

New approaches to a homotopical problem in representation theory



Dissertation

zur

Erlangung des Doktorgrades

der Mathematisch-Naturwissenschaftlichen Fakultät

der Universität zu Köln

vorgelegt von

Maximilian Kaipel

angenommen im Jahr **2025**

Acknowledgements

First of all I would like to thank Sibylle Schroll for her invaluable guidance and constant encouragement throughout my PhD as well as for maintaining such an active and welcoming research environment in Cologne. I am also grateful to Hipolito Treffinger for countless meetings and mathematical discussions as well as his continuous encouragement throughout my studies. In the autumn of 2024, I spent two months at the University of Buenos Aires and I am thankful to Deutscher Akademischer Austauschdienst (DAAD) for providing me with the opportunity for this research visit, and to Hipolito Treffinger for being a wonderful host.

During my PhD, I have been able to travel to many conferences and I want to thank the organisers of these events for their efforts. I owe gratitude to the DAAD for funding my participation at the International Conference on Representations of Algebras in Shanghai, China, in 2024, as well as to the University of Cologne for funding my participation at a conference in Trondheim, Norway, in 2024. A big thank you to everyone who invited me to present my research at conferences and seminars, for the opportunities to share my research with a greater audience.

I also want to thank all the friends I have made in Cologne and everywhere else in the last three years for making my experience a lot more enjoyable. Finally, I want to thank my family and my partner for their endless encouragement and support.

Erklärung

Teilpublikationen:

Kapitel 3 basiert auf dem im *Journal of the London Mathematical Society* veröffentlichten Artikel “The category of a partitioned fan” verfasst von mir. Siehe [120].

Kapitel 4 basiert auf dem Preprint “ τ -cluster morphism categories of factor algebras” verfasst von mir. Siehe [119].

Kapitel 5 enthält einen Ausschnitt aus dem Preprint “Bricks and τ -tilting theory under base field extensions” verfasst in Kollaboration mit Dr. Erlend D. Børve. Siehe [34].

Abstract

Let K be a field. The central object of this thesis is the τ -cluster morphism category $\mathfrak{W}(A)$ of a finite-dimensional K -algebra A . This category encodes the information of all possible τ -tilting reductions in $\text{mod } A$ and encompasses many objects considered in representation theory, for example (semi-)bricks, (τ) -tilting modules and (τ) -exceptional sequences. When A is a hereditary algebra, the category $\mathfrak{W}(A)$ is well-understood and its classifying space $\mathcal{BW}(A)$ is a $K(\pi, 1)$ space in all representation finite and some tame cases. By definition, this means that the fundamental group is the only nontrivial homotopy group. This is insightful, because the fundamental group of this space, known as the picture group, is closely connected to maximal green sequences, a central object in the theory of cluster algebras.

The guiding question of this thesis is based on the conjecture that the classifying space of the τ -cluster morphism category is a $K(\pi, 1)$ space for all τ -tilting finite algebras. It is known that the classifying space of $\mathfrak{W}(A)$ is a cube complex. Thus, the conditions developed by Gromov for cube complexes to be nonpositively curved may be lifted to three conditions which together imply that $\mathcal{BW}(A)$ is a $K(\pi, 1)$ space. The focus of this thesis lies on one of these conditions: the existence of a faithful functor from $\mathfrak{W}(A)$ to a group considered as a groupoid with one object. In fact, given this condition $\mathcal{BW}(A)$ is nonpositively curved if and only if it satisfies the other two conditions.

Such a faithful functor to a group is conjectured to exist for all finite-dimensional algebras which are τ -tilting finite. However, few families of algebras satisfying this condition have been found so far. The first part of this thesis builds on a recently introduced geometric viewpoint of $\mathfrak{W}(A)$ by Schroll–Tattar–Treffinger–Williams. This geometric approach is developed further to obtain a new family of algebras admitting faithful

functors to groups. This is achieved by relaxing a condition in the geometric definition. In this way, a category is defined for any simplicial polyhedral fan with an admissible partition, so that the collection of these categories, for a given fan, forms a lattice. For the g -vector fan of an algebra A , the category $\mathfrak{W}(A)$ is an element of this lattice. If the g -vector fan of A is a finite hyperplane arrangement, it is shown that $\mathfrak{W}(A)$ admits a faithful functor to a group by using the theory of hyperplane arrangements in convex geometry.

In the second part of this thesis, a lattice theoretic approach to $\mathfrak{W}(A)$ is introduced. As a first step, the τ -cluster morphism category is defined using the lattice of torsion classes of $\text{mod } A$. Whenever A is τ -tilting finite, this definition is purely combinatorial, so that the lattice of torsion classes determines $\mathfrak{W}(A)$ up to equivalence. Moreover, the lattice of torsion classes of a τ -tilting finite algebra is isomorphic to that of infinitely many others. Thus, this result extends the families of algebras whose τ -cluster morphism categories admit faithful functors to groups. Let I be an ideal of A . The lattice theoretic approach provides a framework for constructing a functor $F_I : \mathfrak{W}(A) \rightarrow \mathfrak{W}(A/I)$ and various properties of this functor are investigated. In particular, if A is τ -tilting finite, the functor is a regular epimorphism in the category of small categories.

The final chapter contains another application of the lattice theoretic approach. Let $L : K$ be a MacLane separable field extension. After further developing the behaviour of τ -tilting theory under base field extension, a faithful functor $\mathcal{F} : \mathfrak{W}(A) \rightarrow \mathfrak{W}(A \otimes_K L)$ is constructed using lattice theory. The existence of $\mathcal{F}$ leads to the discovery of new families of algebras whose τ -cluster morphism categories admit faithful functors to groups. Thus, this thesis contributes substantially to understanding the relationship between τ -cluster morphism categories of different algebras as well as to answering the question of when their classifying spaces are $K(\pi, 1)$ spaces.

Zusammenfassung

Es sei K ein Körper. Das zentrale Objekt dieser Dissertation ist die τ -Clustermorphismuskategorie $\mathfrak{W}(A)$ einer endlich-dimensionalen K -Algebra A . Diese Kategorie enthält die Information aller möglichen τ -Kippreduktionen in $\text{mod } A$ und umfasst außerdem viele Objekte der Darstellungstheorie, zum Beispiel (Halb-)Ziegel, (τ) -Kippmoduln und (τ) -exzeptionelle Folgen. Wenn A eine erbliche Algebra ist, dann ist $\mathfrak{W}(A)$ gut verstanden und der klassifizierende Raum $\mathcal{BW}(A)$ ist in allen darstellungsendlichen und manchen zahmen Fällen ein $K(\pi, 1)$ -Raum. Per Definition bedeutet das, dass die einzige nicht-triviale Homotopiegruppe die Fundamentalgruppe ist. Das ist aufschlussreich, weil die Fundamentalgruppe, die auch als Bildgruppe bekannt ist, mit maximalen grünen Folgen zusammenhängt, welche zentrale Objekte in der Clustertheorie sind.

Die Leitfrage dieser Dissertation beruht auf der Vermutung, dass der klassifizierende Raum der τ -Clustermorphismuskategorie ein $K(\pi, 1)$ -Raum für alle τ -kippendlichen Algebren ist. Der klassifizierende Raum $\mathcal{BW}(A)$ ist ein kubischer Komplex. Deshalb können die von Gromov entwickelten Bedingungen, an einen kubischen Komplex nichtpositiv gekrümmt zu sein, benutzt werden, um drei Bedingungen zu geben, die zusammen implizieren, dass $\mathcal{BW}(A)$ ein $K(\pi, 1)$ -Raum ist. Der Fokus dieser Dissertation liegt auf einer dieser Bedingungen: der Existenz eines treuen Funktors von $\mathfrak{W}(A)$ zu einer Gruppe, die als Gruppoid mit einem Objekt betrachtet wird. Unter dieser Voraussetzung ist $\mathcal{BW}(A)$ genau dann nichtpositiv gekrümmt wenn die beiden anderen Bedingungen erfüllt sind.

Die Existenz eines solchen treuen Funktors wird für alle endlich-dimensionalen Algebren vermutet, welche τ -kippendlich sind. Bisher sind nur wenige Klassen von Algebren, die diese Bedingung erfüllen, bekannt. Der erste Teil dieser Dissertation baut auf einem kürzlich eingeführten geometrischen Ansatz von Schroll–Tattar–Treffinger–Williams auf.

Diese geometrische Herangehensweise wird weiterentwickelt, um eine neue Familie von Algebren, deren τ -Clustermorphismuskategorien treue Funktoren zu einer Gruppe zu lassen, zu finden. Zu diesem Zweck wird eine Bedingung der geometrischen Definition gelockert. Dadurch wird für jeden glatten, polyedrischen Fächer mit einer zulässigen Partition eine Kategorie definiert, sodass die Sammlung dieser Kategorien eines Fächers einen Verband formt. Für den Fächer der g -Vektoren einer Algebra, enthält dieser Verband die Kategorie $\mathfrak{W}(A)$. Falls der Fächer der g -Vektoren von A eine endliche Anordnung von Hyperebenen ist, wird mit Hilfe der Theorie der Hyperebenenanordnungen ein treuer Funktor von $\mathfrak{W}(A)$ zu einer Gruppe konstruiert.

Der zweite Teil dieser Dissertation führt einen verbandstheoretischen Ansatz ein. Als erster Schritt wird die τ -Clustermorphismuskategorie mit Hilfe des Verbands der Torsionsklassen in $\text{mod } A$ definiert. Wenn A eine τ -kippendliche Algebra ist, ist diese Definition gänzlich kombinatorisch, sodass der Verband der Torsionsklassen die Kategorie $\mathfrak{W}(A)$ bis auf Äquivalenz bestimmt. Außerdem ist der Verband der Torsionsklassen jeder τ -kippendlichen Algebra isomorph zu unendlich vielen solcher Verbände anderer Algebren. Dadurch erweitert dieses Ergebnis die derzeit bekannten Familien von Algebren, deren τ -Clustermorphismuskategorien treue Funktoren zu Gruppen zulassen. Darüber hinaus sei I ein Ideal von A . Die verbandstheoretische Herangehensweise und die Theorie der Verbandskongruenzen ermöglichen es, einen Funktor $F_I : \mathfrak{W}(A) \rightarrow \mathfrak{W}(A/I)$ zu konstruieren und es werden einige Eigenschaften dieses Funktors untersucht. Insbesondere, falls A eine τ -kippendliche Algebra ist, ist dieser Funktor ein regulärer Epimorphismus in der Kategorie der kleinen Kategorien.

Das letzte Kapitel behandelt eine weitere Anwendung des verbandstheoretischen Ansatzes. Es sei $L : K$ eine MacLane separable Körpererweiterung. Nachdem das Verhalten der τ -Kipptheorie unter Körpererweiterungen weiter entwickelt wurde, wird ein treuer Funktor $\mathcal{F} : \mathfrak{W}(A) \rightarrow \mathfrak{W}(A \otimes_K L)$ konstruiert. Die Existenz des Funktors $\mathcal{F}$ führt zur Entdeckung weiterer Familien von Algebren, deren τ -Clustermorphismuskategorien treue Funktoren zu Gruppen zulassen. Zusammenfassend trägt diese Dissertation sowohl zum Verständnis des Verhältnisses zwischen τ -Clustermorphismuskategorien unterschiedlicher Algebren als auch zur Beantwortung der Frage, wann deren klassifizierenden Räume $K(\pi, 1)$ -Räume sind, bei.

Contents

1	Introduction	1
1.1	Generalisations using τ -tilting theory	4
1.2	A geometric approach	7
1.3	A combinatorial approach	8
1.4	Base field extensions of algebras	9
2	Preliminaries	11
2.1	Lattice theory and brick labelling	14
2.2	τ -tilting theory	17
2.3	The τ -cluster morphism category	20
2.4	Cubical categories	23
2.5	Fans and simplicial complexes	28
2.6	The g -vector fan	32
3	The category of a partitioned fan	36
3.1	Definition of the category	37
3.2	Cubical structure	42
3.3	Classifying spaces as a CW-complexes	46
3.4	Picture groups and fan posets	50
3.5	Eilenberg-MacLane spaces in dimension 2	58
3.6	Hyperplane arrangements	61
3.7	Lattice of admissible partitions	68
3.8	Applications to the τ -cluster morphism category	72

4	τ-cluster morphism categories of factor algebras	77
4.1	Lattice congruences	78
4.2	Lattice theoretic definition of the τ -cluster morphism category	79
4.3	Invariance under τ -tilting equivalence	84
4.4	Factor algebras and lattice congruences	90
4.5	Epimorphisms and lifting τ -perpendicular intervals	100
4.6	Classifying spaces and picture groups	105
4.7	Examples	106
5	τ-cluster morphism categories and base field extension	112
5.1	Preliminaries on field extensions	112
5.2	τ -tilting theory under base field extension	115
5.3	Bongartz completions and τ^{-1} -tilting theory	120
5.4	Left and right finite semibricks	123
5.5	Faithful functor	127
5.6	Applications and examples	131
	References	135

Chapter 1

Introduction

In the representation theory of finite-dimensional associative algebras, a central role is played by the class of hereditary algebras. The classification of representation finite hereditary algebras (over algebraically closed fields) as path algebras of quivers whose underlying graph is a Dynkin diagram of the form A_n , D_n , E_6 , E_7 or E_8 establishes a strong relationship between finite-dimensional hereditary algebras, quantum groups and Lie algebras [74]. In fact, indecomposable modules over these algebras correspond bijectively to positive roots of the associated root systems. This connection was developed further to relate the Hall algebra [160] of the hereditary algebra with the universal enveloping algebra of the associated Lie algebra [85, 161].

Moreover, the theory of hereditary algebras has been applied in [49–51, 136] to provide important insights into cluster algebras [71]. Since their introduction, the combinatorial structure underlying cluster algebras has been discovered in various mathematical objects, including some objects arising in theoretical physics [75, 126, 127]. Conversely, this combinatorial structure has been categorified using representations of hereditary algebras, resulting in cluster categories [48]. In a cluster category, the role held by mutable seeds in a cluster algebra is taken by mutable cluster tilting objects. These objects are in bijection with many important objects in representation theory, including functorially finite torsion classes [62] and wide, that is, exactly-embedded abelian and extension-closed, subcategories [98] of the module category [108].

Combinatorially, cluster tilting objects are in bijection with non-crossing partitions

[108, 129]. Other than being of inherent algebraic combinatorial interest, non-crossing partitions arise naturally in the theory of free probability [177] and even in idealised models coming from molecular biology [175].

Additionally, cluster tilting objects are intimately related to exceptional sequences [54, 103, 164]. In fact, there exists an explicit map between these two classes of objects. Generally, this map is neither injective nor surjective, but an extension of the notion to signed exceptional sequences establishes an explicit bijection between these sequences and ordered cluster tilting objects [105]. The study of (unsigned) exceptional sequences dates back 30 years and was initiated in the setting of triangulated categories [30, 82, 83, 167] and shortly after in abelian categories [54, 164]. Exceptional sequences can be used to gain a better understanding of derived categories [128] and they are closely connected to stratifying systems [69].

Direct summands of cluster tilting objects, called partial cluster tilting objects, are equally important. Considering Hom-Ext-orthogonal modules to the modules underlying partial cluster tilting objects yields so-called perpendicular subcategories of the module category. Perpendicular subcategories had previously been studied independently and various of their applications can be found in [78, 131, 137]. Perpendicular subcategories give rise to various reduction techniques, for example, applicable to the classification of representations of tame hereditary algebras via a reduction of the problem to the Kronecker quiver.

Many of the notions discussed above are united in the cluster morphism category [105]. The objects of this category are functorially finite wide subcategories, which are in bijection with non-crossing partitions and functorially finite torsion classes for hereditary algebras [108]. The morphisms of this category are indexed by partial cluster tilting objects, that is, a morphism $[T] : \mathcal{W}_1 \rightarrow \mathcal{W}_2$ is given by a partial cluster tilting object T of the cluster category satisfying $T^\perp \cap \mathcal{W}_1 = \mathcal{W}_2$, where $T^\perp$ denotes the perpendicular subcategory corresponding to T .

Associated to any category is its classifying space [172]. The classifying space of the cluster morphism category of a representation finite hereditary algebra A is a cube complex [106, 171] and homeomorphic to the picture space of A as defined in [105, 107]. In fact, it is a $K(\pi, 1)$ space whose fundamental group is called the picture group.

The picture group $G(K\vec{A}_n)$ of the linearly oriented quiver of type A_n was originally defined as the “Stasheff group” in 1999 [134]. The name “Stasheff group” comes from the fact that a $K(G(K\vec{A}_n), 1)$ space can be obtained as a quotient space of the Stasheff associahedron. Its connection to representation theory follows from the observation that the partially ordered set of tilting modules of $K\vec{A}_n$ corresponds to the 1-skeleton of the Stasheff associahedron [44].

For hereditary algebras of finite type, the picture group contains a “Coxeter element”. In the defining presentation of the group, expressions for the Coxeter element, which do not contain inverse generators, are in bijection with maximal green sequences [104]. These green sequences [124] originate in the study of cluster algebras and arise in the context of Donaldson-Thomas invariants and BPS states in mathematical physics [127].

More generally, if the hereditary algebra is of finite type or of certain tame types, the classifying space exhibits the stronger property of being a locally CAT(0) cube complex [106]. This may be proved using the conditions developed in the celebrated work [86] for cube complexes to be (locally) CAT(0), which have been translated into the language of category theory [101]. In this case, the universal covering space of the classifying space is a CAT(0) cube complex. Since the picture group is isomorphic to the group of deck transformations of the universal cover, it acts on this CAT(0) cube complex [95]. Finitely generated groups with similarly well-behaved actions on CAT(0) cube complexes are known as cubulated (or CAT(0)) groups [183] and have received significant attention in the literature [4, 80, 89, 148, 174] because of their special properties.

In conclusion, the cluster morphism category demonstrates the interlinked structure of many notions within representation theory and their connections to other mathematical areas via hereditary algebras. The aim of this thesis is to expand the understanding of these concepts and relationships beyond the setting of hereditary algebras. Appropriate generalisations of perpendicular subcategories [115], exceptional sequences [45], picture groups [91] and cluster morphisms categories [40] have already been established and provide the foundation for the work herein. The primary focus lies on the question of whether (generalised) cluster morphism categories are CAT(0) and thus $K(\pi, 1)$ spaces for the (generalised) picture group. One crucial part of a sufficient condition for this to be the case is the existence of a faithful functor from the (generalised) cluster morphism

category to the (generalised) picture group, considered as a groupoid with one object. The main contribution of the present thesis lies in the (further) development of two new approaches for constructing such a faithful functor. One approach is of convex geometric nature and one approach is of lattice theoretic nature. Along the way, other new results about (generalised) exceptional sequences and the (generalised) cluster morphism category which are of independent interest as well as new structural results relating (generalised) cluster morphism categories of different algebras are established.

1.1 Generalisations using τ -tilting theory

One decade ago, the combinatorial structure of cluster tilting objects in cluster categories was transferred to module categories of arbitrary finite-dimensional algebras. The central objects of the newly introduced τ -tilting theory are (support) τ -tilting modules, which generalise the mutation behaviour of cluster tilting objects [3]. To understand this relationship, fix a cluster tilting object in a cluster category and consider its endomorphism algebra. Then, there is a bijection between cluster tilting objects in the cluster category and the support τ -tilting modules over the endomorphism algebra which preserves and reflects the mutation of both structures [3].

The name-giving “ τ ” in τ -tilting theory denotes the Auslander–Reiten translation, which is the central notion of Auslander–Reiten theory as initiated in [14]. This theory utilises almost split exact sequences to gain a better understanding of module categories. The importance of the relationship of a module with its Auslander–Reiten translation has been demonstrated repeatedly, see for example [16]. Restricting to the class of hereditary algebras, τ -tilting modules coincide with tilting modules [35], which were introduced to generalise reflection functors [13, 28]. Tilting modules of hereditary algebras are of particular importance, because their endomorphism algebras, known as tilted algebras [31, 93], can be studied using the representation theory of the hereditary algebra. However, tilting modules generally do not admit a well-behaved notion of mutation for arbitrary algebras. Because tilting modules can be viewed as τ -tilting modules of projective dimension at most one [15], the area of τ -tilting theory is often seen as a completion of classical tilting theory from the viewpoint of mutation.

Since its introduction, τ -tilting theory has been widely accepted as the appropriate framework for translating problems and notions from hereditary algebras to arbitrary finite-dimensional algebras. For example, τ -tilting finite algebras are those admitting only finitely many isomorphism classes of basic τ -tilting modules [56] and generalise representation finite algebras. It is remarkable, that even some representation wild algebras can be τ -tilting finite, thus offering a new method for understanding at least some aspects of wild algebras better. While there is no universally agreed upon notion of “ τ -tilting tameness”, various generalisations have been suggested, see [153] and the references therein. Moreover, the classical Brauer-Thrall conjectures [114] have found τ -tilting analogues [144, 170] as well as Hochschild cohomology [53].

More importantly, for the goal of a suitably defined “ τ -cluster morphism category”, the notion of τ -perpendicular subcategories generalise the classical. Using direct summands of τ -tilting modules, so-called τ -rigid modules, gives rise to τ -perpendicular subcategories of the module category which are equivalent to module categories of other finite-dimensional algebras [57, 115]. In particular, these τ -perpendicular subcategories constitute the objects of the τ -cluster morphism category as defined in [40, 46]. Morphisms $[(M, P)] : \mathcal{W}_1 \rightarrow \mathcal{W}_2$ in this category are then given by τ -rigid pairs (M, P) , which consist of a τ -rigid module M and an orthogonal projective module P , of $\mathcal{W}_1$ such that the τ -perpendicular category with respect to (M, P) in $\mathcal{W}_1$ is equal to $\mathcal{W}_2$, see Definition 2.3.1 for details.

Similar to the hereditary setting, factorisations of morphisms in the τ -cluster morphism category into irreducible ones correspond to signed τ -exceptional sequences [45, 46]. In general, (signed) τ -exceptional sequences coincide with (signed) exceptional sequences for hereditary algebras and one of the main motivations for their introduction is the fact that exceptional sequences of maximal length do not always exist for non-hereditary algebras. Whereas, using the reduction process τ -perpendicular subcategories enable, the existence of τ -exceptional sequences of maximal length is guaranteed for any finite-dimensional algebra. Furthermore, a mutation operation for τ -exceptional sequences was recently introduced in [41], which generalises the mutation in the hereditary setting. While transitivity of this mutation is known for τ -tilting finite algebras [41–43, 150], it appears that in most cases this generalisation comes at the cost of the

Braid group action present in the hereditary setting [54, 164].

A generalisation of the picture group to τ -tilting finite algebras was proposed in [91], and shown to be isomorphic to the fundamental group of the classifying space of the τ -cluster morphism category. Moreover, the classifying space is again a cube complex, which enables the use of the sufficient condition developed in [86, 101] for it to be locally CAT(0) and thus a $K(\pi, 1)$ space. Based on the definition in [40], a generalisation of the picture group to arbitrary finite-dimensional algebras is straightforward and given in Definition 2.3.2. Since the classifying spaces of cluster morphism categories of representation finite hereditary algebras are a $K(\pi, 1)$ spaces, it is natural to investigate this question in a more general setting. This gives rise to the guiding conjecture of the present thesis, see also [102, Problem 3.0.5].

Conjecture 1.1.1. Let A be a finite-dimensional algebra over a field K and let $\mathcal{BW}(A)$ denote the classifying space of the τ -cluster morphism category of A . If A is τ -tilting finite, then $\mathcal{BW}(A)$ is a $K(\pi, 1)$ space.

All previous work attempting to answer this question, including the approaches taken in this thesis, relies on the following sufficient condition of [101] which leverages those for cube complexes developed in [86].

Proposition 1.1.2. *If the following three conditions are satisfied, then $\mathcal{BW}(A)$ is locally CAT(0) and thus a $K(\pi, 1)$ space:*

- (1) *There exists a faithful functor $\Psi : \mathfrak{W}(A) \rightarrow G$ for some group G considered as a groupoid with one object;*
- (2) *The category $\mathfrak{W}(A)$ satisfies the pairwise compatibility condition of first factors;*
- (3) *The category $\mathfrak{W}(A)$ satisfies the pairwise compatibility condition of last factors.*

The second and third conditions are defined in Proposition 2.4.4. In the presence of Condition (1), they coincide precisely with those developed in [86]. Moreover, a detailed description of previous work regarding these three conditions is given in Example 2.4.5. It is important to highlight that the second condition is always satisfied due to the structure of τ -rigid pairs, whereas the third condition may not hold [20, 92]. Moreover, it was conjectured in [92], that the first condition, the existence of a faithful (group) functor,

holds for all τ -tilting finite algebras. In this thesis, the focus therefore lies on showing that the first condition holds for various classes of algebras. For this purpose, new approaches to the τ -cluster morphism category are developed, which reveal interesting structural results regarding this category as well as new insights into Conjecture 1.1.1.

1.2 A geometric approach

In the theory of cluster algebras, an important role is played by g -vectors, which were introduced to express the cluster variables of any mutated cluster in terms of the initial seed [72]. Many of the conjectures raised in [72] were later answered using the theory of representations of quivers (with potential) and g -vectors in [61]. It is a fundamental observation that the collection of polyhedral cones generated by compatible g -vectors forms a polyhedral fan. This fan appears in the tropical cluster $\mathcal{X}$ -variety [70] and is a subfan of the cluster scattering diagram [87] and the stability scattering diagram [36] in many cases. Moreover, when the cluster algebra is of finite type, it also arises as the normal fan of the generalised associahedron [52].

Returning to the representation theory of finite-dimensional algebras, g -vector fans find a natural analogue using minimal projective presentations [3, 55, 56, 60]. In this setting, the g -vector fan of an algebra encodes its τ -tilting theory by definition. Thus, if the algebra is τ -tilting finite, its g -vector fan is finite and complete [56]. The completeness of the fan gives rise to a notion of g -tameness proposed in [7]. Moreover, the g -vector fan embeds into the wall-and-chamber structure [9, 38] of the algebra which is the support of the scattering diagram [36].

A new approach to the τ -cluster morphism category was recently initiated in [169], where a category equivalent to the τ -cluster morphism category is constructed from g -vector cones. Due to its geometric definition, this construction simplifies the proof of associativity of morphisms in the category, which had previously been a significant hurdle [33, 40, 46]. Objects of the geometrically defined τ -cluster morphism category are equivalence classes of g -vector cones. In Chapter 3, the rule for identifying cones is relaxed. This defines a category associated to any simplicial polyhedral fan and a choice of (admissible) partition of its cones. The resulting “category of the partitioned fan”

shares many properties with the τ -cluster morphism category since its construction generalises that of [169]. In particular, the simplicial fan structure implies that the classifying spaces of these categories form cube complexes.

As a consequence, the collection of different admissible partitions of the g -vector fan establishes a lattice (in the order theoretic sense) of categories containing the τ -cluster morphism category. If the g -vector fan of an algebra is a finite hyperplane arrangement, the theory of hyperplane arrangements may be applied to show that Condition (1) of Proposition 1.1.2 holds for the maximal element in this lattice. Moreover, since there exists a faithful functor from the τ -cluster morphism category of the algebra to the maximal element, this establishes the existence of a faithful group functor from the τ -cluster morphism category. Two important classes of algebras whose g -vector fan forms a finite hyperplane arrangement are (generalised) preprojective algebras [79, 141, 146] and contraction algebras [12, 182].

Moreover, the ubiquity of polyhedral fans throughout many areas of mathematics [27, 37, 73, 143, 168, 185] suggests that fans arising in other contexts may admit meaningful admissible partitions. This opens up many directions for future research, aiming to obtain new insights from the categories constructed from partitioned fans.

1.3 A combinatorial approach

In an abelian category, a torsion class [62] is a subcategory which axiomatises the properties of torsion groups in the category of abelian groups. Torsion classes are intimately connected to the study of triangulated categories and their t -structures [24]. Moreover, torsion classes are closely related to classical tilting theory [35], as well as τ -tilting theory [3]. Indeed, the subset of functorially finite torsion classes is in bijection with support τ -tilting modules. Moreover, an algebra is τ -tilting finite if and only if every torsion class is functorially finite [56].

The significance of torsion classes in a variety of settings has inspired substantial research. In particular, the partially ordered structure of the set of all torsion classes, which forms a lattice, provides the foundation for a large body of research [18, 57, 77, 109, 111, 162]. It turns out that, in many cases, algebraic information can be recovered

combinatorially from the lattice structure of the poset of torsion classes, especially when the lattice is finite. For example, τ -exceptional sequences [19] and the poset of wide subcategories [67] can be constructed entirely combinatorially. To a certain extent, it is even possible to reconstruct an algebra given its lattice of torsion classes [94, 121].

In Chapter 4 the τ -cluster morphism category of a τ -tilting finite algebra is constructed combinatorially from its (finite) poset of torsion classes. This leads to the observation that two algebras with isomorphic finite posets of torsion classes have equivalent τ -cluster morphism categories. However, for each τ -tilting finite algebra, there are infinitely many nonisomorphic algebras with isomorphic posets of torsion classes [66]. Consequently, this result extends the known families of algebras satisfying the conditions of Proposition 1.1.2. Another consequence is that the signed τ -exceptional sequences of two such algebras are in bijection, and may be constructed combinatorially.

Additionally, using this lattice theoretic construction reveals a relationship between the τ -cluster morphism category of an algebra and that of a quotient algebra by an ideal. More precisely, using the theory of lattice congruences, an explicit “quotient” functor is constructed. Whenever the lattice of torsion classes is finite, this functor is a regular epimorphism in the category of small categories. Thus, in a vague sense, the “quotient relationship” between two algebras is reflected by their τ -cluster morphism categories. This lattice theoretic viewpoint of the τ -cluster morphism category may then be applied to find further examples of algebras whose τ -cluster morphism categories admit faithful group functors.

1.4 Base field extensions of algebras

Recall that representation finite hereditary algebras over algebraically closed fields are precisely path algebras of simply laced Dynkin quivers [74]. Using the theory of field extensions and of valued quivers, also known as species [63, 64], this connection is extended to give a classification of representation finite hereditary algebras over perfect fields as those coming from finite Dynkin diagrams, that is, including types B_n , C_n , F_4 and G_2 . Again indecomposable modules correspond bijectively to roots of the corresponding root systems [118].

Many interesting families of algebras arise from valued quivers and illustrate that the world of finite-dimensional algebras beyond bound quiver algebras should be explored further. In fact, given a field extension $L : K$, many properties of $A \otimes_K L$ are inherited by the finite-dimensional K -algebra A , and vice versa. However, by passing to an algebraic closure of K , for example, certain geometric tools become available, which were not available in the original setting. This idea is common in algebraic geometry where it provides the basis of descent theory, see [97, 140] and the references therein.

In representation theory, the proof of the 2nd Brauer–Thrall conjecture can be extended from algebraically closed fields to perfect fields [22] and many homological dimensions are preserved under field extension [116]. Moreover, representation finiteness of an algebra is preserved [116, 122], as well as the notion of (semi-)generic tameness [123, 138, 152]. Some particular families of algebras are also preserved under base field extension, for example hereditary algebras [116], piecewise hereditary algebras, canonical algebras, tilted algebras [132], and derived-discrete algebras [133]. Furthermore, the techniques of base field extension have been applied to extend proofs over algebraically closed fields to arbitrary fields [56, 110, 133] and to unfold valued quivers to classical quivers [58, 59, 100], offering a close relationship and frequently revealing beautiful symmetries.

In Chapter 5, the study of τ -tilting theory under base field extension is developed further. The chapter culminates in the construction of a faithful functor from the τ -cluster morphism category of a finite-dimensional algebra A over a field K to the τ -cluster morphism category of $A \otimes_K L$, where $L : K$ is a MacLane separable field extension. Consequently, if the latter admits a faithful group functor, so does the former. This result may be applied to find new families of algebras whose τ -cluster morphism categories admit faithful group functors. In particular, this is made precise for quotients of representation finite and hereditary algebras over perfect fields, partially extending results of [92, 106].

Chapter 2

Preliminaries

Most generally, consider the setting of an essentially small abelian length category $\mathcal{A}$, which will later be specialised to the category of finite-dimensional right modules of a finite-dimensional algebra. Let $\mathcal{C}$ be a full subcategory of $\mathcal{A}$ and define the following related subcategories:

- $\text{add } \mathcal{C} := \{X \in \mathcal{A} : \exists \text{ a split epimorphism } C^n \rightarrow X \text{ for some } C \in \mathcal{C} \text{ and } n \geq 1\};$
- $\text{Fac } \mathcal{C} := \{X \in \mathcal{A} : \exists \text{ an epimorphism } C^n \twoheadrightarrow X \text{ for some } C \in \mathcal{C} \text{ and } n \geq 1\};$
- $\text{Sub } \mathcal{C} := \{X \in \mathcal{A} : \exists \text{ a monomorphism } X \hookrightarrow C^n \text{ for some } C \in \mathcal{C} \text{ and } n \geq 1\};$
- $\text{Filt } \mathcal{C} := \{X \in \mathcal{A} : \exists \text{ a filtration } 0 = X_0 \subseteq \cdots \subseteq X_r = X \text{ with } X_{i+1}/X_i \in \text{add } \mathcal{C}\};$
- $\mathcal{C}^\perp := \{X \in \mathcal{A} : \text{Hom}(C, X) = 0 \text{ for all } C \in \mathcal{C}\};$
- ${}^\perp \mathcal{C} := \{X \in \mathcal{A} : \text{Hom}(X, C) = 0 \text{ for all } C \in \mathcal{C}\}.$

Moreover, given an object $C \in \mathcal{A}$, define the subcategories $\text{Fac } C$, $\text{Sub } C$ and $\text{Filt } C$ as $\text{Fac}(\text{add } C)$, $\text{Sub}(\text{add } C)$ and $\text{Filt}(\text{add } C)$ respectively. In cases where explicit reference to the ambient category $\mathcal{A}$ is important to avoid ambiguities, a subscript is added to the subcategories above. For example, $\text{Fac}_{\mathcal{A}} \mathcal{C}$ or $\mathcal{C}^{\perp_{\mathcal{A}}}$.

Among the most well-studied classes of subcategories of an abelian category are torsion classes as introduced in [62], which axiomatise the properties of torsion groups in the category of abelian groups. These will play a central role in this thesis. Torsion classes are closely related to t -structures of triangulated categories [24] and have analogues in a plethora of categorical settings [2, 25, 117, 179]. In representation theory, torsion

classes are essential objects of tilting theory [35], τ -tilting theory [3] and Auslander–Reiten theory [16]. As subcategories, torsion classes admit a partial order with respect to inclusion and arise as the Tamari lattice [181], as Cambrian lattices [108] and as the weak order of Weyl groups [141] in combinatorics. Their purely lattice theoretic properties have also inspired substantial research [18, 57, 77, 109, 111, 162]. Additionally, the study of torsion classes is motivated by their connection to cluster algebras [39].

Definition 2.0.1. [62] A pair $(\mathcal{T}, \mathcal{F})$ of full subcategories $\mathcal{T}, \mathcal{F} \subseteq \mathcal{A}$ is called a *torsion pair* if $\mathcal{F} = \mathcal{T}^\perp$ and $\mathcal{T} = {}^\perp \mathcal{F}$. In this case $\mathcal{T}$ is called a *torsion class* and $\mathcal{F}$ a *torsion-free class*.

An equivalent characterisation [62, Thm. 2.3], states that a full subcategory $\mathcal{T} \subseteq \mathcal{A}$ is a torsion class if and only if it is closed under factor objects and extensions and $\mathcal{F} \subseteq \mathcal{A}$ is a torsion-free class if and only if it is closed under subobjects and extensions. For any subcategory $\mathcal{C}$, the smallest torsion class containing it is denoted by $\mathrm{T}(\mathcal{C})$ and may be expressed explicitly as $\mathrm{T}(\mathcal{C}) = \mathrm{Filt}(\mathrm{Fac} \mathcal{C})$, see [135, Lem. 3.1]. Denote by $\mathrm{tors} \mathcal{A}$ and $\mathrm{torf} \mathcal{A}$ the collections of torsion classes and torsion-free classes of $\mathcal{A}$ respectively. By definition there are mutually inverse bijections

$$\mathrm{tors} \mathcal{A} \xrightleftharpoons[\perp(-)]{(-)^\perp} \mathrm{torf} \mathcal{A}. \quad (2.0.1)$$

Beside torsion classes, this thesis focuses on wide subcategories of abelian categories, which were first considered in [98]. They are intimately related to torsion classes [135] and the study of stability conditions [9, 38, 125, 184]. Again these subcategories admit a partial order with respect to inclusion and in this context wide subcategories arise as non-crossing partitions [108] and as the shard-intersection order of Weyl groups [180]. In fact, the poset of wide subcategories can be reconstructed in a combinatorial way from the partially ordered set of torsion classes [67].

Definition 2.0.2. [98] A full subcategory $\mathcal{W} \subseteq \mathcal{A}$ is called *wide* if it is closed under kernels, cokernels and extensions.

In later sections, a particular focus will lie on wide subcategories which arise as particular intersections of a torsion-free class with a torsion class. As abelian subcategories

of $\mathcal{A}$, wide subcategories $\mathcal{W} \subseteq \mathcal{A}$ have their own (relative) simple objects, therefore let $\text{simp}(\mathcal{W})$ denote the set of all simple objects in $\mathcal{W}$. More generally, a collection $\mathcal{S} \subseteq \mathcal{A}$ of objects such that every morphism $S \rightarrow S'$ in $\mathcal{A}$ is either zero or an isomorphism is called a *semibrick*, see [8]. This generalises the notion of a *brick*, an object $X \in \mathcal{A}$ such that $\text{End}(X)$ is a division algebra. The following bijective correspondence was first observed in [163, Sec. 1.2]:

$$\{\mathcal{W} \subseteq \mathcal{A} : \mathcal{W} \text{ is wide}\} \xrightleftharpoons[\text{Filt}(-)]{\text{simp}(-)} \{\mathcal{S} \subseteq \mathcal{A} : \mathcal{S} \text{ is a semibrick}\}. \quad (2.0.2)$$

The collection of all wide subcategories of $\mathcal{A}$ is denoted by $\text{wide } \mathcal{A}$, that of all semibricks by $\text{sbrick } \mathcal{A}$ and that of all bricks by $\text{brick } \mathcal{A}$. Consider the following two classical definitions which play an important role throughout this thesis, see for example [17].

Definition 2.0.3. Let $\mathcal{C} \subseteq \mathcal{A}$ be a full subcategory and $X \in \mathcal{A}$ an object. A *right $\mathcal{C}$ -approximation* of X is a map $f_X : C_M \rightarrow X$ with $C_M \in \mathcal{C}$ such that for any map $g : C' \rightarrow X$ with $C' \in \mathcal{C}$, there exists a map $g' : C' \rightarrow C$ such that $g = f_X g'$.

Dually, a *left $\mathcal{C}$ -approximation* of X is a map $g_X : X \rightarrow C_X$ with $C_X \in \mathcal{C}$ such that for any map $h : X \rightarrow C'$ with $C' \in \mathcal{C}$, there exists a map $h' : C_X \rightarrow C'$ such that $h = h' g_X$.

Definition 2.0.4. A full subcategory $\mathcal{C} \subseteq \mathcal{A}$ is called *contravariantly finite* (resp. *covariantly finite*) if every object $X \in \mathcal{A}$ admits a right (resp. left) $\mathcal{C}$ -approximation. It is called *functorially finite* if it is both contravariantly finite and covariantly finite.

It is well-known and easy to show that torsion classes are always contravariantly finite. However, both the subcollection of functorially finite torsion classes, denoted by $\text{f-tors } \mathcal{A}$, as well as that of functorially finite wide subcategories are particularly interesting and important. In the setting where $\mathcal{A}$ is the category of finite-dimensional (right) modules of a finite-dimensional algebra, a torsion class $\mathcal{T}$ is functorially finite if and only if $\mathcal{T} = \text{Fac } M$ for some module M by [176]. In particular, M may be chosen to be a so-called τ -rigid module [3]. These form an important class of modules studied in detail in Section 2.2 and throughout this thesis. Functorially finite wide subcategories, on the other hand, are equivalent to categories of finite-dimensional modules of other finite-dimensional algebras, see [68, Prop. 4.12].

2.1 Lattice theory and brick labelling

Recall that torsion classes, torsion-free classes and wide subcategories of $\mathcal{A}$ form partially ordered sets (posets) under inclusion and also denote these posets by $\text{tors } \mathcal{A}$, $\text{torf } \mathcal{A}$ and $\text{wide } \mathcal{A}$, respectively. In this section, these posets are endowed with the additional structure of a labelling of cover relations. This labelling provides a useful tool, used frequently throughout this thesis, in particular in Chapter 4. For a textbook reference on lattices and lattice theory see for example [84].

Definition 2.1.1. Let L be a poset.

- (1) L is called a *join-semilattice* if there exists a unique minimal common upper bound, the *join* $x \vee y \in L$ of x and y for all $x, y \in L$.
- (2) L is called a *meet-semilattice* if there exists a unique maximal common lower bound, the *meet* $x \wedge y \in L$ of x and y for all $x, y \in L$.

A join-semilattice (resp. meet-semilattice) L is called *complete* if every subset $S \subseteq L$ admits a unique minimal upper bound $\bigvee S$ (resp. a unique maximal lower bound $\bigwedge S$).

Definition 2.1.2. A poset L which is both a join-semilattice and a meet-semilattice is called a *lattice*. A lattice is *complete* if it is both a complete join-semilattice and a complete meet-semilattice.

Definition 2.1.3. Let L be a join-semilattice (resp. meet-semilattice). Then a *join-subsemilattice* (resp. *meet-subsemilattice*) K is a subset $K \subseteq L$ such that all $x, y \in K$ satisfy $x \vee y \in K$ (resp. $x \wedge y \in K$). A subset K of a lattice L is a *sublattice* if it is both a join-subsemilattice and meet-subsemilattice.

Lemma 2.1.4. [84, Lem. I.3.14] *The following coincide for a poset L :*

- (1) L is a complete join-semilattice.
- (2) L is a complete meet-semilattice.
- (3) L is complete lattice.

It is well-known and easy to see that torsion classes, torsion-free classes and wide subcategories are closed under taking arbitrary intersections and therefore form com-

plete meet-semilattices. By Lemma 2.1.4 they thus form complete lattices. Moreover, the bijections of Eq. (2.0.1) define an order reversing map between posets.

Definition 2.1.5. Let L_1, L_2 be complete lattices. A map $\eta : L_1 \rightarrow L_2$ is a *morphism of complete lattices* if $\eta(\bigvee_{L_1} S) = \bigvee_{L_2} \eta(S)$ and $\eta(\bigwedge_{L_1} S) = \bigwedge_{L_2} \eta(S)$ for all $S \subseteq L_1$.

The following is an important but rather rare property of a lattice.

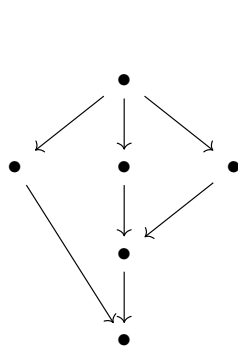
Definition 2.1.6. Let L be a complete lattice, $x \in L$ and $S \subseteq L$. Then L is called *completely semidistributive* if the following two conditions hold:

- If $x \wedge y = x \wedge z$ for all $y, z \in S$, then also $x \wedge (\bigvee S) = x \wedge y$ for all $y \in S$;
- If $x \vee y = x \vee z$ for all $y, z \in S$, then also $x \vee (\bigwedge S) = x \vee y$ for all $y \in S$.

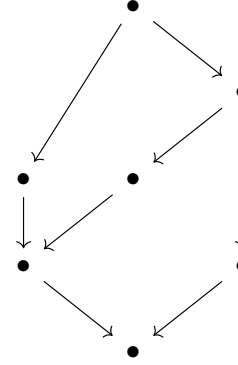
Throughout this thesis, *intervals* of posets play a central role. For two elements $x, y \in L$ in a poset L , denote by $[x, y] \subseteq L$ the set of all elements $z \in L$ such that $x \leq z \leq y$. Let $x, y \in L$ be distinct, then $x \leq y$ is called a *cover relation* and denoted by $x \lessdot y$, if there does not exist an element $z \in L$ such that $x < z < y$. Moreover, a *chain* in L is a sequence of cover relations. Define the Hasse quiver, denoted by $\text{Hasse}(L)$, of L to be the quiver whose vertices are elements of L and which has a unique arrow $y \rightarrow x$ whenever $x \lessdot y$ is a cover relation. This leads to the following notion.

Definition 2.1.7. Let L be a lattice and $x, y \in L$. An interval $[x, y]$ is called a *polygon* if it is the union of two finite chains from x to y which are disjoint except at x and y . Moreover, L is called *polygonal* if for any two cover relations $x \lessdot y_1$ and $x \lessdot y_2$ the interval $[x, y_1 \vee y_2]$ is a polygon and if for any two cover relations $x_1 \lessdot y$ and $x_2 \lessdot y$ the interval $[x_1 \wedge x_2, y]$ is a polygon.

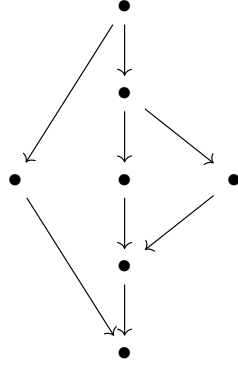
Furthermore, two maximal chains in an interval $[x, y] \subseteq L$ are said to be *related by a polygon move* if the two chains from x to y differ only in that one chain covers one side of a polygon inside $[x, y]$ while the other covers the other side. For further background on polygonal lattices see [157, Sec. 9.6]. Finally, a lattice L is called ℓ -regular for some $\ell \in \mathbb{Z}_{\geq 0}$ if every vertex in $\text{Hasse}(L)$ has ℓ arrows incident to it. Fig. 2.1 displays the Hasse quivers of four different lattices which illustrate the properties described in this section.



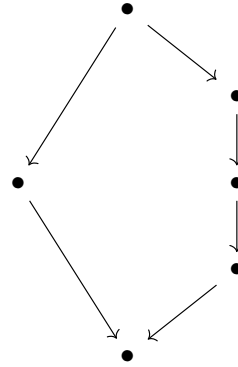
(a) Not polygonal, not semidistributive, not regular lattice



(b) Polygonal, not semidistributive, not regular lattice



(c) Not polygonal, semidistributive, not regular lattice



(d) Polygonal, semidistributive, regular lattice

Figure 2.1: Examples of lattices

Example 2.1.8. Let $\mathcal{A}$ be the category of finite-dimensional (right) modules of a finite-dimensional algebra. Then $\text{tors } \mathcal{A}$ is completely semidistributive, see [77, Thm. 4.5] and [57, Thm. 3.1(a)]. If $\text{tors } \mathcal{A}$ is finite, then it is regular and polygonal, see [57, Cor. 4.6, Prop. 4.21(a)].

In the setting of a general abelian category $\mathcal{A}$ intervals of the form $[\mathcal{U}, \mathcal{T}] \subseteq \text{tors } \mathcal{A}$ where $\mathcal{U}^\perp \cap \mathcal{T}$ is a wide subcategory of $\mathcal{A}$ are of particular interest. Such intervals are called *wide intervals* [10]. They have been studied in great detail in [10] and in the

setting of Example 2.1.8 under the name “polytopes” in [57]. Wide intervals are also sometimes called binuclear intervals [90] since they can be described combinatorially as follows.

Theorem 2.1.9. [10, Thm. 1.6] *Let $[\mathcal{U}, \mathcal{T}] \subseteq \text{tors } \mathcal{A}$. The following are equivalent:*

- $[\mathcal{U}, \mathcal{T}]$ is a wide interval;
- $\mathcal{T} = \bigvee \mathcal{U}_i$, where $\mathcal{U} \leq \mathcal{U}_i$ in $[\mathcal{U}, \mathcal{T}]$ and the torsion classes $\mathcal{U}_i$ are called the atoms of the interval $[\mathcal{U}, \mathcal{T}]$;
- $\mathcal{U} = \bigwedge \mathcal{T}_i$, where $\mathcal{T}_i \leq \mathcal{T}$ in $[\mathcal{U}, \mathcal{T}]$, and the torsion classes $\mathcal{T}_i$ are called the coatoms of the interval $[\mathcal{U}, \mathcal{T}]$.

To conclude this section, $\text{tors } \mathcal{A}$ is endowed with an additional structure called *brick labelling*. By [57, Thm. 3.3(b)], see also [8, 10, 18], there is an arrow $q : \mathcal{T} \rightarrow \mathcal{U}$ in $\text{Hasse}(\text{tors } \mathcal{A})$ if and only if there exists exact one brick S_q contained in $\mathcal{U}^\perp \cap \mathcal{T}$. Thus the brick S_q is assigned as a label to $q : \mathcal{T} \rightarrow \mathcal{U}$. Generally, define $\text{brick}[\mathcal{U}, \mathcal{T}]$ to be the collection of bricks in $\mathcal{U}^\perp \cap \mathcal{T}$ and note that $\mathcal{U}^\perp \cap \mathcal{T} = \text{Filt}(\text{brick}[\mathcal{U}, \mathcal{T}])$ by [57, Lem. 3.10]. It can be checked that Eq. (2.0.1) preserves the brick labelling, see [10, Prop. 3.4].

2.2 τ -tilting theory

Let K be an arbitrary field and A a basic finite-dimensional K -algebra. In this section, the categorical and combinatorial definitions from the previous sections are united via representation theory. Let by $\text{mod } A$ the category of finite-dimensional right A -modules and by $\text{proj } A$ the full subcategory of projective A -modules. Throughout τ denotes the Auslander–Reiten translation. Given $M \in \text{mod } A$, denote by $|M|$ the number of nonisomorphic indecomposable direct summands of M and by $\mathbf{dim } M$ the *dimension vector* defined as $(\mathbf{dim } M)_i := \dim_{\text{End}_A(S(i))}(Me_i)$ where $\{e_i\}_{i=1}^{|A|}$ is a set of primitive orthogonal idempotents of A . The most important class of modules for this thesis is the following.

Definition 2.2.1. [3] A module $M \in \text{mod } A$ is called τ -rigid if $\text{Hom}(M, \tau M) = 0$ and τ -tilting if additionally $|M| = |A|$. A pair $(M, P) \in \text{mod } A \times \text{proj } A$ is called τ -rigid if M is τ -rigid and $\text{Hom}(P, M) = 0$. It is called τ -tilting if additionally $|M| + |P| = |A|$.

Saying that a τ -rigid pair (N, Q) is a direct summand of a τ -rigid pair (M, P) , means that N is a direct summand of M and Q is a direct summand of P . Similarly, a τ -rigid pair (M, P) is called basic if M and P are both basic modules, which means that distinct indecomposable direct summands are nonisomorphic. Thus, denote the set of basic τ -rigid pairs by $\tau\text{-rigidp } A$ and its subset of basic τ -tilting pairs by $\tau\text{-tiltp } A$. These modules and pairs are closely related to torsion classes.

Theorem 2.2.2. *[3, Thm. 2.7] There is a bijection*

$$\tau\text{-tiltp } A \longrightarrow \text{f-tors } A$$

given by sending $(M, P) \in \tau\text{-tiltp } A$ to $\text{Fac } M$.

This bijection equips $\tau\text{-tiltp } A$ with the structure of a partially ordered set which is inherited from $\text{f-tors } A$. Whenever it is finite, this structure can also be seen intrinsically using the process of mutation [3, Sec. 2.3-2.4]. Roughly speaking, if an indecomposable direct summand of a basic τ -tilting pair (M, P) is deleted, there is a unique indecomposable τ -rigid module completing the resulting τ -rigid pair to a different basic τ -tilting pair. This defines the mutation, and the direction is determined by the corresponding torsion classes via Theorem 2.2.2.

Theorem 2.2.3. *The following are equivalent:*

- (1) $\tau\text{-tiltp } A$ is finite, in which case A is called τ -tilting finite;
- (2) $\text{f-tors } A$ is finite;
- (3) $\text{brick } A$ is finite;
- (4) $\text{f-tors } A$ is a complete lattice;
- (5) $\text{f-tors } A = \text{tors } A$.

Proof. (1) $\Leftrightarrow$ (2) is Theorem 2.2.2, (1) $\Leftrightarrow$ (3) is [56, Thm. 1.4], (1) $\Leftrightarrow$ (5) is [56, Thm. 1.2], (1) $\Leftrightarrow$ (4) is [109, Thm. 0.2]. $\square$

The characterisations of τ -tilting finiteness in Theorem 2.2.3 have inspired significant work expanding these equivalent conditions, see for example [8, 76, 135, 170, 173]. Recall from Example 2.1.8 the additional lattice theoretic properties of the poset $\tau\text{-tiltp } A$, which are exhibited whenever it is finite.

Given a basic τ -rigid pair (M, P) , the maximal (in the poset $\tau\text{-tiltp } A$) basic τ -tilting pair containing (M, P) as a direct summand is called the *Bongartz completion* of (M, P) , denoted by (M^+, P) . It is characterised by satisfying $\text{Fac } M^+ = {}^\perp \tau M \cap P^\perp$, see [3, Thm. 2.10] and [57, Thm. 4.4]. On the other hand, the smallest τ -tilting pair containing (M, P) as a direct summand is called the *co-Bongartz completion* of (M, P) , denoted by (M^-, P^-) , and characterised by satisfying $\text{Fac } M = \text{Fac}(M^-)$. The mutation of τ -tilting pairs mentioned above corresponds exactly to changing the Bongartz completion of a basic τ -rigid pair with $|A| - 1$ isomorphism classes of indecomposable direct summands into the co-Bongartz completion.

Definition 2.2.4. Let (M, P) be a τ -rigid pair.

- The interval $[(M^-, P^-), (M^+, P)] \subseteq \tau\text{-tiltp } A$ is called a *τ -perpendicular interval*;
- The interval $[\text{Fac } M, {}^\perp \tau M \cap P^\perp] \subseteq \text{tors } A$ is called a *τ -perpendicular interval*.

It should be highlighted that elements in interval $[\text{Fac } M, {}^\perp \tau M \cap P^\perp] \subseteq \text{tors } A$ are generally not in bijection with elements in the interval $[(M^-, P^-), (M^+, P)] \subseteq \tau\text{-tiltp } A$ if the algebra A is τ -tilting infinite, see Theorem 2.2.3.

In the introduction, it was mentioned that τ -rigid modules and pairs give rise to perpendicular wide subcategories of $\text{mod } A$. This is made precise in the following result, which shows how to construct these subcategories and that they are equivalent to categories of modules of another finite-dimensional algebra.

Theorem 2.2.5. [115, Thm. 3.8]/[57, Thm. 4.12]/[38, Thm. 3.14] *Let (M, P) be a τ -rigid pair. Then the wide subcategory*

$$\mathcal{W}_{(M,P)} := M^\perp \cap {}^\perp \tau M \cap P^\perp \subseteq \text{mod } A$$

is equivalent to a module category $\text{mod } B_{(M,P)}$, for some finite-dimensional algebra $B_{(M,P)}$ with $|A| - |M| - |P|$ isomorphism classes of simples. Moreover, there exists an isomorphism of complete lattices

$$- \cap \mathcal{W}_{(M,P)} : [\text{Fac } M, {}^\perp \tau M \cap P^\perp] \rightarrow \text{tors } \mathcal{W}_{(M,P)}.$$

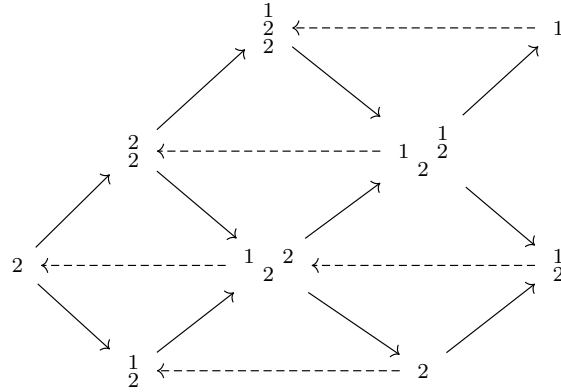
which restricts to an order-preserving bijection of functorially finite torsion classes.

Wide subcategories $\mathcal{W}_{(M,P)}$ of $\text{mod } A$ arising in this way are called τ -perpendicular and $B_{(M,P)}$ is called the τ -tilting reduction of A with respect to (M, P) . If A is hereditary then τ -perpendicular subcategories are precisely the perpendicular subcategories studied in [78]. Moreover, by [108], see also [40, Rmk. 4.10], if A is τ -tilting finite, then every wide subcategory is τ -perpendicular.

Example 2.2.6. Let $A \cong KQ/I$, where

$$Q : 1 \longrightarrow 2 \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} a, \quad I = \langle a^2 \rangle.$$

Then, the Auslander–Reiten quiver of A is given by



The nonzero indecomposable τ -rigid modules are $\frac{1}{2} \ 2$ and $2 \ 2$ since they are projective and thus have zero Auslander–Reiten translation, and 1 and $1 \ \frac{1}{2}$ whose Auslander–Reiten translations are the indecomposable projective A -modules. The Hasse quiver of the poset of τ -tilting pairs is displayed in Fig. 2.2a with the Hasse quiver of its corresponding poset of torsion classes displayed in Fig. 2.2b. The bricks of A are 2 , $\frac{1}{2} \ 2$ and $1 \ 2$, which label the arrows in these Hasse quivers, as discussed in Section 2.1. Observe that the posets coincide with the polygonal, semidistributive, regular lattice of Fig. 2.1d.

2.3 The τ -cluster morphism category

Loosely speaking, the τ -cluster morphism category encodes the possible τ -tilting reductions, in the sense that its objects are τ -perpendicular subcategories. Since each

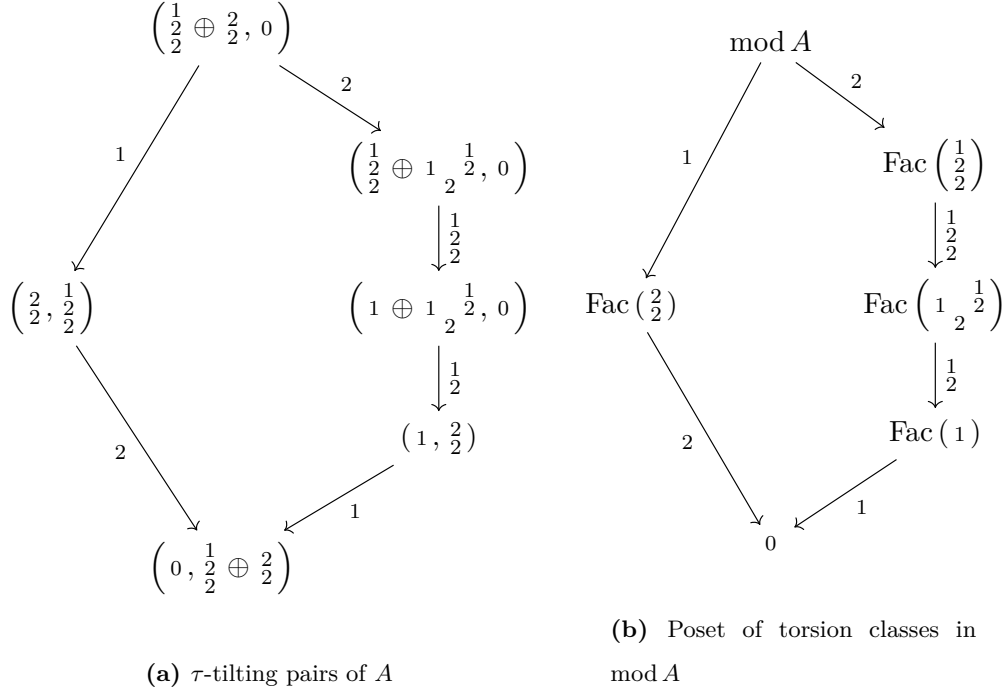


Figure 2.2: Hasse quivers of Example 2.2.6

τ -perpendicular category $\mathcal{W}$ is equivalent to a module category, it has its own (relative) Auslander–Reiten translation $\tau_{\mathcal{W}}$ and (relative) $\tau_{\mathcal{W}}$ -rigid pairs, which may differ from those of $\text{mod } A$. The algebraic definition of the τ -cluster morphism category below holds in the general setting of arbitrary finite-dimensional algebras. For earlier definitions of the category for hereditary and τ -tilting finite algebras see [105] and [46], respectively. For definitions in more general settings see [32, 33].

Definition 2.3.1. [40, Def. 6.1] The τ -cluster morphism category $\mathfrak{W}(A)$ has as its objects the τ -perpendicular wide subcategories of $\text{mod } A$. Given two τ -perpendicular wide subcategories $\mathcal{W}_1, \mathcal{W}_2$ of $\text{mod } A$, define

$$\begin{aligned} & \text{Hom}_{\mathfrak{W}(A)}(\mathcal{W}_1, \mathcal{W}_2) \\ &= \{g_{(M,P)}^{\mathcal{W}_1} : (M, P) \text{ is a basic } \tau_{\mathcal{W}_1}\text{-rigid pair and } \mathcal{W}_2 = M^{\perp_{\mathcal{W}_1}} \cap {}^{\perp_{\mathcal{W}_1}} \tau_{\mathcal{W}_1} M \cap P^{\perp_{\mathcal{W}_1}}\}. \end{aligned}$$

The details on composition of morphisms for this category is omitted for now. The associativity of the composition of morphisms is highly nontrivial to prove, see [46,

Thm. 5.9] and [40, Thm. 6.12]. The generalisation of $\mathfrak{W}(A)$ to nonpositive differential graded algebras with finite-dimensional cohomology in all degrees of [33] specialises to the above for finite-dimensional algebras and replaces τ -perpendicular categories by certain thick subcategories of the bounded derived category $\mathcal{D}^b(\text{mod } A)$. The process of τ -tilting reduction then becomes silting reduction [112], whose functoriality implies the associativity more directly [33, Thm. 4.3].

For hereditary algebras, the τ -cluster morphism category has close connections to exceptional sequences [54, 164]: Factorisations of morphisms $\mathfrak{W}(A)$ into irreducible morphisms are in bijection with the more general *signed exceptional sequences* [105]. As a generalisation of signed exceptional sequences for hereditary algebras, define *signed τ -exceptional sequences* [45] to be factorisations of morphisms in $\mathfrak{W}(A)$ into irreducible ones, see [46, Sec. 11]. These sequences have recently inspired much research [41–43, 47, 139, 145, 149, 150].

An arbitrary category $\mathcal{C}$ defines a topological space, called the *classifying space* $\mathcal{BC}$. This space is the geometric realisation of a simplicial set known as the *simplicial nerve* of the category. Its 0-simplices correspond to objects of $\mathcal{C}$ and its k -simplices correspond to chains of composable nonidentity morphisms $(X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \dots \xrightarrow{f_k} X_k)$ in $\mathcal{C}$.

The classifying space $\mathcal{BW}(A)$ of the τ -cluster morphism category is particularly interesting because, by [91, Thm. B], its fundamental group is isomorphic to the *picture group* as defined in [107] for hereditary algebras and in [91] for τ -tilting finite algebras. The appropriate general definition is as follows.

Definition 2.3.2. Let A be a finite-dimensional algebra. The *picture group* $G(A)$ is defined by having generators

$$\{X_S : S \in \text{brick } A \text{ and } \text{Filt}\{S\} \text{ is } \tau\text{-perpendicular}\} \cup \{g_{\mathcal{T}} : \mathcal{T} \in \text{f-tors } A\}$$

with a relation $g_{\mathcal{T}_1} = X_S g_{\mathcal{T}_2}$ whenever there is an arrow $\mathcal{T}_1 \xrightarrow{S} \mathcal{T}_2$ in $\text{Hasse}(\text{f-tors } A)$ and the relation $g_0 = e$.

Moreover, for hereditary algebras, the picture group is closely connected to maximal green sequences [124] arising in the context of cluster algebras [104]. For many algebras the classifying space $\mathcal{BW}(A)$ is even a $K(\pi, 1)$ space for the picture group [91, 92, 105,

106]. As mentioned in the introduction, Conjecture 1.1.1 poses the question of whether this is always the case. In particular, later chapters establish many new examples of $K(\pi, 1)$ spaces arising in this way. The main tool to achieve this is the fact that the classifying space $\mathcal{BW}(A)$ is a cube complex, see [91, Thm. A].

Example 2.3.3. Let A be the finite-dimensional K -algebra of Example 2.2.6. Recall that the τ -tilting pairs in $\text{mod } A$ are displayed in Fig. 2.2a. The indecomposable direct summands of these pairs may be used to obtain the following description of the τ -cluster morphism category $\mathfrak{W}(A)$ of A . The category is displayed in Fig. 2.3, where the arrows correspond to the irreducible morphisms and the label $(M, P) : \mathcal{W}_1 \rightarrow \mathcal{W}_2$ abbreviates the formal symbol $g_{(M,P)}^{\mathcal{W}_1}$.

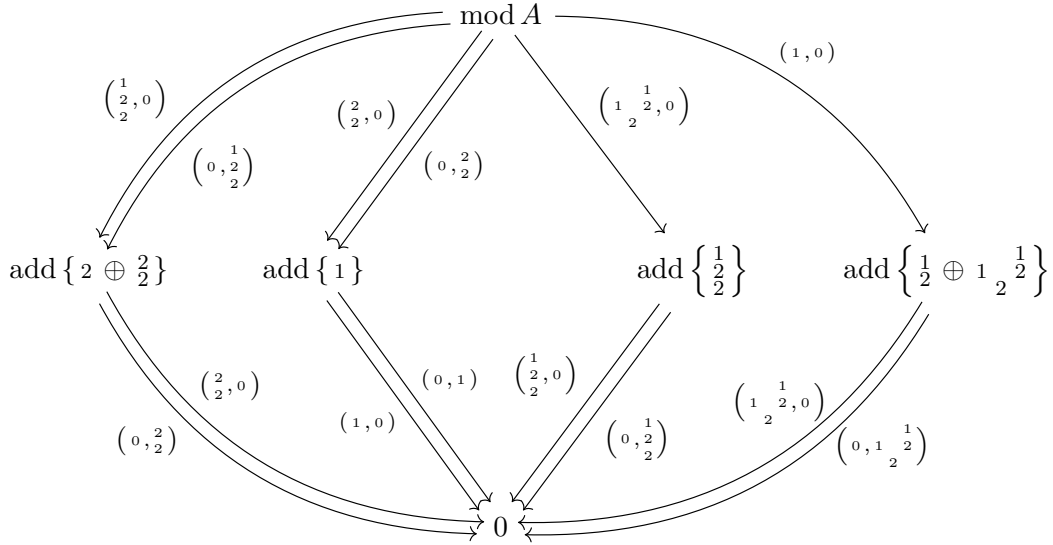


Figure 2.3: The τ -cluster morphism category $\mathfrak{W}(A)$ of Example 2.3.3

2.4 Cubical categories

A cube complex is a metric space made up from cubes of different dimensions which are glued together along their faces. This construction is similar to that of a simplicial complex by replacing ℓ -simplices by ℓ -cubes. Gromov [86] observed that for cube complexes,

being (locally) $\text{CAT}(0)$ and thus a $K(\pi, 1)$ space, is equivalent to a local combinatorial condition. Informally speaking, $\text{CAT}(0)$ spaces are geodesic metric spaces whose geodesic triangles are “no fatter than Euclidean triangles” and locally $\text{CAT}(0)$ spaces, also known as nonpositively curved spaces, are those which admit a $\text{CAT}(0)$ universal cover. Such spaces are important examples of $K(\pi, 1)$ spaces, a class of Eilenberg–MacLane spaces whose only nontrivial homotopy group is its fundamental group π . For more background and an introduction to $\text{CAT}(0)$ cube complexes see [171]. By definition, a connected topological space X is a $K(\pi, 1)$ space if it satisfies the following equivalent conditions:

- (1) The homotopy groups of X above degree 1 are all trivial;
- (2) The universal cover of X is contractible;
- (3) The cohomology of X with arbitrary coefficients is isomorphic to the cohomology of its fundamental group.

Igusa [101] categorified Gromov’s conditions and introduced *cubical categories*, whose classifying spaces are cube complexes. In this setting, the conditions of Gromov for a cube complex to be $\text{CAT}(0)$ constitute two parts of a sufficient conditions for the classifying space of the category to be a $K(\pi, 1)$ space, see [101, Prop. 3.4]. The definition of a cubical category is based on the following two categories:

- The *standard k -cube category* $\mathcal{I}^k$ is the name-giving example of a cubical category. It is the poset category on subsets of $\{1, \dots, k\}$ where morphisms are given by inclusion.
- For any category $\mathcal{C}$ and any morphism $(A \xrightarrow{f} B) \in \mathcal{C}$, the *factorisation category* $\text{Faq}(f)$ is the category whose objects are factorisations $A \xrightarrow{g} C \xrightarrow{h} B$ such that $h \circ g = f$ and whose morphisms

$$\begin{array}{ccccc} & & C_1 & & \\ & g_1 \nearrow & \downarrow \phi & \nwarrow h_1 & \\ A & & & & B \\ & g_2 \searrow & \downarrow & \swarrow h_2 & \\ & & C_2 & & \end{array}$$

are morphisms $\phi : C_1 \rightarrow C_2$ such that $\phi \circ g_1 = g_2$ and $h_1 = h_2 \circ \phi$.

Given an object $(A \xrightarrow{g} C \xrightarrow{h} B)$ in $\text{Faq}(f)$, call g a *first factor of f* if g is irreducible in $\mathcal{C}$ and h a *last factor of f* if h is irreducible in $\mathcal{C}$.

Definition 2.4.1. A *cubical category* is a small category $\mathcal{C}$ with the following properties:

- (1) Every morphism $f : A \rightarrow B$ in $\mathcal{C}$ has a *rank* $\text{rk}(f)$, which is a non-negative integer such that $\text{rk}(g \circ f) = \text{rk}(f) + \text{rk}(g)$ for all composable morphisms $f, g \in \mathcal{C}$;
- (2) If $\text{rk}(f) = k$ then there is an isomorphism $\text{Fa}q(f) \cong \mathcal{I}^k$;
- (3) The forgetful functor $\text{Fa}q(f) \rightarrow \mathcal{C}$ sending $(A \rightarrow C \rightarrow B) \mapsto C$ is an embedding, that is, faithful and injective-on-objects. Thus, every morphism of rank k has k distinct first factors and k distinct last factors;
- (4) Every morphism of rank k is determined by its k first factors;
- (5) Every morphism of rank k is determined by its k last factors.

Condition 2 implies that in a cubical category, the classifying space of any morphism f is a solid cube, that is, $\mathcal{B}\text{Fa}q(f) = [0, 1]^{\text{rk}(f)}$. Moreover, in a cubical category, every morphism is both a monomorphism and an epimorphism.

Lemma 2.4.2. *Let $\mathcal{C}$ be a category satisfying Condition (3) of Definition 2.4.1. Then every morphism in $\mathcal{C}$ is both a monomorphism and an epimorphism.*

Proof. Suppose that the morphism $X \xrightarrow{f} Y$ in $\mathcal{C}$ is not a monomorphism. Then, by definition, there exists a diagram in $\mathcal{C}$ of the form

$$Z \begin{array}{c} \xrightarrow{g_1} \\ \xrightarrow{g_2} \end{array} X \xrightarrow{f} Y, \quad (2.4.1)$$

where the compositions $f \circ g_1$ and $f \circ g_2$ are equal but $g_1 \neq g_2$. Let $h = f \circ g_1 = f \circ g_2$, and consider the two objects $(Z \xrightarrow{g_1} X \xrightarrow{f} Y)$ and $(Z \xrightarrow{g_2} X \xrightarrow{f} Y)$ in the factorisation category $\text{Fa}q(h)$. They are distinct since g_1 and g_2 are, but they are sent to the same object by the forgetful functor $\text{Fa}q(h) \rightarrow \mathcal{C}$ defined in Condition (3) of Definition 2.4.1 above. Thus, the forgetful functor is not injective-on-objects, so that Condition (3) of Definition 2.4.1 cannot hold. As a consequence, the morphisms must satisfy $g_1 = g_2$, which makes f a monomorphism. A similar argument shows that every morphism in $\mathcal{C}$ is an epimorphism, hence the proof is complete. $\square$

Example 2.4.3. The following are examples of cubical categories:

- The standard k -cube category $\mathcal{I}^k$ for any $k \geq 1$;
- For hereditary algebras the category $\mathfrak{W}(A)$ is cubical by [106, Thm. A];

- For τ -tilting finite algebras the category $\mathfrak{W}(A)$ is cubical by [91].

In Chapter 3 a large new class of cubical categories is introduced. The sufficient criteria mentioned previously and stated imprecisely in Proposition 1.1.2 for a cubical category to have a locally CAT(0) classifying space, which would thus be a $K(\pi, 1)$ space, are as follows.

Proposition 2.4.4. *[101, Prop. 3.4, Prop. 3.7] Let $\mathcal{C}$ be a cubical category. If the following properties are satisfied, then the classifying space $\mathcal{BC}$ is locally CAT(0) and thus a $K(\pi, 1)$ space:*

- (1) *There is a faithful (group) functor $\Psi : \mathcal{C} \rightarrow G$ for some group G , viewed as a groupoid with one object;*
- (2) *A set of k rank 1 morphisms $\{f_i\}_{i=1}^k$ forms the set of first factors of a rank k morphism if and only if each pair $\{f_i, f_j\}$ forms the set of first factors of a rank 2 morphism for $i \neq j$. In other words, first factors are given by pairwise compatibility conditions;*
- (3) *A set of k rank 1 morphisms $\{g_i\}_{i=1}^k$ forms the set of last factors of a rank k morphism if and only if each pair $\{g_i, g_j\}$ forms the set of last factors of a rank 2 morphism for $i \neq j$. In other words, last factors are given by pairwise compatibility conditions.*

This sufficient condition highlights the usefulness of cubical categories for investigating fundamental groups of classifying spaces. In particular, Condition (1) of Proposition 2.4.4 implies that the universal cover of $\mathcal{BC}$ is a cube complex, see the proof of [101, Prop. 3.4]. If Condition (1) holds, then Condition (2) and Condition (3) of Proposition 2.4.4 are exactly those of [86] and are equivalent to the universal cover of $\mathcal{BC}$ being CAT(0). Consequently, it makes sense to focus on the existence of a faithful group functor, as Condition (2) and Condition (3) build on Condition (1). Nonetheless, from the viewpoint of representation theory, these conditions are also interesting in their own right [20, 92].

Example 2.4.5. In this example all currently known results regarding the different parts of Proposition 2.4.4 are collected in the setting of the τ -cluster morphism category

$\mathfrak{W}(A)$ of a finite-dimensional K -algebra A . Firstly, Condition (2) of Proposition 2.4.4 always holds for $\mathfrak{W}(A)$ by the compatibility properties of τ -rigid pairs. Furthermore, Condition (1) is conjectured hold for all τ -tilting finite algebras in [92, Conj. 5.10], but is known to hold only for the following classes:

- [106, Thm. 3.7] If A is hereditary of finite or tame type;
- [92, Thm. 5.9] If K is a finite field, the K -algebra A admits Hall polynomials [160] and additionally satisfies the following: Any two bricks $B_1, B_2 \in \text{brick } A$ which form a semibrick together, also satisfy $\text{Ext}^1(B_1, B_1) = 0 = \text{Ext}^1(B_2, B_2)$ and $\text{End}(B_1) = K = \text{End}(B_2)$ or otherwise satisfy $\text{Ext}^1(B_1, B_2) = 0 = \text{Ext}^1(B_2, B_1)$.

Regarding Condition (3) of Proposition 2.4.4, the pairwise compatibility of last factors, the following is known:

- [106, Thm. 2.5] If A is hereditary of finite or tame type, then $\mathfrak{W}(A)$ satisfies Condition (3) if and only if the Auslander–Reiten quiver of A has no tubes of rank ≥ 3 ;
- [91, Thm. 2.1] If A is a Nakayama algebra, then $\mathfrak{W}(A)$ satisfies Condition (3);
- [92, Thm. 4.1] If A is a τ -tilting finite gentle algebra without loops and 2-cycles, then $\mathfrak{W}(A)$ satisfies Condition (3) if and only if every vertex in the underlying quiver has degree at most 2;
- [20, Thm. 1] If A is τ -tilting finite and $\text{mod } A$ has at most three isomorphism classes of simple modules, then $\mathfrak{W}(A)$ satisfies Condition (3) of Proposition 2.4.4;
- [20, Thm. 4] If A is a preprojective algebra of type ADE, then $\mathfrak{W}(A)$ satisfies Condition (3) of Proposition 2.4.4 if and only if it is of type A_1 , A_2 or A_3 .

As can be concluded from the length of the list above, a lot of research has investigated these properties. This thesis contributes to the problem of finding a faithful group functor from $\mathfrak{W}(A)$ using two new approaches and obtains various new families of algebras satisfying Condition (1) of Proposition 2.4.4 in this way.

Example 2.4.6. Let A be the finite-dimensional K -algebra of Example 2.2.6 for which $\mathfrak{W}(A)$ is displayed in Fig. 2.3. In Fig. 2.4 the classifying space $\mathcal{BC}(A)$ is displayed in a way that highlights its cubical structure. The gray arrows, which are labelled by the

τ -tilting pairs in $\text{mod } A$, form diagonals of the 2-cubes of $\mathcal{BC}(A)$. The vertices (labelled by wide subcategories) with the same label and the arrows (labelled by relative τ -rigid pairs corresponding to morphisms in $\mathfrak{W}(A)$) with the same colour and pointing in the same direction are supposed to be identified.

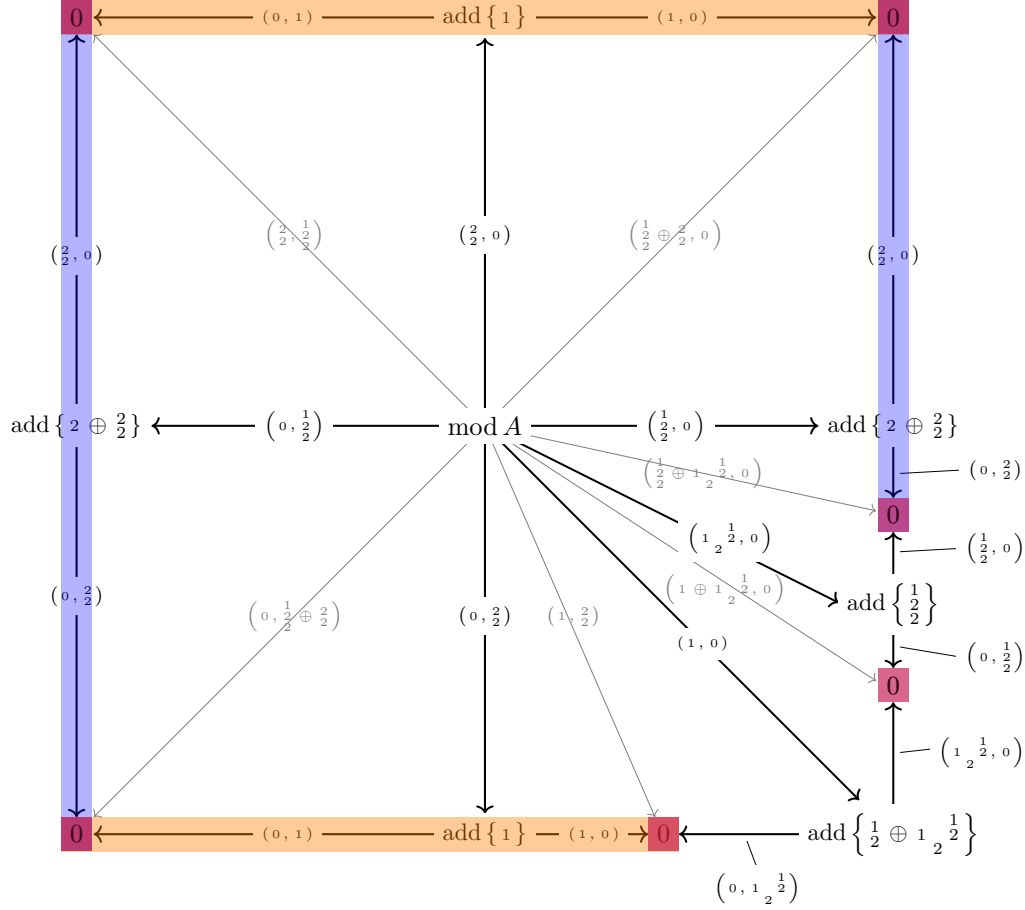


Figure 2.4: The cubical structure of the classifying space $\mathcal{BC}(A)$ of Example 2.4.6

2.5 Fans and simplicial complexes

The first new approach taken in Chapter 3 uses the theory of polyhedral fans in convex geometry. The subsequent Section 2.6 associates a polyhedral fan to the τ -tilting theory of a finite-dimensional algebra. This fan was used in [169] to construct $\mathfrak{W}(A)$ geometrically, and this construction lies at the foundation of Chapter 3.

Polyhedral fans arise naturally in many areas of mathematics. In toric geometry, fans serve as fundamental tools for defining toric varieties [73]. In commutative algebra, the *Gröbner fan* is an invariant associated to an ideal in a commutative polynomial ring [143]. Polytopes give rise to *normal fans* and *face fans* [185] which play an important role in the theory of optimisation [168]. In matroid theory, the *Bergman fan* is a subfan of the normal fan of the matroid polytope [27]. Recently, it was shown that abelian categories define *heart fans* [37]. Within representation theory, in classical tilting theory, tilting modules define fans [96] and the τ -tilting theory of an algebra is encoded by the g -vector fan [56], as described in Section 2.6.

Generally, a *convex polyhedral cone* σ in $\mathbb{R}^n$ is a set of the form

$$\sigma = \left\{ \sum_{i=1}^s \lambda_i v_i \in \mathbb{R}^n : \lambda_i \geq 0 \right\}, \text{ where } v_1, \dots, v_s \in \mathbb{R}^n.$$

Denote such a nonnegative linear combination by $\sigma = \text{cone}\{v_1, \dots, v_s\}$ and call the positive linear combination $\text{span}_{>0}\{v_1, \dots, v_s\} \subseteq \mathbb{R}^n$ the *interior cone* of σ . Given two polyhedral cones $\sigma = \text{cone}\{v_1, \dots, v_s\}$ and $\kappa = \text{cone}\{w_1, \dots, w_t\} \subseteq \mathbb{R}^n$, denote the cone they span together by $\text{cone}\{\sigma, \kappa\} = \text{cone}\{v_1, \dots, v_s, w_1, \dots, w_t\}$. Note that $\{0\}$ is also regarded as a convex polyhedral cone. Unless otherwise specified, in this thesis, a *cone* σ is a convex polyhedral cone with the following two properties:

- σ is *strongly convex*, that is, $\sigma \cap (-\sigma) = \{0\}$ holds;
- σ is *simplicial*, that is, the generating set of σ is linearly independent (up to duplicate generators).

The *dimension* or *rank* $\dim(\sigma)$ of a cone is the dimension of the linear subspace $\text{span}\{\sigma\}$ in $\mathbb{R}^n$. Denote by $\langle -, - \rangle$ the standard inner product in $\mathbb{R}^n$. A *face* of a cone σ is the intersection of σ with a hyperplane $\{v \in \mathbb{R}^n : \langle u, v \rangle = 0\}$ for some $u \in \mathbb{R}^n$ satisfying $\langle u, w \rangle \geq 0$ for all $w \in \sigma$. If $\sigma = \text{cone}\{v_1, \dots, v_s\} \subseteq \mathbb{R}^n$ is a simplicial cone, then a face of σ is simply a cone generated by a proper subset of $\{v_1, \dots, v_s\}$.

Definition 2.5.1. A *fan* Σ in $\mathbb{R}^n$ is a collection of cones in $\mathbb{R}^n$ satisfying the following:

- (1) Each face of a cone in Σ is also a cone contained in Σ ;
- (2) The intersection of two cones in Σ is a face of each of the two cones.

Denote by $\Sigma^i \subseteq \Sigma$ the subset of cones of dimension i . A fan Σ in $\mathbb{R}^n$ is said to be *finite* if it consists of a finite number of cones and *complete* if $\bigcup_{\sigma \in \Sigma} \sigma = \mathbb{R}^n$. There are two different ways of viewing fans. Firstly, fans naturally have the structure of a poset $(\Sigma, \subseteq)$ ordered by inclusion, whose corresponding poset category has cones as its objects and a unique morphism $\kappa \rightarrow \sigma$ whenever $\kappa \subseteq \sigma$. A *maximal cone* of a fan Σ is any cone $\sigma \in \Sigma$ which is not contained in another cone $\rho \in \Sigma$, that is, a maximal element of $(\Sigma, \subseteq)$. The collection of cones containing a cone σ is denoted by $\text{star}(\sigma) := \{\rho \in \Sigma : \sigma \subseteq \rho\}$, and define $\text{star}(\sigma)^i := \text{star}(\sigma) \cap \Sigma^i$. If Σ is a finite and complete fan in $\mathbb{R}^n$, then all maximal cones are of the same dimension equal to n , and this dimension is called the *rank* of Σ . For a thorough introduction to fans in the context of toric geometry see [73].

On the other hand (simplicial) fans can be viewed as simplicial complexes, which are finite sets Δ^0 together with a collection Δ of subsets of Δ^0 such that if $X \in \Delta$ and $Y \subseteq X$, then $Y \in \Delta$. Elements $v \in \Delta^0$ such that $\{v\} \in \Delta$ are called *vertices* and subsets consisting of vertices are called *faces* or more specifically *k-simplices* if they consist of exactly $k + 1$ vertices. A simplicial fan Σ in $\mathbb{R}^n$ defines a simplicial complex $\Delta(\Sigma)$ whose vertices are the dimension 1 cones Σ^1 and whose simplices are sets of vertices which together generate a cone of the fan. When a fan Σ in $\mathbb{R}^n$ is simplicial, finite and complete, the *geometric realisation* of $\Delta(\Sigma)$ is a *simplicial sphere*, that is, homeomorphic to a $(n - 1)$ -sphere. However, this geometric realisation is not necessarily a polytope [185, Ex. 7.5].

Example 2.5.2. Consider the complete fan $\Sigma(\mathbb{F}_a)$ in $\mathbb{R}^2$ as shown in Fig. 2.5a, which gives rise to a common toric variety called the *Hirzebruch surface* $\mathbb{F}_a$ where $\sigma_3 = \text{cone}\{(-1, a)\}$ and a is a positive integer. The simplicial complex is given by the vertex set $\Delta^0 = \{\sigma_i : 1 \leq i \leq 4\}$ and $\Delta = \{\{\sigma_i\}_{i=1}^4, \{\sigma_1, \sigma_2\}, \{\sigma_2, \sigma_3\}, \{\sigma_3, \sigma_4\}, \{\sigma_4, \sigma_1\}\}$ and is depicted in Fig. 2.5b.

The viewpoint of a fan as a simplicial complex is useful when constructing the classifying space of the τ -cluster morphism category. For this purpose the following two constructions for simplicial complexes are necessary:

- (1) The *link* $\text{lk}_\Delta(\sigma)$ of a simplex $\sigma \in \Delta$ is the simplicial subcomplex of Δ given by

$$\text{lk}_\Delta(\sigma) := \{\kappa \in \Delta : \sigma \cap \kappa = \emptyset \text{ and } \sigma \cup \kappa \in \Delta\}.$$

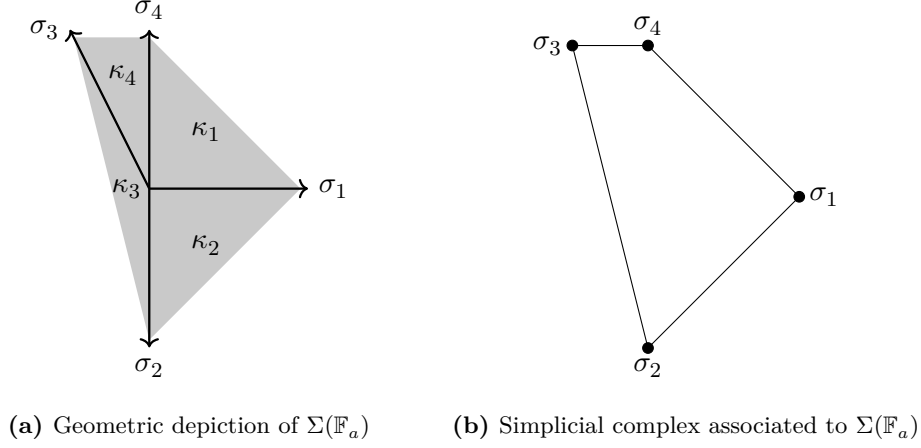


Figure 2.5: The polyhedral fan of a Hirzebruch surface $\mathbb{F}_a$

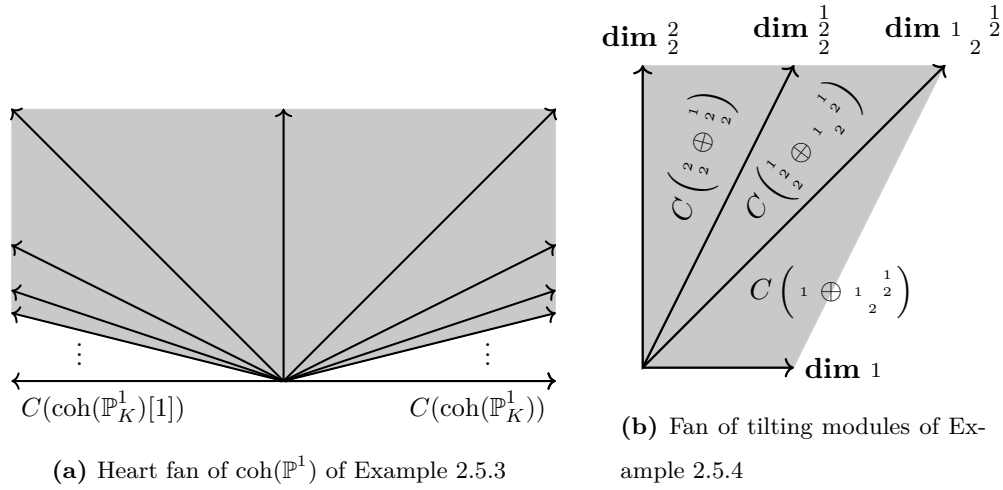


Figure 2.6: Examples of polyhedral fans in $\mathbb{R}^2$

- (2) The *join* $\Delta_1 * \Delta_2$ of two simplicial complexes Δ_1, Δ_2 has vertex set $\Delta_1^0 \cup \Delta_2^0$ and simplices given by

$$\Delta_1 * \Delta_2 := \{\sigma \in \Delta_1^0 \cup \Delta_2^0 : \sigma \cap \Delta_1^0 \in \Delta_1 \text{ and } \sigma \cap \Delta_2^0 \in \Delta_2\}.$$

The join with a simplicial complex consisting of a single vertex is called the (*topological*) *cone* over a simplicial complex.

To conclude this section, consider two more examples of polyhedral fans arising in

different contexts.

Example 2.5.3. [37, Exmp. 4.7] Let $\mathcal{A}$ be an abelian category. Then $\mathcal{A}$ may be viewed as the heart of the standard t -structure in its bounded derived category. The authors of [37] define a cone associated to each heart of a bounded t -structure on a triangulated category. Such hearts admit a partial order via inclusion of the co-aisles. The cones defined by those hearts lying between $\mathcal{A}$ and $\mathcal{A}[1]$ with respect to this partial order define a fan called the *heart fan* in [37]. As an example consider Fig. 2.6a, where $\mathcal{A} = \text{coh}(\mathbb{P}_K^1)$ is the category of coherent sheaves on the projective line $\mathbb{P}_K^1$. Observe that this fan is infinite, incomplete and simplicial.

Example 2.5.4. Let A be the finite-dimensional K -algebra from Example 2.2.6. The (classical) tilting modules of this algebra are

$$\frac{2}{2} \oplus \frac{1}{2}, \quad \frac{1}{2} \oplus 1 \oplus \frac{1}{2}, \quad \text{and} \quad 1 \oplus \frac{1}{2} \oplus 1.$$

By definition these are the modules $M \in \text{mod } A$ whose projective dimension is at most one and which satisfy $\text{Ext}^1(M, M) = 0$ and $|M| = |A|$, see [15, Cor. IV.4.7]. If M is of projective dimension at most one, then [16, Prop. 5.8] and [11, Cor. IV.2.14] imply that $\text{Hom}(M, \tau M) = 0$ if and only if $\text{Ext}^1(M, M) = 0$. Hence classical tilting modules are a subset of τ -tilting modules, and for this algebra, they actually coincide with them. By [96, Thm. 4.1, 4.2] the cones spanned by dimension vectors of indecomposable direct summands of tilting A -modules define a fan, displayed in Fig. 2.6b. Observe that this is a finite, incomplete and simplicial fan.

2.6 The g -vector fan

In this section the g -vector fan of a finite-dimensional algebra is introduced. It is defined as a fan in the real vector space $K_0(\text{proj } A)_{\mathbb{R}} := K_0(\text{proj } A) \otimes_{\mathbb{Z}} \mathbb{R}$ which is isomorphic to $\mathbb{R}^{|A|}$. By construction it encodes the compatibility of τ -rigid pairs.

Definition 2.6.1. Let $M \in \text{mod } A$ and let $P^{-1} \rightarrow P^0 \rightarrow M \rightarrow 0$ be a minimal projective presentation of M , where $P^0 = \bigoplus_{i=1}^{|A|} P(i)^{a_i}$ and $P^{-1} = \bigoplus_{i=1}^{|A|} P(i)^{b_i}$. Define the g -vector of M as

$$g^M := (a_1 - b_1, a_2 - b_2, \dots, a_{|A|} - b_{|A|}) \in \mathbb{R}^{|A|}.$$

This vector corresponds to $[P^0] - [P^{-1}] \in K_0(\text{proj } A)_{\mathbb{R}}$. Furthermore, define the g -vector of a τ -rigid pair (M, P) as $g^{(M,P)} = g^M - g^P$. In this way, associate a polyhedral cone $\bar{\mathcal{C}}_{(M,P)} := \text{cone}\{g^{M_1}, \dots, g^{M_k}, -g^{P_{k+1}}, \dots, -g^{P_t}\} \subseteq \mathbb{R}^{|A|}$ to each τ -rigid pair (M, P) , where $\{M_i\}_{i=1}^k$ and $\{P_j\}_{j=k+1}^t$ are the indecomposable direct summands of M and P respectively. The interior cone of $\bar{\mathcal{C}}_{(M,P)}$ is denoted by $\mathcal{C}_{(M,P)}$.

Now consider the fan whose cones are generated by the g -vectors of τ -rigid pairs.

Definition 2.6.2. Define the g -vector fan of an algebra A to be given by

$$\Sigma(A) := \{\bar{\mathcal{C}}_{(M,P)} \subseteq \mathbb{R}^{|A|} : (M, P) \in \tau\text{-rigidp } A\}.$$

Since the g -vectors of a g -vector cone are linearly independent [3, Thm. 5.1], faces of g -vector cones are g -vector cones of direct summands. Moreover, the intersection of two g -vector cones is a face of both, see [56, Cor. 6.7(b)] and [120, Thm. 6.13]. Therefore $\Sigma(A)$ is a simplicial polyhedral fan.

Example 2.6.3. Let A be the finite-dimensional K -algebra of Example 2.2.6. Its poset of τ -tilting pairs is displayed in Fig. 2.2a. The g -vectors of the indecomposable projective modules $\frac{1}{2}$ and $\frac{2}{2}$ are given by the standard basis vectors e_1 and e_2 of $\mathbb{R}^2$, respectively. Moreover, the minimal projective presentations of the other indecomposable τ -rigid modules are as follows:

$$\frac{2}{2} \rightarrow \left(\frac{1}{2}\right)^2 \rightarrow {}_1\frac{1}{2} \rightarrow 0, \quad \frac{2}{2} \rightarrow \frac{1}{2} \rightarrow {}_1 \rightarrow 0.$$

The g -vector fan $\Sigma(A)$ of A is displayed in Fig. 2.7.

As demonstrated in [169] the τ -cluster morphism category $\mathfrak{W}(A)$ can be constructed from the g -vector fan $\Sigma(A)$. In the remainder of this section, this construction is recalled. Denote by $\Sigma^o(A)$ the poset of interior cones of $\Sigma(A)$ induced by the poset structure of $\Sigma(A)$. That is, $\Sigma^o(A)$ is a category with a unique morphism $f_{\mathcal{C}_{(M,P)}\mathcal{C}_{(N,Q)}}$ whenever $\bar{\mathcal{C}}_{(M,P)} \subseteq \bar{\mathcal{C}}_{(N,Q)}$ in $\Sigma(A)$.

Definition 2.6.4. [169, Def. 3.3] Let A be a finite-dimensional algebra. Define $\mathfrak{C}(A)$ to be the category

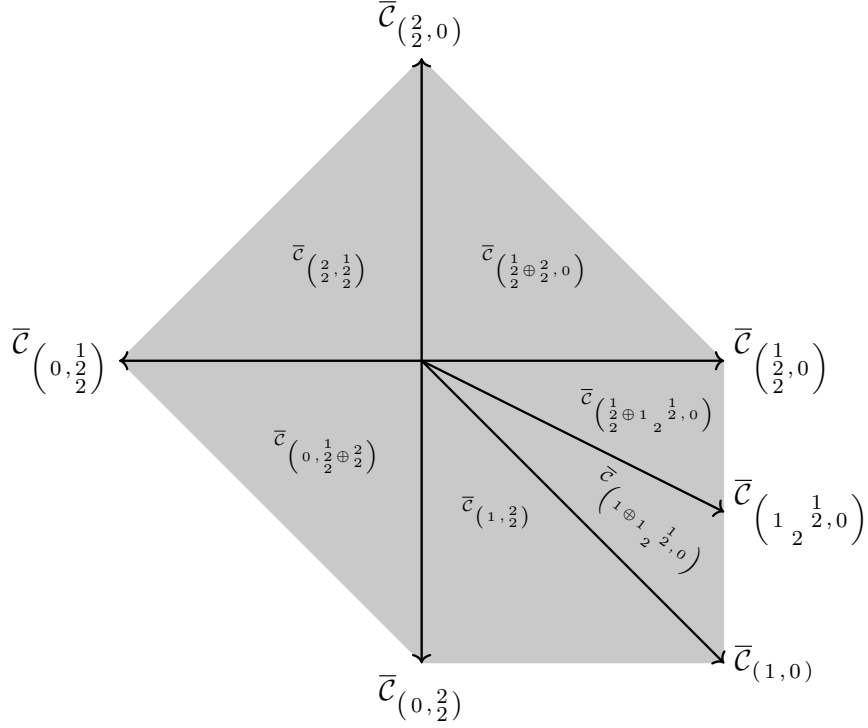


Figure 2.7: The g -vector fan $\Sigma(A)$ of Example 2.6.3

- whose objects are equivalence classes of interior g -vector cones $\mathcal{C}_{(M,P)}$, coming from $\bar{\mathcal{C}}_{(M,P)} \in \Sigma(A)$, under the identification $\mathcal{C}_{(M_1,P_1)} \sim \mathcal{C}_{(M_2,P_2)}$ whenever their corresponding τ -perpendicular subcategories $\mathcal{W}_{(M_1,P_1)}$ and $\mathcal{W}_{(M_2,P_2)}$ coincide;
- whose morphisms $\text{Hom}_{\mathfrak{C}(A)}([\mathcal{C}_{(M,P)}], [\mathcal{C}_{(N,Q)}])$ are given by the set of morphisms

$$\bigcup_{\mathcal{C}_{(M',P')} \in [\mathcal{C}_{(M,P)}], \mathcal{C}_{(N',Q')} \in [\mathcal{C}_{(N,Q)}]} \text{Hom}_{\Sigma^o(A)}(\mathcal{C}_{(M',P')}, \mathcal{C}_{(N',Q')})$$

under the identification $f_{\mathcal{C}_{(M_1,P_1)}\mathcal{C}_{(N_1,Q_1)}} \sim f_{\mathcal{C}_{(M_2,P_2)}\mathcal{C}_{(N_2,Q_2)}}$ whenever

$$\pi_{\mathcal{C}_{(M_1,P_1)}}(\mathcal{C}_{(N_1,Q_1)}) = \pi_{\mathcal{C}_{(M_2,P_2)}}(\mathcal{C}_{(N_2,Q_2)}),$$

where $\pi_{\mathcal{C}_{(M,P)}} : \mathbb{R}^{|A|} \rightarrow \text{span}(\mathcal{C}_{(M,P)})^\perp$ denotes the orthogonal projection of $\mathbb{R}^{|A|}$ onto the orthogonal complement of $\text{span}(\mathcal{C}_{(M,P)})$.

This definition omits the composition of morphisms again, which will be made precise in the subsequent chapter.

Theorem 2.6.5. *[169, Thm. 1] The category $\mathfrak{C}(A)$ defined from the g -vector fan $\Sigma(A)$ is equivalent to $\mathfrak{W}(A)$.*

The advantage of Definition 2.6.4 over Definition 2.3.1 lies in showing that composition of morphisms is associative, but gives rise to certain ambiguities since equivalence classes are involved. This issue is discussed further in the subsequent chapter.

Example 2.6.6. Let A be the finite-dimensional K -algebra of Example 2.2.6 and compare the cubical structure of $\mathcal{BW}(A)$ displayed in Fig. 2.4 with the g -vector fan $\Sigma(A)$ displayed in Fig. 2.7. The similarities between the two figures illustrates Theorem 2.6.5.

Chapter 3

The category of a partitioned fan

The starting point for this chapter is the geometric Definition 2.6.4 of the τ -cluster morphism category from the g -vector fan. In Section 3.1 the necessary geometric properties are extracted from this construction and the category generalised to arbitrary simplicial polyhedral fans. In this general setting, the algebraic information on which cones of the fan should be identified is not available, which leads to the definition of admissible partitions of the fan in Definition 3.1.1. An admissible partition guarantees that when mimicking the construction, the resulting category, called the *the category of a partitioned fan*, is well-defined, see Proposition 3.1.8. This raises the question which of the properties of the τ -cluster morphism category are satisfied by these categories. In Section 3.2, it is shown that all categories constructed from admissible partitions are cubical, see Section 2.4, and that their classifying spaces thus form cube complexes. Moreover, in Section 3.3 these classifying spaces are shown to have the structure of CW-complexes which simplifies studying their fundamental groups. By generalising Definition 2.3.2 in Definition 3.4.4, a geometric analogue of the picture group is defined for any finite complete fan whose maximal cones are equipped with a well-behaved (fan) poset. Then, the relationship between this group and the fundamental group of the category is investigated. In Section 3.5 the focus lies on fans in $\mathbb{R}^2$, where it is characterised when the three parts of the sufficient condition of Proposition 2.4.4 are satisfied. This yields many examples of $K(\pi, 1)$ spaces. In a similar way, in Section 3.6, the existence of a faithful group functor from the category, is proven to hold for the category of a partitioned fan

whenever the fan is a hyperplane arrangement and the partition is maximal. This result is extended to all partitions in Section 3.7 by showing that the collection of admissible partitions, and hence of their associated categories, forms a complete lattice. Moreover, the existence of a faithful group functor is inherited by finer, that is, “smaller” in the lattice, partitions. Finally, in Section 3.8, these general results are applied to the τ -cluster morphism category of an algebra. In particular, to show that whenever the g -vector fan $\Sigma(A)$ is a finite hyperplane arrangement, then the τ -cluster morphism category admits a faithful group functor to the picture group. This gives a new family of algebras admitting a faithful group functor, extending those listed in Example 2.4.5, one particular example is given in Example 3.8.5.

3.1 Definition of the category

Let Σ be a polyhedral fan in $\mathbb{R}^n$. Recall that the collection of cones containing a cone σ of Σ is denoted by $\text{star}(\sigma) = \{\rho \in \Sigma : \sigma \subseteq \rho\}$. Cones which are identified in Definition 2.6.4 “have the same relative fan structure around them”. To make this statement precise, let $\pi_\sigma : \mathbb{R}^n \rightarrow \text{span}\{\sigma\}^\perp$ be the projection onto the orthogonal complement of σ . For each cone $\sigma \in \Sigma$, this defines another fan $\pi_\sigma(\text{star}(\sigma))$. Then two cones σ_1, σ_2 of Σ which are identified in Definition 2.6.4 satisfy both $\text{span}\{\sigma_1\}^\perp = \text{span}\{\sigma_2\}^\perp$ as well as $\pi_{\sigma_1}(\text{star}(\sigma_1)) = \pi_{\sigma_2}(\text{star}(\sigma_2))$. Importantly, not all pairs of cones sharing these properties are identified. However, when generalising from the g -vector fan of a finite-dimensional algebra to an arbitrary simplicial fan, the information of which cones to identify is lost. Therefore, consider for each cone $\sigma_1 \in \Sigma$ the collection of *potential identifications*

$$\mathcal{E}_{\sigma_1} := \{\sigma_2 \in \Sigma : \text{span}\{\sigma_1\}^\perp = \text{span}\{\sigma_2\}^\perp \text{ and } \pi_{\sigma_1}(\text{star}(\sigma_1)) = \pi_{\sigma_2}(\text{star}(\sigma_2))\}.$$

It is clear that this is an equivalence relation and therefore $\mathcal{E}_{\sigma_1} = \mathcal{E}_{\sigma_2}$ for any two cones sharing these properties. Now the set of potential identifications may be partitioned into sets of *actual identifications*. Recall, that a *partition* of a set X is a set P of nonempty pairwise-disjoint subsets, called *blocks*, of X whose union is X . In other words, each $\mathcal{E}_\sigma$ is split into blocks $\mathcal{E}_\sigma^1, \dots, \mathcal{E}_\sigma^{m_\sigma}$ for some $1 \leq m_\sigma \leq |\mathcal{E}_\sigma|$ such that these

coincide for all representatives of $\sigma \in \mathcal{E}_\sigma$. This induces a partition $\mathfrak{P}$ of the fan Σ , for which $\sigma_1 \sim \sigma_2$ whenever $\sigma_1, \sigma_2 \in \mathcal{E}_\sigma^k$ for some $1 \leq k \leq m_\sigma$ and $\sigma \in \Sigma$.

Definition 3.1.1. A partition $\mathfrak{P}$ of Σ as described above is called *admissible* if whenever $\sigma_1 \sim \sigma_2$ are such that $\pi_{\sigma_1}(\rho_1) = \pi_{\sigma_2}(\rho_2)$ for some $\rho_1 \in \text{star}(\sigma_1)$ and $\rho_2 \in \text{star}(\sigma_2)$, then $\rho_1 \sim \rho_2$. A *partitioned fan* is a pair $(\Sigma, \mathfrak{P})$ of a simplicial fan Σ and an admissible partition $\mathfrak{P}$ of Σ .

This says that if two cones σ_1, σ_2 are in the same equivalence class, then any two cones in their respective stars which are “in the same relative position” should be identified. This restriction is necessary to make the composition of morphisms in the category of a partitioned fan well-defined. It is not obvious that nontrivial admissible partitions exist, because the cones ρ_1, ρ_2 in the definition might not satisfy $\mathcal{E}_{\rho_1} = \mathcal{E}_{\rho_2}$. In other words it might not be possible to identify ρ_1 and ρ_2 with the rules defined above. Before proving that nontrivial admissible partitions always exist, recall the following elementary result from linear algebra, whose proof is included for the sake of completeness.

Lemma 3.1.2. *Let $\sigma \subseteq \rho \in \Sigma^n$, then $\pi_\rho \circ \pi_\sigma = \pi_\rho$.*

Proof. Every vector $v \in \mathbb{R}^n$ has a unique orthogonal decomposition $v = \pi_\sigma(v) + p_\sigma(v)$, where $p_\sigma : \mathbb{R}^n \rightarrow \text{span}\{\sigma\}$ is the orthogonal projection onto the subspace of $\mathbb{R}^n$ spanned by σ . Then

$$\pi_\rho(v) = \pi_\rho(\pi_\sigma(v) + p_\sigma(v)) = \pi_\rho \circ \pi_\sigma(v) + \pi_\rho \circ p_\sigma(v).$$

But since $p_\sigma(v) \subseteq \text{span}\{\rho\}$, it follows that $\pi_\rho \circ p_\sigma = 0$. □

The following result shows that admissible partitions exist by showing that the relevant cones lie in the same set of potential identifications.

Lemma 3.1.3. *Let $\sigma_1 \sim \sigma_2$ in $\mathfrak{C}(\Sigma, \mathfrak{P})$. If there exist $\rho_1 \in \text{star}(\sigma_1)$ and $\rho_2 \in \text{star}(\sigma_2)$ such that $\pi_{\sigma_1}(\rho_1) = \pi_{\sigma_2}(\rho_2)$, then $\rho_1, \rho_2 \in \mathcal{E}_\rho$ for some $\rho \in \Sigma$.*

Proof. By definition, one needs to show that in this case $\text{span}\{\rho_1\}^\perp = \text{span}\{\rho_2\}^\perp$ and $\pi_{\rho_1}(\text{star}(\rho_1)) = \pi_{\rho_2}(\text{star}(\rho_2))$ are satisfied, which directly implies $\rho_1, \rho_2 \in \mathcal{E}_{\rho_1} = \mathcal{E}_{\rho_2}$. Because $\pi_{\sigma_1}(\rho_1) \cap \sigma_1 = \{0\}$, it is possible to take any basis B_1 of $\pi_{\sigma_1}(\rho_1) = \pi_{\sigma_2}(\rho_2)$ and

any basis B_2 of $\text{span}(\sigma_1) = \text{span}(\sigma_2)$ and then obtain a basis $B_1 \cup B_2$ of $\text{span}\{\rho_1\}$ and $\text{span}\{\rho_2\}$. This implies $\text{span}\{\rho_1\}^\perp = \text{span}\{\rho_2\}^\perp$, as required.

To show that $\pi_{\rho_1}(\text{star}(\rho_1)) = \pi_{\rho_2}(\text{star}(\rho_2))$ holds, Lemma 3.1.2 is invoked. For $\sigma, \rho \in \Sigma$ such that $\sigma \subseteq \rho$, write $\text{star}_\Sigma(\rho)$ to mean the star of ρ in Σ , and write $\text{star}_{\pi_\sigma(\text{star}(\sigma))}(\pi_\sigma(\rho))$ to mean the star of $\pi_\sigma(\rho)$ in $\pi_\sigma(\text{star}(\sigma))$. It is easy to see that $\text{star}_{\pi_\sigma(\text{star}(\sigma))}(\pi_\sigma(\rho)) = \pi_\sigma(\text{star}_\Sigma(\rho))$ holds. From $\sigma_i \subseteq \rho_i$ for $i = 1, 2$ and $\sigma_1 \sim \sigma_2$ it follows that $\pi_{\sigma_1}(\text{star}(\rho_1)) = \pi_{\sigma_2}(\text{star}(\rho_2))$ holds. In conclusion

$$\pi_{\rho_1}(\text{star}(\rho_1)) = \pi_{\rho_1}(\pi_{\sigma_1}(\text{star}(\rho_1))) = \pi_{\rho_2}(\pi_{\sigma_2}(\text{star}(\rho_2))) = \pi_{\rho_2}(\text{star}(\rho_2)),$$

holds by Lemma 3.1.2 since $\pi_{\rho_1} = \pi_{\rho_2}$. Therefore, $\mathcal{E}_{\rho_1} = \mathcal{E}_{\rho_2}$. $\square$

Thus, admissible partitions exist and throughout this chapter, let $\mathfrak{P}$ denote an admissible partition. Using admissible partitions it is now possible to generalise Definition 2.6.4. This definition and the subsequent discussion make precise the definition of composition of morphisms omitted in Definition 2.6.4.

Definition 3.1.4. Given a partitioned fan $(\Sigma, \mathfrak{P})$, define the *category of the partitioned fan* $\mathfrak{C}(\Sigma, \mathfrak{P})$ as follows:

- (1) The objects of $\mathfrak{C}(\Sigma, \mathfrak{P})$ are equivalence classes $[\sigma]$ of the partition $\mathfrak{P}$ of Σ ;
- (2) The set of morphisms $\text{Hom}_{\mathfrak{C}(\Sigma, \mathfrak{P})}([\sigma], [\rho])$ consists of equivalence classes of objects in

$$\bigcup_{\sigma_i \in [\sigma], \rho_j \in [\rho]} \text{Hom}_\Sigma(\sigma_i, \rho_j)$$

under the equivalence relation where $f_{\sigma_1 \rho_1} \sim f_{\sigma_2 \rho_2}$ if and only if $\pi_{\sigma_1}(\rho_1) = \pi_{\sigma_2}(\rho_2)$;

- (3) Given $[f_{\sigma \kappa}] \in \text{Hom}_{\mathfrak{C}(\Sigma, \mathfrak{P})}([\sigma], [\kappa])$ and $[f_{\kappa \rho}] \in \text{Hom}_{\mathfrak{C}(\Sigma, \mathfrak{P})}([\kappa], [\rho])$, their composition is defined as $[f_{\kappa \rho}] \circ [f_{\sigma \kappa}] = [f_{\sigma \rho}]$.

Because there exists a unique morphism $f_{\sigma \rho}$ in the poset category Σ whenever $\sigma \subseteq \rho$, any two compositions of morphisms $[f_{\kappa_1 \rho}] \circ [f_{\sigma \kappa_1}]$ and $[f_{\kappa_2 \rho}] \circ [f_{\sigma \kappa_2}]$ which map to the same representative of an equivalence class coincide. Similar to [169, Rem. 3.5] it is not clear that the composition of morphisms in $\mathfrak{C}(\Sigma, \mathfrak{P})$ is well-defined for two reasons:

- (1) In order to define the composition of two nonzero morphisms $[f_{\sigma_1 \kappa_1}]$ and $[f_{\kappa_2 \rho_2}]$ where $\kappa_1 \sim \kappa_2$ there must exist a morphism $f_{\kappa_1 \rho_1} \sim f_{\kappa_2 \rho_2}$ so that

$$[f_{\kappa_2 \rho_2}] \circ [f_{\sigma_1 \kappa_1}] = [f_{\sigma_1 \rho_1}].$$

- (2) Given morphisms $f_{\sigma_1 \kappa_1} \sim f_{\sigma_2 \kappa_2}$ and $f_{\kappa_1 \rho_1} \sim f_{\kappa_2 \rho_2}$ it needs to be shown that

$$[f_{\kappa_1 \rho_1}] \circ [f_{\sigma_1 \kappa_1}] = [f_{\kappa_2 \rho_2}] \circ [f_{\sigma_2 \kappa_2}].$$

In an analogous way to [169, Lem. 3.9, Lem. 3.10], these problems are resolved by the following two lemmas.

Lemma 3.1.5. *For any two morphisms $[f_{\sigma_1 \kappa_1}]$ and $[f_{\kappa_2 \rho_2}]$ in $\mathfrak{C}(\Sigma, \mathfrak{P})$ with $\kappa_1 \sim \kappa_2$ there exists a morphism $f_{\kappa_1 \rho_1} \sim f_{\kappa_2 \rho_2}$ with $\rho_1 \sim \rho_2$.*

Proof. Since $\kappa_1 \sim \kappa_2$ it follows by definition that $\pi_{\kappa_1}(\text{star}(\kappa_1)) = \pi_{\kappa_2}(\text{star}(\kappa_2))$. Thus, for each $\rho_2 \in \text{star}(\kappa_2)$ there exists $\rho_1 \in \text{star}(\kappa_1)$ such that $\pi_{\kappa_1}(\rho_1) = \pi_{\kappa_2}(\rho_2)$. Since the partition is admissible, it follows that therefore $\rho_1 \sim \rho_2$. Thus, there exists a morphism $f_{\kappa_1 \rho_1}$ which satisfies $f_{\kappa_1 \rho_1} \sim f_{\kappa_2 \rho_2}$. $\square$

The following is a special case of Lemma 3.1.5 where $\sigma_1 = \kappa_1$, which will be used repeatedly throughout.

Corollary 3.1.6. *Let $\mathfrak{P}$ be an admissible partition and let $f_{\sigma_2 \rho_2}$ be a morphism of the poset category Σ . If $\sigma_1 \sim \sigma_2$ in $\mathfrak{P}$, then there exists a morphism $f_{\sigma_1 \rho_1}$ in the poset category such that $f_{\sigma_1 \rho_1} \sim f_{\sigma_2 \rho_2}$ in $\mathfrak{C}(\Sigma, \mathfrak{P})$.*

The second concern regarding composition of morphisms in $\mathfrak{C}(\Sigma, \mathfrak{P})$ is resolved as follows.

Lemma 3.1.7. *Let $f_{\sigma_1 \kappa_1} \sim f_{\sigma_2 \kappa_2}$ and $f_{\kappa_1 \rho_1} \sim f_{\kappa_2 \rho_2}$ be two pairs of identified morphisms in $\mathfrak{C}(\Sigma, \mathfrak{P})$. Then $[f_{\kappa_1 \rho_1}] \circ [f_{\sigma_1 \kappa_1}] = [f_{\kappa_2 \rho_2}] \circ [f_{\sigma_2 \kappa_2}]$.*

Proof. By assumption the equalities $\pi_{\sigma_1}(\kappa_1) = \pi_{\sigma_2}(\kappa_2)$ and $\pi_{\kappa_1}(\rho_1) = \pi_{\kappa_2}(\rho_2)$ hold. Now take two vectors $w_1 \in \kappa_1$ and $w_2 \in \kappa_2$ such that $\pi_{\sigma_1}(w_1) = \pi_{\sigma_2}(w_2)$ and take $v \in \pi_{\kappa_1}(\rho_1) = \pi_{\kappa_2}(\rho_2)$. Then, similar to the proof of Lemma 3.1.3, a basis of $\text{span}\{\rho_1\}$ consists of the union of a basis of κ_1 and a basis of $\pi_{\kappa_1}(\rho_1)$. Hence there exists a scalar

$\epsilon_1 > 0$ such that $w_1 + \epsilon_1 v \in \rho_1$. Similarly, there exists $\epsilon_2 > 0$ such that $w_2 + \epsilon_2 v \in \rho_2$. Set $\delta = \min\{\epsilon_1, \epsilon_2\}$, then $w_i + \delta v \in \rho_i$ for $i = 1, 2$. Then

$$\pi_{\sigma_1}(w_1 + \delta v) = \pi_{\sigma_1}(w_1) + \delta \pi_{\sigma_1}(v) = \pi_{\sigma_2}(w_2) + \delta \pi_{\sigma_2}(v) = \pi_{\sigma_2}(w_2 + \delta v).$$

Hence $\pi_{\sigma_1}(\rho_1) \cap \pi_{\sigma_2}(\rho_2) \neq \{0\}$. Because of the equivalence of fans between $\pi_{\sigma_1}(\text{star } \sigma_1)$ and $\pi_{\sigma_2}(\text{star } \sigma_2)$, the intersection is either zero or the two projections coincide. This implies $\pi_{\sigma_1}(\rho_1) = \pi_{\sigma_2}(\rho_2)$ and thus $f_{\sigma_1 \rho_1} \sim f_{\sigma_2 \rho_2}$. $\square$

As a consequence of Lemma 3.1.5 and Lemma 3.1.7, the composition of morphisms in $\mathfrak{C}(\Sigma, \mathfrak{P})$ is well-defined and therefore the following holds.

Proposition 3.1.8. *The category of Definition 3.1.4 is well-defined.*

Example 3.1.9. Consider the complete fan $\Sigma(\mathbb{F}_a)$ in Fig. 2.5a. The possible identifications for this fan are:

$$\mathcal{E}_0 = \{0\}, \quad \mathcal{E}_{\sigma_1} = \{\sigma_1\}, \quad \mathcal{E}_{\sigma_2} = \{\sigma_2, \sigma_4\}, \quad \mathcal{E}_{\sigma_3} = \{\sigma_3\}, \quad \mathcal{E}_{\kappa_1} = \{\kappa_1, \kappa_2, \kappa_3, \kappa_4\}.$$

The trivial case of not making any identifications gives the standard poset category $(\Sigma, \subseteq) = \mathfrak{C}(\Sigma, \mathfrak{P}_{\text{poset}})$ whose classifying space is the disk, or more specifically a square. Only two cones of dimension 1 may be identified since their linear spans coincide, namely σ_2 and σ_4 . Moreover, they are such that $\pi_{\sigma_2}(\kappa_2) = \pi_{\sigma_4}(\kappa_1)$ and $\pi_{\sigma_2}(\kappa_3) = \pi_{\sigma_4}(\kappa_4)$ and therefore to make the partition admissible, one must identify the cones in the following way:

$$\mathfrak{P}_1 = \{\{0\}, \{\sigma_1\}, \{\sigma_2, \sigma_4\}, \{\sigma_3\}, \{\kappa_1, \kappa_2\}, \{\kappa_3, \kappa_4\}\}.$$

The category is displayed in Fig. 3.1, where the identified morphisms are given the same label and colour. The classifying space is a cylinder and is obtained from the square by identifying the opposite “sides” $\kappa_4 \leftarrow \sigma_4 \rightarrow \kappa_1$ and $\kappa_3 \leftarrow \sigma_2 \rightarrow \kappa_2$, see Section 3.3 for more details. It is also possible to additionally identify all rank 2 cones whose classifying space would join the two ends of the cylinder in one point. If the rank 1 cones are not identified, arbitrary identifications may be made among the rank 2 cones, giving rise to the topological spaces coming from a square with any combination of vertices identified. Recall for Fig. 3.1 that any two compositions having the same target are identified.

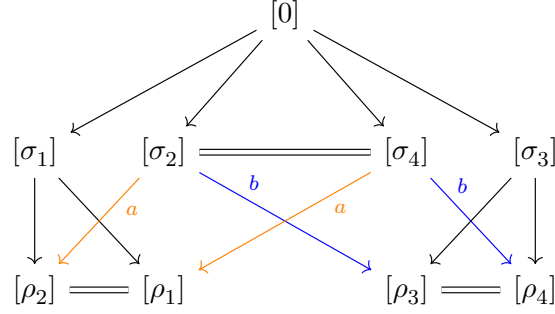


Figure 3.1: The category of the partitioned fan $(\Sigma, \mathfrak{P}_1)$, with $\Sigma(\mathbb{F}_a)$ as in Fig. 2.5a

3.2 Cubical structure

In this section, it is shown that the category $\mathfrak{C}(\Sigma, \mathfrak{P})$ of a partitioned fan is cubical, see Definition 2.4.1. The rank of a morphism $[f_{\sigma\rho}]$ in $\mathfrak{C}(\Sigma, \mathfrak{P})$ is defined to be the difference of the dimensions of the cones, in other words,

$$\text{rk}([f_{\sigma\rho}]) = \dim \rho - \dim \sigma.$$

This is well-defined since two identified cones $\sigma_1 \sim \sigma_2$ in $\mathfrak{C}(\Sigma, \mathfrak{P})$ have the same linear span in $\mathbb{R}^n$ by definition and therefore the same dimension. As a starting point, consider the finest partition, the trivial poset partition $\mathfrak{P}_{\text{poset}}$ of the fan. In this case the category $\mathfrak{C}(\Sigma, \mathfrak{P}_{\text{poset}})$ is just the poset category naturally associated with the fan.

Lemma 3.2.1. *Let Σ be a simplicial fan. The category $\mathfrak{C}(\Sigma, \mathfrak{P}_{\text{poset}})$ satisfies Condition (2) of Definition 2.4.1.*

Proof. Let $\sigma \subseteq \rho \in \Sigma$ be two cones and consider the morphism $[f_{\sigma\rho}] \in \mathfrak{C}(\Sigma, \mathfrak{P}_{\text{poset}})$. Assume that $\dim \sigma = k$ and $\dim \rho = \ell$. Then ρ can be expressed as

$$\rho = \text{cone}\{v_1, \dots, v_{\ell-k}, \sigma\} \in \Sigma,$$

where $v_1, \dots, v_{\ell-k} \in \Sigma$ are the linearly independent dimension 1 cones of Σ generating ρ which are not contained in σ . This way of writing ρ is unique. The following bijection

between objects of $\mathcal{I}^{\ell-k}$ and $\text{Fa}(\mathcal{Q}([f_{\sigma\rho}]))$ induces an isomorphism of categories

$$\begin{aligned} \mathcal{I}^{\ell-k} &\rightarrow \text{Fa}(\mathcal{Q}([f_{\sigma\rho}])) \\ \{1, \dots, \ell-k\} \supseteq S &\mapsto (\sigma \rightarrow \text{cone} \{\{v_i\}_{i \in S}, \sigma\} \rightarrow \rho), \end{aligned}$$

where morphisms are induced by the poset structures, in other words $S \subseteq P \in \mathcal{I}^{\ell-k}$ if and only if $\text{cone} \{\{v_i\}_{i \in S}, \sigma\} \subseteq \text{cone} \{\{v_i\}_{i \in P}, \sigma\}$. It is clear how to define the inverse and that this is an isomorphism. $\square$

Since more general categories of a partitioned fan are constructed from the poset category $\mathfrak{C}(\Sigma, \mathfrak{P}_{\text{poset}})$ via identifications, the following key lemma is essential in understanding how the category changes when cones are identified. In particular, it shows that two morphisms which are identified have coinciding factorisation cubes.

Lemma 3.2.2. *Let $\sigma_1, \sigma_2, \rho_1, \rho_2 \in \Sigma$ be distinct cones such that $f_{\sigma_1\rho_1} \sim f_{\sigma_2\rho_2}$ in $\mathfrak{C}(\Sigma, \mathfrak{P})$. Then for every factorisation $\sigma_1 \xrightarrow{f_{\sigma_1\kappa_1}} \kappa_1 \xrightarrow{f_{\kappa_1\rho_1}} \rho_1$ of $f_{\sigma_1\rho_1}$ there exists a unique factorisation $\sigma_2 \xrightarrow{f_{\sigma_2\kappa_2}} \kappa_2 \xrightarrow{f_{\kappa_2\rho_2}} \rho_2$ of $f_{\sigma_2\rho_2}$ such that $\kappa_1 \sim \kappa_2$. In this case $f_{\sigma_1\kappa_1} \sim f_{\sigma_2\kappa_2}$ and $f_{\kappa_1\rho_1} \sim f_{\kappa_2\rho_2}$ hold in $\mathfrak{C}(\Sigma, \mathfrak{P})$.*

Proof. If σ_1 or σ_2 is a maximal cone, then such ρ_1 and ρ_2 do not exist and the result is trivial. Therefore assume that σ_1 and σ_2 are not maximal. By assumption $\sigma_1 \sim \sigma_2$ and $\rho_1 \sim \rho_2$ and thus $\pi_{\sigma_1}(\text{star}(\sigma_1)) = \pi_{\sigma_2}(\text{star}(\sigma_2))$. So for every $\kappa_1 \in \text{star}(\sigma_1)$ satisfying $\kappa_1 \subseteq \rho_1$ there exists a unique $\kappa_2 \in \text{star}(\sigma_2)$ such that $\pi_{\sigma_1}(\kappa_1) = \pi_{\sigma_2}(\kappa_2)$. Because the partition $\mathfrak{P}$ is admissible and $\sigma_1 \sim \sigma_2$, this implies that $\kappa_1 \sim \kappa_2$. This implies $f_{\sigma_1\kappa_1} \sim f_{\sigma_2\kappa_2}$ by definition. These cones satisfy $\kappa_2 \subseteq \rho_2$, since

$$\pi_{\sigma_2}(\kappa_2) = \pi_{\sigma_1}(\kappa_1) \subseteq \pi_{\sigma_1}(\rho_1) = \pi_{\sigma_2}(\rho_2)$$

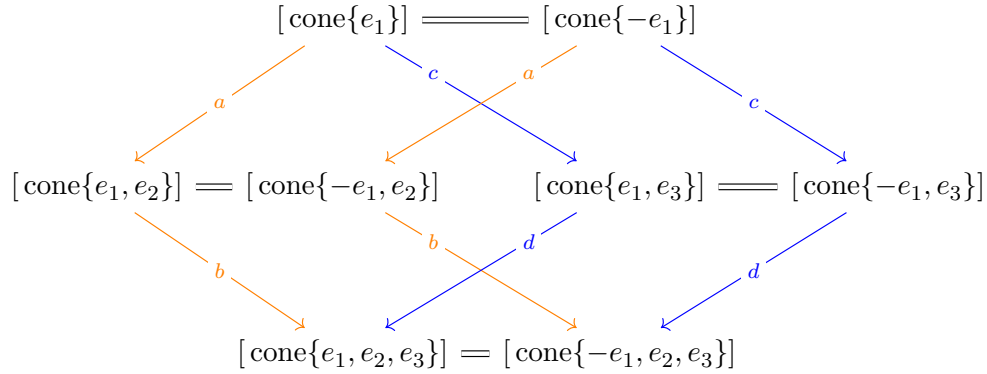
implies that κ_2 is a face of ρ_2 . It is clear that $\sigma_i \subseteq \kappa_i$ implies $\kappa_i^\perp \subseteq \sigma_i^\perp$ for $i = 1, 2$. Hence Lemma 3.1.2 may be applied to show that $f_{\kappa_1\rho_1} \sim f_{\kappa_2\rho_2}$ by observing that

$$\pi_{\kappa_1}(\rho_1) = \pi_{\kappa_1}(\pi_{\sigma_1}(\rho_1)) = \pi_{\kappa_2}(\pi_{\sigma_2}(\rho_2)) = \pi_{\kappa_2}(\rho_2)$$

holds since $\pi_{\sigma_1}(\rho_1) = \pi_{\sigma_2}(\rho_2)$ follows from $f_{\sigma_1\rho_1} \sim f_{\sigma_2\rho_2}$ by definition. $\square$

Therefore, the category $\mathfrak{C}(\Sigma, \mathfrak{P})$ of a partitioned fan satisfies Condition (2) of Definition 2.4.1 for any admissible partition. The following example illustrates the identification of factorisation cubes described in the previous lemma.

Example 3.2.3. Consider the standard coordinate fan Σ in $\mathbb{R}^3$ whose cones are spanned by the linearly independent (nonnegative) combinations of the vectors $\pm e_i \in \mathbb{R}^3$. Choose a partition $\mathfrak{P}$ such that $e_1 \sim -e_1 \in \mathfrak{C}(\Sigma, \mathfrak{P})$. For $\mathfrak{P}$ to be admissible the factorisation cubes of $[f_{e_1, \text{cone}\{e_1, e_2, e_3\}}]$ and $[f_{-e_1, \text{cone}\{-e_1, e_2, e_3\}}]$ must be identified as follows:



Consider the following analogue of [91, Lem. 2.5b] a tool for showing Condition (3) of Definition 2.4.1.

Lemma 3.2.4. *Let $[f_{\sigma\rho}] \in \mathfrak{C}(\Sigma, \mathfrak{P})$. There exists exactly one morphism*

$$([\sigma] \xrightarrow{[g_1]} [\kappa] \xrightarrow{[h_1]} [\rho]) \rightarrow ([\sigma] \xrightarrow{[g_2]} [\lambda] \xrightarrow{[h_2]} [\rho])$$

in $\text{Fa}([f_{\sigma\rho}])$ whenever $\kappa_1 \subseteq \lambda_1$ for some $\kappa_1 \in [\kappa]$ and $\lambda_1 \in [\lambda]$ and none otherwise.

Proof. If there exist two morphisms $[f_{\kappa_1\lambda_1}] \neq [f_{\kappa_2\lambda_2}]$ in $\text{Fa}([f_{\sigma\rho}])$, consider the representatives $\kappa_1 \sim \kappa_2$ and $\lambda_1 \sim \lambda_2$ in the equivalence classes. It is immediate from the rule for identifying morphisms that if $\kappa_1 = \kappa_2$, then $\lambda_1 = \lambda_2$. Conversely, if $\lambda_1 = \lambda_2$ but $\kappa_1 \neq \kappa_2$, then $\text{span}(\kappa_1) = \text{span}(\kappa_2)$ gives a contradiction to the cones being simplicial. Thus we may assume that if the morphisms are distinct so are the cones. Assume there exists only one representative $\sigma \in [\sigma]$, then $\kappa_1, \kappa_2 \in \text{star}(\sigma)$ implies $\pi_\sigma(\kappa_1) \neq \pi_\sigma(\kappa_2)$. Hence $[f_{\sigma\kappa_1}] \neq [f_{\sigma\kappa_2}]$ which is a contradiction since they both equal $[g_1]$. It follows that there need to be distinct representatives $\sigma_1, \sigma_2 \in [\sigma]$ and thus there are two sets of inclusions $\sigma_i \subseteq \kappa_i \subseteq \lambda_i$ satisfying $[f_{\sigma_i\lambda_i}] = [g_2]$ for $i = 1, 2$. Then Lemma 3.2.2 implies that $[f_{\kappa_1\lambda_1}] = [f_{\kappa_2\lambda_2}]$, a contradiction. The (non-)existence is obvious. $\square$

Notice however, that there is not necessarily a unique morphism $[\kappa] \rightarrow [\lambda]$ in $\mathfrak{C}(\Sigma, \mathfrak{P})$ in general. However, if one $\kappa_1 \subseteq \lambda_1$ exists, then there exist $\kappa_i \subseteq \lambda_i$ for every $\kappa_i \in [\kappa]$,

such that $\lambda_i \in [\lambda]$ and $f_{\kappa_i \lambda_i} \in [f_{\kappa \lambda}]$ by Lemma 3.2.2. This sets up the proof of the third condition of the definition of a cubical category.

Lemma 3.2.5. *Let $[f_{\sigma\rho}]$ be a morphism in $\mathfrak{C}(\Sigma, \mathfrak{P})$. The forgetful functor*

$$\begin{aligned} \text{Fa}([f_{\sigma\rho}]) &\rightarrow \mathfrak{C}(\Sigma, \mathfrak{P}) \\ ([\sigma] \rightarrow [\kappa] \rightarrow [\rho]) &\mapsto [\kappa] \end{aligned}$$

is an embedding.

Proof. Consider two distinct factorisations of a morphism $[f_{\sigma\rho}] \in \mathfrak{C}(\Sigma, \mathfrak{P})$ given by

$$[\sigma_1] \xrightarrow{[f_{\sigma_1 \kappa_1}]} [\kappa_1] \xrightarrow{[f_{\kappa_1 \rho_1}]} [\rho_1] \quad \text{and} \quad [\sigma_2] \xrightarrow{[f_{\sigma_2 \kappa_2}]} [\kappa_2] \xrightarrow{[f_{\kappa_2 \rho_2}]} [\rho_2]$$

satisfying $[\kappa_1] = [\kappa_2]$. Without loss of generality, by Lemma 3.2.2, assume that $\sigma_1 = \sigma_2$ and $\rho_1 = \rho_2$. Assume $\kappa_1 \neq \kappa_2$. Then, since κ_1 and κ_2 have the same linear span but are distinct, it follows that there is no generator of one which is linearly independent with respect to the generators of the other and thus they cannot both be contained in the same simplicial cone ρ . Hence ρ_1 and ρ_2 have to be distinct maximal cones, yielding a contradiction. Thus $\kappa_1 = \kappa_2$, so that the functor is injective-on-objects. Additionally, the functor is faithful since there exists at most one morphism between any two objects in $\text{Fa}([f_{\sigma\rho}])$ by Lemma 3.2.4. $\square$

In $\mathfrak{C}(\Sigma, \mathfrak{P})$, like in the poset category of the fan Σ , the irreducible morphisms are exactly the morphisms of rank 1. Hence, given a morphism $[f_{\sigma\rho}] \in \mathfrak{C}(\Sigma, \mathfrak{P})$ where $\dim(\rho) = \ell$ and $\dim(\sigma) = k$, write $\rho = \text{cone}\{\sigma, v_1, \dots, v_{\ell-k}\}$. Then it is easily seen that the $\ell - k$ first factors are the rank 1 morphisms $[f_{\sigma \kappa_i}]$, where $\kappa_i := \text{cone}\{\sigma, v_i\} \subseteq \rho$ and the last factors are the rank 1 morphisms $[f_{\lambda_i \rho}]$, where

$$\lambda_i := \text{cone}\{\sigma, v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_{\ell-k}\}.$$

Putting the above results together yields the following.

Theorem 3.2.6. *The category $\mathfrak{C}(\Sigma, \mathfrak{P})$ of a partitioned fan is cubical.*

Proof. As stated at the beginning of this section, the rank of a morphism is given by $\text{rk}([f_{\sigma\rho}]) = \dim \rho - \dim \sigma$. Condition (2) of Definition 2.4.1 follows from combining Lemma 3.2.1 and Lemma 3.2.2, whereas Condition (3) was shown explicitly in

Lemma 3.2.5. Let $[f_{\sigma\rho}] \in \mathfrak{C}(\Sigma, \mathfrak{P})$ be a morphism. Given a choice of representative $f_{\sigma'\rho'} \in [f_{\sigma\rho}]$, the first factors are given by $\{[f_{\sigma'\kappa'_i}]\}_{i=1}^{\ell-k}$, where $\kappa'_i \subseteq \rho'$ are constructed from σ' and ρ' as above. It is then clear that they uniquely determine the morphism $[f_{\sigma'\rho'}]$ by writing $\rho' = \text{cone}\{\sigma', \kappa'_1, \dots, \kappa'_{\ell-k}\}$. On the other hand, the last factors $[f_{\lambda_i\rho}]$ are easily seen to determine the morphism $[f_{\sigma\rho}]$, where $\sigma = \bigcap_{i=1}^{\ell-k} \lambda_i$. $\square$

3.3 Classifying spaces as a CW-complexes

The previous section established that categories of partitioned fans are cubical categories, hence it is natural to study their topological properties next. Throughout this section, let Σ be a finite and complete fan in $\mathbb{R}^n$ and $\mathfrak{P}$ an admissible partition of Σ . In this case, the classifying space of the category $\mathfrak{C}(\Sigma, \mathfrak{P})$ can be described as a finite CW-complex, similar to [91], [101] and [105].

A *CW-complex* X is a topological space of particular importance in algebraic topology. It is constructed starting with a discrete set X^0 , called *0-cells*. Iteratively, the k -*skeleton* X^k is formed from X^{k-1} by attaching k -cells e_i^k via maps $\varphi_i : S^{k-1} \rightarrow X^{k-1}$ for some index set I . Hence X^k is the quotient space of the disjoint union $X^{k-1} \sqcup \bigsqcup_{i \in I} D_i^k$ of X^{k-1} with a collection of k -disks $\{D_i^k\}_{i \in I}$ under the identification $x \sim \varphi_i(x)$ for $x \in \partial D_i^k$. As a set, X^k is the disjoint union of X^{k-1} with open k -disks. The name CW-complex comes from two properties of such complexes: closure-finiteness and weak topology. Furthermore, the fundamental group of CW-complexes is completely determined by their 1-cells and 2-cells. For more details see [95]. In the following construction of the CW-complex $\mathcal{BC}(\Sigma, \mathfrak{P})$, each cell is the (topological) cone of the simplicial sphere described in the following definition, hence a disk.

Definition 3.3.1. Let $(\Sigma, \mathfrak{P})$ be a partitioned fan and $\sigma \in \Sigma$ be of dimension $k \neq n$. Define $\mathcal{S}(\sigma)$ to be the simplicial complex whose vertices are the cones $\rho \in \text{star}(\sigma)^{k+1}$ and for which $\{\rho_i\}_{i=1}^{\ell-k}$ spans a simplex if and only if

$$\text{cone}\{\pi_\sigma(\rho_1), \dots, \pi_\sigma(\rho_{\ell-k})\} \in \pi_\sigma(\text{star}(\sigma)).$$

Lemma 3.3.2. $\mathcal{S}(\sigma)$ is homeomorphic to an $(n - k - 1)$ -sphere.

Proof. By intersecting a finite and complete simplicial fan in $\mathbb{R}^n$ with a $(n - 1)$ -sphere

centred at the origin, one obtains the geometric realisation of Σ to be an $(n-1)$ -sphere. Similarly for $\mathcal{S}(\sigma)$, the projection $\pi_\sigma(\text{star}(\sigma))$ is a finite and complete simplicial fan and thus its geometric realisation is a $(n - \dim(\sigma) - 1)$ -sphere. $\square$

Note that all representatives $\sigma_i \in [\sigma]$ define isomorphic simplicial complexes because $\sigma_i \sim \sigma_j$ implies that the fans $\pi_{\sigma_1}(\text{star } \sigma_1) = \pi_{\sigma_2}(\text{star } \sigma_2)$ coincide. Therefore, given an equivalence class $[\sigma] \in \mathfrak{P}$, denote by $[\mathcal{S}(\sigma)]$ the isomorphism class of simplicial complexes $\mathcal{S}(\sigma_i)$ for $\sigma_i \in [\sigma]$.

Example 3.3.3. Consider the fan Σ given in Fig. 3.2a which is the complete fan underlying the toric variety $\mathbb{P}^1 \times \mathbb{P}^1$ and at the same time the g -vector fan of a semisimple algebra with two isomorphism classes of simple modules. In Fig. 3.2b (the geometric realisation of) the simplicial complex $\mathcal{S}(0)$ is seen to be homeomorphic to a 1-sphere, where the vertices are labelled by the defining cones. Similarly, in Fig. 3.2c the simplicial complex $\mathcal{S}(\sigma_1)$ consists only of the vertices and is a 0-sphere.

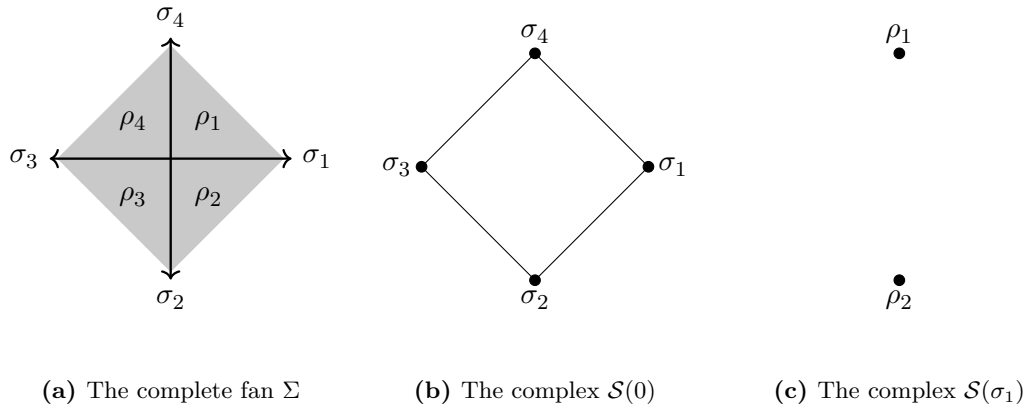


Figure 3.2: An example of the simplicial complexes of a fan.

The remainder of this subsection is devoted to proving the following result. Its proof closely follows that of [91, Prop. 4.7]. Thus, identify each morphism $[f_{\sigma\rho}] \in \mathfrak{C}(\Sigma, \mathfrak{P})$ with its factorisation cube in $\mathcal{BC}(\Sigma, \mathfrak{P})$.

Theorem 3.3.4. *Let Σ be a finite and complete fan in $\mathbb{R}^n$ and $\mathfrak{P}$ an admissible partition of Σ . The classifying space $\mathcal{BC}(\Sigma, \mathfrak{P})$ is a n -dimensional CW-complex having one cell*

$e([\sigma])$ of dimension $k = n - \dim(\sigma)$ for each equivalence class $[\sigma] \in \mathfrak{P}$. The k -cell $e([\sigma])$ is the union of the factorisation cubes of the morphisms $[f_{\sigma\rho}]$, where $\rho \in \text{star}(\sigma)^n$.

Proof. Let $e([\sigma])$ be the union of factorisation cubes of all morphisms $[f_{\sigma\rho}]$, where ρ is in $\text{star}(\sigma)^n$. To obtain the CW-structure of $\mathcal{BC}(\Sigma, \mathfrak{P})$, one must show that each $e([\sigma])$ is a disk of dimension $k = n - \dim(\sigma)$ attached to lower dimensional cells along its boundary. This is achieved in three steps.

Step 1: Starting with the disjoint union of factorisation cubes.

Let $\sigma_1, \sigma_2 \in [\sigma]$ and let $\rho_1 \in \text{star}(\sigma_1)$ and $\rho_2 \in \text{star}(\sigma_2)$ be maximal cones such that $f_{\sigma_1\rho_1} \sim f_{\sigma_2\rho_2}$. From Lemma 3.2.2 it follows that factorisation cubes of the two morphisms $[f_{\sigma_1\rho_1}]$ and $[f_{\sigma_2\rho_2}]$ are identified, hence it suffices to consider only one representative $\sigma \in [\sigma]$ by Corollary 3.1.6. First, define the disjoint union of factorisation cubes to be $X([\sigma]) = \bigsqcup_{\rho \in \text{star}(\sigma)^n} [f_{\sigma\rho}]$. It is clear that a face of the factorisation cube of $[f_{\sigma\rho}]$ corresponds to the factorisation cube of some morphism $[f_{\kappa\lambda}]$ satisfying $\sigma \subseteq \kappa \subseteq \lambda \subseteq \rho$. Consider the equivalence relation $\sim_1$ on $X([\sigma])$ which identifies faces corresponding to the same morphism. By definition this gives $e([\sigma]) \cong X([\sigma]) / \sim_1$. Now split $\sim_1$ into two types of identifications. First, only identify factorisation cubes of morphisms of the form $[f_{\sigma\lambda}]$ and denote this equivalence relation on $X([\sigma])$ by $\sim_2$. After showing that the resulting space is a disk, let $\sim_3$ identify the factorisation cubes of morphisms $[f_{\kappa\rho}]$ not starting at σ for which it is shown that the identifications of $\sim_3$ occur only on the boundary. Thus since $X([\sigma]) / \sim_1 = (X([\sigma]) / \sim_2) / \sim_3$ the desired result follows.

Step 2: Showing that $X([\sigma]) / \sim_2$ is a disk.

The strategy is to compare the quotient space $X([\sigma]) / \sim_2$ with the classifying space of the following category.

Definition 3.3.5. Given an object $[\sigma] \in \mathfrak{C}(\Sigma, \mathfrak{P})$, the *under category* (or *coslice category*), denoted by $[\sigma] \backslash \mathfrak{C}(\Sigma, \mathfrak{P})$, is the category whose objects are morphisms $[\sigma] \rightarrow [\rho]$ in $\mathfrak{C}(\Sigma, \mathfrak{P})$ and whose morphisms $([\sigma] \rightarrow [\rho_1]) \rightarrow ([\sigma] \rightarrow [\rho_2])$ are morphisms $[f_{\rho_1\rho_2}]$

making the following triangle commute:

$$\begin{array}{ccc} & [\sigma] & \\ [f_{\sigma\rho_1}] \swarrow & & \searrow [f_{\sigma\rho_2}] \\ [\rho_1] & \xrightarrow{[f_{\rho_1\rho_2}]} & [\rho_2] \end{array} .$$

The 0-simplices of the classifying space $\mathcal{B}([\sigma] \backslash \mathfrak{C}(\Sigma, \mathfrak{P}))$ are in bijection with (identity morphisms of) elements of $[\sigma] \backslash \mathfrak{C}(\Sigma, \mathfrak{P})$. There exists one 0-simplex for every cone in $\text{star}(\sigma)$. It follows from Lemma 3.2.4 that this category is a poset category, and one can easily observe that $\mathcal{B}([\sigma] \backslash \mathfrak{C}(\Sigma, \mathfrak{P})) \cong X([\sigma]) / \sim_2$.

There is a bijection between those 0-simplices of $\mathcal{B}([\sigma] \backslash \mathfrak{C}(\Sigma, \mathfrak{P}))$, which correspond to cones of dimension $\dim(\sigma) + 1$ in $\text{star}(\sigma)$, and vertices of the simplicial complex $\mathcal{S}(\sigma)$ of Definition 3.3.1. The other 0-simplices of $\mathcal{B}([\sigma] \backslash \mathfrak{C}(\Sigma, \mathfrak{P}))$ correspond to cones $\rho \in \text{star}(\sigma)$ of dimension equal to $\dim(\sigma) + \ell$ for $\ell \in \{2, \dots, k\}$ and are in bijection with the $(\ell - 1)$ -simplices of the simplicial complex $\mathcal{S}(\sigma)$, recalling that $\sigma_1 \sim \sigma_2$ implies $[\mathcal{S}(\sigma_1)] = [\mathcal{S}(\sigma_2)]$. Therefore $\mathcal{B}([\sigma] \backslash \mathfrak{C}(\Sigma, \mathfrak{P}))$ may be viewed as the cone with cone point $[f_{\sigma\sigma}]$ over $\mathcal{S}(\sigma)$. This implies that $\mathcal{B}([\sigma] \backslash \mathfrak{C}(\Sigma, \mathfrak{P}))$ and consequently $X([\sigma]) / \sim_2$ is an $(n - k - 1)$ -disk.

Step 3: Showing that identifications happen on the boundary.

Define $\sim_3$ to be the equivalence relation on $\mathcal{B}([\sigma] \backslash \mathfrak{C}(\Sigma, \mathfrak{P}))$ identifying all faces corresponding to the same morphism $[f_{\kappa\rho}]$ for $[\kappa] \neq [\sigma]$. By definition there is a homeomorphism $e([\sigma]) \cong (\mathcal{B}([\sigma] \backslash \mathfrak{C}(\Sigma, \mathfrak{P}))) / \sim_3$. From the construction of $\mathcal{B}([\sigma] \backslash \mathfrak{C}(\Sigma, \mathfrak{P}))$ as the cone over $\mathcal{S}(\sigma)$, it follows from Lemma 3.3.2 that the link of $[f_{\sigma\sigma}]$ is a $(n - k - 1)$ -sphere. Hence $[f_{\sigma\sigma}]$ is in the interior of the disk. Consider now a different 0-simplex $[f_{\sigma\rho}]$ of $\mathcal{B}([\sigma] \backslash \mathfrak{C}(\Sigma, \mathfrak{P}))$. Then its link $\text{lk}([f_{\sigma\rho}])$ is given by the simplicial join of:

- the link of $[f_{\rho\rho}]$ in $\mathcal{B}([\rho] \backslash \mathfrak{C}(\Sigma, \mathfrak{P}))$, which is the part of $\text{lk}([f_{\sigma\rho}])$ in the boundary of $e([\sigma])$; and
- the link of $[\sigma] \xrightarrow{[f_{\sigma\rho}]} [\rho] \xrightarrow{[f_{\rho\rho}]} [\rho]$ in the factorisation cube $[f_{\sigma\rho}]$, which is the part of $\text{lk}([f_{\sigma\rho}])$ in the interior of $e([\sigma])$.

This is the join of an $(n - \dim(\rho) - 1)$ -sphere with a $(\dim(\rho) - \dim(\sigma) - 1)$ -disk, which is a $(k - 1)$ -disk. Thus $[f_{\sigma\rho}]$ is a boundary vertex of $\mathcal{B}([\sigma] \backslash \mathfrak{C}(\Sigma, \mathfrak{P}))$. Any simplex which

contains the cone point $[f_{\sigma\sigma}]$ has to be the only representative in its equivalence class of $\sim_3$, by definition. It follows that $\sim_3$ identifies simplices containing exclusively simplices on the boundary of $\mathcal{B}([\sigma] \setminus \mathfrak{C}(\Sigma, \mathfrak{P}))$. Moreover, these identified simplices are factorisation cubes of morphisms of rank strictly less than k . Hence $e([\sigma])$ is attached to lower-dimensional cells. This concludes the proof of Theorem 3.3.4. $\square$

Example 3.3.6. Consider the complete fan of Fig. 3.2a with the partition $\mathfrak{P}$ identifying $\sigma_1 \sim \sigma_3$ and $\sigma_2 \sim \sigma_4$. This partition implies the identification of all maximal cones. In Fig. 3.3, the three steps of constructing the 2-cell $e([0])$ are illustrated. To begin, consider the disjoint union of factorisation cubes in Fig. 3.3a. Then $\sim_2$ identifies corresponding factorisation cubes of morphisms starting at 0, which are labelled with the same number. The resulting space is a disk with only one 0-simplex in the interior. Then $\sim_3$ identifies all factorisation cubes of morphisms on the boundary which are identified in $\mathfrak{C}(\Sigma, \mathfrak{P})$. For example, in this partition $[f_{\sigma_4\rho_4}] = [f_{\sigma_2\rho_3}]$, which are labelled “5” in Fig. 3.3b. A detailed picture of $X([0]) / \sim_2 = \mathcal{B}([0] \setminus \mathfrak{C}(\Sigma, \mathfrak{P}))$ is given in Fig. 3.4. These identifications give the 2-cell $e([0])$ of the classifying space $\mathcal{BC}(\Sigma, \mathfrak{P})$ to be a torus.

3.4 Picture groups and fan posets

In this section, let Σ be a finite and complete fan in $\mathbb{R}^n$ and $\mathfrak{P}$ an admissible partition, so that the maximal cones of Σ may be equipped with a well-behaved poset structure following [156, Sec. 3]. This poset is called *weak fan poset* and is a straightforward generalisation of the notion of a fan poset introduced in [156, Sec. 3]. The poset structure gives rise to a geometric generalisation of the picture group.

Definition 3.4.1. A *fan poset* is a pair $(\Sigma, \mathcal{P})$ where Σ is a finite and complete fan in $\mathbb{R}^n$ and $\mathcal{P}$ is a finite poset whose elements are the maximal cones of Σ , subject to the following conditions:

- (1) For every cone $\sigma \in \Sigma$, the set of maximal cones $\text{star}(\sigma)^n$ containing σ is an interval in $\mathcal{P}$, which is denoted by $[\sigma^-, \sigma^+]$ and called a *facial interval*.
- (2) For every interval I of $\mathcal{P}$, the union of all maximal cones in I is a convex polyhedral cone, which is not required to be strongly convex.

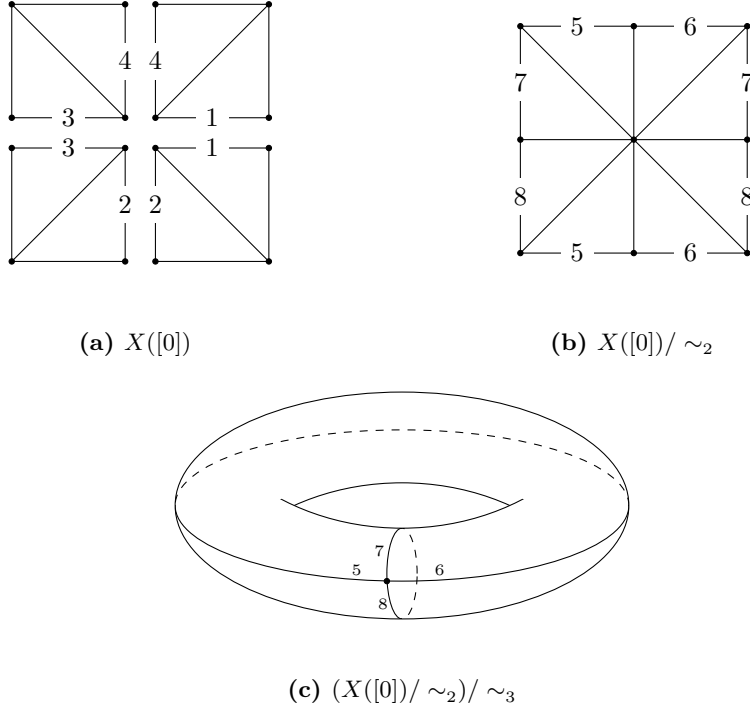


Figure 3.3: The construction of $e([0])$ from factorisation cubes.

For example, the poset of regions of a central simplicial hyperplane arrangement with an arbitrary choice of base region as defined by Edelman [65] defines a fan poset by [156, Thm. 4.2]. Moreover, g -vector fans of τ -tilting finite algebras are equipped with a natural fan poset induced by the poset of torsion classes as will be shown in Proposition 3.8.1. To achieve greater generality the definition above is weakened in the following way.

Definition 3.4.2. A *weak fan poset* is a pair $(\Sigma, \mathcal{P})$ where Σ is a finite and complete fan in $\mathbb{R}^n$ and $\mathcal{P}$ is a finite poset whose elements are the maximal cones of Σ , subject to the following conditions:

- (1) For every cone $\sigma \in \Sigma$, the set of maximal cones $\text{star}(\sigma)^n$ containing σ is an interval in $\mathcal{P}$, which is denoted by $[\sigma^-, \sigma^+]$ and called a *facial interval*.
- (2) Every cover relation $\rho_1 \lessdot \rho_2$ in $\mathcal{P}$ can be viewed as a facial interval $[\sigma^-, \sigma^+]$ for a $(n-1)$ -dimensional cone $\sigma = \rho_1 \cap \rho_2$.

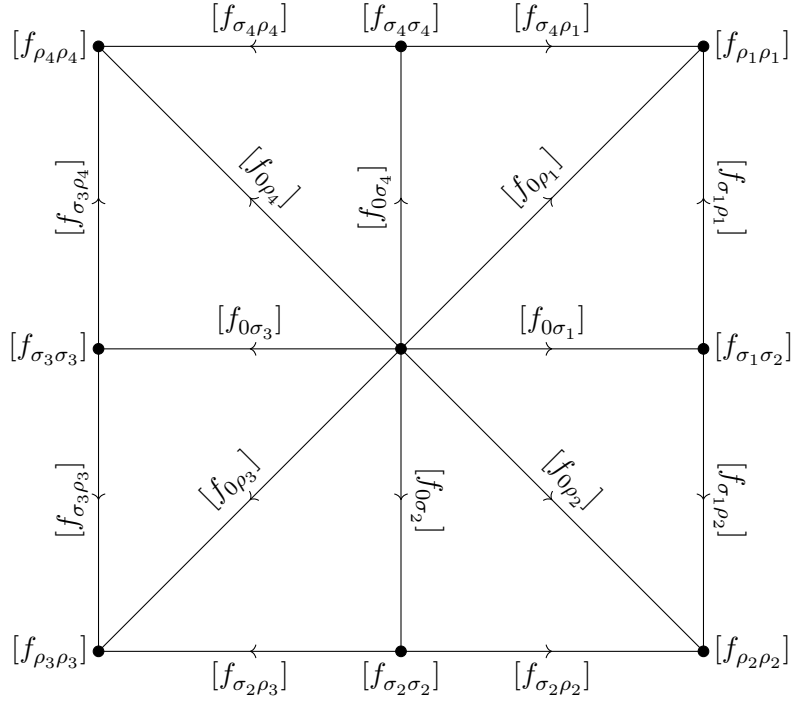


Figure 3.4: The classifying space $\mathcal{B}([0] \setminus \mathfrak{C}(\Sigma, \mathfrak{P}))$.

The result [156, Prop. 3.3] implies that if $(\Sigma, \mathcal{P})$ is a fan poset then it is also a weak fan poset. A triple $(\Sigma, \mathfrak{P}, \mathcal{P})$ is called a *partitioned fan poset* if $(\Sigma, \mathfrak{P})$ is a partitioned fan and $(\Sigma, \mathcal{P})$ is a weak fan poset.

Example 3.4.3. Consider the fan of the Hirzebruch surface in Fig. 2.5a, then the poset $\mathcal{P}$ whose two maximal chains are given by $\kappa_3 \leq \kappa_2 \leq \kappa_1$ and $\kappa_3 \leq \kappa_4 \leq \kappa_1$ with no other cover relations forms a weak fan poset but not a fan poset, since the union $\kappa_2 \cup \kappa_3$ is not a convex polyhedral cone.

Similar to the brick labelling in the lattice of torsion classes, see Section 2.1, if $(\Sigma, \mathcal{P})$ is a weak fan poset, the arrows of $\text{Hasse}(\mathcal{P})$ can be labelled with the cone of codimension 1 giving rise to the cover relation. This idea is used to define the picture group of a partitioned fan with a choice of fan poset.

Definition 3.4.4. Let $(\Sigma, \mathfrak{P}, \mathcal{P})$ be a partitioned fan poset. Define the *picture group* $G(\Sigma, \mathfrak{P}, \mathcal{P})$ to have generators $\{X_{[\sigma]} : \sigma \in \Sigma^{n-1}\}$ and the following sets of relations:

- (1) $X_{[\sigma_1]} \cdots X_{[\sigma_k]} = X_{[\sigma'_1]} \cdots X_{[\sigma'_\ell]}$ whenever $\rho_2 \xrightarrow{\sigma_1} \cdots \xrightarrow{\sigma_k} \rho_1$ and $\rho_2 \xrightarrow{\sigma'_1} \cdots \xrightarrow{\sigma'_\ell} \rho_1$ are

two distinct ordered sequences of cones of codimension 1 labelling the arrows of some maximal chain of an interval $[\rho_1, \rho_2]$ in $\text{Hasse}(\mathcal{P})$. Let this group element be denoted by $X_{[\rho_1, \rho_2]}$, and let $X_{[\rho, \rho]} = e$ for all $\rho \in \Sigma^n$;

- (2) $X_{[\sigma_1^-, \kappa_1^-]} = X_{[\sigma_2^-, \kappa_2^-]}$, whenever $[f_{\sigma_1 \kappa_1}] = [f_{\sigma_2 \kappa_2}]$ in $\mathfrak{C}(\Sigma, \mathfrak{P})$.

Notice that the picture group $G(\Sigma, \mathfrak{P}, \mathcal{P})$ satisfies $X_{[\rho_2 \rho_3]} X_{[\rho_1 \rho_2]} = X_{[\rho_1 \rho_3]}$ for a sequence $\rho_1 \leq \rho_2 \leq \rho_3$ in $\mathcal{P}$. In Section 3.8, it is shown that the picture group defined for τ -tilting finite algebras in Definition 2.3.2 may be recovered through Definition 3.4.4. Given a fan Σ , different choices of a weak fan poset $(\Sigma, \mathcal{P})$ may define nonisomorphic picture groups. Moreover, it is possible that some generators become trivial due to the relation $X_{[\sigma]} = e$ arising from relations of the second type in Definition 3.4.4.

Some trivial due to the relation $X_{[\sigma]} = e$ arising from relations of the second type in Definition 3.4.4.

Example 3.4.5. Consider the algebra A of Example 2.6.3 whose g -vector fan is displayed in Fig. 2.7. Consider the poset $\mathcal{P}$ defined by the cover relations

$$\begin{aligned} \bar{\mathcal{C}}_{\left(\begin{smallmatrix} 1, & 2 \\ & 2 \end{smallmatrix}\right)} &\leq \bar{\mathcal{C}}_{\left(\begin{smallmatrix} 0, & \frac{1}{2} \oplus \frac{2}{2} \\ & 2 \end{smallmatrix}\right)} \leq \bar{\mathcal{C}}_{\left(\begin{smallmatrix} 2, & 1 \\ & 2 \end{smallmatrix}\right)} \leq \bar{\mathcal{C}}_{\left(\begin{smallmatrix} 1 & \oplus & 2 \\ 2 & & 0 \end{smallmatrix}\right)}, \quad \text{and} \\ \bar{\mathcal{C}}_{\left(\begin{smallmatrix} 1, & 2 \\ & 2 \end{smallmatrix}\right)} &\leq \bar{\mathcal{C}}_{\left(\begin{smallmatrix} 1 \oplus 1 & & \frac{1}{2} \\ 2 & & 2, 0 \end{smallmatrix}\right)} \leq \bar{\mathcal{C}}_{\left(\begin{smallmatrix} 1 & \oplus & 1 & & \frac{1}{2} \\ 2 & & 2 & & 0 \end{smallmatrix}\right)} \leq \bar{\mathcal{C}}_{\left(\begin{smallmatrix} 1 & \oplus & 2 \\ 2 & & 0 \end{smallmatrix}\right)}. \end{aligned}$$

Let $\mathfrak{P}$ be any admissible partition of $\Sigma(A)$ which identifies the g -vector cones $\bar{\mathcal{C}}_{\left(\begin{smallmatrix} 2 \\ 2, 0 \end{smallmatrix}\right)}$ and $\bar{\mathcal{C}}_{\left(\begin{smallmatrix} 0, & 2 \\ & 2 \end{smallmatrix}\right)}$. Then the relations of the second type in Definition 3.4.4 impose that

$$X_{\left[\bar{\mathcal{C}}_{\left(\begin{smallmatrix} 2 \\ 2, 2 \end{smallmatrix}\right)}, \bar{\mathcal{C}}_{\left(\begin{smallmatrix} 1 & \oplus & 2 \\ 2 & & 0 \end{smallmatrix}\right)}\right]} = X_{\left[\bar{\mathcal{C}}_{\left(\begin{smallmatrix} 1, & 2 \\ & 2 \end{smallmatrix}\right)}, \bar{\mathcal{C}}_{\left(\begin{smallmatrix} 1, & 2 \\ & 2 \end{smallmatrix}\right)}\right]} = e$$

in $G(\Sigma(A), \mathfrak{P}, \mathcal{P})$.

This prompts the following definition.

Definition 3.4.6. A partitioned fan poset $(\Sigma, \mathfrak{P}, \mathcal{P})$ is called *nondegenerate* if for any two cones $\sigma_1, \sigma_2 \in \Sigma^{n-1}$ such that $\sigma_1 \sim_{\mathfrak{P}} \sigma_2$, the equality $[f_{\sigma_1 \rho_1}] = [f_{\sigma_2 \rho_2}]$ in $\mathfrak{C}(\Sigma, \mathfrak{P})$ implies that either $\sigma_1^- = \rho_1$ and $\sigma_2^- = \rho_2$ or that $\sigma_1^- \neq \rho_1$ and $\sigma_2^- \neq \rho_2$.

In a similar spirit, for a partitioned fan poset $(\Sigma, \mathfrak{P}, \mathcal{P})$, the poset $\mathcal{P}$ is said to be *well-defined on identified stars* whenever the induced fan posets $\mathcal{P}|_{\pi_{\sigma_1}}$ and $\mathcal{P}|_{\pi_{\sigma_2}}$ on

$\pi_{\sigma_1}(\text{star } \sigma_1)$ and $\pi_{\sigma_2}(\text{star } \sigma_2)$ coincide for distinct cones $\sigma_1, \sigma_2 \in \Sigma$ such that $[\sigma_1] = [\sigma_2]$ in $\mathfrak{C}(\Sigma, \mathfrak{P})$. These two notions are equivalent.

Lemma 3.4.7. *A fan poset $(\Sigma, \mathcal{P})$ is nondegenerate if and only if $\mathcal{P}$ is well-defined on identified stars. In this case the second set of relations of Definition 3.4.4 is trivially satisfied.*

Proof. ($\Leftarrow$). If $[f_{\sigma_1 \rho_1}] = [f_{\sigma_2 \rho_2}]$, then by definition $\pi_{\sigma_1}(\rho_1) = \pi_{\sigma_2}(\rho_2)$. If $\sigma_1^- = \rho_1$ holds, then $\pi_{\sigma_2}(\rho_2) = \pi_{\sigma_1}(\rho_1) = \pi_{\sigma_1}(\sigma_1^-) = \pi_{\sigma_2}(\sigma_2^-)$ where the last equality uses the fact that the induced posets on the stars coincide, that is, the fan poset being well-defined on identifies stars implies the following sequence of equalities holds:

$$\pi_{\sigma_1}(\sigma_1^-) = \pi_{\sigma_1}(\sigma_1)^- = \pi_{\sigma_2}(\sigma_2)^- = \pi_{\sigma_2}(\sigma_2^-).$$

Thus $\rho_2 = \sigma_2^-$. If $\sigma_1^- \neq \rho_1$ we must have $\sigma_1^+ = \rho_1$ and the same argument shows $\sigma_2^+ = \rho_2$, so $\sigma_2^- \neq \sigma_2^+ = \rho_2$.

($\Rightarrow$). Since the poset is determined by cover relations, there exists $\sigma_1 \sim_{\mathfrak{P}} \sigma_2$ of codimension 1, such that $\pi_{\sigma_1}(\sigma_1)^{-\sigma_1} \neq \pi_{\sigma_2}(\sigma_2)^{-\sigma_2}$. Here, taking τ^- on the left-hand side is done with respect to $\mathcal{P}|_{\pi_{\sigma_1}}$, whereas it is done with respect to $\mathcal{P}|_{\pi_{\sigma_2}}$ on the right-hand side. From this it follows that $\pi_{\sigma_1}(\sigma_1^-) \neq \pi_{\sigma_2}(\sigma_2^-)$. Since the projections coincide and σ_i is of codimension 1 it follows that then $\pi_{\sigma_1}(\sigma_1^-) = \pi_{\sigma_2}(\sigma_2^+)$. However, then $[f_{\sigma_1 \sigma_1^-}] = [f_{\sigma_2 \sigma_2^+}]$ yields a contradiction to non-degeneracy.

The moreover part holds in this case: By definition $[f_{\sigma_1 \rho_1}] = [f_{\sigma_2 \rho_2}]$ implies that $\pi_{\sigma_1}(\rho_1) = \pi_{\sigma_2}(\rho_2)$ and since the poset is well-defined on identified stars it follows that $\pi_{\sigma_1}(\sigma_1^-) = \pi_{\sigma_2}(\sigma_2^-)$ and similarly $\pi_{\sigma_1}(\rho_1^-) = \pi_{\sigma_2}(\rho_2^-)$. Since the paths from $\pi_{\sigma_i}(\rho_i^-)$ to $\pi_{\sigma_i}(\sigma_i^-)$ coincide in the projection for $i = 1, 2$ and the partition is admissible, the labels of the paths from ρ_i^- to σ_i^- must be pairwise identified. $\square$

The assumption on a (weak) fan poset $\mathcal{P}$ of the following lemma is referred to as $\mathcal{P}$ being *induced by* a linear functional $b \in (\mathbb{R}^n)^\vee$ in [156, Sec. 3].

Lemma 3.4.8. *Let $(\Sigma, \mathcal{P})$ be a weak fan poset. Let $b : \mathbb{R}^n \rightarrow \mathbb{R}$ be a linear map. For any $\rho_1 \leq \rho_2$ in $\mathcal{P}$, let ν be the unit normal vector to the hyperplane $\text{span}\{\rho_1 \cap \rho_2\} \subseteq \mathbb{R}^n$ separating ρ_1 from ρ_2 , oriented to point from ρ_1 to ρ_2 . If every $\rho_1 \leq \rho_2$ in $\mathcal{P}$ with*

corresponding ν is such that $b(\nu) > 0$, then the poset is nondegenerate, equivalently, well-defined on identified stars.

Proof. Any poset is determined by its cover relations. Let $[\sigma_1] = [\sigma_2] \in \mathfrak{C}(\Sigma, \mathfrak{P})$ be an equivalence class of cones of codimension 1. Assume for a contradiction that $\rho_1 \triangleleft \rho_2$ is the cover relation given by the star of σ_1 and $\rho_3 \triangleleft \rho_4$ the covering relation given by the star of σ_2 such that $\pi_{\sigma_1}(\rho_1) = \pi_{\sigma_2}(\rho_4)$ and $\pi_{\sigma_1}(\rho_2) = \pi_{\sigma_2}(\rho_3)$, that is to say not well-defined on identified stars. Then the normal vector ν_1 pointing from ρ_1 to ρ_2 is orthogonal to the hyperplane $\text{span}\{\sigma_1\}$ and satisfies $b(\nu_1) > 0$, but at the same time the normal vector ν_2 pointing from ρ_3 to ρ_4 is orthogonal to the hyperplane $\text{span}\{\sigma_2\}$ and satisfies $b(\nu_2) > 0$. Since $[\sigma_1] = [\sigma_2]$ the cones satisfy $\text{span}\{\sigma_1\} = \text{span}\{\sigma_2\}$, it follows that $\nu_1 = -\nu_2$, but then either $b(\nu_1) < 0$ or $b(\nu_2) < 0$, a contradiction. $\square$

In a similar way to [91, Prop. 4.4d], the following alternative presentation of the picture group makes the connection of Definition 3.4.4 with Definition 2.3.2 more apparent.

Lemma 3.4.9. *If $\mathcal{P}$ is nondegenerate, the picture group $G(\Sigma, \mathfrak{P}, \mathcal{P})$ may be presented with the set of generators $\{X_{[\sigma]} : \sigma \in \Sigma^{n-1}\} \cup \{g_\rho : \rho \in \Sigma^n\}$ and a relation*

$$g_{\rho_2} = X_{[\sigma]} g_{\rho_1}$$

if there is an arrow $\rho_2 \xrightarrow{\sigma} \rho_1$ in $\text{Hasse}(\mathcal{P})$ labelled by σ and the relation $g_{0^-} = e$.

Proof. Let H be a group with presentation given as above and let $\rho_2 \xrightarrow{\sigma_1} \dots \xrightarrow{\sigma_k} \rho_1$ and $\rho_2 \xrightarrow{\sigma'_1} \dots \xrightarrow{\sigma'_\ell} \rho_1$ be two distinct sequences of codimension 1 cones labelling the arrows of some maximal chain in the interval $[\rho_1, \rho_2] \subseteq \mathcal{P}$. These sequences of codimension 1 cones give rise to the relation

$$X_{[\sigma_1]} \dots X_{[\sigma_k]} g_{\rho_1} = g_{\rho_2} = X_{[\sigma'_1]} \dots X_{[\sigma'_\ell]} g_{\rho_1}$$

in H which implies that H satisfies the relation $X_{[\sigma_1]} \dots X_{[\sigma_k]} = X_{[\sigma'_1]} \dots X_{[\sigma'_\ell]}$ as required. Since $\mathcal{P}$ has a minimal element 0^- , there exists a sequence $\rho \xrightarrow{\sigma''_1} \dots \xrightarrow{\sigma''_s} 0^-$ labelling the arrows of a maximal chain in the interval $[0^-, \rho]$ in $\text{Hasse}(\mathcal{P})$. Therefore the generators corresponding to maximal cones can be expressed as

$$g_\rho = X_{[\sigma''_1]} \dots X_{[\sigma''_s]}$$

since $g_{0-} = e$. Hence every generator of H can be written in terms of generators of $G(\Sigma, \mathfrak{P}, \mathcal{P})$. As a consequence of these two observations it is possible to replace the generators $\{g_\rho : \rho \in \mathcal{P}\}$ by expressions using only generators $X_{[\sigma_i]}$ and obtain the presentation of the picture group in Definition 3.4.4. $\square$

The finite CW-structure obtained for $\mathcal{BC}(\Sigma, \mathfrak{P})$ in Theorem 3.3.4 helps to describe its fundamental group in the following way: By definition, the 1-skeleton $\mathcal{BC}(\Sigma, \mathfrak{P})^1$ of the CW-complex $\mathcal{BC}(\Sigma, \mathfrak{P})$ is a graph, which contains a loop whenever two adjacent maximal cones get identified. Since this graph is connected, it contains a maximal tree T . Now every edge e which is not part of the tree determines a loop f_e in the graph and thus a generator of the fundamental group. For more details see [95, Sec. 1.A].

Lemma 3.4.10. *The fundamental group $\pi_1(\mathcal{BC}(\Sigma, \mathfrak{P}))$ is the free group with one generator $[f_e]$ for each edge $e \in \mathcal{BC}(\Sigma, \mathfrak{P})^1 - T$ modulo the relations given by the attaching maps of the 2-cells.*

Remark 3.4.11. Consider any fan Σ with its trivial poset partition $\mathfrak{P}_{\text{poset}}$, since the classifying space $\mathcal{BC}(\Sigma, \mathfrak{P}_{\text{poset}})$ is a ball, its fundamental group is trivial but the picture group is not. So, in contrast to the setting of finite-dimensional algebras, the picture group $G(\Sigma, \mathfrak{P}, \mathcal{P})$ is not necessarily isomorphic to the fundamental group of $\mathcal{BC}(\Sigma, \mathfrak{P})$.

Nonetheless, for a special class of fan posets, the picture group is isomorphic to the fundamental group.

Proposition 3.4.12. *Let $(\Sigma, \mathfrak{P}, \mathcal{P})$ be a nondegenerate partitioned fan poset. If $\mathcal{P}$ is a polygonal lattice, then it suffices to consider facial intervals coming from cones of codimension 2 to obtain all relations of $G(\Sigma, \mathfrak{P}, \mathcal{P})$. If additionally $\mathfrak{P}$ identifies all maximal cones of Σ , then $G(\Sigma, \mathfrak{P}, \mathcal{P})$ is isomorphic to $\pi_1(\mathcal{BC}(\Sigma, \mathfrak{P}))$.*

Proof. The type 2 relations of Definition 3.4.4 are satisfied due to nondegeneracy of the poset by Lemma 3.4.7. Let $[x, y]$ be an interval in $\mathcal{P}$. Since $\mathcal{P}$ is a finite polygonal lattice, any two maximal chains in $[x, y]$ are related by a sequence of polygon moves by [157, Lem. 9-6.3]. Trivially, the labels of two maximal chains which are related by a polygon move differ only in the labels of the two sides of the polygon. Thus it is sufficient

to consider the group relations coming from polygons of $\mathcal{P}$ to give a presentation of $G(\Sigma, \mathfrak{P}, \mathcal{P})$.

Let $\sigma \in \Sigma$ be a cone of codimension 2 and consider the induced weak fan poset $\mathcal{P}|_{\pi_\sigma}$ on $\pi_\sigma(\text{star } \sigma)$, which has a maximal and minimal element, and two disjoint chains similar to Fig. 3.5b. Hence the interval $[\sigma^-, \sigma^+]$ is a polygon of $\mathcal{P}$. Conversely, take a polygon $[\rho_1 \wedge \rho_2, \rho_3] \subseteq \mathcal{P}$ for some $\rho_1 \triangleleft \rho_3$ and $\rho_2 \triangleleft \rho_3$, then $\kappa = \rho_1 \cap \rho_3 \cap \rho_2$ is a cone of codimension 2 since $\rho_1 \cap \rho_3$ and $\rho_2 \cap \rho_3$ are both generated by distinct subsets of $(n-1)$ vectors generating ρ_3 . By the previous, $[\kappa^-, \kappa^+]$ is a polygon which must contain ρ_1, ρ_2 and ρ_3 by construction. More precisely the interval satisfies $\kappa^+ = \rho_3$ and by the uniqueness of the meet in a lattice also $\kappa^- = \rho_1 \wedge \rho_2$. Hence every polygon arises as a facial interval of a cone of codimension 2 and every cone of codimension 2 gives rise to a polygon.

Additionally, if all maximal cones are identified, then there exists a unique 0-cell in $\mathcal{BC}(\Sigma, \mathfrak{P})$ and Lemma 3.4.10 implies that the generators of $G(\Sigma, \mathfrak{P}, \mathcal{P})$ and $\pi_1(\mathcal{BC}(\Sigma, \mathfrak{P}))$ coincide. The relations of the fundamental group of a CW-complex are given exactly by the 2-cells which correspond with the cones of codimension 2. $\square$

The previous result implies that different choices of nondegenerate fan posets define isomorphic picture groups when all maximal cones are identified by $\mathfrak{P}$ and the fan poset is a polygonal lattice. A similar result holds for any fan in $\mathbb{R}^2$.

Lemma 3.4.13. *Let Σ be a finite and complete fan in $\mathbb{R}^2$ and $(\Sigma, \mathcal{P}_1)$ and $(\Sigma, \mathcal{P}_2)$ be nondegenerate weak fan posets. Then $G(\Sigma, \mathfrak{P}, \mathcal{P}_1) \cong G(\Sigma, \mathfrak{P}, \mathcal{P}_2)$.*

Proof. Lemma 3.4.7 and nondegeneracy imply that the type 2 relations are satisfied by both $G(\Sigma, \mathfrak{P}, \mathcal{P}_1)$ and $G(\Sigma, \mathfrak{P}, \mathcal{P}_2)$. Denote the generators of $G(\Sigma, \mathfrak{P}, \mathcal{P}_1)$ by $X_{[\sigma_i]}$ and the generators of $G(\Sigma, \mathfrak{P}, \mathcal{P}_2)$ by $Y_{[\sigma_i]}$ for $\sigma_i \in \Sigma^1$. Given an arbitrary cover relation $\rho_i \triangleleft \rho_j$ in $\mathcal{P}_1$ with $\sigma_k := \rho_i \cap \rho_j$, define

$$\delta^{\sigma_k} := \begin{cases} 1 & \text{if } \rho_i \triangleleft \rho_j \text{ in } \mathcal{P}_2, \\ -1 & \text{if } \rho_j \triangleleft \rho_i \text{ in } \mathcal{P}_2. \end{cases}$$

Then the desired group isomorphism $\phi : G(\Sigma, \mathfrak{P}, \mathcal{P}_1) \rightarrow G(\Sigma, \mathfrak{P}, \mathcal{P}_2)$ is given on the generators by $\phi(X_{[\sigma_i]}) = Y_{[\sigma_i]}^{\delta^{\sigma_i}}$. Note that this is well-defined on equivalence classes by

the assumption that $\mathcal{P}_1$ and $\mathcal{P}_2$ are nondegenerate. $\square$

3.5 Eilenberg-MacLane spaces in dimension 2

Let Σ be a finite and complete fan in $\mathbb{R}^n$ and $\mathfrak{P}$ an admissible partition. While the picture group of the partitioned fan $(\Sigma, \mathfrak{P})$, with respect to a fan poset, is not necessarily isomorphic to the fundamental group of $\mathcal{BC}(\Sigma, \mathfrak{P})$ in general, it still plays an important role in understanding the classifying spaces of the categories of a partitioned fan. Recall, that $\mathfrak{C}(\Sigma, \mathfrak{P})$ is a cubical category by Theorem 3.2.6 and that there are three conditions which together imply that $\mathcal{BC}(\Sigma, \mathfrak{P})$ is a $K(\pi, 1)$ space, see Proposition 2.4.4.

In particular, the picture group appears to be the most natural group to study Condition (1) of Proposition 2.4.4. While there is a recipe for constructing a $K(G, 1)$ space for any finitely presented group G , the result may be an infinite-dimensional CW-complex, see [95, Sec. 1.B.]. Hence it is natural to ask whether the finite CW-complex of Theorem 3.3.4 is a $K(\pi, 1)$ space for its fundamental group. For this purpose, consider the following functor from the category of a partitioned fan to its picture group with respect to some nondegenerate weak fan poset:

$$\begin{aligned} \Psi : \mathfrak{C}(\Sigma, \mathfrak{P}) &\rightarrow G(\Sigma, \mathfrak{P}, \mathcal{P}) \\ [f_{\sigma\kappa}] &\mapsto X_{[\sigma^-, \kappa^-]}. \end{aligned} \tag{3.5.1}$$

It follows from basic hyperplane arrangement theory (i.e. convex geometry) and the definition of admissible partitions that the functor is well-defined. Indeed, take two representatives $\sigma_1, \sigma_2 \in [\sigma]$ and $\kappa_1, \kappa_2 \in [\kappa]$ such that $[f_{\sigma_1\kappa_1}] = [f_{\sigma_2\kappa_2}]$. Now consider the projection of their stars onto the orthogonal complements, $\pi_{\sigma_1}(\text{star } \sigma_1) = \pi_{\sigma_2}(\text{star } \sigma_2)$, then $\pi_{\sigma}(\kappa_1^-) = \pi_{\sigma}(\kappa_2^-)$ follows from $\pi_{\sigma}(\kappa_1) = \pi_{\sigma}(\kappa_2)$ which follows from $[f_{\sigma_1\kappa_1}] = [f_{\sigma_2\kappa_2}]$ by definition. The terms $X_{[\sigma_i^-, \kappa_i^-]}$ for $i = 1, 2$ are determined by the paths $\pi_{\sigma}(\kappa_i^-)$ to $\pi_{\sigma}(\sigma_i^-)$ for $i = 1, 2$, which coincide in $\pi_{\sigma}(\text{star}(\sigma))$, and hence coincide. It is easily seen to be well-defined on identity morphisms $[f_{\sigma\sigma}]$, which get sent to the trivial element $X_{[\sigma^-, \sigma^-]} = e$. Furthermore, since the weak fan poset is nondegenerate, the functor is well-defined on composition of morphisms by construction.

Example 3.5.1. Consider a partitioned fan $(\Sigma, \mathfrak{P})$ as in Fig. 3.5a, then the set of three rank 1 morphisms $\{[f_1], [f_2], [f_3]\}$ cannot be the last factors of a rank 3 morphism

since no such morphism exists when Σ is a fan in $\mathbb{R}^2$. However, the pairs of morphisms $\{[f_1], [f_3]\}$, $\{[f_1], [f_2]\}$ and $\{[f_2], [f_3]\}$ form the last factors of the morphisms $[f_{0\rho_1}]$, $[f_{0\rho_3}]$ and $[f_{0\rho_2}]$ respectively.

Example 3.5.2. Consider a finite and complete fan with only three cones of dimension 1, for example the fan $\Sigma(\mathbb{P}^2)$ whose toric variety is the projective plane $\mathbb{P}^2$, see [73, p. 6-7]. This fan $\Sigma(\mathbb{P}^2)$ has three cones, $\text{cone}\{e_1\}$, $\text{cone}\{e_2\}$ and $\text{cone}\{-e_1 - e_2\}$, of dimension 1, which are all pairwise compatible as first factors. But since there is no morphism of rank 3, this gives an example where the first factors are not given by pairwise compatibility conditions.

Lemma 3.5.3. *Let Σ be a fan in $\mathbb{R}^2$. Then $\mathfrak{C}(\Sigma, \mathfrak{P})$ satisfies the pairwise compatibility of first (resp. last) factors if and only if there is no set of three pairwise compatible first (resp. last) factors.*

Proof. Any category trivially satisfies the pairwise compatibility of first (resp. last) factors for $k = 2$ in Proposition 2.4.4. For $k \geq 3$, there is no morphism of rank k in $\mathfrak{C}(\Sigma, \mathfrak{P})$ and hence the pairwise compatibility condition is equivalent to there being no set of k pairwise compatible first (resp. last) factors. For $k \geq 4$, it is geometrically impossible to have four compatible first (resp. last) factors in $\mathbb{R}^2$ so the result follows. $\square$

The restricted setting of a fan in $\mathbb{R}^2$ enables a detailed understanding of the functor in Eq. (3.5.1). This is used to obtain many examples of $K(\pi, 1)$ spaces.

Theorem 3.5.4. *Let Σ be a fan in $\mathbb{R}^2$ and $\mathcal{P}$ be a nondegenerate weak fan poset, then the functor of Eq. (3.5.1) is faithful. Moreover, if $\mathfrak{C}(\Sigma, \mathfrak{P})$ does not admit a set of three pairwise compatible rank 1 morphisms, then $\mathcal{BC}(\Sigma, \mathfrak{P})$ is a $K(\pi, 1)$ space.*

Proof. Without loss of generality, by Lemma 3.4.13, choose one particular nondegenerate weak fan poset defined as follows: Choose a base region $\rho_A \in \Sigma^2$ and consider the angle bisector of the angle spanned by the two dimension 1 cones defining ρ_A . Then let ρ_D be the region containing the opposite of the angle bisector. If the opposite of the angle bisector is contained in a cone of codimension 1, then choose either of the adjacent maximal cones as ρ_D . The set-up is depicted in Fig. 3.5b, where the angle bisector is

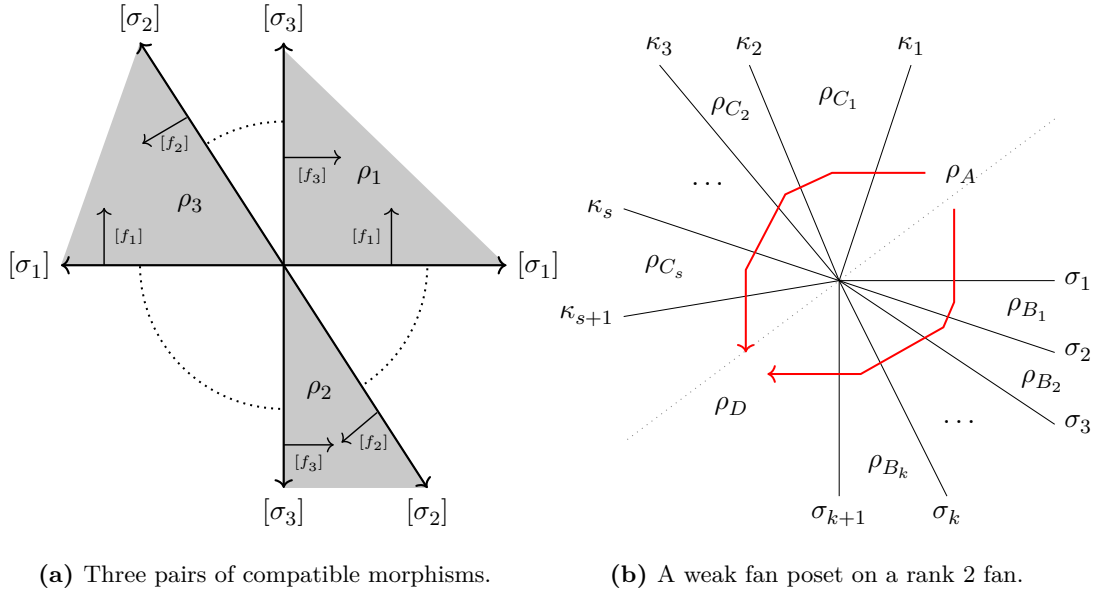


Figure 3.5: Last factors and fan poset of fans in $\mathbb{R}^2$.

the dotted line and the fan poset is indicated in red and given by the following Hasse quiver:

$$\begin{array}{ccccccc}
 & & \rho_{B_1} & \longrightarrow & \dots & \longrightarrow & \rho_{B_k} \\
 \rho_A & \nearrow & & & & & \searrow \\
 & & \rho_{C_1} & \longrightarrow & \dots & \longrightarrow & \rho_{C_s} \\
 & & & & & & \nearrow \\
 & & & & & & \rho_D
 \end{array}$$

Showing that the functor is faithful is equivalent to showing that the induced map $\Psi_{[\sigma][\rho]} : \text{Hom}([\sigma], [\rho]) \rightarrow G(\Sigma, \mathfrak{P}, \mathcal{P})$ is injective. It is sufficient to show that two distinct morphisms $[f_{\sigma\rho_1}]$ and $[f_{\sigma\rho_2}]$ cannot map to the same group element under Ψ , by Corollary 3.1.6 and Lemma 3.2.2. Using the description of the picture group in Definition 3.4.4 it is easily seen that the generators are distinct since the presentation only contains one relation. Consider the different possible dimensions of σ and ρ :

- Dimension 0 to 1: For $0 \in \Sigma^0$ and $\lambda \in \Sigma^1$, any morphism $[f_{0\lambda}] \in \mathfrak{C}(\Sigma, \mathfrak{P})$ gets mapped to $X_{[\rho_D, \lambda^-]}$. All group elements of this form are clearly distinct, except potentially $X_{[\rho_D, \rho_{C_s}]}$ and $X_{[\rho_D, \rho_{B_k}]}$. However, if the cones σ_{k+1} and κ_{s+1} give rise the same generator $X_{[\sigma_{k+1}]}$ then their linear span must be equal, hence the cone ρ_D is a half-plane and not strongly convex. This is also the reason why it is no problem that $[f_{0\sigma_{k+1}}]$ and $[f_{0\kappa_{s+1}}]$ have the same image under Ψ , since σ_{k+1} and

κ_{s+1} cannot be identified in $\mathfrak{C}(\Sigma, \mathfrak{P})$.

- Dimension 0 to 2: For $0 \in \Sigma^0$ and $\rho \in \Sigma^2$, each such morphism $[f_{0\rho_i}] \in \mathfrak{C}(\Sigma, \mathfrak{P})$ maps to $X_{[\rho_D, \rho_i]}$ and by the previous case, all these elements of $G(\Sigma, \mathfrak{P}, \mathcal{P})$ of this form are distinct.
- Dimension 1 to 2: For $\sigma \in \Sigma^1$ and $\rho \in \Sigma^2$, there are exactly two possibilities, either $[f_{\sigma\rho}] \in \mathfrak{C}(\Sigma, \mathfrak{P})$ maps to $X_{[\sigma]}$ or it maps to the identity element e , which are different group elements by the nondegeneracy assumption.

Thus the functor is faithful. Now by Lemma 3.5.3 if $\mathfrak{C}(\Sigma, \mathfrak{P})$ does not admit a set of three pairwise compatible rank 1 morphisms, then it satisfies the pairwise compatibility of first and last factors. In this case, $\mathcal{B}\mathfrak{C}(\Sigma, \mathfrak{P})$ is a $K(\pi, 1)$ space by Proposition 2.4.4. $\square$

Corollary 3.5.5. *In the setting of Theorem 3.5.4. If $\mathfrak{P}$ identifies all maximal cones, then $\mathcal{B}\mathfrak{C}(\Sigma, \mathfrak{P})$ is a $K(G(\Sigma, \mathfrak{P}, \mathcal{P}), 1)$ space.*

Proof. In this case, the graph $\mathcal{B}\mathfrak{C}(\Sigma, \mathfrak{P})^1$ contains 1 vertex and a loop for every equivalence class $[\sigma] \in \mathfrak{P}$ of a dimension 1 cone. Hence the generators of $\pi_1(\mathcal{B}\mathfrak{C}(\Sigma, \mathfrak{P}))$ coincide with those of $G(\mathfrak{C}, \mathfrak{P}, \mathcal{P})$. Moreover, the attaching map of the unique 2-cell induces a homotopy which is equivalent to the unique relation of the picture group, similar to Proposition 3.4.12. $\square$

Example 3.5.6. The main examples of fan posets are the following:

- (1) The poset of regions of a finite central simplicial hyperplane arrangement as introduced by [65] is a fan poset by [156], a lattice by [29] and polygonal by [157]. Furthermore, it is easily seen to be nondegenerate.
- (2) The fan poset induced by tors A on the g -vector fan of a τ -tilting finite algebra is a polygonal lattice [57] and nondegenerate, see Proposition 3.8.1.
- (3) Nondegenerate fan posets in $\mathbb{R}^2$ constructed as in the proof of Theorem 3.5.4.

3.6 Hyperplane arrangements

Recall briefly the theory of hyperplane arrangements, which are collections of subspaces of codimension one. A *central simplicial hyperplane arrangement* $\mathcal{H}$ in $\mathbb{R}^n$ is a collection

of subspaces of codimension 1 which defines a simplicial fan $\Sigma_{\mathcal{H}}$ and dissects the space into *regions*. All fans considered in this section are induced by hyperplane arrangements. The closures of these regions are maximal cones of the fan and two regions are called *adjacent* whenever their closures intersect in a cone of codimension one. Choosing any such region B as the base region and orienting the adjacency graph away from B defines a poset, called the *poset of regions* $\mathcal{P}(\mathcal{H}, B)$ see [65]. For any choice of base region B , this defines a fan poset on $\Sigma_{\mathcal{H}}$ obtained by taking the closure of each region, by [156, Sec. 4]. Denote this induced fan poset of regions by $\mathcal{P}_B$. Given a region R , its *separating set* (with respect to B), denoted by $S(R)$, is the collection of all hyperplanes in $\mathcal{H}$ separating R from the region $-B$. This gives an equivalent way of defining the poset of regions by saying $R_1 \leq R_2$ whenever $S(R_1) \subseteq S(R_2)$.

A *flat* of a hyperplane arrangement $\mathcal{H} \subseteq \mathbb{R}^n$ is an intersection of hyperplanes of $\mathcal{H}$, and thus a linear subspace of the ambient space $\mathbb{R}^n$. In particular, the empty intersection gives the ambient space $\mathbb{R}^n$ as a flat. The *support* of a cone $\sigma \in \Sigma_{\mathcal{H}}$ is the smallest flat $s(\sigma)$ which contains σ . This leads to the *flat-partition* $\mathfrak{P}_{\text{flat}}$ of the simplicial fan $\Sigma_{\mathcal{H}}$ given by $[\sigma_1]_{\mathfrak{P}_{\text{flat}}} = [\sigma_2]_{\mathfrak{P}_{\text{flat}}}$ if and only if $s(\sigma_1) = s(\sigma_2)$. In other words, cones whose support is the same flat are identified in $\mathfrak{P}_{\text{flat}}$.

Proposition 3.6.1. *The partition $\mathfrak{P}_{\text{flat}}$ is an admissible partition of $\Sigma_{\mathcal{H}}$ and thus the category of the flat-partition $\mathfrak{C}(\Sigma_{\mathcal{H}}, \mathfrak{P}_{\text{flat}})$ is a well-defined category.*

Proof. Let $\sigma_1, \sigma_2 \in \Sigma_{\mathcal{H}}$ be two cones such that the flat $X = s(\sigma_1) = s(\sigma_2)$ is the same intersection of hyperplanes. It follows immediately that

$$\text{span}\{\sigma_1\} = X = \text{span}\{\sigma_2\}.$$

It follows from [5, Lem. 1.36] that both $\text{star}(\sigma_1)$ and $\text{star}(\sigma_2)$ are “equivalent” to the *arrangement over the flat X* whose *essentialisation* is precisely the projection onto the orthogonal complement, see [5] for more details. The partition is admissible by Lemma 3.1.3 since it identifies the whole set of possible identifications. $\square$

Shards of hyperplane arrangements were introduced in [155]. Informally speaking a shard of a hyperplane arrangement $\mathcal{H} \subseteq \mathbb{R}^n$ is a “piece of a hyperplane”. They are obtained as follows: Let B be a choice of base region and call the n hyperplanes defining

it *basic*. Each pair of hyperplanes $H_1, H_2 \in \mathcal{H}$ gives rise to a *subarrangement* defined by $\mathcal{H}(H_1, H_2) := \{H \in \mathcal{H} : H_1 \cap H_2 \subseteq H\}$, which has an induced (relative) base region. Say that H_1 *cuts* H_2 if H_1 is basic in $\mathcal{H}(H_1, H_2)$ and H_2 is not basic in $\mathcal{H}(H_1, H_2)$. For each hyperplane $H \in \mathcal{H}$ remove from H all points contained in hyperplanes H' which cut H in the subarrangement $\mathcal{H}(H, H')$. The closures of the remaining connected components are called *shards*.

Remark 3.6.2. Essentially, shards partition hyperplanes in the same way that stability spaces of bricks define parts of hyperplanes in the wall-and-chamber structure of a finite-dimensional algebra, see [9, 38]. In particular, for preprojective algebras whose wall-and-chamber structure is a hyperplane arrangement [141], the shards coincide exactly with stability spaces of bricks [180] and shards were generalised and this result extended to all finite-dimensional algebras in [142].

Following [158], let Ξ denote the set of arbitrary intersections of shards, which has a natural poset structure by inclusion with maximal element the empty intersection $\mathbb{R}^n$. Using this set, define a partition $\mathfrak{P}_{\text{shard}}$ of the cones $\Sigma_{\mathcal{H}}$ of a hyperplane arrangement given by $[\sigma_1]_{\mathfrak{P}_{\text{shard}}} = [\sigma_2]_{\mathfrak{P}_{\text{shard}}}$ if and only if the smallest elements $\xi_i \in \Xi$ which contain σ_i , for $i = 1, 2$ respectively, coincide.

Proposition 3.6.3. *Let $\Sigma_{\mathcal{H}}$ be the fan of a finite central simplicial hyperplane arrangement in $\mathbb{R}^n$. The partition $\mathfrak{P}_{\text{shard}}$ is an admissible partition of $\Sigma_{\mathcal{H}}$ and thus the category of the shard-partition $\mathfrak{C}(\Sigma_{\mathcal{H}}, \mathfrak{P}_{\text{shard}})$ is a well-defined category.*

Proof. Since by definition each intersection of shards is contained in the intersection of corresponding hyperplanes, two such cones σ_1, σ_2 have the same support and hence the proof of Proposition 3.6.1 yields that they are in the same class of potential identifications.

To see that $\mathfrak{P}_{\text{shard}}$ is admissible, let $\sigma_1 \sim \sigma_2$ in $\mathfrak{P}_{\text{shard}}$ be distinct cones, and let cones $\kappa_i \in \text{star}(\sigma_i)$ for $i = 1, 2$ be such that $\pi_{\sigma_1}(\kappa_1) = \pi_{\sigma_2}(\kappa_2)$. It follows from Lemma 3.1.3 that $\text{span}\{\kappa_1\} = \text{span}\{\kappa_2\}$. If κ_1 and κ_2 are maximal, then the minimal intersection of shards they are each contained in is the empty one, which gives $[\kappa_1]_{\mathfrak{P}_{\text{shard}}} = [\kappa_2]_{\mathfrak{P}_{\text{shard}}}$. Otherwise, since $\Sigma_{\mathcal{H}}$ is finite and complete there exists a hyperplane $H \in \mathcal{H}$ containing κ_1, κ_2 . Assume for a contradiction that $\kappa_1 \not\sim \kappa_2$ in $\mathfrak{P}_{\text{shard}}$ which means that the hyper-

plane H is cut by a hyperplane H' separating κ_1 and κ_2 in the sense that the interiors of κ_1 and κ_2 lie in opposite half-spaces defined by H' . Since $\sigma_i \subseteq \kappa_i$, the hyperplane H' also separates σ_1 and σ_2 and in particular the cut on H induced by H' implies that the interiors of σ_1 and σ_2 lie in different shards, a contradiction.

Finally, it is left to verify that $\mathfrak{P}_{\text{shard}}$ is well-defined, in other words, when σ_1 and σ_2 are two distinct cones identified in $\mathfrak{P}_{\text{shard}}$ then given cones $\kappa_i, \kappa'_i \in \text{star}(\sigma_i)$ for $i = 1, 2$ satisfying

$$\pi_{\sigma_1}(\kappa_1) = \pi_{\sigma_2}(\kappa_2), \quad \pi_{\sigma_1}(\kappa'_1) = \pi_{\sigma_2}(\kappa'_2),$$

they are such that $\kappa_1 \sim \kappa'_1$ if and only if $\kappa_2 \sim \kappa'_2$ in $\mathfrak{P}_{\text{shard}}$. Let σ_1, σ_2 be two distinct cones identified in $\mathfrak{P}_{\text{shard}}$ and $\kappa_i, \kappa'_i \in \text{star}(\sigma_i)$ be as above for $i = 1, 2$. Assume for a contradiction that $\kappa_1 \sim \kappa'_1$ but $\kappa_2 \not\sim \kappa'_2$ in $\mathfrak{P}_{\text{shard}}$. By Lemma 3.1.3 these cones satisfy

$$\text{span}\{\kappa_2\} = \text{span}\{\kappa_1\} = \text{span}\{\kappa'_1\} = \text{span}\{\kappa'_2\}.$$

Therefore every hyperplane containing one of $\kappa_1, \kappa'_1, \kappa_2$ and κ'_2 also contains all others and the same holds for σ_1 and σ_2 . Since $\kappa_2 \not\sim \kappa'_2$ in $\mathfrak{P}_{\text{shard}}$, there must exist a shard S that contains κ_2 but not κ'_2 . By definition this means that there exists a hyperplane H containing κ_2 and κ'_2 and a hyperplane H' which separates κ_2 from κ'_2 in such a way that H' is basic in $\mathcal{H}(H, H')$ whereas H is not. By definition if H' separates κ_2 and κ'_2 then it must pass through their intersection, in other words it contains σ_2 and thus σ_1 . However, H also contains σ_1 and σ_2 . Consider now the orthogonal projections $\pi_{\sigma_1}(\text{star}(\sigma_1))$ and $\pi_{\sigma_2}(\text{star}(\sigma_2))$ which coincide by assumption. Then the images of $\pi_{\sigma_1}(\kappa_1) = \pi_{\sigma_2}(\kappa_2)$ lie on the same linear subspace of $\pi_{\sigma_i}(\mathbb{R}^n)$ as $\pi_{\sigma_1}(\kappa'_1)$ and $\pi_{\sigma_2}(\kappa'_2)$. Moreover, $\pi_{\sigma_i}(H)$ and $\pi_{\sigma_i}(H')$ are hyperplanes in $\pi_{\sigma_i}(\mathbb{R}^n)$ and $\pi_i(H')$ separates $\pi_{\sigma_1}(\kappa_1)$ from $\pi_{\sigma_1}(\kappa'_1)$. Since κ_1 and κ'_1 both lie on H , which is cut by H' , and they are separated by H' , they cannot be identified in $\mathfrak{P}_{\text{shard}}$, a contradiction. Swapping the indices proves the reverse direction. $\square$

Besides the importance of establishing the flat-partition and the shard-partition, it is also necessary to understand picture groups of fans coming from hyperplane arrangements.

Lemma 3.6.4. *Let $\mathcal{H}$ be a finite central simplicial hyperplane arrangement. The fan poset of regions $\mathcal{P}_B$ with any choice of base region B is well-defined on identified stars and hence the presentation of the picture group in Lemma 3.4.9 may be used.*

Proof. This follows from the fact that any poset of regions $\mathcal{P}(\mathcal{H}, B)$ of a simplicial hyperplane arrangement is induced by any linear functional $b \in (\mathbb{R}^n)^\vee$ whose minimum on the unit sphere lies inside the base region B of $\mathcal{P}(\mathcal{H}, B)$, see [156, Thm. 4.2], and Lemma 3.4.8. $\square$

For the flat-partition $\mathfrak{P}_{\text{flat}}$, for which all possible identifications are made, equivalence classes $[\sigma]$ of cones $\sigma \in \Sigma_{\mathcal{H}}^{n-1}$ of codimension 1 are given by all such cones contained in the same hyperplane denoted by $H_\sigma = \text{span}\{\sigma\}$. Therefore these equivalence classes may be represented by a unit normal vector $\mathbf{n}_{H_\sigma}$ to the hyperplane. The following is one of the main results of this thesis and the rest of this section concerns its proof. In Section 3.8 its algebraic implications are discussed.

Theorem 3.6.5. *Let $\mathcal{H} \subseteq \mathbb{R}^n$ be a finite central simplicial hyperplane arrangement. Let B be a chosen base region and $\mathcal{P}_B$ the corresponding fan poset of regions on $\Sigma_{\mathcal{H}}$. Assume that the unit normal vectors $\mathbf{n}_H$ to all hyperplanes $H \in \mathcal{H}$ can be taken to lie in the positive orthant $\mathbb{R}_{\geq 0}^n$. Then the functor of Eq. (3.5.1) is faithful.*

Denote by $\mathbb{Z}[[\mathbb{R}^n]]$ the formal power series with generators $\{x^{\mathbf{v}} : \mathbf{v} \in \mathbb{R}^n\}$ over $\mathbb{Z}$ whose multiplication is given by $x^{\mathbf{v}_1} * x^{\mathbf{v}_2} = x^{\mathbf{v}_1 + \mathbf{v}_2}$. This is a commutative associative algebra. The group of units $\mathbb{Z}[[\mathbb{R}^n]]^*$ is therefore an abelian group consisting of all formal sums with constant term equal to 1 or -1.

Lemma 3.6.6. *In the setting of Theorem 3.6.5 there exists a group homomorphism*

$$\begin{aligned} \phi : G(\Sigma_{\mathcal{H}}, \mathfrak{P}_{\text{max}}, \mathcal{P}_B) &\rightarrow \mathbb{Z}[[\mathbb{R}^n]]^* \\ X_{\mathbf{n}_H} &\mapsto 1 + x^{\mathbf{n}_H}. \end{aligned}$$

Proof. As pointed out in Example 3.5.6 the poset of regions $\mathcal{P}_B$ is a polygonal lattice, so by Proposition 3.4.12 it is sufficient to consider the relations coming from facial intervals of cones of codimension 2, which are called *polygon relations*. To show that the polygon relations are preserved is simple because of the commutativity of $\mathbb{Z}[[\mathbb{R}^n]]^*$ and

the hyperplane structure. In particular, because the fan $\Sigma_{\mathcal{H}}$ comes from a hyperplane arrangement and $\mathfrak{P}_{\max}$ is the maximal partition, the labels of the two disjoint chains in any polygon of $\mathcal{P}_{\mathcal{B}}$ correspond to the two sequences $(H_1, \dots, H_r)$ and $(H_r, \dots, H_1)$ of hyperplanes. These clearly satisfy

$$\begin{aligned} \phi(X_{\mathbf{n}_{H_1}} \dots X_{\mathbf{n}_{H_r}}) &= (1 + x^{\mathbf{n}_{H_1}}) * \dots * (1 + x^{\mathbf{n}_{H_r}}) \\ &= (1 + x^{\mathbf{n}_{H_r}}) * \dots * (1 + x^{\mathbf{n}_{H_1}}) \\ &= \phi(X_{\mathbf{n}_{H_r}} \dots X_{\mathbf{n}_{H_1}}) \end{aligned}$$

since $\mathbb{Z}[[\mathbb{R}^n]]^*$ is abelian. $\square$

Remark 3.6.7. In [91, Sec. 4.3] and [92, Sec. 5.2] the authors take a similar approach and find a group homomorphism into the groups of units of different versions of Hall algebras. However, in the geometric setting of this section, the hyperplane arrangement gives the polygon relations a symmetric structure, so that the abelian group $\mathbb{Z}[[\mathbb{R}^n]]^*$ is a more natural candidate to consider.

Corollary 3.6.8. *Let $H_1, H_2 \in \mathcal{H}$ be distinct hyperplanes. Then they satisfy $\mathbf{n}_{H_1} \neq \mathbf{n}_{H_2}$ and hence $e \neq X_{\mathbf{n}_{H_1}} \neq X_{\mathbf{n}_{H_2}} \in G(\Sigma_{\mathcal{H}}, \mathfrak{P}_{\max}, \mathcal{P}_B)$.*

Proof. This immediately follows from the existence of the group homomorphism defined in Lemma 3.6.6 because

$$\phi(X_{\mathbf{n}_{H_1}}) = 1 + x^{\mathbf{n}_{H_1}} \neq 1 + x^{\mathbf{n}_{H_2}} = \phi(X_{\mathbf{n}_{H_2}}).$$

In other words, the generators $X_{\mathbf{n}_{H_1}}$ and $X_{\mathbf{n}_{H_2}}$ of $G(\Sigma_{\mathcal{H}}, \mathfrak{P}_{\max}, \mathcal{P}_B)$ are mapped to distinct elements in $\mathbb{Z}[[\mathbb{R}^n]]^*$ by ϕ , hence they are distinct. $\square$

Since the poset of regions is nondegenerate by Lemma 3.6.4, consider the additional generators of the alternative presentation of the picture group given in Lemma 3.4.9.

Lemma 3.6.9. *In the setting of Theorem 3.6.5, let all unit normal vector $\mathbf{n}_H$ to the hyperplanes lie in the positive orthant. Let $R_1 \neq R_2 \in \mathbb{R}^n \setminus \mathcal{H}$ be distinct regions, then $g_{\overline{R}_1} \neq g_{\overline{R}_2}$ in $G(\mathcal{H}, \mathfrak{P}_{\max}, \mathcal{P}_B)$.*

Proof. Let $\overline{R}_1 \xrightarrow{\mathbf{n}_{H_1}} \dots \xrightarrow{\mathbf{n}_{H_s}} -B$ and $\overline{R}_2 \xrightarrow{\mathbf{n}_{H'_1}} \dots \xrightarrow{\mathbf{n}_{H'_r}} -B$ be two (labelled) chains in $\text{Hasse}(\mathcal{P}_B)$. Assume for a contradiction that $g_{\overline{R}_1} = g_{\overline{R}_2}$. The inequality $\max(r, s) > 0$

must hold as otherwise $R_1 = -B = R_2$. Let $\mathbf{m} \in \{\mathbf{n}_{H_1}, \dots, \mathbf{n}_{H_s}, \mathbf{n}_{H'_1}, \dots, \mathbf{n}_{H'_r}\}$ be any vector. Then, the assumption $g_{\overline{R}_1} = g_{\overline{R}_2}$ implies

$$X_{\mathbf{n}_{H_1}} \dots X_{\mathbf{n}_{H_s}} e = g_{\overline{R}_1} = g_{\overline{R}_2} = X_{\mathbf{n}_{H'_1}} \dots X_{\mathbf{n}_{H'_r}} e.$$

After applying the group homomorphism ϕ of Lemma 3.6.6, this yields

$$(1 + x^{\mathbf{n}_{H_1}}) * \dots * (1 + x^{\mathbf{n}_{H_s}}) = (1 + x^{\mathbf{n}_{H'_1}}) * \dots * (1 + x^{\mathbf{n}_{H'_r}}).$$

Therefore $x^{\mathbf{m}}$ arises with positive coefficient on one of the sides. Since the sum of the entries of $\mathbf{m}$ is minimal (and all unit normal vectors have nonnegative entries) $x^{\mathbf{m}}$ cannot be written as the product of other terms (since $\mathbf{m}$ cannot be written as a sum of the other unit normal vectors). Hence it must appear on both sides of the equality. Since $\mathbf{m}$ was chosen arbitrarily, this holds for all (unit normal) vectors in $\{\mathbf{n}_{H_1}, \dots, \mathbf{n}_{H_s}, \mathbf{n}_{H'_1}, \dots, \mathbf{n}_{H'_r}\}$. In other words, the sets $\{\mathbf{n}_{H_1}, \dots, \mathbf{n}_{H_s}\}$ and $\{\mathbf{n}_{H'_1}, \dots, \mathbf{n}_{H'_r}\}$ coincide, and hence do the sets $\{H_1, \dots, H_s\}$ and $\{H'_1, \dots, H'_r\}$ of hyperplanes.

By [157, Prop. 9-1.15], each cover relation $R \triangleleft R'$ in $\mathcal{P}(\mathcal{H}, B)$ comes from adjacent regions separated by a hyperplane H and is such that $S(R') = S(R) \cup \{H\}$. Consequently, there is an equality

$$S(R_1) = \{H_1, \dots, H_s\} = \{H'_1, \dots, H'_r\} = S(R_2),$$

which implies $R_1 = R_2$, a contradiction. Therefore $g_{\overline{R}_1} \neq g_{\overline{R}_2}$. □

These intermediate results are combined in the following proof.

Proof of Theorem 3.6.5. Consider the functor

$$\begin{aligned} \Psi : \mathfrak{C}(\Sigma_{\mathcal{H}}, \mathfrak{P}_{\max}) &\rightarrow G(\Sigma_{\mathcal{H}}, \mathfrak{P}_{\max}, \mathcal{P}_B) \\ [f_{\sigma\kappa}] &\mapsto g_{\kappa}^- \cdot g_{\sigma}^{-1} \end{aligned} \tag{3.6.1}$$

which is simply an alternative way of writing the one in Eq. (3.5.1) using the presentation of the picture group given in Lemma 3.4.9, which is possible by Lemma 3.6.4. To show that Ψ is faithful it is required to show that the induced map $\Psi_{\sigma\kappa}$ from $\text{Hom}_{\mathfrak{C}(\Sigma_{\mathcal{H}}, \mathfrak{P}_{\text{flat}})}([\sigma], [\kappa])$ to $\text{Hom}_{G(\Sigma_{\mathcal{H}}, \mathfrak{P}_{\max}, \mathcal{P}_B)}(\bullet, \bullet)$ is injective. By Corollary 3.1.6 and

Lemma 3.2.2, it suffices to prove this for one representative $\sigma \in [\sigma]$. Hence, take distinct morphisms $[f_{\sigma\kappa_1}] \neq [f_{\sigma\kappa_2}] \in \text{Hom}_{\mathfrak{C}(\Sigma_{\mathcal{H}}, \mathfrak{P}_{\text{flat}})}([\sigma], [\kappa])$ which are such that $\kappa_1 \neq \kappa_2$ but $\kappa_1 \sim \kappa_2$ in $\mathfrak{P}_{\text{max}}$. Applying the functor yields

$$\Psi([f_{\sigma\kappa_1}]) = g_{\kappa_1^-} \cdot g_{\sigma^-}^{-1} \quad \text{and} \quad \Psi([f_{\sigma\kappa_2}]) = g_{\kappa_2^-} \cdot g_{\sigma^-}^{-1}.$$

Then Lemma 3.6.9 implies that these group elements coincide if and only if $\kappa_1^- = \kappa_2^-$, which would mean that $\kappa_1^- = \kappa_2^- \in \text{star}(\kappa_1) \cap \text{star}(\kappa_2)$. This leads to a contradiction because $\text{span}\{\kappa_1\} = \text{span}\{\kappa_2\}$ and $\kappa_1 \neq \kappa_2$ imply that $\text{star}(\kappa_1)$ and $\text{star}(\kappa_2)$ intersect only at the boundary. Therefore the functor is faithful. $\square$

Corollary 3.6.10. *In the setting of Theorem 3.6.5, if $\mathfrak{C}(\Sigma_{\mathcal{H}}, \mathfrak{P}_{\text{flat}})$ satisfies the pairwise compatibility of first and last factors, then the classifying space is a $K(\pi, 1)$ space for π the picture group.*

Proof. It follows from Proposition 2.4.4 and Theorem 3.6.5 that $\mathcal{BC}(\Sigma_{\mathcal{H}}, \mathfrak{P}_{\text{flat}})$ is a $K(\pi, 1)$ space and from Proposition 3.4.12 that the fundamental group is isomorphic to the picture group, in this case. $\square$

3.7 Lattice of admissible partitions

It is well-known that the collection of partitions of a set, ordered by refinement, forms a complete lattice, see [84, Sec. IV.4]. As a first step in this section, it is proven that the restriction to admissible partitions of a fan preserves the lattice structure, thus establishing a lattice of categories (one for each admissible partition) of a fan. In the special case where the underlying fan is the g -vector fan of a finite-dimensional algebra, see Section 3.8, the τ -cluster morphism category is an element of this lattice. Similarly if the underlying fan is a hyperplane arrangement, the category of the flat-partition and the category of the shard-partition lie in this lattice. Recall that an equivalence relation induces a partition on a set X and vice versa. Define the following poset relation on partitions.

Definition 3.7.1. Let P_1 and P_2 be partitions of X . The partition P_1 is called a *finer* partition than P_2 if

$$x \sim_{P_1} y \implies x \sim_{P_2} y, \quad \text{or equivalently,} \quad \{x, y\} \subseteq a \in P_1 \implies \{x, y\} \subseteq b \in P_2$$

for $x, y \in X$ and some $a \in P_1$ and $b \in P_2$. In this case, write $P_1 \leq P_2$ and say that P_2 is *coarser* than P_1 .

Denote by $\text{Part}(X)$ the set of all partitions of a set X . The set $\text{Part}(X)$ is partially ordered by refinement and forms a lattice with meet and join described as follows. Given a (possibly infinite) subset $\mathcal{S} = \{P_i\}_{i \in I} \subseteq \text{Part}(X)$, define the meet $\bigwedge \mathcal{S}$ to be the partition satisfying

$$x \sim_{\bigwedge \mathcal{S}} y \quad \text{if and only if} \quad x \sim_{P_i} y \text{ for all } i \in I.$$

Define the join $\bigvee \mathcal{S}$ to be the partitioned satisfying $x \sim_{\bigvee \mathcal{S}} y$ if and only if there exists a natural number m , indices $i_0, \dots, i_m \in I$ and $x_0, \dots, x_{m+1} \in X$ such that $x = x_0$, $y = x_{m+1}$ and $x_j \sim_{P_{i_j}} x_{j+1}$ for $0 \leq j \leq m$. In view of Definition 3.1.1, denote by $\text{APart}(\Sigma)$ the set of all admissible partitions of a fan Σ . The following shows that $\text{APart}(\Sigma)$ is a complete sublattice of $\text{Part}(\Sigma)$.

Proposition 3.7.2. *The partially ordered set $\text{APart}(\Sigma)$ forms a complete lattice.*

Proof. Since $\text{Part}(\Sigma)$ is a complete lattice, it is sufficient to show that $\text{APart}(\Sigma)$ is closed under the lattice operations. Recall from Section 4.2 that this means showing that the join and meet partitions preserve the sets of potential identifications and are admissible. Regarding the meet, take a subset $\mathcal{S} = \{\mathfrak{P}_i\}_{i \in I} \subseteq \text{APart}(\Sigma)$ of admissible partitions and take a block $\mathcal{E}_\sigma^\ell \in \bigwedge \mathcal{S}$. If $|\mathcal{E}_\sigma^\ell| = 1$, then there is nothing to show, so assume $|\mathcal{E}_\sigma^\ell| > 1$ and take $\sigma_1, \sigma_2 \in \mathcal{E}_\sigma^\ell$. By definition $\sigma_1 \sim_{\mathfrak{P}_i} \sigma_2$ for all $i \in I$, so the partition $\bigwedge \mathcal{S}$ consists of possible identifications. Similarly, whenever $\pi_{\sigma_1}(\rho_1) = \pi_{\sigma_2}(\rho_2)$, then $\rho_1 \sim_{\mathfrak{P}_i} \rho_2$ for all $i \in I$ and thus $\rho_1 \sim_{\bigwedge \mathcal{S}} \rho_2$, hence $\bigwedge \mathcal{S}$ is an admissible partition.

For the join, take $\mathcal{E}_\sigma^\ell \in \bigvee \mathcal{S}$ and $\sigma_1, \sigma_2 \in \mathcal{E}_\sigma^\ell$, then if both σ_1 and σ_2 are contained in one block $\mathcal{E}_\lambda^h \in \mathfrak{P}_i$ for some $i \in I$, the result is immediate. Therefore assume that σ_1 and σ_2 are not contained in the same block in any $\mathfrak{P}_i$. However, by the construction of the join, there exists a sequence

$$\sigma_1 \sim_{\mathfrak{P}_{i_1}} \sigma_{i_1} \sim_{\mathfrak{P}_{i_2}} \dots \sim_{\mathfrak{P}_{i_{r-1}}} \sigma_{i_{r-1}} \sim_{\mathfrak{P}_{i_r}} \sigma_2$$

for some $r \geq 1$, such that each term is contained in $\mathcal{E}_{\sigma_1} = \mathcal{E}_{\sigma_2}$, hence $\bigvee \mathcal{S}$ consists of possible identifications. This sequence of possible identifications implies that

$$\pi_{\sigma_1}(\text{star}(\sigma_1)) = \pi_{\sigma_{i_j}}(\text{star}(\sigma_{i_j})) = \pi_{\sigma_2}(\text{star}(\sigma_2)),$$

for $j = 1, \dots, r-1$. To show that $\bigvee \mathcal{S}$ is admissible, assume $\pi_{\sigma_1}(\rho_1) = \pi_{\sigma_2}(\rho_2)$ holds for some $\rho_1 \in \text{star}(\sigma_1)$ and $\rho_2 \in \text{star}(\sigma_2)$. Then by the above there exists a sequence of $\rho_{i_j} \in \text{star}(\sigma_{i_j})$ for $j = 1, \dots, r-1$ such that

$$\pi_{\sigma_1}(\rho_1) = \pi_{\sigma_{i_j}}(\rho_{i_j}) = \pi_{\sigma_2}(\rho_2).$$

Since each $\mathfrak{P}_{i_j}$ is admissible for $j = 1, \dots, r$ this yields a sequence

$$\rho_1 \sim_{\mathfrak{P}_{i_1}} \rho_{i_1} \sim_{\mathfrak{P}_{i_2}} \dots \sim_{\mathfrak{P}_{i_{r-1}}} \rho_{i_{r-1}} \sim_{\mathfrak{P}_{i_r}} \rho_2,$$

as required. Hence $\bigvee \mathcal{S}$ is an admissible partition. $\square$

It is easily seen that this is a bounded lattice, that is, there exists a maximal and a minimal element. The minimal element is the partition $\mathfrak{P}_{\text{poset}}$ mentioned in previous sections, which is just the finest partition with trivial equivalence classes. The maximal element is the coarsest partition, whose equivalence classes are exactly the $\mathcal{E}_\sigma$ described following Definition 3.1.1. When the fan is a hyperplane arrangement the coarsest partition is exactly the flat-partition, see Proposition 3.6.1. The main result of this section is the following description of the relationship between comparable partitions.

Theorem 3.7.3. *Let $(\Sigma, \mathcal{P})$ be a fan poset and $\mathfrak{P}_1, \mathfrak{P}_2$ be two admissible partitions of Σ such that $\mathfrak{P}_1$ is finer than $\mathfrak{P}_2$. Then the following hold:*

- (1) *There exists a faithful surjective-on-objects functor $F : \mathfrak{C}(\Sigma, \mathfrak{P}_1) \rightarrow \mathfrak{C}(\Sigma, \mathfrak{P}_2)$.*
- (2) *The classifying spaces satisfy $\mathcal{BC}(\Sigma, \mathfrak{P}_2) \cong \mathcal{BC}(\Sigma, \mathfrak{P}_1) / \sim$, where $\sim$ identifies the cells $e([\sigma_1]_{\mathfrak{P}_1})$ and $e([\sigma_2]_{\mathfrak{P}_1})$ whenever $[\sigma_1]_{\mathfrak{P}_2} = [\sigma_2]_{\mathfrak{P}_2}$.*
- (3) *If $(\Sigma, \mathcal{P})$ is nondegenerate, then the groups $G(\Sigma, \mathfrak{P}_2, \mathcal{P})$ and $G(\Sigma, \mathfrak{P}_1, \mathcal{P})/I$ are isomorphic, where the normal subgroup I is generated by the relations $X_{[\sigma_1]} = X_{[\sigma_2]}$ whenever $[\sigma_1]_{\mathfrak{P}_1} \neq [\sigma_2]_{\mathfrak{P}_1}$ but $[\sigma_1]_{\mathfrak{P}_2} = [\sigma_2]_{\mathfrak{P}_2}$.*

Proof. (1) The functor $F : \mathfrak{C}(\Sigma, \mathfrak{P}_1) \rightarrow \mathfrak{C}(\Sigma, \mathfrak{P}_2)$ is given on objects by the map sending $[\sigma]_{\mathfrak{P}_1} \mapsto [\sigma]_{\mathfrak{P}_2}$ for all cones $\sigma \in \Sigma$ and given on morphisms by sending $[f_{\sigma\rho}]_{\mathfrak{P}_1} \mapsto [f_{\sigma\rho}]_{\mathfrak{P}_2}$ for all $\sigma, \rho \in \Sigma$. Objects of $\mathfrak{C}(\Sigma, \mathfrak{P}_i)$ are exactly the blocks of $\mathfrak{P}_i$ for both $i = 1, 2$. By definition, $\mathfrak{P}_2$ being coarser than $\mathfrak{P}_1$ means $[\sigma]_{\mathfrak{P}_1} \subseteq [\sigma]_{\mathfrak{P}_2}$. Hence $\mathfrak{P}_2$ has at most as many blocks as $\mathfrak{P}_1$. Thus, F is surjective-on-objects.

To see that the functor is faithful, one observes that the identification of morphisms in the construction of the category does not depend on a choice but is induced when certain cones are identified. Take two different morphisms $[f_{\sigma_1 \rho_1}]_{\mathfrak{P}_1} \neq [f_{\sigma_2 \rho_2}]_{\mathfrak{P}_1}$ in $\text{Hom}_{\mathfrak{C}(\Sigma, \mathfrak{P}_1)}([\sigma]_{\mathfrak{P}_1}, [\rho]_{\mathfrak{P}_1})$ satisfying $\sigma_1 \neq \sigma_2$ or $\rho_1 \neq \rho_2$ but $\sigma_1 \sim_{\mathfrak{P}_1} \sigma_2$ and $\rho_1 \sim_{\mathfrak{P}_1} \rho_2$. Assume for a contradiction that $[f_{\sigma_1 \rho_1}]_{\mathfrak{P}_2} = [f_{\sigma_2 \rho_2}]_{\mathfrak{P}_2}$ then the cones must satisfy $\pi_{\sigma_1}(\rho_1) = \pi_{\sigma_2}(\rho_2)$ by definition. However, then they should have also been identified in $\mathfrak{P}_1$, a contradiction. Hence the induced map

$$F_{[\sigma], [\rho]} : \text{Hom}_{\mathfrak{C}(\Sigma, \mathfrak{P}_1)}([\sigma]_{\mathfrak{P}_1}, [\rho]_{\mathfrak{P}_1}) \rightarrow \text{Hom}_{\mathfrak{C}(\Sigma, \mathfrak{P}_2)}([\sigma]_{\mathfrak{P}_2}, [\rho]_{\mathfrak{P}_2})$$

is injective and hence the functor faithful.

- (2) This follows from the description of the CW-complex in Theorem 3.3.4 whose cells are the union of factorisation cubes, and the fact that the factorisation cubes of two identified morphisms get identified, by Lemma 3.2.2.
- (3) Let $\kappa_1, \kappa_2, \kappa_3 \in \Sigma$. The quotient group homomorphism

$$H : G(\Sigma, \mathfrak{P}_1, \mathcal{P}) \rightarrow G(\Sigma, \mathfrak{P}_1, \mathcal{P})/I$$

identifies the elements $X_{[\kappa_1]_{\mathfrak{P}_1}}$ and $X_{[\kappa_2]_{\mathfrak{P}_1}}$ of $G(\Sigma, \mathfrak{P}_1, \mathcal{P})$ whenever $\kappa_1 \sim_{\mathfrak{P}_2} \kappa_2$ but $\kappa_1 \not\sim_{\mathfrak{P}_1} \kappa_2$. More precisely, in this case $X_{[\kappa_1]_{\mathfrak{P}_1}} + I = X_{[\kappa_2]_{\mathfrak{P}_1}} + I \in G(\Sigma, \mathfrak{P}_1, \mathcal{P})/I$. Furthermore if $\kappa_3 \sim_{\mathfrak{P}_1} \kappa_1$, then also $X_{[\kappa_3]_{\mathfrak{P}_1}} + I = X_{[\kappa_2]_{\mathfrak{P}_1}} + I \in G(\Sigma, \mathfrak{P}_1, \mathcal{P})/I$. By construction the generators $X_{[\kappa_i]_{\mathfrak{P}_1}} + I$ and $X_{[\kappa_j]_{\mathfrak{P}_1}} + I$ coincide in $G(\Sigma, \mathfrak{P}_1, \mathcal{P})/I$ exactly when $X_{[\kappa_i]_{\mathfrak{P}_2}}$ and $X_{[\kappa_j]_{\mathfrak{P}_2}}$ coincide in $G(\Sigma, \mathfrak{P}_2, \mathcal{P})$ for any $\kappa_i, \kappa_j \in \Sigma$. It is easily checked that the canonical assignment $X_{[\kappa]_{\mathfrak{P}_1}} + I \mapsto X_{[\kappa]_{\mathfrak{P}_2}}$ provides a group homomorphism between $G(\Sigma, \mathfrak{P}_1, \mathcal{P})/I$ and $G(\Sigma, \mathfrak{P}_2, \mathcal{P})$. Indeed, there is clearly a bijection between the generators and the relations of the quotient group are preserved as they are inherited from $G(\Sigma, \mathfrak{P}_1, \mathcal{P})$ by construction.

□

Corollary 3.7.4. *Let $(\Sigma, \mathcal{P})$ be a fan poset and $\mathfrak{P}_1 \leq \mathfrak{P}_2$ be two admissible partitions of Σ . If there exists a faithful group functor $H : \mathfrak{C}(\Sigma, \mathfrak{P}_2) \rightarrow G(\Sigma, \mathfrak{P}_2, \mathcal{P})$, then there exists a faithful group functor $\mathfrak{C}(\Sigma, \mathfrak{P}_1) \rightarrow G(\Sigma, \mathfrak{P}_2, \mathcal{P})$.*

Proof. Let $F : \mathfrak{C}(\Sigma, \mathfrak{P}_1) \rightarrow \mathfrak{C}(\Sigma, \mathfrak{P}_2)$ be the faithful functor of Theorem 3.7.3(1). The

desired group functor is given by the composition $H \circ F$ of faithful functors, which is faithful. $\square$

Corollary 3.7.5. *In the setting of Theorem 3.6.5, the category $\mathfrak{C}(\Sigma_{\mathcal{H}}, \mathfrak{P})$ admits a faithful functor to the group $G(\Sigma_{\mathcal{H}}, \mathfrak{P}_{\text{flat}}, \mathcal{P})$ for all admissible partitions $\mathfrak{P}$.*

Proof. The flat-partition $\mathfrak{P}_{\text{flat}}$ is the maximal partition, so that any other partition $\mathfrak{P}$ is finer, that is, $\mathfrak{P} \leq \mathfrak{P}_{\text{flat}}$. The result follows from Corollary 3.7.4 and Theorem 3.6.5. $\square$

3.8 Applications to the τ -cluster morphism category

In this final section, the results of this chapter are applied to the τ -cluster morphism category of a finite-dimensional K -algebra A . First, it is shown that $\text{tors } A$ induces a natural fan poset structure on the g -vector fan $\Sigma(A)$ whenever $\Sigma(A)$ is τ -tilting finite.

Proposition 3.8.1. *Let A be a τ -tilting finite algebra, then the poset of torsion classes $\text{tors } A$ induces a fan poset $(\Sigma, \leq_{\text{tors } A})$ on $\Sigma(A)$ via the bijection of Theorem 2.2.2.*

Proof. By definition, every cone $\sigma \in \Sigma(A)$ is the g -vector cone of a τ -rigid pair (M, P) . Since A is τ -tilting finite, the maximal cones containing $\bar{\mathcal{C}}_{(M,P)}$ are exactly those corresponding to the interval $[\text{Fac } M, {}^\perp \tau M \cap P^\perp] \subseteq \text{tors } A$. This is because maximal g -vector cones containing $\bar{\mathcal{C}}_{(M,P)}$ are exactly those $\bar{\mathcal{C}}_{(N,Q)} \in \Sigma(A)$ such that (N, Q) is a τ -tilting pair with $M \in \text{add } N$ and $P \in \text{add } Q$. Hence the interval of maximal cones corresponds to the τ -perpendicular interval between the co-Bongartz and Bongartz completions of (M, P) , see also Definition 2.2.4.

It remains to show that each interval is a cone. For a torsion class $\mathcal{T} \in \text{tors } A$ and a torsion-free class $\mathcal{F} \in \text{torf } A$, respectively, define the following subspaces of $\mathbb{R}^n$:

$$\mathcal{H}_{\mathcal{T}}^+ := \bigcap_{T \in \mathcal{T}} \{v \in \mathbb{R}^n : \langle v, \mathbf{dim } T \rangle \geq 0\}, \quad \mathcal{H}_{\mathcal{F}}^- := \bigcap_{F \in \mathcal{F}} \{v \in \mathbb{R}^n : \langle v, \mathbf{dim } F \rangle \leq 0\}.$$

Both spaces are intersections of half-spaces and thus convex cones. Take an interval $I = [\sigma_1, \sigma_2] \in (\Sigma(A), \leq_{\text{tors } A})$ corresponding to the interval $[\mathcal{T}_{\sigma_1}, \mathcal{T}_{\sigma_2}] \subseteq \text{tors } A$, and define $\mathcal{T}_i := \mathcal{T}_{\sigma_i}$. Proving that $(\Sigma(A), \leq_{\text{tors } A})$ is a fan poset requires showing that

$$\bigcup_{\sigma_i \in I} \sigma_i = \mathcal{H}_{\mathcal{T}_1}^+ \cap \mathcal{H}_{\mathcal{T}_2^\perp}^-.$$

Take $v \in \bigcup_{\sigma_i \in I} \sigma_i$. From [38, Lem. 3.12] it follows that $\langle v, \mathbf{dim} T \rangle \geq 0$ for all $T \in \mathcal{T}_1$ since $\mathcal{T}_1 \subseteq \mathcal{T}_i$ for all $\sigma_i \in I$. Therefore $v \in \mathcal{H}_{\mathcal{T}_1}^+$. Dually, it follows for the torsion-free class $\mathcal{T}_2^\perp$ that $\langle v, \mathbf{dim} F \rangle \leq 0$ for all $F \in \mathcal{T}_2^\perp$. Hence $\bigcup_{\sigma_i \in I} \sigma_i \subseteq \mathcal{H}_{\mathcal{T}_1}^+ \cap \mathcal{H}_{\mathcal{T}_2^\perp}^-$.

Conversely, assume for a contradiction that there exists $v \in \mathcal{H}_{\mathcal{T}_1}^+ \cap \mathcal{H}_{\mathcal{T}_2^\perp}^-$ such that there is no $\sigma_i \in I$ satisfying $v \in \sigma_i$. Since the g -vector fan is finite and complete by [9, Thm. 4.7], there exists some other maximal cone $\sigma' \in \Sigma(A)$ with $\sigma' \notin I$ containing v . By construction of $\mathcal{H}_{\mathcal{T}_1}^+$, the torsion class $\mathcal{T}_v$ corresponding to the maximal cone σ' via Theorem 2.2.2 contains $\mathcal{T}_1$ since $\mathcal{T}_v$ can be written as

$$\mathcal{T}_v = \{T \in \text{mod } A : \langle v, \mathbf{dim} X \rangle \geq 0, \text{ for every quotient } X \text{ of } T\}$$

by [21] and [38, Prop. 3.27, Rmk. 3.28]. Therefore $\mathcal{T}_1 \subseteq \mathcal{T}_{\sigma'}$. Dually, it follows that $(\mathcal{T}_{\sigma'})^\perp \subseteq (\mathcal{T}_2)^\perp$ and hence $\sigma' \in I$. $\square$

Therefore any properties of the poset $\text{tors } A$ also hold for the induced fan poset $(\Sigma(A), \leq_{\text{tors } A})$. In particular it is a polygonal completely semidistributive lattice, see Example 2.1.8. Moreover, the fan poset is well-defined on identified stars as the projection onto stars coincide with the process of τ -tilting reduction, which is an order-preserving operation [115, Thm. 3.12], see also [57, Thm. 4.12]. Hence $(\Sigma(A), \leq_{\text{tors } A})$ is nondegenerate by Lemma 3.4.7.

In order to apply results from previous sections it is necessary to understand where the τ -cluster morphism category fits into the construction of the categories of partitioned fans. Comparing Definition 3.1.4 with Definition 2.6.4 suggests the definition of a partition $\mathfrak{P}_{\text{WAC}}$ via the identification

$$\overline{\mathcal{C}}_{(M_1, P_1)} \sim_{\mathfrak{P}_{\text{WAC}}} \overline{\mathcal{C}}_{(M_2, P_2)} \quad \text{whenever} \quad \mathcal{W}_{(M_1, P_1)} = \mathcal{W}_{(M_2, P_2)}.$$

By [169, Cor. 3.7, Lem. 3.8], the partition $\mathfrak{P}_{\text{WAC}}$ is admissible. Then the categories defined in Definition 3.1.4 and Definition 2.6.4 differ only in whether objects are closed cones or interior cones. It follows directly that the τ -cluster morphism category $\mathfrak{C}(A)$ is equivalent to the category of the partitioned fan $\mathfrak{C}(\Sigma(A), \mathfrak{P}_{\text{WAC}})$.

Now compare the picture group $G(A)$ of the finite-dimensional algebra A , as defined in Definition 2.3.2, with the picture group of the partitioned g -vector fan $(\Sigma(A), \mathfrak{P}_{\text{WAC}})$ with respect to the fan poset $(\Sigma(A), \leq_{\text{tors } A})$ as in Proposition 3.8.1. The following proof

uses the notion of a stable module with respect to a stability condition [125] in the context of the wall-and-chamber structure [9, 38]. The definitions of these notions are omitted for the sake of brevity since they are not important for the proof.

Proposition 3.8.2. *Let A be τ -tilting finite. Let $\mathfrak{C}(A) \cong \mathfrak{C}(\Sigma, \mathfrak{P}_{\text{WAC}})$, then the group $G(\Sigma(A), \mathfrak{P}_{\text{WAC}}, \leq_{\text{tors } A})$ is isomorphic to $G(A)$, where $(\Sigma(A), \leq_{\text{tors } A})$ is the fan poset induced by $\text{tors } A$.*

Proof. Since $(\Sigma(A), \leq_{\text{tors } A})$ is nondegenerate, consider the presentation of the picture group $G(\Sigma(A), \mathfrak{P}_{\text{WAC}}, \leq_{\text{tors } A})$ of the partitioned fan poset as in Lemma 3.4.9. The equivalence classes of $\mathfrak{P}_{\text{WAC}}$ are determined by bricks for the following reason: Let $\bar{\mathcal{C}}_{(M,P)} \in \Sigma(A)$ be of codimension 1, then by [38, Prop. 3.17] and [166, Thm. 1] there exists a brick B associated to $\bar{\mathcal{C}}_{(M,P)}$, which is obtained as a v -stable module for all $v \in \mathcal{C}_{(M,P)}$. Moreover, by [38, Prop. 3.13, Thm. 3.14] is the unique v -stable module in $\mathcal{W}_{(M,P)}$. Take now a distinct cone $\bar{\mathcal{C}}_{(M',P')} \sim \bar{\mathcal{C}}_{(M,P)} \in \mathfrak{C}(\Sigma, \mathfrak{P}_{\text{WAC}})$, then since $\mathcal{W}_{(M',P')} = \mathcal{W}_{(M,P)}$, it follows that the brick B' corresponding to $\bar{\mathcal{C}}_{(M',P')}$ is isomorphic to B . Hence the generators $X_{[\bar{\mathcal{C}}_{(M,P)}]}$ of $G(\Sigma(A), \mathfrak{P}_{\text{WAC}}, \leq_{\text{tors } A})$ are in bijection with the generators X_B of $G(A)$.

There are bijections $\bar{\mathcal{C}}_{(M,P)} \leftrightarrow (M,P) \leftrightarrow \text{Fac } M$ between maximal g -vector cones, basic τ -tilting pairs and functorially finite torsion classes, hence the generators $g_{\bar{\mathcal{C}}_{(M,P)}}$ of $G(\Sigma(A), \mathfrak{P}_{\text{WAC}}, \leq_{\text{tors } A})$ and the generators $g_{\mathcal{T}_{(M,P)}}$ of $G(A)$ are also in bijection.

The fact that the relations of the two groups coincide follows immediately from [56, Thm. 6.11]. Hence they are isomorphic. $\square$

Therefore, Definition 3.4.4 is indeed a generalisation of Definition 2.3.2 in the τ -tilting finite case. Recall from [91, Thm. 4.10] that, the picture group $G(A)$ is isomorphic to the fundamental group of $\mathcal{BC}(A)$. Through this connection, an infinite new class of algebras is obtained whose τ -cluster morphism category admits a faithful group functor.

Theorem 3.8.3. *Let A be a finite-dimensional K -algebra such that $\Sigma(A)$ is a finite hyperplane arrangement. Then $\mathfrak{C}(A)$ admits a faithful functor to $G(\Sigma(A), \mathfrak{P}_{\text{flat}}, \leq_{\text{tors } A})$, and hence to $G(A)$.*

Proof. Let σ be a cone of codimension 1 of $\Sigma(A)$. Then σ is contained in a hyperplane whose normal vector is given by the dimension vector of a brick $B \in \text{brick } A$ see [38,

Prop. 3.13]. Since the dimension vector lies in the positive orthant, it can be normalised and then $\Sigma(A)$ satisfies the assumptions of Theorem 3.6.5. Since the τ -cluster morphism category $\mathfrak{C}(A)$ is equivalent to the category $\mathfrak{C}(\Sigma(A), \mathfrak{P}_{\text{WAC}, \leq_{\text{tors}} A})$, Corollary 3.7.5 gives the desired result. Since $G(A)$ is the fundamental group of $\mathcal{BC}(A)$ by [91, Thm. 4.10], the faithful functor to $G(\Sigma(A), \mathfrak{P}_{\text{flat}, \leq_{\text{tors}} A})$ factors through a faithful functor to $G(A)$, see the proof of [101, Prop. 3.7]. $\square$

Corollary 3.8.4. *Let A be such that $\Sigma(A)$ is a finite hyperplane arrangement. Then the classifying space $\mathcal{BC}(A)$ is a $K(\pi, 1)$ space if $\mathfrak{C}(A)$ satisfies the pairwise compatibility condition of last factors. In particular, if $\Sigma(A) \subseteq \mathbb{R}^3$, then $\mathcal{BC}(A)$ is a $K(\pi, 1)$ space.*

Proof. The τ -cluster morphism category always satisfies the pairwise compatibility condition of first factors. The first part of the statement is then the remaining sufficient condition of Proposition 2.4.4, and the second part follows from [20], where it is shown that the pairwise compatibility of last factors always holds for algebras with three isomorphism classes of simple modules. $\square$

As a consequence of Theorem 3.8.3, the family of algebras whose g -vector fan is a finite hyperplane arrangement yields a new family of algebras whose τ -cluster morphism category admits a faithful group functor, extending Example 2.4.5. For example, preprojective algebras of Dynkin type ADE [141]. More generally, the generalised preprojective algebras coming from Cartan matrices of finite (Dynkin) type, as introduced in [79], have g -vector fans which define finite hyperplane arrangements by [146, Thm. 3.19]. The theorem states that their g -vector fans come from the root systems of the corresponding Weyl groups. On the other hand, conjecturally, the finite hyperplane arrangements coming from crepant resolutions as in [182] define contraction algebras whose g -vector fan would then be the finite hyperplane arrangement by [12].

To conclude this chapter consider the following new example of an algebra A for which $\mathcal{BC}(A)$ is a $K(\pi, 1)$ space.

Example 3.8.5. Let A be the generalised preprojective algebra of type C_3 , see [79, Sec. 13.8]. That is, the algebra coming from the Cartan matrix C with symmetrizer D

given as follows:

$$C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}, \quad D = \text{diag}(1, 1, 2).$$

In other words, write $A \cong KQ/I$ with quiver given by

$$Q : 1 \begin{array}{c} \xrightarrow{\alpha_{21}} \\ \xleftarrow{\alpha_{12}} \end{array} 2 \begin{array}{c} \xrightarrow{\alpha_{32}} \\ \xleftarrow{\alpha_{23}} \end{array} 3 \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \epsilon_3$$

and $I = \langle \alpha_{21}\alpha_{12}, \alpha_{32}\alpha_{23} - \alpha_{12}\alpha_{21}, \epsilon_3\alpha_{23}\alpha_{32} + \alpha_{23}\alpha_{32}\epsilon_3, \epsilon_3^2 \rangle$. Similar to [141, Thm. 3.9] for preprojective algebras of type ADE, the result [146, Thm. 3.19] states that up to a base change the g -vector cones of generalised preprojective algebras of arbitrary Dynkin type of [79] coincide with the (Weyl) chambers in the hyperplane arrangement obtained by taking orthogonal hyperplanes to the roots of the corresponding root system. Moreover, by [79, Thm. 1.3] these roots are a positive scalar multiple of the dimension vectors of certain (τ -locally free) modules. For this example of type C_3 , it can be read off from [79, Sec. 13.8] that the dimension vectors of these modules are the following 9 integer vectors:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix},$$

Since $\text{Ext}^1(S(3), S(3)) \neq 0$, the algebra A is not a K -stone algebra, so that [92, Thm. 5.9] does not apply to yield a faithful group functor. Instead, since $\Sigma(A)$ is a finite hyperplane arrangement, Theorem 3.8.3 gives the existence of a faithful group functor. Furthermore, by [20] the category $\mathfrak{C}(A)$ satisfies the pairwise compatibility condition of last factors since A has 3 isomorphism classes of simples. Therefore $\mathcal{BC}(A)$ is a new example of a $K(G(A), 1)$ space obtained from τ -cluster morphism categories, by Proposition 2.4.4.

Chapter 4

τ -cluster morphism categories of factor algebras

This chapter approaches the τ -cluster morphism category from a new perspective. Using lattice theory, a relationship between $\mathfrak{W}(A)$ and $\mathfrak{W}(A/I)$ for some finite-dimensional K -algebra A and its quotient algebra A/I by some ideal I of A is established. To achieve this, some additional lattice theoretic notions and techniques are introduced in Section 4.1. In particular the notion of lattice congruences plays an important role in this chapter. The new perspective on $\mathfrak{W}(A)$, introduced in Section 4.2, defines the τ -cluster morphism category via τ -perpendicular intervals of tors A . As shown in Section 4.3, the necessary information for this construction is encoded purely combinatorially in the poset $\text{tors } A$ when it is finite. Consequently, Corollary 4.3.8 shows that if two finite-dimensional algebras A and B are such that $\text{tors } A \cong \text{tors } B$ and such that both are finite lattices, then $\mathfrak{W}(A) \cong \mathfrak{W}(B)$. Subsequently, the study of the τ -cluster morphism category of factor algebras is initiated in Section 4.4. The lattice congruence from $\text{tors } A$ to $\text{tors } A/I$, established in [57, Sec. 5.2], forms the main ingredient for constructing a functor $F_I : \mathfrak{W}(A) \rightarrow \mathfrak{W}(A/I)$ in Section 4.4. In that section and the following, categorical properties of F_I are investigated, leading to the conclusion that F_I is a regular epimorphism in the category of small categories. To conclude, a comparison of the different approaches to the τ -cluster morphism category is presented in Section 4.7 using some examples.

4.1 Lattice congruences

In this chapter, let A denote a finite-dimensional K -algebra over an arbitrary field K , unless stated otherwise. When studying factor algebras some additional lattice theory is required. In particular, the following notion plays a central role.

Definition 4.1.1. An equivalence relation $\equiv$ on a complete lattice L is called a *complete lattice congruence* if for an indexing set I and families $\{x_i\}_{i \in I}, \{y_i\}_{i \in I} \subseteq L$ the following holds:

$$\begin{aligned} & x_i \equiv y_i \text{ for all } i \in I \\ \implies & \bigvee \{x_i : i \in I\} \equiv \bigvee \{y_i : i \in I\} \text{ and } \bigwedge \{x_i : i \in I\} \equiv \bigwedge \{y_i : i \in I\}. \end{aligned}$$

Given the corresponding set of partitions $\Phi_{\equiv}$ of the equivalence relation $\equiv$, define the *quotient lattice* $L/\equiv$ as the complete lattice whose elements are elements of $\Phi_{\equiv}$ and such that for a subset $S \subseteq \Phi_{\equiv}$, the element $\bigvee S \in L/\equiv$ is the equivalence class containing $\bigvee S' \in L$, where $S' \subseteq L$ is constructed from S as a union of representatives $c \in C$ for every equivalence class $C \in S$. The meet $C_1 \wedge C_2$ is defined dually.

The focus of this chapter lies on lattice congruences on the lattice of torsion classes $\text{tors } A$. In particular, those lattice congruences which arise from quotient algebras in the following way.

Theorem 4.1.2. [57, Thm. 5.12] *For any ideal $I \in \text{ideal } A$, the map*

$$\overline{(-)}_I : \mathcal{T} \mapsto \overline{(\mathcal{T})}_I := \mathcal{T} \cap \text{mod } A/I$$

is a surjective morphism of complete lattices $\text{tors } A \rightarrow \text{tors } A/I$ and the induced equivalence relation Φ_I on $\text{tors } A$ is a complete lattice congruence.

For convenience, write $\overline{(-)} := \overline{(-)}_I$ and $\overline{\mathcal{T}} := \overline{(\mathcal{T})}_I$ when no confusion about the ideal I of A may arise. The following properties of lattice congruences are frequently used.

Proposition 4.1.3. *Let Φ be a complete lattice congruence on L , then the equivalence classes $[x] \in \Phi$ correspond to intervals which will be denoted by*

$$[\pi_{\downarrow}^{\Phi} x, \pi_{\uparrow}^{\Phi} x] \subseteq L.$$

Define $\pi_{\downarrow}^{\Phi}(L)$ to be the subset of elements which are minimal in their equivalence class and define $\pi_{\uparrow}^{\Phi}(L)$ to be the subset of elements which are maximal in their equivalence class. Then, $\pi_{\downarrow}^{\Phi}(L)$ and $\pi_{\uparrow}^{\Phi}(L)$ are isomorphic to the quotient lattice L/Φ and $\pi_{\downarrow}^{\Phi} : L \rightarrow \pi_{\downarrow}^{\Phi}(L)$ and $\pi_{\uparrow}^{\Phi} : L \rightarrow \pi_{\uparrow}^{\Phi}(L)$, which send x to the minimal and maximal elements in its equivalence class respectively, are morphisms of complete lattices.

Proof. If L is a finite lattice, then [157, Prop. 9-5.2] implies that an equivalence class $[x] \in \Phi$ corresponds to an interval $[\pi_{\downarrow}^{\Phi} x, \pi_{\uparrow}^{\Phi} x] \subseteq L$ and [157, Prop. 9-5.5] states that $\pi_{\downarrow}^{\Phi} : L \rightarrow \pi_{\downarrow}^{\Phi} L$ is a morphism of lattices and that L/Φ is equivalent to $\pi_{\downarrow}^{\Phi} L$. The same result holds for $\pi_{\uparrow}^{\Phi}$ by the dual of [157, Prop. 9-5.5]. By [157, Exercise 9.42] these result hold analogously for complete lattices and complete lattice congruences. $\square$

For a complete lattice congruence Φ on a complete lattice L , the surjective morphisms of lattices $L \rightarrow L/\Phi$ is sometimes also denoted by Φ . The first part of Proposition 4.1.3 then states that for any $x \in L/\Phi$, the preimage $\Phi^{-1}(x)$ is an interval, which will be denoted by $[\pi_{\downarrow} \Phi^{-1}(x), \pi_{\uparrow} \Phi^{-1}(x)] \subseteq L$.

4.2 Lattice theoretic definition of the τ -cluster morphism category

In contrast to the geometric approaches of [169] and Chapter 3 to the τ -cluster morphism category, in this section the τ -cluster morphism category is defined from the lattice of torsion classes. Let $\text{itv}(\text{tors } A)$ denote the poset of intervals of $\text{tors } A$ ordered (by convention) by reverse containment. That means, if $[\mathcal{U}, \mathcal{T}], [\mathcal{S}, \mathcal{V}] \in \text{itv}(\text{tors } A)$ are two intervals, then $[\mathcal{U}, \mathcal{T}] \leq [\mathcal{S}, \mathcal{V}]$ whenever $\mathcal{U} \subseteq \mathcal{S}$ and $\mathcal{V} \subseteq \mathcal{T}$. The connection of this construction with the geometric one comes from the correspondence

$$\begin{aligned} \Sigma(A) &\rightarrow \text{itv}(\text{tors } A) \\ \bar{\mathcal{C}}_{(M,P)} &\mapsto [\text{Fac } M, {}^{\perp}\tau M \cap P^{\perp}]. \end{aligned} \tag{4.2.1}$$

For the sake of brevity, given a τ -rigid pair (M, P) denote by $[\mathcal{U}_{(M,P)}, \mathcal{T}_{(M,P)}]$ the interval $[\text{Fac } M, {}^{\perp}\tau M \cap P^{\perp}]$. In particular, considering only maximal cones $\bar{\mathcal{C}}_{(M,P)}$ of $\Sigma(A)$ in Eq. (4.2.1), which means that (M, P) is τ -tilting and ${}^{\perp}\tau M \cap P^{\perp} = \text{Fac } M$ by [3, Prop. 2.16b], Eq. (4.2.1) is the bijection between τ -tilting modules (M, P) and functorially finite torsion classes $\text{Fac } M \in \text{f-tors } A$ of Theorem 2.2.2.

The analogue of the projection map $\pi_{\mathcal{C}_{(M,P)}}$ in the geometric construction of Definition 3.1.4, which defines the identification of morphisms corresponding to “the same relative τ -tilting reduction”, is the following result, which is a direct consequence of Theorem 2.2.5.

Lemma 4.2.1. *Let $[\mathcal{U}_{(M_1,P_1)}, \mathcal{T}_{(M_1,P_1)}]$ and $[\mathcal{U}_{(M_2,P_2)}, \mathcal{T}_{(M_2,P_2)}]$ be τ -perpendicular intervals of tors A such that $\mathcal{W}_{(M_1,P_1)} = \mathcal{W}_{(M_2,P_2)}$. Then there are three lattice isomorphisms*

$$\begin{array}{ccc} [\mathcal{U}_{(M_1,P_1)}, \mathcal{T}_{(M_1,P_1)}] & \xrightarrow{\cong} & [\mathcal{U}_{(M_2,P_2)}, \mathcal{T}_{(M_2,P_2)}] \\ & \searrow \text{---} \cap \mathcal{W}_{(M_1,P_1)} & \swarrow \text{---} \cap \mathcal{W}_{(M_2,P_2)} \\ & \text{tors } \mathcal{W}_{(M_i,P_i)} & \end{array} \quad (4.2.2)$$

The isomorphism between the τ -perpendicular intervals factors through the lattice of torsion classes of their shared τ -perpendicular wide subcategory. The following is a key feature of these isomorphisms.

Proposition 4.2.2. *[57, Prop. 4.13] The three lattice isomorphisms in Eq. (4.2.2) preserve the brick labelling of Hasse quivers.*

Using the isomorphisms of Eq. (4.2.2) as analogues of the projection onto the orthogonal complement in Definition 3.1.4, the τ -cluster morphism category may be defined from the lattice of torsion classes as follows.

Definition 4.2.3. The (lattice theoretic) τ -cluster morphism category $\mathfrak{T}(A)$ has as its objects equivalence classes $[\mathcal{U}_{(M,P)}, \mathcal{T}_{(M,P)}]_{\sim}$ of τ -perpendicular intervals of tors A under the equivalence relation

$$[\mathcal{U}_{(M_1,P_1)}, \mathcal{T}_{(M_1,P_1)}]_{\sim} [\mathcal{U}_{(M_2,P_2)}, \mathcal{T}_{(M_2,P_2)}]_{\sim}$$

whenever $\mathcal{W}_{(M_1,P_1)} = \mathcal{W}_{(M_2,P_2)}$. The morphisms of $\mathfrak{T}(A)$ are given by equivalence classes of morphisms in the poset category $\text{itv}(\text{tors } A)$. More precisely,

$$\begin{aligned} & \text{Hom}_{\mathfrak{T}(A)}([\mathcal{U}_{(M,P)}, \mathcal{T}_{(M,P)}]_{\sim}, [\mathcal{U}_{(N,Q)}, \mathcal{T}_{(N,Q)}]_{\sim}) \\ &= \bigcup_{\substack{[\mathcal{U}_{(M',P')}, \mathcal{T}_{(M',P')}] \in [\mathcal{U}_{(M,P)}, \mathcal{T}_{(M,P)}]_{\sim} \\ [\mathcal{U}_{(N',Q')}, \mathcal{T}_{(N',Q')}] \in [\mathcal{U}_{(N,Q)}, \mathcal{T}_{(N,Q)}]_{\sim}}} \text{Hom}_{\text{itv}(\text{tors } A)}([\mathcal{U}_{(M',P')}, \mathcal{T}_{(M',P')}], [\mathcal{U}_{(N',Q')}, \mathcal{T}_{(N',Q')}]) \end{aligned}$$

modulo the equivalence relation

$$f_{[\mathcal{U}_{(M_1, P_1)}, \mathcal{T}_{(M_1, P_1)}][\mathcal{U}_{(N_1, Q_1)}, \mathcal{T}_{(N_1, Q_1)}]} \sim f_{[\mathcal{U}_{(M_2, P_2)}, \mathcal{T}_{(M_2, P_2)}][\mathcal{U}_{(N_2, Q_2)}, \mathcal{T}_{(N_2, Q_2)}]}$$

whenever

$$[\mathcal{U}_{(N_1, Q_1)}, \mathcal{T}_{(N_1, Q_1)}] \cap \mathcal{W}_{(M_1, P_1)} = [\mathcal{U}_{(N_2, Q_2)}, \mathcal{T}_{(N_2, Q_2)}] \cap \mathcal{W}_{(M_2, P_2)}.$$

Composition of morphisms in $\mathfrak{T}(A)$ is defined analogously to the geometric realisation of the τ -cluster morphism category in Definition 3.1.4. In Lemma 4.2.6, ambiguities regarding different representatives of equivalence classes are resolved, which makes the composition well-defined.

It is not difficult to see that there are bijections between the objects of the category $\mathfrak{C}(A)$ and those of $\mathfrak{T}(A)$ via the correspondence Eq. (4.2.1). When the reference to the specific τ -rigid pair is not important in later sections, τ -perpendicular intervals will simply be denoted by $[\mathcal{U}, \mathcal{T}]$ suppressing explicit reference to the corresponding τ -rigid pair, which further shortens the notation. As a first step, it is easy to connect objects and morphisms of $\mathfrak{C}(A)$ and $\mathfrak{T}(A)$, showing that identifications happen in the same way in $\mathfrak{C}(A)$ and $\mathfrak{T}(A)$.

Lemma 4.2.4. *The assignment of Eq. (4.2.1) induces a well-defined bijection between the objects of $\mathfrak{C}(A)$ and $\mathfrak{T}(A)$.*

Proof. It is clear that there exists a bijection between g -vector cones $\bar{\mathcal{C}}_{(M, P)} \in \Sigma(A)$ and τ -perpendicular intervals $[\mathcal{U}_{(M, P)}, \mathcal{T}_{(M, P)}]$ as they are both in bijection with basic τ -rigid pairs (M, P) by definition. It then suffices to show that the identifications happen the same way in $\mathfrak{C}(A)$ and $\mathfrak{T}(A)$. From the respective definitions it follows immediately that

$$\begin{aligned} \mathcal{C}_{(M_1, P_1)} \sim_{\mathfrak{C}(A)} \mathcal{C}_{(M_2, P_2)} &\Leftrightarrow \mathcal{W}_{(M_1, P_1)} = \mathcal{W}_{(M_2, P_2)} \\ &\Leftrightarrow [\mathcal{U}_{(M_1, P_1)}, \mathcal{T}_{(M_1, P_1)}] \sim_{\mathfrak{T}(A)} [\mathcal{U}_{(M_2, P_2)}, \mathcal{T}_{(M_2, P_2)}]. \end{aligned}$$

Therefore the equivalence relation is preserved on objects. $\square$

Lemma 4.2.5. *The assignment of Eq. (4.2.1) induces a well-defined map between morphisms of $\mathfrak{C}(A)$ and $\mathfrak{T}(A)$. That is, the identification of morphisms is preserved, or in*

other words, two inclusions of g -vector cones $\overline{\mathcal{C}}_{(M_i, P_i)} \subseteq \overline{\mathcal{C}}_{(N_i, Q_i)} \in \Sigma(A)$ for $i = 1, 2$ with corresponding τ -perpendicular intervals $[\mathcal{U}_{(M_i, P_i)}, \mathcal{T}_{(M_i, P_i)}] \leq [\mathcal{U}_{(N_i, Q_i)}, \mathcal{T}_{(N_i, Q_i)}]$ satisfy

$$\begin{aligned} \pi_{\overline{\mathcal{C}}_{(M_1, P_1)}}(\overline{\mathcal{C}}_{(N_1, Q_1)}) &= \pi_{\overline{\mathcal{C}}_{(M_2, P_2)}}(\overline{\mathcal{C}}_{(N_2, Q_2)}) \\ \Leftrightarrow ([\mathcal{U}_{(N_1, Q_1)}, \mathcal{T}_{(N_1, Q_1)}]) \cap \mathcal{W}_{(M_1, P_1)} &= [\mathcal{U}_{(N_2, Q_2)}, \mathcal{T}_{(N_2, Q_2)}] \cap \mathcal{W}_{(M_2, P_2)}. \end{aligned}$$

Proof. This follows from [169, Lem. 3.8] and [9, Lem. 4.4(3)]. More precisely, the result [169, Lem. 3.8] relates the projection map $\pi_{\overline{\mathcal{C}}_{(M, P)}} : \mathbb{R}^{|A|} \rightarrow \text{span}\{\overline{\mathcal{C}}_{(M, P)}\}^\perp$ defined in Chapter 3 with the map $\pi : K_0(\text{proj } A) \rightarrow K_0(\text{proj } \mathcal{W}_{(M, P)})$ defined in [9, p. 33]. Then [9, Lem. 4.4(3)] implies that

$$\begin{aligned} \pi_{\overline{\mathcal{C}}_{(M_1, P_1)}}(\overline{\mathcal{C}}_{(N_1, Q_1)}) &= \pi_{\overline{\mathcal{C}}_{(M_2, P_2)}}(\overline{\mathcal{C}}_{(N_2, Q_2)}) \\ \Leftrightarrow \mathcal{T}_{(N_1, Q_1)} \cap \mathcal{W}_{(M_1, P_1)} &= \mathcal{T}_{(N_2, Q_2)} \cap \mathcal{W}_{(M_2, P_2)} \\ \text{and } \mathcal{U}_{(N_1, Q_1)} \cap \mathcal{W}_{(M_1, P_1)} &= \mathcal{U}_{(N_2, Q_2)} \cap \mathcal{W}_{(M_2, P_2)} \\ \Leftrightarrow ([\mathcal{U}_{(N_1, Q_1)}, \mathcal{T}_{(N_1, Q_1)}]) \cap \mathcal{W}_{(M_1, P_1)} &= [\mathcal{U}_{(N_2, Q_2)}, \mathcal{T}_{(N_2, Q_2)}] \cap \mathcal{W}_{(M_2, P_2)}. \end{aligned}$$

Whence the result follows. $\square$

Next, it is shown that composition of morphisms in $\mathfrak{T}(A)$ is well-defined. The first concern, see also Lemma 3.1.5, is that it is not clear how to compose two morphisms $[f_{[\mathcal{U}, \mathcal{T}][\mathcal{V}_1, \mathcal{S}_1]}]$ and $[f_{[\mathcal{V}_2, \mathcal{S}_2][\mathcal{X}_2, \mathcal{Y}_2]}]$ when $[\mathcal{V}_1, \mathcal{S}_1] \sim [\mathcal{V}_2, \mathcal{S}_2]$. The following lemma implies the existence of a morphism $f_{[\mathcal{V}_1, \mathcal{S}_1][\mathcal{X}_1, \mathcal{Y}_1]} \sim f_{[\mathcal{V}_2, \mathcal{S}_2][\mathcal{X}_2, \mathcal{Y}_2]}$ so that composition of the two morphisms is simply $[f_{[\mathcal{U}, \mathcal{T}][\mathcal{X}_1, \mathcal{Y}_1]}]$.

Lemma 4.2.6. *Let (M_1, P_1) and (M_2, P_2) be τ -rigid pairs with the property that $\mathcal{W}_{(M_1, P_1)} = \mathcal{W}_{(M_2, P_2)}$. Then, for every τ -perpendicular interval*

$$[\mathcal{U}_{(N_1, Q_1)}, \mathcal{T}_{(N_1, Q_1)}] \subseteq [\mathcal{U}_{(M_1, P_1)}, \mathcal{T}_{(M_1, P_1)}],$$

there exists a τ -perpendicular interval

$$[\mathcal{U}_{(N_2, Q_2)}, \mathcal{T}_{(N_2, Q_2)}] \subseteq [\mathcal{U}_{(M_2, P_2)}, \mathcal{T}_{(M_2, P_2)}]$$

such that $\mathcal{W}_{(N_1, Q_1)} = \mathcal{W}_{(N_2, Q_2)}$ and such that

$$[\mathcal{U}_{(N_1, Q_1)}, \mathcal{T}_{(N_1, Q_1)}] \cap \mathcal{W}_{(M_1, P_1)} = [\mathcal{U}_{(N_2, Q_2)}, \mathcal{T}_{(N_2, Q_2)}] \cap \mathcal{W}_{(M_2, P_2)}.$$

By definition, this means that

$$f_{[\mathcal{U}_{(M_1, P_1)}, \mathcal{T}_{(M_1, P_1)}][\mathcal{U}_{(N_1, Q_1)}, \mathcal{T}_{(N_1, Q_1)}]} \sim f_{[\mathcal{U}_{(M_2, P_2)}, \mathcal{T}_{(M_2, P_2)}][\mathcal{U}_{(N_2, Q_2)}, \mathcal{T}_{(N_2, Q_2)}]}$$

in $\mathfrak{T}(A)$.

Proof. By Lemma 4.2.1, there is an isomorphism of lattices

$$\phi : [\mathcal{U}_{(M_1, P_1)}, \mathcal{T}_{(M_1, P_1)}] \rightarrow [\mathcal{U}_{(M_2, P_2)}, \mathcal{T}_{(M_2, P_2)}],$$

which, by Proposition 4.2.2, preserves the brick labelling of the intervals. Therefore, given a subinterval $[\mathcal{U}_{(N_1, Q_1)}, \mathcal{T}_{(N_1, Q_1)}] \subseteq [\mathcal{U}_{(M_1, P_1)}, \mathcal{T}_{(M_1, P_1)}]$ consider the image

$$[\mathcal{V}, \mathcal{S}] := \phi([\mathcal{U}_{(N_1, Q_1)}, \mathcal{T}_{(N_1, Q_1)}]) \subseteq [\mathcal{U}_{(M_2, P_2)}, \mathcal{T}_{(M_2, P_2)}].$$

By definition of ϕ , the interval $[\mathcal{V}, \mathcal{S}]$ is such that

$$[\mathcal{V}, \mathcal{S}] \cap \mathcal{W}_{(M_2, P_2)} = [\mathcal{U}_{(M_1, P_1)}, \mathcal{T}_{(M_1, P_1)}] \cap \mathcal{W}_{(M_1, P_1)}.$$

The brick labelling of $[\mathcal{V}, \mathcal{S}]$ coincides with that of $[\mathcal{U}_{(N_1, Q_1)}, \mathcal{T}_{(N_1, Q_1)}]$ by Proposition 4.2.2. It follows from [10, Thm. 5.2, Prop. 5.3] that $\mathcal{V}^\perp \cap \mathcal{S} = \mathcal{W}_{(N_1, Q_1)}$, making $[\mathcal{V}, \mathcal{S}]$ a τ -perpendicular interval. Indeed, the corresponding basic τ -rigid pair (N_2, Q_2) such that $[\mathcal{U}_{(N_2, Q_2)}, \mathcal{T}_{(N_2, Q_2)}] = [\mathcal{V}, \mathcal{S}]$ is obtained as follows.

There exists an explicit bijection $E_{(M_i, P_i)}$ from basic τ -rigid pairs in $\text{mod } A$ containing (M_i, P_i) as a direct summand to basic τ -rigid pairs in the subcategory $\mathcal{W}(M_i, P_i)$ for $i = 1, 2$, see [45, Sec. 5] and [43, Def. 3.5]. The desired basic τ -rigid pair is then obtained as $(N_2, Q_2) = E_{(M_2, P_2)}^{-1}(E_{(M_1, P_1)}(N_1, Q_1))$, by [40, Lem. 6.5, 6.6]. $\square$

The second concern is whether the composition of pairwise-identified morphisms gives identified morphisms, see [169, Lem. 3.10] and Lemma 3.1.7. However, this is easily seen to hold via the lattice isomorphism of Lemma 4.2.1 similar to Lemma 4.2.6.

Theorem 4.2.7. *The category $\mathfrak{T}(A)$ is well-defined and equivalent to $\mathfrak{C}(A)$.*

Proof. The composition of morphisms is well-defined by Lemma 4.2.6, therefore $\mathfrak{T}(A)$ is a well-defined category. The equivalence $\Psi : \mathfrak{C}(A) \rightarrow \mathfrak{T}(A)$ is induced by Eq. (4.2.1). From Lemma 4.2.4 and Lemma 4.2.5 it follows that Ψ is a well-defined on objects and

morphisms. Lemma 4.2.6 is the lattice theoretic analogue of Lemma 3.1.5, making Ψ well-defined on compositions of morphisms. By definition, every τ -perpendicular interval comes from a τ -rigid pair (M, P) and hence from a g -vector cone $\bar{\mathcal{C}}_{(M,P)}$ via Eq. (4.2.1), so Ψ is essentially surjective. For the same reason, two τ -rigid pairs (M, P) and (N, Q) satisfy $\bar{\mathcal{C}}_{(N,Q)} \subseteq \bar{\mathcal{C}}_{(M,P)}$ if and only if $[\mathcal{U}_{(N,Q)}, \mathcal{T}_{(N,Q)}] \leq [\mathcal{U}_{(M,P)}, \mathcal{T}_{(M,P)}]$. This determines the morphisms of $\mathfrak{C}(A)$ and $\mathfrak{T}(A)$ respectively, therefore there are bijections between Hom-sets. So Ψ is fully faithful and hence an equivalence. $\square$

Remark 4.2.8. The structure of the lattice of torsion classes and its brick labelling is encoded in the morphism spaces of the τ -cluster morphism category. For example, there is a bijection

$$\mathrm{Hom}_{\mathfrak{T}(A)}([0, \mathrm{mod} A]_{\sim}, [0, 0]_{\sim}) \longleftrightarrow \mathcal{T} \in \mathrm{f-tors} A$$

since $[0, 0]_{\sim} = [\mathcal{T}, \mathcal{T}]_{\sim}$ for all $\mathcal{T} \in \mathrm{f-tors} A$. Each such interval is τ -perpendicular since functorially finite torsion classes correspond bijectively to basic τ -tilting pairs (M, P) for which $\mathrm{Fac} M = {}^{\perp}\tau M \cap P^{\perp}$ by [3, Prop. 2.16]. Moreover, let $\mathcal{T}_2 \xleftarrow{B} \mathcal{T}_1$ be an arrow of $\mathrm{Hasse}(\mathrm{f-tors} A)$ labelled by the brick B , then there is a bijection

$$\mathrm{Hom}_{\mathfrak{T}(A)}([0, \mathrm{mod} A]_{\sim}, [\mathcal{T}_2, \mathcal{T}_1]_{\sim}) \longleftrightarrow \{\text{arrows labelled by } B \text{ in } \mathrm{Hasse}(\mathrm{f-tors} A)\}.$$

And more generally for an arbitrary τ -perpendicular interval $[\mathcal{T}_4, \mathcal{T}_3] \subseteq \mathrm{tors} A$:

$$\begin{aligned} & \mathrm{Hom}_{\mathfrak{T}(A)}([0, \mathrm{mod} A]_{\sim}, [\mathcal{T}_4, \mathcal{T}_3]_{\sim}) \\ & \longleftrightarrow \{\tau\text{-perpendicular intervals with brick label preserving isomorphism to } [\mathcal{T}_4, \mathcal{T}_3]\}. \end{aligned}$$

4.3 Invariance under τ -tilting equivalence

In this section the focus lies on families of algebras for which there exists an isomorphism between their respective posets of τ -tilting pairs. The idea of using only lattice theoretic information to study τ -tilting theory is common throughout the literature, see for example [10, 18, 19, 57, 67, 121]. Similarly, Definition 4.2.3 can be rephrased so that the τ -cluster morphism category is defined using only the underlying poset of (functorially finite) torsion classes when $\mathrm{tors} A$ is finite.

Definition 4.3.1. Two finite-dimensional K -algebras A and B are called τ -tilting equivalent if there exists a poset isomorphism $\text{f-tors } A \cong \text{f-tors } B$.

Example 4.3.2. The following algebras are τ -tilting equivalent:

- (1) Any two Brauer graph algebras with the same underlying ribbon graph [1, Thm. 1.1(1)].
- (2) Any algebra A and $A/\langle c \rangle$, where $c \in A$ is a central element contained in the Jacobson radical [66, Thm. 1]. See also [57, Cor. 5.20].
- (3) Two algebras A and B which geometrically have coinciding g -vector fans, by the duality of the poset $\text{f-tors } A$ with the chambers of the g -vector fan, see [56].
- (4) The algebras $\mathcal{K}_2 = K(1 \rightrightarrows 2)$ and $\mathcal{K}_m = K(1 \xrightarrow[m]{\dots\dots\dots} 2)$, for any $m \geq 3$, by the description of $\text{f-tors}(\mathcal{K}_2)$ and $\text{f-tors}(\mathcal{K}_m)$ as polygons with one infinite side.

Note that $\text{f-tors } A$ is a complete lattice if and only if A is τ -tilting finite by Theorem 2.2.3. However, since $\text{f-tors } A \subseteq \text{tors } A$ it is possible to consider the join and meet of elements of $\text{f-tors } A$ in $\text{tors } A$. The first example of some of the τ -tilting theory of $\text{mod } A$ being encoded lattice theoretically in $\text{f-tors } A$ is that of τ -perpendicular intervals.

Proposition 4.3.3. [57, Prop. 4.19 and 4.20] *Let $\mathcal{U} \in \text{f-tors } A$. Consider ℓ atoms $\mathcal{U}_i \rightarrow \mathcal{U} \subseteq \text{Hasse}(\text{f-tors } A)$. Then $\mathcal{T} := \bigvee_{i=1}^{\ell} \mathcal{U}_i$ is functorially finite and $[\mathcal{U}, \mathcal{T}] \subseteq \text{tors } A$ is a τ -perpendicular interval of $\text{tors } A$. Moreover, every τ -perpendicular interval is a so-called join-interval of this form.*

Theorem 2.1.9 states that the construction of Proposition 4.3.3 gives all intervals $[\mathcal{U}, \mathcal{T}] \subseteq \text{tors } A$ for which $\mathcal{U}^{\perp} \cap \mathcal{T}$ is a wide subcategory of $\text{mod } A$. For this construction to yield τ -perpendicular subcategories, the bottom torsion class $\mathcal{U}$ has to be functorially finite. Thus, knowledge of the subset $\text{f-tors } A \subseteq \text{tors } A$ is necessary. Determining this subset of $\text{tors } A$ using lattice theoretic techniques is not known to be possible. Therefore, this section restricts itself to the case when A is τ -tilting finite, which implies $\text{f-tors } A = \text{tors } A$ by Theorem 2.2.3.

In this case, τ -perpendicular intervals and join-intervals (and the dually defined meet-intervals) all coincide. In order to distinguish between intervals, which admit a poset isomorphism but which correspond to distinct subcategories, the brick labelling

of $\text{tors } A$ needs to be encoded combinatorially. Since the brick labelling determines the corresponding τ -perpendicular subcategory by [57, Thm. 4.16] and [10, Prop. 5.3], this would enable the correct identification of intervals in an entirely combinatorial construction of the τ -cluster morphism category.

Let L be a complete lattice, an element $j \in L$ is called *completely join-irreducible* if there does not exist a subset $S \subseteq L$ such that $j = \bigvee S$ and $j \notin S$. A join-irreducible element $j \in L$ covers exactly one other element j_* . Take $x < y$ then the set

$$\{t \in L : t \vee x = y\} \subseteq L$$

has a minimum element $\ell \in L$ which is completely join-irreducible and which satisfies $\ell_* \leq x$ by [159, Lem. 3.7]. For an interval $[x, y] \subseteq L$, write $\text{j-irr}^c[x, y]$ for the set of completely join-irreducible elements $j \in L$ which arise as labels of some cover relation $x \leq a < b \leq y$. This inspires the following definition, similar to [159, Rmk. 3.8].

Definition 4.3.4. Let L be a completely semidistributive lattice. By definition, the *join-irreducible labelling* of L associates to each arrow $x \leftarrow y \subseteq \text{Hasse}(L)$ the unique completely join-irreducible element j which is minimal in the set $\{t \in L : x \vee t = y\} \subseteq L$.

Lemma 4.3.5. [57, Thm. 3.11] *Let L be an abstract lattice isomorphic to $\text{tors } A$, then the join-irreducible labelling on L corresponds to the brick labelling of $\text{tors } A$. In other words, the brick labelling is determined combinatorially by the underlying lattice structure.*

Remark 4.3.6. This fact is used in [67] to construct the poset of wide subcategories and the poset of and image-cokernel-extension-closed (ICE-closed) subcategories of $\text{mod } A$ from the lattice theoretic information of $\text{tors } A$. The formulation of Theorem 4.3.7 is inspired by the phrasing of [67, Thm. A].

The following is the main result of this section and follows the spirit of the previous remark.

Theorem 4.3.7. *Let L be an abstract finite lattice which is isomorphic to $\text{tors } A$ for some finite-dimensional algebra A . Then $\mathfrak{T}(A)$ can be constructed combinatorially from L without using any algebraic information of A or $\text{mod } A$. More precisely, there is an equivalence of categories $\mathfrak{T}(L) \cong \mathfrak{T}(A)$, where $\mathfrak{T}(L)$ is defined to be the category:*

- whose objects are equivalence classes of join-intervals $[\mathcal{U}, \mathcal{T}] = [\mathcal{U}, \bigvee_{i=1}^{\ell} \mathcal{U}_i] \subseteq L$ under the equivalence

$$[\mathcal{U}_1, \mathcal{T}_1] \sim [\mathcal{U}_2, \mathcal{T}_2]$$

whenever $\text{j-irr}^c[\mathcal{U}_1, \mathcal{T}_1] = \text{j-irr}^c[\mathcal{U}_2, \mathcal{T}_2]$. In this case there exists an isomorphism $\phi_{[\mathcal{U}_1, \mathcal{T}_1][\mathcal{U}_2, \mathcal{T}_2]} : [\mathcal{U}_1, \mathcal{T}_1] \rightarrow [\mathcal{U}_2, \mathcal{T}_2]$ preserving the join-irreducible labels (in L) of the arrows in the respective Hasse diagrams;

- whose morphisms are given by equivalence classes of morphisms in the poset category $\text{itv}(L)$ of intervals of L partially-ordered by reverse containment. More precisely

$$\text{Hom}_{\mathfrak{T}(L)}([\mathcal{U}, \mathcal{T}]_{\sim}, [\mathcal{V}, \mathcal{S}]_{\sim}) = \bigcup_{\substack{[\mathcal{U}', \mathcal{T}'] \in [\mathcal{U}, \mathcal{T}]_{\sim} \\ [\mathcal{V}', \mathcal{S}'] \in [\mathcal{V}, \mathcal{S}]_{\sim}}} \text{Hom}_{\text{itv}(L)}([\mathcal{U}', \mathcal{T}'], [\mathcal{V}', \mathcal{S}'])$$

under the equivalence relation $f_{[\mathcal{U}_1, \mathcal{T}_1][\mathcal{V}_1, \mathcal{S}_1]} \sim f_{[\mathcal{U}_2, \mathcal{T}_2][\mathcal{V}_2, \mathcal{S}_2]}$ whenever

$$\phi_{[\mathcal{U}_1, \mathcal{T}_1][\mathcal{U}_2, \mathcal{T}_2]}([\mathcal{V}_1, \mathcal{S}_1]) = [\mathcal{V}_2, \mathcal{S}_2];$$

- and for which composition of morphisms is given by

$$\begin{aligned} & [f_{[\mathcal{V}_2, \mathcal{S}_2][\mathcal{X}_2, \mathcal{Y}_2]}] \circ [f_{[\mathcal{U}_1, \mathcal{T}_1][\mathcal{V}_1, \mathcal{S}_1]}] \\ &:= [f_{\phi_{[\mathcal{V}_2, \mathcal{S}_2][\mathcal{V}_1, \mathcal{S}_1]}([\mathcal{V}_2, \mathcal{S}_2])\phi_{[\mathcal{V}_2, \mathcal{S}_2][\mathcal{V}_1, \mathcal{S}_1]}([\mathcal{X}_2, \mathcal{Y}_2])}] \circ [f_{[\mathcal{U}_1, \mathcal{T}_1][\mathcal{V}_1, \mathcal{S}_1]}]. \end{aligned}$$

Proof. Denote by $\psi : L \rightarrow \text{tors } A$ the isomorphism of complete lattices. Since L is finite it follows that $\text{tors } A$ is finite and hence $\text{f-tors } A = \text{tors } A$ by Theorem 2.2.3. Let $[\mathcal{U}_1, \mathcal{T}_1]$ and $[\mathcal{U}_2, \mathcal{T}_2]$ be two distinct join-intervals of L satisfying $\text{j-irr}^c[\mathcal{U}_1, \mathcal{T}_1] = \text{j-irr}^c[\mathcal{U}_2, \mathcal{T}_2]$. By Proposition 4.3.3 and Lemma 4.3.5, this is equivalent to saying that $\psi([\mathcal{U}_1, \mathcal{T}_1])$ and $\psi([\mathcal{U}_2, \mathcal{T}_2])$ are τ -perpendicular intervals of $\text{tors } A$ with the additional property that $\text{brick}(\psi([\mathcal{U}_1, \mathcal{T}_1])) = \text{brick}(\psi([\mathcal{U}_2, \mathcal{T}_2]))$. Moreover, using [57, Lem. 3.10] gives

$$\psi(\mathcal{U}_1)^\perp \cap \psi(\mathcal{T}_1) = \text{Filt}(\text{brick}(\psi([\mathcal{U}_1, \mathcal{T}_1]))) = \text{Filt}(\text{brick}(\psi([\mathcal{U}_2, \mathcal{T}_2]))) = \psi(\mathcal{U}_2)^\perp \cap \psi(\mathcal{T}_2).$$

It follows that $[\mathcal{U}_1, \mathcal{T}_1] \sim_{\mathfrak{T}(L)} [\mathcal{U}_2, \mathcal{T}_2]$ implies $\psi([\mathcal{U}_1, \mathcal{T}_1]) \sim_{\mathfrak{T}(A)} \psi([\mathcal{U}_2, \mathcal{T}_2])$. Moreover, then Lemma 4.2.1 and Proposition 4.2.2 imply that there is an isomorphism $\varphi : \psi([\mathcal{U}_1, \mathcal{T}_1]) \rightarrow \psi([\mathcal{U}_2, \mathcal{T}_2])$ which preserves the brick labelling. By Lemma 4.3.5, the isomorphism φ lifts to an isomorphism

$$\phi_{[\mathcal{U}_1, \mathcal{T}_1][\mathcal{U}_2, \mathcal{T}_2]} := \psi^{-1} \varphi \psi : [\mathcal{U}_1, \mathcal{T}_1] \rightarrow [\mathcal{U}_2, \mathcal{T}_2],$$

which preserves the join-irreducible labelling.

Let $[\mathcal{V}_i, \mathcal{S}_i] \subseteq L$ be join-intervals such that $[\mathcal{V}_i, \mathcal{S}_i] \subseteq [\mathcal{U}_i, \mathcal{T}_i]$ for $i = 1, 2$. Then $f_{[\mathcal{U}_1, \mathcal{T}_1][\mathcal{V}_1, \mathcal{S}_1]} \sim_{\mathfrak{T}(L)} f_{[\mathcal{U}_2, \mathcal{T}_2][\mathcal{V}_2, \mathcal{S}_2]}$ means $\phi_{[\mathcal{U}_1, \mathcal{T}_1][\mathcal{U}_2, \mathcal{T}_2]}([\mathcal{V}_1, \mathcal{S}_1]) = [\mathcal{V}_2, \mathcal{S}_2]$ which, by definition, implies

$$\psi([\mathcal{V}_1, \mathcal{S}_1]) \cap (\psi(\mathcal{U}_1)^\perp \cap \psi(\mathcal{T}_1)) = \psi([\mathcal{V}_2, \mathcal{S}_2]) \cap (\psi(\mathcal{U}_2)^\perp \cap \psi(\mathcal{T}_2))$$

as φ factors through $\text{tors}(\psi(\mathcal{U}_1)^\perp \cap \psi(\mathcal{T}_1)) = \text{tors}(\psi(\mathcal{U}_2)^\perp \cap \psi(\mathcal{T}_2))$. Thus

$$f_{[\mathcal{U}_1, \mathcal{T}_1][\mathcal{V}_1, \mathcal{S}_1]} \sim_{\mathfrak{T}(L)} f_{[\mathcal{U}_2, \mathcal{T}_2][\mathcal{V}_2, \mathcal{S}_2]} \implies f_{\psi([\mathcal{U}_1, \mathcal{T}_1])\psi([\mathcal{V}_1, \mathcal{S}_1])} \sim_{\mathfrak{T}(A)} f_{\psi([\mathcal{U}_2, \mathcal{T}_2])\psi([\mathcal{V}_2, \mathcal{S}_2])}.$$

Conversely, let $[\mathcal{U}'_1, \mathcal{T}'_1]$ and $[\mathcal{U}'_2, \mathcal{T}'_2]$ be τ -perpendicular intervals of $\text{tors } A$ such that $(\mathcal{U}'_1)^\perp \cap \mathcal{T}'_1 = (\mathcal{U}'_2)^\perp \cap \mathcal{T}'_2$, then Lemma 4.2.1 and Proposition 4.2.2 imply that there exists an isomorphism $\varphi' : [\mathcal{U}'_1, \mathcal{T}'_1] \rightarrow [\mathcal{U}'_2, \mathcal{T}'_2]$, which preserves the brick labelling, and which lifts to an isomorphism

$$\phi'_{[\mathcal{U}'_1, \mathcal{T}'_1][\mathcal{U}'_2, \mathcal{T}'_2]} := \psi^{-1} \varphi' \psi : \psi^{-1}([\mathcal{U}'_1, \mathcal{T}'_1]) \rightarrow \psi^{-1}([\mathcal{U}'_2, \mathcal{T}'_2]),$$

which preserves the join-irreducible labelling.

From Lemma 4.3.5 it follows that $\text{j-irr}^c(\psi^{-1}([\mathcal{U}'_1, \mathcal{T}'_1])) = \text{j-irr}^c(\psi^{-1}([\mathcal{U}'_2, \mathcal{T}'_2]))$. Therefore

$$[\mathcal{U}'_1, \mathcal{T}'_1] \sim_{\mathfrak{T}(A)} [\mathcal{U}'_2, \mathcal{T}'_2] \implies \psi^{-1}([\mathcal{U}'_1, \mathcal{T}'_1]) \sim_{\mathfrak{T}(L)} \psi^{-1}([\mathcal{U}'_2, \mathcal{T}'_2]).$$

Consequently the objects of $\mathfrak{T}(A)$ and $\mathfrak{T}(L)$ are in bijection. Now let $[\mathcal{V}'_i, \mathcal{S}'_i]$ be τ -perpendicular intervals of $\text{tors } A$ such that $[\mathcal{V}'_i, \mathcal{S}'_i] \subseteq [\mathcal{U}'_i, \mathcal{T}'_i]$ for $i = 1, 2$. Then $f_{[\mathcal{U}'_1, \mathcal{T}'_1][\mathcal{V}'_1, \mathcal{S}'_1]} \sim_{\mathfrak{T}(A)} f_{[\mathcal{U}'_2, \mathcal{T}'_2][\mathcal{V}'_2, \mathcal{S}'_2]}$ means $[\mathcal{V}'_1, \mathcal{S}'_1] \cap ((\mathcal{U}'_1)^\perp \cap \mathcal{T}'_1) = [\mathcal{V}'_2, \mathcal{S}'_2] \cap ((\mathcal{U}'_2)^\perp \cap \mathcal{T}'_2)$. Since φ' factors through $\text{tors}((\mathcal{U}'_1)^\perp \cap \mathcal{T}'_1) = \text{tors}((\mathcal{U}'_2)^\perp \cap \mathcal{T}'_2)$, it follows that

$$\phi_{[\mathcal{U}'_1, \mathcal{T}'_1][\mathcal{U}'_2, \mathcal{T}'_2]}([\mathcal{V}'_1, \mathcal{S}'_1]) = [\mathcal{V}'_2, \mathcal{S}'_2].$$

Therefore $\phi'_{[\mathcal{U}'_1, \mathcal{T}'_1][\mathcal{U}'_2, \mathcal{T}'_2]}(\psi^{-1}([\mathcal{V}'_1, \mathcal{S}'_1])) = \psi^{-1}([\mathcal{V}'_2, \mathcal{S}'_2])$, which implies

$$f_{[\mathcal{U}'_1, \mathcal{T}'_1][\mathcal{V}'_1, \mathcal{S}'_1]} \sim_{\mathfrak{T}(A)} f_{[\mathcal{U}'_2, \mathcal{T}'_2][\mathcal{V}'_2, \mathcal{S}'_2]} \implies f_{\psi^{-1}([\mathcal{U}'_1, \mathcal{T}'_1])\psi^{-1}([\mathcal{V}'_1, \mathcal{S}'_1])} \sim_{\mathfrak{T}(L)} f_{\psi^{-1}([\mathcal{U}'_2, \mathcal{T}'_2])\psi^{-1}([\mathcal{V}'_2, \mathcal{S}'_2])}.$$

Hence morphism sets of $\mathfrak{T}(A)$ and $\mathfrak{T}(L)$ are in bijection. It is clear that Lemma 4.2.6 may be transferred to the lattice theoretic setting via the lattice isomorphism ψ and the interval isomorphism of Lemma 4.2.1, making $\mathfrak{T}(L)$ a well-defined category and showing

that composition of morphisms is defined in the same way in $\mathfrak{T}(L)$ as in $\mathfrak{T}(A)$. Thus the construction of the $\mathfrak{T}(A)$ and $\mathfrak{T}(L)$ coincide. As a consequence, the categories $\mathfrak{T}(A)$ and $\mathfrak{T}(L)$ are equivalent. $\square$

Corollary 4.3.8. *Let A be τ -tilting finite and B be τ -tilting equivalent to A . Then there exists an equivalence of categories $\mathfrak{T}(A) \cong \mathfrak{T}(B)$.*

Proof. Since A is τ -tilting finite $\text{tors } A = \text{f-tors } A \cong \text{f-tors } B$ and therefore B is τ -tilting finite. Then both are isomorphic to an abstract finite lattice $L \cong \text{f-tors } A$ and the equivalences $\mathfrak{T}(A) \cong \mathfrak{T}(L) \cong \mathfrak{T}(B)$ follow from Theorem 4.3.7. $\square$

As an alternative definition, see [45, Prop. 11.7], *signed τ -exceptional sequences* may be defined as factorisations of morphisms in $\mathfrak{T}(A)$ into irreducible morphisms. As a consequence of Theorem 4.3.7, the following holds.

Corollary 4.3.9. *Let A be τ -tilting finite and B be τ -tilting equivalent to A . Then there exists a bijection*

$$\begin{aligned} & \{\text{signed } \tau\text{-exceptional sequences of mod } A\} \\ & \longleftrightarrow \{\text{signed } \tau\text{-exceptional sequences of mod } B\}. \end{aligned}$$

Proof. The fully faithful functor of the equivalence of categories $\mathfrak{T}(A) \cong \mathfrak{T}(B)$ of Corollary 4.3.8 induces a bijection between Hom-sets of $\mathfrak{T}(A)$ and $\mathfrak{T}(B)$, which consequently gives a bijection between all factorisations of morphisms in the Hom-sets and thus all signed τ -exceptional sequences corresponding with those. $\square$

Remark 4.3.10. Recently, a mutation for τ -exceptional sequences was defined [41]. This mutation generalises that of exceptional sequences of hereditary algebras [54, 164]. If A and B are two finite-dimensional algebras such that $\text{tors } A \cong \text{tors } B$ and both are finite, then there is a bijection between τ -exceptional sequences of $\text{mod } A$ and $\text{mod } B$ by [19, Thm. 8.10, Rmk. 8.11]. However, the mutation of τ -exceptional sequences relies heavily on signed τ -exceptional sequences. Since Corollary 4.3.9 shows that these are in bijection as well, it seems plausible that the mutation of τ -exceptional sequences may be determined by the lattice of torsion classes, whenever it is finite.

4.4 Factor algebras and lattice congruences

In this section the lattice theoretic definition of the τ -cluster morphism category of Definition 4.2.3 is used to gain new insights into its structure. In particular, comparing it with previous approaches in Example 4.7.1, this approach appears to be the most natural one to study factor algebras of A by an ideal I . An intuitive reason for this is that $\text{tors } A \cap \text{mod } A/I = \text{tors } A/I$ but the same behaviour is generally not exhibited by τ -rigid pairs or wide subcategories. More precisely, recall from Theorem 4.1.2 that an ideal I of A induces a surjective morphism of lattices $\text{tors } A \twoheadrightarrow \text{tors } A/I$ given by $\mathcal{T} \mapsto \mathcal{T} \cap \text{mod } A/I$. Throughout this section, let A be a finite-dimensional K -algebra, I an ideal of A and Φ_I the lattice congruence on $\text{tors } A$ induced by I as in Theorem 4.1.2. The relationship between the brick labelling of $\text{Hasse}(\text{tors } A)$ and $\text{Hasse}(\text{tors } A/I)$ is made explicit in the following result.

Theorem 4.4.1. *[57, Thm. 5.15] The brick labelling of $\text{Hasse}(\text{tors } A)$ satisfies the following:*

- (1) *An arrow $q \in \text{Hasse}(\text{tors } A)$ is not contracted by Φ_I if and only if its brick label B_q lies in $\text{mod } A/I$. In this case, the arrow q has the same label in $\text{Hasse}(\text{tors } A)$ and $\text{Hasse}(\text{tors } A/I)$.*
- (2) *Let $\mathcal{U} \subseteq \mathcal{T} \in \text{tors } A$, then $\mathcal{U} \equiv_{\Phi_I} \mathcal{T}$ if and only if, $IB \neq 0$ for all bricks $B \in \mathcal{U}^\perp \cap \mathcal{T}$.*

The following two results illustrate the nice behaviour of torsion and torsion-free classes under intersection.

Lemma 4.4.2. *Let $\mathcal{U} \in \text{tors } A$, then $(\mathcal{U} \cap \text{mod } A/I)^{\perp_{A/I}} = \mathcal{U}^{\perp_A} \cap \text{mod } A/I$ in $\text{mod } A/I$.*

Proof. To show the inclusion $\supseteq$, take $X \in \mathcal{U}^{\perp_A} \cap \text{mod } A/I$. Assume for a contradiction that there exists $Y \in \mathcal{U} \cap \text{mod } A/I$ such that $\text{Hom}(Y, X) \neq 0$. Since $Y \in \mathcal{U}$ it follows immediately that $X \notin \mathcal{U}^{\perp_A}$, a contradiction. To show the reverse inclusion $\subseteq$, take $X \in (\mathcal{U} \cap \text{mod } A/I)^{\perp_{A/I}} \subseteq \text{mod } A/I$. Assume for a contradiction that there exists $0 \neq f \in \text{Hom}(Y, X)$ with $Y \in \mathcal{U}$. By the assumption on X , it follows that $Y \notin \text{mod } A/I$. However, $\text{im } f$ is a submodule of X , and thus lies in $\text{mod } A/I$. Moreover, since $\mathcal{U}$ is a torsion class and hence closed under quotients it follows that $\text{im } f \in \mathcal{U}$. In conclusion,

$\text{im } f \in \mathcal{U} \cap \text{mod } A/I$. By assumption, there does not exist a nonzero morphism $\text{im } f \rightarrow X$. This is a contradiction and equality follows. $\square$

Corollary 4.4.3. *Let $[\mathcal{U}, \mathcal{T}] \subseteq \text{tors } A$. Then $\overline{\mathcal{U}}^{\perp_{A/I}} \cap \overline{\mathcal{T}} = \mathcal{U}^{\perp_A} \cap \mathcal{T} \cap \text{mod } A/I$.*

Proof. By Lemma 4.4.2 the chain of equalities

$$\overline{\mathcal{U}}^{\perp_{A/I}} \cap \overline{\mathcal{T}} = (\mathcal{U} \cap \text{mod } A/I)^{\perp_{A/I}} \cap (\mathcal{T} \cap \text{mod } A/I) = \mathcal{U}^{\perp_A} \cap \mathcal{T} \cap \text{mod } A/I$$

holds, as required. $\square$

The following result is a simple observation, but since wide subcategories play a central role in this dissertation it is included.

Lemma 4.4.4. *Let $\mathcal{W}$ be a wide subcategory of $\text{mod } A$, then $\mathcal{W} \cap \text{mod } A/I$ is a wide subcategory of $\text{mod } A/I$.*

Proof. Let $L, M, N \in \text{mod } A/I$ lie in a short exact sequence $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ with $L, N \in \mathcal{W} \cap \text{mod } A/I \subseteq \text{mod } A$. Since $L, N \in \mathcal{W} \subseteq \text{mod } A$, it follows that $M \in \mathcal{W}$ since $\mathcal{W}$ is wide. Hence $M \in \mathcal{W} \cap \text{mod } A/I$ as required. Similarly, let $f : M \rightarrow N$ be a morphism with $M, N \in \mathcal{W} \cap \text{mod } A/I$, then from $M, N \in \mathcal{W}$ it follows that $\ker f, \text{coker } f \in \mathcal{W}$. Moreover $\ker f, \text{coker } f \in \text{mod } A/I$ since $\text{mod } A/I$ is an abelian full subcategory of $\text{mod } A$. $\square$

Corollary 4.4.5. *Let $[\mathcal{U}, \mathcal{T}]$ be a wide interval of $\text{tors } A$, then $[\overline{\mathcal{U}}, \overline{\mathcal{T}}]$ is a wide interval of $\text{tors } A/I$.*

Proof. By assumption $\mathcal{U}^{\perp_A} \cap \mathcal{T}$ is a wide subcategory of $\text{mod } A$, then by Corollary 4.4.3, the equality $\overline{\mathcal{U}}^{\perp_{A/I}} \cap \overline{\mathcal{T}} = \mathcal{U}^{\perp_A} \cap \mathcal{T} \cap \text{mod } A/I$ holds. As $\mathcal{U}^{\perp_A} \cap \mathcal{T}$ is a wide subcategory of $\text{mod } A$, Lemma 4.4.4 gives the desired result.. $\square$

The following is the starting point for relating $\mathfrak{T}(A)$ and $\mathfrak{T}(A/I)$.

Lemma 4.4.6. *If $[\mathcal{U}, \mathcal{T}]$ is a τ -perpendicular interval of $\text{tors } A$, then the image $[\overline{\mathcal{U}}, \overline{\mathcal{T}}]$ is a τ -perpendicular interval of $\text{tors } A/I$.*

Proof. Let $[\mathcal{U}, \mathcal{T}]$ be a τ -perpendicular interval, in particular, it is wide. By Corollary 4.4.5 the interval $[\overline{\mathcal{U}}, \overline{\mathcal{T}}]$ is a wide interval of $\text{tors } A/I$. Thus, Theorem 2.1.9 implies

that the interval $[\overline{\mathcal{U}}, \overline{\mathcal{T}}] \subseteq \text{tors } A/I$ is a join interval. Since $\mathcal{U} \in \text{f-tors } A$ it follows that $\overline{\mathcal{U}} \in \text{f-tors } A/I$. In conclusion, it follows that $[\overline{\mathcal{U}}, \overline{\mathcal{T}}]$ is a τ -perpendicular interval of $\text{tors } A/I$ by Proposition 4.3.3. $\square$

This implies that the lattice isomorphisms between the τ -perpendicular intervals of Lemma 4.2.1 may be extended to their quotients in a natural way.

Proposition 4.4.7. *The following diagram extending Lemma 4.2.1 is commutative:*

$$\begin{array}{ccccc}
 [\mathcal{U}_{(M_1, P_1)}, \mathcal{T}_{(M_1, P_1)}] & \xrightarrow{\cong} & & & [\mathcal{U}_{(M_2, P_2)}, \mathcal{T}_{(M_2, P_2)}] \\
 \downarrow (-) \cap \text{mod } A/I & \searrow (-) \cap \mathcal{W}_{(M_1, P_1)} & \text{tors } \mathcal{W}_{(M_1, P_1)} = \text{tors } \mathcal{W}_{(M_2, P_2)} & \swarrow (-) \cap \mathcal{W}_{(M_2, P_2)} & \downarrow (-) \cap \text{mod } A/I \\
 [\overline{\mathcal{U}_{(M_1, P_1)}}, \overline{\mathcal{T}_{(M_1, P_1)}}] & \xrightarrow{\cong} & & & [\overline{\mathcal{U}_{(M_2, P_2)}}, \overline{\mathcal{T}_{(M_2, P_2)}}] \\
 \searrow (-) \cap (\mathcal{W}_{(M_1, P_1)} \cap \text{mod } A/I) & & \downarrow (-) \cap \text{mod } A/I & & \swarrow (-) \cap (\mathcal{W}_{(M_2, P_2)} \cap \text{mod } A/I) \\
 & & \text{tors}(\mathcal{W}_{(M_i, P_i)} \cap \text{mod } A/I) & &
 \end{array}$$

The top and bottom parts are lattice isomorphisms coming from Lemma 4.2.1, and the downward-facing arrows are given by intersecting with $\text{mod } A/I$.

Proof. Let $\mathcal{W} := \mathcal{W}_{(M_1, P_1)} = \mathcal{W}_{(M_2, P_2)}$. By Lemma 4.4.4, $\mathcal{W} \cap \text{mod } A/I$ is a wide subcategory of $\text{mod } A/I$. For $\mathcal{T} \in \text{tors } \mathcal{W}$ it follows trivially that $\mathcal{T} \cap \text{mod } A/I$ is a torsion class of the wide subcategory $\mathcal{W} \cap \text{mod } A/I$ of $\text{mod } A/I$. By Lemma 4.4.6, the intervals

$$[\overline{\mathcal{U}_{(M_1, P_1)}}, \overline{\mathcal{T}_{(M_1, P_1)}}] \text{ and } [\overline{\mathcal{U}_{(M_2, P_2)}}, \overline{\mathcal{T}_{(M_2, P_2)}}]$$

are τ -perpendicular intervals of $\text{tors } A/I$ which correspond, by Corollary 4.4.3, to the wide subcategory $\mathcal{W} \cap \text{mod } A/I = \mathcal{W} \cap \text{mod } A/I$. Thus Lemma 4.2.1 implies the existence of the lattice isomorphisms in the bottom half of the diagram and the downward-facing arrows are well-defined. The commutativity of the squares is obvious from the descriptions of the maps. $\square$

As a consequence the following relationship can now be established.

Theorem 4.4.8. *Let $I \in \text{ideal } A$. There exists a functor $F_I : \mathfrak{T}(A) \rightarrow \mathfrak{T}(A/I)$ induced by $\overline{(-)}_I$, that is, F_I is given on objects by $[\mathcal{U}, \mathcal{T}]_{\sim} \mapsto [\overline{\mathcal{U}}, \overline{\mathcal{T}}]_{\sim}$ and on morphisms by $[f_{[\mathcal{U}, \mathcal{T}][\mathcal{V}, \mathcal{S}]}] \mapsto [f_{[\overline{\mathcal{U}}, \overline{\mathcal{T}}][\overline{\mathcal{V}}, \overline{\mathcal{S}}]}]$.*

Proof. F_I maps τ -perpendicular intervals of tors A to τ -perpendicular intervals of tors A/I by Lemma 4.4.6. Then, Corollary 4.4.3 implies that F_I is well-defined on objects since

$$\begin{aligned} [\mathcal{U}_1, \mathcal{T}_1] \sim [\mathcal{U}_2, \mathcal{T}_2] &\Leftrightarrow \mathcal{U}_1^{\perp A} \cap \mathcal{T}_1 = \mathcal{U}_2^{\perp A} \cap \mathcal{T}_2 \\ &\Rightarrow \mathcal{U}_1^{\perp A} \cap \mathcal{T}_1 \cap \text{mod } A/I = \mathcal{U}_2^{\perp A} \cap \mathcal{T}_2 \cap \text{mod } A/I \\ &\Leftrightarrow [\overline{\mathcal{U}}_1, \overline{\mathcal{T}}_1] \sim [\overline{\mathcal{U}}_2, \overline{\mathcal{T}}_2]. \end{aligned}$$

It is clear that containment of intervals is preserved by $\overline{(-)}_I$. Let $[\mathcal{U}_i, \mathcal{T}_i] \leq [\mathcal{V}_i, \mathcal{S}_i]$ for $i = 1, 2$ be such that $f_{[\mathcal{U}_1, \mathcal{T}_1][\mathcal{V}_1, \mathcal{S}_1]} \sim_{\mathfrak{T}(A)} f_{[\mathcal{U}_2, \mathcal{T}_2][\mathcal{V}_2, \mathcal{S}_2]}$. The commutativity of the diagram in Proposition 4.4.7 implies that

$$\begin{aligned} [\mathcal{V}_1, \mathcal{S}_1] \cap (\mathcal{U}_1^{\perp A} \cap \mathcal{T}_1) &= [\mathcal{V}_2, \mathcal{S}_2] \cap (\mathcal{U}_2^{\perp A} \cap \mathcal{T}_2) \\ \implies [\overline{\mathcal{V}}_1, \overline{\mathcal{S}}_1] \cap (\mathcal{U}_1^{\perp A} \cap \mathcal{T}_1 \cap \text{mod } A/I) &= [\overline{\mathcal{V}}_2, \overline{\mathcal{S}}_2] \cap (\mathcal{U}_2^{\perp A} \cap \mathcal{T}_2 \cap \text{mod } A/I). \end{aligned}$$

Thus F_I is well-defined on morphisms, that is, $f_{[\overline{\mathcal{U}}_1, \overline{\mathcal{T}}_1][\overline{\mathcal{V}}_1, \overline{\mathcal{S}}_1]} \sim_{\mathfrak{T}(A)} f_{[\overline{\mathcal{U}}_2, \overline{\mathcal{T}}_2][\overline{\mathcal{V}}_2, \overline{\mathcal{S}}_2]}$. To show that composition is preserved it is sufficient, by Lemma 4.2.6, to consider any three τ -perpendicular intervals $[\mathcal{X}, \mathcal{Y}] \subseteq [\mathcal{V}, \mathcal{S}] \subseteq [\mathcal{U}, \mathcal{T}] \subseteq \text{tors } A$. The following is obvious:

$$\begin{aligned} F_I(f_{[\mathcal{U}, \mathcal{T}][\mathcal{X}, \mathcal{Y}]}) &= f_{[\overline{\mathcal{U}}, \overline{\mathcal{T}}][\overline{\mathcal{X}}, \overline{\mathcal{Y}}]} \\ &= f_{[\overline{\mathcal{V}}, \overline{\mathcal{S}}][\overline{\mathcal{X}}, \overline{\mathcal{Y}}]} \circ f_{[\overline{\mathcal{U}}, \overline{\mathcal{T}}][\overline{\mathcal{V}}, \overline{\mathcal{S}}]} \\ &= F_I(f_{[\mathcal{V}, \mathcal{S}][\mathcal{X}, \mathcal{Y}]}) \circ F_I(f_{[\mathcal{U}, \mathcal{T}][\mathcal{V}, \mathcal{S}]}). \end{aligned}$$

It is clear that identity morphisms are preserved. Thus, F_I is a well-defined functor. $\square$

The remainder of this section is dedicated to characterising properties of F_I .

Lemma 4.4.9. *The functor F_I is faithful if and only if the restriction*

$$\overline{(-)} : \text{f-tors } A \rightarrow \text{f-tors } A/I$$

is injective.

Proof. ($\Leftarrow$). Let $[f_{[\mathcal{V}, \mathcal{S}][\mathcal{U}_1, \mathcal{T}_1]}], [f_{[\mathcal{V}, \mathcal{S}][\mathcal{U}_2, \mathcal{T}_2]}] \in \text{Hom}_{\mathfrak{T}(A)}([\mathcal{V}, \mathcal{S}]_{\sim}, [\mathcal{U}, \mathcal{T}]_{\sim})$ be distinct morphisms, which is to say that either $\mathcal{U}_1 \neq \mathcal{U}_2$ or $\mathcal{T}_1 \neq \mathcal{T}_2$. Then applying F_I gives

$$[f_{[\overline{\mathcal{V}}, \overline{\mathcal{S}}][\overline{\mathcal{U}}_1, \overline{\mathcal{T}}_1]}], [f_{[\overline{\mathcal{V}}, \overline{\mathcal{S}}][\overline{\mathcal{U}}_2, \overline{\mathcal{T}}_2]}] \in \text{Hom}_{\mathfrak{T}(A/I)}([\overline{\mathcal{V}}, \overline{\mathcal{S}}]_{\sim}, [\overline{\mathcal{U}}, \overline{\mathcal{T}}]_{\sim}),$$

which coincide if and only if $[\overline{\mathcal{U}}_1, \overline{\mathcal{T}}_1] = [\overline{\mathcal{U}}_2, \overline{\mathcal{T}}_2]$, which would imply that the restriction $\overline{(-)} : \text{f-tors } A \rightarrow \text{f-tors } A/I$ is not injective. Thus F_I is injective on Hom-sets, hence faithful.

($\Rightarrow$). Let F_I be faithful and assume that the map $\overline{(-)} : \text{f-tors } A \rightarrow \text{f-tors } A/I$ is not an injection. There exists a distinct morphism

$$[f_{[0, \text{mod } A]}[\mathcal{T}, \mathcal{T}]] : [0, \text{mod } A]_{\sim} \rightarrow [\mathcal{T}, \mathcal{T}]_{\sim}$$

in $\mathfrak{T}(A)$ for every functorially finite torsion class $\mathcal{T} \in \text{f-tors } A$, see also Remark 4.2.8. If $\overline{(-)}$ is not injective, then $\mathcal{T}_1 \cap \text{mod } A/I = \mathcal{T}_2 \cap \text{mod } A/I$ for two distinct $\mathcal{T}_1, \mathcal{T}_2 \in \text{f-tors } A$. However, then the distinct morphisms

$$[f_{[0, \text{mod } A]}[\mathcal{T}_1, \mathcal{T}_1]] \neq [f_{[0, \text{mod } A]}[\mathcal{T}_2, \mathcal{T}_2]]$$

of $\mathfrak{T}(A)$ have the same image under F_I and hence F_I would not be faithful. $\square$

Remark 4.4.10. If A is τ -tilting finite, then $\text{tors } A = \text{f-tors } A$ by Theorem 2.2.3. In this case, $\overline{(-)} : \text{f-tors } A \rightarrow \text{f-tors } A/I$ is surjective by [57, Prop. 5.7(d)] for all $I \in \text{ideal } A$. Thus, Lemma 4.4.9 holds for τ -tilting finite algebras if and only if there is a lattice isomorphism $\text{tors } A \cong \text{tors } A/I$. The map $\overline{(-)}$ is generally not surjective when A is τ -tilting infinite, see Example 4.7.2 or [57, Exmp. 5.11].

Let $\mathcal{S} = \{B_1, \dots, B_r\} \subseteq \text{mod } A/I$ be a semibrick, then $\mathcal{S} \subseteq \text{mod } A$ is also a semibrick in $\text{mod } A$. Thus, for any $I \in \text{ideal } A$, define the following map using Eq. (2.0.2) by factoring through the canonical inclusion of semibricks:

$$\begin{aligned} \iota : \text{wide } A/I &\rightarrow \text{wide } A \\ \text{Filt}_{A/I}\{B_1, \dots, B_r\} &\mapsto \text{Filt}_A\{B_1, \dots, B_r\} \end{aligned} \tag{4.4.1}$$

If the algebra A is τ -tilting finite, then every wide subcategory is τ -perpendicular [108, Cor. 2.17], see also [40, Rmk. 4.10] and A/I is τ -tilting finite [57, Cor. 1.9]. In this setting, it is possible to lift semibricks, which in turn lifts τ -perpendicular subcategories. This is the key idea of the following result.

Lemma 4.4.11. *Let A be τ -tilting finite and $I \in \text{ideal } A$, then the functor*

$$F_I : \mathfrak{T}(A) \rightarrow \mathfrak{T}(A/I)$$

in Theorem 4.4.8 is surjective-on-objects.

Proof. Let $[\mathcal{U}, \mathcal{T}]_\sim \in \mathfrak{T}(A/I)$, then $\mathcal{U}^\perp \cap \mathcal{T} = \text{Filt}_{A/I}\{B_1, \dots, B_r\}$ for some semibrick $\{B_1, \dots, B_r\} \subseteq \text{mod } A/I \subseteq \text{mod } A$, by Eq. (2.0.2). Consider the lifted wide subcategory $\iota(\text{Filt}_{A/I}\{B_1, \dots, B_r\}) \subseteq \text{mod } A$. Since A is τ -tilting finite, the wide subcategory $\text{Filt}_A\{B_1, \dots, B_r\}$ is also a τ -perpendicular. By [40, Thm. 4.5], there then exists some τ -perpendicular interval $[\mathcal{A}, \mathcal{B}] \subseteq \text{tors } A$ such that $\mathcal{A}^\perp \cap \mathcal{B} = \text{Filt}_A\{B_1, \dots, B_r\}$. It is clear that $[\overline{\mathcal{A}}, \overline{\mathcal{B}}] \sim [\mathcal{U}, \mathcal{T}]$ since

$$\begin{aligned} \overline{\mathcal{A}}^\perp \cap \overline{\mathcal{B}} &= \mathcal{A}^\perp \cap \mathcal{B} \cap \text{mod } A/I \\ &= \text{Filt}_A\{B_1, \dots, B_r\} \cap \text{mod } A/I \\ &= \text{Filt}_{A/I}\{B_1, \dots, B_r\} \\ &= \mathcal{U}^\perp \cap \mathcal{T}. \end{aligned}$$

Hence every object $[\mathcal{U}, \mathcal{T}]_\sim \in \mathfrak{T}(A/I)$ lies in the image of F_I . $\square$

Example 4.7.2 illustrates that the assumption of τ -tilting finiteness is necessary in Lemma 4.4.11. To understand in which cases $F_I : \mathfrak{T}(A) \rightarrow \mathfrak{T}(A/I)$ is full, the Cartesian product of posets is required.

Definition 4.4.12. Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be two posets. Define the product of posets $(P_1 \times P_2, \leq)$ via the partial order

$$(a, b) \leq (c, d) \quad \Leftrightarrow \quad a \leq_1 c \text{ and } b \leq_2 d.$$

If $(P_1, \leq_1)$ and $(P_2, \leq_2)$ are lattices, define the join and meet componentwise, then $(P_1 \times P_2, \leq)$ is a lattice.

The following intermediate result is similar to [6, Thm. 4.19(a)], where a similar relationship is established for g -vector fans.

Lemma 4.4.13. *Let $A \cong A_1 \times A_2$ where A_1, A_2 are finite-dimensional algebras. Then as posets*

$$\text{tors } A \cong \text{tors } A_1 \times \text{tors } A_2, \quad \text{and} \quad \text{f-tors } A \cong \text{f-tors } A_1 \times \text{f-tors } A_2.$$

Proof. It is well-known that $\text{mod } A \cong \text{mod } A_1 \times \text{mod } A_2$ in this case. This implies that

the inverse bijections are given by

$$\begin{aligned} \text{tors } A &\cong \text{tors } A_1 \times \text{tors } A_2 \\ \mathcal{T} &\mapsto (\mathcal{T} \cap \text{mod } A_1, \mathcal{T} \cap \text{mod } A_2) \\ \text{add}(\mathcal{T}_1 \cup \mathcal{T}_2) &\leftrightarrow (\mathcal{T}_1, \mathcal{T}_2). \end{aligned}$$

If $\mathcal{T} \subseteq \text{mod } A$ is a torsion class, it is clear that $\mathcal{T} \cap \text{mod } A_i \in \text{tors } A_i$ for $i = 1, 2$ since both terms of the intersection are closed under extensions and quotients. In particular, if $\mathcal{T}$ is functorially finite then $\mathcal{T} \cap \text{mod } A_i$ is functorially finite, for $i = 1, 2$ by [57, Prop. 5.6(b)]. Conversely, given $\mathcal{T}_i \in \text{mod } A_i$ for $i = 1, 2$, the full subcategory $\text{add}(\mathcal{T}_1 \cup \mathcal{T}_2)$ is a torsion class. This follows because the equality $\text{Ext}_A^1(X_1^a, X_2^b) = 0 = \text{Ext}_A^1(X_2^c, X_1^d)$ for all $X_1 \in \text{mod } A_1$, $X_2 \in \text{mod } A_2$ and $a, b, c, d \geq 1$ implies closure under extensions. Because $\text{Hom}_A(X_1, X_2) = 0 = \text{Hom}_A(X_2, X_1)$ it follows that $\text{add}(\mathcal{T}_1 \cup \mathcal{T}_2)$ is functorially finite, since any module $M \in \text{mod } A$ admits left and right $\text{add}(\mathcal{T}_1 \cup \mathcal{T}_2)$ -approximations. Indeed, a $\text{add}(\mathcal{T}_1 \cup \mathcal{T}_2)$ -approximation is given by a $\mathcal{T}_1$ -approximation of the direct summand of M which is in $\text{mod } A_1$, and a $\mathcal{T}_2$ -approximation of the direct summand of M which is in $\text{mod } A_2$. One sees directly that these are inverse assignments and order preserving. $\square$

For the remainder of this section, assume for simplicity that the field K is algebraically closed. This assumption implies that $A \cong B \times C$ for some finite-dimensional algebras B and C if and only if there exist two sets of simple A -modules $\mathcal{S}_1$ and $\mathcal{S}_2$ such that $\text{Ext}_A^1(S_1, S_2) = \text{Ext}_A^1(S_2, S_1)$ for $S_i \in \mathcal{S}_i$ and $i = 1, 2$. This is because then $A \cong KQ/I$ for some ideal I , where Q is the Ext-quiver of the algebra, see [11, Lem. II.2.5, Lem. III.2.12]. The following result is a converse to Lemma 4.4.13.

Lemma 4.4.14. *Let A be a finite-dimensional algebra such that there is a lattice isomorphism $\text{tors } A \cong \text{tors } B' \times \text{tors } C'$, for some finite-dimensional algebras B' and C' . Then $A \cong B \times C$ for some finite-dimensional algebras B and C .*

Proof. Denote the lattice isomorphism by $\phi : \text{tors } B' \times \text{tors } C' \rightarrow \text{tors } A$. Take X_1 to be a simple B' -module and X_2 to be a simple C' -module, then by definition of the join

$$(\text{Filt}_{B'}\{X_1\}, 0) \vee (0, \text{Filt}_{C'}\{X_2\}) = (\text{Filt}_{B'}\{X_1\}, \text{Filt}_{C'}\{X_2\}). \quad (4.4.2)$$

Applying ϕ to both sides of the equation, there are A -modules, say $\widehat{X}_1$ and $\widehat{X}_2$, labelling the cover relations $\phi(\text{Filt}_{B'}\{X_1\}, 0) \rightarrow 0$ and $\phi(0, \text{Filt}_{C'}\{X_2\}) \rightarrow 0$ in $\text{Hasse}(\text{tors } A)$

respectively, which are simple by [57, Prop. 3.16(a)]. By Eq. (4.4.2) and [57, Thm. 4.16(b)], the simples $\widehat{X}_1$ and $\widehat{X}_2$ are such that $\text{Filt}_A\{\widehat{X}_1, \widehat{X}_2\}$ contains no bricks other than $\widehat{X}_1$ and $\widehat{X}_2$ themselves. It follows that $\text{Ext}_A^1(\widehat{X}_1, \widehat{X}_2) = 0 = \text{Ext}_A^1(\widehat{X}_2, \widehat{X}_1)$ by [57, Lem. 4.26].

Repeating this process for all such pairs of cover relations, it follows that there exist two sets of simple modules $\mathcal{S}_1, \mathcal{S}_2 \subseteq \text{mod } A$ such that $\mathcal{S}_1 \cup \mathcal{S}_2 = \text{simp}(\text{mod } A)$ and such that $\text{Ext}_A^1(S_1, S_2) = 0 = \text{Ext}_A^1(S_2, S_1)$ for $S_i \in \mathcal{S}_i$ for $i = 1, 2$. Define

$$B \cong A / \left\langle \sum_{i: S(i) \in \mathcal{S}_1} e_i \right\rangle, \quad \text{and} \quad C \cong A / \left\langle \sum_{i: S(i) \in \mathcal{S}_2} e_i \right\rangle.$$

Then $A \cong B \times C$ as required. $\square$

To avoid technicalities involving isomorphisms between infinite lattices of functorially finite torsion classes like in Example 4.3.2(3), the following result assumes that the lattice of torsion classes is finite.

Proposition 4.4.15. *Let A be τ -tilting finite. The functor F_I is full if and only if $\text{tors } A \cong \text{tors } A/I \times \text{tors } B$ for some finite-dimensional algebra B .*

Proof. ($\Leftarrow$). It follows from (the proof of) Lemma 4.4.14 that $A \cong C \times B$ where C is a quotient algebra of A by an ideal generated by the primitive orthogonal idempotents, such that $\text{tors } C \cong \text{tors } A/I$. By Theorem 4.3.7 there is a (fully faithful) equivalence $G : \mathfrak{T}(C) \xrightarrow{\cong} \mathfrak{T}(A/I)$. Moreover, there is a sequence of surjective algebra morphisms $A \twoheadrightarrow C \twoheadrightarrow A/I$ and hence the lattice congruences on $\text{tors } A$ induced by $A \twoheadrightarrow C$ and $A \twoheadrightarrow A/I$ coincide. Furthermore, the functor F_I factors through this equivalence, that is, $F_I \cong G \circ F_C$, where $F_C : \mathfrak{T}(A) \rightarrow \mathfrak{T}(C)$ is induced by the epimorphism $A \twoheadrightarrow C$. Since the composition of full functors is full, it is sufficient to show that F_C is full.

Let $[\mathcal{V}', \mathcal{S}']_{\sim}, [\mathcal{U}', \mathcal{T}']_{\sim} \in \mathfrak{T}(A)$ be such that there exists a pair of representatives $[\mathcal{V}, \mathcal{S}] \in [\mathcal{V}', \mathcal{S}']_{\sim}$ and $[\mathcal{U}, \mathcal{T}] \in [\mathcal{U}', \mathcal{T}']_{\sim}$ satisfying $[\mathcal{V}, \mathcal{S}] \leq [\mathcal{U}, \mathcal{T}]$. It remains to show that the induced map

$$\begin{aligned} & \text{Hom}_{\mathfrak{T}(A)}([\mathcal{V}, \mathcal{S}]_{\sim}, [\mathcal{U}, \mathcal{T}]_{\sim}) \\ & \rightarrow \text{Hom}_{\mathfrak{T}(C)}([\mathcal{V} \cap \text{mod } C, \mathcal{S} \cap \text{mod } C]_{\sim}, [\mathcal{U} \cap \text{mod } C, \mathcal{T} \cap \text{mod } C]_{\sim}) \end{aligned}$$

is surjective. Thus, take an arbitrary morphism $f_{[\mathcal{V}_1, \mathcal{S}_1][\mathcal{U}_1, \mathcal{T}_1]} \in \mathfrak{T}(C)$ in the codomain, where $[\mathcal{V}_1, \mathcal{S}_1] \sim [\mathcal{V} \cap \text{mod } C, \mathcal{S} \cap \text{mod } C]$ and $[\mathcal{U}_1, \mathcal{T}_1] \sim [\mathcal{U} \cap \text{mod } C, \mathcal{T} \cap \text{mod } C]$. Define

$$[\widehat{\mathcal{U}}, \widehat{\mathcal{T}}] := [\text{add}(\mathcal{U}_1 \cup (\mathcal{U} \cap \text{mod } B)), \text{add}(\mathcal{T}_1 \cup (\mathcal{T} \cap \text{mod } B))],$$

which is an interval of tors A because both boundary terms are additive closures of a torsion classes in $\text{mod } B$ and a torsion class in $\text{mod } C$. Since $\text{mod } A \cong \text{mod } B \times \text{mod } C$ the result is a torsion class by Lemma 4.4.13. Then, the simple objects of the wide subcategories satisfy

$$\begin{aligned} \text{simp}(\mathcal{U}^\perp \cap \mathcal{T}) &= (\text{simp}(\mathcal{U}^\perp \cap \mathcal{T}) \cap \text{mod } C) \cup (\text{simp}(\mathcal{U}^\perp \cap \mathcal{T}) \cap \text{mod } B) \\ &= \text{simp}(\mathcal{U}_1^\perp \cap \mathcal{T}_1) \cup (\text{simp}(\mathcal{U}^\perp \cap \mathcal{T} \cap \text{mod } B)) \\ &= \text{simp}(\widehat{\mathcal{U}}^\perp \cap \widehat{\mathcal{T}}), \end{aligned}$$

where the second equality follows from Corollary 4.4.3. By [57, Thm. 4.16(a)] this implies $\mathcal{U}^\perp \cap \mathcal{T} = \widehat{\mathcal{U}}^\perp \cap \widehat{\mathcal{T}}$ and thus $[\widehat{\mathcal{U}}, \widehat{\mathcal{T}}]_\sim = [\mathcal{U}, \mathcal{T}]_\sim \in \mathfrak{T}(A)$. Similarly, define

$$[\widehat{\mathcal{V}}, \widehat{\mathcal{S}}] := [\text{add}(\mathcal{V}_1 \cup (\mathcal{V} \cap \text{mod } B)), \text{add}(\mathcal{S}_1 \cup (\mathcal{S} \cap \text{mod } B))]$$

from which it follows that $[\widehat{\mathcal{V}}, \widehat{\mathcal{S}}]_\sim = [\mathcal{V}, \mathcal{S}]_\sim \in \mathfrak{T}(A)$ using an analogous argument. Then $[f_{[\widehat{\mathcal{V}}, \widehat{\mathcal{S}}][\widehat{\mathcal{U}}, \widehat{\mathcal{T}}]}] \in \text{Hom}_{\mathfrak{T}(A)}([\mathcal{V}, \mathcal{S}]_\sim, [\mathcal{U}, \mathcal{T}]_\sim)$ gets mapped to $[f_{[\mathcal{V}_1, \mathcal{S}_1][\mathcal{U}_1, \mathcal{T}_1]}] \in \mathfrak{T}(C)$ as required. Therefore the induced map between Hom-sets is surjective. Hence, F_C is full and so is $F_I = G \circ F_C$.

($\Rightarrow$). Assume F_I is full. If $\text{tors } A \cong \text{tors } A/I$ there is nothing to show, otherwise let $\mathcal{T}_1 \xleftarrow{X} \mathcal{T}_2 \subseteq \text{Hasse}(\text{tors } A)$ be an arrow contracted by the lattice congruence. In particular, since F_I is full the following induced map of Hom-sets is surjective:

$$\text{Hom}_{\mathfrak{T}(A)}([0, \text{mod } A]_\sim, [\mathcal{T}_1, \mathcal{T}_2]_\sim) \rightarrow \text{Hom}_{\mathfrak{T}(A/I)}([0, \text{mod } A/I]_\sim, [0, 0]_\sim).$$

Let $\mathcal{X} \in \text{tors } A/I$ be any (functorially finite) torsion class, then the corresponding morphism $[f_{[0, \text{mod } A/I][\mathcal{X}, \mathcal{X}]}] \in \mathfrak{T}(A/I)$ lies in the codomain of the map above. Thus the above map is surjective only if there exists an arrow $\mathcal{T}_3 \xleftarrow{X} \mathcal{T}_4 \subseteq \text{Hasse}([\pi_\downarrow \mathcal{X}, \pi_\uparrow \mathcal{X}])$. Applying this idea to all contracted arrows in a congruence class of Φ_I , it follows that

$$\text{brick}[\pi_\downarrow \mathcal{X}_1, \pi_\uparrow \mathcal{X}_1] = \text{brick}[\pi_\downarrow \mathcal{X}_2, \pi_\uparrow \mathcal{X}_2], \quad (4.4.3)$$

for all $\mathcal{X}_1, \mathcal{X}_2 \in \text{tors } A/I$. Let $\mathcal{S}_2 := \text{simp } A/I$ and $\mathcal{S}_1 := \text{simp } A \setminus \mathcal{S}_2$ and. The aim is to show that $\text{Ext}_A^1(S_1, S_2) = 0 = \text{Ext}_A^1(S_2, S_1)$ for all simple modules $S_1 \in \mathcal{S}_1$ and $S_2 \in \mathcal{S}_2$. This implies that the two corresponding idempotents $\epsilon_1 := \sum_{i:S(i) \in \mathcal{S}_1} e_i$ and $\epsilon_2 := \sum_{i:S(i) \in \mathcal{S}_2} e_i$ are central and thus that A is not connected. Let $S_1 \in \mathcal{S}_1$ and $S_2 \in \mathcal{S}_2$. Since A is τ -tilting finite, the descriptions of the lattice congruence in [57, Prop. 4.21, Thm. 4.23] and [157, Thm. 9-6.5] imply that all cover relations arising in $[0, \text{Filt}_A\{S_1, S_2\}] \subseteq \text{tors } A$ except two, which are labelled by S_2 , are contracted by the congruence Φ_I . Hence the polygon consists of two halves, one half is the side lying in the congruence class $\Phi_I^{-1}(0) \subseteq \text{tors } A$, and the other half lies in the congruence class $\Phi_I^{-1}(\text{Filt}_{A/I}\{S_2\}) \subseteq \text{tors } A$. These halves of the polygon are connected by the two arrows labelled by S_2 .

Let $0 \rightarrow S_2 \rightarrow M \rightarrow S_1 \rightarrow 0$ be a non-split short exact sequence, so in particular $M \notin S_2^\perp$. Then, by [57, Thm. 4.26] the module M is a brick and therefore arises as a label of an arrow in the polygon $[0, \text{Filt}_A\{S_1, S_2\}] \subseteq \text{tors } A$ by [57, Thm. 4.21(b)]. More precisely M arises in the half of the polygon which lies in $\Phi_I^{-1}(0) = [0, \pi_\uparrow 0]$ since $M \notin S_2^\perp$. However, from Eq. (4.4.3) it follows that M must arise as a label of some arrow in $\Phi_I^{-1}(\text{Filt}_{A/I}\{S_2\}) = [\text{Filt}_A\{S_2\}, \pi_\uparrow \text{Filt}_{A/I}\{S_2\}]$, which is a contradiction, since this requires $M \in \text{Filt}_A\{S_2\}^\perp = S_2^\perp$ by definition of the brick labelling. Thus $\text{Ext}_A^1(S_1, S_2) = 0$.

Let $0 \rightarrow S_1 \rightarrow M \rightarrow S_2 \rightarrow 0$ be a non-split short exact sequence. Like in the previous paragraph, it follows that M arises as a label of an arrow in the half of the polygon

$$[0, \text{Filt}_A\{S_1, S_2\}] \subseteq \text{tors } A$$

which is contained in $\Phi_I^{-1}(\text{Filt}_{A/I}\{S_2\})$. However, by Eq. (4.4.3) it must also arise in $\Phi_I^{-1}(0)$. Since $M \notin S_1^\perp$, M cannot label an arrow above $\text{Filt}_A\{S_1\}$ in $\text{Hasse}(\text{tors } A)$. Consequently, there must exist another simple module $S_3 \in \text{mod } A$ with the property that $M \in \text{brick}[\text{Filt}_A\{S_3\}, \pi_\uparrow 0]$. As $\text{Filt}_A\{S_3\} \in \Phi_I^{-1}(0)$ it follows that $S_3 \in \mathcal{S}_2$. In other words, there exists an arrow $\mathcal{T}' \xleftarrow{M} \mathcal{T} \subseteq \text{Hasse}([\text{Filt}_A\{S_3\}, \pi_\uparrow 0])$ labelled by M . Because $M \in \mathcal{T}$ it follows that $S_2 \in \text{Fac } M \subseteq \mathcal{T}$. Moreover, it follows that $S_2 \notin (\mathcal{T}')^\perp$, as otherwise there would be two bricks M and S_2 in the intersection $(\mathcal{T}')^\perp \cap \mathcal{T}$, which is a contradiction to [57, Thm. 3.3(b)]. Thus, there exists a morphism $f : T \rightarrow S_2$ for some

$T \in \mathcal{T}'$. However, any morphism to the simple module S_2 is an epimorphism, and since torsion classes are closed under quotients, it follows that $S_2 \in \mathcal{T}'$. As a consequence the join $\text{Filt}_A\{S_2\} \vee \text{Filt}_A\{S_3\} = \text{Filt}_A\{S_2, S_3\}$ is contained in $\mathcal{T}'$. In particular, there exists an arrow $\mathcal{T}'' \xleftarrow{S_2} \text{Filt}_A\{S_2, S_3\}$ in $\text{Hasse}(\text{tors } A)$ for some $\mathcal{T}''$ containing $\text{Filt}_A\{S_3\}$. Combining these observations, it follows that

$$0 \subseteq \text{Filt}_A\{S_3\} \subseteq \mathcal{T}'' \subseteq \text{Filt}_A\{S_2, S_3\} \subseteq \mathcal{T}' \subseteq \pi_{\uparrow} 0.$$

In conclusion, the interval $[0, \pi_{\uparrow} 0]$ contains arrows labelled by S_2 which are not contracted by Φ_I . Therefore the equivalence class $[0]$ of Φ_I is not an interval, a contradiction to Proposition 4.1.3. Hence, $\text{Ext}_A^1(S_2, S_1) = 0$.

In conclusion, $A \cong (A/\langle \epsilon_1 \rangle) \times (A/\langle \epsilon_2 \rangle)$. By construction $\text{tors}(A/\langle \epsilon_2 \rangle) \cong \text{tors}(A/I)$ and from Lemma 4.4.13 it follows that $\text{tors } A \cong \text{tors}(A/\langle \epsilon_1 \rangle) \times \text{tors}(A/I)$ as required. $\square$

4.5 Epimorphisms and lifting τ -perpendicular intervals

In this section, let A be τ -tilting finite K -algebra, implying that $\text{tors } A = \text{f-tors } A$ is finite. Let I denote an ideal of A . Under this assumption it is to define the lifting of τ -perpendicular intervals of $\text{tors } A/I$ to τ -perpendicular intervals of $\text{tors } A$ in a more precise way than in the proof of Lemma 4.4.11. Recall from Proposition 4.1.3 the isomorphism of lattices $\pi_{\uparrow}^{\Phi_I}(\text{tors } A) \cong \text{tors } A/I$. By a slight abuse of notation, given $\mathcal{T} \in \text{tors } A/I$ denote by $\pi_{\uparrow}^{\Phi_I} \mathcal{T}$, or $\pi_{\uparrow} \mathcal{T}$ for short, the top element $\pi_{\uparrow}^{\Phi_I} \Phi_I^{-1}(\mathcal{T})$ of the preimage of $\mathcal{T}$ under the lattice congruence Φ_I . As a first step, consider the lemma which makes it possible to lift τ -perpendicular intervals of $\text{tors } A/I$ to τ -perpendicular intervals of $\text{tors } A$ explicitly.

Lemma 4.5.1. *Let $I \in \text{ideal } A$. For every τ -perpendicular interval $[\mathcal{U}, \mathcal{T}] \subseteq \text{tors } A/I$ there exists a τ -perpendicular interval $[\mathcal{A}_{\mathcal{U}}, \mathcal{B}_{\mathcal{T}}] \subseteq [\pi_{\uparrow} \mathcal{U}, \pi_{\uparrow} \mathcal{T}] \subseteq \text{tors } A$. The interval is such that $\overline{[\mathcal{A}_{\mathcal{U}}, \mathcal{B}_{\mathcal{T}}]} = [\mathcal{U}, \mathcal{T}]$ and satisfies $\mathcal{A}_{\mathcal{U}}^{\perp A} \cap \mathcal{B}_{\mathcal{T}} = \iota(\mathcal{U}^{\perp A/I} \cap \mathcal{T})$, where ι is the inclusion of Eq. (4.4.1). Moreover, it is given by*

$$[\mathcal{A}_{\mathcal{U}}, \mathcal{B}_{\mathcal{T}}] = [\pi_{\uparrow} \mathcal{U}, \pi_{\uparrow} \mathcal{U} \vee \text{T}(\mathcal{W})],$$

where $\mathcal{W} = \iota(\mathcal{U}^{\perp A/I} \cap \mathcal{T})$.

Proof. Let $[\mathcal{U}, \mathcal{T}]$ be a τ -perpendicular interval of $\text{tors } A/I$ whose corresponding wide subcategory is given by $\mathcal{U}^{\perp_{A/I}} \cap \mathcal{T} = \text{Filt}_{A/I}\{S_1, \dots, S_k\} \in \text{wide } A/I$, for some semibrick $\{S_1, \dots, S_k\} \in \text{sbrick } A$ by Eq. (2.0.2). This implies that

$$\{S_1, \dots, S_k\} \subseteq \mathcal{U}^{\perp_{A/I}} \cap \mathcal{T} = (\pi_{\uparrow} \mathcal{U})^{\perp_A} \cap \pi_{\uparrow} \mathcal{T} \cap \text{mod } A/I \subseteq (\pi_{\uparrow} \mathcal{U})^{\perp_A} \cap \pi_{\uparrow} \mathcal{T} \quad (4.5.1)$$

by using Corollary 4.4.3 to obtain the equality. Since $(\pi_{\uparrow} \mathcal{U})^{\perp_A}$ and $\pi_{\uparrow} \mathcal{T}$ are a torsion-free and a torsion class of $\text{mod } A$, respectively, they are closed under extensions. As a consequence the lifted τ -perpendicular subcategory $\mathcal{W} = \text{Filt}_A\{S_1, \dots, S_k\} \in \text{wide } A$ satisfies $\mathcal{W} \subseteq (\pi_{\uparrow} \mathcal{U})^{\perp_A} \cap \pi_{\uparrow} \mathcal{T}$, as $\mathcal{W} = \text{Filt}_A\{S_1, \dots, S_k\}$ consists of iterated extensions of modules contained in $(\pi_{\uparrow} \mathcal{U})^{\perp_A} \cap \pi_{\uparrow} \mathcal{T}$.

By the dual of [157, Prop. 9-5.10] and Proposition 4.1.3, the intersection with $\text{mod } A/I$ induces a bijection from the elements $\mathcal{V} \in \text{tors } A$ covering $\pi_{\uparrow} \mathcal{U}$ in $\text{tors } A$ to the elements $\mathcal{V}'$ covering $\mathcal{U}$ in $\text{tors } A/I$. Thus, using Proposition 4.2.2 and Theorem 4.4.1, there is a brick label preserving bijection between arrows

$$\begin{aligned} & \left\{ \pi_{\uparrow} \mathcal{U} \vee T_A(S_i) \xrightarrow{S_i} \pi_{\uparrow} \mathcal{U} \text{ in } \text{Hasse}([\pi_{\uparrow} \mathcal{U}, \pi_{\uparrow} \mathcal{T}]) \subseteq \text{Hasse}(\text{tors } A) \right\}_{i=1}^k \\ & \longleftrightarrow \left\{ \mathcal{U} \vee T_{A/I}(S_i) \xrightarrow{S_i} \mathcal{U} \text{ in } \text{Hasse}([\mathcal{U}, \mathcal{T}]) \subseteq \text{Hasse}(\text{tors } A/I) \right\}_{i=1}^k, \end{aligned} \quad (4.5.2)$$

which are labelled by the bricks $\{S_1, \dots, S_k\}$ generating $\mathcal{U}^{\perp_{A/I}} \cap \mathcal{T}$ by [57, Thm. 4.16].

It is easy to see that $T_A(\mathcal{W}) = T_A(S_1) \vee \dots \vee T_A(S_k)$, so that Proposition 4.3.3 gives the following τ -perpendicular interval by taking the join of all atoms:

$$[\mathcal{A}_{\mathcal{U}}, \mathcal{B}_{\mathcal{T}}] := [\pi_{\uparrow} \mathcal{U}, \pi_{\uparrow} \mathcal{U} \vee T(\mathcal{W})] \subseteq \text{tors } A.$$

By [57, Thm. 4.16] this interval satisfies $\mathcal{A}^{\perp_A} \cap \mathcal{B}_{\mathcal{T}} = \mathcal{W}$. Since both $\mathcal{W} \subseteq \pi_{\uparrow} \mathcal{T}$ and $\pi_{\uparrow} \mathcal{U} \subseteq \pi_{\uparrow} \mathcal{T}$, the interval satisfies $[\mathcal{A}_{\mathcal{U}}, \mathcal{B}_{\mathcal{T}}] \subseteq [\mathcal{U}, \mathcal{T}] \subseteq \text{tors } A/I$. By Corollary 4.4.3,

$$\overline{\mathcal{A}_{\mathcal{U}}}^{\perp_{A/I}} \cap \overline{\mathcal{B}_{\mathcal{T}}} = \text{Filt}_A\{S_1, \dots, S_k\} \cap \text{mod } A/I = \text{Filt}_{A/I}\{S_1, \dots, S_k\}.$$

Hence $\overline{\mathcal{A}_{\mathcal{U}}}^{\perp_{A/I}} \cap \overline{\mathcal{B}_{\mathcal{T}}} = \mathcal{U}^{\perp_{A/I}} \cap \mathcal{T}$ and by Lemma 4.2.1 there is a lattice isomorphism between $[\mathcal{A}_{\mathcal{U}}, \mathcal{B}_{\mathcal{T}}]$ and $[\mathcal{U}, \mathcal{T}]$. Since $[\mathcal{A}_{\mathcal{U}}, \mathcal{B}_{\mathcal{T}}] \subseteq [\mathcal{U}, \mathcal{T}]$ it follows that $[\mathcal{A}_{\mathcal{U}}, \mathcal{B}_{\mathcal{T}}] = [\mathcal{U}, \mathcal{T}]$. $\square$

Using Lemma 4.5.1, define a map of τ -perpendicular intervals:

$$\begin{aligned} \mathbf{i} : \tau\text{-itv}(\text{tors } A/I) & \rightarrow \tau\text{-itv}(\text{tors } A) \\ [\mathcal{U}, \mathcal{T}] & \mapsto \mathbf{i}[\mathcal{U}, \mathcal{T}] = [\pi_{\uparrow} \mathcal{U}, \pi_{\uparrow} \mathcal{U} \vee T(\mathcal{W})]. \end{aligned} \quad (4.5.3)$$

Example 4.5.2. The map of τ -perpendicular intervals $i : \tau\text{-itv}(\text{tors } A/I) \rightarrow \tau\text{-itv}(A)$ of Eq. (4.5.3) is not inclusion-preserving. For example, take the surjective algebra morphism $A \cong K(1 \rightarrow 2) \twoheadrightarrow K(1 \rightarrow 2) \cong K^2$. The inclusion $[\text{Fac}(1), \text{Fac}(1)] \subseteq [0, \text{Fac}(1)]$ of τ -perpendicular intervals of $\text{tors } K^2$ maps to $[\text{Fac}(\frac{1}{2}), \text{Fac}(\frac{1}{2})] \not\subseteq [0, \text{Fac}(1)]$.

Nonetheless, there is a way of resolving this problem by restricting the lattice congruence to the desired interval.

Proposition 4.5.3. *For every inclusion of τ -perpendicular intervals $[\mathcal{V}, \mathcal{S}] \subseteq [\mathcal{U}, \mathcal{T}]$ in $\text{tors } A/I$, there exists a τ -perpendicular interval $\bar{i}_{\mathcal{U}}^{\mathcal{T}}[\mathcal{V}, \mathcal{S}] \subseteq i[\mathcal{U}, \mathcal{T}]$ whose corresponding wide subcategory is $\iota(\mathcal{V}^{\perp_{A/I}} \cap \mathcal{S})$ and such that it satisfies $(\bar{i}_{\mathcal{U}}^{\mathcal{T}}[\mathcal{V}, \mathcal{S}]) \cap \text{mod } A/I = [\mathcal{V}, \mathcal{S}]$. These intervals are such that*

$$F_I([f_{(i[\mathcal{U}, \mathcal{T}])(\bar{i}_{\mathcal{U}}^{\mathcal{T}}[\mathcal{V}, \mathcal{S}])}]) = [f_{[\mathcal{U}, \mathcal{T}][\mathcal{V}, \mathcal{S}]}].$$

Thus, every morphism of $\mathfrak{T}(A/I)$ lies in the image of $F_I : \mathfrak{T}(A) \rightarrow \mathfrak{T}(A/I)$.

Proof. Let $\mathcal{U}, \mathcal{V}, \mathcal{S}, \mathcal{T} \in \text{tors } A/I$ have the property that $[\mathcal{V}, \mathcal{S}] \subseteq [\mathcal{U}, \mathcal{T}]$ is an inclusion of τ -perpendicular intervals of $\text{tors } A/I$. By definition of $\mathfrak{T}(A/I)$ every nonzero morphism in $\mathfrak{T}(A/I)$ is of the form $[f_{[\mathcal{U}, \mathcal{T}][\mathcal{V}, \mathcal{S}]}] \in \mathfrak{T}(A/I)$. Consider the representative $f_{[\mathcal{U}, \mathcal{T}][\mathcal{V}, \mathcal{S}]}$ of its equivalence class. Using the map i of Eq. (4.5.3) gives a τ -perpendicular interval $i[\mathcal{U}, \mathcal{T}]$ of $\text{tors } A$ such that $F_I((i[\mathcal{U}, \mathcal{T}])_{\sim}) = [\mathcal{U}, \mathcal{T}]_{\sim}$. However, the τ -perpendicular interval $i[\mathcal{V}, \mathcal{S}]$ may not be contained in $i[\mathcal{U}, \mathcal{T}]$, see Example 4.5.2.

Thus, consider the restriction of the lattice congruence $\Phi := \Phi_I$ to the interval lattice $i[\mathcal{U}, \mathcal{T}]$, and denote it by $\bar{\Phi}_{\mathcal{U}}^{\mathcal{T}}$. By the dual of [157, Lem. 9-5.7], the interval $\overline{i[\mathcal{U}, \mathcal{T}]} = [\mathcal{U}, \mathcal{T}]$ of $\text{tors } A/I$, which contains $[\mathcal{V}, \mathcal{S}]$, is isomorphic to the quotient lattice $i[\mathcal{U}, \mathcal{T}]/(\bar{\Phi}_{\mathcal{U}}^{\mathcal{T}})$. Let $\mathcal{W} := \iota(\mathcal{V}^{\perp_{A/I}} \cap \mathcal{S})$ and let $\{S_1, \dots, S_k\}$ be the relative simple modules of $\mathcal{V}^{\perp_{A/I}} \cap \mathcal{S}$, so that $\mathcal{W} = \text{Filt}_A\{S_1, \dots, S_k\}$ by Eq. (2.0.2).

Restricting Proposition 4.1.3 to $[\mathcal{U}, \mathcal{T}]$ yields that the preimage of $\mathcal{V}$ under $\bar{\Phi}_{\mathcal{U}}^{\mathcal{T}}$ is an interval of $i[\mathcal{U}, \mathcal{T}]$ which is denoted by $[(\bar{\pi}_{\mathcal{U}}^{\mathcal{T}})_{\downarrow}(\mathcal{V}), (\bar{\pi}_{\mathcal{U}}^{\mathcal{T}})_{\uparrow}(\mathcal{V})] \subseteq i[\mathcal{U}, \mathcal{T}]$. Now, let $\mathcal{V}' = (\bar{\pi}_{\mathcal{U}}^{\mathcal{T}})_{\uparrow}(\mathcal{V}) \in \text{tors } A$. Similar to the proof of Lemma 4.5.1, one applies the dual of [157, Prop. 9-5.10] and Proposition 4.2.2 to obtain a brick label preserving bijection

$$\begin{aligned} & \left\{ \mathcal{V}' \vee T_A(S_i) \xrightarrow{S_i} \mathcal{V}' \text{ in } \text{Hasse}((\bar{\pi}_{\mathcal{U}}^{\mathcal{T}})_{\uparrow}(\mathcal{V}), (\bar{\pi}_{\mathcal{U}}^{\mathcal{T}})_{\uparrow}(\mathcal{S})) \subseteq \text{Hasse}(i[\mathcal{U}, \mathcal{T}]) \right\}_{i=1}^k \\ & \longleftrightarrow \left\{ \mathcal{V} \vee T_{A/I}(S_i) \xrightarrow{S_i} \mathcal{V} \text{ in } \text{Hasse}([\mathcal{V}, \mathcal{S}]) \subseteq \text{Hasse}([\mathcal{U}, \mathcal{T}]) \right\}_{i=1}^k, \end{aligned} \quad (4.5.4)$$

between covering relations which are labelled by the bricks $\{S_1, \dots, S_k\}$ generating $\mathcal{V}^{\perp_{A/I}} \cap \mathcal{S}$ by [57, Thm. 4.16]. Since $T(\mathcal{W}_1) = \bigvee_{i=1}^k T(S_i)$ it follows from Proposition 4.3.3 that

$$\bar{i}_{\mathcal{U}}^{\mathcal{T}}[\mathcal{V}, \mathcal{S}] := [(\bar{\pi}_{\mathcal{U}}^{\mathcal{T}})_{\uparrow}(\mathcal{V}), (\bar{\pi}_{\mathcal{U}}^{\mathcal{T}})_{\uparrow}(\mathcal{V}) \vee T(\mathcal{W})] \quad (4.5.5)$$

is a τ -perpendicular interval of tors A such that the corresponding wide subcategory is $\mathcal{W}$ by [57, Thm. 4.16]. Write $\mathcal{S}' = (\bar{\pi}_{\mathcal{U}}^{\mathcal{T}})_{\uparrow}(\mathcal{S})$. Because $(\bar{\Phi}_{\mathcal{U}}^{\mathcal{T}})^{-1}(\mathcal{V}) \subseteq \Phi^{-1}(\mathcal{V})$ and $(\bar{\Phi}_{\mathcal{U}}^{\mathcal{T}})^{-1}(\mathcal{S}) \subseteq \Phi^{-1}(\mathcal{S})$, it follows that $\overline{\mathcal{V}'} = \mathcal{V}$ and $\overline{\mathcal{S}'} = \mathcal{S}$. Moreover, it follows from Corollary 4.4.3 that therefore $(\overline{\mathcal{V}'})^{\perp_A} \cap \mathcal{S}' \cap \text{mod } A/I = \mathcal{V}^{\perp_{A/I}} \cap \mathcal{S}$. Now the argument following Eq. (4.5.1) in the proof of Lemma 4.5.1 applies to give $\mathcal{W} \subseteq \mathcal{S}'$. Since furthermore $\mathcal{V}' \subseteq \mathcal{S}'$, it follows that $(\bar{i}_{\mathcal{U}}^{\mathcal{T}}[\mathcal{V}, \mathcal{S}]) \cap \text{mod } A/I \subseteq [\mathcal{V}, \mathcal{S}]$.

Again, analogous to the proof of Lemma 4.5.1, it follows from Lemma 4.2.1 and Corollary 4.4.3 that actually $(\bar{i}_{\mathcal{U}}^{\mathcal{T}}[\mathcal{V}, \mathcal{S}]) \cap \text{mod } A/I = [\mathcal{V}, \mathcal{S}]$ because there exists a lattice isomorphism between the two. To complete the proof it is left to show that

$$\mathcal{V}' \vee T(\mathcal{W}) \subseteq \mathcal{V}' \vee T(\iota(\mathcal{U}^{\perp_{A/I}} \cap \mathcal{T})) \subseteq \pi_{\uparrow} \mathcal{U} \vee T(\iota(\mathcal{U}^{\perp_{A/I}} \cap \mathcal{T})).$$

Indeed, the first inclusion holds since the inclusion $\mathcal{V}^{\perp_{A/I}} \cap \mathcal{S} \subseteq \mathcal{U}^{\perp_{A/I}} \cap \mathcal{T}$ and the inclusion $T(\mathcal{W}) \subseteq T(\mathcal{U}^{\perp_{A/I}} \cap \mathcal{S})$ in $\text{mod } A/I$ lift to inclusions in $\text{mod } A$. The second inclusion follows from the same observation and the fact that $\mathcal{V}' \subseteq \pi_{\uparrow} \mathcal{U} \vee T(\iota(\mathcal{U}^{\perp_{A/I}} \cap \mathcal{T}))$ by construction.

In conclusion, given an inclusion of τ -perpendicular intervals $[\mathcal{V}, \mathcal{S}] \subseteq [\mathcal{U}, \mathcal{T}]$ in tors A/I , there exists an inclusion of τ -perpendicular intervals

$$\bar{i}_{\mathcal{U}}^{\mathcal{T}}[\mathcal{V}, \mathcal{S}] \subseteq i[\mathcal{U}, \mathcal{T}]$$

such that $((\bar{i}_{\mathcal{U}}^{\mathcal{T}}[\mathcal{V}, \mathcal{S}]) \cap \text{mod } A/I = [\mathcal{V}, \mathcal{S}]$ and $i[\mathcal{U}, \mathcal{T}] \cap \text{mod } A/I = [\mathcal{U}, \mathcal{T}]$. Consequently, $F_I((\bar{i}_{\mathcal{U}}^{\mathcal{T}}[\mathcal{V}, \mathcal{S}])_{\sim}) = [\mathcal{V}, \mathcal{S}]_{\sim}$ and $F_I((i[\mathcal{U}, \mathcal{T}])_{\sim}) = [\mathcal{U}, \mathcal{T}]_{\sim}$. Therefore, it is possible to obtain any morphism $[f_{[\mathcal{U}, \mathcal{T}][\mathcal{V}, \mathcal{S}]}] \in \mathfrak{T}(A/I)$ by applying F_I to $[f_{(i[\mathcal{U}, \mathcal{T}])(\bar{i}_{\mathcal{U}}^{\mathcal{T}}[\mathcal{V}, \mathcal{S}])}]$. $\square$

Recall that an epimorphism e in a category $\mathcal{C}$ is called *extremal* if whenever one can write $e = m \circ f$, with m a monomorphism, then m is an isomorphism. Let Cat denote the category of small categories.

Corollary 4.5.4. *The smallest subcategory of $\mathfrak{T}(A/I)$ containing the image of F_I is $\mathfrak{T}(A/I)$ itself. Thus F_I is an extremal epimorphism in the category Cat .*

Proof. It follows immediately from Proposition 4.5.3 that the image of F_I is $\mathfrak{T}(A/I)$. Thus, the functor F_I of Theorem 4.4.8 is an extremal epimorphism in $\mathcal{Cat}$ by [23, Thm. 3.4]. $\square$

Moreover, a functor $G : \mathcal{A} \rightarrow \mathcal{B}$ is said to *reflect composition* if given two morphisms f and g in $\mathcal{A}$ such that $G(f) \circ G(g)$ is defined in $\mathcal{B}$, there exist morphisms f' and g' in $\mathcal{A}$ such that $G(f) = G(f')$, $G(g) = G(g')$ and $f' \circ g'$ is defined in $\mathcal{A}$.

Lemma 4.5.5. *The functor $F_I : \mathfrak{T}(A) \rightarrow \mathfrak{T}(A/I)$ reflects composition. Moreover, if $[f_{[\mathcal{V}, \mathcal{S}][\mathcal{X}, \mathcal{Y}]}] \circ [f_{[\mathcal{U}, \mathcal{T}][\mathcal{V}', \mathcal{S}']}]$ is defined in $\mathfrak{T}(A/I)$ then the composition*

$$[f_{(i[\mathcal{V}, \mathcal{S}])(\tilde{i}_{\mathcal{V}}^{\mathcal{S}}[\mathcal{X}, \mathcal{Y}])}] \circ [f_{(i[\mathcal{U}, \mathcal{T}])(\tilde{i}_{\mathcal{U}}^{\mathcal{T}}[\mathcal{V}', \mathcal{S}'])}]$$

is defined in $\mathfrak{T}(A)$, where the maps of τ -perpendicular intervals are as in Eq. (4.5.3) and Eq. (4.5.5).

Proof. By Lemma 4.2.6, assume that $\mathcal{V} = \mathcal{V}'$ and $\mathcal{S} = \mathcal{S}'$ without loss of generality. From Lemma 4.5.1 it follows that the interval $i[\mathcal{V}, \mathcal{S}] \subseteq \text{tors } A$ corresponds to the wide subcategory $\iota(\mathcal{V}^{\perp A/I} \cap \mathcal{S}) \in \text{wide } A$. On the other hand, Proposition 4.5.3 implies that the interval $\tilde{i}_{\mathcal{U}}^{\mathcal{T}}[\mathcal{V}, \mathcal{S}]$ corresponds to the wide subcategory $\iota(\mathcal{V}^{\perp A/I} \cap \mathcal{S})$ as well. Therefore $(\tilde{i}_{\mathcal{U}}^{\mathcal{T}}[\mathcal{V}, \mathcal{S}])_{\sim} = (i[\mathcal{V}, \mathcal{S}])_{\sim}$ in $\mathfrak{T}(A)$. It follows from Lemma 4.2.6 that the composition is defined in $\mathfrak{T}(A)$. It is clear from Proposition 4.5.3 that the functor F_I sends this composition in $\mathfrak{T}(A)$ to the desired one in $\mathfrak{T}(A/I)$. $\square$

Corollary 4.5.6. *The functor $F_I : \mathfrak{T}(A) \rightarrow \mathfrak{T}(A/I)$ is a regular epimorphism in $\mathcal{Cat}$, that is, it is the coequaliser of a pair of morphisms in $\mathcal{Cat}$.*

Proof. The result [23, Prop. 5.1] states that an extremal epimorphism in $\mathcal{Cat}$ which reflects composition is regular. Therefore, the result follows from Corollary 4.5.4 and Lemma 4.5.5. $\square$

Example 4.5.7. Continuing with the surjective algebra homomorphism of Example 4.5.2, one sees that the assignment from morphisms of $\mathfrak{T}(A/I)$ to morphisms of $\mathfrak{T}(A)$ given by

$$[f_{[\mathcal{U}, \mathcal{T}][\mathcal{V}, \mathcal{S}]}] \mapsto [f_{(i[\mathcal{U}, \mathcal{T}])(\tilde{i}_{\mathcal{U}}^{\mathcal{T}}[\mathcal{V}, \mathcal{S}])}] \quad (4.5.6)$$

is not well-defined on composition. Consider the morphism $[f_{[0, \text{mod } A/I][\text{Fac}(1), \text{Fac}(1)]}]$ of $\mathfrak{T}(A/I)$ which may be decomposed as

$$[f_{[0, \text{Fac}(1)][\text{Fac}(1), \text{Fac}(1)]}] \circ [f_{[0, \text{mod } A][0, \text{Fac}(1)]}] = [f_{[0, \text{mod } A][\text{Fac}(1), \text{Fac}(1)]}].$$

However, applying Eq. (4.5.6) componentwise to the left-hand side yields the composition

$$[f_{[0, \text{Fac}(1)][\text{Fac}(1), \text{Fac}(1)]}] \circ [f_{[0, \text{mod } A][0, \text{Fac}(1)]}],$$

which composes to $[f_{[0, \text{mod } A][\text{Fac}(1), \text{Fac}(1)]}]$, whereas applying Eq. (4.5.6) to the right-hand side gives $[f_{[0, \text{mod } A][\text{Fac}(\frac{1}{2}), \text{Fac}(\frac{1}{2})]}]$.

4.6 Classifying spaces and picture groups

Having established a certain quotient relationship between $\mathfrak{T}(A)$ and $\mathfrak{T}(A/I)$ in Section 4.5, this section considers the classifying space and picture group. This section demonstrates that they exhibit similar quotient relationships.

Proposition 4.6.1. *Let A be τ -tilting finite and $I \in \text{ideal } A$, then the classifying space $\mathcal{B}\mathfrak{T}(A/I)$ is a quotient space of $\mathcal{B}\mathfrak{T}(A)$ and the quotient map is induced by F_I from Theorem 4.4.8.*

Proof. The τ -cluster morphism category $\mathfrak{T}(A)$ is a cubical category by Theorem 3.2.6. This means that every morphism can be seen as (the diagonal of) a cube, whose edges correspond to factorisations of the morphism into irreducible ones. It is shown in Theorem 3.3.4 that the classifying space $\mathcal{B}\mathfrak{T}(A)$ of the τ -cluster morphism category is a CW-complex with one k -cell $e([\mathcal{U}, \mathcal{T}]_{\sim})$ for each equivalence class of wide intervals, where k is the number of isomorphism classes of (relative) simple modules of $\mathcal{U}^{\perp} \cap \mathcal{T}$. This cell is the union of factorisation cubes of morphisms $[f_{[\mathcal{U}, \mathcal{T}][\mathcal{X}, \mathcal{X}]}]$ for $\mathcal{X} \in [\mathcal{U}, \mathcal{T}]$. Given an ideal $I \in \text{ideal } A$, define an equivalence $\asymp$ relation on factorisation cubes by setting

$$[f_{[\mathcal{U}, \mathcal{T}][\mathcal{X}, \mathcal{X}]}] \asymp [f_{[\mathcal{U}', \mathcal{T}'][\mathcal{X}', \mathcal{X}']}]$$

whenever F_I applied to these morphisms coincides. Since factorisation cubes are simply geometric realisations of morphisms, this identification coincides with the generalised

congruence $\asymp_I$ on $\mathfrak{T}(A)$ induced by the functor F_I , as defined in [23, Sec. 3]. By [23, Cor. 3.11] there is a monomorphism $(\mathfrak{T}(A)/\asymp) \rightarrow \mathfrak{T}(A/I)$ in $\mathcal{Cat}$ from the quotient category $\mathfrak{T}(A)/\asymp$ as defined in [23, Sec. 3.9]. Since F is an extremal epimorphism by Corollary 4.5.4, it follows that $(\mathfrak{T}(A)/\asymp) \cong \mathfrak{T}(A/I)$. Clearly, $\mathcal{B}(\mathfrak{T}(A)/\asymp)$ is a quotient space of $\mathcal{B}\mathfrak{T}(A)$ and the result follows. $\square$

Likewise, the picture groups $G(A)$ and $G(A/I)$ are related by a surjective group homomorphism, when the underlying algebras are τ -tilting finite.

Proposition 4.6.2. *Let A be τ -tilting finite and $I \in \text{ideal } A$. Then there is a surjective group homomorphism $G(A) \rightarrow G(A/I)$ induced by $\overline{(-)} : \text{tors } A \rightarrow \text{tors } A/I$.*

Proof. Define a map $\phi : G(A) \rightarrow G(A/I)$ given by

$$X_S \mapsto \begin{cases} X_S & \text{if } S \in \text{brick}(A/I), \\ e & \text{otherwise,} \end{cases} \quad \text{and} \quad g_{\mathcal{T}} \mapsto g_{\overline{\mathcal{T}}}.$$

Since $\text{mod } A/I$ is a full subcategory of $\text{mod } A$, identify $\text{brick}(A/I)$ with the subset $\{S \in \text{brick } A : IS = 0\} \subseteq \text{brick } A$, similar to [57, Sec. 5.2]. Thus, ϕ obviously induces a surjection on the generators $X_S \in G(A/I)$. Similarly, by Theorem 4.1.2, the map $\overline{(-)} : \text{tors } A \rightarrow \text{tors } A/I$ is surjective and hence ϕ is surjective on the generators $g_{\overline{\mathcal{T}}}$. To show that the group relations $g_{\mathcal{T}_1} = X_S g_{\mathcal{T}_2}$ are preserved distinguish between two cases: If $S \in \text{brick}(A/I)$, the corresponding arrow of $\text{Hasse}(\text{tors } A)$ is not contracted, by Theorem 4.4.1. Then, the group relation becomes

$$\phi(g_{\mathcal{T}_1}) = g_{\overline{\mathcal{T}_1}} = X_S g_{\overline{\mathcal{T}_2}} = \phi(X_S) \phi(g_{\mathcal{T}_2}).$$

If $S \notin \text{brick}(A/I)$, then $\phi(X_S) = e$ and the corresponding arrow of $\text{Hasse}(\text{tors } A)$ is contracted, so $\overline{\mathcal{T}_1} = \overline{\mathcal{T}_2}$ and the group relation becomes

$$\phi(g_{\mathcal{T}_1}) = g_{\overline{\mathcal{T}_1}} = g_{\overline{\mathcal{T}_2}} = e g_{\overline{\mathcal{T}_2}} = \phi(X_S) \phi(g_{\mathcal{T}_2}).$$

Hence ϕ is a well-defined group homomorphism and surjective. $\square$

4.7 Examples

The final section of this chapter illustrates the obtained results using some examples. In particular, the following example illustrates the different realisations $\mathfrak{W}(A)$, $\mathfrak{C}(A)$ and

$\mathfrak{T}(A)$ of the τ -cluster morphism category.

Example 4.7.1. Let $A := K(1 \xrightarrow{a} 2)$ and consider $K^2 \cong A/\langle a \rangle \cong K(1 \rightarrow 2)$. The objects of the different realisations of the τ -cluster morphism categories are displayed in Table 4.1.

$\mathfrak{W}(A)$	$\mathfrak{C}(A)$	$\mathfrak{T}(A)$
$\text{mod } A$	$[\mathcal{C}_{(0,0)}]$	$[0, \text{mod } A]_{\sim}$
$\text{Filt}\{1\}$	$[\mathcal{C}_{(2,0)}] = [\mathcal{C}_{(0,2)}]$	$[0, \text{Fac}(1)]_{\sim} = [\text{Fac}(2), \text{mod } A]_{\sim}$
$\text{Filt}\{2\}$	$[\mathcal{C}_{(\frac{1}{2},0)}] = [\mathcal{C}_{(0,\frac{1}{2})}]$	$[0, \text{Fac}(2)]_{\sim} = [\text{Fac}(\frac{1}{2}), \text{mod } A]_{\sim}$
$\text{Filt}\{\frac{1}{2}\}$	$[\mathcal{C}_{(1,0)}]$	$[\text{Fac}(1), \text{Fac}(\frac{1}{2})]_{\sim}$
0	$[\mathcal{C}_{(M,P)}]$	$[\mathcal{T}, \mathcal{T}]_{\sim}$
	$(M, P) \in \tau\text{-tilt } A$	$\mathcal{T} \in \text{tors } A$

Table 4.1: Objects of the τ -cluster morphism categories of A

Recall that the correspondence between the objects of the different realisations of the τ -cluster morphism categories are given as follows:

- (1) $\mathfrak{C}(A) \ni [\mathcal{C}_{(M,P)}] \mapsto M^{\perp} \cap {}^{\perp}\tau M \cap P^{\perp} \in \mathfrak{W}(A)$.
- (2) $\mathfrak{T}(A) \ni [\mathcal{U}, \mathcal{T}]_{\sim} \mapsto \mathcal{U}^{\perp} \cap \mathcal{T} \in \mathfrak{W}(A)$.
- (3) $\mathfrak{C}(A) \ni [\mathcal{C}_{(M,P)}] \mapsto [\text{Fac } M, {}^{\perp}\tau M \cap P^{\perp}]_{\sim} \in \mathfrak{T}(A)$.

These translations are used to describe the image of the functor $F_I : \mathfrak{T}(A) \rightarrow \mathfrak{T}(A/I)$ in $\mathfrak{W}(A)$ and $\mathfrak{C}(A)$ in Table 4.2. Each entry in Table 4.2 represents the image under $F_{\langle a \rangle} : A \rightarrow K^2$ of the entry in the same position of Table 4.1. Moreover, the image of the morphism $[f_{[0, \text{mod } A][\text{Fac}(1), \text{Fac}(\frac{1}{2})]}] \in \mathfrak{T}(A)$ under F_I is displayed in Table 4.3. This example suggests that describing the functor $F_I : \mathfrak{T}(A) \rightarrow \mathfrak{T}(A/I)$ is most natural in the lattice theoretic realisation $\mathfrak{T}(A)$ of the τ -cluster morphism category.

The assumption of τ -tilting finiteness in Lemma 4.4.11 is justified by the following.

$\mathfrak{W}(K^2)$	$\mathfrak{C}(K^2)$	$\mathfrak{T}(K^2)$
$\text{mod } K^2$	$[\mathcal{C}_{(0,0)}]$	$[0, \text{mod } K^2]_{\sim}$
$\text{Filt}\{ \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \}$	$[\mathcal{C}_{(2,0)}] = [\mathcal{C}_{(0,2)}]$	$[0, \text{Fac}(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix})]_{\sim} = [\text{Fac}(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}), \text{mod } K^2]_{\sim}$
$\text{Filt}\{ \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \}$	$[\mathcal{C}_{(1,0)}] = [\mathcal{C}_{(0,1)}]$	$[0, \text{Fac}(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix})]_{\sim} = [\text{Fac}(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}), \text{mod } K^2]_{\sim}$
0	$[\mathcal{C}_{(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}, 2)}]$	$[\text{Fac}(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}), \text{Fac}(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix})]_{\sim}$
0	$[\mathcal{C}_{(M,P)}]$	$[\mathcal{T}, \mathcal{T}]_{\sim}$
	$(M, P) \in \tau\text{-tilt } K^2$	$\mathcal{T} \in \text{tors } K^2$

Table 4.2: The corresponding images under F_I of objects in Table 4.1.

category / algebra	A	K^2
$\mathfrak{W}(-)$	$[(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}, 0)] : \text{mod } A \rightarrow \text{Filt}\{\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\}$	$[(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}, 2)] : \text{mod } K^2 \rightarrow 0$
$\mathfrak{C}(-)$	$[f_{\mathcal{C}_{(0,0)}\mathcal{C}_{(1,0)}}]$	$[f_{\mathcal{C}_{(0,0)}\mathcal{C}_{(1,2)}}]$
$\mathfrak{T}(-)$	$[f_{[0, \text{mod } A][\text{Fac}(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}), \text{Fac}(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix})]}]$	$[f_{[0, \text{mod } K^2][\text{Fac}(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}), \text{Fac}(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix})]}]$

Table 4.3: The image of a morphism under F_I in the different presentations of the τ -cluster morphism category

Example 4.7.2. Let $A := K(\begin{smallmatrix} 1 & \xrightarrow{b} & 2 \end{smallmatrix})$ and $B := A/\langle b \rangle \cong K(\begin{smallmatrix} 1 & \longrightarrow & 2 \end{smallmatrix})$ then A is well-known to be τ -tilting infinite and B to be τ -tilting finite. Consider the interval $[\text{Fac}_B(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}), \text{Fac}_B(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix})] \subseteq \text{tors } B$, which is a τ -perpendicular interval and gives rise to the τ -perpendicular subcategory $\text{Filt}_B\{\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\} \subseteq \text{mod } B$.

The preimage $[\pi_{\downarrow} \text{Fac}_B(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}), \pi_{\uparrow} \text{Fac}_B(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix})] \subseteq \text{tors } A$ of this interval under the surjection $\overline{(-)} : \text{tors } A \rightarrow \text{tors } B$ is $[\text{Fac}_A(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}), \text{Fac}_A(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix})] \subseteq \text{tors } A$ which is not τ -perpendicular.

Moreover the τ -perpendicular intervals contained in this preimage come in four families of the forms

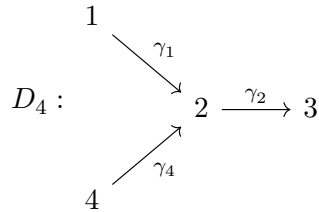
$$\begin{aligned} & [\text{Fac}_A(\tau_A^{-m}(\frac{1}{22})), \text{Fac}_A(\tau_A^{-(m+1)}(2))], \quad [\text{Fac}_A(\tau_A^{-(m+1)}(2)), \text{Fac}_A(\tau_A^{-(m+1)}(\frac{1}{22}))], \\ & [\text{Fac}_A(\tau_A^m(1)), \text{Fac}_A(\tau_A^m(\frac{11}{2}))], \quad [\text{Fac}_A(\tau_A^m(\frac{11}{2})), \text{Fac}_A(\tau_A^{m+1}(2))] \end{aligned}$$

for $m \geq 0$, where $\tau^{-m} := \tau^{-1} \circ \dots \circ \tau^{-1}$ is a composition of m terms. Importantly, the image of the intervals under $\overline{(-)} : \text{tors } A \rightarrow \text{tors } B$ in the top row is the trivial interval $[\text{Fac}_B(\frac{1}{2}), \text{Fac}_B(\frac{1}{2})]$ and the image of the intervals in the bottom row is the trivial interval $[\text{Fac}_B(1), \text{Fac}_B(1)]$. Hence no τ -perpendicular interval of $\text{tors } A$ maps onto the interval $[\text{Fac}_B(1), \text{Fac}_B(\frac{1}{2})] \subseteq \text{tors } B$. In particular, the module $\frac{1}{2}$ is τ -rigid in $\text{mod } B$ but not in $\text{mod } A$, where it is a regular module.

The example [57, Exmp. 5.11] shows that the map $\overline{(-)} : \text{f-tors } A \rightarrow \text{f-tors } A/I$ may also not be a surjection when both A and A/I are τ -tilting infinite.

Recall from Proposition 2.4.4 that besides the existence of a faithful group functor, the pairwise compatibility condition of last factors of the τ -cluster morphism category is a sufficient condition for $\mathcal{BT}(A/I)$ to be a $K(\pi, 1)$ space. The following example illustrates that this pairwise compatibility condition is independent of taking quotient algebras.

Example 4.7.3. Let $A = KD_4$ be the representation finite hereditary algebra of Dynkin type D_4 with orientation



It is shown in [106, Thm. 2.5] that $\mathfrak{T}(A)$ satisfies the pairwise compatibility property since A is hereditary of finite type. The quotient $A/\langle \gamma_4 \gamma_2 \rangle$ is gentle with no loops and 2-cycles, thus [92, Thm. 4.1] implies that A/I does not satisfy the pairwise compatibility condition since there exists a vertex of valency greater than 2. Hence taking quotients does not preserve the pairwise compatibility condition. Moreover, A is a quotient of the preprojective algebra Π_{D_4} of type D_4 which is shown in [20, Thm. 4] to not satisfy the

pairwise compatibility condition of last factors, so taking quotient also does not preserve the failure of the pairwise compatibility condition. In conclusion, there is a sequence of surjective algebra morphisms:

$$\Pi_{D_4} \twoheadrightarrow KD_4 \twoheadrightarrow KD_4/\langle \gamma_4\gamma_2 \rangle \twoheadrightarrow K^4$$

where K^4 is the semisimple algebra on 4 vertices, which does satisfy the pairwise compatibility property.

By Theorem 4.3.7, the purely combinatorial definition of $\mathfrak{T}(A)$ yields that when $\text{f-tors } A$ is finite, it determines $\mathfrak{T}(A)$ up to equivalence. This reduces the problem of obtaining faithful group functors for some classes of algebras. For example, by [1, Thm. 1.1(1)], two Brauer graph algebras with the same underlying ribbon graph are τ -tilting equivalent. Thus, if a Brauer graph algebra is τ -tilting finite and its τ -cluster morphism category admits a faithful group functor, the same holds for all other Brauer graph algebras defined by the same ribbon graph.

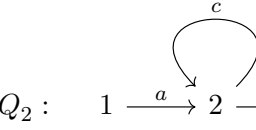
Similarly, if A is a τ -tilting finite K -algebra such that $\mathfrak{T}(A)$ admits a faithful group functor, consider $A^{(r)} := A \otimes_K K[\epsilon]/(\epsilon^r)$ for some $r \geq 2$. Then since $1 \otimes \epsilon$, is a central element contained in the Jacobson radical, it follows that $\text{f-tors } A \cong \text{f-tors } A^{(r)}$. The special case, where A is hereditary and $r = 2$, has been shown to be of special interest in [165]. When K is algebraically closed, a similar statement holds when replacing $K[\epsilon]/(\epsilon^r)$ with a finite-dimensional local commutative K -algebra R , see also [149]. Thus every known example of a finite-dimensional algebra such that $\mathfrak{T}(A)$ admits a faithful group functor gives rise to an infinite family of finite-dimensional algebras with the same property.

Moreover, the following example displays two algebras with similar properties, but which are not connected via a relationship as described above.

Example 4.7.4. Consider the path K -algebras $A \cong KQ_1/I_1$ and $A' \cong KQ_2/I_2$ defined by

$$Q_1 : \quad \begin{array}{c} \text{c} \\ \curvearrowright \\ 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \end{array} \quad \begin{array}{c} \text{d} \\ \curvearrowright \\ 3 \end{array} \quad \text{and} \quad I_1 = \langle c^2, d^2, cab - abd \rangle$$

and

$$Q_2 : \quad 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \quad \text{and} \quad I_2 = \langle ab, c^2 \rangle.$$


It can be checked, using, for example, the applet [81], that there is a lattice isomorphism $\text{tors } A \cong \text{tors } A'$. Therefore, there is an equivalence of categories $\mathfrak{T}(A) \cong \mathfrak{T}(A')$ by Corollary 4.3.8. Thus, if one of the algebras admits a faithful group functor, so does the other. Furthermore, the signed τ -exceptional sequences of $\text{mod } A$ and $\text{mod } A'$ are in bijection by Corollary 4.3.9 and it can be verified that the mutation of τ -exceptional sequences coincides for these algebras, see Remark 4.3.10.

Finally, let K and K' be two different fields, possibly of different characteristic. Let A be a finite-dimensional K -algebra and let B be a finite-dimensional K' -algebra. One may have that $\text{tors } A \cong L \cong \text{tors } B$, for some finite lattice L . Then, if $\mathfrak{T}(A)$ admits a faithful group functor, so does $\mathfrak{T}(L)$ and hence $\mathfrak{T}(B)$, by Theorem 4.3.7. It follows that the result [92, Thm. 5.9] may be extended beyond finite fields in some cases. In the following final chapter, the case where $K' : K$ is a field extension is considered.

Chapter 5

τ -cluster morphism categories and base field extension

In this chapter, the lattice theoretic approach developed in Chapter 4, is applied to study how the τ -cluster morphism category behaves under base field extension. Let $L : K$ be a field extension and A a finite-dimensional K -algebra, then the tensor product $A \otimes_K L$ is a finite-dimensional L -algebra. The main result of this section, Theorem 5.5.1, is the construction of a faithful functor $\mathcal{F} : \mathfrak{W}(A) \rightarrow \mathfrak{W}(A \otimes_K L)$. As a consequence of this result, the τ -cluster morphism category is shown to admit a faithful group functor for a new family of algebras in Section 5.6. The following Section 5.1 introduces different types of fields and field extensions used commonly throughout this chapter. Subsequently, Section 5.2 investigates τ -tilting theory under base field extension in a general context and Section 5.3 dually studies τ^{-1} -tilting theory in order to describe both Bongartz and co-Bongartz completions under base field extension. These two sections are combined in Section 5.4 to investigate τ -perpendicular intervals in this setting, which leads to the proof of Theorem 5.5.1 in Section 5.5.

5.1 Preliminaries on field extensions

Let A be a finite-dimensional algebra and let $L : K$ be a field extension. Denote by A_L the finite-dimensional L -algebra $A \otimes_K L$. Let $\overline{K}$ denote an algebraic closure of the base

field K . Moreover, the field extension $L : K$ defines a *scalar extension* functor

$$\begin{aligned} - \otimes_K L &: \text{mod } A \rightarrow \text{mod } A_L \\ M &\mapsto M_L := M \otimes_K L \end{aligned}$$

which is faithful and preserves and reflects exact sequences since L is a faithfully flat K -module. The following lemma collects essential properties of the scalar extension functor $- \otimes_K L$.

Lemma 5.1.1. *Let $L : K$ be a field extension and let $M, N \in \text{mod } A$ and $P \in \text{proj } A$.*

- (1) *Let M be indecomposable. Then, M is a direct summand of N if and only if M_L and N_L have a common nonzero direct summand.*
- (2) *Let M and N be indecomposable. Then M_L and N_L share a common nonzero direct summand if and only if $M \cong N$.*
- (3) *$M \cong N$ if and only if $M_L \cong N_L$. In other words, $- \otimes_K L$ is injective-on-objects.*
- (4) *Let M and N be basic. If $\text{add}(M_L) = \text{add}(N_L)$, then $M \cong N$.*
- (5) *$\bigcup_{X \in \text{add}(M) \cap \text{add}(N)} \text{add}(X_L) = \{X : X \in \text{add}(M_L) \cap \text{add}(N_L)\}$.*
- (6) *$P_L \in \text{proj } A_L$. Moreover, each projective A_L -module arises as a direct summand of $(P')_L$ for some $P' \in \text{proj } A$.*

Proof. (1)-(3) are [122, Lem. 2.5], where (2) is also known as the Noether-Deuring Theorem, see [130, Thm. 19.25].

(4) Let X be an indecomposable direct summand of M . From $\text{add}(M_L) = \text{add}(N_L)$ it follows that X_L is a direct summand of $(N_L)^r$ for some $r \geq 1$. It follows from (1) that X is a direct summand of N^r , and since X is indecomposable X is a direct summand of N . Repeating this for all indecomposable direct summands of M yields that every indecomposable direct summand of M is also direct summand of N . Reversing the argument similarly gives every indecomposable direct summand of N as a direct summand of M . It follows that $\text{add}(M) = \text{add}(N)$. Since M and N are basic, $M \cong N$ follows.

(5) To show the inclusion $(\subseteq)$, let $X \in \text{add}(M) \cap \text{add}(N)$ be indecomposable. Then X is a direct summand of M and a direct summand of N . It follows from (1) that X_L is a direct summand of M_L and of N_L , hence $X_L \in \text{add}(M_L) \cap \text{add}(N_L)$. Thus

$$\text{add}(X_L) \subseteq \text{add}(M_L) \cap \text{add}(N_L).$$

Conversely, to show the inclusion ($\supseteq$), let $Y \in \text{add}(M_L) \cap \text{add}(N_L)$ be indecomposable. Then $Y \in \text{add}(Z_L) \subseteq \text{add}(M_L)$ and $Y \in \text{add}(Z'_L) \subseteq \text{add}(N_L)$, for some indecomposable modules $Z, Z' \in \text{mod } A$. From (1) it follows that $Z \in \text{add}(M)$ and $Z' \in \text{add}(M)$, and moreover it follows from (2) that $Z \cong Z'$. In conclusion, $Y \in \text{add}(Z_L)$, where $Z \in \text{add}(M) \cap \text{add}(N)$ as required.

(6) is [122, Lem. 2.1]. □

Crucially, the homomorphisms and extensions between two modules behave well under base field extension.

Lemma 5.1.2. [122, Lem. 2.2] *Let $L : K$ be a field extension. Let $X, Y \in \text{mod } A$ and let $i \in \mathbb{Z}_{\geq 0}$. Then the canonical homomorphism of L -vector spaces*

$$\text{Ext}_A^i(X, Y) \otimes_K L \rightarrow \text{Ext}_{A_L}^i(X_L, Y_L)$$

is an isomorphism which is natural in both arguments, where $\text{Ext}_A^0(-, ?) = \text{Hom}_A(-, ?)$.

The remainder of this section introduces different types of fields and field extensions.

Definition 5.1.3. A field K is *perfect* if every algebraic field extension of K is separable.

For example, by [113, Thm. IV.3], all fields of characteristic zero and all finite fields are perfect. Moreover, the following type of field extension is of great importance.

Definition 5.1.4. Let $L : K$ be a field extension and let $L_1 : K$ and $L_2 : K$ be intermediate field extensions of $L : K$. The field extensions $L_1 : K$ and $L_2 : K$ are called *linearly disjoint over K in L* if the K -homomorphism

$$\begin{array}{ccc} L_1 \otimes_K L_2 & \xrightarrow{\mu} & L_1 L_2 \quad \subseteq \quad L \\ \downarrow & & \downarrow \\ \lambda_1 \otimes \lambda_2 & \longmapsto & \lambda_1 \lambda_2 \end{array}$$

is an isomorphism of K -algebras, where $L_1 L_2$ is the smallest subalgebra of L containing L_1 and L_2 . A field extension $L : K$ is said to be *MacLane separable* if either $\text{char}(K) = 0$ or $\text{char}(K) = p \neq 0$ and the field extensions L and $K^{p^{-1}}$ are linearly disjoint over L in an algebraic closure $\bar{L}$. Here $K^{p^{-1}}$ denotes the subset $\{\lambda \in \bar{L} : \lambda^p \in K\}$.

For example, an algebraic field extension is separable if and only if it is MacLane separable by [113, Thm. IV.9]. And generally every separable field extension is MacLane separable [113, p. 163, MacLane's Criterion]. By the same result, a finite field extension is MacLane separable if and only if it is separable. An equivalent characterisation for $L : K$ to be MacLane separable was given in [116, Thm. 2.4], which states that $L : K$ is MacLane separable if and only if $\text{gl. dim } A = \text{gl. dim } A_L$ holds for any finite-dimensional K -algebra A .

Example 5.1.5. [113, Exercise IV.5.1] Let K be a field of characteristic $p \neq 0$ and let $L = K(\xi, \xi^{p^{-1}}, \xi^{p^{-2}}, \dots)$ where ξ is transcendental over K . Then $L : K$ is MacLane separable but not separable.

MacLane separable field extensions interact particularly well with homological algebra as can be seen by the results of [116]. Further properties in relation to the radical and the socle of a module are summarised in the following lemma.

Lemma 5.1.6. [122, Lem. 3.3, 3.5] *Let $L : K$ be a field extension and $M \in \text{mod } A$.*

- (1) *There is an inclusion $(\text{rad } A)_L \subseteq \text{rad } A_L$.*
- (2) *There is an inclusion $(\text{rad}_A M)_L \subseteq \text{rad}_{A_L} M_L$.*
- (3) *There is an inclusion $(\text{soc}_A M)_L \subseteq \text{rad}_{A_L} M_L$.*

Moreover, if $L : K$ is MacLane separable, then all of the above are equalities.

5.2 τ -tilting theory under base field extension

Let $L : K$ be a field extension. Since $\text{mod } A$ and $\text{mod } A_L$ are categories of modules of different algebras, they have their respective Auslander-Reiten translations, denoted by τ_A and τ_{A_L} . As a first step in establishing a connection between the two, consider the following result concerning the Nakayama functor $\nu_A : \text{mod } A \rightarrow \text{mod } A$ in relation to $\nu_{A_L} : \text{mod } A_L \rightarrow \text{mod } A_L$.

Lemma 5.2.1. *Let $L : K$ be a field extension and $M \in \text{mod } A$. There is an isomorphism of A_L -modules $(\nu_A M)_L \cong \nu_{A_L} M_L$.*

Proof. Let $M \in \text{mod } A$. The result follows from applying Lemma 5.1.2 twice and using the definition $\nu_{\gamma}(-) = \text{Hom}_K(\text{Hom}_{\gamma}(-, ?), K)$ as follows:

$$\begin{aligned} \nu_A(M) \otimes_K L &= \text{Hom}_K(\text{Hom}_A(M, A), K) \otimes_K L \\ &\cong \text{Hom}_L(\text{Hom}_A(M, A) \otimes_K L, L) \\ &\cong \text{Hom}_L(\text{Hom}_{A_L}(M_L, A_L), L) \\ &= \nu_{A_L}(M_L). \end{aligned}$$

This completes the proof. $\square$

This gives rise to an analogous result about Auslander-Reiten translations.

Lemma 5.2.2. *Let $L : K$ be a field extension and $M \in \text{mod } A$. There is an isomorphism of A_L -modules $(\tau_A M)_L \cong \tau_{A_L} M_L$.*

Proof. Let $P^1 \xrightarrow{p^1} P^0 \xrightarrow{p^0} M \rightarrow 0$ be a minimal projective presentation of M in $\text{mod } A$. By Lemma 5.1.1(6), the modules P_L^1 and P_L^0 are projective A_L -modules and by Lemma 5.1.6(2) the exact sequence $P_L^1 \xrightarrow{p_L^1} P_L^0 \xrightarrow{p_L^0} M_L \rightarrow 0$, is a minimal projective presentation of M_L in $\text{mod } A_L$. After applying the (exact) Nakayama functor to these sequences, consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tau_{A_L} M_L & \longrightarrow & \nu_{A_L} P_L^0 & \longrightarrow & \nu_{A_L} P_L^1 & \longrightarrow & \nu_{A_L} M_L & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\ 0 & \longrightarrow & (\tau_A M)_L & \longrightarrow & (\nu_A P^0)_L & \longrightarrow & (\nu_A P^1)_L & \longrightarrow & (\nu_A M)_L & \longrightarrow & 0 \end{array}$$

The three maps on the right are isomorphisms by Lemma 5.2.1, and the relevant squares commute. By the Five Lemma, there is an induced isomorphism $\tau_{A_L} M_L \cong (\tau_A M)_L$ as required. $\square$

Therefore, it is possible to relate the τ -rigid modules and pairs of the two algebras.

Lemma 5.2.3. *Let $L : K$ be a field extension, $M \in \text{mod } A$ and $P \in \text{proj } A$. Then M is τ -rigid in $\text{mod } A$ if and only if M_L is τ -rigid in $\text{mod } A_L$. Moreover, (M, P) is a τ -rigid pair in $\text{mod } A$ if and only if (M_L, P_L) is a τ -rigid pair in $\text{mod } A_L$.*

Proof. The first statement follows directly from the sequence of isomorphisms

$$\text{Hom}_A(M, \tau_A M) \otimes_K L \cong \text{Hom}_{A_L}(M_L, (\tau_A M)_L) \cong \text{Hom}_{A_L}(M_L, \tau_{A_L} M_L),$$

where the first isomorphism follows from Lemma 5.1.2 and the second one follows from Lemma 5.2.2. Moreover, the module P_L is projective by Lemma 5.1.1(6) and Lemma 5.1.2 implies that $\text{Hom}_A(P, M) = 0$ if and only if $\text{Hom}_{A_L}(P_L, M_L)$ which gives the desired result. $\square$

The following is then an easy observation.

Lemma 5.2.4. *Let $L : K$ be a field extension and let (M, P) be a τ -tilting pair in $\text{mod } A$ and $N \in \text{mod } A$. Then, $N \in \mathcal{W}_{(M, P)}$ if and only if $N_L \in \mathcal{W}_{(M_L, P_L)}$.*

Proof. The notation $\mathcal{W}_{(M_L, P_L)} \subseteq \text{mod } A_L$ is well-defined since (M_L, P_L) is a τ -rigid pair in $\text{mod } A_L$ by Lemma 5.2.3. Then, using Lemma 5.1.2 the following hold:

- (1) $\text{Hom}_A(M, N) = 0$ if and only if $\text{Hom}_{A_L}(M_L, N_L) = 0$;
- (2) $\text{Hom}_A(P, N) = 0$ if and only if $\text{Hom}_{A_L}(P_L, N_L) = 0$;
- (3) $\text{Hom}_A(N, \tau_A M) = 0$ if and only if $\text{Hom}_{A_L}(N_L, (\tau_A M)_L) = 0$ if and only if $\text{Hom}_{A_L}(N_L, \tau_{A_L} M_L) = 0$ by Lemma 5.2.2.

Therefore the result follows from the definitions of $\mathcal{W}_{(M, P)}$ and $\mathcal{W}_{(M_L, P_L)}$. $\square$

Since the number of indecomposable direct summands of modules is difficult to control under field extension, it is necessary to pass to the bounded homotopy category $K^b(\text{proj } A)$ of $\text{proj } A$. For $i \in \mathbb{Z}$, denote by $[i]$ the i -th power of the suspension functor.

Definition 5.2.5. Let $P^\bullet \in K^b(\text{proj } A)$.

- (1) The complex $P^\bullet$ is called *presilting* if $\text{Hom}(P^\bullet, P^\bullet[i]) = 0$ for all $i > 0$.
- (2) The complex $P^\bullet$ is called *silting* if it is presilting and additionally satisfies $\text{thick}(P^\bullet) = K^b(\text{proj } A)$, where $\text{thick}(P^\bullet)$ is the smallest triangulated subcategory of $K^b(\text{proj } A)$ containing $P^\bullet$.

Finally, $P^\bullet$ is called 2-term if it is isomorphic to an object (P^i, d^i) in $K^b(\text{proj } A)$ such that $P^i = 0$ for $i \neq -1, 0$.

The collections of basic 2-term presilting objects and basic silting objects of $K^b(\text{proj } A)$ are denoted by 2-presilt A and 2-silt A respectively. The induced scalar extension functor $-\otimes_K L : K^b(\text{proj } A) \rightarrow K^b(\text{proj } A_L)$ behaves well with respect to silting and presilting complexes.

Lemma 5.2.6. *Let $L : K$ be a field extension and let $P^\bullet \in K^b(\text{proj } A)$ be a presilting complex. Then $P_L^\bullet \in K^b(\text{proj } A_L)$ is a presilting complex. Moreover, if $P^\bullet$ is silting, then $P_L^\bullet$ is silting.*

Proof. The first part of the statement follows directly from Lemma 5.1.2 adapted to this setting, where $\text{Ext}^i(-, ?)$ becomes $\text{Hom}(-, ?[i])$. Assume that $P^\bullet \in K^b(\text{proj } A)$ is silting then $A \in \text{thick}(P^\bullet) \subseteq K^b(\text{proj } A)$. Since $- \otimes_K L$ is a triangle functor it follows that $A_L \in \text{thick}(P_L^\bullet) \subseteq K^b(\text{proj } A_L)$, which implies that $P_L^\bullet$ is silting in $K^b(\text{proj } A_L)$. $\square$

The study of 2-term (pre)silting objects is the main focus of [60]. By [3, Thm. 3.2] there exist mutually inverse bijections between basic 2-term silting objects of $K^b(\text{proj } A)$ and basic τ -tilting pairs given by the following maps:

$$\tau\text{-tiltp } A \xrightleftharpoons[G]{F} 2\text{-silt } A$$

where $F(M, P) = (P^1 \oplus P \xrightarrow{(f \ 0)} P^0)$ and $P^1 \xrightarrow{f} P^0 \rightarrow M$ is a minimal projective presentation of M and $G(P^\bullet) = (H^0(P^\bullet), (P^1)^\prime)$, where $P^\bullet = P^1 \xrightarrow{d} P^0$ is decomposed into $P^\bullet = (P^1)^\prime \oplus (P^1)^\prime \xrightarrow{(d' \ 0)} P^0$ with d' right minimal, see [3, Prop. 3.6(b)]. This relationship commutes with taking field extensions.

Lemma 5.2.7. *Let $L : K$ be a field extension. If (M, P) is a τ -tilting pair in $\text{mod } A$, then (M_L, P_L) is a τ -tilting pair in $\text{mod } A_L$. In particular, if M is a τ -tilting module in $\text{mod } A$, then M_L is a τ -tilting module in $\text{mod } A_L$.*

Proof. Let $(M, P) \in \tau\text{-tiltp } A$ and $P^1 \xrightarrow{p^1} P^0 \rightarrow M \rightarrow 0$ be a minimal projective presentation. By Lemma 5.1.6(2), the sequence $P_L^1 \xrightarrow{p_L^1} P_L^0 \rightarrow M_L \rightarrow 0$ is a minimal projective presentation of M_L . Since (M, P) is τ -tilting in $\text{mod } A$, [3, Thm. 3.2] states that the corresponding 2-term complex $P^1 \oplus P \xrightarrow{(p^1 \ 0)} P^0$ is a silting object of $K^b(\text{proj } A)$. By Lemma 5.2.6, the complex $P_L^1 \oplus P_L \xrightarrow{(p_L^1 \ 0)} P_L^0$ is a silting complex in $K^b(\text{proj } A_L)$. Moreover, its corresponding τ -tilting pair in $\text{mod } A_L$ is clearly (M_L, P_L) , completing the proof. $\square$

Moreover, base field extension preserves the partial order on $\tau\text{-tiltp } A$ inherited from $\text{tors } A$ via Theorem 2.2.2.

Lemma 5.2.8. *Let $L : K$ be a field extension and let (M, P) and (N, Q) be two τ -tilting pairs in $\text{mod } A$ such that $\text{Fac } M \subseteq \text{Fac } N \subseteq \text{mod } A$. Then $\text{Fac } M_L \subseteq \text{Fac } N_L \subseteq \text{mod } A_L$.*

Proof. Take $X \in \text{Fac } M_L$, then by definition there is an epimorphism $M_L^r \twoheadrightarrow X \rightarrow 0$ in $\text{mod } A_L$ for some $r \geq 1$. Since $\text{Fac } M \subseteq \text{Fac } N \subseteq \text{mod } A$, it follows that in particular $M^r \in \text{Fac } N$ which gives rise to an epimorphism $N^s \twoheadrightarrow M^r \rightarrow 0$ in $\text{mod } A$ for some $s \geq 1$. Applying the exact scalar extension functor then gives an epimorphism $N_L^s \twoheadrightarrow M_L^r \rightarrow 0$ in $\text{mod } A_L$. In conclusion, there is a chain of epimorphisms $N_L^s \twoheadrightarrow M_L^r \twoheadrightarrow X$ which yields $X \in \text{Fac } N_L$ as required. $\square$

Consequently, the image of τ -tilting pairs in $\text{mod } A$ under base field extension gives a subposet of τ -tilting pairs in $\text{mod } A_L$, see also [110, Thm. 2.14(a)].

Proposition 5.2.9. *Let $L : K$ be a field extension. For an A_L -module N , let N^b denote a (choice of) basic direct summand of N with $\text{add } N = \text{add } N^b$. There is an embedding of posets*

$$\tau\text{-tiltp } A \rightarrow \tau\text{-tiltp } A_L$$

sending (M, P) to (M_L^b, P_L^b) . This map is well-defined up to isomorphism.

Proof. Since $- \otimes_K L$ is injective-on-objects by Lemma 5.1.1(3), the result follows from Lemma 5.2.7 and Lemma 5.2.8. $\square$

Similarly, it is possible to obtain the following result concerning functorially finite torsion classes.

Corollary 5.2.10. *There is an embedding of posets*

$$\text{f-tors } A \rightarrow \text{f-tors } A_L$$

sending $\mathcal{T}$ to $\mathcal{T}_L = \text{Fac } M_L$, where (M, P) is the τ -tilting pair such that $\mathcal{T} = \text{Fac } M$ under Theorem 2.2.2. Moreover, if $\mathcal{T} = \text{Fac } N$, for some τ -rigid module N , then $\mathcal{T}_L = \text{Fac } N_L$.

Proof. The map $\text{f-tors } A \rightarrow \text{f-tors } A_L$ is defined as follows:

$$\mathcal{T} \xrightarrow[\text{bijection}]{\text{Theorem 2.2.2}} (M, P) \xrightarrow[\text{injection}]{\text{Proposition 5.2.9}} (M_L^b, P_L^b) \xrightarrow[\text{bijection}]{\text{Theorem 2.2.2}} \mathcal{T}_L. \quad (5.2.1)$$

All maps are order preserving, so that the results cited along the arrows in (5.2.1) yield a well-defined embedding. Let N be τ -rigid and assume $\mathcal{T} = \text{Fac } N$. Consider the image of

$\mathcal{T}$ under Theorem 2.2.2, and denote it by (N', P') . It follows that $\text{Fac } N = \text{Fac } N'$. From Lemma 5.2.8 it follows that $\text{Fac } N_L = \text{Fac}(N')_L$, and therefore $\mathcal{T}_L = \text{Fac}(N')_L = \text{Fac } N_L$ as required. $\square$

It should be remarked that $\mathcal{T}_L$ is generally not equal to the full subcategory of $\text{mod } \Lambda_L$ obtained by applying $- \otimes_K L$ to every object of $\mathcal{T} \in \text{f-tors } \Lambda$. This illustrates why it is usually preferable to study the behaviour of individual objects under base field extension, rather than the behaviour of subcategories.

5.3 Bongartz completions and τ^{-1} -tilting theory

The aim of this section is to describe τ -perpendicular intervals of τ -tiltp A under base field extension. Recall from Section 2.2 that the co-Bongartz completion of a τ -rigid pair (M, P) is the unique basic τ -tilting pair (M^-, P^-) such that $\text{Fac } M = \text{Fac } M^-$ and the Bongartz completion of (M, P) is the unique basic τ -tilting pair (M^+, P) such that $\text{Fac } M^+ = {}^\perp \tau M \cap P^\perp$, see [57, Thm. 4.4]. By Definition 2.2.4, a τ -perpendicular interval of τ -tiltp A is precisely the interval $[(M^-, P^-), (M^+, P)] \subseteq \tau\text{-tiltp } A$ for some τ -rigid pair (M, P) .

Lemma 5.3.1. *Taking co-Bongartz completions commutes with base field extension. More precisely, let $L : K$ be a field extension and $(M, P) \in \tau\text{-rigidp } A$. Then*

$$((M_L)^-, (P_L)^-) = ((M^-)_L, (P^-)_L).$$

Proof. By definition, the co-Bongartz completion of (M, P) is the unique basic τ -tilting pair (M^-, P^-) in $\text{mod } A$ such that $\text{Fac}(M^-) = \text{Fac}(M)$. By Lemma 5.2.3, (M_L, P_L) is a τ -rigid pair in $\text{mod } A_L$ and thus $((M_L)^-, (P_L)^-)$ is a likewise defined to be the unique basic τ -tilting pair in $\text{mod } A_L$ such that $\text{Fac}((M_L)^-) = \text{Fac}(M_L)$. However, by Corollary 5.2.10 the equality $\text{Fac}(M^-) = \text{Fac}(M)$ implies $\text{Fac}((M^-)_L) = \text{Fac}(M_L)$. This implies that $\text{Fac}((M_L)^-) = \text{Fac}(M_L) = \text{Fac}((M^-)_L)$. Finally, since (M^-, P^-) is also a τ -tilting pair in $\text{mod } A$, it follows that $((M^-)_L, (P^-)_L)$ is a τ -tilting pair in $\text{mod } A_L$ by Lemma 5.2.7. Thus, Theorem 2.2.2 implies that $((M_L)^-, (P_L)^-) = ((M^-)_L, (P^-)_L)$ as desired. $\square$

A similar result for Bongartz completions requires the dualisation to torsion-free classes and τ^{-1} -tilting modules which also play an important role in later sections. Denote by τ^{-1} the inverse Auslander-Reiten translation.

Definition 5.3.2. Let $M \in \text{mod } A$ and $Q \in \text{inj } A$.

- (1) The module M is called τ^{-1} -rigid if $\text{Hom}(\tau^{-1}M, M) = 0$. If additionally $|M| = |A|$ then M is called τ^{-1} -tilting.
- (2) The pair (M, Q) is called τ^{-1} -rigid if M is τ^{-1} -rigid and $\text{Hom}(Q, M) = 0$. If additionally $|M| + |Q| = |A|$ then (M, Q) is called τ^{-1} -tilting.

Denote by $\tau^{-1}\text{-tiltp } A$ the collection of basic τ^{-1} -tilting pairs. It is clear that dual statements of Lemma 5.2.3 and Lemma 5.2.7 hold for τ^{-1} -rigid and τ^{-1} -tilting pairs.

Proposition 5.3.3. [3, p. 12] *There exists an explicit bijection*

$$H : \tau\text{-tiltp } A \rightarrow \tau^{-1}\text{-tiltp } A \quad (5.3.1)$$

given by $H(M, P) = (\tau M \oplus \nu P, \nu M_{\text{pr}})$ where M_{pr} is the largest projective direct summand of M . It fits into a commutative square

$$\begin{array}{ccc} \tau\text{-tiltp } A & \xrightarrow{H} & \tau^{-1}\text{-tiltp } A \\ \downarrow \text{Fac} & & \downarrow \text{Sub} \\ \text{f-tors } A & \xrightarrow{(-)^\perp} & \text{f-torf } A. \end{array}$$

The dual of Theorem 2.2.2 establishes a bijection $\tau^{-1}\text{-tiltp } A \rightarrow \text{f-torf } A$ given by $M \mapsto \text{Sub } M$. The interaction of the bijection in Proposition 5.3.3 with extension of scalars is described in the following.

Lemma 5.3.4. *Let $L : K$ be a field extension. The following square is commutative:*

$$\begin{array}{ccc} \tau\text{-tiltp } A & \xrightarrow{H} & \tau^{-1}\text{-tiltp } A \\ -\otimes_K L \downarrow & & \downarrow -\otimes_K L \\ \tau\text{-tiltp } A_L & \xrightarrow{H} & \tau^{-1}\text{-tiltp } A_L. \end{array}$$

Proof. Let $(M, P) \in \tau\text{-tiltp } A$. By Lemma 5.2.2 there is an isomorphism of A_L -modules $(\tau_A M)_L \cong \tau_{A_L} M_L$ and by Lemma 5.2.1 there is an isomorphism $(\nu_A P)_L \cong \nu_{A_L} P_L$. In combination, there is an isomorphism $(\tau_A M \oplus \nu_A P)_L \cong \tau_{A_L} M_L \oplus \nu_{A_L} P_L$ as the scalar extension functor is additive. Finally, Lemma 5.1.1(6) implies that $(M_{\text{pr}})_L \cong (M_L)_{\text{pr}}$ and using Lemma 5.2.1 again implies the desired result. $\square$

Consequently, the dual of Lemma 5.3.1 is obtained in the following.

Lemma 5.3.5. *Taking Bongartz completions commutes with base field extension. More precisely, let $L : K$ be a field extension and $(M, P) \in \tau\text{-rigidp } A$. Then*

$$((M_L)^+, P_L) = ((M^+)_L, P_L).$$

Proof. By definition, the Bongartz completion of (M, P) is the unique basic τ -tilting pair (M^+, P) in $\text{mod } A$ with the property that $\text{Fac}(M^+) = {}^\perp \tau_A M \cap P^\perp$. Since ${}^\perp \tau_A M \cap P^\perp = {}^\perp (\tau_A M \oplus \nu_A P)$, this is equivalent to saying $\text{Fac}(M^+) = {}^\perp (\tau_A M \oplus \nu_A P)$, and thus equivalent to $\text{Fac}(M^+)^\perp = \text{Sub}(\tau_A M \oplus \nu_A P)$ since $\tau_A M \oplus \nu_A P$ is τ^{-1} -rigid in $\text{mod } A$. Because torsion classes uniquely determine their corresponding torsion-free classes, the Bongartz completion (M^+, P) is also the unique basic τ -tilting pair in $\text{mod } A$ such that $\text{Fac}(M^+)^\perp = \text{Sub}(\tau_A M \oplus \nu_A P)$. Consequently, $\text{Sub}(\tau_A M \oplus \nu_A P) = \text{Sub}(\tau_A(M^+) \oplus \nu_A P)$ holds. Now, Lemma 5.3.4 and the dual of Corollary 5.2.10 imply that

$$\text{Sub}(\tau_{A_L} M_L \oplus \nu_{A_L} P_L) = \text{Sub}(\tau_{A_L}(M^+)_L \oplus \nu_{A_L} P_L). \quad (5.3.2)$$

On the other hand, (M_L, P_L) is a τ -rigid pair in $\text{mod } A_L$ by Lemma 5.2.3. Therefore, $((M_L)^+, P_L)$ may also be defined as the unique basic τ -tilting pair in $\text{mod } A_L$ such that $\text{Fac}((M_L)^+)^\perp = \text{Sub}(\tau_{A_L} M_L \oplus \nu_{A_L} P_L)$. However, since $H(M^+, P)$ is a τ^{-1} -tilting pair in $\text{mod } A$, the dual of Lemma 5.2.7 implies that $(H(M^+, P))_L$ is a τ^{-1} -tilting pair in $\text{mod } A_L$. By Eq. (5.3.2), the τ -tilting pair $(H(M^+, P))_L$ in $\text{mod } A_L$ also corresponds to the torsion-free class $\text{Sub}(\tau_{A_L} M_L \oplus \nu_{A_L} P_L)$ under the dual of Theorem 2.2.2. This implies $(H(M^+, P))_L = H((M_L)^+, P_L)$. By Lemma 5.3.4 it thus follows that

$$H((M^+)_L, P_L) = (H(M^+, P))_L = H((M_L)^+, P_L),$$

so that applying H^{-1} to both sides yields $((M^+)_L, P_L) = ((M_L)^+, P_L)$ as required. $\square$

The results of this section are combined to obtain the following result.

Proposition 5.3.6. *Let $L : K$ be a field extension. There is a well-defined map*

$$\begin{aligned} - \otimes_K L : \tau\text{-itv } A &\rightarrow \tau\text{-itv } A_L \\ [\mathcal{U}_{(M,P)}, \mathcal{T}_{(M,P)}] &\mapsto [\mathcal{U}_{(M_L,P_L)}, \mathcal{T}_{(M_L,P_L)}]. \end{aligned} \quad (5.3.3)$$

Proof. Let (M, P) be a τ -tilting pair in $\text{mod } A$. By definition $\mathcal{U}_{(M, P)} \in \text{f-tors } A$ corresponds to the τ -tilting pair (M^-, P) and $\mathcal{T}_{(M, P)} \in \text{f-tors } A$ corresponds to the τ -tilting pair (M^+, P^+) via Theorem 2.2.2. The result then follows from Lemma 5.3.1, Lemma 5.3.5 and the fact that the order of τ -tilting pairs is preserved by Lemma 5.2.8. $\square$

5.4 Left and right finite semibricks

The brick labelling of $\text{Hasse}(\text{tors } A)$ has already played a central role throughout this thesis. In this section the focus lies on those brick labels which arise as labels of the subquiver $\text{Hasse}(\text{f-tors } A)$. It was shown in [8], see also [57, Prop. 4.9], that it is possible to calculate the labels adjacent to a torsion class $\mathcal{T} \in \text{f-tors } A$ via the corresponding τ -tilting pair and τ^{-1} -tilting pair. Given a module $M \in \text{mod } A$, denote by $\text{ind}(M)$ the set of isomorphism classes of indecomposable direct summands of M .

Proposition 5.4.1. *[8, Thm. 1.3]/[57, Prop. 4.9] Let $\mathcal{T} \in \text{f-tors } A$ correspond to $(M, P) \in \tau\text{-tiltp } A$ under the bijection of Theorem 2.2.2. Then the cover relations $\mathcal{T} \rightarrow \mathcal{T}_i$ in $\text{Hasse}(\text{tors } A)$ are labelled by distinct isomorphism classes of objects in the (left-finite) semibrick*

$$\mathcal{S} = \text{ind}(M / \text{rad}_{\text{End}_A(M)} M). \quad (5.4.1)$$

Dually, let $(N, Q) = H(M, P) \in \tau^{-1}\text{-tiltp } A$ be the corresponding τ^{-1} -tilting pair. Then the cover relations $\mathcal{T}'_j \rightarrow \mathcal{T}$ in $\text{Hasse}(\text{tors } A)$ are labelled by the distinct isomorphism classes of objects in the (right-finite) semibrick

$$\mathcal{S}' = \text{ind}(\text{soc}_{\text{End}_A(N)} N). \quad (5.4.2)$$

Denote the collection of left-finite semibricks, that is, semibricks arising as in Eq. (5.4.1) for some τ -tilting pair (M, P) , by $\text{f}_{\mathcal{L}}\text{-sbrick } A$ and the collection of right-finite semibricks, that is, semibricks arising as in Eq. (5.4.2) for some τ^{-1} -tilting pair (N, Q) , by $\text{f}_{\mathcal{R}}\text{-sbrick } A$. Given a set $\mathcal{X}$ of indecomposable A -modules, denote by $\mathcal{X}_L$ or $\text{ind}(\mathcal{X} \otimes_K L)$ the set $\bigcup_{X \in \mathcal{X}} \text{ind}(X_L)$ of A_L -modules. In particular, this notation is used for semibricks.

Lemma 5.4.2. *Let $L : K$ be a MacLane separable field extension. Then the following square is commutative*

$$\begin{array}{ccc} \tau\text{-tiltp } A & \longrightarrow & \text{f}_{\mathcal{L}}\text{-sbrick } A \\ \downarrow -\otimes_K L & & \downarrow \text{ind}(-\otimes_K L) \\ \tau\text{-tiltp } A_L & \longrightarrow & \text{f}_{\mathcal{L}}\text{-sbrick } A_L \end{array} \quad (5.4.3)$$

where the horizontal map at the top is given by $(M, P) \mapsto \text{ind}(M/\text{rad}_{\text{End}_A(M)} M)$ and the horizontal map at the bottom is given by $(M', P') \mapsto \text{ind}(M'/\text{rad}_{\text{End}_{A_L}(M')} M')$.

Proof. Let $(M, P) \in \tau\text{-tiltp } A$. As a first step to showing the desired equality

$$\text{ind}(\text{ind}(M/\text{rad}_{\text{End}_A(M)} M) \otimes_K L) = \text{ind}(M_L/\text{rad}_{\text{End}_{A_L}(M_L)} M_L),$$

observe that since $-\otimes_K L$ is additive the left-hand side may be simplified as

$$\text{ind}(\text{ind}(M/\text{rad}_{\text{End}_A(M)} M) \otimes_K L) = \text{ind}((M/\text{rad}_{\text{End}_A(M)} M) \otimes_K L). \quad (5.4.4)$$

Moreover, since $-\otimes_K L$ is exact any quotient module M/N satisfies $(M/N)_L \cong M_L/N_L$. Now, Lemma 5.1.2 gives that $\text{End}_A(M) \otimes_K L \cong \text{End}_{A_L}(M_L)$ and since $L : K$ is MacLane separable, Lemma 5.1.6(2) yields that $(\text{rad}_A M)_L \cong \text{rad}_{A_L} M_L$. Combining these observations it follows that

$$\begin{aligned} \text{ind}((M/\text{rad}_{\text{End}_A(M)} M) \otimes_K L) &= \text{ind}(M_L/(\text{rad}_{\text{End}_A(M)} M)_L) \quad (\text{as } -\otimes_K L \text{ is exact}) \\ &= \text{ind}(M_L/\text{rad}_{\text{End}_A(M) \otimes_K L} M_L) \quad (\text{by Lemma 5.1.6(2)}) \\ &= \text{ind}(M_L/\text{rad}_{\text{End}_{A_L}(M_L)} M_L) \quad (\text{by Lemma 5.1.2}) \end{aligned}$$

so that Eq. (5.4.4) completes the proof. $\square$

The dual statement follows from a similar proof. It is included for the sake of completeness and for later reference.

Lemma 5.4.3. *Let $L : K$ be a MacLane separable field extension. Then the following square is commutative*

$$\begin{array}{ccc} \tau^{-1}\text{-tiltp } A & \longrightarrow & \text{f}_{\mathcal{R}}\text{-sbrick } A \\ \downarrow -\otimes_K L & & \downarrow \text{ind}(-\otimes_K L) \\ \tau^{-1}\text{-tiltp } A_L & \longrightarrow & \text{f}_{\mathcal{R}}\text{-sbrick } A_L \end{array} \quad (5.4.5)$$

where the horizontal map at the top is given by $(M, P) \mapsto \text{ind}(\text{soc}_{\text{End}_A(M)} M)$ and the horizontal map at the bottom is given by $(M', P') \mapsto \text{ind}(\text{soc}_{\text{End}_{A_L}(M')} M')$.

Proof. Since $L : K$ is MacLane separable, it follows that $(\text{soc}_A M)_L \cong \text{soc}_{A_L} M_L$ by Lemma 5.1.6(3). Similar to the previous lemma it follows that

$$\begin{aligned} \text{ind}(\text{ind}(\text{soc}_{\text{End}_A(M)} M) \otimes_K L) &= \text{ind}((\text{soc}_{\text{End}_A(M)}) \otimes_K L) \quad (\text{as } - \otimes_K L \text{ is additive}) \\ &= \text{ind}(\text{soc}_{\text{End}_A(M) \otimes_K L} M_L) \quad (\text{by Lemma 5.1.6(3)}) \\ &= \text{ind}(\text{soc}_{\text{End}_{A_L}(M_L)} M_L) \quad (\text{by Lemma 5.1.2}). \end{aligned}$$

This completes the proof. $\square$

Remark 5.4.4. Let $L : K$ be any field extension. Then there exist two injective maps

$$\mathcal{L} : \text{f}_{\mathcal{L}\text{-sbrick}}(A) \rightarrow \text{f}_{\mathcal{L}\text{-sbrick}}(A_L), \quad \mathcal{R} : \text{f}_{\mathcal{R}\text{-sbrick}}(A) \rightarrow \text{f}_{\mathcal{R}\text{-sbrick}}(A_L).$$

This observation follows from the fact that the horizontal maps in the commutative diagrams Eq. (5.4.3) and Eq. (5.4.5) are bijections by [8, Thm. 1.3] and the fact that $- \otimes_K L$ is injective-on-objects and lifts τ -tilting pairs and τ^{-1} -tilting pairs by Lemma 5.2.7 and its dual. However, it is not clear whether $\mathcal{L}$ and $\mathcal{R}$ are given by $\text{ind}(- \otimes_K L)$ as demonstrated in Lemma 5.4.2 and Lemma 5.4.3 when $L : K$ is not MacLane separable.

In the following result the importance of studying left-finite and right-finite semibricks for the aim of studying the τ -cluster morphism category becomes apparent.

Lemma 5.4.5. *Let (M, P) be a τ -rigid pair. Write $\mathcal{B}_{\mathcal{L}}^{(M,P)}$ for the left-finite semibrick corresponding to (M^+, P) as in Eq. (5.4.1) and $\mathcal{B}_{\mathcal{R}}^{(M,P)}$ for the right-finite semibrick corresponding to the τ^{-1} -tilting pair $H(M^-, P^-)$ as in Eq. (5.4.2), with H as in Eq. (5.3.1). Then the τ -perpendicular subcategory $\mathcal{W}_{(M,P)} \subseteq \text{mod } A$ as defined in Theorem 2.2.5 is given by*

$$\mathcal{W}_{(M,P)} = \text{Filt}_A \left\{ \mathcal{B}_{\mathcal{L}}^{(M,P)} \cap \mathcal{B}_{\mathcal{R}}^{(M,P)} \right\}.$$

Proof. Write $[\mathcal{U}, \mathcal{T}] = [\text{Fac } M, {}^\perp \tau M \cap P^\perp] \subseteq \text{tors } A$ for the τ -perpendicular interval corresponding to (M, P) . By Eq. (2.0.2) it is possible to write $\mathcal{W}_{(M,P)} = \text{Filt}_A \{\mathcal{S}\}$ for some semibrick $\mathcal{S} \in \text{sbrick } A$ and by [57, Thm. 4.16(a)(c)], $\mathcal{S}$ consists of labels of arrows incident to $\mathcal{U}$ in $\text{Hasse}[\mathcal{U}, \mathcal{T}]$. By Proposition 5.4.1, the right-finite semibrick

$\mathcal{B}_{\mathcal{R}}^{(M,P)}$ consists of labels of all arrows going into $\mathcal{U}$ in $\text{Hasse}(\text{tors } A)$, it follows that $\mathcal{S} \subseteq \mathcal{B}_{\mathcal{R}}^{(M,P)}$. Dually by [57, Thm. 4.16(a)(d)], $\mathcal{S}$ consists of labels of arrows incident to $\mathcal{T}$ in $\text{Hasse}[\mathcal{U}, \mathcal{T}]$. Again by Proposition 5.4.1 the left-finite semibrick $\mathcal{B}_{\mathcal{L}}^{(M,P)}$ consists of labels of all arrows going out of $\mathcal{U}$ in $\text{Hasse}(\text{tors } A)$. It follows that $\mathcal{S} \subseteq \mathcal{B}_{\mathcal{L}}^{(M,P)} \cap \mathcal{B}_{\mathcal{R}}^{(M,P)}$.

Conversely, assume that there is a brick $S \in \mathcal{B}_{\mathcal{L}}^{(M,P)} \cap \mathcal{B}_{\mathcal{R}}^{(M,P)}$, then by Proposition 5.4.1 the brick S labels an arrow $\mathcal{U}' \rightarrow \mathcal{U}$ in $\text{Hasse}(\text{tors } A)$ and an arrow $\mathcal{T} \rightarrow \mathcal{T}'$ in $\text{Hasse}(\text{tors } A)$. However, by definition of the brick labelling this means that $S \in \mathcal{U}^\perp$ and $S \in \mathcal{T}$, so that $S \in \mathcal{U}^\perp \cap \mathcal{T} = \mathcal{W}_{(M,P)}$. In particular, $\mathcal{U}' = \mathcal{U} \vee \text{T}(S) \subseteq \mathcal{T}$, so S is the label of an arrow adjacent to $\mathcal{U}$ in $\text{Hasse}[\mathcal{U}, \mathcal{T}]$, and thus $S \in \mathcal{S}$ by [57, Thm. 4.16]. $\square$

Let $\mathcal{T} \in \text{f-tors } A$, then define $\mathcal{B}_{\mathcal{L}}^{\mathcal{X}} := \mathcal{B}_{\mathcal{L}}^{(M,P)}$, where (M, P) is the basic τ -tilting pair corresponding to $\mathcal{T}$ via Theorem 2.2.2.

Proposition 5.4.6. *Let $L : K$ be a MacLane separable field extension, and (M, P) be a τ -rigid pair in $\text{mod } A$ whose τ -perpendicular subcategory is given by $\mathcal{W}_{(M,P)} = \text{Filt}_A\{\mathcal{S}\}$ for some semibrick $\mathcal{S}$, then*

$$\mathcal{W}_{(M_L, P_L)} = \text{Filt}_{A_L}\{\mathcal{S}_L\} \subseteq \text{mod } A_L,$$

where $\mathcal{S}_L = \bigcup_{X \in \mathcal{S}} \text{ind}(X_L)$.

Proof. Using Lemma 5.4.5, write $\mathcal{S} = \mathcal{B}_{\mathcal{L}}^{(M,P)} \cap \mathcal{B}_{\mathcal{R}}^{(M,P)}$. By Lemma 5.1.1(5) the equality

$$\mathcal{S}_L = \bigcup_{X \in \mathcal{S}} \text{ind}(X_L) = \bigcup_{X \in \mathcal{B}_{\mathcal{L}}^{(M,P)} \cap \mathcal{B}_{\mathcal{R}}^{(M,P)}} \text{ind}(X_L) = \text{ind}((\mathcal{B}_{\mathcal{L}}^{(M,P)})_L) \cap \text{ind}((\mathcal{B}_{\mathcal{R}}^{(M,P)})_L)$$

holds. Combining Lemma 5.3.5 and Lemma 5.4.2 gives the equality

$$\text{ind}((\mathcal{B}_{\mathcal{L}}^{(M,P)})_L) = \mathcal{B}_{\mathcal{L}}^{(M_L, P_L)} \in \text{f}_{\mathcal{L}}\text{-sbrick } A_L,$$

and similarly combining Lemma 5.3.1, Lemma 5.3.4 and Lemma 5.4.3 gives the equality

$$\text{ind}((\mathcal{B}_{\mathcal{R}}^{(M,P)})_L) = \mathcal{B}_{\mathcal{R}}^{(M_L, P_L)} \in \text{f}_{\mathcal{R}}\text{-sbrick } A_L.$$

By Lemma 5.4.5, it follows that

$$\begin{aligned} \mathcal{W}_{(M_L, P_L)} &= \text{Filt}_{A_L} \left\{ \mathcal{B}_{\mathcal{L}}^{(M_L, P_L)} \cap \mathcal{B}_{\mathcal{R}}^{(M_L, P_L)} \right\} \\ &= \text{Filt}_{A_L} \left\{ \text{ind}((\mathcal{B}_{\mathcal{L}}^{(M,P)})_L) \cap \text{ind}((\mathcal{B}_{\mathcal{R}}^{(M,P)})_L) \right\} \\ &= \text{Filt}_{A_L} \{\mathcal{S}_L\}, \end{aligned}$$

as required. $\square$

5.5 Faithful functor

The understanding of τ -tilting theory under base field extension developed in the previous sections culminates in the following main theorem of this chapter.

Theorem 5.5.1. *Let $L : K$ be a MacLane separable field extension. There exists a well-defined faithful functor $\mathcal{F} : \mathfrak{T}(A) \rightarrow \mathfrak{T}(A_L)$ given by*

$$\mathcal{F} : \mathfrak{T}(A) \rightarrow \mathfrak{T}(A_L)$$

$$[\mathcal{U}_{(M,P)}, \mathcal{T}_{(M,P)}]_{\sim} \mapsto [\mathcal{U}_{(M_L, P_L)}, \mathcal{T}_{(M_L, P_L)}]_{\sim} \quad (5.5.1)$$

$$[f_{[\mathcal{U}_{(N,Q)}, \mathcal{T}_{(N,Q)}][\mathcal{U}_{(M,P)}, \mathcal{T}_{(M,P)}]}] \mapsto [f_{[\mathcal{U}_{(N_L, Q_L)}, \mathcal{T}_{(N_L, Q_L)}][\mathcal{U}_{(M_L, P_L)}, \mathcal{T}_{(M_L, P_L)}]}].$$

Consequently, if $\mathfrak{W}(A_L)$ admits a faithful group functor, so does $\mathfrak{W}(A)$.

It is not difficult to see that Proposition 5.4.6 may be used to show that $\mathcal{F}$ is well-defined on objects. To understand the morphisms of $\mathfrak{T}(A)$ under $\mathcal{F}$ the following equivalent condition for the identification of morphisms in Definition 4.2.3 is necessary.

Lemma 5.5.2. *Let (N, Q) and (N', Q') be two τ -rigid pairs which give rise to the same wide subcategory $\mathcal{W}_{(N,Q)} = \mathcal{W}_{(N',Q')}$. Similarly, let $(M, P) \in \text{add}(N, Q)$ and let $(M', P') \in \text{add}(N', Q')$ be τ -rigid pairs such that $\mathcal{W} := \mathcal{W}_{(M,P)} = \mathcal{W}_{(M',P')}$. Then, the following are equivalent:*

- (1) $\{\mathcal{V} \cap \mathcal{W} : \mathcal{V} \in [\mathcal{U}_{(N,Q)}, \mathcal{T}_{(N,Q)}]\} = \{\mathcal{V}' \cap \mathcal{W} : \mathcal{V}' \in [\mathcal{U}_{(N',Q')}, \mathcal{T}_{(N',Q')}]\};$
- (2) $\mathcal{U}_{(N,Q)} \cap \mathcal{W} = \mathcal{U}_{(N',Q')} \cap \mathcal{W}$ and $\mathcal{T}_{(N,Q)} \cap \mathcal{W} = \mathcal{T}_{(N',Q')} \cap \mathcal{W};$
- (3) $\mathcal{B}_{\mathcal{R}}^{(N,Q)} \cap \mathcal{W} = \mathcal{B}_{\mathcal{R}}^{(N',Q')} \cap \mathcal{W}$ and $\mathcal{B}_{\mathcal{L}}^{(N,Q)} \cap \mathcal{W} = \mathcal{B}_{\mathcal{L}}^{(N',Q')} \cap \mathcal{W}.$

Proof. (1) $\iff$ (2). This is immediate.

(2) $\implies$ (3). Let $\mathcal{X} \in \text{f-tors } \mathcal{W}$, then there exists a unique (relative) left-finite semibrick $\tilde{\mathcal{B}}_{\mathcal{L}}^{\mathcal{X}} \in \text{f}_{\mathcal{L}}\text{-sbrick } \mathcal{W}$ corresponding to it by [8, Thm. 1.3] and Theorem 2.2.2. By Proposition 5.4.1 it labels the arrows going out of $\mathcal{X}$ in $\text{Hasse}(\text{tors } \mathcal{W})$ and by Theorem 2.2.5, there exists a unique $\mathcal{Y} \in [\mathcal{U}_{(N,Q)}, \mathcal{T}_{(N,Q)}] \subseteq \text{tors } A$ such that now $\mathcal{X} = \mathcal{Y} \cap \mathcal{W}$. It follows that $\tilde{\mathcal{B}}_{\mathcal{L}}^{\mathcal{X}} = \mathcal{B}_{\mathcal{L}}^{\mathcal{Y}} \cap \mathcal{W}$, since the intersection with $\mathcal{W}$ preserves the brick labels by Proposition 4.2.2. Consequently, since $\mathcal{T}_{(N,Q)} \cap \mathcal{W} = \mathcal{T}_{(N',Q')} \cap \mathcal{W} \in \text{tors } \mathcal{W}$ it follows that

$$\mathcal{B}_{\mathcal{L}}^{(N,Q)} \cap \mathcal{W} = \tilde{\mathcal{B}}_{\mathcal{L}}^{\mathcal{T}_{(N,Q)} \cap \mathcal{W}} = \tilde{\mathcal{B}}_{\mathcal{L}}^{\mathcal{T}_{(N',Q')} \cap \mathcal{W}} = \mathcal{B}_{\mathcal{L}}^{(N',Q')} \cap \mathcal{W}.$$

An entirely analogous argument shows that the equality $\mathcal{U}_{(N,Q)} \cap \mathcal{W} = \mathcal{U}_{(N',Q')} \cap \mathcal{W}$ implies $\mathcal{B}_{\mathcal{R}}^{(N,Q)} \cap \mathcal{W} = \mathcal{B}_{\mathcal{R}}^{(N',Q')} \cap \mathcal{W}$, as required.

(3) $\implies$ (2). Assume now $\mathcal{B}_{\mathcal{L}}^{(N,Q)} \cap \mathcal{W} = \mathcal{B}_{\mathcal{L}}^{(N',Q')} \cap \mathcal{W}$. Using the same notation and reasoning as above, this implies

$$\tilde{\mathcal{B}}_{\mathcal{L}}^{\mathcal{T}_{(N,Q)} \cap \mathcal{W}} = \mathcal{B}_{\mathcal{L}}^{(N,Q)} \cap \mathcal{W} = \mathcal{B}_{\mathcal{L}}^{(N',Q')} \cap \mathcal{W} = \tilde{\mathcal{B}}_{\mathcal{L}}^{\mathcal{T}_{(N',Q')} \cap \mathcal{W}}.$$

Then it follows directly from [8, Thm. 1.3] that $\mathcal{T}_{(N,Q)} \cap \mathcal{W} = \mathcal{T}_{(N',Q')} \cap \mathcal{W}$ as required. Again, an analogous argument shows that $\mathcal{B}_{\mathcal{R}}^{(N,Q)} \cap \mathcal{W} = \mathcal{B}_{\mathcal{R}}^{(N',Q')} \cap \mathcal{W}$ implies that $\mathcal{U}_{(N,Q)} \cap \mathcal{W} = \mathcal{U}_{(N',Q')} \cap \mathcal{W}$ as required. $\square$

The following lemma, in combination with the previous, is important to control the identification of morphisms when proving Theorem 5.5.1.

Lemma 5.5.3. *Let $L : K$ be a MacLane separable field extension and (M, P) be a τ -rigid pair in $\text{mod } A$. Then the square*

$$\begin{array}{ccc} \left\{ \begin{array}{l} \mathcal{B}_{\mathcal{L}}^{(N,Q)} \in \text{f}_{\mathcal{L}}\text{-sbrick } A \\ \text{s.t. } (N, Q) \in [(M^-, P^-), (M^+, P)] \subseteq \tau\text{-tiltp } A \end{array} \right\} & \xrightarrow{-\cap \mathcal{W}_{(M,P)}} & \text{f}_{\mathcal{L}}\text{-sbrick } \mathcal{W}_{(M,P)} \\ \downarrow \text{ind}(-\otimes_K L) & & \downarrow \text{ind}(-\otimes_K L) \\ \left\{ \begin{array}{l} \mathcal{B}_{\mathcal{L}}^{(N',Q')} \in \text{f}_{\mathcal{L}}\text{-sbrick } A_L \\ \text{s.t. } (N', Q') \in [(M_L^-, P_L^-), (M_L^+, P_L)] \subseteq \tau\text{-tiltp } A_L \end{array} \right\} & \xrightarrow{-\cap \mathcal{W}_{(M_L, P_L)}} & \text{f}_{\mathcal{L}}\text{-sbrick } \mathcal{W}_{(M_L, P_L)} \end{array}$$

commutes and the horizontal maps are bijections.

Proof. The vertical maps are well-defined by Proposition 5.3.6 and Lemma 5.4.2. Moreover, the horizontal map given by intersection with $\mathcal{W}_{(M,P)}$ defines an isomorphism from $\text{Hasse}[(M^-, P^-), (M^+, P)]$ to $\text{Hasse}(\tau\text{-tiltp } \mathcal{W}_{(M,P)})$ which preserves the brick labelling by Proposition 4.2.2. As discussed in the proof of Lemma 5.5.2, the horizontal map is therefore well-defined, since the brick labelling is preserved and left-finite semibricks can be read off as the labels of arrows going out of a τ -tilting pair in the Hasse quiver by Proposition 5.4.1. The same holds for the bottom horizontal map. The horizontal maps are bijections because the intervals of τ -tilting pairs are in bijection by Theorem 2.2.5 and basic τ -tilting pairs are in bijection with left-finite semibricks by [8, Thm. 1.3].

Let $S \in \mathcal{B}_{\mathcal{L}}^{(N,Q)}$. Then $S \in \mathcal{W}_{(M,P)}$ if and only if $S_L \in \mathcal{W}_{(M_L, P_L)}$ by Lemma 5.2.4. In particular, this is the case if and only if every indecomposable direct summand $S' \in \text{add}(S_L)$ is contained in $\mathcal{W}_{(M_L, P_L)}$. Therefore, the square commutes. $\square$

Now all preliminary results have been collected, and this section is concluded by proving Theorem 5.5.1.

Proof of Theorem 5.5.1. It is shown in Proposition 5.3.6 that if $[\mathcal{U}_{(M,P)}, \mathcal{T}_{(M,P)}]$ is a τ -perpendicular interval of tors A , then $[\mathcal{U}_{(M_L,P_L)}, \mathcal{T}_{(M_L,P_L)}]$ is a τ -perpendicular interval of tors A_L . Moreover, consider two τ -perpendicular intervals $[\mathcal{U}_{(M,P)}, \mathcal{T}_{(M,P)}]$ and $[\mathcal{U}_{(M',P')}, \mathcal{T}_{(M',P')}]$ of tors A such that $\mathcal{W}_{(M,P)} = \mathcal{W}_{(M',P')} = \text{Filt}_A\{\mathcal{S}\}$ for some $\mathcal{S} \in \text{sbrick } A$. It follows that

$$\mathcal{W}_{(M_L,P_L)} = \text{Filt}_{A_L}\{\mathcal{S}_L\} = \mathcal{W}_{((M')_L,(P')_L)}$$

by Proposition 5.4.6. Therefore, if $[\mathcal{U}_{(M,P)}, \mathcal{T}_{(M,P)}]_{\sim} = [\mathcal{U}_{(M',P')}, \mathcal{T}_{(M',P')}]_{\sim}$ in $\mathfrak{T}(A)$, then $[\mathcal{U}_{(M_L,P_L)}, \mathcal{T}_{(M_L,P_L)}]_{\sim} = [\mathcal{U}_{((M')_L,(P')_L)}, \mathcal{T}_{((M')_L,(P')_L)}]_{\sim}$ in $\mathfrak{T}(A_L)$. Thus $\mathcal{F}$ is well-defined on objects. To investigate morphisms, consider two τ -perpendicular intervals of tors A such that $[\mathcal{U}_{(N,Q)}, \mathcal{T}_{(N,Q)}] \subseteq [\mathcal{U}_{(M,P)}, \mathcal{T}_{(M,P)}]$. Then

$$[\mathcal{U}_{(N_L,Q_L)}, \mathcal{T}_{(N_L,Q_L)}] \subseteq [\mathcal{U}_{(M_L,P_L)}, \mathcal{T}_{(M_L,P_L)}] \subseteq \text{tors } A_L,$$

follows from the fact that $- \otimes_K L : \tau\text{-tiltp } A \rightarrow \tau\text{-tiltp } A_L$ preserves the partial order of τ -perpendicular intervals by Lemma 5.2.8.

Now, consider four τ -perpendicular intervals $[\mathcal{U}_{(N,Q)}, \mathcal{T}_{(N,Q)}] \subseteq [\mathcal{U}_{(M,P)}, \mathcal{T}_{(M,P)}]$ and $[\mathcal{U}_{(N',Q')}, \mathcal{T}_{(N',Q')}] \subseteq [\mathcal{U}_{(M',P')}, \mathcal{T}_{(M',P')}]$ of tors A whose corresponding wide subcategories are such that $\text{Filt}_A\{\mathcal{S}\} = \mathcal{W}_{(M,P)} = \mathcal{W}_{(M',P')}$ for some $\mathcal{S} \in \text{sbrick } A$ and moreover such that $\mathcal{W}_{(N,Q)} = \mathcal{W}_{(N',Q')}$ and

$$\{\mathcal{V} \cap \mathcal{W}_{(M,P)} : \mathcal{V} \in [\mathcal{U}_{(N,Q)}, \mathcal{T}_{(N,Q)}]\} = \{\mathcal{V}' \cap \mathcal{W}_{(M',P')} : \mathcal{V}' \in [\mathcal{U}_{(N',Q')}, \mathcal{T}_{(N',Q')}] \}$$

as subsets of $\text{tors}(\text{Filt}_A\{\mathcal{S}\})$. Then there is a chain of implications

$$\begin{aligned} \mathcal{T}_{(N,Q)} \cap \mathcal{W}_{(M,P)} &= \mathcal{T}_{(N',Q')} \cap \mathcal{W}_{(M',P')} \\ \Rightarrow \mathcal{B}_{\mathcal{L}}^{(N,Q)} \cap \text{Filt}_A\{\mathcal{S}\} &= \mathcal{B}_{\mathcal{L}}^{(N',Q')} \cap \text{Filt}_A\{\mathcal{S}\} && \text{(by Lemma 5.5.2)} \\ \Rightarrow \text{ind}((\mathcal{B}_{\mathcal{L}}^{(N,Q)} \cap \text{Filt}_A\{\mathcal{S}\})_L) &= \text{ind}((\mathcal{B}_{\mathcal{L}}^{(N',Q')} \cap \text{Filt}_A\{\mathcal{S}\})_L) \\ \Rightarrow \mathcal{B}_{\mathcal{L}}^{(N_L,Q_L)} \cap \text{Filt}_{A_L}\{\mathcal{S}_L\} &= \mathcal{B}_{\mathcal{L}}^{((N')_L,(Q')_L)} \cap \text{Filt}_{A_L}\{\mathcal{S}_L\} && \text{(by Lemma 5.5.3)} \\ \Rightarrow \mathcal{T}_{(N_L,Q_L)} \cap \text{Filt}_{A_L}\{\mathcal{S}_L\} &= \mathcal{T}_{((N')_L,(Q')_L)} \cap \text{Filt}_{A_L}\{\mathcal{S}_L\} && \text{(by Lemma 5.5.2)} \\ \Rightarrow \mathcal{T}_{(N_L,Q_L)} \cap \mathcal{W}_{(M_L,P_L)} &= \mathcal{T}_{((N')_L,(Q')_L)} \cap \mathcal{W}_{((M')_L,(P')_L)} && \text{(by Proposition 5.4.6).} \end{aligned}$$

An analogous argument shows that

$$\begin{aligned}\mathcal{U}_{(N,Q)} \cap \mathcal{W}_{(M,P)} &= \mathcal{U}_{(N',Q')} \cap \mathcal{W}_{(M',P')} \\ &\Rightarrow \mathcal{U}_{(N_L,Q_L)} \cap \mathcal{W}_{(M_L,P_L)} = \mathcal{U}_{((N')_L,(Q')_L)} \cap \mathcal{W}_{((M')_L,(P')_L)}.\end{aligned}$$

Therefore, Lemma 5.5.2 yields that

$$\begin{aligned}&\{\mathcal{V} \cap \mathcal{W}_{(M_L,P_L)} : \mathcal{V} \in \mathcal{U}_{(N_L,Q_L)}, \mathcal{T}_{(N_L,Q_L)}\} \\ &= \left\{ \mathcal{V}' \cap \mathcal{W}_{((M')_L,(P')_L)} : \mathcal{V}' \in [\mathcal{U}_{((N')_L,(Q')_L)}, \mathcal{T}_{((N')_L,(Q')_L)}] \right\}\end{aligned}$$

in $\text{tors}(\text{Filt}_{A_L}\{\mathcal{S}_L\})$. In other words, identification of morphisms is preserved. To investigate the composition of morphisms, let

$$[f_{[\mathcal{U}_{(N',Q')}, \mathcal{T}_{(N',Q')}] [\mathcal{U}_{(N'',Q'')}, \mathcal{T}_{(N'',Q'')}] \circ [f_{[\mathcal{U}_{(M,P)}, \mathcal{T}_{(M,P)}] [\mathcal{U}_{(M',P')}, \mathcal{T}_{(M',P')}]]$$

be two composable morphisms in $\mathfrak{T}(A)$, that is, such that their composition is nonzero.

Then by Lemma 4.2.6 there exists an interval $[\mathcal{U}_{(M'',P'')}, \mathcal{T}_{(M'',P'')}] \subseteq [\mathcal{U}_{(M',P')}, \mathcal{T}_{(M',P')}]$ such that $[f_{[\mathcal{U}_{(N',Q')}, \mathcal{T}_{(N',Q')}] [\mathcal{U}_{(N'',Q'')}, \mathcal{T}_{(N'',Q'')}] = [f_{[\mathcal{U}_{(M',P')}, \mathcal{T}_{(M',P')}] [\mathcal{U}_{(M'',P'')}, \mathcal{T}_{(M'',P'')}]]$ so that their composition is given by

$$[f_{[\mathcal{U}_{(M,P)}, \mathcal{T}_{(M,P)}] [\mathcal{U}_{(M'',P'')}, \mathcal{T}_{(M'',P'')}]].$$

It is immediate that

$$\begin{aligned}&\mathcal{F}([f_{[\mathcal{U}_{(M',P')}, \mathcal{T}_{(M',P')}] [\mathcal{U}_{(M'',P'')}, \mathcal{T}_{(M'',P'')}]] \circ \mathcal{F}([f_{[\mathcal{U}_{(M,P)}, \mathcal{T}_{(M,P)}] [\mathcal{U}_{(M',P')}, \mathcal{T}_{(M',P')}]]]) \\ &= [f_{[\mathcal{U}_{((M')_L,(P')_L)}, \mathcal{T}_{((M')_L,(P')_L)}] [\mathcal{U}_{((M'')_L,(P'')_L)}, \mathcal{T}_{((M'')_L,(P'')_L)}]] \\ &\quad \circ [f_{[\mathcal{U}_{(M_L,P_L)}, \mathcal{T}_{(M_L,P_L)}] [\mathcal{U}_{((M')_L,(P')_L)}, \mathcal{T}_{((M')_L,(P')_L)}]] \\ &= [f_{[\mathcal{U}_{(M_L,P_L)}, \mathcal{T}_{(M_L,P_L)}] [\mathcal{U}_{((M'')_L,(P'')_L)}, \mathcal{T}_{((M'')_L,(P'')_L)}]] \\ &= \mathcal{F}([f_{[\mathcal{U}_{(M,P)}, \mathcal{T}_{(M,P)}] [\mathcal{U}_{(M'',P'')}, \mathcal{T}_{(M'',P'')}]]]).\end{aligned}$$

Thus $\mathcal{F}$ preserves composition of morphisms. It is immediate that $\mathcal{F}$ preserves identity morphisms. Therefore $\mathcal{F} : \mathfrak{T}(A) \rightarrow \mathfrak{T}(A_L)$ is a well-defined functor. Finally, to see that $\mathcal{F}$ is faithful, take two distinct morphisms

$$\begin{aligned}&[f_{[\mathcal{U}_{(M,P)}, \mathcal{T}_{(M,P)}] [\mathcal{U}_{(N,Q)}, \mathcal{T}_{(N,Q)}]]], [f_{[\mathcal{U}_{(M,P)}, \mathcal{T}_{(M,P)}] [\mathcal{U}_{(N',Q')}, \mathcal{T}_{(N',Q')}]] \\ &\in \text{Hom}_{\mathfrak{T}(A)}([\mathcal{U}_{(M,P)}, \mathcal{T}_{(M,P)}]_{\sim}, [\mathcal{U}_{(N,Q)}, \mathcal{T}_{(N,Q)}]_{\sim}).\end{aligned}$$

By Lemma 4.2.6, see also Corollary 3.1.6, they may be taken to be represented by two morphisms in $\text{itv}(\text{tors } A)$ with the same domain without loss of generality. Assume for a contradiction that

$$[f[\mathcal{U}_{(M_L, P_L)}, \mathcal{T}_{(M_L, P_L)}][\mathcal{U}_{(N_L, Q_L)}, \mathcal{T}_{(N_L, Q_L)}] = [f[\mathcal{U}_{(M_L, P_L)}, \mathcal{T}_{(M_L, P_L)}][\mathcal{U}_{((N')_L, (Q')_L)}, \mathcal{T}_{((N')_L, (Q')_L)}]$$

in $\mathfrak{T}(A_L)$. This would mean that the intervals

$$[\mathcal{U}_{(N_L, Q_L)}, \mathcal{T}_{(N_L, Q_L)}] \quad \text{and} \quad [\mathcal{U}_{((N')_L, (Q')_L)}, \mathcal{T}_{((N')_L, (Q')_L)}]$$

coincide since their intersections with $\mathcal{W}_{(M_L, P_L)}$ coincide and the intervals $\text{tors}(\mathcal{W}_{(M_L, P_L)})$ and $[\mathcal{U}_{(M_L, P_L)}, \mathcal{T}_{(M_L, P_L)}] \subseteq \text{tors } A_L$ are in bijection by Theorem 2.2.5. Using Lemma 5.3.1 and Lemma 5.3.5, this implies that

$$\begin{aligned} \mathcal{U}_{(N_L, Q_L)} &= \mathcal{U}_{((N')_L, (Q')_L)} \quad \text{and} \quad \mathcal{T}_{(N_L, Q_L)} = \mathcal{T}_{((N')_L, (Q')_L)} \\ &\Rightarrow ((N_L)^-, (Q_L)^-) = (((N')_L)^-, ((Q')_L)^-) \quad \text{and} \quad ((N_L)^+, (Q_L)^+) = (((N')_L)^+, ((Q')_L)^+) \\ &\Rightarrow ((N^-)_L, (Q^-)_L) = (((N')^-)_L, ((Q')^-)_L) \quad \text{and} \quad ((N^+)_L, (Q^+)_L) = (((N')^+)_L, ((Q')^+)_L). \end{aligned}$$

Since $-\otimes_K L$ defines an injective map of τ -tilting pairs by Proposition 5.2.9 it follows that $(N^-, Q^-) = ((N')^-, (Q')^-)$ and $(N^+, Q) = ((N')^+, Q')$. Since two distinct τ -rigid pairs cannot have both the same Bongartz completion and the same co-Bongartz completion this yields $(N, Q) = (N', Q')$. However, this is a contradiction to the assumption that the original morphisms are distinct. As a consequence the functor $\mathcal{F}$ is faithful as required. $\square$

Corollary 5.5.4. *Let $L : K$ be a MacLane separable field extension. If $\mathfrak{T}(A_L)$ admits a faithful group functor, then so does $\mathfrak{T}(A)$.*

Proof. This is an immediate consequence of Theorem 5.5.1 since the composition of faithful functors is again a faithful functor. $\square$

5.6 Applications and examples

To conclude this chapter and this thesis, the τ -cluster morphism categories of a new class of algebras are shown to admit faithful group functors. This partially extends the

two classes of algebras admitting faithful group functors listed in Example 2.4.5, see also [106, Thm. 3.7] and [92, Thm. 5.9]. To give a specific example, it is necessary to first introduce the notion of species, a generalisation of quivers, which are designed to encode information about field extensions.

Definition 5.6.1. Let K be any field. A K -species $\mathbb{S} = (Q^{\mathbb{S}}, D_i, X_{\alpha})$ consists of the data of:

- A quiver $Q^{\mathbb{S}} = (Q_0^{\mathbb{S}}, Q_1^{\mathbb{S}})$;
- A division K -algebra D_i for each $i \in Q_0^{\mathbb{S}}$;
- A D_i - D_j -bimodule X_{α} for each $\alpha : i \rightarrow j$ in $Q_1^{\mathbb{S}}$.

If K algebraically closed, then every division K -algebra is isomorphic to K itself. Choosing every division K -algebra in the definition of a K -species to be K itself and choosing moreover every K - K -bimodule to be K as well, then the resulting K -species is simply a quiver. The *tensor path algebra* of a K -species $\mathbb{S} = (Q^{\mathbb{S}}, D_i, X_{\alpha})$ is defined as the K -algebra

$$K\mathbb{S} := \bigoplus_{\ell \geq 0} X^{\otimes \ell}, \quad \text{where } X^{\otimes 0} = \bigoplus_{i \in Q_0^{\mathbb{S}}} D_i, \quad X = \bigoplus_{\alpha \in Q_1^{\mathbb{S}}} X_{\alpha}.$$

The tensor products in this definition are taken over $X^{\otimes 0}$, with the bimodules $\{X_{\alpha}\}_{\alpha \in Q_1^{\mathbb{S}}}$ and X becoming $X^{\otimes 0}$ - $X^{\otimes 0}$ -bimodules in the standard way. For algebras over non-algebraically closed fields, the endomorphism rings of bricks may not all be isomorphic to the ground field. The following definition and subsequent theorem illustrate the usefulness of particularly well-behaved classes of bricks.

Definition 5.6.2. Let A be a finite-dimensional K -algebra. A brick $B \in \text{brick } A$ is called a *stone* if $\text{Ext}_A^1(B, B) = 0$. Moreover, it is called a K -*stone* if it additionally satisfies $\text{End}_A(B) \cong K$. If every brick of A is a (K) -stone, then the algebra A is called a (K) -*stone algebra*.

If the division ring D_i is assigned to a vertex $i \in Q_0^{\mathbb{S}}$ of a K -species, then the corresponding simple representation/module is a brick whose endomorphism ring is isomorphic to D_i . In the remainder of this section the scope of the following result is expanded using the theory of base field extensions and Theorem 5.5.1.

Theorem 5.6.3. [92, Thm. 5.9] *Let K be a finite field and let A be τ -tilting finite K -algebra which admits Hall polynomials. Assume that A is such that for every semibrick $\{S, T\} \in \text{sbrick } A$ one of the following holds:*

- *S and T are both K -stones;*
- *$\text{Ext}_A^1(S, T) = 0 = \text{Ext}_A^1(T, S)$.*

Then there exists a faithful group functor $\mathfrak{T}(A) \rightarrow G(A)$.

Remark 5.6.4. While the result [92, Thm. 5.9] is stated in greater generality in the original text, the adaptation of the result as above is necessary to guarantee that the result holds. In the original text the existence of Hall polynomials [160, 161] is assumed. However, this notion is generally only defined for algebras over finite fields, and the existence of Hall polynomials a highly nontrivial property, known to hold only for few classes of algebras, see [88, 99, 147, 151, 160] and the references therein.

Using Corollary 5.5.4, the result Theorem 5.6.3 may be extended to yield new families of algebras for which the τ -cluster morphism category admits a faithful group functor.

Theorem 5.6.5. *Let K be a perfect field and let A be a quotient of a representation finite and hereditary K -algebra. Then $\mathfrak{T}(A)$ admits a faithful group functor.*

Proof. Since K is a perfect field, the result [26, Cor. 4.1.11] states that any K -algebra A is of the form $K\mathbb{S}/I$, where $\mathbb{S}$ is a K -species known as the Ext- K -species of A and I is an ideal of $K\mathbb{S}$. By assumption, it is possible to choose $A \cong K\mathbb{S}/I$ with $K\mathbb{S}$ a representation finite and hereditary algebra. It then follows from the isomorphisms

$$A_{\overline{K}} \cong (K\mathbb{S}/I)_{\overline{K}} \cong (K\mathbb{S})_{\overline{K}}/(I_{\overline{K}}),$$

that $A_{\overline{K}}$ is isomorphic to a quotient of a representation finite and hereditary algebra. That is, because $K\mathbb{S}$ is representation finite, so is $(K\mathbb{S})_{\overline{K}}$ by [116, Thm. 3.2], and because $K\mathbb{S}$ is hereditary, so is $(K\mathbb{S})_{\overline{K}}$ by [116, Thm. 2.4].

As a quotient of a representation finite hereditary $\overline{K}$ -algebra, it is possible to write $\text{mod } A_{\overline{K}} \cong \text{mod } (\overline{K}Q/\langle P \rangle)$ for some quiver Q of Dynkin type ADE and a set of monomials P , generating an admissible ideal. As a consequence of Gabriel's Theorem [74], finite-dimensional $\overline{K}Q$ -modules are in bijection with finite-dimensional $\mathbb{F}_2Q$ -modules.

It then follows from [108, Thm. 4.3], that both $\text{tors}(\mathbb{F}_2 Q)$ and $\text{tors}(\overline{K} Q)$ are isomorphic to a Cambrian lattice of type Q , as introduced in [154]. Moreover, since the ideal $\langle P \rangle$ of $\overline{K} Q$ is generated by monomials, it makes sense to also consider $\mathbb{F}_2 Q / \langle P \rangle$.

The lattice congruence on $\text{tors}(\overline{K} Q)$ induced by $\langle P \rangle$ is determined by the bricks the ideal annihilates, by Theorem 4.4.1. The bijection between indecomposable $\overline{K} Q$ -modules and indecomposable $\mathbb{F}_2 Q$ -modules yields directly that $\text{tors}(\overline{K} Q / \langle P \rangle) \cong \text{tors}(\mathbb{F}_2 Q / \langle P \rangle)$. The corresponding lattice congruence on the Cambrian lattice of type Q may be described using the join-irreducible elements corresponding to the annihilated bricks. It follows from Theorem 4.3.7 that there is an equivalence of categories

$$\mathfrak{T}(A_{\overline{K}}) \cong \mathfrak{T}(\overline{K} Q / \langle P \rangle) \cong \mathfrak{T}(\mathbb{F}_2 Q / \langle P \rangle). \quad (5.6.1)$$

To conclude, it remains to show that $\mathfrak{T}(\mathbb{F}_2 Q / \langle P \rangle)$ admits a faithful group functor. It is well-known that quotients of representation finite hereditary algebras are directed, so that $\mathbb{F}_2 / \langle P \rangle$ is a directed algebra. Hence, the algebra $\mathbb{F}_2 Q / \langle P \rangle$ admits Hall polynomial by [160, Thm. 1]. Similarly, since $\mathbb{F}_2 Q / \langle P \rangle$ is the quotient of a representation finite hereditary path algebra, it is an $\mathbb{F}_2$ -stone algebra. Thus, $\mathfrak{T}(\mathbb{F}_2 Q / \langle P \rangle)$ admits a faithful group functor by Theorem 5.6.3, and so does $\mathfrak{T}(A_{\overline{K}})$ by Eq. (5.6.1). Consequently, $\mathfrak{T}(A)$ admits a faithful group functor by Corollary 5.5.4. $\square$

Example 5.6.6. Let $K = \mathbb{R}$ and let $\mathbb{S}$ be the following K -species

$$\mathbb{R} \xrightarrow{\mathbb{R}\mathbb{R}} \mathbb{R} \xrightarrow{\mathbb{R}\mathbb{R}} \mathbb{R} \xrightarrow{\mathbb{R}\mathbb{C}} \mathbb{C}$$

of Dynkin type C_4 . Let A be the radical square zero quotient of $K\mathbb{S}$, so in the previous notation $A \cong K\mathbb{S} / \langle X^{\otimes 2} \rangle$. Thus, $\mathfrak{T}(A)$ admits a faithful group functor by Theorem 5.6.5. It can be checked, that $A_{\mathbb{C}}$ is isomorphic to the radical square zero quotient of the path algebra

$$\mathbb{C} \left(\begin{array}{c} \bullet \longrightarrow \bullet \longrightarrow \bullet \begin{array}{l} \nearrow \bullet \\ \searrow \bullet \end{array} \end{array} \right)$$

of Dynkin type D_5 .

Similar types of (un)foldings of quivers and species and their associated representations have been studied in the literature, see [58, 59, 100, 178].

References

- [1] T. Adachi, T. Aihara, and A. Chan. “Classification of two-term tilting complexes over Brauer graph algebras”. *Math. Z.* 290 (2018), pp. 1–36 (cit. on pp. 85, 110).
- [2] T. Adachi, H. Enomoto, and M. Tsukamoto. “Intervals of s -torsion pairs in extriangulated categories with negative first extensions”. *Math. Proc. Cambridge Philos. Soc.* 174.3 (2023), pp. 451–469 (cit. on p. 11).
- [3] T. Adachi, O. Iyama, and I. Reiten. “ τ -tilting theory”. *Compos. Math.* 150.3 (2014), pp. 415–452 (cit. on pp. 4, 7, 8, 12, 13, 17–19, 33, 79, 84, 118, 121).
- [4] I. Agol. “The virtual Haken conjecture (with an appendix by I. Agol, D. Groves and J. Manning).” *Doc. Math.* 18 (2013), pp. 1045–1087 (cit. on p. 3).
- [5] M. Aguiar and S. Mahajan. *Topics in hyperplane arrangements*. Vol. 226. Math. Surv. Monogr. American Mathematical Society (AMS), 2017 (cit. on p. 62).
- [6] T. Aoki, A. Higashitani, O. Iyama, R. Kase, and Y. Mizuno. *Fans and polytopes in tilting theory I: Foundations*. 2022. arXiv: 2203.15213 (cit. on p. 95).
- [7] T. Aoki and T. Yurikusa. “Complete gentle and special biserial algebras are g -tame”. *J. Algebr. Comb.* 57.4 (2023), pp. 1103–1137 (cit. on p. 7).
- [8] S. Asai. “Semibricks”. *Int. Math. Res. Not.* 2020 (16 2020), pp. 4993–5054 (cit. on pp. 13, 17, 18, 123, 125, 127, 128).
- [9] S. Asai. “The wall-chamber structures of the real Grothendieck groups”. *Adv. Math.* 381 (2021), Paper No. 107615 (cit. on pp. 7, 12, 63, 73, 74, 82).
- [10] S. Asai and C. Pfeifer. “Wide subcategories and lattices of torsion classes”. *Algebr. Represent. Theory* 25 (2022), pp. 1611–1629 (cit. on pp. 16, 17, 83, 84, 86).

- [11] I. Assem, A. Skowroński, and D. Simson. *Elements of the representation theory of associative algebras: Techniques of representation theory*. Vol. 1. Lond. Math. Soc. Stud. Texts. Cambridge University Press, 2006 (cit. on pp. 32, 96).
- [12] J. August. “The tilting theory of contraction algebras”. *Adv. Math.* 374 (2020), Paper No. 107372 (cit. on pp. 8, 75).
- [13] M. Auslander, M. I. Platzzeck, and I. Reiten. “Coxeter functors without diagrams”. *Trans. Am. Math. Soc.* 250 (1979), pp. 1–46 (cit. on p. 4).
- [14] M. Auslander and I. Reiten. “Representation theory of Artin algebras. III: Almost split sequences”. *Commun. Algebra* 3 (1975), pp. 239–294 (cit. on p. 4).
- [15] M. Auslander, I. Reiten, and S. O. Smalø. *Representation Theory of Artin Algebras*. Camb. Stud. Adv. Math. Cambridge University Press, 1995 (cit. on pp. 4, 32).
- [16] M. Auslander and S. O. Smalø. “Almost split sequences in subcategories”. *J. Algebra* 69.2 (1981), pp. 426–454 (cit. on pp. 4, 12, 32).
- [17] M. Auslander and S. O. Smalø. “Preprojective modules over Artin algebras”. *J. Algebra* 66.1 (1980), pp. 61–122 (cit. on p. 13).
- [18] E. Barnard, A. Carroll, and S. Zhu. “Minimal inclusions of torsion classes”. *Algebr. Comb.* 2.5 (2019), pp. 879–901 (cit. on pp. 8, 12, 17, 84).
- [19] E. Barnard and E. J. Hanson. “Exceptional sequences in semidistributive lattices and the poset topology of wide subcategories”. *J. Algebra Appl.* (2024) (cit. on pp. 9, 84, 89).
- [20] E. Barnard and E. J. Hanson. “Pairwise compatibility for 2-simple minded collections II: Preprojective Aagebras and semibrick pairs of full rank”. *Ann. Comb.* 26.4 (2022), pp. 803–855 (cit. on pp. 6, 26, 27, 75, 76, 109).
- [21] P. Baumann, J. Kamnitzer, and P. Tingley. “Affine Mirković-Vilonen polytopes”. *Publ. Math., Inst. Hautes Étud. Sci.* 120 (2014), pp. 113–205 (cit. on p. 73).

- [22] R. Bautista and L. Salmerón. “On discrete and inductive algebras.” In: *Representations of algebras and related topics. Proceedings from the 10th international conference, ICRA X, Toronto, Canada, July 15–August 10, 2002*. American Mathematical Society (AMS), 2005, pp. 17–35 (cit. on p. 10).
- [23] M. A. Bednarczyk, A. M. Borzyszkowski, and W. Pawłowski. “Generalized congruences – epimorphisms in $\mathcal{Cat}$ ”. *Theory Appl. Categ.* 5 (1999), No. 11, 266–280 (cit. on pp. 104, 106).
- [24] A. A. Beilinson, J. Bernstein, and P. Deligne. “Faisceaux pervers”. In: *Analysis and topology on singular spaces, I (Luminy, 1981)*. Vol. 100. Astérisque. Soc. Math. France, Paris, 1982, pp. 5–171 (cit. on pp. 8, 11).
- [25] A. Beligiannis and I. Reiten. “Homological and homotopical aspects of torsion theories”. *Mem. Amer. Math. Soc.* 188.883 (2007), pp. viii+207 (cit. on p. 11).
- [26] D. J. Benson. *Representations and cohomology. I: Basic representation theory of finite groups and associative algebras*. Vol. 30. Camb. Stud. Adv. Math. Cambridge University Press, 1991 (cit. on p. 133).
- [27] G. M. Bergman. “The logarithmic limit-set of an algebraic variety”. *Trans. Am. Math. Soc.* 157 (1971), pp. 459–469 (cit. on pp. 8, 29).
- [28] I. N. Bernstein, I. M. Gel’fand, and V. A. Ponomarev. “Coxeter functors and Gabriel’s theorem”. *Russ. Math. Surv.* 28.2 (1973), pp. 17–32 (cit. on p. 4).
- [29] A. Björner, P. H. Edelman, and G. M. Ziegler. “Hyperplane arrangements with a lattice of regions”. *Discrete Comput. Geom.* 5 (1990), pp. 263–288 (cit. on p. 61).
- [30] A. I. Bondal. “Representations of associative algebras and coherent sheaves”. *Izv. Akad. Nauk SSSR Ser. Mat.* 53.1 (1989), pp. 25–44 (cit. on p. 2).
- [31] K. Bongartz. “Tilted algebras”. In: *Representations of algebras (Puebla, 1980)*. Vol. 903. Lect. Notes Math.. Springer, 1981, pp. 26–38 (cit. on p. 4).
- [32] E. D. Børve. *Silting reduction and picture categories of 0-Auslander extriangulated categories*. 2024. arXiv: 2405.00593 (cit. on p. 21).
- [33] E. D. Børve. *Two-term silting and τ -cluster morphism categories*. 2021. arXiv: 2110.03472 (cit. on pp. 7, 21, 22).

- [34] E. D. Børve and M. Kaipel. *Bricks and τ -tilting theory under base field extensions (with an appendix by E. J. Hanson)*. 2025. arXiv: 2508.01040 (cit. on p. ii).
- [35] S. Brenner and M. C. R. Butler. “Generalizations of the Bernstein-Gel’fand-Ponomarev reflection functors”. In: *Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979)*. Vol. 832. Lect. Notes Math. Springer, 1980, pp. 103–169 (cit. on pp. 4, 8, 12).
- [36] T. Bridgeland. “Scattering diagrams, Hall algebras and stability conditions”. *Algebr. Geom.* 4.5 (2017), pp. 523–561 (cit. on p. 7).
- [37] N. Broomhead, D. Pauksztello, D. Ploog, and J. Woolf. *The heart fan of an abelian category*. 2023. arXiv: 2310.02844 (cit. on pp. 8, 29, 32).
- [38] T. Brüstle, D. Smith, and H. Treffinger. “Wall and chamber structure for finite-dimensional algebras”. *Adv. Math.* 354 (2019), Paper No. 106746 (cit. on pp. 7, 12, 19, 63, 73, 74).
- [39] T. Brüstle and D. Yang. “Ordered exchange graphs”. In: *Advances in representation theory of algebras*. European Mathematical Society (EMS), 2014, pp. 135–193 (cit. on p. 12).
- [40] A. B. Buan and E. J. Hanson. “ τ -perpendicular wide subcategories”. *Nagoya Math. J.* 252 (2023), pp. 959–984 (cit. on pp. 3, 5–7, 20–22, 83, 94, 95).
- [41] A. B. Buan, E. J. Hanson, and B. R. Marsh. *Mutation of τ -exceptional pairs and sequences*. 2024. arXiv: 2402.10301 (cit. on pp. 5, 22, 89).
- [42] A. B. Buan, E. J. Hanson, and B. R. Marsh. *Transitivity of mutation of τ -exceptional sequences in the τ -tilting finite case*. 2025. arXiv: 2506.21372 (cit. on pp. 5, 22).
- [43] A. B. Buan, M. Kaipel, and H. U. Terland. *Mutating ordered τ -rigid modules with applications to Nakayama algebras*. 2025. arXiv: 2501.13694 (cit. on pp. 5, 22, 83).
- [44] A. B. Buan and H. Krause. “Tilting and cotilting for quivers and type $\tilde{A}_n$ ”. *J. Pure Appl. Algebra* 190.1-3 (2004), pp. 1–21 (cit. on p. 3).

- [45] A. B. Buan and B. R. Marsh. “ τ -exceptional sequences”. *J. Algebra* 585 (2021), pp. 36–68 (cit. on pp. 3, 5, 22, 83, 89).
- [46] A. B. Buan and B. R. Marsh. “A category of wide subcategories”. *Int. Math. Res. Not.* 2021 (13 2021), pp. 10278–10338 (cit. on pp. 5, 7, 21, 22).
- [47] A. B. Buan and B. R. Marsh. “Mutating signed τ -exceptional sequences”. *Glasg. Math. J.* 65.3 (2023), pp. 716–729 (cit. on p. 22).
- [48] A. B. Buan, B. R. Marsh, M. Reineke, I. Reiten, and G. Todorov. “Tilting theory and cluster combinatorics”. *Adv. Math.* 204.2 (2006), pp. 572–618 (cit. on p. 1).
- [49] A. B. Buan, B. R. Marsh, I. Reiten, and G. Todorov. “Clusters and seeds in acyclic cluster algebras. With an appendix coauthored in addition by P. Caldero and B. Keller.” *Proc. Am. Math. Soc.* 135.10 (2007), pp. 3049–3060 (cit. on p. 1).
- [50] P. Caldero and B. Keller. “From triangulated categories to cluster algebras”. *Invent. Math.* 172.1 (2008), pp. 169–211 (cit. on p. 1).
- [51] P. Caldero and B. Keller. “From triangulated categories to cluster algebras. II.” *Ann. Sci. Éc. Norm. Supér. (4)* 39.6 (2006), pp. 983–1009 (cit. on p. 1).
- [52] F. Chapoton, S. Fomin, and A. Zelevinsky. “Polytopal realizations of generalized associahedra”. *Can. Math. Bull.* 45.4 (2002), pp. 537–566 (cit. on p. 7).
- [53] C. Cibils, M. Lanzilotta, E. N. Marcos, and A. Solotar. *On the first τ -tilting Hochschild cohomology of an algebra*. 2024. arXiv: 2404.06916 (cit. on p. 5).
- [54] W. Crawley-Boevey. “Exceptional sequences of representations of quivers”. In: *Representations of algebras. (Ottawa, ON, 1992)*. CMS Conf. Proc., vol. 14. American Mathematical Society (AMS), 1993, pp. 117–124 (cit. on pp. 2, 6, 22, 89).
- [55] R. Dehy and B. Keller. “On the combinatorics of rigid objects in 2-Calabi-Yau categories”. *Int. Math. Res. Not.* 2008 (2008). ID/No. rnn029 (cit. on p. 7).
- [56] L. Demonet, O. Iyama, and G. Jasso. “ τ -tilting finite algebras, bricks, and g -vectors”. *Int. Math. Res. Not.* 2019.3 (2019), pp. 852–892 (cit. on pp. 5, 7, 8, 10, 18, 29, 33, 74, 85).

- [57] L. Demonet, O. Iyama, N. Reading, I. Reiten, and H. Thomas. “Lattice theory of torsion classes: Beyond τ -tilting theory”. *Trans. Am. Math. Soc., Ser. B* 10 (2023), pp. 542–612 (cit. on pp. 5, 8, 12, 16, 17, 19, 61, 73, 77, 78, 80, 84–87, 90, 94, 96–99, 101, 103, 106, 109, 120, 123, 125, 126).
- [58] B. Deng and J. Du. “Folding derived categories with Frobenius functors.” *J. Pure Appl. Algebra* 208.3 (2007), pp. 1023–1050 (cit. on pp. 10, 134).
- [59] B. Deng and J. Du. “Frobenius morphisms and representations of algebras.” *Trans. Am. Math. Soc.* 358.8 (2006), pp. 3591–3622 (cit. on pp. 10, 134).
- [60] H. Derksen and J. Fei. “General presentations of algebras”. *Adv. Math.* 278 (2015), pp. 210–237 (cit. on pp. 7, 118).
- [61] H. Derksen, J. Weyman, and A. Zelevinsky. “Quivers with potentials and their representations. II: Applications to cluster algebras”. *J. Am. Math. Soc.* 23.3 (2010), pp. 749–790 (cit. on p. 7).
- [62] S. E. Dickson. “A torsion theory for abelian categories”. *Trans. Am. Math. Soc.* 121.1 (1966), pp. 223–235 (cit. on pp. 1, 8, 11, 12).
- [63] V. Dlab and C. M. Ringel. *Indecomposable representations of graphs and algebras*. Vol. 173. Mem. Am. Math. Soc. American Mathematical Society (AMS), 1976 (cit. on p. 9).
- [64] V. Dlab and C. M. Ringel. “On algebras of finite representation type”. *J. Algebra* 33.2 (1975), pp. 306–394 (cit. on p. 9).
- [65] P. H. Edelman. “A partial order on the regions of $\mathbb{R}^n$ dissected by hyperplanes”. *Trans. Am. Math. Soc.* 283.2 (1984), pp. 617–631 (cit. on pp. 51, 61, 62).
- [66] F. Eisele, G. Janssens, and T. Raedschelders. “A reduction theorem for τ -rigid modules”. *Math. Z.* 290 (2018), pp. 1377–1413 (cit. on pp. 9, 85).
- [67] H. Enomoto. “From the lattice of torsion classes to the posets of wide subcategories and ICE-closed subcategories”. *Algebr. Represent. Theory* 26.6 (2023), pp. 3223–3253 (cit. on pp. 9, 12, 84, 86).
- [68] H. Enomoto. “Rigid modules and ICE-closed subcategories in quiver representations”. *J. Algebra* 594 (2022), pp. 364–388 (cit. on p. 13).

- [69] K. Erdmann and C. Sáenz. “On standardly stratified algebras”. *Commun. Algebra* 31.7 (2003), pp. 3429–3446 (cit. on p. 2).
- [70] V. V. Fock and A. B. Goncharov. “Cluster ensembles, quantization and the dilogarithm”. *Ann. Sci. Éc. Norm. Supér. (4)* 42.6 (2009), pp. 865–930 (cit. on p. 7).
- [71] S. Fomin and A. Zelevinsky. “Cluster algebras. I: Foundations”. *J. Am. Math. Soc.* 15.2 (2002), pp. 497–529 (cit. on p. 1).
- [72] S. Fomin and A. Zelevinsky. “Cluster algebras. IV: Coefficients.” *Compos. Math.* 143.1 (2007), pp. 112–164 (cit. on p. 7).
- [73] W. Fulton. *Introduction to toric varieties*. Vol. 131. Ann. Math. Stud. Princeton University Press, 1993 (cit. on pp. 8, 29, 30, 59).
- [74] P. Gabriel. “Unzerlegbare Darstellungen I”. *Manuscr. Math.* 6 (1972), pp. 71–103 (cit. on pp. 1, 9, 133).
- [75] D. Gaiotto, G. W. Moore, and A. Neitzke. “Wall-crossing in coupled 2d-4d systems”. *J. High Energy Phys.* 2012.12 (2012). Id/No 82 (cit. on p. 1).
- [76] M. Garcia. *On g -finiteness in the category of projective presentations*. 2024. arXiv: 2406.04134 (cit. on p. 18).
- [77] A. Garver and T. McConville. “Lattice properties of oriented exchange graphs and torsion classes”. *Algebr. Represent. Theory* 22.1 (2019), pp. 43–78 (cit. on pp. 8, 12, 16).
- [78] W. Geigle and H. Lenzing. “Perpendicular categories with applications to representations and sheaves”. *J. Algebra* 144.2 (1991), pp. 273–343 (cit. on pp. 2, 20).
- [79] C. Geiss, B. Leclerc, and J. Schröer. “Quivers with relations for symmetrizable Cartan matrices I: Foundations”. *Invent. Math.* 209.1 (2017), pp. 61–158 (cit. on pp. 8, 75, 76).
- [80] S. M. Gersten. “The automorphism group of a free group is not a CAT(0) group”. *Proc. Am. Math. Soc.* 121.4 (1994), pp. 999–1002 (cit. on p. 3).

- [81] J. Geuenich. *String Applet*. accessed 2025. URL: <https://www.math.uni-bielefeld.de/~jgeuenich/string-applet/> (cit. on p. 111).
- [82] A. L. Gorodentsev. “Exceptional bundles on surfaces with a moving anticanonical class”. *Izv. Akad. Nauk SSSR Ser. Mat.* 52.4 (1988), pp. 740–757, 895 (cit. on p. 2).
- [83] A. L. Gorodentsev and A. N. Rudakov. “Exceptional vector bundles on projective spaces”. *Duke Math. J.* 54.1 (1987), pp. 115–130 (cit. on p. 2).
- [84] G. Grätzer. *General Lattice Theory: Second edition*. Birkhäuser Basel, 2002 (cit. on pp. 14, 68).
- [85] J. A. Green. “Hall algebras, hereditary algebras and quantum groups”. *Invent. Math.* 120.2 (1995), pp. 361–377 (cit. on p. 1).
- [86] M. Gromov. “Hyperbolic Groups”. In: *Essays in Group Theory*. Ed. by S. M. Gersten. Springer, 1987, pp. 75–263 (cit. on pp. 3, 6, 23, 26).
- [87] M. Gross, P. Hacking, S. Keel, and M. Kontsevich. “Canonical bases for cluster algebras”. *J. Am. Math. Soc.* 31.2 (2018), pp. 497–608 (cit. on p. 7).
- [88] J. Y. Guo. “The Hall polynomials of a cyclic serial algebra”. *Commun. Algebra* 23.2 (1995), pp. 743–751 (cit. on p. 133).
- [89] F. Haglund and D. T. Wise. “Special cube complexes”. *Geom. Funct. Anal.* 17.5 (2008), pp. 1551–1620 (cit. on p. 3).
- [90] E. J. Hanson. “A facial order for torsion classes”. *Int. Math. Res. Not.* 2024 (12 2024), pp. 9849–9874 (cit. on p. 17).
- [91] E. J. Hanson and K. Igusa. “ τ -cluster morphism categories and picture groups”. *Commun. Algebra* 49.10 (2021), pp. 4376–4415 (cit. on pp. 3, 6, 22, 23, 26, 27, 44, 46, 47, 55, 66, 74, 75).
- [92] E. J. Hanson and K. Igusa. “Pairwise compatibility for 2-simple minded collections”. *J. Pure Appl. Algebra* 225.6 (2021), Paper No. 106598 (cit. on pp. 6, 10, 22, 26, 27, 66, 76, 109, 111, 132, 133).
- [93] D. Happel and C. M. Ringel. “Tilted algebras”. *Trans. Am. Math. Soc.* 274 (1982), pp. 399–443 (cit. on p. 4).

- [94] D. Happel and L. Unger. “Reconstruction of path algebras from their posets of tilting modules.” *Trans. Am. Math. Soc.* 361.7 (2009), pp. 3633–3660 (cit. on p. 9).
- [95] A. Hatcher. *Algebraic Topology*. Cambridge University Press, 2002 (cit. on pp. 3, 46, 56, 58).
- [96] L. Hille. “On the volume of a tilting module.” *Abh. Math. Semin. Univ. Hamb.* 76 (2006), pp. 261–277 (cit. on pp. 29, 32).
- [97] A. Hohl. *An introduction to field extensions and Galois descent for sheaves of vector spaces*. 2023. arXiv: 2302.14837 (cit. on p. 10).
- [98] M. Hovey. “Classifying subcategories of modules”. *Trans. Am. Math. Soc.* 353.8 (2001), pp. 3181–3191 (cit. on pp. 1, 12).
- [99] A. Hubery. “Hall polynomials for affine quivers.” *Represent. Theory* 14 (2010), pp. 355–378 (cit. on p. 133).
- [100] A. Hubery. “Quiver representations respecting a quiver automorphism: A generalisation of a theorem of Kac”. *J. Lond. Math. Soc., II. Ser* 69.1 (2004), pp. 79–96 (cit. on pp. 10, 134).
- [101] K. Igusa. *A category of noncrossing partitions*. 2014. arXiv: 1411.0196 (cit. on pp. 3, 6, 24, 26, 46, 75).
- [102] K. Igusa. *Are finite type picture groups virtually special?* accessed 2025. URL: <https://people.brandeis.edu/~igusa/Papers/lowa2017.pdf> (cit. on p. 6).
- [103] K. Igusa and R. Schiffler. “Exceptional sequences and clusters. (With an appendix with H. Thomas).” *J. Algebra* 323.8 (2010), pp. 2183–2202 (cit. on p. 2).
- [104] K. Igusa and G. Todorov. “Picture groups and maximal green sequences”. *Electron Res. Arch.* 29.5 (2021), pp. 3031–3068 (cit. on pp. 3, 22).
- [105] K. Igusa and G. Todorov. *Signed exceptional sequences and the cluster morphism category*. 2017. arXiv: 1706.02041 (cit. on pp. 2, 21, 22, 46).
- [106] K. Igusa and G. Todorov. *Which cluster morphism categories are CAT(0)*. 2022. arXiv: 2203.16679 (cit. on pp. 2, 3, 10, 22, 25, 27, 109, 132).

- [107] K. Igusa, G. Todorov, and J. Weyman. *Picture groups of finite type and cohomology in type A_n* . 2016. arXiv: 1609.02636 (cit. on pp. 2, 22).
- [108] C. Ingalls and H. Thomas. “Noncrossing partitions and representations of quivers”. *Compos. Math.* 145.6 (2009), pp. 1533–1562 (cit. on pp. 1, 2, 12, 20, 94, 134).
- [109] O. Iyama, I. Reiten, H. Thomas, and G. Todorov. “Lattice structure of torsion classes for path algebras”. *Bull. Lond. Math. Soc.* 47.4 (2015), pp. 639–650 (cit. on pp. 8, 12, 18).
- [110] O. Iyama and Y. Kimura. “Classifying subcategories of modules over Noetherian algebras”. *Adv. Math.* 446 (2024), Paper No. 109631 (cit. on pp. 10, 119).
- [111] O. Iyama, N. Reading, I. Reiten, and H. Thomas. “Lattice structure of Weyl groups via representation theory of preprojective algebras”. *Compos. Math.* 154.6 (2018), pp. 1269–1305 (cit. on pp. 8, 12).
- [112] O. Iyama and D. Yang. “Silting reduction and Calabi-Yau reduction of triangulated categories”. *Trans. Am. Math. Soc.* 370.11 (2018), pp. 7861–7898 (cit. on p. 22).
- [113] N. Jacobson. *Lectures in abstract algebra. Vol. 3: Theory of fields and Galois theory. Reprint of the 1951-1964 Van Nostrand ed.* Vol. 32. Grad. Texts Math. Springer, 1976 (cit. on pp. 114, 115).
- [114] J. P. Jans. “On the indecomposable representations of algebras”. *Ann. Math. (2)* 66 (1957), pp. 418–429 (cit. on p. 5).
- [115] G. Jasso. “Reduction of τ -tilting modules and torsion pairs”. *Int. Math. Res. Not.* 2015.16 (2015), pp. 7190–7237 (cit. on pp. 3, 5, 19, 73).
- [116] C. U. Jensen and H. Lenzing. “Homological dimension and representation type of algebras under base field extension”. *Manuscripta Math.* 39.1 (1982), pp. 1–13 (cit. on pp. 10, 115, 133).
- [117] P. Jørgensen. “Torsion classes and t-structures in higher homological algebra”. *Int. Math. Res. Not.* 2016 (13 2016), pp. 3880–3905 (cit. on p. 11).

- [118] V. G. Kac. “Infinite root systems, representations of graphs and invariant theory”. *Invent. Math.* 56 (1980), pp. 57–92 (cit. on p. 9).
- [119] M. Kaipel. τ -cluster morphism categories of factor algebras. 2024. arXiv: 2408.03818 (cit. on p. ii).
- [120] M. Kaipel. “The category of a partitioned fan”. *J. Lond. Math. Soc., II. Ser.* 111.2 (2025), Paper No. e70071 (cit. on pp. ii, 33).
- [121] R. Kase. *From support τ -tilting posets to algebras I*. 2024 (cit. on pp. 9, 84).
- [122] S. Kasjan. “Auslander-Reiten sequences under base field extension”. *Proc. Am. Math. Soc.* 128.10 (2000), pp. 2885–2896 (cit. on pp. 10, 113–115).
- [123] S. Kasjan. “Base field extensions and generic modules over finite dimensional algebras”. *Arch. Math.* 77.2 (2001), pp. 155–162 (cit. on p. 10).
- [124] B. Keller. “On cluster theory and quantum dilogarithm identities”. In: *Representations of algebras and related topics*. EMS Ser. Congr. Rep. Eur. Math. Soc., Zürich, 2011, pp. 85–116 (cit. on pp. 3, 22).
- [125] A. D. King. “Moduli of representations of finite dimensional algebras”. *Q. J. Math.* 45.4 (1994), pp. 515–530 (cit. on pp. 12, 74).
- [126] Y. Kodama and L. K. Williams. “KP solitons, total positivity, and cluster algebras”. *Proc. Natl. Acad. Sci. USA* 108.22 (2011), pp. 8984–8989 (cit. on p. 1).
- [127] M. Kontsevich and Y. Soibelman. *Stability structures, motivic Donaldson-Thomas invariants and cluster transformations*. 2008. arXiv: 0811.2435 (cit. on pp. 1, 3).
- [128] H. Krause. “Highest weight categories and recollements”. *Ann. Inst. Fourier* 67.6 (2017), pp. 2679–2701 (cit. on p. 2).
- [129] G. Kreweras. “Sur les partitions non croisées d’un cycle”. *Discrete Math.* 1 (1972), pp. 333–350 (cit. on p. 2).
- [130] T. Y. Lam. *A first course in noncommutative rings*. Second. Vol. 131. Graduate Texts in Mathematics. Springer, 2001, pp. xx+385 (cit. on p. 113).
- [131] H. Lenzing and J. A. de la Peña. “Wild canonical algebras”. *Math. Z.* 224.3 (1997), pp. 403–425 (cit. on p. 2).

- [132] J. Li. “Piecewise hereditary algebras under field extensions.” *Czech. Math. J.* 71.4 (2021), pp. 1025–1034 (cit. on p. 10).
- [133] J. Li. “The derived-discrete algebras over the real numbers”. *Algebr. Represent. Theory* 26.4 (2023), pp. 1141–1162 (cit. on p. 10).
- [134] J.-L. Loday. “Homotopical syzygies”. In: *Une dégustation topologique: homotopy theory in the Swiss Alps (Arolla, 1999)*. Vol. 265. Contemp. Math. American Mathematical Society (AMS), 2000, pp. 99–127 (cit. on p. 3).
- [135] F. Marks and J. Šťovíček. “Torsion classes, wide subcategories and localisations”. *Bull. Lond. Math. Soc.* 49.3 (2017), pp. 405–416 (cit. on pp. 12, 18).
- [136] B. R. Marsh, M. Reineke, and A. Zelevinsky. “Generalized associahedra via quiver representations”. *Trans. Am. Math. Soc.* 355.10 (2003), pp. 4171–4186 (cit. on p. 1).
- [137] H. Meltzer. “Auslander-Reiten components for concealed-canonical algebras”. *Colloq. Math.* 71.2 (1996), pp. 183–202 (cit. on p. 2).
- [138] G. Méndez and E. Pérez. “A remark on generic tameness preservation under base field extension”. *J. Algebra Appl.* 12.4 (2013), Paper No. 1250183 (cit. on p. 10).
- [139] O. Mendoza and H. Treffinger. “Stratifying systems through τ -tilting theory”. *Doc. Math.* 25 (2020), pp. 701–720 (cit. on p. 22).
- [140] J. S. Milne. *Descent for Algebraic Schemes*. 2024. arXiv: 2406.05550 (cit. on p. 10).
- [141] Y. Mizuno. “Classifying τ -tilting modules over preprojective algebras of Dynkin type”. *Math. Z.* 277 (2014), pp. 665–690 (cit. on pp. 8, 12, 63, 75, 76).
- [142] Y. Mizuno. “Shard theory for g -fans”. *Int. Math. Res. Not.* 2024 (19 2024), pp. 13106–13126 (cit. on p. 63).
- [143] T. Mora and L. Robbiano. “The Gröbner fan of an ideal”. *J. Symb. Comput.* 6.2 (1988), pp. 183–208 (cit. on pp. 8, 29).
- [144] K. Mousavand. “ τ -tilting finiteness of non-distributive algebras and their module varieties”. *J. Algebra* 608 (2022), pp. 673–690 (cit. on p. 5).

- [145] D. Msapato. “Counting the number of τ -exceptional sequences over Nakayama algebras”. *Algebr. Represent. Theory* 25.5 (2022), pp. 1071–1105 (cit. on p. 22).
- [146] K. Murakami. “PBW parametrizations and generalized preprojective algebras”. *Adv. Math.* 395 (2022), Paper No. 108144 (cit. on pp. 8, 75, 76).
- [147] A. R. Nasr-Isfahani. “Hall polynomials and composition algebra of representation finite algebras”. *Algebr. Represent. Theory* 17.4 (2014), pp. 1155–1161 (cit. on p. 133).
- [148] G. A. Niblo and L. D. Reeves. “Coxeter groups act on CAT(0) cube complexes.” *J. Group Theory* 6.3 (2003), pp. 399–413 (cit. on p. 3).
- [149] I. Nonis. *τ -exceptional sequences for representations of quivers over local algebras*. 2025. arXiv: 2502.15417 (cit. on pp. 22, 110).
- [150] I. Nonis. *Mutation of τ -exceptional sequences for acyclic quivers over local algebras*. 2025. arXiv: 2505.22770 (cit. on pp. 5, 22).
- [151] L. Peng. “Some Hall polynomials for representation-finite trivial extension algebras”. *J. Algebra* 197.1 (1997), pp. 1–13 (cit. on p. 133).
- [152] E. Pérez. “On semigeneric tameness and base field extension”. *Glasg. Math. J.* 58.1 (2016), pp. 39–53 (cit. on p. 10).
- [153] C. Pfeifer. *On τ -tilted representation types and affine GLS algebras*. [Ph.D. thesis, SDU]. Syddansk Universitet. Det Naturvidenskabelige Fakultet. accessed 2025. 2023. URL: <https://doi.org/10.21996/tdt8-ge05> (cit. on p. 5).
- [154] N. Reading. “Cambrian lattices.” *Adv. Math.* 205.2 (2006), pp. 313–353 (cit. on p. 134).
- [155] N. Reading. “Lattice and order properties of the poset of regions in a hyperplane arrangement”. *Algebra Univers.* 50.2 (2003), pp. 179–205 (cit. on p. 62).
- [156] N. Reading. “Lattice congruences, fans and Hopf algebras”. *J. Comb. Theory, Ser. A* 110.2 (2005), pp. 237–273 (cit. on pp. 50–52, 54, 61, 62, 65).
- [157] N. Reading. “Lattice theory of the poset of regions”. In: *Lattice theory: special topics and applications. Volume 2*. Birkhäuser/Springer, 2016, pp. 399–487 (cit. on pp. 15, 56, 61, 67, 79, 99, 101, 102).

- [158] N. Reading. “Noncrossing partitions and the shard intersection order”. *J. Algebraic Combin.* 33.4 (2011), pp. 483–530 (cit. on p. 63).
- [159] N. Reading, D. E. Speyer, and H. Thomas. “The fundamental theorem of finite semidistributive lattices”. *Sel. Math., New Ser.* 27.4 (2021). Id/No 59 (cit. on p. 86).
- [160] C. M. Ringel. “Hall algebras”. In: *Topics in algebra, Part 1 (Warsaw, 1988)*. Vol. 26, Part 1. Banach Center Publ. PWN, Warsaw, 1990, pp. 433–447 (cit. on pp. 1, 27, 133, 134).
- [161] C. M. Ringel. “Hall algebras and quantum groups”. *Invent. Math.* 101.3 (1990), pp. 583–591 (cit. on pp. 1, 133).
- [162] C. M. Ringel. “Lattice structure of torsion classes for hereditary Artin algebras”. *Nagoya Math. J.* 231 (2018), pp. 89–100 (cit. on pp. 8, 12).
- [163] C. M. Ringel. “Representations of K -species and bimodules”. *J. Algebra* 41.2 (1976), pp. 269–302 (cit. on p. 13).
- [164] C. M. Ringel. “The braid group action on the set of exceptional sequences of a hereditary Artin algebra”. In: *Abelian group theory and related topics. (Oberwolfach Conference, 1993)*. Contemp. Math., vol. 171. American Mathematical Society (AMS), 1994, pp. 339–352 (cit. on pp. 2, 6, 22, 89).
- [165] C. M. Ringel and P. Zhang. “Representations of quivers over the algebra of dual numbers”. *J. Algebra* 475 (2017), pp. 327–360 (cit. on p. 110).
- [166] A. Rudakov. “Stability for an abelian category”. *J. Algebra* 197.1 (1997), pp. 231–245 (cit. on p. 74).
- [167] A. N. Rudakov, ed. *Helices and vector bundles: seminaire Rudakov. Transl. by A. D. King, P. Kobak and A. Maciocia*. Vol. 148. Lond. Math. Soc. Lect. Note Ser. Cambridge University Press, 1990 (cit. on p. 2).
- [168] A. Schrijver. *Combinatorial Optimization: Polyhedra and Efficiency*. Algorithms Comb. v. 1. Springer, 2003 (cit. on pp. 8, 29).

- [169] S. Schroll, A. Tattar, H. Treffinger, and N. J. Williams. *A geometric perspective on the τ -cluster morphism category*. 2023. arXiv: 2302.12217 (cit. on pp. 7, 8, 28, 33, 35, 39, 40, 73, 79, 82, 83).
- [170] S. Schroll and H. Treffinger. “A τ -tilting approach to the first Brauer-Thrall conjecture”. *Proc. Am. Math. Soc.* 150.11 (2022), pp. 4567–4574 (cit. on pp. 5, 18).
- [171] P. Schwer. *CAT(0) Cube Complexes: An Introduction*. Vol. 1. Lect. Notes Math. Springer, 2023 (cit. on pp. 2, 24).
- [172] G. Segal. “Classifying spaces and spectral sequences”. *Publ. Math., Inst. Hautes Étud. Sci.* 34 (1968), pp. 105–112 (cit. on p. 2).
- [173] F. Sentieri. “A brick version of a theorem of Auslander”. *Nagoya Math. J.* 249 (2023), pp. 88–106 (cit. on p. 18).
- [174] S. Shepherd. “Semistability of cubulated groups”. *Math. Ann.* 387.3-4 (2023), pp. 1481–1511 (cit. on p. 3).
- [175] R. Simion. “Noncrossing partitions”. *Discrete Math.* 217.1-3 (2000), pp. 367–409 (cit. on p. 2).
- [176] S. O. Smalø. “Torsion theories and tilting Modules”. *Bull. Lond. Math. Soc.* 16.5 (1984), pp. 518–522 (cit. on p. 13).
- [177] R. Speicher. “Free probability theory and non-crossing partitions”. *Sémin. Lothar. Comb.* 39 (1997), b39c, 38 (cit. on p. 2).
- [178] T. Tanisaki. “Foldings of root systems and Gabriel’s theorem”. *Tsukuba J. Math.* 4 (1980), pp. 89–97 (cit. on p. 134).
- [179] A. Tattar. “Torsion pairs and quasi-abelian categories”. *Algebr. Represent. Theory* 24.6 (2021), pp. 1557–1581 (cit. on p. 11).
- [180] H. Thomas. “Stability, shards, and preprojective algebras”. In: *Representations of algebras*. Vol. 705. Contemp. Math. American Mathematical Society (AMS), 2018, pp. 251–262 (cit. on pp. 12, 63).

- [181] H. Thomas. “The Tamari lattice as it arises in quiver representations”. In: *Associahedra, Tamari lattices and related structures*. Vol. 299. Progr. Math. Birkhäuser/Springer, 2012, pp. 281–291 (cit. on p. 12).
- [182] M. Wemyss. “Flops and clusters in the homological minimal model programme”. *Invent. Math.* 211.2 (2018), pp. 435–521 (cit. on pp. 8, 75).
- [183] D. T. Wise. *From riches to raags: 3-manifolds, right-angled Artin groups, and cubical geometry*. Vol. 117. CBMS Reg. Conf. Ser. Math. American Mathematical Society (AMS), 2012 (cit. on p. 3).
- [184] T. Yurikusa. “Wide subcategories are semistable”. *Doc. Math.* 23 (2018), pp. 35–47 (cit. on p. 12).
- [185] G. M. Ziegler. *Lectures on polytopes*. Vol. 152. Grad. Texts Math. Springer, 1995 (cit. on pp. 8, 29, 30).