Real contact geometry

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Kurzzusammenfassung

Die vorliegende Arbeit ist dem Studium von *reellen* Kontaktmannigfaltigkeiten gewidmet. Dies sind Kontaktmannigfaltigkeiten (M, ξ), die mit einer zusätzlichen Involution f versehen sind, von der man die Antisymmetrie-Bedingung $f_*\xi = -\xi$ verlangt. Reelle Kontaktmannigfaltigkeiten tauchen unter anderem in der Hamiltonschen Beschreibung mechanischer Systeme als Hyperebenen in Phasenräumen auf, beispielweise im Drei-Körper-Problem. Diese Arbeit bietet zunächst eine Einfürung sowohl in die Theorie der Kontakmannigfaltigkeiten, als auch die der Involutionen. Im ersten Kapitel werden grundsätzliche Eigenschaften untersucht anhand einiger Beispiele. Dabei stellen wir fest, dass reelle Kontaktmannigfaltigkeiten eine Verbindung zwischen zwei großen offenen Problemen der Kontaktgeometrie herstellen, nämlich der Weinstein- und der Arnold-Vermutung. Das zweite Kapitel beinhaltet eine Sammlung von Struktursätzen für reelle Kontaktmannigfaltigkeiten wie beispielsweise die Gray-Stabilität, den Satz von Darboux, Umgebungssätze für Untermannigfaltigkeiten und eine Klassifikationsaussage für reelle Strukturen zur Standard-Kontaktform im euklidischen Raum. Im dritten Kapitel untersuchen wir dann zwei Methoden zur Konstruktion von reellen Kontaktmannigfaltigkeiten, namentlich reelle offene Bücher und reelle Chirurgien. Letztere werden im Rahmen von reellen symplektischen Kobordismen eingeführt.

Abstract

The present thesis is devoted to the study of *real* contact manifolds. These are contact manifolds that carry an additional involution f, of which one requires the anti-symmetry condition $f_*\xi = -\xi$. Amongst others, real contact manifolds appear in Hamilton's description of mechanical systems as hyperplanes in phase spaces, for example in the three body problem. This text offers an introduction both to the theories of contact manifolds and involutions. In the first chapter, fundamental properties are studied on the basis of various examples. We observe that real contact manifolds establish a connection between two outstanding problems in contact geometry, specifically the Weinstein conjecture and Arnold's chord conjecture. The second chapter contains a collection of structure theorems for real contact manifolds, including Gray stability, Darboux's theorem, neighbourhood theorems for submanifolds, and a classification result for real structures compatible with the standard contact form in euclidean space. In the third chapter, we investigate two methods for constructing real contact manifolds, namely real open books and real surgery. The latter are introduced in the framework of real symplectic cobordisms.

Contents

Introduction

The motions of celestial bodies fascinate mankind since ancient times. Early contemplations date back as far as Plato's days, but it was not until the early 17th century when Kepler, Newton and others placed celestial mechanics on the solid ground of physical laws. In spirit of Hamilton's work, these mechanical systems are nowadays embedded into symplectic geometry: The corresponding phase space's model is a symplectic manifold, while the dynamics is encoded in a smooth function. Contact manifolds enter the stage as hypersurfaces in these symplectic manifolds, and studying the symplectic dynamics translates into a problem entirely phrased in terms of contact manifolds – this is the birthplace of contact geometry. But, as 1977's Nobel laureate P. W. Anderson remarks: 'It is only slightly overstating the case to say that physics is the study of symmetry.'[1] Thus, obeying his credo, we ought to find some kind of symmetry in symplectic manifolds representing mechanical systems. What is a symmetry? In a rather broad sense, an isometry on a Riemannian manifold may be regarded as a symmetry. And this applies to a particularly handy class of symmetries, namely involutions. These are diffeomorphisms that square to the identity – rotations by the angle π, reflections

[1] in: 'More Is Different', Science, 1972.

on a plane, and the like. For the scope of this text, we will settle for these symmetries, and define a *real* contact manifold to be a contact manifold equipped with an involution. Requiring that the involution pulls the contact form back to *minus* itself leads us into the realm of anti-equivariant contact manifolds: These are the involutions we will attend to. The insight that various mechanical systems bring along anti-equivariant involutions justifies our choice.

Realising hypersurfaces as contact manifolds can be an intricate task. Albers et al. showed in [AFvKP12] that certain energy levels in the restricted 3-body problem carry induced contact structures – and all of them are real, as we shall see later. This fuelled the interest in real contact manifolds greatly. Recent results in that area include the works of Öztürk and Salepci [ÖS11] and [ÖS15]. These contain a classification of tight contact structures on the 3-ball as well as an approach towards a real version of Giroux's correspondence in the context of open books.

In the theory of involutions, one studies involutions by investigating their fixed point sets; a typical problem is the classification of involutions on a given manifold up to conjugacy. The 'modern era' of this field was ushered in by Smith's work on transformations of finite period. In [Smi38], he showed that fixed point sets of non-trivial orientation-preserving smooth involutions on the 3-sphere S^3 consist of a single circle respectively, and raised the question whether this circle is necessarily unknotted. An affirmative answer was finally given by Waldhausen in [Wal69]. In its full generality, the classification problem for involutions remains unsolved, even in the 3-dimensional case. Aside from the topology of fixed point sets, a major challenge is the existence of involutions. Here, a number of obstructions are known, see for example [Pup07].

Nowadays, contact geometers name Lie as the founding father of their subject, who introduced contact manifolds in the guise of *Berührungstransformationen* (contact transformations). As for involutions, predominant problems are the existence and classification of contact structures. A significant progress in the 3-dimensional case was made by Eliashberg in [Eli89], and, more recently and for arbitrary dimensions, by [BEM15].

The aim of this thesis is to supply a multitude of methods for producing real contact manifolds and to establish structure theorems similar to the ones found in classical contact topology. The text is divided into three parts: The first chapter introduces the reader to contact manifolds and real structures, and, besides several examples, the topology of an involution's fixed point set is discussed. The second chapter is a collection of said structure theorems for real contact manifolds. We prove real versions of Gray's and Darboux's theorems as well as neighbourhood theorems for submanifolds. This chapter also features a classification of real structures for the standard contact structure on $\mathbb{R}^{2n+1}$: All involutions that pull the standard contact form back to minus itself are isotopic. The third chapter presents two methods of producing new real contact manifolds out of old ones, via real open books and real surgery. The latter is related to the study of real symplectic fillings; these are highlighted in connection with real contact structures on Brieskorn manifolds.

In the study of real contact manifolds, there are two possible points of view: On the one hand, one may consider a given involution and investigate contact forms that are compatible with that involution. On the other hand, one may look for compatible involutions, once a contact form is fixed. In this text, we will usually find ourselves in the first situation. A notable exception is the classification theorem mentioned above. Also, throughout the text, we will remark on similarities and differences between the anti-equivariant and the equivariant theory of involutions on contact manifolds.

The figures in the text were created with Ti*k*Z.

1 First steps in the real world

This chapter serves as an introduction both to the study of involutions as well as the theory of contact manifolds. Its first section combines these concepts, and various examples are accompanied by elementary observations about real contact manifolds. In classical mechanics, contact manifolds appear as hypersurfaces in phase spaces, interpreted as symplectic manifolds. We will extend this discussion to real symplectic manifolds in Section 1.2. This then relates two outstanding open problems in symplectic topology, namely Arnold's chord conjecture and the Weinstein conjecture. Following that, we investigate fixed point sets of involutions; the results obtained there mark a notable difference to the theory of equivariant involutions on contact manifolds. The last section exhibits examples of contact manifolds not admitting an involution, and vice versa.

Usually, we require the objects of our considerations – manifolds, functions, vector fields, differntial forms, ... – to be smooth. *So from now on, and unless otherwise stated, all these objects are assumed to be smooth whenever this term applies.*

1.1 Real contact manifolds

The present section offers a first look at real contact manifolds, introducing the basic notions used throughout this text. A collection of illustrating examples will be given, as well as a quick review of fundamental concepts in contact topology. We begin with the discussion of certain involutions on manifolds.

Definition 1.1.1. Let M be an oriented manifold of dimension d.

- An **involution** on M is a diffeomorphism f of M of order 2, i.e. we require $f \circ f = \mathrm{id}_M$.
- Now suppose $d = 2n + 1$ is odd. An involution f on M is called a **real structure** on M if its fixed point set
  $$\operatorname{Fix} f = \{p \in M \mid f(p) = p\}$$
  is an n-dimensional submanifold of M (unless it is empty) and if, additionally,
  $$f \text{ is } \begin{cases} \text{orientation-preserving} & \text{for } n \text{ odd and} \\ \text{orientation-reversing} & \text{for } n \text{ even.} \end{cases}$$
  The pair (M, f) is then called a **real manifold**.
- Similarly, one defines a real structure on a $(2n + 2)$-dimensional manifold as an involution with fixed point set either empty or an $(n + 1)$-dimensional submanifold which is orientation-preserving precisely when n is odd.

Example 1.1.2. (1) The prototypical example in even dimensions is given by the complex conjugation,

$$\begin{array}{rccc} f: & \mathbb{C} & \longrightarrow & \mathbb{C} \\ & (x,y) & \longmapsto & (x,-y), \end{array}$$

to which the theory of real structures owes its name. Here, the fixed points are precisely the real numbers. In this spirit, the fixed point set of a real manifold is sometimes referred to as its *real part*.

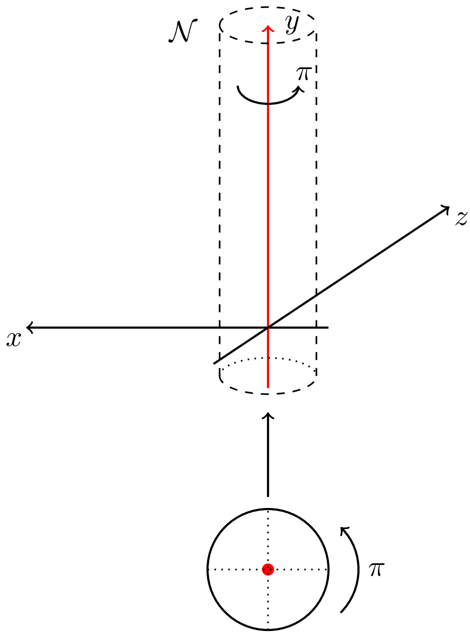

Figure 1.1: A real structure in $\mathbb{R}^3$. The fixed point set is depicted in red.

(2) Rotation about the y-axis by the angle π, i.e. the mapping

$$f_{\mathrm{st}}\colon \begin{array}{ccc} \mathbb{R}^3 & \longmapsto & \mathbb{R}^3 \\ (x,y,z) & \longmapsto & (-x,y,-z), \end{array}$$

defines a real structure on the manifold $\mathbb{R}^3$. In a tubular neighbourhood $\mathcal{N}$ of its fixed point set

$$\left\{(x,y,z) \in \mathbb{R}^3 \mid x = z = 0\right\},$$

the involution is given as the antipodal map on the fibres D^2. This is true for all involutions – a suitable modification concerning dimensions understood –, and a primary task of this text will be to detect trivial neighbourhoods of fixed point sets in the presence of additional geometric structures.

More generally, the linear involution

$$\begin{aligned}(x_1,\ldots,x_n,x_{n+1},\ldots,x_{2n+1})& \\ \mapsto (x_1,\ldots,x_n,&-x_{n+1},\ldots,-x_{2n+1})\end{aligned}$$

turns $\mathbb{R}^{2n+1}$ into a real manifold.

(3) On $T^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$, the involution

$$(\theta_1,\theta_2,\theta_3) \mapsto (\theta_1+\pi,\theta_2+\pi,\theta_3)$$

defines a real structure with empty fixed point set.

(4) Let (W^{2n+2}, f) be an even-dimensional real manifold. Suppose that M^{2n+1} is a submanifold of W invariant under f. Moreover, let Y be a vector field defined near and transverse to M such that $T_pf(Y_p) = Y_{f(p)}$. This expression is meant to hold for all p where the relevant terms are defined, and T_pf denotes the differential of f in p. Then

$(M, f|_M)$ is a real manifold: Under the given assumptions, the direction corresponding to Y is preserved by the involution f; consequently, f is orientation-preserving if and only if $f|_M$ is orientation-preserving. This is a common technique of producing odd-dimensional real manifolds out of even-dimensional ones; later on, we will use this method frequently.

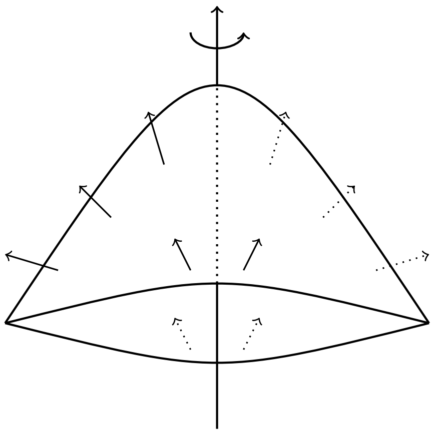

Figure 1.2: Invariant hypersurface.

Remark. In the presence of fixed points, the two conditions on the involution in Definition 1.1.1 – orientation behaviour and dimension of the fixed point set – are not independent. We shall discuss that matter in Section 1.3.

In the study of real manifolds, we will be particularly interested in objects compatible with the given involution.

Definition 1.1.3. Let (M, f) be a real manifold.

(1) A vector field or a differential form on M is **symmetric** if it maps to itself under the involution f. It is said to be **antisymmetric** if f sends it to minus itself.

(2) An oriented submanifold M' of M invariant under f is **symmetric** if f preserves the orientation of M'. In symbols: $f(M') = M'$. If f reverses its orientation, we will call it **antisymmetric** and write $f(M') = -M'$.

(3) Let (N, g) be real manifold. A map $\varphi\colon M \to N$ is said to be **symmetric** (or **equivariant**) if $\varphi \circ f = g \circ \varphi$.

When considering multiple involutions on a manifold, say f and g, we will speak of f-symmetric or g-symmetric objects to clarify the meaning.

Example 1.1.4. Let η be an arbitrary k-form on a real manifold (M, f). Define a new k-form by

$$\eta^+ = \frac{1}{2}\left(\eta + f^*\eta\right).$$

Then η^+ is symmetric. Putting

$$\eta^- = \frac{1}{2}\left(\eta - f^*\eta\right)$$

instead produces an antisymmetric k-form. Similar constructions apply to functions $M \to \mathbb{R}$ and vector fields on M.

Observation. Let M be a manifold, and denote by $\mathcal{I}(M)$ the set of real structures on M. An action of the diffeomorphism group of M on $\mathcal{I}(M)$ is given by conjugation. That is, for f an involution on M and φ any diffeomorphism, the composition $\varphi \circ f \circ \varphi^{-1}$ defines an involution on M. The stabiliser of an element $f \in \mathcal{I}(M)$ consists precisely of f-symmetric self-diffeomorphisms of M. Determining the orbits of this action for certain manifolds is a much-studied problem in the theory of involutions.

Now consider a $(2n+1)$-dimensional manifold M. A hyperplane distribution ξ on M is a codimension 1 subbundle of TM. Locally, such a distribution can always be written as the kernel of a 1-form α. We say that the pair (M, ξ) is a **contact manifold** if, for any 1-form α locally defining ξ, the contact condition

$$\alpha \wedge (d\alpha)^n \neq 0$$

is satisfied. There exists a globally defined 1-form α with $\ker \alpha = \xi$ if and only if ξ is coorientable, i.e. the quotient bundle TM/ξ is trivial. Such a 1-form is called a **contact form** for the **contact structure** ξ. A manifold admitting a coorientable contact structure is necessarily orientable.

All manifolds appearing in this text will be assumed to be orientable, and the term 'contact structure' always refers to a coorientable contact structure.

Notation. Consider a contact manifold (M, ξ) and a diffeomorphism f of M, and let Tf denote its differential. If ξ is invariant under Tf, we will write $Tf(\xi) = \xi$ if Tf preserves the coorientation of ξ and $Tf(\xi) = -\xi$ if Tf reverses its coorientation.

Definition 1.1.5. A **real contact manifold** is a triple (M, ξ, f) consisting of a contact structure ξ on a $(2n+1)$-dimensional manifold M and a real structure f on M such that

$$Tf(\xi) = -\xi.$$

Observation. Let (M^{2n+1}, ξ, f) be a real contact manifold.

(1) Pick any 1-form $\widetilde{\alpha}$ defining the contact structure ξ on M and put

$$\alpha = \frac{1}{2}\left(\widetilde{\alpha} - f^*\widetilde{\alpha}\right).$$

This 1-form satisfies $f^*\alpha = -\alpha$. Using that Tf maps ξ into itself, we conclude $f^*\widetilde{\alpha} = \widetilde{\lambda}\alpha$ for some function $\widetilde{\lambda}$ on M. Since f is coorientation-reversing, the function $\widetilde{\lambda}$ is everywhere negative, and thus $\alpha = \lambda\widetilde{\alpha}$, $\lambda = (1-\widetilde{\lambda})/2 > 0$, meaning that α is a contact form for ξ. Therefore, in a real contact manifold (M, ξ, f), we may always choose a contact form α for ξ with $f^*\alpha = -\alpha$. In that case, the triple (M, α, f) is called a **real contact manifold** as well.

On the other hand, if ξ is given as the kernel of a 1-form α and $g^*\alpha = -\alpha$ for any diffeomorphism g of M, then $Tg(\xi) = -\xi$, too.

(2) Suppose that $\xi = \ker\alpha$, and let g be any diffeomorphism of M such that $g^*\alpha = -\alpha$. Then

$$g^*\left(\alpha \wedge (d\alpha)^n\right) = (-1)^{n+1}\,\alpha \wedge (d\alpha)^n.$$

Therefore, g is orientation-preserving if and only if n is odd.

Example 1.1.6. (1) Let $M = \mathbb{R}^{2n+1}$ with coordinates $(x_1, y_1, \ldots, x_n, y_n, z)$. As a shorthand notation, we will use $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$. Consider the 1-form

$$\alpha_{\text{st}} = dz + \sum_{k=1}^{n} x_k\, dy_k = dz + \mathbf{x}\, d\mathbf{y}.$$

Then $\xi_{\mathrm{st}} := \ker \alpha_{\mathrm{st}}$ is a contact structure on $\mathbb{R}^{2n+1}$, and an involution f_{st} on $\mathbb{R}^{2n+1}$ satisfying $f_{\mathrm{st}}^*\alpha_{\mathrm{st}} = -\alpha_{\mathrm{st}}$ is given by $f_{\mathrm{st}}(\mathbf{x},\mathbf{y},z) = (-\mathbf{x},\mathbf{y},-z)$. This real contact manifold will be referred to as the **standard real contact manifold**, denoted succinctly by $\mathbb{R}^{2n+1}_{\mathrm{st}}$. Its fixed point set, given as $\operatorname{Fix} f_{\mathrm{st}} = \{\mathbf{x} = \mathbf{0}, z = 0\}$, is a diffeomorphic copy of $\mathbb{R}^n$.

(2) In cylindrical coordinates (r,φ,z) on $\mathbb{R}^3$, the involution f_{st} is given by

$$f_{\mathrm{st}}(r,\varphi,z) = (r, \pi - \varphi, -z).$$

So, incidentally, f_{st} is a real structure for the *overtwisted contact structure* ξ_{ot}, defined as the kernel of the 1-form $\cos r\, dz + r \sin r\, d\varphi$, as well.

(3) Again on $M = \mathbb{R}^{2n+1}$, put $\alpha = dz + \mathbf{x}\, d\mathbf{y} - \mathbf{y}\, d\mathbf{x}$ and $f(\mathbf{x},\mathbf{y},z) = (\mathbf{y},\mathbf{x},-z)$. Then $f^*\alpha = -\alpha$. As in the previous examples, the fixed point set $\{\mathbf{x} = \mathbf{y}, z = 0\}$ is diffeomorphic to $\mathbb{R}^n$.

(4) A contact structure on the unit sphere $M = S^{2n+1}$ in $\mathbb{R}^{2n+2}$ is given as the kernel of the 1-form

$$\alpha = \mathbf{x}\, d\mathbf{y} - \mathbf{y}\, d\mathbf{x},$$

where again $\mathbf{x} = (x_1,\ldots,x_{n+1})$ and $\mathbf{y} = (y_1,\ldots,y_{n+1})$. Define an involution f on S^{2n+1} by $f(\mathbf{x},\mathbf{y}) = (-\mathbf{x},\mathbf{y})$. Then (M,α,f) is a real contact manifold with fixed point set $\operatorname{Fix} f = \{\mathbf{x} = \mathbf{0}\} \cong S^n$.

(5) For an example of a real structure with empty fixed point set, consider the manifold $M = S^1 \times S^2 \subset S^1 \times \mathbb{R}^3$. Denote coordinates on M by $(\theta,(x,y,z))$, equip it with the

contact structure $\xi = \ker\alpha$, $\alpha = z\,d\theta + x\,dy - y\,dx$, and define an involution on M by setting

$$f(\theta, (x, y, z)) = (\theta + \pi, (-x, y, -z)).$$

Then $f^*\alpha = -\alpha$ and $\operatorname{Fix} f = \emptyset$.

If one uses the involution $f'(\theta, (x, y, z)) = (\theta, (-x, y, -z))$ on M instead, one obtains $\operatorname{Fix} f' = S^1 \times \{y = \pm 1\} \cong S^1 \sqcup S^1$, a fixed point set consisting of two components.

(6) Real contact structures on T^2-bundles over S^1 can be constructed as follows: Write $T^2 \times \mathbb{R} = (\mathbb{R}^2/\mathbb{Z}^2) \times \mathbb{R}$ with coordinates $(\mathbf{x}, t)$, and consider a matrix

$$A_k = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}, k \in \mathbb{Z}.$$

The quotient of $T^2 \times \mathbb{R}$ by the relation $(\mathbf{x}, t) \sim (A\mathbf{x}, t + 1)$, denoted by M_k, is a T^2-bundle over S^1. For any function $\varphi\colon \mathbb{R} \to \mathbb{R}$ with strictly positive derivative, the equation

$$\cos\varphi(t)\,dx - \sin\varphi(t)\,dy = 0$$

defines a contact structure ξ_φ on $\mathbb{R}^3$. According to [DG01], for any integer k, there exists a function $\varphi\colon \mathbb{R} \to \mathbb{R}$ such that ξ_φ descends to a contact structure on M_k. The involution

$$\begin{array}{rccc} f'\colon & \mathbb{R}^2 \times \mathbb{R} & \longrightarrow & \mathbb{R}^2 \times \mathbb{R} \\ & (\mathbf{x}, t) & \longmapsto & (-\mathbf{x}, t) \end{array}$$

is compatible with the equivalence relation defined above, and thus defines an involution f on the quotient M_k. From $f_*\xi_\varphi = -\xi_\varphi$ we conclude that (M_k, ξ_φ, f) is a real contact manifold.

Observation. Suppose that (M^{2n+1}, α, f) is a real contact manifold, and let X be a vector tangent to the submanifold $\operatorname{Fix} f$. Then

$$\alpha(X) = \alpha(Tf(X)) = (f^*\alpha)(X) = -\alpha(X),$$

and hence $\alpha(X) = 0$. A submanifold M' in a contact manifold M with $\alpha|_{TM'} = 0$ is called **isotropic**. It is **Legendre** if, additionally, $\dim M' = n$. In Proposition 1.3.1 we will show that this dimension condition is in fact a consequence of the condition $f^*\alpha = -\alpha$ alone. More precisely, if g is any involution on a contact manifold (M, α) with $g^*\alpha = -\alpha$, then the fixed point set of g is either empty or a Legendrian submanifold.

Next, we define a notion of equivalence for real contact manifolds.

Definition 1.1.7. Let (M_1, ξ_1, f_1) and (M_2, ξ_2, f_2) be two real contact manifolds. We will call them **isomorphic** (or **equivariantly contactomorphic**) if there exists a diffeomorphism $\varphi\colon M_1 \to M_2$ with

$$T\varphi(\xi_1) = \pm\xi_2 \quad \text{and} \quad \varphi \circ f_1 = f_2 \circ \varphi.$$

If we write $\xi_1 = \ker \alpha_1$ and $\xi_2 = \ker \alpha_2$, the first condition translates into the existence of a nowhere zero function λ on M_1 such that $\varphi^*\alpha_2 = \lambda\alpha_1$. Such a diffeomorphism φ is called an **isomorphism** between the two real contact manifolds.

If contact forms α_i with $\ker \alpha_i = \xi_i$, $i = 1, 2$, are chosen and the isomorphism φ satisfies $\varphi^*\alpha_2 = \alpha_1$, then φ is a **strict isomorphism**, and (M_1, α_1, f_1) and (M_2, α_2, f_2) are called **strictly isomorphic**.

Example 1.1.8. (1) The real contact manifolds in Example 1.1.6 (1) and (3) are strictly isomorphic. An isomorphism $\varphi\colon \mathbb{R}^{2n+1}_{\mathrm{st}} \to (\mathbb{R}^{2n+1}, \alpha, f)$ is given by

$$\varphi(\mathbf{x}, \mathbf{y}, z) = \left(\frac{\mathbf{x}+\mathbf{y}}{2}, \frac{\mathbf{y}-\mathbf{x}}{2}, z + \frac{\mathbf{x}\mathbf{y}}{2}\right).$$

(2) For $p = (0, \dots, 0, 1) \in S^{2n+1}$, the real contact manifolds $(S^{2n+1} \setminus \{p\}, \ker \alpha, f)$ and $\mathbb{R}^{2n+1}_{\mathrm{st}}$ in Example 1.1.6 (4) are isomorphic: For $\psi : S^{2n+1} \setminus \{p\} \to \mathbb{R}^{2n+1}$ the stereographic projection from p, one has $\psi \circ f = f_{\mathrm{st}} \circ \psi$. As in the classical proof in [Gei08, Proposition 2.1.8], put

$$\begin{array}{cccc} \varphi\colon & \mathbb{R}^{2n+1} & \to & \mathbb{R}^{2n+1} \\ & (\mathbf{r}, \boldsymbol{\theta}, w) & \mapsto & \left(\mathbf{r}, \boldsymbol{\theta} - w, \frac{1}{2}w\left(1 + \frac{1}{3}w^2 + \sum_j r_j^2\right)\right), \end{array}$$

where (r_j, θ_j) denote polar coordinates in the (x_j, y_j)-plane. This diffeomorphism satisfies $(\varphi \circ \psi)_* \ker \alpha = \ker \alpha_{\mathrm{st}}$. Furthermore, since $f_{\mathrm{st}}(\mathbf{r}, \boldsymbol{\theta}, w) = (\mathbf{r}, \pi - \boldsymbol{\theta}, -w)$, we compute $\varphi \circ f_{\mathrm{st}} = f_{\mathrm{st}} \circ \varphi$. Hence $\varphi \circ \psi$ is the desired isomorphism

$$\left(S^{2n+1} \setminus \{p\}, \ker \alpha, f\right) \to \mathbb{R}^{2n+1}_{\mathrm{st}}.$$

In fact, by the same method, $\left(S^{2n+1} \setminus \{p\}, \ker \alpha, f\right)$ and $\mathbb{R}^{2n+1}_{\mathrm{st}}$ are isomorphic for all $p \in \operatorname{Fix} f$.

Observation. If (M_1, ξ_1, f_1) and (M_2, ξ_2, f_2) are isomorphic (strictly or not), then the fixed point sets of f_1 and f_2 are diffeomorphic: Since $f_1 = \varphi^{-1} \circ f_2 \circ \varphi$, we have $f_2(p) = p$ if and only if $f_1(\varphi^{-1}(p)) = \varphi^{-1}(p)$, and therefore we conclude that $\operatorname{Fix}(f_2) \cong \varphi^{-1}(\operatorname{Fix} f_1)$.

Example 1.1.9. By this observation, the real structures on $S^1 \times S^2$ as in Example 1.1.6 (5) are not isomorphic. The same is true for the involutions

$$(t, x, y) \mapsto (t, -x, -y) \quad \text{and} \quad (t, x, y) \mapsto (t + \frac{\pi}{2}, -y, x)$$

on the 3-torus $T^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$ with contact structure given by $\ker(\sin t\, dx - \cos t\, dy)$.

Now we will take a closer look at $\mathbb{R}^{2n+1}_{\mathrm{st}}$, giving a preview of three fundamental theorems appearing in Chapter 2.

(1) Consider the family of contact structures $\xi_t = \ker \alpha_t$ on $\mathbb{R}^{2n+1}$ given by the 1-forms $\alpha_t = dz + \mathbf{x}\, d\mathbf{y} - t\mathbf{y}\, d\mathbf{x}$. For $t = 0$, we have $\alpha_0 = \alpha_{\mathrm{st}}$. The involution f_{st} turns $(\mathbb{R}^{2n+1}, \alpha_t)$ into a real contact manifold for all $t \in [0, 1]$. Define a vector field X_t on $\mathbb{R}^{2n+1}$ by

$$X_t = -\frac{\mathbf{y}}{1+t}\partial_{\mathbf{y}} + \frac{\mathbf{x}\mathbf{y}}{1+t}\partial_z.$$

We have $X_t \in \xi_t$ for all t, and we compute its flow as

$$\varphi_t(\mathbf{x}, \mathbf{y}, z) = \left(\mathbf{x}, \frac{1}{1+t}\mathbf{y}, \frac{t\mathbf{x}\mathbf{y}}{1+t} + z\right).$$

Then $T\varphi_t(\xi_0) = \xi_t$ and $\varphi_t \circ f_{\mathrm{st}} = f_{\mathrm{st}} \circ \varphi_t$ for all $t \in [0, 1]$, i.e. φ_t is symmetric. Said differently, the deformation of contact structures ξ_t embeds into an ambient isotopy φ_t. The existence of such a symmetric contact isotopy is known as *Gray stability* and holds for all closed real contact manifolds. We will give a proof in Theorem 2.1.6.

(2) Let $p = (\mathbf{x}_0, \mathbf{y}_0, z)$ be a point in $\mathbb{R}^{2n+1}$. A contactomorphism φ with $\varphi(p) = \mathbf{0}$ is given by

$$\varphi(\mathbf{x}, \mathbf{y}, z) = (\mathbf{x} - \mathbf{x}_0, \mathbf{y} - \mathbf{y}_0, z - z_0 + \mathbf{x}_0(\mathbf{y} - \mathbf{y}_0)).$$

If $p \in \operatorname{Fix} f_{\mathrm{st}}$, then φ is symmetric. In that case we have found a symmetric contactomorphism that identifies a neighbourhood of p with a neighbourhood of $\mathbf{0}$ in $\mathbb{R}^{2n+1}$. *Darboux's theorem*, proved in Theorem 2.3.1, states that this is true for all real contact manifolds, as long as we work around the fixed point set of the involution.

(3) The real structures f_{st} and $f(x, y, z) = (x, -y, -z)$ on $\mathbb{R}^3$ are isotopic as real structures for α_{st}: Put

$$\begin{aligned} f_t \colon \quad \mathbb{R}^3 \quad &\to \quad \mathbb{R}^3 \\ (x, y, z) \quad &\mapsto \quad \big(-x \cos t + y \sin t, x \sin t + y \cos t, -z \\ &\qquad + \tfrac{1}{2} x^2 \sin t \cos t - xy \sin^2 t - \tfrac{1}{2} y^2 \sin t \cos t \big) \,. \end{aligned}$$

Then $f_0 = f_{\mathrm{st}}$, $f_1 = f$, and for all $t \in [0, 1]$, we have that $f_t^* \alpha_{\mathrm{st}} = -\alpha_{\mathrm{st}}$ as well as $f_t^2 = \mathrm{id}_{\mathbb{R}^3}$. This is a version of the *contact disc theorem* which we deal with in Theorem 2.7.7.

Sometimes, one is interested in an explicit trivialisation of TM/ξ, that is, a vector field defining the coorientation of ξ. This leads to the notion of Reeb vector fields. These vector fields depend on a particular choice of a contact form for the contact structure ξ. We will discuss the real version of Reeb vector fields, after a notational convention.

Notation. Let f be a diffeomorphism on a manifold M. Given a vector field X on M, we define the vector field $f_* X$ as the pushforward of X by f, that is,

$$(f_* X)_p = T_{f^{-1}(p)} f \left(X_{f^{-1}(p)} \right)$$

for $p \in M$.

Let (M^{2n+1}, α) be a contact manifold. The contact condition $\alpha \wedge (d\alpha)^n \neq 0$ guarantees the existence and uniqueness of a vector field R_α, the so-called **Reeb vector field**, defined by the equations

$$i_{R_\alpha}\alpha \equiv 1 \quad \text{and} \quad i_{R_\alpha}d\alpha \equiv 0.$$

In the real case, Reeb vector fields are always antisymmetric:

Proposition 1.1.10. *Let (M^{2n+1}, α, f) be a real contact manifold. Then*

$$f_* R_\alpha = -R_\alpha.$$

Before we proceed to prove the proposition, we observe the following general fact:

Lemma 1.1.11. *Let f be a diffeomorphism on a manifold M. Then for any differential k-form η on M, we have*

$$f^*\left(i_X\eta\right) = i_{(f^{-1})_* X} f^*\eta \quad \text{and} \quad i_{f_* X}\eta = \left(f^{-1}\right)^*\left(i_X f^*\eta\right).$$

In particular, for an involution f, this reads

$$f^*\left(i_X\eta\right) = i_{f_* X} f^*\eta \quad \text{and} \quad i_{f_* X}\eta = f^*\left(i_X f^*\eta\right).$$

Proof. We compute

$$\begin{aligned} f^*\left(i_X\eta\right)(X_1, \ldots, X_{k-1}) &= \eta\left(X, f_* X_1, \ldots, f_* X_{k-1}\right) \\ &= \eta\left(f_*\left(f^{-1}\right)_* X, f_* X_1, \ldots, f_* X_{k-1}\right) \\ &= i_{(f^{-1})_* X} f^*\eta\left(X_1, \ldots, X_{k-1}\right). \end{aligned}$$

The second equation follows from the first one. $\square$

Proof of Proposition 1.1.10. We have

$$i_{-f_* R_\alpha} \alpha = -f^* i_{R_\alpha} f^* \alpha = f^* i_{R_\alpha} \alpha \equiv 1$$

and

$$i_{-f_* R_\alpha} d\alpha = -f^* i_{R_\alpha} f^* d\alpha = f^* i_{R_\alpha} d\alpha \equiv 0$$

by the preceding lemma. As above, the contact condition implies $-f_* R_\alpha = R_\alpha$. □

The next section features Reeb vector fields in the context of classical mechanics.

1.2 Symplectic manifolds and classical mechanics

The even-dimensional counterpart of contact manifolds are symplectic manifolds. We carry the concept of real structures over to symplectic manifolds and highlight an application to physics. This exhibits a first link between real symplectic and real contact manifolds.

Definition 1.2.1. A **symplectic manifold** is a $2n$-dimensional manifold (W, ω) equipped with a closed 2-form ω that satisfies $\omega^n \neq 0$.

The condition that ω^n be a volume form on W is equivalent to requiring that ω_p is non-degenerate for all $p \in W$. This turns the pair $(T_p W, \omega_p)$ into a **symplectic vector space** for any $p \in W$. As in the contact case, we have a notion of real structures for symplectic manifolds:

Definition 1.2.2. A triple (W, ω, f) consisting of a symplectic manifold (W, ω) and a real structure f on W is called a **real symplectic manifold** if $f^*\omega = -\omega$.

Example 1.2.3. The **standard real symplectic structure** on $W = \mathbb{R}^{2n}$ is given by $\omega_{\mathrm{st}} = d\mathbf{x} \wedge d\mathbf{y}$ (with coordinates $(\mathbf{x}, \mathbf{y})$ on $\mathbb{R}^{2n}$ as before) and $f_{\mathrm{st}}(\mathbf{x}, \mathbf{y}) = (-\mathbf{x}, \mathbf{y})$.

Symplectic manifolds make a prominent appearance in classical mechanics. In the spirit of Hamilton, the phase space of a mechanical system is given as the cotangent bundle T^*Q over a manifold Q. A symplectic form on T^*Q is defined as follows: Denote the projection $T^*Q \to Q$ by π, and define the **Liouville form** λ on T^*Q by $\lambda_\eta = \eta \circ T\pi$ for $\eta \in T^*Q$. In local coordinates $(\mathbf{q}, \mathbf{p})$, this form is given by $\lambda = \mathbf{p}\, d\mathbf{q}$. Put $\omega = d\lambda$. Locally, we have $\omega = d\mathbf{p} \wedge d\mathbf{q}$, and therefore ω is a symplectic form on T^*Q. Consider the involution f given by multiplication with -1 in the fibre, that is, in local coordinates $(\mathbf{q}, \mathbf{p})$, we have $f(\mathbf{q}, \mathbf{p}) = (\mathbf{q}, -\mathbf{p})$. Then $f^*\lambda = -\lambda$, turning the triple $(T^*Q, \omega = d\lambda, f)$ into a real symplectic manifold.

The dynamics of the system in question is encoded in a function on the phase space in the following way. Let $H : W \to \mathbb{R}$ be any function on an arbitrary symplectic manifold (W, ω). Then there exists a vector field X_H on W uniquely defined by the condition

$$i_{X_H}\omega = -dH.$$

This vector field is called the **Hamiltonian vector field** for the function H. In the situation as above, the Hamiltonian vector field provides a coordinate-free description of the

Hamiltonian equations for H. Observe that since the equality $i_{X_H} dH = 0$ holds, the flow of X_H preserves the level sets of H. This is the mathematical incarnation of conservation of energy.

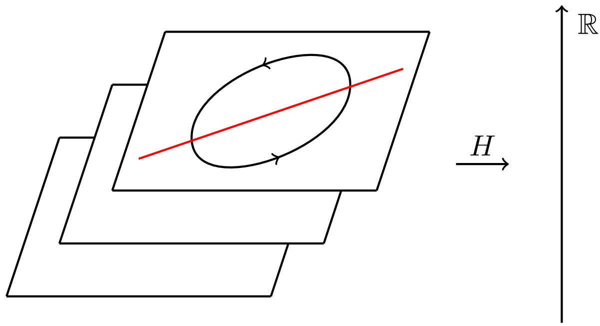

Figure 1.3: Energy levels of a Hamilton function.

Now, let (W^{2n+2}, ω, f) be a real symplectic manifold. Further suppose that M is a $(2n+1)$-dimensional submanifold of W with $f(M) = (-1)^{n+1}M$ – for example $M = H^{-1}(c)$ for an f-invariant function H on W with regular value c. In order to equip M with a contact structure, we make the following defintion. A vector field Y on W is called **Liouville** if its Lie derivative satisfies $L_Y\omega = \omega$. By Cartan's formula and using the fact that $d\omega = 0$, this reads $d(i_Y\omega) = \omega$. If Y is a Liouville vector field transverse to M, then

$$\alpha = (i_Y\omega)|_{TM}$$

defines a contact form on M. In this situation, we say that the hypersurface M is of **contact type**. This information is sufficient to prove that f descends to a real structure on the manifold M:

Proposition 1.2.4. *Let M be a hypersurface in a real symplectic manifold (W, ω, f), $\dim W = 2n+2$. Suppose that M is symmetric for n odd and antisymmetric for n even. Suppose further that there exists a Liouville vector field on W transverse to M. Then there is an induced contact form α on M such that $(M, \alpha, f|_M)$ is a real contact manifold.*

Proof. Let $\widetilde{Y}$ be the said Liouville vector field, and put

$$Y = \frac{1}{2}\left(\widetilde{Y} + f_*\widetilde{Y}\right).$$

This produces a symmetric vector field Y. Using the assumption $f^*\omega = -\omega$ and applying Lemma 1.1.11, we find $L_Y\omega = \omega$. The symmetry condition on M guarantees that Y is transverse to M, and for $\alpha = (i_Y\omega)|_{TM}$, we have $f^*\alpha = -\alpha$. Additionally, from the assumptions on f and M it follows that $f|_M$ is a real structure on M. □

An application of this method in classical contact topology can be found in [AFvKP12]. There, the authors exhibit a Liouville vector field for certain level sets in the restricted 3-body problem. Specifically, for $W = T^*\left(\mathbb{R}^2 \setminus \{E, M\}\right)$, $\omega_{\mathrm{st}} = d\mathbf{p} \wedge d\mathbf{q}$ the standard symplectic form and

$$H(\mathbf{q}, \mathbf{p}) = \frac{1}{2}\left((p_1+q_2)^2 + (p_2-q_1)^2\right) - \frac{1-\mu}{|\mathbf{q}-E|} - \frac{\mu}{|\mathbf{q}-M|} - \frac{1}{2}|\mathbf{q}|^2,$$

$E = (\mu, 0), M = (-(1-\mu), 0)$ and $\mu \in [0,1]$, they have

$$Y = (\mathbf{q} - M)\,\partial_{\mathbf{q}}.$$

This Liouville vector field is symmetric with respect to the *Birkhoff involution*

$$\begin{array}{rccc} f\colon & T^*\mathbb{R}^2 & \to & T^*\mathbb{R}^2 \\ & (q_1, q_2, p_1, p_2) & \mapsto & (q_1, -q_2, -p_1, p_2), \end{array}$$

and the Hamilton function is invariant under f. This is true even after regularisation. Thus, all level sets of contact type carry a real structure induced by the Birkhoff involution.

As a particular instance of Proposition 1.2.4, we construct the following class of real contact manifolds:

Example 1.2.5. As seen above, the cotangent bundle T^*Q of a manifold Q carries a real symplectic structure, locally given as $\omega = d\mathbf{p} \wedge d\mathbf{q}$ and $f(\mathbf{q}, \mathbf{p}) = (\mathbf{q}, -\mathbf{p})$. A Riemannian metric g on Q induces a bundle metric g^* on T^*Q. Given this metric, the unit cotangent bundle ST^*Q is defined fibrewise by

$$ST^*_{\mathbf{q}}Q = \left\{X \in T^*_{\mathbf{q}}Q \mid g^*_{\mathbf{q}}(X, X) = 1\right\}.$$

The condition $i_Y\omega = \lambda$ defines a Liouville vector field Y on T^*Q. Locally, we have $Y = \mathbf{p}\partial_{\mathbf{p}}$, i.e. Y is the radial vector field in fibre direction. Furthermore, Y is transverse to ST^*Q, so $\alpha = (i_Y\omega)\,|_{T(ST^*Q)}$ is a contact form on ST^*Q. Since the antipodal map on S^n is orientation-preserving precisely for n odd,

$$(ST^*Q, \alpha, f|_{ST^*Q})$$

is a real contact manifold. Its involution does not have any fixed points.

Now suppose that a hypersurface M in a symplectic manifold (W, ω) is both the level set of a Hamiltonian function H on W and of contact type. In classical contact geometry, one observes that in this case, the Reeb flow is a reparametrisation of the Hamiltonian flow. Thus the study of mechanical systems translates into understanding the Reeb dynamics in contact manifolds. As usual, one is particularly interested in finding closed orbits for the corresponding vector field. This problem is known as the Weinstein conjecture:

Conjecture (Weinstein)**.** *In a closed manifold M with contact structure ξ, for any contact form α with $\ker \alpha = \xi$, the corresponding Reeb vector field R_α has a closed orbit.*

This conjecture is known to be true in dimension 3 by works of Taubes (see for example the survey article [Hut10] by Hutchings). In its full generality, the question remains unanswered. We shall see presently that in real contact manifolds, the task of finding closed orbits can be rephrased.

Proposition 1.2.6. *Let (M, α, f) be a real contact manifold with non-empty fixed point set. Suppose that γ is an integral curve for the Reeb vector field R_α through $p \in \operatorname{Fix} f$. If the curve γ returns to the fixed point set of f for some $T \in \mathbb{R}$, then γ is a closed orbit.*

The proposition follows from the next lemma.

Lemma 1.2.7. *Let f be an involution on a closed manifold M. Suppose that X is an antisymmetric vector field on M. Then for any integral curve γ of X with $\gamma(0) \in \operatorname{Fix} f$, we have $f(\gamma(t)) = \gamma(-t)$ for all $t \in \mathbb{R}$.*

Proof. Since $\dot\gamma(t) = X_{\gamma(t)}$, we compute

$$\begin{aligned}\frac{d}{dt}(f\circ\gamma)(-t) &= -T_{\gamma(-t)}f\cdot\dot\gamma(-t)\\ &= -T_{\gamma(-t)}f\cdot X_{\gamma(-t)}\\ &= X_{f\circ\gamma(-t)}.\end{aligned}$$

Furthermore, $(f\circ\gamma)(0) = \gamma(0)$ by assumption. From the uniqueness of solution curves to a vector field we conclude that $f(\gamma(-t)) = \gamma(t)$ for all $t\in\mathbb{R}$. □

Proof of Proposition 1.2.6. If $\gamma(T) = \gamma(0)$, we are done. Otherwise, from the foregoing lemma we have the equality $f(\gamma(t+T)) = \gamma(T-t)$ for all t, and therefore it follows that $f(\gamma(2T)) = \gamma(0)$. This implies $\gamma(2T) = \gamma(0)$. □

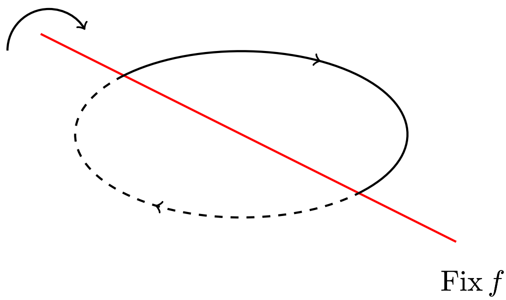

Figure 1.4: Orbit returning to the fixed point set. The involution reverses the orientation of the curve.

Now, let $L\subset M$ be a Legendrian submanifold of a contact manifold (M,α). A **Reeb chord** for L is an integral curve γ of the Reeb vector field R_α with both $\gamma(0), \gamma(T) \in L$ for some $T > 0$. Arnold's chord conjecture deals with the existence of Reeb chords:

Conjecture (Arnold)**.** *Let (M, α) be a closed contact manifold. Then any Legendrian submanifold $L \subset M$ has a Reeb chord.*

Again, only partial answers are known; Arnold's chord conjecture is true in dimension 3 due to Hutchings and Taubes, [HT11]. In the next section, we will prove that, in a real contact manifold (M, ξ, f), the fixed point set of f is always a Legendrian submanifold. Therefore, we have:

Observation. In a real contact manifold (M, α, f) with non-empty fixed point set, Arnold's chord conjecture implies Weinstein's conjecture.

Remark. Of course, the conclusion of Proposition 1.2.6 holds in the symplectic case as well: Let (W, ω, f) be a real symplectic manifold. The Hamiltonian vector field of a symmetric function $H\colon W \to \mathbb{R}$ is antisymmetric, and so are its integral curves. Therefore, they are necessarily closed in case they ever return to the fixed point set of f.

We conclude this section with two further constructions producing real contact manifolds out of real symplectic ones and vice versa, the contactification and the symplectisation. The latter one will return in the context of symplectic fillings in Chapter 3.

Let $(W^{2n}, \omega = d\lambda, g)$ be a real exact symplectic manifold, i.e. we have $g^*\lambda = -\lambda$. On $M := \mathbb{R} \times W$ with coordinate z on the $\mathbb{R}$-factor, put $\alpha = dz + \lambda$. Then (M, α) is a contact manifold. Define an involution f on M by $f(z, p) = (-z, g(p))$. We have $\operatorname{Fix} f \cong \operatorname{Fix} g$, and f is orientation-preserving if and only if g is orientation-reversing. Therefore, (M, α, f) is a real contact manifold, called the **real contactification of** (W, ω, g).

On the other hand, let (M^{2n+1}, α, g) be a real contact manifold. Denote the coordinate on the $\mathbb{R}$-factor in the manifold $W := \mathbb{R} \times M$ by t. Then $\omega = d(\mathrm{e}^t \alpha)$ defines a symplectic form on W. A real structure f on W is given by $f(t, p) = (t, g(p))$: It has $\operatorname{Fix} f \cong \mathbb{R} \times \operatorname{Fix} g$, and the orientation behaviour of f is as desired. Thus (W, ω, f) is a real symplectic manifold. We say that (W, ω, f) is the **real symplectisation of** (M, α, g). The vector field ∂_t is a symmetric Liouville vector field for ω that turns $M \equiv \{0\} \times M$ into a hypersurface of contact type.

1.3 Fixed point sets

In this section, we investigate the relation between anti-contact involutions and real structures. It turns out that the first of these properties suffices:

Proposition 1.3.1. *Let (M^{2n+1}, ξ) be a contact manifold. Suppose that f is an involution on M with non-empty fixed point set such that $Tf(\xi) = -\xi$. Then* $\operatorname{Fix} f$ *is a Legendrian submanifold of M.*

Thus, in view of our observation following Definition 1.1.5, the anti-equivariance condition $Tf(\xi) = -\xi$ alone implies that f is a real structure. Since we already know that $\mathrm{Fix}\, f$ is isotropic (assuming, for the moment, that $\mathrm{Fix}\, f$ is indeed a submanifold), all that remains to be done is to compute the dimension of the fixed point set. As a first step towards a proof, we note the following elementary observation:

Lemma 1.3.2. *Let A be a linear involution on a d-dimensional real vector space V, i.e. $A \in GL_d(V)$ and $A^2 = \mathrm{id}_V$. Then A is diagonalisable, and we have a splitting*

$$V = E_1 \oplus E_{-1}$$

of V into the eigenspaces corresponding to the eigenvalues $+1$ and -1, respectively, of A.

Proof. Without loss of generality, we may assume $V = \mathbb{R}^d$ and identify A with a matrix in $GL_d(\mathbb{R})$. Let J be the Jordan normal form of A and write $A = T^{-1}JT$ for a matrix T in $GL_d(\mathbb{R})$. The property of being an involution is preserved under conjugation, so we have $J^2 = E$, the $d \times d$ identity matrix. This implies that J is, in fact, a diagonal matrix.

If λ is an eigenvalue of A, then, by $A^2 = E$, we have $\lambda^2 = 1$. $\square$

In general, fixed point sets of diffeomorphisms can behave wildly. For example, any closed subset of $\mathbb{R}^d$ may be realised as the fixed point set of a diffeomorphism. (Simply take a constant non-zero vector field on $\mathbb{R}^d$, multiply it with a function that is 0 precisely on the given subset, and take the time 1 flow.) However, fixed point sets of involutions are tame in the following sense:

Lemma 1.3.3. *Let f be an involution on a manifold M^d. Then the components of the fixed point set of f are submanifolds of M.*

Proof. Pick any Riemannian metric $\widetilde{g}$ on M and put

$$g = \frac{1}{2}\left(\widetilde{g} + f^*\widetilde{g}\right).$$

Then g is an f-invariant metric on M, i.e. f is an isometry for the Riemannian manifold (M, g). Denote by exp the exponential map associated to the metric g. If the fixed point set of f is empty, we are done. Otherwise, let p be a point in $\operatorname{Fix} f$. Since f is an isometry for g, we have $\exp_p \circ T_p f = f \circ \exp_p$, or, equivalently,

$$\exp_p \circ T_p f \circ \exp_p^{-1} = f.$$

The differential $T_p f$ is a linear involution of $T_p M \equiv \mathbb{R}^d$. By Lemma 1.3.2, there exists a matrix $T \in GL_d(\mathbb{R})$ such that

$$T \circ T_p f \circ T^{-1} = \begin{pmatrix} E_k & 0 \\ 0 & -E_{d-k} \end{pmatrix}$$

for some number $0 \leq k \leq d$. This implies that $T^{-1} \circ \exp_p^{-1}$ is a submanifold chart for $\operatorname{Fix} f$. $\square$

Remark. (1) With the Riemannian metric g as in the proof above, f is an isometry of (M, g). Thus f may be regarded as a symmetry of M, satisfying our postulation in the introduction.

(2) Let f be an involution on a manifold M^{2n+1} whose fixed point set $\operatorname{Fix} f$ is an n-dimensional submanifold of M,

and let p be a fixed point of f. Then the differential T_pf is conjugate to the map

$$(x_1, \ldots, x_{2n+1}) \mapsto (x_1, \ldots, x_n, -x_{n+1}, \ldots, -x_{2n+1}).$$

So by the foregoing proof, f is orientation-preserving if and only if n is odd.

Let $p \in \operatorname{Fix} f$, and decompose the tangent space to M at p as $T_pM = E_1 \oplus E_{-1}$. Here, $E_{\pm 1}$ are the eigenspaces of T_pf. The dimension of the component of $\operatorname{Fix} f$ containing p equals the dimension of E_1. The fixed point set may contain components of different dimensions, however, as can be seen in the following example: Consider $M = \mathbb{R}P^3$ and, using homogeneous coordinates, define an involution f on M by $(x_0 : x_1 : x_2 : x_3) \mapsto (-x_0 : x_1 : x_2 : x_3)$. Then $\operatorname{Fix} f \cong \{\text{point}\} \sqcup \mathbb{R}P^2$. According to Proposition 1.3.1, this involution cannot satisfy $f^*\alpha = -\alpha$ for any contact form α on $\mathbb{R}P^3$. (This example also shows that the fixed point set of an involution on an orientable manifold does not have to be orientable itself.)

Proof of Proposition 1.3.1. Let $p \in \operatorname{Fix} f$. Pick a contact form α for ξ that satisfies $f^*\alpha = -\alpha$. In the last section, we already observed that for X tangent to $\operatorname{Fix} f$, we have $\alpha(X) = 0$. Combining this with Lemma 1.3.3, we conclude that the components of the fixed point set are isotropic submanifolds of M. It remains to show that the dimension of any component is n.

With R_p denoting the Reeb vector field of α in the point p, we have a splitting $T_pM = \xi_p \oplus \langle R_p \rangle$. Since $f^*\alpha = -\alpha$, the involution f respects this splitting. The claim now follows from applying the subsequent lemma to $V = \xi_p$, $\Omega = (d\alpha)_p$ and $A = T_p f|_{\xi_p}$. $\square$

A similar proof of Lemma 1.3.4 can be found in [Mey81].

Lemma 1.3.4. *Suppose that* (V, Ω) *is a* $(2n)$*-dimensional symplectic vector space, and let* A *be a linear anti-symplectic involution on* V*, i.e.* $A^*\Omega = -\Omega$*. Then the fixed point set of* A *is a Lagrangian subspace of* (V, Ω)*, that is,* $\Omega|_{\operatorname{Fix} A} \equiv 0$ *and* $\dim \operatorname{Fix} A = n$*.*

Proof. As before, decompose $V = E_1 \oplus E_{-1}$ into the direct sum of the eigenspaces of A. Let $v, w \in E_1$. Then

$$\begin{aligned}\Omega(v, w) &= \Omega(Av, Aw) \\ &= A^*\Omega(v, w) \\ &= -\Omega(v, w).\end{aligned}$$

Therefore, $\Omega(v, w) = 0$, and thus

$$E_1 \subset E_1^{\perp} = \{v \in V \mid \Omega(v, w) = 0 \text{ for all } w \in E_1\}.$$

Similarly, we have $E_{-1} \subset E_{-1}^{\perp}$. Now, if $V = U_1 \oplus U_2$ with $U_i \subset U_i^{\perp}$ for both $i = 1, 2$, then $U_i = U_i^{\perp}$. (This follows from $U_i^{\perp} \cap U_j \subset U_i^{\perp} \cap U_j^{\perp} = \{0\}$, $i \neq j$.) Lastly, a subspace $U \subset V$ is Lagrange if and only if $U = U^{\perp}$. $\square$

Remark. The quotient of a real manifold M by its involution f is a (smooth) manifold (without boundary), provided $\operatorname{Fix} f = \emptyset$. If f has fixed points, however, then M/f may or may not be a manifold (with or without boundary). To see this, consider the following examples.

(1) The quotient of T^3 by the real structure

$$\begin{array}{rccc} f\colon & T^3 & \longrightarrow & T^3 \\ & (\theta, \varphi, \psi) & \longmapsto & (-\theta, -\varphi, \psi) \end{array}$$

is diffeomorphic to $S^2 \times S^1$.

(2) The real structure

$$\begin{array}{rccc} f\colon & S^1 \times S^2 & \longrightarrow & S^1 \times S^2 \\ & (\theta, (x, y, z)) & \longmapsto & (-\theta, (-x, y, z)) \end{array}$$

produces a quotient diffeomorphic to $[-1, 1] \times D^2 \cong D^3$.

(3) By regarding $\mathbb{R}^3$ as $\{\mathbf{0}\} \cup \big(\mathbb{R}^+ \times S^2\big)$, one sees that the quotient X of $\mathbb{R}^3$ by the antipodal map is an open cone over $\mathbb{R}P^2$, and so, in particular, X is not a manifold. Therefore, the quotient of $\mathbb{R}^7$ by the real structure

$$\begin{array}{rccc} f\colon & \mathbb{R}^7 = \mathbb{R}^3 \times \mathbb{R}^4 & \longrightarrow & \mathbb{R}^7 = \mathbb{R}^3 \times \mathbb{R}^4 \\ & (\mathbf{x}, \mathbf{y}) & \longmapsto & (-\mathbf{x}, \mathbf{y}) \end{array}$$

is not a manifold.

Note that all the examples above may be endowed with a real contact structure.

Remark. In terms of fixed point sets, the theory of real contact manifolds differs significantly from the symmetric case. We will collect some of these features below. To that end, let (M, ξ) be a contact manifold of dimension $2n + 1$, and suppose that f is an involution on M with $f_*\xi = \xi$. As before, there exists a contact form α for ξ with $f^*\alpha = \alpha$: simply replace any contact form $\widetilde{\alpha}$ for ξ by $\alpha := (\widetilde{\alpha} + f^*\widetilde{\alpha})/2$. Such an involution is called **symmetric** (with respect to α).

(1) The involution f is necessarily orientation-preserving, no matter the dimension of M. This follows from the equality $f^*(\alpha \wedge (d\alpha)^n) = \alpha \wedge (d\alpha)^n$.

(2) The foregoing point implies that the components of the fixed point set of f are either empty or of odd dimension. The antipodal map on S^{2n+1}, equipped with the contact form as in Example 1.1.6 (4), is an involution without fixed points that preserves the contact form. On $\mathbb{R}^{2n+1}$ with contact form $\alpha_{\mathrm{st}} = dz + \mathbf{x}\, d\mathbf{y}$, for any $0 \leq k \leq n$, write the cartesian coordinates as $\mathbf{x} = (\mathbf{x}', \mathbf{x}'') \in \mathbb{R}^{n-k} \times \mathbb{R}^k$ and $\mathbf{y} = (\mathbf{y}', \mathbf{y}'') \in \mathbb{R}^{n-k} \times \mathbb{R}^k$, and define an involution by $f((\mathbf{x}', \mathbf{x}''), (\mathbf{y}', \mathbf{y}''), z) = ((-\mathbf{x}', \mathbf{x}''), (-\mathbf{y}', \mathbf{y}''), z)$. Then f is a symmetric involution with $\dim \operatorname{Fix} f = 2k+1$. Therefore, any number in $\{0, 1, 3, \ldots, 2n+1\}$ may be realised as the dimension of $\operatorname{Fix} f$ for a suitable triple (M, α, f).

(3) The dimensions of the components of $\operatorname{Fix} f$ may vary, as can be seen in the following example. The standard contact form $\mathbf{x}\, d\mathbf{y} - \mathbf{y}\, d\mathbf{x}$ on S^{2n+1} is invariant under the antipodal map, so it descends to a contact form α on $\mathbb{R}P^{2n+1}$. Define an involution f on $\mathbb{R}P^{2n+1}$ by

$$f(x_0 : y_0 : \ldots : x_n : y_n) := (-x_0 : -y_0 : \ldots : -x_k : -y_k : \\ x_{k+1} : y_{k+1} : \ldots : x_n : y_n)$$

for any $0 \leq k \leq n$. Then $f^*\alpha = \alpha$, and its fixed point set is given by $\operatorname{Fix} f \cong \mathbb{R}P^{2k+1} \sqcup \mathbb{R}P^{2(n-k)-1}$. In particular, for $n = 2$ and $k = 0$, the involution f has fixed point set diffeomorphic to $\mathbb{R}P^3 \sqcup S^1$.

(4) Since $f^*\alpha = \alpha$, we have $f_* R_\alpha = R_\alpha$ for R_α the Reeb vector field of α, and thus $R_\alpha \in T \operatorname{Fix} f$. This implies that

the fixed point set of a symmetric involution is never Legendrian.

1.4 One out of two

Do contact manifolds always carry an involution? Can one deduce the existence of a contact structure from the presence of an involution? In both cases, the answer is no. In this section, we will discuss examples of manifolds carrying only one of the two structures in question. We begin with a contact manifold on which there is no involution.

The question about the existence of contact structures on 3-manifolds is settled due to a theorem by Martinet:

Theorem 1.4.1 (Martinet, [Gei08, Theorem 4.1.1]). *Every closed, orientable 3-manifold admits a contact structure.*

In view of our agenda, a suitable candidate is therefore any 3-manifold that does not admit an involution at all. First, some terminology. A manifold M is called **aspherical** if $\pi_k M = 0$ for all $k \geq 2$. If no compact Lie group acts effectively on M, then M is called **asymmetric**. (This is the same as requiring that $\mathrm{Isom}(M, g)$ be trivial for any Riemannian metric g on M.) Following [Pup07], a theorem by Borel relates these notions:

Theorem 1.4.2 (Borel). *If M is an aspherical manifold with centreless fundamental group such that the group of outer automorphisms of $\pi_1 M$ is torsion-free, then the manifold M is asymmetric.*

Realising these assumptions proved to be difficult. An explicit construction of an asymmetric 3-manifold, due to Conner and Raymond and using Borel's theorem, can be found in [Edm85, Section 2.F]. In this example, the manifold is given as a bundle over S^1 with fibre a surface of genus ≥ 3. As of today, it is unknown whether there exist asymmetric simply-connected manifolds.

For the other direction, again by Martinet's theorem, a real manifold without a contact structure has to be at least 5-dimensional. According to [Gei08, Section 8.1], a necessary condition for the existence of a contact structure on a closed, oriented, simply-connected 5-manifold is the equality $W_3(M) = 0$. Here, $W_3(M)$ is the third integral Stiefel-Whitney class of M. Consider the inclusion of $SO(3)$ into $SU(3)$. Since $SO(3)$ is a closed subgroup of the Lie group $SU(3)$, the quotient $M = SU(3)/SO(3)$ is a manifold of dimension $8 - 3 = 5$. It is commonly referred to as the *Wu manifold*, and it appears as a component in Barden's classification of simply-connected 5-manifolds [Bar65]. As a quotient of a connected Lie group by a connected subgroup, M is orientable. Using the exact sequence induced by the fibration $SO(3) \to SU(3) \to M$ and the fact that $\pi_1 SU(3) = 0$, we have $\pi_1 M = 0$. By [Özb16, Exercise 1.34], $W_3(M) \neq 0$, so M does not carry a contact structure. On $SU(3)$, an involution is given by complex conjugation $A \mapsto \overline{A}$. If $B^{-1}A \in SO(3)$, then $\overline{B^{-1}A} = B^{-1}A$. Denote the image of A under the projection $SU(3) \to M$ by $[A]$. Then we have

$$\left[\overline{B}\right] = \left[\overline{B}\,\overline{B^{-1}A}\right] = \left[\overline{A}\right],$$

and therefore complex conjugation descends to an involution

$$\begin{array}{rccc} f\colon & M & \to & M \\ & [A] & \mapsto & [\overline{A}] \end{array}$$

on the quotient $SU(3)/SO(3)$.

By analysing the effect of the differential of complex conjugation on $T_E SU(3)$ (which can be identified with the space of anti-Hermitian traceless matrices), one deduces that f is orientation-reversing.

On the other hand, it is unknown whether the mere existence of an involution on a contact manifold implies the existence of a real contact structure on that manifold.

Question. Do there exist manifolds admitting a contact structure as well as a real structure but no real contact structure?

A possible approach to this problem, at least in the 3-dimensional case, will be presented in Chapter 3.

2 Structure theorems

The very basis of contact topology consists of various structure theorems for contact manifolds. In the present chapter, we establish real versions of these theorems, providing an extensive toolbox for the real contact topologist's everyday work. First, Gray's stability theorem – a result about deformations of contact structures on closed manifolds – is translated into the real contact setting. In addition, a generalisation of Gray's theorem to certain decorated group actions is discussed. Following that, we introduce real contact Hamiltonians; this allows an identification of certain vector fields with functions on a manifold. Darboux's theorem states that, locally, all contact manifolds look the same. This is still true in the presence of a real structure around its fixed point set, and we will give a proof in Section 2.3. The subsequent section covers neighbourhood theorems for isotropic submanifolds in the fixed point set and invariant contact submanifolds. These lead to a local model for Legendrian submanifolds which will be used for the definition of real contact surgeries. In 3-dimensional contact topology, numerous results are obtained through convex surface theory. Real analogues may be found in Section 2.5. After proving the real isotopy extension theorem, we classify real structures for the standard contact structure on $\mathbb{R}^{2n+1}$ in

Section 2.7. With similar methods, a real contact disc theorem is derived. The majority of arguments used to prove the aforementioned theorems is inspired by classical methods, found in [Gei08]. We will indicate where our techniques differ from that source.

2.1 Gray stability

A remarkable feature of closed manifolds is that there are no non-trivial deformations of contact structures. This is Gray's stability theorem, and we will prove a real version of it in this section. Before we proceed to Gray's theorem, we set up some terminology. In homotopy theory, one usually denotes the (free) path space of a topological space X by

$$P_*X := \{\gamma\colon [0,1] \to X \text{ continuous}\}\,.$$

In the same vein, for a manifold M, we refer to the space of time-dependent vector fields on M by $P_*\Gamma(TM)$. (This includes ordinary vector fields on M, thought of as vector fields constant in time.) The space of differential k-forms will be abbreviated by $\Omega^k(M)$.

Definition 2.1.1. An **expression** on M is a map

$$E\colon P_*\Gamma(TM) \to \Omega^k(TM)$$

for some $k \geq 0$. A **solution** of an expression E is a vector field X_t such that $E(X_t) = 0$.

For example, in a contact manifold (M, α), the expressions $E_1(X) = i_X\alpha - 1$ and $E_2(X) = i_X d\alpha$ uniquely determine the Reeb vector field of α.

Now, let (M, f) be a real manifold.

Definition 2.1.2. An expression E on M is called **symmetric** if

$$f^*(E(X_t)) = \pm E(f_* X_t)$$

for all $X_t \in P_*\Gamma(TM)$. It is called **antisymmetric** if

$$f^*(E(X_t)) = \pm E(-f_* X_t)$$

for all $X_t \in P_*\Gamma(TM)$.

In Proposition 1.1.10, we actually checked that the expressions E_1 and E_2 are antisymmetric. The nomenclature is motivated by the elementary observation that symmetric expressions produce symmetric vector fields:

Proposition 2.1.3. *Let (M, f) be a real manifold, and let $S \subset P_*\Gamma(TM)$ be any subset. Suppose that an expression E on M has a unique solution X_t when restricted to S. Further assume that $f_* X_t \in S$. Then if E is (anti-)symmetric, so is its solution X_t.*

Furthermore, let $E_1, \ldots, E_k$ be a system of expressions all of which are (anti-)symmetric. Suppose that this system admits a unique common solution X_t in S with $f_* X_t \in S$. Then X_t is (anti-)symmetric as well.

Proof. Let E be a symmetric expression with unique solution X_t in S such that $f_* X_t \in S$. Then $E(X_t) = 0$, and therefore

$$0 = f^* E(X_t) = \pm E(f_* X_t),$$

which implies that we have $f_* X_t = X_t$. The other cases are similar. $\square$

Symmetric vector fields are particularly useful since their flows commute with the given real structure:

Proposition 2.1.4. *Let (M, f) be a real manifold. Suppose that φ_t is the flow of a time-dependent vector field X_t on M, i.e. $\dot{\varphi}_t = X_t \circ \varphi_t$ for all $t \in \mathbb{R}$. If X_t is symmetric, then*

$$\varphi_t \circ f = f \circ \varphi_t$$

for all $t \in \mathbb{R}$, that is, φ_t is symmetric. The flow of an antisymmetric vector field satisfies

$$\varphi_{-t} \circ f = f \circ \varphi_t$$

for all $t \in \mathbb{R}$.

Proof. For X_t symmetric, we have

$$\begin{aligned} \frac{d}{dt} (f \circ \varphi_t \circ f) &= f_* (\dot{\varphi}_t \circ f) \\ &= f_* (X_t \circ \varphi_t \circ f) \\ &= X_t \circ (f \circ \varphi_t \circ f) . \end{aligned}$$

From the uniqueness of solutions of the given differential equation, we deduce that $f \circ \varphi_t \circ f = \varphi_t$. The antisymmetric case is similar. □

For later use, we note the following consequence of this proposition:

Corollary 2.1.5. *If X_t is a symmetric time-dependent vector field on a real manifold (M, f), then the flow φ_t of X_t preserves the fixed point set of f.* □

With these tools at hand, we are ready to prove the real version of Gray's theorem. The goal is to find a symmetric isotopy. We will simply copy the classical proof using the *Moser trick*; this will produce a unique vector field. In order for its flow to be symmetric, by Proposition 2.1.4, the vector field ought to be symmetric itself. This is indeed the case, as we shall see presently.

Theorem 2.1.6 (Real Gray stability)**.** *Let (M, f) be a closed real manifold, and suppose that ξ_t, $t \in [0,1]$, is a smooth family of real contact structures, i.e. we have $f_*\xi_t = -\xi_t$. Then there exists a symmetric isotopy φ_t of M such that $(\varphi_t)_* \xi_0 = \xi_t$ for all $t \in [0,1]$.*

Proof. Choose a smooth family of contact forms α_t for ξ_t for which $f^*\alpha_t = -\alpha_t$. The equation we are trying to solve can be rewritten as

$$(\varphi_t)^* \alpha_t = \lambda_t \alpha_0 \tag{2.1}$$

where λ_t is a smooth family of functions on M. Assume that φ_t is the flow of a time-dependent vector field X_t on M. Differentiating Equation 2.1 with respect to t, we obtain

$$\varphi_t^* \left(\dot{\alpha}_t + d\left(i_{X_t}\alpha_t\right) + i_{X_t} d\alpha_t\right) = \varphi_t^* \left(\mu_t \alpha_t\right)$$

with $\mu_t = \frac{d}{dt}\left(\log \lambda_t\right) \circ \varphi_t^{-1}$. For $X_t \in \xi_t$, this simplifies to

$$\dot{\alpha}_t + i_{X_t} d\alpha_t = \mu_t \alpha_t. \tag{2.2}$$

Plugging in the Reeb vector field R_t for α_t yields

$$i_{R_t} \dot{\alpha}_t = \mu_t.$$

Define μ_t by this equation. By assumption, $f_* R_t = -R_t$ and $f^* \dot{\alpha}_t = -\dot{\alpha}_t$, so the functions μ_t are symmetric. Thanks to the non-degeneracy of $d\alpha_t$ and the fact that the Reeb vector fields satisfy $R_t \in \ker \mu_t\alpha_t - \dot{\alpha}_t$, Equation 2.2 has a unique solution $X_t \in \xi_t$. The expression

$$E(X_t) = \dot{\alpha}_t + i_{X_t} d\alpha_t - \mu_t\alpha_t$$

is symmetric, and so is its unique solution in

$$S = \{X_t \in P_*\Gamma(TM) \mid X_t \in \xi_t\}$$

by Proposition 2.1.3. Finally, Proposition 2.1.4 tells us that the isotopy φ_t, defined as the flow of the vector field X_t, is symmetric. □

Remark. In points $p \in M$ where $\dot{\alpha}_t$ is zero, we have that $\mu_t(p) = 0$, and therefore $X(p) = \mathbf{0}$. This implies that p stays fixed under φ_t for all t.

Observation. Let ξ_t, $t \in [0, 1]$ be a family of contact structures on a closed manifold M, and suppose that f is an involution on M such that $f_*\xi_0 = -\xi_0$. By *classical* Gray stability, there exists an isotopy φ_t of M with $(\varphi_t)_*\,\xi_0 = \xi_t$. Define a time-dependent involution via $f_t := \varphi_t \circ f \circ (\varphi_t)^{-1}$. Then every f_t is an involution on M that satisfies $(f_t)_*\,\xi_t = -\xi_t$. That is, (M, ξ_t, f_t) is a real contact manifold. Therefore, being real is a property not of a contact structure but of its isotopy class.

Remark. One may wonder whether there is a version of real Gray stability for time-dependent involutions. However, the Moser trick argument *cannot* be applied in that situation: In case of time-dependent involutions f_t, the assumption $f_t^*\alpha_t = -\alpha_t$ does not imply $f_t^*\dot{\alpha}_t = -\dot{\alpha}_t$ in general, as can be seen in the following example. Take $M = \mathbb{R}^3$, and put $\alpha_t = (1+t)dz + x\,dy$, $0 \leq t \leq 1$. A family of real structures for α_t is given by

$$\begin{array}{cccc} f_t\colon & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ & (x,y,z) & \longmapsto & \left(-x-1, y, -z + \frac{1}{1+t}y\right). \end{array}$$

It is easy to check that all the maps f_t are indeed involutions, and that $f_t^*\alpha_t = -\alpha_t$ for all t. Furthermore, we compute $\dot{\alpha}_t = dz$. However,

$$f_t^*\dot{\alpha}_t = f_t^* dz = -dz + \frac{1}{1+t}dy \neq -dz = -\dot{\alpha}_t.$$

As in the classical set-up, Gray stability does not hold for contact forms (rather than contact structures):

Example 2.1.7. On $S^3 \subset \mathbb{R}^4$ with involution given by $f(x_1, y_1, x_2, y_2) = (-x_1, y_1, -x_2, y_2)$, consider a smooth family of contact forms given by

$$\alpha_t = (x_1 dy_1 - y_1 dx_1) + (1+t)(x_2 dy_2 - y_2 dx_2), \quad t \geq 0.$$

We have $f^*\alpha_t = -\alpha_t$ for all t, so each triple (M, α_t, f) is a real contact manifold. The Reeb vector field of α_0 defines the Hopf fibration (and therefore all of its orbits are closed), while for $t \in \mathbb{R}^+ \setminus \mathbb{Q}$, the Reeb vector field of α_t has precisely two closed orbits. However, a diffeomorphism φ with the property $\varphi^*\alpha_t = \alpha_0$ would have to satisfy $\varphi_* R_0 = R_t$.

In general, Gray's theorem also fails for open manifolds. On $S^1 \times \mathbb{R}^2$ with polar coordinates (ρ, φ) on $\mathbb{R}^2$ and $\theta \in S^1$ and for a real number $t > 0$, define contact forms α_t by

$$\alpha_t = d\theta - \frac{\arctan^2 \rho}{t} d\varphi.$$

The involution

$$\begin{array}{rccc} f\colon & S^1 \times \mathbb{R}^2 & \longrightarrow & S^1 \times \mathbb{R}^2 \\ & (\theta, (\rho, \varphi)) & \longmapsto & (-\theta, (\rho, -\varphi)) \end{array}$$

defines a real structure for all α_t. But according to [Eli93, Theorem 1.C], the contact structures $\ker \alpha_t$ and $\ker \alpha_{t'}$ are isotopic if and only if $t - t' = \pi^2 k/4$ for a $k \in \mathbb{Z}$.

The remainder of this section addresses a certain generalisation of Gray's stability theorem. An involution on a manifold is the same as a smooth $\mathbb{Z}_2$-action on that manifold. Since the proof of Gray's theorem in the real case does not rely on any special features of the group $\mathbb{Z}_2$, the stability theorem generalises to arbitrary compact (topological) groups G as follows: Suppose we are given a compact group G together with a group homomorphism $\rho\colon G \to \mathbb{Z}_2 = \{\pm 1\}$, and let G act smoothly on a contact manifold (M, ξ). Then any element $g \in G$ induces a diffeomorphism of M via $p \mapsto g \cdot p$, denoted by g as well.

Definition 2.1.8. The triple $(M, \xi, (G, \rho))$ is called a **(G, ρ)-contact manifold** if

$$g_* \xi = \rho(g) \xi \quad \text{for all } g \in G.$$

In that terminology, a real contact manifold is nothing but a $(\mathbb{Z}_2, \mathrm{id}_{\mathbb{Z}_2})$-contact manifold. Gray's stability theorem for (anti-)equivariant group actions then reads:

Theorem 2.1.9 (Anti-equivariant Gray stability)**.** *Let ξ_t, $t \in [0, 1]$, be a family of contact structures on a closed manifold M and suppose that, for a pair (G, ρ) with G a compact* *group and $\rho\colon G \to \mathbb{Z}_2$ a group homomorphism, we have $g_*\xi_t = \rho(g)\xi_t$ for all $g \in G$ and $t \in [0,1]$. Then there exists an isotopy φ_t of M with $(\varphi_t)_*\,\xi_0 = \xi_t$ for each $t \in [0,1]$ and $\varphi_t \circ g = g \circ \varphi_t$ for all $g \in G$, $t \in [0,1]$.*

Proof. Pick a smooth family of contact forms $\widetilde{\alpha_t}$ for the contact structures ξ_t. There exists a version of the Haar integral for differential forms, see for example [CE48, Section I.3]. Put

$$\alpha_t = \int_G \rho(g) g^* \widetilde{\alpha_t}.$$

The assumption $g_*\xi = \rho(g)\xi$ implies that α_t is a contact form with kernel ξ_t. Furthermore, we have $g^*\alpha_t = \rho(g)\alpha_t$ for all $g \in G$:

$$\begin{aligned}
g_0^*\alpha_t &= \int_G g_0^*\left(\rho(g) g^*\widetilde{\alpha_t}\right)\\
&= \int_G \rho(g)\,(g \circ g_0)^*\,\widetilde{\alpha_t}\\
&= \int_G \rho(g \circ g_0)\rho(g_0)(g \circ g_0)^*\widetilde{\alpha_t}\\
&= \rho(g_0)\int_G \rho(g \circ g_0)(g \circ g_0)^*\widetilde{\alpha_t}\\
&= \rho(g_0)\alpha_t
\end{aligned}$$

For R_t the Reeb vector field of α_t, we have $g_*R_t = \rho(g)R_t$ for all $g \in G$. Thus, the remaining portion of the proof in Theorem 2.1.6 may be copied verbatim. The vector field X_t

constructed in this way is symmetric with respect to G, and so is φ_t. □

Remark. This theorem includes the equivariant case by picking the trivial homomorphism $\rho\colon G \to \mathbb{Z}_2$.

2.2 Contact Hamiltonians

In this section, we cover the real version of contact Hamiltonians, that is, functions $M \to \mathbb{R}$ on a real contact manifold (M, ξ, f). We establish a one-to-one correspondence between antisymmetric functions and symmetric contact vector fields.

Definition 2.2.1. Let $(M, \xi = \ker \alpha, f)$ be a real contact manifold. Denote by φ_t the flow of a vector field X on M. (In general, this flow will be defined only locally in M in case M is not compact.) The vector field X is a **real infinitesimal automorphism of** ξ (or a **real contact vector field**) if φ_t is symmetric and $T\varphi_t(\xi) = \pm\xi$ for all $t \in \mathbb{R}$. It is called a **real infinitesimal automorphism of** α (or a **real strict contact vector field**) if φ_t is symmetric and $\varphi_t^*\alpha = \alpha$ for all $t \in \mathbb{R}$.

The following lemma characterises infinitesimal automorphisms in terms of the Lie derivative $L_X\alpha$ of α with respect to X.

Lemma 2.2.2. *Let X be a vector field on a real contact manifold $(M, \xi = \ker \alpha, f)$.*

(a) The vector field X is a real infinitesimal automorphism of α if and only if X is symmetric and $L_X\alpha \equiv 0$.

(b) The vector field X is a real infinitesimal automorphism of ξ if and only if X is symmetric and $L_X\alpha = \mu\alpha$ for some function μ on M. This condition is independent of the choice of the contact form for ξ.

Proof. Combine the proof in the classical case [Gei08, Lemma 1.5.8] with Proposition 2.1.4. It remains to show the reverse direction of Proposition 2.1.4, namely that symmetric flows have symmetric vector fields: Suppose that we have $\varphi_t \circ f = f \circ \varphi_t$. Differentiation with respect to t implies $\dot{\varphi}_t = f_*\dot{\varphi}_t$. Assuming $\dot{\varphi}_t = X \circ \varphi_t$, we find $f_*X = X$. □

The following proposition enables us to replace arbitrary contact vector fields by (anti-)symmetric ones.

Proposition 2.2.3. *Let X be a (strict) contact vector field on a real contact manifold $(M, \xi = \ker\alpha, f)$. Then*

$$X^+ := \frac{1}{2}\left(X + f_*X\right)$$

defines a (strict) real contact vector field. The (strict) contact vector field

$$X^- := \frac{1}{2}\left(X - f_*X\right)$$

is antisymmetric with respect to f.

Proof. Suppose that $L_X\alpha = \mu\alpha$. Then $L_{f_*X}\alpha = -\left(\mu \circ f\right)\alpha$, and

$$L_{X^+}\alpha = \frac{1}{2}\left(\mu - \left(\mu \circ f\right)\right)\alpha.$$

The second case is similar. □

Now we prove the real version of [Gei08, Theorem 2.3.1] about contact Hamiltonians.

Theorem 2.2.4. *Let (M, ξ, f) be a real contact manifold and fix a contact form α for ξ such that $f^*\alpha = -\alpha$. Then there is a one-to-one correspondence between real infinitesimal automorphisms of ξ and smooth antisymmetric functions $M \to \mathbb{R}$, given by*

$$X \mapsto H_X = i_X\alpha \quad \textit{and} \quad H \mapsto X_H.$$

Here, X_H is defined uniquely by the conditions

$$i_{X_H}\alpha = H \quad \textit{and} \quad i_{X_H}d\alpha = (i_{R_\alpha}dH)\,\alpha - dH.$$

Proof. If X is a symmetric vector field, then $H_X \circ f = -H_X$. For an antisymmetric function H, the expressions

$$E_1(X) = i_X\alpha - H \quad \text{and} \quad E_2(X) = i_X d\alpha - (i_{R_\alpha}dH)\,\alpha - dH$$

are symmetric. Thus the unique solution X_H of the system $E_1 = 0, E_2 = 0$ is symmetric. To conclude, combine the proof in [Gei08, Theorem 2.3.1] with Lemma 2.2.2. □

Remark. Similarly, one has a one-to-one correspondence between symmetric functions on M and antisymmetric infinitesimal automorphisms of ξ.

A consequence of this theorem is a time-dependent variant of Lemma 2.2.2:

Corollary 2.2.5. *Let $H_t \colon M \to \mathbb{R}$, $t \in [0,1]$, be a smooth family of antisymmetric functions on a closed real contact manifold $(M, \xi = \ker \alpha, f)$. Denote by φ_t the flow of the time-dependent vector field X_{H_t}. Then φ_t is a symmetric contact isotopy, i.e. $\varphi_t^* \alpha = \lambda_t \alpha$ for some smooth family of functions λ_t on M and $\varphi_t \circ f = f \circ \varphi_t$ for all $t \in [0,1]$.*

The proof follows from the proof given in [Gei08, Corollary 2.3.2] together with Proposition 2.1.4. This corollary provides a valuable means for proving isotopy extension theorems in Section 2.6.

2.3 Darboux's theorem

This section features a real version of Darboux's theorem: Locally, in the neighbourhood of a fixed point, all contact forms on a real manifold (M, f) look the same.

Theorem 2.3.1 (Real Darboux)**.** *Let α_0 and α_1 be contact forms on a real manifold (M, f) such that (M, α_i, f) is a real contact manifold for $i = 0, 1$. Suppose that $p \in \operatorname{Fix} f$. Then there exists an invariant neighbourhood U of p in M and a diffeomorphism $\varphi \colon U \to \varphi(U) \subset M$ such that $\varphi(p) = p$, $\varphi^* \alpha_1 = \alpha_0$ and $f \circ \varphi = \varphi \circ f$.*

In order to prove Darboux's theorem, we need the following lemma.

Lemma 2.3.2. *Suppose we are in the situation of Theorem 2.3.1. Then there exists a symmetric diffeomorphism*

$\psi\colon V \to V$, defined on an invariant neighbourhood V of p, with $\psi(p) = p$ and

$$\psi^*(\alpha_1)_p = (\alpha_0)_p \quad \textit{and} \quad \psi^*(d\alpha_1)_p = (d\alpha_0)_p\,. \tag{2.3}$$

Proof. Put $V_0 = V_1 = T_pM$. First, we construct a linear map $A : V_0 \to V_1$ that satisfies both conditions 2.3 and commutes with T_pf. As in Section 1.3, there are splittings $V_i = \xi_i \oplus \langle R_i\rangle = E_1 \oplus E^i_{-1} \oplus \langle R_i\rangle$, $i = 0, 1$, where $\xi_i = \ker \alpha_i$ and R_i denotes the Reeb vector field of α_i in the point p. (Since the Reeb vector fields may be different, the spaces E^i_{-1} will be different, in general. The eigenspace to the eigenvalue 1, i.e. E_1, however, does not depend on the contact form.) Pick bases (e^i_j, f^i_j) of ξ_i, respectively, such that

$$(d\alpha_i)_p = \sum_{j=1}^{n} de^i_j \wedge df^i_j, \quad i = 0, 1, \quad \text{and} \quad e^i_j \in E_1, f^i_j \in E^i_{-1}.$$

Such bases are found using the inductive method as in the classical proof, see for example [MS98, Theorem 2.3]. Observing the second condition is possible thanks to the fact that both E_1 and E^i_{-1} are Lagrangian subspaces of ξ_i, equipped with the symplectic form $(d\alpha_i)_p$. Now define the linear map A by

$$Ae^0_j = e^1_j, \quad Af^0_j = f^1_j, \quad j = 1, \ldots, n, \quad \text{and} \quad AR_0 = R_1.$$

Then we have

$$A^*(\alpha_1)_p = (\alpha_0)_p\,, \quad A^*(d\alpha_1)_p = (d\alpha_0)_p \qquad \text{and } A \circ T_pf = T_pf \circ A.$$

Pick an f-invariant Riemannian metric on M and define $\psi := \exp_p \circ A \circ \exp_p^{-1}$. □

Proof of Darboux's theorem. For $t \in [0,1]$, consider the family $\alpha_t := (1-t)\alpha_0 + t(\psi^*\alpha_1)$ of 1-forms. In the point p, we have $\alpha_t = \alpha_0$ and $d\alpha_t = d\alpha_0$, and therefore α_t is a contact form on some neighbourhood of p for all $t \in [0,1]$. As in the proof of Gray's stability theorem, we would like to employ the Moser trick. That is, we try to solve the equation $\phi_t^*\alpha_t = \alpha_0$ by assuming that ϕ_t is the flow of a time-dependent vector field X_t on M. By differentiation, we obtain the equation

$$\dot{\alpha}_t + L_{X_t}\alpha_t = 0. \tag{2.4}$$

Write X_t as $X_t = H_t R_t + Y_t$, where R_t is the Reeb vector field of α_t, H_t some function on M and $Y_t \in \ker \alpha_t$. Plugging in R_t into 2.4 yields

$$i_{R_t}\dot{\alpha}_t + i_{R_t} dH_t = 0. \tag{2.5}$$

Find a solution $\widetilde{H_t}$ of 2.5 with $\widetilde{H_t}(p) = 0$ and $d\widetilde{H_t}|_p = 0$ for all $t \in [0,1]$, and replace $\widetilde{H_t}$ by

$$H_t := \frac{1}{2}\left(\widetilde{H_t} - \widetilde{H_t} \circ f\right).$$

Then H_t still satisfies 2.5, and we have $H_t(p) = 0$, $dH_t|_p = 0$ for all $t \in [0,1]$. Additionally, H_t is antisymmetric. Now define the vector field Y_t by Equation 2.4, so that we have

$$\dot{\alpha}_t + dH_t + i_{Y_t} d\alpha_t = 0.$$

Since the expression $E(Y_t) := \dot{\alpha}_t + dH_t + i_{Y_t} d\alpha_t$ is symmetric, so is Y_t. Consequently, X_t is symmetric. Let ϕ_t be the flow of X_t. We have $\phi_t(p) = p$ for all $t \in [0,1]$. Since a local flow's domain of definition is always open, by shrinking the neighbourhood, if necessary, we may assume that ϕ_t is defined for all t. Put $\varphi := \psi \circ \phi_1$. Then $\varphi(p) = p$, $\varphi^* \alpha_1 = \alpha_0$ and $f \circ \varphi = \varphi \circ f$, as required. $\square$

2.4 Neighbourhood theorems for submanifolds

In this section, we establish real versions of neighbourhood theorems for isotropic submanifolds contained in the fixed point set as well as for contact submanifolds. A discussion of these theorems, employing similar methods as presented below, may be found in [ÖS11]. This article contains a proof of the neighbourhood theorem for isotropic submanifolds in the fixed point set; we supply some details omitted there.

As a preparation, the first part of the present section is devoted to real symplectic linear algebra.

Definition 2.4.1. Let V be a real vector space, and let F be an involution on V. A **real complex structure** on (V, F) is an automorphism J of V satisfying $J^2 = -\operatorname{id}_V$ and $J \circ F = -F \circ J$. If (V, ω, F) is a real symplectic vector space, then J is called ω-**compatible** if, additionally,

$$\omega(J\mathbf{u}, J\mathbf{v}) = \omega(\mathbf{u}, \mathbf{v}) \text{ for all } \mathbf{u}, \mathbf{v} \in V$$

and

$$\omega(\mathbf{v}, J\mathbf{v}) > 0 \text{ for all non-zero } \mathbf{v} \in V.$$

Denote by $\mathcal{RJ}(\omega, F)$ the space of real complex structures on (V, ω, F) compatible with ω, with its topology induced by the inclusion of $\mathcal{RJ}(\omega, F)$ into the vector space of endomorphisms of V. As in the classical case, $\mathcal{RJ}(\omega, F)$ can be characterised as follows.

Proposition 2.4.2. *The space $\mathcal{RJ}(\omega, F)$ is non-empty and contractible.*

Proof. We follow the proof in [Gei08, Proposition 1.3.10] and observe that the constructions suit the involution F. Pick an F-invariant inner product g on V. Define a matrix A by the requirement $\omega(\mathbf{u}, \mathbf{v}) = g(A\mathbf{u}, \mathbf{v})$. From that equation, we conclude

$$g\left((F \circ A)\mathbf{u}, \mathbf{v}\right) = g(A\mathbf{u}, F\mathbf{v}) = -g\left((A \circ F)\mathbf{u}, \mathbf{v}\right)$$

for all $\mathbf{u}, \mathbf{v} \in V$, and therefore $F \circ A = -A \circ F$. Furthermore, F preserves the eigenspaces of $-A^2$, so $F \circ Q = Q \circ F$ for Q the positive square root of $-A^2$. Put $J = AQ^{-1}$. Then $F \circ J = -J \circ F$, implying that $\mathcal{RJ}(\omega, F)$ is non-empty. Denote by $\mathcal{SG}$ the space of F-invariant inner products on the vector space V. The above construction defines a surjective map $\mathcal{SG} \to \mathcal{RJ}(\omega, F)$ that has a section. Since $\mathcal{SG}$ is a convex space, so is $\mathcal{RJ}(\omega, F)$. $\square$

Now, before proving a real version of the neighbourhood theorem for isotropic submanifolds, we collect some statements about real symplectic vector bundles. We will follow the discussion in Section 2.5 of [Gei08] closely.

Definition 2.4.3. A **real symplectic vector bundle** over a manifold B is a triple (E, ω, F) that consists of a symplectic vector bundle $\pi\colon E \to B$ and a fibre-preserving involution

F on E such that $(\pi^{-1}(b), \omega_b, F|_{\pi^{-1}(b)})$ is a real symplectic vector space for all $b \in B$.

Example 2.4.4. Let (M, ξ, f) be a real contact manifold. Then $(\xi, d\alpha, Tf|_\xi)$ is a real symplectic vector bundle over M.

Suppose now that $(M, \xi = \ker \alpha, f)$ is a real contact manifold, and let L be an isotropic submanifold of M contained in $\operatorname{Fix} f$. Since $Tf|_{TL} = \mathrm{id}_{TL}$, the following is well defined.

Definition 2.4.5. The quotient bundle

$$\mathrm{RCSN}_M(L) := (TL)^\perp / TL,$$

equipped with the conformal symplectic structure induced by $d\alpha$ and the involution

$$\begin{array}{rccc} Tf\colon & \mathrm{RCSN}_M(L) & \to & \mathrm{RCSN}_M(L) \\ & [X] & \mapsto & [Tf(X)]\,, \end{array}$$

is called the **real conformal symplectic normal bundle** of L in M.

As in the classical set-up, we decompose the normal bundle of L in M as

$$NL \cong (TM|_L)/(\xi|_L) \oplus (\xi|_L)/(TL)^\perp \oplus \mathrm{RCSN}_M(L). \quad (2.6)$$

Observe that Tf descends to an involution on the quotient $(\xi|_L)/(TL)^\perp$: For if $X - X' = Y \in (TL)^\perp$, then for any $Z \in TL$,

$$d\alpha(Z, Tf(Y)) = i_Y i_{Tf(Z)} f^* d\alpha = -d\alpha(Z, Y) = 0,$$

i.e. $Tf(Y) \in (TL)^\perp$. We will continue to call this involution Tf.

Lemma 2.4.6. *The real bundle* $\left((\xi|_L)/(TL)^\perp, Tf|_\xi\right)$ *is equivariantly isomorphic to the cotangent bundle* T^*L *via the symmetric bundle isomorphism*

$$\begin{array}{rccc} \Psi\colon & (\xi|_L)/(TL)^\perp & \to & T^*L \\ & [Y] & \mapsto & i_Y d\alpha|_{TL}. \end{array}$$

Here, the real structure on the contangent bundle T^*L *is given by multiplication with* -1 *in the fibres, i.e.* $\eta \mapsto -\eta$.

Proof. It only remains to check that the map Ψ is symmetric with respect to the given involutions. We have the identity $(-\operatorname{id}_{T^*L}) \circ \Psi([Y]) = -i_Y d\alpha|_{TL}$, and we compute

$$\begin{aligned} \Psi \circ Tf([Y]) &= i_{Tf(Y)} d\alpha|_{TL} \\ &= \left(f^*\left(i_Y f^* d\alpha\right)\right)|_{TL} \\ &= -\left(f^*\left(i_Y d\alpha\right)\right)|_{TL} \\ &= -i_Y d\alpha|_{TL}, \end{aligned}$$

using that $Tf|_{TL} = \operatorname{id}_{TL}$. □

We aim for a symmetric identification of the summands in Equation 2.6 with sub-bundles of TM. To that end, we note:

Proposition 2.4.7. *Let* $J\colon \xi \to \xi$ *be a real complex bundle structure on* $(\xi, Tf|_\xi)$ *compatible with the symplectic bundle structure given by* $d\alpha$. *Then there are symmetric bundle isomorphisms*

$$\left((\xi|_L)/(TL)^\perp, Tf|_\xi\right) \to \left(J(TL), Tf|_{J(TL)}\right)$$

and

$$(\mathrm{RCSN}_M(L), Tf) \to \left((TL \oplus J(TL))^{\perp}, Tf|_{(TL\oplus J(TL))^{\perp}}\right).$$

Proof. From the condition $J \circ Tf = Tf \circ J$, we conclude that Tf restricts to an involution on $J(TL)$ and $(TL \oplus J(TL))^{\perp}$, respectively. It is a straightforward calculation to check that both maps are equivariant. □

Since $L \subset \operatorname{Fix} f$, we have $Tf|_{J(TL)} = -\operatorname{id}_{J(TL)}$.

Lemma 2.4.8. *The bundle map*

$$\begin{aligned} \operatorname{id}_{TL} \oplus \Psi \colon & (TL \oplus J(TL), d\alpha, Tf) \\ & \to (TL \oplus T^*L, \Omega_L, \operatorname{id}_{TL} \oplus (-\operatorname{id}_{T^*L})) \end{aligned}$$

is a symmetric isomorphism of real symplectic vector bundles. Here,

$$\Omega_{L,p}(X + \eta, X' + \eta') = \eta(X') - \eta'(X)$$

for $X, X' \in T_pL$ and $\eta, \eta' \in T_p^*L$.

Proof. Since $Tf|_{TL} = \operatorname{id}_{TL}$, this differential restricts to an involution on $TL \oplus J(TL)$. It follows from the definitions that $(\operatorname{id}_{TL} \oplus (-\operatorname{id}_{T^*L}))^* \Omega_L = -\Omega_L$. Finally, for $X \in TL$ and $Y \in J(TL)$, we have

$$\begin{aligned} (\operatorname{id}_{TL} \oplus \Psi) \circ Tf(X, Y) &= (\operatorname{id}_{TL} \oplus \Psi)(X, -Y) \\ &= (X, -i_Y d\alpha|_{TL}) \end{aligned}$$

and

$$((\operatorname{id}_{TL} \oplus (-\operatorname{id}_{T^*L})) \circ (\operatorname{id}_{TL} \oplus \Psi))(X, Y) = (X, -i_Y d\alpha|_{TL}).$$

□

Combining the previous observations, we can prove a real version of the neighbourhood theorem for isotropic submanifolds.

Theorem 2.4.9. *Let (M_i, ξ_i, f_i), $i = 0, 1$, be real contact manifolds with closed isotropic submanifolds $L_i \subset \operatorname{Fix} f_i$. Suppose there is a symmetric isomorphism of real conformal symplectic normal bundles $\Phi\colon \mathrm{RCSN}_{M_0}(L_0) \to \mathrm{RCSN}_{M_1}(L_1)$ that covers a diffeomorphism $\phi\colon L_0 \to L_1$. Then this diffeomorphism ϕ extends to a symmetric contactomorphism $\psi\colon \mathcal{N}(L_0) \to \mathcal{N}(L_1)$ of suitable f_i-invariant neighbourhoods $\mathcal{N}(L_i)$ of L_i such that the bundle maps $T\psi|_{\mathrm{RCSN}_{M_0}(L_0)}$ and Φ are equivariantly bundle homotopic (as real conformal symplectic bundle isomorphisms).*

Proof. Since Φ is symmetric, we can choose contact forms α_i for ξ_i, $i = 0, 1$, with $f_i^*\alpha_i = -\alpha_i$ such that Φ is an isomorphism

$$(\mathrm{RCSN}_{M_0}(L_0), d\alpha_0, Tf_0) \to (\mathrm{RCSN}_{M_1}(L_1), d\alpha_1, Tf_1)$$

of real symplectic vector bundles. We have an identification of real bundles

$$\begin{aligned}(NL_i, Tf_i) = {} & (\langle R_{\alpha_i}\rangle, Tf_i) \\ & \oplus (J(TL_i), Tf_i) \oplus (\mathrm{RCSN}_{M_i}(L_i), Tf_i)\,.\end{aligned}$$

Define $\Phi_R : (\langle R_{\alpha_0}\rangle, Tf_0) \to (\langle R_{\alpha_1}\rangle, Tf_1)$ by sending $R_{\alpha_0}(p)$ to $R_{\alpha_1}(p)$. This map satisfies $\Phi_R \circ Tf_0 = Tf_1 \circ \Phi_R$. Put

$$\begin{array}{rccc}\Psi_i\colon & (J(TL_i), Tf_i) & \longrightarrow & (T^*L_i, -\mathrm{id}_{T^*L_i}) \\ & Y & \longmapsto & i_Y d\alpha_i|_{TL_i}.\end{array}$$

As in Lemma 2.4.8, this map is symmetric. Furthermore, we have that

$$T\phi \circ (\phi^*)^{-1} \colon (TL_0 \oplus T^*L_0, \Omega_{L_0}, \mathrm{id} \oplus (-\mathrm{id})) \\ \to (TL_1 \oplus T^*L_1, \Omega_{L_1}, \mathrm{id} \oplus (-\mathrm{id}))$$

is a symmetric isomorphism of real symplectic vector bundles. Therefore,

$$T\phi \oplus \left(\Psi_1^{-1} \circ (\phi^*)^{-1} \circ \Psi_0\right) \colon (TL_0 \oplus J_0(TL_0), d\alpha_0, \mathrm{id} \oplus (-\mathrm{id})) \\ \to (TL_1 \oplus J_1(TL_1), d\alpha_1, \mathrm{id} \oplus (-\mathrm{id}))$$

is a symmetric isomorphism of real symplectic vector bundles. Then define a symmetric bundle isomorphism

$$\widetilde{\Phi} \colon (NL_0, Tf_0) \to (NL_1, Tf_1)$$

that covers ϕ by

$$\widetilde{\Phi} := \Phi_R \oplus \left(\Psi_1^{-1} \circ (\phi^*)^{-1} \circ \Psi_0\right) \oplus \Phi.$$

Let $\tau_i \colon NL_i \to M_i$ be tubular maps, constructed by using f_i-invariant Riemannian metrics. Then we have $\tau_i \circ Tf_i = f_i \circ \tau_i$. With these,

$$\tau_1 \circ \widetilde{\Phi} \circ \tau_0^{-1} \colon \mathcal{N}(L_0) \to \mathcal{N}(L_1)$$

is a symmetric diffeomorphism of suitable neighbourhoods $\mathcal{N}(L_i)$ of L_i in M_i that induces the bundle map

$$T\phi \oplus \widetilde{\Phi} \colon TM_0|_{L_0} \to TM_1|_{L_1}.$$

Consider the family of 1-forms

$$\beta_t = (1-t)\alpha_0 + t\left(\tau_1 \circ \widetilde{\Phi} \circ \tau_0^{-1}\right)^* \alpha_1, \quad t \in [0,1].$$

By construction, we have $f_0^*\beta_t = -\beta_t$ for all t. As in the classical proof, β_t is a contact form on M_0 in some neighbourhood $\mathcal{N}(L_0)$ of L_0 in M_0. Using Gray's theorem 2.1.6 and the remark following its proof, there is a symmetric isotopy ψ_t of $\mathcal{N}(L_0)$ fixing L_0 with $\psi_t^*\beta_t = \lambda_t\alpha_0$ for some positive functions λ_t on M. By arguing as in the proof of Darboux's theorem 2.3.1, we can arrange that $\psi_t^*\beta_t = \alpha_0$. Then define a symmetric contactomorphism by $\psi := \tau_1 \circ \widetilde{\Phi} \circ \tau_0^{-1} \circ \psi_1$. $\square$

As a special case, we have:

Corollary 2.4.10. *Let (M_i, ξ_i, f_i) be real contact manifolds, and let $L_i \subset \operatorname{Fix} f_i$ be equivariantly diffeomorphic closed Legendrian submanifolds of M_i. Then L_0 and L_1 have equivariantly contactomorphic neighbourhoods.*

Proof. As in [Gei08, Corollary 2.5.9]. $\square$

Example 2.4.11. Let (M^3, α, f) be a 3-dimensional real contact manifold. A model for the neighbourhood of a Legendrian submanifold $L \subset \operatorname{Fix} f$ (i.e. a component of the fixed point set) is given as follows. Denote the coordinates on $S^1 \times \mathbb{R}^2$ by $(\theta, (x, y))$, and put $f_{\mathrm{st}}(\theta, (x, y)) = (\theta, (-x, -y))$. Then a neighbourhood of L is isomorphic to

$$\left(S^1 \times \mathbb{R}^2, \ker\left(\cos\theta\, dx - \sin\theta\, dy\right), f_{\mathrm{st}}\right).$$

This contactomorphism identifies L with $S^1 \times \{\mathbf{0}\} = \operatorname{Fix} f_{\mathrm{st}}$.

Observation. A real model for components of fixed point sets L in real contact manifolds of arbitrary dimension is given as the contactification of the cotangent bundle T^*L, introduced in Section 1.2: Let (M, ξ, f) be a real contact manifold, and suppose that L is a component of $\operatorname{Fix} f$. Consider the Liouville form λ on T^*L, and denote the coordinate on $\mathbb{R}$ by z. Define an involution f_{st} on $\mathbb{R} \times T^*L$ by $(z, \eta) \mapsto (-z, -\eta)$. Then

$$(\mathbb{R} \times T^*L, dz + \lambda, f_{\mathrm{st}})$$

provides a model for a neighbourhood of L in M. Here, the submanifold L is identified with $\{0\} \times \mathbf{0}_{T^*L}$, where $\mathbf{0}_{T^*L}$ denotes the zero section of T^*L.

This observation allows us to decompose a real contact manifold (M^{2n+1}, α, f) into the real contact manifolds

$$([0,1] \times DT^*L, dz + \lambda, f_{\mathrm{st}}) \cup (N, \beta, g)\,,$$

where N is a $(2n+1)$-dimensional manifold with boundary $\{\pm 1\} \times ST^*L$, and g is a fixed point free involution on N such that g coincides with f_{st} in a neighbourhood of ∂N. This is the real contact version of the *equivariant tubular neighbourhood theorem* for fixed point sets.

Next, we turn our attention to contact submanifolds. Let (M, ξ) be a contact manifold, and consider a submanifold $M' \subset M$, equipped with a contact structure ξ'. Then the pair (M', ξ') is called a **contact submanifold** of (M, ξ) if it satisfies $TM' \cap \xi|_{M'} = \xi'$. Now, suppose that (M, ξ, f) is a real contact manifold, and let (M', ξ') be a contact submanifold of (M, ξ) with M' invariant under f. Then $f' := f|_{M'}$ defines a real contact structure on M'. This can be seen as follows: First, Tf' maps ξ' into itself. Let α be a contact form for ξ such that $f^*\alpha = -\alpha$, and denote by R its Reeb vector field. If R' is the Reeb vector field for any contact form α' defining ξ', we have $R' = \lambda R + Y$ with $Y \in \xi$ and λ a non-vanishing function on M'. Then $Tf'(R') = -(\lambda \circ f')R + \widetilde{Y}$, where $\widetilde{Y} \in \xi$, and $\lambda \circ f$ has the same sign as λ. Therefore, $Tf'(\xi') = -\xi'$. We will call (M', ξ', f') a **real contact submanifold** of (M, ξ, f).

Let (M', ξ', f') be a real contact submanifold of a real contact manifold (M, ξ, f). We have a splitting

$$TM' \oplus \left(\xi'\right)^{\perp} = TM|_{M'},$$

and the differential Tf respects this splitting.

Definition 2.4.12. The bundle $\mathrm{RCSN}_M(M') := (\xi')^{\perp}$, equipped with the conformal symplectic structure induced by $d\alpha$ and the involution Tf, restricted to $(\xi')^{\perp}$, is called the **real conformal symplectic normal bundle** of M' in M.

Theorem 2.4.13. *Let (M_i, ξ_i, f_i), $i = 0, 1$, be real contact manifolds containing the compact real contact submanifolds (M_i', ξ_i', f_i'). Suppose there is an equivariant isomorphism of real conformal symplectic normal bundles*

$$\Phi\colon \mathrm{RCSN}_{M_0}(M_0') \to \mathrm{RCSN}_{M_1}(M_1')$$

that covers an equivariant contactomorphism

$$\phi\colon (M_0', \xi_0', f_0') \to (M_1', \xi_1', f_1').$$

Then ϕ extends to an equivariant contactomorphism ψ of suitable f_i-invariant neighbourhoods $\mathcal{N}(M_i')$ of M_i' such that $T\psi|_{\mathrm{RCSN}_{M_0}(M_0')}$ and Φ are bundle homotopic (as real conformal symplectic bundle isomorphisms).

Proof. We follow the proof in [Gei08, Theorem 2.5.15] closely. As there, the first steps in the proof are accomplished independently on M_0 and M_1, so we will suppress any indices. Pick contact forms α and α' for ξ and ξ', respectively, such that we have $f^*\alpha = -\alpha$ and $(f')^* \alpha' = -\alpha'$. The function $g\colon M' \to \mathbb{R}$, $g(p) = \alpha_p(R'_p)$, is invariant under f'. Therefore, α/g pulls back to minus itself under f as well. By abuse of notation, we will call this new 1-form α, too. Next, we want to find a symmetric function $h\colon M \to \mathbb{R}^+$ with $h|_{M'} \equiv 1$ and $i_{R'} d(h\alpha) = 0$ on $TM|_{M'}$. Pick any such function $\widetilde{h}$, and replace it by

$$h := \frac{1}{2}\left(\widetilde{h} + \widetilde{h} \circ f\right).$$

Then the function h is symmetric and satisfies all of our requirements. The remainder of the proof works just as in [Gei08, Theorem 2.5.15]. □

2.5 Surfaces in real contact manifolds

This section introduces characteristic foliations and dividing sets in the context of real contact structures. First, we describe certain coordinates near hypersurfaces in real contact manifolds.

Lemma 2.5.1. *Let S be a compact orientable hypersurface in a real manifold (M, f), $\dim M = 2n+1$, and suppose that $f(S) = \pm S$. Then there exists a vector field Z transverse to S such that $f_* Z = \pm Z$. Consequently, there are coordinates (p, r) on $S \times \mathbb{R}$ in a neighbourhood of S with*

- *$S \equiv S \times \{0\}$ and*
- *$f(S \times \{r_0\}) = S \times \{\pm r_0\}$ for all $r_0 \in \mathbb{R}$.*

The second condition implies that $f^*dr = \pm dr$. We have $f_*Z = Z$ and $f^*dr = dr$ if and only if $f(S) = S$, n odd or $f(S) = -S$, n even. Otherwise, we have $f_*Z = -Z$ and $f^*dr = -dr$.

Proof. Pick a vector field $\widetilde{Z}$ that trivialises the normal bundle of S in M. In case that $f(S) = S$, n odd or $f(S) = -S$, n even, put

$$Z^+ = \frac{1}{2}\left(\widetilde{Z} + f_*\widetilde{Z}\right).$$

Otherweise, define

$$Z^- = \frac{1}{2}\left(\widetilde{Z} - f_*\widetilde{Z}\right).$$

Since f is a real structure, the new vector field $Z^\pm$ is transverse to S, and $Z^\pm$ is symmetric (respectively, antisymmetric) with respect to f. Denote by φ_t the flow of $Z^\pm$. Let $\varepsilon > 0$ be sufficiently small such that $(p, r) \mapsto \varphi_r(p)$, $p \in S$, $|r| < \varepsilon$, defines an embedding into M. Then these coordinates have the desired properties. □

Now, let S be a compact orientable hypersurface in a real contact manifold $(M^{2n+1}, \xi = \ker\alpha, f)$ with $f(S) = \pm S$. In a neighbourhood $S \times \mathbb{R}$ of S as in the preceding lemma, we can write the contact form α as

$$\alpha = \beta_r + u_r dr.$$

Here, β_r is a family of 1-forms on S, and the u_r are functions $S \to \mathbb{R}$. Since $f^*\alpha = -\alpha$, we have $f^*\beta_r = -\beta_r$ and $u_r \circ f = \pm u_r$ for all $r \in \mathbb{R}$; the minus sign appears precisely for $f(S) = S$, n odd or $f(S) = -S$, n even. The distribution $(TS \cap \xi|_S)^\perp$ defines a singular 1-dimensional foliation of S, called the **characteristic foliation** S_ξ of S. This distribution is preserved by f. Furthermore, we have:

Lemma 2.5.2. *Let Ω be a volume form on S with $f^*\Omega = \Omega$ in case $f(S) = S$, and $f^*\Omega = -\Omega$, otherwise. Then S_ξ is defined by the vector field X satisfying*

$$i_X\Omega = \beta_0 \wedge (d\beta_0)^{n-1}\,. \tag{2.7}$$

The vector field X is symmetric if and only if $f(S) = S$, n even or $f(S) = -S$, n odd. Otherwise, it is antisymmetric.

Proof. The relationship between X and S_ξ is discussed in [Gei08, Lemma 2.5.20]. The transformation behaviour of X under f then follows from counting minus signs on both sides of Equation 2.7. □

Theorem 2.5.3 (Giroux)**.** *Let S_i be invariant closed orientable surfaces in real contact 3-manifolds (M_i, ξ_i, f_i). Suppose that $\phi\colon S_0 \to S_1$ is a symmetric diffeomorphism with $\phi(S_{0,\xi_0}) = S_{1,\xi_1}$ as oriented characteristic foliations. Then there is a symmetric contactomorphism $\psi\colon \mathcal{N}(S_0) \to \mathcal{N}(S_1)$ of suitable invariant neighbourhoods $\mathcal{N}(S_i)$ of S_i with the property $\psi(S_0) = S_1$ such that $\psi|_{S_0} = \phi$.*

Proof. As in Theorems 2.5.22 and 2.5.23 in [Gei08] using Lemma 2.5.1. The vector field produced there is symmetric by the same argument as in Darboux's Theorem 2.3.1. □

Now we turn our attention to surfaces in 3-dimensional manifolds. A surface $S \subset (M, \xi)$ is called **convex** if there exists a contact vector field, defined in a neighbourhood of S and transverse to S.

Lemma 2.5.4. *Suppose that S is an invariant convex surface in a real contact 3-manifold (M, ξ, f). Then there exists a contact vector field X defined near S, transverse to S and $f_*X = \pm X$. We have $f_*X = X$ if and only if f preserves the orientation of S.*

Proof. Let $\widetilde{X}$ be a contact vector field defined near and transverse to S, and define an (anti-)symmetric vector field by

$$X = \frac{1}{2}\left(\widetilde{X} \pm f_*\widetilde{X}\right).$$

Here, we choose the minus sign in case that f reverses the orientation of S. This new vector field is still transverse to S, thanks to the orientation behaviour of $f|_S$ and f. Furthermore, X is a contact vector field by Proposition 2.2.3. □

Let S be an invariant closed convex surface in a real contact manifold (M, ξ, f), and suppose that the corresponding contact vector field X is compatible with f as in the foregoing lemma. The set of points $p \in S$ where X is tangent to ξ is called the **dividing set** Γ_S of S. The dividing set Γ_S is a 1-dimensional submanifold of S, i.e. a finite collection of circles.

Proposition 2.5.5. *The dividing set Γ_S is invariant under the real structure f. Furthermore, unless it is empty, the intersection $\Gamma_S \cap \operatorname{Fix} f$ consists of discrete points in case* *that f preserves the orientation of S, and is a collection of circles if f reverses the orientation of S.*

Proof. Let $p \in S$ be such that $X(p) \in \xi_p$. Then we have

$$X(f(p)) = \pm (f_* X)(f(p)) = \pm T_p f \cdot X(p) \in \xi_p,$$

which implies the first assertion. Denote by φ_t the flow of the contact vector field X as above. Let $f|_S$ be orientation-preserving. By Corollary 2.1.5, φ_t preserves the fixed point set of f. Since X is transverse to S, so is the fixed point set, and since the fixed point set is 1-dimensional, the claim follows. Now suppose that $f|_S$ is orientation-reversing. Put

$$S_\varepsilon^\pm = \left\{ \varphi_{\pm\varepsilon(p)} \mid p \in S \right\}.$$

For $\varepsilon > 0$ sufficiently small, we have $S_\varepsilon^+ \cap S_\varepsilon^- = \emptyset$. Since X is antisymmetric, we have $\varphi_t \circ f = f \circ \varphi_{-t}$, and thus $f(S_\varepsilon^\pm) = S_\varepsilon^\mp$. Thus, it follows that $\operatorname{Fix} f \cap S_\varepsilon^\pm = \emptyset$. $\square$

2.6 The isotopy extension theorem

This section features a real version of the isotopy extension theorem. As a consequence, we show that the group of symmetric contactomorphisms acts transitively on the fixed point set of a real contact manifold (M, ξ, f).

Theorem 2.6.1 (Real isotopy extension theorem)**.** *Suppose $j_t \colon L \to (M, \xi = \ker \alpha, f)$, $t \in [0,1]$, is an isotopy of isotropic embeddings of a closed manifold L in a real contact manifold (M, ξ, f) such that $j_t(L) \subset \operatorname{Fix} f$ for all $t \in [0,1]$. Then there is a compactly supported symmetric contact isotopy ψ_t of (M, ξ, f) satisfying $\psi_t \circ j_0 = j_t$.*

Proof. First, define a time-dependent vector field X_t along $j_t(L)$ by

$$X_t \circ j_t = \frac{d}{dt} j_t.$$

Since $f \circ j_t = j_t$, we have $f_* X_t = X_t$, i.e. X_t is symmetric. Observe that, from the assumption, we have $X_t \in \ker \alpha$. Our task is now to extend this vector field symmetrically over all of M. By the discussion in Section 2.2, such a vector field corresponds to a (time-dependent) antisymmetric function $\widetilde{H}$ on M. We will copy the construction from [Gei08, Theorem 2.6.2] and take care that the objects involved are compatible with the given real structure.

Set $\hat{M} = M \times [0,1]$, and put

$$\hat{L} = \bigcup_{q \in L, t \in [0,1]} (j_t(q), t).$$

The real structure f extends over $\hat{M}$ by $\hat{f}(p,t) := (f(p), t)$. We have $\hat{f}|_{\hat{L}} = \mathrm{id}_{\hat{L}}$. Let g be an f-invariant Riemannian metric on $\hat{M}$ such that R_α is orthogonal to $\ker \alpha$. Let $\tau \colon N\hat{L} \to \hat{M}$ be a tubular map, constructed using the Riemannian metric g. Then τ satisfies $\tau \circ T\hat{f} = \hat{f} \circ \tau$, or, equivalently, $T\hat{f} \circ \tau^{-1} = \tau^{-1} \circ \hat{f}$. Define a function $H' \colon N\hat{L} \to \mathbb{R}$ as follows. Here, (p,t) denotes a point in $\hat{L} \subset N\hat{L}$.

- $H'(p,t) = \alpha(X_t(p)) = 0$,
- $dH'_{(p,t)}(R_\alpha) = 0$,
- $dH'_{(p,t)}(\mathbf{v}) = -d\alpha(X_t, \mathbf{v}), \mathbf{v} \in \ker \alpha_p \subset T_pM \subset T_{(p,t)}\hat{M}$,
- H' is linear on the fibres of $N\hat{L} \to \hat{L}$.

Replace H' by the function

$$\hat{H} = \frac{1}{2}\left(H' - H' \circ T\hat{f}\right).$$

Then $\hat{H}$ is antisymmetric and satisfies the four conditions as well. Let $\chi' \colon \hat{M} \to [0,1]$ be a function with $\chi' \equiv 0$ outside a small invariant neighbourhood $\mathcal{N}_0 \subset \tau(N\hat{L})$ of $\hat{L}$ and $\chi' \equiv 1$ in a smaller invariant neighbourhood $\mathcal{N}_1 \subset \mathcal{N}_0$ of $\hat{L}$. Replace χ' by

$$\chi := \frac{1}{2}\left(\chi' + \chi' \circ f\right).$$

Then χ is symmetric and has the same properties as χ'. For $(p,t) \in \hat{M}$, set

$$\widetilde{H}_t(p) = \begin{cases} \chi(p,t)\hat{H}\left(\tau^{-1}(p,t)\right) & \text{for } (p,t) \in \tau(N\hat{L}), \\ 0 & \text{for } (p,t) \notin \tau(N\hat{L}). \end{cases}$$

We have

$$\begin{aligned} \widetilde{H}_t\left(f(p)\right) &= \chi(f(p),t)\hat{H}\left(\tau^{-1}(f(p),t)\right) \\ &= \chi(p,t)\hat{H}\left(\tau^{-1} \circ \hat{f}(p,t)\right) \\ &= \chi(p,t)\hat{H}\left(T\hat{f} \circ \tau^{-1}(p,t)\right) \\ &= -\chi(p,t)\hat{H}\left(\tau^{-1}(p,t)\right) \\ &= -\widetilde{H}_t(p). \end{aligned}$$

Therefore, the Hamiltonian flow ψ_t of $\widetilde{H}_t$ is symmetric. □

A particular instance of this theorem is the case where L is a single point:

Corollary 2.6.2. *Let (M, ξ, f) be a real contact manifold, and let $\gamma\colon [0,1] \to M$ be a path with $\gamma([0,1]) \subset \operatorname{Fix} f$ connecting two points $p = \gamma(0)$ and $q = \gamma(1)$. Then there is a compactly supported symmetric contact isotopy $(\psi_t)_{t\in[0,1]}$ with $\psi(p) = q$. This implies that the group of symmetric contactomorphisms of (M, ξ, f) acts transitively on the fixed point set M.* □

For a family of real contact structures, one has the following proposition.

Proposition 2.6.3. *Let ξ_t, $t \in [0,1]$, be a smooth family of contact structures on a closed real manifold (M, f) such that $f_*\xi_t = -\xi_t$ for all t. Suppose that we are given a smooth path $\gamma\colon [0,1] \to \operatorname{Fix} f \subset M$ in the fixed point set of f. Then there is a symmetric isotopy ψ_t of (M, f) with $T\psi_t(\xi_0) = \xi_t$ and $\psi_t(\gamma(0)) = \gamma(t)$ for all $t \in [0,1]$.*

Proof. Replace the reference to Gray's theorem in [Gei08, Proposition 2.6.4] by its real counterpart, Theorem 2.1.6. □

2.7 The contact disc theorem

In the present section, we will prove a real contact version of the disc theorem, stating that all real contact embeddings of the standard disc into a real contact manifold are isotopic. The methods of the proof in the classical case can be used to deduce a more general classification result about involutions for the standard contact structure ξ_{st} on $\mathbb{R}^{2n+1}$. First, we turn our attention to that classification. Put

$$\mathcal{RCI}(2n+1) = \big\{f \in \mathrm{Diff}(\mathbb{R}^{2n+1}) \mid f^2 = \mathrm{id}_{\mathbb{R}^{2n+1}},\\ f^*\alpha_{\mathrm{st}} = -\alpha_{\mathrm{st}}, f(\mathbf{0}) = \mathbf{0}\big\}\,.$$

The condition on $f(\mathbf{0})$ is included merely for technical reasons. As a subspace of $\mathrm{Diff}(\mathbb{R}^{2n+1})$, $\mathcal{RCI}(2n+1)$ inherits its topology. The homotopy type of $\mathcal{RCI}(2n+1)$ computes as follows.

Theorem 2.7.1. *$\mathcal{RCI}(2n+1) \simeq U(n)/O(n)$.*

As a consequence of Theorem 2.7.1, we have:

Corollary 2.7.2. *Every involution f of $\mathbb{R}^{2n+1}$ with the property $f^*\alpha_{\mathrm{st}} = -\alpha_{\mathrm{st}}$ is isotopic to f_{st} through such involutions.*

This corollary follows from the observation that every involution of $\mathbb{R}^d$ has a fixed point.

Proposition 2.7.3. *Let f be an involution of $\mathbb{R}^d$. Then f has at least one fixed point.*

Proof. Assume that an involution f on $\mathbb{R}^d$ is fixed point free. Then the quotient space $M = \mathbb{R}^d/f$ is a d-dimensional manifold, and the quotient map $\mathbb{R}^d \to M$ is a 2-fold covering. Thus $\pi_k M = 0$ for all k except $k = 1$ where we have that $\pi_1 M = \mathbb{Z}_2$. This implies that M is an Eilenberg–MacLane space $K(\mathbb{Z}_2, 1)$. A CW model for this space is $\mathbb{R}P^\infty$, so we have $M \simeq \mathbb{R}P^\infty$. But $\mathbb{R}P^\infty$ has non-trivial homology in infinitely many dimensions, contradicting the assumption that M is a d-dimensional manifold.[1] □

[1] Proof taken from `https://mathoverflow.net/questions/18192/`.

Alternatively, as a proper map, such an f would extend to an involution $\overline{f}$ of S^{2n+1} with a single fixed point. But for involutions on closed manifolds, the congruence

$$\chi(\mathrm{Fix}\,\overline{f}) \equiv \chi(S^{2n+1}) \mod 2$$

holds (see [Bre72, Theorem III.4.3]); contradiction.

According to [Bre97, Problems VI.7.3 and VI.7.4], even more is true: Any continuous map $\mathbb{R}^d \to \mathbb{R}^d$ of prime period has at least one fixed point.

Proof of Corollary 2.7.2. Let f be as in the corollary, and let $(\mathbf{x}_0, \mathbf{y}_0, z_0)$ be a fixed point of f. Define a family of strict contactomorphisms of $\mathbb{R}^{2n+1}_{\mathrm{st}}$ by

$$\varphi_t(\mathbf{x}, \mathbf{y}, z) = (\mathbf{x} - t\mathbf{x}_0, \mathbf{y} - t\mathbf{y}_0, z - tz_0 + t\mathbf{x}_0(\mathbf{y} - \mathbf{y}_0))\,,$$

$t \in [0,1]$. Then $\varphi_0 = \mathrm{id}_{\mathbb{R}^{2n+1}}$ and $\varphi_1(\mathbf{x}_0, \mathbf{y}_0, z_0) = \mathbf{0}$. Put

$$f_t = \varphi_t^{-1} \circ f \circ \varphi_t.$$

Then f_t is an isotopy through real involutions for $\mathbb{R}^{2n+1}_{\mathrm{st}}$ with $f_0 = f$ and $f_1(\mathbf{0}) = \mathbf{0}$, i.e. $f_1 \in \mathcal{RCI}(2n+1)$. Since $U(n)$ is connected, so is its quotient $U(n)/O(n)$. From that we conclude that f is isotopic to f_{st}. $\square$

Now, we return to Theorem 2.7.1. Its proof will be divided into two parts – first a linearisation, and then a discussion of the linear problem. The inspiration for the first step is drawn from the Alexander trick, known from the proof of the disc theorem in differential topology: Let f be a diffeomorphism of $\mathbb{R}^d$ with $f(\mathbf{0}) = \mathbf{0}$. Define an isotopy of f by

$$f_t(p) = \begin{cases} f(tp)/t & \text{for } t > 0, \\ T_{\mathbf{0}} f(p) & \text{for } t = 0. \end{cases}$$

By a lemma of Morse, see [Kos93, Lemma A.2.1], this indeed defines a smooth family of diffeomorphisms connecting f with its differential in $\mathbf{0}$. (In particular, this argument shows that the space of diffeomorphisms of $\mathbb{R}^d$ fixing the origin is homotopy equivalent to $GL_d(\mathbb{R})$, and thus to $O(d)$.) With $\varepsilon_t(p) = tp$ for $t > 0$, we have $f_t = \varepsilon_t^{-1} \circ f \circ \varepsilon_t$. By this description we see that if f is not only a diffeomorphism but an involution of $\mathbb{R}^d$, then f_t defines an isotopy through involutions to the involution's differential in the origin. Therefore, we have:

Observation. The space of involutions of $\mathbb{R}^d$ fixing the origin is homotopy equivalent to the space of linear involutions on $\mathbb{R}^d$.

As a by-product of the second step in our argument, we will compute the homotopy groups of the space of linear involutions on $\mathbb{R}^d$.

Unfortunately, since $\varepsilon_t^* \alpha_{\mathrm{st}} \neq \alpha_{\mathrm{st}}$, the Alexander trick does not carry over directly to anti-contact involutions. Even worse: In general, the differential of an anti-contactomorphism is not an anti-contactomorphism of $\mathbb{R}^{2n+1}_{\mathrm{st}}$. In view of that, a better choice seems to be the family of diffeomorphisms δ_t given by

$$\delta_t(\mathbf{x}, \mathbf{y}, z) = (t\mathbf{x}, t\mathbf{y}, t^2 z),$$

for $t > 0$. Here, we have $\delta_t^* \alpha_{\mathrm{st}} = t^2 \alpha_{\mathrm{st}}$. Then for a given involution $f \in \mathcal{RCI}(2n+1)$, put $f_t = \delta_t^{-1} \circ f \circ \delta_t$, $t > 0$. This defines a smooth family of involutions on $\mathbb{R}^{2n+1}$ such that $f_t^* \alpha_{\mathrm{st}} = -\alpha_{\mathrm{st}}$. Of course, one has to wonder what the limit of f_t for $t \to 0$ might be – if it exists at all. Our next task will be to determine this limit.

We proceed as in the classical case in [Gei08, Section 2.6]. For an involution $f \in \mathcal{RCI}(2n+1)$, write

$$f(\mathbf{x}, \mathbf{y}, z) = (\mathbf{X}(\mathbf{x}, \mathbf{y}, z), \mathbf{Y}(\mathbf{x}, \mathbf{y}, z), Z(\mathbf{x}, \mathbf{y}, z))$$

and define $n \times n$ matrices by

$$\begin{aligned} A &= \left(\frac{\partial \mathbf{X}}{\partial \mathbf{x}}(\mathbf{0}) \right), \\ B &= \left(\frac{\partial \mathbf{X}}{\partial \mathbf{y}}(\mathbf{0}) \right), \\ C &= \left(\frac{\partial \mathbf{Y}}{\partial \mathbf{x}}(\mathbf{0}) \right), \\ D &= \left(\frac{\partial \mathbf{Y}}{\partial \mathbf{y}}(\mathbf{0}) \right). \end{aligned}$$

The contact condition then translates into

$$\begin{cases} A^T C &= C^T A, \\ B^T D &= D^T B, \\ A^T D - C^T B &= -E; \end{cases} \tag{2.8}$$

the assumption $f^2 = \mathrm{id}_{\mathbb{R}^{2n+1}}$ yields

$$\begin{cases} A^2 + BC &= E, \\ AB + BD &= 0, \\ CA + DC &= 0, \\ CB + D^2 &= E. \end{cases} \tag{2.9}$$

These equations are equivalent to requiring that the linear map

$$M_f \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}$$

is both an involution of $\mathbb{R}^{2n}$ and antisymplectic with respect to the standard symplectic form on $\mathbb{R}^{2n}$, given by

$$\Omega \left(\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}, \begin{pmatrix} \mathbf{x}' \\ \mathbf{y}' \end{pmatrix} \right) = \mathbf{x}\mathbf{y}' - \mathbf{y}\mathbf{x}'.$$

We will denote the space of linear antisymplectic involutions on $\mathbb{R}^{2n}$ by $\mathcal{LRSI}(2n)$. A map from $\mathcal{LRSI}(2n)$ into $\mathcal{RCI}(2n+1)$ is given as follows:

Lemma 2.7.4. *Let*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be an element of $\mathcal{LRSI}(2n)$. *Then the map defined via*

$$\begin{aligned} \mathbf{X} &= A\mathbf{x} + B\mathbf{y} \\ \mathbf{Y} &= C\mathbf{x} + D\mathbf{y} \\ Z &= -z - \tfrac{1}{2}\langle A\mathbf{x}, C\mathbf{x} \rangle - \langle C\mathbf{x}, B\mathbf{y} \rangle - \tfrac{1}{2}\langle B\mathbf{y}, D\mathbf{y} \rangle \end{aligned}$$

is an element of $\mathcal{RCI}(2n+1)$.

Proof. The proof in [Gei08, Lemma 2.6.9] shows that the map in question is a diffeomorphism that pulls back α_{st} to minus itself. It remains to check that this diffeomorphism squares to the identity. By [MS98, Exercise 1.13], we know that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix},$$

so conditions 2.8 and 2.9 imply that

$$A = D^T, \quad B = -B^T, \quad C = -C^T \quad \text{and} \quad D = A^T. \tag{2.10}$$

(In fact, any two of the three sets of equations 2.8, 2.9 and 2.10 imply the third.) Using 2.10, a direct computation shows the claim. □

For an involution $f \in \mathcal{RCI}(2n+1)$, $(\mathbf{x}, \mathbf{y}, z) \mapsto (\mathbf{X}, \mathbf{Y}, Z)$, denote by $S_{\mathbf{0}}f$ the element of $\mathcal{RCI}(2n+1)$ from the preceeding lemma. Then $S_{\mathbf{0}}f$ is precisely the limit of the isotopy f_t we were looking for:

Proposition 2.7.5. *Let $f \in \mathcal{RCI}(2n+1)$. Then*

$$f_t(\mathbf{x}, \mathbf{y}, z) = \begin{cases} \delta_t^{-1} \circ f \circ \delta_t(\mathbf{x}, \mathbf{y}, z) & \textit{for } t \in (0,1]\,, \\ S_{\mathbf{0}}f(\mathbf{x}, \mathbf{y}, z) & \textit{for } t = 0 \end{cases}$$

defines an isotopy in $\mathcal{RCI}(2n+1)$ from f to $S_{\mathbf{0}}f$. Consequently, we have a homotopy equivalence

$$\mathcal{RCI}(2n+1) \simeq \mathcal{LRSI}(2n)$$

given by the assignment $f \mapsto M_f$.

Proof. There is nothing more to prove than what is already done in [Gei08, Proposition 2.6.10]. □

Remark. Observe that the isotopy in Proposition 2.7.5 fixes involutions of the form $S_{\mathbf{0}}f$. So in fact, $\mathcal{LRSI}(2n)$ is a strong deformation retract of $\mathcal{RCI}(2n+1)$.

This concludes the linearisation part. We will now continue by investigating the space $\mathcal{LRSI}(2n)$. As a start, we will compute the homotopy type of

$$\mathcal{LI}_k(d) = \left\{ A \in GL_d(\mathbb{R}) \mid A^2 = E \text{ and } \dim \operatorname{Fix} A = k \right\}.$$

This will guide us in how to deal with the antisymplectic case. Note that an element in $\mathcal{LI}_k(d)$ is nothing else but a splitting $\mathbb{R}^d = U \oplus V$ with $\dim U = k$: Simply define a linear map on $\mathbb{R}^d$ by setting it to 1 on U and to -1 on V. For a matrix $A \in \mathcal{LI}_k(d)$, we have a splitting $\mathbb{R}^d = E_1(A) \oplus E_{-1}(A)$, where $E_{\pm 1}(A)$ are the eigenspaces of A. By assumption, $E_1(A)$ is a k-dimensional subspace of $\mathbb{R}^d$. Fix an inner product on $\mathbb{R}^d$, and let $C(A)$ denote the orthogonal complement of $E_1(A)$ in $\mathbb{R}^d$. Now any complement C to $E_1(A)$ in $\mathbb{R}^d$ is a graph over $C(A)$. That is, if $\mathrm{pr}^{C(A)}$ denotes the orthogonal projection onto $C(A)$, then $\mathrm{pr}^{C(A)}|_C$ is bijective.

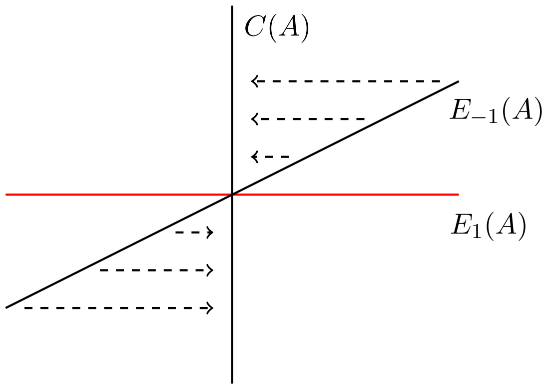

Figure 2.1: The homotopy type of $\mathcal{LI}_k(d)$.

Define a homotopy of $\mathcal{LI}_k(d)$ by

$$\begin{array}{rcll} h\colon & \mathcal{LI}_k(d) \times [0,1] & \to & \mathcal{LI}_k(d) \\ & (A,t) & \mapsto & h_t(A), \end{array}$$

where $h_t(A)$ is the map

$$\begin{array}{rcl} \mathbb{R}^d = E_1(A) \oplus E_{-1}(A) & \to & \mathbb{R}^d \\ (\mathbf{x}, \mathbf{y}) & \mapsto & \big(\mathbf{x}, (1-t)\mathbf{y} + t\,\mathrm{pr}^{C(A)}(\mathbf{y})\big)\,. \end{array}$$

Then we have $h_0 = \mathrm{id}_{\mathcal{LI}_k(d)}$, and

$$h_1(\mathcal{LI}_k(d)) = \{A \in \mathcal{LI}_k(d) \mid E_{-1}(A) \perp E_1(A)\}\,.$$

This subspace can be identified with the Grassmannian manifold $\mathcal{G}_k(d)$ of k-planes in $\mathbb{R}^d$ by mapping A to $E_1(A)$. Furthermore, notice that the homotopy h_t fixes $\mathcal{G}_k(d)$ pointwise. Therefore, $\mathcal{G}_k(d)$ is a strong deformation retract of $\mathcal{LI}_k(d)$, and we have shown:

Proposition 2.7.6. *$\mathcal{LI}_k(d) \simeq \mathcal{G}_k(d)$.* □

A similar reasoning applies to the antisymplectic case. Let $A \in \mathcal{LRSI}(2n)$, and write $\mathbb{R}^{2n} = E_1(A) \oplus E_{-1}(A)$ as before. Recall that, by the proof of Lemma 1.3.4, both $E_{\pm 1}$ are Lagrangian subspaces of $(\mathbb{R}^{2n}, \Omega)$. Instead of working with an inner product, the canonical complement to $E_1(A)$ will now be defined as

$$C(A) = J_0\left(E_1(A)\right), \quad \text{where} \quad J_0 = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix} \in GL_{2n}(\mathbb{R}).$$

One checks that if $U \subset (\mathbb{R}^{2n}, \Omega)$ is a Lagrangian subspace, then so is $J_0(U)$. Again, any complement of $E_1(A)$ in $\mathbb{R}^{2n}$ is a graph over $J_0\,(E_1(A))$. By [MS98, Lemma 2.30], the space of Lagrangian graphs over a given subspace can be identified with the space of symmetric matrices. This space is convex, and therefore the homotopy h_t defined earlier (and with the new complement $C(A)$ understood) restricts to a homotopy $\mathcal{LRSI}(2n) \times [0,1] \to \mathcal{LRSI}(2n)$. Thus, we have

$$\begin{aligned} \mathcal{LRSI}(2n) &\simeq h_1(\mathcal{LRSI}(2n)) \\ &\qquad \cong \mathcal{L}(n) := \left\{U \subset (\mathbb{R}^{2n}, \Omega) \text{ Lagrangian}\right\}. \end{aligned}$$

According to [MS98, Lemma 2.31], there is a homeomorphism $\mathcal{L}(n) \to U(n)/O(n)$. This completes the proof of Theorem 2.7.1.

Remark. In fact, even more is true: The space $\mathcal{LRSI}(2n)$ is diffeomorphic to $T\mathcal{L}(n)$, the tangent bundle of $\mathcal{L}(n)$. This is proved in [AF12].

Remark. Suppose f is an involution of $\mathbb{R}^{2n+1}$ with the property $f_*\xi_{\mathrm{st}} = -\xi_{\mathrm{st}}$, i.e. $f^*\alpha_{\mathrm{st}} = -\lambda\alpha_{\mathrm{st}}$ for some positive function λ on $\mathbb{R}^{2n+1}$. Then the same reasoning as before can be applied to f, and the linearisation produces a matrix

$$M_f = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with $M_f^2 = E$ and $M_f^*\Omega = -\lambda(\mathbf{0})\Omega$. But since M_f is an involution, we conclude that $\lambda(\mathbf{0})^2 = 1$, and therefore we find $\lambda(\mathbf{0}) = -1$. Thus the map

$$\begin{aligned} \big\{f \in \mathrm{Diff}(\mathbb{R}^{2n+1}) \mid f^2 = \mathrm{id}_{\mathbb{R}^{2n+1}}, f_*\xi_{\mathrm{st}} = -\xi_{\mathrm{st}}, \\ f(\mathbf{0}) = \mathbf{0}\} \to \mathcal{LRSI}(2n), \end{aligned}$$

given by $f \mapsto M_f$ as in the discussion above, is a homotopy equivalence, too. So in fact Theorem 2.7.1 is a statement about the standard contact structure ξ_{st} rather than the standard contact form α_{st}.

In dimension 3, there is a stronger version of Theorem 2.7.1. Let ξ be any contact structure on $\mathbb{R}^{2n+1}$ and set

$$\begin{aligned}\mathcal{RCI}(\xi) = \big\{ f \in \mathrm{Diff}(\mathbb{R}^{2n+1}) \mid f^2 = \mathrm{id}_{\mathbb{R}^{2n+1}}, \\ f_*\xi = -\xi, f(\mathbf{0}) = \mathbf{0} \big\}\,.\end{aligned}$$

In what follows, we refer to a contact structure as *tight* if it is tight in the usual sense, with no additional assumptions from a given involution. For any positive tight contact structure ξ on $\mathbb{R}^3$, we have:

Theorem 2.7.7. *$\mathcal{RCI}(\xi) \simeq U(1)/O(1) \cong S^1$.*

Proof. Let ξ be any tight contact structure on $\mathbb{R}^3$. According to Eliashberg's theorem [Gei08, Theorem 4.10.1], there exists a unique positive tight contact structure on $\mathbb{R}^3$ up to isotopy. In particular, there is a diffeomorphism $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$ with $\varphi_*\xi = \xi_{\mathrm{st}}$. By the proof of Corollary 2.7.2, we may assume that $\varphi(\mathbf{0}) = \mathbf{0}$. Then the map

$$\begin{array}{ccc} \mathcal{RCI}(\xi_{\mathrm{st}}) & \to & \mathcal{RCI}(\xi) \\ f & \mapsto & \varphi^{-1} \circ f \circ \varphi \end{array}$$

is a homeomorphism, and therefore Theorem 2.7.7 follows from Theorem 2.7.1. $\square$

Remark. With exactly the same argument as in the preceding proof, one has

$$\mathcal{RCI}(\xi) \simeq U(n)/O(n)$$

for any contact structure ξ on $\mathbb{R}^{2n+1}$ contactomorphic to ξ_{st}.

So far, we only allowed manipulations of the involution, keeping the contact structure fixed. The next theorem involves deformations of real contact structures. A **deformation** of a real contact structure (ξ, f) on a manifold M^{2n+1} is a (smooth) family (ξ_t, f_t) of contact structures ξ_t and involutions f_t on M, $t \in [0,1]$, with $(\xi_0, f_0) = (\xi, f)$ and $(f_t)_* \xi_t = -\xi_t$ for all t.

Theorem 2.7.8. *Let (ξ, f) be any positive tight real contact structure on $\mathbb{R}^3$. Then (ξ, f) can be deformed into $(\xi_{\mathrm{st}}, f_{\mathrm{st}})$.*

Proof. Let φ be a diffeomorphism of $\mathbb{R}^3$ with $\varphi_*\xi = \xi_{\mathrm{st}}$. Choose an isotopy φ_t from $\varphi_0 = \mathrm{id}_{\mathbb{R}^3}$ to $\varphi_1 = \varphi$ and put $\xi_t = (\varphi_t)_* \xi$. Set $f_t = \varphi_t^{-1} \circ f \circ \varphi_t$. Then $(f_t)_* \xi_t = -\xi_t$, i.e. $(\mathbb{R}^3, \xi_t, f_t)$ is a real contact manifold for all t. This defines a deformation from $(\xi_0, f_0) = (\xi, f)$ into $(\xi_1, f_1) = (\xi_{\mathrm{st}}, f_1)$. Now use Theorem 2.7.1 – or rather the remark following its proof – to deform (ξ_{st}, f_1) into $(\xi_{\mathrm{st}}, f_{\mathrm{st}})$. $\square$

Theorem 2.7.1 does not allow any conclusions about the topology of the fixed point set of an involution. This question is addressed in *Smith theory*. Since an involution f of $\mathbb{R}^d$ is a proper map, it extends to an involution $\overline{f}$ of S^d, the one-point compactification of $\mathbb{R}^d$. By a theorem due to Smith, see [Bre72, Section III.5], the fixed point set of $\overline{f}$

is a mod 2 homology k-sphere for some $0 \leq k \leq d$. For $f \in \mathcal{RCI}(2n+1)$, the fixed point set of the extension $\overline{f}$ is a mod 2 homology n-sphere. Therefore, $\operatorname{Fix} f$ is a mod 2 homology n-sphere minus a point.

For $n = 1$, i.e. in $\mathbb{R}^3_{\text{st}}$, a little bit more can be said. Since the only mod 2 homology 1-sphere is S^1, the fixed point set of f is diffeomorphic to the real line $\mathbb{R}$. Following his results, Smith conjectured that the fixed point set of a non-trivial orientation-preserving involution on S^3 – a diffeomorphic copy of S^1 – is unknotted. This conjecture is now known to be true thanks to [Wal69]. Combining this result with Theorem 2.7.1, we have:

Theorem 2.7.9. *Let* $(\mathbb{R}^3, \xi_{\text{st}}, f)$ *be a real contact manifold. Then* f *is isotopic to* f_{st} *through real structures for* $\mathbb{R}^3_{\text{st}}$*, and* $\operatorname{Fix} f$ *is an unknotted copy of* $\mathbb{R}$. □

Thanks to Example 1.1.8 (2) and the homeomorphism in the proof of Theorem 2.7.7, there is a one-to-one correspondence between $\mathcal{RCI}(2n+1)$ and

$$\begin{aligned}\big\{f \in \operatorname{Diff}(S^{2n+1}) \mid f^2 = \mathrm{id}_{S^{2n+1}},\\ f_*\left(\mathbf{x}d\mathbf{y} - \mathbf{y}d\mathbf{x}\right) = -\left(\mathbf{x}d\mathbf{y} - \mathbf{y}d\mathbf{x}\right), f(N) = N\big\},\end{aligned}$$

where N is any point on the sphere S^{2n+1}. Thus, Theorem 2.7.1 yields a classification for real structures on the standard sphere with non-empty fixed point set. To that end, write $\xi_{\text{st}} = \ker(\mathbf{x}\, d\mathbf{y} - \mathbf{y}\, d\mathbf{x})$ for the standard contact structure on S^3, and denote the standard real structure by $f_{\text{st}}(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, -\mathbf{y})$.

Theorem 2.7.10. *Let* $(S^3, \xi_{\text{st}}, f)$ *be a real contact manifold with non-empty fixed point set. Then* f *is isotopic to* f_{st}

through real structures for ξ_{st}, and $\operatorname{Fix} f$ *is an unknotted copy of S^1.* □

The case of fixed point free involutions on S^{2n+1} cannot be attacked with this approach, however.

By studying real contact structures on solid tori, the following classification result for real contact structures on the 3-ball was obtained by Öztürk and Salepci:

Theorem 2.7.11 ([ÖS11])**.** *Up to isotopy through real contact structures for which ∂D^3 is convex, there is a unique tight real contact structure on D^3.*

First, the authors reduce the problem to the case where the involution on D^3 is given as the standard one from Example 1.1.6 (1). Thanks to Theorem 2.4.9, a neighbourhood of the fixed point set $\{x = z = 0\}$ is trivial. The complement of this neighbourhood is then investigated using a partial classification of real contact structures on solid tori. Certain real contact structures on solid tori will be featured in the next chapter in the context of Dehn surgery.

To conclude this section, we formulate a real version of the disc theorem, stating that all symmetric contact embeddings into a real contact manifold are isotopic, given they intersect the same component of the fixed point set. Its proof follows easily from the discussion above.

Denote by B_{st} the unit ball in $\mathbb{R}^{2n+1}_{\mathrm{st}}$.

Theorem 2.7.12 (Real contact disc theorem)**.** *Let*

$$j_t \colon B_{\mathrm{st}} \to (M, \xi, f), t = 0, 1,$$

be symmetric contact embeddings into a connected real contact manifold (M, ξ, f) *such that* $j_0(\mathbf{0})$ *and* $j_1(\mathbf{0})$ *lie in the same component of* $\operatorname{Fix} f$. *Further assume that* $Tj_i(\xi_{\mathrm{st}}) = \xi$. *Then* j_0 *and* j_1 *are isotopic as symmetric contact embeddings.*

Proof. By Corollary 2.6.2, we may assume that, after an isotopy, $j_0(\mathbf{0}) = j_1(\mathbf{0})$. Precomposing j_i with $\delta_{1-t+t\varepsilon}$, $i = 0, 1$, $t \in [0, 1]$, $\varepsilon > 0$ sufficiently small, and δ as in the preceding discussion, defines an isotopy to symmetric contact embeddings $(B_{\mathrm{st}}, \mathbf{0}) \to (\mathbb{R}^{2n+1}_{\mathrm{st}}, \mathbf{0})$. Such an embedding j is isotopic to $S_{\mathbf{0}} j$ through symmetric contact embeddings by the contact Alexander trick. Then use that the space $\mathcal{LRSI}(2n)$ is connected. $\square$

3 Constructions of real contact manifolds

Surgery provides a method of producing manifolds by attaching handles. In the contact setting, surgery may be performed along isotropic spheres; this construction can be adapted to real contact manifolds, given that the attaching sphere is contained in the fixed point set. In order to introduce real contact surgery, we provide a technical framework, namely real symplectic fillings. These will be illustrated by a class of examples, the so-called Brieskorn manifolds. Following that, there is a discussion of real contact Dehn surgery, providing yet another device of constructing real contact 3-manifolds. Open books, on the other hand, constitute a way to decompose manifolds into lower-dimensional pieces. In classical theory, open books can be used to construct contact structures on all closed (oriented) 3-manifolds. We will discuss a real version of open books and show how to build real contact 3-manifolds out of real surfaces.

3.1 Brieskorn manifolds, fillings and surgery

This section is devoted to the study of symplectic fillings in the context of real manifolds, leading to a real version of contact surgery. A particular instance of fillable contact manifolds are links of algebraic varieties, known in the literature as Brieskorn manifolds. We begin by endowing these manifolds with real contact structures.

Definition 3.1.1. For $\mathbf{a} = (a_0, \ldots, a_n)$ an $(n+1)$-tupel of integers $a_k > 1$, put

$$V(\mathbf{a}) = \left\{ \mathbf{z} \in \mathbb{C}^{n+1} \mid z_0^{a_0} + \ldots + z_n^{a_n} = 0 \right\},$$

that is, $V(\mathbf{a})$ is the zero set of the polynomial $z_0^{a_0} + \ldots + z_n^{a_n}$. The **Brieskorn manifold** $M(\mathbf{a})$ is defined as the intersection of $V(\mathbf{a})$ with the sphere S_2^{2n+1} of radius 2 around the origin in $\mathbb{C}^{n+1}$:

$$M(\mathbf{a}) := V(\mathbf{a}) \cap S_2^{2n+1}$$

This naming is justified by [Gei08, Lemma 7.1.1], where it is shown that $M(\mathbf{a})$ is a $(2n-1)$-dimensional manifold. For example, the Brieskorn manifold $M(d, 2, 2)$ is diffeomorphic to the lens space $L(d, 1)$, see, for example, the book [HM68]. A contact form on $M(\mathbf{a})$ is given by

$$\alpha = \frac{\mathrm{i}}{4} \sum_{k=0}^{n} \left(z_k \, d\overline{z_k} - \overline{z_k} \, dz_k \right),$$

see [Gei08, Theorem 7.1.2]. Alternatively, in real coordinates $\mathbf{z} = \mathbf{x} + \mathrm{i}\mathbf{y}$, we have

$$\alpha = \frac{1}{2} \left(\mathbf{x} \, d\mathbf{y} - \mathbf{y} \, d\mathbf{x} \right).$$

An obvious candidate for a real structure f on $(M(\mathbf{a}), \alpha)$ is complex conjugation. Since both the sphere S_2^{2n+1} and the variety $V(\mathbf{a})$ are invariant under the map $(\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{x}, -\mathbf{y})$, we have:

Proposition 3.1.2. *With involution $f(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, -\mathbf{y})$, the triple $(M(\mathbf{a}), \alpha, f)$ is a real contact manifold.* □

In fact, any involution g of $\mathbb{C}^{n+1} \equiv \mathbb{R}^{2n+2}$ with the properties $g(M(\mathbf{a})) = M(\mathbf{a})$ and

$$g^* \left(\mathbf{x} d\mathbf{y} - \mathbf{y} d\mathbf{x} \right) = - \left(\mathbf{x} d\mathbf{y} - \mathbf{y} d\mathbf{x} \right)$$

turns $(M(\mathbf{a}), \alpha)$ into a real contact manifold.

The fixed point set of $(M(\mathbf{a}), \alpha, f)$ consists of points $(\mathbf{x}, \mathbf{0})$ in $\mathbb{C}^{n+1}$ that satisfy

$$x_0^2 + \ldots + x_n^2 = 2 \quad \text{and} \quad x_0^{a_0} + \ldots + x_n^{a_n} = 0.$$

This set is non-empty if and only if at least one integer a_k is odd.

Example 3.1.3. Consider the contact structure $\alpha' = \mathbf{y} d\mathbf{x}$ on the Brieskorn manifold $M(2, 2, 2)$ with real structure g given by complex conjugation. Then $(M(2, 2, 2), \alpha', g)$ and $(ST^* S^2, \alpha, f)$ from Example 1.2.5 are strictly isomorphic. An isomorphism $M(2, 2, 2) \to ST^* S^2$ is given by

$$(x_0, y_0, x_1, y_1, x_2, y_2) \mapsto ((x_0, x_1, x_2), (y_0, y_1, y_2)).$$

Here, (y_0, y_1, y_2) is read as the restriction of the corresponding linear map $\mathbb{R}^3 \to \mathbb{R}$ to $T_{(x_0,x_1,x_2)}S^2 \equiv \langle(x_0, x_1, x_2)\rangle^{\perp}$. Topologically, both manifolds are diffeomorphic to $\mathbb{R}P^3$.

As in the classical case, Proposition 3.1.2 generalises to a larger class of complex submanifolds V of $\mathbb{C}^{n+1}$:

Theorem 3.1.4. *Let V be a complex submanifold of $\mathbb{C}^{n+1}$ that intersects S_2^{2n+1} transversely. Put $\alpha = (\mathbf{x}d\mathbf{y} - \mathbf{y}d\mathbf{x})/2$. Assume that f is an involution of $\mathbb{C}^{n+1}$ with $f^*\alpha = -\alpha$ such that $M := V \cap S_2^{2n+1}$ is invariant under f. Then $(M, \alpha|_{TM}, f|_M)$ is a real contact manifold.* $\square$

Section 1.2 dealt with real contact manifolds that appear as hypersurfaces in real symplectic manifolds. Conversely, one may ask in which ways a given real contact manifold can be realised as a hypersurface in a real symplectic manifold, with contact structure induced by the symplectic structure. This question leads to the notion of symplectic fillings. A closed symplectic manifold (W, ω) is called a **strong (symplectic) filling** of a contact manifold (M, ξ) if $\partial W = M$, and there exists a Liouville vector field Y, defined near M and pointing outwards along the boundary, such that $\xi = \ker(i_Y\omega|_{TM})$.

If, on the other hand, Y is a vector field on a symplectic manifold (W, ω) as above, then $\ker(i_Y\omega|_{TM})$ defines a contact structure on the boundary $M = \partial W$.

Proposition 3.1.5. *Let (W, ω) be a strong filling of (M, ξ'), and suppose that f is a real structure on W, i.e. we require $f^*\omega = -\omega$. Then there exists a contact structure ξ on M such that $(M, \xi, f|_M)$ is a real contact manifold.*

Furthermore, there exists an involution g on M for which (M, ξ', g) is a real contact manifold. The real contact structures $(\xi, f|_M)$ and (ξ', g) are isotopic through real contact structures on M.

Proof. Pick a Liouville vector field Y' inducing the contact structure $\xi' = \ker(i_{Y'}\omega|_{TM})$. Setting

$$Y = \frac{1}{2}\left(Y' + f_* Y'\right)$$

defines a symmetric Liouville vector field for ω that points outwards along ∂W. Define a contact structure α on M by $\alpha := i_Y\omega|_{TM}$. Then $(f|_M)^* \alpha = -\alpha$, i.e. $(M, \xi = \ker\alpha, f|_M)$ is a real contact manifold. The space of Liouville vector fields Y defined near ∂W such that Y points outwards along ∂W is convex. Thus, putting $\xi_t = \ker(i_{Y_t}\omega|_{TM})$ for the vector fields $Y_t = tY' + (1-t)Y$ defines a family of contact structures on M. Then use classical Gray stability to find an isotopy φ_t with $(\varphi_t)_* \xi_t = \xi_0$. For $f_t := \varphi_t^{-1} \circ f|_M \circ \varphi_t$, we have $(f_t)_* \xi_t = -\xi_t$. Thus, we may pick $g = f_1$. □

If the Liouville vector field in question is symmetric in the first place, we may choose $g = f|_M$. This motivates the following definition.

Definition 3.1.6. A closed real symplectic manifold (W, ω, f) is a **real strong (symplectic) filling** of a real contact manifold (M, ξ, g) if (W, ω) is a strong filling of (M, ξ), there exists a Liouville vector field associated to this filling which is symmetric, and $g = f|_M$. In that case, we say that (M, ξ, g) is **strongly fillable in the real sense**.

Example 3.1.7. The standard real contact structure on S^3, given by the 1-form $\alpha_{\mathrm{st}} = \mathbf{x}\,d\mathbf{y} - \mathbf{y}\,d\mathbf{x}$, $(\mathbf{x}, \mathbf{y}) \in S^3 \subset \mathbb{C}^2$, and involution g induced from the (coordinatewise) complex conjugation on $\mathbb{C}^2$, is strongly fillable in the real sense. Simply take $W = D^4$, $\omega = d\mathbf{x} \wedge d\mathbf{y}$ and f the complex conjugation in $\mathbb{C}^2$.

This example hints at how to find real strong fillings for Brieskorn manifolds. Before we tackle these, we need some preparations.

Lemma 3.1.8. *Let (W, ω, f) be a real strong symplectic filling of $(M, \xi, f|_M)$, with symmetric Liouville vector field Y. The flow φ_t of Y defines collar coordinates $(t, p) \in [0, \varepsilon] \times M$, $\varepsilon > 0$, in a neighbourhood of M in W. In these coordinates, f is given as $(t, p) \mapsto (t, f|_M(p))$.*

Proof. By Proposition 2.1.4, the flow φ_t is symmetric, and therefore $f(t, p) \equiv f(\varphi_t(p)) = \varphi_t(f(p)) \equiv (t, f(p))$. $\square$

Lemma 3.1.9. *Let (W_i, ω_i, f_i) be real strong symplectic fillings of (M_i, α_i, g_i) for $i = 0, 1$, and denote the associated Liouville vector fields by Y_i. Suppose that*

$$\phi\colon (M_0, \alpha_0, g_0) \to (M_1, \alpha_1, g_1)$$

is a symmetric contactomorphism. Extend it to a diffeomorphism $\widetilde{\phi}$ between collar neighbourhoods of M_i in W_i by sending the flow lines of Y_0 to those of Y_1. Then $\widetilde{\phi}$ is a symmetric symplectomorphism.

Proof. Let φ_t^i be the flow of the vector field Y_i, $i = 0, 1$. By the defintion of $\widetilde{\phi}$, we have

$$\widetilde{\phi}(\varphi_t^0(p)) = \varphi_t^1(\phi(p)).$$

Lemma 3.1.8 implies that

$$\begin{aligned}
\widetilde{\phi} \circ f_0(\varphi_t^0(p)) &= \widetilde{\phi}\left(\varphi_t^0(f_0(p))\right) \\
&= \varphi_t^1(\phi(f_0(p))) \\
&= \varphi_t^1(f_1(\phi(p))) \\
&= f_1 \circ \varphi_t^1(\phi(p)) \\
&= f_1 \circ \widetilde{\phi}(\varphi_t^0(p)),
\end{aligned}$$

i.e. $\widetilde{\phi}$ is symmetric. The remaining part is proved in [Gei08, Lemma 5.2.4]. □

Lemma 3.1.10. *Let ξ_t, $t \in [0, 1]$, be a family of contact structures on a closed manifold M, and let g be an involution on M such that (M, ξ_t, g) is a real contact manifold for all t. If (M, ξ_0, g) is strongly fillable in the real sense, then there exists a number $\varepsilon > 0$ such that (M, ξ_t, g) is strongly fillable in the real sense for all $t \leq \varepsilon$.*

Proof. Let (W, ω, f) be a real strong filling of (M, ξ_0, g), and denote a corresponding Liouville vector field by Y. By real Gray stability, there exists a symmetric contactomorphism $\varphi_t \colon M \to M$ with $(\varphi_t)_* \xi_0 = \xi_t$ for all t. Extend φ_t to a diffeomorphism $\overline{\varphi_t}$, defined on an f-invariant neighbourhood $\mathcal{N}$ of $\partial W = M$ in W, by sending flow lines of Y to flow lines of Y. Then $\overline{\varphi_t}$ is a symmetric symplectomorphism of the symplectic manifold $(\mathcal{N}, \omega|_{\mathcal{N}}, f|_{\mathcal{N}})$. This can be seen by an argument similar to the one in the proof of Lemma 3.1.9. Let $\alpha = i_Y\omega$, such that $\alpha|_{TM}$ is a contact form for ξ_0. Put $\alpha_t := \left(\overline{\varphi_t}^{-1}\right)^* \alpha$. Then $\ker \alpha_t|_{TM} = \xi_t$. Define the time-dependent vector field Y_t via the equation $i_{Y_t}\omega = \alpha_t$. Since both ω and α_t are antisymmetric with respect to f, the vector field Y_t has to be symmetric. We have $Y_0 = Y$, which is transverse to M. Therefore, by the compactness of M, there exists a number $\varepsilon > 0$ such that Y_t is transverse to M for all $t \leq \varepsilon$. Thus, the real symplectic manifold (W, ω, f), together with the Liouville vector field Y_t, constitutes a real symplectic filling of (M, ξ_t, g), $t \leq \varepsilon$. $\square$

We are now in the position to prove that the Brieskorn manifolds $M(\mathbf{a})$, equipped with the real contact structures introduced above, are strongly fillable in the real sense. As before, we write $\alpha_{\mathrm{st}} = (\mathbf{x}d\mathbf{y} - \mathbf{y}d\mathbf{x})/2$ and f for the complex conjugation $(\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{x}, -\mathbf{y})$ on $\mathbb{C}^{n+1}$. In what follows, we will denote the restriction of f to certain subspaces of $\mathbb{C}^{n+1}$ by f as well.

Proposition 3.1.11. *The Brieskorn manifold*

$$(M(\mathbf{a}), \xi = \ker \alpha, f)$$

is strongly fillable in the real sense.

Proof. At first glance, a reasonable choice for a filling of $M(\mathbf{a})$ seems to be the manifold $W(\mathbf{a}) = V(\mathbf{a}) \cap D_2^{2n+1}$, where D_2^{2n+1} denotes the ball of radius 2 around the origin in $\mathbb{C}^{n+1}$. But since $V(\mathbf{a})$ has a singularity in the origin, $W(\mathbf{a})$ is not a manifold. Instead, one works with $W_s(\mathbf{a}) = V_s(\mathbf{a}) \cap D_2^{2n+1}$, where

$$V_s(\mathbf{a}) = \left\{\mathbf{z} \in \mathbb{C}^{n+1} \mid z_0^{a_0} + \ldots + z_n^{a_n} = s\right\}.$$

Then $W_s(\mathbf{a})$ is a (smooth) manifold with symplectic form ω induced by $d\mathbf{x} \wedge d\mathbf{y}$, and $(W_s(\mathbf{a}), \omega, f)$ is a real strong filling of $(M_s(\mathbf{a}), \xi_s = \ker\alpha, f)$ for any $s > 0$. According to the result [HM68, Satz 14.3], there exists an $\varepsilon > 0$ such that each manifold $M_s(\mathbf{a})$, $s \leq \varepsilon$, is diffeomorphic to $M(\mathbf{a})$. In fact, the $M_s(\mathbf{a})$ are the fibres of the fibre bundle

$$\begin{array}{rccc} \psi\colon & S_2^{2n+1} & \longrightarrow & \mathbb{C} \\ & \mathbf{z} & \longmapsto & z_0^{a_0} + \ldots + z_n^{a_n}. \end{array}$$

This map commutes with complex conjugation, that is, if f_1 denotes complex conjugation on $\mathbb{C}$, we have $\psi \circ f = f_1 \circ \psi$. Therefore, we have a symmetric diffeomorphism

$$M_s(\mathbf{a}) \to M(\mathbf{a}).$$

Pick a connection on this bundle such that the parallel transport induced by this connection commutes with the given involution. Then this allows us to view the contact structures ξ_s as contact structures on $M(\mathbf{a})$, compatible with the real structure f. Now use Lemma 3.1.10. $\square$

A concept related to fillings are cobordisms.

Definition 3.1.12. Let $(M_\pm, \xi_\pm, g_\pm)$ be closed real contact manifolds of dimension $2n+1$. A **real symplectic cobordism** from (M_-, ξ_-, g_-) to (M_+, ξ_+, g_+) is a compact $(2n+2)$-dimensional real symplectic manifold (W, ω, f), oriented by the volume form ω^{n+1}, such that

- the oriented boundary of W equals

$$\partial W = M_+ \sqcup (-M_-),$$

- in a neighbourhood of ∂W, there is an f-symmetric Liouville vector field Y for ω, transverse to the boundary and pointing outwards along M_+, inwards along M_-,
- $f|_{M_\pm} = g_\pm$, and
- the 1-form $\alpha := i_Y\omega$ restricts to $TM_\pm$ as a contact form for $\xi_\pm$.

Note that the second condition already implies that we have $f(M_\pm) = M_\pm$. A real symplectic filling of a manifold is a real symplectic cobordism from the empty set to the manifold.

Remark. The relation of being real symplectically cobordant is reflexive: Let $(M, \xi = \ker\alpha, f)$ be a real contact manifold, and simply take the real symplectisation

$$W = [0,1] \times M, \quad \omega = d(\mathrm{e}^t\alpha), \quad \overline{f} = \mathrm{id}_{[0,1]} \times f.$$

Next, we show that this relation is transitive.

Proposition 3.1.13. *Let (W_-, ω_-, f_-) be a real symplectic cobordism from the real contact manifold (M_-, ξ_-, g_-) to (M, ξ, g), and (W_+, ω_+, f_+) a real symplectic cobordism from (M, ξ, g) to (M_+, ξ_+, g_+). Then there is a real symplectic cobordism from (M_-, ξ_-, g_-) to (M_+, ξ_+, g_+).*

Proof. We follow the lines of the proof in [Gei08, Proposition 5.2.5] and realise the cobordism as $W = W_- \cup_M W_0 \cup_M W_+$. Define a real structure h on W_0 by $(t, p) \mapsto (t, g(p))$. By Lemma 3.1.8, we can glue this involution to a (smooth) real structure $f_- \cup h \cup f_+$ on W. $\square$

However, this relation is not symmetric. This follows from the non-symmetry in the classical case.

The remainder of this section is devoted to surgery constructions. First, we show how to perform real contact surgeries along isotropic submanifolds fixed by the real structure. As in the classical case, this involves the use of symplectic handles, now with real structures added. Following that, there is a brief discussion of real contact Dehn surgery. In order to define real contact surgery, we need a stronger version of the Neighbourhood Theorem 2.4.9. For an isotropic submanifold L contained in the fixed point set of a real contact manifold (M, α, f), the quotient bundle

$$\mathrm{RSN}_M(L) := (TL)^{\perp} / TL,$$

with symplectic bundle structure induced by $d\alpha$ and involution Tf (cf. Section 2.4), is called the **real symplectic normal bundle** of L in M.

Theorem 3.1.14. *Let (M_i, α_i, f_i), $i = 0, 1$, be real contact manifolds with closed isotropic submanifolds $L_i \subset \mathrm{Fix}\, f_i$ contained in the corresponding fixed point sets. Further suppose there is a symmetric isomorphism of real symplectic normal bundles $\Phi : \mathrm{RSN}_{M_0}(L_0) \to \mathrm{RSN}_{M_1}(L_1)$ that covers a symmetric diffeomorphism $\phi\colon L_0 \to L_1$. Then ϕ extends to a strict real contactomorphism $\psi\colon \mathcal{N}(L_0) \to \mathcal{N}(L_1)$*

of suitable invariant neighbourhoods $\mathcal{N}(L_i)$ *of* L_i *such that* $T\psi|_{\mathrm{RSN}_{M_0}(L_0)} = \Phi$.

The proof of the preceding theorem requires a real version of the generalised Poincaré Lemma:

Lemma 3.1.15. *Let* L *be a submanifold contained in the fixed point set of a real manifold* (M, f), *and denote by* $j_L : L \to M$ *its inclusion into* M. *Let* $\eta \in \Omega^k(M)$ *be a closed antisymmetric* k*-form with* $j_L^*\eta = 0$. *Then there is a symmetric open neighbourhood* U *of* L *in* M *and an antisymmetric* $(k-1)$*-form* ζ *on* U *vanishing on* L *and with* $d\zeta = \eta$ *on* U.

Proof. Use the generalised Poincaré Lemma [Gei08, Corollary A.4] to obtain a differential $(k-1)$-form $\widetilde{\zeta}$ on a symmetric neighbourhood U of L vanishing on L with $d\widetilde{\zeta} = \eta$, and put

$$\zeta = \frac{1}{2}\left(\widetilde{\zeta} - f^*\widetilde{\zeta}\right).$$

Then ζ is an antisymmetric $(k-1)$-form that satisfies both $j_L^*\zeta = 0$ and $d\zeta = \eta$. □

Proof of Theorem 3.1.14. Literally the same as in [Gei08, Theorem 6.2.2]. Except for the hypersurface Σ (which we may assume to be symmetric), there are no choices made, and all expressions involved are symmetric, thus producing symmetric flows. Simply replace all references to the generalised Poincaré lemma by citations of Lemma 3.1.15. □

Our goal now is to define handle attachments in the real contact case. But to begin with, consider a manifold M^d

without any additional data, and let $S^k \times D^{d-k} \to M$ be an embedding. Then we can define a new manifold M' by

$$M' := \left(M \setminus \left(S^k \times \operatorname{Int} D^{d-k}\right)\right) \cup_{S^k \times S^{d-k-1}} \left(D^{k+1} \times S^{d-k-1}\right).$$

We say that M' is obtained from M by **surgery along** S^k. The diffeomorphism type of M' depends not only on S^k, but also on the embedding $S^k \times D^{d-k} \to M$, i.e. a framing of S^k in M. Surgery embeds into the theory of cobordisms as follows: Consider the map $S^k \times D^{d-k} \to M$ as an inclusion into $M \equiv \{1\} \times M \subset [-1,1] \times M$, and put

$$W = ([-1,1] \times M) \cup_{S^k \times D^{d-k}} \left(D^{k+1} \times D^{d-k}\right),$$

i.e. W is the manifold $[-1,1] \times M$ with a $(k+1)$-**handle** attached. The manifold W represents a cobordism from $M \equiv \{-1\} \times M$ to

$$M' \equiv \left((\{1\} \times M) \setminus \left(S^k \times \operatorname{Int} D^{d-k}\right)\right) \cup_{S^k \times S^{d-k-1}} \left(D^{k+1} \times S^{d-k-1}\right).$$

This description has the disadvantage that W is a manifold with corners – this would cause problems as soon as we include additional geometric objects (such as contact forms and involutions) into the surgery construction. Thus, we turn to a different approach of attaching handles, the so-called *symplectic handles*. An extensive discussion may be found in [Gei08, Section 6.2]. For notational reasons, we will consider an isotropic $(k-1)$-sphere L contained in the fixed point set of a real contact manifold (M^{2n-1}, α, f), $0 \leq k \leq n$. Write coordinates on $\mathbb{R}^{2n} = \mathbb{R}^k \times \mathbb{R}^{2n-k}$ as

$$((q_1, \ldots, q_k), (q_{k+1}, \ldots, q_n, p_1, \ldots, p_n)).$$

On $\mathbb{R}^{2n}$, we have the standard symplectic form $\omega_{\mathrm{st}} = d\mathbf{p} \wedge d\mathbf{q}$, and the real structure is given by $f_{\mathrm{st}}(\mathbf{q}, \mathbf{p}) = (\mathbf{q}, -\mathbf{p})$. The Liouville vector field

$$Y := \sum_{j=1}^{k} \left(-q_j \partial_{q_j} + 2p_j \partial_{p_j}\right) + \frac{1}{2} \sum_{j=k+1}^{n} \left(q_j \partial_{q_j} + p_j \partial_{p_j}\right)$$

is symmetric with respect to f_{st}. With respect to the standard euclidean metric on $\mathbb{R}^{2n}$, the vector field Y is the gradient vector field of the function

$$g(\mathbf{q}, \mathbf{p}) = \sum_{j=1}^{k} \left(-\frac{1}{2} q_j^2 + p_j^2\right) + \frac{1}{4} \sum_{j=k+1}^{n} \left(q_j^2 + p_j^2\right).$$

Note that f_{st} preserves the level sets of g. Now, denote by $\mathcal{N}_H \cong S^{k-1} \times \operatorname{Int} D^{2n-k}$ an open symmetric neighbourhood in the hypersurface $g^{-1}(-1)$ of the (symmetric) $(k-1)$-sphere

$$S_H^{k-1} := \left\{ \sum_{j=1}^{k} q_j^2 = 2, q_{k+1} = \ldots = q_n = p_1 = \ldots = p_n = 0 \right\}.$$

This $\mathcal{N}_H$ is called the lower boundary of H. We will later specify which neighbourhood to choose. Similarly, there is an upper boundary in $g^{-1}(1)$. The **real symplectic handle** H is now defined to be the set of points $(\mathbf{q}, \mathbf{p}) \in (\mathbb{R}^{2n}, \omega_{\mathrm{st}}, f)$ that satisfy the inequality $-1 \leq g(\mathbf{q}, \mathbf{p}) \leq 1$ and lie on a gradient flow line of g through $\mathcal{N}_H$. Observe that, since

$g \circ f_{\mathrm{st}} = g$, the handle H is invariant under f_{st}. The Liouville vector field Y is transverse to both the lower and the upper boundary, and thus

$$\alpha_0 := i_Y \omega_{\mathrm{st}} = \sum_{j=1}^{k} (q_j dp_j + 2p_j dq_j) + \frac{1}{2} \sum_{j=k+1}^{n} (-q_j dp_j + p_j dq_j)$$

induces a contact form there. For this form, S_H^{k-1} is an isotropic sphere in the lower boundary. Its real symplectic normal bundle $\mathrm{RSN}_{\partial H}(S_H^{k-1})$ is trivialised by the vector fields $\partial_{q_j}, \partial_{p_j}$, $j = k+1, \ldots, n$.

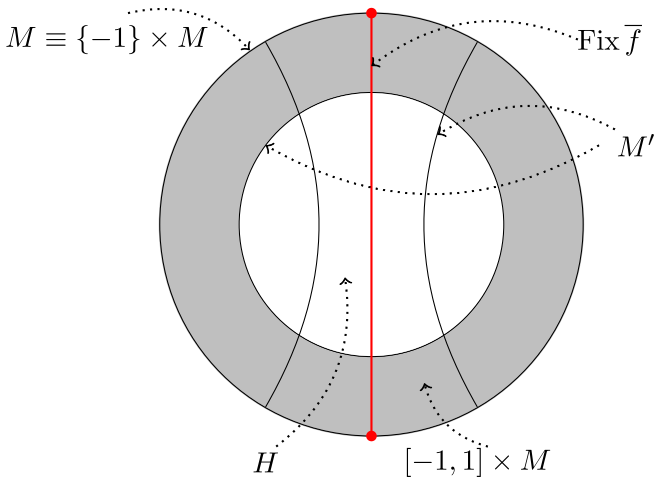

Figure 3.1: A real cobordism corresponding to real surgery.

Morally, with the help of Lemma 3.1.9, we may now glue the real symplectic handle H to $[-1,1] \times M$. As before, we will denote the resulting manifold by W and its symplectic form by ω. Since the Liouville vector fields ∂_t on $[-1,1]\times M$ and Y on H are symmetric with respect to the given real structures, the latter patch together to a real structure f' on (W,ω). Figure 3.1 shows a handle H attached to the manifold $[-1,1]\times M$ (in gray). The spheres we perform surgery along are depicted as red circles, and the vertical red line represents the fixed point set of the extension of f to $[-1,1]\times M$. The surgered manifold M' can be identified with the boundary of the 'lakes' in the interior.

More precisely: Consider (M^{2n-1},α,f), a real contact manifold (with non-empty fixed point set), and let the map $\phi\colon S^{k-1}\times D^{2n-k}\to M$ be an embedding into M for which $\phi(S^{k-1}\times\{\mathbf{0}\})$ is contained in the fixed point set of f. By Theorem 3.1.14, there exists an equivariant strict contactomorphism $\psi\colon \mathcal{N}(S^{k-1})\to\mathcal{N}(S^{k-1}_H)$ between symmetric open neighbourhoods of $S^{k-1}\subset M$ and $S^{k-1}_H\subset\partial H$, respectively. Put $\mathcal{N}_H=\psi(\mathcal{N}(S^{k-1}))$. Then, for $\mathbf{u}\in S^{k-1}$, $\mathbf{v}\in S^{2n-k-1}$, $0<r<1$ and $c\in[-1,1]$, we identify points

$$(c,\phi(\mathbf{u},r\mathbf{v}))\sim(\mathbf{q},\mathbf{p})$$

if and only if

$$g(\mathbf{q},\mathbf{p})=c \text{ and } (\mathbf{q},\mathbf{p}) \text{ lies on} \\ \text{a flow line of } Y \text{ through } \psi\left(\phi(\mathbf{u},r\mathbf{v})\right).$$

Denote the extension of f to the cylinder over M by the map $\overline{f}\colon[-1,1]\times M\to[-1,1]\times M$, $\overline{f}(c,p)=(c,f(p))$. In order to prove that the given involutions piece together, we have to show that

$$
\begin{aligned}
&(c, \phi(\mathbf{u}, r\mathbf{v})) \sim (\mathbf{q}, \mathbf{p}) \\
&\qquad\qquad \Rightarrow \overline{f}(c, \phi(\mathbf{u}, r\mathbf{v})) \sim f_{\mathrm{st}}(\mathbf{q}, \mathbf{p}) = (\mathbf{q}, -\mathbf{p}).
\end{aligned}
$$

Since $g(\mathbf{q}, -\mathbf{p}) = g(\mathbf{q}, \mathbf{p})$, it remains to show that $(\mathbf{q}, -\mathbf{p})$ lies on a flow line of Y through $\psi\left(f \circ \phi(\mathbf{u}, r\mathbf{v})\right)$. Denote the flow of Y by φ_t. Sine $(\mathbf{q}, \mathbf{p})$ lies on a flow line of Y through $\psi(\phi(\mathbf{u}, r\mathbf{v}))$, there exists a time t_0 such that

$$
(\mathbf{q}, \mathbf{p}) = \varphi_{t_0}\left(\psi(\phi(\mathbf{u}, r\mathbf{v}))\right).
$$

Thus, using the equivariance of ψ and φ_t, we have:

$$
\begin{aligned}
\varphi_{t_0}\left(\psi\left(f \circ \phi(\mathbf{u}, r\mathbf{v})\right)\right) &= \varphi_{t_0}\left(f_{\mathrm{st}} \circ \psi(\phi(\mathbf{u}, r\mathbf{v}))\right) \\
&= f_{\mathrm{st}} \circ \varphi_{t_0}\left(\psi(\phi(\mathbf{u}, r\mathbf{v}))\right) \\
&= f_{\mathrm{st}}(\mathbf{q}, \mathbf{p}) \\
&= (\mathbf{q}, -\mathbf{p}).
\end{aligned}
$$

Therefore, $(\mathbf{q}, -\mathbf{p})$ lies on a flow line of the vector field Y through $\psi\left(f \circ \phi(\mathbf{u}, r\mathbf{v})\right)$, i.e. the point $(\mathbf{q}, -\mathbf{p})$ is identified with $(c, f \circ \phi(\mathbf{u}, r\mathbf{v}))$.

As in the discussion in [Gei08, Section 6.2], the **natural framing** of an isotropic sphere S^{k-1} contained in the fixed point set of a real contact manifold (M, α, f) is given by the natural trivialisation of $\langle -R \rangle \oplus J(TS^{k-1})$ (coming from the inclusion $S^{k-1} \to \mathbb{R}^k$) and a real symplectic trivialisation of $\mathrm{RSN}_M(S^{k-1})$. Here, R denotes the Reeb vector field of α, and J is a complex bundle structure on $\ker \alpha$ compatible with $d\alpha$ and Tf.

Theorem 3.1.16. *Let S^{k-1} be an isotropic sphere in the fixed point set of a real contact manifold $(M, \xi = \ker \alpha, f)$ with a trivialisation of the real conformal symplectic normal bundle $\mathrm{RCSN}_M(S^{k-1})$. Then there is a real symplectic cobordism from (M, ξ, f) to the manifold M' obtained from M by surgery along S^{k-1} using the natural framing. In particular, the surgered manifold M' carries a real contact structure (ξ', f') that coincides with (ξ, f) away from the surgery region.* □

In dimension 3, a construction related to the above is Dehn surgery. The idea is to remove a tubular neighbourhood of a knot in a 3-manifold, and then to glue back a solid torus by a diffeomorphism between the boundaries. Under certain assumptions on this diffeomorphism, a real contact version of Dehn surgery can be performed. Suppose that (M, ξ, f) is a real contact 3-manifold, and let $L \subset \operatorname{Fix} f$ be a component of the fixed point set of f. By Example 2.4.11, an invariant neighbourhood $\mathcal{N}(L)$ of L can be equivariantly identified with the model $(D^2 \times S^1, \xi_1, f_{\mathrm{st}})$. Here, we have $f_{\mathrm{st}}((x,y),\theta) = ((-x,-y),\theta)$ in cartesian coordinates on the D^2-factor. But rather than working with this standard model, we fix a real contact identification $\Phi\colon D^2 \times S^1 \to \mathcal{N}(L)$, where the contact structure on $\mathcal{N}(L)$ is given as

$$\xi_k = \ker(\cos(k\theta)dx - \sin(k\theta)dy);$$

the number k is to be chosen later. Notice that we have $(f_{\mathrm{st}})_* \xi_k = -\xi_k$. This defines a meridian $\Phi(\partial D^2 \times \{0\})$ on $\partial\mathcal{N}(L)$ in M, as well as a longitude $\Phi(\{(1,0)\} \times S^1)$. Now, we want to cut out the interior of this neighbourhood $\mathcal{N}(L)$

and glue back a real standard model of $D^2 \times S^1$ (with contact structure ξ_1) via a map

$$\begin{array}{rccc} \psi\colon & \partial\left(D^2\times S^1\right) & \longrightarrow & \partial\left(M\setminus \operatorname{Int}\mathcal{N}(L)\right) \\ & (\varphi,\theta) & \longmapsto & (p\varphi+s\theta, q\varphi+t\theta), \end{array}$$

where

$$\begin{pmatrix} p & s \\ q & t \end{pmatrix} \in SL_2(\mathbb{Z}).$$

In order to find matching real contact structures, we restrict our attention to the case

$$\begin{pmatrix} p & s \\ q & t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$$

for some $n \in \mathbb{Z}$. According to classical convex surface theory, we can glue the contact manifolds along their boundary as soon as their diving sets coincide. A computation shows that this is precisely the case for $k = n-1$.

Next, we check under which conditions on the coefficient n the given involutions piece together. We have

$$\psi \circ f_{\mathrm{st}}(\varphi,\theta) = (\varphi+\pi, n\varphi+n\pi+\theta)$$

and

$$f_{\mathrm{st}} \circ \psi(\varphi,\theta) = (\varphi+\pi, n\varphi+\theta).$$

Thus, it follows that $\psi\circ f_{\mathrm{st}} = f_{\mathrm{st}}\circ\psi$ if and only if n is even. Note that, since both involutions are trivial in the radial direction of D^2 near the boundary, they may be glued together to a real structure f' on the manifold

$$M' := \left(D^2\times S^1\right)\cup_\psi \left(M\setminus \operatorname{Int}\mathcal{N}(L)\right).$$

So, let n be an even integer. According to the preceding discussion, there exist a contact structure ξ' and a real structure f' on M' such that (M', ξ', f') is a real contact manifold. We call it the result of a **real contact Dehn surgery (with fixed points)** along L with coefficient $1/n$.

A similar reasoning applies to the case where we glue back a real solid torus without fixed points. Denote the map $(\varphi, \theta) \mapsto (\varphi, \theta + \pi)$ by $g\colon D^2 \times S^1 \to D^2 \times S^1$. This involution defines a real contact structure on $(D^2 \times S^1, \xi_1)$.

With notation as above, we have $f_{\mathrm{st}} \circ \psi = \psi \circ g$ if and only if s is odd and t is even. In particular, this implies that q is odd. A possible solution is given by

$$\begin{pmatrix} 1+n & 2+n \\ n & 1+n \end{pmatrix}$$

where n is odd. Again, one computes that the contact structures piece together for the choice $k = 1 + 2n$. Consequently, for n odd, we can perform a **real contact Dehn surgery (without fixed points)** along L with coefficient $1/n$; this removes a component of the fixed point set of f. Summarising, we have shown:

Theorem 3.1.17. *Let (M, ξ, f) be a real contact manifold of dimension 3, and suppose that L is a component of the fixed point set of the involution f.*

(1) For any even integer n, we can perform a real contact Dehn surgery along L with contact coefficient $1/n$, preserving the number of components of the fixed point set.

(2) For any odd integer n, we can perform a real contact Dehn surgery along L with contact coefficient $1/n$, reducing

the number of components of the fixed point set by one. □

Remark. In the 3-dimensional case, real contact surgery along S^1 (that is, a component of the fixed point set of the involution) is the same as real contact Dehn surgery *without* fixed points and with coefficient -1.

Example 3.1.18. Consider the 3-sphere $S^3 \subset \mathbb{R}^4$, equipped with the real contact structure given by

$$\alpha = x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2$$

and

$$f(x_1, y_1, x_2, y_2) = (x_1, -y_1, x_2, -y_2).$$

Its fixed point set, $\operatorname{Fix} f = \{y_1 = y_2 = 0\}$, is a copy of S^1. Performing a real contact Dehn surgery without fixed points with coefficient -1 yields a real contact structure on $\mathbb{R}P^3$ without fixed points.

Remark. According to [Smi38], the fixed point set of an involution on S^3 is either empty or a single circle. On the other hand, the fixed point set of an involution on $\mathbb{R}P^3$ is either empty, or it has two components, as is shown in [Smi60]. Therefore, no matter which involution or coefficient we choose, a real contact Dehn surgery *with* fixed points on S^3 along the fixed circle never produces $\mathbb{R}P^3$.

3.2 Open books

Open books constitute a method for decomposing manifolds that proved particularly useful in 3-dimensional contact topology: An argument by Thurston and Winkelnkemper shows that any open book on a 3-manifold induces a contact structure. The same implication holds for real open books; relevant definitions and discussions are given in the present section. In the context of contact manifolds, real open books were introduced by Öztürk and Salepci, [ÖS15].

Denote complex conjugation $\varphi \mapsto -\varphi$ on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ by κ.

Definition 3.2.1. Let (M, f) be a real 3-dimensional manifold. A **real open book decomposition** of (M, f) is a pair (B, p), consisting of a codimension 2 submanifold B (called the **binding** of the open book) and a locally trivial fibration $p\colon M \setminus B \to S^1$, satisfying the following properties:

(1) The binding B has a trivial tubular neighbourhood $B \times D^2$ in which p is given by the angular coordinate in the D^2-factor. (That is, (B, π) is an open book decomposition of M.)

(2) The binding B is invariant under the real structure f.

(3) We have $\kappa \circ p = p \circ f$.

The closures of the fibres $p^{-1}(\varphi)$, $\varphi \in S^1$, are called the **pages** of the open book.

The third condition implies that fixed points of f are necessarily contained in the pages $p^{-1}(0)$ and $p^{-1}(\pi)$.

Example 3.2.2. A real open book decomposition of S^3 with pages open discs is depicted in Figure 3.2. Here, we identify S^3 with $\mathbb{R}^3 \cup \{\infty\}$; in order to arrive there, rotate the picture as indicated. After rotation, the blue dots become a circle and represent the binding of the open book. Each arc connecting the blue dots embodies a page, including the red arc through ∞. The dotted horizontal line – a circle in S^3 – intersects each page exactly once and may be thought of as a section of the fibration p. Thus, the condition that the real structure act on the set of pages as complex conjugation translates into reflection through the red axis. As demanded, this operation leaves two pages invariant while exchanging the others, by twos. Moreover, f reverses the direction invisible in this picture. On the invariant pages D^2, thought of as subsets of $\mathbb{C}$, the real structure is given as complex conjugation. Thus, in total, f is an orientation-preserving involution with fixed point set precisely the red line (which represents a copy of S^1).

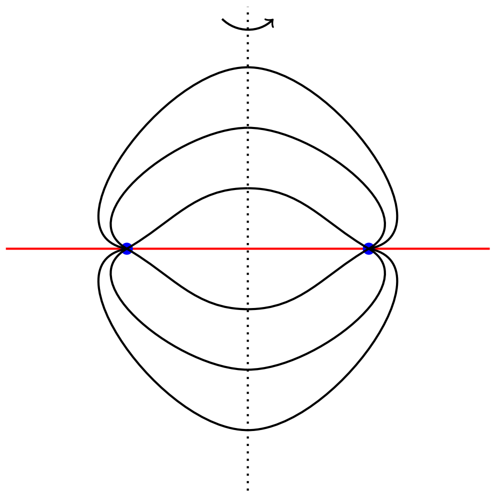

Figure 3.2: A real open book decomposition for S^3.

Related to open book decompositions are abstract open books. An **abstract open book** is a pair (Σ, ϕ) where Σ is a compact, oriented surface (with boundary), and the **monodromy** ϕ is an orientation-preserving diffeomorphism of Σ that equals the identity near the boundary $\partial\Sigma$. An abstract open book uniquely determines a closed 3-dimensional manifold $M(\phi)$ by gluing copies of $\partial\Sigma \times D^2$ to the mapping torus $\Sigma(\phi)$ along its boundary. The mapping torus is defined to be the quotient

$$\Sigma(\phi) = (\Sigma \times [0, 2\pi]) \,/\sim$$

under the relation $(x, 2\pi) \sim (\phi(x), 0)$. Here is the real version of abstract open books.

Definition 3.2.3. An **abstract real open book** is a triple (Σ, ϕ, g) with (Σ, ϕ) an abstract open book and $g\colon \Sigma \to \Sigma$ a real structure on Σ such that $\phi \circ g = g \circ \phi^{-1}$.

A real open book decomposition (B, p) of a real manifold (M^3, f) induces an abstract real open book: First, observe that, due to the definition, both the pages $\Sigma_0 := \overline{p^{-1}(0)}$ and $\Sigma_\pi = \overline{p^{-1}(\pi)}$ are invariant under the real structure f. Let Σ be any of these two. By condition (3) above, f reverses a direction transverse to Σ, and therefore $g := f|_\Sigma$ defines a real structure on Σ. Denote the coordinate in S^1 by φ. Let $\widetilde{Y}$ be a vector field on $M \setminus B$ transverse to the pages that projects to ∂_φ under the projection p. Such a vector field extends over all of M by setting it equal to zero on B. Put

$$Y := \frac{1}{2}\left(\widetilde{Y} - f_*\widetilde{Y}\right).$$

Again by condition (3), we compute $\pi_* Y = \partial_\varphi$. Moreover, Y is antisymmetric with respect to f. Thus its time 2π flow ϕ_t

satisfies $\phi_t^{-1} \circ f = f \circ \phi_t$, see Proposition 2.1.4. We conclude that $(\Sigma, \phi_{2\pi}, g)$ is an abstract real open book. Conversely, we have:

Lemma 3.2.4. *An abstract real open book induces a real open book decomposition of $M(\phi)$. This operation, however, is unique only up to equivariant isotopy.*

Proof. For details, see [ÖS15, Lemma 1]. Rather than working with the classical mapping torus, one first defines a mapping torus $\Sigma'(\phi)$ as the quotient

$$\begin{aligned}&(\Sigma \times [0,\pi] \cup \Sigma \times [-\pi, 0]) \,/ \sim, \\ &\qquad \text{where } (p,0) \sim (g(x), -\pi) \text{ and } (p,\pi) \sim (\phi \circ g(p), 0).\end{aligned}$$

Observe that this mapping torus has the monodromy map $g \circ \phi^{-1} \circ g = \phi$. A real structure f_ϕ on $\Sigma'(\phi)$ is given by

$$\Sigma \times [0,\pi] \ni (p,t) \mapsto (p, t-\pi) \in \Sigma \times [-\pi, 0]$$

and

$$\Sigma \times [-\pi, 0] \ni (p,t) \mapsto (p, t+\pi) \in \Sigma \times [0,\pi]\,.$$

Then f_ϕ extends over $M(\phi)$, still denoted by f_ϕ, in the following way: Either f_ϕ exchanges two solid tori $\partial\Sigma \times D^2$ glued to the boundary of $\Sigma'(\phi)$, or f_ϕ acts by rotation by π on a single solid torus. Beware that this extension is unique only up to isotopy. $\square$

Note that, in the construction above, the fixed point set of f_ϕ on $\Sigma'(\phi)$ is given as $\operatorname{Fix} g \times \{0,\pi\}$, where we identify the

S^1-direction with $\mathbb{R}/2\pi\mathbb{Z}$. This implies that real manifolds realised as open books always have non-empty fixed point set.

As in the classical case, abstract real open books can be stabilised: Either glue a 1-handle to the surface where the attaching region is invariant under g, or glue two 1-handles to regions that are interchanged by g. Concatenate the monodromy with suitable Dehn twists. Then there exists an involution on the new surface that yields an abstract real open book. Details may again be found in [ÖS15].

Lemma 3.2.5. *Let (Σ, ϕ_0, g) be an abstract real open book, and let (Σ, ϕ_1) be an abstract open book such that ϕ_0 and ϕ_1 are isotopic as diffeormophisms of Σ fixing the boundary. Then there exists a real structure f on $M(\phi_1)$ such that $(M(\phi_0), f_{\phi_0})$ and $(M(\phi_0), f)$ are equivariantly diffeomorphic.*

Proof. There exists a diffeomorphism $\psi\colon M(\phi_0) \to M(\phi_1)$ that preserves the pages – this can be shown just as in the classical case. Then put $f := \psi \circ f_{\phi_0} \circ \psi^{-1}$. $\square$

Example 3.2.6. (1) Let $\Sigma = D^2$ be the 2-disc. By the work of Smale [Sma59], every diffeomorphism of D^2 fixing the boundary is isotopic to the identity. According to the preceding lemma, it then suffices to consider the case $\phi = \mathrm{id}_{D^2}$. The open book (D^2, id_{D^2}) can be made real by the involution $g =$ complex conjugation. Topologically, this open book yields S^3. On $\Sigma'(\mathrm{id}_{D^2}) \equiv D^2 \times S^1$, the involution $f_{\mathrm{id}_{D^2}}$ is given as $((x, y), t) \mapsto ((x, -y), -t)$. Its fixed point set, $\{(x, 0)\} \times \{0, \pi\}$, is depicted red in the upper half of Figure 3.3. When gluing $D^2 \times S^1$ to $\partial\Sigma'(\mathrm{id}_{D^2})$ along their common boundary, the coloured curves are identified accordingly. This produces an involution on $M(\mathrm{id}_{D^2}) = S^3$ with fixed point set a circle – as was to be expected in view of Smith's results, forcing an involution's fixed point set on S^3 to be diffeomorphic so S^1.

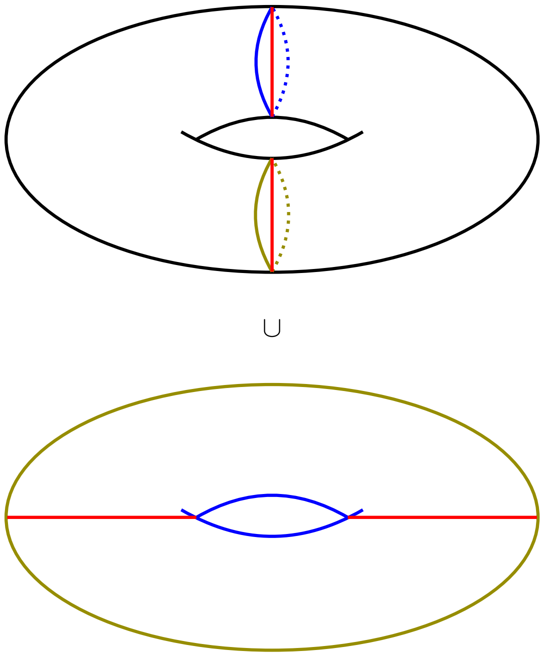

Figure 3.3: An abstract real open book for S^3.

This real abstract open book defines the same real structure as the real open book decomposition depicted in Figure

3.2. In fact, this real structure on S^3 is equivariantly diffeomorphic to the standard real structure $f(\mathbf{x},\mathbf{y}) = (-\mathbf{x},\mathbf{y})$ on $S^3 \subset \mathbb{R}^4$. To see this, split S^3 as

$$H_1 = \left\{x_1^2 + y_1^2 \geq \frac{1}{\sqrt{2}}\right\} \quad \text{and} \quad H_2 = \left\{x_2^2 + y_2^2 \geq \frac{1}{\sqrt{2}}\right\}.$$

Both handlebodies are invariant under f, and

$$\begin{aligned}\operatorname{Fix} f \cap H_1 = \Big\{ &(x_1, y_1, x_2, y_2) \in S^3 \mid y_1 = y_2 = 0, \\ &x_1 \in \left[-1, -\frac{1}{\sqrt{2}}\right] \sqcup \left[\frac{1}{\sqrt{2}}, 1\right]\Big\}.\end{aligned}$$

This is the disjoint union of two intervals, as before. In polar coordinates $((r,\vartheta),\theta)$ on $D^2 \times S^1$, an explicit identification $\psi\colon D^2 \times S^1 \to H_1$ is given by

$$\psi((r,\vartheta),\theta) = \left(\left(\sqrt{1 - \frac{r^2}{2}}, \theta\right), \left(\frac{1}{\sqrt{2}} r, \vartheta\right)\right).$$

It follows that

$$\begin{aligned}f \circ \psi\left((r,\vartheta),\theta\right) &= \left(\left(\sqrt{1 - \frac{r^2}{2}}, -\theta\right), \left(\frac{1}{\sqrt{2}} r, -\vartheta\right)\right) \\ &= \psi \circ f_{\mathrm{id}_{D^2}}\left((r,\vartheta),\theta\right).\end{aligned}$$

Similarly for H_2.

This real abstract open book yields the same open book decomposition of S^3 as the book in Example 3.2.2.

(2) (taken from [ÖS15].) Every open book with page an annulus $\Sigma = S^1 \times [1,2]$, considered as a subset of $\mathbb{C}$, extends

to a real open book. This can be seen as follows. The mapping class group of Σ is generated by the Dehn twist

$$\begin{array}{rccc} \tau\colon & \Sigma & \longrightarrow & \Sigma \\ & (\theta,t) & \longmapsto & (\theta+2\pi t,t). \end{array}$$

Note that, for $n \in \mathbb{Z}$, we have $\tau^n(\theta,t) = (\theta + 2\pi nt, t)$.

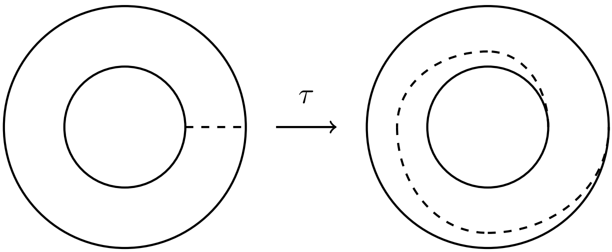

Figure 3.4: Dehn twist.

Consider the involutions

$$g_n(\theta,t) = (-2\pi nt - \theta, t), n \in \mathbb{Z},$$

on Σ. All of them are orientation-reversing, and consequently, they define real structures on the annulus Σ. Furthermore, we have $\tau^n = g_0 \circ g_n$. Thus if (Σ,ϕ) is an open book with $\phi = \tau^k$ for some $k \in \mathbb{Z}$, choose any of the real structures $g = g_n$, $n \in \mathbb{Z}$. Then we have $\phi \circ g \circ \phi = g$, i.e. (Σ,ϕ,g) is a real open book.

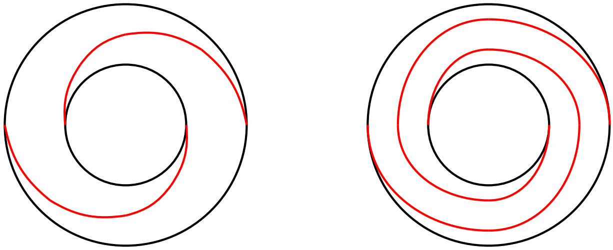

Figure 3.5: Fixed point sets of g_1 (left) and g_2 (right).

According to [EÖ08, Lemma 5.1], $M(\tau^n)$ is diffeomorphic to $L(n, n-1)$ for $n > 0$ and to $L(-n, 1)$ for $n \leq 0$.

Let (B, p) be an open book decomposition of a contact manifold $(M, \xi = \ker \alpha)$. The contact structure ξ is **supported** by the open book if α is positive on the binding (with orientation induced as the boundary of the pages) and $d\alpha$ is a symplectic form on each page of the open book. A real open book decomposition (B, p) of a real contact manifold $(M, \xi = \ker \alpha, f)$ is said to **support** ξ if (B, p) supports ξ in the classical sense. In [ÖS15], the authors show that every abstract real open book (Σ, ϕ, g) supports a real contact structure on the manifold $M(\phi)$. In Section 1.4, we raised the question whether every manifold that admits both a real and a contact structure admits a real contact structure, too. This question could be answered affirmatively if one knew that every real 3-manifold admits an abstract real open book. Moreover, Öztürk and Salepci conjecture that a real Giroux correspondence holds:

Conjecture. *Let (M, f) be a real 3-dimensional manifold.*

Then there is a one-to-one correspondence between real contact structures on (M, f) up to equivariant contact isotopy and real open book decompositions of (M, f) up to positive real stabilisation and equivariant isotopy.

As an application of real open books, here is a real version of Bourgeois's theorem:

Theorem 3.2.7. *Let $(M, \xi = \ker \alpha, f)$ be a closed real contact manifold of dimension three. Suppose that (B, p) is a real open book decomposition that supports ξ. Then $M \times T^2$ admits a real contact structure $(\widetilde{\xi}, \widetilde{f})$.*

Proof. Let (r, φ) be polar coordinates on the D^2-factor of a neighbourhood of the binding B such that the projection p is given by φ in that neighbourhood. Thanks to the compatibility condition (3) in the definition of real open books, we have $\varphi \circ f = -\varphi$. Let ρ be a symmetric function of the variable r on $B \times D^2$ such that

- $\rho(r) = r$ near $B \times \{\mathbf{0}\}$,
- $\rho'(r) \geq 0$, and
- $\rho \equiv 1$ near $B \times \partial D^2$.

Extend ρ to a symmetric function on M, equal to 1 outside $B \times D^2$. Then, setting $x_1 := \rho \cos \varphi$ and $x_2 := \rho \sin \varphi$, we have $x_1 \circ f = x_1$ and $x_2 \circ f = -x_2$. Define a 1-form $\widetilde{\alpha}$ on $M \times T^2$ by

$$\widetilde{\alpha} = x_1 d\theta_1 - x_2 d\theta_2 + \alpha,$$

and let $\widetilde{f}\colon M \times T^2 \to M \times T^2$ be the real structure given by

$$\widetilde{f}\,(p, (\theta_1, \theta_2)) = ((f(p), (-\theta_1, \theta_2))\,.$$

Then we have $\widetilde{f}^*\widetilde{\alpha} = -\widetilde{\alpha}$, i.e. $(M \times T^2, \widetilde{\xi} = \ker \widetilde{\alpha}, \widetilde{f})$ is a real contact manifold. The verification of the the contact condition is accomplished as in [Gei08, Theorem 7.3.6]. $\square$

In conjunction with the preceding discussion, this theorem implies:

Corollary 3.2.8. *There are real contact structures on the 5-manifolds $S^3 \times T^2$, $L(n, n-1) \times T^2$, $n > 0$, and $L(n, 1) \times T^2$, $n \geq 0$.* $\square$

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Erklärung

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Köln, 1. September 2017 Christian Evers