_u^{-1}[\partial_u \begin{pmatrix} v(u) \\ w(u) \end{pmatrix} - (\partial_u \mathcal{N}_u) \begin{pmatrix} \theta(u) \\ \psi(u) \\ \lambda(u) \end{pmatrix}]h \right]_{\ker \mathcal{M}_u} \\
&\quad + \sum_{k=1}^{\infty} \begin{pmatrix} e_k(u) t_\xi(u) \left\langle \begin{pmatrix} \theta(u) \\ \psi(u) \end{pmatrix}, \partial_u [e_k(u) t_\xi(u)] \right\rangle_{L^2(\mathbb{R}^2)} \\ 0 \end{pmatrix} h \Bigg|_{H^2(\mathbb{R}^2) \oplus H^1(\mathbb{R}^2) \oplus H^2(\mathbb{R})} .
\end{aligned}$$

We consider the terms separately:

$$\begin{aligned}
&\frac{1}{h}\|\hat{\mathcal{N}}_u^{-1}\| \Bigg| \hat{\mathcal{N}}_u \left[ \begin{pmatrix} \theta(u+h) \\ \psi(u+h) \\ \lambda(u+h) \end{pmatrix} - \begin{pmatrix} \theta(u) \\ \psi(u) \\ \lambda(u) \end{pmatrix} \right]_{(\ker M_u)^{\perp}} \\
&\quad - \left[ \partial_u \begin{pmatrix} v(u) \\ w(u) \end{pmatrix} - (\partial_u \mathcal{N}_u) \begin{pmatrix} \theta(u) \\ \psi(u) \\ \lambda(u) \end{pmatrix} \right]_{(\ker M_u)^{\perp}} h \Bigg|_{H^1(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)} \\
&\leq \frac{1}{h}\|\hat{\mathcal{N}}_u^{-1}\| \Bigg| \hat{\mathcal{N}}_u \begin{bmatrix} \theta(u+h) \\ \psi(u+h) \\ \lambda(u+h) \end{bmatrix}_{(\ker M_u)^{\perp}} - \hat{\mathcal{N}}_u \begin{pmatrix} \theta(u) \\ \psi(u) \\ \lambda(u) \end{pmatrix} \\
&\quad - \left[ \partial_u \begin{pmatrix} v(u) \\ w(u) \end{pmatrix} - (\partial_u \mathcal{N}_u) \begin{pmatrix} \theta(u) \\ \psi(u) \\ \lambda(u) \end{pmatrix} \right]_{(\ker M_u)^{\perp}} h \Bigg|_{H^1(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{h}\|\hat{\mathcal{N}}_u^{-1}\| \Bigg| \mathcal{N}_u \Big[ \begin{pmatrix}\theta(u+h)\\ \psi(u+h)\\ \lambda(u+h)\end{pmatrix} - \begin{pmatrix}\theta(u)\\ \psi(u)\\ \lambda(u)\end{pmatrix} \Big] \\
&\quad - \mathcal{N}_{u+h} \Big[ \begin{pmatrix}\theta(u+h)\\ \psi(u+h)\\ \lambda(u+h)\end{pmatrix} - \begin{pmatrix}\theta(u)\\ \psi(u)\\ \lambda(u)\end{pmatrix} \Big] \Bigg|_{H^1(\mathbb{R}^2)\oplus L^2(\mathbb{R}^2)} \\
&\quad + \frac{1}{h}\|\hat{\mathcal{N}}_u^{-1}\| \Bigg| \mathcal{N}_{u+h} \Big[ \begin{pmatrix}\theta(u+h)\\ \psi(u+h)\\ \lambda(u+h)\end{pmatrix} - \begin{pmatrix}\theta(u)\\ \psi(u)\\ \lambda(u)\end{pmatrix} \Big] \\
&\quad - [\partial_u \begin{pmatrix}v(u)\\ w(u)\end{pmatrix} - (\partial_u \mathcal{N}_u) \begin{pmatrix}\theta(u)\\ \psi(u)\\ \lambda(u)\end{pmatrix} ]h \Bigg|_{H^1(\mathbb{R}^2)\oplus L^2(\mathbb{R}^2)} \\
&= \frac{1}{h}\|\hat{\mathcal{N}}_u^{-1}\| \Bigg| \Big(\mathcal{N}_u - \mathcal{N}_{u+h}\Big) \Big[ \begin{pmatrix}\theta(u+h)\\ \psi(u+h)\\ \lambda(u+h)\end{pmatrix} - \begin{pmatrix}\theta(u)\\ \psi(u)\\ \lambda(u)\end{pmatrix} \Big] \Bigg|_{H^1(\mathbb{R}^2)\oplus L^2(\mathbb{R}^2)} \\
&\quad + \frac{1}{h}\|\hat{\mathcal{N}}_u^{-1}\| \Bigg| \mathcal{N}_{u+h} \begin{pmatrix}\theta(u+h)\\ \psi(u+h)\\ \lambda(u+h)\end{pmatrix} - \mathcal{N}_u \begin{pmatrix}\theta(u)\\ \psi(u)\\ \lambda(u)\end{pmatrix} - \mathcal{N}_{u+h} \begin{pmatrix}\theta(u)\\ \psi(u)\\ \lambda(u)\end{pmatrix} \\
&\quad + \mathcal{N}_u \begin{pmatrix}\theta(u)\\ \psi(u)\\ \lambda(u)\end{pmatrix} - [\partial_u \begin{pmatrix}v(u)\\ w(u)\end{pmatrix} - (\partial_u \mathcal{N}_u) \begin{pmatrix}\theta(u)\\ \psi(u)\\ \lambda(u)\end{pmatrix} ]h \Bigg|_{H^1(\mathbb{R}^2)\oplus L^2(\mathbb{R}^2)} \\
&= \frac{1}{h}\|\hat{\mathcal{N}}_u^{-1}\| \Bigg| \Big(\mathcal{N}_u - \mathcal{N}_{u+h}\Big) \Big[ \begin{pmatrix}\theta(u+h)\\ \psi(u+h)\\ \lambda(u+h)\end{pmatrix} - \begin{pmatrix}\theta(u)\\ \psi(u)\\ \lambda(u)\end{pmatrix} \Big] \Bigg|_{H^1(\mathbb{R}^2)\oplus L^2(\mathbb{R}^2)} \\
&\quad + \frac{1}{h}\|\hat{\mathcal{N}}_u^{-1}\| \Bigg| \begin{pmatrix}v(u+h)\\ w(u+h)\end{pmatrix} - \begin{pmatrix}v(u)\\ w(u)\end{pmatrix} - \Big(\mathcal{N}_{u+h} - \mathcal{N}_u\Big) \begin{pmatrix}\theta(u)\\ \psi(u)\\ \lambda(u)\end{pmatrix} \\
&\quad - [\partial_u \begin{pmatrix}v(u)\\ w(u)\end{pmatrix} - (\partial_u \mathcal{N}_u) \begin{pmatrix}\theta(u)\\ \psi(u)\\ \lambda(u)\end{pmatrix} ]h \Bigg|_{H^1(\mathbb{R}^2)\oplus L^2(\mathbb{R}^2)} \\
&\overset{h\to 0}{\to} 0\,.
\end{aligned}
$$

We use the notation $e_k(\xi,u)t_\xi(\xi,u,x) = b_k(u)$ for the next term.

$$
\begin{aligned}
&\frac{1}{h}\Bigg|\Bigg[\begin{pmatrix}\theta(u+h)\\ \psi(u+h)\\ \lambda(u+h)\end{pmatrix} - \begin{pmatrix}\theta(u)\\ \psi(u)\\ \lambda(u)\end{pmatrix} - \hat{\mathcal{N}}_u^{-1}[\partial_u \begin{pmatrix}v(u)\\ w(u)\end{pmatrix} - (\partial_u \mathcal{N}_u)\begin{pmatrix}\theta(u)\\ \psi(u)\\ \lambda(u)\end{pmatrix}]\Bigg]_{\ker\mathcal{M}_u}\\
&\quad+\sum_{k=1}^{\infty}\begin{pmatrix} b_k(u)\left\langle\begin{pmatrix}\theta(u)\\ \psi(u)\end{pmatrix}, \partial_u b_k(u)\right\rangle_{L^2(\mathbb{R}^2)\oplus L^2(\mathbb{R}^2)}\\ 0\end{pmatrix} h\Bigg|_{H^2(\mathbb{R}^2)\oplus H^1(\mathbb{R}^2)\oplus H^2(\mathbb{R})}\\
&=\frac{1}{h}\Bigg|\Bigg[\begin{pmatrix}\theta(u+h)\\ \psi(u+h)\\ \lambda(u+h)\end{pmatrix}\Bigg]_{\ker\mathcal{M}_u}\\
&\quad+\sum_{k=1}^{\infty}\begin{pmatrix} b_k(u)\left\langle\begin{pmatrix}\theta(u)\\ \psi(u)\end{pmatrix}, \partial_u b_k(u)\right\rangle_{L^2(\mathbb{R}^2)\oplus L^2(\mathbb{R}^2)}\\ 0\end{pmatrix} h\Bigg|_{H^2(\mathbb{R}^2)\oplus H^1(\mathbb{R}^2)\oplus H^2(\mathbb{R})}\\
&=\frac{1}{h}\Bigg|\sum_{k=1}^{\infty}\begin{pmatrix} b_k(u)\left\langle\begin{pmatrix}\theta(u+h)-\theta(u)+\theta(u)\\ \psi(u+h)-\psi(u)+\psi(u)\end{pmatrix}, b_k(u)-b_k(u+h)\right\rangle_{L^2(\mathbb{R}^2)\oplus L^2(\mathbb{R}^2)}\\ 0\end{pmatrix}\\
&\quad+\sum_{k=1}^{\infty}\begin{pmatrix} b_k(u)\left\langle\begin{pmatrix}\theta(u)\\ \psi(u)\end{pmatrix}, \partial_u b_k(u)\right\rangle_{L^2(\mathbb{R}^2)\oplus L^2(\mathbb{R}^2)}\\ 0\end{pmatrix} h\Bigg|_{H^2(\mathbb{R}^2)\oplus H^1(\mathbb{R}^2)\oplus H^2(\mathbb{R})}\\
&=\Bigg|\sum_{k=1}^{\infty}\begin{pmatrix} b_k(u)\left\langle\begin{pmatrix}\theta(u+h)-\theta(u)\\ \psi(u+h)-\psi(u)\end{pmatrix}, \tfrac{1}{h}\Big[b_k(u)-b_k(u+h)\Big]\right\rangle_{L^2(\mathbb{R}^2)\oplus L^2(\mathbb{R}^2)}\\ 0\end{pmatrix}\\
&\quad+\sum_{k=1}^{\infty}\begin{pmatrix} b_k(u)\left\langle\begin{pmatrix}\theta(u)\\ \psi(u)\end{pmatrix}, \partial_u b_k(u)\right\rangle_{L^2(\mathbb{R}^2)\oplus L^2(\mathbb{R}^2)}\\ 0\end{pmatrix}
\end{aligned}
$$

$$
+ \sum_{k=1}^{\infty} \left( \begin{matrix} b_k(u) \left\langle \begin{pmatrix} \theta(u) \\ \psi(u) \end{pmatrix}, \tfrac{1}{h}\Big[b_k(u) - b_k(u+h)\Big] \right\rangle_{L^2(\mathbb{R}^2)\oplus L^2(\mathbb{R}^2)} \\ 0 \end{matrix} \right) \Bigg\|_{H^2(\mathbb{R}^2)\oplus H^1(\mathbb{R}^2)\oplus H^2(\mathbb{R})}
$$

$$
\leq \sum_{k=1}^{\infty} \left| \left\langle \begin{pmatrix} \theta(u+h) - \theta(u) \\ \psi(u+h) - \psi(u) \end{pmatrix}, \frac{1}{h}\Big[b_k(u) - b_k(u+h)\Big] \right\rangle_{L^2(\mathbb{R}^2)\oplus L^2(\mathbb{R}^2)} \right|
$$

$$
\cdot \left\| \begin{pmatrix} b_k(u) \\ 0 \end{pmatrix} \right\|_{H^2(\mathbb{R}^2)\oplus H^1(\mathbb{R}^2)\oplus H^2(\mathbb{R})}
$$

$$
+ \sum_{k=1}^{\infty} \left| \left\langle \begin{pmatrix} \theta(u) \\ \psi(u) \end{pmatrix}, \partial_u \Big[b_k(u)\Big] + \frac{1}{h}\Big[b_k(u) - b_k(u+h)\Big] \right\rangle_{L^2(\mathbb{R}^2)\oplus L^2(\mathbb{R}^2)} \right|
$$

$$
\cdot \left\| \begin{pmatrix} b_k(u) \\ 0 \end{pmatrix} \right\|_{H^2(\mathbb{R}^2)\oplus H^1(\mathbb{R}^2)\oplus H^2(\mathbb{R})}
$$

$$
\overset{h\to 0}{\to} 0\,.
$$

Higher regularity and the continuity of the derivatives can be shown in a similar way. The linear operator $\widetilde{\mathfrak{M}}_n$ is obviously bounded. The inverse mapping theorem yields the claim. □

In the following, we want to show that an analogous operator to $\mathfrak{M}$ is invertible in spaces of higher regularity in all variables $u$,$\xi$ and $x$. Thus we define analogous spaces to those from Section 20.1.1 with higher regularity and and analogous operators.

**Definition 20.29.**

*(a)* $H^3_\perp(\mathbb{R}^2) := \{\theta \in H^3(\mathbb{R}^2) \mid \forall \lambda \in H^2(\mathbb{R}) : \ \langle \theta(\xi, Z), \lambda(\xi)\theta_K'(Z)\rangle_{L^2(\mathbb{R}^2)} = 0\}$.

*(b)* $\underline{\hat{L}} : H^3_\perp(\mathbb{R}^2) \subset L^2(\mathbb{R}^2) \to H^1(\mathbb{R}^2)$, *where*

$$
(\underline{\hat{L}}\theta)(\xi, Z) = -\partial_Z^2\theta(\xi, Z) + \cos(\theta_K(Z))\theta(\xi, Z).
$$

*(c)* $\underline{\hat{M}} : H^3_\perp(\mathbb{R}^2) \oplus H^2(\mathbb{R}) \to H^1(\mathbb{R}^2)$, *where*

$$
\left(\underline{\hat{M}}\begin{pmatrix} \theta \\ \lambda \end{pmatrix}\right)(\xi, Z) = (\underline{\hat{L}}^{\alpha}\theta)(\xi, Z) + \lambda(\xi)\theta_K'(Z).
$$

*(d)* $\underline{\hat{\mathcal{L}}_u} : \left[H^3(\mathbb{R}^2) \oplus H^2(\mathbb{R}^2)\right] \cap \ker \mathcal{L}_u^{\perp, L^2 \oplus L^2} \to H^2(\mathbb{R}^2) \oplus H^1(\mathbb{R}^2)$, *where*

$$\left(\underline{\hat{\mathcal{L}}_u} \begin{pmatrix} \theta \\ \psi \end{pmatrix}\right)(\xi, x) = \begin{pmatrix} -u\partial_x \theta(\xi, x) - \psi(\xi, x) \\ -\partial_x^2 \theta(\xi, x) + \cos(\theta_K(\gamma(x-\xi)))\theta(\xi, x) - u\partial_x \psi(\xi, x) \end{pmatrix}.$$

*(e)* $\underline{\hat{\mathcal{M}}_u} : \left[H^3(\mathbb{R}^2) \oplus H^2(\mathbb{R}^2)\right] \cap \ker \mathcal{L}_u^{\perp, L^2 \oplus L^2} \oplus H^2(\mathbb{R}) \to H^2(\mathbb{R}^2) \oplus H^1(\mathbb{R}^2)$, *where*

$$\left(\underline{\hat{\mathcal{M}}_u} \begin{pmatrix} \theta \\ \psi \\ \lambda \end{pmatrix}\right)(\xi, x) = \left(\underline{\hat{\mathcal{L}}_u} \begin{pmatrix} \theta \\ \psi \end{pmatrix}\right)(\xi, x) + \lambda(\xi) t_u(\xi, u, x).$$

**Definition 20.30.** *Let* $\underline{C} : (-1, 1) \to \mathbb{R}$ *be a fixed continuous function such that*

$$\left\| \left[\underline{\hat{\mathcal{M}}_u}\right]^{-1} \right\| \le \underline{C}(u).$$

*Set*

$$\underline{u}^* = \underline{u}^*(\|\underline{\hat{M}}^{-1}\|) = \sup\{u \in (-1, 1) \mid \forall s, t \in \mathbb{R} : \ |s|, |t| \le |u| : |s|\underline{C}(t) < 1\}.$$

**Remark 20.31.** Using Remark 20.11, (20.4) and Remark 20.19 we obtain Lemma 20.12, Lemma 20.22, Lemma 20.23, Lemma 20.24 with higher regularity in $(\xi, x)$, which ensures the existence of the function $\underline{C}$ in Definition 20.30.

**Definition 20.32.** *(a)* $\bar{Y}' = H^3(\mathbb{R}^2) \oplus H^2(\mathbb{R}^2) \oplus H^2(\mathbb{R})$ *with the finite norm*

$$|y|_{\bar{Y}'} = |\theta|_{H^3(\mathbb{R}^2)} + |\psi|_{H^2(\mathbb{R}^2)} + |\lambda|_{H^2(\mathbb{R})}.$$

*(b)* $\bar{Z}' = H^2(\mathbb{R}^2) \oplus H^1(\mathbb{R}^2)$ *with the finite norm*

$$|z|_{\bar{Z}'} = |v|_{H^2(\mathbb{R}^2)} + |w|_{H^1(\mathbb{R}^2)}.$$

*(c)*
$$\begin{aligned} Y_n &= Y_n(u_*) \\ &= \Bigg\{ y = (\theta, \psi, \lambda_u) \in C^n(I(u_*), \bar{Y}') : \|y\|_{Y_n(u_*)} < \infty; \ \forall\ u \in I(u_*), \ \forall\ \lambda \in H^2(\mathbb{R}) : \\ &\quad \left\langle \begin{pmatrix} \theta(u)(\xi, x) \\ \psi(u)(\xi, x) \end{pmatrix}, \lambda(\xi) \begin{pmatrix} \theta_K'(\gamma(x-\xi)) \\ -u\gamma\theta_K''(\gamma(x-\xi)) \end{pmatrix} \right\rangle_{L^2_{\xi,x}(\mathbb{R}^2) \oplus L^2_{\xi,x}(\mathbb{R}^2)} = 0 \Bigg\}, \end{aligned}$$

*with the finite norm*

$$\|y\|_{Y_n(u_*)} = \sup_{u \in I(u_*)} \left( \sum_{i=0}^{n} |\partial_u^i y(u)|_{\bar{Y}'} \right).$$

*(d)* $Z_n = Z_n(u_*) = \left\{ z = (v,w) \in C^n(I(u_*), \bar{Z}') : \|z\|_{Z_n(u_*)} < \infty \right\}$ *with the finite norm*

$$\|z\|_{Z_n(u_*)} = \sup_{u \in I(u_*)} \left( \sum_{i=0}^{n} |\partial_u^i z(u)|_{\bar{Z}'} \right) .$$

*(e)* $\underline{\hat{\mathcal{K}}_u} : \left[ H^3(\mathbb{R}^2) \oplus H^2(\mathbb{R}^2) \right] \cap \ker \mathcal{L}_u^{\perp, L^2 \oplus L^2} \to H^2(\mathbb{R}^2) \oplus H^1(\mathbb{R}^2)$, *where*

$$\left( \underline{\hat{\mathcal{K}}_u} \begin{pmatrix} \theta \\ \psi \end{pmatrix} \right)(\xi, x) = \begin{pmatrix} u\partial_\xi \theta(\xi,x) - \psi(\xi,x) \\ -\partial_x^2 \theta(\xi,x) + \cos(\theta_K(\gamma(x-\xi)))\theta(\xi,x) + u\partial_\xi \psi(\xi,x) \end{pmatrix} .$$

*(f)* $\underline{\hat{\mathcal{N}}_u} : \left[ H^3(\mathbb{R}^2) \oplus H^2(\mathbb{R}^2) \right] \cap \ker \mathcal{L}_u^{\perp, L^2 \oplus L^2} \oplus H^2(\mathbb{R}) \to H^2(\mathbb{R}^2) \oplus H^1(\mathbb{R}^2)$, *where*

$$\left( \underline{\hat{\mathcal{N}}_u} \begin{pmatrix} \theta \\ \psi \\ \lambda \end{pmatrix} \right)(\xi, x) = \left( \underline{\hat{\mathcal{K}}_u} \right) \begin{pmatrix} \theta \\ \psi \end{pmatrix} (\xi, x) + \lambda(\xi) t_u(\xi, u, x) .$$

The following Proposition provides that the desired operator is invertible in spaces of higher regularity in all variables $u$, $\xi$ and $x$.

**Proposition 20.33.** *The linear operator*

$$\mathfrak{M}_n : Y_n(u_*) \to Z_n(u_*),$$

$$\begin{pmatrix} \tilde{\theta} \\ \tilde{\psi} \\ \tilde{\lambda}_u \end{pmatrix} \mapsto \mathfrak{M}_n \begin{pmatrix} \tilde{\theta} \\ \tilde{\psi} \\ \tilde{\lambda}_u \end{pmatrix} ,$$

*given by*

$$\mathfrak{M}_n \begin{pmatrix} \tilde{\theta} \\ \tilde{\psi} \\ \tilde{\lambda}_u \end{pmatrix} (u) = \underline{\hat{\mathcal{N}}_u} \begin{pmatrix} \tilde{\theta}(u) \\ \tilde{\psi}(u) \\ \tilde{\lambda}_u(u) \end{pmatrix} ,$$

*is invertible if* $u_* < \underline{u}^*$.

**Proof.** $\mathfrak{M}_n$ is well defined. Using Remark 20.11, (20.4) and Remark 20.19 we obtain Lemma 20.12, Lemma 20.22, Lemma 20.23, Lemma 20.24, Corollary 20.26, Lemma 20.28 with higher regularity in $(\xi, x)$. □

The higher regularity in $u$ and $\xi$ will be needed in Chapter 21.

## 20.2 Inverse Operator in Weighted Sobolev Spaces

Let $\alpha, n \in \mathbb{N}$. We consider some spaces $Y_n^\alpha(u_*) \subset C^n(I(u_*), (\bar{Y}^\alpha)')$, $Z_n^\alpha(u_*) \subset C^n(I(u_*), (\bar{Z}^\alpha)')$, analogous to the spaces $Y_n(u_*)$, $Z_n(u_*)$ from Definition 20.32. The main difference to the previous section is that $(\bar{Y}^\alpha)', (\bar{Z}^\alpha)'$ are weighted Sobolev spaces. The goal of this section is to show that certain operators

$$\mathfrak{M}_n^\alpha : Y_n^\alpha(u_*) \to Z_n^\alpha(u_*)\,,$$

which are defined analogous to the operators

$$\mathfrak{M}_n : Y_n(u_*) \to Z_n(u_*)\,,$$

are invertible for small $u_*$. The reason for working with the spaces $Y_n^\alpha(u_*)$ and $Z_n^\alpha(u_*)$ will become clear in Chapter 21. Since this section has the same structure as Section 20.1 we will refrain from comment the way of proceeding.

### 20.2.1 Preliminary Decomposition

**Definition 20.34.** *We define the following spaces.*

*(a)* $L^{2,\alpha}(\mathbb{R}) := H^{0,\alpha}(\mathbb{R})$.

*(b)* $H^{2,\alpha}_{\xi,u,\perp}(\mathbb{R}) := \{\theta \in H^{2,\alpha}(\mathbb{R}) \;:\; \langle \theta(\cdot), \theta_K'(\gamma(\cdot - \xi))\rangle_{L^{2,\alpha}(\mathbb{R})} = 0\}$.

*(c)* $L^{2,\alpha}_{\xi,u,\perp}(\mathbb{R}) := \{\theta \in L^{2}(\mathbb{R}) \;:\; \langle \theta(\cdot), \theta_K'(\gamma(\cdot - \xi))\rangle_{L^{2,\alpha}(\mathbb{R})} = 0\}$.

*We define the following operators.*

*(a)* $L^\alpha_{\xi,u} : H^{2,\alpha}(\mathbb{R}) \subset L^{2,\alpha}(\mathbb{R}) \to L^{2,\alpha}(\mathbb{R})$, *where*

$$(L^\alpha_{\xi,u}\theta)(x) = -(1-u^2)\partial_x^2\theta(x) + \cos(\theta_K(\gamma(x-\xi)))\theta(x).$$

*(b)* $\hat{L}^\alpha_{\xi,u} : H^{2,\alpha}_{\xi,u,\perp}(\mathbb{R}) \subset L^{2,\alpha}(\mathbb{R}) \to L^{2,\alpha}(\mathbb{R})$, *where*

$$(\hat{L}^\alpha_{\xi,u}\theta)(x) = -(1-u^2)\partial_x^2\theta(x) + \cos(\theta_K(\gamma(x-\xi)))\theta(x).$$

*(c)* $M^\alpha_{\xi,u} : H^{2,\alpha}(\mathbb{R}) \oplus \mathbb{R} \to L^{2,\alpha}(\mathbb{R})$, *where*

$$\left(M^\alpha_{\xi,u}\begin{pmatrix}\theta\\ \lambda\end{pmatrix}\right)(x) = (L_{\xi,u}\theta)(x) + \lambda\theta_K'(\gamma(x-\xi)).$$

*(d)* $\hat{M}^\alpha_{\xi,u} : H^{2,\alpha}_{\xi,u,\perp}(\mathbb{R}) \oplus \mathbb{R} \to L^{2,\alpha}(\mathbb{R})$, *where*

$$\left(\hat{M}^\alpha_{\xi,u}\begin{pmatrix}\theta\\ \lambda\end{pmatrix}\right)(x) = (\hat{L}_{\xi,u}\theta)(x) + \lambda\theta_K'(\gamma(x-\xi)).$$

**Lemma 20.35.** $\operatorname{ran} L^{\alpha}_{\xi,u}$ *is closed with respect to* $L^{2,\alpha}(\mathbb{R})$.

**Proof.** We consider the case $(\xi, u) = (0,0)$. The proof works for a general $(\xi, u) \in \mathbb{R} \times (-1,1)$ in the same way.
Let $v_n \in \operatorname{ran} L^{\alpha}_{0,0}$ and $v_n = L^{\alpha}_{0,0}\theta_n \overset{L^{2,\alpha}}{\to} v$, s.t. $\theta_n \in H^{2,\alpha}_{0,0,\perp}(\mathbb{R})$. Therefore $v \in L^{2,\alpha}(\mathbb{R})$. Lemma 20.2 implies the existence of $\theta \in H^2_{0,0,\perp}(\mathbb{R})$ such that $v = L_{0,0}\theta$. We claim that $\theta \in H^{2,\alpha}(\mathbb{R})$.
Lemma 20.3 (d) yields $\theta_n \overset{H^2}{\to} \theta$. It holds that $\tilde{\theta}_n(x) = x\theta_n(x) \in H^2(\mathbb{R})$ and

$$\begin{aligned} & L_{0,0}[\tilde{\theta}_n(x)] \\ =& -2\theta_n'(x) - x\theta_n''(x) + x\cos(\theta_K(x))[\theta_n(x)] \\ =& -2\theta_n'(x) + xL_{0,0}\theta_n(x) \overset{L^2}{\to} -2\theta'(x) + xL_{0,0}\theta(x) =: \tilde{v}(x) \in L^2(\mathbb{R}) \,. \end{aligned}$$

Lemma 20.2 implies the existence of $\tilde{\theta} \in H^2_{0,0,\perp}(\mathbb{R})$ such that $\tilde{v} = L_{0,0}\tilde{\theta}$. Lemma 20.3 (d) yields $\tilde{\theta}_n \overset{H^2}{\to} \tilde{\theta}$. Using

$$\begin{aligned} & \left|\chi(x)[\tilde{\theta}(x) - x\theta(x)]\right|_{H^2_x(\mathbb{R})} \\ \leq & \left|\chi(x)[\tilde{\theta}(x) - x\theta_n(x)]\right|_{H^2_x(\mathbb{R})} + |\chi(x)[x\theta_n(x) - x\theta(x)]|_{H^2_x(\mathbb{R})} \,, \end{aligned}$$

with any smooth cut-off function $\chi$ we obtain $\tilde{\theta}(x) = x\theta(x)$ almost everywhere, since

$$\begin{aligned} \theta_n &\overset{H^2}{\to} \theta, \\ \tilde{\theta}_n &\overset{H^2}{\to} \tilde{\theta} \,. \end{aligned}$$

Thus $v = L_{0,0}\theta = L^{\alpha}_{0,0}\theta$. We get the claim by application of the same argument on $\hat{\theta}(x) = x^2\theta(x)$ etc. □

**Lemma 20.36.** *(a)* $\ker L^{\alpha}_{\xi,u} = \langle \theta_K'(\gamma(\cdot - \xi))\rangle$, $L^{2,\alpha}(\mathbb{R}) = \operatorname{ran} L^{\alpha}_{\xi,u} \overset{L^{2,\alpha}}{\oplus} \langle \theta_K'(\gamma(\cdot - \xi))\rangle$.

*(b)* $L^{2,\alpha}(\mathbb{R}) = \operatorname{ran} \hat{L}^{\alpha}_{\xi,u} \overset{L^{2,\alpha}}{\oplus} \langle \theta_K'(\gamma(\cdot - \xi))\rangle$.

*(c)* $\hat{L}^{\alpha}_{\xi,u} \in L(H^{2,\alpha}_{\xi,u,\perp}(\mathbb{R}), L^{2,\alpha}_{\xi,u,\perp}(\mathbb{R}))$.

*(d)* $[\hat{L}^{\alpha}_{\xi,u}]^{-1} \in L(L^{2,\alpha}_{\xi,u,\perp}(\mathbb{R}), H^{2,\alpha}_{\xi,u,\perp}(\mathbb{R}))$.

*(e)* $\hat{M}^{\alpha}_{\xi,u} \in L(H^{2,\alpha}_{\xi,u,\perp}(\mathbb{R}) \oplus \mathbb{R}, L^{2,\alpha}(\mathbb{R}))$ *and* $\hat{M}^{\alpha}_{\xi,u}$ *is one-to-one and onto.*

**Proof.** Analogous to the proof of Lemma 20.3. □

**Lemma 20.37.** *Let* $0 \le k \le \alpha$ *and* $0 < |\lambda| < 1$. *Then*

$$(L^{\alpha}_{0,0} - \lambda)^{-1} \in L(L^{2,k}(\mathbb{R}), L^{2,k}(\mathbb{R}))\,.$$

**Proof.** It follows from the fact that the operator $-\partial_Z^2 + \cos\theta_K(Z)$ is nonnegative and it has a one dimensional null space spanned by $\theta'_K(\cdot)$ and the essential spectrum $[1, \infty)$. □

**Lemma 20.38.** $H^{2,\alpha}(\mathbb{R}^2)$ *lies dense in* $L^{2,\alpha}(\mathbb{R}^2)$.

**Definition 20.39.** *We define the following spaces.*

*(a)* $L^{2,\alpha}(\mathbb{R}^2) := H^{0,\alpha}(\mathbb{R}^2)$.

*(b)* $H^{2,\alpha}_{\perp}(\mathbb{R}^2) := \{\theta \in H^{2,\alpha}(\mathbb{R}^2) \mid \forall\lambda \in H^2(\mathbb{R}) : \ \langle\theta(\xi, Z), \lambda(\xi)\theta'_K(Z)\rangle_{L^{2,\alpha}_{\xi,Z}(\mathbb{R}^2)} = 0\}$.

*(c)* $H^{2,\alpha}_{u,\perp}(\mathbb{R}^2) := \{\theta \in H^{2,\alpha}(\mathbb{R}^2) \mid \forall\ \lambda \in H^{2,\alpha}(\mathbb{R}) : \ \langle\theta(\xi, x), \lambda(\xi)\theta'_K(\gamma(x - \xi))\rangle_{L^{2,\alpha}_{\xi,x}(\mathbb{R}^2)} = 0\}$.

*(d)* $L^{2,\alpha}_{u,\perp}(\mathbb{R}^2) := \{\theta \in L^{2,\alpha}(\mathbb{R}^2) \mid \forall\ \lambda \in H^{2,\alpha}(\mathbb{R}) : \ \langle\theta(\xi, x), \lambda(\xi)\theta'_K(\gamma(x - \xi))\rangle_{L^{2,\alpha}_{\xi,x}(\mathbb{R}^2)} = 0\}$.

*(e)* $\bar{\mathcal{Y}}^{\alpha} = H^{2,\alpha}(\mathbb{R}^2) \oplus H^{1,\alpha}(\mathbb{R}^2)$ *with the finite norm*

$$|y|_{\bar{\mathcal{Y}}^{\alpha}} = |\theta|_{H^{2,\alpha}(\mathbb{R}^2)} + |\lambda|_{H^{1,\alpha}(\mathbb{R}^2)}.$$

*(f)* $\bar{\mathcal{Z}}^{\alpha} = L^{2,\alpha}(\mathbb{R}^2)$ *with the finite norm*

$$|z|_{\bar{\mathcal{Z}}^{\alpha}} = |z|_{L^{2,\alpha}(\mathbb{R}^2)}.$$

*(g)*
$$\begin{aligned}\mathcal{Y}^{\alpha} &= \mathcal{Y}^{\alpha}(u_*)\\ &= \left\{ y = (\theta, \lambda_u) \in C(I(u_*), \bar{\mathcal{Y}}^{\alpha}) : \|y\|_{\mathcal{Y}^{\alpha}(u_*)} < \infty,\ \forall u \in I(u_*) : \theta(u) \in H^{2,\alpha}_{u,\perp}(\mathbb{R}^2) \right\}\end{aligned}$$

*with the finite norm*

$$\|y\|_{\mathcal{Y}^{\alpha}(u_*)} = \sup_{u \in I(u_*)} |y|_{\bar{\mathcal{Y}}^{\alpha}}\,.$$

*(h)* $\mathcal{Z}^{\alpha} = \mathcal{Z}^{\alpha}(u_*) = \left\{ z \in C(I(u_*), \bar{\mathcal{Z}}^{\alpha}) : \|z\|_{\mathcal{Z}^{\alpha}(u_*)} < \infty \right\}$ *with the finite norm*

$$\|z\|_{\mathcal{Z}^{\alpha}(u_*)} = \sup_{u \in I(u_*)} |z|_{\bar{\mathcal{Z}}^{\alpha}}\,.$$

*We define the following operators.*

*(a)* $L^{\alpha} : H^{2,\alpha}(\mathbb{R}^2) \subset L^{2,\alpha}(\mathbb{R}^2) \to L^{2,\alpha}(\mathbb{R}^2)$, *where*

$$(L^{\alpha}\theta)(\xi, Z) = -\partial_Z^2\theta(\xi, Z) + \cos(\theta_K(Z))\theta(\xi, Z)\,.$$

*(b)* $\hat{L}^\alpha : H^{2,\alpha}_\perp(\mathbb{R}^2) \subset L^{2,\alpha}(\mathbb{R}^2) \to L^{2,\alpha}(\mathbb{R}^2)$, *where*

$$(\hat{L}^\alpha\theta)(\xi, Z) = -\partial_Z^2\theta(\xi, Z) + \cos(\theta_K(Z))\theta(\xi, Z)\,.$$

*(c)* $L^\alpha_u : H^{2,\alpha}(\mathbb{R}^2) \subset L^{2,\alpha}(\mathbb{R}^2) \to L^{2,\alpha}(\mathbb{R}^2)$, *where*

$$(L^\alpha_u\theta)(\xi, x) = -(1-u^2)\partial_x^2\theta(\xi, x) + \cos(\theta_K(\gamma(x-\xi)))\theta(\xi, x).$$

*(d)* $\hat{L}^\alpha_u : H^{2,\alpha}_{u,\perp}(\mathbb{R}^2) \subset L^{2,\alpha}(\mathbb{R}^2) \to L^{2,\alpha}(\mathbb{R}^2)$, *where*

$$(\hat{L}^\alpha_u\theta)(\xi, x) = -(1-u^2)\partial_x^2\theta(\xi, x) + \cos(\theta_K(\gamma(x-\xi)))\theta(\xi, x).$$

*(e)* $M^\alpha_u : H^{2,\alpha}(\mathbb{R}^2) \oplus H^{2,\alpha}(\mathbb{R}) \to L^{2,\alpha}(\mathbb{R})$, *where*

$$\left(M^\alpha_u \begin{pmatrix}\theta\\ \lambda\end{pmatrix}\right)(\xi, x) = (L^\alpha_u\theta)(\xi, x) + \lambda(\xi)\theta'_K(\gamma(x-\xi)).$$

*(f)* $\hat{M}^\alpha_u : H^{2,\alpha}_{u,\perp}(\mathbb{R}^2) \oplus H^{2,\alpha}(\mathbb{R}) \to L^{2,\alpha}(\mathbb{R}^2)$, *where*

$$\left(\hat{M}^\alpha_u \begin{pmatrix}\theta\\ \lambda\end{pmatrix}\right)(\xi, x) = (\hat{L}^\alpha_u\theta)(\xi, x) + \lambda(\xi)\theta'_K(\gamma(x-\xi)).$$

*(g)* $M^\alpha : H^{2,\alpha}(\mathbb{R}^2) \oplus H^{2,\alpha}(\mathbb{R}) \to L^{2,\alpha}(\mathbb{R})$, *where*

$$\left(M^\alpha \begin{pmatrix}\theta\\ \lambda\end{pmatrix}\right)(\xi, Z) = (L^\alpha\theta)(\xi, Z) + \lambda(\xi)\theta'_K(Z).$$

*(h)* $\hat{M}^\alpha : H^{2,\alpha}_\perp(\mathbb{R}^2) \oplus H^{2,\alpha}(\mathbb{R}) \to L^{2,\alpha}(\mathbb{R}^2)$, *where*

$$\left(\hat{M}^\alpha \begin{pmatrix}\theta\\ \lambda\end{pmatrix}\right)(\xi, Z) = (\hat{L}^\alpha\theta)(\xi, Z) + \lambda(\xi)\theta'_K(Z).$$

**Lemma 20.40.** *(a)* $\ker L^\alpha_u = \{\, \theta \in H^{2,\alpha}(\mathbb{R}^2) \mid \theta(\xi, x) = \lambda(\xi)\theta'_K(\gamma(x-\xi)),\ \lambda \in H^{2,\alpha}(\mathbb{R})\}$.

*(b)* $0$ *is an isolated eigenvalue of* $L^\alpha_u$.

**Proof.** Let $u \in (-1, 1)$.
(a) Let $w \in H^{2,\alpha}(\mathbb{R}^2)$ and $L^\alpha_u w = 0$.

$$\begin{aligned}
& L^\alpha_u w = 0 \\
\Rightarrow\quad & \text{for a.e. } (\xi, x) \in \mathbb{R}^2 : \ \Big[u^2\partial_x^2 - \partial_x^2 + \cos(\theta_K(\gamma(x-\xi)))\Big]w(\xi, x) = 0 \\
\overset{\text{Lemma 20.36}}{\Rightarrow}\quad & \text{for a.e. } \xi \in \mathbb{R} : \ w(\xi, \cdot) = \lambda(\xi)\theta'_K(\gamma(\cdot - \xi)) \\
\Rightarrow\quad & \ker L^\alpha_u = \{\, \theta \in H^{2,\alpha}(\mathbb{R}^2) \mid \theta(\xi, x) = \lambda(\xi)\theta'_K(\gamma(x-\xi)),\ \lambda \in H^{2,\alpha}(\mathbb{R})\}\,.
\end{aligned}$$

This implies that 0 is an eigenvalue of $L_u^\alpha$.

(b) 0 is an isolated eigenvalue of $L_u^\alpha$:
Let $w \in H^{2,\alpha}(\mathbb{R}^2)$, $0 < |\lambda| < 1$ and $(L_u^\alpha - \lambda)w = 0$.

$$\begin{aligned}
&(L_u^\alpha - \lambda)w = 0 \\
&\Rightarrow \text{for a.e. } (\xi, x) \in \mathbb{R}^2 : \ \Big(\Big[u^2\partial_x^2 - \partial_x^2 + \cos(\theta_K(\gamma(x-\xi)))\Big] - \lambda\Big)w(\xi, x) = 0 \\
&\Rightarrow \text{for a.e. } \xi \in \mathbb{R} : \ w(\xi, \cdot) = 0 \\
&\Rightarrow \ker(L_u^\alpha - \lambda) = \{0\}
\end{aligned}$$

The second implication holds because of the spectral gap of $L_{\xi,u}^\alpha$ (see: [Stu92]). Therefore

$$L^{2,\alpha}(\mathbb{R}^2) = \overline{\operatorname{ran}(L_u^\alpha - \lambda)} \oplus \ker(L_u^\alpha - \lambda) = \overline{\operatorname{ran}(L_u^\alpha - \lambda)}\,.$$

$(L_u^\alpha - \lambda)^{-1} : \operatorname{ran} L_u^\alpha \cap L^{2,\alpha}(\mathbb{R}^2) \to L^{2,\alpha}(\mathbb{R}^2)$ is bounded:
Let $v \in \operatorname{ran} L_u^\alpha$, $|v(\xi, x)|_{L^{2,\alpha}(\mathbb{R}^2)} \le 1$. Let $\theta \in H^{2,\alpha}(\mathbb{R}^2)$ be such that

$$\begin{aligned}
&& (L_u^\alpha - \lambda)\theta = v \\
\Rightarrow \text{for a.e. } \xi \in \mathbb{R} : && (L_{\xi,u} - \lambda)\theta(\xi, \cdot) = v(\xi, \cdot) \\
\Rightarrow \text{for a.e. } \xi \in \mathbb{R} : && (L_{0,0}^\alpha - \lambda)\theta(\xi, \frac{\cdot}{\gamma} + \xi) = v(\xi, \frac{\cdot}{\gamma} + \xi)\,.
\end{aligned}$$

This implies the following identities

$$\begin{aligned}
&& \theta = (L_u^\alpha - \lambda)^{-1}v\,, \\
\text{for a.e. } \xi \in \mathbb{R} : && \theta(\xi, \cdot) = (L_{\xi,u}^\alpha - \lambda)^{-1}v(\xi, \cdot)\,, \\
\text{for a.e. } \xi \in \mathbb{R} : && \theta(\xi, \frac{\cdot}{\gamma} + \xi) = (L_{0,0}^\alpha - \lambda)^{-1}v(\xi, \frac{\cdot}{\gamma} + \xi)\,.
\end{aligned} \tag{20.10}$$

Since $(L_{0,0}^k - \lambda)^{-1} \in L(L^{2,k}(\mathbb{R}), L^{2,k}(\mathbb{R}))$ for $0 \le k \le \alpha$ and $0 < |\lambda| < 1$ due to Lemma 20.37, we obtain for almost all $\xi \in \mathbb{R}$:

$$\begin{aligned}
&\gamma^{\frac{1}{2}} \left|\left(1 + |\xi|^2 + |\cdot|^2\right)^{\frac{\alpha}{2}} \theta(\xi, \cdot)\right|_{L^2(\mathbb{R})} \\
&= \left|\left(1 + |\xi|^2 + \left|\frac{\cdot}{\gamma} + \xi\right|^2\right)^{\frac{\alpha}{2}} \theta(\xi, \frac{\cdot}{\gamma} + \xi)\right|_{L^2(\mathbb{R})} \\
&= \left|\left(1 + |\xi|^2 + \left|\frac{\cdot}{\gamma} + \xi\right|^2\right)^{\frac{\alpha}{2}} (L_{0,0}^\alpha - \lambda)^{-1}v(\xi, \frac{\cdot}{\gamma} + \xi)\right|_{L^2(\mathbb{R})}
\end{aligned}$$

$$\begin{aligned}
&\le c(\alpha)\sum_{k=0}^{\alpha}\binom{\alpha}{k}^{\frac{1}{2}}|\xi|^{\alpha-k}\left|\left(1+\left|\frac{\cdot}{\gamma}\right|^2\right)^{\frac{k}{2}}(L_{0,0}^{\alpha}-\lambda)^{-1}v(\xi,\frac{\cdot}{\gamma}+\xi)\right|_{L^2(\mathbb{R})}\\
&\le c(\alpha)\sum_{k=0}^{\alpha}\binom{\alpha}{k}^{\frac{1}{2}}|\xi|^{\alpha-k}\left|\left(1+|\cdot|^2\right)^{\frac{k}{2}}(L_{0,0}^{\alpha}-\lambda)^{-1}v(\xi,\frac{\cdot}{\gamma}+\xi)\right|_{L^2(\mathbb{R})}\\
&\le c(\alpha)\sum_{k=0}^{\alpha}\binom{\alpha}{k}^{\frac{1}{2}}|\xi|^{\alpha-k}\|(L_{0,0}^{\alpha}-\lambda)^{-1}\|_k\left(\int\left(1+|Z|^2\right)^k v(\xi,\frac{Z}{\gamma}+\xi)^2\,dZ\right)^{\frac{1}{2}}\\
&\le \gamma 2^{\frac{\alpha}{2}}c(\alpha)\sum_{k=0}^{\alpha}\binom{\alpha}{k}^{\frac{1}{2}}|\xi|^{\alpha-k}\|(L_{0,0}^{\alpha}-\lambda)^{-1}\|_k\left(\int\left(1+|\frac{Z}{\gamma}+\xi|^2+|\xi|^2\right)^k v(\xi,\frac{Z}{\gamma}+\xi)^2\,dZ\right)^{\frac{1}{2}}\\
&\le \gamma^{\frac{3}{2}}2^{\frac{\alpha}{2}}c(\alpha)\sum_{k=0}^{\alpha}\binom{\alpha}{k}|\xi|^{\alpha-k}\|(L_{0,0}^{\alpha}-\lambda)^{-1}\|_k\left|\left(1+|\xi|^2+|\cdot|^2\right)^{\frac{k}{2}}v(\xi,\cdot)\right|_{L^2(\mathbb{R})}\\
&\le \gamma^{\frac{3}{2}}C(\alpha)\left|\left(1+|\xi|^2+|\cdot|^2\right)^{\frac{\alpha}{2}}v(\xi,\cdot)\right|_{L^2(\mathbb{R})},
\end{aligned}$$

where $\|(L_{0,0}^{\alpha}-\lambda)^{-1}\|_k := \|(L_{0,0}^{\alpha}-\lambda)^{-1}\|_{L(L^{2,k}(\mathbb{R}),L^{2,k}(\mathbb{R}))}$. Integration with respect to $\xi$ yields due to (20.10):

$$\begin{aligned}
&\gamma\left|(L_u^{\alpha}-\lambda)^{-1}v\right|^2_{L^{2,\alpha}(\mathbb{R}^2)}\\
&=\gamma\left|\left(1+|\xi|^2+|x|^2\right)^{\frac{\alpha}{2}}(L_u^{\alpha}-\lambda)^{-1}v(\xi,x)\right|^2_{L^2_{\xi,x}(\mathbb{R}^2)}\\
&\le\gamma^3C(\alpha)^2\left|\left(1+|\xi|^2+|x|^2\right)^{\frac{\alpha}{2}}v(\xi,x)\right|^2_{L^2_{\xi,x}(\mathbb{R}^2)}\\
&=\gamma^3C(\alpha)^2\,|v|^2_{L^{2,\alpha}(\mathbb{R}^2)}\ .
\end{aligned}$$

$\operatorname{ran}(L_u^{\alpha}-\lambda)$ is closed:
Let $y_n=(L_u^{\alpha}-\lambda)x_n\overset{L^{2,\alpha}(\mathbb{R}^2)}{\rightarrow}y$. Boundness of $(L_u^{\alpha}-\lambda)^{-1}$ yields $x_n=(L_u^{\alpha}-\lambda)^{-1}y_n\overset{L^{2,\alpha}(\mathbb{R}^2)}{\rightarrow}\overline{(L_u^{\alpha}-\lambda)^{-1}y}$, where $\overline{(L_u^{\alpha}-\lambda)^{-1}}$ denotes the bounded extension of $(L_u^{\alpha}-\lambda)^{-1}$ on the closure $\overline{\operatorname{ran}(L_u^{\alpha}-\lambda)}$. Since $(L_u-\lambda)$ is a closed operator we obtain $(L_u^{\alpha}-\lambda)(\overline{(L_u^{\alpha}-\lambda)^{-1}}y)=y$.

□

**Lemma 20.41.** $\operatorname{ran}L_u^{\alpha}$ *is closed with respect to* $L^{2,\alpha}(\mathbb{R}^2)$.

**Proof.** Let $v_n\in\operatorname{ran}L_u^{\alpha}$ and $v_n=L_u^{\alpha}\theta_n\overset{L^{2,\alpha}}{\rightarrow}v$, s.t. $\theta_n\in H^{2,\alpha}_{u,\perp}(\mathbb{R}^2)$. Therefore $v\in L^{2,\alpha}(\mathbb{R}^2)$. Lemma 20.9 implies the existence of $\theta\in H^2_{u,\perp}(\mathbb{R}^2)$ such that $v=L_u\theta$. We claim that

$\theta \in H^{2,\alpha}(\mathbb{R}^2)$. Lemma 20.10 (d) yields $\theta_n \overset{H^2}{\to} \theta$. It holds that $\tilde{\theta}_n(\xi, x) = x\theta_n(\xi, x) \in H^2(\mathbb{R}^2)$ and

$$\begin{aligned} & L_u[\tilde{\theta}_n(\xi, x)] \\ = & -2(1-u^2)\partial_x\theta_n(\xi, x) - x(1-u^2)\partial_x^2\theta_n(\xi, x) + x\cos(\theta_K(\gamma(x-\xi))[\theta_n(\xi, x)] \\ = & -2(1-u^2)\partial_x\theta_n(\xi, x) + xL_u\theta_n(\xi, x) \overset{L^2}{\to} -2(1-u^2)\partial_x\theta(\xi, x) + xL_u\theta(\xi, x) =: \tilde{v}(\xi, x) \in L^2(\mathbb{R}^2)\,. \end{aligned}$$

Lemma 20.9 implies the existence of $\tilde{\theta} \in H^2_{u,\perp}(\mathbb{R}^2)$ such that $\tilde{v} = L_u\tilde{\theta}$. Lemma 20.10 (d) yields $\tilde{\theta}_n \overset{H^2}{\to} \tilde{\theta}$. Using

$$\begin{aligned} & \left|\chi(\xi, x)[\tilde{\theta}(\xi, x) - x\theta(\xi, x)]\right|_{H^2_{\xi,x}(\mathbb{R}^2)} \\ \leq & \left|\chi(\xi, x)[\tilde{\theta}(\xi, x) - x\theta_n(\xi, x)]\right|_{H^2_{\xi,x}(\mathbb{R}^2)} + |\chi(\xi, x)[x\theta_n(\xi, x) - x\theta(\xi, x)]|_{H^2_{\xi,x}(\mathbb{R}^2)}\ , \end{aligned}$$

with any smooth cut-off function $\chi$ we obtain $\tilde{\theta}(\xi, x) = x\theta(\xi, x)$ almost everywhere, since

$$\begin{aligned} \theta_n &\overset{H^2}{\to} \theta, \\ \tilde{\theta}_n &\overset{H^2}{\to} \tilde{\theta}\,. \end{aligned}$$

Thus $v = L_u\theta = L_u^\alpha\theta$. We get the claim by application of the same argument on $\hat{\theta}(\xi, x) = x^2\theta(\xi, x)$ etc. □

**Lemma 20.42.** *(a)* $L^{2,\alpha}(\mathbb{R}^2) = \operatorname{ran} L_u^\alpha \overset{L^{2,\alpha}}{\oplus} \ker L_u^\alpha$.

*(b)* $L^{2,\alpha}(\mathbb{R}^2) = \operatorname{ran} \hat{L}_u^\alpha \overset{L^{2,\alpha}}{\oplus} \ker L_u^\alpha$.

*(c)* $\hat{L}_u^\alpha \in L(H^{2,\alpha}_{u,\perp}(\mathbb{R}^2), L^{2,\alpha}_{u,\perp}(\mathbb{R}^2))$.

*(d)* $\left[\hat{L}_u^\alpha\right]^{-1} \in L(L^{2,\alpha}_{u,\perp}(\mathbb{R}^2), H^{2,\alpha}_{u,\perp}(\mathbb{R}^2))$.

*(e)* $\hat{M}_u^\alpha, \hat{M}^\alpha$ *are one-to-one, onto, bounded and the inverse mappings are also bounded.*

**Proof.** Analogous to the proof of Lemma 20.10. □

**Lemma 20.43.** *Let* $\mathfrak{m}^\alpha$ *be the linear operator, defined by*

$$\begin{aligned} & \mathfrak{m}^\alpha : \mathcal{Y}^\alpha \to \mathcal{Z}^\alpha, \\ & \qquad (\theta, \lambda) \mapsto \mathfrak{m}^\alpha(\theta, \lambda)\,, \quad s.t.\ \ \mathfrak{m}^\alpha(\theta, \lambda)(u) = \hat{M}_u^\alpha(\theta(u), \lambda(u)). \end{aligned}$$

$\mathfrak{m}^\alpha$ *is one-to-one, onto and bounded, i.e.,* $[\mathfrak{m}^\alpha]^{-1}$ *is bounded.*

**Proof.** $\mathfrak{m}^\alpha$ is well defined: clear.
$\mathfrak{m}^\alpha$ is one-to-one: Let $(\theta, \lambda) \in \mathcal{Y}^\alpha$ with $\mathfrak{m}^\alpha(\theta,\lambda) = 0$. It follows

$$\forall u \in I: \quad \mathfrak{m}^\alpha(\theta,\lambda)(u) = \hat{M}_u^\alpha(\theta(u),\lambda(u)) = \hat{L}_u^\alpha\theta(u) + \lambda(u)\theta_K'(\gamma(x-\xi)) = 0\,.$$

Lemma 20.42 yields

$$\forall u \in I: \quad \theta(u) = 0, \lambda(u) = 0 \Rightarrow \theta = 0, \lambda = 0\,.$$

$\mathfrak{m}^\alpha$ is onto: Let $v \in \mathcal{Z}^\alpha$. Due to Lemma 20.42 there exists for all $u \in I$ a $(\theta(u),\lambda(u)) \in H_{u,\perp}^{2,\alpha}(\mathbb{R}^2) \oplus H^{2,\alpha}(\mathbb{R})$ such that

$$\forall u \in I: \quad v(u)(\xi,x) = \hat{M}_u^\alpha \begin{pmatrix}\theta(u)\\ \lambda(u)\end{pmatrix}(\xi,x) = \left(\hat{L}_u^\alpha\theta(u)\right)(\xi,x) + \lambda(u)(\xi)\theta_K'(\gamma(x-\xi))\,.$$

Thus abusing notation in the same way as in (20.2), we write

$$\begin{aligned} & v(u)(\xi, \frac{Z}{\gamma} + \xi) \\ = {} & \hat{M}^\alpha \begin{pmatrix}\theta(u)(\xi, \frac{Z}{\gamma}+\xi)\\ \lambda(u)(\xi)\end{pmatrix}(\xi,Z) = \left(\hat{L}^\alpha\,\theta(u)(\xi,\frac{Z}{\gamma}+\xi)\right)(\xi,Z) + \lambda(u)(\xi)\theta_K'(Z)\,. \end{aligned} \tag{20.11}$$

Using (20.3) for $h(\xi,x) = (1+|\xi|^2+|x|^2)^{\frac{\alpha}{2}}\left[\theta(u)-\theta(\bar{u})\right](\xi,x)$, (20.11) and Lemma 20.10 we obtain:

$$\begin{aligned} & \left|\begin{pmatrix}\theta(u)-\theta(\bar{u})\\ \lambda(u)-\lambda(\bar{u})\end{pmatrix}\right|_{\bar{\mathcal{Y}}^\alpha} \\ = {} & |\theta(u)(\xi,x)-\theta(\bar{u})(\xi,x)|_{H_{\xi,x}^{2,\alpha}(\mathbb{R}^2)} + |\lambda(u)(\xi)-\lambda(\bar{u})(\xi)|_{H_\xi^{2,\alpha}(\mathbb{R})} \\ \leq {} & c\gamma(u)^{\frac{3}{2}}\left|\left(1+|\xi|^2+\left|\frac{Z}{\gamma(u)}+\xi\right|^2\right)^{\frac{\alpha}{2}}\left[\theta(u)(\xi,\frac{Z}{\gamma(u)}+\xi)-\theta(\bar{u})(\xi,\frac{Z}{\gamma(u)}+\xi)\right]\right|_{H_{\xi,Z}^2(\mathbb{R}^2)} \\ & + |\lambda(u)(\xi)-\lambda(\bar{u})(\xi)|_{H_\xi^{2,\alpha}(\mathbb{R})} \\ \leq {} & c\gamma(u)^{\frac{3}{2}}\left|\left(1+|\xi|^2+\left|\frac{Z}{\gamma(u)}+\xi\right|^2\right)^{\frac{\alpha}{2}}\left[\theta(u)(\xi,\frac{Z}{\gamma(u)}+\xi)-\theta(\bar{u})(\xi,\frac{Z}{\gamma(\bar{u})}+\xi)\right]\right|_{H_{\xi,Z}^2(\mathbb{R}^2)} \end{aligned}$$

$$+ c\gamma(u)^{\frac{3}{2}} \left| \left(1+|\xi|^2+\left|\frac{Z}{\gamma(u)}+\xi\right|^2\right)^{\frac{\alpha}{2}} \left[\theta(\bar{u})(\xi, \frac{Z}{\gamma(\bar{u})}+\xi) - \theta(\bar{u})(\xi, \frac{Z}{\gamma(u)}+\xi)\right]\right|_{H^2_{\xi,Z}(\mathbb{R}^2)}$$

$$+ |\lambda(u)(\xi)-\lambda(\bar{u})(\xi)|_{H^{2,\alpha}_{\xi}(\mathbb{R})}$$

$$\le \gamma(u)^{\frac{3}{2}} C(\alpha) \Big| \left(1+|\xi|^2+|Z|^2\right)^{\frac{\alpha}{2}}$$

$$\cdot \left[ \left[\hat{M}^\alpha\right]^{-1} \left(v(u)(\xi, \frac{Z}{\gamma(u)}+\xi)\right) - \left[\hat{M}^\alpha\right]^{-1} \left(v(\bar{u})(\xi, \frac{Z}{\gamma(\bar{u})}+\xi)\right)\right]\Big|_{H^{2,\alpha}_{\xi,Z}(\mathbb{R}^2)\oplus H^{2,\alpha}_{\xi}(\mathbb{R})}$$

$$+ \gamma(u)^{\frac{3}{2}} C(\alpha) \left| \left(1+|\xi|^2+|Z|^2\right)^{\frac{\alpha}{2}} \left[\theta(\bar{u})(\xi, \frac{Z}{\gamma(\bar{u})}+\xi) - \theta(\bar{u})(\xi, \frac{Z}{\gamma(u)}+\xi)\right]\right|_{H^2_{\xi,Z}(\mathbb{R}^2)}$$

$$\le \gamma(u)^{\frac{3}{2}} C(\alpha) \| \left[\hat{M}^\alpha\right]^{-1} \| \left| \left(1+|\xi|^2+|Z|^2\right)^{\frac{\alpha}{2}} \left[v(u)(\xi, \frac{Z}{\gamma(u)}+\xi) - v(\bar{u})(\xi, \frac{Z}{\gamma(\bar{u})}+\xi)\right]\right|_{L^2_{\xi,Z}(\mathbb{R}^2)}$$

$$+ \gamma(u)^{\frac{3}{2}} C(\alpha) \left| \left(1+|\xi|^2+|Z|^2\right)^{\frac{\alpha}{2}} \left[\theta(\bar{u})(\xi, \frac{Z}{\gamma(\bar{u})}+\xi) - \theta(\bar{u})(\xi, \frac{Z}{\gamma(u)}+\xi)\right]\right|_{H^2_{\xi,Z}(\mathbb{R}^2)}$$

$$\le \gamma(u)^{\frac{3}{2}} C(\alpha) \| \left[\hat{M}^\alpha\right]^{-1} \| \left| \left(1+|\xi|^2+|Z|^2\right)^{\frac{\alpha}{2}} \left[v(u)(\xi, \frac{Z}{\gamma(u)}+\xi) - v(u)(\xi, \frac{Z}{\gamma(\bar{u})}+\xi)\right]\right|_{L^2_{\xi,Z}(\mathbb{R}^2)}$$

$$+ \gamma(u)^{\frac{3}{2}} C(\alpha) \| \left[\hat{M}^\alpha\right]^{-1} \| \left| \left(1+|\xi|^2+|Z|^2\right)^{\frac{\alpha}{2}} \left[v(u)(\xi, \frac{Z}{\gamma(\bar{u})}+\xi) - v(\bar{u})(\xi, \frac{Z}{\gamma(\bar{u})}+\xi)\right]\right|_{L^2_{\xi,Z}(\mathbb{R}^2)}$$

$$+ \gamma(u)^{\frac{3}{2}} C(\alpha) \left| \left(1+|\xi|^2+|Z|^2\right)^{\frac{\alpha}{2}} \left[\theta(\bar{u})(\xi, \frac{Z}{\gamma(\bar{u})}+\xi) - \theta(\bar{u})(\xi, \frac{Z}{\gamma(u)}+\xi)\right]\right|_{H^2_{\xi,Z}(\mathbb{R}^2)}$$

This implies that $(\theta, \lambda) \in \mathcal{Y}^\alpha$, since $v \in \mathcal{Z}^\alpha$. The inverse mapping theorem yields that $[\mathfrak{m}^\alpha]^{-1}$ is bounded, since $\mathfrak{m}^\alpha$ is bounded. □

**Definition 20.44.** *(a)* $\mathcal{L}^\alpha_u : H^{2,\alpha}(\mathbb{R}^2) \oplus H^{1,\alpha}(\mathbb{R}^2) \to H^{1,\alpha}(\mathbb{R}^2) \oplus L^{2,\alpha}(\mathbb{R}^2)$, *where*

$$\left(\mathcal{L}^\alpha_u \begin{pmatrix} \theta \\ \psi \end{pmatrix}\right)(\xi, x) = \begin{pmatrix} -u\partial_x\theta(\xi,x) - \psi(\xi,x) \\ -\partial_x^2\theta(\xi,x) + \cos(\theta_K(\gamma(x-\xi)))\theta(\xi,x) - u\partial_x\psi(\xi,x) \end{pmatrix}.$$

*(b)* $\hat{\mathcal{L}}^\alpha_u : \left[H^{2,\alpha}(\mathbb{R}^2) \oplus H^{1,\alpha}(\mathbb{R}^2)\right] \cap (\ker \mathcal{L}_u)^{\perp, L^{2,\alpha}\oplus L^{2,\alpha}} \to H^{1,\alpha}(\mathbb{R}^2) \oplus L^{2,\alpha}(\mathbb{R}^2)$, *where*

$$\left(\hat{\mathcal{L}}^\alpha_u \begin{pmatrix} \theta \\ \psi \end{pmatrix}\right)(\xi, x) = \begin{pmatrix} -u\partial_x\theta(\xi,x) - \psi(\xi,x) \\ -\partial_x^2\theta(\xi,x) + \cos(\theta_K(\gamma(x-\xi)))\theta(\xi,x) - u\partial_x\psi(\xi,x) \end{pmatrix}.$$

*(c)* $\mathcal{M}_u^\alpha : H^{2,\alpha}(\mathbb{R}^2) \oplus H^{1,\alpha}(\mathbb{R}^2) \oplus H^{2,\alpha}(\mathbb{R}) \to H^{1,\alpha}(\mathbb{R}^2) \oplus L^{2,\alpha}(\mathbb{R}^2)$, *where*

$$\left(\mathcal{M}_u^\alpha \begin{pmatrix} \theta \\ \psi \\ \lambda \end{pmatrix}\right)(\xi, x) = \left(\mathcal{L}_u^\alpha \begin{pmatrix} \theta \\ \psi \end{pmatrix}\right)(\xi, x) + \lambda(\xi) t_u(\xi, u, x)\,.$$

*(d)* $\hat{\mathcal{M}}_u^\alpha : \left[H^{2,\alpha}(\mathbb{R}^2) \oplus H^{1,\alpha}(\mathbb{R}^2)\right] \cap (\ker \mathcal{L}_u^\alpha)^{\perp, L^{2,\alpha} \oplus L^{2,\alpha}} \oplus H^{2,\alpha}(\mathbb{R}) \to H^{1,\alpha}(\mathbb{R}^2) \oplus L^{2,\alpha}(\mathbb{R}^2)$, *where*

$$\left(\hat{\mathcal{M}}_u^\alpha \begin{pmatrix} \theta \\ \psi \\ \lambda \end{pmatrix}\right)(\xi, x) = \left(\hat{\mathcal{L}}_u^\alpha \begin{pmatrix} \theta \\ \psi \end{pmatrix}\right)(\xi, x) + \lambda(\xi) t_u(\xi, u, x)\,.$$

*(e)* $\mathcal{K}_u^\alpha : H^{2,\alpha}(\mathbb{R}^2) \oplus H^{1,\alpha}(\mathbb{R}^2) \to H^{1,\alpha}(\mathbb{R}^2) \oplus L^{2,\alpha}(\mathbb{R}^2)$, *where*

$$\left(\mathcal{K}_u^\alpha \begin{pmatrix} \theta \\ \psi \end{pmatrix}\right)(\xi, x) = \begin{pmatrix} u\partial_\xi \theta(\xi, x) - \psi(\xi, x) \\ -\partial_x^2 \theta(\xi, x) + \cos(\theta_K(\gamma(x - \xi)))\theta(\xi, x) + u\partial_\xi \psi(\xi, x) \end{pmatrix}\,.$$

*(f)* $\hat{\mathcal{K}}_u^\alpha : \left[H^{2,\alpha}(\mathbb{R}^2) \oplus H^{1,\alpha}(\mathbb{R}^2)\right] \cap (\ker \mathcal{L}_u^\alpha)^{\perp, L^{2,\alpha} \oplus L^{2,\alpha}} \to H^{1,\alpha}(\mathbb{R}^2) \oplus L^{2,\alpha}(\mathbb{R}^2)$, *where*

$$\left(\hat{\mathcal{K}}_u^\alpha \begin{pmatrix} \theta \\ \psi \end{pmatrix}\right)(\xi, x) = \begin{pmatrix} u\partial_\xi \theta(\xi, x) - \psi(\xi, x) \\ -\partial_x^2 \theta(\xi, x) + \cos(\theta_K(\gamma(x - \xi)))\theta(\xi, x) + u\partial_\xi \psi(\xi, x) \end{pmatrix}.$$

*(g)* $\mathcal{N}_u^\alpha : H^{2,\alpha}(\mathbb{R}^2) \oplus H^{1,\alpha}(\mathbb{R}^2) \oplus H^{2,\alpha}(\mathbb{R}) \to H^{1,\alpha}(\mathbb{R}^2) \oplus L^{2,\alpha}(\mathbb{R}^2)$, *where*

$$\left(\mathcal{N}_u^\alpha \begin{pmatrix} \theta \\ \psi \\ \lambda \end{pmatrix}\right)(\xi, x) = \left(\mathcal{K}_u^\alpha \begin{pmatrix} \theta \\ \psi \end{pmatrix}\right)(\xi, x) + \lambda(\xi) t_u(\xi, u, x)\,.$$

*(h)* $\hat{\mathcal{N}}_u^\alpha : \left[H^{2,\alpha}(\mathbb{R}^2) \oplus H^{1,\alpha}(\mathbb{R}^2)\right] \cap (\ker \mathcal{L}_u^\alpha)^{\perp, L^{2,\alpha} \oplus L^{2,\alpha}} \oplus H^{2,\alpha}(\mathbb{R}) \to H^{1,\alpha}(\mathbb{R}^2) \times L^{2}(\mathbb{R}^2)$, *where*

$$\left(\hat{\mathcal{N}}_u^\alpha \begin{pmatrix} \theta \\ \psi \\ \lambda \end{pmatrix}\right)(\xi, x) = \left(\hat{\mathcal{K}}_u^\alpha \begin{pmatrix} \theta \\ \psi \end{pmatrix}\right)(\xi, x) + \lambda(\xi) t_u(\xi, u, x)\,.$$

**Lemma 20.45.**

*(a)* $\ker \mathcal{L}_u^\alpha = \{\lambda t_\xi(u), \lambda \in H^{2,\alpha}(\mathbb{R})\}\,.$

*(b)* $\ker\, [\mathcal{L}_u^\alpha]^* = \{\lambda \mathbb{J} t_\xi(u), \lambda \in H^{2,\alpha}(\mathbb{R})\}\,.$

**Proof.** Analogous to the proof of Lemma 20.14. □

**Lemma 20.46 (orthogonal sum).**

$$H^{1,\alpha}(\mathbb{R}^2)\oplus L^{2,\alpha}(\mathbb{R}^2) = \hat{\mathcal{L}}_u^\alpha\left(\left[H^{2,\alpha}(\mathbb{R}^2)\oplus H^{1,\alpha}(\mathbb{R}^2)\right]\cap(\ker\mathcal{L}_u^\alpha)^{\perp,L^{2,\alpha}\oplus L^{2,\alpha}}\right)\overset{L^{2,\alpha}\oplus L^{2,\alpha}}{\oplus}\ker\,[\mathcal{L}_u^\alpha]^*\,.$$

**Proof.** ”$\supset$”: clear.
”$\subset$”: Let $\begin{pmatrix}\bar v\\ \bar w\end{pmatrix}\in H^{1,\alpha}(\mathbb{R}^2)\oplus L^{2,\alpha}(\mathbb{R}^2)$. Since $L^{2,\alpha}(\mathbb{R}^2)\oplus L^{2,\alpha}(\mathbb{R}^2)=\overline{\mathcal{L}_u^\alpha(H^{2,\alpha}(\mathbb{R}^2)\oplus H^{1,\alpha}(\mathbb{R}^2))}\oplus\ker[\mathcal{L}_u^\alpha]^*$, using Lemma 20.45 there exists $\mu=\mu(u)\in H^{2,\alpha}(\mathbb{R})$ such that

$$\begin{pmatrix}\bar v\\ \bar w\end{pmatrix} = {}_{L^{2,\alpha}\oplus L^{2,\alpha}-}\lim_{n\to\infty}\mathcal{L}_u^\alpha\begin{pmatrix}\theta_n\\ \psi_n\end{pmatrix}+\mu\mathbb{J}t_\xi\,,$$

where

$$\begin{pmatrix}\theta_n\\ \psi_n\end{pmatrix}\in H^{2,\alpha}(\mathbb{R}^2)\oplus H^{1,\alpha}(\mathbb{R}^2)\,.$$

Hence $\begin{pmatrix}v\\ w\end{pmatrix}:={}_{L^{2,\alpha}\oplus L^{2,\alpha}-}\lim_{n\to\infty}\mathcal{L}_u^\alpha\begin{pmatrix}\theta_n\\ \psi_n\end{pmatrix}\in(\ker[\mathcal{L}_u^\alpha]^*)^{\perp,L^{2,\alpha}\oplus L^{2,\alpha}}\cap\left[H^{1,\alpha}(\mathbb{R}^2)\oplus L^{2,\alpha}(\mathbb{R}^2)\right]$.
There exists $\begin{pmatrix}\theta\\ \psi\end{pmatrix}\in H^2(\mathbb{R}^2)\oplus H^1(\mathbb{R}^2)$ such that

$$\mathcal{L}_u^\alpha\begin{pmatrix}\theta\\ \psi\end{pmatrix}=\begin{pmatrix}v\\ w\end{pmatrix}\,.$$

We are going to show that

$$\begin{pmatrix}\theta\\ \psi\end{pmatrix}\in H^{2,\alpha}(\mathbb{R}^2)\oplus H^{1,\alpha}(\mathbb{R}^2)\,.$$

Let $Z=\gamma(x-\xi)$. Notice that

$$\begin{aligned}
&\mathcal{L}_u^\alpha\begin{pmatrix}x\theta_n\\ x\psi_n\end{pmatrix}\\
&=\begin{pmatrix}-u\partial_x(x\theta_n)-x\psi_n\\ -\partial_x^2(x\theta_n)+\cos(\theta_K(Z))(x\theta_n)-u\partial_x(x\theta_n)\end{pmatrix}\\
&=\begin{pmatrix}-u\partial_x\theta_n-ux\partial_x\theta_n-x\psi_n\\ -\partial_x^2\theta_n-x\partial_x^2\theta_n-\partial_x\theta_n+x\cos(\theta_K(Z))\theta_n-u\partial_x\theta_n-ux\partial_x\theta_n\end{pmatrix}\\
&=x\mathcal{L}_u^\alpha\begin{pmatrix}\theta_n\\ \psi_n\end{pmatrix}+\begin{pmatrix}-u\partial_x\theta_n\\ -\partial_x^2\theta_n-\partial_x\theta_n-u\partial_x\theta_n\end{pmatrix}\\
&\overset{L^2}{\to}\quad x\mathcal{L}_u^\alpha\begin{pmatrix}\theta\\ \psi\end{pmatrix}+\begin{pmatrix}-u\partial_x\theta\\ -\partial_x^2\theta-\partial_x\theta-u\partial_x\theta\end{pmatrix}\in L^2(\mathbb{R}^2)\oplus L^2(\mathbb{R}^2)\,.
\end{aligned}$$

Considering $\mathcal{L}_u^\alpha \begin{pmatrix} \xi\theta_n \\ \xi\psi_n \end{pmatrix}$ etc. and applying the same type of arguments as in Lemma 20.41 yield

$$\begin{pmatrix} \theta \\ \psi \end{pmatrix} \in H^{2,\alpha}(\mathbb{R}^2) \oplus H^{1,\alpha}(\mathbb{R}^2) \,.$$

Set $\begin{pmatrix} \tilde{\theta} \\ \tilde{\psi} \end{pmatrix} := \begin{pmatrix} \theta \\ \psi \end{pmatrix}_{(\ker \mathcal{L}_u^\alpha)^{\perp, L^{2,\alpha} \oplus L^{2,\alpha}}}$, so $\mathcal{L}_u^\alpha \begin{pmatrix} \tilde{\theta} \\ \tilde{\psi} \end{pmatrix} = \begin{pmatrix} v \\ w \end{pmatrix}$. □

**Corollary 20.47 (direct sum).**

$$\begin{aligned} &H^{1,\alpha}(\mathbb{R}^2) \oplus L^{2,\alpha}(\mathbb{R}^2) \\ = &\hat{\mathcal{L}}_u^\alpha \left( \Big[ H^{2,\alpha}(\mathbb{R}^2) \oplus H^{1,\alpha}(\mathbb{R}^2) \Big] \cap (\ker \mathcal{L}_u^\alpha)^{\perp, L^{2,\alpha} \oplus L^{2,\alpha}} \right) \overset{L^{2,\alpha} \oplus L^{2,\alpha}}{\oplus} \left\{ \lambda t_u(u), \lambda \in H_\xi^{2,\alpha}(\mathbb{R}) \right\} . \end{aligned}$$

**Proof.** "$\supset$": clear.
"$\subset$": Let $(v, w) \in H^{1,\alpha}(\mathbb{R}^2) \oplus L^{2,\alpha}(\mathbb{R}^2)$ then there exists due to Lemma 20.46 $\begin{pmatrix} \theta \\ \psi \end{pmatrix} = \begin{pmatrix} \theta(u) \\ \psi(u) \end{pmatrix} \in H^{2,\alpha}(\mathbb{R}^2) \oplus H^{1,\alpha}(\mathbb{R}^2) \cap (\ker \mathcal{L}_u^\alpha)^{\perp, L^{2,\alpha} \oplus L^{2,\alpha}}$ and $\lambda = \lambda(u) \in H^{2,\alpha}(\mathbb{R})$ s.t.

$$\begin{pmatrix} v \\ w \end{pmatrix} = \hat{\mathcal{L}}_u^\alpha \begin{pmatrix} \theta \\ \psi \end{pmatrix} + \lambda \mathbb{J} t_\xi(u) \,.$$

Let $Z = \gamma(x - \xi)$. Assume without loss of generality $|\lambda|_{H^{2,\alpha}(\mathbb{R})} \neq 0$, then

$$\begin{aligned} &\langle \lambda t_\xi(u), \lambda \mathbb{J} t_u(u) \rangle_{L^{2,\alpha}(\mathbb{R}^2) \oplus L^{2,a}(\mathbb{R}^2)} \\ = &\int \lambda(\xi)^2 (1 + |\xi|^2 + |x|^2)^{\frac{\alpha}{2}} \Big( -\gamma^4 \Big[\theta_K'(Z)\Big]^2 - u^2 \gamma^5 (x - \xi) \theta_K''(Z) \theta_K'(Z) \\ &+ u^2 \gamma^5 \theta_K''(Z)(x - \xi) \theta_K'(Z) \Big) \, d\xi \, dx \\ = &-\gamma^4 \int (1 + |\xi|^2 + |x|^2)^{\frac{\alpha}{2}} \lambda(\xi)^2 \Big[ \theta_K'(\gamma(x - \xi)) \Big]^2 \, d\xi \, dx \\ \neq &\, 0 \,. \end{aligned}$$

Since

$$\langle \lambda t_u(u), \lambda \mathbb{J} t_\xi(u) \rangle_{L^{2,\alpha}(\mathbb{R}^2) \oplus L^{2,a}(\mathbb{R}^2)} \neq 0,$$

due to Lemma 20.46 there exist

$$\begin{aligned}&\begin{pmatrix}\bar\theta\\ \bar\psi\end{pmatrix}=\begin{pmatrix}\bar\theta(u)\\ \bar\psi(u)\end{pmatrix}\in H^{2,\alpha}(\mathbb{R}^2)\oplus H^{1,\alpha}(\mathbb{R}^2)\cap(\ker\mathcal{L}_u^\alpha)^{\perp,L^{2,\alpha}\oplus L^{2,\alpha}}\quad\text{and}\\ &0\neq\mu=\mu(u)\in H^{2,\alpha}(\mathbb{R})\ \text{ s.t. }\ \lambda(u)t_u(u)=\mathcal{L}_u^\alpha\begin{pmatrix}\bar\theta(u)\\ \bar\psi(u)\end{pmatrix}+\mu(u)\mathbb{J}t_\xi(u)\,.\end{aligned}\tag{20.12}$$

This is an identity in $H^{1,\alpha}(\mathbb{R}^2)\oplus L^{2,\alpha}(\mathbb{R}^2)$. We fix $\xi$ and pair this identity with $\mathbb{J}t_\xi(\xi,u,\cdot)$ in $L^2_x(\mathbb{R})\oplus L^2_x(\mathbb{R})$ yields due to Lemma 20.5 for a.e. $\xi\in\mathbb{R}$:

$$\lambda(\xi,u)\gamma(u)^3m=\mu(\xi,u)\|\mathbb{J}t_\xi(\xi,u,\cdot)\|^2_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})}=\mu(\xi,u)(u^2\gamma^3\,|\theta_K''|^2_{L^2(\mathbb{R})}+\gamma\,|\theta_K'|^2_{L^2(\mathbb{R})}),$$

since

$$\begin{aligned}&\|\mathbb{J}t_\xi(\xi,u,\cdot)\|^2_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})}\\ &=\|\begin{pmatrix}-u\gamma^2\theta_K''(\gamma(\cdot-\xi))\\ -\gamma\theta_K'(\gamma(\cdot-\xi))\end{pmatrix}\|^2_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})}\\ &=u^2\gamma^3\,|\theta_K''|^2_{L^2(\mathbb{R})}+\gamma\,|\theta_K'|^2_{L^2(\mathbb{R})}\ .\end{aligned}$$

Consequently

$$\lambda(u)=\eta(u)\mu(u)\,,$$

where

$$\eta(u):=\frac{u^2\gamma^3\,|\theta_K''|^2_{L^2(\mathbb{R})}+\gamma\,|\theta_K'|^2_{L^2(\mathbb{R})}}{\gamma(u)^3m}\in\mathbb{R}\,.$$

Thus using (20.12) we obtain

$$\begin{aligned}&\lambda(u)\mathbb{J}t_\xi(u)\\ &=\eta(u)\mu(u)\mathbb{J}t_\xi(u)\\ &=\eta(u)\lambda(u)t_u(u)-\eta(u)\mathcal{L}_u^\alpha\begin{pmatrix}\bar\theta(u)\\ \bar\psi(u)\end{pmatrix}\end{aligned}$$

and

$$\begin{pmatrix}v\\ w\end{pmatrix}=\hat{\mathcal{L}}_u^\alpha\begin{pmatrix}\theta(u)\\ \psi(u)\end{pmatrix}+\lambda(u)\mathbb{J}t_\xi(u)=\hat{\mathcal{L}}_u^\alpha\left(\begin{pmatrix}\theta(u)\\ \psi(u)\end{pmatrix}-\eta(u)\begin{pmatrix}\bar\theta(u)\\ \bar\psi(u)\end{pmatrix}\right)+\eta(u)\lambda(u)t_u(u)\,.$$

The sum is direct, i.e.,

$$\{\lambda t_u(u);\lambda\in H^{2,\alpha}(\mathbb{R})\}\cap\mathcal{L}_u^\alpha\left(\left[H^{2,\alpha}(\mathbb{R}^2)\oplus H^{1,\alpha}(\mathbb{R}^2)\right]\cap(\ker\mathcal{L}_u^\alpha)^{\perp,L^{2,\alpha}\oplus L^{2,\alpha}}\right)=0$$

due to (20.12). □

**Corollary 20.48.** *(a)* $\ker \mathcal{M}_u^\alpha = \left\{ \mu(\xi) \begin{pmatrix} -\theta_K'(\gamma(x-\xi)) \\ u\gamma\theta_K''(\gamma(x-\xi)) \\ 0 \end{pmatrix},\ \mu \in H^{2,\alpha}(\mathbb{R}) \right\}.$

*(b)* $\hat{\mathcal{M}}_u^\alpha$ *is one-to-one, onto, bounded and the inverse mapping is also bounded.*

**Proof.** Follows from Lemma 20.14, Corollary 20.18 and the inverse mapping theorem. □

### 20.2.2 Inverse Operator

**Definition 20.49.**

*(a)* $X = \mathbb{R}$ *with the absolute value on* $\mathbb{R}$ *as a norm.*

*(b)* $\bar{Y}^\alpha = H^{2,\alpha}(\mathbb{R}^2) \oplus H^{1,\alpha}(\mathbb{R}^2) \oplus H^{2,\alpha}(\mathbb{R})$ *with the finite norm*

$$|y|_{\bar{Y}^\alpha} = |\theta|_{H^{2,\alpha}(\mathbb{R}^2)} + |\psi|_{H^{1,\alpha}(\mathbb{R}^2)} + |\lambda|_{H^{2,\alpha}(\mathbb{R})}\,.$$

*(c)* $\bar{Z}^\alpha = H^{1,\alpha}(\mathbb{R}^2) \oplus L^{2,\alpha}(\mathbb{R}^2)$ *with the finite norm*

$$|z|_{\bar{Z}^\alpha} = |v|_{H^{1,\alpha}(\mathbb{R}^2)} + |w|_{L^{2,\alpha}(\mathbb{R}^2)}\,.$$

*(d)*
$$\begin{aligned} Y^\alpha &= Y^\alpha(u_*) \\ &= \Bigg\{ y = (\theta, \psi, \lambda_u) \in C(I(u_*), \bar{Y}^\alpha) : \|y\|_{Y^\alpha(u_*)} < \infty;\ \forall\ u \in I(u_*),\ \forall\ \lambda \in H^{2,\alpha}(\mathbb{R}) : \\ &\quad \left\langle \begin{pmatrix} \theta(u)(\xi,x) \\ \psi(u)(\xi,x) \end{pmatrix}, \lambda(\xi) \begin{pmatrix} \theta_K'(\gamma(x-\xi)) \\ -u\gamma\theta_K''(\gamma(x-\xi)) \end{pmatrix} \right\rangle_{L^{2,\alpha}_{\xi,x}(\mathbb{R}^2) \oplus L^{2,\alpha}_{\xi,x}(\mathbb{R}^2)} = 0 \Bigg\} \end{aligned}$$

*with the finite norm*

$$\|y\|_{Y^\alpha(u_*)} = \sup_{u \in I(u_*)} |y|_{\bar{Y}^\alpha}.$$

*(e)* $Z^\alpha = Z^\alpha(u_*) = \left\{ z = (v, w) \in C(I(u_*), \bar{Z}^\alpha) : \|z\|_{Z^\alpha(u_*)} < \infty \right\}$ *with the finite norm*

$$\|z\|_{Z^\alpha(u_*)} = \sup_{u \in I(u_*)} |z|_{\bar{Z}^\alpha}\,.$$

**Lemma 20.50.** *The linear operator*

$$\tilde{\mathfrak{M}}^\alpha : Y^\alpha \to Z^\alpha ,$$

$$\begin{pmatrix} \tilde{\theta} \\ \tilde{\psi} \\ \tilde{\lambda}_u \end{pmatrix} \mapsto \tilde{\mathfrak{M}}^\alpha \begin{pmatrix} \tilde{\theta} \\ \tilde{\psi} \\ \tilde{\lambda}_u \end{pmatrix} ,$$

*given by*

$$\tilde{\mathfrak{M}}^\alpha \begin{pmatrix} \tilde{\theta} \\ \tilde{\psi} \\ \tilde{\lambda}_u \end{pmatrix} (u) = \hat{\mathcal{M}}^\alpha_u \begin{pmatrix} \tilde{\theta}(u) \\ \tilde{\psi}(u) \\ \tilde{\lambda}_u(u) \end{pmatrix} ,$$

*is invertible.*

**Proof.** Analogous to the proof of Lemma 20.22 by using Corollary 20.47 and Lemma 20.43.
□

**Lemma 20.51 (Norm of $\left[\hat{M}^\alpha_u\right]^{-1}$).** *There exists a constant $c^\alpha > 0$ such that*

$$\left\| \left[\hat{M}^\alpha_u\right]^{-1} \right\|_{L(L^{2,\alpha}(\mathbb{R}^2), H^{2,\alpha}(\mathbb{R}^2) \oplus H^{2,\alpha}(\mathbb{R}))} \leq \gamma(u) c^\alpha \left\| \left[\hat{M}^\alpha\right]^{-1} \right\|_{L(L^{2,\alpha}(\mathbb{R}^2), H^{2,\alpha}(\mathbb{R}^2) \oplus H^{2,\alpha}(\mathbb{R}))} .$$

**Proof.** Analogous to the proof of Lemma 20.23. □

**Lemma 20.52 (Norm of $\left[\hat{\mathcal{M}}^\alpha_u\right]^{-1}$).** *There exists a continuous function $C^\alpha : (-1, 1) \to \mathbb{R}$ such that*

$$\left\| \left[\hat{\mathcal{M}}^\alpha_u\right]^{-1} \right\| \leq C^\alpha(u) .$$

**Proof.** Analogous to the proof of Lemma 20.24 by using Corollary 20.47 and Lemma 20.51.
□

**Definition 20.53.** *Let $C^\alpha$ be a specific fixed function from Lemma 20.52. Set*

$$[\tilde{u}^\alpha]^* = \tilde{u}^*(\|[\hat{M}^\alpha]^{-1}\|) = \sup\{u \in (-1,1) \mid \forall s, t \in \mathbb{R} : \ |s|, |t| \leq |u| : |s| C^\alpha(t) < 1\} .$$

**Corollary 20.54.** *The linear operator*

$$\mathfrak{M}^{\alpha} : Y^{\alpha}(u_*) \to Z^{\alpha}(u_*),$$

$$\begin{pmatrix} \tilde{\theta} \\ \tilde{\psi} \\ \tilde{\lambda}_u \end{pmatrix} \mapsto \mathfrak{M}^{\alpha} \begin{pmatrix} \tilde{\theta} \\ \tilde{\psi} \\ \tilde{\lambda}_u \end{pmatrix} ,$$

*given by*

$$\mathfrak{M}^{\alpha} \begin{pmatrix} \tilde{\theta} \\ \tilde{\psi} \\ \tilde{\lambda}_u \end{pmatrix} (u) = \hat{\mathcal{N}}_u^{\alpha} \begin{pmatrix} \tilde{\theta}(u) \\ \tilde{\psi}(u) \\ \tilde{\lambda}_u(u) \end{pmatrix} ,$$

*is invertible if* $u_* < [\tilde{u}^{\alpha}]^*$.

**Proof.** Analogous to the proof of Lemma 20.26 by using Lemma 20.50 and Lemma 20.52. □

### 20.2.3 Inverse Operator in Spaces of Higher Regularity

Let $n \in \mathbb{N}$.

**Definition 20.55.**

*(a)* $H_{\perp}^{3,\alpha}(\mathbb{R}^2) := \{\theta \in H^{3,\alpha}(\mathbb{R}^2) \mid \forall \lambda \in H^2(\mathbb{R}) : \ \langle \theta(\xi, Z), \lambda(\xi)\theta_K'(Z)\rangle_{L^{2,\alpha}_{\xi,Z}(\mathbb{R}^2)} = 0\}$.

*(b)* $\underline{\hat{L}}^{\alpha} : H_{\perp}^{3,\alpha}(\mathbb{R}^2) \subset L^{2,\alpha}(\mathbb{R}^2) \to H^{1,\alpha}(\mathbb{R}^2)$, *where*

$$(\underline{\hat{L}}^{\alpha}\theta)(\xi, Z) = -\partial_Z^2\theta(\xi, Z) + \cos(\theta_K(Z))\theta(\xi, Z) \,.$$

*(c)* $\underline{\hat{M}}^{\alpha} : H_{\perp}^{3,\alpha}(\mathbb{R}^2) \oplus H^{2,\alpha}(\mathbb{R}) \to H^{1,\alpha}(\mathbb{R}^2)$, *where*

$$\left(\underline{\hat{M}}^{\alpha} \begin{pmatrix} \theta \\ \lambda \end{pmatrix}\right)(\xi, Z) = (\underline{\hat{L}}^{\alpha}\theta)(\xi, Z) + \lambda(\xi)\theta_K'(Z) \,.$$

*(d)* $\underline{\hat{\mathcal{L}}_u^{\alpha}} : \left[H^{3,\alpha}(\mathbb{R}^2) \oplus H^{2,\alpha}(\mathbb{R}^2)\right] \cap \ker \mathcal{L}_u^{\perp, L^{2,\alpha} \oplus L^{2,\alpha}} \to H^{2,\alpha}(\mathbb{R}^2) \oplus H^{1,\alpha}(\mathbb{R}^2)$, *where*

$$\left(\underline{\hat{\mathcal{L}}_u^{\alpha}} \begin{pmatrix} \theta \\ \psi \end{pmatrix}\right)(\xi, x) = \begin{pmatrix} -u\partial_x\theta(\xi, x) - \psi(\xi, x) \\ -\partial_x^2\theta(\xi, x) + \cos(\theta_K(\gamma(x-\xi)))\theta(\xi, x) - u\partial_x\psi(\xi, x) \end{pmatrix} .$$

*(e)* $\underline{\hat{\mathcal{M}}^\alpha_u} : \left[H^{3,\alpha}(\mathbb{R}^2) \oplus H^{2,\alpha}(\mathbb{R}^2)\right] \cap \ker \mathcal{L}_u^{\perp, L^{2,\alpha}\oplus L^{2,\alpha}} \oplus H^{2,\alpha}(\mathbb{R}) \to H^{2,\alpha}(\mathbb{R}^2) \oplus H^{1,\alpha}(\mathbb{R}^2)$, *where*

$$\left(\underline{\hat{\mathcal{M}}^\alpha_u} \begin{pmatrix} \theta \\ \psi \\ \lambda \end{pmatrix}\right)(\xi, x) = \left(\underline{\hat{\mathcal{L}}^\alpha_u} \begin{pmatrix} \theta \\ \psi \end{pmatrix}\right)(\xi, x) + \lambda(\xi) t_u(\xi, u, x)\,.$$

**Definition 20.56.** *Let* $\underline{C}^\alpha : (-1,1) \to \mathbb{R}$ *be a specific fixed continuous function such that*

$$\left\| \left[\underline{\hat{\mathcal{M}}^\alpha_u}\right]^{-1} \right\| \le \underline{C}^\alpha(u)\,.$$

*Set*

$$[\underline{u}^\alpha]^* = \underline{u}^*(\| \left[\underline{\hat{M}}^\alpha\right]^{-1} \|) = \sup\{u \in (-1,1) \mid \forall s, t \in \mathbb{R} :\ |s|, |t| \le |u| : |s| \underline{C}^\alpha(t) < 1\}\,.$$

**Remark 20.57.** The existence of the function $\underline{C}^\alpha$ in Definition 20.56 is ensured analogously to Section 20.1.3 by using the statements of Section 20.2.2 with higher regularity in $(\xi, x)$.

**Definition 20.58.** *(a)* $(\bar{Z}^\alpha)' = H^{2,\alpha}(\mathbb{R}^2) \oplus H^{1,\alpha}(\mathbb{R}^2)$ *with the finite norm*

$$|z|_{(\bar{Z}^\alpha)'} = |v|_{H^{2,\alpha}(\mathbb{R}^2)} + |w|_{H^{1,\alpha}(\mathbb{R}^2)}\,.$$

*(b)* $Z^\alpha_n = Z^\alpha_n(u_*) = \left\{ z = (v, w) \in C^n(I(u_*), (\bar{Z}^\alpha)') : \|z\|_{Z^\alpha_n(u_*)} < \infty \right\}$
*with the finite norm*

$$\|z\|_{Z^\alpha_n(u_*)} = \sup_{u \in I(u_*)} \left( \sum_{i=0}^n |\partial^i_u y(u)|_{(\bar{Z}^\alpha)'} \right)\,.$$

*(c)* $\underline{\hat{\mathcal{K}}^\alpha_u} : \left[H^{3,\alpha}(\mathbb{R}^2) \oplus H^{2,\alpha}(\mathbb{R}^2)\right] \cap \ker \mathcal{L}_u^{\perp, L^{2,\alpha}\oplus L^{2,\alpha}} \to H^{2,\alpha}(\mathbb{R}^2) \oplus H^{1,\alpha}(\mathbb{R}^2)$, *where*

$$\underline{\hat{\mathcal{K}}^\alpha_u} \begin{pmatrix} \theta \\ \psi \end{pmatrix}(\xi, x) = \begin{pmatrix} u\partial_\xi \theta(\xi, x) - \psi(\xi, x) \\ -\partial_x^2 \theta(\xi, x) + \cos(\theta_K(\gamma(x - \xi)))\theta(\xi, x) + u\partial_\xi \psi(\xi, x) \end{pmatrix}\,.$$

*(d)* $\underline{\hat{\mathcal{N}}^\alpha_u} : \left[H^{3,\alpha}(\mathbb{R}^2) \oplus H^{2,\alpha}(\mathbb{R}^2)\right] \cap \ker \mathcal{L}_u^{\perp, L^{2,\alpha}\oplus L^{2,\alpha}} \oplus H^{2,\alpha}(\mathbb{R}) \to H^{2,\alpha}(\mathbb{R}^2) \oplus H^{1,\alpha}(\mathbb{R}^2)$, *where*

$$\underline{\hat{\mathcal{N}}^\alpha_u} \begin{pmatrix} \theta \\ \psi \\ \lambda \end{pmatrix}(\xi, x) = \left(\underline{\hat{\mathcal{K}}^\alpha_u} \begin{pmatrix} \theta \\ \psi \end{pmatrix}\right)(\xi, x) + \lambda(\xi) t_u(\xi, u, x)\,.$$

**Remark 20.59.** Recall that the spaces $(\bar{Y}^\alpha)'$, $Y_n^\alpha(u_*)$ were defined in Chapter 19 by

(a) $(\bar{Y}^\alpha)' = H^{3,\alpha}(\mathbb{R}^2) \oplus H^{2,\alpha}(\mathbb{R}^2) \oplus H^{2,\alpha}(\mathbb{R})$ with the finite norm

$$|y|_{(\bar{Y}^\alpha)'} = |\theta|_{H^{3,\alpha}(\mathbb{R}^2)} + |\psi|_{H^{2,\alpha}(\mathbb{R}^2)} + |\lambda|_{H^{2,\alpha}(\mathbb{R})} \, ,$$

(b)
$$\begin{aligned} Y_n^\alpha &:= Y_n^\alpha(u_*) \\ &= \Bigg\{ y = (\theta, \psi, \lambda_u) \in C^n(I(u_*), (\bar{Y}^\alpha)') : \|y\|_{Y_n^\alpha(u_*)} < \infty; \ \forall\, u \in I(u_*), \ \forall\, \lambda \in H^{2,\alpha}(\mathbb{R}) : \\ &\qquad \left\langle \begin{pmatrix} \theta(u)(\xi, x) \\ \psi(u)(\xi, x) \end{pmatrix}, \lambda(\xi) \begin{pmatrix} \theta_K'(\gamma(x-\xi)) \\ -u\gamma\theta_K''(\gamma(x-\xi)) \end{pmatrix} \right\rangle_{L^{2,\alpha}_{\xi,x}(\mathbb{R}^2) \oplus L^{2,\alpha}_{\xi,x}(\mathbb{R}^2)} = 0 \Bigg\} \end{aligned}$$

with the finite norm

$$\|y\|_{Y_n^\alpha(u_*)} = \sup_{u \in I(u_*)} \left( \sum_{i=0}^{n} |\partial_u^i y(u)|_{(\bar{Y}^\alpha)'} \right) .$$

**Proposition 20.60.** *The linear operator*

$$\mathfrak{M}_n^\alpha : Y_n^\alpha(u_*) \to Z_n^\alpha(u_*),$$
$$\begin{pmatrix} \tilde{\theta} \\ \tilde{\psi} \\ \tilde{\lambda}_u \end{pmatrix} \mapsto \mathfrak{M}_n^\alpha \begin{pmatrix} \tilde{\theta} \\ \tilde{\psi} \\ \tilde{\lambda}_u \end{pmatrix} ,$$

*given by*

$$\mathfrak{M}_n^\alpha \begin{pmatrix} \tilde{\theta} \\ \tilde{\psi} \\ \tilde{\lambda}_u \end{pmatrix} (u) = \underline{\hat{\mathcal{N}}_u^\alpha} \begin{pmatrix} \tilde{\theta}(u) \\ \tilde{\psi}(u) \\ \tilde{\lambda}_u(u) \end{pmatrix} ,$$

*is invertible if* $u_* < [\underline{u}^\alpha]^*$*.*

**Proof.** Analogous to the proof of Lemma 20.28 (using Corollary 20.54) and to the proof of Proposition 20.33.

□

## 20.3 Implicit Function Theorem

Let $\alpha, n \in \mathbb{N}$. Let $\tilde{F} : (-1,1) \to H^{1,\alpha}(\mathbb{R}^2)\,, \varepsilon \mapsto \tilde{F}(\varepsilon)$ be a $C^n$ function and $\tilde{F}(0) = 0$. $\tilde{F}(\varepsilon)$ depends on $(\xi, x)$. As mentioned in Chapter 19, we are going to solve certain equations successively by using the implicit function theorem. $(\theta_0, \psi_0)$, defined in (3) dependes on $(\xi, u, x)$ and solves the equation characterizing the classical solitons:

$$\underbrace{u\partial_\xi \begin{pmatrix} \theta \\ \psi \end{pmatrix} - \begin{pmatrix} \psi \\ \partial_x^2\theta - \sin\theta \end{pmatrix}}_{=:\, \mathcal{G}_0(\theta,\psi)} = 0\,.$$

Using $(\theta_0, \psi_0)$ we define the function $\mathcal{G}_1$ and solve the equation

$$\underbrace{u\partial_\xi \begin{pmatrix} \theta \\ \psi \end{pmatrix} - \begin{pmatrix} \psi \\ \theta_{xx} - \sin\theta + \tilde{F}(\varepsilon) \end{pmatrix} + \lambda_u \partial_u \begin{pmatrix} \theta_0 \\ \psi_0 \end{pmatrix}}_{=:\, \mathcal{G}_1^\varepsilon(\theta,\psi,\lambda_u)} = 0\,, \qquad (20.13)$$

implicitly for $(\theta, \psi, \lambda_u)$ in terms of $\varepsilon$. We call the implicit solution $(\theta_1^\varepsilon, \psi_1^\varepsilon, \lambda_{u,1}^\varepsilon)$. The functions $\theta_1^\varepsilon, \psi_1^\varepsilon$ depend on $(\xi, u, x)$ and the function $\lambda_{u,1}^\varepsilon$ depends on $(\xi, u)$. Using $(\theta_1^\varepsilon, \psi_1^\varepsilon, \lambda_{u,1}^\varepsilon)$ we define the function $\mathcal{G}_2$ and solve the equation

$$\underbrace{u\partial_\xi \begin{pmatrix} \theta \\ \psi \end{pmatrix} - \begin{pmatrix} \psi \\ \theta_{xx} - \sin\theta + \tilde{F}(\varepsilon) \end{pmatrix} + \lambda_u \partial_u \begin{pmatrix} \theta_1^0 + \partial_\varepsilon \theta_1^0 \varepsilon \\ \psi_1^0 + \partial_\varepsilon \psi_1^0 \varepsilon \end{pmatrix}}_{=:\, \mathcal{G}_2^\varepsilon(\theta,\psi,\lambda_u)} = 0\,, \qquad (20.14)$$

implicitly for $(\theta, \psi, \lambda_u)$ in terms of $\varepsilon$. We call the implicit solution $(\theta_2^\varepsilon, \psi_2^\varepsilon, \lambda_{u,2}^\varepsilon)$. The functions $\theta_2^\varepsilon, \psi_2^\varepsilon$ depend on $(\xi, u, x)$ and the function $\lambda_{u,2}^\varepsilon$ depends on $(\xi, u)$. Using $(\theta_2^\varepsilon, \psi_2^\varepsilon, \lambda_{u,2}^\varepsilon)$ we define the function $\mathcal{G}_3$ and solve the equation

$$\underbrace{u\partial_\xi \begin{pmatrix} \theta \\ \psi \end{pmatrix} - \begin{pmatrix} \psi \\ \theta_{xx} - \sin\theta + \tilde{F}(\varepsilon) \end{pmatrix} + \lambda_u \partial_u \begin{pmatrix} \theta_2^0 + \partial_\varepsilon \theta_2^0 \varepsilon + \frac{\partial_\varepsilon^2 \theta_2^0}{2}\varepsilon^2 \\ \psi_2^0 + \partial_\varepsilon \psi_2^0 \varepsilon + \frac{\partial_\varepsilon^2 \psi_2^0}{2}\varepsilon^2 \end{pmatrix}}_{=:\, \mathcal{G}_3^\varepsilon(\theta,\psi,\lambda_u)} = 0\,.$$

$$\vdots$$

After solving the $n$ preceding equations we define the function $\mathcal{G}_n$ and solve the equation

$$\underbrace{u\partial_\xi \begin{pmatrix} \theta \\ \psi \end{pmatrix} - \begin{pmatrix} \psi \\ \theta_{xx} - \sin\theta + \tilde{F}(\varepsilon) \end{pmatrix} + \lambda_u \partial_u \begin{pmatrix} \sum_{i=0}^{n-1} \frac{\partial_\varepsilon^i \theta_{n-1}^0}{i!}\varepsilon^i \\ \sum_{i=0}^{n-1} \frac{\partial_\varepsilon^i \psi_{n-1}^0}{i!}\varepsilon^i \end{pmatrix}}_{=:\, \mathcal{G}_n^\varepsilon(\theta,\psi,\lambda_u)} = 0\,. \qquad (20.15)$$

implicitly for $(\theta, \psi, \lambda_u)$ in terms of $\varepsilon$. We formalize the procedure in the following theorem. This part of the thesis emerged during my visit at Brown University, where we had discussions on this topic with Justin Holmer.

**Theorem 20.61.** *Let* $0 \in I_\varepsilon \subset \mathbb{R}$, $u_* < [\underline{u}^\alpha]^*$. *Let* $\tilde{F} : (-1,1) \to H^{1,\alpha}(\mathbb{R}^2)\,, \varepsilon \mapsto \tilde{F}(\varepsilon)$ *be a* $C^n$ *function and* $\tilde{F}(0) = 0$. *Let* $(\theta_0, \psi_0)$ *be given by* (3). *There exists* $\varepsilon^* > 0$ *such that the following holds. Let* $\tilde{\mathcal{G}}_1$ *be given by*

$$\begin{aligned}&\tilde{\mathcal{G}}_1 : I_\varepsilon \times Y^\alpha_{n+1}(u_*) \to Z^\alpha_{n+1}(u_*),\\&(\varepsilon, \hat{\theta}, \hat{\psi}, \lambda_u) \mapsto \tilde{\mathcal{G}}^\varepsilon_1(\hat{\theta}, \hat{\psi}, \lambda_u) := \mathcal{G}^\varepsilon_1(\theta_0 + \hat{\theta}, \psi_0 + \hat{\psi}, \lambda_u),\end{aligned}$$

*where* $\mathcal{G}_1$ *is defined by* (20.13). *There exists a map*

$$\begin{aligned}&(-\varepsilon^*, +\varepsilon^*) \to Y^\alpha_{n+1}(u_*),\\&\varepsilon \mapsto (\hat{\theta}^\varepsilon_1, \hat{\psi}^\varepsilon_1, \lambda^\varepsilon_{u,1}),\end{aligned}$$

*of class* $C^n$ *such that* $\tilde{\mathcal{G}}^\varepsilon_1(\hat{\theta}^\varepsilon_1, \hat{\psi}^\varepsilon_1, \lambda^\varepsilon_{u,1}) = 0$. *Let* $\tilde{\mathcal{G}}_2$ *be given by*

$$\begin{aligned}&\tilde{\mathcal{G}}_2 : I_\varepsilon \times Y^\alpha_{n}(u_*) \to Z^\alpha_{n}(u_*),\\&(\varepsilon, \hat{\theta}, \hat{\psi}, \lambda_u) \mapsto \tilde{\mathcal{G}}^\varepsilon_2(\hat{\theta}, \hat{\psi}, \lambda_u) := \mathcal{G}^\varepsilon_2(\theta_0 + \hat{\theta}, \psi_0 + \hat{\psi}, \lambda_u),\end{aligned}$$

*where* $\mathcal{G}_2$ *is defined by* (20.14) *with* $(\theta^\varepsilon_1, \psi^\varepsilon_1, \lambda^\varepsilon_{u,1}) = (\theta_0 + \hat{\theta}^\varepsilon_1, \psi_0 + \hat{\psi}^\varepsilon_1, \lambda^\varepsilon_{u,1})$. *There exists a map*

$$\begin{aligned}&(-\varepsilon^*, +\varepsilon^*) \to Y^\alpha_{n}(u_*),\\&\varepsilon \mapsto (\hat{\theta}^\varepsilon_2, \hat{\psi}^\varepsilon_2, \lambda^\varepsilon_{u,2}),\end{aligned}$$

*of class* $C^n$ *such that* $\tilde{\mathcal{G}}^\varepsilon_2(\hat{\theta}^\varepsilon_2, \hat{\psi}^\varepsilon_2, \lambda^\varepsilon_{u,2}) = 0$. *This process can be continued successively to arrive at* $\tilde{\mathcal{G}}_n$ *be given by*

$$\begin{aligned}&\tilde{\mathcal{G}}_n : I_\varepsilon \times Y^\alpha_{2}(u_*) \to Z^\alpha_{2}(u_*),\\&(\varepsilon, \hat{\theta}, \hat{\psi}, \lambda_u) \mapsto \tilde{\mathcal{G}}^\varepsilon_n(\hat{\theta}, \hat{\psi}, \lambda_u) := \mathcal{G}^\varepsilon_n(\theta_0 + \hat{\theta}, \psi_0 + \hat{\psi}, \lambda_u),\end{aligned}$$

*where* $\mathcal{G}_n$ *is defined by* (20.15) *with* $(\theta^\varepsilon_{n-1}, \psi^\varepsilon_{n-1}, \lambda^\varepsilon_{u,n-1}) = (\theta_0 + \hat{\theta}^\varepsilon_{n-1}, \psi_0 + \hat{\psi}^\varepsilon_{n-1}, \lambda^\varepsilon_{u,n-1})$. *There exists a map*

$$\begin{aligned}&(-\varepsilon^*, +\varepsilon^*) \to Y^\alpha_{2}(u_*),\\&\varepsilon \mapsto (\hat{\theta}^\varepsilon_n, \hat{\psi}^\varepsilon_n, \lambda^\varepsilon_{u,n}),\end{aligned}$$

*of class* $C^n$ *such that* $\tilde{\mathcal{G}}^\varepsilon_n(\hat{\theta}^\varepsilon_n, \hat{\psi}^\varepsilon_n, \lambda^\varepsilon_{u,n}) = 0$. *We set* $(\theta^\varepsilon_n, \psi^\varepsilon_n, \lambda^\varepsilon_{u,n}) = (\theta_0 + \hat{\theta}^\varepsilon_n, \psi_0 + \hat{\psi}^\varepsilon_n, \lambda^\varepsilon_{u,n})$.

**Proof.** We skip $u_*$ in the notation. Notice that the functions $\tilde{\mathcal{G}}_k$ for $1 \leq k \leq n$ are well defined. We are going to consider the first equation. Notice:

$$\tilde{\mathcal{G}}_1^0(0,0,0) = \mathcal{G}_1^0(\theta_0, \psi_0, 0) = 0\,.$$

Let $Z = \gamma(x-\xi)$. The derivative of

$$\tilde{\mathcal{G}}_1 : I_\varepsilon \times Y_{n+1}^\alpha \to Z_{n+1}^\alpha$$

with respect to $(\hat{\theta}, \hat{\psi}, \lambda_u)$ evaluated at $(\varepsilon, \hat{\theta}, \hat{\psi}, \lambda_u) = (0,0,0,0)$, applied to $\begin{bmatrix} \tilde{\theta} \\ \tilde{\psi} \\ \tilde{\lambda_u} \end{bmatrix}$ is:

$$\begin{aligned} & D\mathcal{G}_1^\varepsilon(\theta_0, \psi_0, 0)\Big|_{\varepsilon\,=\,0} \begin{bmatrix} \tilde{\theta} \\ \tilde{\psi} \\ \tilde{\lambda_u} \end{bmatrix} \\ &= \underbrace{\begin{pmatrix} u\partial_\xi \tilde{\theta} - \tilde{\psi} \\ -\partial_x^2 \tilde{\theta} + \cos(\theta_K(Z))\tilde{\theta} + u\partial_\xi \tilde{\psi} \end{pmatrix}}_{= \underline{\hat{\mathcal{K}}_u^\alpha} \begin{pmatrix} \tilde{\theta} \\ \tilde{\psi} \end{pmatrix}} + \tilde{\lambda_u} \begin{pmatrix} u\gamma^3(x-\xi)\theta_K'(Z) \\ -\gamma^3\theta_K'(Z) - u^2\gamma^4(x-\xi)\theta_K''(Z) \end{pmatrix} \end{aligned}$$

$D\mathcal{G}_1^\varepsilon(\theta_0, \psi_0, 0)\Big|_{\varepsilon\,=\,0}$ is invertible due to Proposition 20.60. By the implicit function theorem there exists a $\varepsilon_1^* > 0$ and a map

$$\begin{aligned} &(-\varepsilon_1^*, +\varepsilon_1^*) \to Y_{n+1}^\alpha, \\ &\varepsilon \mapsto (\hat{\theta}_1^\varepsilon, \hat{\psi}_1^\varepsilon, \lambda_{u,1}^\varepsilon) \end{aligned}$$

of class $C^n$ such that

$$\tilde{\mathcal{G}}_1^\varepsilon(\hat{\theta}_1^\varepsilon, \hat{\psi}_1^\varepsilon, \lambda_{u,1}^\varepsilon) = 0\,.$$

Now we consider the second equation. Notice that

$$\tilde{\mathcal{G}}_2^0(0,0,0) = \mathcal{G}_2^0(\theta_0, \psi_0, 0) = 0\,.$$

We compute the derivative of

$$\tilde{\mathcal{G}}_2 : I_\varepsilon \times Y_n^\alpha \to Z_n^\alpha$$

with respect to $(\hat{\theta}, \hat{\psi}, \lambda_u)$ evaluated at $(\varepsilon, \hat{\theta}, \hat{\psi}, \lambda_u) = (0, 0, 0, 0)$, applied to $\begin{bmatrix} \tilde{\theta} \\ \tilde{\psi} \\ \tilde{\lambda_u} \end{bmatrix}$:

$$
\begin{aligned}
& D\mathcal{G}_2^\varepsilon(\theta_0, \psi_0, 0)\Big|_{\varepsilon = 0} \begin{bmatrix} \tilde{\theta} \\ \tilde{\psi} \\ \tilde{\lambda_u} \end{bmatrix} \\
&= \underbrace{\begin{pmatrix} u\partial_\xi \tilde{\theta} - \tilde{\psi} \\ -\partial_x^2 \tilde{\theta} + \cos(\theta_K(Z))\tilde{\theta} + u\partial_\xi \tilde{\psi} \end{pmatrix}}_{= \underline{\hat{\mathcal{K}}_u^\alpha} \begin{pmatrix} \tilde{\theta} \\ \tilde{\psi} \end{pmatrix}} + \tilde{\lambda_u} \begin{pmatrix} u\gamma^3(x - \xi)\theta_K'(Z) \\ -\gamma^3\theta_K'(Z) - u^2\gamma^4(x - \xi)\theta_K''(Z) \end{pmatrix}
\end{aligned}
$$

$D\mathcal{G}_2^\varepsilon(\theta_0, \psi_0, 0)\Big|_{\varepsilon = 0}$ is invertible due to Proposition 20.60. By the implicit function theorem there exists a $\varepsilon_2^* > 0$ and a map

$$
\begin{aligned}
&(-\varepsilon_2^*, +\varepsilon_2^*) \to Y_n^\alpha, \\
&\varepsilon \mapsto (\hat{\theta}_2^\varepsilon, \hat{\psi}_2^\varepsilon, \lambda_{u,2}^\varepsilon)
\end{aligned}
$$

of class $C^n$ such that

$$\tilde{\mathcal{G}}_2^\varepsilon(\hat{\theta}_2^\varepsilon, \hat{\psi}_2^\varepsilon, \lambda_{u,2}^\varepsilon) = 0\,.$$

By the same argument we find a $\varepsilon_n^* > 0$ and a map

$$
\begin{aligned}
&(-\varepsilon_n^*, +\varepsilon_n^*) \to Y_2^\alpha, \\
&\varepsilon \mapsto (\hat{\theta}_n^\varepsilon, \hat{\psi}_n^\varepsilon, \lambda_{u,n}^\varepsilon)
\end{aligned}
$$

of class $C^n$ such that

$$\tilde{\mathcal{G}}_n^\varepsilon(\hat{\theta}_n^\varepsilon, \hat{\psi}_n^\varepsilon, \lambda_{u,n}^\varepsilon) = 0\,.$$

Set $\varepsilon^* = \min\{\varepsilon_1^*, \varepsilon_2^*, \dots, \varepsilon_n^*\}$. □

The following theorem shows a relation between the iterative solutions from the previous theorem. In addition it gives equations that are satisfied by the iterative solutions up to a certain order in $\varepsilon$.

**Theorem 20.62.** *Let the assumptions of Theorem 20.61 hold. Then*

*(a)* *(1)* $(\theta_1^0, \psi_1^0, \lambda_{u,1}^0) = (\theta_2^0, \psi_2^0, \lambda_{u,2}^0)$.

*(2)* $(\partial_\varepsilon \theta_1^0, \partial_\varepsilon \psi_1^0, \partial_\varepsilon \lambda_{u,1}^0) = (\partial_\varepsilon \theta_2^0, \partial_\varepsilon \psi_2^0, \partial_\varepsilon \lambda_{u,2}^0)$.

*(3)* $\forall u \in I$ :

$$u\partial_\xi \begin{pmatrix} \theta_2^\varepsilon \\ \psi_2^\varepsilon \end{pmatrix} - \begin{pmatrix} \psi_2^\varepsilon \\ [\theta_2^\varepsilon]_{xx} - \sin\theta_2^\varepsilon + \tilde{F}(\varepsilon) \end{pmatrix} + \lambda_{u,2}^\varepsilon \begin{pmatrix} \partial_u \theta_2^\varepsilon \\ \partial_u \psi_2^\varepsilon \end{pmatrix}$$

$$+ \lambda_{u,2}^\varepsilon \partial_u \begin{pmatrix} \theta_2^0 + \partial_\varepsilon \theta_2^0 \varepsilon - \theta_2^\varepsilon \\ \psi_2^0 + \partial_\varepsilon \psi_2^0 \varepsilon - \psi_2^\varepsilon \end{pmatrix} = 0,$$

*where the following rates of convergence hold:*

$$\left\| \begin{pmatrix} \theta_2^0 + \partial_\varepsilon \theta_2^0 \varepsilon - \theta_2^\varepsilon \\ \psi_2^0 + \partial_\varepsilon \psi_2^0 \varepsilon - \psi_2^\varepsilon \\ 0 \end{pmatrix} \right\|_{Y_n^\alpha(u_*)} = \mathcal{O}(\varepsilon^2)\,,$$

$$\left\| \begin{pmatrix} 0 \\ 0 \\ \lambda_{u,2}^\varepsilon \end{pmatrix} \right\|_{Y_n^\alpha(u_*)} = \mathcal{O}(\varepsilon)\,.$$

*(b)* *(1)* $(\theta_2^0, \psi_2^0, \lambda_{u,2}^0) = (\theta_3^0, \psi_3^0, \lambda_{u,3}^0)$.

*(2)* $(\partial_\varepsilon \theta_2^0, \partial_\varepsilon \psi_2^0, \partial_\varepsilon \lambda_{u,2}^0) = (\partial_\varepsilon \theta_3^0, \partial_\varepsilon \psi_3^0, \partial_\varepsilon \lambda_{u,3}^0)$.

*(3)* $(\partial_\varepsilon^2 \theta_2^0, \partial_\varepsilon^2 \psi_2^0, \partial_\varepsilon^2 \lambda_{u,2}^0) = (\partial_\varepsilon^2 \theta_3^0, \partial_\varepsilon^2 \psi_3^0, \partial_\varepsilon^2 \lambda_{u,3}^0)$.

*(4)* $\forall u \in I$ :

$$u\partial_\xi \begin{pmatrix} \theta_3^\varepsilon \\ \psi_3^\varepsilon \end{pmatrix} - \begin{pmatrix} \psi_3^\varepsilon \\ [\theta_3^\varepsilon]_{xx} - \sin\theta_3^\varepsilon + \tilde{F}(\varepsilon) \end{pmatrix} + \lambda_{u,3}^\varepsilon \begin{pmatrix} \partial_u \theta_3^\varepsilon \\ \partial_u \psi_3^\varepsilon \end{pmatrix}$$

$$+ \lambda_{u,3}^\varepsilon \partial_u \begin{pmatrix} \theta_3^0 + \partial_\varepsilon \theta_3^0 \varepsilon + \frac{\partial_\varepsilon^2 \theta_3^0}{2}\varepsilon^2 - \theta_3^\varepsilon \\ \psi_3^0 + \partial_\varepsilon \psi_3^0 \varepsilon + \frac{\partial_\varepsilon^2 \psi_3^0}{2}\varepsilon^2 - \psi_3^\varepsilon \end{pmatrix} = 0,$$

*where the following rates of convergence hold:*

$$\left\| \begin{pmatrix} \theta_3^0 + \partial_\varepsilon \theta_3^0 \varepsilon + \frac{\partial_\varepsilon^2 \theta_3^0}{2}\varepsilon^2 - \theta_3^\varepsilon \\ \psi_3^0 + \partial_\varepsilon \psi_3^0 \varepsilon + \frac{\partial_\varepsilon^2 \psi_3^0}{2}\varepsilon^2 - \psi_3^\varepsilon \\ 0 \end{pmatrix} \right\|_{Y_{n-1}^\alpha(u_*)} = \mathcal{O}(\varepsilon^3)\,,$$

$$\left\| \begin{pmatrix} 0 \\ 0 \\ \lambda_{u,3}^\varepsilon \end{pmatrix} \right\|_{Y_{n-1}^\alpha(u_*)} = \mathcal{O}(\varepsilon)\,.$$

*(c)* *(1)* $(\partial_\varepsilon^k \theta_{n-1}^0, \partial_\varepsilon^k \psi_{n-1}^0, \partial_\varepsilon^k \lambda_{u,n-1}^0) = (\partial_\varepsilon^k \theta_n^0, \partial_\varepsilon^k \psi_n^0, \partial_\varepsilon^k \lambda_{u,n}^0)$ *for* $k = 0, \ldots, n-1$.

*(2)* $\forall u \in I$ :

$$u\partial_\xi \begin{pmatrix} \theta_n^\varepsilon \\ \psi_n^\varepsilon \end{pmatrix} - \begin{pmatrix} \psi_n^\varepsilon \\ [\theta_n^\varepsilon]_{xx} - \sin\theta_n^\varepsilon + \tilde{F}(\varepsilon) \end{pmatrix} + \lambda_{u,n}^\varepsilon \partial_u \begin{pmatrix} \theta_n^\varepsilon \\ \psi_n^\varepsilon \end{pmatrix} \tag{20.16}$$

$$\underbrace{+\lambda_{u,n}^\varepsilon \partial_u \begin{pmatrix} \sum_{i=0}^{n-1} \frac{\partial_\varepsilon^i \theta_n^0}{i!} \varepsilon^i - \theta_n^\varepsilon \\ \sum_{i=0}^{n-1} \frac{\partial_\varepsilon^i \psi_n^0}{i!} \varepsilon^i - \psi_n^\varepsilon \end{pmatrix}}_{:= \mathcal{R}_n^\varepsilon} = 0\,,$$

*where the following rates of convergence hold:*

$$\left\| \begin{pmatrix} \sum_{i=0}^{n-1} \frac{\partial_\varepsilon^i \theta_n^0}{i!} \varepsilon^i - \theta_n^\varepsilon \\ \sum_{i=0}^{n-1} \frac{\partial_\varepsilon^i \psi_n^0}{i!} \varepsilon^i - \psi_n^\varepsilon \\ 0 \end{pmatrix} \right\|_{Y_2^\alpha(u_*)} = \mathcal{O}(\varepsilon^n)\,,$$

$$\left\| \begin{pmatrix} 0 \\ 0 \\ \lambda_{u,n}^\varepsilon \end{pmatrix} \right\|_{Y_2^\alpha(u_*)} = \mathcal{O}(\varepsilon)\,.$$

**Remark 20.63.** In further computations the term $\mathcal{R}_n^\varepsilon$ in Theorem 20.62 (c)(2) is going to be of order $n+1$ in $\varepsilon$ in an appropriate norm.

**Remark 20.64.** Provided the assumptions of Theorem 20.62 are satisfied then we obtain the following relations:

$$(\partial_\varepsilon^k \theta_1^0, \partial_\varepsilon^k \psi_1^0, \partial_\varepsilon^k \lambda_{u,1}^0) = (\partial_\varepsilon^k \theta_2^0, \partial_\varepsilon^k \psi_2^0, \partial_\varepsilon^k \lambda_{u,2}^0), \quad k = 0, 1\ ;$$

$$(\partial_\varepsilon^k \theta_2^0, \partial_\varepsilon^k \psi_2^0, \partial_\varepsilon^k \lambda_{u,2}^0) = (\partial_\varepsilon^k \theta_3^0, \partial_\varepsilon^k \psi_3^0, \partial_\varepsilon^k \lambda_{u,3}^0), \quad k = 0, 1, 2\ ;$$

$$\vdots$$

$$(\partial_\varepsilon^k \theta_{n-1}^0, \partial_\varepsilon^k \psi_{n-1}^0, \partial_\varepsilon^k \lambda_{u,n-1}^0) = (\partial_\varepsilon^k \theta_n^0, \partial_\varepsilon^k \psi_n^0, \partial_\varepsilon^k \lambda_{u,n}^0), \quad k = 0 \ldots, n-1.$$

**Proof** (of Theorem 20.62)**.** Let $1 \le k \le n$. Notice that the solutions $(\hat{\theta}_k^\varepsilon, \hat{\psi}_k^\varepsilon, \lambda_{u,k}^\varepsilon)$, from Theorem 20.61 satisfy

$$\forall u \in I(u_*) : \quad (\theta_k^\varepsilon(u), \psi_k^\varepsilon(u), \lambda_{u,k}^\varepsilon(u)) \in (\bar{Y}^\alpha)' = H^{3,\alpha}(\mathbb{R}^2) \oplus H^{2,\alpha}(\mathbb{R}^2) \oplus H^{2,\alpha}(\mathbb{R})\,.$$

Due to [Bre11, Corollary 9.13]

$$H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})\,,$$

$$H^2(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2)\,,$$

and these injections are continuous. This allows us to draw conclusions regarding to the differentiability in the $L^\infty$ spaces and regarding to the form of the derivatives.

(a)
(1) clear.
(2)

$$0 = \mathcal{G}_1^\varepsilon(\theta_1^\varepsilon, \psi_1^\varepsilon, \lambda_{u,1}^\varepsilon) = u\partial_\xi \begin{pmatrix} \theta_1^\varepsilon \\ \psi_1^\varepsilon \end{pmatrix} - \begin{pmatrix} \psi_1^\varepsilon \\ [\theta_1^\varepsilon]_{xx} - \sin\theta_1^\varepsilon + \tilde{F}(\varepsilon) \end{pmatrix} + \lambda_{u,1}^\varepsilon \begin{pmatrix} \partial_u \theta_1^0 \\ \partial_u \psi_1^0 \end{pmatrix} .$$

Differentiate with respect to $\varepsilon$:

$$\begin{aligned} 0 &= \partial_\varepsilon \mathcal{G}_1^\varepsilon(\theta_1^\varepsilon, \psi_1^\varepsilon, \lambda_{u,1}^\varepsilon) \\ &= \begin{pmatrix} 0 \\ \partial_\varepsilon \tilde{F}(\varepsilon) \end{pmatrix} + \begin{pmatrix} u\partial_\xi\partial_\varepsilon\theta_1^\varepsilon - \partial_\varepsilon\psi_1^\varepsilon \\ u\partial_\xi\partial_\varepsilon\psi_1^\varepsilon - \partial_x^2\partial_\varepsilon\theta_1^\varepsilon + \cos(\theta_1^\varepsilon)\partial_\varepsilon\theta_1^\varepsilon \end{pmatrix} + \partial_\varepsilon\lambda_{u,1}^\varepsilon \begin{pmatrix} \partial_u \theta_1^0 \\ \partial_u \psi_1^0 \end{pmatrix} . \end{aligned}$$

We get for $\varepsilon = 0$:

$$\begin{aligned} 0 &= \partial_\varepsilon \mathcal{G}_1^0(\theta_K(Z), -\gamma u \theta_K'(Z), 0) \\ &= \begin{pmatrix} 0 \\ \partial_\varepsilon \tilde{F}(0) \end{pmatrix} + \begin{pmatrix} u\partial_\xi\partial_\varepsilon\theta_1^0 - \partial_\varepsilon\psi_1^0 \\ u\partial_\xi\partial_\varepsilon\psi_1^0 - \partial_x^2\partial_\varepsilon\theta_1^0 + \cos(\theta_K(Z))\partial_\varepsilon\theta_1^0 \end{pmatrix} + \partial_\varepsilon\lambda_{u,1}^0 \begin{pmatrix} \partial_u \theta_1^0 \\ \partial_u \psi_1^0 \end{pmatrix} , \end{aligned} \tag{20.17}$$

$$\begin{aligned} 0 &= \mathcal{G}_2^\varepsilon(\theta_2^\varepsilon, \psi_2^\varepsilon, \lambda_{u,2}^\varepsilon) \\ &= u\partial_\xi \begin{pmatrix} \theta_2^\varepsilon \\ \psi_2^\varepsilon \end{pmatrix} - \begin{pmatrix} \psi_2^\varepsilon \\ [\theta_2^\varepsilon]_{xx} - \sin\theta_2^\varepsilon + \tilde{F}(\varepsilon) \end{pmatrix} + \lambda_{u,2}^\varepsilon \begin{pmatrix} \partial_u \theta_1^0 + \partial_u\partial_\varepsilon\theta_1^0\varepsilon \\ \partial_u \psi_1^0 + \partial_u\partial_\varepsilon\psi_1^0\varepsilon \end{pmatrix} . \end{aligned}$$

Differentiate with respect to $\varepsilon$:

$$\begin{aligned} 0 &= \partial_\varepsilon \mathcal{G}_2^\varepsilon(\theta_2^\varepsilon, \psi_2^\varepsilon, \lambda_{u,2}^\varepsilon) \\ &= \begin{pmatrix} 0 \\ \partial_\varepsilon \tilde{F}(\varepsilon) \end{pmatrix} + \begin{pmatrix} u\partial_\xi\partial_\varepsilon\theta_2^\varepsilon - \partial_\varepsilon\psi_2^\varepsilon \\ u\partial_\xi\partial_\varepsilon\psi_2^\varepsilon - \partial_x^2\partial_\varepsilon\theta_2^\varepsilon + \cos(\theta_2^\varepsilon)\partial_\varepsilon\theta_2^\varepsilon \end{pmatrix} \\ &\quad + \partial_\varepsilon\lambda_{u,2}^\varepsilon \begin{pmatrix} \partial_u \theta_1^0 + \partial_u\partial_\varepsilon\theta_1^0\varepsilon \\ \partial_u \psi_1^0 + \partial_u\partial_\varepsilon\psi_1^0\varepsilon \end{pmatrix} + \lambda_{u,2}^\varepsilon \begin{pmatrix} \partial_u\partial_\varepsilon\theta_1^0 \\ \partial_u\partial_\varepsilon\psi_1^0 \end{pmatrix} . \end{aligned}$$

We get for $\varepsilon = 0$:

$$\begin{aligned} 0 &= \partial_\varepsilon \mathcal{G}_2^0(\theta_K(Z), -\gamma u \theta_K'(Z), 0) \\ &= \begin{pmatrix} 0 \\ \partial_\varepsilon \tilde{F}(0) \end{pmatrix} + \begin{pmatrix} u\partial_\xi\partial_\varepsilon\theta_2^0 - \partial_\varepsilon\psi_2^0 \\ u\partial_\xi\partial_\varepsilon\psi_2^0 - \partial_x^2\partial_\varepsilon\theta_2^0 + \cos(\theta_K(Z))\partial_\varepsilon\theta_2^0 \end{pmatrix} + \partial_\varepsilon\lambda_{u,2}^0 \begin{pmatrix} \partial_u \theta_1^0 \\ \partial_u \psi_1^0 \end{pmatrix} , \end{aligned} \tag{20.18}$$

(20.17) and (20.18) yield the claim due to Proposition 20.60.
(3) follows from (1), (2), Theorem 20.61 and Taylor's formula.

(b)
(1) clear.
(2)

$$\begin{aligned}0 &= \mathcal{G}_3^\varepsilon(\theta_3^\varepsilon, \psi_3^\varepsilon, \lambda_{u,3}^\varepsilon)\\ &= u\partial_\xi \begin{pmatrix}\theta_3^\varepsilon\\ \psi_3^\varepsilon\end{pmatrix} - \begin{pmatrix}\psi_3^\varepsilon\\ [\theta_3^\varepsilon]_{xx} - \sin\theta_3^\varepsilon + \tilde{F}(\varepsilon)\end{pmatrix}\\ &\quad + \lambda_{u,3}^\varepsilon \partial_u \begin{pmatrix}\theta_1^0 + \partial_\varepsilon\theta_1^0\varepsilon + \frac{\partial_\varepsilon^2\theta_2^0}{2}\varepsilon^2\\ \psi_1^0 + \partial_\varepsilon\psi_1^0\varepsilon + \frac{\partial_\varepsilon^2\psi_2^0}{2}\varepsilon^2\end{pmatrix}.\end{aligned}$$

Differentiate with respect to $\varepsilon$:

$$\begin{aligned}0 &= \partial_\varepsilon\mathcal{G}_3^\varepsilon(\theta_3^\varepsilon, \psi_3^\varepsilon, \lambda_{u,3}^\varepsilon)\\ &= \begin{pmatrix}0\\ \partial_\varepsilon\tilde{F}(\varepsilon)\end{pmatrix} + \begin{pmatrix}u\partial_\xi\partial_\varepsilon\theta_3^\varepsilon - \partial_\varepsilon\psi_3^\varepsilon\\ u\partial_\xi\partial_\varepsilon\psi_3^\varepsilon - \partial_x^2\partial_\varepsilon\theta_3^\varepsilon + \cos(\theta_3^\varepsilon)\partial_\varepsilon\theta_3^\varepsilon\end{pmatrix}\\ &\quad + \partial_\varepsilon\lambda_{u,3}^\varepsilon\partial_u \begin{pmatrix}\theta_1^0 + \partial_\varepsilon\theta_1^0\varepsilon + \frac{\partial_\varepsilon^2\theta_2^0}{2}\varepsilon^2\\ \psi_1^0 + \partial_\varepsilon\psi_1^0\varepsilon + \frac{\partial_\varepsilon^2\psi_2^0}{2}\varepsilon^2\end{pmatrix}\\ &\quad + \lambda_{u,3}^\varepsilon\partial_u \begin{pmatrix}\partial_\varepsilon\theta_1^0 + \partial_\varepsilon^2\theta_2^0\varepsilon\\ \partial_\varepsilon\psi_1^0 + \partial_\varepsilon^2\psi_2^0\varepsilon\end{pmatrix}.\end{aligned}$$

We get for $\varepsilon = 0$:

$$\begin{aligned}0 &= \partial_\varepsilon\mathcal{G}_3^0(\theta_K(Z), -\gamma u\theta_K'(Z), 0) \qquad (20.19)\\ &= \begin{pmatrix}0\\ \partial_\varepsilon\tilde{F}(0)\end{pmatrix} + \begin{pmatrix}u\partial_\xi\partial_\varepsilon\theta_3^0 - \partial_\varepsilon\psi_3^0\\ u\partial_\xi\partial_\varepsilon\psi_3^0 - \partial_x^2\partial_\varepsilon\theta_3^0 + \cos(\theta_3^0)\partial_\varepsilon\theta_3^0\end{pmatrix}\\ &\quad + \partial_\varepsilon\lambda_{u,3}^0\partial_u \begin{pmatrix}\theta_1^0\\ \psi_1^0\end{pmatrix}.\end{aligned}$$

(20.18) and (20.19) yield the claim due to Proposition 20.60.

(3) Compute $\partial_\varepsilon^2 \mathcal{G}_3^\varepsilon$:

$$\begin{aligned}
0 &= \partial_\varepsilon^2 \mathcal{G}_3^\varepsilon(\theta_3^\varepsilon, \psi_3^\varepsilon, \lambda_{u,3}^\varepsilon) \\
&= \begin{pmatrix} 0 \\ \partial_\varepsilon^2 \tilde{F}(\varepsilon) \end{pmatrix} + \begin{pmatrix} u\partial_\xi \partial_\varepsilon^2 \theta_3^\varepsilon - \partial_\varepsilon^2 \psi_3^\varepsilon \\ u\partial_\xi \partial_\varepsilon^2 \psi_3^\varepsilon - \partial_x^2 \partial_\varepsilon^2 \theta_3^\varepsilon - \sin(\theta_3^\varepsilon)\partial_\varepsilon \theta_3^\varepsilon \partial_\varepsilon \theta_3^\varepsilon + \cos(\theta_3^\varepsilon)\partial_\varepsilon^2 \theta_3^\varepsilon \end{pmatrix} \\
&\quad + \partial_\varepsilon^2 \lambda_{u,3}^\varepsilon \partial_u \begin{pmatrix} \theta_1^0 + \partial_\varepsilon \theta_1^0 \varepsilon + \frac{\partial_\varepsilon^2 \theta_2^0}{2}\varepsilon^2 \\ \psi_1^0 + \partial_\varepsilon \psi_1^0 \varepsilon + \frac{\partial_\varepsilon^2 \psi_2^0}{2}\varepsilon^2 \end{pmatrix} \\
&\quad + \partial_\varepsilon \lambda_{u,3}^\varepsilon \partial_u \begin{pmatrix} \partial_\varepsilon \theta_1^0 + \partial_\varepsilon^2 \theta_2^0 \varepsilon \\ \partial_\varepsilon \psi_1^0 + \partial_\varepsilon^2 \psi_2^0 \varepsilon \end{pmatrix} \\
&\quad + \partial_\varepsilon \lambda_{u,3}^\varepsilon \partial_u \begin{pmatrix} \partial_\varepsilon \theta_1^0 + \partial_\varepsilon^2 \theta_2^0 \varepsilon \\ \partial_\varepsilon \psi_1^0 + \partial_\varepsilon^2 \psi_2^0 \varepsilon \end{pmatrix} + \lambda_{u,3}^\varepsilon \partial_u \begin{pmatrix} \partial_\varepsilon \theta_1^0 + \partial_\varepsilon^2 \theta_2^0 \varepsilon \\ \partial_\varepsilon \psi_1^0 + \partial_\varepsilon^2 \psi_2^0 \varepsilon \end{pmatrix} .
\end{aligned}$$

We get for $\varepsilon = 0$:

$$\begin{aligned}
0 &= \partial_\varepsilon^2 \mathcal{G}_3^0(\theta_K(Z), -\gamma u \theta_K'(Z), 0) \qquad (20.20) \\
&= \begin{pmatrix} 0 \\ \partial_\varepsilon^2 F(0) \end{pmatrix} + \begin{pmatrix} -u\partial_x \partial_\varepsilon^2 \theta_3^0 - \partial_\varepsilon^2 \psi_3^0 \\ -u\partial_x \partial_\varepsilon^2 \psi_3^0 - \partial_x^2 \partial_\varepsilon^2 \theta_3^0 - \sin(\theta_3^0)\partial_\varepsilon \theta_3^0 \partial_\varepsilon \theta_3^0 + \cos(\theta_3^0)\partial_\varepsilon^2 \theta_3^0 \end{pmatrix} \\
&\quad + \partial_\varepsilon^2 \lambda_{u,3}^0 \partial_u \begin{pmatrix} \theta_1^0 \\ \psi_1^0 \end{pmatrix} \\
&\quad + \partial_\varepsilon \lambda_{u,3}^0 \partial_u \begin{pmatrix} \partial_\varepsilon \theta_1^0 \\ \partial_\varepsilon \psi_1^0 \end{pmatrix} \\
&\quad + \partial_\varepsilon \lambda_{u,3}^0 \partial_u \begin{pmatrix} \partial_\varepsilon \theta_1^0 \\ \partial_\varepsilon \psi_1^0 \end{pmatrix} .
\end{aligned}$$

Compute $\partial_\varepsilon^2 \mathcal{G}_2^\varepsilon$:

$$\begin{aligned}
0 &= \partial_\varepsilon^2 \mathcal{G}_2^\varepsilon(\theta_2^\varepsilon, \psi_2^\varepsilon, \lambda_{u,2}^\varepsilon) \\
&= \begin{pmatrix} 0 \\ \partial_\varepsilon^2 \tilde{F}(\varepsilon) \end{pmatrix} + \begin{pmatrix} u\partial_\xi \partial_\varepsilon^2 \theta_2^\varepsilon - \partial_\varepsilon^2 \psi_2^\varepsilon \\ u\partial_\xi \partial_\varepsilon^2 \psi_2^\varepsilon - \partial_x^2 \partial_\varepsilon^2 \theta_2^\varepsilon - \sin(\theta_2^\varepsilon)\partial_\varepsilon \theta_2^\varepsilon \partial_\varepsilon \theta_2^\varepsilon + \cos(\theta_2^\varepsilon)\partial_\varepsilon^2 \theta_2^\varepsilon \end{pmatrix} \\
&\quad + \partial_\varepsilon^2 \lambda_{u,3}^\varepsilon \partial_u \begin{pmatrix} \theta_1^0 + \partial_\varepsilon \theta_1^0 \varepsilon \\ \psi_1^0 + \partial_\varepsilon \psi_1^0 \varepsilon \end{pmatrix} + \partial_\varepsilon \lambda_{u,3}^\varepsilon \partial_u \begin{pmatrix} \partial_\varepsilon \theta_1^0 \\ \partial_\varepsilon \psi_1^0 \end{pmatrix} \\
&\quad + \partial_\varepsilon \lambda_{u,3}^\varepsilon \partial_u \begin{pmatrix} \partial_\varepsilon \theta_1^0 \\ \partial_\varepsilon \psi_1^0 \end{pmatrix} .
\end{aligned}$$

We get for $\varepsilon = 0$:

$$
\begin{aligned}
0 &= \partial_\varepsilon^2 \mathcal{G}_2^0(\theta_K(Z), -\gamma u \theta_K'(Z), 0) \qquad (20.21)\\
&= \begin{pmatrix} 0 \\ \partial_\varepsilon^2 F(0) \end{pmatrix} + \begin{pmatrix} u\partial_\xi \partial_\varepsilon^2 \theta_2^0 - \partial_\varepsilon^2 \psi_2^0 \\ u\partial_\xi \partial_\varepsilon^2 \psi_2^0 - \partial_x^2 \partial_\varepsilon^2 \theta_2^0 - \sin(\theta_2^0)\partial_\varepsilon \theta_2^0 \partial_\varepsilon \theta_2^0 + \cos(\theta_2^0)\partial_\varepsilon^2 \theta_2^0 \end{pmatrix}\\
&\quad + \partial_\varepsilon^2 \lambda_{u,2}^0 \partial_u \begin{pmatrix} \theta_1^0 \\ \psi_1^0 \end{pmatrix} + \partial_\varepsilon \lambda_{u,2}^0 \partial_u \begin{pmatrix} \partial_\varepsilon \theta_1^0 \\ \partial_\varepsilon \psi_1^0 \end{pmatrix}\\
&\quad + \partial_\varepsilon \lambda_{u,2}^0 \partial_u \begin{pmatrix} \partial_\varepsilon \theta_1^0 \\ \partial_\varepsilon \psi_1^0 \end{pmatrix} .
\end{aligned}
$$

(20.20) and (20.21) yield the claim due to Proposition 20.60.
(4) follows from (1), (2), (3), Theorem 20.61 and Taylor's formula.

(c)
(1) Assume

$$
\begin{aligned}
&(\partial_\varepsilon^k \theta_1^0, \partial_\varepsilon^k \psi_1^0, \partial_\varepsilon^k \lambda_{u,1}^0) = (\partial_\varepsilon^k \theta_2^0, \partial_\varepsilon^k \psi_2^0, \partial_\varepsilon^k \lambda_{u,2}^0), \quad k = 0, 1 \ ;\\
&(\partial_\varepsilon^k \theta_2^0, \partial_\varepsilon^k \psi_2^0, \partial_\varepsilon^k \lambda_{u,2}^0) = (\partial_\varepsilon^k \theta_3^0, \partial_\varepsilon^k \psi_3^0, \partial_\varepsilon^k \lambda_{u,3}^0), \quad k = 0, 1, 2 \ ;\\
&\vdots\\
&(\partial_\varepsilon^k \theta_{n-2}^0, \partial_\varepsilon^k \psi_{n-2}^0, \partial_\varepsilon^k \lambda_{u,n-2}^0) = (\partial_\varepsilon^k \theta_{n-1}^0, \partial_\varepsilon^k \psi_{n-1}^0, \partial_\varepsilon^k \lambda_{u,n-1}^0), \quad k = 0 \ldots, n-2.
\end{aligned}
$$

The Sobolev embeddings mentioned above yield the justification for using the Leibniz's formula and Faà di Bruno's formula.
Leibniz's formula:

$$
(f \cdot g)^{(k)}(\varepsilon) = \sum_{l=0}^{k} \binom{k}{l} f^{(k-l)}(\varepsilon) \cdot g^{(l)}(\varepsilon) .
$$

Faà di Bruno's formula (see A Primer of Real Analytic Functions, Steven G Krantz, Harold R. Parks, p. 17):

$$
(g \circ f)^{(k)}(\varepsilon) = \sum_{I_k} \frac{k!}{l_1! l_2! \ldots l_k!} g^{(l)}(f(\varepsilon)) \left(\frac{f^{(1)}(\varepsilon)}{1!}\right)^{l_1} \left(\frac{f^{(2)}(\varepsilon)}{2!}\right)^{l_2} \cdots \left(\frac{f^{(k)}(\varepsilon)}{k!}\right)^{l_k} ,
$$

where $l = l_1 + l_2 + \ldots + l_k$ and the sum is taken over all $l_1, l_2, \ldots, l_k$ for which $l_1 + 2l_2 + \ldots + kl_k = k$.

$k$-th derivative with respect to $\varepsilon$, where $0 \leq k \leq n-1$:

$$
\begin{aligned}
&\partial_\varepsilon^k \mathcal{G}_n^\varepsilon(\theta_n^\varepsilon, \psi_n^\varepsilon, \lambda_{u,n}^\varepsilon) \\
&= \begin{pmatrix} u\partial_\xi \partial_\varepsilon^k \theta_n^\varepsilon - \partial_\varepsilon^k \psi_n^\varepsilon \\ u\partial_\xi \partial_\varepsilon^k \psi_n^\varepsilon - \partial_x^2 \partial_\varepsilon^k \theta_n^\varepsilon + \partial_\varepsilon^k[\sin\theta_n^\varepsilon] - \partial_\varepsilon^k \tilde{F}(\varepsilon) \end{pmatrix} \\
&\quad + \partial_\varepsilon^k \begin{pmatrix} \sum_{i=0}^{n-1} \lambda_{u,n}^\varepsilon \frac{\partial_u \partial_\varepsilon^i \theta_{n-1}^0}{i!} \varepsilon^i \\ \sum_{i=0}^{n-1} \lambda_{u,n}^\varepsilon \frac{\partial_u \partial_\varepsilon^i \psi_{n-1}^0}{i!} \varepsilon^i \end{pmatrix} \\
&= \begin{pmatrix} u\partial_\xi \partial_\varepsilon^k \theta_n^\varepsilon - \partial_\varepsilon^k \psi_n^\varepsilon \\ u\partial_\xi \partial_\varepsilon^k \psi_n^\varepsilon - \partial_x^2 \partial_\varepsilon^k \theta_n^\varepsilon \end{pmatrix} + \begin{pmatrix} 0 \\ \sum_{I_k} \frac{k!}{l_1! l_2! \ldots l_k!} \partial_\varepsilon^l \sin(\theta_n^\varepsilon) \left(\frac{\partial_\varepsilon^1 \theta_n^\varepsilon}{1!}\right)^{l_1} \left(\frac{\partial_\varepsilon^2 \theta_n^\varepsilon}{2!}\right)^{l_2} \ldots \left(\frac{\partial_\varepsilon^k \theta_n^\varepsilon}{k!}\right)^{l_k} \end{pmatrix} \\
&\quad - \begin{pmatrix} 0 \\ \partial_\varepsilon^k \tilde{F}(\varepsilon) \end{pmatrix} + \begin{pmatrix} \sum_{i=0}^{n-1} \sum_{l=0}^{k} \binom{k}{l} \partial_\varepsilon^{k-l} \lambda_{u,n}^\varepsilon \partial_\varepsilon^l \left[\frac{\partial_u \partial_\varepsilon^i \theta_{n-1}^0}{i!} \varepsilon^i\right] \\ \sum_{i=0}^{n-1} \sum_{l=0}^{k} \binom{k}{l} \partial_\varepsilon^{k-l} \lambda_{u,n}^\varepsilon \partial_\varepsilon^l \left[\frac{\partial_u \partial_\varepsilon^i \psi_{n-1}^0}{i!} \varepsilon^i\right] \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
&\partial_\varepsilon^k \mathcal{G}_{n-1}^\varepsilon(\theta_{n-1}^\varepsilon, \psi_{n-1}^\varepsilon, \lambda_{u,n-1}^\varepsilon) \\
&= \begin{pmatrix} u\partial_\xi \partial_\varepsilon^k \theta_{n-1}^\varepsilon - \partial_\varepsilon^k \psi_{n-1}^\varepsilon \\ u\partial_\xi \partial_\varepsilon^k \psi_{n-1}^\varepsilon - \partial_x^2 \partial_\varepsilon^k \theta_{n-1}^\varepsilon + \partial_\varepsilon^k[\sin\theta_{n-1}^\varepsilon] - \partial_\varepsilon^k \tilde{F}(\varepsilon) \end{pmatrix} \\
&\quad + \partial_\varepsilon^k \begin{pmatrix} \sum_{i=0}^{n-2} \lambda_{u,n-1}^\varepsilon \frac{\partial_u \partial_\varepsilon^i \theta_{n-2}^0}{i!} \varepsilon^i \\ \sum_{i=0}^{n-2} \lambda_{u,n-1}^\varepsilon \frac{\partial_u \partial_\varepsilon^i \psi_{n-2}^0}{i!} \varepsilon^i \end{pmatrix} \\
&= \begin{pmatrix} u\partial_\xi \partial_\varepsilon^k \theta_{n-1}^\varepsilon - \partial_\varepsilon^k \psi_{n-1}^\varepsilon \\ u\partial_\xi \partial_\varepsilon^k \psi_{n-1}^\varepsilon - \partial_x^2 \partial_\varepsilon^k \theta_{n-1}^\varepsilon \end{pmatrix} + \begin{pmatrix} 0 \\ \sum_{I_k} \frac{k!}{l_1! l_2! \ldots l_k!} \partial_\varepsilon^l \sin(\theta_{n-1}^\varepsilon) \left(\frac{\partial_\varepsilon^1 \theta_{n-1}^\varepsilon}{1!}\right)^{l_1} \left(\frac{\partial_\varepsilon^2 \theta_{n-1}^\varepsilon}{2!}\right)^{l_2} \ldots \left(\frac{\partial_\varepsilon^k \theta_{n-1}^\varepsilon}{k!}\right)^{l_k} \end{pmatrix} \\
&\quad - \begin{pmatrix} 0 \\ \partial_\varepsilon^k \tilde{F}(\varepsilon) \end{pmatrix} + \begin{pmatrix} \sum_{i=0}^{n-2} \sum_{l=0}^{k} \binom{k}{l} \partial_\varepsilon^{k-l} \lambda_{u,n-1}^\varepsilon \partial_\varepsilon^l \left[\frac{\partial_u \partial_\varepsilon^i \theta_{n-2}^0}{i!} \varepsilon^i\right] \\ \sum_{i=0}^{n-2} \sum_{l=0}^{k} \binom{k}{l} \partial_\varepsilon^{k-l} \lambda_{u,n-1}^\varepsilon \partial_\varepsilon^l \left[\frac{\partial_u \partial_\varepsilon^i \psi_{n-2}^0}{i!} \varepsilon^i\right] \end{pmatrix}.
\end{aligned}
$$

We obtain for $\varepsilon = 0$:

$$
\begin{aligned}
&\partial_\varepsilon^k \mathcal{G}_n^0(\theta_n^0, \psi_n^0, \lambda_{u,n}^0) \qquad\qquad (20.22)\\
&= \begin{pmatrix} u\partial_\xi \partial_\varepsilon^k \theta_n^0 - \partial_\varepsilon^k \psi_n^0 \\ u\partial_\xi \partial_\varepsilon^k \psi_n^0 - \partial_x^2 \partial_\varepsilon^k \theta_n^0 \end{pmatrix} + \begin{pmatrix} 0 \\ \sum_{I_k} \frac{k!}{l_1! l_2! \ldots l_k!} \partial_\varepsilon^l \sin(\theta_n^0) \left(\frac{\partial_\varepsilon^1 \theta_n^0}{1!}\right)^{l_1} \left(\frac{\partial_\varepsilon^2 \theta_n^0}{2!}\right)^{l_2} \ldots \left(\frac{\partial_\varepsilon^k \theta_n^0}{k!}\right)^{l_k} \end{pmatrix}\\
&\quad - \begin{pmatrix} 0 \\ \partial_\varepsilon^k \tilde{F}(0) \end{pmatrix} + \begin{pmatrix} \sum_{i=0}^k \binom{k}{i} \partial_\varepsilon^{k-i} \lambda_{u,n}^0 \partial_u \partial_\varepsilon^i \theta_{n-1}^0 \\ \sum_{i=0}^k \binom{k}{i} \partial_\varepsilon^{k-i} \lambda_{u,n}^0 \partial_u \partial_\varepsilon^i \psi_{n-1}^0 \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
&\partial_\varepsilon^k \mathcal{G}_{n-1}^0(\theta_{n-1}^0, \psi_{n-1}^0, \lambda_{u,n-1}^0) \qquad\qquad (20.23)\\
&= \begin{pmatrix} u\partial_\xi \partial_\varepsilon^k \theta_{n-1}^0 - \partial_\varepsilon^k \psi_{n-1}^0 \\ u\partial_\xi \partial_\varepsilon^k \psi_{n-1}^0 - \partial_x^2 \partial_\varepsilon^k \theta_{n-1}^0 \end{pmatrix} + \begin{pmatrix} 0 \\ \sum_{I_k} \frac{k!}{l_1! l_2! \ldots l_k!} \partial_\varepsilon^l \sin(\theta_{n-1}^0) \left(\frac{\partial_\varepsilon^1 \theta_{n-1}^0}{1!}\right)^{l_1} \left(\frac{\partial_\varepsilon^2 \theta_{n-1}^0}{2!}\right)^{l_2} \ldots \left(\frac{\partial_\varepsilon^k \theta_{n-1}^0}{k!}\right)^{l_k} \end{pmatrix}\\
&\quad - \begin{pmatrix} 0 \\ \partial_\varepsilon^k \tilde{F}(0) \end{pmatrix} + \begin{pmatrix} \sum_{i=0}^k \binom{k}{i} \partial_\varepsilon^{k-i} \lambda_{u,n-1}^0 \partial_u \partial_\varepsilon^i \theta_{n-2}^0 \\ \sum_{i=0}^k \binom{k}{i} \partial_\varepsilon^{k-i} \lambda_{u,n-1}^0 \partial_u \partial_\varepsilon^i \psi_{n-2}^0 \end{pmatrix}.
\end{aligned}
$$

Subtracting (20.23) from (20.22) yield the claim due to Proposition 20.60.
(2) follows from (1), Theorem 20.61 and Taylor's formula.
□

## 20.4 Virtual Solitary Manifold

From now on we set $\alpha := 1$. We apply Theorem 20.61 on a specific $\tilde{F}$ and define by the obtained solution the virtual solitary manifold.

**Definition 20.65.** *Let $F, \xi_s$ be from Theorem 19.3 and $\Xi > |\xi_s| + 2$. We set*

$$\tilde{F}(\varepsilon, \xi, x) := F(\varepsilon, x)\chi(\xi),$$

*where $\chi(\xi)$ is a smooth cutoff function such that*

$$\chi(\xi) = \begin{cases} 1, & \xi \in [-\Xi, \Xi] \\ 0, & |\xi| \geq \Xi + 1 \end{cases}$$

*and $\chi$ is constructed as in [AE09, Chapter X,Proposition 7.14].*

**Lemma 20.66.** *Let $F$ be from Theorem 19.3, $\tilde{F}$ from Definition 20.65. Then it holds that*

*(a)* $\forall\ \ (\varepsilon,\xi,x)\in(-1,1)\times[-\Xi,\Xi]\times\mathbb{R}:\tilde{F}(\varepsilon,\xi,x)=F(\varepsilon,x)\,,$

*(b)* $\tilde{F}\in C^n((-1,1),H^{1,1}(\mathbb{R}^2))$ *and* $\partial_\varepsilon^l\tilde{F}(0,\cdot,\cdot)=0$ *for* $0\le l\le k\,,$

*(c)*

$$\left\|\begin{pmatrix}0\\0\\\lambda_{u,n}^{\varepsilon}\end{pmatrix}\right\|_{Y_2^\alpha(u_*)}=\mathcal{O}(\varepsilon^{k+1})\,.$$

**Proof.** (a) Follows from Definition 20.65.
(b) We show the continuity of $\tilde{F}$.

$$\begin{aligned}
&|F(\varepsilon,x)\chi(\xi)-F(0,x)\chi(\xi)|^2_{H^{1,1}_{\xi,x}(\mathbb{R}^2)}\\
\le&\left|(1+|x|^2+|\xi|^2)^{\frac12}\Big(F(\varepsilon,x)\chi(\xi)-F(0,x)\chi(\xi)\Big)\right|^2_{H^1_{\xi,x}(\mathbb{R}^2)}\\
=&\left|(1+|x|^2+|\xi|^2)\Big(F(\varepsilon,x)\chi(\xi)-F(0,x)\chi(\xi)\Big)^2\right|_{L^1_{\xi,x}(\mathbb{R}^2)}\\
&+\left|\Big(\partial_x\Big[(1+|x|^2+|\xi|^2)^{\frac12}\Big(F(\varepsilon,x)\chi(\xi)-F(0,x)\chi(\xi)\Big)\Big]\Big)^2\right|_{L^1_{\xi,x}(\mathbb{R}^2)}\\
&+\left|\Big(\partial_\xi\Big[(1+|x|^2+|\xi|^2)^{\frac12}\Big(F(\varepsilon,x)\chi(\xi)-F(0,x)\chi(\xi)\Big)\Big]\Big)^2\right|_{L^1_{\xi,x}(\mathbb{R}^2)}\\
\le&\Big|(1+|x|^2)\Big(F(\varepsilon,x)\chi(\xi)-F(0,x)\chi(\xi)\Big)^2\\
&+\Big(F(\varepsilon,x)\chi(\xi)-F(0,x)\chi(\xi)\Big)^2|\xi|^2\Big|_{L^1_{\xi,x}(\mathbb{R}^2)}\\
&+\Big|\Big(x\Big(F(\varepsilon,x)\chi(\xi)-F(0,x)\chi(\xi)\Big)\\
&+(1+|x|^2+|\xi|^2)^{\frac12}\partial_x\Big(F(\varepsilon,x)\chi(\xi)-F(0,x)\chi(\xi)\Big)\Big)^2\Big|_{L^1_{\xi,x}(\mathbb{R}^2)}
\end{aligned}$$

$$+ \left| \Big( \xi \Big( F(\varepsilon,x)\chi(\xi) - F(0,x)\chi(\xi) \Big) \right.$$

$$\left. + (1+|x|^2+|\xi|^2)^{\frac{1}{2}} \partial_\xi \Big( F(\varepsilon,x)\chi(\xi) - F(0,x)\chi(\xi) \Big) \Big)^2 \right|_{L^1_{\xi,x}(\mathbb{R}^2)} .$$

We consider the first term.

$$\left| (1+|x|^2) \Big( F(\varepsilon,x)\chi(\xi) - F(0,x)\chi(\xi) \Big)^2 \right|_{L^1_{\xi,x}(\mathbb{R}^2)}$$

$$= \int_{\mathbb{R}} (1+|x|^2)F(\varepsilon,x)^2 - (1+|x|^2)2F(\varepsilon,x)F(0,x) + (1+|x|^2)F(0,x)^2 \, dx$$

$$\cdot \int_{\mathbb{R}} \chi(\xi)^2 \, d\xi$$

$$\overset{\varepsilon \to 0}{\to} 0 .$$

The other terms can be treated analogously. The proof for the differentiability works in the same way.
(c) Follows from Theorem 20.62 (c) and the assumption (a) in Theorem 19.3. □

**Lemma 20.67.** *Let $v \in H^1(\mathbb{R}^2)$. Then there exists $b > 0$ such that*

$$\forall \ \xi \in \mathbb{R} \ \ |v(\xi,x)|_{L^2_x(\mathbb{R})} \leq b \, |v(\xi,x)|_{H^1_{\xi,x}(\mathbb{R}^2)} .$$

**Proof.** We obtain by Morrey's embedding Theorem $\forall \ \xi \in \mathbb{R}$ :

$$|v(\xi,x)|^2_{H^1_{\xi,x}(\mathbb{R}^2)}$$

$$= \int v(\xi,x)^2 + (\partial_\xi v(\xi,x))^2 + (\partial_x v(\xi,x))^2 \, d\xi \, dx$$

$$\geq \int |v(\xi,x)|^2_{H^1_\xi(\mathbb{R})} \, dx$$

$$\geq \frac{1}{b} \int |v(\xi,x)|^2_{L^\infty_\xi(\mathbb{R})} \, dx$$

$$\geq \frac{1}{b} \, |v(\xi,x)|^2_{L^2_x(\mathbb{R})} .$$

□

We solve iteratively the equations in Theorem 20.61 with the specific $\tilde{F}(\varepsilon,\xi,x) := F(\varepsilon,x)\chi(k,\xi)$ from Definition 20.65. and define by the nth solution $(\theta^\varepsilon_n, \psi^\varepsilon_n, \lambda^\varepsilon_n)$ the solitary manifold.

**Definition 20.68.** *Let* $[\underline{u}^1]^*$ *be from Proposition 20.60 (case* $\alpha = 1$*). We fix a specific* $u_*$ *such that* $0 < u_* < [\underline{u}^1]^*$. *Let* $\tilde{F}$ *be from Definition 20.65. Let* $(\theta_n^\varepsilon, \psi_n^\varepsilon, \lambda_n^\varepsilon)$ *be the nth solution obtained from application of Theorem 20.61 to* $\tilde{F}$, *where* $\varepsilon \in (-\varepsilon^*, \varepsilon^*)$. *We set*

$$\mathcal{S}_n^\varepsilon := \left\{ \begin{pmatrix} \theta_n^\varepsilon(\xi, u, \cdot) \\ \psi_n^\varepsilon(\xi, u, \cdot) \end{pmatrix} \; : \; u \in (-u_*, u_*), \; \xi \in \mathbb{R} \right\} , \qquad \varepsilon \in (-\varepsilon^*, \varepsilon^*) \, ,$$

*and call* $\mathcal{S}_n^\varepsilon$ *the virtual solitary manifold.*

**Remark 20.69.** Let $\tilde{F}$ be from Definition 20.65. Notice that $\tilde{F} \in C^n((-1,1), H^{1,1}(\mathbb{R}^2))$ due to Lemma 20.66. Thus Theorem 20.61 is applicable to $\tilde{F}$.

From now on we always assume that $\varepsilon \in (-\varepsilon^*, \varepsilon^*)$ with $\varepsilon^*$ from Theorem 20.61 and we always denote by $(\theta_n^\varepsilon, \psi_n^\varepsilon, \lambda_n^\varepsilon)$ the nth solution obtained from Theorem 20.61 applied to $\tilde{F}$ from Definition 20.65.

## 20.5 Tangent Vectors

**Definition 20.70.** *We set*

*(a)* $t_{\xi,n}^\varepsilon(\xi, u, x) = \begin{pmatrix} \partial_\xi \theta_n^\varepsilon(\xi, u, x) \\ \partial_\xi \psi_n^\varepsilon(\xi, u, x) \end{pmatrix}$ ,

*(b)* $t_{u,n}^\varepsilon(\xi, u, x) = \begin{pmatrix} \partial_u \theta_n^\varepsilon(\xi, u, x) \\ \partial_u \psi_n^\varepsilon(\xi, u, x) \end{pmatrix}$ .

**Remark 20.71.** (a) One should understand the subscripts $\xi$ and $u$ in our notation of $t_{\xi,n}^\varepsilon$ and $t_{u,n}^\varepsilon$ just as symbols. $t_{\xi,n}^\varepsilon$ and $t_{u,n}^\varepsilon$ always really depend on $(\xi, u, x)$.

(b) The vectors $t_{\xi,n}^\varepsilon(\xi, u, \cdot)$ and $t_{u,n}^\varepsilon(\xi, u, \cdot)$ are tangent vectors of the manifold $\mathcal{S}_n^\varepsilon$ at the point $(\theta_n^\varepsilon(\xi, u, \cdot), \psi_n^\varepsilon(\xi, u, \cdot))$ and form a basis of the tangent space at this point.

# Chapter 21

# Symplectic Orthogonal Decomposition

Let $u_*$ be from Theorem 20.61. In the following we recall some definitions from Chapter 19.

**Definition 21.1.** *Let* $0 < U < u_*$, $\Xi > 0$. *We set*

*(a)* $U(l) := U(l, u_*) := \frac{u_* - U}{l}$,

*(b)* $\Sigma(l, U) := \Sigma(l, U, u_*) := \Big\{ (\xi, u) \in \mathbb{R} \times (-1, 1) : u \in (-U - U(l), U + U(l)) \Big\}$,

*(c)* $\Sigma(l, U, \Xi) := \Big\{ (\xi, u) \in (-\Xi + 1 - U(l), \Xi - 1 + U(l)) \times (-U - U(l), U + U(l)) \Big\}$.

*(d) We define the map* $\mathcal{N}^\varepsilon : L^\infty(\mathbb{R}) \times L^2(\mathbb{R}) \times \Sigma(2, U) \to \mathbb{R}^2$ *by*

$$\mathcal{N}^\varepsilon(\theta, \psi, \xi, u) := \begin{pmatrix} \mathcal{C}_1^\varepsilon(\theta, \psi, \xi, u) \\ \mathcal{C}_2^\varepsilon(\theta, \psi, \xi, u) \end{pmatrix}$$

$$:= \begin{pmatrix} \Omega\left( \begin{pmatrix} \partial_\xi \theta_n^\varepsilon(\xi, u, \cdot) \\ \partial_\xi \psi_n^\varepsilon(\xi, u, \cdot) \end{pmatrix}, \begin{pmatrix} \theta(\cdot) - \theta_n^\varepsilon(\xi, u, \cdot) \\ \psi(\cdot) - \psi_n^\varepsilon(\xi, u, \cdot) \end{pmatrix} \right) \\ \Omega\left( \begin{pmatrix} \partial_u \theta_n^\varepsilon(\xi, u, \cdot) \\ \partial_u \psi_n^\varepsilon(\xi, u, \cdot) \end{pmatrix}, \begin{pmatrix} \theta(\cdot) - \theta_n^\varepsilon(\xi, u, \cdot) \\ \psi(\cdot) - \psi_n^\varepsilon(\xi, u, \cdot) \end{pmatrix} \right) \end{pmatrix}.$$

**Remark 21.2.** In Theorem 20.61 we have solved the equations defining $(\theta_n^\varepsilon, \psi_n^\varepsilon, \lambda_{u,n}^\varepsilon)$ in weighted spaces. One of the reasons for working in weighted Sobolev spaces was to make sure that $\mathcal{N}^\varepsilon$ is going to be well defined.

**Remark 21.3.** Notice that $U(l)$ and $\Sigma(l, U)$ have been already defined in Definition 1.1. In Part IV we use always the notation from Definition 21.1.

In this chapter we will choose a sufficiently small $\varepsilon_0$ and consider $\varepsilon \in (0, \varepsilon_0]$. As mentioned in Chapter 19 we show that if $(\theta, \psi) \in L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R})$ is close enough (in the $L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R})$ norm) to the region

$$\mathcal{S}_n^\varepsilon(U) := \left\{ \begin{pmatrix} \theta_n^\varepsilon(\xi, u, \cdot) \\ \psi_n^\varepsilon(\xi, u, \cdot) \end{pmatrix} \ : \ (\xi, u) \in \Sigma(4, U) \right\} ,$$

of the virtual solitary manifold $\mathcal{S}_n^\varepsilon$, then there exists a unique $(\xi, u) \in \Sigma(2, U)$ such that we are able to decompose

$$\begin{pmatrix} \theta(\cdot) \\ \psi(\cdot) \end{pmatrix} = \begin{pmatrix} \theta_n^\varepsilon(\xi, u, \cdot) \\ \psi_n^\varepsilon(\xi, u, \cdot) \end{pmatrix} + \begin{pmatrix} v(\cdot) \\ w(\cdot) \end{pmatrix} ,$$

in a point on the virtual solitary manifold $(\theta_n^\varepsilon(\xi, u, \cdot), \psi_n^\varepsilon(\xi, u, \cdot))$ and a transversal component $(v(\cdot), w(\cdot))$, which is symplectic orthogonal to the tangent vectors $\begin{pmatrix} \partial_\xi \theta_n^\varepsilon(\xi, u, \cdot) \\ \partial_\xi \psi_n^\varepsilon(\xi, u, \cdot) \end{pmatrix}$ and $\begin{pmatrix} \partial_u \theta_n^\varepsilon(\xi, u, \cdot) \\ \partial_u \psi_n^\varepsilon(\xi, u, \cdot) \end{pmatrix}$ of the virtual solitary manifold $\mathcal{S}_n^\varepsilon$, i.e., the orthogonality condition

$$\mathcal{N}^\varepsilon(\theta, \psi, \xi, u) = 0$$

is satisfied. We prove that the symplectic decomposition is possible in a small uniform distance to the manifold $\mathcal{S}_n^\varepsilon$, where the distance might depend on $\varepsilon_0$ but does not depend on $\varepsilon$. We start with a definition and some technical lemmas.

**Definition 21.4.** *Let $\varepsilon \in (0, \varepsilon^*)$ and let $(\hat{\theta}_n^\varepsilon, \hat{\psi}_n^\varepsilon, \lambda_{u,n}^\varepsilon)$ be the nth iterative solution from Theorem 20.61. We set*

*(a)* $(v_n^\varepsilon, w_n^\varepsilon) := (\hat{\theta}_n^\varepsilon, \hat{\psi}_n^\varepsilon)$,

*(b)*
$$\begin{aligned} k_n^\varepsilon(\xi, u) := \int & -\partial_\xi \psi_0(\xi, u, x) \partial_u v_n^\varepsilon(\xi, u, x) - \partial_u \theta_0(\xi, u, x) \partial_\xi w_n^\varepsilon(\xi, u, x) \\ & + \partial_\xi \theta_0(\xi, u, x) \partial_u w_n^\varepsilon(\xi, u, x) + \partial_u \psi_0(\xi, u, x) \partial_\xi v_n^\varepsilon(\xi, u, x) \\ & - \partial_\xi w_n^\varepsilon(\xi, u, x) \partial_u v_n^\varepsilon(\xi, u, x) + \partial_\xi v_n^\varepsilon(\xi, u, x) \partial_u w_n^\varepsilon(\xi, u, x) \, dx , \end{aligned}$$

*(c)*
$$m_n^\varepsilon(\xi, u) := \int -\partial_\xi \psi_n^\varepsilon(\xi, u, x) \partial_u \theta_n^\varepsilon(\xi, u, x) + \partial_\xi \theta_n^\varepsilon(\xi, u, x) \partial_u \psi_n^\varepsilon(\xi, u, x) \, dx .$$

**Lemma 21.5.** *Let $\varepsilon \in (0, \varepsilon^*)$. It holds that*

$$\forall \ (\xi, u) \in \mathbb{R} \times [-U - U(2), U + U(2)] : \quad m_n^\varepsilon(\xi, u) = \gamma^3(u) m + k_n^\varepsilon(\xi, u) ,$$

*where $m$ is from Definition 2.2.*

**Proof.** Notice that

$$\int \gamma^4 [\theta_K'(Z)]^2 \, dx = \gamma^3 \int [\theta_K'(Z)]^2 \, dZ = \gamma^3 m \, .$$

Let $Z = \gamma(x-\xi)$.

$$\begin{aligned}
& m_n^\varepsilon(\xi,u) \\
= & \int -\partial_\xi \psi_n^\varepsilon(\xi,u,x)\partial_u \theta_n^\varepsilon(\xi,u,x) + \partial_\xi \theta^\varepsilon(\xi,u,x)\partial_u \psi^\varepsilon(\xi,u,x)\, dx \\
= & \int -[\partial_\xi \psi_0(\xi,u,x) + \partial_\xi w_n^\varepsilon(\xi,u,x)][\partial_u \theta_0(\xi,u,x) + \partial_u v_n^\varepsilon(\xi,u,x)] \\
& + [\partial_\xi \theta_0(\xi,u,x) + \partial_\xi v_n^\varepsilon(\xi,u,x)][\partial_u \psi_0(\xi,u,x) + \partial_u w_n^\varepsilon(\xi,u,x)]\, dx| \\
= & \int -\partial_\xi \psi_0(\xi,u,x)\partial_u \theta_0(\xi,u,x) + \partial_\xi \theta_0(\xi,u,x)\partial_u \psi_0(\xi,u,x) \\
& - \partial_\xi \psi_0(\xi,u,x)\partial_u v_n^\varepsilon(\xi,u,x) - \partial_u \theta_0(\xi,u,x)\partial_\xi w_n^\varepsilon(\xi,u,x) \\
& + \partial_\xi \theta_0(\xi,u,x)\partial_u w_n^\varepsilon(\xi,u,x) + \partial_u \psi_0(\xi,u,x)\partial_\xi v_n^\varepsilon(\xi,u,x) \\
& - \partial_\xi w_n^\varepsilon(\xi,u,x)\partial_u v_n^\varepsilon(\xi,u,x) + \partial_\xi v_n^\varepsilon(\xi,u,x)\partial_u w_n^\varepsilon(\xi,u,x)\, dx \\
= & \int -u^2\gamma^5 \theta_K''(Z)(x-\xi)\theta_K'(Z) + \gamma^4 \theta_K'(Z)\theta_K'(Z) + u^2\gamma^5 (x-\xi)\theta_K''(Z)\theta_K'(Z) \\
& - \partial_\xi \psi_0(\xi,u,x)\partial_u v_n^\varepsilon(\xi,u,x) - \partial_u \theta_0(\xi,u,x)\partial_\xi w_n^\varepsilon(\xi,u,x) \\
& + \partial_\xi \theta_0(\xi,u,x)\partial_u w_n^\varepsilon(\xi,u,x) + \partial_u \psi_0(\xi,u,x)\partial_\xi v_n^\varepsilon(\xi,u,x) \\
& - \partial_\xi w_n^\varepsilon(\xi,u,x)\partial_u v_n^\varepsilon(\xi,u,x) + \partial_\xi v_n^\varepsilon(\xi,u,x)\partial_u w_n^\varepsilon(\xi,u,x)\, dx \\
= & \gamma(u)^3 m + \int -\partial_\xi \psi_0(\xi,u,x)\partial_u v_n^\varepsilon(\xi,u,x) - \partial_u \theta_0(\xi,u,x)\partial_\xi w_n^\varepsilon(\xi,u,x) \\
& + \partial_\xi \theta_0(\xi,u,x)\partial_u w_n^\varepsilon(\xi,u,x) + \partial_u \psi_0(\xi,u,x)\partial_\xi v_n^\varepsilon(\xi,u,x) \\
& - \partial_\xi w_n^\varepsilon(\xi,u,x)\partial_u v_n^\varepsilon(\xi,u,x) + \partial_\xi v_n^\varepsilon(\xi,u,x)\partial_u w_n^\varepsilon(\xi,u,x)\, dx \, .
\end{aligned}$$

□

**Lemma 21.6.** *Let $0 < U < u_*$. Let $\varepsilon_0 > 0$ be sufficiently small. There exist constants*

$$c = c(U) > 0, \; C = C(U) > 0,$$

*such that*

$$\forall \, \varepsilon \in (0, \varepsilon_0], \; (\xi,u) \in \mathbb{R} \times [-U - U(2), U + U(2)] : \; c \le \frac{\gamma^3(u)m}{2} \le m_n^\varepsilon(\xi,u) \le 2\gamma^3(u)m \le C \, .$$

**Proof.** Let $Z = \gamma(x-\xi)$. Since

$$\begin{aligned}
&|k_n^\varepsilon(\xi,u)| \\
\leq\ & \left|u\gamma^3(x-\xi)\theta'_K(Z)\right|_{L^2_x(\mathbb{R})} |\partial_\xi w_n^\varepsilon(\xi,u,x)|_{L^2_x(\mathbb{R})} + |\partial_\xi w_n^\varepsilon(\xi,u,x)|_{L^2_x(\mathbb{R})} |\partial_u v_n^\varepsilon(\xi,u,x)|_{L^2_x(\mathbb{R})} \\
&+ |\gamma\theta'_K(Z)|_{L^2_x(\mathbb{R})} |\partial_u w_n^\varepsilon(\xi,u,x)|_{L^2_x(\mathbb{R})} + \left|\gamma^3\theta'_K(Z)\right|_{L^2_x(\mathbb{R})} |\partial_\xi v_n^\varepsilon(\xi,u,x)|_{L^2_x(\mathbb{R})} \\
&+ \left|u^2\gamma^4(x-\xi)\theta''_K(Z)\right|_{L^2_x(\mathbb{R})} |\partial_\xi v_n^\varepsilon(\xi,u,x)|_{L^2_x(\mathbb{R})} + |\partial_\xi v_n^\varepsilon(\xi,u,x)|_{L^2_x(\mathbb{R})} |\partial_u w_n^\varepsilon(\xi,u,x)|_{L^2_x(\mathbb{R})} \ ,
\end{aligned}$$

we obtain for sufficiently small $\varepsilon_0$ using Lemma 20.67 and the continuity of $\varepsilon \mapsto (v_n^\varepsilon, w_n^\varepsilon, \lambda_{u,n}^\varepsilon)$ (see Theorem 20.61):

$$\forall\ \varepsilon \in (0,\varepsilon_0],\ (\xi,u) \in \mathbb{R} \times [-U-U(2), U+U(2)] : \quad |k_n^\varepsilon(\xi,u)| < \frac{m}{2}. \tag{21.1}$$

Since $\frac{m}{2} \leq \frac{\gamma^3(u)}{2} m$ we obtain for all $\varepsilon \in (0,\varepsilon_0]$, $(\xi,u) \in \mathbb{R} \times [-U-U(2), U+U(2)]$ :

$$\begin{aligned}
m_n^\varepsilon(\xi,u) &= \gamma^3(u)m + k_n^\varepsilon(\xi,u) \geq \gamma^3(u)m - \frac{m}{2} \geq \frac{\gamma^3(u)m}{2}\,, \\
m_n^\varepsilon(\xi,u) &= \gamma^3(u)m + k_n^\varepsilon(\xi,u) \leq \gamma^3(u)m + \frac{m}{2} \leq 2\gamma^3(u)m\,.
\end{aligned}$$

□

The next lemma provides that the symplectic decomposition described above is possible. In the proof we will take derivatives of $(\theta_n^\varepsilon, \psi_n^\varepsilon)$ up to second order with respect to $\xi$ and $u$. This was the reason for solving the equations defining $(\theta_n^\varepsilon, \psi_n^\varepsilon, \lambda_{u,n}^\varepsilon)$ in spaces of higher regularity in $\xi$ and $u$.

**Lemma 21.7.** *Let $0 < U < u_*$. Let $\varepsilon_0 > 0$ be sufficiently small. Let*

$$\mathcal{O} = \mathcal{O}_{U,p}^\varepsilon = \left\{(\theta,\psi) \in L^\infty(\mathbb{R}) \times L^2(\mathbb{R}) : \inf_{(\xi,u)\in\Sigma(4,U)} \left|\begin{pmatrix}\theta(\cdot)\\ \psi(\cdot)\end{pmatrix} - \begin{pmatrix}\theta_n^\varepsilon(\xi,u,\cdot)\\ \psi_n^\varepsilon(\xi,u,\cdot)\end{pmatrix}\right|_{L^\infty(\mathbb{R})\oplus L^2(\mathbb{R})} < p\right\}.$$

*There exists $r > 0$ such that if $\varepsilon \in (0,\varepsilon_0]$ and $p \leq r$ then for any $(\theta,\psi) \in \mathcal{O}_{U,p}^\varepsilon$ there exists a unique $(\xi,u) \in \Sigma(2,U)$ such that*

$$\mathcal{N}^\varepsilon(\theta,\psi,\xi,u) = 0$$

*and the map*

$$(\theta,\psi) \mapsto (\xi(\theta,\psi), u(\theta,\psi))$$

*is in $C^1(\mathcal{O}_{U,p}^\varepsilon, \Sigma(2,U))$.*

**Proof.** Let $\varepsilon_0 \in (0, \varepsilon^*)$ with $\varepsilon^*$ from Theorem 20.61. $\varepsilon_0$ will be specified later in this proof. Let $\varepsilon \in (0, \varepsilon_0]$. Notice that the map $\varepsilon \mapsto (v_n^\varepsilon, w_n^\varepsilon, \lambda_{u,n}^\varepsilon)$ from Theorem 20.61 is continuous. Notice that

$$U(4) \leq U(3) \leq U(2)\,,$$
$$\Sigma(4, U) \subset \Sigma(3, U) \subset \Sigma(2, U)\,.$$

$\mathcal{N}^\varepsilon$ is given by

$$\mathcal{N}^\varepsilon(\theta, \psi, \xi, u) = \begin{pmatrix} \int \partial_\xi \psi_n^\varepsilon(\xi, u, x)\Big[\theta(x) - \theta_n^\varepsilon(\xi, u, x)\Big] - \partial_\xi \theta_n^\varepsilon(\xi, u, x)\Big[\psi(x) - \psi_n^\varepsilon(\xi, u, x)\Big]\, dx \\ \int \partial_u \psi_n^\varepsilon(\xi, u, x)\Big[\theta(x) - \theta_n^\varepsilon(\xi, u, x)\Big] - \partial_u \theta_n^\varepsilon(\xi, u, x)\Big[\psi(x) - \psi_n^\varepsilon(\xi, u, x)\Big]\, dx \end{pmatrix}.$$

Consider $(\xi_0, u_0) \in \Sigma(3, U)$.

**i)** It holds that $\mathcal{N}^\varepsilon(\theta_n^\varepsilon(\xi_0, u_0, \cdot), \psi_n^\varepsilon(\xi_0, u_0, \cdot), \xi_0, u_0) = (0, 0)$. Lemma 21.5 yields

$$\begin{aligned} &D_{\xi,u}\mathcal{N}^\varepsilon(\theta_n^\varepsilon(\xi_0, u_0, \cdot), \psi_n^\varepsilon(\xi_0, u_0, \cdot), \xi_0, u_0) \\ &= \begin{pmatrix} \partial_\xi \mathcal{N}^1_{\xi_0,u_0}(\theta_n^\varepsilon(\xi_0, u_0, \cdot), \psi_n^\varepsilon(\xi_0, u_0, \cdot), \xi_0, u_0) & \partial_u \mathcal{N}^1_{\xi_0,u_0}(\theta_n^\varepsilon(\xi_0, u_0, \cdot), \psi_n^\varepsilon(\xi_0, u_0, \cdot), \xi_0, u_0) \\ \partial_\xi \mathcal{N}^2_{\xi_0,u_0}(\theta_n^\varepsilon(\xi_0, u_0, \cdot), \psi_n^\varepsilon(\xi_0, u_0, \cdot), \xi_0, u_0) & \partial_u \mathcal{N}^2_{\xi_0,u_0}(\theta_n^\varepsilon(\xi_0, u_0, \cdot), \psi_n^\varepsilon(\xi_0, u_0, \cdot), \xi_0, u_0) \end{pmatrix} \\ &= \begin{pmatrix} 0 & \gamma^3(u_0)m + k_n^\varepsilon(\xi_0, u_0) \\ -(\gamma^3(u_0)m + k_n^\varepsilon(\xi_0, u_0)) & 0 \end{pmatrix}. \end{aligned}$$

Using Lemma 20.67 we obtain for sufficiently small $\varepsilon_0$ for all $\varepsilon \in (0, \varepsilon_0]$:

$$|k_n^\varepsilon(\xi_0, u_0)| \leq \frac{m}{2}$$

and thus

$$\det D_{\xi,u}\mathcal{N}^\varepsilon(\theta^\varepsilon(\xi_0, u_0, \cdot), \psi^\varepsilon(\xi_0, u_0, \cdot), \xi_0, u_0) \neq 0.$$

It follows by the implicit function theorem that there exist balls

$$B_r(\theta_n^\varepsilon(\xi_0, u_0, \cdot), \psi_n^\varepsilon(\xi_0, u_0, \cdot)) \subset L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R})\,, \quad B_{\bar\delta}(\xi_0, u_0) \subset \Sigma(2, U)\,,$$

and exactly one map

$$T^\varepsilon_{\xi_0,u_0} : B_r(\theta_n^\varepsilon(\xi_0, u_0, \cdot), \psi_n^\varepsilon(\xi_0, u_0, \cdot)) \to B_{\bar\delta}(\xi_0, u_0)$$

such that

$$\mathcal{N}^\varepsilon(\theta, \psi, T_{\xi_0,u_0}(\theta, \psi)) = 0$$

on $B_r(\theta_n^\varepsilon(\xi_0, u_0, \cdot), \psi_n^\varepsilon(\xi_0, u_0, \cdot))$.

**ii)** We refer to [Dei85, Theorem 15.1] and we are going to prove:
There exist $r > 0, \bar{\delta} > 0, \varepsilon_0 > 0$ s.t. $\forall\ \varepsilon \in (0, \varepsilon_0], (\xi_0, u_0) \in \Sigma(3, U)$ there exist balls

$$B_r(\theta_n^\varepsilon(\xi_0, u_0, \cdot), \psi_n^\varepsilon(\xi_0, u_0, \cdot)) \subset L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R}), \quad B_{\bar{\delta}}(\xi_0, u_0) \subset \Sigma(2, U),$$

and a map

$$T^\varepsilon_{\xi_0,u_0} : B_r(\theta_n^\varepsilon(\xi_0, u_0, \cdot), \psi_n^\varepsilon(\xi_0, u_0, \cdot)) \to B_{\bar{\delta}}(\xi_0, u_0)$$

such that

$$\mathcal{N}^\varepsilon(\theta, \psi, T^\varepsilon_{\xi_0,u_0}(\theta, \psi)) = 0$$

on $B_r(\theta_n^\varepsilon(\xi_0, u_0, \cdot), \psi_n^\varepsilon(\xi_0, u_0, \cdot))$. The claim follows from this statement. In order to obtain the same setting as in [Dei85, Theorem 15.1] we introduce

$$\bar{\mathcal{N}}^\varepsilon_{\xi_0,u_0}(\theta, \psi, \xi, u) := \mathcal{N}^\varepsilon(\theta(\cdot) + \theta_n^\varepsilon(\xi_0, u_0, \cdot), \psi(\cdot) + \psi_n^\varepsilon(\xi_0, u_0, \cdot), \xi + \xi_0, u + u_0).$$

Then

$$\mathcal{N}^\varepsilon(\theta_n^\varepsilon(\xi_0, u_0, \cdot), \psi_n^\varepsilon(\xi_0, u_0, \cdot), \xi_0, u_0) = \bar{\mathcal{N}}^\varepsilon_{\xi_0,u_0}(0, 0, 0, 0) = (0, 0).$$

Set

$$\begin{aligned} K^\varepsilon_{\xi_0,u_0} &:= D_{(\xi,u)}\mathcal{N}^\varepsilon_{\xi_0,u_0}(\theta_n^\varepsilon(\xi_0, u_0, \cdot), \psi_n^\varepsilon(\xi_0, u_0, \cdot), \xi_0, u_0) \\ &= D_{(\xi,u)}\bar{\mathcal{N}}^\varepsilon_{\xi_0,u_0}(0, 0, 0, 0) \\ &= \begin{pmatrix} 0 & \gamma(u_0)^3 m + k_n^\varepsilon(\xi_0, u_0) \\ -(\gamma(u_0)^3 m + k_n^\varepsilon(\xi_0, u_0)) & 0 \end{pmatrix}. \end{aligned}$$

Hence

$$\left[K^\varepsilon_{\xi_0,u_0}\right]^{-1} = \frac{1}{\gamma(u_0)^3 m + k_n^\varepsilon(\xi_0, u_0)} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let $Z = \gamma(x - \xi)$. We define

$$S^\varepsilon_{\xi_0,u_0}(\theta, \psi, \xi, u) = \left[K^\varepsilon_{\xi_0,u_0}\right]^{-1} \bar{\mathcal{N}}^\varepsilon_{\xi_0,u_0}(\theta, \psi, \xi, u) - I(\xi, u).$$

We are able to control all derivatives of $\theta^\varepsilon$ and $\psi^\varepsilon$ with respect to $\xi$ and $u$ up to order 2 in the $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ norms. Some preliminary calculations:

$$\begin{aligned} \partial_\xi \theta_n^\varepsilon(\xi, u, x) &= -\gamma\theta_K'(Z) + \partial_\xi v_n^\varepsilon(\xi, u, x) \\ \partial_\xi \psi_n^\varepsilon(\xi, u, x) &= u\gamma^2\theta_K''(Z) + \partial_\xi w_n^\varepsilon(\xi, u, x) \\ \partial_u \theta_n^\varepsilon(\xi, u, x) &= u\gamma^3(x - \xi)\theta_K'(Z) + \partial_u v_n^\varepsilon(\xi, u, x) \\ \partial_u \psi_n^\varepsilon(\xi, u, x) &= -\gamma^3\theta_K'(Z) - u^2\gamma^4(x - \xi)\theta_K''(Z) + \partial_\xi w_n^\varepsilon(\xi, u, x) \end{aligned}$$

$$\begin{aligned}
\partial_\xi^2 \theta_n^\varepsilon(\xi,u,x) &= \gamma^2 \theta_K'(Z) + \partial_\xi^2 v_n^\varepsilon(\xi,u,x) \\
\partial_\xi^2 \psi_n^\varepsilon(\xi,u,x) &= -u\gamma^3 \theta_K'''(Z) + \partial_\xi^2 w_n^\varepsilon(\xi,u,x) \\
\partial_u^2 \theta_n^\varepsilon(\xi,u,x) &= \Big[\gamma^3 + 3u\gamma^2 u\gamma^3\Big](x-\xi)\theta_K'(Z) + u\gamma^3 u\gamma^3 (x-\xi)^2 \theta_K''(Z) + \partial_u^2 v_n^\varepsilon(\xi,u,x) \\
\partial_u^2 \psi_n^\varepsilon(\xi,u,x) &= -3\gamma^2 u\gamma^3 \theta_K'(Z) - \gamma^3 u\gamma^3 (x-\xi)\theta_K''(Z) \\
&\quad - \Big[2u\gamma^4 + u^2 4\gamma^3 u\gamma^3\Big](x-\xi)\theta_K''(Z) \\
&\quad - u^2\gamma^4 u\gamma^3 (x-\xi)^2 \theta_K'''(Z) + \partial_u^2 w_n^\varepsilon(\xi,u,x)
\end{aligned}$$

$$\begin{aligned}
\partial_\xi \partial_u \theta_n^\varepsilon(\xi,u,x) &= -u\gamma^3 \theta_K'(Z) - u\gamma^4 (x-\xi)\theta_K''(Z) + \partial_\xi \partial_u v_n^\varepsilon(\xi,u,x) \\
\partial_\xi \partial_u \psi_n^\varepsilon(\xi,u,x) &= u\gamma^4 \theta_K''(Z) + u^2\gamma^4 \theta_K''(Z) + u^2\gamma^5 (x-\xi)\theta_K'''(Z) + \partial_\xi \partial_u w_n^\varepsilon(\xi,u,x)
\end{aligned}$$

Hence for any $(\xi,u) \in \Sigma(2,U)$ we obtain by using Lemma 20.67 that

$$\partial_\xi^2 \psi_n^\varepsilon(\xi,u,\cdot), \partial_\xi \partial_u \theta_n^\varepsilon(\xi,u,\cdot), \partial_\xi \partial_u \psi_n^\varepsilon(\xi,u,\cdot), \partial_\xi \partial_u \psi_n^\varepsilon(\xi,u,\cdot) \in L^1(\mathbb{R}),$$

since :

$$\begin{aligned}
&\Big|\partial_\xi^2 \psi_n^\varepsilon(\xi,u,x)\Big|_{L_x^1(\mathbb{R})} \\
&\leq \Big|-u\gamma^3 \theta_K'''(Z)\Big|_{L_x^1(\mathbb{R})} + \left|\frac{(1+|x|^2)^{\frac{1}{2}} \partial_\xi^2 w_n^\varepsilon(\xi,u,x)}{(1+|x|^2)^{\frac{1}{2}}}\right|_{L_x^1(\mathbb{R})} \\
&\leq \Big|-u\gamma^3 \theta_K'''(Z)\Big|_{L_x^1(\mathbb{R})} + \Big|(1+|x|^2)^{\frac{1}{2}} \partial_\xi^2 w_n^\varepsilon(\xi,u,x)\Big|_{L_x^2(\mathbb{R})} \left|\frac{1}{(1+|x|^2)^{\frac{1}{2}}}\right|_{L_x^2(\mathbb{R})},
\end{aligned}$$

$$\begin{aligned}
&\Big|\partial_u^2 \psi_n^\varepsilon(\xi,u,x)\Big|_{L_x^1(\mathbb{R})} \\
&\leq \Big| -3\gamma^2 u\gamma^3 \theta_K'(Z) - \gamma^3 u\gamma^3 (x-\xi)\theta_K''(Z) \\
&\quad - \Big[2u\gamma^4 + u^2 4\gamma^3 u\gamma^3\Big](x-\xi)\theta_K''(Z) - u^2\gamma^4 u\gamma^3 (x-\xi)^2 \theta_K'''(Z)\Big|_{L_x^1(\mathbb{R})} \\
&\quad + \left|\frac{(1+|x|^2)^{\frac{1}{2}} \partial_u^2 w_n^\varepsilon(\xi,u,x)}{(1+|x|^2)^{\frac{1}{2}}}\right|_{L_x^1(\mathbb{R})}
\end{aligned}$$

$$\leq \Big| -3\gamma^2 u\gamma^3\theta_K'(Z) - \gamma^3 u\gamma^3(x-\xi)\theta_K''(Z)$$

$$- \Big[2u\gamma^4 + u^2 4\gamma^3 u\gamma^3\Big](x-\xi)\theta_K''(Z) - u^2\gamma^4 u\gamma^3(x-\xi)^2\theta_K'''(Z)\Big|_{L^1_x(\mathbb{R})}$$

$$+ \left|(1+|x|^2)^{\frac{1}{2}}\partial_u^2 w_n^\varepsilon(\xi,u,x)\right|_{L^2_x(\mathbb{R})} \left|\frac{1}{(1+|x|^2)^{\frac{1}{2}}}\right|_{L^2_x(\mathbb{R})},$$

$$|\partial_\xi\partial_u\theta_n^\varepsilon(\xi,u,x)|_{L^1_x(\mathbb{R})}$$

$$\leq \left|-u\gamma^3\theta_K'(Z) - u\gamma^4(x-\xi)\theta_K''(Z)\right|_{L^1_x(\mathbb{R})} + \left|\frac{(1+|x|^2)^{\frac{1}{2}}\partial_\xi\partial_u v_n^\varepsilon(\xi,u,x)}{(1+|x|^2)^{\frac{1}{2}}}\right|_{L^1_x(\mathbb{R})}$$

$$\leq \left|-u\gamma^3\theta_K'(Z) - u\gamma^4(x-\xi)\theta_K''(Z)\right|_{L^1_x(\mathbb{R})} + \left|(1+|x|^2)^{\frac{1}{2}}\partial_\xi\partial_u v_n^\varepsilon(\xi,u,x)\right|_{L^2_x(\mathbb{R})} \left|\frac{1}{(1+|x|^2)^{\frac{1}{2}}}\right|_{L^2_x(\mathbb{R})},$$

$$|\partial_\xi\partial_u\psi_n^\varepsilon(\xi,u,x)|_{L^1_x(\mathbb{R})}$$

$$\leq \left|u\gamma^4\theta_K''(Z) + u^2\gamma^4\theta_K''(Z) + u^2\gamma^5(x-\xi)\theta_K'''(Z)\right|_{L^1_x(\mathbb{R})} + \left|\frac{(1+|x|^2)^{\frac{1}{2}}\partial_\xi\partial_u w_n^\varepsilon(\xi,u,x)}{(1+|x|^2)^{\frac{1}{2}}}\right|_{L^1_x(\mathbb{R})}$$

$$\leq \left|u\gamma^4\theta_K''(Z) + u^2\gamma^4\theta_K''(Z) + u^2\gamma^5(x-\xi)\theta_K'''(Z)\right|_{L^1_x(\mathbb{R})}$$

$$+ \left|(1+|x|^2)^{\frac{1}{2}}\partial_\xi\partial_u w_n^\varepsilon(\xi,u,x)\right|_{L^2_x(\mathbb{R})} \left|\frac{1}{(1+|x|^2)^{\frac{1}{2}}}\right|_{L^2_x(\mathbb{R})}.$$

The first order derivatives can be treated in the same way. Thus for a sufficiently small $\varepsilon_0$ the following norms

$$|\partial_\xi\theta_n^\varepsilon(\xi,u,x)|_{L^2_x(\mathbb{R})}, |\partial_u\theta_n^\varepsilon(\xi,u,x)|_{L^2_x(\mathbb{R})}, |\partial_\xi\psi_n^\varepsilon(\xi,u,x)|_{L^2_x(\mathbb{R})}, |\partial_u\psi_n^\varepsilon(\xi,u,x)|_{L^2_x(\mathbb{R})},$$

$$|\partial_\xi\theta_n^\varepsilon(\xi,u,x)|_{L^1_x(\mathbb{R})}, |\partial_u\theta_n^\varepsilon(\xi,u,x)|_{L^1_x(\mathbb{R})}, |\partial_\xi\psi_n^\varepsilon(\xi,u,x)|_{L^1_x(\mathbb{R})}, |\partial_u\psi_n^\varepsilon(\xi,u,x)|_{L^1_x(\mathbb{R})},$$

$$\left|\partial_\xi^2\theta_n^\varepsilon(\xi,u,x)\right|_{L^2_x(\mathbb{R})}, \left|\partial_u^2\theta_n^\varepsilon(\xi,u,x)\right|_{L^2_x(\mathbb{R})}, |\partial_\xi\partial_u\theta_n^\varepsilon(\xi,u,x)|_{L^2_x(\mathbb{R})}, |\partial_\xi\partial_u\psi_n^\varepsilon(\xi,u,x)|_{L^2_x(\mathbb{R})},$$

$$\left|\partial_\xi^2\psi_n^\varepsilon(\xi,u,x)\right|_{L_x^1(\mathbb{R})},\left|\partial_u^2\psi_n^\varepsilon(\xi,u,x)\right|_{L_x^1(\mathbb{R})},\left|\partial_\xi\partial_u\theta_n^\varepsilon(\xi,u,x)\right|_{L_x^1(\mathbb{R})},\left|\partial_\xi\partial_u\psi_n^\varepsilon(\xi,u,x)\right|_{L_x^1(\mathbb{R})},$$

are bounded from above, uniformly in $\varepsilon\in(0,\varepsilon_0]$, $(\xi,u)\in\mathbb{R}\times[-U-U(2),U+U(2)]$ by

$$B:=\max\left\{\sup_{u\in(-U-U(2),U+U(2))}\left|\partial_\xi^{\beta_1}\partial_u^{\beta_2}\theta_0(\xi,u,x)\right|_{L_x^p(\mathbb{R})},\right.$$
$$\left.\sup_{u\in(-U-U(2),U+U(2))}\left|\partial_\xi^{\beta_1}\partial_u^{\beta_2}\psi_0(\xi,u,x)\right|_{L_x^p(\mathbb{R})}\;\middle|\;\beta_1+\beta_2\le 2,\ p=1,2\right\}+1,$$

i.e.,

$$\begin{aligned}&\forall\varepsilon\in(0,\varepsilon_0],(\xi,u)\in\mathbb{R}\times[-U-U(2),U+U(2)],\ \beta_1+\beta_2\le 2,\ p=1,2:\\&\left|\partial_\xi^{\beta_1}\partial_u^{\beta_2}\theta^\varepsilon(\xi,u,x)\right|_{L_x^p(\mathbb{R})}\le B,\qquad\left|\partial_\xi^{\beta_1}\partial_u^{\beta_2}\psi^\varepsilon(\xi,u,x)\right|_{L_x^p(\mathbb{R})}\le B.\end{aligned}\tag{21.2}$$

Notice that

$$\forall\ \varepsilon\in(0,\varepsilon_0],\ (\xi,u)\in\mathbb{R}\times[-U-U(2),U+U(2)]:\qquad\frac{1}{|m_n^\varepsilon(\xi,u)|}\le\frac{1}{c},$$

due to Lemma 21.6.

In this proof we denote by $\|\cdot\|$the maximum row sum norm of a $2\times 2$ matrix induced by the maximum norm $|\cdot|_\infty$ in $\mathbb{R}^2$.

We show that $\exists k\in(0,1),\bar{\delta}>0,\varepsilon_0>0\quad\forall\varepsilon\in(0,\varepsilon_0],(\xi_0,u_0)\in\Sigma(3,U)$
$\forall\ ((\theta,\psi),(\xi,u))\in B_{\bar{\delta}}(0)\times B_{\bar{\delta}}(0)\ :\ \|D_{(\xi,u)}S^\varepsilon_{\xi_0,u_0}(\theta,\psi,\xi,u)\|\le k<1:$

We set

$$k:=\frac{1}{2},\quad K:=\frac{k}{30}$$

Consider $|\theta_n^\varepsilon(\xi_0,u_0,x)-\theta_n^\varepsilon(\bar{\xi},\bar{u},x)|_{L_x^\infty(\mathbb{R})}$ and $|\psi_n^\varepsilon(\xi_0,u_0,x)-\psi_n^\varepsilon(\bar{\xi},\bar{u},x)|_{L_x^2(\mathbb{R})}$,
where $(\xi_0,u_0),(\bar{\xi},\bar{u})\in\overline{\Sigma(2,U)}$:
Since

$$\theta_K(\gamma(u_0)(x-\xi_0))-\theta_K(\gamma(u_0)(x-\bar{\xi}))=\theta_K'(\gamma(u_0)(x-\hat{\xi}[\xi_0,\bar{\xi},u_0,x]))\cdot\gamma(u_0)(\xi_0-\bar{\xi}),$$
$$\theta_K(\gamma(u_0)(x-\bar{\xi}))-\theta_K(\gamma(\bar{u})(x-\bar{\xi}))=\theta_K'(\hat{\gamma}[\gamma(u_0),\gamma(\bar{u}),\bar{\xi},x](x-\bar{\xi}))\cdot(\gamma(u_0)-\gamma(\bar{u})),$$

by the mean value theorem, there exist a $\eta>0$ s.t. for all $(\xi_0,u_0),(\bar{\xi},\bar{u})\in\overline{\Sigma(2,U)}$ with $|(\xi_0,u_0)-(\bar{\xi},\bar{u})|<\eta$, we obtain:

$$\left|\theta_0(\xi_0,u_0,x)-\theta_0(\bar{\xi},\bar{u},x)\right|_{L_x^\infty(\mathbb{R})}\le\frac{Kc}{B}.$$

Since

$$\begin{aligned}
& -u_0\gamma(u_0)\theta'_K(\gamma(u_0)(x-\xi_0)) + \bar{u}\gamma(\bar{u})\theta'_K(\gamma(\bar{u})(x-\bar{\xi})) \\
= & -u_0\gamma(u_0)\theta'_K(\gamma(u_0)(x-\xi_0)) + u_0\gamma(u_0)\theta'_K(\gamma(u_0)(x-\bar{\xi})) \\
& -u_0\gamma(u_0)\theta'_K(\gamma(u_0)(x-\bar{\xi})) + u_0\gamma(u_0)\theta'_K(\gamma(\bar{u})(x-\bar{\xi})) \\
& -u_0\gamma(u_0)\theta'_K(\gamma(\bar{u})(x-\bar{\xi})) + \bar{u}\gamma(\bar{u})\theta'_K(\gamma(\bar{u})(x-\bar{\xi}))
\end{aligned}$$

and

$$\begin{aligned}
&\theta'_K(\gamma(u_0)(x-\xi_0)) - \theta'_K(\gamma(u_0)(x-\bar{\xi})) = \theta''_K(\gamma(u_0)(x-\hat{\xi}[\xi_0,\bar{\xi},u_0,x]))\cdot\gamma(u_0)(\xi_0-\bar{\xi})\,, \\
&\theta'_K(\gamma(u_0)(x-\bar{\xi})) - \theta'_K(\gamma(\bar{u})(x-\bar{\xi})) = \theta''_K(\hat{\gamma}[\gamma(u_0),\gamma(\bar{u}),\bar{\xi},x](x-\bar{\xi}))\cdot(\gamma(u_0)-\gamma(\bar{u}))\,,
\end{aligned}$$

by the mean value theorem, there exist a $\eta > 0$ s.t. for all $(\xi_0,u_0),(\bar{\xi},\bar{u}) \in \overline{\Sigma(2,U)}$ with $|(\xi_0,u_0)-(\bar{\xi},\bar{u})| < \eta$, we obtain:

$$\left|\psi_0(\xi_0,u_0,x) - \psi_0(\bar{\xi},\bar{u},x)\right|_{L^\infty_x(\mathbb{R})} \le \frac{Kc}{B}$$

and

$$|(\gamma(u_0)^3 - \gamma(\bar{u})^3)m| \le \frac{Kc}{B}\,.$$

Using Lemma 20.67 we choose $\varepsilon_0$ so small that

$$\begin{aligned}
&\forall\ \varepsilon \in (0,\varepsilon_0],\ (\xi,u) \in \mathbb{R}\times[-U-U(2),U+U(2)]: \\
&|v^\varepsilon_n(\xi,u,x)|_{L^2_x(\mathbb{R})} \le \frac{Kc}{B}, \quad |w^\varepsilon_n(\xi,u,x)|_{L^2_x(\mathbb{R})} \le \frac{Kc}{B}
\end{aligned} \tag{21.3}$$

and so small that

$$\forall\ \varepsilon \in (0,\varepsilon_0],\ (\xi,u) \in \mathbb{R}\times[-U-U(2),U+U(2)]: \qquad |k^\varepsilon_n(\xi,u)| \le \frac{Kc}{B}\,. \tag{21.4}$$

Notice:

$$\begin{aligned}
&|m^\varepsilon_n(\bar{\xi},\bar{u}) - m^\varepsilon_n(\xi_0,u_0)| \\
&\le |(\gamma(u_0)^3-\gamma(\bar{u})^3)m| + |k^\varepsilon_n(\xi_0,u_0) - k^\varepsilon_n(\bar{\xi},\bar{u})|\,.
\end{aligned}$$

We set

$$\bar{\delta} := \min\left\{\eta, \frac{Kc}{B}, U(7)\right\}$$

and get back to our case where $(\xi_0, u_0) \in \Sigma(3, U)$. Notice that

$$\begin{aligned}
&D_{(\xi,u)} S^{\varepsilon}_{\xi_0,u_0}(\theta, \psi, \xi, u) \\
&= \frac{1}{\gamma(u_0)^3 m + k_n^{\varepsilon}(\xi_0, u_0)} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} D_{(\xi,u)} \bar{\mathcal{N}}^{\varepsilon}_{\xi_0,u_0}(\theta, \psi, \xi, u) - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \frac{1}{\gamma(u_0)^3 m + k_n^{\varepsilon}(\xi_0, u_0)} \begin{pmatrix} -\partial_\xi \bar{\mathcal{N}}^{\varepsilon,2}_{\xi_0,u_0}(\theta, \psi, \xi, u) & -\partial_u \bar{\mathcal{N}}^{\varepsilon,2}_{\xi_0,u_0}(\theta, \psi, \xi, u) \\ \partial_\xi \bar{\mathcal{N}}^{\varepsilon,1}_{\xi_0,u_0}(\theta, \psi, \xi, u) & \partial_u \bar{\mathcal{N}}^{\varepsilon,1}_{\xi_0,u_0}(\theta, \psi, \xi, u) \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

The assertion follows from the following estimates. For all $(\theta, \psi), (\xi, u) \in B_{\bar{\delta}}(0) \times B_{\bar{\delta}}(0)$:

$$\begin{aligned}
&|\frac{1}{m_n^{\varepsilon}(\xi_0, u_0)} \partial_\xi \bar{\mathcal{N}}^{\varepsilon,1}_{\xi_0,u_0}(\theta, \psi, \xi, u)| \\
&= |\frac{1}{m_n^{\varepsilon}(\xi_0, u_0)} \partial_\xi \mathcal{N}^{\varepsilon,1}((\theta(\cdot) + \theta_n^{\varepsilon}(\xi_0, u_0, \cdot), \psi(\cdot) + \psi_n^{\varepsilon}(\xi_0, u_0, \cdot)), (\underbrace{\xi + \xi_0}_{= \bar{u}}, \underbrace{u + u_0}_{= \bar{\xi}}))| \\
&\leq \frac{1}{|m_n^{\varepsilon}(\xi_0, u_0)|} \Big( |\partial_\xi^2 \psi_n^{\varepsilon}(\bar{\xi}, \bar{u}, x)|_{L^2_x(\mathbb{R})} |v_n^{\varepsilon}(\xi_0, u_0, x) - v_n^{\varepsilon}(\bar{\xi}, \bar{u}, x)|_{L^2_x(\mathbb{R})} \\
&\quad + |\partial_\xi^2 \psi_n^{\varepsilon}(\bar{\xi}, \bar{u}, x)|_{L^1_x(\mathbb{R})} |\theta(x) + \theta_0(\xi_0, u_0, x) - \theta_0(\bar{\xi}, \bar{u}, x)|_{L^\infty_x(\mathbb{R})} \\
&\quad + |\partial_\xi^2 \theta_n^{\varepsilon}(\bar{\xi}, \bar{u}, x)|_{L^2_x(\mathbb{R})} |\psi(x) + w_n^{\varepsilon}(\xi_0, u_0, x) - w_n^{\varepsilon}(\bar{\xi}, \bar{u}, x)|_{L^2_x(\mathbb{R})} \\
&\quad + |\partial_\xi^2 \theta_n^{\varepsilon}(\bar{\xi}, \bar{u}, x)|_{L^1_x(\mathbb{R})} |\psi_0(\xi_0, u_0, x) - \psi_0(\bar{\xi}, \bar{u}, x)|_{L^\infty_x(\mathbb{R})} \\
&\quad + |\int -\partial_\xi \psi_n^{\varepsilon}(\bar{\xi}, \bar{u}, x) \partial_\xi \theta_n^{\varepsilon}(\bar{\xi}, \bar{u}, x) + \partial_\xi \theta_n^{\varepsilon}(\bar{\xi}, \bar{u}, x) \partial_\xi \psi_n^{\varepsilon}(\bar{\xi}, \bar{u}, x) \, dx| \Big) \\
&= \frac{1}{|m_n^{\varepsilon}(\xi_0, u_0)|} \Big( |\partial_\xi^2 \psi_n^{\varepsilon}(\bar{\xi}, \bar{u}, x)|_{L^2_x(\mathbb{R})} |v_n^{\varepsilon}(\xi_0, u_0, x) - v_n^{\varepsilon}(\bar{\xi}, \bar{u}, x)|_{L^2_x(\mathbb{R})} \\
&\quad + |\partial_\xi^2 \psi_n^{\varepsilon}(\bar{\xi}, \bar{u}, x)|_{L^1_x(\mathbb{R})} |\theta(x) + \theta_0(\xi_0, u_0, x) - \theta_0(\bar{\xi}, \bar{u}, x)|_{L^\infty_x(\mathbb{R})} \\
&\quad + |\partial_\xi^2 \theta_n^{\varepsilon}(\bar{\xi}, \bar{u}, x)|_{L^2_x(\mathbb{R})} |\psi(x) + w_n^{\varepsilon}(\xi_0, u_0, x) - w_n^{\varepsilon}(\bar{\xi}, \bar{u}, x)|_{L^2_x(\mathbb{R})} \\
&\quad + |\partial_\xi^2 \theta_n^{\varepsilon}(\bar{\xi}, \bar{u}, x)|_{L^1_x(\mathbb{R})} |\psi_0(\xi_0, u_0, x) - \psi_0(\bar{\xi}, \bar{u}, x)|_{L^\infty_x(\mathbb{R})} \Big) \\
&< \frac{k}{2},
\end{aligned}$$

$$
\begin{aligned}
&|-\frac{1}{m_n^\varepsilon(\xi_0,u_0)}\partial_\xi\bar{\mathcal{N}}^{\varepsilon,2}_{\xi_0,u_0}(\theta,\psi,\xi,u)-1|\\
&=|-\frac{1}{m_n^\varepsilon(\xi_0,u_0)}\partial_\xi\mathcal{N}^{\varepsilon,2}((\theta(\cdot)+\theta_n^\varepsilon(\xi_0,u_0,\cdot),\psi(\cdot)+\psi_n^\varepsilon(\xi_0,u_0,\cdot)),(\underbrace{\xi+\xi_0}_{=\bar u},\underbrace{u+u_0}_{=\bar\xi}))-1|\\
&\le\frac{1}{|m_n^\varepsilon(\xi_0,u_0)|}\Big(|\partial_\xi\partial_u\psi_n^\varepsilon(\bar\xi,\bar u,x)|_{L^2_x(\mathbb{R})}|v_n^\varepsilon(\xi_0,u_0,x)-v_n^\varepsilon(\bar\xi,\bar u,x)|_{L^2_x(\mathbb{R})}\\
&\quad+|\partial_\xi\partial_u\psi_n^\varepsilon(\bar\xi,\bar u,x)|_{L^1_x(\mathbb{R})}|\theta(x)+\theta_0(\xi_0,u_0,x)-\theta_0(\bar\xi,\bar u,x)|_{L^\infty_x(\mathbb{R})}\\
&\quad+|\partial_\xi\partial_u\theta_n^\varepsilon(\bar\xi,\bar u,x)|_{L^2_x(\mathbb{R})}|\psi(x)+w_n^\varepsilon(\xi_0,u_0,x)-w_n^\varepsilon(\bar\xi,\bar u,x)|_{L^2_x(\mathbb{R})}\\
&\quad+|\partial_\xi\partial_u\theta_n^\varepsilon(\bar\xi,\bar u,x)|_{L^1_x(\mathbb{R})}|\psi_0(\xi_0,u_0,x)-\psi_0(\bar\xi,\bar u,x)|_{L^\infty_x(\mathbb{R})}\Big)\\
&\quad+|-\frac{1}{m_n^\varepsilon(\xi_0,u_0)}\int-\partial_u\psi_n^\varepsilon(\bar\xi,\bar u,x)\partial_\xi\theta_n^\varepsilon(\bar\xi,\bar u,x)+\partial_u\theta_n^\varepsilon(\bar\xi,\bar u,x)\partial_\xi\psi_n^\varepsilon(\bar\xi,\bar u,x)\,dx-1|\\
&\le\frac{1}{|m_n^\varepsilon(\xi_0,u_0)|}\Big(|\partial_\xi\partial_u\psi_n^\varepsilon(\bar\xi,\bar u,x)|_{L^2_x(\mathbb{R})}|v_n^\varepsilon(\xi_0,u_0,x)-v_n^\varepsilon(\bar\xi,\bar u,x)|_{L^2_x(\mathbb{R})}\\
&\quad+|\partial_\xi\partial_u\psi_n^\varepsilon(\bar\xi,\bar u,x)|_{L^1_x(\mathbb{R})}|\theta(x)+\theta_0(\xi_0,u_0,x)-\theta_0(\bar\xi,\bar u,x)|_{L^\infty_x(\mathbb{R})}\\
&\quad+|\partial_\xi\partial_u\theta_n^\varepsilon(\bar\xi,\bar u,x)|_{L^2_x(\mathbb{R})}|\psi(x)+w_n^\varepsilon(\xi_0,u_0,x)-w_n^\varepsilon(\bar\xi,\bar u,x)|_{L^2_x(\mathbb{R})}\\
&\quad+|\partial_\xi\partial_u\theta_n^\varepsilon(\bar\xi,\bar u,x)|_{L^1_x(\mathbb{R})}|\psi_0(\xi_0,u_0,x)-\psi_0(\bar\xi,\bar u,x)|_{L^\infty_x(\mathbb{R})}\\
&\quad+|m_n^\varepsilon(\bar\xi,\bar u)-m_n^\varepsilon(\xi_0,u_0)|\Big)\\
&<\frac{k}{2},
\end{aligned}
$$

$$
\begin{aligned}
&|\frac{1}{m_n^\varepsilon(\xi_0,u_0)}\partial_u\bar{\mathcal{N}}^{\varepsilon,1}_{\xi_0,u_0}(\theta,\psi,\xi,u)-1|\\
&=|\frac{1}{m_n^\varepsilon(\xi_0,u_0)}\partial_u\mathcal{N}^{\varepsilon,1}((\theta(\cdot)+\theta_n^\varepsilon(\xi_0,u_0,\cdot),\psi(\cdot)+\psi_n^\varepsilon(\xi_0,u_0,\cdot),(\underbrace{\xi+\xi_0}_{=\bar u},\underbrace{u+u_0}_{=\bar\xi}))-1|\\
&\le\frac{1}{|m_n^\varepsilon(\xi_0,u_0)|}\Big(|\partial_u\partial_\xi\psi_n^\varepsilon(\bar\xi,\bar u,x)|_{L^2_x(\mathbb{R})}|v_n^\varepsilon(\xi_0,u_0,x)-v_n^\varepsilon(\bar\xi,\bar u,x)|_{L^2_x(\mathbb{R})}\\
&\quad+|\partial_u\partial_\xi\psi_n^\varepsilon(\bar\xi,\bar u,x)|_{L^1_x(\mathbb{R})}|\theta(x)+\theta_0(\xi_0,u_0,x)-\theta_0(\bar\xi,\bar u,x)|_{L^\infty_x(\mathbb{R})}\\
&\quad+|\partial_u\partial_\xi\theta_n^\varepsilon(\bar\xi,\bar u,x)|_{L^2_x(\mathbb{R})}|\psi(x)+w_n^\varepsilon(\xi_0,u_0,x)-w_n^\varepsilon(\bar\xi,\bar u,x)|_{L^2_x(\mathbb{R})}\\
&\quad+|\partial_u\partial_\xi\theta_n^\varepsilon(\bar\xi,\bar u,x)|_{L^1_x(\mathbb{R})}|\psi_0(\xi_0,u_0,x)-\psi_0(\bar\xi,\bar u,x)|_{L^\infty_x(\mathbb{R})}\Big)\\
&\quad+|\frac{1}{m_n^\varepsilon(\xi_0,u_0)}\int-\partial_\xi\psi_n^\varepsilon(\bar\xi,\bar u,x)\partial_u\theta_n^\varepsilon(\bar\xi,\bar u,x)+\partial_\xi\theta_n^\varepsilon(\bar\xi,\bar u,x)\partial_u\psi_n^\varepsilon(\bar\xi,\bar u,x)\,dx-1|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{|m_n^\varepsilon(\xi_0,u_0)|}\Big(|\partial_u\partial_\xi\psi_n^\varepsilon(\bar\xi,\bar u,x)|_{L^2_x(\mathbb{R})}|v_n^\varepsilon(\xi_0,u_0,x)-v_n^\varepsilon(\bar\xi,\bar u,x)|_{L^2_x(\mathbb{R})}\\
&\quad+|\partial_u\partial_\xi\psi_n^\varepsilon(\bar\xi,\bar u,x)|_{L^1_x(\mathbb{R})}|\theta(x)+\theta_0(\xi_0,u_0,x)-\theta_0(\bar\xi,\bar u,x)|_{L^\infty_x(\mathbb{R})}\\
&\quad+|\partial_u\partial_\xi\theta_n^\varepsilon(\bar\xi,\bar u,x)|_{L^2_x(\mathbb{R})}|\psi(x)+w_n^\varepsilon(\xi_0,u_0,x)-w_n^\varepsilon(\bar\xi,\bar u,x)|_{L^2_x(\mathbb{R})}\\
&\quad+|\partial_u\partial_\xi\theta_n^\varepsilon(\bar\xi,\bar u,x)|_{L^1_x(\mathbb{R})}|\psi_0(\xi_0,u_0,x)-\psi_0(\bar\xi,\bar u,x)|_{L^\infty_x(\mathbb{R})}\\
&\quad+|m_n^\varepsilon(\bar\xi,\bar u)-m_n^\varepsilon(\xi_0,u_0)|\Big)\\
&<\frac{k}{2}\,,
\end{aligned}
$$

$$
\begin{aligned}
&|\frac{1}{m_n^\varepsilon(\xi_0,u_0)}\partial_u\bar{\mathcal{N}}^{\varepsilon,2}_{\xi_0,u_0}(\theta,\psi,\xi,u)|\\
&=|\frac{1}{m_n^\varepsilon(\xi_0,u_0)}\partial_u\mathcal{N}^{\varepsilon,2}((\theta(\cdot)+\theta_n^\varepsilon(\xi_0,u_0,\cdot),\psi(\cdot)+\psi_n^\varepsilon(\xi_0,u_0,\cdot)),(\underbrace{\xi+\xi_0}_{=\bar u},\underbrace{u+u_0}_{=\bar\xi}))|\\
&\leq \frac{1}{|m_n^\varepsilon(\xi_0,u_0)|}\Big(|\partial_u^2\psi_n^\varepsilon(\bar\xi,\bar u,x)|_{L^2_x(\mathbb{R})}|v_n^\varepsilon(\xi_0,u_0,x)-v_n^\varepsilon(\bar\xi,\bar u,x)|_{L^2_x(\mathbb{R})}\\
&\quad+|\partial_u^2\psi_n^\varepsilon(\bar\xi,\bar u,x)|_{L^1_x(\mathbb{R})}|\theta(x)+\theta_0(\xi_0,u_0,x)-\theta_0(\bar\xi,\bar u,x)|_{L^\infty_x(\mathbb{R})}\\
&\quad+|\partial_u^2\theta_n^\varepsilon(\bar\xi,\bar u,x)|_{L^2_x(\mathbb{R})}|\psi(x)+w_n^\varepsilon(\xi_0,u_0,x)-w_n^\varepsilon(\bar\xi,\bar u,x)|_{L^2_x(\mathbb{R})}\\
&\quad+|\partial_u^2\theta_n^\varepsilon(\bar\xi,\bar u,x)|_{L^1_x(\mathbb{R})}|\psi_0(\xi_0,u_0,x)-\psi_0(\bar\xi,\bar u,x)|_{L^\infty_x(\mathbb{R})}\\
&\quad+|\int-\partial_u\psi_n^\varepsilon(\bar\xi,\bar u,x)\partial_u\theta_n^\varepsilon(\bar\xi,\bar u,x)+\partial_u\theta_n^\varepsilon(\bar\xi,\bar u,x)\partial_u\psi_n^\varepsilon(\bar\xi,\bar u,x)\,dx|\Big)\\
&=\frac{1}{|m_n^\varepsilon(\xi_0,u_0)|}\Big(|\partial_u^2\psi_n^\varepsilon(\bar\xi,\bar u,x)|_{L^2_x(\mathbb{R})}|v_n^\varepsilon(\xi_0,u_0,x)-v_n^\varepsilon(\bar\xi,\bar u,x)|_{L^2_x(\mathbb{R})}\\
&\quad+|\partial_u^2\psi_n^\varepsilon(\bar\xi,\bar u,x)|_{L^1_x(\mathbb{R})}|\theta(x)+\theta_0(\xi_0,u_0,x)-\theta_0(\bar\xi,\bar u,x)|_{L^\infty_x(\mathbb{R})}\\
&\quad+|\partial_u^2\theta_n^\varepsilon(\bar\xi,\bar u,x)|_{L^2_x(\mathbb{R})}|\psi(x)+w_n^\varepsilon(\xi_0,u_0,x)-w_n^\varepsilon(\bar\xi,\bar u,x)|_{L^2_x(\mathbb{R})}\\
&\quad+|\partial_u^2\theta_n^\varepsilon(\bar\xi,\bar u,x)|_{L^1_x(\mathbb{R})}|\psi_0(\xi_0,u_0,x)-\psi_0(\bar\xi,\bar u,x)|_{L^\infty_x(\mathbb{R})}\Big)\\
&<\frac{k}{2}\,.
\end{aligned}
$$

We show that $\exists\ r \le \bar{\delta}, \varepsilon_0 > 0 \quad \forall \varepsilon \in (0, \varepsilon_0], (\xi_0, u_0) \in \Sigma(3, U)$
$\forall\ (\theta, \psi) \in B_r(0):\ |S^\varepsilon_{\xi_0,u_0}(\theta, \psi, 0, 0)|_\infty < \bar{\delta}(1-k)$ :

We set

$$r := \min\left\{\frac{\bar{\delta}}{2}, \frac{\bar{\delta}C}{3B}\right\}.$$

Notice that

$$S^\varepsilon_{\xi_0,u_0}(\theta,\psi,0,0) = \frac{1}{\gamma(u_0)^3 m + k^\varepsilon_n(\xi_0,u_0)} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \bar{\mathcal{N}}^\varepsilon_{\xi_0,u_0}(\theta,\psi,0,0).$$

The assertion follows from the following estimates. For all $(\theta, \psi) \in B_r(0)$:

$$\begin{aligned}
&|\frac{1}{m^\varepsilon_n(\xi_0,u_0)} \bar{\mathcal{N}}^{\varepsilon,1}_{\xi_0,u_0}(\theta,\psi,0,0)| \\
&= |\frac{1}{m^\varepsilon_n(\xi_0,u_0)} \mathcal{N}^{\varepsilon,1}_{\xi_0,u_0}((\theta(\cdot) + \theta^\varepsilon_n(\xi_0,u_0,\cdot), \psi(\cdot) + \psi^\varepsilon_n(\xi_0,u_0,\cdot), (\xi_0,u_0))| \\
&= \frac{1}{|m^\varepsilon_n(\xi_0,u_0)|} \left| \int \partial_\xi \psi^\varepsilon_n(\xi_0,u_0,x)\theta(x) - \partial_\xi \theta^\varepsilon_n(\xi_0,u_0,x)\psi(x)\, dx \right| \\
&\le \frac{1}{|m^\varepsilon_n(\xi_0,u_0)|} \Big( |\partial_\xi \psi^\varepsilon_n(\xi_0,u_0,x)|_{L^1(\mathbb{R})} |\theta(x)|_{L^\infty_x(\mathbb{R})} + |\partial_\xi \theta^\varepsilon_n(\xi_0,u_0,x)|_{L^2_x(\mathbb{R})} |\psi(x)|_{L^2_x(\mathbb{R})} \Big),
\end{aligned}$$

$$\begin{aligned}
&|\frac{1}{m^\varepsilon_n(\xi_0,u_0)} \bar{\mathcal{N}}^{\varepsilon,2}_{\xi_0,u_0}(\theta,\psi,0,0)| \\
&= |\frac{1}{m^\varepsilon_n(\xi_0,u_0)} \mathcal{N}^{\varepsilon,2}((\theta(\cdot) + \theta^\varepsilon_n(\xi_0,u_0,\cdot), \psi(\cdot) + \psi^\varepsilon_n(\xi_0,u_0,\cdot), (\xi_0,u_0))| \\
&= \frac{1}{|m^\varepsilon_n(\xi_0,u_0)|} \left| \int \partial_u \psi^\varepsilon_n(\xi_0,u_0,x)\theta(x) - \partial_u \theta^\varepsilon_n(\xi_0,u_0,x)\psi(x)\, dx \right| \\
&\le \frac{1}{|m^\varepsilon_n(\xi_0,u_0)|} \Big( |\partial_u \psi^\varepsilon_n(\xi_0,u_0,x)|_{L^1_x(\mathbb{R})} |\theta(x)|_{L^\infty_x(\mathbb{R})} + |\partial_u \theta^\varepsilon_n(\xi_0,u_0,x)|_{L^2_x(\mathbb{R})} |\psi(x)|_{L^2_x(\mathbb{R})} \Big).
\end{aligned}$$

□

**Remark 21.8.** In this chapter we have chosen $\varepsilon_0$ as small as (21.1), (21.2), (21.3), and (21.4) are satisfied.

# Chapter 22

# Existence of Dynamics and the Orthogonal Component

Let $\varepsilon \in (0, \varepsilon_0]$, where $\varepsilon_0$ is from Lemma 21.7. We argue similar to [Stu98, Proof of theorem 2.1]. In order to be able to make use of existence theory we set

$$\bar{v}(t,x) = \theta(t,x) - \theta_n^\varepsilon(\xi_s, u_s, x)\,,$$
$$\bar{w}(t,x) = \psi(t,x) - \psi_n^\varepsilon(\xi_s, u_s, x)\,,$$

and consider the problem

$$\begin{pmatrix} \bar{v}(0,x) \\ \bar{w}(0,x) \end{pmatrix} = \begin{pmatrix} \theta(0,x) - \theta_n^\varepsilon(\xi_s, u_s, x) \\ \psi(0,x) - \psi_n^\varepsilon(\xi_s, u_s, x) \end{pmatrix}, \tag{22.1}$$

$$\partial_t \begin{pmatrix} \bar{v}(t,x) \\ \bar{w}(t,x) \end{pmatrix} = \begin{pmatrix} \bar{w}(t,x) - \psi_n^\varepsilon(\xi_s, u_s, x) \\ [\bar{v}(t,x) + \theta_n^\varepsilon(\xi_s, u_s, x)]_{xx} - \sin(\bar{v}(t,x) + \theta_n^\varepsilon(\xi_s, u_s, x)) + F(\varepsilon, x) \end{pmatrix}.$$

By [Mar76, Theorem VIII 2.1, Theorem VIII 3.2 ] there exists a local solution (see also [Stu98, Proof of theorem 2.1], [Stu92, p.434 ]) with

$$(\bar{v}, \bar{w}) \in C^1([0, T_{loc}], H^1(\mathbb{R}) \oplus L^2(\mathbb{R}))\,.$$

$(\theta, \psi)$ given by $\theta(t,x) = \bar{v}(t,x) + \theta_n^\varepsilon(\xi_s, u_s, x)$ and $\psi(t,x) = \bar{w}(t,x) + \psi_n^\varepsilon(\xi_s, u_s, x)$ solves obviously locally the Cauchy problem (19.2)-(19.4) and $(\theta, \psi) \in C^1([0, T_{loc}], L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R}))$ due to Morrey's embedding theorem.

We are going to obtain a bound in Chapter 26 which will imply that the local solutions are indeed continuable.

So from now we assume that $(\bar{v}, \bar{w}) \in C^1([0, \overline{T}], H^1(\mathbb{R}) \oplus L^2(\mathbb{R}))$ is a solution of (22.1)-(**??**) and $(\theta, \psi)$ is a solution of (19.2)-(19.4) such that $(\theta, \psi) \in C^1([0, \overline{T}], L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R}))$. Given

$(\theta, \psi)$ we choose the parameters $(\xi(t), u(t))$ according to Lemma 21.7 and define $(v, w)$ as follows:

$$v(t,x) = \theta(t,x) - \theta_n^\varepsilon(\xi(t), u(t), x)\,, \tag{22.2}$$

$$w(t,x) = \psi(t,x) - \psi_n^\varepsilon(\xi(t), u(t), x)\,. \tag{22.3}$$

$(v(t,x), w(t,x))$ is well defined for $t \geq 0$ so small that

$$|v(t)|_{L^\infty(\mathbb{R})} + |w(t)|_{L^2(\mathbb{R})} \leq r$$

and

$$(\xi(t), u(t)) \in \Sigma(4, U, \Xi)\,,$$

where $r, U$ are from Lemma 21.7 and $\Xi$ from Definition 20.65. We formalize this by the following definition.

**Definition 22.1.** *Let $t^*$ be the "exit time":*

$$t^* := \sup\Big\{T > 0 : |v|_{L^\infty(\mathbb{R})L^\infty([0,t])} + |w|_{L^\infty([0,t],L^2(\mathbb{R}))} \leq r,$$

$$(\xi(t), u(t)) \in \Sigma(4, U, \Xi),\ 0 \leq t \leq T\Big\}\,,$$

*where r,U are from Lemma 21.7 and $\Xi$ is from Definition 20.65.*

Notice that $(\xi_s, u_s) = (\xi(0), u(0)) \in \Sigma(4, U, \Xi)$. We will choose $\varepsilon$ such that, among others, the following conditions are fulfilled:

(a) $\varepsilon \in (0, \varepsilon_0]$, where $\varepsilon_0$ is from Lemma 21.7,

(b) $|v(0)|_{L^\infty(\mathbb{R})} + |w(0)|_{L^2(\mathbb{R})} \leq \frac{r}{2}$, where $(v(0), w(0))$ is given by (19.3)-(19.4).

Thus $(v(t,x), w(t,x))$ is well defined for $0 \leq t \leq t^*$. The next lemma provides more information on $(v, w)$.

**Lemma 22.2.** *Let $T = \min\{t^*, \overline{T}\}$ and let $(v, w)$ be defined by* (22.2)-(22.3)*. Then $(v, w) \in C^1([0,T], H^1(\mathbb{R}) \oplus L^2(\mathbb{R}))$.*

**Proof.** Using (22.2)-(22.3), Lemma 4.2 and the fact that $(\bar{v}, \bar{w}) \in C^1([0, T_{loc}], H^1(\mathbb{R}) \oplus L^2(\mathbb{R}))$ , we obtain

$$\begin{aligned} v(t,x) &= \theta(t,x) - \theta_n^\varepsilon(\xi_0, u_0, x) + \theta_n^\varepsilon(\xi_0, u_0, x) - \theta_n^\varepsilon(\xi, u, x) \\ &= \bar{v}(t,x) + \theta_n^\varepsilon(\xi_0, u_0, x) - \theta_n^\varepsilon(\xi, u, x) \in H_x^1(\mathbb{R})\,, \end{aligned}$$

$$w(t,x) = \psi(t,x) - \psi_n^\varepsilon(\xi, u, x) \in L_x^2(\mathbb{R})\,.$$

This implies the claim. □

In the following remark we point out the relation between $F$ and $(\theta_n^\varepsilon, \psi_n^\varepsilon, \lambda_n^\varepsilon)$.

**Remark 22.3.** Due to (20.16) and Lemma 20.66 for a. e. $(\xi, u, x) \in \Sigma(4, U, \Xi) \times \mathbb{R}$

$$u\partial_\xi \begin{pmatrix} \theta_n^\varepsilon(\xi, u, x) \\ \psi_n^\varepsilon(\xi, u, x) \end{pmatrix} - \begin{pmatrix} \psi_n^\varepsilon(\xi, u, x) \\ \partial_x^2 \theta_n^\varepsilon(\xi, u, x) - \sin \theta_n^\varepsilon(\xi, u, x) + F(\varepsilon, x) \end{pmatrix}$$

$$+ \lambda_{u,n}^\varepsilon(\xi, u)\partial_u \begin{pmatrix} \theta_n^\varepsilon(\xi, u, x) \\ \psi_n^\varepsilon(\xi, u, x) \end{pmatrix} + \mathcal{R}_n^\varepsilon(\xi, u, x) = 0\,.$$

Notice that there appears $F$ instead of $\tilde{F}$ in the equation above. The following rates of convergence hold:

$$\left\| \begin{pmatrix} \sum_{i=0}^{n-1} \frac{\partial_\varepsilon^i \theta_n^0}{i!} \cdot \varepsilon^i - \theta_n^\varepsilon \\ \sum_{i=0}^{n-1} \frac{\partial_\varepsilon^i \psi_n^0}{i!} \cdot \varepsilon^i - \psi_n^\varepsilon \\ 0 \end{pmatrix} \right\|_{Y_2^\alpha(u_*)} = \mathcal{O}(\varepsilon^n)\,,$$

$$\left\| \begin{pmatrix} 0 \\ 0 \\ \lambda_{u,n}^\varepsilon \end{pmatrix} \right\|_{Y_2^\alpha(u_*)} = \mathcal{O}(\varepsilon^{1+k})\,.$$

The next two lemmas give rates of convergence of $\mathcal{R}_n^\varepsilon(\xi, u, \cdot)$ and $\lambda_{u,n}^\varepsilon(\xi, u)$ which will be needed in the proof of the modulation equations for the parameters $(\xi(t), u(t))$ in the next chapter and in the proof of the main result in Chapter 26. We start with $\mathcal{R}_n^\varepsilon(\xi, u, \cdot)$.

**Lemma 22.4.** *It holds that*

$$|[\mathcal{R}_n^\varepsilon(\xi, u, \cdot)]_1|_{L^2(\mathbb{R})} = \mathcal{O}(\varepsilon^{n+k+1})\,,$$

$$|[\mathcal{R}_n^\varepsilon(\xi, u, \cdot)]_2|_{L^2(\mathbb{R})} = \mathcal{O}(\varepsilon^{n+k+1})$$

*uniformly in* $(\xi, u) \in \Sigma(4, U, \Xi)$.

**Proof.** Using Remark 22.3, Lemma 20.67 and Morrey's embedding theorem we obtain for all $(\xi, u) \in \Sigma(4, U, \Xi)$:

$$\begin{aligned} &|[\mathcal{R}_n^\varepsilon(\xi, u, \cdot)]_1|_{L^2(\mathbb{R})} \\ \le\ & \left|\lambda_{u,n}^\varepsilon(\cdot, u)\right|_{L^\infty(\mathbb{R})} \left| \partial_u \left( \sum_{i=0}^{n-1} \frac{\partial_\varepsilon^i \theta_n^0(\xi, u, \cdot)}{i!} \varepsilon^i - \theta_n^\varepsilon(\xi, u, \cdot) \right) \right|_{L^2(\mathbb{R})} \\ \le c & \left\| \begin{pmatrix} 0 \\ 0 \\ \lambda_{u,n}^\varepsilon \end{pmatrix} \right\|_{Y_2^\alpha(u_*)} \left\| \begin{pmatrix} \sum_{i=0}^{n-1} \frac{\partial_\varepsilon^i \theta_n^0}{i!} \cdot \varepsilon^i - \theta_n^\varepsilon \\ \sum_{i=0}^{n-1} \frac{\partial_\varepsilon^i \psi_n^0}{i!} \cdot \varepsilon^i - \psi_n^\varepsilon \\ 0 \end{pmatrix} \right\|_{Y_2^\alpha(u_*)} \\ = & \ \mathcal{O}(\varepsilon^{n+k+1}) \end{aligned}$$

and

$$
\begin{aligned}
&\left|[\mathcal{R}_n^\varepsilon(\xi,u,\cdot)]_2\right|_{L^2(\mathbb{R})} \\
&\leq \left|\lambda_{u,n}^\varepsilon(\cdot,u)\right|_{L^\infty(\mathbb{R})}\left|\partial_u\left(\sum_{i=0}^{n-1}\frac{\partial_\varepsilon^i\psi_n^0(\xi,u,\cdot)}{i!}\cdot\varepsilon^i-\psi_n^\varepsilon(\xi,u,\cdot)\right)\right|_{L^2(\mathbb{R})} \\
&\leq c\left\|\begin{pmatrix}0\\0\\\lambda_{u,n}^\varepsilon\end{pmatrix}\right\|_{Y_2^\alpha(u_*)}\left\|\begin{pmatrix}\sum_{i=0}^{n-1}\frac{\partial_\varepsilon^i\theta_n^0}{i!}\cdot\varepsilon^i-\theta_n^\varepsilon\\\sum_{i=0}^{n-1}\frac{\partial_\varepsilon^i\psi_n^0}{i!}\cdot\varepsilon^i-\psi_n^\varepsilon\\0\end{pmatrix}\right\|_{Y_2^\alpha(u_*)} \\
&=\mathcal{O}(\varepsilon^{n+k+1})\,.
\end{aligned}
$$

□

The following lemma gives rates of convergence of $\lambda_{u,n}^\varepsilon(\xi,u)$ and its derivatives.

**Lemma 22.5.** *It holds that*

$$
\begin{aligned}
|\lambda_{u,n}^\varepsilon(\xi,u)| &= \mathcal{O}(\varepsilon^{k+1})\,,\\
|\partial_1\lambda_{u,n}^\varepsilon(\xi,u)| &= \mathcal{O}(\varepsilon^{k+1})\,,\\
|\partial_2\lambda_{u,n}^\varepsilon(\xi,u)| &= \mathcal{O}(\varepsilon^{k+1})\,.
\end{aligned}
$$

*uniformly in* $(\xi,u)\in\mathbb{R}\times I(u_*)$.

**Proof.** Using Remark 22.3 and Morreys embedding Theorem we obtain for all $(\xi,u)\in\Sigma(4,U,\Xi)$:

$$
\begin{aligned}
&|\partial_1\lambda_{u,n}^\varepsilon(\xi,u)| \\
&\leq \left|\partial_1\lambda_{u,n}^\varepsilon(\cdot,u)\right|_{L^\infty(\mathbb{R})} \\
&\leq c\left|\partial_1\lambda_{u,n}^\varepsilon(\cdot,u)\right|_{H^1(\mathbb{R})} \\
&\leq c\left\|\begin{pmatrix}0\\0\\\lambda_{u,n}^\varepsilon\end{pmatrix}\right\|_{Y_2^\alpha(u_*)} \\
&=\mathcal{O}(\varepsilon^{k+1})
\end{aligned}
$$

and

$$
\begin{aligned}
&|\partial_2 \lambda^\varepsilon_{u,n}(\xi, u)| \\
&\leq \left|\partial_2 \lambda^\varepsilon_{u,n}(\cdot, u)\right|_{L^\infty(\mathbb{R})} \\
&\leq c \left|\partial_2 \lambda^\varepsilon_{u,n}(\cdot, u)\right|_{H^1(\mathbb{R})} \\
&\leq c \left\| \begin{pmatrix} 0 \\ 0 \\ \lambda^\varepsilon_{u,n} \end{pmatrix} \right\|_{Y_2^\alpha(u_*)} \\
&= \mathcal{O}(\varepsilon^{k+1}) \,.
\end{aligned}
$$

$|\lambda^\varepsilon_{u,n}(\xi, u)|$ can be treated analogously. □

We compute the time derivatives of $v$ and $w$ which will be needed in the following chapters.

**Lemma 22.6.** *The equations for* $(v, w)$ *defined by* (22.2)-(22.3), *are*

$$
\begin{aligned}
\dot{v}(x) &= w(x) - \dot{\xi}\partial_\xi \theta^\varepsilon_n(\xi, u, x) - \dot{u}\partial_u \theta^\varepsilon_n(\xi, u, x) \\
&\quad + u\partial_\xi \theta^\varepsilon_n(\xi, u, x) + \lambda^\varepsilon_{u,n}(\xi, u)\partial_u \theta^\varepsilon_n(\xi, u, x) + [\mathcal{R}^\varepsilon_n(\xi, u, x)]_1 \,, \\
\dot{w}(x) &= \partial_x^2 v(x) - \cos\theta^\varepsilon_n(\xi, u, x) v(x) + \frac{\sin\theta^\varepsilon_n(\xi, u, x) v^2(x)}{2} \\
&\quad + \tilde{R}(v)(x) + u\partial_\xi \psi^\varepsilon_n(\xi, u, x) + \lambda^\varepsilon_{u,n}(\xi, u)\partial_u \psi^\varepsilon_n(\xi, u, x) \\
&\quad + [\mathcal{R}^\varepsilon_n(\xi, u, x)]_2 - \dot{\xi}\partial_\xi \psi^\varepsilon_n(\xi, u, x) - \dot{u}\partial_u \psi^\varepsilon_n(\xi, u, x) \,,
\end{aligned}
$$

*for times* $t \in [0, t^*]$, *where* $\tilde{R}(v) = \mathcal{O}(|v|^3_{H^1_x(\mathbb{R})})$ *and* $\mathcal{R}^\varepsilon_n(\xi, u, x)$ *is from Theorem 20.62 (c).*

**Proof.** We take the time derivatives of $(v, w)$ and use Remark 22.3, (19.2):

$$
\begin{aligned}
\dot{v}(x) &= w(x) + \psi^\varepsilon_n(\xi, u, x) \\
&\quad - \dot{\xi}\partial_\xi \theta^\varepsilon_n(\xi, u, x) - \dot{u}\partial_u \theta^\varepsilon_n(\xi, u, x) \\
&= w(x) - \dot{\xi}\partial_\xi \theta^\varepsilon_n(\xi, u, x) - \dot{u}\partial_u \theta^\varepsilon_n(\xi, u, x) \\
&\quad + u\partial_\xi \theta^\varepsilon_n(\xi, u, x) + \lambda^\varepsilon_{u,n}(\xi, u)\partial_u \theta^\varepsilon_n(\xi, u, x) + [\mathcal{R}^\varepsilon_n(\xi, u, x)]_1 \,,
\end{aligned}
$$

$$\begin{aligned}\dot{w}(x) &= \partial_x^2\theta(x) - \sin\theta(x) + F(\varepsilon, x) - \dot{\xi}\partial_\xi\psi_n^\varepsilon(\xi, u, x) - \dot{u}\partial_u\psi_n^\varepsilon(\xi, u, x) \\ &= \partial_x^2\theta_n^\varepsilon(\xi, u, x) + \partial_x^2 v(x) - \sin\theta_n^\varepsilon(\xi, u, x) \\ &\quad - \cos\theta_n^\varepsilon(\xi, u, x)v(x) + \frac{\sin\theta_n^\varepsilon(\xi, u, x)v^2(x)}{2} \\ &\quad + \tilde{R}(v)(x) + F(\varepsilon, x) - \dot{\xi}\partial_\xi\psi_n^\varepsilon(\xi, u, x) - \dot{u}\partial_u\psi_n^\varepsilon(\xi, u, x) \\ &= \partial_x^2 v(x) - \cos\theta_n^\varepsilon(\xi, u, x)v(x) + \frac{\sin\theta_n^\varepsilon(\xi, u, x)v^2(x)}{2} \\ &\quad + \tilde{R}(v)(x) + u\partial_\xi\psi_n^\varepsilon(\xi, u, x) + \lambda_{u,n}^\varepsilon(\xi, u)\partial_u\psi_n^\varepsilon(\xi, u, x) \\ &\quad + [\mathcal{R}_n^\varepsilon(\xi, u, x)]_2 - \dot{\xi}\partial_\xi\psi_n^\varepsilon(\xi, u, x) - \dot{u}\partial_u\psi_n^\varepsilon(\xi, u, x)\,,\end{aligned}$$

where we have expanded the term $\sin(\theta_n^\varepsilon(\xi, u, x) + v(x))$. □

# Chapter 23

# Modulation Equations

In the following lemma we derive modulation equations for the parameters $(\xi(t), u(t))$.

**Lemma 23.1.** *There exists an* $\varepsilon_0 > 0$ *such that the following statement holds. Let* $\varepsilon \in (0, \varepsilon_0]$ *and let* $(v, w)$ *be given as in* (22.2)-(22.3) *by*

$$v(t,x) = \theta(t,x) - \theta_n^\varepsilon(\xi(t), u(t), x)\,,$$
$$w(t,x) = \psi(t,x) - \psi_n^\varepsilon(\xi(t), u(t), x)\,,$$

*with* $(\xi, u)$ *obtained from Lemma 21.7. Let*

$$|v|_{L^\infty([0,t^*],H^1(\mathbb{R}))}\,, |w|_{L^\infty([0,t^*],L^2(\mathbb{R}))} \leq \varepsilon_0\,,$$

*where* $t^*$ *is from Definition 22.1. Then it holds for* $t \in [0, t^*]$ *that*

$$|\dot{\xi}(t) - u(t)| \leq C[|v(t)|_{H^1(\mathbb{R})} + |w(t)|_{L^2(\mathbb{R})}]\varepsilon^{k+1} + C\,|v(t)|^2_{H^1(\mathbb{R})} + C\varepsilon^{n+k+1},$$
$$|\dot{u}(t) - \lambda^\varepsilon_{u,n}(\xi(t), u(t))| \leq C[|v(t)|_{H^1(\mathbb{R})} + |w(t)|_{L^2(\mathbb{R})}]\varepsilon^{k+1} + C\,|v(t)|^2_{H^1(\mathbb{R})} + C\varepsilon^{n+k+1}\,,$$

*where* $C$ *depends on* $F$ *and* $\xi_s$.

**Proof.** The technique we use is similar to that in the proof of [IKV12, Lemma 6.2]. Let $\varepsilon_0 \in (0, \varepsilon^*)$ with $\varepsilon^*$ from Theorem 20.61 and let $\varepsilon \in (0, \varepsilon_0)$. Further in the proof we will make some more assumptions on $\varepsilon_0$. Using Definition 21.1 and (22.2)-(22.3), we write the orthogonality conditions as follows:

$$0 = \mathcal{C}_1^\varepsilon(\theta, \psi, \xi, u) = \int \partial_\xi \psi_n^\varepsilon(\xi, u, x) v(x) - \partial_\xi \theta_n^\varepsilon(\xi, u, x) w(x)\, dx\,,$$
$$0 = \mathcal{C}_2^\varepsilon(\theta, \psi, \xi, u) = \int \partial_u \psi_n^\varepsilon(\xi, u, x) v(x) - \partial_u \theta_n^\varepsilon(\xi, u, x) w(x)\, dx\,.$$

In the following we skip $(\theta, \psi, \xi, u)$ for simplicity of further notation and take the derivatives of $\mathcal{C}_1^\varepsilon$, $\mathcal{C}_2^\varepsilon$ with respect to $t$ and use Lemma 22.6.

$$
\begin{aligned}
\frac{d}{dt}\mathcal{C}_1^\varepsilon = & \int \partial_t[\partial_\xi \psi_n^\varepsilon(\xi,u,x)]v(x) + \partial_\xi \psi_n^\varepsilon(\xi,u,x)\partial_t v(x) \\
& - \partial_t[\partial_\xi \theta_n^\varepsilon(\xi,u,x)]w(x) - \partial_\xi \theta_n^\varepsilon(\xi,u,x)\partial_t w(x)\, dx \\
= & \int \left\{ \dot{\xi}\partial_\xi^2 \psi_n^\varepsilon(\xi,u,x) + \dot{u}\partial_u \partial_\xi \psi_n^\varepsilon(\xi,u,x) \right\} v(x) \\
& + \left\{ \partial_\xi \psi_n^\varepsilon(\xi,u,x) \right\} \left\{ w(x) - \dot{\xi}\partial_\xi \theta_n^\varepsilon(\xi,u,x) - \dot{u}\partial_u \theta_n^\varepsilon(\xi,u,x) \right. \\
& \left. + u\partial_\xi \theta_n^\varepsilon(\xi,u,x) + \lambda_{u,n}^\varepsilon(\xi,u)\partial_u \theta_n^\varepsilon(\xi,u,x) + [\mathcal{R}_n^\varepsilon(\xi,u,x)]_1 \right\} \\
& - \left\{ \dot{\xi}\partial_\xi^2 \theta_n^\varepsilon(\xi,u,x) + \dot{u}\partial_u \partial_\xi \theta_n^\varepsilon(\xi,u,x) \right\} w(x) \\
& - \left\{ \partial_\xi \theta_n^\varepsilon(\xi,u,x) \right\} \left\{ \partial_x^2 v(x) - \cos\theta_n^\varepsilon(\xi,u,x)v(x) + \frac{\sin\theta_n^\varepsilon(\xi,u,x)v^2(x)}{2} \right. \\
& + \tilde{R}(v)(x) + u\partial_\xi \psi_n^\varepsilon(\xi,u,x) + \lambda_{u,n}^\varepsilon(\xi,u)\partial_u \psi_n^\varepsilon(\xi,u,x) \\
& \left. + [\mathcal{R}_n^\varepsilon(\xi,u,x)]_2 - \dot{\xi}\partial_\xi \psi_n^\varepsilon(\xi,u,x) - \dot{u}\partial_u \psi_n^\varepsilon(\xi,u,x) \right\} dx \\
= & \underbrace{\int -\partial_\xi \psi_n^\varepsilon(\xi,u,x)\partial_u \theta_n^\varepsilon(\xi,u,x) + \partial_\xi \theta_n^\varepsilon(\xi,u,x)\partial_u \psi_n^\varepsilon(\xi,u,x)\, dx}_{= \Omega\left(t_{\xi,n}^\varepsilon(\xi,u,\cdot),\, t_{u,n}^\varepsilon(\xi,u,\cdot)\right)} \cdot (\dot{u} - \lambda_{u,n}^\varepsilon(\xi,u)) \\
& + \underbrace{\int \partial_\xi \psi_n^\varepsilon(\xi,u,x)\partial_\xi \theta_n^\varepsilon(\xi,u,x) - \partial_\xi \psi_n^\varepsilon(\xi,u,x)\partial_\xi \theta_n^\varepsilon(\xi,u,x)\, dx}_{= \Omega\left(t_{\xi,n}^\varepsilon(\xi,u,\cdot),\, t_{\xi,n}^\varepsilon(\xi,u,\cdot)\right)} \cdot (u - \dot{\xi}) \\
& + \underbrace{\int \partial_u \partial_\xi \psi_n^\varepsilon(\xi,u,x)v(x) - \partial_u \partial_\xi \theta_n^\varepsilon(\xi,u,x)w(x)\, dx}_{= \left[M_n^\varepsilon(\xi,u,v,w)\right]_{12}} \cdot (\dot{u} - \lambda_{u,n}^\varepsilon(\xi,u)) \\
& - \underbrace{\int \partial_\xi^2 \psi_n^\varepsilon(\xi,u,x)v(x) - \partial_\xi^2 \theta_n^\varepsilon(\xi,u,x)w(x)\, dx}_{= \left[M_n^\varepsilon(\xi,u,v,w)\right]_{11}} \cdot (u - \dot{\xi})
\end{aligned}
$$

$$+\int \partial_\xi \psi^\varepsilon_n(\xi,u,x)w(x) - \partial_\xi \theta^\varepsilon_n(\xi,u,x)\left(\partial_x^2 v(x) - \cos\theta^\varepsilon_n(\xi,u,x)v(x)\right) dx$$

$$+\int u\partial_\xi^2 \psi^\varepsilon_n(\xi,u,x)v(x) - u\partial_\xi^2 \theta^\varepsilon_n(\xi,u,x)w(x)\, dx$$

$$+\int \partial_u\partial_\xi \psi^\varepsilon_n(\xi,u,x)v(x) - \partial_u\partial_\xi \theta^\varepsilon_n(\xi,u,x)w(x)\, dx \cdot \lambda^\varepsilon_{u,n}(\xi,u)$$

$$+\int \left\{\partial_\xi \psi^\varepsilon_n(\xi,u,x)\right\}\left\{[\mathcal{R}^\varepsilon_n(\xi,u,x)]_1\right\} dx$$

$$\underbrace{-\int \left\{\partial_\xi \theta^\varepsilon_n(\xi,u,x)\right\}\left\{\frac{\sin\theta^\varepsilon_n(\xi,u,x)v^2(x)}{2} + \tilde{R}(v)(x) + [\mathcal{R}^\varepsilon_n(\xi,u,x)]_2\right\} dx}_{=\left[P^\varepsilon_n(\xi,u,v,w)\right]_1}$$

and

$$\begin{aligned}
\frac{d}{dt}\mathcal{C}^\varepsilon_2 &= \int \partial_t[\partial_u \psi^\varepsilon_n(\xi,u,x)]v(x) + \partial_u \psi^\varepsilon_n(\xi,u,x)\partial_t v(x) \\
&\quad - \partial_t[\partial_u \theta^\varepsilon_n(\xi,u,x)]w(x) - \partial_u \theta^\varepsilon_n(\xi,u,x)\partial_t w(x)\, dx \\
&= \int \left\{\dot{\xi}\partial_\xi\partial_u \psi^\varepsilon_n(\xi,u,x) + \dot{u}\partial_u^2 \psi^\varepsilon_n(\xi,u,x)\right\}v(x) \\
&\quad + \left\{\partial_u \psi^\varepsilon_n(\xi,u,x)\right\}\left\{w(x) - \dot{\xi}\partial_\xi \theta^\varepsilon_n(\xi,u,x) - \dot{u}\partial_u \theta^\varepsilon_n(\xi,u,x)\right. \\
&\quad \left. + u\partial_\xi \theta^\varepsilon_n(\xi,u,x) + \lambda^\varepsilon_{u,n}(\xi,u)\partial_u \theta^\varepsilon_n(\xi,u,x) + [\mathcal{R}^\varepsilon_n(\xi,u,x)]_1\right\} \\
&\quad - \left\{\dot{\xi}\partial_\xi\partial_u \theta^\varepsilon_n(\xi,u,x) + \dot{u}\partial_u^2 \theta^\varepsilon_n(\xi,u,x)\right\}w(x)
\end{aligned}$$

$$
\begin{aligned}
&- \left\{ \partial_u \theta_n^\varepsilon(\xi,u,x) \right\} \left\{ \partial_x^2 v(x) - \cos\theta_n^\varepsilon(\xi,u,x) v(x) + \frac{\sin\theta_n^\varepsilon(\xi,u,x) v^2(x)}{2} \right. \\
&+ \tilde{R}(v)(x) + u \partial_\xi \psi_n^\varepsilon(\xi,u,x) + \lambda_{u,n}^\varepsilon(\xi,u) \partial_u \psi_n^\varepsilon(\xi,u,x) \\
&\left. + [\mathcal{R}_n^\varepsilon(\xi,u,x)]_2 - \dot{\xi} \partial_\xi \psi_n^\varepsilon(\xi,u,x) - \dot{u} \partial_u \psi_n^\varepsilon(\xi,u,x) \right\} dx \\
= \; & \underbrace{\int \partial_u \theta_n^\varepsilon(\xi,u,x) \partial_u \psi_n^\varepsilon(\xi,u,x) - \partial_u \theta_n^\varepsilon(\xi,u,x) \partial_u \psi_n^\varepsilon(\xi,u,x)\, dx}_{= \Omega\left(t_{u,n}^\varepsilon(\xi,u,\cdot),\, t_{u,n}^\varepsilon(\xi,u,\cdot)\right)} \cdot (\dot{u} - \lambda_{u,n}^\varepsilon(\xi,u)) \\
&+ \underbrace{\int \partial_\xi \theta_n^\varepsilon(\xi,u,x) \partial_u \psi_n^\varepsilon(\xi,u,x) - \partial_\xi \psi_n^\varepsilon(\xi,u,x) \partial_u \theta_n^\varepsilon(\xi,u,x)\, dx}_{= \Omega\left(t_{u,n}^\varepsilon(\xi,u,\cdot),\, t_{\xi,n}^\varepsilon(\xi,u,\cdot)\right)} \cdot (u - \dot{\xi}) \\
&+ \underbrace{\int \partial_u^2 \psi_n^\varepsilon(\xi,u,x) v(x) - \partial_u^2 \theta_n^\varepsilon(\xi,u,x) w(x)\, dx}_{= \left[M_n^\varepsilon(\xi,u,v,w)\right]_{22}} \cdot (\dot{u} - \lambda_{u,n}^\varepsilon(\xi,u)) \\
&- \underbrace{\int \partial_\xi \partial_u \psi_n^\varepsilon(\xi,u,x) v(x) - \partial_\xi \partial_u \theta_n^\varepsilon(\xi,u,x) w(x)\, dx}_{= \left[M_n^\varepsilon(\xi,u,v,w)\right]_{21}} \cdot (u - \dot{\xi}) \\
&\underbrace{
\begin{aligned}
&+ \int \partial_u \psi_n^\varepsilon(\xi,u,x) w(x) - \partial_u \theta_n^\varepsilon(\xi,u,x) \left( \partial_x^2 v(x) - \cos\theta_n^\varepsilon(\xi,u,x) v(x) \right) dx \\
&+ \int u \partial_\xi \partial_u \psi_n^\varepsilon(\xi,u,x) v(x) - u \partial_\xi \partial_u \theta_n^\varepsilon(\xi,u,x) w(x)\, dx \\
&+ \int \partial_u^2 \psi_n^\varepsilon(\xi,u,x) v(x) - \partial_u^2 \theta_n^\varepsilon(\xi,u,x) w(x)\, dx \cdot \lambda_{u,n}^\varepsilon(\xi,u) \\
&+ \int \left\{ \partial_u \psi_n^\varepsilon(\xi,u,x) \right\} \left\{ [\mathcal{R}_n^\varepsilon(\xi,u,x)]_1 \right\} dx \\
&- \int \left\{ \partial_u \theta_n^\varepsilon(\xi,u,x) \right\} \left\{ \frac{\sin\theta_n^\varepsilon(\xi,u,x) v^2(x)}{2} + \tilde{R}(v)(x) + [\mathcal{R}_n^\varepsilon(\xi,u,x)]_2 \right\} dx
\end{aligned}
}_{= \left[P_n^\varepsilon(\xi,u,v,w)\right]_2} .
\end{aligned}
$$

We set

$$\begin{aligned}\Omega_n^\varepsilon(\xi,u) :=& \begin{pmatrix}\Omega\left(t^\varepsilon_{\xi,n}(\xi,u,\cdot),t^\varepsilon_{\xi,n}(\xi,u,\cdot)\right) & \Omega\left(t^\varepsilon_{\xi,n}(\xi,u,\cdot),t^\varepsilon_{u,n}(\xi,u,\cdot)\right)\\ \Omega\left(t^\varepsilon_{u,n}(\xi,u,\cdot),t^\varepsilon_{\xi,n}(\xi,u,\cdot)\right) & \Omega\left(t^\varepsilon_{u,n}(\xi,u,\cdot),t^\varepsilon_{u,n}(\xi,u,\cdot)\right)\end{pmatrix}\\ =& \begin{pmatrix}0 & \Omega\left(t^\varepsilon_{\xi,n}(\xi,u,\cdot),t^\varepsilon_{u,n}(\xi,u,\cdot)\right)\\ \Omega\left(t^\varepsilon_{u,n}(\xi,u,\cdot),t^\varepsilon_{\xi,n}(\xi,u,\cdot)\right) & 0\end{pmatrix}\\ =& \begin{pmatrix}0 & \gamma(u)^3m+m_n^\varepsilon(\xi,u)\\ -(\gamma(u)^3m+m_n^\varepsilon(\xi,u)) & 0\end{pmatrix}\\ =& \left(\gamma(u)^3m+m_n^\varepsilon(\xi,u)\right)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.\end{aligned}$$

Now we consider for any $(\bar\xi,\bar u)\in\mathbb{R}\times[-U-U(2),U+U(2)]$, $(\bar v,\bar w)\in H^1(\mathbb{R})\times L^2(\mathbb{R})$ the matrix:

$$\begin{aligned}&M_n^\varepsilon(\bar\xi,\bar u,\bar v,\bar w)\\ =&\begin{pmatrix}\left\langle\begin{pmatrix}\partial_\xi^2\psi_n^\varepsilon(\bar\xi,\bar u,\cdot)\\ -\partial_\xi^2\theta_n^\varepsilon(\bar\xi,\bar u,\cdot)\end{pmatrix},\begin{pmatrix}\bar v(\cdot)\\ \bar w(\cdot)\end{pmatrix}\right\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} & \left\langle\begin{pmatrix}\partial_u\partial_\xi\psi_n^\varepsilon(\bar\xi,\bar u,\cdot)\\ -\partial_u\partial_\xi\theta_n^\varepsilon(\bar\xi,\bar u,\cdot)\end{pmatrix},\begin{pmatrix}\bar v(\cdot)\\ \bar w(\cdot)\end{pmatrix}\right\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})}\\ \left\langle\begin{pmatrix}\partial_\xi\partial_u\psi_n^\varepsilon(\bar\xi,\bar u,\cdot)\\ -\partial_\xi\partial_u\theta_n^\varepsilon(\bar\xi,\bar u,\cdot)\end{pmatrix},\begin{pmatrix}\bar v(\cdot)\\ \bar w(\cdot)\end{pmatrix}\right\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} & \left\langle\begin{pmatrix}\partial_u^2\psi_n^\varepsilon(\bar\xi,\bar u,\cdot)\\ -\partial_u^2\theta_n^\varepsilon(\bar\xi,\bar u,\cdot)\end{pmatrix},\begin{pmatrix}\bar v(\cdot)\\ \bar w(\cdot)\end{pmatrix}\right\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})}\end{pmatrix}.\end{aligned}$$

We use Lemma 21.6, Lemma 20.67 and Hölder's inequality similar to the proof of Lemma 21.7 and obtain for all $(\bar\xi,\bar u)\in\mathbb{R}\times[-U-U(2),U+U(2)]$, $(\bar v,\bar w)\in H^1(\mathbb{R})\times L^2(\mathbb{R})$:

$$\|\left[\Omega_n^\varepsilon(\bar\xi,\bar u)\right]^{-1}M_n^\varepsilon(\bar\xi,\bar u,\bar v,\bar w)\|\le C(|\bar v|_{H^1(\mathbb{R})}+|\bar w|_{L^2(\mathbb{R})}), \tag{23.1}$$

where we denote by $\|\cdot\|$ a matrix norm. Let $I=I_2$ be the identety matrix of dimension 2. Due to (23.1) we are to choose $\varepsilon_0>0$ such that if $|\bar v|_{H^1(\mathbb{R})},|\bar w|_{L^2(\mathbb{R})}\le\varepsilon_0$ then the matrix

$$I+\left[\Omega_n^\varepsilon(\bar\xi,\bar u)\right]^{-1}M_n^\varepsilon(\bar\xi,\bar u,\bar v,\bar w)$$

is invertible by Neumann's theorem. We write the time derivatives of $(\mathcal{C}_1^\varepsilon,\mathcal{C}_2^\varepsilon)$ in matrix form and use the notation $P_n^\varepsilon(\xi,u,v,w)=P$, $M_n^\varepsilon(\xi,u,v,w)=M$, $\Omega_n^\varepsilon(\xi,u)=\Omega$:

$$\begin{aligned}0=&\frac{d}{dt}\begin{pmatrix}\mathcal{C}_1^\varepsilon\\ \mathcal{C}_2^\varepsilon\end{pmatrix}\\ =&\,\Omega\begin{pmatrix}\dot\xi-u\\ \dot u-\lambda^\varepsilon_{u,n}(\xi,u)\end{pmatrix}+M\begin{pmatrix}\dot\xi-u\\ \dot u-\lambda^\varepsilon_{u,n}(\xi,u)\end{pmatrix}+P\,.\end{aligned}$$

This implies

$$-\Omega^{-1}P = \Big(I + \Omega^{-1}M\Big)\begin{pmatrix} \dot{\xi} - u \\ \dot{u} - \lambda^{\varepsilon}_{u,n}(\xi, u)\end{pmatrix}.$$

If $|v|_{H^1(\mathbb{R})}, |w|_{L^2(\mathbb{R})} \leq \varepsilon_0$ then we obtain as mentioned above by Neumann's theorem that

$$\begin{pmatrix} \dot{\xi} - u \\ \dot{u} - \lambda^{\varepsilon}_{u,n}(\xi, u)\end{pmatrix} = -\Big(I + \Omega^{-1}M\Big)^{-1}[\Omega^{-1}P].$$

We make a further assumption on $\varepsilon_0$, namely that $\varepsilon_0$ should be so small that the convergence rates in Lemma 22.4 and in Lemma 22.5 are satisfied.
We consider $P_1$ and $P_2$:

$$\begin{aligned}
P_1 = & \int \partial_\xi \psi^{\varepsilon}_n(\xi, u, x)w(x) - \partial_\xi \theta^{\varepsilon}_n(\xi, u, x)\Bigg(\partial_x^2 v(x) - \cos\theta^{\varepsilon}_n(\xi, u, x)v(x)\Bigg)\,dx \\
& + \int u\partial_\xi^2 \psi^{\varepsilon}_n(\xi, u, x)v(x) - u\partial_\xi^2 \theta^{\varepsilon}_n(\xi, u, x)w(x)\,dx \\
& + \int \partial_u\partial_\xi \psi^{\varepsilon}_n(\xi, u, x)v(x) - \partial_u\partial_\xi \theta^{\varepsilon}_n(\xi, u, x)w(x)\,dx \cdot \lambda^{\varepsilon}_{u,n}(\xi, u) \\
& + \int \Bigg\{\partial_\xi \psi^{\varepsilon}_n(\xi, u, x)\Bigg\}\Bigg\{[\mathcal{R}^{\varepsilon}_n(\xi, u, x)]_1\Bigg\}\,dx \\
& - \int \Bigg\{\partial_\xi \theta^{\varepsilon}_n(\xi, u, x)\Bigg\}\Bigg\{\frac{\sin\theta^{\varepsilon}_n(\xi, u, x)v^2(x)}{2} + \tilde{R}(v)(x) + [\mathcal{R}^{\varepsilon}_n(\xi, u, x)]_2\Bigg\}\,dx.
\end{aligned}$$

Consequently using Corollary 20.6, Lemma 22.4, Lemma 22.5 and similar arguments as above we obtain

$$|P_1| \leq C[|v|_{H^1(\mathbb{R})} + |w|_{L^2(\mathbb{R})}]\varepsilon^{k+1} + C\,|v|^2_{H^1(\mathbb{R})} + C\varepsilon^{n+k+1}.$$

$$\begin{aligned}
P_2 = & \int \partial_u \psi^{\varepsilon}_n(\xi, u, x)w(x) - \partial_u \theta^{\varepsilon}_n(\xi, u, x)\Bigg(\partial_x^2 v(x) - \cos\theta^{\varepsilon}_n(\xi, u, x)v(x)\Bigg)\,dx \\
& + \int u\partial_\xi\partial_u \psi^{\varepsilon}_n(\xi, u, x)v(x) - u\partial_\xi\partial_u \theta^{\varepsilon}_n(\xi, u, x)w(x)\,dx \\
& + \int \partial_u^2 \psi^{\varepsilon}_n(\xi, u, x)v(x) - \partial_u^2 \theta^{\varepsilon}_n(\xi, u, x)w(x)\,dx \cdot \lambda^{\varepsilon}_{u,n}(\xi, u) \\
& + \int \Bigg\{\partial_u \psi^{\varepsilon}_n(\xi, u, x)\Bigg\}\Bigg\{[\mathcal{R}^{\varepsilon}_n(\xi, u, x)]_1\Bigg\}\,dx
\end{aligned}$$

$$-\int \left\{ \partial_u \theta_n^\varepsilon(\xi, u, x) \right\} \left\{ \frac{\sin \theta_n^\varepsilon(\xi, u, x) v^2(x)}{2} + \tilde{R}(v)(x) + [\mathcal{R}_n^\varepsilon(\xi, u, x)]_2 \right\} dx\,.$$

Consequently using Corollary 20.6, Lemma 22.4, Lemma 22.5 and similar arguments as above we obtain

$$|P_2| \leq C[|v|_{H^1(\mathbb{R})} + |w|_{L^2(\mathbb{R})}]\varepsilon^{k+1} + C\,|v|_{H^1(\mathbb{R})}^2 + C\varepsilon^{n+k+1}\,.$$

□

# Chapter 24

# Lyapunov Functional

We introduce the Lyapunov functions.

**Definition 24.1.** *Let $(v, w)$ be given by* (22.2)-(22.3)*, with $(\xi, u)$ obtained from Lemma 21.7. We set*

$$L(t) = \int \frac{w^2(x)}{2} + \frac{(\partial_x v(x))^2}{2} + \frac{\cos(\theta_K(\gamma(x-\xi)))v^2(x)}{2} + uw(x)\partial_x v(x)\, dx\,,$$

$$L^\varepsilon(t) = \int \frac{w^2(x)}{2} + \frac{(\partial_x v(x))^2}{2} + \frac{\cos(\theta_n^\varepsilon(\xi,u,x))v^2(x)}{2} + uw(x)\partial_x v(x)\, dx\,. \tag{24.1}$$

**Remark 24.2.** $L^\varepsilon$ is the quadratic part of

$$H(\theta, \psi) + u\Pi(\theta, \psi)$$

above the virtual solitary manifold $\mathcal{S}_n^\varepsilon$, since due to (7)

$$\begin{aligned} & H\Big(\theta_n^\varepsilon(\xi,u,\cdot) + v(\cdot), \psi_n^\varepsilon(\xi,u,\cdot) + w(\cdot)\Big) \\ &= \frac{1}{2}\int (\psi_n^\varepsilon(\xi,u,x) + w(x))^2 + (\partial_x\theta_n^\varepsilon(\xi,u,x) + \partial_x v(x))^2 \\ &\quad + 2(1-\cos(\theta_n^\varepsilon(\xi,u,x) + v(x)))\, dx \\ &= \frac{1}{2}\int (\psi_n^\varepsilon(\xi,u,x))^2 + 2\psi_n^\varepsilon(\xi,u,x)w(x) + w^2(x) \\ &\quad + (\partial_x\theta_n^\varepsilon(\xi,u,x))^2 + 2\partial_x\theta_n^\varepsilon(\xi,u,x)\partial_x v(x) + (\partial_x v(x))^2 \\ &\quad + 2 - 2\cos\theta_n^\varepsilon(\xi,u,x) + 2\sin(\theta_n^\varepsilon(\xi,u,x))v(x) \\ &\quad + \cos(\theta_n^\varepsilon(\xi,u,x))v^2(x)\, dx + \mathcal{O}(v^3) + \mathcal{O}(w^3) \end{aligned}$$

and due to (8)

$$\begin{aligned}
&\Pi\Big(\theta_n^\varepsilon(\xi,u,\cdot)+v(\cdot),\psi_n^\varepsilon(\xi,u,\cdot)+w(\cdot)\Big)\\
&=\int(\psi_n^\varepsilon(\xi,u,x)+w(x))(\partial_x\theta_n^\varepsilon(\xi,u,x)+\partial_x v(x))\,dx\\
&=\int\psi_n^\varepsilon(\xi,u,x)\partial_x\theta_n^\varepsilon(\xi,u,x)+\psi_n^\varepsilon(\xi,u,x)\partial_x v(x)\\
&\quad+\partial_x\theta_n^\varepsilon(\xi,u,x)\ w(x)+w(x)\partial_x v(x)\,dx\,.
\end{aligned}$$

We compute the time derivative of $L^\varepsilon(t)$ in the following lemma. This will be one of the main ingredients in the proof of the main result.

**Lemma 24.3.**

$$\begin{aligned}
\frac{d}{dt}L^\varepsilon(t)=(u-\dot\xi)\Bigg[\int &-u\partial_x v(x)\Big\{-\partial_\xi\psi_n^\varepsilon(\xi,u,x)\Big\}-w(x)\Big\{-\partial_\xi\psi_n^\varepsilon(\xi,u,x)\Big\}\\
&+[\cos(\theta_n^\varepsilon(\xi,u,x))v(x)-\partial_x^2v(x)]\partial_\xi\theta_n^\varepsilon(\xi,u,x)+uw(x)\partial_x\partial_\xi\theta_n^\varepsilon(\xi,u,x)\,dx\Bigg]\\
&-(\dot u-\lambda_{u,n}^\varepsilon(\xi,u))\Bigg[\int-u\partial_x v(x)\Big\{-\partial_u\psi_n^\varepsilon(\xi,u,x)\Big\}-w(x)\Big\{-\partial_u\psi_n^\varepsilon(\xi,u,x)\Big\}\\
&+[\cos(\theta_n^\varepsilon(\xi,u,x))v(x)-\partial_x^2v(x)]\partial_u\theta_n^\varepsilon(\xi,u,x)+uw(x)\partial_x\partial_u\theta_n^\varepsilon(\xi,u,x)\,dx\Bigg]\\
&-\dot u\int\frac{\sin(\theta_n^\varepsilon(\xi,u,x))}{2}\partial_u\theta_n^\varepsilon(\xi,u,x)v^2(x)\,dx\\
&+(\dot\xi-u)\int\cos(\theta_n^\varepsilon(\xi,u,x))v(x)\partial_x v(x)\,dx\\
&+\dot u\int w(x)\partial_x v(x)\,dx\\
&+\int w(x)[\frac{\sin\theta_n^\varepsilon(\xi,u,x)v^2(x)}{2}+\tilde R(v)(x)+[\mathcal{R}_n^\varepsilon(\xi,u,x)]_2]\\
&+u\partial_x v(x)[\frac{\sin\theta_n^\varepsilon(\xi,u,x)v^2(x)}{2}+\tilde R(v)(x)+[\mathcal{R}_n^\varepsilon(\xi,u,x)]_2]+\partial_x v(x)\partial_x[\mathcal{R}_n^\varepsilon(\xi,u,x)]_1\\
&+\cos(\theta_n^\varepsilon(\xi,u,x))v(x)[\mathcal{R}_n^\varepsilon(\xi,u,x)]_1+uw(x)\partial_x[\mathcal{R}_n^\varepsilon(\xi,u,x)]_1\,dx\,.
\end{aligned}$$

**Proof.** We use a similar technique as in the proof of [KSK97, Lemma 2.1]. We can assume that the initial data of our problem has compact support. This allows us to do the following computations (integration by parts etc.). The claim for non-compactly supported initial data follows by density arguments.

Some preliminary calculations:

$$
\begin{aligned}
&\int \frac{\partial_t[\cos\theta_n^\varepsilon(\xi,u,x)]}{2} v^2(x)\,dx\\
={}& -\int \frac{\sin(\theta_n^\varepsilon(\xi,u,x))}{2}\dot{\xi}\partial_\xi\theta_n^\varepsilon(\xi,u,x)v^2(x)\,dx\\
&-\int \frac{\sin(\theta_n^\varepsilon(\xi,u,x))}{2}\dot{u}\partial_u\theta_n^\varepsilon(\xi,u,x)v^2(x)\,dx\\
={}& -\int \frac{\sin(\theta_n^\varepsilon(\xi,u,x))}{2}\dot{\xi}\Big[\partial_\xi\theta_n^\varepsilon(\xi,u,x)+\partial_x\theta_n^\varepsilon(\xi,u,x)\Big]v^2(x)\,dx\\
&+\int \frac{\sin(\theta_n^\varepsilon(\xi,u,x))}{2}\dot{\xi}\partial_x\theta_n^\varepsilon(\xi,u,x)v^2(x)\,dx\\
&-\int \frac{\sin(\theta_n^\varepsilon(\xi,u,x))}{2}\dot{u}\partial_u\theta_n^\varepsilon(\xi,u,x)v^2(x)\,dx\\
={}& -\int \frac{\sin(\theta_n^\varepsilon(\xi,u,x))}{2}\dot{\xi}\Big[\partial_\xi\theta_n^\varepsilon(\xi,u,x)+\partial_x\theta_n^\varepsilon(\xi,u,x)\Big]v^2(x)\,dx\\
&-\int \dot{\xi}\frac{\partial_x(\cos(\theta_n^\varepsilon(\xi,u,x)))}{2}v^2(x)\\
&-\int \frac{\sin(\theta_n^\varepsilon(\xi,u,x))}{2}\dot{u}\partial_u\theta_n^\varepsilon(\xi,u,x)v^2(x)\,dx\\
={}& -\int \frac{\sin(\theta_n^\varepsilon(\xi,u,x))}{2}\dot{\xi}\Big[\partial_\xi\theta_n^\varepsilon(\xi,u,x)+\partial_x\theta_n^\varepsilon(\xi,u,x)\Big]v^2(x)\,dx\\
&+\int \dot{\xi}\cos(\theta_n^\varepsilon(\xi,u,x))v(x)\partial_x v(x)-\dot{u}\frac{\sin(\theta_n^\varepsilon(\xi,u,x))}{2}\partial_u\theta_n^\varepsilon(\xi,u,x)v^2(x)\,dx\,.
\end{aligned}
$$

Partial integration yields $\int \partial_x v(x)\partial_x^2 v(x)+w(x)\partial_x w(x)\,dx=0$. We differentiate the Lyapunov function (24.1) with respect to $t$:

$$
\begin{aligned}
\frac{d}{dt}L^{\varepsilon}(t) = &\int w(x)\dot{w}(x) + \partial_x v(x)\partial_x \dot{v}(x) + \cos\theta_n^{\varepsilon}(\xi,u,x)v(x)\dot{v}(x) + \frac{\partial_t[\cos\theta_n^{\varepsilon}(\xi,u,x)]}{2}v^2(x)\\
&+ u\dot{w}(x)\partial_x v(x) + uw(x)\partial_x\dot{v}(x) + \dot{u}w(x)\partial_x v(x)\,dx\\
= &\int w(x)\Bigg[\partial_x^2 v(x) - \cos\theta_n^{\varepsilon}(\xi,u,x)v(x) + \frac{\sin\theta_n^{\varepsilon}(\xi,u,x)v^2(x)}{2}\\
&+ \tilde{R}(v)(x) + u\partial_\xi\psi_n^{\varepsilon}(\xi,u,x) + \lambda_{u,n}^{\varepsilon}(\xi,u)\partial_u\psi_n^{\varepsilon}(\xi,u,x)\\
&+ [\mathcal{R}_n^{\varepsilon}(\xi,u,x)]_2 - \dot{\xi}\partial_\xi\psi_n^{\varepsilon}(\xi,u,x) - \dot{u}\partial_u\psi_n^{\varepsilon}(\xi,u,x)\Bigg]\\
&+ \partial_x v(x)\partial_x\Bigg[w(x) - \dot{\xi}\partial_\xi\theta_n^{\varepsilon}(\xi,u,x) - \dot{u}\partial_u\theta_n^{\varepsilon}(\xi,u,x)\\
&+ u\partial_\xi\theta_n^{\varepsilon}(\xi,u,x) + \lambda_{u,n}^{\varepsilon}(\xi,u)\partial_u\theta_n^{\varepsilon}(\xi,u,x) + [\mathcal{R}_n^{\varepsilon}(\xi,u,x)]_1\Bigg]\\
&+ \cos(\theta_n^{\varepsilon}(\xi,u,x))v(x)\Bigg[w(x) - \dot{\xi}\partial_\xi\theta_n^{\varepsilon}(\xi,u,x) - \dot{u}\partial_u\theta_n^{\varepsilon}(\xi,u,x)\\
&+ u\partial_\xi\theta_n^{\varepsilon}(\xi,u,x) + \lambda_{u,n}^{\varepsilon}(\xi,u)\partial_u\theta_n^{\varepsilon}(\xi,u,x) + [\mathcal{R}_n^{\varepsilon}(\xi,u,x)]_1\Bigg]\\
&- \frac{\sin(\theta_n^{\varepsilon}(\xi,u,x))}{2}\dot{\xi}\Big[\partial_\xi\theta_n^{\varepsilon}(\xi,u,x) + \partial_x\theta_n^{\varepsilon}(\xi,u,x)\Big]v^2(x)\\
&+ \dot{\xi}\cos(\theta_n^{\varepsilon}(\xi,u,x))v(x)\partial_x v(x) - \dot{u}\frac{\sin(\theta_n^{\varepsilon}(\xi,u,x))}{2}\partial_u\theta_n^{\varepsilon}(\xi,u,x)v^2(x)\\
&+ u\partial_x v(x)\Bigg[\partial_x^2 v(x) - \cos\theta_n^{\varepsilon}(\xi,u,x)v(x) + \frac{\sin\theta_n^{\varepsilon}(\xi,u,x)v^2(x)}{2}\\
&+ \tilde{R}(v)(x) + u\partial_\xi\psi_n^{\varepsilon}(\xi,u,x) + \lambda_{u,n}^{\varepsilon}(\xi,u)\partial_u\psi_n^{\varepsilon}(\xi,u,x)\\
&+ [\mathcal{R}_n^{\varepsilon}(\xi,u,x)]_2 - \dot{\xi}\partial_\xi\psi_n^{\varepsilon}(\xi,u,x) - \dot{u}\partial_u\psi_n^{\varepsilon}(\xi,u,x)\Bigg]\\
&+ \dot{u}\int w(x)\partial_x v(x)\,dx\\
&+ \int uw(x)\partial_x\Bigg[w(x) - \dot{\xi}\partial_\xi\theta_n^{\varepsilon}(\xi,u,x) - \dot{u}\partial_u\theta_n^{\varepsilon}(\xi,u,x)
\end{aligned}
$$

$$
\begin{aligned}
&+ u\partial_\xi \theta^\varepsilon_n(\xi,u,x) + \lambda^\varepsilon_{u,n}(\xi,u)\partial_u\theta^\varepsilon_n(\xi,u,x) + [\mathcal{R}^\varepsilon_n(\xi,u,x)]_1\Bigg]\,dx \\
= (u-\dot{\xi})&\Bigg[\int -u\partial_x v(x)\Big\{-\partial_\xi\psi^\varepsilon_n(\xi,u,x)\Big\} - w(x)\Big\{-\partial_\xi\psi^\varepsilon_n(\xi,u,x)\Big\} \\
&+ [\cos(\theta^\varepsilon_n(\xi,u,x))v(x) - \partial_x^2 v(x)]\partial_\xi\theta^\varepsilon_n(\xi,u,x) + uw(x)\partial_x\partial_\xi\theta^\varepsilon_n(\xi,u,x)\,dx\Bigg] \\
&- (\dot{u} - \lambda^\varepsilon_{u,n}(\xi,u))\Bigg[\int -u\partial_x v(x)\Big\{-\partial_u\psi^\varepsilon_n(\xi,u,x)\Big\} - w(x)\Big\{-\partial_u\psi^\varepsilon_n(\xi,u,x)\Big\} \\
&+ [\cos(\theta^\varepsilon_n(\xi,u,x))v(x) - \partial_x^2 v(x)]\partial_u\theta^\varepsilon_n(\xi,u,x) + uw(x)\partial_x\partial_u\theta^\varepsilon_n(\xi,u,x)\,dx\Bigg] \\
&- \dot{u}\int \frac{\sin(\theta^\varepsilon_n(\xi,u,x))}{2}\partial_u\theta^\varepsilon_n(\xi,u,x)v^2(x)\,dx \\
&- \dot{\xi}\int \frac{\sin(\theta^\varepsilon_n(\xi,u,x))}{2}\Big[\partial_\xi\theta^\varepsilon_n(\xi,u,x) + \partial_x\theta^\varepsilon_n(\xi,u,x)\Big]v^2(x)\,dx \\
&+ (\dot{\xi}-u)\int \cos(\theta^\varepsilon_n(\xi,u,x))v(x)\partial_x v(x)\,dx \\
&+ \dot{u}\int w(x)\partial_x v(x)\,dx \\
&+ \int w(x)[\frac{\sin\theta^\varepsilon_n(\xi,u,x)v^2(x)}{2} + \tilde{R}(v)(x) + [\mathcal{R}^\varepsilon_n(\xi,u,x)]_2] \\
&+ u\partial_x v(x)[\frac{\sin\theta^\varepsilon_n(\xi,u,x)v^2(x)}{2} + \tilde{R}(v)(x) + [\mathcal{R}^\varepsilon_n(\xi,u,x)]_2] \\
&+ \partial_x v(x)\partial_x[\mathcal{R}^\varepsilon_n(\xi,u,x)]_1 + \cos(\theta^\varepsilon_n(\xi,u,x))v(x)[\mathcal{R}^\varepsilon_n(\xi,u,x)]_1 + uw(x)\partial_x[\mathcal{R}^\varepsilon_n(\xi,u,x)]_1\,dx\,.
\end{aligned}
$$

□

# Chapter 25

# Lower Bound

We use the notation from Chapter 7. In this chapter we will prove the lower bound on the Lyapunov function $L$. This will be one of the main ingredients in the proof of the main result. Due to (22.2)-(22.3) $(v, w)$ is given by

$$v(t,x) = \theta(t,x) - \theta_n^\varepsilon(\xi(t), u(t), x)\,,$$
$$w(t,x) = \psi(t,x) - \psi_n^\varepsilon(\xi(t), u(t), x)\,.$$

Thus the orthogonality conditions

$$\mathcal{C}_1^\varepsilon(\theta, \psi, \xi, u) = 0\,,$$
$$\mathcal{C}_2^\varepsilon(\theta, \psi, \xi, u) = 0\,,$$

can be expressed in terms of the variables $(\theta, \psi, \xi, u)$ and in terms of the variables $(v, w, \xi, u)$. First we introduce a notation in order to be able to express the orthogonality conditions in terms of the variables $(v, w, \xi, u)$.

**Definition 25.1.**

$$\check{\mathcal{C}}_1^\varepsilon(v, w, \xi, u) = \int \partial_\xi \psi_n^\varepsilon(\xi, u, x) v(x) - \partial_\xi \theta_n^\varepsilon(\xi, u, x) w(x)\, dx\,,$$
$$\check{\mathcal{C}}_2^\varepsilon(v, w, \xi, u) = \int \partial_u \psi_n^\varepsilon(\xi, u, x) v(x) - \partial_u \theta_n^\varepsilon(\xi, u, x) w(x)\, dx\,.$$

Now we prove a lower bound on the functional $\mathcal{E}$ which will be related to $L$ later.

**Lemma 25.2.** *Let $\varepsilon_0 > 0$ be sufficiently small. There exists $c > 0$ such that if $0 < \varepsilon \leq \varepsilon_0$, $(\xi, u) \in [-\Xi, \Xi] \times [-U - U(2), U + U(2)] \subset \mathbb{R} \times (-1, 1)$ and $(v, w) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ satisfies*

$$\check{\mathcal{C}}_2^\varepsilon(v, w, \xi, u) = 0$$

*then*

$$\mathcal{E}(v, w, \xi, u) \geq c(|v|_{H^1(\mathbb{R})}^2 + |w|_{L^2(\mathbb{R})}^2)\,.$$

**Proof.** We follow closely [Stu12] and [Stu98]. This proof is a slight modification of the proof of [Stu12, Lemma 4.3]. First of all we choose $\varepsilon_0$ such that $\varepsilon_0 \in (0, \varepsilon^*)$ with $\varepsilon^*$ from Theorem 20.61. We will specify $\varepsilon_0$ later. Notice that the operator $-\partial_Z^2 + \cos\theta_K(Z)$ is nonnegative. It has (see [Stu92]) an one dimensional null space spanned by $\theta_K'(\cdot)$ and the essential spectrum $[1, \infty)$. We argue by contradiction and asume that the result claimed is false:

$$\forall j \in \mathbb{N}\ \exists\ \varepsilon_j \in (0, \varepsilon_0],\quad (\xi_j, u_j) \in [-\Xi, \Xi] \times [-U - U(2), U + U(2)], (\bar{v}_j, \bar{w}_j) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})\ :$$

$$\check{\mathcal{C}}_2^{\varepsilon_j}(\bar{v}_j, \bar{w}_j, \xi_j, u_j) = 0\,, \quad \mathcal{E}(\bar{v}_j, \bar{w}_j, \xi, u_j) < \frac{1}{j}(|\bar{v}_j|^2_{H^1(\mathbb{R})} + |\bar{w}_j|^2_{L^2(\mathbb{R})}). \tag{25.1}$$

This statement is also true for the sequences

$$v_j := \frac{\bar{v}_j}{(|\bar{v}_j|^2_{H^1(\mathbb{R})} + |\bar{w}_j|^2_{L^2(\mathbb{R})})^{\frac{1}{2}}} \quad \text{and} \quad w_j := \frac{\bar{w}_j}{(|\bar{v}_j|^2_{H^1(\mathbb{R})} + |\bar{w}_j|^2_{L^2(\mathbb{R})})^{\frac{1}{2}}}\,.$$

Assuming that $|v_j|_{L^2(\mathbb{R})} \overset{j\to\infty}{\to} 0$ we get: $|(v_j)_x|_{L^2(\mathbb{R})} \overset{j\to\infty}{\to} 0$ and $|w_j|_{L^2(\mathbb{R})} \overset{j\to\infty}{\to} 0$. This is a contradiction to the fact that $|v_j|^2_{H^1(\mathbb{R})} + |w_j|^2_{L^2(\mathbb{R})} = 1\ \forall j \in \mathbb{N}$. By passing to a subsequence we may assume (without loss of generality) that there exists an $\bar{\delta} > 0$ s.t.

$$|v_j|^2_{L^2(\mathbb{R})} \geq \bar{\delta}\ \forall j \in \mathbb{N}\,. \tag{25.2}$$

Since $(v_j, w_j)$ is bounded in $H^1(\mathbb{R}) \times L^2(\mathbb{R})$ we may assume that $v_j \overset{H^1(\mathbb{R})}{\rightharpoonup} v$ and $w_j \overset{L^2(\mathbb{R})}{\rightharpoonup} w$ by taking subsequances. Due to Rellich's theorem we may assume again by passing to subsequences that $v_j \overset{L^2(\Omega)}{\to} v$ for $\Omega \subset \mathbb{R}$ bounded and open. Passing to a further subsequence we may assume almost everywhere convergence. The fact that

$$\exists\ r > 0 \quad \text{s.t.} \quad |\cos(\theta_K(Z))| > \frac{1}{2} \quad \text{for} \quad |Z| > r \tag{25.3}$$

and that $-\partial_Z^2 + \cos\theta_K(Z)$ is a nonnegative operator yields the following estimate if we assume that $(v_j, w_j) \in H^2(\mathbb{R}) \times L^2(\mathbb{R})$.

$$\begin{aligned}
&\mathcal{E}(v_j, w_j, \xi_j, u_j) \\
&= \frac{1}{2}\int (w_j(x) + u_j v_j'(x))^2 + [\frac{1}{\gamma(u_j)}(v_j)'(x)]^2 \\
&\quad + \cos(\theta_K(\gamma(u_j)(x - \xi_j)))v_j^2(x)\,dx \\
&= \frac{1}{2\gamma(u_j)}\int (w_j(\frac{Z}{\gamma(u_j)} + \xi_j) + u_j v_j'(\frac{Z}{\gamma(u_j)} + \xi_j))^2 + [\frac{1}{\gamma(u_j)}(v_j)'(\frac{Z}{\gamma(u_j)} + \xi_j)]^2 \\
&\quad + \cos(\theta_K(Z))v_j^2(\frac{Z}{\gamma(u_j)} + \xi_j)\,dZ
\end{aligned}$$

$$= \frac{1}{2\gamma(u_j)} \int (w_j(\frac{Z}{\gamma(u_j)} + \xi_j) + u_j v_j'(\frac{Z}{\gamma(u_j)} + \xi_j))^2 - \partial_Z^2 v_j(\frac{Z}{\gamma(u_j)} + \xi_j) v_j(\frac{Z}{\gamma(u_j)} + \xi_j)$$

$$+ \cos(Z) v_j^2(\frac{Z}{\gamma(u_j)} + \xi_j)\, dZ$$

$$\geq \frac{1}{2\gamma(u_j)} \int_{\{Z\in\mathbb{R}:|Z|\geq r\}} (w_j(\frac{Z}{\gamma(u_j)} + \xi_j) + u_j v_j'(\frac{Z}{\gamma(u_j)} + \xi_j))^2 + [\partial_Z v_j(\frac{Z}{\gamma(u_j)} + \xi_j)]^2$$

$$+ \cos(Z) v_j^2(\frac{Z}{\gamma(u_j)} + \xi_j)\, dZ$$

$$\geq \frac{1}{4\gamma(u_j)} \int_{\{Z\in\mathbb{R}:|Z|\geq r\}} v_j^2(\frac{Z}{\gamma(u_j)} + \xi_j)\, dZ$$

$$= \frac{1}{4} \int_{-\infty}^{\frac{-r}{\gamma(u_j)}+\xi_j} v_j^2(x)\, dx + \frac{1}{4} \int_{\frac{r}{\gamma(u_j)}+\xi_j}^{\infty} v_j^2(x)\, dx\,.$$

It follows by approximation

$$\mathcal{E}(v_j, w_j, \xi_j, u_j) \geq \frac{1}{4} \int_{-\infty}^{\frac{-r}{\gamma(u_j)}+\xi_j} v_j^2(x)\, dx + \frac{1}{4} \int_{\frac{r}{\gamma(u_j)}+\xi_j}^{\infty} v_j^2(x)\, dx$$

for $(v_j, w_j) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ and $(\xi_j, u_j)$ as above. We may extract a subsequence such that $u_j \overset{\mathbb{R}}{\to} u$, $\xi_j \overset{\mathbb{R}}{\to} \xi$ and $\varepsilon_j \overset{\mathbb{R}}{\to} \hat{\varepsilon}$. It follows from (25.1) and from the previous estimate that

$$\int_{\{x\in\mathbb{R}:|x|\geq\tilde{r}\}} v_j^2(x)\, dx \overset{j\to\infty}{\to} 0$$

for a sufficiently large $\tilde{r}$. As a consequence (25.2) and the strong convergence on bounded domains yield

$$\int_{\{x\in\mathbb{R}:|x|\leq\tilde{r}\}} v^2(x)\, dx \geq \bar{\delta},$$

from which it follows that $v \not\equiv 0$. Weak convergence and the continuity of $\varepsilon \mapsto (\hat{\theta}_n^\varepsilon, \hat{\psi}_n^\varepsilon, \lambda_{u,n}^\varepsilon)$ (see Theorem 20.61) imply using the triangle inequality that

$$\check{\mathcal{C}}_2^{\hat{\varepsilon}}(v, w, \xi, u) = 0 \tag{25.4}$$

and

$$\frac{1}{2} \int \left(w(x) + u v'(x)\right)^2\, dx \leq \liminf_{j\to\infty} \frac{1}{2} \int \left(w_j(x) + u_j v_j'(x)\right)^2\, dx, \tag{25.5}$$

$$\frac{1}{2} \int \left(\frac{1}{\gamma(u)} (v)'(x)\right)^2\, dx \leq \liminf_{j\to\infty} \frac{1}{2} \int \left(\frac{1}{\gamma(u_j)} (v_j)'(x)\right)^2\, dx. \tag{25.6}$$

Due to (25.3) we are able to apply Fatou's lemma for a sufficiently large $\tilde{r}$ and obtain:

$$\begin{aligned} &\frac{1}{2}\int_{\{x\in\mathbb{R}:|x|>\tilde{r}\}} \cos(\theta_K(\gamma(u)(x-\xi)))v^2(x)\,dx \\ &\leq \liminf_{j\to\infty}\frac{1}{2}\int_{\{x\in\mathbb{R}:|x|>\tilde{r}\}} \cos(\theta_K(\gamma(u_j)(x-\xi_j)))v_j^2(x)\,dx, \end{aligned} \tag{25.7}$$

where we have used that $(v_j)$ converges almost everywhere. Dominated convergence theorem yields:

$$\begin{aligned} &\frac{1}{2}\int_{\{x\in\mathbb{R}:|x|\leq\tilde{r}\}} \cos(\theta_K(\gamma(u)(x-\xi)))v^2(x)\,dx \\ &= \lim_{j\to\infty}\frac{1}{2}\int_{\{x\in\mathbb{R}:|x|\leq\tilde{r}\}} \cos(\theta_K(\gamma(u_j)(x-\xi_j)))v_j^2(x)\,dx\,. \end{aligned} \tag{25.8}$$

(25.1) together with (25.5)-(25.8) imply:

$$\mathcal{E}(v,w,\xi,u)=0\,. \tag{25.9}$$

Since $v\not\equiv 0$, (25.9) yields $(v(x),w(x))=\eta(\theta_K'(\gamma(u)(x-\xi)),-u\gamma(u)\theta_K''(\gamma(u)(x-\xi)))$ for some $\eta\neq 0$. Using Lemma 20.67 and the notation from Definition 21.4 we choose $\varepsilon_0$ sufficiently small so that for all $(\xi,u)$

$$\frac{1}{\gamma(u)}\left(\left|\partial_u v_n^{\hat{\varepsilon}}(\xi,u,x)\right|_{L^2(\mathbb{R})}|\partial_\xi\psi_0(\xi,u,x)|_{L^2(\mathbb{R})}+\left|\partial_u w_n^{\hat{\varepsilon}}(\xi,u,x)\right|_{L^2(\mathbb{R})}|\partial_\xi\theta_0(\xi,u,x)|_{L^2(\mathbb{R})}\right)\leq\frac{m}{2}\,.$$

We obtain a contradiction to (25.4), since

$$\begin{aligned} 0&=\check{\mathcal{C}}_2^{\hat{\varepsilon}}(v,w,\xi,u)\\ &=\int\partial_u\psi_n^{\hat{\varepsilon}}(\xi,u,x)v(x)-\partial_u\theta_n^{\hat{\varepsilon}}(\xi,u,x)w(x)\,dx\\ &=\eta\left(\gamma^3(u)m+\int\partial_u v_n^{\hat{\varepsilon}}(\xi,u,x)\frac{\partial_\xi\psi_0(\xi,u,x)}{\gamma(u)}-\partial_u w_n^{\hat{\varepsilon}}(\xi,u,x)\frac{\partial_\xi\theta_0(\xi,u,x)}{\gamma(u)}\,dx\right)\\ &\neq 0\,. \end{aligned}$$

□

**Remark 25.3.** Let $(v,w)$ be given by (22.2)-(22.3), with $(\xi,u)$ obtained from Lemma 21.7. It holds that

$$L(t)=\mathcal{E}(v(t),w(t),\xi(t),u(t))\,.$$

# Chapter 26

# Proof of Theorem 19.3

## 26.1 Dynamics with Approximate Equations for the Parameters $(\xi, u)$

The goal of this section is to prove the following theorem. We consider again the Cauchy problem defined by (19.2)-(19.4).

**Theorem 26.1.** *Let $n \geq 1$, $k+1 \leq n$. Assume*

*(a) $F \in C^n((-1,1), H^{1,1}(\mathbb{R}))$, $\partial_\varepsilon^l F(0,\cdot) = 0$, where $0 \leq l \leq k$;*

*(b) $\xi_s \in \mathbb{R}$.*

*There exist $\varepsilon_0 > 0$, $u_* > 0$ and a map*

$$(-\varepsilon_0, \varepsilon_0) \to Y_2^1(u_*),\ \varepsilon \mapsto (\hat{\theta}_n^\varepsilon, \hat{\psi}_n^\varepsilon, \lambda_{u,n}^\varepsilon) \tag{26.1}$$

*of class $C^n$ such that the following holds. We consider the initial value problem*

$$\partial_t \begin{pmatrix} \theta(t,x) \\ \psi(t,x) \end{pmatrix} = \begin{pmatrix} \psi(t,x) \\ \partial_x^2\theta(t,x) - \sin\theta(t,x) + F(\varepsilon,x) \end{pmatrix}, \tag{26.2}$$

$$\theta(0,x) = \theta_n^\varepsilon(\xi_s, u_s, x) + v(0,x)\,, \tag{26.3}$$

$$\psi(0,x) = \psi_n^\varepsilon(\xi_s, u_s, x) + w(0,x)\,, \tag{26.4}$$

*where $(\theta_n^\varepsilon, \psi_n^\varepsilon, \lambda_{u,n}^\varepsilon) = (\theta_0 + \hat{\theta}_n^\varepsilon, \psi_0 + \hat{\psi}_n^\varepsilon, \lambda_{u,n}^\varepsilon)$, $(v(0,x), w(0,x)) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ and $(\xi_s, u_s) = (\xi(0), u(0)) \in \mathbb{R} \times (-1,1)$ such that the following assumptions are satisfied:*

*(c) $\varepsilon \in (0, \varepsilon_0]$,*

*(d) $|u_s| \leq \varepsilon^{\frac{k+1}{2}}$,*

*(e) $\mathcal{N}^\varepsilon(\theta(0,\cdot), \psi(0,\cdot), \xi_s, u_s) = 0\,.$*

*(f)* $|v(0)|^2_{H^1(\mathbb{R})} + |w(0)|^2_{L^2(\mathbb{R})} \le \varepsilon^{2n}$, *where* $(v(0,\cdot), w(0,\cdot))$ *is given by* (26.3)-(26.4).

*Then the Cauchy problem defined by* (26.2)-(26.4) *has a unique solution on the time interval*

$$0 \le t \le T, \text{ where } T = T(\varepsilon, k, \delta) = \frac{1}{\varepsilon^{\beta(k,\delta)}}, \quad \beta(k,\delta) = \frac{k+1-\delta}{2}.$$

*The solution may be written in the form*

$$\theta(t,x) = \theta_n^\varepsilon(\xi(t), u(t), x) + v(t,x)\,,$$
$$\psi(t,x) = \psi_n^\varepsilon(\xi(t), u(t), x) + w(t,x)\,,$$

*where* $v, w, u, \xi$ *have regularity*

$$(\xi(t), u(t)) \in C^1([0,T], \mathbb{R} \times (-1,1)),$$
$$(v(t), w(t)) \in C^1([0,T], H^1(\mathbb{R}) \oplus L^2(\mathbb{R})),$$

*such that the symplectic orthogonality condition*

$$\mathcal{N}^\varepsilon(\theta(t,\cdot), \psi(t,\cdot), \xi(t), u(t)) = 0$$

*is satisfied. There exist positive constants* $c, C$ *such that*

$$|\dot{\xi}(t) - u(t)| \le C\varepsilon^{n+k+1},$$
$$|\dot{u}(t) - \lambda^\varepsilon_{u,n}(\xi(t), u(t))| \le C\varepsilon^{n+k+1}\,,$$

*and*

$$|v|^2_{L^\infty([0,T],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,T],L^2(\mathbb{R}))} \le c\varepsilon^{2n}\,.$$

*The constants* $c, C$ *depend on* $F$ *and* $\xi_s$*.*

The previous theorem yields us only approximate equations for the parameters $(\xi, u)$ whereas Theorem 19.3 provides ODE's (19.5)-(19.6) which describe the dynamics more precisely. In the previous theorem the orthogonality conditions are satisfied which do not have to hold in Theorem 19.3.

The existence of $\varepsilon_0 > 0$, $u_* > 0$ and the map (26.1) is ensured by Theorem 20.61. Now we suppose that (26.2)-(26.4) has a solution and we make some assumptions on $(v, w)$ given by (22.2)-(22.3) and on $(\xi, u)$ obtained from Lemma 21.7. Then the following lemma yields us more accurate information about $(v, w)$ and $(\xi, u)$.

**Lemma 26.2.** *Let $n \geq 1$, $k+1 \leq n$. Assume that the assumptions (a),(b) of Theorem 26.1 are satisfied. Let $\Xi$ be from Definition 20.65 and $U$ from Lemma 21.7. There exists $\bar{\varepsilon}_0 > 0$ such that the following statement holds. Let $\varepsilon \in (0, \bar{\varepsilon}_0)$. Assume that* (26.2)-(26.4) *has a solution $(\theta, \psi)$ on $\left[0, \overline{T}\right]$ such that*

$$(\theta, \psi) \in C^1([0, \overline{T}], L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R})) .$$

*Let the assumptions (d),(e) of Theorem 26.1 be satisfied. Suppose that*

$$0 \leq T \leq t^* \leq \overline{T} ,$$

*where $t^*$ is from Definition 22.1. Suppose that $(v, w)$ is given by* (22.2)-(22.3), *with $(\xi, u)$ obtained from Lemma 21.7 such that*

$$|v|^2_{L^\infty([0,T],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,T],L^2(\mathbb{R}))} \leq \varepsilon^{2n-\delta} .$$

*Then , provided*

$$0 \leq T \leq \frac{1}{\varepsilon^{\beta(k,\delta)}}, \quad \beta(k, \delta) = \frac{k+1-\delta}{2} ,$$

*it holds that*

- *(a)* $\forall t \in [0, T]$ $(\xi(t), u(t)) \in \Sigma(5, U, \Xi)$ ,
- *(b)* $|v|^2_{L^\infty([0,T],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,T],L^2(\mathbb{R}))} \leq \frac{1}{c}(L(0) + C\varepsilon^{2n})$ , *where $c$ is from Lemma 25.2 and $C$ depends on $F, \xi_s$.*

**Remark 26.3.** Notice that the assumption $T \leq t^*$ yields us the information:
$\forall t \in [0, T]$ $(\xi(t), u(t)) \in \Sigma(4, U, \Xi)$ .

**Proof.** Choose $\bar{\varepsilon}_0$ such that the following holds:

1. $\bar{\varepsilon}_0 > 0$ and $\bar{\varepsilon}_0$ is less than the minimum of the $\varepsilon_0$'s from Lemma 21.7, Lemma 23.1 and Lemma 25.2;
2. $\forall \varepsilon \in (0, \bar{\varepsilon}_0) :$ $\varepsilon^{\frac{k+1}{2}} < U$ with $U$ from Lemma 21.7;
3. $\bar{\varepsilon}_0$ is so small that

$$\forall \varepsilon \in (0, \bar{\varepsilon}_0) : \quad C\varepsilon^{k+1} \left[ \frac{1}{\varepsilon^{\frac{k+1-\delta}{2}}} \right] + |u(0)| \leq \frac{U(5)}{2} + U ,$$

   where C is a constant that appears in (26.5) further in this proof which depends on the bounds of the derivatives (order 0 to n) of the mapping $\varepsilon \mapsto (\theta^\varepsilon_n, \psi^\varepsilon_n, \lambda^\varepsilon_{u,n})$ (which depend on $F$ and $\xi_s$);

(4) $\bar{\varepsilon}_0$ is so small that

$$\forall \varepsilon \in (0, \bar{\varepsilon}_0): \quad C\varepsilon^{n+k+1-\delta}\left[\frac{1}{\varepsilon^{\frac{k+1-\delta}{2}}}\right] + C\varepsilon^{k+1}\left[\frac{1}{\varepsilon^{\frac{k+1-\delta}{2}}}\right]^2 + \varepsilon^{\frac{k+1}{2}}\left[\frac{1}{\varepsilon^{\frac{k+1-\delta}{2}}}\right] \leq \frac{1}{2},$$

where C is a constant that appears in (26.6) further in this proof which depends on the bounds of the derivatives (order 0 to n) of the mapping $\varepsilon \mapsto (\theta_n^\varepsilon, \psi_n^\varepsilon, \lambda_{u,n}^\varepsilon)$ (which depend on $F$ and $\xi_s$).

(5) $\bar{\varepsilon}_0$ is so small that $\forall \varepsilon \in (0, \bar{\varepsilon}_0)$ the following statement holds: if $(v, w) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ satisfies $|v|^2_{H^1(\mathbb{R})} + |w|^2_{L^2(\mathbb{R})} \leq \varepsilon^{2n-\delta}$ then it holds that $|v|_{L^\infty(\mathbb{R})} + |w|_{L^2(\mathbb{R})} \leq \frac{r}{2}$, where $r$ is from Lemma 21.7. This can be ensured by Morrey's embedding theorem.

Let $\varepsilon \in (0, \bar{\varepsilon}_0)$. Notice that we have chosen $\varepsilon_0$ in Lemma 23.1 such that the convergence rates in Lemma 22.4 and in Lemma 22.5 are satisfied.
Lemma 23.1 yields $\forall t \in [0, T]$:

$$\begin{aligned}|\dot{\xi}(t) - u(t)| &\leq C[|v|_{H^1(\mathbb{R})} + |w|_{L^2(\mathbb{R})}]\varepsilon^{k+1} + C\,|v|^2_{H^1(\mathbb{R})} + C\varepsilon^{n+k+1} \\ &\leq C\varepsilon^{n+k+1-\frac{\delta}{2}} + C\varepsilon^{2n-\delta} + C\varepsilon^{n+k+1} \\ &\leq C\varepsilon^{n+k+1-\delta},\end{aligned}$$

$$\begin{aligned}|\dot{u}(t) - \lambda_{u,n}^\varepsilon(\xi(t), u(t))| &\leq C[|v|_{H^1(\mathbb{R})} + |w|_{L^2(\mathbb{R})}]\varepsilon^{k+1} + C\,|v|^2_{H^1(\mathbb{R})} + C\varepsilon^{n+k+1} \\ &\leq C\varepsilon^{n+k+1-\frac{\delta}{2}} + C\varepsilon^{2n-\delta} + C\varepsilon^{n+k+1} \\ &\leq C\varepsilon^{n+k+1-\delta}.\end{aligned}$$

Thus we obtain using Lemma 22.5 $\forall t \in [0, T]$:

$$\begin{aligned}|u(t) - u(0)| &\leq \int_0^t |\dot{u}(s)|\, ds \\ &\leq \int_0^t |\dot{u}(s) - \lambda_{u,n}^\varepsilon(\xi(s), u(s))| + |\lambda_{u,n}^\varepsilon(\xi(s), u(s))|\, ds \\ &\leq C\varepsilon^{k+1} t\end{aligned}$$

$$\Rightarrow \quad |u(t)| \leq C\varepsilon^{k+1} t + |u(0)|, \tag{26.5}$$

$$\begin{aligned}|\xi(t)-\xi(0)| &\leq \int_0^t |\dot{\xi}(s)|\,ds\\ &\leq \int_0^t |\dot{\xi}(s)-u(s)|+|u(s)|\,ds\\ &\leq C\varepsilon^{n+k+1-\delta}t+C\varepsilon^{k+1}t^2+|u(0)|t\end{aligned}$$

$$\Rightarrow \quad |\xi(t)| \leq C\varepsilon^{n+k+1-\delta}t+C\varepsilon^{k+1}t^2+|u(0)|t+|\xi(0)|\,. \tag{26.6}$$

This implies (a) due to (3), (4) and assumption (d) of Theorem 26.1. Using Lemma 25.2, Lemma 7.2, Lemma 24.3 and Lemma 22.4 we obtain for times

$$0 \leq t \leq T \leq \frac{1}{\varepsilon^{\beta(k,\delta)}},$$

the following estimate

$$\begin{aligned}&c(|v(t)|^2_{H^1(\mathbb{R})}+|w(t)|^2_{L^2(\mathbb{R})})\\ &\leq L(t)\\ &= L^\varepsilon(t)+C\varepsilon^{k+1}\,|v|^2_{L^\infty([0,t],H^1(\mathbb{R}))}\\ &= L^\varepsilon(0)+\int_0^t \frac{d}{dt}L^\varepsilon(t)\,dt+C\varepsilon\,|v|^2_{L^\infty([0,t],H^1(\mathbb{R}))}\\ &= L^\varepsilon(0)+\int_0^t (u-\dot{\xi})\Bigg[\int -u\partial_x v(x)\Big\{-\partial_\xi\psi_n^\varepsilon(\xi,u,x)\Big\}-w(x)\Big\{-\partial_\xi\psi_n^\varepsilon(\xi,u,x)\Big\}\\ &\quad+[\cos(\theta_n^\varepsilon(\xi,u,x))v(x)-\partial_x^2 v(x)]\partial_\xi\theta_n^\varepsilon(\xi,u,x)+uw(x)\partial_x\partial_\xi\theta_n^\varepsilon(\xi,u,x)\,dx\Bigg]\\ &\quad-(\dot{u}-\lambda_{u,n}^\varepsilon(\xi,u))\Bigg[\int -u\partial_x v(x)\Big\{-\partial_u\psi_n^\varepsilon(\xi,u,x)\Big\}-w(x)\Big\{-\partial_u\psi_n^\varepsilon(\xi,u,x)\Big\}\\ &\quad+[\cos(\theta_n^\varepsilon(\xi,u,x))v(x)-\partial_x^2 v(x)]\partial_u\theta_n^\varepsilon(\xi,u,x)+uw(x)\partial_x\partial_u\theta_n^\varepsilon(\xi,u,x)\,dx\Bigg]\\ &\quad-\dot{u}\int \frac{\sin(\theta_n^\varepsilon(\xi,u,x))}{2}\partial_u\theta_n^\varepsilon(\xi,u,x)v^2(x)\,dx\\ &\quad+(\dot{\xi}-u)\int \frac{\cos(\theta_n^\varepsilon(\xi,u,x))}{2}v(x)\partial_x v(x)\,dx\end{aligned}$$

$$
\begin{aligned}
&+ \dot{u} \int w(x)\partial_x v(x)\, dx \\
&+ \int w(x)[\frac{\sin \theta_n^\varepsilon(\xi,u,x) v^2(x)}{2} + \tilde{R}(v)(x) + [\mathcal{R}_n^\varepsilon(\xi,u,x)]_2] \\
&+ u\partial_x v(x)[\frac{\sin \theta_n^\varepsilon(\xi,u,x) v^2(x)}{2} + \tilde{R}(v)(x) + [\mathcal{R}_n^\varepsilon(\xi,u,x)]_2] \\
&+ \partial_x v(x)\partial_x[\mathcal{R}_n^\varepsilon(\xi,u,x)]_1 + \cos(\theta_n^\varepsilon(\xi,u,x))v(x)[\mathcal{R}_n^\varepsilon(\xi,u,x)]_1 + uw(x)\partial_x[\mathcal{R}_n^\varepsilon(\xi,u,x)]_1\, dx\, dt \\
&+ C\varepsilon^{2n+1+k-\delta} \\
\le{}& L(0) + C\varepsilon^{2n}\,.
\end{aligned}
$$

□

**Theorem 26.4.** *Let $n \ge 1$, $k+1 \le n$. Assume that the assumptions (a),(b) of Theorem 26.1 are satisfied. Let $\Xi$ be from Definition 20.65 and $U$ from Lemma 21.7. There exists $\bar{\varepsilon}_0 > 0$ such that the following statement holds. Let $\varepsilon \in (0, \bar{\varepsilon}_0)$. Assume that* (26.2)-(26.4) *has a solution $(\theta, \psi)$ on $\left[0, \overline{T}\right]$ such that*

$$(\theta, \psi) \in C^1([0,\overline{T}], L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R}))\,.$$

*Suppose that*

$$0 \le T \le \overline{T}\,.$$

*and that the assumptions (d),(e),(f) of Theorem 26.1 are satisfied. Then, provided*

$$0 \le T \le \frac{1}{\varepsilon^{\beta(k,\delta)}}, \quad \beta(k,\delta) = \frac{k+1-\delta}{2}\,,$$

*it holds that $(v, w)$ given by* (22.2)-(22.3) *is well defined for times* $[0,T]$ *and there exists a constant $\hat{c}$ such that*

*(a)* $|v|^2_{L^\infty([0,T],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,T],L^2(\mathbb{R}))} \le \hat{c}\varepsilon^{2n}\,,$

*(b)* $\forall t \in [0,T]\ (\xi(t), u(t)) \in \Sigma(5, U, \Xi)\,.$

**Proof.** Choose $\bar{\varepsilon}_0$ such that the following holds:

(1) $\bar{\varepsilon}_0$ satisfies all smallness assumtions of Lemma 26.2;

(2) $\bar{\varepsilon}_0$ is so small that

$$\forall \varepsilon \in (0, \bar{\varepsilon}_0):\ \frac{2}{c}(L(0) + C\varepsilon^{2n}) < \varepsilon^{2n-\delta}\,,$$

where $L(0) = \mathcal{E}(v(0), w(0), \xi_s, u_s)$ and the constants $c, C$ are from Lemma 26.2 (b);

Let $\varepsilon \in (0, \bar{\varepsilon}_0)$. Notice that $\Sigma(5, U, \Xi) \subset \Sigma(4, U, \Xi)$. We define an exit time

$$t_* := \sup \Big\{ T > 0 : |v|^2_{L^\infty([0,t],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,t],L^2(\mathbb{R}))} \leq \frac{2}{c}(L(0) + C\varepsilon^{2n}),$$

$$(\xi(t), u(t)) \in \Sigma(5, U, \Xi),\ 0 \leq t \leq T \Big\} .$$

Suppose $t_* < \frac{1}{\varepsilon^{\beta(k,\delta)}}$. Then there exists a time $\hat{t}$ such that

$$\frac{1}{\varepsilon^{\beta(k,\delta)}} > \hat{t} > t_* \,,$$

with

$$\forall t \in [0, \hat{t}] : \ (\xi(t), u(t)) \in \Sigma(4, U, \Xi), \qquad (\xi(\hat{t}), u(\hat{t})) \notin \Sigma(5, U, \Xi)$$

or

$$\frac{1}{c}(L(0) + C\varepsilon^{2n}) < \frac{2}{c}(L(0) + C\varepsilon^{2n}) < |v|^2_{L^\infty([0,\hat{t}],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,\hat{t}],L^2(\mathbb{R}))} < \varepsilon^{2n-\delta} \,.$$

This leads a contradiction to the previous lemma. Thus

$$|v|^2_{L^\infty([0,T],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,T],L^2(\mathbb{R}))} \leq \frac{2}{c}(L(0) + C\varepsilon^{2n}) \leq \hat{c}\varepsilon^{2n}$$

and

$$\forall t \in [0, T] \ (\xi(t), u(t)) \in \Sigma(5, U, \Xi) \,.$$

□

The previous theorem implies that the local solution of (26.2)-(26.4) discussed in Chapter 22 is indeed continuable up to times $\frac{1}{\varepsilon^{\beta(k,\delta)}}$ for $\varepsilon \in (0, \bar{\varepsilon}_0)$. Theorem 26.4 and Lemma 23.1 yield the approximate equations for the parameters $(\xi, u)$. This verifies the claim of Theorem 26.1.

## 26.2 ODE Analysis

In this section we lay the groundwork for passing from the approximate equations for the parameters $(\xi, u)$ in Theorem 26.1 to the ODE's in (19.5)-(19.6). We start with a preparing lemma.

**Lemma 26.5.** *Let* $n \geq 1$, $k + 1 \leq n$. *There exists* $\varepsilon_0 > 0$ *such that the following statement holds. Let* $\varepsilon \in (0, \varepsilon_0)$. *Let* $\beta(k, \delta) = \frac{k+1-\delta}{2}$. *Let* $\tilde{\xi} = \tilde{\xi}(s)$, $\tilde{u} = \tilde{u}(s)$, $\epsilon_1 = \epsilon_1(s)$, $\epsilon_2 = \epsilon_2(s)$ *be* $C^1$ *real-valued functions. Suppose that*

$$|\epsilon_j(s)| \leq \bar{c}\varepsilon^{n+\delta}$$

*on* $[0,T]$ *for* $j=1,2$. *Assume that on* $[0,T]$,

$$\frac{d}{ds}\tilde{\xi}(s)=\tilde{u}(s)+\epsilon_1(s)\,,\quad \tilde{\xi}(0)=\tilde{\xi}_0\,,$$
$$\frac{d}{ds}\tilde{u}(s)=\frac{1}{\varepsilon^{2\beta(k,\delta)}}\lambda^{\varepsilon}_{u,n}(\tilde{\xi}(s),\varepsilon^{\beta(k,\delta)}\tilde{u}(s))+\epsilon_2(s)\,,\quad \tilde{u}(0)=\tilde{u}_0\,.$$

*Let* $\hat{\xi}=\hat{\xi}(s)$ *and* $\hat{u}=\hat{u}(s)$ *be* $C^1$ *real-valued functions which satisfy the exact equations*

$$\frac{d}{ds}\hat{\xi}(s)=\hat{u}(s)\,,\quad \hat{\xi}(0)=\tilde{\xi}_0\,,$$
$$\frac{d}{ds}\hat{u}(s)=\frac{1}{\varepsilon^{2\beta(k,\delta)}}\lambda^{\varepsilon}_{u,n}(\hat{\xi}(s),\varepsilon^{\beta(k,\delta)}\hat{u}(s))\,,\quad \hat{u}(0)=\tilde{u}_0\,.$$

*Then provided* $T\leq 1$, *there exists a* $c>0$ *such that the estimates*

$$|\tilde{\xi}(s)-\hat{\xi}(s)|\leq c\varepsilon^{n+\delta},\qquad |\tilde{u}(s)-\hat{u}(s)|\leq c\varepsilon^{n+\delta}\,,$$

*hold on* $[0,T]$.

**Proof.** In the following proof we follow very closely [HZ08, Lemma 6.1]. We choose $\varepsilon_0$ so small that the convergence rates in Lemma 22.5 are satisfied for all $\varepsilon\in(0,\varepsilon_0)$. Let $\varepsilon\in(0,\varepsilon_0)$. Let $x=x(s)$ and $y=y(s)$ be $C^1$ real-valued functions, $C\geq 1$, and let $(x,y)$ satisfy the differential inequalities:

$$\begin{cases}|\dot{x}|\leq|y| \\ |\dot{y}|\leq C|x|+C|y|\end{cases},\qquad \begin{matrix}x(0)=x_0\\ y(0)=y_0\end{matrix}\,.$$

We are going to apply the Gronwall lemma. Let $z(s)=x^2+y^2$. Then

$$|\dot{z}|=|2x\dot{x}+2y\dot{y}|\leq 2|x||y|+2C|x||y|+2C|y||y|\leq 4C(x^2+y^2)=4Cz\,,$$

and hence $z(s)\leq z(0)e^{4Cs}$. Thus

$$\begin{aligned}|x(s)|&\leq\sqrt{2}\max(|x_0|,|y_0|)\exp(2Cs)\,,\\ |y(s)|&\leq\sqrt{2}\max(|x_0|,|y_0|)\exp(2Cs)\,.\end{aligned}\tag{26.7}$$

Now we recall the Duhamel's formula. Let $X(s):\mathbb{R}\to\mathbb{R}^2$ be a two-vector function, $X_0\in\mathbb{R}^2$ a two-vector, and $A(s):\mathbb{R}\to(2\times 2\text{ matrices})$ a $2\times 2$ matrix function. We consider the ODE system

$$\dot{X}(s)=A(s)X(s),\qquad X(s')=X_0$$

and denote its solution by $X(s)=S(s,s')X_0$ such that

$$\frac{d}{ds}S(s,s')X_0=A(s)S(s,s')X_0\,,\quad S(s',s')X_0=X_0\,.$$

Let $F(s) : \mathbb{R} \to \mathbb{R}^2$ be a two-vector function. We can express the solution to the inhomogeneous ODE system

$$\dot{X}(s) = A(s)X(s) + F(s)$$

with initial condition $X(0) = 0$ by the Duhamel's formula

$$X(s) = \int_0^s S(s, s')F(s')ds' .$$

Let $U = \hat{u} - \tilde{u}$ and $\Xi = \hat{\xi} - \tilde{\xi}$. These functions satisfy

$$\begin{aligned}
&\frac{d}{ds}\Xi(s) = U(s) + \epsilon_1(s) , \\
&\frac{d}{ds}U(s) = \frac{1}{\varepsilon^{2\beta(k,\delta)}}\Big[\lambda^{\varepsilon}_{u,n}(\hat{\xi}(s), \varepsilon^{\beta(k,\delta)}\hat{u}(s)) - \lambda^{\varepsilon}_{u,n}(\tilde{\xi}(s), \varepsilon^{\beta(k,\delta)}\tilde{u}(s))\Big] + \epsilon_2(s) .
\end{aligned}$$

Notice that

$$\begin{aligned}
&\frac{1}{\varepsilon^{2\beta(k,\delta)}}\lambda^{\varepsilon}_{u,n}(\hat{\xi}(s), \varepsilon^{\beta(k,\delta)}\hat{u}(s)) - \frac{1}{\varepsilon^{2\beta(k,\delta)}}\lambda^{\varepsilon}_{u,n}(\tilde{\xi}(s), \varepsilon^{\beta(k,\delta)}\tilde{u}(s)) \\
&= \frac{1}{\varepsilon^{2\beta(k,\delta)}}\frac{\lambda^{\varepsilon}_{u,n}(\hat{\xi}(s), \varepsilon^{\beta(k,\delta)}\hat{u}(s)) - \lambda^{\varepsilon}_{u,n}(\tilde{\xi}(s), \varepsilon^{\beta(k,\delta)}\hat{u}(s))}{\hat{\xi}(s) - \tilde{\xi}(s)}\Big[\hat{\xi}(s) - \tilde{\xi}(s)\Big] \\
&\quad + \frac{1}{\varepsilon^{2\beta(k,\delta)}}\frac{\lambda^{\varepsilon}_{u,n}(\tilde{\xi}(s), \varepsilon^{\beta(k,\delta)}\hat{u}(s)) - \lambda^{\varepsilon}_{u,n}(\tilde{\xi}(s), \varepsilon^{\beta(k,\delta)}\tilde{u}(s))}{\tilde{u}(s) - \hat{u}(s)}\Big[\tilde{u}(s) - \hat{u}(s)\Big] .
\end{aligned}$$

Let

$$g(s) = \begin{cases} \dfrac{1}{\varepsilon^{2\beta(k,\delta)}}\dfrac{\lambda^{\varepsilon}_{u,n}(\hat{\xi}(s), \varepsilon^{\beta(k,\delta)}\hat{u}(s)) - \lambda^{\varepsilon}_{u,n}(\tilde{\xi}(s), \varepsilon^{\beta(k,\delta)}\hat{u}(s))}{\hat{\xi}(s) - \tilde{\xi}(s)} & \text{if } \hat{\xi}(s) \neq \tilde{\xi}(s) \\ \dfrac{1}{\varepsilon^{2\beta(k,\delta)}}\partial_1\lambda^{\varepsilon}_{u,n}(\tilde{\xi}(s), \varepsilon^{\beta(k,\delta)}\hat{u}(s)) & \text{if } \hat{\xi}(s) = \tilde{\xi}(s) \end{cases} ,$$

$$h(s) = \begin{cases} \dfrac{1}{\varepsilon^{2\beta(k,\delta)}}\dfrac{\lambda^{\varepsilon}_{u,n}(\tilde{\xi}(s), \varepsilon^{\beta(k,\delta)}\hat{u}(s)) - \lambda^{\varepsilon}_{u,n}(\tilde{\xi}(s), \varepsilon^{\beta(k,\delta)}\tilde{u}(s))}{\tilde{u}(s) - \hat{u}(s)} & \text{if } \hat{u}(s) \neq \tilde{u}(s) \\ \dfrac{1}{\varepsilon^{\beta(k,\delta)}}\partial_2\lambda^{\varepsilon}_{u,n}(\tilde{\xi}(s), \varepsilon^{\beta(k,\delta)}\hat{u}(s)) & \text{if } \tilde{u}(s) = \tilde{u}(s) \end{cases} .$$

We set

$$A(s) = \begin{bmatrix} 0 & 1 \\ g(s) & h(s) \end{bmatrix}, \quad F(s) = \begin{bmatrix} \epsilon_1(s) \\ \epsilon_2(s) \end{bmatrix}, \quad X(s) = \begin{bmatrix} \Xi(s) \\ U(s) \end{bmatrix}$$

and obtain by Duhamel's formula:

$$\begin{bmatrix}\Xi(s)\\ U(s)\end{bmatrix} = \int_0^s S(s,s')\begin{bmatrix}\epsilon_1(s')\\ \epsilon_2(s')\end{bmatrix}\,ds'\,. \tag{26.8}$$

We use Lemma 22.5 and apply (26.7) with

$$\begin{bmatrix}x(s)\\ y(s)\end{bmatrix} = S(s+s',s')\begin{bmatrix}\epsilon_1(s')\\ \epsilon_2(s')\end{bmatrix}, \quad \begin{bmatrix}x_0\\ y_0\end{bmatrix} = \begin{bmatrix}\epsilon_1(s')\\ \epsilon_2(s')\end{bmatrix}.$$

It follows that

$$\left|S(s,s')\begin{bmatrix}\epsilon_1(s')\\ \epsilon_2(s')\end{bmatrix}\right| \le \sqrt{2}\begin{bmatrix}\exp(2C(s-s'))\\ \exp(2C(s-s'))\end{bmatrix}\max(|\epsilon_1(s')|,|\epsilon_2(s')|)\,.$$

Using (26.8) we obtain that on $[0,T]$

$$|\Xi(s)| \le \sqrt{2}\,T\mathrm{exp}(2CT)\sup_{0\le s\le T}\max(|\epsilon_1(s)|,|\epsilon_2(s)|)\,,$$

$$|U(s)| \le \sqrt{2}\,T\exp(2CT)\sup_{0\le s\le T}\max(|\epsilon_1(s)|,|\epsilon_2(s)|)\,,$$

which yields the claim. □

In the following we show the relation between the parameters $(\xi,u)$ selected by the implicit function theorem according to Lemma 21.7 and the solutions $(\hat{\xi},\hat{u})$ of the exact ODE's from the previous lemma.

**Lemma 26.6.** *Let $n\ge 1$, $k+1\le n$. There exists $\hat{\varepsilon}_0>0$ such that the following statement holds. Let $\varepsilon\in(0,\hat{\varepsilon}_0)$, $\beta(k,\delta)=\frac{k+1-\delta}{2}$ and*

$$s=\varepsilon^{\beta(k,\delta)}t,$$

*where*

$$0\le s\le 1,\quad 0\le t\le \frac{1}{\varepsilon^{\beta(k,\delta)}}\,.$$

*Let $(\xi,u)$ be the parameters selected according to Lemma 21.7 and $(\hat{\xi},\hat{u})$ from Lemma 26.5. Then it holds that*

$$|\xi(t)-\hat{\xi}(\varepsilon^{\beta(k,\delta)}t)| \le \varepsilon^{n+\delta}\,,$$

$$|u(t)-\varepsilon^{\beta(k,\delta)}\hat{u}(\varepsilon^{\beta(k,\delta)}t)| \le \varepsilon^{n+\delta+\beta(k,\delta)}\,.$$

**Proof.** We choose $\hat{\varepsilon}_0$ as the minimum of $\bar{\varepsilon}_0$ from Theorem 26.4 and of the $\varepsilon_0$'s from Lemma 23.1 and Lemma 26.5. Let $\varepsilon \in (0, \hat{\varepsilon}_0)$.

$$\tilde{\xi}(s) = \xi(s/\varepsilon^{\beta(k,\delta)}), \quad \tilde{u}(s) = \frac{1}{\varepsilon^{\beta(k,\delta)}} u(s/\varepsilon^{\beta(k,\delta)})\,.$$

For times

$$0 \leq t \leq \frac{1}{\varepsilon^{\beta(k,\delta)}}$$

Lemma 23.1 and Theorem 26.4 yield:

$$\begin{aligned} |\dot{\xi}(t) - u(t)| &\leq C[|v|_{H^1(\mathbb{R})} + |w|_{L^2(\mathbb{R})}]\varepsilon^{k+1} + C\,|v|^2_{H^1(\mathbb{R})} + C\varepsilon^{n+k+1}, \\ &\leq C\varepsilon^{n+k+1} + C\varepsilon^{2n} + C\varepsilon^{n+k+1} \\ &\leq C\varepsilon^{n+k+1}\,, \end{aligned}$$

$$\begin{aligned} |\dot{u}(t) - \lambda^{\varepsilon}_{u,n}(\xi(t), u(t))| &\leq C[|v|_{H^1(\mathbb{R})} + |w|_{L^2(\mathbb{R})}]\varepsilon^{k+1} + C\,|v|^2_{H^1(\mathbb{R})} + C\varepsilon^{n+k+1} \\ &\leq C\varepsilon^{n+k+1} + C\varepsilon^{2n} + C\varepsilon^{n+k+1} \\ &\leq C\varepsilon^{n+k+1}\,. \end{aligned}$$

Thus $(\tilde{\xi}, \tilde{u})$ satisfy the assumptions of Lemma 26.5, since

$$\begin{aligned} &\frac{d}{ds}\tilde{\xi}(s) \\ =&\frac{1}{\varepsilon^{\beta(k,\delta)}}\xi'\left(\frac{s}{\varepsilon^{\beta(k,\delta)}}\right) \\ =&\frac{1}{\varepsilon^{\beta(k,\delta)}}u\left(\frac{s}{\varepsilon^{\beta(k,\delta)}}\right) + \frac{1}{\varepsilon^{\beta(k,\delta)}}\mathcal{O}(\varepsilon^{n+k+1}) \\ =&\tilde{u}(s) + \mathcal{O}(\varepsilon^{n+\frac{k+1+\delta}{2}})\,, \end{aligned}$$

$$\begin{aligned} &\frac{d}{ds}\tilde{u}(s) \\ =&\frac{1}{\varepsilon^{2\beta(k,\delta)}}u'\left(\frac{s}{\varepsilon^{\beta(k,\delta)}}\right) \\ =&\frac{1}{\varepsilon^{2\beta(k,\delta)}}\lambda^{\varepsilon}_{u,n}\left(\xi\left(\frac{s}{\varepsilon^{\beta(k,\delta)}}\right), u\left(\frac{s}{\varepsilon^{\beta(k,\delta)}}\right)\right) + \frac{1}{\varepsilon^{2\beta(k,\delta)}}\mathcal{O}(\varepsilon^{n+k+1}) \\ =&\frac{1}{\varepsilon^{2\beta(k,\delta)}}\lambda^{\varepsilon}_{u,n}\left(\tilde{\xi}\,(s)\,, \varepsilon^{\beta(k,\delta)}\tilde{u}\,(s)\right) + \mathcal{O}(\varepsilon^{n+\delta})\,. \end{aligned}$$

Hence Lemma 26.5 yields:

$$
\begin{aligned}
&|\tilde{\xi}(s) - \hat{\xi}(s)| = |\xi(t) - \hat{\xi}(\varepsilon^{\beta(k,\delta)}t)| \leq c\varepsilon^{n+\delta} \\
\Rightarrow\ &|\xi(t) - \hat{\xi}(\varepsilon^{\beta(k,\delta)}t)| \leq c\varepsilon^{n+\delta}\,,
\end{aligned}
$$

$$
\begin{aligned}
&|\tilde{u}(s) - \hat{u}(s)| = |\frac{u(t)}{\varepsilon^{\beta(k,\delta)}} - \hat{u}(\varepsilon^{\beta(k,\delta)}t)| \leq c\varepsilon^{n+\delta}\,, \\
\Rightarrow\ &|u(t) - \varepsilon^{\beta(k,\delta)}\hat{u}(\varepsilon^{\beta(k,\delta)}t)| \leq c\varepsilon^{n+\delta+\beta(k,\delta)}\,.
\end{aligned}
$$

□

## 26.3 Completion of the Proof of Theorem 19.3

Theorem 26.1 yields the dynamics with the parameters $(\xi, u)$ selected by the implicit function theorem according to Lemma 21.7 on the time interval

$$0 \leq t \leq \frac{1}{\varepsilon^{\beta(k,\delta)}}\,.$$

Using Lemma 26.6 and the the triangle inequality we can replace

$$(\xi(t), u(t))$$

with

$$(\hat{\xi}(\varepsilon^{\beta(k,\delta)}t), \varepsilon^{\beta(k,\delta)}\hat{u}(\varepsilon^{\beta(k,\delta)}t))\,.$$

We set

$$(\bar{\xi}(t), \bar{u}(t)) = (\hat{\xi}(\varepsilon^{\beta(k,\delta)}t), \varepsilon^{\beta(k,\delta)}\hat{u}(\varepsilon^{\beta(k,\delta)}t))$$

and conclude that the equations claimed are satisfied. □

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