On Divisors, Congruences, and Symmetric Powers of Modular Forms

Inaugural-Dissertation

zur
Erlangung des Doktorgrades
der Mathematisch-Naturwissenschaftlichen Fakultät
der Universität zu Köln

vorgelegt von
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aus
Lich

Köln, 2017

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Tag der mündlichen Prüfung: 19. Dezember 2017

Kurzzusammenfassung

Diese Dissertation umfasst Arbeiten zu verschiedenen Themen aus der Theorie der Modulformen. Als Erstes untersuchen wir Divisoren von meromorphen Modulformen höherer Stufe mithilfe von polaren harmonischen Maaßformen vom Gewicht 2. Danach setzen wir diese Maaßformen setzen in Beziehung zu Modulformen, die imaginär-quadratischen Zahlkörpern zugeordnet sind. Wir zeigen, dass Fourierkoeffizienten dieser Funktionen durch Spuren singulärer Moduln gegeben sind und berechnen ihre regularisierten inneren Produkte. Danach untersuchen wir die p-adischen Eigenschaften der Fourierkoeffizienten von verallgemeinerten Eta-Quotienten. Insbesondere zeigen wir, dass diese Koeffizienten keine linearen Kongruenzen erfüllen können, deren Reste nicht bestimmten quadratischen Gleichungen genügen. Schließlich konstruieren wir Polynome für motivische L-Funktionen von ungeradem motivischen Gewicht, die die bekannten Periodenpolynome für L-Funktionen von Hecke-Eigenformen verallgemeinern. Wir zeigen, dass fast alle Nullstellen dieser Polynome auf dem komplexen Einheitskreis liegen und gegen eine Gleichverteilung streben, falls die Stufe oder das Gewicht des entsprechenden Motivs genügend groß sind.

Abstract

This thesis contains research articles on various topics in the theory of modular forms. First we investigate divisors of meromorphic modular forms of higher level using polar harmonic Maass forms of weight 2. We continue by relating these Maass forms to modular forms associated to imaginary quadratic fields. We show that the Fourier coefficients of these functions are given by traces of singular moduli and compute their regularized inner products. After that we investigate p-adic properties of the Fourier coefficients of generalized eta-quotients. In particular, we show that these coefficients cannot satisfy any linear congruences whose residues do not fulfill certain quadratic equations. Finally we construct polynomials for motivic L-functions that generalize the well-known period polynomials for Hecke eigenforms. We show that these polynomials have almost all of their zeros on the complex unit circle and that the zeros tend to be equidistributed as the level or the weight of the motive are sufficiently large.

Danksagung

First of all, I would like to thank my adviser Prof. Dr. Kathrin Bringmann for giving me the opportunity to be part of her research group. I appreciate all her valuable guidance and advice that made my Ph.D. a fruitful and productive experience.

I also want to thank Prof. Dr. Ken Ono for his constant support and all the enlightening ideas he shared with me.

I am grateful to Prof. Dr. Sander Zwegers for being the secondary adviser and to Dr. Michael Mertens and Dr. Larry Rolen for proofreading an earlier version of this thesis.

I thank the Deutsche Forschungsgemeinschaft and the Fulbright Commission for their extensive financial support.

Moreover, I would like to thank all the current and former members of the number theory group in Köln for their permanent support and many interesting conversations. The group has been a source for good advice and collaboration, but also for precious friendships. Because of all the extraordinary people I had the chance to meet in the last years, I will always keep my time as a Ph.D. student in a good memory.

Besonderer Dank gilt meinen Eltern Ingrid und Werner Löbrich, die mich in jeder Lebensphase dazu ermutigt und dabei unterstützt haben, diesen Weg zu gehen.

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Chapter I

Introduction

I.1 Preliminary Definitions and Results

I.1.1 Meromorphic modular forms and their expansions

Modular forms, and more generally harmonic Maass forms, are of fundamental significance in modern number theory. To define them, we have to introduce some standard notation. Let $\mathbb{H}$ denote the complex upper half-plane $\{\tau \in \mathbb{C} : \operatorname{Im}(\tau) > 0\}$ and let

$$\Gamma_0(N) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : N|c\}$$

be the Hecke congruence subgroup of level N. For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and $\tau \in \mathbb{H}$, we define

$$M\tau := \frac{a\tau + b}{c\tau + d} \quad \text{and} \quad j(M, \tau) := c\tau + d.$$

Then, for a function $f : \mathbb{H} \to \mathbb{C} \cup \{\infty\}$ and $k \in \mathbb{Z}$, the *weight k Petersson slash operator* is given by

$$(f\,|_k\, M)(\tau) := j(M, \tau)^{-k} f(M\tau).$$

A *meromorphic modular form of weight k and level N* is a meromorphic function f on $\mathbb{H}$ satisfying $f\,|_k\, M = f$ for every $M \in \Gamma_0(N)$ which satisfies certain growth conditions at the cusps of $\Gamma_0(N)$ (see [34], Chapter III, §3 for a precise definition).

In particular, modular forms are meromorphic and invariant under the translation $\tau \mapsto \tau + 1$, and therefore can be expanded into a Fourier series

$$f(\tau) = \sum_{n \gg -\infty} a_f(n) q^n, \tag{I.1.1}$$

where here and throughout we write $q := e^{2\pi i \tau}$. The expansion (I.1.1) is associated to the cusp $i\infty$ and similar expansions exist for the other cusps of $\Gamma_0(N)$. A modular form is called *cusp form* if all non-positive Fourier coefficients at all cusps vanish. The Fourier coefficients $a_f(n)$ often encode interesting sequences of numbers, which allows for far reaching applications to various areas of mathematics and mathematical physics such as number theory, algebraic geometry, representation theory, combinatorics, and string theory.

An interesting example of this is the *partition function* $p(n)$, which counts the number of ways to write a positive integer n as a sum of positive integers. The values $p(n)$ are Fourier coefficients of the inverse of the *Dedekind eta-function*, which is a modular form of weight $-\frac{1}{2}$ (see [34], Chapter IV., §1 for a definition of half-integral weight modular forms).

$$\eta(\tau)^{-1} := q^{-\frac{1}{24}} \prod_{n\geq 1} (1-q^n)^{-1} = q^{-\frac{1}{24}} \sum_{n\geq 0} p(n)q^n.$$

A variety of information on the partition function, such as asymptotic growth [30, 53], p-adic properties [47, 55], or even explicit formulas [13] can be deduced from the modularity of η^{-1}.

The most prominent example of the significance of Fourier coefficients of modular forms in group theory is the *(normalized) modular j-invariant.* This function can be defined as the unique modular form of weight 0 and level 1 that is holomorphic on $\mathbb{H}$ and has a Fourier expansion of the form

$$J(\tau) = q^{-1} + \sum_{n>0} c(n)q^n = q^{-1} + 196884q + 21493760q^2 + 864299970q^3 + \dots.$$

A phenomenon called *Monstrous Moonshine* relates the coefficients $c(n)$ to dimensions of irreducible representations of the sporadic Monster group (see [23] for a recent survey). It was proven by Borcherds [5] and has now been extended and generalized to other groups and types of modular forms, see e.g. [15], [24], and [27] for *Umbral Moonshine.*

To state a product expansion for J, we define a certain system of modular functions. Namely, for every positive integer n, we let j_n be the unique $\mathrm{SL}_2(\mathbb{Z})$-invariant holomorphic function on $\mathbb{H}$ that has a Fourier expansion of the form

$$j_n(\tau) = q^{-n} + \sum_{m>0} c_n(m)q^m. \tag{I.1.2}$$

These functions can be explicitly given as so-called *Faber polynomials* of J or acting on J with normalized Hecke operators.

The modular function $J(\tau) - J(z)$ has a Borcherds product expansion, also known as the *denominator formula* for the Monster Lie algebra [5]. It can also be given in terms of the functions j_n evaluated at z [2].

$$J(\tau)-J(z) = e(-\tau) \prod_{\substack{m>0 \\ n\in\mathbb{Z}}} (1-e(m\tau)e(nz))^{c(mn)} = q^{-1} \exp\left(- \sum_{n\geq 1} \frac{j_n(z)}{n} q^n \right),$$

where we let $e(w) := e^{2\pi i w}$ for $w \in \mathbb{C}$. Taking the logarithmic derivative with respect to τ, we obtain a weight 2 meromorphic modular form

$$H_z(\tau) := -\frac{1}{2\pi i} \frac{J'(\tau)}{J(\tau) - J(z)} = \sum_{n\geq 1} j_n(z) q^n. \tag{I.1.3}$$

with simple poles at $\mathrm{SL}_2(\mathbb{Z})$-equivalent points to z. The series on the right-hand side converges for $\mathrm{Im}(\tau) > \mathrm{Im}(z)$.

In general, if a weight k modular form f has a pole in $\mathbb{H}$, then the expansion (I.1.1) can only converge if $\mathrm{Im}(\tau)$ is sufficiently large. Therefore, it can be more natural to consider *elliptic expansions* around a point $\rho \in \mathbb{H}$. These are of the form

$$f(\tau) = (\tau - \overline{\rho})^{-k} \sum_{n\gg -\infty} a_{f,\rho}(n) X_\rho^n(\tau)$$

with

$$X_\rho(\tau) := \frac{\tau - \rho}{\tau - \overline{\rho}} \tag{I.1.4}$$

and converge if $X_\rho(\tau)$ is sufficiently small. It is common to choose ρ to be a pole of f. The elliptic expansion of the function H_z from (I.1.3) around the pole z has the form

$$H_z(\tau) = \frac{1}{(\tau - \overline{z})^2} \left(-\frac{w_z \,\mathrm{Im}(z)}{\pi} X_z(\tau)^{-1} + \sum_{n\geq 0} a_z(n) X_z(\tau)^n \right),$$

where

$$w_z := \begin{cases} 3 & \text{if } z \text{ is } \mathrm{SL}_2(\mathbb{Z})\text{-equivalent to } \frac{-1+\sqrt{3}i}{2}, \\ 2 & \text{if } z \text{ is } \mathrm{SL}_2(\mathbb{Z})\text{-equivalent to } i, \\ 1 & \text{otherwise.} \end{cases} \tag{I.1.5}$$

I.1.2 Modular forms associated to imaginary quadratic fields

For a fixed $z \in \mathbb{H}$ and $k \in \mathbb{Z}$, $k > 1$, Petersson [51] defined meromorphic elliptic Poincaré series given by

$$\Psi_{k,z}(\tau) := \mathrm{Im}(z)^k \sum_{M \in \mathrm{SL}_2(\mathbb{Z})} \left((\tau - \overline{z})^{-k}(\tau - z)^{-k}\right) |_{2k,\tau} M,$$

where $|_{2k,\tau}$ means that we apply the slash-operator with respect to the variable τ. The functions $\Psi_{k,z}$ are meromorphic modular forms of weight $2k$ for $\mathrm{SL}_2(\mathbb{Z})$ with poles of order k at every point that is $\mathrm{SL}_2(\mathbb{Z})$-equivalent to z. Although it is straightforward to generalize these functions to higher level, we will restrict ourselves to the full modular group in this section.

The functions $\Psi_{k,z}$ can be used to construct meromorphic modular forms associated to imaginary quadratic fields as follows. Let $\Delta < 0$ be a discriminant, i.e. an integer congruent to 0 or 1 (mod 4). The group $\mathrm{SL}_2(\mathbb{Z})$ acts on the set $\mathscr{Q}_\Delta$ of all binary integral quadratic forms

$$Q(X, Y) = aX^2 + bXY + cY^2$$

that satisfy $b^2 - 4ac = \Delta$ via

$$\left(Q \circ \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)\right)(X, Y) := Q(\alpha X + \beta Y, \gamma X + \delta Y).$$

We denote by z_Q the *CM-point of* Q, which is the unique zero of $Q(\tau, 1)$ in $\mathbb{H}$. One can show that the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathscr{Q}_\Delta$ has only finitely many orbits. Now tracing the functions $\Psi_{k,z}$ in the z-variable over a (finite) set of representatives of CM-points of $\mathscr{Q}_\Delta / \mathrm{SL}_2(\mathbb{Z})$, we obtain

$$\sum_{Q \in \mathscr{Q}_\Delta / \mathrm{SL}_2(\mathbb{Z})} \frac{\Psi_{k,z_Q}(\tau)}{w_{z_Q}} = \frac{|\Delta|^{\frac{k}{2}}}{2^{k-1}} \sum_{Q \in \mathscr{Q}_\Delta} Q(\tau, 1)^{-k} =: f_{k,\Delta}(\tau) \tag{I.1.6}$$

with w_{z_Q} as in (I.1.5). Note that the function Ψ_{k,z_Q} does not depend on the choice of the representative Q, since the CM-point of $M \circ Q$ is $M^{-1}z_Q$. The functions $f_{k,\Delta}$ are meromorphic modular forms of weight $2k$ with poles of order k at the CM-points of discriminant Δ.

The sum on the right-hand side in (I.1.6) is also defined for $\Delta > 0$ and yields cusp forms of weight $2k$ that were introduced by Zagier [60]. Kohnen and Zagier [37] used these functions to construct the kernel function for the Shimura lift and to prove non-negativity of twisted central L-values. The $f_{k,\Delta}$ for negative discriminants were introduced by Bengoechea [3], who showed that their Fourier coefficients are algebraic for small values of k.

Bringmann, Kane, and von Pippich [11] computed regularized inner products of the $f_{k,\Delta}$ and related them to evaluations of *higher Green's functions* at CM-points (see Subsection 2.6 of loc. cit. for a definition of higher Green's functions). Namely, for $k > 1$, two different negative fundamental discriminants d, δ and the higher Green's function G_k, we have

$$\langle f_{k,d}, f_{k,\delta} \rangle = \frac{(-1)^k \Gamma\left(k - \frac{1}{2}\right)\sqrt{\pi}}{2^k (k-1)!} \sum_{\substack{Q \in \mathcal{Q}_d / \mathrm{SL}_2(\mathbb{Z}) \\ \mathfrak{Q} \in \mathcal{Q}_\delta / \mathrm{SL}_2(\mathbb{Z})}} \frac{G_k\left(z_Q, z_{\mathfrak{Q}}\right)}{w_{z_Q} w_{z_{\mathfrak{Q}}}}. \qquad \text{(I.1.7)}$$

I.1.3 L-functions and period polynomials

Let $f(\tau) = \sum_{n \geq 1} a_f(n) q^n$ be a cusp form of weight k and level N. Then the *L-function associated to* f is given by

$$L(s, f) := \sum_{n \geq 1} \frac{a_f(n)}{n^s},$$

which converges for $\mathrm{Re}(s) > \frac{k+1}{2}$. The completed L-function

$$\Lambda(s, f) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s) L(s, f)$$

has an analytic continuation to all of $\mathbb{C}$ and can be written as a *period integral*

$$\Lambda(s, f) = N^{s/2} \int_0^\infty f(iy) y^{s-1} dy, \qquad \text{(I.1.8)}$$

which converges for $\mathrm{Re}(s) \geq 1$. One then defines the *period polynomial of* f by

$$r_f(z) := \int_0^{i\infty} f(\tau)(\tau - z)^{k-2} d\tau,$$

which is a polynomial of degree $k-2$ in z. Expanding $(\tau - z)^{k-2}$ we obtain, using (I.1.8),

$$r_f(z) = \left(\frac{i}{\sqrt{N}}\right)^{k-1} \sum_{j=0}^{k-2} \binom{k-2}{j} (iz\sqrt{N})^j \Lambda(k-1-j, f). \qquad \text{(I.1.9)}$$

Therefore, up to simple factors, the coefficients of the period polynomial r_f are given by the special values $L(1, f)$, $L(2, f), \ldots, L(k-1, f)$. These are precisely the *critical values for* $L(s, f)$ (see Subsection V.2.2).

If f is a *Hecke eigenform* (see [34], Chapter III, §5 for a definition), meaning that the Fourier coefficients of f are multiplicative, then $L(s, f)$ has an Euler product expansion given by

$$L(s,f) = \left(\prod_{p \mid N} \frac{1}{1 - a_f(p)p^{-s}}\right)\left(\prod_{p \nmid N} \frac{1}{1 - a_f(p)p^{-s} + p^{k-1-2s}}\right) \tag{I.1.10}$$

and the completed L-function satisfies a functional equation

$$\Lambda(s, f) = \varepsilon(f)\Lambda(k - s, f) \tag{I.1.11}$$

for some $\varepsilon(f) \in \{-1, 1\}$.

More generally, an important class of L-functions attached to modular forms are the *symmetric power L-functions*, which are constructed from the expansion (I.1.10) as follows. By Deligne's proof of the Weil Conjectures [19], we know that $|a_f(p)| \leq 2p^{(k-1)/2}$. We denote the roots of the polynomial

$$X^2 - a_f(p)X + p^{k-1}$$

by $\alpha_p p^{(k-1)/2}$ and $\beta_p p^{(k-1)/2}$ and rewrite the Euler product of $L(s, f)$ as

$$L(s,f) = \left(\prod_{p \mid N} \frac{1}{1 - a_f(p)p^{-s}}\right)\left(\prod_{p \nmid N} \prod_{j=0}^{1} \frac{1}{1 - \alpha_p^j \beta_p^{1-j} p^{(k-1)/2-s}}\right).$$

If N is square-free, the Euler product of the n-th symmetric power of f, which we denote by $\mathrm{Sym}^n f$, is given by

$$L(s, \mathrm{Sym}^n f) = \left(\prod_{p \mid N} \frac{1}{1 - a_f(p)^n p^{-s}}\right)\left(\prod_{p \nmid N} \prod_{j=0}^{n} \frac{1}{1 - \alpha_p^j \beta_p^{n-j} p^{n(k-1)/2-s}}\right) \tag{I.1.12}$$

and converges for $\mathrm{Re}(s) > \frac{n(k-1)}{2} + 1$ The symmetric power L-functions are conjectured to have an analytic continuation to all of $\mathbb{C}$ and to satisfy a functional equation like (I.1.11). This is part of the more fundamental *Langlands Program* (see Subsection V.6.1 for more details and [14] for an introduction to the Langlands Program).

I.1.4 Polar Harmonic Maass Forms

Harmonic Maass forms are real-analytic generalizations of holomorphic modular forms that become increasingly important in number theory. We refer the reader to [7] for an introduction to the theory and various applications of these functions. *Polar harmonic Maass forms* were introduced by Bringmann and Kane [8], but already studied by Fay [26] in a broader setting. They generalize both meromorphic modular forms and harmonic Maass forms. To define them, we let $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ be the Riemann sphere, equipped with the usual topology as the one-point compactification of $\mathbb{C}$.

Definition I.1.1. For $k \in \mathbb{Z}$, a *polar harmonic Maass form of weight k on $\Gamma_0(N)$* is a continuous function $F\colon \mathbb{H} \to \widehat{\mathbb{C}}$ with the following properties ($\tau = u + iv$).

- $F^{-1}(\infty)$ is discrete with respect to the hyperbolic measure $\frac{dudv}{v^2}$ on $\mathbb{H}$ and F is real-analytic on $\mathbb{H} \backslash F^{-1}(\infty)$.
- For every $M \in \Gamma_0(N)$, we have $F|_k M = F$.
- The function F is annihilated by the *weight k hyperbolic Laplacian*

$$\Delta_k := -v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + ikv \left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right).$$

- For every $\rho \in \mathbb{H}$, there exists an $n \in \mathbb{N}_0$ such that $(\tau - \rho)^n F(\tau)$ is bounded in some punctured neighborhood of z.
- The function F grows at most linearly exponentially at the cusps of $\Gamma_0(N)$.

Just like meromorphic modular forms, polar harmonic Maass forms have a Fourier expansion at every cusp as well as elliptic expansions around every point $\rho \in \mathbb{H}$. However, as in the case of harmonic Maass forms, these expansions split into a meromorphic and a non-meromorphic part.

If $k \leq 0$, then the Fourier expansion of F at $i\infty$ has the shape (convergent for $\operatorname{Im} \tau$ sufficiently large)

$$F(\tau) = F_{i\infty}^{+}(\tau) + F_{i\infty}^{-}(\tau),$$

where

$$F_{i\infty}^{+}(\tau) := \sum_{n \gg -\infty} a_{F,i\infty}^{+}(n) e(n\tau)$$

and

$$F_{i\infty}^{-}(\tau) := a_F^{-}(0)\,\mathrm{Im}(\tau)^{1-k} + \sum_{\substack{n \ll \infty \\ n \neq 0}} a_F^{-}(n)\Gamma\left(1-k, -4\pi n v\right) e(n\tau)$$

for some $a_{F,i\infty}^{\pm}(n) \in \mathbb{C}$. Here $\Gamma(a,w)$ denotes the incomplete Gamma-function, defined as the analytic continuation of $s \mapsto \int_w^\infty t^{s-1}e^{-t}dt$ to $s = a$. Again there exist similar expansions at the other cusps of $\Gamma_0(N)$.

A polar harmonic Maass form F of weight $k \leq 0$ also has an elliptic expansion around each point $\rho \in \mathbb{H}$ of the form

$$F(\tau) = F_\rho^{+}(\tau) + F_\rho^{-}(\tau)$$

with a *meromorphic part* F_ρ^+ given by

$$F_\rho^{+}(\tau) := (\tau - \overline{\rho})^{-k} \sum_{n \gg -\infty} a_{F,\rho}^{+}(n) X_\rho^n(\tau) \tag{I.1.13}$$

with X_ρ as in (I.1.4), and a *non-meromorphic part* F_ρ^-. In case that $k = 0$, the non-meromorphic part has the form

$$F_\rho^{-}(\tau) := a_{F,\rho}^{-}(0) \log(|X_\rho(\tau)|^2) + \sum_{\substack{n \ll \infty \\ n \neq 0}} a_{F,\rho}^{-}(n) \overline{X_\rho(\tau)}^{-n}. \tag{I.1.14}$$

For $k < 0$, the non-meromorphic part has a more complicated shape given in Proposition III.6. The series in (I.1.13) and (I.1.14) converge if $X_\rho(\tau)$ is sufficiently small. For a survey on polar harmonic Maass forms and their applications, we refer the reader to Section 13.3 of [7] and to [9], where also some of the results of Chapter II are presented.

I.2 Scope of this thesis

This thesis is cumulative and contains research in various areas related to modular forms. In the following, we give a motivation for the different research projects contained in this thesis.

I.2.1 On divisors of modular forms

Given any modular form, it is a problem of general interest to determine its zeros and poles. The picture can be entirely different in different situations.

For example, Rankin and Swinnerton-Dyer [56] showed that the zeros in the fundamental domain of Eisenstein series for the full modular group all lie on the complex unit circle, whereas it follows from holomorphic *Quantum Unique Ergodicity*, proven by Holowinsky and Soundarajan [32], that the zeros of Hecke eigenforms tend to be equidistributed as the weight increases. A useful tool in complex analysis for studying divisors of the logarithmic derivative, which converts the points in the divisor of a meromorphic function into simple poles with the respective orders as residues.

Using the functions H_z from (I.1.3), Bruinier, Kohnen, and Ono [12] associated to a meromorphic modular form f of level k for $\mathrm{SL}_2(\mathbb{Z})$ a divisor modular form

$$f^{\mathrm{div}}(\tau) := \sum_{z\in\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \frac{\mathrm{ord}_z(f)}{w_z} H_z(\tau),$$

with w_z as in (I.1.5), and showed the identity

$$f^{\mathrm{div}}(\tau) = -\frac{\Theta(f\,(\tau))}{f(\tau)} + \frac{kE_2(\tau)}{12}, \tag{I.2.1}$$

where $\frac{\Theta(f(\tau))}{f(\tau)}$ is the logarithmic derivative of f and E_2 is the holomorphic weight 2 Eisenstein series defined in Section II.1. These results rely on the fact that the modular curve $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ has genus 0, because weakly holomorphic analogues of the j-function, so-called *Hauptmoduln*, do not exist on modular curves of positive genus.

The work presented in Chapter II is joint work with Kathrin Bringmann, Ben Kane, Ken Ono, and Larry Rolen and treats the divisor problem in higher levels. In Theorem II.1 we construct higher level generalizations $H^*_{N,z}$ to the functions H_z. These are polar harmonic Maass forms of weight 2 that also play a crucial role in work of Bringmann and Kane [8] on the explicit Riemann-Roch Theorem in weight 0. We obtain in Theorem II.3 a generalization of (I.2.1) to the compactified higher level modular curves $X_0(N)$. From this, one can determine the points in the divisor of f, because these are exactly the poles where the coefficients of $H^*_{N,z}$ grow exponentially.

I.2.2 Niebur-Poincaré series and traces of singular moduli

The right-hand side of (I.1.6) does not converge for $k = 1$. However, one can use *Hecke's trick*, introduced in [31] to show modularity properties of the weight

2 Eisenstein series, to obtain weight 2 analogues of the functions $f_{k,\Delta}$. For this we define

$$f_{s,\Delta}(\tau) := \sum_{Q\in\mathscr{Q}_\Delta} Q(\tau,1)^{-1}\left|Q(\tau,1)\right|^{-s},$$

which converges for $\mathrm{Re}(s) > 0$ and defines an analytic function in s. We then define f_Δ^* to be the analytic continuation of $f_{s,\Delta}$ to $s = 0$, which has been shown to exist for $\Delta > 0$ by Kohnen [35]. For a split negative discriminant $-dD$ and a level N, there are straightforward generalizations $f_{d,D,N}^*$ of f_d^* given in Definition III.3.1.

In Chapter III, we show that the Fourier coefficients of $f_{d,D,N}^*$ are given by twisted traces of singular moduli of the Niebur-Poincaré series $j_{N,n}$, which makes them algebraic integers in $\mathbb{Q}(\sqrt{D})$ if $X_0(N)$ has genus 0 (see Theorem III.4). We also compute regularized inner products $\langle f_d, f_\delta\rangle$ for two (not necessarily distinct) negative discriminants d, δ. Here f_d denotes a meromorphic modular normalization of $f_{d,1,1}^*$ (see (III.1.2)). This gives an analogue for formula (I.1.7) in weight 2.

I.2.3 Linear congruences for modular forms

Many p-adic properties of the Fourier coefficients of modular forms, such as the partition function, are still unknown. For example, it is conjectured that the asymptotic densities of $n \in \mathbb{N}$ for which $p(n)$ is even, resp. odd, are both $\frac{1}{2}$ [50], but it is still unknown whether either of these sets even has a positive density. However, Subbarao [58] conjectured that for every arithmetic progression $t \pmod{m}$, there are infinitely many even and odd values of the partition function. This was proven by Ono [48] for even and by Radu [54] for odd values. Adapting the methods of Radu's proof, Ahlgren and Kim [1] showed analogous results for the mock theta functions $f(q)$ and $\omega(q)$, as well as for other classes of modular forms, including (classical) eta-quotients.

By considering instead the number of partitions in which all parts are congruent to a certain residue $\pm g \pmod{\delta}$ for fixed $\delta \in \mathbb{Z}^+$ and $g \in \{0,\dots,\delta\}$, we obtain the functions

$$\eta_{\delta,g}(z) := q^{\frac{\delta}{2}P_2\left(\frac{g}{\delta}\right)} \prod_{\substack{m>0 \\ m\equiv g \pmod{\delta}}} (1-q^m) \prod_{\substack{m>0 \\ m\equiv -g \pmod{\delta}}} (1-q^m),$$

where P_2 denotes the second Bernoulli polynomial (see Section IV.1). Quotients of products of these functions are called *generalized eta-quotients*.

In Chapter IV, we apply Radu's approach to show that, for a given generalized eta-quotient, the Fourier coefficients on linear progressions of the form $t \pmod{m}$ cannot vanish modulo any prime if t satisfies certain quadratic equations depending on m and certain parameters of the eta-quotient. This recovers known results for classical eta-quotients like the partition function, but also yields linear incongruences for more general weakly holomorphic modular forms, such as the Rogers-Ramanujan functions. The precise statement of the theorem and several examples and applications are given in Section IV.1.

I.2.4 Motivic L-functions and period polynomials

Using the functional equation (I.1.11) for the completed L-function of a Hecke eigenform f, one can show that the period polynomial r_f defined in (I.1.9) satisfies a symmetry of the form

$$r_f(z) = -i^k \varepsilon(f) \left(\sqrt{N} z\right)^{\frac{k-2}{2}} r_f\left(-\frac{1}{Nz}\right).$$

Jin, Ma, Ono, and Soundarajan [33] showed that for all Hecke eigenforms f of weight k and level N, all zeros of r_f lie on the circle $|z| = \frac{1}{\sqrt{N}}$ and they tend to be equidistributed as $k \cdot N \to \infty$ (meaning that k or N goes to infinity). This result is known as the "Riemann hypothesis for period polynomials" (RHPP) and was shown for the odd part of the period polynomial by Conrey, Farmer, and Imamoglu [18] and for $N = 1$ by El-Guindy and Raji [25].

Chapter V is joint work with Wenjun Ma and Jesse Thorner and concerns polynomials associated to motivic L-functions, in particular the symmetric power L-functions defined in (I.1.12). We associate to a motivic L-function of odd motivic weight a polynomial whose coefficients are given by special values (including all the critical values) of the L-function and certain data associated to the motive (see Section V.2 for details). We then show that, under certain automorphicity assumptions on the motive stated in Hypothesis V.5, this polynomial has all (resp. almost all) its zeros equidistributed on the complex unit circle as the conductor (resp. weight) of the motive is sufficiently large.

For simplicity, we now focus on odd symmetric power L-functions of rational elliptic curves. Let E is an elliptic curve of conductor N over $\mathbb{Q}$ and $L(s, \mathrm{Sym}^n E)$ be its symmetric power L-function (see for example [45]). Then E corresponds to a Hecke eigenform f_E of weight 2 and level N by the Modularity Theorem [6] and we have

$$L(s, \mathrm{Sym}^n E) = L(s, \mathrm{Sym}^n f_E).$$

In this case, our polynomial takes the form (see Section V.3 for more general motives)

$$p_{\mathrm{Sym}^{2m+1}E}(z) := \sum_{j=0}^{2m} \left[\prod_{\ell=0}^{m} \binom{2m-\ell}{m-|m-j|} \right] \Lambda(2m+1-j, \mathrm{Sym}^{2m+1}E) z^j.$$

For example, we consider the odd symmetric powers of the weight 2 and and level 11 Hecke eigenform

$$f(\tau) := \eta(\tau)^2\eta(11\tau)^2 = q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 - 2q^9 + \ldots$$

associated to the elliptic curve with Weierstrass equation

$$E : Y^2 + Y = X^3 - X^2 - 10X - 20.$$

The central values of the completed symmetric power L-functions stored in the L-functions and Modular Forms Database [39] are given by

$$\begin{aligned}
&\Lambda(\mathrm{Sym}^1(f), 1) = \Lambda(f, 1) \approx 0.267985,\\
&\Lambda(\mathrm{Sym}^3(f), 2) \approx 24.473227,\\
&\Lambda(\mathrm{Sym}^5(f), 3) = 0,\\
&\Lambda(\mathrm{Sym}^7(f), 4) = 0.
\end{aligned}$$

We also have to assume that the symmetric power L-functions satisfy Hypothesis V.5, which is predicted to be true by the Langlangs Program. Then we obtain the polynomials

$$\begin{aligned}
p_{\mathrm{Sym}^1(f)}(z) &\approx 0.267985,\\
p_{\mathrm{Sym}^3(f)}(z) &\approx 44.903139 + 48.946454z + 44.903139z^2,\\
p_{\mathrm{Sym}^5(f)}(z) &\approx 7.668627 \cdot 10^6 + 5.161982 \cdot 10^6 z\\
&\qquad - 5.161982 \cdot 10^6 z^3 - 7.668627 \cdot 10^6 z^4,\\
p_{\mathrm{Sym}^7(f)}(z) &\approx 1.814679 \cdot 10^{16} + 6.551373 \cdot 10^{15} z + 1.538730 \cdot 10^{14} z^2\\
&\qquad - 1.538730 \cdot 10^{14} z^4 - 6.551373 \cdot 10^{15} z^5 - 1.814679 \cdot 10^{16} z^6.
\end{aligned}$$

These were computed in SAGE [57] and have zero sets

$$\begin{aligned}
Z_{\mathrm{Sym}^1(f)} &= \{ \},\\
Z_{\mathrm{Sym}^3(f)} &\approx \{-0.545023 \pm 0.838421i\},\\
Z_{\mathrm{Sym}^5(f)} &\approx \{\pm 1, -0.336565 \pm 0.941660i\},\\
Z_{\mathrm{Sym}^7(f)} &\approx \{\pm 1, -0.596246 \pm 0.802802i, 0.415735 \pm 0.909486i, \}.
\end{aligned}$$

We observe that all of these zeros lie in the unit circle, which illustrates Theorem V.1.

Chapter II

On Divisors of Modular Forms

This chapter is based on a manuscript submitted for publication and is joint work with Prof. Dr. Kathrin Bringmann, Prof. Dr. Ben Kane, Prof. Dr. Ken Ono and Prof. Dr. Larry Rolen [10].

II.1 Introduction and statement of results

As usual, let $J(\tau)$ be the $\mathrm{SL}_2(\mathbb{Z})$ Hauptmodul defined by

$$J(\tau) = \sum_{n=-1}^{\infty} c(n)e^{2\pi in\tau} := \frac{E_4(\tau)^3}{\Delta(\tau)} - 744 = e^{-2\pi i\tau} + 196884e^{2\pi i\tau} + \cdots,$$

where $E_k(\tau) := 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)e^{2\pi in\tau}$ is the weight $k \in 2\mathbb{N}$ Eisenstein series, $\sigma_\ell(n) := \sum_{d|\ell} d^\ell$, B_k is the kth Bernoulli number, and $\Delta(\tau) := (E_4(\tau)^3 - E_6(\tau)^2)/1728$. By Moonshine (for example, see [DGO15]), $J(\tau)$ is the McKay-Thompson series for the identity (i.e., its coefficients are the graded dimensions of the Monster module $V^\natural$). Moonshine also offers the striking infinite product

$$J(z) - J(\tau) = e^{-2\pi iz}\prod_{m>0,\ n\in\mathbb{Z}}\left(1 - e^{2\pi imz}e^{2\pi in\tau}\right)^{c(mn)},$$

the *denominator formula* for the Monster Lie algebra. Here we let $\tau, z \in \mathbb{H}$. This formula is equivalent to the following identity of Asai, Kaneko, and Ninomiya (see Theorem 3 of [AKN97])

$$H_z(\tau) := \sum_{n=0}^{\infty} j_n(z)e^{2\pi in\tau} = \frac{E_4(\tau)^2E_6(\tau)}{\Delta(\tau)}\frac{1}{J(\tau)-J(z)} = -\frac{1}{2\pi i}\frac{J'(\tau)}{J(\tau)-J(z)}. \tag{II.1.1}$$

The functions $j_n(\tau)$ form a Hecke system. Namely, if we let $j_0(\tau) := 1$ and $j_1(\tau) := J(\tau)$, then the others are obtained by applying the normalized Hecke operator $T(n)$

$$j_n(\tau) := j_1(\tau) \mid T(n). \tag{II.1.2}$$

Remark 1. The functions $H_z(\tau)$ and $j_n(\tau)$ played central roles in Zagier's [Zag02] seminal paper on traces of singular moduli and the Duncan-Frenkel work [DF11] on the Moonshine Tower. Carnahan [Car12] has obtained similar denominator formulas for completely replicable modular functions.

If $z \in \mathbb{H}$, then $H_z(\tau)$ is a weight 2 meromorphic modular form on $\mathrm{SL}_2(\mathbb{Z})$ with a single pole (modulo $\mathrm{SL}_2(\mathbb{Z})$) at the point z. Using these functions, the *divisor modular form* of a normalized weight k meromorphic modular form $f(\tau)$ on $\mathrm{SL}_2(\mathbb{Z})$ was defined in [BKO04] as[1]

$$f^{\mathrm{div}}(\tau) := \sum_{z \in \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} e_z \mathrm{ord}_z(f) H_z(\tau), \tag{II.1.3}$$

where $e_z := 2/\#\,\mathrm{Stab}_z\,(\mathrm{SL}_2(\mathbb{Z}))$ With $\Theta := \frac{1}{2\pi i}\frac{d}{d\tau}$, Theorem 1 of [BKO04] asserts that

$$f^{\mathrm{div}}(\tau) = -\frac{\Theta(f\,(\tau))}{f(\tau)} + \frac{kE_2(\tau)}{12}. \tag{II.1.4}$$

Although these results rely on the fact that $X_0(1)$ has genus 0, there is a natural extension for congruence subgroups. This extension requires polar harmonic Maass forms, which are harmonic Maass forms with poles in the upper half-plane (see [BFOR] for details). Here we consider the modular curves $X_0(N)$. For $n \in \mathbb{N}$, we define a Hecke system of $\Gamma_0(N)$ harmonic Maass functions $j_{N,n}(\tau)$ in Section II.3 which generalize the $j_n(\tau)$.

In Section III.2.3 we construct weight 2 polar harmonic Maass forms $H^*_{N,z}(\tau)$ which generalize the $H_z(\tau)$. We have two cases for the $H^*_{N,z}(\tau)$, according to whether $z \in \mathbb{H}$ or z is a cusp, which we consider separately. The following theorem summarizes the essential properties of these functions when $z \in \mathbb{H}$.

Theorem II.1. *If $z \in \mathbb{H}$, then $H^*_{N,z}(\tau)$ is a weight 2 polar harmonic Maass form on $\Gamma_0(N)$ which vanishes at all cusps and has a single simple pole at z. Moreover, the following are true:*

(1) *If $z \in \mathbb{H}$ and $\mathrm{Im}(\tau) > \max\{\mathrm{Im}(z), \frac{1}{\mathrm{Im}(z)}\}$, then we have that*

$$H^*_{N,z}(\tau) = \frac{3}{\pi\,[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]\,\mathrm{Im}(\tau)} + \sum_{n=1}^{\infty} j_{N,n}(z) e^{2\pi i n\tau}.$$

(2) *For $\gcd(N, n) = 1$, we have $j_{N,n}(\tau) = j_{N,1}(\tau) \mid T(n)$.*

(3) *For $n \mid N$, we have $j_{N,n}(\tau) = j_{\frac{N}{n},1}(n\tau)$.*

[1]Note that this summation does not include the cusp $i\infty$.

(4) *As $n \to \infty$, we have*

$$j_{N,n}(\tau) = \sum_{\substack{\lambda \in \Lambda_\tau \\ \lambda \leq n}} \sum_{(c,d) \in S_\lambda} e\left(-\frac{n}{\lambda} r_\tau(c,d)\right) e^{\frac{2\pi n \mathrm{Im}(\tau)}{\lambda}} + O_\tau(n) \tag{II.1.5}$$

for some real numbers $r_\tau(c,d)$ (see (II.3.2)*), Λ_τ a lattice in $\mathbb{R}$ (see* (II.3.3)*), and S_λ the set of solutions to $Q_\tau(c,d) = \lambda$ for a certain positive-definite binary quadratic form Q_τ (see* (II.3.4)*).*

Four Remarks.

(1) In Theorem II.1 (1), the inequality on $\mathrm{Im}(\tau)$ is required for convergence.

(2) For $N = 1$, we have that $H^*_{1,z}(\tau) = H_z(\tau) - E^*_2(\tau)$, where $E^*_2(\tau) := -\frac{3}{\pi \mathrm{Im}(\tau)} + E_2(\tau)$ is the usual weight 2 nonholomorphic Eisenstein series, and we have that $j_{1,n}(\tau) = j_n(\tau) + 24\sigma_1(n)$.

(3) The sums (II.1.5) were introduced by Hardy and Ramanujan [HR18] (see also [BBY02, Bia95]) to study the Fourier coefficients of $1/E_6$. Their formulas have been generalized [BK2, BK3] to negative weight meromorphic modular forms. Theorem II.1 (4) extends these results to weight 0 where the series are not absolutely convergent.

(4) Theorem II.1 (4) gives asymptotics for $j_{N,n}(z)$ in the n-aspect. If $\mathrm{Im}(z) \geq \mathrm{Im}(Mz)$ for all $M \in \Gamma_0(N)$, then

$$j_{N,n}(z) = e^{-2\pi i n z} + \sum_{\substack{c \geq 1 \\ N|c}} \sum_{\substack{d \in \mathbb{Z} \\ \gcd(c,d)=1 \\ |cz+d|^2=1}} e\left(n\frac{d-a}{c}\right) e^{2\pi i n \overline{z}} + O_z(n) \tag{II.1.6}$$

as $n \to \infty$.

The second case we consider are those $H^*_{N,\rho}(\tau)$ where ρ is a cusp of $X_0(N)$. These functions are compatible with the $H^*_{N,z}(\tau)$ considered in Theorem II.1. More precisely, since $z \mapsto H^*_{N,z}(\tau)$ is continuous (even harmonic) and $\Gamma_0(N)$-invariant, it follows that

$$H^*_{N,\rho}(\tau) := \lim_{z \to \rho} H^*_{N,z}(\tau) \tag{II.1.7}$$

is well-defined and only depends on the equivalence class of ρ. The next result summarizes these functions' properties. We use the Kloosterman sums $K_{i\infty,\rho}(0,-n;c)$ of (II.2.4) and the weight 2 harmonic Eisenstein series $E^*_{2,N,\rho}(\tau)$ for $\Gamma_0(N)$ defined in Section III.2.3. These have constant term 1 at ρ and vanish at all other cusps.

Theorem II.2. *We have that* $H^*_{N,\rho}(\tau) = -E^*_{2,N,\rho}(\tau)$. *Moreover, the following are true:*

(1) *We have*

$$H^*_{N,\rho}(\tau) = \frac{3}{\pi\,[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]\,\mathrm{Im}(\tau)} - \delta_{\rho,\infty} + \sum_{n=1}^{\infty} j_{N,n}(\rho) e^{2\pi i n\tau}, \qquad \textit{with}$$
$$j_{N,n}(\rho) := \lim_{\tau\to\rho} j_{N,n}(\tau) = \frac{4\pi^2 n}{\ell_\rho} \sum_{\substack{c\geq 1\\ N\mid c}} \frac{K_{i\infty,\rho}(0,-n;c)}{c^2},$$

where ℓ_ρ *denotes the cusp width of* ρ *and* $\delta_{\rho,\infty} := 1$ *if* $\rho = i\infty$ *and* 0 *otherwise.*

(2) *For* $\gcd(N,n) = 1$, *we have* $j_{N,n}(\rho) = \lim_{\tau\to\rho} j_{N,1}(\tau) \mid T(n)$.

(3) *For* $n \mid N$, *we have* $j_{N,n}(\rho) = \lim_{\tau\to\rho} j_{\frac{N}{n},1}(n\tau)$.

Two Remarks.
(1) Recall that the Fourier expansion in Theorem II.1 (1) is not valid as $z \to i\infty$.

(2) The $j_{N,n}(\rho)$ are divisor sums, which we leave to the interested reader to verify. From a generalization of the Weil bound (II.3.9) one can obtain $j_{N,n}(\rho) = O(n^{\frac{3}{2}})$.

We turn to the task of extending (II.1.4) to generic $\Gamma_0(N)$. Suppose that f is a weight k meromorphic modular form on $\Gamma_0(N)$. In analogy with (II.1.3), we define the *divisor polar harmonic Maass form*

$$f^{\mathrm{div}}(\tau) := \sum_{z\in X_0(N)} e_{N,z}\mathrm{ord}_z(f) H^*_{N,z}(\tau), \tag{II.1.8}$$

where $e_{N,z} := 2/\#\,\mathrm{Stab}_z(\Gamma_0(N))$ and $e_{N,\rho} := 1$ when ρ is a cusp. Generalizing (II.1.4), we show the following.

Theorem II.3. *If* $S_2(\Gamma_0(N))$ *denotes the space of weight 2 cusp forms on* $\Gamma_0(N)$, *then*

$$f^{\mathrm{div}}(\tau) \equiv \frac{k}{4\pi\,\mathrm{Im}(\tau)} - \frac{\Theta(f(\tau))}{f(\tau)} \pmod{S_2(\Gamma_0(N))}. \tag{II.1.9}$$

Three Remarks.
(1) The coefficient of $1/\operatorname{Im}(\tau)$ in $H^*_{N,z}(\tau)$ is independent of z. By the valence formula, summing over every element of $X_0(N)$ in the definition of $f^{\mathrm{div}}(\tau)$ multiplies this constant by $\frac{k}{12}\,[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]$, giving the nonholomorphic correction term on the right-hand side of Theorem II.3.

(2) At first glance, definitions (II.1.3) and (II.1.8) might appear different for $N = 1$. Indeed, $H^*_{1,z}(\tau) = H_z(\tau) - E^*_2(\tau)$, and the sum in $(II.1.8)$ includes the cusp $i\infty$ whereas $(II.1.3)$ omits it. The quasimodular Eisenstein series $E_2(\tau)$ in (II.1.4) and the valence formula guarantee that they coincide.

(3) The formula in Theorem II.3 has already been obtained by Choi using a regularized inner product due to Petersson, but without relating the Fourier coefficients of f^{div} to the polar harmonic Maass forms $H^*_{N,z}$ (see Theorem 1.4 of [Cho10]).

Theorem II.3 can be used to numerically compute divisors of meromorphic modular forms $f(\tau)$, which, in general, is a difficult task (for example, see [Del05]). The series $-\frac{\Theta(f(\tau))}{f(\tau)}$ is the logarithmic derivative of $f(\tau)$, and this fact converts the points $z \in \mathbb{H}$ in the divisor of $f(\tau)$ into simple poles. These can be identified by the asymptotic properties of the coefficients of $H^*_{N,z}(\tau)$ given in Theorem II.1. This follows from Theorem II.3 and the fact that coefficients of cusp forms satisfy Deligne's bound. In the case of the modular functions $j(\tau) - \alpha$, where $\alpha \in \mathbb{C}$, this has been carried out recently by Alwaise [Alw]. The method is based on the following immediate corollary to Theorems II.1–II.3.

Corollary II.4. *Suppose that $f(\tau)$ is a meromorphic modular form of weight k on $\Gamma_0(N)$ whose divisor is not supported at cusps. Let y_1 be the largest imaginary part of any points in the divisor of $f(\tau)$ lying in $\mathbb{H}$. Then if $-\frac{\Theta(f(\tau))}{f(\tau)} =: \sum_{n \gg -\infty} a(n)q^n$ $(q = e^{2\pi i\tau})$, we have that*

$$y_1 = \limsup_{n\to\infty} \frac{\log|a(n)|}{2\pi n}. \tag{II.1.10}$$

Two Remarks.
(1) We require lim sup in Corollary II.4 because the $a(n)$ can vanish on arithmetic progressions.

(2) It would be interesting to develop a practical algorithm for numerically computing modular form divisors. The idea would be to carefully peel away poles of $f^{\mathrm{div}}(\tau)$ in descending order until one is left with a linear combination of functions $E^*_{N,\rho}(\tau)$.

Example II.5. *For the Eisenstein series $E_4(\tau)$, we have*

$$-\frac{\Theta(E_4(\tau))}{E_4(\tau)} = -240q + 53280q^2 - 12288960q^3 + 2835808320q^4 + \cdots .$$

The sequence $\{b(n)\}_{n\geq 1} = \left\{\frac{\log|a(n)|}{2\pi n}\right\}_{n\geq 1}$ converges rapidly. Indeed, $b(2) = 0.866066794\ldots$, and $b(10) = 0.866025404\ldots$ matches the first 16 digits of the limiting value. The divisor of $E_4(\tau)$ is supported on a zero at $\omega := (-1+\sqrt{-3})/2$. By (II.1.6)*, since ω lies on the unit circle (implying that the second term on the right-hand side of* (II.1.6) *appears) for large n, $a(n)$ should very nearly be $\frac{1}{3}\left(e^{-2\pi in\omega} + 2e^{2\pi in\overline{\omega}}\right) = e^{-2\pi in\omega}$, which is very easily seen numerically.*

Example II.6. *We consider $f(\tau) := E_4(2\tau) + \frac{\eta^{16}(2\tau)}{\eta^8(\tau)}$, where $\eta(\tau)$ is Dedekind's eta-function. By the valence formula for $\Gamma_0(2)$, it has a single zero, say z_0, in $X_0(2)$. We find that*

$$-\frac{\Theta(f(\tau))}{f(\tau)} = -q - 495q^2 + 659q^3 + 113233q^4 - 261211q^5 + \cdots .$$

After the first 3000 *terms the sequence $\left\{\frac{\log|a(n)|}{2\pi n}\right\}_{n\geq 1}$ stabilizes and offers* $\mathrm{Im}(z_0) \approx 0.4357$*. As $f(\tau)$ has real coefficients and there is only one zero, $-\overline{z}_0$ must be $\Gamma_0(2)$-equivalent to z_0. We choose the fundamental domain*

$$\Bigg\{z \in \mathbb{H}\colon -\frac{1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2} \text{ and } \forall M \in \Gamma_0(2)\colon$$
$$\Big(\mathrm{Im}\,(Mz) \geq \mathrm{Im}(z) \text{ and } \mathrm{Im}\,(Mz) > \mathrm{Im}(z) \text{ if } \mathrm{Re}(z) < 0\Big)\Bigg\}.$$

Thus, either $\mathrm{Re}(z) \in \{0, \frac{1}{2}\}$*, or z lies on the arc $|2z-1| = 1$. The first two cases are easily excluded by the sign patterns of $a(n)$, and the zero on the arc is easily approximated as $z_0 \approx 0.2547 + 0.4357i$.*

This paper is organized as follows. In Section III.2.3 we construct the weight 2 polar harmonic Maass forms $H^*_{N,z}(\tau)$. In Section II.3 we relate their Fourier coefficients to the values of the weight 0 weak Maass forms at $\tau = z$, proving Theorems II.1, II.2, and II.3.

II.2 Weight 2 Polar Harmonic Maass forms

II.2.1 The $H^*_{N,z}(\tau)$ when $z \in \mathbb{H}$

Define for $z, \tau \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$

$$P_{N,s}(\tau, z) := \sum_{M \in \Gamma_0(N)} \frac{\varphi_s\left(M\tau, z\right)}{j\left(M, \tau\right)^2 \left|j\left(M, \tau\right)\right|^{2s}} \tag{II.2.1}$$

with $j\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \tau\right) := c\tau + d$ and

$$\varphi_s(\tau, z) := \mathrm{Im}(z)^{1+s}(\tau - z)^{-1}(\tau - \overline{z})^{-1} \left|\tau - \overline{z}\right|^{-2s}.$$

These functions were introduced and investigated in the z-variable in [BK], where it was shown that these are *polar harmonic Maass forms*. These functions are allowed to have poles in the upper half plane instead of only at the cusps. In this paper, we are interested in properties of $P_{N,s}(\tau, z)$ as functions of τ. A direct calculation shows that for $L \in \Gamma_0(N)$

$$P_{N,s}\left(L\tau, z\right) = j\left(L, \tau\right)^2 \left|j\left(L, \tau\right)\right|^{2s} P_{N,s}(\tau, z).$$

In [BK] it was shown, by a lengthy calculation, that the function $P_{N,s}(\tau, z)$ has an analytic continuation to $s = 0$, which we denote by $\mathrm{Im}(z)\Psi_{2,N}(\tau, z)$. Let $\mathcal{H}_k(\Gamma_0(N))$ be the space of weight k polar harmonic Maass forms with respect to $\Gamma_0(N)$. Lemma 4.4 of [BK] then states that $z \mapsto \mathrm{Im}(z)\Psi_{2,N}(\tau, z) \in \mathcal{H}_0(\Gamma_0(N))$. In the τ variable, these functions are also polar harmonic Maass forms, as the next proposition shows. For this, for $w \in \mathbb{C}$, let $e(w) := e^{2\pi i w}$, and

$$K(m, n; c) := \sum_{\substack{a,d \pmod c \\ ad \equiv 1 \pmod c}} e\left(\frac{md + na}{c}\right).$$

Moreover, I_k and J_k denote the usual I- and J-Bessel functions. The following proposition can be obtained by a careful inspection of the proof of Theorem 3.1 of [BK].

Proposition II.7. *We have that* $\tau \mapsto \mathrm{Im}(z)\Psi_{2,N}(\tau, z) \in \mathcal{H}_2(\Gamma_0(N))$. *For* $\mathrm{Im}(\tau) > \max\{\mathrm{Im}(z), \frac{1}{\mathrm{Im}(z)}\}$, *its Fourier expansion (in* τ*) has the form*

$$\mathrm{Im}(z)\Psi_{2,N}(\tau, z) = -\frac{6}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]\,\mathrm{Im}(\tau)}$$

$$- 2\pi \sum_{m\geq 1} \left(e^{-2\pi imz} - e^{-2\pi im\overline{z}}\right) e^{2\pi im\tau} - 8\pi^3 \sum_{m\geq 1} m \sum_{\substack{c\geq 1\\ N|c}} \frac{K(m,0;c)}{c^2} e^{2\pi im\tau}$$

$$- 4\pi^2 \sum_{m\geq 1} \sum_{\substack{n,c\geq 1\\ N|c}} \sqrt{\frac{m}{n}} \frac{K(m,-n;c)}{c} I_1\left(\frac{4\pi\sqrt{mn}}{c}\right) e^{2\pi inz} e^{2\pi im\tau}$$

$$- 4\pi^2 \sum_{m\geq 1} \sum_{\substack{n,c\geq 1\\ N|c}} \sqrt{\frac{m}{n}} \frac{K(m,n;c)}{c} J_1\left(\frac{4\pi\sqrt{mn}}{c}\right) e^{-2\pi in\overline{z}} e^{2\pi im\tau}.$$

We then set

$$H^*_{N,z}(\tau) := -\frac{\mathrm{Im}(z)}{2\pi}\Psi_{2,N}(\tau,z). \tag{II.2.2}$$

Remark 2. We have, as $\tau \to z$,

$$H^*_{N,z}(\tau) = \frac{1}{2\pi i e_{N,z}} \frac{1}{\tau - z} + O(1) \tag{II.2.3}$$

with $e_{N,z}$ as defined after (II.1.8).

II.2.2 The $H^*_{N,z}(\tau)$ for cusps

We require the Fourier expansion of the functions $H^*_{N,\rho}(\tau)$ defined in (II.1.7). For any cusp ρ of $\Gamma_0(N)$, denote by ℓ_ρ the cusp width and let M_ρ be a matrix in $\mathrm{SL}_2(\mathbb{Z})$ with $\rho = M_\rho i\infty$. For two cusps $\mathfrak{a}, \mathfrak{b}$ of $\Gamma_0(N)$, the generalized Kloosterman sums are

$$K_{\mathfrak{a},\mathfrak{b}}(m,n;c) := \sum_{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \Gamma_\infty^{\ell_\mathfrak{a}} \backslash M_\mathfrak{a}^{-1}\Gamma_0(N)M_\mathfrak{b}/\Gamma_\infty^{\ell_\mathfrak{b}}} e\left(\frac{md}{\ell_\mathfrak{b} c} + \frac{na}{\ell_\mathfrak{a} c}\right) \tag{II.2.4}$$

with $\Gamma_\infty := \{\pm \left(\begin{smallmatrix} 1 & n\\ 0 & 1\end{smallmatrix}\right) : n \in \mathbb{Z}\}$. Note that we have $K_{i\infty,i\infty}(m,n;c) = K(m,n;c)$.

Lemma II.8 (Lemma 5.4 of [BK])**.** *We have*

$$H^*_{N,\rho}(\tau) = \frac{3}{\pi\left[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)\right]\mathrm{Im}(\tau)} - \delta_{\rho,\infty} + \frac{4\pi^2}{\ell_\rho} \sum_{n\geq 1} n \sum_{c\geq 1} \frac{K_{\rho,i\infty}(n,0;c)}{c^2} e^{2\pi in\tau}.$$

The Fourier expansions in Lemma II.8 yield a relation with the harmonic weight 2 Eisenstein series $E^*_{2,N,\rho}(\tau)$ for $\Gamma_0(N)$. For $\mathrm{Re}(s) > 0$, define

$$E^*_{2,N,\rho,s}(\tau) := \sum_{M\in\Gamma_\rho\backslash\Gamma_0(N)} j\left(M_\rho M, \tau\right)^{-2} \left|j\left(M_\rho M, \tau\right)\right|^{-2s}. \tag{II.2.5}$$

Using the Hecke trick, it is well-known (cf. Satz 6 of [Hec24]) that $E^*_{2,N,\rho,s}(\tau)$ has an analytic continuation to $s = 0$, denoted by $E^*_{2,N,\rho}(\tau)$. Applying equations (5.3) and (5.4) in Theorem 1 of [Sma65] with $v = 1$, $A_j = M_\rho$, $\Gamma{=}\Gamma_0(N)$, and $\mu = 0$ to obtain the Fourier expansion of $E^*_{2,N,\rho}$, we see that

$$H^*_{N,\rho}(\tau) = -E^*_{2,N,\rho}(\tau). \tag{II.2.6}$$

II.3 The $j_{N,n}(z)$ and the proofs of Theorems II.1 and II.2

II.3.1 The functions $j_{N,n}(z)$

The functions $j_{N,n}(z)$ are constructed as analytic continuations of Niebur's Poincaré series [Nie73]. To be more precise, set for $n \in \mathbb{N}$ and $\operatorname{Re}(s) > 1$

$$F_{N,-n,s}(z) := \sum_{M \in \Gamma_\infty \backslash \Gamma_0(N)} e\left(-n \operatorname{Re}(Mz)\right) \operatorname{Im}(Mz)^{\frac{1}{2}} I_{s-\frac{1}{2}}\left(2\pi n \operatorname{Im}(Mz)\right).$$

These functions are *weak Maass forms* of weight 0; instead of being annihilated by Δ_0, they have eigenvalue $s(1-s)$. To obtain an analytic continuation to $s = 1$, one computes the Fourier expansion of $F_{N,-n,s}(z)$ and uses

$$\lim_{s\to 1} y^{\frac{1}{2}} I_{s-\frac{1}{2}}\left(2\pi n y\right) = y^{\frac{1}{2}} I_{\frac{1}{2}}\left(2\pi n y\right) = \frac{1}{\pi\sqrt{n}} \sinh\left(2\pi n y\right) = \frac{e^{2\pi n y} - e^{-2\pi n y}}{2\pi\sqrt{n}}.$$

Proposition II.9 (Theorem 1 of [Nie73]). *The function $F_{N,-n,s}(z)$ has an analytic continuation $F_{N,-n}(z)$ to $s = 1$, and $F_{N,-n}(z) \in \mathcal{H}_0(\Gamma_0(N))$. It has the Fourier expansion*

$$F_{N,-n}(z) = \frac{e^{-2\pi i n z} - e^{-2\pi i n \bar{z}}}{2\pi\sqrt{n}} + c_N(n,0) + \sum_{m\geq 1} \left(c_N(n,m) e^{2\pi i m z} + c_N(n,-m) e^{-2\pi i m \bar{z}}\right),$$

where the coefficients are given by

$$c_N(n,m) := \sum_{\substack{c\geq 1\\ N|c}} \frac{K(m,-n;c)}{c} \times \begin{cases} \frac{1}{\sqrt{m}} I_1\left(\frac{4\pi\sqrt{mn}}{c}\right) & \text{if } m > 0,\\ \frac{2\pi\sqrt{n}}{c} & \text{if } m = 0,\\ \frac{1}{\sqrt{|m|}} J_1\left(\frac{4\pi\sqrt{|m|n}}{c}\right) & \text{if } m < 0. \end{cases}$$

We then define the functions $j_{N,n}(z)$ by

$$j_{N,n}(z) := 2\pi\sqrt{n}F_{N,-n}(z). \tag{II.3.1}$$

For $N = 1$, we recover the $j_n(z)$ from the introduction up to the constant $2\pi\sqrt{n}c_1(n, 0) = 24\sigma_1(n)$.

II.3.2 Proofs of Theorems II.1 and II.2

In order to formally state Theorem II.1 (4), for an arbitrary solution $a, b \in \mathbb{Z}$ to $ad - bc = 1$, we define

$$\begin{aligned} r_z(c,d) &:= ac|z|^2 + (ad+bc)\operatorname{Re}(z) + bd, && \text{(II.3.2)}\\ \Lambda_z &:= \left\{\alpha^2|z|^2 + \beta\operatorname{Re}(z) + \gamma^2 > 0 : \alpha,\beta,\gamma \in \mathbb{Z}\right\}, && \text{(II.3.3)}\\ Q_z(c,d) &:= c^2|z|^2 + 2cd\operatorname{Re}(z) + d^2,\\ S_\lambda &:= \{(c,d) \in N\mathbb{Z}\times\mathbb{Z} :\ c \geq 0,\ \gcd(c,d) = 1,\ \text{and } Q_z(c,d) = \lambda\}. && \text{(II.3.4)} \end{aligned}$$

Note that although $r_z(c, d)$ is not uniquely determined, $e(-nr_z(c, d)/Q_z(c, d))$ is well-defined.

Proof of Theorem II.1. (1) For $n \in \mathbb{N}$, inspecting the expansions in Propositions II.7 and II.9 yields that $2\pi\sqrt{n}F_{N,-n}(z)$ is the coefficient of $e^{2\pi in\tau}$ in $-\frac{1}{2\pi}\operatorname{Im}(z)\Psi_{2,N}(\tau, z)$, yielding the claim.
(2) Since $\gcd(N, n) = 1$, $T(n)$ commutes with the action of $\Gamma_0(N)$, and so it suffices to show that (by analytic continuation) $f_n(z) = f_1(z) \mid T(n)$, where

$$f_n(z) = f_{n,s}(z) := e\left(-n\operatorname{Re}(z)\right)\left(n\operatorname{Im}(z)\right)^{\frac{1}{2}} I_{s-\frac{1}{2}}\left(2\pi n\operatorname{Im}(z)\right).$$

Let f be a nonholomorphic modular form of weight 0 with Fourier expansion

$$f(z) = \sum_{m\in\mathbb{Z}} a(\operatorname{Im}(z), m)e^{2\pi imz}.$$

Then for $\gcd(n, N) = 1$, the action of $T(n)$ on f is given by

$$f(z) \mid T(n) = n\sum_{m\in\mathbb{Z}}\sum_{d\mid\gcd(m,n)} \frac{a\left(\frac{d^2}{n}\operatorname{Im}(z), \frac{mn}{d^2}\right)}{d}e^{2\pi imz}. \tag{II.3.5}$$

Write $f_n(z) = f_n^*(\operatorname{Im}(z))e^{-2\pi inz}$ with $f_n^*(y) := (ny)^{\frac{1}{2}}I_{s-\frac{1}{2}}(2\pi ny)e^{-2\pi ny}$. The mth coefficient in (II.3.5) vanishes unless $m = -n$. Moreover, only $d = n$ contributes, giving

$$f_1(z) \mid T(n) = f_n^*\left(n\,\mathrm{Im}(z)\right) e^{-2\pi i n z}$$
$$= \left(n\,\mathrm{Im}(z)\right)^{\frac{1}{2}} I_{s-\frac{1}{2}}\left(2\pi n\,\mathrm{Im}(z)\right) e^{-2\pi i n\,\mathrm{Re}(z)} = f_n(z).$$

(3) For $n \mid N$, we rewrite

$$\sum_{M\in\Gamma_\infty\backslash\Gamma_0(N)} f_n(Mz) = \sum_{M\in\Gamma_\infty\backslash\Gamma_0(N)} f_1(nMz).$$

Now, with $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, we have $nMz = \frac{anz+bn}{\frac{c}{n}nz+d}$ and $\left(\begin{smallmatrix} a & bn \\ \frac{c}{n} & d \end{smallmatrix}\right)$ runs through $\Gamma_\infty \setminus \Gamma_0(\frac{N}{n})$ if M runs through $\Gamma_\infty \setminus \Gamma_0(N)$, implying the claim for $n \mid N$.

(4) We first rewrite the claimed asymptotic formula in terms of the corresponding points Mz with $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty\backslash\Gamma_0(N)$. Directly plugging in and simplifying yields $r_z(c,d)/Q_z(c,d) = \mathrm{Re}(Mz)$ and $\mathrm{Im}(z)/Q_z(c,d) = \mathrm{Im}(Mz)$, so the claim in Theorem II.1 (4) is equivalent to

$$j_{N,n}(z) = \sum_{\substack{M\in\Gamma_\infty\backslash\Gamma_0(N)\\ n\,\mathrm{Im}(Mz)\geq\mathrm{Im}(z)}} e^{-2\pi i n Mz} + O_z(n). \tag{II.3.6}$$

In order to show (II.3.6), we only expand the Fourier expansion for large c. That is to say, we write

$$\begin{aligned} j_{N,n}(z) = 2 &\sum_{\substack{1\leq c\leq\frac{\sqrt{n}}{\mathrm{Im}(z)}\\ N\mid c}} \sum_{\substack{d\in\mathbb{Z}\\ \gcd(c,d)=1}} e(-n\,\mathrm{Re}(Mz))\sinh(2\pi n\,\mathrm{Im}(Mz)) \\ &+ 2\pi\sqrt{n} \sum_{\substack{c>\frac{\sqrt{n}}{\mathrm{Im}(z)}\\ N\mid c}} \sum_{m\geq1} \frac{K(m,-n;c)}{\sqrt{m}c} I_1\left(\frac{4\pi\sqrt{mn}}{c}\right) e^{2\pi i m z} \\ &+ 4\pi^2 n \sum_{\substack{c>\frac{\sqrt{n}}{\mathrm{Im}(z)}\\ N\mid c}} \frac{K(0,-n;c)}{c^2} \\ &+ 2\pi\sqrt{n} \sum_{\substack{c>\frac{\sqrt{n}}{\mathrm{Im}(z)}\\ N\mid c}} \sum_{m\geq1} \frac{K(-m,-n;c)}{\sqrt{m}c} J_1\left(\frac{4\pi\sqrt{mn}}{c}\right) e^{-2\pi i m\overline{z}}. \end{aligned} \tag{II.3.7}$$

In order to obtain (II.3.6), we split the main terms with $n\,\mathrm{Im}(Mz) \geq \mathrm{Im}(z)$ off and rewrite

$$2\sinh(2\pi n\,\mathrm{Im}(Mz)) = e^{2\pi n\,\mathrm{Im}(Mz)} - e^{-2\pi n\,\mathrm{Im}(Mz)}. \tag{II.3.8}$$

The second term above is obviously bounded. Since

$$\mathrm{Im}(z) \le n\,\mathrm{Im}(Mz) = \frac{n\,\mathrm{Im}(z)}{c^2\,\mathrm{Im}(z)^2 + (d + c\,\mathrm{Re}(z))^2}$$

implies that $c \le \sqrt{n}/\,\mathrm{Im}(z) \ll_z \sqrt{n}$ and $|d| \le |c\,\mathrm{Re}(z)| + \sqrt{n\,\mathrm{Im}(z)} \ll_z \sqrt{n}$, the contribution to the error from the sum of the second terms in (II.3.8) yields an error of at most $O_z(n)$.

For the second, third, and fourth sums in (II.3.7), we use the Weil bound for Kloosterman sums

$$|K(m,-n;c)| \le \sqrt{\gcd(m,n,c)}\sigma_0(c)\sqrt{c} \ll \begin{cases} \sqrt{n}c^{\frac{1}{2}+\varepsilon} & \text{if } m = 0, \\ \sqrt{|m|}c^{\frac{1}{2}+\varepsilon} & \text{if } m \neq 0. \end{cases} \tag{II.3.9}$$

For the third sum in (II.3.7), this gives

$$2\pi\sqrt{n} \sum_{\substack{c > \frac{\sqrt{n}}{\mathrm{Im}(z)} \\ N|c}} \frac{K(0,-n;c)}{c^2} \ll n \sum_{\substack{c > \frac{\sqrt{n}}{\mathrm{Im}(z)} \\ N|c}} c^{-\frac{3}{2}+\varepsilon} \ll_z n^{\frac{3}{4}+\varepsilon}. \tag{II.3.10}$$

Next note that for $x \ge 0$ we have $|J_1(x)| \le I_1(x)$ by their series expansions. Since $x \mapsto \frac{I_1(x)}{x}$ is monotonically increasing and grows at most exponentially, the contribution from the second and fourth terms in (II.3.7) may be bounded by, using (II.3.9),

$$\begin{aligned} &\ll \sum_{\substack{c > \frac{\sqrt{n}}{\mathrm{Im}(z)} \\ N|c}} \sum_{m \ge 1} \frac{|K(\pm m,-n;c)|}{\sqrt{m}c} I_1\left(\frac{4\pi\sqrt{mn}}{c}\right) e^{-2\pi m\,\mathrm{Im}(z)} \\ &\ll \sqrt{n} \sum_{\substack{c > \frac{\sqrt{n}}{\mathrm{Im}(z)} \\ N|c}} \sum_{m \ge 1} \frac{|K(\pm m,-n;c)|}{c^2} \frac{I_1\left(4\pi\,\mathrm{Im}(z)\sqrt{m}\right)}{4\pi\,\mathrm{Im}(z)\sqrt{m}} e^{-2\pi m\,\mathrm{Im}(z)} \qquad \text{(II.3.11)} \\ &\ll \sqrt{n} \sum_{m \ge 1} I_1\left(4\pi\,\mathrm{Im}(z)\sqrt{m}\right) e^{-2\pi m\,\mathrm{Im}(z)} \ll \sqrt{n}. \end{aligned}$$

It remains to bound the terms in the first sum in (II.3.7) with $|cz + d|^2 > n$. Since each term gives a constant contribution, the terms with $|d| < \sqrt{n} + |c\,\mathrm{Re}(z)|$ give an error term of at most $O_z(n)$.

We finally assume that $|d| \ge \sqrt{n} + |c\,\mathrm{Re}(z)|$. Since $x \mapsto \frac{\sinh(x)}{x}$ is monotonically increasing and $|cz + d|^2 > n$, the remaining terms contribute

$$\left| \sum_{\substack{c \le \frac{\sqrt{n}}{\mathrm{Im}(z)} \\ N|c}} \sum_{\substack{|d| \ge \sqrt{n} + |c\,\mathrm{Re}(z)| \\ \gcd(c,d)=1}} e\left(-n\,\mathrm{Re}\left(Mz\right)\right) \sinh\left(2\pi n\,\mathrm{Im}\left(Mz\right)\right) \right|$$

$$\leq \sum_{c\leq \frac{\sqrt{n}}{\operatorname{Im}(z)}} \sum_{|d|\geq\sqrt{n}+|cx|} \sinh\left(\frac{2\pi n \operatorname{Im}(z)}{|cz+d|^2}\right) \leq \sum_{c\leq \frac{\sqrt{n}}{\operatorname{Im}(z)}} \sum_{|d|\geq\sqrt{n}} \sinh\left(\frac{2\pi n \operatorname{Im}(z)}{d^2}\right)$$
$$\leq 2\pi\sqrt{n} \sum_{d\geq\sqrt{n}} \frac{n}{d^2} \frac{\sinh(2\pi \operatorname{Im}(z))}{2\pi \operatorname{Im}(z)} = O_z(n).$$

This implies that the terms in the first sum in (II.3.7) with $|cz+d|^2 > n$ contribute $O_z(n)$. □

Remark.
By replacing $c > \sqrt{n}/\operatorname{Im}(z)$ with $c > C$ in (II.3.10) and (II.3.11), one finds that the terms decay like $C^{-\frac{1}{2}+\varepsilon}$ times a power of n. For $c \leq C$, the expansions in Proposition II.9 decay exponentially in m.

Proof of Theorem II.2. (1) Let K_s denote the usual K-Bessel function. Expanding $F_{N,-n,s}(z)$ at the cusp ρ as in Section 3.4 of [Iwa02], we obtain

$$F_{N,-n,s}(M_\rho z) = \frac{c_{\rho,s}(n,0)}{2s-1} (\operatorname{Im}(z))^{1-s} + \sum_{m\in\mathbb{Z}\setminus\{0\}} c_{\rho,s}(n,m) e^{2\pi i m \frac{\operatorname{Re}(z)}{\ell_\rho}} (\operatorname{Im}(z))^{\frac{1}{2}} K_{s-\frac{1}{2}}\left(\frac{2\pi|m|\operatorname{Im}(z)}{\ell_\rho}\right),$$

with

$$c_{\rho,s}(n,m) := \sum_{c\geq 1} K_{i\infty,\rho}(m,-n;c) \times \begin{cases} \frac{2}{c\sqrt{\ell_\rho}} I_{2s-1}\left(\frac{4\pi\sqrt{mn}}{\ell_\rho c}\right) & \text{if } m>0, \\ \frac{2\pi^s n^{s-\frac{1}{2}}}{\ell_\rho^s c^{2s}\Gamma(s)} \text{if } m=0, \\ \frac{2}{c\sqrt{\ell_\rho}} J_{2s-1}\left(\frac{4\pi\sqrt{|m|n}}{\ell_\rho c}\right) & \text{if } m<0, \end{cases}$$

The right-hand side is analytic at $s=1$, which gives the expansion of $F_{N,-n}(z)$ at ρ. Plugging in $K_{\frac{1}{2}}(y) = \sqrt{\frac{\pi}{2y}} e^{-y}$ and taking the limit $z \to i\infty$, we obtain

$$j_{N,n}(\rho) = 2\pi\sqrt{n} \lim_{s\to 1^+} c_{\rho,s}(n,0) = \frac{4\pi^2 n}{\ell_\rho} \sum_{c\geq 1} \frac{K_{i\infty,\rho}(0,-n;c)}{c^2}. \tag{II.3.12}$$

We have $K_{i\infty,\rho}(0,-n;c) = K_{\rho,i\infty}(n,0;c)$, since $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ runs through $\Gamma_0(N)M_\rho/\Gamma_\infty^{\ell_\rho}$ iff $-M^{-1} = \left(\begin{smallmatrix} -d & b \\ c & -a \end{smallmatrix}\right)$ runs through $\Gamma_\infty^{\ell_\rho}\backslash M_\rho^{-1}\Gamma_0(N)$ in (II.2.4). Hence (II.2.6) yields the claim.

Parts (2) and (3) follow by taking limits $\tau \to \rho$ in Theorem II.1 (2) and (3), respectively. Using the growth in n of $j_{N,n}(\rho)$ from (II.3.12), these limits may be taken termwise.

$\square$

Proof of Theorem II.3. We show that the difference of both sides has no poles in $\mathbb{H}$ and decays towards the cusps. We start by considering the points in $\mathbb{H}$. One easily computes that the residue of $-\frac{\Theta(f(\tau))}{f(\tau)}$ at $\tau = z$ equals $\frac{1}{2\pi i}\mathrm{ord}_z(f)$. Using (II.2.3) gives that the principal part at z agrees. At a cusp ρ one similarly sees that $\frac{\Theta(f(\tau))}{f(\tau)}$ has no pole and its constant term equals $\mathrm{ord}_\rho(f)$. Using that the constant term of $H^*_{N,z}(\tau)$ at ρ is -1 then gives the claim. $\square$

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Chapter III

Niebur-Poincaré Series and Traces of Singular Moduli

This chapter is based on a manuscript submitted for publication [41].

III.1 Introduction

For a positive discriminant Δ and an integer $k > 1$, Zagier [Zag75] introduced the weight $2k$ cusp forms (in a different normalization)

$$f_{k,\Delta}(\tau) := \frac{\Delta^{k-\frac{1}{2}}}{2\pi} \sum_{Q \in \mathscr{Q}_\Delta} Q(\tau, 1)^{-k}, \tag{III.1.1}$$

where $\mathscr{Q}_\Delta$ denotes the set of binary integral quadratic forms of discriminant Δ. The functions $f_{k,\Delta}$ were extensively studied by Kohnen and Zagier and have several applications. For example, they used these functions to construct the kernel function for the Shimura and Shintani lifts and to prove the non-negativity of twisted central L-values [KZ81]. Furthermore, the even periods

$$\int_0^\infty f_{k,\Delta}(it) t^{2n} dt, \quad (0 \leq n \leq k-1)$$

of the $f_{k,\Delta}$ are rational [KZ84]. Bengoechea [Ben15] introduced analogous functions for negative discriminants and showed that their Fourier coefficients are algebraic for small k. These functions are no longer holomorphic, but have poles at the CM-points of discriminant Δ. They were realized as regularized theta lifts by Bringmann, Kane, and von Pippich [BKvP] and Zemel [Zem16]. Moreover, Bringmann, Kane, and von Pippich related regularized inner products of the $f_{k,\Delta}$ to evaluations of higher Green's functions at CM-points.

The right-hand side of (III.1.1) does not converge for $k = 1$. However, one can use Hecke's trick to obtain weight 2 analogues of the $f_{k,\Delta}$. These were introduced by Zagier [Zag75] and further studied by Kohnen [KO92]. The aim of this paper is to analyze these weight 2 analogues for negative discriminants. Here we deal with generalizations $f^*_{d,D,N}$ for a level N, a discriminant d, and a fundamental discriminant D of opposite sign (see Definition III.3.1). The $f^*_{d,D,N}$ transform like modular forms of weight 2 for $\Gamma_0(N)$ and have simple poles at the Heegner points of discriminant dD and level N. Let $\mathscr{Q}_{dD,N}$ denote the set of quadratic forms $[a,b,c]$ of discriminant dD with $a>0$ and $N|a$, χ_D the generalized genus character associated to D, and $H(d,D,N)$ the twisted Hurwitz class number of discriminants d, D and level N (see Subsection III.2.2 for precise definitions). Then we obtain the following Fourier expansion ($v := \mathrm{Im}(\tau)$ throughout).

Theorem III.1. *For $v > \frac{\sqrt{|dD|}}{2}$, we have*

$$f^*_{d,D,N}(\tau) = -\frac{3H(d,D,N)}{\pi\,[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]\,v} - 2\sum_{n\geq 1}\sum_{\substack{a>0\\ N|a}} S_{d,D}(a,n)\sinh\left(\frac{\pi n\sqrt{|dD|}}{a}\right)e(n\tau),$$

where $e(w) := e^{2\pi i w}$ for all $w\in\mathbb{C}$ and

$$S_{d,D}(a,n) := \sum_{\substack{b \pmod{2a}\\ b^2\equiv dD \pmod{4a}}} \chi_D\left(\left[a,b,\frac{b^2-dD}{4a}\right]\right)e\left(\frac{nb}{2a}\right).$$

Remark 3. The exponential sums $S_{d,D}$ also occur for example in [DIT11] and [KO92].

Note that we obtain a non-holomorphic term in the Fourier expansion of $f^*_{d,D,N}$, just like in the case of the non-holomorphic weight 2 Eisenstein series E_2^* (see Subsection III.2.1). Therefore, in contrast to the higher weight case, the $f^*_{d,D,N}$ are in general no longer meromorphic modular forms, but *polar harmonic Maass forms*. This class of functions is defined and studied in Subsection III.2.3.

We also use a different approach to compute the coefficients of the $f^*_{d,D,N}$, writing them as traces of certain Poincaré series denoted by $H^*_N(z,\cdot)$ (see Proposition III.8). The $H^*_N(z,\cdot)$ are weight 2 analogues of Petersson's Poincaré series and were introduced by Bringmann and Kane [BK16] to obtain an explicit version of the Riemann-Roch Theorem in weight 0. We obtain the following different Fourier expansion of the $f^*_{d,D,N}$, realizing their coefficients as twisted traces of the Niebur-Poincaré series $j_{N,n}$ (see Definition III.2.1 and Theorem III.7).

Theorem III.2. *For* $v > \max\left\{\frac{\sqrt{|dD|}}{2}, 1\right\}$, *we have*

$$f^*_{d,D,N}(\tau) = -\frac{3H(d,D,N)}{\pi\,[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(N)]\,v} - \sum_{n>0} \mathrm{tr}_{d,D,N}\left(j_{N,n}\right) e(n\tau).$$

An interesting phenomenon occurs when $\Gamma_0(N)$ has genus 0. Subgroups of this type and their Hauptmoduln play a fundamental role in Monstrous Moonshine (see for example [CN79] for a classical and [DF11] for a more modern treatment). When we apply the suitably normalized n-th Hecke operator T_n to the Hauptmodul J_N for $\Gamma_0(N)$, then the Niebur-Poincaré series $j_{N,n}$ coincides with T_nJ_N, up to an additive constant. Zagier [Zag02] showed that, for discriminants $d < 0$ and $D > 0$, the functions

$$q^d + \sum_{D>0} \mathrm{tr}_{d,D,N}(T_nJ_N)q^D \quad \text{and} \quad q^{-D} + B_n(D,0) + \sum_{d>0} \mathrm{tr}_{d,D,N}(T_nJ_N)q^d$$

are weakly holomorphic modular forms for $\Gamma_0(4N)$ of weight $\frac{1}{2}$ resp. $\frac{3}{2}$ in the Kohnen plus-space. Now summing over n instead of D or d, Theorem III.2 states that the twisted Hecke traces $\{\mathrm{tr}_{d,D,N}(T_nJ_N)\}_{n>0}$ give rise to Fourier coefficients of the meromorphic modular forms $f^*_{d,D,N} - \frac{H(d,D,N)}{[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(N)]}E^*_2$.

We give three applications of Theorem III.2. First, comparing Theorems III.1 and III.2, we obtain explicit series expressions for traces of Niebur-Poincaré series.

Corollary III.3. *We have*

$$\mathrm{tr}_{d,D,N}\left(j_{N,n}\right) = 2\sum_{\substack{a>0\\ N\mid a}} S_{d,D}(a,n)\sinh\left(\frac{\pi n\sqrt{|dD|}}{a}\right).$$

Corollary III.3 was obtained by Duke for $N = D = 1$ ([Du06], Proposition 4) and Jenkins for $N = 1$ and $D > 1$ ([Jen06], Theorems 1.5 and 2.2). Choi, Jeon, Kang, and Kim [CJKK08] obtained an analogous formula for the subgroups $\Gamma_0(p)^+$ for p prime, later generalized by Kang and Kim [KK10] to $\Gamma_0(N)^+$ for arbitrary N.

Next we examine algebraicity properties of the Fourier coefficients of $f^*_{d,D,N}$. Bengoechea [Ben15] showed that for $\Delta < 0$ and $k \in \{2,3,4,5,7\}$ (so that $S_{2k} = \{0\}$), the Fourier coefficients of $f_{k,\Delta}$ lie in the Hilbert class field of $\mathbb{Q}(\sqrt{\Delta})$. We have the following extension to the weight 2 case.

Theorem III.4. *If $\Gamma_0(N)$ has genus 0, then the Fourier coefficients of the meromorphic part of $f^*_{d,D,N}$ are real algebraic integers in the field $\mathbb{Q}\left(\sqrt{D}\right)$.*

Eventually, we compute regularized inner products of meromorphic analogues of the $f^*_{d,D,N}$. For this we restrict to the case $D = N = 1$ and consider the meromorphic modular forms

$$f_d(\tau) := f^*_{d,1,1}(\tau) - H(d,1,1)E^*_2(\tau). \tag{III.1.2}$$

The usual inner product $\langle f_d, f_\delta\rangle$ for negative discriminants d, δ does not converge, so we need to use a regularization by Bringmann, Kane, and von Pippich. Moreover, since the f_d do not decay like cusp forms towards $i\infty$, we also have to apply Borcherds's regularization near the cusp $i\infty$ (see Section III.4 for a precise definition). We obtain the following evaluations, where $J(z) := j_{1,1}(z) - 24$ denotes the *normalized modular j-invariant.*

Theorem III.5. *Let d be a negative discriminant and $\mathcal{Q}_d := \mathcal{Q}_{d,1}$.*

(i) If $\delta < d$ is another negative discriminant such that $\frac{\delta}{d}$ is not a square, then

$$\langle f_d, f_\delta\rangle = \frac{1}{2\pi}\sum_{\substack{Q\in\mathcal{Q}_d/\operatorname{SL}_2(\mathbb{Z})\\ \mathcal{Q}\in\mathcal{Q}_\delta/\operatorname{SL}_2(\mathbb{Z})}} \frac{1}{w_Q w_{\mathcal{Q}}}\log\left|J(z_Q) - J(z_{\mathcal{Q}})\right|.$$

(ii) If neither $-\frac{d}{3}$ nor $-\frac{d}{4}$ is a square, then

$$\langle f_d, f_d\rangle = \frac{1}{2\pi}\sum_{Q\in\mathcal{Q}_d/\operatorname{SL}_2(\mathbb{Z})}\log\left|\sqrt{|d|}\frac{J'(z_Q)}{Q(1,0)}\right| + \frac{1}{2\pi}\sum_{\substack{Q,\mathcal{Q}\in\mathcal{Q}_d/\operatorname{SL}_2(\mathbb{Z})\\ Q\neq\mathcal{Q}}}\log\left|J(z_Q) - J(z_{\mathcal{Q}})\right|.$$

(iii) We have

$$\langle f_{-3}, f_{-3}\rangle = \frac{1}{18\pi}\log\left|\frac{\sqrt{3}}{2}J'''\left(\frac{1+i\sqrt{3}}{2}\right)\right|,$$
$$\langle f_{-4}, f_{-4}\rangle = \frac{1}{8\pi}\log\left|2J''(i)\right|.$$

Note that $\log|J(z) - J(\mathfrak{z})|$ is a *Green's function* on the modular curve $X_0(1)$. The double traces over CM-values of Green's functions occurring in Theorem

III.5 have been related to heights of Heegner points on modular curves by Gross and Zagier [GZ86]. Since the f_d are modular forms of weight 2, it would be enlightening to find a geometric interpretation of their inner products and see how they relate to height functions. In higher weight, Bringmann, Kane, and von Pippich [BKvP] wrote regularized inner products of the functions $f_{k,\Delta}$ for $\Delta < 0$ in terms of double traces over CM-values of *higher Green's functions*, so we can see Theorem III.5 as an extension of their result to the weight 2 case.

The paper is organized as follows: In Section III.2, we introduce the necessary notation and definitions. In Section III.3, we prove Theorems III.1, III.2, and III.4. Eventually, in Section III.4, we compute the regularized inner products, proving Theorem III.5.

Acknowledgements

We thank Kathrin Bringmann, Stephan Ehlen, and Markus Schwagenscheidt for valuable advice on writing the paper. We are also grateful to John Duncan, Michael Mertens, Ken Ono, Dillon Reihill, Larry Rolen, Tonghai Yang, Shaul Zemel, and the referee for helpful comments and conversations.

III.2 Definitions and Preliminaries

III.2.1 General notation

Throughout this paper, we denote variables in the complex upper half-plane $\mathbb{H}$ by τ, z, and ϱ with $v := \mathrm{Im}(\tau), y := \mathrm{Im}(z), \eta := \mathrm{Im}(\varrho)$ and for $w \in \mathbb{C}$ we write $e(w) := e^{2\pi i w}$. For a matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and $\tau \in \mathbb{H}$, we set

$$M\tau := \frac{a\tau + b}{c\tau + d} \quad \text{and} \quad j(M, \tau) := c\tau + d.$$

For each point $\varrho \in \mathbb{H}$ we let $\Gamma_{N,\varrho}$ denote the stabilizer of ϱ in $\Gamma_0(N)$ and set $w_{N,\varrho} := \frac{1}{2}\#\Gamma_{N,\varrho}$. Note that if $\rho := \frac{1+i\sqrt{3}}{2}$ denotes the sixth order root of unity in $\mathbb{H}$, then we have

$$w_\varrho := w_{1,\varrho} = \begin{cases} 3, & \text{if } \varrho \in \mathrm{SL}_2(\mathbb{Z})\rho, \\ 2, & \text{if } \varrho \in \mathrm{SL}_2(\mathbb{Z})i, \\ 1, & \text{otherwise.} \end{cases}$$

Furthermore, we define the divisor sum function $\sigma(m) := \sum_{d|m} d$ and the weight 2 Eisenstein series

$$E_2(\tau) := 1 - 24 \sum_{m \geq 1} \sigma(m) e(m\tau),$$

as well as its non-holomorphic completion

$$E_2^*(\tau) := -\frac{3}{\pi v} + E_2(\tau),$$

which transforms like a weight 2 modular form for $\mathrm{SL}_2(\mathbb{Z})$. In general, we will use a star to denote non-holomorphic modular forms (cf. Proposition III.8 and Definition III.3.1).

III.2.2 Quadratic forms and traces of singular moduli

We denote an integral binary quadratic form $Q(X, Y) = aX^2 + bXY + cY^2 \in \mathbb{Z}[X, Y]$ by $Q = [a, b, c]$. The group $\mathrm{SL}_2(\mathbb{Z})$ acts on the set of binary quadratic forms via

$$\left(Q \circ \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)\right)(X, Y) := Q(\alpha X + \beta Y, \gamma X + \delta Y), \tag{III.2.1}$$

leaving the discriminant $\Delta = b^2 - 4ac$ invariant. For a positive integer N and a discriminant Δ, we write $\mathcal{Q}_{\Delta,N}$ for the set of all binary integral quadratic forms $Q = [a, b, c]$ of discriminant Δ with $a > 0$ and $N|a$. Then the group $\Gamma_0(N)$ acts on $\mathcal{Q}_{\Delta,N}$. For $\Delta < 0$ and $Q \in \mathcal{Q}_{\Delta,N}$, we denote by z_Q the *Heegner point of* Q, which is the unique zero of $Q(\tau, 1)$ in $\mathbb{H}$.

For $\Delta < 0$, we consider a splitting $\Delta = d \cdot D$ into a discriminant d and a fundamental disriminant D that are both congruent to squares modulo $4N$ (meaning that $d, D \equiv 0$ or $1 \pmod 4$ and D is not a proper square multiple of an integer congruent to 0 or $1 \pmod 4$) and denote by χ_D the *generalized genus character* corresponding to the decomposition $\Delta = d \cdot D$ as defined in [GKZ87].

Definition III.2.1. For a $\Gamma_0(N)$-invariant function $g : \mathbb{H} \to \mathbb{C}$, we define the *twisted trace of singular moduli of discriminants* d *and* D *of* g as

$$\mathrm{tr}_{d,D,N}(g) := \sum_{Q \in \mathcal{Q}_{dD,N}/\Gamma_0(N)} \frac{\chi_D(Q)}{w_{N,Q}} g(z_Q),$$

where $w_{N,Q} := w_{N,z_Q}$. Moreover, we call

$$H(d, D, N) := \mathrm{tr}_{d,D,N}(1) = \sum_{Q \in \mathscr{Q}_{dD,N}/\Gamma_0(N)} \frac{\chi_D(Q)}{w_{N,Q}}$$

the *Hurwitz class number of discriminants d and D and level N.*

III.2.3 Polar Harmonic Maass Forms

Now we define polar harmonic Maass forms and study their elliptic expansions, which we will need to compute the regularized inner products in Section III.4. See [BFOR], Section 13.3 for an introduction to polar harmonic Maass forms and their applications.

Definition III.2.2. For $k \in \mathbb{Z}$, a *polar harmonic Maass form of weight k for* $\Gamma_0(N)$ is a continuous function $F\colon \mathbb{H} \to \mathbb{C} \cup \{\infty\}$ which is real-analytic outside a discrete set of points and satisfies the following conditions:

i) For every $M \in \Gamma_0(N)$ and $\tau \in \mathbb{H}$, we have

$$F(M\tau) = j(M, \tau)^k F(\tau).$$

ii) The function F is annihilated by the *weight k hyperbolic Laplacian*

$$\Delta_k := -v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + ikv \left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right).$$

iii) For every $z \in \mathbb{H}$, there exists an $n \in \mathbb{N}_0$ such that $(\tau - z)^n F(\tau)$ is bounded in some neighborhood of z.

iv) The function F grows at most linearly exponentially at the cusps of $\Gamma_0(N)$.

We denote by $\mathcal{H}_k(N)$ the space of weight k polar harmonic Maass forms for $\Gamma_0(N)$.

Polar harmonic Maass forms have *elliptic expansions* around every point $\varrho \in \mathbb{H}$, which converge if

$$X_\varrho(\tau) := \frac{\tau - \varrho}{\tau - \overline{\varrho}} \tag{III.2.2}$$

is sufficiently small. These can be seen as counterparts to the more common q-series expansions at the cusps and also break into two pieces.

Proposition III.6 (Proposition 2.2 of [BK16], see also Subsection 2.3 of [BKvP])**.** *A polar harmonic Maass form F of weight $k \leq 0$ has an expansion around each point $\varrho \in \mathbb{H}$ of the form $F = F_\varrho^+ + F_\varrho^-$, where the meromorphic part F_ϱ^+ is given by*

$$F_\varrho^+(\tau) := (\tau - \overline{\varrho})^{-k} \sum_{n \gg -\infty} A_{F,\varrho}^+(n) X_\varrho^n(\tau) \tag{III.2.3}$$

and the non-meromorphic part F_ϱ^- by

$$F_\varrho^-(\tau) := (\tau - \overline{\varrho})^{-k} \sum_{n \ll \infty} A_{F,\varrho}^-(n) \beta_0 \left(1 - |X_\varrho(\tau)|^2; 1-k, -n\right) X_\varrho^n(\tau). \tag{III.2.4}$$

These expressions converge for $|X_\varrho(\tau)| \ll 1$. Here, we have that

$$\beta_0\left(w; a, b\right) := \beta\left(w; a, b\right) - \mathcal{C}_{a,b} \qquad \textit{with} \qquad \mathcal{C}_{a,b} := \sum_{\substack{0 \leq j \leq a-1 \\ j \neq -b}} \binom{a-1}{j} \frac{(-1)^j}{j+b},$$

where the incomplete β-function *is defined by $\beta(w; a, b) := \int_0^w t^{a-1}(1-t)^{b-1} dt$.*

We refer to the terms in (III.2.3) and (III.2.4) which grow as $\tau \to \varrho$ as the *principal part of F around ϱ.*

Remark 4. The hyperbolic Laplacian splits as

$$\Delta_k = -\xi_{2-k} \circ \xi_k, \quad \text{where} \quad \xi_k := 2iv^k \overline{\frac{\partial}{\partial \overline{\tau}}}. \tag{III.2.5}$$

If F satisfies weight k modularity, then $\xi_k(F)$ is modular of weight $2-k$. Moreover, ξ_k annihilates the meromorphic part of a polar harmonic Maass form, so that it maps weight k polar harmonic Maass forms to weight $2-k$ meromorphic modular forms and its kernel is given by the space of weight k meromorphic modular forms.

III.2.4 Niebur-Poincaré series

Here we introduce *Niebur-Poincaré series* and give their explicit Fourier expansion. Following [Nie73], we define for $n > 0$

$$F_{N,-n,s}(z) := 2\pi\sqrt{n} \sum_{M \in \Gamma_\infty \backslash \Gamma_0(N)} e(-n \operatorname{Re}(Mz)) \operatorname{Im}(Mz)^{\frac{1}{2}} I_{s-\frac{1}{2}}(2\pi n \operatorname{Im}(Mz)),$$

where $I_{s-\frac{1}{2}}$ denotes the *I-Bessel function* and $\Gamma_\infty := \{\pm \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) : n \in \mathbb{Z}\}$. This series converges absolutely and locally uniformly for $\mathrm{Re}(s) > 1$. The function $F_{N,-n,s}$ is a $\Gamma_0(N)$-invariant eigenfunction of the hyperbolic Laplacian with eigenvalue $s(1-s)$. Niebur showed that $F_{N,-n,s}$ is analytic in s and has an analytic continuation to $s = 1$.

Proposition III.7 (Theorem 1 of [Nie73])**.** *The function $F_{N,-n,s}$ has an analytic continuation $j_{N,n}$ to $s = 1$, and $j_{N,n} \in \mathcal{H}_0(N)$. It has a Fourier expansion of the form*

$$j_{N,n}(z) = e(-nz) - e(-n\overline{z}) + c_N(n,0) \\ + \sum_{m \geq 1} \left(c_N(n,m)e(mz) + c_N(n,-m)e(-m\overline{z})\right).$$

The coefficients are given by

$$c_N(n,m) := 2\pi\sqrt{n} \sum_{\substack{c \geq 1 \\ N|c}} \frac{K(m,-n;c)}{c} \times \begin{cases} \frac{1}{\sqrt{m}} I_1\left(\frac{4\pi\sqrt{mn}}{c}\right), & \text{if } m > 0, \\ \frac{2\pi\sqrt{n}}{c}, & \text{if } m = 0, \\ \frac{1}{\sqrt{|m|}} J_1\left(\frac{4\pi\sqrt{|m|n}}{c}\right), & \text{if } m < 0, \end{cases}$$

where

$$K(m,n;c) := \sum_{\substack{a,d \pmod c \\ ad \equiv 1 \pmod c}} e\left(\frac{md + na}{c}\right)$$

denotes the Kloosterman sum *and I_1, J_1 are the* first order I- and J-Bessel functions*, respectively.*

The constants $c_N(n,0)$ can be explicitly evaluated. For example, for $N = 1$ we obtain

$$c_1(n,0) = 24\sigma(n)$$

and for p prime we have

$$c_p(n,0) = -\frac{24}{p^2-1}\left(\sigma(n) - p^2\sigma\left(\frac{n}{p}\right)\right), \tag{III.2.6}$$

where $\sigma(\ell) := 0$ if $\ell \notin \mathbb{Z}$ (see [CJKK08] for a similar calculation).

III.2.5 Petersson's Poincaré series

For $w, s \in \mathbb{C}$, $w \neq 0$, we let

$$\phi_s(w) := w^{-1}|w|^{-s} \tag{III.2.7}$$

and define

$$\begin{aligned} H_{N,s}(z,\tau) &:= -\frac{v^s}{2\pi} \sum_{M \in \Gamma_0(N)} \phi_s\left(j(M,\tau)^2 \frac{(M\tau - z)(M\tau - \overline{z})}{y}\right) \\ &= -\frac{v^s}{2\pi} \sum_{M \in \Gamma_0(N)} \phi_s\left(\frac{(\tau - Mz)(\tau - M\overline{z})}{\mathrm{Im}(Mz)}\right). \end{aligned} \tag{III.2.8}$$

These sums converge locally uniformly for $\mathrm{Re}(s) > 0$ and define analytic functions in s. They satisfy modularity of weight 0 in z and of weight 2 in τ. Bringmann and Kane showed that they have an analytic continuation H_N^* to $s = 0$, which are a polar harmonic Maass forms of weight 2 with simple poles at $\Gamma_0(N)$-equivalent points to z. For this, they used a splitting of the sum due to Petersson and obtained an analytic continuation of the Fourier expansion of every part by Poisson summation and locally uniform estimates.

Proposition III.8 (Lemma 4.4 of [BK16])**.** *The function $H_{N,s}$ has an analytic continuation H_N^* to $s = 0$. We have*

$$z \mapsto H_N^*(z,\tau) \in \mathcal{H}_0(N) \quad \textit{and} \quad \tau \mapsto H_N^*(z,\tau) \in \mathcal{H}_2(N).$$

Furthermore, the function

$$\tau \mapsto H_N(z,\tau) := H_N^*(z,\tau) + \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]} E_2^*(\tau)$$

is a meromorphic modular form of weight 2 *for* $\Gamma_0(N)$ *with only simple poles at points that are* $\Gamma_0(N)$*-equivalent to* z*.*

Remark 5. Note that $H_N^*(z,\cdot)$ has principal part $-\frac{w_{N,z}}{2\pi i}\frac{1}{\tau - z}$ at $\tau = z$.

The Fourier coefficients of $H_N^*(z,\cdot)$ were computed in [BKLOR], where it was shown that they are given by the Niebur-Poincaré series $j_{N,n}$ of Proposition III.7, evaluated at z.

Proposition III.9 (Theorem 1.1 of [BKLOR])**.** *For $v > \max\left\{y, \frac{1}{y}\right\}$, we have*

$$H_N^*(z,\tau) = \frac{3}{\pi[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]v} + \sum_{n>0} j_{N,n}(z) e(n\tau).$$

III.3 Weight 2 Modular Forms Associated to Imaginary Quadratic Fields

Now we define and study the weight 2 analogues of the functions $f_{k,\Delta}$.

Definition III.3.1. For $N \in \mathbb{N}$, discriminants d, D that are congruent to squares modulo $4N$ with D fundamental and dD negative, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$, we let

$$f_{d,D,N,s}(\tau) := \frac{(|dD|)^{\frac{1+s}{2}} v^s}{2^{1+s}\pi} \sum_{Q \in \mathcal{Q}_{dD,N}} \chi_D(Q)\phi_s(Q(\tau,1))$$

with ϕ_s as in (III.2.7) and define $f^*_{d,D,N}$ to be the analytic continuation of $f_{d,D,N,s}$ to $s = 0$.

Remark 6. The existence of the analytic continuation is established by combining Lemma III.10 with the analytic continuation of $H_{N,s}$ stated in Proposition III.8.

With the trace operation from Definition III.2.1, we obtain the following relation.

Lemma III.10. *We have*

$$f_{d,D,N,s}(\tau) = -\operatorname{tr}_{d,D,N}\left(H_{N,s}(\cdot,\tau)\right).$$

Proof. For $M \in \Gamma_0(N)$, $Q \in \mathcal{Q}_{dD,N}$, and the group action defined in (III.2.1), we have $z_{Q\circ M} = M^{-1}z_Q$ and

$$Q(\tau,1) = \frac{\sqrt{|dD|}}{2}\frac{(\tau - z_Q)(\tau - \overline{z}_Q)}{\mathrm{Im}(z_Q)},$$

since $\mathrm{Im}(z_Q) = \frac{\sqrt{|dD|}}{2a}$ for $Q = [a,b,c]$. Thus it follows

$$\begin{aligned} H_{N,s}(z_Q,\tau) &= -\frac{v^s}{2\pi} \sum_{M \in \Gamma_0(N)} \phi_s\left(\frac{(\tau - Mz_Q)(\tau - M\overline{z}_Q)}{\mathrm{Im}(Mz_Q)}\right) \\ &= -\frac{v^s}{2\pi} \sum_{M \in \Gamma_0(N)} \phi_s\left(\frac{(\tau - z_{Q\circ M})(\tau - \overline{z}_{Q\circ M})}{\mathrm{Im}(z_{Q\circ M})}\right) \\ &= -\frac{v^s}{2\pi} \sum_{M \in \Gamma_0(N)} \phi_s\left(\frac{2}{\sqrt{|dD|}}(Q\circ M)(\tau,1)\right) \end{aligned}$$

$$= -\frac{v^s(|dD|)^{\frac{1+s}{2}}}{2^{2+s}\pi} \sum_{M \in \Gamma_0(N)} \phi_s\left((Q \circ M)(\tau, 1)\right)$$

Taking the twisted trace we obtain

$$\begin{aligned}
\operatorname{tr}_{d,D,N}\left(H_{N,s}(\cdot,\tau)\right) &= -\frac{v^s|dD|^{\frac{1+s}{2}}}{2^{2+s}\pi} \sum_{Q \in \mathcal{Q}_{dD,N}/\Gamma_0(N)} \frac{\chi_D(Q)}{w_{N,Q}} \\
&\qquad\qquad \times \sum_{M \in \Gamma_0(N)} \phi_s\left((Q \circ M)(\tau,1)\right) \\
&= -\frac{v^s(|dD|)^{\frac{1+s}{2}}}{2^{2+s}\pi} \sum_{Q \in \mathcal{Q}_{dD,N}} \frac{\chi_D(Q)}{w_{N,Q}} \cdot 2w_{N,Q}\phi_s\left(Q(\tau,1)\right) \\
&= -f_{d,D,N,s}(\tau)
\end{aligned}$$

□

Theorem III.2 now follows directly from Proposition III.9 and taking the analytic continuation to $s = 0$ in Lemma III.10.

We now move on to compute the Fourier expansion of $f^*_{d,D,N}$ directly and prove Theorem III.1.

Proof of Theorem III.1. We follow the approach of Appendix 2 of [Zag75]. For $v > \frac{\sqrt{|dD|}}{2}$, we obtain by Poisson summation

$$\begin{aligned}
f_{d,D,N,s}(\tau) &= \frac{(|dD|)^{\frac{1+s}{2}} v^s}{2^{1+s}\pi} \sum_{\substack{a>0 \\ N|a}} \sum_{\substack{b \in \mathbb{Z} \\ b^2 \equiv dD \pmod{4a}}} \chi_D\left(\left[a, b, \frac{b^2 - dD}{4a}\right]\right) \\
&\qquad \times \phi_s\left(a\tau^2 + b\tau + \frac{b^2 - dD}{4a}\right) \\
&= \frac{(|dD|)^{\frac{1+s}{2}} v^s}{2^{1+s}\pi} \sum_{\substack{a>0 \\ N|a}} \sum_{n \in \mathbb{Z}} \sum_{\substack{b \pmod{2a} \\ b^2 \equiv dD \pmod{4a}}} \chi_D\left(\left[a, b, \frac{b^2 - dD}{4a}\right]\right) \\
&\qquad \times \int_{\mathbb{R}} \phi_s\left(a(\tau + t)^2 + b(\tau + t) + \frac{b^2 - dD}{4a}\right) e(-nt)dt.
\end{aligned}$$

Here we used that

$$a\tau^2 + (b + 2an)\tau + \frac{(b + 2an)^2 - dD}{4a} = a(\tau + n)^2 + b(\tau + n) + \frac{b^2 - dD}{4a}$$

and that χ_D is invariant under translation. Together with

$$\int_{\mathbb{R}} \phi_s\left(a(\tau+t)^2 + b(\tau+t) + \frac{b^2-dD}{4a}\right) e(-nt)dt$$
$$= a^{-1-s}e(n\tau)\int_{\mathbb{R}+iv} \phi_s\left(t^2 - \frac{dD}{4a^2}\right) e(-nt)dt$$

we obtain

$$f_{d,D,N,s}(\tau) = \frac{(|dD|)^{\frac{1+s}{2}}v^s}{2^{1+s}\pi}\sum_{\substack{a>0\\N|a}}\sum_{n\in\mathbb{Z}} S_{d,D}(a,n)a^{-1-s}e(n\tau)$$
$$\times \int_{\mathbb{R}+iv} \phi_s\left(t^2 - \frac{dD}{4a^2}\right) e(-nt)dt.$$

First we consider terms with $n \neq 0$. We have to show locally uniform convergence in s of the double sum. For this we will bound the integral locally uniformly for $\sigma := \mathrm{Re}(s) > -\varepsilon$ for some $\varepsilon > 0$ and independently of a and n. First we write

$$\int_{\mathbb{R}+iv} \phi_s\left(t^2 - \frac{dD}{4a^2}\right) e(-nt)dt$$
$$= \int_{\mathbb{R}} \left(\left(t^2 - v^2 - \frac{dD}{4a^2}\right)^2 + 4v^2t^2\right)^{-\frac{s}{2}} \frac{e(-nt)dt}{(t+iv)^2 - \frac{dD}{4a^2}}.$$

Note that the integrand is holomorphic in t in the region $\mathrm{Im}(t) > \frac{\sqrt{|dD|}}{2} - v$. Thus for $n < 0$, we may shift the path of integration to $\mathbb{R}+i\infty$ and the integral vanishes.

For $n > 0$, we may fix $\alpha \in \left(0, v - \frac{\sqrt{|dD|}}{2}\right)$ and shift the path of integration to $\mathbb{R} - i\alpha$. This yields

$$\left|\int_{\mathbb{R}-i\alpha} \phi_s\left((t+iv)^2 - \frac{dD}{4a^2}\right) e(-nt)dt\right|$$
$$\leq 2e^{-2\pi n\alpha}\int_0^\infty \left(\left(t^2 - (v-\alpha)^2 - \frac{dD}{4a^2}\right)^2 + 4(v-\alpha)^2t^2\right)^{-\frac{1+\sigma}{2}} dt.$$

Now we apply the estimates

$$\left(t^2 - (v-\alpha)^2 - \frac{dD}{4a^2}\right)^2 + 4(v-\alpha)^2t^2 \geq \begin{cases} \left((v-\alpha)^2 + \frac{dD}{4}\right)^2, & \text{for every } t,\\ \left(t^2 + (v-\alpha)^2\right)^2, & \text{for } t > v-\alpha, \end{cases}$$

to obtain

$$\int_0^\infty \left(\left(t^2-(v-\alpha)^2-\frac{dD}{4a^2}\right)^2+4(v-\alpha)^2t^2\right)^{-\frac{1+\sigma}{2}} dt$$
$$\leq \int_0^{v-\alpha}\left((v-\alpha)^2+\frac{dD}{4}\right)^{-1-\sigma} dt+\int_{v-\alpha}^\infty \left(t^2+(v-\alpha)^2\right)^{-1-\sigma} dt.$$

The last bound is locally uniform for $\sigma > -\frac{1}{2}$ and independent of a and n. Thus the overall sum is uniformly bounded by

$$\ll \sum_{n\geq 1}\sum_{\substack{a>0\\ N|a}} S_{d,D}(a,n)a^{-1-\sigma}e^{-2\pi n\alpha}.$$

We define the half-integral weight Kloosterman sum as

$$K^*(m,n,c) := \sum_{\substack{a,d \pmod c)^* \\ ad\equiv 1 \pmod c}} \left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{3/2} e\left(\frac{na+md}{c}\right),$$

where $\left(\frac{\cdot}{\cdot}\right)$ denotes the Kronecker symbol. Plugging $c \mapsto 4a$ into Proposition 3 of [DIT11] and noting that the definition of S given there differs from ours by a factor 2, we obtain

$$S_{d,D}(a,n)=\frac{1-i}{4}\sum_{r|(a,n)}\left(\frac{D}{r}\right)\sqrt{\frac{r}{a}}\left(1+\left(\frac{4}{a/r}\right)\right)K^*\left(d,\frac{n^2D}{r^2},\frac{4a}{r}\right),$$

and hence

$$\sum_{\substack{a>0\\ N|a}} S_{d,D}(a,n)a^{-1-\sigma}=\frac{1-i}{4}\sum_{r|n}r^{-1-\sigma}\left(\frac{D}{r}\right)\sum_{\substack{a>0\\ N|a}}\left(1+\left(\frac{4}{a}\right)\right)\frac{K^*\left(d,\frac{n^2D}{r^2},4a\right)}{a^{\frac{3}{2}+\sigma}}.$$

It has been observed in the remark following Theorem 2.1 of [FO08] that the Selberg-Kloosterman zeta function

$$S_{m,n}(s) := \sum_{a>0}\frac{K^*(m,n,a)}{a^s}$$

has an analytic continuation to $s=\frac{3}{2}$ for $mn<0$. Since $S_{m,n}$ has only finitely many poles in $[1,2]$, there is an $\varepsilon > 0$ such that the function $S_{d,\frac{n^2D}{r^2}}\left(\frac{3}{2}+\sigma\right)$

has an analytic continuation to $\sigma > -\varepsilon$. This gives a locally uniform bound for $\sigma > -\varepsilon$ and we obtain the analytic continuation for the sum over the positive n by just plugging in $s = 0$.

Now we have, for $v > \frac{\sqrt{|dD|}}{2}$ and $n > 0$,

$$\int_{\mathbb{R}+iv} \left(t^2 - \frac{dD}{4a^2}\right)^{-1} e(-nt)dt = -\frac{4\pi a}{\sqrt{|dD|}} \sinh\left(\frac{\pi n\sqrt{|dD|}}{a}\right),$$

since $t \mapsto \sinh(\kappa t)$ is the inverse Laplace transform of $s \mapsto \frac{\kappa}{s^2-\kappa^2}$ (see for example (29.3.17) of [AS64]). So all in all we obtain that for $n > 0$, the n-th Fourier coefficient of $f^*_{d,D,N}$ equals

$$-2\sum_{\substack{a>0\\ N|a}} S_{d,D}(a,n) \sinh\left(\frac{\pi n\sqrt{|dD|}}{a}\right).$$

Finally, it follows from Proposition III.9 and Lemma III.10 that the remaining part of the Fourier expansion, i.e. the $n = 0$ term, equals

$$-\operatorname{tr}_{d,D,N}\left(\frac{3}{\pi[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]v}\right) = -\frac{3H(d,D,N)}{\pi[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]v}.$$

□

Proof of Theorem III.4. By Theorem III.2, the n-th Fourier coefficient of $f^*_{d,D,N}$ is $-\operatorname{tr}_{d,D,N}(j_{N,n})$. If $\Gamma_0(N)$ has genus 0, then $j_{N,n}$ is weakly holomorphic on the modular curve $X_0(N)$. Lemma 5.1 (v) of [BO10] states that the twisted Heegner divisor

$$Z_{d,D,N} := \sum_{Q\in\mathscr{Q}_{dD,N}/\Gamma_0(N)} \frac{\chi_D(Q)}{w_{N,Q}} z_Q$$

is defined over $\mathbb{Q}(\sqrt{D})$. This means that

$$\langle Z_{d,D,N}, j_{N,n}\rangle := \sum_{Q\in\mathscr{Q}_{dD,N}/\Gamma_0(N)} \frac{\chi_D(Q)}{w_{N,Q}} j_{N,n}(z_Q) = \operatorname{tr}_{d,D,N}(j_{N,n}) \in \mathbb{Q}(\sqrt{D}).$$

By Theorem I of [CY96], $j_{N,1}(z_Q)$ is an algebraic integer for every quadratic form $Q \in \mathscr{Q}_{dD,N}$. Now $j_{N,n}$ is a polynomial in $j_{N,1}$, so the twisted sum $\operatorname{tr}_{d,D,N}(j_{N,n})$ is also an algebraic integer, which implies the statement. □

III.4 Regularized Inner Products

In this section we restrict to the full modular group and therefore drop the subscript N throughout. Let f, g be meromorphic modular forms of weight k which decay like cusp forms at $i\infty$ and have poles at $\mathfrak{z}_1, \dots, \mathfrak{z}_r \in \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. We choose a fundamental domain $\mathcal{F}$ such that for every $j \in \{1, \dots, r\}$, the representative of $\mathfrak{z}_j$ in $\mathcal{F}$ lies in the interior of $\Gamma_{\mathfrak{z}_j}\mathcal{F}$. We identify the $\mathfrak{z}_1, \dots, \mathfrak{z}_r \in \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ with their representatives in $\mathcal{F}$.

For an analytic function $A(s)$ in $s = (s_1, \dots, s_r)$, denote by $\mathrm{CT}_{s=0}A(s)$ the constant term of the meromorphic continuation of $A(s)$ around $s_1 = \cdots = s_r = 0$. Then the regularized inner product introduced in [BKvP] is given by

$$\langle f, g\rangle := \mathrm{CT}_{s=0}\left(\int_{\mathcal{F}} f(\tau)\prod_{\ell=1}^{r}|X_{\mathfrak{z}_\ell}(\tau)|^{2s_\ell}\,\overline{g(\tau)}v^k\frac{dudv}{v^2}\right). \tag{III.4.1}$$

Note that, as $z \to \mathfrak{z}_\ell$, we have $X_{\mathfrak{z}_\ell}(z) \to 0$, so the integral in (III.4.1) converges if we have $\mathrm{Re}(s_\ell) \gg 0$ for every $1 \leq \ell \leq r$. One can show that the regularization is independent of the choice of fundamental domain. Since the functions we integrate do not decay like cusp forms, we need to use another regularization by Borcherds [Bor98]. Namely for holomorphic modular forms f, g of weight k, we define

$$\langle f, g\rangle := \mathrm{CT}_{s=0}\left(\lim_{T\to\infty}\int_{\mathcal{F}_T} f(\tau)\overline{g(\tau)}v^{k-s}\frac{dudv}{v^2}\right), \tag{III.4.2}$$

whenever it exists. Here, for a fundamental domain $\mathcal{F}$ and $T > 0$, we set

$$\mathcal{F}_T := \{z \in \mathcal{F} : \mathrm{Im}(z) \leq T\}.$$

To compute the inner products in Theorem III.5, we split the domain of integration into a part which contains all the poles of the integrands, where we apply the regularization of Bringmann, Kane, and von Pippich, and a part around the cusp $i\infty$, where we apply Borcherds's regularization. Therefore, the regularized integral will look like

$$\begin{aligned}\langle f, g\rangle = \mathrm{CT}_{s=0}&\left(\int_{\mathcal{F}_Y} f(\tau)\prod_{\ell=1}^{r}|X_{\mathfrak{z}_\ell}(\tau)|^{2s_\ell}\,\overline{g(\tau)}v^k\frac{dudv}{v^2}\right)\\ &+ \mathrm{CT}_{s=0}\left(\lim_{T\to\infty}\int_{\mathcal{F}_T\backslash\mathcal{F}_Y} f(\tau)\overline{g(\tau)}v^{k-s}\frac{dudv}{v^2}\right),\end{aligned} \tag{III.4.3}$$

where f, g are meromorphic modular forms of weight k and $Y > 1$ a fixed constant, such that all poles of f and g lie in $\mathcal{F}_Y$. Here we can assume that $\mathcal{F}$ contains all the poles of f and g as well as $[0,1] + i[Y, \infty]$.

To prepare the proof, we first look at elliptic expansions of the polar harmonic Maass forms $H_z(\tau) := H_1(z, \tau)$. For $X_\varrho(\tau) \ll 1$, we have

$$\begin{aligned} H_z(\tau) &= -\frac{1}{2\pi i}\frac{J'(\tau)}{J(\tau) - J(z)} \\ &= \frac{1}{(\tau - \overline{\varrho})^2}\left(-\frac{\delta_{z,\varrho} w_z y}{\pi} X_\varrho(\tau)^{-1} + \sum_{n \geq 0} a_{z,\varrho}(n) X_\varrho(\tau)^n\right), \end{aligned} \tag{III.4.4}$$

where $\delta_{z,\varrho}$ is defined to be 1 if $z \in \mathrm{SL}_2(\mathbb{Z})\varrho$ and 0 otherwise (cf. the remark following Proposition III.8). Furthermore, let

$$G_z(\tau) := -\frac{1}{2\pi}\log|J(\tau) - J(z)|,$$

so that $\xi_0(G_z) = H_z$ with ξ_0 as in (III.2.5).

Lemma III.11. *The function G_z is a weight 0 polar harmonic Maass form. For every $\varrho \in \mathbb{H}$, it has an elliptic expansion*

$$G_z(\tau) = -\frac{\delta_{z,\varrho} w_z}{2\pi}\log(|X_\varrho(\tau)|) + \sum_{n \geq 0} A^+_{z,\varrho}(n) X_\varrho(\tau)^n + \sum_{n > 0} A^-_{z,\varrho}(n)\overline{X_\varrho(\tau)}^n, \tag{III.4.5}$$

which converges for $|X_\varrho(\tau)| \ll 1$. Moreover, we have

$$A^-_{z,\varrho}(n) = \frac{\overline{a_{z,\varrho}(n-1)}}{4\eta n}$$

with $a_{z,\varrho}(n)$ as in (III.4.4) *and*

$$A^+_{z,\varrho}(0) = -\frac{1}{2\pi} \times \begin{cases} \log|J(\varrho) - J(z)|, & \text{if } \varrho \notin \mathrm{SL}_2(\mathbb{Z})z, \\ \log|2yJ'(z)|, & \text{if } \varrho \in \mathrm{SL}_2(\mathbb{Z})z \text{ and } i, \rho \notin \mathrm{SL}_2(\mathbb{Z})z, \\ \log|2J''(i)|, & \text{if } \varrho, z \in \mathrm{SL}_2(\mathbb{Z})i, \\ \log\left|\frac{\sqrt{3}}{2}J'''(\rho)\right|, & \text{if } \varrho, z \in \mathrm{SL}_2(\mathbb{Z})\rho. \end{cases} \tag{III.4.6}$$

Proof. One easily checks that G_z is a polar harmonic Maass form of weight 0. By Proposition III.6, it has for every $\varrho \in \mathbb{H}$ an elliptic expansion

$$G_z(\tau) = \sum_{n \gg -\infty} A^+_{z,\varrho}(n) X_\varrho(\tau)^n + \sum_{n \ll \infty} A^-_{z,\varrho}(n) \beta_0 \left(1 - |X_\varrho(\tau)|^2, 1, -n\right) X_\varrho(\tau)^n$$

for $X_\varrho(\tau) \ll 1$. Noting that

$$\beta_0(1-r^2;1,-n) = \int_0^{1-r^2} (1-t)^{-n-1} dt + \delta_{n\neq 0} \cdot \frac{1}{n} = \begin{cases} \frac{r^{-2n}}{n}, & \text{if } n \neq 0, \\ -\log(r^2), & \text{if } n = 0, \end{cases}$$

we obtain an elliptic expansion of the shape

$$G_z(\tau) = A^-_{z,\varrho}(0)\log(|X_\varrho(\tau)|^2) + \sum_{n \gg -\infty} A^+_{z,\varrho}(n) X_\varrho(\tau)^n + \sum_{\substack{n \ll \infty \\ n \neq 0}} A^-_{z,\varrho}(n) \overline{X_\varrho(\tau)}^{-n}$$

Now using

$$\xi_0 \left(\overline{X_\varrho(\tau)}\right) = 2i \overline{\partial_{\overline{\tau}} \frac{\overline{\tau} - \overline{\varrho}}{\overline{\tau} - \varrho}} = -\frac{4\eta}{(\tau - \overline{\varrho})^2}$$

and

$$\begin{aligned} \xi_0 \left(\log(|X_\varrho(\tau)|^2)\right) &= 2i \overline{\partial_{\overline{\tau}} \log(|X_\varrho(\tau)|^2)} = 2i \overline{\frac{\partial_{\overline{\tau}} |X_\varrho(\tau)|^2}{|X_\varrho(\tau)|^2}} \\ &= 2i \overline{X_\varrho(\tau)} \frac{\overline{\partial_{\overline{\tau}} \overline{X_\varrho(\tau)}}}{|X_\varrho(\tau)|^2} = -\frac{4\eta}{(\tau - \overline{\varrho})^2} X_\varrho(\tau)^{-1} \end{aligned}$$

gives

$$\xi_0\left(G_z(\tau)\right) = -\frac{4\eta}{(\tau - \overline{\varrho})^2} \left(\overline{A^-_{z,\varrho}(0)} X_\varrho(\tau)^{-1} + \sum_{\substack{n \ll \infty \\ n \neq 0}} n \overline{A^-_{z,\varrho}(n)} X_\varrho(\tau)^{-n-1} \right).$$

We compare with (III.4.4) and obtain $A^-_{z,\varrho}(n) = 0$ for $n > 0$,

$$A^-_{z,\varrho}(0) = -\frac{\delta_{z,\varrho} w_z}{2\pi}, \quad \text{and} \quad -4\eta n \overline{A_{\varrho,z}(n)} = a_{\varrho,z}(-n-1).$$

for $n < 0$. We also have $A^+_{z,\varrho}(n) = 0$ for $n < 0$ since the principal part of G_z at ϱ comes entirely from the $n = 0$ term.

For the evaluation of $A^+_{z,\varrho}(0)$, note that

$$A^+_{z,\varrho}(0) = -\frac{1}{2\pi}\lim_{\tau\to\varrho}\left(\log|J(\tau)-J(z)| - \delta_{z,\varrho}w_z\log(|X_\varrho(\tau)|)\right),$$

which equals

$$G_z(\varrho) = -\frac{1}{2\pi w_z}\log|J(\tau)-J(z)|$$

if $\varrho \neq z$. If $\varrho = z$, note that

$$\begin{aligned}
&\lim_{\tau\to z}\left(\log|J(\tau)-J(z)| - w_z\log(|X_\varrho(\tau)|)\right)\\
&\quad= \lim_{\tau\to z}\left(\log|J(\tau)-J(z)| + \log\left(\left|\frac{\tau-\overline{z}}{\tau-z}\right|^{w_z}\right)\right)\\
&\quad= \lim_{\tau\to z}\log\left|\frac{J(\tau)-J(z)}{(\tau-z)^{w_z}}\right| + \log\left((2y)^{w_z}\right) = \log\left|\frac{(2y)^{w_z}}{w_z!}J^{(w_z)}(z)\right|,
\end{aligned}$$

which implies the statement. □

Lemma III.12. *For every $z, \mathfrak{z} \in \mathbb{H}$, we have*

$$\langle H_z, H_{\mathfrak{z}}\rangle = -A^+_{\mathfrak{z},z}(0)$$

with $A^+_{\mathfrak{z},z}(0)$ as in (III.4.6).

Proof. Applying Stokes's Theorem to the second summand of (III.4.3), we obtain for $\mathrm{Re}(s) > 1$

$$\begin{aligned}
&\int_{\mathcal{F}_T\setminus\mathcal{F}_Y} H_z(\tau)\overline{H_{\mathfrak{z}}(\tau)}v^{-s}dudv = -\int_{\mathcal{F}_T\setminus\mathcal{F}_Y} H_z(\tau)\xi_0\left(\overline{G_{\mathfrak{z}}(\tau)}\right)v^{-s}dudv\\
&= -\int_{\mathcal{F}_T\setminus\mathcal{F}_Y}\xi_0\left(\overline{H_z(\tau)G_{\mathfrak{z}}(\tau)v^{-s}}\right)dudv - s\int_{\mathcal{F}_T\setminus\mathcal{F}_Y} H_z(\tau)G_{\mathfrak{z}}(\tau)v^{-s-1}dudv\\
&= -\int_{\partial(\mathcal{F}_T\setminus\mathcal{F}_Y)} H_z(\tau)G_{\mathfrak{z}}(\tau)v^{-s}d\tau - s\int_{\mathcal{F}_T\setminus\mathcal{F}_Y}\overline{H_{\mathfrak{z}}(\tau)G_z(\tau)}v^{-s-1}dudv\\
&= \int_0^1 H_z(u+iT)G_{\mathfrak{z}}(u+iT)T^{-s}du - \int_0^1 H_z(u+iY)G_{\mathfrak{z}}(u+iY)Y^{-s}du\\
&\qquad + s\int_Y^T\int_0^1\overline{H_z(u+iv)G_{\mathfrak{z}}(u+iv)}v^{-s-1}dudv.
\end{aligned}$$

Now we have $G_{\mathfrak{z}}(\tau) = v + O(1)$ and $H_z(\tau) = O(1)$ as $v\to\infty$. Hence, the first summand vanishes as $T\to\infty$, provided that $\mathrm{Re}(s)$ is sufficiently large, and the limit as $T\to\infty$ of the third integral has a meromorphic continuation to $\mathbb{C}$ with the only pole at $s = 1$. Therefore the contributions of the first and third summand vanish in the analytic continuation to $s = 0$. The second summand is analytic in $s = 0$, so in total we have

$$\mathrm{CT}_{s=0} \int_{\mathcal{F}\setminus\mathcal{F}_Y} H_z(\tau)\overline{H_{\mathfrak{z}}(\tau)} v^{-s} d\tau = -\int_0^1 H_z(u+iY)G_{\mathfrak{z}}(u+iY)du. \quad \text{(III.4.7)}$$

To compute the first summand of (III.4.3), note that the functions H_z and $H_{\mathfrak{z}}$ have simple poles only at z and $\mathfrak{z}$. Thus the regularized inner product equals

$$\begin{aligned}\langle H_z, H_{\mathfrak{z}}\rangle &= \mathrm{CT}_{(s_1,s_2)=(0,0)} \int_{\mathcal{F}} H_z(\tau)|X_z(\tau)|^{2s_1}|X_{\mathfrak{z}}(\tau)|^{2s_2}\overline{H_{\mathfrak{z}}(\tau)}dudv \\ &= -\mathrm{CT}_{(s_1,s_2)=(0,0)} \int_{\mathcal{F}} H_z(\tau)|X_z(\tau)|^{2s_1}|X_{\mathfrak{z}}(\tau)|^{2s_2}\xi_0\left(\overline{G_{\mathfrak{z}}(\tau)}\right) dudv,\end{aligned}$$

By Stokes's Theorem, the integral equals

$$\begin{aligned}&-\int_{\mathcal{F}} H_z(\tau)\overline{\xi_0\left(|X_z(\tau)|^{2s_1}|X_{\mathfrak{z}}(\tau)|^{2s_2}\right)}G_{\mathfrak{z}}(\tau)dudv \\ &\qquad\qquad -\int_{\partial\mathcal{F}_Y} H_z(\tau)|X_z(\tau)|^{2s_1}|X_{\mathfrak{z}}(\tau)|^{2s_2}G_{\mathfrak{z}}(\tau)d\tau. \quad \text{(III.4.8)}\end{aligned}$$

Since there are no poles on $\mathcal{F}_Y$, the analytic continuation of the second summand is given by just plugging in $(s_1, s_2) = (0, 0)$. Now note that

$$\int_{\partial\mathcal{F}_Y} H_z(\tau)G_{\mathfrak{z}}(\tau)d\tau = -\int_0^1 H_z(u+iY)G_{\mathfrak{z}}(u+iY)du,$$

which cancels with (III.4.7). Therefore the contribution from the cusp $i\infty$ vanishes.

We are left to compute the analytic continuation of the first summand of (III.4.8). For this, we closely follow the proof of Theorem 6.1 in [BKvP]. For $\delta > 0$ and $\varrho \in \mathbb{H}$, we let $B_\delta(\varrho)$ denote the closed disc of radius δ around ϱ and split the domain of integration into $B_\delta(z)\cap\mathcal{F}$, $B_\delta(\mathfrak{z})\cap\mathcal{F}$, and $\mathcal{F}\setminus(B_\delta(z)\cup B_\delta(\mathfrak{z}))$. The integral over $\mathcal{F}\setminus(B_\delta(z)\cup B_\delta(\mathfrak{z}))$, away from the poles, vanishes at $(s_1, s_2) = (0, 0)$. Similarly, in the integral over $B_\delta(z)$ (resp. $B_\delta(\mathfrak{z})$) we can plug in $s_2 = 0$ (resp. $s_1 = 0$). By construction of $\mathcal{F}$, the points z and $\mathfrak{z}$ lie in the interior of $\Gamma_z\mathcal{F}$, resp. $\Gamma_{\mathfrak{z}}\mathcal{F}$. For $\varrho \in \{z, \mathfrak{z}\}$, we decompose

$$B_\delta(\varrho) = \bigcup_{M\in\Gamma_\varrho} M\left(B_\delta(\varrho)\cap\mathcal{F}\right)$$

to write

$$-\int_{B_\delta(\varrho)\cap\mathcal{F}} H_z(\tau)\overline{\xi_0\left(|X_\varrho(\tau)|^{2s}\right)}G_{\mathfrak{z}}(\tau)dudv$$
$$= -\frac{1}{w_\varrho}\int_{B_\delta(\varrho)} H_z(\tau)\overline{\xi_0\left(|X_\varrho(\tau)|^{2s}\right)}G_{\mathfrak{z}}(\tau)dudv.$$

Now using

$$\xi_0\left(|X_\varrho(\tau)|^{2s}\right) = -4s\eta|X_\varrho(\tau)|^{2s-2}\frac{\overline{X_\varrho(\tau)}}{(\tau-\overline{\varrho})^2},$$

we have to compute

$$\mathrm{Res}_{s=0}\left(\frac{4\eta}{w_\varrho}\int_{B_\delta(\varrho)} G_z(\tau)|X_\varrho(\tau)|^{2s-2}\frac{X_\varrho(\tau)}{(\overline{\tau}-\varrho)^2}H_{\mathfrak{z}}(\tau)dudv\right) \tag{III.4.9}$$

for $\varrho \in \{z, \mathfrak{z}\}$. Plugging in the elliptic expansions (III.4.4) and (III.4.5) around ϱ, we obtain

$$\frac{4\eta}{w_\varrho}\int_{B_\delta(\varrho)} H_z(\tau)|X_\varrho(\tau)|^{2s-2}\frac{X_\varrho(\tau)}{(\overline{\tau}-\varrho)^2}G_{\mathfrak{z}}(\tau)dudv$$
$$= \frac{4\eta}{w_\varrho}\int_{B_\delta(\varrho)}\left(-\frac{\delta_{z,\varrho}w_z\eta}{\pi}X_\varrho(\tau)^{-1} + \sum_{n\geq 0} a_{z,\varrho}(n)X_\varrho(\tau)^n\right)\frac{|X_\varrho(\tau)|^{2s-2}X_\varrho(\tau)}{(\overline{\tau}-\varrho)^2(\tau-\overline{\varrho})^2}$$
$$\times\left(-\frac{w_{\mathfrak{z}}}{2\pi}\log(|X_\varrho(\tau)|) + \sum_{n\geq 0} A^+_{\mathfrak{z},\varrho}(n)X_\varrho(\tau)^n + \sum_{n>0} A^-_{\mathfrak{z},\varrho}(n)\overline{X_\varrho(\tau)}^n\right)dudv$$

We substitute $X_\varrho(\tau) = Re(\theta)$ and use $\frac{4\eta^2}{|\tau-\overline{\varrho}|^4}dudv = 2\pi R\, d\theta dR$ to obtain

$$\frac{2\pi}{\eta w_\varrho}\int_0^\delta\int_0^1\left(-\frac{\delta_{z,\varrho}w_z\eta}{\pi} + \sum_{n\geq 0} a_{z,\varrho}(n)R^{n+1}e((n+1)\theta)\right)R^{2s-1}$$
$$\times\left(-\frac{\delta_{\mathfrak{z},\varrho}w_{\mathfrak{z}}}{2\pi}\log(R) + \sum_{n\geq 0} A^+_{\mathfrak{z},\varrho}(n)R^n e(n\theta) + \sum_{n>0} A^-_{\mathfrak{z},\varrho}(n)R^n e(-n\theta)\right)d\theta dR$$
$$= \int_0^\delta\left(\frac{\delta_{z,\varrho}\delta_{\mathfrak{z},\varrho}w_z w_{\mathfrak{z}}}{\pi w_\varrho}\log(R) - \frac{2\delta_{z,\varrho}w_z}{w_\varrho}A^+_{\mathfrak{z},\varrho}(0)\right.$$
$$\left. + \frac{2\pi}{\eta w_\varrho}\sum_{n\geq 0} A^-_{\mathfrak{z},\varrho}(n+1)a_{z,\varrho}(n)R^{2n+2}\right)R^{2s-1}dR$$

$$= \frac{\delta_{z,\varrho}\delta_{\mathfrak{z},\varrho}w_z}{\pi}\int_0^\delta \log(R)R^{2s-1}dR - \delta_{z,\varrho}A^+_{\mathfrak{z},\varrho}(0)\frac{\delta^{2s}}{s}$$
$$+ \frac{2\pi}{\eta w_\varrho}\sum_{n\geq 0} A^-_{\mathfrak{z},\varrho}(n+1)a_{z,\varrho}(n)\frac{\delta^{2(n+s+1)}}{2(n+s+1)}.$$

The last sum is analytic in $s=0$ and we have

$$\int_0^\delta \log(R)R^{2s-1}dR = \frac{\delta^{2s}\log(\delta)}{2s} - \frac{\delta^{2s}}{4s^2} = -\frac{1}{4s^2} + O(1),$$

so only the second term contributes to the residue in (III.4.9), yielding the statement. □

Proof of Theorem III.5. It follows from Lemmas III.10 and III.12 that

$$\langle f_d, f_\delta\rangle = \sum_{\substack{Q\in\mathcal{Q}_d/\operatorname{SL}_2(\mathbb{Z})\\ \mathcal{Q}\in\mathcal{Q}_\delta/\operatorname{SL}_2(\mathbb{Z})}} \frac{1}{w_{z_Q}w_{z_\mathcal{Q}}}\left\langle H_{z_Q}, H_{z_\mathcal{Q}}\right\rangle = -\sum_{\substack{Q\in\mathcal{Q}_d/\operatorname{SL}_2(\mathbb{Z})\\ \mathcal{Q}\in\mathcal{Q}_\delta/\operatorname{SL}_2(\mathbb{Z})}} \frac{1}{w_{z_Q}w_{z_\mathcal{Q}}}A^+_{z_\mathcal{Q},z_Q}(0).$$

(i) Note that if two quadratic forms Q, $\mathcal{Q}$ have the same CM-point, then one has to be an integer multiple of the other. The factor has to be $\sqrt{\frac{\delta}{d}}$. So conversely, if $\frac{\delta}{d}$ is not a square, then we have

$$A^+_{z_\mathcal{Q},z_Q}(0) = -\frac{1}{2\pi}\log\left(|J(z_Q)-J(z_\mathcal{Q})|^{\frac{1}{w_Q w_\mathcal{Q}}}\right)$$

for any $Q\in\mathcal{Q}_d$ and $\mathcal{Q}\in\mathcal{Q}_\delta$ by the first case of (III.4.6).

(ii) By the same argument as above, if neither $\frac{d}{3}$ nor $\frac{d}{4}$ is a square, then neither ρ nor i is a CM-point of any quadratic form of discriminant d. Thus by the first two cases of (III.4.6), we have for any $Q\neq\mathcal{Q}\in\mathcal{Q}_d$

$$A^+_{z_\mathcal{Q},z_Q}(0) = -\frac{1}{2\pi}\log|J(z_Q)-J(z_\mathcal{Q})|$$

and

$$A^+_{z_Q,z_Q}(0) = -\frac{1}{2\pi}\log|2\operatorname{Im}(z_Q)J'(z_Q)| = -\frac{1}{2\pi}\log\left|\sqrt{|d|}\frac{J'(z_Q)}{Q(1,0)}\right|.$$

(iii) This follows directly from the last two cases of $A^+_{z_Q,z_\mathcal{Q}}(0)$ given in (III.4.6).

□

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Chapter IV

Linear Incongruences for Generalized Eta-Quotients

This chapter is based on a manuscript published in *Research in Number Theory* [40].

IV.1 Introduction and Statement of Results

Ever since Ramanujan established his famous linear congrucences

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5} \\ p(7n+5) &\equiv 0 \pmod{7} \\ p(11n+6) &\equiv 0 \pmod{11} \end{aligned} \tag{IV.1.1}$$

for the partition function $p(n)$, the challenge of proving and generalizing them triggered a vast amount of research. For instance, Ono [Ono00] found analogues of (IV.1.1) for every modulus coprime to 6. See also [AO01] and the sources contained therein for further results. However, recent work of Radu [Rad12] proved that there are no linear congruences of $p(n)$ modulo 2 and 3, affirming a famous conjecture of Subbarao. The main ingredients of his proof are skillful computations and the *q-expansion principle* due to Deligne and Rapoport [DR73]. Adapting the methods of Radu's proof, Ahlgren and Kim [AK15] showed analogous results for the mock theta functions $f(q)$ and $\omega(q)$, as well as for certain classes of weakly holomorphic modular forms, including (classical) eta-quotients. In this paper, we extend their approach to generalized eta-quotients.

These functions are defined as follows: For $\delta \in \mathbb{Z}^+$ and a residue class $g \pmod{\delta}$, we set

$$\eta_{\delta,g}(z) := q^{\frac{\delta}{2}P_2\left(\frac{g}{\delta}\right)} \prod_{\substack{m>0 \\ m\equiv g \pmod{\delta}}} (1-q^m) \prod_{\substack{m>0 \\ m\equiv -g \pmod{\delta}}} (1-q^m),$$

where $z \in \mathbb{H}$ and $q := e^{2\pi i z}$ throughout. Here, for $x \in \mathbb{R}$ and $\{x\} := x - \lfloor x \rfloor$, we let

$$P_2(x) := \{x\}^2 - \{x\} + \frac{1}{6}$$

be the *second Bernoulli function.*

Note that if

$$\eta(z) := q^{1/24} \prod_{m>0} (1 - q^m)$$

denotes the usual Dedekind eta-function, then

$$\eta_{\delta,0}(z) = \eta(\delta z)^2 \qquad \text{and} \qquad \eta_{\delta,\frac{\delta}{2}}(z) = \frac{\eta\left(\frac{\delta}{2} z\right)^2}{\eta(\delta z)^2}.$$

Furthermore, for $g \notin \left\{0, \frac{\delta}{2}\right\}$ we have

$$\eta_{\delta,g}(z)^{-1} = q^{-\frac{\delta}{2} P_2\left(\frac{g}{\delta}\right)} \sum_{n \geq 0} p_{\delta,g}(n) q^n,$$

where $p_{\delta,g}(n)$ denotes the number of partitions of n with all parts congruent to $\pm g \pmod{\delta}$.

For $\delta = 5$, these functions occur in the well-known *Rogers-Ramanujan identities*, which state that

$$q^{\frac{1}{60}} \eta_{5,1}^{-1}(z) = \sum_{n \geq 0} \frac{q^{n^2}}{(q;q)_n} = 1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + \ldots$$

and

$$q^{-\frac{11}{60}} \eta_{5,2}^{-1}(z) = \sum_{n \geq 0} \frac{q^{n^2+n}}{(q;q)_n} = 1 + q^2 + q^3 + q^4 + q^5 + 2q^6 + \ldots,$$

where $(q;q)_n := \prod_{j=1}^{n} (1 - q^j)$.

For $N \in \mathbb{Z}^+$, a residue class $a \pmod{N}$, let $r := (r_{\delta,g})_{\delta | N, g \pmod{\delta}}$ be a tuple of half-integers, indexed by the divisors of N and their residue classes, with $r_{\delta,g} \in \mathbb{Z}$ unless $g = 0$ or $g = \frac{\delta}{2}$. In this paper, we study *generalized eta-quotients* of the form

$$H_r(z) := \prod_{\substack{\delta | N \\ g \pmod{\delta}}} \eta_{\delta,g}(z)^{r_{\delta,g}} =: q^{P(r)} \sum_{n \geq 0} c_r(n) q^n,$$

where

$$P(r) := \frac{1}{2} \sum_{\substack{\delta | N \\ g \pmod{\delta}}} \delta r_{\delta,g} P_2\left(\frac{g}{\delta}\right).$$

Note that the denominator of $P(r)$ divides $12N$.

For every modulus $m \in \mathbb{Z}^+$ and residue class $t \pmod{m}$, we give conditions on prime numbers p that guarantee that the linear progression $t \pmod{m}$ does not satisfy a linear congruence mod p for the generalized eta-quotient H_r. Here, for any residue class $a \pmod{N}$, we denote by r_a the tuple $(r_{\delta,ag})_{\delta|N, g \pmod{\delta}}$.

Theorem IV.1. *Let $m \in \mathbb{Z}^+$ and $t \in \{0, \ldots, m-1\}$. For $a, d \in \mathbb{Z}$ with $ad \equiv 1 \pmod{24Nm}$, let n be the smallest nonnegative integer for which*

$$d^2(n + P(r_a)) - P(r) \equiv t \pmod{m}.$$

Then for every prime p not dividing $c_{r_a}(n)$, we have

$$\sum_{n \geq 0} c_r(mn + t)q^n \not\equiv 0 \pmod{p}.$$

Remark 7. Since we always have $c_r(0) = 1$, the linear incongruence is satisfied for any prime p if

$$d^2 P(r_a) - P(r) \equiv t \pmod{m}.$$

Remark 8. By work of Ahlgren and Boylan [AB08], if the conditions of Theorem IV.1 are satisfied, we even have that

$$\#\{n \leq X : c_r(mn + t) \not\equiv 0 \pmod{p}\} \gg_{p,r,m,t,K} \frac{\sqrt{X}}{\log X} (\log\log X)^K$$

for every positive integer K.

Theorem IV.1 has several immediate applications.

Example IV.2. *Let $N = a = 1$ and $r = -\frac{1}{2}$. Then*

$$H_{-\frac{1}{2}}(z) = \eta^{-1}(z) = q^{-\frac{1}{24}} \sum_{n \geq 0} p(n)q^n.$$

Since $p(0) = p(1) = 1$, Theorem IV.1 then implies that

$$\sum_{n \geq 0} p(mn + t)q^n \not\equiv 0 \pmod{\ell}$$

for every prime ℓ if there is a d coprime to $6m$ with

$$t \equiv \frac{1-d^2}{24} \pmod{m} \quad \text{or} \quad t \equiv \frac{1+23d^2}{24} \pmod{m}.$$

Now assume that $\ell \geq 5$ is prime with $\left(\frac{-23}{\ell}\right) = -1$. Then the classes $d^2 \pmod{\ell}$ and $-23d^2 \pmod{\ell}$ together run over all residue classes except for 0 as d runs over residue classes coprime to ℓ. Since $(\ell, 24) = 1$, the classes $\frac{1-d^2}{24}$ and $\frac{1+23d^2}{24}$ cover every residue class modulo ℓ except for $\frac{1-\ell^2}{24}$. It follows that we can only have a linear congruence

$$\sum_{n\geq 0} p(\ell n + t)q^n \equiv 0 \pmod{\ell}$$

if $t \equiv \frac{1-\ell^2}{24} \pmod{\ell}$. This result was shown by Kiming and Olsson for every prime ℓ [KO92]. In particular, for $\ell \in \{5, 7, 11\}$, this implies that the residues in (IV.1.1) are the only ones for which such a congruence can hold.

Example IV.3. *More generally, Theorem IV.1 specializes to classical eta-quotients if $r_{\delta,g} = 0$ for $g \neq 0$. Then we have $P(r_a) = \frac{1}{12}\sum_{\delta|N} \delta r_\delta$ for all a. Since we always have $c_a(0) = 1$, we obtain that for every prime p, we have*

$$\sum_{n\geq 0} c_r(mn+t)q^n \not\equiv 0 \pmod{p} \qquad \text{if} \qquad t \equiv \frac{d^2-1}{12}\sum_{\delta|N} \delta r_\delta \pmod{m}$$

for some d coprime to $6Nm$.

Example IV.4. *Another interesting example are partitions occurring in Schur's Theorem [Sch26]. These are given by*

$$q^{\frac{1}{12}}\eta_{6,1}^{-1}(z) = \sum_{n\geq 0} p_{6,1}(n)q^n = 1 + q + q^2 + q^3 + q^4 + 2q^5 + 2q^6 + \dots$$

with $N = 6$, $r_{6,1} = -1$ and $r_{\delta,g} = 0$ otherwise, and $P(r_a) = -\frac{1}{12}$ for every a coprime to 6. Thus Theorem 1 implies that

$$\sum_{n\geq 0} p_{6,1}(mn+t)q^n \not\equiv 0 \pmod{p}$$

for any prime p if there is a d coprime to $6m$ and $j \in \{-1, 11, 23, 35, 47\}$ with

$$t \equiv \frac{1+jd^2}{12} \pmod{m}.$$

As in Example 1, if $\ell \geq 5$ is a prime with at least one of $\left(\frac{-11}{\ell}\right)$, $\left(\frac{-23}{\ell}\right)$, $\left(\frac{-35}{\ell}\right)$, or $\left(\frac{-47}{\ell}\right)$ equal to -1, then there can only be a linear congruence if $t \equiv \frac{1-\ell^2}{12} \pmod{\ell}$.

Example IV.5. *Now we take a closer look at the Rogers-Ramanujan functions* $\eta_{5,1}^{-1}$ *and* $\eta_{5,2}^{-1}$*. If* $H_{r_1} = H_{r_4} = \eta_{5,1}^{-1}$*, then we have* $N = 5$*,* $r_{5,1} = -1$*,* $r_{5,2} = 0$*,* $H_{r_2} = H_{r_3} = \eta_{5,2}^{-1}$*, and*

$$P(r_a) = \begin{cases} -\frac{1}{60} & \text{if } a \equiv 1 \text{ or } 4 \pmod 5, \\ \frac{11}{60} & \text{if } a \equiv 2 \text{ or } 3 \pmod 5. \end{cases}$$

Hence Theorem 1 states that $\sum_{n\geq 0} p_{5,1}(mn+t)q^n \not\equiv 0 \pmod p$ *for every prime* p*, if*

$$t \equiv nd^2 + \frac{1-d^2}{60} \pmod m,$$

for $n \in \{0,1,2,3\}$ *and* $d \equiv 1,4 \pmod 5$ *coprime to* $6m$*, or*

$$t \equiv nd^2 + \frac{11d^2+1}{60} \pmod m,$$

for $n \in \{0,2,3,4,5\}$ *and* $d \equiv 2,3 \pmod 5$ *coprime to* $6m$*.*

If we switch the roles of $\eta_{5,1}^{-1}$ *and* $\eta_{5,2}^{-1}$*, we obtain that* $\sum_{n\geq 0} p_{5,2}(mn+t)q^n \not\equiv 0 \pmod p$ *for every prime* p*, if*

$$t \equiv nd^2 - \frac{d^2+11}{60} \pmod m,$$

for $n \in \{0,1,2,3\}$ *and* $d \equiv 2,3 \pmod 5$ *coprime to* $6m$*, or*

$$t \equiv nd^2 + \frac{11(d^2-1)}{60} \pmod m,$$

for $n \in \{0,2,3,4,5\}$ *and* $d \equiv 1,4 \pmod 5$ *coprime to* $6m$*.*

In contrast, applying work of Gordon [Gor12]*, Hirschhorn* [Hir16] *found linear congruences* $\pmod 2$ *for* $p_{5,1}$ *and* $p_{5,2}$*. For example, Theorem 3 of* [Hir16] *states that*

$$p_{5,1}(98n+t) \equiv 0 \pmod 2$$

for $t \in \{23,37,51,65,79,93\}$ *and*

$$p_{5,2}(98n+t) \equiv 0 \pmod 2$$

for $t \in \{6,20,34,62,76,90\}$*. The above discussion precludes all the other residues* $\pmod{98}$ *except for* $t \in \{9,16,58,72,86\}$ *resp.* $t \in \{13,27,48,55,97\}$ *from satisfying these congruences.*

The paper is organized as follows: In Section 2 we define generalized eta-quotients and study their transformation behavior under $\Gamma_0(12N)$, slightly adapting a result of Robins [Rob94]. This will lead to modularity properties for the functions $H_{m,r,t}$ whose Fourier coefficients are given by those of H_r on the arithmetic progression $t \pmod{m}$. In Section 3 we prove Theorem IV.1 using the q-expansion principle.

Acknowledgements

We thank Ken Ono for suggesting this project and for his advice and support. We also thank Min-Joo Jang, Wenjun Ma, and the referee for helpful comments, the Fulbright Commission for their generous support, and the Department of Mathematics and Computer Science at Emory University for their hospitality.

IV.2 Transformation properties of Eta-Quotients

We begin by studying modularity properites of $\eta_{\delta,g}$. For $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(\delta)$ we define $\mu_{A,g,\delta}$ by

$$\eta_{\delta,g}\left(Az\right) = e\left(\mu_{A,g,\delta}\right) j(A,z)^{\delta_{g,0}} \eta_{\delta,ag}(z),$$

where $j(A,z) := cz + d$ and $e(w) := e^{2\pi i w}$ throughout. An analogue of the following proposition for the subgroup $\Gamma_1(\delta)$ was shown in Theorem 2 of [Rob94].

Proposition IV.6. *For $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(12\delta)$ we have*

$$\mu_{A,g,\delta} \equiv \frac{1}{2} db\delta P_2\left(\frac{ag}{\delta}\right) - \frac{a-1}{4} + \frac{1}{2}\left\lfloor \frac{ag}{\delta} \right\rfloor \pmod{1}.$$

Proof. An equation on p. 122 of [Rob94] states that (note the different normalization of $\mu_{A,g,\delta}$)

$$\mu_{A,g,\delta} = \sum_{\mu=1}^{a-1} \left(\left(\frac{\mu}{a}\right)\right)\left(\left(\frac{c}{\delta}\frac{\mu}{a} + \frac{g}{\delta}\right)\right) + \frac{\delta b}{2a} P_2\left(\frac{ag}{\delta}\right) - \frac{c}{12\delta a}$$

with

$$((x)) := \begin{cases} \{x\} - \frac{1}{2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

By Eqn. (34) of [Sch74], Ch. VIII §4, the denominator of $\mu_{A,g,\delta}$ divides 12δ. This implies that for $A \in \Gamma_0(12\delta)$ we have, using that $ad \equiv 1 \pmod{12\delta}$,

$$\mu_{A,g,\delta} \equiv ad \sum_{\mu=1}^{a-1} \left(\left(\frac{\mu}{a}\right)\right) \left(\left(\frac{c}{\delta}\frac{\mu}{a} + \frac{g}{\delta}\right)\right) + \frac{\delta db}{2} P_2\left(\frac{ag}{\delta}\right) \pmod{1}.$$

We compute

$$\begin{aligned} ad \sum_{\mu=1}^{a-1} \left(\left(\frac{\mu}{a}\right)\right) \left(\left(\frac{c}{\delta}\frac{\mu}{a} + \frac{g}{\delta}\right)\right) &= d \sum_{\mu=1}^{a-1} \left(\mu - \frac{a}{2}\right) \left(\left(\frac{c}{\delta}\frac{\mu}{a} + \frac{g}{\delta}\right)\right) \\ &= d \sum_{\mu=1}^{a-1} \mu \left(\left(\frac{c}{\delta}\frac{\mu}{a} + \frac{g}{\delta}\right)\right) - \frac{ad}{2} \sum_{\mu=1}^{a-1} \left(\left(\frac{\mu}{a} + \frac{g}{\delta}\right)\right). \end{aligned}$$

Now

$$d \sum_{\mu=1}^{a-1} \mu \left(\left(\frac{c}{\delta}\frac{\mu}{a} + \frac{g}{\delta}\right)\right) \equiv d \sum_{\mu=1}^{a-1} \mu \left(\frac{c}{\delta}\frac{\mu}{a} + \frac{g}{\delta} - \frac{1}{2}\right) \equiv \frac{a-1}{2}\left(\frac{g}{\delta} - \frac{1}{2}\right) \pmod{1}$$

and

$$\begin{aligned} \frac{ad}{2} \sum_{\mu=1}^{a-1} \left(\left(\frac{\mu}{a} + \frac{g}{\delta}\right)\right) &\equiv \frac{1}{2} \sum_{\mu=1}^{a-1} \left(\frac{\mu}{a} + \frac{g}{\delta} - \left\lfloor \frac{\mu}{a} + \frac{g}{\delta} \right\rfloor - \frac{1}{2}\right) \\ &\equiv \frac{a-1}{4} + \frac{a-1}{2}\left(\frac{g}{\delta} - \frac{1}{2}\right) - \frac{1}{2}\left\lfloor \frac{ag}{\delta} \right\rfloor \pmod{1}, \end{aligned}$$

using that $\sum_{\mu=0}^{a-1} \left\lfloor \frac{\mu}{a} + x \right\rfloor = \lfloor ax \rfloor$.

□

Let

$$H_{r,m,t}(z) := \frac{1}{m} \sum_{\lambda \pmod{m}} e\left(-\frac{\lambda}{m}(t + P(r))\right) H_r\left(\begin{pmatrix} 1 & \lambda \\ & m \end{pmatrix} z\right), \qquad \text{(IV.2.1)}$$

so that

$$H_{r,m,t}(z) = q^{\frac{t+P(r)}{m}} \sum_{n \geq 0} c_r(mn + t) q^n$$

and for every $\lambda \pmod{m}$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(m)$ choose λ' with

$$a\lambda' \equiv b + d\lambda \pmod{m}.$$

Note that λ' runs over all residue classes modulo m with λ.

Moreover, let $k := \sum_{\delta | N} r_{\delta,0}$, so that k is the weight of H_r.

Proposition IV.7. *For $A \in \Gamma_0(24Nm)$, we have*

$$H_{r,m,t}(Az) = j(A,z)^k \frac{\zeta}{m} \sum_{\lambda \pmod m} e\left(\frac{\lambda}{m}\left(d^2 P(r_a) - P(r) - t\right) - \frac{\lambda'}{m} P(r_a)\right) \times H_{r_a}\left(\begin{pmatrix} 1 & \lambda' \\ & m \end{pmatrix} z\right),$$

where ζ is a $24Nm$-th root of unity depending on r, m, and A. In particular, $H_{r,m,t}^{24Nm}$ is a weakly holomorphic modular form of weight $24Nmk$ for $\Gamma_1(24Nm)$, i.e. a meromorphic modular form whose poles are supported at the cusps.

Proof. Let

$$A_\lambda := \begin{pmatrix} a + c\lambda & \frac{1}{m}\left(b + d\lambda - \lambda'(a + c\lambda)\right) \\ mc & d - c\lambda' \end{pmatrix},$$

so that

$$\begin{pmatrix} 1 & \lambda \\ & m \end{pmatrix} A = A_\lambda \begin{pmatrix} 1 & \lambda' \\ & m \end{pmatrix}$$

Then for $A \in \Gamma_0(24Nm)$, we have by Proposition IV.6

$$\begin{aligned}
\eta_{\delta,g}\left(\begin{pmatrix} 1 & \lambda \\ & m \end{pmatrix} Az\right)^{r_{\delta,g}} &= \eta_{\delta,g}\left(A_\lambda \begin{pmatrix} 1 & \lambda' \\ & m \end{pmatrix} z\right)^{r_{\delta,g}} \\
&= e\left(r_{\delta,g}\mu_{A_\lambda,g,\delta}\right) j\left(A_\lambda, \begin{pmatrix} 1 & \lambda' \\ & m \end{pmatrix} z\right)^{\delta_{g,0} r_{\delta,g}} \eta_{\delta,ag}\left(\begin{pmatrix} 1 & \lambda' \\ & m \end{pmatrix} z\right)^{r_{\delta,g}} \\
&= \zeta_0 e\left(\frac{r_{\delta,g}\delta}{2} P_2\left(\frac{ag}{\delta}\right)(d - c\lambda')\frac{1}{m}\left(b + d\lambda - \lambda'(a + c\lambda)\right)\right) \\
&\qquad \times j(A,z)^{\delta_{g,0} r_{\delta,g}} \eta_{\delta,ag}\left(\begin{pmatrix} 1 & \lambda' \\ & m \end{pmatrix} z\right)^{r_{\delta,g}} \\
&= \zeta_0 e\left(\frac{r_{\delta,g}\delta}{2m} P_2\left(\frac{ag}{\delta}\right)\left(db + d^2\lambda - \lambda'\right)\right) j(A,z)^{\delta_{g,0} r_{\delta,g}} \eta_{\delta,ag}\left(\begin{pmatrix} 1 & \lambda' \\ & m \end{pmatrix} z\right)^{r_{\delta,g}},
\end{aligned}$$

where ζ_0 is a fourth root of unity depending on $r_{\delta,g}$ and A. Thus,

$$H_r\left(\begin{pmatrix} 1 & \lambda \\ & m \end{pmatrix} Az\right) = \zeta j(A,z)^k e\left(\frac{P(r_a)}{m}\left(d^2\lambda - \lambda'\right)\right) H_{r_a}\left(\begin{pmatrix} 1 & \lambda' \\ & m \end{pmatrix} z\right).$$

Together with (IV.2.1), this yields the formula in the proposition.

Moreover, note that for $A \in \Gamma_1(24Nm)$, we have $\lambda' \equiv \lambda + b \pmod m$ and

$$
\begin{aligned}
H_{r,m,t}(Az) &= j(A,z)^k \frac{\zeta}{m} \sum_{\lambda' \pmod m} e\left(\frac{\lambda' - b}{m} \left((d^2-1)P(r) - t \right) - \frac{\lambda'}{m} P(r) \right) \\
&\qquad \times H_r \left(\begin{pmatrix} 1 & \lambda' \\ & m \end{pmatrix} z \right) \\
&= j(A,z)^k \frac{\zeta_1}{m} \sum_{\lambda' \pmod m} e\left(-\frac{\lambda'}{m} (t + P(r)) \right) H_r \left(\begin{pmatrix} 1 & \lambda' \\ & m \end{pmatrix} z \right) \\
&= \zeta_1 j(A,z)^k H_{r,m,t}(z)
\end{aligned}
$$

with $\zeta_1 := e\left(\frac{bt}{m}\right)\zeta$. Since ζ_1 is a $24Nm$-th root of unity, we conclude that $H_{r,m,t}^{24Nm}$ is a weakly holomorphic modular form of weight $24Nmk$ for $\Gamma_1(24Nm)$. □

IV.3 Proof of Theorem IV.1

Proof. For j large enough, we have that $H_{r,m,t}^{24Nm}\Delta^j$ is a holomorphic modular form for $\Gamma_1(24Nm)$. Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(24Nm)$ and let $|_k$ denote the Petersson slash-operator, i.e. $(f|_k A)(z) := j(A,z)^{-k} f(Az)$. Then since

$$
H_{r_a}\left(\begin{pmatrix} 1 & \lambda' \\ & m \end{pmatrix} z \right) = e\left(\frac{\lambda'}{m} P(r_a) \right) q^{\frac{P(r_a)}{m}} \sum_{n \geq 0} c_{r_a}(n) e\left(\frac{\lambda'}{m} n \right) q^{\frac{n}{m}},
$$

we obtain by Proposition IV.7

$$
\begin{aligned}
\left(H_{r,m,t}|_k A \right)(z) &= \frac{\zeta}{m} \sum_{\lambda \pmod m} e\left(\frac{\lambda}{m} \left(d^2 P(r_a) - P(r) - t \right) - \frac{\lambda'}{m} P(r_a) \right) \\
&\qquad \times H_{r_a} \left(\begin{pmatrix} 1 & \lambda' \\ & m \end{pmatrix} z \right) \\
&= \frac{\zeta}{m} \sum_{\lambda \pmod m} e\left(\frac{\lambda}{m} \left(d^2 P(r_a) - P(r) - t \right) \right) q^{\frac{P(r_a)}{m}} \sum_{n\geq 0} c_{r_a}(n) e\left(\frac{\lambda'}{m} n \right) q^{\frac{n}{m}} \\
&= \frac{\zeta}{m} q^{\frac{P(r_a)}{m}} \sum_{n \geq 0} c_{r_a}(n) e\left(\frac{dbn}{m} \right) \sum_{\lambda \pmod m} e\left(\frac{\lambda}{m} \left(d^2 (P(r_a) + n) - P(r) - t \right) \right) q^{\frac{n}{m}},
\end{aligned}
$$

since $\lambda' \equiv db + d^2\lambda \pmod m$.

Now assume that n is the smallest nonnegative integer with $d^2(P(r_a)+n) - P(r) \equiv t \pmod m$. Then we have

$$
\left(H_{r,m,t}|_k A \right)(z) = \zeta_2 c_{r_a}(n) q^{\frac{P(r_a)+n}{m}} \left(1 + O\left(q^{\frac{1}{m}} \right) \right)
$$

with $\zeta_2 := e\left(\frac{dbn}{m}\right)\zeta$.

Suppose that $H_{r,m,t} \equiv 0 \pmod{p}$. Then

$$(p^{-1}H_{r,m,t})^{24Nm}\Delta^j \in M_{24Nmk+12j}(\Gamma_1(24Nm)) \cap \mathbb{Z}[\zeta_{24Nm}][q].$$

The q-expansion principle from Corollaire 3.12 of [DR73], Ch. VII states that if f is a modular form of weight κ for $\Gamma_1(N)$ whose Fourier coefficients at $i\infty$ lie in $\mathbb{Z}[\zeta_N]$, then for any $A \in \Gamma_0(N)$, also $f|_\kappa A$ has Fourier coefficients in $\mathbb{Z}[\zeta_N]$ (see also Corollary 5.3 of [Rad12]). Thus it follows that

$$\left((p^{-1}H_{r,m,t})^{24Nm}\Delta^j\right)|_{24Nmk+12j}A \in \mathbb{Z}[\zeta_{24Nm}][q]$$

for every $A \in \Gamma_0(24Nm)$. By the above computation we have

$$\begin{aligned}\left(\left((p^{-1}H_{r,m,t})^{24Nm}\Delta^j\right)|_{24Nmk+12j}A\right) &= p^{-24Nm}((H_{r,m,t}|A)(z))^{24Nm}\Delta(z)^j\\ &= \left(\frac{c_{r_a}(n)}{p}\right)^{24Nm} q^{24N(P(r_a)+n)+j}\,(1+O(q)) \in \mathbb{Z}[\zeta_{24Nm}][q].\end{aligned}$$

This can only hold if p divides $c_{r_a}(n)$. □

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Chapter V

Special Values of Motivic L-functions

This chapter is based on a manuscript published in *Research in the Mathematical Sciences* and is joint work with Dr. Wenjun Ma and Dr. Jesse Thorner [42].

V.1 Introduction and statement of results

Let $f(z) = \sum_{n=1}^{\infty} a_f(n)q^n$ be a normalized holomorphic cuspidal modular form of even weight $k \geq 2$ and level N, and trivial nebentypus. Assume further that f is an eigenform for the Hecke operators T_p for $p \nmid N$ and U_p for all $p \mid N$. We call such a modular form a newform. The L-function $L(s, f)$ associated to a newform f, which is given by

$$L(s,f) := \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \Big(\prod_{p|N} \frac{1}{1-a_f(p)p^{-s}}\Big) \prod_{p\nmid N} \frac{1}{1-a_f(p)p^{-s}+p^{k-1-2s}}, \tag{V.1.1}$$

has an analytic continuation to $\mathbb{C}$. The completed L-function

$$\Lambda(s,f) = \Big(\frac{\sqrt{N}}{2\pi}\Big)^s \Gamma(s) L(s,f) \tag{V.1.2}$$

is an entire function of order one and satisfies the functional equation $\Lambda(s, f) = \varepsilon(f)\Lambda(k - s, f)$, where $\varepsilon(f) \in \{-1, 1\}$. The completed L-function arises as a period integral of f:

$$\Lambda(s,f) = N^{s/2} \int_0^{\infty} f(iy) y^{s-1} dy. \tag{V.1.3}$$

One defines the period polynomial associated to f by $r_f(z) := \int_0^{i\infty} f(\tau)(\tau - z)^{k-2} d\tau$, which is a polynomial of degree at most $k-2$ in z. Using (V.1.3), we expand $(\tau - z)^{k-2}$ to obtain

$$r_f(z) = \Big(\frac{i}{\sqrt{N}}\Big)^{k-1} \sum_{j=0}^{k-2} \binom{k-2}{j} (iz\sqrt{N})^j \Lambda(k-1-j, f). \tag{V.1.4}$$

By expressing $\Lambda(s, f)$ in terms of $L(s, f)$ via (V.1.2), we see that $r_f(z)$ is a generating function for the critical values $L(1, f)$, $L(2, f), \ldots, L(k-1, f)$. For additional background and details, see [JMOS16] and the sources contained therein.

It follows from the functional equation for $\Lambda(s, f)$ that $r_f(z)$ satisfies a functional equation of its own, relating $r_f(\frac{z}{i\sqrt{N}})$ to $r_f(\frac{1}{iz\sqrt{N}})$ and fixing the unit circle $\mathbb{S}^1 = \{z \in \mathbb{C}\colon |z| = 1\}$. In analogy with the expected behavior of the nontrivial zeros of the Riemann zeta function $\zeta(s)$ or the nontrivial zeros of $L(s, f)$, one might expect that all of the zeros of $r_f(\frac{z}{i\sqrt{N}})$ lie on $\mathbb{S}^1$. Because of the similarity with the Riemann hypothesis, this has been called the *Riemann hypothesis for period polynomials.* Conrey, Farmer, and Imamoglu [CFI13] proved a result of this sort for the odd part of $r_f(\frac{z}{i\sqrt{N}})$, and the Riemann hypothesis for the period polynomials associated to newforms of level 1 and even weight $k \geq 2$ was established by El-Guindy and Raji [EGR14]. The Riemann hypothesis for period polynomials is now a theorem due to Jin, Ma, Ono, and Soundararajan [JMOS16] for all newforms of weight $k \geq 2$ with trivial nebentypus; furthermore, they proved that if either k or N is sufficiently large, then the zeros of $r_f(\frac{z}{i\sqrt{N}})$ are equidistributed on $\mathbb{S}^1$.

The truth of the Riemann hypothesis for period polynomials, along with the statement of equidistribution, introduces strong conditions on the sizes of the critical values $L(1, f)$, $L(2, f), \ldots, L(k-1, f)$; these values have significance in algebraic number theory and arithmetic geometry. For newforms f of weight 2 associated to elliptic curves, $r_f(z)$ is a constant polynomial with a non-zero factor of $L(1, f)$. If the Birch and Swinnerton-Dyer conjecture is true, then $L(1, f)$ encapsulates much of the arithmetic of the elliptic curve, including order of the Tate-Shafarevich group and whether or not the rank of the Mordell-Weil group is positive. Unfortunately, the results in [JMOS16] cannot provide insight into the Birch and Swinnerton-Dyer conjecture, because for $k = 2$, the period polynomial is constant. Thus the Riemann hypothesis for period polynomials when $k = 2$ is trivially satisfied without shedding light on $L(1, f)$. If $k \geq 4$, the critical values hold similar importance in the context of the Bloch-Kato conjecture [BK90], which generalizes of the Birch and Swinnerton-Dyer conjecture.

In this paper, we use the ideas in [JMOS16] to study critical values of motivic L-functions. It is well-known that each modular L-function $L(s, f)$ is attached to a certain pure motive over $\mathbb{Q}$ of weight $k-1$, conductor N, and rank 2; furthermore, $L(s, f)$ is the L-function of a certain cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. (Here, $\mathbb{A}_\mathbb{Q}$ denotes the ring of adeles of $\mathbb{Q}$.) The critical values of motivic L-functions carry similar arithmetic significance in the context of the Bloch-Kato conjecture. When motivic L-functions coincide with automorphic L-functions, they have important analytic properties which generalize those of $L(s,f)$. However, there does not appear to be a canonical generating polynomial for critical values of motivic L-functions that generalizes the properties of $r_f(z)$. Thus we construct a polynomial $p_{\mathcal{M}}(z)$ (see (V.3.1)) which mimics $r_f(\frac{z}{i\sqrt{N}})$ and prove the following.

Theorem V.1. *Let $\mathcal{M}$ be a pure motive over $\mathbb{Q}$ of odd motivic weight $w = 2m+1 \geq 3$, even rank $d \geq 2$, global conductor N, and Hodge numbers h_ν for $0 \leq \nu \leq m$ (see Section V.2). Suppose that the L-function $L(s,\mathcal{M})$ of $\mathcal{M}$ coincides with the L-function of an algebraic, tempered, cuspidal symplectic representation of $\mathrm{GL}_d(\mathbb{A}_\mathbb{Q})$. Let $p_{\mathcal{M}}(z)$ be the polynomial defined in* (V.3.1)*.*

1. *If $m = 1$ and $h_0 \in \{0,1\}$, then the zeros of $p_{\mathcal{M}}(z)$ lie on $\mathbb{S}^1$ and tend to be equidistributed as $N \to \infty$.*
2. *If $m \geq 2$, $2m^{h_m} \geq (1+1/m)^{h_0}$, and $N > A_m^d$ (where A_m is defined by* (V.4.4)*), then the zeros of $p_{\mathcal{M}}(z)$ lie on $\mathbb{S}^1$ and tend to be equidistributed as $N \to \infty$.*
3. *If m is sufficiently large, then nearly all of the zeros of $p_{\mathcal{M}}(z)$ lie on $\mathbb{S}^1$. (See Theorem V.9 for a more precise statement.)*

Remark 9. If $L(s,\mathcal{M})$ is the L-function of a newform of (modular) weight $k \geq 4$, then $p_{\mathcal{M}}(z)$ reduces to a constant multiple of $r_f(\frac{z}{i\sqrt{N}})$, whose zeros are studied in [JMOS16].

It is unclear how to ensure that all of the zeros lie on $\mathbb{S}^1$ while maintaining uniformity in d when $m \geq 2$ and d is large compared to $\log N$. Despite this setback, we already have a result that is strong enough to address a natural family of examples, namely the odd symmetric power L-functions $L(s, \mathrm{Sym}^n f)$ of the newforms f considered in [JMOS16] that do not have complex multiplication (CM). The next result follows from Theorem V.1 in case of $\mathcal{M} = \mathrm{Sym}^n f$ and n odd.

Corollary V.2. *Let $n \geq 3$ be an odd integer and f a non-CM newform of even integral weight $k \geq 2$, squarefree level $N \geq 13$, trivial nebentypus, and integral Fourier coefficients. We assume that $N \geq 46$ if $(k,n) = (2,5)$ and $N \geq 17$ if $(k,n) \in \{(2,7),(4,3)\}$. If $L(s,\mathrm{Sym}^n f)$ is the L-function of an algebraic tempered cuspidal symplectic representation of $\mathrm{GL}_{n+1}(\mathbb{A}_\mathbb{Q})$, then all of the zeros of $p_{\mathrm{Sym}^n f}(z)$ lie on $\mathbb{S}^1$. The zeros tend to be equidistributed as n or N goes to ∞.*

We find the most interesting case of Corollary V.2 to be where $k = 2$. In this case, the period polynomial of f is constant, and the results in [JMOS16] are trivial. When considering the odd symmetric power L-functions $L(s, \mathrm{Sym}^n f)$, we see that $L(s, \mathrm{Sym}^n f)$ has only one critical value at $s = \frac{n+1}{2}$ but many special values. By numerically checking the cases that are not covered by Corollary V.2, we obtain the following result.

Theorem V.3. *Let $E/\mathbb{Q}$ be a non-CM elliptic curve of squarefree conductor N, and let $n \geq 3$ be an odd integer. If $L(s, \mathrm{Sym}^n E)$ is the L-function of an algebraic, tempered, cuspidal symplectic representation of $\mathrm{GL}_{n+1}(\mathbb{A}_\mathbb{Q})$, then all of the zeros of $p_{\mathrm{Sym}^n E}(z)$ given by (V.3.1) lie on $\mathbb{S}^1$. The zeros tend to be equidistributed as n or N goes to ∞.*

In [Man16], Manin speculated on the existence of *zeta-polynomials* $Z(s)$ which (in analogy with expected behavior of the Riemann zeta function and $L(s, f)$) satisfy a functional equation of the form $Z(s) = \pm Z(1-s)$ and have all of their zeros lie on the line $\Re(s) = 1/2$. Furthermore, there should be a "nice" generating function for the sequence $\{Z(-n)\}_{n=1}^{\infty}$ along with an arithmetic-geometric interpretation of $Z(-n)$. Manin constructed zeta-polynomials by applying the "Rodriguez-Villegas transform" [RV02] to the odd part of the period polynomial of a newform using the results in [CFI13]; he suggests that these polynomials arise from non-Tate motives and geometric objects lying below Spec $\mathbb{Z}$ but not over $\mathbb{F}_1$.

Manin asked whether there exist zeta-polynomials which can be canonically constructed from the full period polynomial. Ono, Rolen, and Sprung [ORS17] recently used the results in [JMOS16] to address this question, producing a large class of zeta-polynomials canonically constructed from the critical values of classical newforms f. Assuming the Bloch-Kato conjecture, these zeta-polynomials encode further Galois cohomological structure of Selmer groups for Tate-twists that have been assembled as Stirling complexes. Moreover, in analogy with the Maclaurin expansion

$$\frac{t}{e^t - 1} = 1 - \frac{t}{2} + t \sum_{\ell=1}^{\infty} \zeta(-n) \cdot \frac{(-t)^\ell}{\ell!},$$

the zeta-polynomials $Z_f(s)$ constructed in [ORS17] satisfy

$$\frac{(\frac{\sqrt{N}}{i})^{k-1} r_f(\frac{z}{i\sqrt{N}})}{(1-z)^{k-1}} = \sum_{\ell=0}^{\infty} Z_f(-\ell) z^\ell.$$

Using Theorem V.3, we construct zeta-polynomials arising from the special values of odd symmetric power L-functions of semistable elliptic curves over $\mathbb{Q}$.

Using the Bloch-Kato conjecture, one can express the coefficients of these zeta-polynomials in terms of Tamagawa numbers and generalized Shafarevich-Tate groups of the symmetric powers.

Theorem V.4. *Let $E/\mathbb{Q}$ be a non-CM elliptic curve, and let $n \geq 3$ be odd. Suppose that $L(s, \mathrm{Sym}^n E)$ is the L-function of an algebraic, tempered, cuspidal symplectic representation of $\mathrm{GL}_{n+1}(\mathbb{A}_{\mathbb{Q}})$. Let $Z_{\mathrm{Sym}^n E}(s)$ be the polynomial defined by* (V.7.1)*. The following are true.*

1. *For all $s \in \mathbb{C}$, we have that $Z_{\mathrm{Sym}^n E}(s) = \varepsilon(\mathrm{Sym}^n E) Z_{\mathrm{Sym}^n E}(1-s)$, where $\varepsilon(\mathrm{Sym}^n E)$ is the sign of the functional equation for $L(s, \mathrm{Sym}^n E)$.*
2. *If $Z_{\mathrm{Sym}^n E}(\rho) = 0$, then $\Re(\rho) = 1/2$.*
3. *We have the Maclaurin expansion*

$$\frac{p_{\mathrm{Sym}^n E}(z)}{(1-z)^n} = \sum_{\ell=0}^{\infty} Z_{\mathrm{Sym}^n E}(-\ell) z^{\ell}.$$

We review motivic L-functions and their conjectured analytic properties in Section V.2. In Section V.3, we prove some lemmas that are needed for the proofs of Theorem V.1, which we prove in Sections V.4 and V.5. We then discuss symmetric power L-functions and prove Theorems V.3 and V.4 in Sections V.6 and V.7.

Acknowledgements

This project was suggested by Ken Ono; the authors thank him for his comments and support. The authors also thank Seokho Jin, Robert Lemke Oliver, Akshay Venkatesh, and Don Zagier for helpful conversations. S.L. was supported by the Fulbright Commission and the Deutsche Forschungsgemeinschaft (DFG) Grant No. BR 4082/3-1. He thanks the Department of Mathematics and Computer Science at Emory University for their hospitality. W.M. thanks the Chinese Scholarship Council for its generous support and the Department of Mathematics and Computer Science at Emory University for their hospitality. J.T. is supported by a NSF Mathematical Sciences Postdoctoral Research Fellowship.

V.2 Motivic L-functions

We begin by recalling the conjectural properties of motivic L-functions. For more details, see Serre [Ser94] and Iwaniec and Kowalski [IK04, Chapter 5].

V.2.1 Conjectured analytic properties

Define a pure motive $\mathcal{M}$ over $\mathbb{Q}$ of weight w, rank d, and global conductor N by specifying Betti, de Rham, and ℓ-adic realizations (for each prime ℓ)

$$H_B(\mathcal{M}), \quad H_{dR}(\mathcal{M}), \quad H_\ell(\mathcal{M})$$

which are vector spaces of dimension d over $\mathbb{Q}$, $\mathbb{Q}$, and $\mathbb{Q}_\ell$, respectively; each is endowed with additional structures and comparison isomorphisms as in [CPR89, Del79]. In particular, $H_B(\mathcal{M})$ admits an involution ρ_B, $H_\ell(\mathcal{M})$ is a $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-module, and there is a Hodge decomposition into $\mathbb{C}$-vector spaces

$$H_B(\mathcal{M}) \otimes \mathbb{C} = \bigoplus_{\substack{i+j=w \\ i,j \geq 0}} H^{i,j}(\mathcal{M}).$$

The involution ρ_B acts on $H^{i,j}(\mathcal{M})$ by $\rho_B(H^{i,j}(\mathcal{M})) = H^{j,i}(\mathcal{M})$. When w is even, this tells us that $H^{w/2,w/2}(\mathcal{M})$ is invariant under ρ_B; when w is odd, we take $H^{w/2,w/2}(\mathcal{M}) = \{0\}$. If w is even and $H^{w/2,w/2}(\mathcal{M}) \neq \{0\}$, then the involution ρ_B acts on $H^{w/2,w/2}(\mathcal{M})$ by $\alpha \in \{-1, 1\}$; we then define the quantity $b^{\pm}(\mathcal{M})$ by

$$b^{\alpha}(\mathcal{M}) := \dim_{\mathbb{C}}\{x \in H^{w/2,w/2}(\mathcal{M}) : \rho_B(x) = \alpha(-1)^{w/2}x\}, \qquad \alpha \in \{-1, 1\}.$$

We denote by ρ_ℓ the representation which induces the $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-module structure on $H_\ell(\mathcal{M})$.

For any prime p, let $\mathrm{Frob}_p \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ be the Frobenius element at p, which is defined modulo conjugation and modulo the inertia subgroup $I_p \subset G_p \subset \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ of the decomposition group G_p. Define

$$L_{\ell,p}(X, \mathcal{M}) := \det(1 - X \cdot \rho_\ell(\mathrm{Frob}_p^{-1})|_{H_\ell(\mathcal{M})^{I_p}})^{-1} = \prod_{j=1}^{d}(1 - \alpha_{\mathcal{M}}(j, \ell, p)X)^{-1}.$$

One typically assumes (and expects) that $L_{\ell,p}(X, \mathcal{M})$ and $\alpha_{\mathcal{M}}(j, \ell, p)$ are in fact independent of ℓ; as such, we write $L_p(X, \mathcal{M})$ and $\alpha_{\mathcal{M}}(j, p)$ instead of $L_{\ell,p}(X, \mathcal{M})$ and $\alpha_{\mathcal{M}}(j, \ell, p)$ for convenience. (If this is not true, our results are only affected notationally.) The Euler product and Dirichlet series representations of $L(s, \mathcal{M})$ are now given as

$$L(s, \mathcal{M}) := \prod_p L_p(p^{-s}, \mathcal{M}) =: \sum_{n \geq 1} \frac{\lambda_{\mathcal{M}}(n)}{n^s}$$

with $\lambda_{\mathcal{M}}(n) \in \mathbb{C}$. Both the Euler product and the Dirichlet series converge absolutely in the half-plane $\mathrm{Re}(s) > w/2 + 1$.

Define the ν-th Hodge number of $\mathcal{M}$ by $h_\nu := \dim_{\mathbb{C}} H^{\nu, w-\nu}(\mathcal{M})$. Let $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$, and define

$$L_\infty(s, \mathcal{M}) = \Gamma_{\mathbb{R}}(s - w/2)^{b^+(\mathcal{M})}\Gamma_{\mathbb{R}}(s + 1 - w/2)^{b^-(\mathcal{M})} \prod_{0 \le \nu < w/2} \Gamma_{\mathbb{C}}(s - \nu)^{h_\nu}.$$

Because we consider $\mathcal{M}$ over $\mathbb{Q}$, the degree of $L(s, \mathcal{M})$ also equals

$$d = b^+(\mathcal{M}) + b^-(\mathcal{M}) + 2 \sum_{0 \le \nu < w/2} h_\nu. \tag{V.2.1}$$

We now describe the hypotheses for $L(s, \mathcal{M})$ which are crucial to our arguments.

Hypothesis V.5. *Let $\mathcal{M}$ be a self-dual motive of weight $w \ge 1$, rank $d \ge 1$, and global conductor N. Let $L(s, \mathcal{M})$ be the L-function of $\mathcal{M}$. The following are true.*

1. *Self-duality: For all $n \ge 1$, we have that $\lambda_{\mathcal{M}}(n) \in \mathbb{R}$.*
2. *The generalized Ramanujan conjecture (GRC): We have that $|\lambda_{\mathcal{M}}(n)| \le d(n)n^{w/2}$ for every $n \ge 1$, where $d(n)$ is the usual divisor function.*
3. *Analytic continuation: The function $\Lambda(s, \mathcal{M}) := N^{s/2}L_\infty(s, \mathcal{M})L(s, \mathcal{M})$ is entire of order 1.*
4. *Functional equation: There exists $\varepsilon(\mathcal{M}) \in \{-1, 1\}$ such that for every $s \in \mathbb{C}$, we have that $\Lambda(s, \mathcal{M}) = \varepsilon(\mathcal{M})\Lambda(w + 1 - s, \mathcal{M})$. We call $\varepsilon(\mathcal{M})$ the root number of $\mathcal{M}$.*
5. *We have $\Lambda(\frac{w+1}{2}, \mathcal{M}) \ge 0$.*

Property 5 follows from the Generalized Riemann Hypothesis for $L(s, \mathcal{M})$, and it is known unconditionally in many cases. Every other property of Hypothesis V.5 is immediately satisfied when $L(s, \mathcal{M})$ coincides with the L-function $L(s, \pi_{\mathcal{M}})$ of an algebraic, self-dual, tempered, cuspidal automorphic representation $\pi_{\mathcal{M}}$ of $\mathrm{GL}_d(\mathbb{A}_{\mathbb{Q}})$, where d is the rank of $\mathcal{M}$. This is predicted by the Langlands program but is known unconditionally for a small (though highly important and useful) collection of motivic L-functions, such as the L-functions associated to newforms. In what follows, we will always assume that $L(s, \mathcal{M}) = L(s, \pi_{\mathcal{M}})$ for some $\pi_{\mathcal{M}}$ in $\mathcal{A}_d(\mathbb{Q})$, the set of all algebraic, self-dual, tempered, cuspidal automorphic representations of $\mathrm{GL}_d(\mathbb{A}_{\mathbb{Q}})$, where d is the rank of $\mathcal{M}$.

V.2.2 Critical values and Hodge numbers

Following Deligne [Del79], we define an integer n to be *critical* for $\mathcal{M}$ if neither $L_\infty(s,\mathcal{M})$ nor $L_\infty(w+1-s,\mathcal{M})$ has a pole at $s=n$; if n is critical for $\mathcal{M}$, then we call $L(n,\mathcal{M})$ a *critical value* of $L(s,\mathcal{M})$. With this definition, the critical integers are purely dictated by the Hodge numbers. The simplest situation occurs when $b^+(\mathcal{M})$ and $b^-(\mathcal{M})$ both equal zero; then the set of integers n which are critical for $\mathcal{M}$ are precisely those which lie in the interval

$$\Big(\max_{\substack{h_\nu\neq 0\\ 0\leq\nu<w/2}}\nu,\, w-\max_{\substack{h_\nu\neq 0\\ 0\leq\nu<w/2}}\nu\Big]. \tag{V.2.2}$$

(When $\mathcal{M}$ corresponds with a newform f of (modular) weight k, then $w=k-1$, $h_0=1$, and $h_\nu=0$ for all $1\leq\nu<\frac{k-1}{2}$. Thus the critical values of $L(s,f)$ are $L(n,f)$ for integers $1\leq n\leq k-1$.) On the other hand, if at least one of $b^+(\mathcal{M})$ and $b^-(\mathcal{M})$ is nonzero, then the distribution of critical integers is slightly more complicated. Briefly stated, if just one of $b^+(\mathcal{M})$ and $b^-(\mathcal{M})$ are nonzero, then the critical integers of $\mathcal{M}$ will not be consecutive integers; if both $b^+(\mathcal{M})$ and $b^-(\mathcal{M})$ are nonzero, then $L(s,\mathcal{M})$ has no critical values. For simplicity, we only consider motives $\mathcal{M}$ such that w is odd and $h_\nu\geq 1$ for some $0\leq\nu<w/2$. Thus $b^+(\mathcal{M})=b^-(\mathcal{M})=0$, the integers that are critical for $\mathcal{M}$ are symmetric about the critical line for $L(s,\mathcal{M})$, and $d\geq 2$. We will study polynomials that generate the special values $L(1,\mathcal{M})$, $L(2,\mathcal{M}),\ldots,L(w,\mathcal{M})$, which, by our hypotheses, includes all of the critical values.

When w is odd, we see that d must be even (see (V.2.1)). Now, consider now the exterior square representation $\mathrm{Ext}^2(\pi_\mathcal{M})$ and the Euler product

$$L(s,\mathrm{Ext}^2(\pi_\mathcal{M}))=\prod_p L_p(p^{-s},\mathrm{Ext}^2(\pi_\mathcal{M})),$$

where at each prime $p\nmid N$ we have

$$L_p(p^{-s},\mathrm{Ext}^2(\mathcal{M}))=\prod_{1\leq j<k\leq n}(1-\alpha_\mathcal{M}(j,p)\alpha_\mathcal{M}(k,p)p^{-s})^{-1}. \tag{V.2.3}$$

We know that $L(s,\mathrm{Ext}^2(\pi_\mathcal{M}))$ has a meromorphic continuation to $\mathbb{C}$ with no poles outside of the set $\{\frac{w'}{2},\frac{w'}{2}+1\}$, where w' is the weight of $\mathrm{Ext}^2(\pi_\mathcal{M})$ [MS12]. If $L(s,\mathrm{Ext}^2(\pi_\mathcal{M}))$ has a pole at $s=\frac{w'}{2}+1$, then $\pi_\mathcal{M}$ is a cuspidal symplectic representation of $\mathrm{GL}_d(\mathbb{A}_\mathbb{Q})$; let $\mathcal{A}_d^{\mathrm{s}}(\mathbb{Q})$ denote the set of such representations. For any $\pi_\mathcal{M}\in\mathcal{A}_d^{\mathrm{s}}(\mathbb{Q})$, Lapid and Rallis [LR03] proved that $\Lambda(\frac{w+1}{2},\pi_\mathcal{M})\geq 0$. (This vastly generalizes a result of Waldspurger [Wal85] for L-functions of newforms.) Therefore, the hypotheses of Theorem V.1 succinctly describe the most natural class of motivic L-functions for which the methods in [JMOS16] can be used for studying special and critical values.

In Theorem V.1, we require that $2m^{h_m} \geq (1+1/m)^{h_0}$. This is not true of all $\mathcal{M}$. In fact, for *any* integer $m \geq 0$ and *any* collection of nonnegative integers $h_0, \ldots, h_m$, there exists a motive of weight $2m+1$ with Hodge numbers $h_0, \ldots, h_m$; see Arapura [Ara16] and Schreieder [Sch15] for explicit constructions. However, for newforms and their symmetric powers (see Section V.6) as well as many other interesting cases, we have $h_\nu \in \{0,1\}$ for each $1 \leq \nu \leq m$.

V.3 Preliminary Lemmas and Setup

Let $\mathcal{M}$ be a pure motive over $\mathbb{Q}$ of rank $d \geq 2$ with global conductor N, odd weight $w = 2m+1 \geq 3$, root number $\varepsilon = \varepsilon(\mathcal{M})$, and Hodge numbers h_ν for $0 \leq \nu \leq m$. (It will be more notationally convenient for us to use m instead of w.) For convenience, we let $\mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\}$ and $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.

We now define our first analogue of (V.1.4) by letting

$$p_{\mathcal{M}}(z) := \sum_{j=0}^{2m} \Big[\prod_{\nu=0}^{m} \binom{2m-\nu}{m-|m-j|}^{h_\nu} \Big] \Lambda(2m+1-j, \mathcal{M}) z^j. \tag{V.3.1}$$

Using the functional equation of $\Lambda(s, \mathcal{M})$ in Part (3) of Hypothesis V.5, we have that

$$p_{\mathcal{M}}(z) = \varepsilon z^m (P_{\mathcal{M}}(z) + \varepsilon P_{\mathcal{M}}(1/z)), \tag{V.3.2}$$

where

$$P_{\mathcal{M}}(z) := \frac{1}{2} \Big[\prod_{\nu=0}^{m} \binom{2m-\nu}{m}^{h_\nu} \Big] \Lambda(m+1, \mathcal{M}) + \sum_{j=1}^{m} \Big[\prod_{\nu=0}^{m} \binom{2m-\nu}{m-j}^{h_\nu} \Big] \Lambda(m+1+j, \mathcal{M}) z^j.$$

If $z = e^{i\theta} \in \mathbb{S}^1$, then $P_{\mathcal{M}}(z) + \varepsilon P_{\mathcal{M}}(1/z)$ is a trigonometric polynomial in either $\cos(\theta)$ or $\sin(\theta)$ (depending on the sign of ε). Therefore, to prove that the zeros of $p_{\mathcal{M}}(z)$ are equidistributed on $\mathbb{S}^1$, we find the correct number and placement of sign changes of $P_{\mathcal{M}}(z) + \varepsilon P_{\mathcal{M}}(1/z)$ as θ varies along $[0, 2\pi)$.

Since $\Lambda(s, \mathcal{M})$ is an entire function of order one, there exist constants $A = A_{\mathcal{M}}$ and $B = B_{\mathcal{M}}$ such that $\Lambda(s, \mathcal{M})$ has the Hadamard factorization

$$\Lambda(s, \mathcal{M}) = e^{A+Bs} \prod_{\rho} \Big(1 - \frac{s}{\rho}\Big) e^{s/\rho}, \tag{V.3.3}$$

where the product runs over the zeros ρ of $\Lambda(s, \mathcal{M})$. Self-duality and the functional equation of $\Lambda(s, \mathcal{M})$ imply that if ρ is a zero of $\Lambda(s, \mathcal{M})$, then so are $\bar{\rho}$ and $w + 1 - \rho$. Self-duality also implies that $\Lambda(s, \mathcal{M})$ is real-valued on the real line, and in view of the functional equation of $\Lambda(s, \mathcal{M})$, we have that B is real-valued and $B = -\sum_\rho \operatorname{Re}(\rho^{-1}) = -\sum_\rho \operatorname{Re}(\rho)|\rho|^{-2}$. Thus if $s \in \mathbb{R}$, then

$$\Lambda(s, \mathcal{M}) = e^A \Big[\prod_{\rho \in \mathbb{R}} \Big(1 - \frac{s}{\rho}\Big)\Big] \cdot \Big[\prod_{\operatorname{Im}(\rho) > 0} \Big|1 - \frac{s}{\rho}\Big|^2\Big]. \qquad \text{(V.3.4)}$$

Lemma V.6. *The function $\Lambda(s, \mathcal{M})$ is monotonically increasing for $s \geq m + 3/2$; moreover,*

$$0 \leq \Lambda(m+1, \mathcal{M}) \leq \Lambda(m+2, \mathcal{M}) \leq \Lambda(m+3, \mathcal{M}) \leq \Lambda(m+4, \mathcal{M}) \leq \dots$$

If $\varepsilon = -1$, then $\Lambda(m+1, \mathcal{M}) = 0$ and

$$0 \leq \Lambda(m+2, \mathcal{M}) \leq \frac{1}{2}\Lambda(m+3, \mathcal{M}) \leq \frac{1}{3}\Lambda(m+4, \mathcal{M}) \leq \dots$$

Proof. All of the zeros in the product (V.3.4) lie in the vertical strip $|m + 1 - \operatorname{Re}(s)| < 1/2$, and we see that $|1 - s/\rho|$ is increasing for $s \geq m + 3/2$. Thus by (V.3.4), we have that $\Lambda(s, \mathcal{M})$ is increasing for $s \geq m + 3/2$. Moreover, $|1 - \frac{m+1}{\rho}| \leq |1 - \frac{m+2}{\rho}|$, so $\Lambda(m+1, \mathcal{M}) \leq \Lambda(m+2, \mathcal{M})$. When $\varepsilon = -1$, we apply the same reasoning and take into account that $\Lambda(s, \mathcal{M})$ has a zero of odd order at $s = m + 1$. □

Lemma V.7. *For $0 < a < b$, we have*

$$\frac{L(m + 3/2 + a, \mathcal{M})}{L(m + 3/2 + b, \mathcal{M})} \leq \Big(\frac{\zeta(1+a)}{\zeta(1+b)}\Big)^d,$$

where $\zeta(s)$ is the Riemann zeta function.

Proof. The Euler product for $L(s, \mathcal{M})$ gives rise to the function $\Lambda_{\mathcal{M}}(n)$ which is defined by the Dirichlet series identity

$$-\frac{L'}{L}(s, \mathcal{M}) = \sum_{n=1}^{\infty} \frac{\Lambda_{\mathcal{M}}(n)}{n^s}.$$

One sees that $|\Lambda_{\mathcal{M}}(n)| \leq d n^{w/2} \Lambda(n)$ for all $n \geq 1$, where $\Lambda(n)$ is the usual von Mangoldt function; this estimate follows from Part (4) of Hypothesis V.5.

Let $0 < a \leq t \leq b$. By the above discussion,

$$\Big| -\frac{L'}{L}(m+3/2+t,\mathcal{M})\Big| \leq \sum_{n=1}^{\infty}\Big|\frac{\Lambda_{\mathcal{M}}(n)}{n^{1+t+w/2}}\Big| \leq d\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{1+t}} = -d\frac{\zeta'}{\zeta}(1+t).$$

Consequently,

$$\frac{L(m+3/2+a,\mathcal{M})}{L(m+3/2+b,\mathcal{M})} = \exp\Big(\int_a^b -\frac{L'}{L}(m+3/2+t,\mathcal{M})dt\Big)$$
$$\leq \exp\Big(-d\int_a^b \frac{\zeta'}{\zeta}(1+t)dt\Big),$$

which equals the right hand side of the desired inequality. $\square$

We will also use the following lemma due to Pólya [Pol18] and Szegö [Sze36] on the zeros of trigonometric polynomials.

Lemma V.8. *If $0 \leq a_0 \leq a_1 \leq \cdots \leq a_{n-1} < a_n$, then the polynomial $\sum_{j=0}^{n} a_n\cos(n\theta)$ has exactly one zero in each interval $(\frac{2j-1}{2n+1}\pi, \frac{2j+1}{2n+1}\pi)$ for $1 \leq j \leq n$. Also, the polynomial $\sum_{j=1}^{n} a_n\sin(n\theta)$ has a zero at $\theta = 0$ and exactly one zero in each interval $(\frac{2j}{2n+1}\pi, \frac{2(j+1)}{2n+1}\pi)$ for $1 \leq j \leq n-1$.*

V.4 Proof of Theorem V.1 when N is large

Our proof of Theorem V.1 is broken into two cases. First we consider the case when $m = 1$, in which case $P_{\mathcal{M}}(z)$ is linear. Then we consider the case where $m \geq 2$.

V.4.1 Case 1: $m = 1$

We have $P_{\mathcal{M}}(z) = \Lambda(3,\mathcal{M})z + 2^{h_0-1}\Lambda(2,\mathcal{M})$. If $\varepsilon = -1$, then

$$p_{\mathcal{M}}(z) = z^m(P_{\mathcal{M}}(z) + \varepsilon P_{\mathcal{M}}(1/z)) = (z^2-1)\Lambda(3,\mathcal{M}).$$

Since -1 and 1 are the roots and they are clearly equidistributed on $\mathbb{S}^1$, Theorem V.1 is proven for all d and all N.

On the other hand, if $\varepsilon = 1$ and $z = e^{i\theta}$ for some $\theta \in [0, 2\pi)$, then

$$z^m(P_{\mathcal{M}}(z) + \varepsilon P_{\mathcal{M}}(1/z)) = 2e^{i\theta}(\cos(\theta)\Lambda(3,\mathcal{M}) + 2^{h_0-1}\Lambda(2,\mathcal{M})). \quad \text{(V.4.1)}$$

Since $\Lambda(2,\mathcal{M}) < \Lambda(3,\mathcal{M})$ by Lemma V.6, (V.4.1) has two roots for $\theta \in [0, 2\pi)$; these are the two values of θ for which $\cos\theta = -2^{h_0-1}\Lambda(2,\mathcal{M})/\Lambda(3,\mathcal{M})$, provided that $h_0 \in \{0,1\}$. This places the roots of $p_{\mathcal{M}}(z)$ on $\mathbb{S}^1$.

We now show that the zeros of (V.4.1) are equidistributed when N is large. By the definition of $\Lambda(s,\mathcal{M})$ and Lemma V.6, we have that $\Lambda(3,\mathcal{M}) \gg N^{3/2}$, whereas

$$\Lambda(2,\mathcal{M}) \leq \sup_{t\in\mathbb{R}} |\Lambda(5/2+\epsilon+it,\mathcal{M})| \ll N^{5/4+\epsilon}$$

for any $\epsilon > 0$. (This uses the Phragmén-Lindelöf convexity bound for $L(s,\mathcal{M})$ in the critical strip is given by [IK04, Equation 5.21].) Therefore, $\Lambda(2,\mathcal{M})/\Lambda(3,\mathcal{M}) \ll N^{-1/4+\epsilon}$, and so the corresponding values of θ tend to $\pi/2$ and $3\pi/2$. Thus if $\varepsilon = 1$, then the zeros of $p_{\mathcal{M}}(z)$ are $\pm i + O(N^{-1/4+\epsilon})$.

V.4.2 Case 2: $m \geq 2$

We will show that if N is sufficiently large and $2m^{h_m} \geq (1+1/m)^{h_0}$, then the zeros of $p_{\mathcal{M}}(z)$ are equidistributed on $\mathbb{S}^1$. This follows as soon as we show that we can apply Lemma V.8 to the real and imaginary parts of $P_{\mathcal{M}}(e^{i\theta})+\varepsilon P_{\mathcal{M}}(e^{-i\theta})$. So that we may apply Lemma V.8, we will verify that

$$\Big[\prod_{\nu=0}^{m}\binom{2m-\nu}{m-j}^{h_\nu}\Big]\Lambda(m+1+j,\mathcal{M}) < \Big[\prod_{\nu=0}^{m}\binom{2m-\nu}{m-(j+1)}^{h_\nu}\Big]\Lambda(m+2+j,\mathcal{M})$$

for all $1 \leq j \leq m-1$ and

$$\frac{1}{2}\Big[\prod_{\nu=0}^{m}\binom{2m-\nu}{m}^{h_\nu}\Big]\Lambda(m+1,\mathcal{M}) \leq \Big[\prod_{\nu=0}^{m}\binom{2m-\nu}{m-1}^{h_\nu}\Big]\Lambda(m+2,\mathcal{M}).$$

By the definitions of $\Lambda(s,\mathcal{M})$ and d, this is equivalent to

$$\frac{1}{(m-j)^{d/2}}L(m+j+1,\mathcal{M}) < \Big(\frac{N}{(2\pi)^d}\Big)^{1/2}L(m+j+2,\mathcal{M}) \qquad \text{(V.4.2)}$$

for each $1 \leq j \leq m-1$ and

$$\frac{1}{2}\Big[\prod_{\nu=0}^{m}\frac{1}{m^{h_\nu}}\Big]\Lambda(m+1,\mathcal{M}) \leq \Big[\prod_{\nu=0}^{m}\frac{1}{(m+1-\nu)^{h_\nu}}\Big]\Lambda(m+2,\mathcal{M}). \qquad \text{(V.4.3)}$$

By Lemma V.7 we have

$$\frac{L(m+j+1,\mathcal{M})}{L(m+j+2,\mathcal{M})} \leq \Big(\frac{\zeta(j+1/2)}{\zeta(j+3/2)}\Big)^d.$$

Therefore, (V.4.2) is satisfied when $N > A_m^d$, where

$$A_m := \max_{1\leq j\leq m-1} \frac{2\pi}{m-j} \cdot \Big(\frac{\zeta(j+1/2)}{\zeta(j+3/2)}\Big)^2. \tag{V.4.4}$$

Since $\Lambda(m{+}1,\mathcal{M}) \leq \Lambda(m{+}2,\mathcal{M})$, (V.4.3) is satisfied when $2m^{h_m} \geq (1{+}1/m)^{h_0}$, as can be seen using term-by-term comparison. This completes the proof.

It is straightforward to compute $A_2 \leq 23.83$, $A_3 \leq 11.92$, $A_m \leq 8$ for $m \geq 4$, and $\lim_{m\to\infty} A_m = 2\pi$. Thus the above proof cannot produce a lower bound for N better than $(2\pi)^d$; we must handle the cases where $N \leq A_m^d$ differently.

V.5 Proof of Theorem V.1 when m is large

On the unit circle, $r_f(z)$ is well-approximated by an exponential function [JMOS16, Section 6], but if $\mathcal{M}$ is arbitrary, then $p_{\mathcal{M}}(z)$ is well-approximated on the unit circle by a certain generalized hypergeometric function. Unfortunately, it is computationally intractable to locate the zeros of the real and imaginary parts of generalized hypergeometric functions, and Rouché's Theorem only gives us the zeros of the real and imaginary part simultaneously. Therefore, we can only prove that "most" zeros (depending on d and N) lie on the unit circle as the weight becomes large.

Let d be fixed. If we define

$$\begin{aligned} Q_{\mathcal{M}}(z) := z^m \sum_{j=0}^{m-1} \frac{1}{(j!)^{\frac{d}{2}}} \frac{(2\pi)^{\frac{dj}{2}}}{(\sqrt{N}z)^j} \frac{L(2m+1-j,\mathcal{M})}{L(2m+1,\mathcal{M})} \\ + \frac{1}{2(m!)^{d/2}} \Big(\frac{(2\pi)^{d/2}}{\sqrt{N}}\Big)^m \frac{L(m+1,\mathcal{M})}{L(2m+1,\mathcal{M})}, \end{aligned} \tag{V.5.1}$$

then we may write $P_{\mathcal{M}}(z)$ as

$$P_{\mathcal{M}}(z) = \Big[\prod_{\nu=0}^{m} ((2m-\nu)!)^{h_\nu}\Big] \Big(\frac{\sqrt{N}}{(2\pi)^{d/2}}\Big)^{2m+1} L(2m+1,\mathcal{M}) Q_{\mathcal{M}}(z). \tag{V.5.2}$$

Define

$$F_{d,N}(z) := \sum_{j=0}^{\infty} \frac{1}{(j!)^{d/2}} \Big(\frac{(2\pi)^{d/2}}{\sqrt{N}} z\Big)^j, \tag{V.5.3}$$

which we approximate by its partial sums $T_{m,d,N}(z) := \sum_{j=0}^{m} \frac{1}{(j!)^{d/2}} (\frac{(2\pi)^{d/2}}{\sqrt{N}} z)^j$.

Now we decompose $Q_{\mathcal{M}}(z)$ into the sum

$$Q_{\mathcal{M}}(z) = z^m T_{m,d,N}(1/z) + S(z) + \frac{1}{2(m!)^{d/2}} \Big(\frac{(2\pi)^{d/2}}{\sqrt{N}}\Big)^m \frac{L(m+1,\mathcal{M})}{L(2m+1,\mathcal{M})} \quad \text{(V.5.4)}$$

with

$$S(z) := z^m \sum_{j=0}^{m-1} \frac{1}{(j!)^{d/2}} \Big(\frac{(2\pi)^{d/2}}{\sqrt{N}z}\Big)^j \Big(\frac{L(2m+1-j,\mathcal{M})}{L(2m+1,\mathcal{M})} - 1\Big).$$

It follows from [EGR14, Theorem 2.2] that $p_{\mathcal{M}}(z)$ has as many zeros on $\mathbb{S}^1$ as $Q_{\mathcal{M}}(z)$ has inside $\mathbb{D}$. Thus Part 3 of Theorem V.1 follows from the following statement.

Theorem V.9. *Let $c_{d,N}$ denote the number of zeros of $F_{d,N}(z)$ inside $\mathbb{D}$. If m is sufficiently large, then $Q_{\mathcal{M}}(z)$ has $m - c_{d,N}$ zeros inside $\mathbb{D}$.*

Proof. We use Rouché's Theorem. First, for $|z| = 1$, we estimate with Lemma V.7

$$\begin{aligned} |S(z)| &\le \sum_{j=0}^{m-1} \frac{1}{(j!)^{d/2}} \Big(\frac{(2\pi)^{d/2}}{\sqrt{N}}\Big)^j \Big(\frac{L(2m+1-j,\mathcal{M})}{L(2m+1,\mathcal{M})} - 1\Big) \\ &\le \sum_{j=0}^{m-1} \frac{1}{(j!)^{d/2}} \Big(\frac{(2\pi)^{d/2}}{\sqrt{N}}\Big)^j \Big(\zeta\Big(m + \frac{1}{2} - j\Big)^d - 1\Big) \end{aligned}$$

The function $x \mapsto 2^x(\zeta(\frac{1}{2} + x)^d - 1)$ is monotonically decreasing for $x \ge 1$, so

$$|S(z)| \le \sum_{j=0}^{m-1} \frac{4}{(j!)^{d/2}} \Big(\frac{(2\pi)^{d/2}}{\sqrt{N}}\Big)^j 2^{j-m}(\zeta(3/2)^d - 1) < 2^{2-m}(\zeta(3/2)^d - 1) F_{d,N}(2). \quad \text{(V.5.5)}$$

Furthermore,

$$\frac{1}{2(m!)^{d/2}} \Big(\frac{(2\pi)^{d/2}}{\sqrt{N}}\Big)^m \frac{L(m+1,\mathcal{M})}{L(2m+1,\mathcal{M})} \ll \frac{1}{(m!)^{d/2}} \Big(\frac{(2\pi)^{d/2}}{\sqrt{N}}\Big)^m. \quad \text{(V.5.6)}$$

If d is fixed, then both (V.5.5) and (V.5.6) can be made arbitrarily small if m is sufficiently large.

We first assume that $F_{d,N}$ has no zeros on $\mathbb{S}^1$. Since $T_{m,d,N}(z)$ converges to $F_{d,N}(z)$ locally uniformly as m tends to infinity, we have

$$\min_{z \in \mathbb{S}^1} |z^m T_{m,d,N}(1/z)| = \min_{z \in \mathbb{S}^1} |T_{m,d,N}(z)| > \frac{1}{2} \min_{z \in \mathbb{S}^1} |F_{d,N}(z)|$$

for m large enough. We conclude for these m, the functions $Q_{\mathcal{M}}(z)$ and $z^m T_{m,d,N}(1/z)$ have the same number of zeros inside $\mathbb{D}$ by Rouché's Theorem. Every zero of $z^m T_{m,d,N}(1/z)$ inside $\mathbb{D}$ is the inverse of a zero of $T_{m,d,N}(z)$ outside $\mathbb{D}$. Again using locally uniform convergence, we see that, if m is sufficiently large, then $F_{d,N}(z)$ and $T_{m,d,N}(z)$ have the same number of zeros inside $\mathbb{D}$, namely $c_{d,N}$. This implies that $z^m T_{m,d,N}(1/z)$, and hence $Q_{\mathcal{M}}(z)$, has $m - c_{d,N}$ zeros inside $\mathbb{D}$.

If $F_{d,N}$ has zeros on $\mathbb{S}^1$, then we choose an $r > 1$, such that all the zeros of $F_{d,N}$ in the region $\{r^{-1} \leq |z| \leq r\}$ lie on $\mathbb{S}^1$ and slightly modify the argument above by applying Rouché's Theorem to the circle $\{|z| = r\}$. $\square$

By taking $d = 2$, we have that $F_{2,N}(z) = \exp(\frac{2\pi}{\sqrt{N}} z)$. Since $F_{2,N}(z)$ has no zeros in $\mathbb{D}$, we have that $c_{d,N} = 0$; thus $p_{\mathcal{M}}(z)$ has all of its zeros on $\mathbb{S}^1$, as shown in [JMOS16]. However, for $d = 4$, the situation already becomes noticeably more complicated; when $d = 4$, we have that $F_{4,N}(z) = I_0(4\pi N^{-1/4}\sqrt{z})$, where I_0 denotes the I-Bessel function. When $d \geq 6$, $F_{d,N}(z)$ is a generalized hypergeometric function. To illustrate the difficulty when $d \geq 4$, we directly compute

$$c_{4,N} = \begin{cases} 4 & \text{if } N = 1, \\ 3 & \text{if } 2 \leq N \leq 4, \\ 2 & \text{if } 5 \leq N \leq 26, \\ 1 & \text{if } 27 \leq N \leq 745, \\ 0 & \text{if } 746 \leq N, \end{cases} \qquad c_{6,N} = \begin{cases} 5 & \text{if } N = 1, \\ 4 & \text{if } 2 \leq N \leq 6, \\ 3 & \text{if } 7 \leq N \leq 37, \\ 2 & \text{if } 38 \leq N \leq 494, \\ 1 & \text{if } 495 \leq N \leq 45606, \\ 0 & \text{if } 45607 \leq N. \end{cases}$$

To see how these compare with those of the previous section, we observe that $746 \approx \frac{1}{2}(2\pi)^4$ and $45607 \approx \frac{3}{4}(2\pi)^6$. Thus it appears that the weight aspect of the results in [JMOS16] do not readily generalize to our setting when d is large.

V.6 Symmetric Power L-functions and the Proof of Theorem V.3

V.6.1 Symmetric power L-functions of non-CM newforms

Let f be a non-CM newform of even weight $k \geq 2$, squarefree level N, and trivial nebentypus. It is well-known that $L(s, f)$ is a motivic L-function satisfying

Hypothesis V.5 with weight $w = k - 1$, rank $d = 2$, and global conductor N. (See [JMOS16] and the sources contained therein).

For each prime ℓ, Deligne proved that there exists a representation $\rho_\ell : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Z}_\ell)$ with the property that if p is a prime not dividing ℓN and Frob_p is the Frobenius automorphism of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ at p, then the characteristic polynomial of $\rho_d(\mathrm{Frob}_p)$ is $x^2 - a_f(p)x + p^{k-1}$. By Deligne's proof of the Weil Conjectures (which establishes Part 2 of Hypothesis V.5), we know that $|a_f(p)| \leq 2p^{(k-1)/2}$. Thus the roots of the characteristic polynomial are $\alpha_p p^{(k-1)/2}$ and $\beta_p p^{(k-1)/2}$, where $\beta_p = \bar{\alpha}_p$ and $\alpha_p\beta_p = 1$. We recast the Euler product of $L(s, f)$ in (V.1.1) as

$$L(s,f) = \Big(\prod_{p|N} \frac{1}{1 - a_f(p)p^{-s}}\Big) \prod_{p\nmid N} \prod_{j=0}^{1} \frac{1}{1 - \alpha_p^j \beta_p^{1-j} p^{(k-1)/2-s}},$$

When N is squarefree, the Euler product of the n-th symmetric power of f, which we denote by $\mathrm{Sym}^n f$, is given by

$$L(s, \mathrm{Sym}^n f) = \Big(\prod_{p|N} \frac{1}{1 - a_f(p)^n p^{-s}}\Big) \prod_{p\nmid N} \prod_{j=0}^{n} \frac{1}{1 - \alpha_p^j \beta_p^{n-j} p^{n(k-1)/2-s}}.$$

(See Cogdell and Michel [CM04, Section 1.1].) This is the L-function attached to the ℓ-adic realizations of $\mathcal{M} = \mathrm{Sym}^n H^1(f)$; note that $L(s, \mathrm{Sym}^0 f) = \zeta(s)$ and $L(s, \mathrm{Sym}^1 f) = L(s, f)$. The symmetric power L-functions of newforms determine the distribution of $a_f(p)/(2p^{(k-1)/2})$ in $[-1, 1]$, but very little is known about their analytic properties (cf. [BLGHT11, Maz08], for example). Their critical values are important in the context of the Bloch-Kato conjecture, much like those of $L(s, f)$. (See [DW09] for an accessible overview along with some convincing computations.) The weight of $\mathrm{Sym}^n f$ is $n(k - 1)$, the rank is $n + 1$, and the global conductor is N^n. (It is for this reason, and this reason alone, that we restrict N to be squarefree.) When $n = 2r + 1$ is odd, the integers which are critical for $\mathrm{Sym}^{2r+1} f$ are $r(k - 1) + j$ for $1 \leq j \leq k - 1$. The Hodge numbers all lie in $\{0, 1\}$; see [CM04] for an exact expression for $L_\infty(s, \mathrm{Sym}^n f)$. From this we can check that the conditions of Theorem V.1 (1) or (2) are satisfied under the assumptions of Corollary V.2.

Conjecturally, we have $\mathrm{Sym}^n f \in \mathcal{A}_{n+1}(\mathbb{Q})$ for each $n \geq 0$, and $\mathrm{Sym}^n f \in \mathcal{A}^{\mathrm{s}}_{n+1}(\mathbb{Q})$ for each odd $n \geq 1$. Unconditionally, we know that $\mathrm{Sym}^n f \in \mathcal{A}_{n+1}(\mathbb{Q})$ for each $n \leq 8$ (see Clozel and Thorne [CT17], [CM04], and the sources contained therein). Moreover, as part of the celebrated proof of the Sato-Tate conjecture [BLGHT11], we know that $L(s, \mathrm{Sym}^n f)$ can be analytically continued to the line $\Re(s) = 1$ for each $n \geq 1$. It follows from the Euler product representation of $L(s, \mathrm{Sym}^n f)$ and (V.2.3) that if $n \geq 1$ is odd, then

$$L(s, \mathrm{Ext}^2(\mathrm{Sym}^n f)) = \zeta(s) \prod_{j=1}^{\frac{n-1}{2}} L(s, \mathrm{Sym}^{4j} f).$$

In particular, if n is odd and $\mathrm{Sym}^n f \in \mathcal{A}_{n+1}(\mathbb{Q})$, then $L(s, \mathrm{Ext}^2(\mathrm{Sym}^n f))$ has a pole at $s = 1$. Thus by Lapid and Rallis [LR03], we have that $\Lambda(\frac{n(k-1)+1}{2}, f) \geq 0$. For $n = 1$ and $n = 3$, these results were proved by Waldspurger [Wal85] and Kim [Kim03], respectively. Regardless of whether N is squarefree, we expect that $L(s, \mathrm{Ext}^2(\mathrm{Sym}^n f))$ has a pole at $s = 1$ for all odd $n \geq 1$, in which case $\mathrm{Sym}^n f \in \mathcal{A}^s_{n+1}(\mathbb{Q})$ and we obtain the desired nonvanishing at the central critical point.

V.6.2 Proof of Theorem V.3

By the modularity theorem, if E is a semistable elliptic curve of squarefree conductor N, then E corresponds to a weight 2 newform of level N, trivial nebentypus, and integral Fourier coefficients. Thus $L(s, \mathrm{Sym}^n E) = L(s, \mathrm{Sym}^n f)$. By Corollary V.2, the only cases left to check are

$$n = 5, \ 11 \leq N \leq 43$$

and

$$n = 7, \ 11 \leq N \leq 15.$$

We observe that in all of these exceptional cases except for $(n, N) \in \{(5, 37), (5, 43)\}$, corresponding to the isogeny classes 37.a and 43.a in Cremona's table, the root number $\varepsilon(\mathrm{Sym}^n f)$ is -1; these are stored on the L-function and Modular Form Database (LMFDB) website at `http://www.lmfdb.org`.

In the cases with $\varepsilon(\mathrm{Sym}^n f) = 1$ (resp. $n = 7$), we explicitly compute the zeros of $P_{\mathrm{Sym}^5 f}$ (resp. $P_{\mathrm{Sym}^7 f}$) and observe that all of them lie in the open unit disc. For this, we use the critical value $L(3, \mathrm{Sym}^5 f)$ and the Dirichlet coefficients of $L(s, \mathrm{Sym}^5 f)$ (resp. $L(s, \mathrm{Sym}^7 f)$), which are stored in the Lcalc files on `http://www.lmfdb.org`.

If $n = 5$ and $\varepsilon(\mathrm{Sym}^5 f) = -1$, we have

$$P_{\mathrm{Sym}^5 f}(z) = \Lambda(5, \mathrm{Sym}^5 f) z^2 + 24\Lambda(4, \mathrm{Sym}^5 f) z,$$

so $P_{\mathrm{Sym}^5 f}$ has all zeros inside the unit disc, if

$$\left|\frac{24\Lambda(4,\mathrm{Sym}^5 f)}{\Lambda(5,\mathrm{Sym}^5 f)}\right| \leq 1.$$

This can again be checked by computing $L(4,\mathrm{Sym}^5 f)$ and $L(5,\mathrm{Sym}^5 f)$ in these cases.

V.7 Proof of Theorem V.4

We first present some corollaries of the results in [RV02]. Let $U(z)$ be a polynomial of degree e with $U(1) \neq 0$. Consider the rational function $V(z) := U(z)(1-z)^{-(e+1)}$. It is easily shown that there exists a polynomial $H(z)$ of degree e such that $H(\ell) = \frac{1}{\ell!}\frac{d^\ell}{dz^\ell}V(z)\big|_{z=0}$ for each integer $\ell \geq 0$. Define $Z(s) := H(-s)$.

Theorem V.10 (Rodriguez-Villegas)**.** *If all of the roots of U lie on $\mathbb{S}^1$, then all of the roots of $Z(s)$ lie on the line $\Re(s) = 1/2$. Moreover, if U has real coefficients and $U(1) \neq 0$, then $Z(s)$ satisfies the functional equation $Z(1-s) = (-1)^e Z(s)$.*

We now show that under the hypotheses of Theorem V.3, $p_{\mathrm{Sym}^n E}(z)$ satisfies the hypotheses of Theorem V.10.

Lemma V.11. *Let $E/\mathbb{Q}$ be a semistable elliptic curve, and suppose that $\mathrm{Sym}^n E$ satisfies the hypotheses of Theorem V.3. If $\varepsilon(\mathrm{Sym}^n E) = 1$, then $p_{\mathrm{Sym}^n E}(1) \neq 0$. If $\varepsilon(\mathrm{Sym}^n E) = -1$, then $p_{\mathrm{Sym}^n E}(z)$ has a simple zero at $z = 1$.*

Proof. Let $n \geq 3$ be odd, let $m = \frac{n-1}{2}$, and let $\varepsilon = \varepsilon(\mathrm{Sym}^n E)$. By (V.3.2) and the fact that $L(s,\mathrm{Sym}^n E)$ is self-dual, we have that $p_{\mathrm{Sym}^n E}(1)$ equals

$$\Big[\prod_{\nu=0}^{m}\binom{2m-\nu}{m}^{h_\nu}\Big]\Lambda(m+1,\mathrm{Sym}^n E) + 2\sum_{j=1}^{m}\Big[\prod_{\nu=0}^{m}\binom{2m-\nu}{m-j}^{h_\nu}\Big]\Lambda(m+1+j,\mathrm{Sym}^n E)$$

if $\varepsilon = 1$ and $p_{\mathrm{Sym}^n E}(1) = 0$ if $\varepsilon = -1$.

When $\varepsilon = 1$, it follows from Lemma V.6 and Hypothesis V.5 (both of which hold whenever $\mathrm{Sym}^n E$ satisfies the hypotheses of Theorem V.3) that the sum defining $p_{\mathrm{Sym}^n E}(1)$ has only nonnegative terms. If $p_{\mathrm{Sym}^n E}(1) = 0$, then it would follow that all Deligne periods of $\mathrm{Sym}^n E$ would equal zero. This implies that the Deligne periods of E are both zero, which is not true. (For the relationship between the periods of E and the periods of $\mathrm{Sym}^n E$, see [DW09], for example.) Thus $p_{\mathrm{Sym}^n E}(1) \neq 0$.

Now, suppose that $\varepsilon = -1$. Note that the sum defining $p'_{\mathrm{Sym}^n E}(z)$ is a sum of nonpositive terms. Much like the case where $\varepsilon = 1$, if all of these terms equal zero simultaneously, then all of the Deligne periods of E are zero, which cannot happen. Thus $p_{\mathrm{Sym}^n E}(z)$ has a simple zero at $z = 1$. □

Define $\mathfrak{s}(m,n)$ by $\prod_{j=0}^{n}(x-j) = \sum_{m=0}^{n} \mathfrak{s}(n,m)x^m$. Let

$$\mathfrak{M}_{\mathrm{Sym}^n E}(j) := \frac{1}{(n-1)!}\sum_{m=0}^{n-1}\Big[\prod_{\nu=0}^{\frac{n-1}{2}}\binom{n-1-\nu}{\frac{n-1}{2}-|\frac{n-1-2m}{2}|}^{h_\nu}\Big]\Lambda(m+1,\mathrm{Sym}^n E)m^j$$

and

$$Z_{\mathrm{Sym}^n E}(s) := \varepsilon\sum_{h=0}^{n-1}(-s)^h\sum_{j=0}^{n-1-h}\binom{h+j}{h}\mathfrak{s}(n-1,h+j)\mathfrak{M}_{\mathrm{Sym}^n E}(j). \quad \text{(V.7.1)}$$

Proof of Theorem V.4. If $n \geq 1$ is an integer, then we have the Maclaurin expansion

$$(1-z)^{-n} = \sum_{\ell=0}^{\infty}\binom{n-1+\ell}{n-1}z^\ell.$$

Sending j to $n-1-j$ in the sum defining $p_{\mathrm{Sym}^n E}(z)$, using the functional equation for $\Lambda(s,\mathrm{Sym}^n E)$, and sending ℓ to $\ell + j - (n-1)$ yields the identity

$$\frac{p_{\mathrm{Sym}^n E}(z)}{(1-z)^n} = \varepsilon\sum_{\ell=0}^{\infty}z^\ell\Big(\sum_{j=0}^{n-1}\Big[\prod_{\nu=0}^{\frac{n-1}{2}}\binom{n-1-\nu}{\frac{n-1}{2}-|\frac{n-1-2j}{2}|}^{h_\nu}\Big]\Lambda(j{+}1,\mathrm{Sym}^n E)\binom{\ell+j}{n-1}\Big). \quad \text{(V.7.2)}$$

Let h_ℓ be the coefficient of z^ℓ in (V.7.2). With $\mathfrak{s}(n-1,m)$ defined above, we have

$$h_\ell = \frac{\varepsilon}{(n-1)!}\sum_{h=0}^{n-1}\Big[\prod_{\nu=0}^{\frac{n-1}{2}}\binom{n-1-\nu}{\frac{n-1}{2}-|\frac{n-1}{2}-j|}^{h_\nu}\Big] \times \Lambda(j+1,\mathrm{Sym}^n E)\sum_{m=0}^{n-1}\mathfrak{s}(n-1,m)(\ell+j)^m$$

which equals $Z_{\mathrm{Sym}^n E}(-\ell)$ (see [ORS17] for a similar manipulation). This proves Part 3.

Let

$$\hat{p}_{\mathrm{Sym}^n E}(z) = \frac{p_{\mathrm{Sym}^n E}(z)}{(1-z)^{-\delta_{-1,\varepsilon}}},$$

where $\delta_{i,j}$ is the Kronecker delta function. By Theorem V.3 and Lemma V.11, we see that $\hat{p}_{\mathrm{Sym}^n E}(z)$ is a polynomial of degree $n-1-\delta_{-1,\varepsilon}$, all of whose roots lie on $\mathbb{S}^1$. Moreover, $\hat{p}_{\mathrm{Sym}^n E}(1) \neq 0$. Thus

$$\frac{p_{\mathrm{Sym}^n E}(z)}{(1-z)^n} = \frac{\hat{p}_{\mathrm{Sym}^n E}(z)}{(1-z)^{n-\delta_{-1,\varepsilon}}}.$$

Parts 1 and 2 follow from an application of Part 3 and Theorem V.10 with $e = n-1-\delta_{-1,\varepsilon}$. $\square$

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Chapter VI

Summary and Discussion

In this section, we summarize the work presented in this thesis and discuss possible future directions to pursue.

Divisors of modular forms

In Chapter II we constructed generalizations $H^*_{N,z}$ of the functions H_z from (I.1.3) to higher level. For this we used Poincaré series $j_{N,n}$ introduced by Niebur [46] to take over the role of the functions j_n from (I.1.2). The $j_{N,n}$ are harmonic Maass forms of weight 0, but in general not weakly holomorphic anymore. However, the evaluations $j_{N,n}(z)$ at a point $z \in \mathbb{H}$ still constitute the n-th Fourier coefficients at $i\infty$ of the weight 2 polar harmonic Maass forms $H^*_{N,z}$. Together with functions $H^*_{N,\rho}$ for every cusp ρ of $\Gamma_0(N)$, which are shown to be weight 2 Eisenstein series in Theorem II.2, we generalized formula (I.2.1) to higher level. From the growth of Fourier coefficients of polar harmonic Maass forms, Eisenstein series, and cusp forms, we then deduced a procedure to determine the imaginary part of the highest zero or pole of a meromorphic modular form for $\Gamma_0(N)$ in the standard fundamental domain.

It would be interesting for make Corollary II.4 into an algorithm that effectively computes the divisor of any given meromorphic modular form. The idea would be to determine the imaginary part of the highest zero or pole z_0 of f and then find its real part using linear algebra. After that one can subtract the corresponding summand $e_{N,z_0} \operatorname{ord}_{z_0}(f) H^*_{N,z_0}(\tau)$ from (II.1.9) and then repeat the process until only the cuspidal components $H^*_{N,\rho}$ are left in the sum. However, this algorithm is susceptible to rounding errors if there is more than one zero or pole in $\mathbb{H}$ and the convergence of (II.1.10) seems to be slow when the imaginary part of the zero or pole is small. Moreover, if z_0 lies on the boundary of the standard fundamental domain, non-trivial matrices may occur on the right-hand side of (II.3.6), so that the main asymptotic term on the left hand-side of (II.1.9) in Theorem II.3 will exhibit a more delicate growth behavior.

An interesting application would be to derive results on the divisors of theta functions, about which very little is known in general. For example, [17] asks for the vanishing of certain theta series at the point i. Note that the results in Chapter II can also be applied to half-integral weight modular forms, since squaring them does not change the locus of their zeros and poles.

Both sides of (II.1.9) differ by a cusp form depending on f, which would be worthwhile to describe explicitly. In Theorem 1.4 of [16], the coefficients of this cusp form have been rewritten as a regularized inner product of $\frac{\Theta f}{f}$ and $\xi_0(j_{N,n}) = P_{2,N,n}$, the exponential Poincaré series of weight 2, degree n and level N. This series was introduced by Petersson [52] and is defined to be the analytic continuation of

$$P_{2,N,n,s}(\tau) := \sum_{M \in \Gamma_\infty \backslash \Gamma_0(N)} |j(M,\tau)|^{-2s} \left(q^n \mid_2 M\right)$$

to $s = 0$. The Fourier expansion of this Poincaré series converges absolutely and locally uniformly and can also be obtained by applying the ξ_0-operator to the Fourier expansion of $j_{N,n}$ given in Theorem II.9. Theorem II.3, and also Lemma 3.1 of [16], then follows by applying Stokes' Theorem to these regularized inner products.

Niebur-Poincaré series and traces of singular moduli

In Chapter III we showed that twisted analogues of the functions $f_{k,\Delta}$ from (I.1.6) of weight 2 and level N are given by twisted traces of singular moduli in the z-variable of the weight 2 polar harmonic Maass forms $H^*_{N,z}$ from Chapter II. This allowed us to deduce known formulas for traces of singular moduli and show algebraicity results of the Fourier coefficients that correspond to the higher weight case.

Furthermore, we applied the approach of Bringmann, Kane, and von Pippich to compute regularized inner products of the normalized meromorphic modular forms f_d of level 1 and showed that they are given by double traces over CM-values of

$$G_z(\tau) = \log |j(\tau) - j(z)|,$$

which is a Green's function for the modular curve $X_0(1)$. To make the inner products converge, we had to use two regularizations, one by Borcherds [4] to account for the non-vanishing of f_d at $i\infty$ and one by Bringmann, Kane, and von Pippich [11] to account for the poles of f_d.

Evaluations of Green's functions at Heegner points have been related to heights of Heegner points on modular curves by Gross and Zagier [28]. Since the f_d are modular forms of weight 2, it would be enlightening to find a geometric interpretation of their inner products and see how they fit into this picture. Furthermore, in light of the results of Duke, Imamoḡlu, and Tóth [22] on cycle integrals of the j-invariant, it might be worthwhile to also consider cycle integrals of the Niebur Poincaré series $j_{N,n}$, which can be seen as traces for positive discriminants.

Theorem III.5 implies that not only the Fourier coefficients of f_d, but also their inner products $\langle f_d, f_\delta \rangle$ have interesting algebraicity properties. For example, if no two CM-points of discriminants d and δ coincide, meaning that none of discriminants is a square multiple of the other, then the inner products are essentially logarithms of algebraic numbers, namely norms of differences of singular moduli. This is case (i) of Theorem III.5. Norms of singular moduli were studied by Gross and Zagier [29], who showed that their prime factorization contains only small primes and gave an explicit formula.

On the other hand, if some CM-points coincide and none of them are equivalent to i or $\frac{-1+\sqrt{3}i}{2}$, we obtain contributions from the weight 2 modular form j' evaluated at CM-points, which are algebraic numbers up to the square of the Chowla-Selberg period of discriminant D (see [59], Section 6.3 for details). In case that at least one of the discriminants have i or $\frac{-1+\sqrt{3}i}{2}$ as CM-points, we obtain higher derivatives of j evaluated at CM-points and their algebraicity properties can be determined by applying suitable differential operators.

Although we restricted ourselves to the Hecke congruence subgroups $\Gamma_0(N)$, one can define analogues of the $H^*_{N,z}$ for other Fuchsian groups Γ in the same way. This gives a method to explicitly construct weight 2 meromorphic modular forms with prescribed simple poles on other modular curves. Especially in the case when Γ has genus 0, the weight 2 polar harmonic Maass forms $H^*_{\Gamma,z}$ can still be written in terms of the logarithmic derivative of a Hauptmodul for Γ. In the proof of Theorem III.5 we needed the description of the functions $H^*_{1,z}$ as logarithmic derivatives in order to construct an explicit preimage G_z under the ξ_0-operator. This approach should generalize to other modular curves of genus 0, but it would be interesting to compute the regularized inner products also in case of positive genus.

Linear congruences for modular forms

In Chapter IV we gave an explicit condition on an arithmetic progression that prevents a the Fourier coefficients of a generalized eta-quotient from satisfying any linear congruence. Namely, the residue class $t \pmod m$ of the progression $(mn+t)_{n\in\mathbb{N}}$ has to be given by a certain quadratic expression with coefficients depending on the parameters of the eta-quotient. We gave many examples that follow from Theorem IV.1 in case of the partition function, classical eta-quotients, and Schur and Rogers-Ramanujan partitions.

Radu's and our proof rely on the *q-expansion principle* by Deligne and Rapoport, stated in [21], Section 5.VII, Corollaire 3.12 and 3.13. The version of the q-expansion principle that we used is Corollary 5.3 of [54]. It states that if the Fourier coefficients of a modular form for $\Gamma(N)$ at $i\infty$ are algebraic integers, then so are the Fourier coefficients at cusps associated to $\Gamma_0(N)$.

The primes 2 and 3 play a special role in linear congruences for the partition function. These are the residues treated in the papers of Radu [54] and Ahlgren and Kim [1] and also the prime divisors of the denominator 24 in the rational exponent of η^{-1}. It would be interesting to see whether this role can be taken over by larger primes if we study generalized eta-quotients. These primes should be the ones that divide the denominator $P_2\left(\frac{g}{\delta}\right)$ of the eta-quotient $\eta_{g,\delta}$, for example the primes 2, 3, and 5 in case of the Rogers-Ramanujan functions.

One might also be able to extend Radu's method and establish linear incongruences for other kinds of (mock) modular forms. On the other hand, results like Theorem IV.1 could be used to systematically find linear congruences by narrowing down possible residues or to prove further theorems on linear congruences for a generalized eta-quotient of interest.

Motivic L-functions and period polynomials

In Chapter V we showed that, for every motivic L-function of odd motivic weight that satisfies the automorphicity assumptions in Hypothesis V.5, the zeros of the polynomial $p_{\mathcal{M}}$ defined in (V.3.1) all lie on the unit circle if the level is large enough and all but a fixed number lie on the unit circle if the weight is large enough. Furthermore, they tend to be equidistributed as at least one of these parameters tends to infinity.

The results in [33] have recently been employed by Ono, Rolen, and Sprung [49] to study zeta-polynomials of Manin [43] for Hecke eigenforms, where also a connection to Ehrhart polynomials has been drawn. Our results imply that such a theory of zeta-polynomials also exists for odd symmetric power L-functions of elliptic curves (see Theorem V.4). Namely, their zeta-polynomials satisfy a functional equation and have all of their zeros on the "critical strip" $\mathrm{Re}(s) = \frac{1}{2}$.

An obvious task would be to show an similar statement for motives of even motivic weight. However, in this case already in the decomposition (V.3.2) of the polynomial $p_{\mathcal{M}}$, the polynomials $P_{\mathcal{M}}$ are not trigonometric anymore. Moreover, one has to consider real Gamma-factors, which do not occur for odd motivic weight and render the shape of $p_{\mathcal{M}}$ considerably more difficult. In that sense, the assumption of odd motivic weight is crucial to our work.

Since the critical values modular L-functions satisfy interesting transcendence properties, it would be enlightening to determine to what extent the polynomials $p_{\mathcal{M}}$ generalize period polynomials of Hecke eigenforms. Manin [44] showed that for any Hecke eigenform f of weight k and level 1 with integral Fourier coefficients, there are two real numbers $\omega_{\pm}$, such that the even periods $\Lambda(1, f), \Lambda(3, f), \ldots, \Lambda(k-1, f)$ are rational multiples of ω_+ and the odd periods $\Lambda(2, f), \Lambda(4, f), \ldots, \Lambda(k-2, f)$ are rational multiples of ω_- (the shift in the argument of Λ comes from the expression in (I.1.8)). Kohnen and Zagier [36] showed that the even periods of the cusp forms $f_{k,\Delta}$ for $\Delta > 0$ introduced in (I.1.6) are rational (using a different normalization).

If f is a weight 2 modular form, then by the Modularity Theorem f corresponds to an elliptic curve and the geometric interpretation of the periods is clear. For higher weights. Kontsevich and Zagier [38] wrote the critical L-values of $L(\Delta, s)$ for the weight 12 cusp form Δ and $s = 1, \ldots, 12$ as integrals of rational functions over rational domains. Deligne [20] conjectured that critical values of symmetric power L-functions can also be given in terms of periods.

Conjecture (Deligne)**.** *Let f be a Hecke eigenform of weight k and level* 1 *with integral Fourier coefficients. If j is a critical value for* $\mathrm{Sym}^{2m+1} f$*, then* $L\left(\mathrm{Sym}^{2m+1} f, j\right)$ *is a non-zero rational multiple of*

$$(2\pi i)^{1-k+j(m+1)} \omega_{\pm}^{\frac{(m+1)(m+2)}{2}} \omega_{\mp}^{\frac{m(m+1)}{2}},$$

where $\pm = (-1)^m$.

The more complicated original form of the conjecture also involves higher levels and even symmetric powers. In light of Deligne's conjecture, Don Zagier suggested of find an interpretation of the coefficients of $p_{\mathrm{Sym}^n f}$ as period integrals in the sense of [38].

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Declaration

I hereby declare that the article *On Divisors of Modular Forms* [10] was jointly written with Prof. Dr. Kathrin Bringmann, Dr. Ben Kane, Prof. Dr. Ken Ono, and Dr. Larry Rolen and my share of work amounted to 20%. The article *Special Values of Motivic L-functions and zeta-polynomials for symmetric powers of elliptic curves* [42] was jointly written with Dr. Wenjun Ma and Dr. Jesse Thorner and my share of work amounted to 33%. The articles *Niebur-Poincaré Series and Traces of Singular Moduli* [41] and *Linear Incongruences for Generalized Eta-Quotients* [40] are entirely my own work.

Köln, 19. Oktober 2017

Steffen Löbrich

Erklärung

Ich versichere, dass ich die von mir vorgelegte Dissertation selbstständig angefertigt, die benutzten Quellen und Hilfsmittel vollständig angegeben und die Stellen der Arbeit – einschließlich Tabellen, Karten und Abbildungen –, die anderen Werken entnommen sind, in jedem Einzelfall als Entlehnung kenntlich gemacht habe; dass diese Dissertation noch keiner anderen Fakultät oder Universität zur Prüfung vorgelegen hat; dass sie – abgesehen von unten angegebenen Teilpublikationen – noch nicht veröffentlicht worden ist, sowie, dass ich eine solche Veröffentlichung vor Abschluss des Promotionsverfahrens nicht vornehmen werde. Die Bestimmungen der Promotionsordnung sind mir bekannt. Die von mir vorgelegte Dissertation ist von Prof. Dr. Kathrin Bringmann betreut worden.

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3. *On Divisors of Modular Forms* mit Kathrin Bringmann, Ben Kane, Ken Ono und Larry Rolen, zur Publikation angenommen bei Advances in Mathematics.

4. *Linear Incongurences for Generalized Eta-Quotients*, Research in Number Theory **3** (2017), Art. 18, 8.

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6. *Radial Limits of the Universal Mock Theta Function* g_3 mit Min–Joo Jang, Proceedings of the American Mathematical Society **145** (2017), no. 3, 925—935.

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Vorträge

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