Toric degenerations: a bridge between representation theory, tropical geometry and cluster algebras

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Kurzzusammenfassung: In dieser Arbeit untersuchen wir torische Degenerierungen projektiver Varietäten. Wir interessieren uns für Konstruktionen solcher aus der Darstellungstheorie, der tropischen Geometrie und der Theorie von Cluster Algebren. Ziel ist es, durch analysieren bestimmter Spezialfälle die Zusammenhänge der verschiedenen Theorien besser zu verstehen.

Im Fokus sind deshalb Varietäten, auf die eine Vielzahl von Methoden angewandt werden können: Grassmannsche, Fahnenvarietäten und Schubertvarietäten.

Wir vergleichen als ersten Schritt die torischen Varietäten, die als Degenrierungen erhalten werden. Vor allem interessiert uns ob isomorphe torische Varietäten von verschiedenen Konstruktionen erhalten werden. Dies ist häufig der Fall, z.B. für die Grassmannsche von Geraden im $\mathbb{C}^n$ können alle torischen Varietäten, die man mit Methoden der tropischen Geometrie erhält (bis auf Isomorphie) auch mit Hilfe der Darstellungstheorie konstruiert werden.

Ein erstes allgemeines Resultat (für projektive Varietäten) lässt auf weitere tiefere Zusammenhänge hoffen: torische Degenerierungen, die mit Hilfe einer Bewertung und der Theorie von Newton-Okounkov Körpern erzeugt werden lassen sich (unter gewissen Bedingungen) mit Hilfe der tropischen Geometrie realisieren.

Abstract: In this thesis we study toric degenerations of projective varieties. We compare different constructions to understand how and why they are related. In focus are toric degenerations obtained from representation theory, tropical geometry or cluster algebras. Often those rely on valuations and the theory of Newton-Okounkov bodies. Toric degenerations can be seen as a combinatorial shadow of the original objects. The goal is therefore to understand why the different theories are so closely related, by understanding the toric degenerations they yield first. We choose Grassmannians, flag varieties and Schubert varieties as starting point as here many different constructions are applicable. One of our main results shows how toric degenerations obtained using full-rank valuations, independent of how these are constructed, can (under certain conditions) be realized using tropical geometry.

Contents

Chapter 1

Introduction

Toric varieties are popular objects in algebraic geometry due to a dictionary between their geometric properties (e.g. dimension, degree) and properties of associated combinatorial objects (e.g. fans, polytopes), which exists in "nice" cases. This dictionary can be extended from toric varieties to varieties admitting a *toric degeneration*. A toric degeneration is a (flat) family of varieties that share many properties with each other. We mostly consider 1-parameter toric degenerations of certain projective varieties X. These are flat families $\varphi : \mathcal{F} \to \mathbb{A}^1$, where the fiber over zero (also called *special* fiber) is a toric variety and all other fibers are isomorphic to X. Once we have such a degeneration, some of the algebraic invariants of X are the same for all fibers (e.g. dimension, degree, Hilbert-polynomial), hence the computation can be done on the toric fiber. In the case of a toric variety such invariants are easier to compute than in the case of a general variety due to a nice combinatorial description.

Example 1. Consider the Grassmannian $\mathrm{Gr}(2,\mathbb{C}^4)$ of 2-dimensional subspaces of $\mathbb{C}^4$. It is given by the vanishing of the ideal $I = \langle p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}\rangle$ in the polynomial ring on Plücker variables. A toric degeneration of $\mathrm{Gr}(2,\mathbb{C}^4)$ is given by the family $I_t = \langle p_{12}p_{34} - p_{13}p_{24} + tp_{14}p_{23}\rangle$ for $t \in \mathbb{C}$. Setting $t = 1$ we obtain $\mathrm{Gr}(2,\mathbb{C}^4)$, and setting $t = 0$ we get the toric variety defined by the vanishing of $I_0 = \langle p_{12}p_{34} - p_{13}p_{24}\rangle$.

The study of toric degenerations has various applications in pure and applied mathematics, for example in mirror symmetry and statistics. Tailored to the variety of interest, it is a great challenge to decide which toric degeneration has the desired properties. The task is therefore to study and compare all possible constructions. In this context, varieties from representation theory can be thought of as a fertile ground to develop different techniques and test for their fitness. Three main fields intersect here: *representation theory*, *tropical geometry* and the *theory of cluster algebras*. All three can be applied to these varieties and yield toric degenerations with the associated combinatorial data encoding geometric properties.

One can think about the combinatorics appearing in this setting (e.g. cones, polytopes, semi-groups) as a shadow of a deeper connection between the theories. The aim is to understand this connection and develop a global framework into which all three settings can be embedded. In the process of doing so, this thesis is concerned with understanding first special cases to obtain an intuition for the global picture.

1.1 Background and Motivation

We explain in more detail the constructions of toric degenerations that are in focus. The first is the framework of *birational sequences* by Fang, Fourier and Littelmann introduced in [20], see §1.1.1. This work has its origin in representation theory or Lie theory. Second, we consider toric degenerations arising in tropical geometry by *tropicalizing* projective varieties, summarized in §1.1.2. For background on this topic we refer to the textbook by Maclagan and Sturmfels [53]. The third context in which toric degenerations arise that is of great interest to us is the theory of cluster algebras introduced by Fomin and Zelevinsky [26] with constructions of toric degenerations due to Gross, Hacking, Keel and Kontsevich in [37]. We summarize it briefly in §1.1.3.

In all three settings the notion of *Newton-Okounkov body* appears. Let X be a projective variety and $\mathbb{C}[X] =: A$ its homogeneous coordinate ring. For a valuation $\mathfrak{v} : A\setminus\{0\} \to \mathbb{Z}^N$ (see Definition 7), by $S(A,\mathfrak{v})$ we denote the value semi-group (the image of the valuation). The *Newton-Okounkov cone* is the closure of the convex hull of $S(A,\mathfrak{v}) \cup \{0\}$. After intersecting this cone with a particular hyperplane one obtains a convex body, called *Newton-Okounkov body*. These objects have been introduced in a series of papers by different authors ([58], [50], [44], [2]) as a far generalization of Newton polytopes to study the asymptotics of line bundles on X. If $S(A,\mathfrak{v})$ is finitely generated, the Newton-Okounkov body is a polytope and there exists a flat degeneration of X into a toric variety Y. The Newton-Okounkov body in this setting is the polytope associated to the normalization of Y.

1.1.1 Birational sequences

In [20] Fang, Fourier and Littelmann introduce the notion of a *birational sequence*. They work with partial flag varieties and spherical varieties associated to a connected complex reductive algebraic group G. For simplicity we explain the case of flag varieties here. Fixing Borel subgroup $B \subset G$ and a maximal torus $T \subset B$, let R^+ be the set of positive roots for G and N the cardinality of R^+. The main idea is to use the representation theory of G to obtain coordinates on G/B such that $\mathbb{C}(G/B) \cong \mathbb{C}(x_1,\dots,x_N)$. On the right hand side by choosing a monomial order (resp. a total order on $\mathbb{Z}^N$) one can define *lowest-term valuations* in a straight forward way (more details in §3.2.1). This idea is used frequently, we encounter it again when considering valuations constructed using cluster algebra structures.

To every positive root $\beta \in R^+$ there exists a one-parameter root subgroup $U_{-\beta} \subset U^-$, where U^- is the unipotent radical of the opposite Borel subgroup $B^- \subset G$.

Definition 1. A *birational sequence* is a sequence of positive roots $S = (\beta_1,\dots,\beta_N)$ such that the multiplication map $U_{-\beta_1} \times \cdots \times U_{-\beta_N} \to U^-$ is a birational morphism.

In particular, as $U_{-\beta_1}\times\cdots\times U_{-\beta_N} \cong \mathbb{A}^N$, a birational sequence yields $\mathbb{C}(G/B) \cong \mathbb{C}(U^-) \cong \mathbb{C}(x_1,\dots,x_N)$. Fixing a total order $\prec$ on $\mathbb{Z}^N$, they construct a valuation $\mathfrak{v}_S$ on the ring of U-invariant functions on G that restricts to the homogeneous coordinate ring of the flag variety G/B. Further, they study the associated Newton-Okounkov body. Their methods generalize constructions in representation theory (Lie theory) and use ideas from *PBW-filtrations*. For example, consider S_n the Weyl group of SL_n and a reduced expression $\underline{w}_0$ of the longest word $w_0 \in S_n$. Choosing the birational sequence consisting of the simple roots associated to $\underline{w}_0$ and the reverse lexicographic order on $\mathbb{Z}^N$, they recover the toric degeneration of SL_n/B

by Gonciulea and Lakshmibai [34] and the Gelfand-Tsetlin polytope [29]. This degeneration was further studied by Kogan and Miller in [49]. Generalizing to arbitrary flag varieties and arbitrary reduced expressions of w_0, the degenerations by Caldero [13] and Alexeev-Brion [1] are recovered. They give degenerations of flag varieties to toric varieties associated to string polytopes introduced by Littelmann [52] and Berenstein-Zelevinsky [5]. The string polytopes parametrize elements of Lusztig's dual canonical basis. In [43] Kaveh showed how string polytopes are realized as Newton-Okounkov bodies. Another well-studied basis for G-representations (and so for the homogeneous coordinate rings of flag varieties) was introduced in a series of papers by Feigin, Fourier and Littelmann ([23] and [24]), generalizing Feigin's work [22]. This basis is parametrized by the *Feigin-Fourier-Littelmann-Vinberg polytope.* The polytope exists in types A and C, and its lattice points parametrize the above mentioned basis of G-representations. Analogously to the case of string polytopes, FFLV-polytopes are realizable as Newton-Okounkov bodies as shown by Kiritschenko [47]. The FFLV-degeneration can also be recovered in the framework of [20], by choosing the birational sequence to consist of all postive roots in a particular *good ordering* (see [20]).

One starting point for this thesis was to answer the following question:

Question 1. Does the framework of birational sequences extend beyond known toric degenerations in representation theory?

To make this question more precise let us briefly introduce how toric degenrations arise from tropical geometry and cluster algebras.

1.1.2 Tropical Geometry

Tropical geometry is a relatively new field at the intersection of algebraic geometry and polyhedral geometry. We are mostly interested in the *tropicalization* of complex projective varieties, which essentially means studying the algebraic variety over the *tropical semiring* instead of over $\mathbb{C}$. The tropical semiring is $\mathbb{R}\cup\{-\infty\}$ with multiplication in $\mathbb{R}$ being replaced by addition and addition in $\mathbb{R}$ being replaced by taking the minimum.

In this sense, the tropicalization of a projective variety $X = V(I) \subset \mathbb{P}^{n-1}$, denoted $\mathrm{trop}(X) \subset \mathbb{R}^n$, is the support of a rational polyhedral complex of dimension $\dim X$ (see [53, Theorem 3.3.5]). It can be interpreted as a combinatorial shadow of its algebraic counterpart X. This computational approach to tropical geometry is closely related to commuative algebra and Gröbner theory. In fact, $\mathrm{trop}(X)$ is contained in the Gröbner fan associated to $X = V(I)$, i.e. every cone $C \subset \mathrm{trop}(X)$ has an associated (monomial-free) *initial ideal* $\mathrm{in}_C(I)$, a deformation of the ideal I defining X. Using Gröbner theory, every cone $C \subset \mathrm{trop}(X)$ yields a flat degeneration of X into $V(\mathrm{in}_C(I))$. In particular, if $\mathrm{in}_C(I)$ is a binomial prime ideal this yields a toric degeneration of X. In this case C is called a *maximal prime* cone of $\mathrm{trop}(X)$.

The tropicalization of $\mathrm{Gr}(2,\mathbb{C}^n)$ has been studied in [64] by Speyer and Sturmfels. They show that $\mathrm{trop}(\mathrm{Gr}(2,\mathbb{C}^n))$ is parametrized by trivalent trees with n leaves, i.e. to every maximal cone they associate a trivalent tree that encodes the initial ideal corresponding to the cone. Further, they prove that every inital ideal coming from a maximal cone in $\mathrm{trop}(\mathrm{Gr}(2,\mathbb{C}^n))$ is prime, hence yields a toric degeneration of $\mathrm{Gr}(2,\mathbb{C}^n)$. This enables us to formulate a more precise version of Question 1 in this case:

Question 2. Can we find a birational sequence for $\mathrm{Gr}(2,\mathbb{C}^n)$ corresponding to every maximal cone of $\mathrm{trop}(\mathrm{Gr}(2,\mathbb{C}^n))$?

Besides $\mathrm{Gr}(2,\mathbb{C}^n)$ very little is known about the tropicalization of (partial) flag varieties and more generally spherical varieties. The cases of $\mathrm{Gr}(3,\mathbb{C}^6)$ and $\mathrm{Gr}(3,\mathbb{C}^7)$ were computed in [39] (see also [64]) and more conceptually Mohammadi and Shaw study $\mathrm{trop}(\mathrm{Gr}(3,\mathbb{C}^n))$ in [57].

The tropicalization of a variety X is closely related to valuations on its homogeneous coordinate ring $\mathbb{C}[X]$ as studied in [45] by Kaveh and Manon. They associate a full-rank valuation to every maximal prime cone C in $\mathrm{trop}(X)$. They further introduce the notion of *Khovanskii basis*, a set of algebra generators for $\mathbb{C}[X]$, whose images under the valuation generate the value semi-group. Kaveh and Manon show, that the existance of a maximal prime cone in the tropicalization is equivalent to the existance of a finite Khovanksii basis for the associated valuation. This further implies that the corresponding Newton-Okounkov body is a polytope and they show how it can be computed from C.

From a representation theoretic point of view the full flag variety $\mathcal{F}\ell_n$ is generally an object of great interest. Inspired by [45] and keen on applying their methods this lead us to the question:

Question 3. What is the tropicalization of $\mathcal{F}\ell_n$? Can it be computed in small cases, can we find finite Khovanskii bases from it, and if so are the associated Newton-Okounkov bodies related to those from birational sequences?

1.1.3 Cluster algebras

Cluster algebras were introduced in [26] by Fomin and Zelevisnky and quickly grew to become a research area on their own. They are commutative rings endowed with *seeds* (maximal sets of algebraically independent generators) related by *mutation* (local transformations exchanging one seed by another). At their origin they are closely related to the representation theory of finite dimensional algebras, but also many objects related to algebraic groups have a cluster structure. For example, the homogeneous coordinate ring of Grassmannians (see [26], [63]), double Bruhat cells (see [4]), (partial) flag varieties (see [28]) or Richardson varieties (see [51]).

A geometric appraoch to cluster algebras was introduced by Fock and Goncharov in [25]. In this setting they work with *cluster varieties*, schemes glued from algebraic tori (one for every seed) with gluing given by the birational transformations induced by mutation. They come in two flavours, $\mathcal{A}$- and $\mathcal{X}$-cluster varieties, one being the *mirror dual* to the other as developed by Gross, Hacking, Keel and Kontsevich in [37]. Among other things, they define *ϑ-bases* for cluster algebras and toric degenerations of (partial compactifications of) cluster varieties. The $\mathcal{X}$-cluster variety comes endowed with a Laurent polynomial, the *superpotential*, whose tropicalization gives a polyhedral cone and a polytope as a slice of this cone. The superpotential polytope is the polytope associated to the special fibre of the toric degeneration.

A similar approach for Grassmannians can be found in recent work of Rietsch and Williams [62]. They consider the $\mathcal{A}$- and $\mathcal{X}$-cluster varieties contained in the Grassmannian and combine ideas from Newton-Okounkov bodies with cluster duality and mirror symmetry. Using $\mathcal{X}$-cluster coordinates as coordinates for the Grassmannian they construct lowest term valuations on the homogeneous coordinate ring for every seed. On the $\mathcal{A}$-cluster variety they consider a potential function that was defined by Marsh and Rietsch in [55]. Its tropicalization yields a polytope. They show that the Newton-Okounkov body associated with the valuation is the polytope given by the potential. In particular, they obtain explicit inequalities describing the Newton-Okounkov polytope.

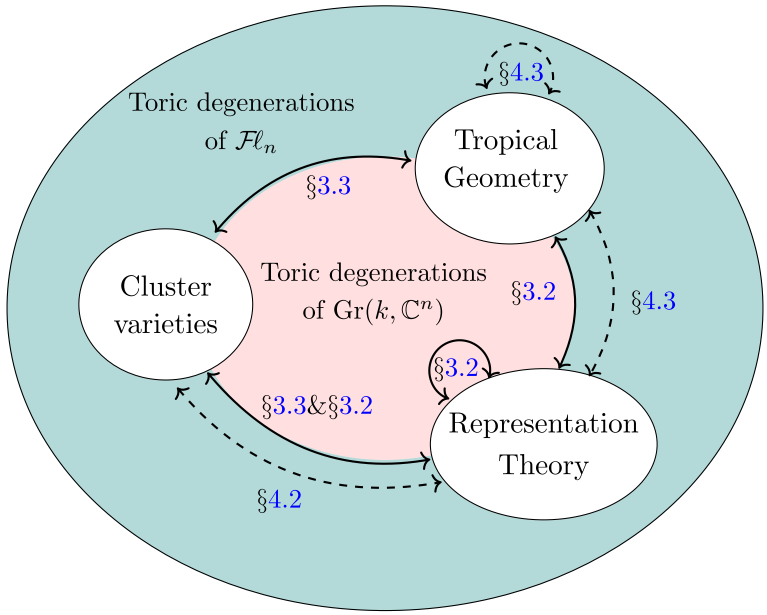

Figure 1.1: The landscape of toric degenerations subject in this thesis.

Question 4. Can the toric degenerations of $\mathcal{F}\ell_n$ (resp. $\mathrm{Gr}(k, \mathbb{C}^n)$) arising from tropicalizing a superpotential be recovered as toric degenerations from the tropicalization of $\mathcal{F}\ell_n$ (resp. $\mathrm{Gr}(k, \mathbb{C}^n)$) or birational sequences?

A first hint towards a positive answer to this question for $\mathcal{F}\ell_n$ was given by Magee in [54]. He recovers the Gelfand-Tsetlin polytope as a superpotential polytope in a particular seed. Further results in this direction are obtained by Genz-Koshevoy-Schumann in [30] and [31], who generalize Magee's result to flag varieties of simple, simply connected, simply laced algebraic groups. They recover the classical string and Lusztig parametrizations from the superpotential.

1.2 Results

We summarize below the results in this thesis and explain which of the above questions could be answered in which generality. In Chapter 2 we recall the necessary general background on the representation theory of $SL_n(\mathbb{C})$, tropical geometry, valuations and cluster algebras. We explain quasi-valuations with weighting matrices and prove a general result relating arbitrary full-rank valuations with such in §2.4. Chapter 3 studies different constructions of toric degenerations of the Grassmannians. More specifically, in §3.2 the class of *iterated* birational sequences is defined. As an application one obtains for $\mathrm{Gr}(2, \mathbb{C}^n)$ a precise relation between birational sequences and the tropical Grassmannian. In §3.3 the connection between the tropical Grassmannian and the cluster combinatorics given by plabic graphs is studied for $\mathrm{Gr}(2, \mathbb{C}^n)$ and $\mathrm{Gr}(3, \mathbb{C}^6)$. The last part of the thesis is Chapter 4, which focusses on flag and Schubert varieties in type A. In §4.2 we show how string polytopes arise from the [37]-superpotential for flag and Schubert varieties in type A. Then in §4.3 we compute the tropical flag varieties $\mathrm{trop}(\mathcal{F}\ell_4)$ and $\mathrm{trop}(\mathcal{F}\ell_5)$ together with the Newton-Okounkov bodies obtained from the Kaveh-Manon construction. The resulting toric degenerations are compared with those from string polytopes and the FFLV polytope.

1.2.1 Valuations and their weighting matrices

In §2.4 we study (quasi-)valuations with weighting matrices (see Definition 13) as introduced in [45]. Our leading example is $A = \mathbb{C}[X]$ the homogeneous coordinate ring of a projective variety. An embedding of $X \hookrightarrow \mathbb{P}^{n-1}$ yields a presentation $A = \mathbb{C}[x_1, \dots, x_n]/I$. Given a full-rank valuation $\mathfrak{v} : A \setminus \{0\} \to \mathbb{Z}^d$ we define the weighting matrix $M_{\mathfrak{v}} \in \mathbb{Z}^{n\times d}$ of $\mathfrak{v}$ (see Definiton 14). Given some additional technical assumptions (that are fulfilled when dealing with the homogeneous coordinate ring of a projective variety) we obtain the following key-theorem:

Theorem 1. Under the assumptions described above, if $\mathrm{in}_{M_{\mathfrak{v}}}(I)$ (see Definiton 12) is prime, then the value semigroup $S(A, \mathfrak{v})$ is generated by the images $\mathfrak{v}(\bar{x}_i)$ for $\bar{x}_i \in A$. Moreover, the Newton-Okounkov body is the convex hull of these images and the $\bar{x}_i$ form a Khovanskii basis.

A precise formulation is Theorem 10 in §2.4. It is in fact a very powerful tool: many of the following results are applications or direct consequences of Theorem 1. The key idea is to use higher-dimensional Gröbner theory and methods of Kaveh-Manon in [45] for arbitrary full-rank valuations. In particular, this links any toric degeneration induced by a valuation (independent from how the valuation is obtained, e.g. using birational sequences or cluster algebras) to those from tropical geometry.

1.2.2 Toric degenerations of Grassmannians

Consider the Grassmannian $\mathrm{Gr}(k, \mathbb{C}^n)$ embedded in the projective space $\mathbb{P}(\bigwedge^k \mathbb{C}^n)$ via the Plücker embedding. We define in Definition 28 a new class of birational sequences for Grassmannians called *iterated sequences*. More specifically, for $\mathrm{Gr}(2, \mathbb{C}^n)$, in Algorithm 3 we reveal their close connection to labelled trivalent trees parametrizing maximal prime cones of $\mathrm{trop}(\mathrm{Gr}(2, \mathbb{C}^n))$. Let S be an iterated sequence for $\mathrm{Gr}(2, \mathbb{C}^n)$ and T_S the trivalent tree that is the output of the algorithm. We consider a valuation $\mathfrak{v}_S$ (see Definition 29) associated with S and the weighting matrix it defines. In a key-proposition (Proposition 2) we show that the initial ideal of the Plücker ideal $I_{2,n}$ with respect to the weighting matrix coincides with the initial ideal with respect to the cone defined by T_S. As the latter is prime (see [64]) this enables us to apply Theorem 1 to obtain:

Theorem 2. (i) For every iterated sequence S for $\mathrm{Gr}(2, \mathbb{C}^n)$ the value semigroup of the associated valuation is generated by the images of Plücker coordinates. That is, the Plücker coordinates form a Khovanskii basis.

(ii) For every iterated sequence S for $\mathrm{Gr}(2, \mathbb{C}^n)$ there exists a maximal prime cone C_S in $\mathrm{trop}(\mathrm{Gr}(2, \mathbb{C}^n))$ such that the associated toric degenerations of $\mathrm{Gr}(2, \mathbb{C}^n)$ are isomorphic.

(iii) For every maximal prime cone C in $\mathrm{trop}(\mathrm{Gr}(2, \mathbb{C}^n))$ there exists an iterated sequence S_C, such that the induced toric degenerations of $\mathrm{Gr}(2, \mathbb{C}^n)$ are isomorphic.

A more precise formulation that implies all of the above results can be found in Theorem 11. Note that this gives an answer to Question 2 for $\mathrm{Gr}(2, \mathbb{C}^n)$.

Having in mind Question 4 we would like to combine techniques from the tropical Grassmannian with the cluster algebra structure on $\mathbb{C}[\mathrm{Gr}(k,\mathbb{C}^n)]$. Similar ideas are discussed in [65], where they show how the two settings are related combinatorially.

In §3.3 we apply Theorem 1 to the valuations defined by Rietsch-Williams [62] for every seed of the cluster algebra $\mathbb{C}[\mathrm{Gr}(k,\mathbb{C}^n)]$ mentioned above. More precisely, we focus on seeds that are encoded by *plabic graphs* (introduced by Postnikov [60]). Here the $\mathcal{A}$-cluster variables consist of only Plücker coordinates. Let $\mathfrak{v}_\mathcal{G}$ be the valuation associated with a plabic graph $\mathcal{G}$ and $M_\mathcal{G}$ the weighting matrix of $\mathfrak{v}_\mathcal{G}$. By $I_{k,n}$ we denote the Plücker ideal describing the Grassmannians with respect to the Plücker embedding (see §3.1). Summarizing Proposition 4, Theorem 12 and Corollary 6 we obtain:

Theorem 3. If the initial ideal $\mathrm{in}_{M_\mathcal{G}}(I_{k,n})$ is prime, then the toric degeneration obtained from the valuation $\mathfrak{v}_\mathcal{G}$ can be realized as a degeneration from the tropicalization of $\mathrm{Gr}(k,\mathbb{C}^n)$. In this case, the associated Newton-Okounkov body is the convex hull of the valuation images of Plücker coordinates and the Plücker coordinates form a Khovanskii basis.

Moreover, if the Newton-Okounkov body of $\mathfrak{v}_\mathcal{G}$ is not integral (see [62, §8]), then the initial ideal $\mathrm{in}_{M_\mathcal{G}}(I_{k,n})$ is not prime.

We analyze $\mathrm{Gr}(3,\mathbb{C}^6)$ computationally and study in more detail $\mathrm{Gr}(2,\mathbb{C}^n)$ in §3.3.3[1]. In the latter case, we define a weight vector $\mathbf{w}_\mathcal{G} \in \mathbb{R}^{\binom{n}{2}}$ for every plabic graph $\mathcal{G}$ and show that it lies in the relative interior of a maximal prime cone of $\mathrm{trop}(\mathrm{Gr}(2,\mathbb{C}^n))$. We use the bijection of labelled triangulations of a disk with n marked points with plabic graphs for $\mathrm{Gr}(2,\mathbb{C}^n)$ (see [48, Algorithm 12.1]). Let T be the labelled trivalent tree that is the dual graph to the triangulation, which is mapped to $\mathcal{G}$ under the bijection. Consider a weight vector $\mathbf{w}_T$ in the relative interior of the maximal cone $C \subset \mathrm{trop}(\mathrm{Gr}(2,\mathbb{C}^n))$ with associated tree T. The following theorem shows how the combinatorial bijections in this case have in fact a deeper meaning and lead us to an answer of Question 4 for $\mathrm{Gr}(2,\mathbb{C}^n)$.

Theorem 4. Let $\mathcal{G}$ be a plabic graph for $\mathrm{Gr}(2,\mathbb{C}^n)$ and T the corresponding labelled trivalent tree. Then the associated initial ideals $\mathrm{in}_{\mathbf{w}_\mathcal{G}}(I_{2,n})$ and $\mathrm{in}_{\mathbf{w}_T}(I_{2,n})$ are equal.

In combination with Theorem 3 this proves the expectation of Kaveh-Manon (see [45, page 6]) that the construction of Rietsch-Williams (based on the cluster structure) agrees essentially with theirs (based on the tropicalization). Further, in combination with Theorem 11 we now have a complete picture for toric degenerations of $\mathrm{Gr}(2,\mathbb{C}^n)$: (up to isomorphism) the constructions using birational sequences, the tropical Grassmannian, and the cluster structure yield the same toric varieties as flat degenerations of $\mathrm{Gr}(2,\mathbb{C}^n)$.

1.2.3 Toric degenerations of flag and Schubert varieties

In §4.3 and §4.2 we consider the variety $\mathcal{F}\ell_n$ of full flags $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = \mathbb{C}^n$ of vector subspaces of $\mathbb{C}^n$ with $\dim_\mathbb{C}(V_i) = i$. In view of Question 4 we want to understand if Newton-Okounkov bodies from birational sequences in [20] are related to the cluster structure on flag varieties in §4.2[2]. As a starting point we decided to study the special case of string polytopes. In [32] Gleizer and Postnikov use *pseudoline arrangements* (see §4.2 Defintion 42) associated to reduced expressions of $w_0 \in S_n$ and *rigorous paths* in these to paramatrize the

[1]Based on joint work with Xin Fang, Ghislain Fourier, Milena Hering and Martina Lanini in [8].
[2]Based on joint work with Ghislain Fourier.

inequalities for string cones $C_{\underline{w}_0} \subset \mathbb{R}^N$. We extend their result by adding weight inequalities encoded combinatorially in the pseudoline arrangement and obtain the *weighted string cones* $\mathcal{C}_{\underline{w}_0} \subset \mathbb{R}^{N+n-1}$ as defined in [52]. Intersecting $\mathcal{C}_{\underline{w}_0}$ with the preimage of a weight $\lambda \in \mathbb{R}^{n-1}$ of an appropriate projection $\pi : \mathbb{R}^{N+n-1} \to \mathbb{R}^{n-1}$ yields the string polytope $\pi^{-1}(\lambda) \cap \mathcal{C}_{\underline{w}_0}$. Generalizing to arbitrary $w \in S_n$ and following Caldero [14] we obtain similarly the string cone, weighted string cone and string polytope for the Schubert variety $X(w) \subset \mathcal{F}\ell_n$.

We introduce a second polyhedral cone $S_{\underline{w}_0} \subset \mathbb{R}^N$ associated to a pseudoline arrangement in a dual way: the variables are associated to the faces of the diagram as opposed to the vertices in case of the string cone. From additional weight inequalities and a second projection $\tau : \mathbb{R}^{N+n-1} \to \mathbb{R}^{n-1}$ we get a weighted cone $\mathcal{S}_{\underline{w}_0}$ and polytopes $\tau^{-1}(\lambda) \cap \mathcal{S}_{\underline{w}_0}$ for $\lambda \in \mathbb{R}^{n-1}$. As in the case of string cones, we obtain these also for arbitrary $w \in S_n$. For simplicity we denote for now the corresponding projection also by π and τ. The first combinatorial result of our study is the following (see Theorem 17).

Theorem 5. For every $\underline{w} \in S_n$, the two cones $\mathcal{C}_{\underline{w}}$ and $\mathcal{S}_{\underline{w}}$ are unimodularly equivalent and the lattice-preserving linear map is given by the duality of faces and vertices in the pseudoline arrangement. Moreover, this linear map restricts to linear bijections between the polytopes $\pi^{-1}(\lambda) \cap \mathcal{C}_{\underline{w}} \cong \tau^{-1}(\lambda) \cap \mathcal{S}_{\underline{w}}$ and the cones $S_{\underline{w}} \cong C_{\underline{w}}$.

The cone $\mathcal{S}_{\underline{w}_0}$ appears in the framework of mirror symmetry for cluster varieties [37]. Recall that $\mathcal{F}\ell_n = SL_n/B$ for the Borel subgroup of upper triangular matrices B. Denote by $B^- \subset SL_n$ the Borel subgroup of lower triangular matrices and by $U \subset B$ (resp. $U^- \subset B^-$) the unipotent radical with all diagonal entries being 1. The double Bruhat cell $G^{e,w_0} = B^- \cap Bw_0B$ is an $\mathcal{A}$-cluster variety (see [4]) and can be identified with an open subset of Bw_0B/U. By [52] the weighted string cone parametrizes a basis of $\mathbb{C}[Bw_0B/U]$. Let $\mathcal{X}$ be the mirror dual of the $\mathcal{A}$-cluster variety G^{e,w_0} and let $s_0 = s_{\underline{\hat{w}}_0}$ be the seed of the cluster algebra $\mathbb{C}[G^{e,w_0}]$ corresponding to the reduced expression $\underline{\hat{w}}_0 = s_1\, s_2 s_1 \cdots s_{n-1} \cdots s_2 s_1$. Let W be the superpotential defined by the sum of the ϑ-functions for every frozen variable in s_0 as introduced in [37]. Then W^{trop} denotes the tropicalization of the superpotential. Magee has shown in [54] (see also Goncharov-Shen in [33]) that

$$\mathcal{S}_{\underline{\hat{w}}_0} = \{x \in \mathbb{R}^{N+n-1} \mid W^{\text{trop}}|_{\mathcal{X}_{s_0}}(x) \geq 0\} =: \Xi_{s_0}.$$

We show that the mutation of the pseudoline arrangement and hence of the cone $\mathcal{S}_{\underline{w}_0}$, is compatible with the mutation of the superpotential [36] by introducing mutation of the rigorous paths defining the cone. We obtain the following (see also [30], where Genz-Koshevoy-Schumann obtain a similar result in the context of crystal graphs):

Theorem 6. Let $\underline{w}_0$ be an arbitrary reduced expression of $w_0 \in S_n$ and $s_{\underline{w}_0}$ be the seed corresponding to the pseudoline arrangement, $\mathcal{X}_{s_{\underline{w}_0}}$ the toric chart of the seed $s_{\underline{w}_0}$. Then

$$\mathcal{S}_{\underline{w}_0} = \{x \in \mathbb{R}^{N+n-1} \mid W^{\text{trop}}|_{\mathcal{X}_{s_{\underline{w}_0}}}(x) \geq 0\} =: \Xi_{s_{\underline{w}_0}},$$

the polyhedral cone defined by the tropicalization of W expressed in the seed $s_{\underline{w}_0}$.

Consider $w \in S_n$ arbitrary and $\underline{w}$ a reduced expression of w. Let W be as above and consider its restriction $\text{res}_{\underline{w}}(W|_{\mathcal{X}_{s_{\underline{w}_0}}})$ to the mirror dual of the $\mathcal{A}$-cluster variety $G^{e,w}$. Let $s_{\underline{w}}$ be the corresponding seed in the cluster algebra (see Definition 43). Then the tropicalization of the restriction yields again a cone $\Xi_{s_{\underline{w}}}$. The last result of this section establishes an answer to Question 4 for Schubert varieties.

Theorem 7. Let $\underline{w} \in S_n$, and fix $\underline{w}_0 = \underline{w} s_{i_{\ell(w)+1}} \dots s_{i_N}$ a reduced expression of $w_0 \in S_n$. Let $s_{\underline{w}}$ resp. $s_{\underline{w}_0}$ be the corresponding seeds. Then $\mathcal{S}_{\underline{w}}$ is the cone $\Xi_{s_{\underline{w}}}$ defined by the tropicalization of the restricted superpotential $\mathrm{res}_{\underline{w}}(W|_{\mathcal{X}_{s_{\underline{w}_0}}})$.

In view of Question 3 we study in §4.3[3] the tropicalization of the flag variety. Consider therefore the natural embeddeding of $\mathcal{F}\ell_n$ in a product of Grassmannians using the Plücker coordinates. We denote by I_n the defining ideal of $\mathcal{F}\ell_n$ with respect to this embedding. We produce toric degenerations of $\mathcal{F}\ell_n$ as Gröbner degenerations coming from the initial ideals associated to the maximal cones of $\mathrm{trop}(\mathcal{F}\ell_n)$. For the case of maximal cones with non-prime associated initial ideal we suggest a procedure (see §4.3 Procedure 7) of how to recover prime cones from reembedding the variety. We successfully apply it to $\mathcal{F}\ell_4$.

The following is our main results on the tropicalization of the flag varieties $\mathcal{F}\ell_4$ and $\mathcal{F}\ell_5$. More detailed formulations can be found in §4.3 Theorem 22, Theorem 23, and Proposition 20.

Theorem 8. The tropical variety $\mathrm{trop}(\mathcal{F}\ell_4) \subset \mathbb{R}^{14}/\mathbb{R}^3$ is a 6-dimensional fan with 78 maximal cones. From prime cones we obtain four non-isomorphic toric degenerations. After applying Procedure 7 we obtain at least two additional non-isomorphic toric degenerations.

The tropical variety $\mathrm{trop}(\mathcal{F}\ell_5) \subset \mathbb{R}^{30}/\mathbb{R}^4$ is a 10-dimensional fan with 69780 maximal cones. From prime cones we obtain 180 non-isomorphic toric degenerations.

In view of Question 3 and following [45], we further compute the Newton-Okounkov polytopes associated to maximal prime cones. These are the polytopes associated to the normalizations of the toric varieties we obtain. We compare these with certain Newton-Okounkov polytopes arising in the setting of [20], more precisely to string polytopes and the FFLV polytope.

Theorem 9. For $\mathcal{F}\ell_4$ there is at least one new toric degeneration arising from prime cones of $\mathrm{trop}(\mathcal{F}\ell_4)$ in comparison to those obtained from string polytopes and the FFLV polytope.

For $\mathcal{F}\ell_5$ there are at least 168 new toric degenerations arising from prime cones of $\mathrm{trop}(\mathcal{F}\ell_5)$ in comparison to those obtained from string polytopes and the FFLV polytope.

Applying Theorem 1 to valuations for string polytopes (this is a particular case of valuations using birational sequences) we further obtain a surprising connection to the *Minkowski property* of string polytopes (see Defintion 64) in Theorem 24 and Corollary 15.

[3]Based on joint work with Sara Lamboglia, Kalina Mincheva and Fatemeh Mohammdi in [9].

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Chapter 2

General Theory

2.1 Representation Theory of $SL_n(\mathbb{C})$

In this section we recall basic notions of the representation theory of $SL_n(\mathbb{C})$ (or SL_n for short) that we need throughout this thesis.

We fix as Borel subgroup the upper triangular matrices $B \subset SL_n$ and diagonal matrices as maximal torus $T \subset B$. We denote the Borel subgroup of lower triangulat matrices B^- (it is also called the *opposite* Borel subgroup of B). Inside of B (resp. B^-) we have the subgroup of unipotent matrices U (resp. U^-) with all diagonal entries being 1. They are the *unipotent radical* of B (resp. B^-).

Consider the Lie algebra $\mathrm{Lie}(SL_n) = \mathfrak{sl}_n = \{n \times n\text{-matrices with trace zero}\}$. The Lie bracket $[\cdot,\cdot] : \mathfrak{sl}_n \times \mathfrak{sl}_n \to \mathfrak{sl}_n$ is given by the commutator

$$[A, B] := AB - BA.$$

We fix the Cartan decomposition $\mathfrak{sl}_n = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ with $\mathfrak{h}$ diagonal matrices as Cartan (maximal abelian Lie-subalgebra) and $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) upper (resp. lower) triangular matrices in $\mathfrak{sl}_n$. Note that with these choices we have $\mathrm{Lie}(B) = \mathfrak{h} \oplus \mathfrak{n}^+ = \mathfrak{b}$, $\mathrm{Lie}(T) = \mathfrak{h}$ and $\mathfrak{n}^- = \mathrm{Lie}(U^-)$. Let us denote the *root system* of SL_n by $R \subset \mathbb{R}^n$. It is the root system of type $\mathtt{A}_{n-1}$. Denoting the standard basis of $\mathbb{R}^n$ by $\{\epsilon_i\}_{i=1,\dots,n}$ we fix the the simple roots of R to be $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i = 1, \dots, n-1$. They generate the *root lattice*. The positive roots are denoted $R^+ = \{\beta \in R \mid \beta > 0\}$. They are of form $\alpha_{i,j} := \alpha_i + \cdots + \alpha_j$ for $i \leq j < n$. With our choice of simple roots we have $\alpha_{i,j} = \epsilon_i - \epsilon_{j+1}$. The number of positive roots is denoted by $N = \frac{n(n-1)}{2}$.

For a positive root $\beta = \alpha_{i,j}$ let $f_\beta \in \mathfrak{n}^-$ be the root vector of weight $-\beta$. In other words, f_β is the lower triangular $n \times n$-matrix with all entries being zero besides the $(i+1, j)$'th entry, which is 1. Similarly we have $e_\beta \in \mathfrak{n}^+$ a root vector for β of weight β. With our choice of $\mathfrak{b}$ it is the transpose of f_β. We define a third element in $\mathfrak{sl}_n$ associated to $\beta \in R^+$, namely $h_\beta = [e_\beta, f_\beta] \in \mathfrak{h}$.

For the weight lattice we choose the notation Λ with generators the fundamental weights being $\omega_1, \dots, \omega_{n-1}$.

Let Λ^+ denote the dominant integral weights in Λ, i.e. those $\lambda = \sum_{i=1}^{n-1} a_i\omega_i$ with $a_i \in \mathbb{Z}_{\geq 0}$. Dominant integral weights are the lattice points in the *dominant Weyl chamber*, the positive span of the fundamental weights. By Λ^{++} we denote the set of *regular dominant weights*, i.e. those $\lambda = \sum_{i=1}^{n-1} a_i\omega_i$ with $a_i \in \mathbb{Z}_{>0}$.

The roots and weights live in the same space $\mathbb{R}^n$, to which we have associated the basis $\{\epsilon_i\}_i$. We can express ω_i as follows in this basis

$$n\omega_i = \sum_{k=1}^{i}(n-i+1)\epsilon_k - 2i\epsilon_{i+1} - \sum_{k=i+2}^{n} i\epsilon_k.$$

Then $\frac{1}{2}(\alpha_1 + \cdots + \alpha_{n-1}) = (\omega_1 + \cdots + \omega_{n-1}) =: \rho \in \lambda^{++}$ is the smallest regular dominant weight.

For every $\lambda \in \Lambda^+$ there is a (finite-dimensional) irreducible representation of $\mathfrak{sl}_n$ of highest weight λ, denote it by $V(\lambda)$. It is cyclically generated by a highest weight vector $v_\lambda \in V(\lambda)$ (unique up to scaling) over $U(\mathfrak{n}^-)$, the *universal enveloping algebra* of $\mathfrak{n}^-$ defined as follows.

For $\mathfrak{g}$ any Lie algebra, $U(\mathfrak{g})$ is a quotient of the tensor algebra $T(\mathfrak{g}) = \bigoplus_{k\geq 0} \mathfrak{g}^{\otimes k}$. The ideal by which we quotient by is generated by relations induced by the Lie bracket, i.e. relations of form $w \otimes v - v \otimes w - [w, v]$ for $v, w \in \mathfrak{g}$. The *PBW-basis-Theorem* states the following: let $\{v_1, \ldots, v_d\}$ be a ordered basis of $\mathfrak{g}$, then as a vector space $U(\mathfrak{g})$ is generated by monomials of the form

$$v_1^{a_1} v_2^{a_2} \cdots v_{d-1}^{a_{d-1}} v_d^{a_d},\ a_i \in \mathbb{Z}_{\geq 0}.$$

As the irreducible highest weight representation $V(\lambda)$ for $\lambda \in P^+$ are cyclically generated by v_λ over $U(\mathfrak{n}^-)$, we are particularly interested in a PBW-basis for $U(\mathfrak{n}^-)$. This is given, for example, by fixing an order on all positive roots, e.g. $\beta_1, \ldots, \beta_N$. Then for a chosen highest weight vector $v_\lambda \in V(\lambda)$ we have

$$V(\lambda) = U(\mathfrak{n}^-) \cdot v_\lambda = \langle f_{\beta_1}^{m_1} \cdots f_{\beta_N}^{m_N} \cdot v_\lambda \mid m_i \in \mathbb{Z}_{\geq 0} \rangle_{\mathbb{C}}. \tag{2.1.1}$$

Example 2. We have $V(\omega_1) = \mathbb{C}^n$ and $V(\omega_k) = \bigwedge^k \mathbb{C}^n$. The root operators $f_{\alpha_{i,j}} = f_{i,j} \in \mathfrak{n}^-$ act on $\mathbb{C}^n$ with standard basis $\{e_i\}_{i=1,\ldots,n}$ by $f_{i,j} \cdot e_l = \delta_{i,l} e_{j+1}$. The highest weight vector v_{ω_1} can be chosen as e_1. For $V(\omega_2)$ fix the basis $\{e_k \wedge e_l \mid 1 \leq k < l \leq n\}$. Then the action of $\mathfrak{n}^-$ is given by

$$f_{i,j} \cdot (e_k \wedge e_l) = f_{i,j} \cdot e_k \wedge e_l + e_k \wedge f_{i,j} \cdot e_l = \begin{cases} e_{j+1} \wedge e_l, & \text{if } k = i, \\ e_k \wedge e_{j+1}, & \text{if } l = i, \\ 0, & \text{otherwise.} \end{cases}$$

We can chose $e_1 \wedge e_2$ as the highest weight vector v_{ω_2}.

The Weyl group of SL_n is the symmetric group S_n generated by the simple transpositions $s_i = (i, i+1)$ for $1 \leq i < n$. By w_0 we denote the longest element in S_n. For every $w \in S_n$, we denote by $\ell(w)$ the minimal length of w as a word in the generators s_i. Further, $\underline{w}$ denotes a reduced expression

$$\underline{w} = s_{i_1} \cdots s_{i_{\ell(w)}}.$$

Such an expression is not unique. For any two reduced expressions of w there is a sequence of local transformations leading from one to the other. These local transformations are either swapping orthogonal reflections $s_i s_j = s_j s_i$ if $|i - j| > 1$ or exchanging consecutive $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$.

The symmetric group acts on the weight lattice as follows. Consider $\lambda \in \Lambda$ and $s_i \in S_n$. Then $s_i(\lambda) \in \Lambda$ is obtained from λ by reflection on the hyperplane H_{α_i} perpendicular to the simple root $\alpha_i = \epsilon_i - \epsilon_{i+1}$.

Fix $w \in S_n$ and $\lambda \in \Lambda^+$. Then the weight space of weight $w(\lambda)$ in $V(\lambda)$, denoted $V(\lambda)_{w(\lambda)}$, is called *extremal* and it is one-dimensional.

Definition 2. For $w \in S_n$ and $\lambda \in \Lambda^+$ we fix a generator $v_{w\lambda} \in V(\lambda)_{w\lambda}$ we consider $U(\mathfrak{b}) \cdot v_{w\lambda} =: V_w(\lambda)$. This is a $\mathfrak{b}$-module called the *Demazure module*.

Note that though $V_w(\lambda)$ is a $\mathfrak{b}$-submodule of $V(\lambda)$, it is not an $\mathfrak{sl}_n$-module. Let $\underline{w} = s_{i_1} \cdots s_{i_{\ell(w)}}$ be a reduced expression of w. Then for any $\lambda \in \Lambda^+$ similarly to the PBW-basis theorem we have that

$$\{f_{\alpha_{i_1}}^{m_{i_1}} \cdots f_{\alpha_{i_{\ell(w)}}}^{m_{i_{\ell(w)}}} \cdot v_\lambda \in V(\lambda) \mid m_{i_j} \geq 0\} \tag{2.1.2}$$

forms a spanning set of $V_w(\lambda)$ as a vector space. In particular, if $w = w_0$ then $V_{w_0}(\lambda) = V(\lambda)$. For a Demazure module $V_w(\lambda)$ we denote by $V_w(\lambda)^\perp$ its orthogonal complement in $V(\lambda)^*$.

2.2 Tropical Geometry

In this section we recall basic notions of tropical geometry that we assume throughout the rest of the thesis. Tropical geometry comes in many flavours, our approach follows closely the book [53] by Maclagan-Sturmfels and we invite the reader to have a look there for a more detailed introduction. This approach to tropical geometry is closely related to Gröbner theory.

Definition 3. Let $f = \sum a_{\mathbf{u}} x^{\mathbf{u}}$ with $\mathbf{u} \in \mathbb{Z}^n, a_{\mathbf{u}} \in \mathbb{C}$ be a polynomial in $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, where $x^{\mathbf{u}}$ denotes the monomial $x_1^{u_1} \dots x_n^{u_n}$. The *initial form* of f with respect to a fixed weight vector $\mathbf{w} \in \mathbb{R}^n$ is given by

$$\mathrm{in}_{\mathbf{w}}(f) := \sum_{\substack{\mathbf{w}^T \cdot \mathbf{u} \text{ is minimal,} \\ a_{\mathbf{u}} \neq 0}} a_{\mathbf{u}} x^{\mathbf{u}}. \tag{2.2.1}$$

This definition can be extended to ideals. For an ideal $I \subset \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ we have *initial ideal* with respect to $\mathbf{w} \in \mathbb{R}^n$

$$\mathrm{in}_{\mathbf{w}}(I) := \langle \mathrm{in}_{\mathbf{w}}(f) \mid f \in I \rangle. \tag{2.2.2}$$

Example 3. Consider the ideal $I = \langle x_1^2 + x_2, x_1 - x_2 \rangle \subset \mathbb{C}[x_1, x_2]$ and $\mathbf{w} = (1, 0)$. Then $\mathrm{in}_{\mathbf{w}}(x_1^2 + x_2) = x_2$ and $\mathrm{in}_{\mathbf{w}}(x_1 - x_2) = -x_2$. In particular, $\langle \mathrm{in}_{\mathbf{w}}(x_1^2 + x_2), \mathrm{in}_{\mathbf{w}}(x_1 - x_2) \rangle = \langle x_2 \rangle \subset \mathbb{C}[x_1, x_2]$. But we also have $x_1^2 + x_1 = (x_1^2 + x_2) + (x_1 - x_2) \in I$, so by definition $\mathrm{in}_{\mathbf{w}}(x_1^2 + x_1) = x_1 \in \mathrm{in}_{\mathbf{w}}(I)$. We deduce

$$I = \langle f_1, \dots, f_s \rangle \not\Rightarrow \mathrm{in}_{\mathbf{w}}(I) = \langle \mathrm{in}_{\mathbf{w}}(f_1), \dots, \mathrm{in}_{\mathbf{w}}(f_s) \rangle. \tag{2.2.3}$$

By [18, Theorem 15.17], there exists a flat family over $\mathbb{A}^1$ whose fiber over $t \neq 0$ is isomorphic to $V(I)$ and whose fiber over $t = 0$ is isomorphic to $V(\mathrm{in}_{\mathbf{w}}(I))$. For t the coordinate in $\mathbb{A}^1$ it is given by the following family of ideals

$$\tilde{I}_t := \left\langle t^{-\min_{\mathbf{u}}\{\mathbf{w} \cdot \mathbf{u}\}} f(t^{w_1} x_1, \dots, t^{w_n} x_n) \,\middle|\, f = \sum a_{\mathbf{u}} x^{\mathbf{u}} \ \in I \right\rangle \subset \mathbb{C}[t, x_1^{\pm 1}, \dots, x_n^{\pm 1}]. \tag{2.2.4}$$

More precisely, for a projective variety $X = V(I) \subset \mathbb{P}^{n-1}$, where $I \subset \mathbb{C}[x_1, \dots, x_n]$ is a homogeneous ideal, there is a flat degeneration over $\mathbb{A}^1$ with generic fibre (i.e. fibre over $t \neq 0$) isomorphic to $V(I)$ and special fibre (i.e. fibre over $t = 0$) $V(\mathrm{in}_{\mathbf{w}}(I))$. Let I_s denote the ideal $\tilde{I}_t|_{t=s}$. For $s \neq 0$ the isomorphism $V(I_s) \cong V(I_1) = V(I)$ is given by a ring automorphism of $\mathbb{C}[x_1, \dots, x_n]$ sending I_s to I. If $\mathrm{in}_{\mathbf{w}}(I)$ is *toric*, i.e. a binomial prime ideal, then $V(\mathrm{in}_{\mathbf{w}}(I))$ is a toric variety (see e.g. [53, Lemma 2.4.14]).

In order to look for these toric degenerations we study the tropicalization of $V(I)$.

Definition 4. Let $f = \sum a_{\mathbf{u}} x^{\mathbf{u}} \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. The *tropicalization* of f is the function $f^{\mathrm{trop}} : \mathbb{R}^n \to \mathbb{R}$ given by

$$f^{\mathrm{trop}}(\mathbf{w}) := \min\{\mathbf{w} \cdot \mathbf{u} \mid \mathbf{u} \in \mathbb{Z}^n \text{ and } a_{\mathbf{u}} \neq 0\}.$$

If $\mathbf{w} - \mathbf{v} = m \cdot \mathbf{1}$, for some $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^n$ and $m \in \mathbb{R}$, we have that the minimum in $f^{\mathrm{trop}}(\mathbf{w})$ and $f^{\mathrm{trop}}(\mathbf{v})$ is achieved for the same $\mathbf{u} \in \mathbb{Z}^n$ with $a_{\mathbf{u}} \neq 0$.

Definition 5. ([53, Definition 3.1.1 and Definition 3.2.1]) Let $f = \sum a_{\mathbf{u}} x^{\mathbf{u}} \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ and $V(f)$ the associated hypersurface in the algebraic torus $T^n = (\mathbb{C}^*)^n$. Then the *tropical hypersurface* of f is

$$\mathrm{trop}(V(f)) := \left\{ \mathbf{w} \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \text{the minimum in } f^{\mathrm{trop}}(\mathbf{w}) \\ \text{is achieved at least twice} \end{array} \right\}.$$

Let I be an ideal in $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. The *tropicalization* of the variety $V(I) \subset T^n$ is defined as

$$\mathrm{trop}(V(I)) := \bigcap_{f \in I} \mathrm{trop}(V(f)) \subset \mathbb{R}^n.$$

For a projective variety $V(I) \subset \mathbb{P}^{n-1}$ with I a homogeneous ideal in $\mathbb{C}[x_1, \dots, x_n]$ we consider the ideal $\hat{I} := I\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Then $V(\hat{I}) = V(I) \cap T^n$. We consider the tropicalization of projective varieties defined as $\mathrm{trop}(V(I)) := \mathrm{trop}(V(\hat{I}))$.

By the *Fundamental Theorem of Tropical Geometry* [53, Theorem 3.2.3] we have

$$\mathrm{trop}(V(I)) = \{\mathbf{w} \in \mathbb{R}^n \mid \mathrm{in}_{\mathbf{w}}(I) \text{ is monomial-free}\}.$$

Further, the *Structure Theorem* [53, Theorem 3.3.5] tells us that if $X \subset T^n$ is an irreducible d-dimensional variety, then $\mathrm{trop}(X)$ is the support of a pure rational d-dimensional polyhedral complex, connected in codimension 1. We do not recall notions from polyhedral geometry but refer the interested reader to [53, §2.3]. To us, most importantly, the structure theorem implies that we can associate a fan-structure with $\mathrm{trop}(V(I))$. We choose the fan structure in such a way that $\mathrm{trop}(V(I))$ is a subfan of the Gröbner fan of I. If $\mathbf{w}, \mathbf{v}$ lie in the relative interior of a cone C (also denoted C°) in the Gröbner fan, then $\mathrm{in}_{\mathbf{w}}(I) = \mathrm{in}_{\mathbf{v}}(I)$. Adopting this fan structure for $\mathrm{trop}(V(I))$ we therefore use the notation $\mathrm{in}_C(I) := \mathrm{in}_{\mathbf{w}}(I)$ for some $\mathbf{w} \in C^\circ$.

For an ideal $I \subset \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ there may exist some $\mathbf{w} \in \mathbb{R}^n$ with $\mathrm{in}_{\mathbf{w}}(I) = I$. For example, if I is homogeneous this is always the case for $\mathbf{1} := (1, \dots, 1) \in \mathbb{R}^n$. The linear subspace $L_I := \{\mathbf{w} \in \mathbb{R}^n \mid \mathrm{in}_{\mathbf{w}}(I) = I\} \subset \mathrm{trop}(V(I))$ is called the *lineality space* of I.

In §4.3 we tropicalize the *flag variety* (see §4.1). Although the flag variety is a projective variety and hence, by the above we have a recipe to tropicalize it, for computational convenience we choose an embedding into a product of projective spaces (instead of just one projective space). The procedure of tropicalization can also be done in this setting, replacing $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ by S, the *total coordinate ring* (see [16, page 207] for a definition) of $\mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_s}$. Then S has a $\mathbb{Z}^s$-grading given by $\deg : \mathbb{Z}^n \to \mathbb{Z}^s$, where $k_1 + \cdots + k_s = n - 1$. An ideal $I \subset S$ defining an irreducible subvariety $V(I)$ of $\mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_s}$ is homogeneous with respect to this grading. The tropicalization of $V(I)$ is contained in $\mathbb{R}^{k_1+\dots+k_s+s}/H$, where H is an s-dimensional linear space spanned by the rows of a matrix D defining deg. Similarly to the projective case, if $V(I)$ is a d-dimensional irreducible subvariety of $\mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_s}$, then $\operatorname{trop}(V(I))$ is the support of a fan, which is the quotient by H of a rational $(d+s)$-dimensional subfan F of the Gröbner fan of I. Here the Krull dimension of S/I is $d + s$.

In the following we always consider $\operatorname{trop}(V(I))$ with the fan structure defined above.

Remark 1. A detailed definition of the tropicalization of a general toric variety X_Σ and of its subvarieties can be found in [53, Chapter 6]. Note that we only consider the tropicalization of the intersection of $V(I)$ with the torus of X_Σ while in [53, Chapter 6] they introduce a generalized version of $\operatorname{trop}(V(I))$ which includes the tropicalization of the intersection of $V(I)$ with each orbit of X_Σ.

Another property of $\operatorname{trop}(V(I))$ is that any fan structure on it can be *balanced* assigning a positive integer weight to every maximal cell. We do not explain the notion of balancing in detail and we consider an adapted version of the multiplicity defined in [53, Definition 3.4.3].

Definition 6. Let $I \subset S$ be a homogeneous ideal and Σ a fan with support $|\Sigma| = |\operatorname{trop}(V(I))|$, such that for every cone $C \subset \Sigma$ the ideal $\operatorname{in}_{\mathbf{w}}(I)$ is constant for $\mathbf{w} \in C^\circ$. For a maximal dimensional cone $C \subset \Sigma$ we define the *multiplicity* as

$$\operatorname{mult}(C) := \sum_P \operatorname{mult}(P, \operatorname{in}_C(I)).$$

Here the sum is taken over the minimal associated primes P of $\operatorname{in}_C(I)$ that do not contain monomials (see [18, §3] or [15, §4.7]).

As we have seen, each cone of $\operatorname{trop}(V(I))$ corresponds to an initial ideal which contains no monomials. We now explain why *good candidates* for toric degenerations are the initial ideals corresponding to the relative interior of maximal cones in $\operatorname{trop}(V(I))$. We say a maximal cone is *prime* if the corresponding initial ideal $\operatorname{in}_C(I)$ is a prime ideal.

Lemma 1. Let $I \subset S$ be a homogeneous ideal and C a maximal cone of $\operatorname{trop}(V(I))$. If $\operatorname{in}_C(I)$ is toric then C has multiplicity one. Moreover, if C has multiplicity one then $\operatorname{in}_C(I)$ has a unique toric ideal in its primary decomposition.

Proof. We first prove the lemma for $s = 1$, i.e. S the homogeneous coordinate ring of $\mathbb{P}^{n-1}$. Let $I' = \operatorname{in}_C(I)\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ and consider $V(I') \subset T^n$. Then by [53, Remark 3.4.4] the multiplicity of a maximal cone C is counting the number of d-dimensional torus orbits whose union is $V(I')$. If $\operatorname{in}_C(I)$ is toric, then $V(I')$ is an irreducible toric variety, hence it has a unique d-dimensional torus orbit. So C has multiplicity one.

Suppose now C has multiplicity one. Then $\operatorname{in}_C(I)$ contains one associated prime J, not containing any monomials. The ideal J is further binomial since it is the ideal of the unique d-dimensional torus orbit contained in $V(I')$.

When $s > 1$ and so S is the total coordinate ring of $\mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_s}$, the torus is given by $T^{k_1} \times \cdots \times T^{k_s} \cong T^{k_1+\cdots+k_s}$. We may assume that for each i,

$$T^{k_i} = \{[1 : a_2 : \ldots : a_{k_i}] \in \mathbb{P}^{k_i} \mid a_j \neq 0 \text{ for all } j\}.$$

The variables for $\mathbb{P}^{k_i}$ are denoted by $x_{i,0}, \ldots, x_{i,k_i}$ for each i. We fix the Laurent polynomial ring

$$S' = \mathbb{C}[x_{1,0}^{\pm 1}, \ldots, x_{1,k_1}^{\pm 1}, x_{2,0}^{\pm 1}, \ldots, x_{2,k_2}^{\pm 1}, \ldots, x_{s,0}^{\pm 1}, \ldots, x_{s,k_s}^{\pm 1}].$$

Then consider the ideal $I' = \text{in}_C(I)S' \subset S'$ and the variety $V(I') \subset T^{k_1+\ldots+k_s}$ and the proof proceeds as before. $\square$

Remark 2. From Lemma 1 we conclude the multiplicity being one is a necessary but not sufficient condition for toric initial ideals. A cone can have multiplicity one but its associated initial ideal might be neither prime nor binomial. There may be associated primes that contain monomials in the decomposition of $\text{in}_{\mathbf{w}}(I)$ and these do not contribute to the multiplicity. We list examples of such cones in $\text{trop}(\mathcal{F}\ell_5)$ (for more details see Theorem 23).

Let I be a homogeneous ideal in S such that the Krull dimension of S/I is d. Consider $\text{trop}(V(I)) \subset \mathbb{R}^n/H$ and the d-dimensional subfan $F \subset \mathbb{R}^n$ of the Gröbner fan of I with $F/H \cong \text{trop}(V(I))$. When $V(I) \subset \mathbb{P}^{k_1-1} \times \cdots \times \mathbb{P}^{k_s-1}$ the linear space H is spanned by the rows of the matrix D. In particular, when $V(I) \subset \mathbb{P}^{n-1}$ we have that H is equal to the span of $(1, \ldots, 1)$. We now describe some properties of the toric initial ideals corresponding to maximal cones of $\text{trop}(V(I))$. Let C be a cone in $\text{trop}(V(I))$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_d\}$ be d linearly independent vectors in F generating the maximal cone C', such that $C'/H \cong C$. We can assume that the $\mathbf{w}_i$'s have integer entries since F is a rational fan. We define a matrix associated to C by

$$W_C := [\mathbf{w}_1, \ldots, \mathbf{w}_d]^t. \tag{2.2.5}$$

Consider a sublattice L of $\mathbb{Z}^n$ and the standard basis $e_1, \ldots, e_n$ of $\mathbb{Z}^n$. Given $\ell = (\ell_1, \ldots, \ell_{n+1}) \in L$ we set $\ell^+ = \sum_{\ell_i > 0} \ell_i e_i$ and $\ell^- = -\sum_{\ell_j < 0} \ell_j e_j$. Note that $\ell = \ell^+ - \ell^-$ and $\ell^+, \ell^- \in \mathbb{N}^{n+1}$. We use the same notation as in [16, page 15].

Lemma 2. Let I be a homogeneous ideal in S and C a maximal cone in $\text{trop}(V(I))$. If $\text{in}_C(I)$ is toric, then there exists a sublattice L of $\mathbb{Z}^n$ and constants $c_\ell \in \mathbb{C}^*$ with $\ell \in L$ such that

$$\text{in}_C(I) = I(W_C) := \langle x^{\ell^+} - c_\ell x^{\ell^-} \mid \ell \in L \rangle. \tag{2.2.6}$$

In particular, L is the kernel of the map $f : \mathbb{Z}^n \to \mathbb{Z}^d$ defined by the matrix W_C. If C has multiplicity one and $\text{in}_C(I)$ is not toric, then the unique toric ideal in the primary decomposition of $\text{in}_C(I)$ is of the form $I(W_C)$.

Proof. Let $\text{in}_C(I) \subset S$ be a toric initial ideal and let C' be the corresponding cone in F. The fan structure is defined on $\text{trop}(V(I))$ so that for every $\mathbf{w}', \mathbf{w}$ in the relative interior of C' we have $\text{in}_{\mathbf{w}'}(I) = \text{in}_C(I) = \text{in}_{\mathbf{w}}(I)$. This implies $\text{in}_C(I)$ is W_C-homogeneous with respect to the $\mathbb{Z}^d$-grading on S given by the matrix W_C. By [67, Lemma 10.12] there exists an automorphism ϕ of S sending x_i to $\lambda_i x_i$ for some $\lambda_i \in \mathbb{C}$, such that the ideal $\text{in}_C(I)$ is isomorphic to an ideal

$$I_L := \langle x^{\ell^+} - x^{\ell^-} \mid \ell \in L \rangle.$$

Here L is the sublattice of $\mathbb{Z}^{n+1}$ given by the kernel of the map $f : \mathbb{Z}^{n+1} \to \mathbb{Z}^d$. Applying ϕ^{-1} to $\mathrm{in}_C(I)$ we can write each toric initial ideal as

$$\langle x^{\ell^+} - c_\ell x^{\ell^-} \mid \ell \in L\rangle = I(W_C),$$

for some $c_\ell \in \mathbb{C}^*$, L and W_C as defined above.

Let C be a cone of multiplicity one and suppose $\mathrm{in}_C(I)$ is not prime. Then by Lemma 1 there exists a unique toric ideal J in the primary decomposition of $\mathrm{in}_C(I)$. This toric ideal J contains $\mathrm{in}_C(I)$ and we show that it can be expressed as $I(W_C)$. The variety $V(I)$ is considered as a subvariety of $\mathbb{P}^{n-1}$. As in Lemma 1, the case $V(I) \subset \mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_s}$ can be treated similarly.

The tropical variety depends only on the intersection of $V(I)$ with the torus, and J is equal to $\mathrm{in}_C(I)\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Hence, J is a prime ideal that is homogeneous with respect to W_C so we can proceed as above to show J can be written as $\langle x^{\ell^+} - c_\ell x^{\ell^-} \mid \ell \in L\rangle = I(W_C)$. $\square$

Remark 3. Note that the lattice L and the ideal $I(W_C)$ only depend on the linear space spanned by the rays of the cone C'. Hence they are the same for every set of d independent vectors in C' chosen to define W_C.

2.3 Valuations

Another construction of toric degenerations can be obtained from valuations as we explain in this section. We recall basic notions of the theory of Newton-Okounkov bodies as presented in [44].

We fix a linear order $\prec$ on the additive abelian group $\mathbb{Q}^r$, where $r \leq d$.

Definition 7. A map $\mathfrak{v} : A \setminus \{0\} \to (\mathbb{Q}^r, \prec)$ is a *valuation*, if it satisfies for all $f, g \in A \setminus \{0\}$ and $c \in \mathbb{C}^*$

(i) $\mathfrak{v}(f+g) \succeq \min\{\mathfrak{v}(f), \mathfrak{v}(g)\}$,

(ii) $\mathfrak{v}(fg) = \mathfrak{v}(f) + \mathfrak{v}(g)$ and

(iii) $\mathfrak{v}(cf) = \mathfrak{v}(f)$.

If we replace (ii) by $\mathfrak{v}(fg) \succeq \mathfrak{v}(f) + \mathfrak{v}(g)$ then $\mathfrak{v}$ is called a *quasi-valuation* (also called *loose valuation* in [68]).

It is not hard to show, that in (i) if $\mathfrak{v}(f) \neq \mathfrak{v}(g)$ then $\mathfrak{v}(f+g) = \min_\prec\{\mathfrak{v}(f), \mathfrak{v}(g)\}$.

Example 4. Consider $\mathbb{C}\{\{t\}\}$ the field of *Piusseux series*. Elements are formal power series $c(t) = c_1 t^{a_1} + c_2 t^{a_2} + \dots$, with $c_i \in \mathbb{C}$, $a_i \in \mathbb{Q}$ sharing a common denominator and increasingly ordered $a_1 < a_2 < \dots$. It is the algebraic closure of the field of Laurent series $\mathbb{C}((t))$ (see e.g. [53, Theorem 2.1.5]). Moreover, we have

$$\mathbb{C}\{\{t\}\} = \bigcup_{n \in \mathbb{Z}_{>0}} \mathbb{C}((t^{\frac{1}{n}})).$$

It comes with a natural valuation $\mathrm{val} : \mathbb{C}\{\{t\}\} \setminus \{0\} \to \mathbb{Q}$ sending an element $0 \neq c(t) \in \mathbb{C}\{\{t\}\}$ to the lowest exponent a_1 in the series expansion. As the field of rational functions $\mathbb{C}(t) \subset \mathbb{C}\{\{t\}\}$ is a subfield, we can consider the restriction $\mathrm{val} : \mathbb{C}(t) \setminus \{0\} \to \mathbb{Q}$. For $0 \neq q(t) \in \mathbb{C}(t)$, the valuation $\mathrm{val}(q(t))$ is the order of the zero (resp. pole) $q(t)$ has at $t = 0$.

Let $\mathfrak{v} : A \setminus \{0\} \to (\mathbb{Z}^r, \prec)$ be a (quasi-)valuation, where we replace $\mathbb{Q}^r$ by $\mathbb{Z}^r$ for simplicity. One naturally defines a $\mathbb{Z}^r$-filtration on A by $F_{\mathfrak{v}\succeq a} := \{f \in A \setminus \{0\} | \mathfrak{v}(f) \succeq a\} \cup \{0\}$ (and similarly $F_{\mathfrak{v}\succ a}$). The associated graded algebra is defined as

$$\mathrm{gr}_{\mathfrak{v}}(A) := \bigoplus_{a\in\mathbb{Z}^r} F_{\mathfrak{v}\succeq a}/F_{\mathfrak{v}\succ a}. \tag{2.3.1}$$

For $f \in A \setminus \{0\}$ denote by $\overline{f}$ its image in the quotient $F_{\mathfrak{v}\succeq\mathfrak{v}(f)}/F_{\mathfrak{v}\succ\mathfrak{v}(f)}$, hence $\overline{f} \in \mathrm{gr}_{\mathfrak{v}}(A)$. If the filtered components $F_{\mathfrak{v}\succeq a}/F_{\mathfrak{v}\succ a}$ are at most one-dimensional for all $a \in \mathbb{Z}^r$, we say $\mathfrak{v}$ has *one-dimensional leaves*.

The filtration induced by a valuation allows to define the following property of vector space bases for A. As stated it can be found in [45], but bases with this property are also studied, for example in [20] where they are called *essential bases*. More details on essential bases can be found in §3.2.

Definition 8. A vector space basis $\mathbb{B} \subset A$ is called *adapted* to a valuation $\mathfrak{v} : A \setminus \{0\} \to \mathbb{Z}^r$, if for every $a \in \mathbb{Z}^r$ $F_{\mathfrak{v}\succeq a} \cap \mathbb{B}$ is a vector space basis for $F_{\mathfrak{v}\succeq a}$.

If a valuation $\mathfrak{v}$ has one-dimensional leaves, an adapted basis $\mathbb{B}$ is particularly useful as by [45, Remark 2.19] there is a bijection between $\mathbb{B}$ and the set of values $\mathfrak{v}(\mathbb{B})$ given by $b \mapsto \mathfrak{v}(b)$. We use this fact in the proof of Theorem 10.

The image $\{\mathfrak{v}(f) \mid f \in A \setminus \{0\}\} \subset \mathbb{Z}^r$ forms by the definition an additive semi-group, that is a subsemi-group of $\mathbb{Z}^r$. We denote it by $S(A, \mathfrak{v})$ and refer to it as the *value semi-group*. The *rank* of the valuation is the rank of the sublattice generated by $S(A, \mathfrak{v})$ in $\mathbb{Z}^r$. If $\mathrm{rank}(\mathfrak{v}) = d$, we say $\mathfrak{v}$ is of *full rank*. By [45, Theorem 2.3] the one-dimensional leaves property holds for valuations of full rank. The value semi-group is of great interest because of the following Lemma that can be found, for example, in [11, Remark 4.13].

Lemma. ([11, Remark 4.13]) If $\mathfrak{v}$ has one-dimensional leaves, then $\mathrm{gr}_{\mathfrak{v}}(A)$ is isomorphic to the semi-group algebra $\mathbb{C}[S(A, \mathfrak{v})]$.

The following defintion introduced by Kaveh and Manon in [45] is closely related. It generalizes the notion of SAGBI basis (a Gröbner basis analogue for subalgebras of polynomial algebras).

Definition 9. A set of algebra generators $\mathcal{B} \subset A$ is called a *Khovanskii basis* for $(A, \mathfrak{v})$ if the image of $\mathcal{B}$ in $\mathrm{gr}_{\mathfrak{v}}(A)$ forms a set of algebra generators.

If $\mathcal{B}$ is a Khovanskii basis for $(A, \mathfrak{v})$ then (independent of the one-dimensional leaves property) by [45, Lemma 2.10] the image $\mathfrak{v}(\mathcal{B})$ generates $S(A, \mathfrak{v})$.

Assume for now that $\mathrm{gr}_{\mathfrak{v}}(A)$ is finitely generated and that $\mathfrak{v}$ has one-dimensional leaves. Hence, $\mathrm{gr}_{\mathfrak{v}}(A) \cong \mathbb{C}[S(A, \mathfrak{v})]$ by the above lemma. Further, the value semigroup $S(A, \mathfrak{v})$ is generated by $\{\mathfrak{v}(b_1), \dots, \mathfrak{v}(b_n)\}$, for some $\{b_1, \dots, b_n\}$ forming a Khovanskii basis for $(A, \mathfrak{v})$. In this case $\mathrm{Proj}(\mathrm{gr}_{\mathfrak{v}}(A)) = \mathrm{Proj}(\mathbb{C}[S(A, v)])$ is a toric variety. In fact, $\mathrm{Proj}(\mathrm{gr}_{\mathfrak{v}}(A))$ is a flat degeneratipon of $\mathrm{Proj}(A)$. To describe the corresponding family, we use the following Lemma due to Caldero.

Lemma. ([13, Lemma 3.2]) Let S be a finite subset of $\mathbb{Z}^r$. Then there exists a linear form $e : \mathbb{Z}^r \to \mathbb{Z}_{\geq 0}$ such that for all $\mathbf{m}, \mathbf{n} \in S$

$$\mathbf{m} \prec \mathbf{n} \Rightarrow e(\mathbf{n}) < e(\mathbf{m}).$$

In [13] the lemma is stated with $\mathbb{N}^r$ in place of $\mathbb{Z}^r$. By adding a large multiple of $(1,\dots,1)$ to every element in S we obtain the lemma as stated above. Examples of such a linear forms can be found throughout the thesis, in particular in §4.3.2.

Consider a linear form $e:\mathbb{Z}^r\to\mathbb{Z}$ as in [13, Lemma 3.2] for $S:=\{\mathfrak{v}(b_1),\dots,\mathfrak{v}(b_n))\}$. We construct a $\mathbb{Z}_{>0}$-filtration on A by $\mathcal{F}_{\le m}:=\mathcal{F}^{\mathfrak{v},e}_{\le m}=\{f\in A\setminus\{0\}\mid e(\mathfrak{v}(f))\le m\}\cup\{0\}$ for $m\in\mathbb{Z}_{>0}$. The filtration $\{\mathcal{F}_{\le m}\}_m$ has the property that $\bigoplus_{m\ge 0}\mathcal{F}_{\le m}/\mathcal{F}_{<m}\cong \operatorname{gr}_{\mathfrak{v}}(A)$ and we obtain a family of $\mathbb{C}$-algebras (see e.g. [2, Proposition 5.1]) that can be defined as follows.

Definition 10. The *Rees algebra* associated with the valuation $\mathfrak{v}$ and the filtration $\{\mathcal{F}_{\le m}\}_m$ is the flat $\mathbb{C}[t]$-subalgebra of $A[t]$ defined as

$$R_{\mathfrak{v},e}:=\bigoplus_{m\ge 0}(\mathcal{F}_{\le m})t^m. \tag{2.3.2}$$

It has the properties that $R_{\mathfrak{v},e}/tR_{\mathfrak{v},e}\cong \operatorname{gr}_{\mathfrak{v}}(A)$ and $R_{\mathfrak{v},e}/(1-t)R_{\mathfrak{v},e}\cong A$. In particular, it defines a flat family over $\mathbb{A}^1$ (the coordinate on $\mathbb{A}^1$ given by t) with generic fibre isomorphic to $\operatorname{Proj}(A)=X$ and special fibre the toric variety $\operatorname{Proj}(\operatorname{gr}_{\mathfrak{v}}(A))$.

More details on Rees algebras can be found in [2, §5], [68, §2], and [45, §7].

Introduced by Lazarsfeld-Mustaţă [50] and Kaveh-Khovanskii [44] we recall the definition of Newton-Okounkov body. The way we present it follows closely [44].

Definition 11. Let $\mathfrak{v}:A\setminus\{0\}\to(\mathbb{Z}^r,\prec)$ be a valuation. The *Newton-Okounkov cone* is

$$C(A,\mathfrak{v}):=\overline{\operatorname{conv}(S(A,\mathfrak{v})\cup\{0\})}\subset\mathbb{R}^r. \tag{2.3.3}$$

One defines the corresponding *Newton-Okounkov body* as

$$\Delta(A,\mathfrak{v}):=\overline{\bigcup_{i>0}\{\mathfrak{v}(f)/i\mid 0\ne f\in A_i\}} \tag{2.3.4}$$

We are mostly interested in projective varieties of subvarieties of a product of projective spaces as seen in the last section. Let X be such a variety of dimension d and A its homogeneous coordinate ring. Recall that the total coordinate ring S of $\mathbb{P}^{k_1}\times\cdots\times\mathbb{P}^{k_s}$ is of form $S=\mathbb{C}[x_{1,0},\dots,x_{1,k_1},x_{2,0},\dots,x_{2,k_2},\dots,x_{s,k_s}]$. On coordinates the degree is given by $\deg x_{i,j}:=\varepsilon_i\in\mathbb{Z}^s$ (see e.g. [16, Example 5.2.2]) for all $i\in[s],j\in[k_i]$, where $\{\varepsilon_i\}_{i\in[s]}$ denotes the standard basis on $\mathbb{Z}^s$. For $f=\sum a_{\mathbf{u}}x^{\mathbf{u}}\in S$ we choose the lexicographic order on $\mathbb{Z}^s$ and set $\deg f:=\max_{\text{lex}}\{\deg x^{\mathbf{u}}\mid a_{\mathbf{u}}\ne 0\}$. The $\mathbb{Z}^s_{\ge 0}$-grading on the induces a $\mathbb{Z}^s_{\ge 0}$-grading on the homogeneous coordinate ring A of X, which we denote $A=\bigoplus_{\mathbf{m}\in\mathbb{Z}^s_{\ge 0}}A_{\mathbf{m}}$.

It is sometimes desirable to have a valuation that encodes the grading of A, i.e.

$$\hat{\mathfrak{v}}:A\setminus\{0\}\to(Z^s_{\ge 0}\times\mathbb{Z}^{r-s},\prec)\ \text{ of form }\ \hat{\mathfrak{v}}(f)=(\deg f,\cdot),\ \forall f\in A\setminus\{0\}. \tag{2.3.5}$$

Examples of valuations that have this form can be found in §4.3 where we consider valuations constructed from maximal prime cones in $\operatorname{trop}(\mathcal{F}\ell_n)$ (as in [45]). In this case the Newton-Okounkov cone $C(A,\hat{\mathfrak{v}})$ is contained in $\mathbb{R}^s_{\ge 0}\times\mathbb{R}^{r-s}$. The Newton-Okounkov body $\Delta(A,\hat{\mathfrak{v}})$ can be defined as the intersection of $C(A,\hat{\mathfrak{v}})$ with the hyperplane $\{(1,\dots,1)\}\times\mathbb{R}^{r-s}$. More generally let $P_{\mathfrak{v}}(\lambda):=C(A,\hat{\mathfrak{v}})\cap\{\lambda\}\times\mathbb{R}^{r-s}$ for $\lambda\in\mathbb{R}^s_{\ge 0}$.

Dealing with polytopes throughout the thesis we need the notion of *Minkowski sum*. For two polytopes $A, B \subset \mathbb{R}^r$ it is defined as

$$A + B := \{a + b \mid a \in A, b \in B\}. \tag{2.3.6}$$

For example, if A is $\mathbb{Z}^s_{\geq 0}$-graded for $s > 1$ and $\mathfrak{v}$ is of form $\hat{\mathfrak{v}}$ as in (2.3.5) it is an interesting question if

$$P_{\mathfrak{v}}(\varepsilon_1) + \cdots + P_{\mathfrak{v}}(\varepsilon_s) = \Delta(A, \mathfrak{v}),$$

where ε_i are standard basis vectors in $\mathbb{R}^s$. We investigate this question in the case where A is the homogeneous cooridnate ring of the flag variety in §4.3.2.

The main result and reason for the popularity of Newton-Okounkov bodies is the following Theorem. This version is closest to the one in [44], but the same result in varying generalities was obtained, for example, in [2] and [50].

Theorem. ([44]) Let A be the homogeneous coordinate ring of a projective variety X of dimension d and $\mathfrak{v}$ a valuation with one-dimensional leaves on A. Then $\Delta(A, \mathfrak{v})$ is a convex body. Moreover, if $S(A, \mathfrak{v})$ is finitely generated, then $\Delta(A, \mathfrak{v})$ is a rational polytope whose volume $\mathrm{Vol}(\Delta(A, \mathfrak{v}))$ (up to rescaling by $d!$) equals the degree of X. In this case, the normalization of the (not necessarily normal) toric variety $Y = \mathrm{Proj}(\mathbb{C}[S(A, \mathfrak{v})])$ is the toric variety associated to $\Delta(A, \mathfrak{v})$.

2.4 Quasi-valuations with weighting matrices

We briefly recall some background on higher-dimensional Gröbner theory and quasi-valuations with weighting matrices as in [45, §3.1&4.1]. Then we define for a given valuation an associated quasi-valuation with weighting matrix. This enables us to use the Kaveh-Manon's machinery for more general valuations. For example, we can test whether a given valuation has a finitely genarated value semi-group and if so, compute the associated Newton-Okounkov body. A central result of this thesis is Theorem 10. It is proved in full generality here, but appears in more specialized formulations in the following chapters. Most of our results are in fact implications of this theorem.

The notions of initial form and initial ideal with respect to a weight vector (as seen in §2.2) can be generalized to *weighting matrices* as follows.

Definition 12. Let $f = \sum a_{\mathbf{u}} x^{\mathbf{u}} \in \mathbb{C}[x_1, \dots, x_n]$ with $\mathbf{u} \in \mathbb{Z}^n$, where $x^{\mathbf{u}} = x_1^{u_1} \cdots x_n^{u_n}$. For $M \in \mathbb{Q}^{r \times n}$ and a linear order on $\prec$ on $\mathbb{Z}^r$ we define

$$\mathrm{in}_M(f) := \sum_{M\mathbf{m} = \min_{\prec}\{M\mathbf{u} \mid a_{\mathbf{u}} \neq 0\}} a_{\mathbf{m}} x^{\mathbf{m}}. \tag{2.4.1}$$

Similar to the case of weight vectors we extend this definition to ideals $I \subset \mathbb{C}[x_1, \dots, x_n]$ by

$$\mathrm{in}_M(I) := \langle \mathrm{in}_M(f) \mid f \in I \rangle. \tag{2.4.2}$$

A weighting matrix $M \in \mathbb{Q}^{r \times n}$ lies in the *Gröbner region* $\mathrm{GR}^r(I)$ of an ideal $I \subset \mathbb{C}[x_1, \dots, x_n]$, if there exists a monomial order $<$ on $\mathbb{C}[x_1, \dots, x_n]$ such that

$$\mathrm{in}_<(\mathrm{in}_M(I)) = \mathrm{in}_<(I).$$

By a *positive grading* we mean a $\mathbb{Z}^s_{\geq 0}$-grading for $s \geq 1$ as in the case of the total coordinate ring S of a product of projective spaces (see below Remark 2) or the usual polynomial ring. If an ideal I is (multi-)homogeneous with respect to a positive grading then the lineality space L_I of I contains $(1, \dots, 1) \in \mathbb{R}^n$. Kaveh-Manon show (see [45, Lemma 3.7]) that in this case $\mathbb{Q}^{r\times n}$ is entirely contained in $\mathrm{GR}^r(I)$.

To a given matrix $M \in \mathbb{Q}^{r\times n}$ in [45] they associate a quasi-valuation as follows. As above, fix a group ordering $\prec$ on $\mathbb{Q}^r$.

Definition 13. Let $\tilde{f} = \sum a_{\mathbf{u}} x^{\mathbf{u}} \in \mathbb{C}[x_1, \dots, x_n]$ and define $\tilde{\mathfrak{v}}_M : \mathbb{C}[x_1, \dots, x_n] \setminus \{0\} \to (\mathbb{Q}^r, \prec)$ by

$$\tilde{\mathfrak{v}}_M(\tilde{f}) := \min{}_{\prec}\{M\mathbf{u} \mid a_{\mathbf{u}} \neq 0\}.$$

Let $A = \mathbb{C}[x_1, \dots, x_n]/I$ with I the kernel of $\pi : \mathbb{C}[x_1, \dots, x_n] \to A$. Then by [45, Lemma 4.2] there exists a quasi-valuation $\mathfrak{v}_M : A \setminus \{0\} \to (\mathbb{Q}^r, \prec)$ given for $f \in A$ by

$$\mathfrak{v}_M(f) := \max{}_{\prec}\{\tilde{\mathfrak{v}}_M(\tilde{f}) \mid \tilde{f} \in \mathbb{C}[x_1, \dots, x_n], \pi(\tilde{f}) = f\}.$$

It is called the *quasi-valuation with weighting matrix M*.

From the definition, it is usually hard to explicitly compute the values of a quasi-valuation $\mathfrak{v}_M$. The following proposition makes it more computable, given that M lies in the Gröbner region of I.

Proposition. ([45, Proposition 4.3]) Let $M \in \mathrm{GR}^r(I)$ and $\mathbb{B} \subset A$ be a standard monomial basis for the corresponding monomial order $<$ on $\mathbb{C}[x_1, \dots, x_n]$. Then $\mathbb{B}$ is adapted to $\mathfrak{v}_M$. Moreover, for every element $f \in A$ written as $f = \sum b_\alpha a_\alpha$ with $\pi(x^\alpha) = b_\alpha \in \mathbb{B}$ and $a_\alpha \in \mathbb{C}$ we have

$$\mathfrak{v}_M(f) = \min{}_{\prec}\{M\alpha \mid a_\alpha \neq 0\}.$$

Recall from above how to associate a filtration to a (quasi-)valuation. We denote the associated graded algebra of a quasi-valuation $\mathfrak{v}_M$ by $\mathrm{gr}_M(A)$.

From our point of view, quasi-valuations with weighting matrices are not the primary object of interest. In most cases we are given a valuation $\mathfrak{v} : A \setminus \{0\} \to (\mathbb{Q}^r, \prec)$ whose properties we would like to know. In particular, we are interested in the generators of the value semi-group and if there are only finitely many. The next definition establishes a connection between a given valuations and weighting matrices. It allows us later to apply techniques from Kaveh-Manon for quasi-valuations with weighting matrices to other valuations of our interest.

From now on let A be a finitely generated algebra and domain with presentation a fixed $\pi : \mathbb{C}[x_1, \dots, x_n] \to A$, such that $A = \mathbb{C}[x_1, \dots, x_n]/\ker(\pi)$. Let $I := \ker(\pi)$ and $\pi(x_i) =: b_i$ for $i \in [n]$. The polynomial ring may be replaced by S the total coordinate ring of the product of projective spaces, but for simplicity we just write $\mathbb{C}[x_1, \dots, x_n]$.

Definition 14. Given a valuation $\mathfrak{v} : A \setminus \{0\} \to (\mathbb{Q}^r, \prec)$. We define the *weighting matrix of* $\mathfrak{v}$ by

$$M_{\mathfrak{v}} := (\mathfrak{v}(b_1), \dots, \mathfrak{v}(b_n)) \in \mathbb{Q}^{r\times n}.$$

That is, the columns of $M_{\mathfrak{v}}$ are given by the images $\mathfrak{v}(b_i)$ for $i \in [n]$.

Assume that the ideal I is homogeneous with respect to a positive grading. We need the following key-lemma from [45].

Lemma. ([45, Lemma 4.4]) The associated graded algebra of the quasi-valuation with weighting matrix M satisfies

$$\mathrm{gr}_M(A) \cong \mathbb{C}[x_1, \dots, x_n]/\operatorname{in}_M(I). \tag{2.4.3}$$

By a similar argument as in the proof of [45, Proposition 5.2] we obtain the following corollary with assumptions being as above.

Corollary 1. Let $M \in \mathbb{Q}^{d\times n}$ with d the Krull-dimension of A (i.e. $\mathfrak{v}_M$ has full rank). If $\operatorname{in}_M(I)$ is prime, then $\mathfrak{v}_M$ is a valuation whose value semi-group $S(A, \mathfrak{v}_M)$ is generated by $\{\mathfrak{v}_M(b_i)\}_{i\in[n]}$. In particular, the associated Newton-Okounkov body is given by

$$\Delta(A, \mathfrak{v}_M) = \operatorname{conv}(\mathfrak{v}_M(b_i) \mid i \in [n]).$$

Proof. As $\operatorname{in}_M(I)$ is prime, we have by [45, Lemma 4.4] $\mathrm{gr}_M(A) \cong \mathbb{C}[x_1, \dots, x_n]/\operatorname{in}_M(I)$ is a domain. Assume $\mathfrak{v}_M$ is not a valuation, i.e. there exist $f, g \in A \setminus \{0\}$ with $\mathfrak{v}_M(fg) \succ \mathfrak{v}_M(f) + \mathfrak{v}_M(g)$. If $\mathfrak{v}_M(f) = a$ and $\mathfrak{v}_M(g) = b$, then $\overline{f} \in F_{\succeq a}/F_{\succ a}$ and $\overline{g} \in F_{\succeq b}/F_{\succ b}$, where $F_{\succ a}$ denotes a filtered piece of the filtration $\mathcal{F}_{\mathfrak{v}_M}$ on A. Then $\overline{fg} = \overline{0} \in \mathrm{gr}_M(A)$ as by the grading we have $\overline{fg} \in F_{\succeq a+b}/F_{\succ a+b}$ but $\mathfrak{v}_M(fg) \succ a + b$, a contradiction to being a domain.

In particular, $\mathrm{gr}_M(A)$ is generated by $\overline{b_i} = \overline{\pi(x_i)}$ for $i \in [n]$. As by [45, Theorem 2.3] $\mathfrak{v}_M$ has one-dimensional leaves, then by [45, Proposition 2.4] we have $\mathrm{gr}_M(A) \cong \mathbb{C}[S(A, \mathfrak{v}_M)]$. The rest of the claim follows. □

As mentioned before, we want to use the results on (quasi-)valuations with weighting matrices to analyze arbitrary given valuations. The next lemma makes a first connection between the (quasi-)valuation with weighting matrix $M_\mathfrak{v}$ as in Definition 14 and the valuation $\mathfrak{v}$ defining it.

Denote by $\{\varepsilon_i\}_{i\in[s]}$ the standard basis of $\mathbb{R}^s$. We consider the (partial) order $>$ on $\mathbb{Z}^s$:

$$(m_1, \dots, m_s) > (n_1, \dots, n_s) \;:\Leftrightarrow\; \sum_{i=1}^s m_i > \sum_{i=1}^s n_i.$$

Lemma 3. Let I be a homogeneous ideal with respect to a $\mathbb{Z}^s_{\geq 0}$-grading for $s \geq 1$ generated by elements $f \in I$ with $\deg f > \varepsilon_i$ for all $i \in [s]$. Then $\mathfrak{v}_{M_\mathfrak{v}}(b_i) = \mathfrak{v}(b_i)$.

Proof. Denote by $\{e_i\}_{i\in[n]}$ the standard basis of $\mathbb{R}^n$. Recall that $b_i = \pi(x_i)$ for all $i \in [n]$. Using the assumption that I is homogeneous, we have by definition of $\mathfrak{v}_{M_\mathfrak{v}}$

$$\begin{aligned} \mathfrak{v}_{M_\mathfrak{v}}(b_i) &= \max{}_{\prec}\{\tilde{\mathfrak{v}}_{M_\mathfrak{v}}(x_i + f) \mid f \in I\} \\ &\overset{\deg f > \varepsilon_i \forall i}{=} \max{}_{\prec}\{\min{}_{\prec}\{M_\mathfrak{v} e_i, \tilde{\mathfrak{v}}_M(f)\} \mid f \in I\}. \end{aligned}$$

As $\min_{\prec}\{M_\mathfrak{v} e_i, \tilde{\mathfrak{v}}_M(f)\} \preceq M_\mathfrak{v} e_i = \mathfrak{v}(b_i)$ we deduce $\mathfrak{v}_{M_\mathfrak{v}}(b_i) = \mathfrak{v}(b_i)$. □

The following theorem relating a given valuation $\mathfrak{v}$ with the (quasi-)valuation with weighting matrix $M_\mathfrak{v}$ is our main result on Newton-Okounkov bodies. In the rest of the thesis we use it to prove our main results. We apply it to valuations from birational sequences ([20]) for $\mathrm{Gr}(2, \mathbb{C}^n)$ in §3.2 and to Rietsch-Williams [62] valuation in §3.3. In §4.3 we use it to make a connection between string valuations on the homogeneous coordinate ring of the flag variety and the topical flag variety.

For simplicity we assume the image of our valuation lies in $\mathbb{Z}^d$ instead of $\mathbb{Q}^d$. This is the case for all valuations we are interested in.

Theorem 10. Assume I is homogeneous with respect to the $\mathbb{Z}^s_{\geq 0}$-grading and generated by elements $f \in I$ with $\deg f > \varepsilon_i$ for all $i \in [s]$. Let $\mathfrak{v} : A \setminus \{0\} \to (\mathbb{Z}^d, \prec)$ be a full-rank valuation with $M_\mathfrak{v} \in \mathbb{Z}^{d\times n}$ the weighting matrix of $\mathfrak{v}$ and assume $\mathrm{in}_{M_\mathfrak{v}}(I)$ is prime.

Then $S(A, \mathfrak{v})$ is generated by $\{\mathfrak{v}(b_i)\}_{i\in[n]}$, where $b_i = \pi(x_i)$. In particular,

$$\Delta(A, \mathfrak{v}) = \mathrm{conv}(\mathfrak{v}(b_i) \mid i \in [n]),$$

and $\{b_1, \dots, b_n\}$ is a Khovanskii basis for $(A, \mathfrak{v})$.

Proof. As $\mathfrak{v}$ and $\mathfrak{v}_{M_\mathfrak{v}}$ (by Corollary 1) are full-rank valuations, they have one-dimensional leaves by [45, Theorem 2.3]. We apply [45, Proposition 2.4] and obtain $\mathrm{gr}_\mathfrak{v}(A) \cong \mathbb{C}[S(A, \mathfrak{v})]$ and $\mathrm{gr}_{M_\mathfrak{v}}(A) \cong \mathbb{C}[S(A, \mathfrak{v}_{M_\mathfrak{v}})]$.

Claim: For all $g \in A$ we have $\mathfrak{v}(g) = \mathfrak{v}_{M_\mathfrak{v}}(g)$.

That is, $S(A, \mathfrak{v}) = S(A, \mathfrak{v}_{M_\mathfrak{v}})$. The latter is generated by $\{\mathfrak{v}(b_i)\}_{i\in[n]}$ by Corollary 1 and Lemma 3. All other statements of the theorem are direct consequences.

Proof of claim: By Corollary 1 $\mathfrak{v}_{M_\mathfrak{v}}$ is a valuation. In particular, then by Lemma 3 we have $\mathfrak{v}(b_\mathbf{u}) = \mathfrak{v}_{M_\mathfrak{v}}(b_\mathbf{u})$ for monomials $b_\mathbf{u} = b_1^{u_1} \cdots b_n^{u_n} \in A$.

As I is homogeneous with respect to a positive grading, $M_\mathfrak{v}$ lies in the Gröbner region of I. Let $\mathbb{B} \subset A$ be the standard monomial basis adapted to $\mathfrak{v}_{M_\mathfrak{v}}$ as in [45, Proposition 4.3] restated above. Then we can write every $g \in A$ as $g = \sum_{i=1}^k b_{\alpha_i} a_i$ for $b_{\alpha_i} = \pi(x^{\alpha_i}) \in \mathbb{B}$ and $a_i \in \mathbb{C}$. We compute

$$\begin{aligned}
\mathfrak{v}_{M_\mathfrak{v}}(g) \quad &= \quad \mathfrak{v}_{M_\mathfrak{v}}\left(\sum_{i=1}^k b_{\alpha_i} a_i\right) \\
&\succeq \quad \min{}_\prec\{\mathfrak{v}_{M_\mathfrak{v}}(b_{\alpha_i}) \mid a_i \neq 0\} \\
&\overset{\text{Lemma 3}}{=} \quad \min{}_\prec\{\mathfrak{v}(b_{\alpha_i}) \mid a_i \neq 0\} \\
&\overset{\text{Def. } M_\mathfrak{v}}{=} \quad \min{}_\prec\{M_\mathfrak{v}\alpha_i \mid a_i \neq 0\} \\
&\overset{\text{[45, Proposition 4.3]}}{=} \quad \mathfrak{v}_{M_\mathfrak{v}}(g).
\end{aligned}$$

As $\mathfrak{v}_{M_\mathfrak{v}}$ has one-dimensional leaves $b \mapsto \mathfrak{v}_{M_\mathfrak{v}}(b)$ for $b \in \mathbb{B}$ (adapted to $\mathfrak{v}_{M_\mathfrak{v}}$) defines a bijection between $\mathbb{B}$ and the set of values of $\mathfrak{v}_{M_\mathfrak{v}}(\mathbb{B})$ by [45, Remark 2.29]. In particular, we have $\mathfrak{v}(b_{\alpha_i}) = \mathfrak{v}_{M_\mathfrak{v}}(b_{\alpha_i}) \neq \mathfrak{v}_{M_\mathfrak{v}}(b_{\alpha_i}) = \mathfrak{v}(b_{\alpha_j})$ for all $i \neq j$. This implies

$$\mathfrak{v}(g) = \min{}_\prec\{\mathfrak{v}(b_{\alpha_i}) \mid a_i \neq 0\} = \mathfrak{v}_{M_\mathfrak{v}}(g).$$

□

From weighting matrix to weight vector

The assumption $\mathrm{in}_{M_\mathfrak{v}}(I)$ being prime is quite strong, as this is in general hard to verify. However, taking the initial ideal with respect to a weighting matrix is closely related to taking the initial ideal with respect to a weight vector, which makes the computation easier. For example, [45, Proposition 3.10] says that for every $M \in \mathbb{Q}^{r\times n}$ there exists $\mathbf{w} \in \mathbb{Q}^n$ such that $\mathrm{in}_M(I) = \mathrm{in}_\mathbf{w}(I)$. We want to make this more explicit using [13, Lemma 3.2] restated in §2.3. The lemma allows us to associate a weight vector to a weighting matrix as follows.

Definition 15. Let $M \in \mathbb{Z}^{r\times n}$ and choose $e : \mathbb{Z}^r \to \mathbb{Z}$ as in [13, Lemma 3.2] for $S = \{M_1, \dots, M_n\}$ the set of columns of M. We define the *weight vector associated to* M as

$$e(M) := (e(M_1), \dots, e(M_n)) \in \mathbb{Z}^n.$$

The following lemma shows that for the initial ideal, $e(M)$ is independent of the choice of e. It is applied throughout the thesis whenever we apply Theorem 10.

Lemma 4. Let $f \in \mathbb{C}[x_1, \dots, x_n]$. Then $\mathrm{in}_M(f) = \mathrm{in}_{e(M)}(f)$. In particular, for every ideal $I \subset \mathbb{C}[x_1, \dots, x_n]$ we have $\mathrm{in}_M(I) = \mathrm{in}_{e(M)}(I)$.

Proof. Let $M = (m_{i,j})_{i\in[r],j\in[n]} \in \mathbb{Z}^{r\times n}$ and $e : \mathbb{Z}^r \to \mathbb{Z}$ given by $e(x_1, \dots, x_r) = \sum_{i=1^r} x_i c_i$ for $c_i \in \mathbb{Z}_{>0}$. Let $\cdot$ denote the usual dot-product in $\mathbb{R}^n$. We compute for $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{Z}^n_{\geq 0}$

$$e(M\mathbf{u}) = \sum_{i=1}^{r} \sum_{j=1}^{n} m_{ij} u_j c_i = e(M) \cdot \mathbf{u}.$$

Now for $f = \sum a_{\mathbf{u}} x^{\mathbf{u}} \in \mathbb{C}[x_1, \dots, x_n]$ by defintion we have

$$\begin{aligned}
\mathrm{in}_M(f) &= \sum_{\mathbf{m}:\ M\mathbf{m}=\min_{\prec}\{M\mathbf{u}|a_{\mathbf{u}}\neq 0\}} a_{\mathbf{m}} x^{\mathbf{m}} \\
&\overset{[13,\text{ Lemma 3.2}]}{=} \sum_{\mathbf{m}:\ e(M\mathbf{m})=\min\{e(M\mathbf{u})|a_{\mathbf{u}}\neq 0\}} a_{\mathbf{m}} x^{\mathbf{m}} \\
&\overset{e(M)\cdot\mathbf{u}=e(M\mathbf{u})}{=} \sum_{\mathbf{m}:\ e(M)\cdot\mathbf{m}=\min\{e(M)\cdot\mathbf{u}|a_{\mathbf{u}}\neq 0\}} a_{\mathbf{m}} x^{\mathbf{m}} \\
&= \mathrm{in}_{e(M)}(f).
\end{aligned}$$

□

With assumptions as in the Lemma let $S_M := \{e(M) \mid e \text{ as in [13, Lemma 3.2] }\} \cup \{0\} \subset \mathbb{Q}^n$. We define a polyhedral cone given the set S_M and the lineality space $L_I \subset \mathbb{R}^n$ of the ideal $I \subset \mathbb{C}[x_1, \dots, x_n]$ by

$$C_M := \mathrm{cone}(S_M) + L_I \subset \mathbb{R}^n.$$

Then the following corollary is a reformulation of Lemma 4.

Corollary 2. There exists a cone C in the Gröbner fan of I with $C_M \subseteq C$. Moreover, if $\mathrm{in}_M(I)$ is monomial-free, $C_M \subset \mathrm{trop}(V(I))$.

2.5 Cluster algebras

We recall here the basic notions and definitions from cluster theory. This section follows [69, §2] for quivers and quiver mutation and [36, §2] for the review of $\mathcal{A}$- and $\mathcal{X}$-cluster varieties.

A *quiver* Q is a tupel (Q_0, Q_1) with Q_0 a finite set of vertices and Q_1 a finite set of arrows between the vertices in Q_0. A *loop* is an arrow whose source and target vertex coincide, a *2-cycle* is an oriented cycle consisting of two arrows. We consider quivers with neither loops nor 2-cycles.

Definition 16. Let Q be a finite quiver without loops and 2-cycles and $k \in Q_0$. Then we define $\mu_k(Q)$ to be the quiver obtained from Q by the following recipe called *mutation at vertex* k:

Step 1: for every configuration of arrows $i \to k \to j$ add a new arrow $i \to j$;
Step 2: reverse all arrows incident to k;
Step 3: delete a maximal set of 2-cycles that may have appeared as a result of Steps 1&2.

It is a fact that mutation defines an involution and we have $\mu_k(\mu_k(Q)) = Q$. For an example see Figure 2.1.

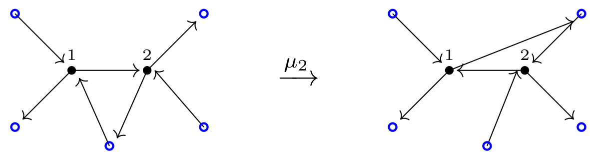


Figure 2.1: An example of quiver mutation at the vertex 2.

We divide the vertex set $Q_0 = \{1, \ldots, m\}$, into two parts $\{1, \ldots n\}$ and $\{n+1, \ldots, m\}$ for $n \leq m$. We call $\{1, \ldots, n\}$ *mutable* vertices and $\{n+1, \ldots, m\}$ *frozen* vertices. From now on we only allow mutation at mutable vertices. Further, we ignore arrows between frozen vertices as they are irrelevant for the mutation. To Q we associate its incidence matrix $(\epsilon_{ik})_{i,k \in Q_0, k \text{ mutable}} \in M_{m \times n}$ given by

$$\epsilon_{ik} := \#\{\text{arrows } i \to k \in Q_1\} - \#\{\text{arrows } k \to i \in Q_1\}. \tag{2.5.1}$$

We fix $\mathcal{F}$ as our ambient field of rational functions in n variables defined over the field $\mathbb{Q}(A_{n+1}, \ldots, A_m)$.

Definition 17. A *labelled seed* in $\mathcal{F}$ is a pair $s := (\mathbf{A}_s, Q_s)$, where $\mathbf{A}_s := (A_{1,s}, \ldots, A_{m,s})$ is a free generating set for $\mathcal{F}$ and Q_s a quiver with mutable vertices $\{1, \ldots, n\}$ and frozen vertices $\{n+1, \ldots, m\}$. We call $\mathbf{A}_s$ an *extended cluster* with *cluster variables* $\{A_{1,s}, \ldots, A_{n,s}\}$ and *frozen variables* $\{A_{n+1,s}, \ldots, A_{m,s}\}$.

Definition 18. Let $s = (\mathbf{A}_s, Q_s)$ be a labelled seed in $\mathcal{F}$ and $k \in \{1, \ldots, n\}$. We define the *seed mutation* (also called $\mathcal{A}$-mutation) in direction k to be the operation that takes s to $s' = (\mathbf{A}_{s'}, Q_{s'})$, where $Q_{s'} = \mu_k(Q_s)$ and $\mathbf{A}_{s'} = (A_{1,s'}, \ldots, A_{m,s'})$ is given by

$$A_{k,s'} A_{k,s} := \prod_{i \to k \in Q} A_{i,s} + \prod_{k \to j \in Q} A_{j,s}. \tag{2.5.2}$$

Note that when s' is obtained from s by mutation at k, then also s is obtained from s' by mutation at k. That is mutation is an involution on seeds. Observe that frozen variables are not affected by mutation. For any two seeds s and s' we have $A_{k,s} = A_{k,s'}$ for all $k \in [n+1, m]$. We therefore drop the index of the seed from frozen variables and have $\mathbf{A}_s = (A_{1,s}, \ldots, A_{n,s}, A_{n+1}, \ldots, A_m)$. If it is clear from the context which seed we are considering we also drop the s completely in our notation.

Consider the n-regular infinite tree $\mathbb{T}_n$ whose edges at every vertex are labelled by $1, \ldots, n$. An assignment of a seed s_t to every vertex $t \in \mathbb{T}_n$ is called a *seed pattern*, if two seeds $s_t, s_{t'}$ associated to adjacent vertices $t \overset{k}{\text{---}} t'$ in $\mathbb{T}_n$ are obtained from each other by mutation at

k. Let $\mathcal{V} := \bigcup_{t\in\mathbb{T}_n}\{A_{1,s_t},\ldots,A_{n,s_t}\}$ be the union of all cluster variables for all seeds in the seed pattern. Note that allthough the tree has infinitely many vertices $\mathcal{V}$ might be a finite set as through repetition some seeds might coincide. For example, in Figure 2.2 there is a seed pattern for $\mathbb{T}_2$ and we observe that the cluster variables for $s_{t_{i-2}}$ coincide with those for $s_{t_{i+3}}$.

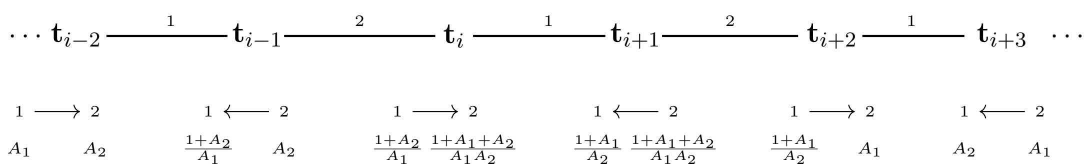


Figure 2.2: A seed pattern for $\mathbb{T}_2$.

Definition 19. The *($\mathcal{A}$-)cluster algebra* associated with a given seed pattern is the algebra

$$\mathcal{Y}(\mathbf{A},Q) := \mathbb{Z}[A_{n+1},\ldots,A_m][\mathcal{V}], \tag{2.5.3}$$

where $(\mathbf{A},Q)$ is any seed in the given seed pattern. We say it has *rank* n, as every cluster contains n cluster variables. It is called a *skew-symmetric cluster algebra of geometric type.* We also define the *upper cluster algebra* following [4, Definition 1.6] as the $\mathcal{F}$-subalgebra of all Laurent polynomials in the variables of any seed in the given seed pattern. We denote it by $\overline{\mathcal{Y}}(\mathbf{A},Q)$.

Example 5. Consider $\mathbb{C}[\mathrm{Gr}(2,4)]$ the homogeneous coordinate ring of the Grassmannian $\mathrm{Gr}(2,4)$. Recall (or see §3.1) that

$$\mathbb{C}[\mathrm{Gr}(2,4)] = \mathbb{C}[p_{12},p_{13},p_{23},p_{14},p_{24},p_{34}]/\langle p_{12}p_{34}-p_{13}p_{24}+p_{14}p_{23}\rangle.$$

Then $\{p_{12},p_{13},p_{23},p_{14},p_{34}\}$ is a set of algebraically independent generators as

$$p_{24}p_{13}^{-1} = p_{12}p_{34}+p_{14}p_{23}.$$

Observe that this relation is strikingly reminiscent with the mutation formula in (2.5.2). In fact, considering the quiver $Q=(Q_0,Q_1)$ with $Q_0=\{1,\ldots,5\}, Q_1=\varnothing$ and 1 being the only mutable vertex we obtain a cluster algebra $\mathcal{Y}$ with $\mathcal{Y}\otimes_{\mathbb{Z}}\mathbb{C}\cong\mathbb{C}[\mathrm{Gr}(2,4)]$. The quiver Q is of type $\mathtt{A}_1$. A more general statement holds due to Scott [63] and Fomin-Zelevinsky [26]: $\mathbb{C}[\mathrm{Gr}(2,n)]$ has the structure of a cluster algebra of *type* $\mathtt{A}_{n-3}$: i.e. among all mutation equivalent quivers defining the cluster algebra, there exists one whose full subquiver on all mutable vertices is an orientation of an $\mathtt{A}_{n-3}$-Dynkin diagram. For example, in Figure 2.1 there are two quivers of type $\mathtt{A}_2$ for $\mathbb{C}[\mathrm{Gr}(2,5)]$.

Very important results in the theory of cluster algebras are the *Laurent phenomenon* [26, Theorem 3.1] and the *Positivity of the Laurent phenomenon* [37, Corollary 0.4]. We state the latter below.

Theorem. ([37, Corollary 0.4]) Each cluster variable of an $\mathcal{A}$-cluster algebra is a Laurent polynomial with nonnegative integer coefficients in the cluster variables of any given seed.

In order to define cluster varieties we slightly change our perspective from this algebraic point of view to a more geometric one. To a seed s we associate a lattice $N = \mathbb{Z}^m$ with basis $\{e_{1,s}, \dots, e_{n,s}, e_{n+1}, \dots, e_m\}$. We sometimes write N_s to refer to N with the associated basis. It comes equipped with a (global) bilinear form on N. For a fixed seed s we have

$$\{\cdot,\cdot\}_s : N \times N \to \mathbb{Z}, \tag{2.5.4}$$

is (locally) induced by the exchange matrix of Q_s (for details see [36, §2]). Let $M = \mathrm{Hom}(N, \mathbb{Z})$ be the dual lattice with dual basis $\{f_{1,s}, \dots, f_{n,s}, f_{n+1}, \dots, f_m\}$. To each lattice we associate a torus $T_N \cong (\mathbb{C}^*)^m \cong T_M$ by

$$\mathcal{X}_s := T_M = \mathrm{Spec}(\mathbb{C}[N]) \text{ and } \mathcal{A}_s := T_N = \mathrm{Spec}(\mathbb{C}[M]). \tag{2.5.5}$$

We denote the coordinates on $\mathcal{X}_s$ by $X_{1,s}, \dots, X_{m,s}$. Corresponding to the basis of the lattice we have $X_{i,s} := z^{e_{i,s}}$. When the seed we are working in is clear we drop it from the notation. We define *mutation* at k on the basis $\{e_{i,s}\}$ of the lattice N for seed s by

$$e_{i,s'} := \begin{cases} e_{i,s} + \max\{\epsilon_{ik}, 0\} e_{k,s}, & \text{for } i \neq k, \\ -e_{k,s}, & \text{for } i = k. \end{cases} \tag{2.5.6}$$

Then $\{e_{1,s'}, \dots, e_{n,s'}, e_{n+1}, \dots, e_m\}$ forms again a basis for N associated with the seed $s' = \mu_k(s)$. The dual basis for M transforms as

$$f_{i,s'} := \begin{cases} -f_{i,s}, & \text{for } i \neq k, \\ f_{k,s} + \sum_j \max\{-\epsilon_{kj}, 0\} f_{j,s}, & \text{for } i = k. \end{cases}$$

Then $\{f_{1,s'}, \dots, f_{n,s'}, f_{n+1}, \dots, f_m\}$ is the dual basis for M associated with $s' = \mu_k(s)$. Mutation induces birational maps between the tori

$$\mu_k : \mathcal{X}_s \to \mathcal{X}_{\mu_k(s)} \text{ and } \mu_k : \mathcal{A}_s \to \mathcal{A}_{\mu_k(s)}.$$

defined by the pullback of functions. We have for $\mathcal{X}$-tori

$$\mu_k^*(z^n) := z^n (1 + z^{e_{k,s}})^{-\{n, e_{k,s}\}_s}, \text{ for } n \in N. \tag{2.5.7}$$

For the $\mathcal{A}$-tori the birational map is induced from the seed mutation defined in (2.5.2), we recover

$$\mu_k^*(A_{k,s'}) = \begin{cases} A_{i,s}, & \text{for } i \neq k, \\ \frac{\prod_{i \to k \in Q_s} A_{i,s} + \prod_{k \to j \in Q_s} A_{j,s}}{A_{k,s}}, & \text{for } i = k. \end{cases}$$

To be consistent with the $\mathcal{X}$-notation, we set $A_{i,s} = z^{f_{i,s}}$ for $1 \leq i \leq n$ and $A_l = z^{f_l}$ for $n+1 \leq l \leq m$ coordinates for $\mathcal{A}_s$. Consider again a given seed pattern, then by [36, Proposition 2.4] we can give the following definition.

Definition 20. Given a seed pattern the $\mathcal{X}$- (resp. $\mathcal{A}$-) cluster variety is defined as the scheme

$$\mathcal{X} := \bigcup_{t \in \mathbb{T}_n} \mathcal{X}_{s_t} \text{ (resp. } \mathcal{A} := \bigcup_{t \in \mathbb{T}_n} \mathcal{A}_{s_t}) \tag{2.5.8}$$

obtained by glueing the tori $\mathcal{X}_{s_t}$ (resp. $\mathcal{A}_{s_t}$) along the birational maps induced by mutation.

Sometimes $\mathcal{X}$ is called the *Fock-Goncharov dual* to the cluster variety $\mathcal{A}$. The relation to cluster algebras is the following. The global sections of the structure sheaf on $\mathcal{A}$ are related to the upper cluster algebra associated to the given seed pattern by

$$H^0(\mathcal{A}, \mathcal{O}(\mathcal{A})) = \overline{\mathcal{Y}}(s_t) \otimes_{\mathbb{Z}} \mathbb{C}.$$

A natural (partial) compactification $\bar{\mathcal{A}}$ of $\mathcal{A}$ (an $\mathcal{A}$-cluster variety) is given by allowing the frozen variables $A_{n+1}, \dots, A_m$ to vanish. We denote the resulting *boudary disivor* in $\bar{\mathcal{A}}$ by

$$D := \sum_{f=n+1}^{m} D_f, \text{ where } D_f := \{A_f = 0\} \subset \bar{\mathcal{A}}. \tag{2.5.9}$$

Example 6. Recall Example 5. The $\mathcal{A}$-cluster variety for this type $\mathtt{A}_1$-cluster algebra with four frozen vertices is given by glueing two tori

$$T_{p_{13},p_{12},p_{14},p_{23},p_{34}} \cup_\mu T_{p_{24},p_{12},p_{14},p_{23},p_{34}}$$

along the birational map μ induced by mutation. The irreducible components of the boundary divisor are

$$\{p_{12} = 0\}, \{p_{14} = 0\}, \{p_{23} = 0\}, \{p_{34} = 0\}.$$

One can show that up to codimension two $\bar{\mathcal{A}}$ is $\mathrm{Gr}(2, 4)$. As the Picard group of $\mathrm{Gr}(2, 4)$ has rank one, all four divisors are linearly equivalent and the boundary divisor D is in fact the anticanonical divisor for $\mathrm{Gr}(2, 4)$. We recall later (in §4.1) how to associate very ample line bundles L_λ on SL_n/B to weights $\lambda \in \Lambda^{++}$. The same construction works for $\mathrm{Gr}(2, 4)$ and one obatians $\mathcal{O}(L_{4\omega_2}) = D$ (up to linear equivalence).

Every component D_f of the boundary divisor induces a (rank 1) valuation $\mathrm{ord}_{D_f} : \mathbb{C}[\mathcal{A}] \to \mathbb{Z}$ by sending a function $g \in \mathbb{C}[\mathcal{A}]$ to its order of vanishing along D_f. If g has a pole along D_f, then $\mathrm{ord}_{D_f}(g) < 0$ is the order of the pole. These valuations are called *divisorial discrete valuations* in [37].

A main result of [37] is the definition and parametrization of the ϑ-basis for $\mathbb{C}[\mathcal{A}]$. One central question is: *When is a basis element of* $\mathbb{C}[\mathcal{A}]$ *also a basis element for* $\mathbb{C}[\bar{\mathcal{A}}]$*?*

The *full Fock-Goncharov conjecture* (see [37, Definition 0.6]) suggests that basis elements for $\mathbb{C}[\mathcal{A}]$ are parametrized by *tropical points* in $\mathcal{X}^{\mathrm{trop}}(\mathbb{Z})$ (see [37, §2]). We don't go into detail about this tropical space due to the following fact: fixing a seed s we have an isomorphism

$$\mathcal{X}^{\mathrm{trop}}(\mathbb{Z})|_s \cong N_s \cong \mathbb{Z}^m.$$

For the purpose of this thesis we always work in a fixed seed and therefore have an identification of lattice points in N_s with basis elements for $\mathbb{C}[\mathcal{A}]$. From now on we assume that the cluster variety $\mathcal{A}$ satisfies the full FG-conjecture, as this is the case for the cluster varieties we are interested in. For example, Magee showed in [54] that this is the case for the cluster variety inside SL_n/U which are of interest in §4.2. A number of criteria for the full Fock-Goncharov conjecture to hold are discussed in [37, §8.4] and we refer the interested reader there for more details.

Associated to each component of the boundary divisor there exists a function ϑ_f on the dual cluster variety $\mathcal{X}$. Assuming the full FG-conjecture we can compute and expression for ϑ_f in $\mathcal{X}_{s_0}$ (s_0 being a fixed initial seed) as described by the Algorithm 1, which we consider as definition.

Definition 21. Let $\mathcal{A}$ be a cluster variety associated to an $\mathcal{A}$-cluster algebra $\mathcal{Y}(\mathbf{A}, Q)$ satisfying the full Fock-Goncharov conjecture. Then we define the *superpotential* $W : \mathcal{X} \to \mathbb{C}$ on the dual cluster variety X as

$$W := \sum_{f \text{ frozen vertex in } Q} \vartheta_f.$$

Algorithm 1: Computing an expression for the superpotential in a given inital seed.

Input: A cluster variety $\mathcal{A}$ with initial seed s_0 satisfying the full FG-conjecture.

for *every frozen vertex* $f \in Q_{s_0}$ **do**
  find a sequence of mutations $\overline{\mu}$ from s_0 to a seed s_f where f is a sink.
  if $s_f = s_0$ **then**
    Output: $\vartheta_f|_{\mathcal{X}_{s_0}} = z^{-e_{f,s_0}}$.
  else
    apply the pullback of the reverse mutation sequence to $z^{-e_{f,s_f}}$.
    Output: $\vartheta_f|_{\mathcal{X}_{s_0}} = (\overline{\mu})^*(z^{-e_{f,s_f}})$.

Output: The superpotential $W|_{\mathcal{X}_{s_0}} = \sum_{f \text{ frozen in } Q_{s_0}} \vartheta_f|_{\mathcal{X}_{s_0}}$.

A seed s_f for which a frozen vertex f is a sink (as in the first step of Algorithm 1) is called *optimized* for f.

Remark 4. Finding an optimized seed for a frozen vertex is in general a hard problem as there might be infinitely many seeds. Further, doing these computations by hand is already after a few mutation quite frustrating due to the recursive formulas. An excellent tool for such computations is provided by Keller's *quiver mutation applet* [46].

Coming back to $\mathbb{C}[\mathcal{A}]$, note that a basis element $\vartheta \in \mathbb{C}[\mathcal{A}]$ gives an element in $\mathbb{C}[\bar{\mathcal{A}}]$ if $\mathrm{ord}_{D_f}(\vartheta) \geq 0$ for every component D_f of the boundary divisor. In particular,

$$\vartheta \in \mathbb{C}[\bar{\mathcal{A}}] \text{ if and only if } \min_{f \text{ frozen}} \{\mathrm{ord}_{D_f}(\vartheta)\} \geq 0.$$

Let $g_\vartheta \in N_s$ be the lattice point associated to ϑ for a fixed seed s. Then using the fact that $\vartheta_f^{\mathrm{trop}}(g_\vartheta) = \mathrm{ord}_{D_f}(\vartheta)$, this translates to

$$\vartheta \in \mathbb{C}[\bar{\mathcal{A}}] \text{ if and only if } g_\vartheta \in \{\mathbf{x} \in \mathbb{R}^m \mid W|_{\mathcal{X}_s}^{\mathrm{trop}}(\mathbf{x}) \geq 0\} \cap N_s. \tag{2.5.10}$$

In particular, the lattice points in $\{\mathbf{x} \in \mathbb{R}^m \mid W|_{\mathcal{X}_s}^{\mathrm{trop}}(\mathbf{x}) \geq 0\}$ parametrize a basis for $\mathbb{C}[\bar{\mathcal{A}}]$.

Chapter 3

Grassmannians

3.1 Preliminary notions

The Grassmannian $\mathrm{Gr}(k, \mathbb{C}^n)$ for integers $k \leq n$ is the space of k-dimensional subspaces of $\mathbb{C}^n$. It has the structure of a projective variety given by the Plücker embedding $\mathrm{Gr}(k, \mathbb{C}^n) \hookrightarrow \mathbb{P}(\bigwedge^k \mathbb{C}^n)$ sending the generators $v_1, \ldots, v_k \in \mathbb{C}^n$ of a k-dimensional vector subspace $V \subset \mathbb{C}^n$ to $[v_1 \wedge \cdots \wedge v_k] \in \mathbb{P}(\bigwedge^k \mathbb{C}^n)$. In many cases it is useful to describe $\mathrm{Gr}(k, \mathbb{C}^n)$ as a vanishing set $V(I_{k,n})$. We denote the standard basis of $\mathbb{C}^n$ by $\{e_1, \ldots, e_n\}$ and choose a subset $I = \{i_1, \ldots, i_k\}$ of $\{1, \ldots, n\} =: [n]$.

Definition 22. The *Plücker coordinate* $\bar{p}_I$ is the basis element in $(\bigwedge^k \mathbb{C}^n)^*$ dual to $e_{i_1} \wedge \cdots \wedge e_{i_k}$.

Plücker coordinates generate the homogeneous coordinate ring of $\mathrm{Gr}(k, \mathbb{C}^n)$ satisfying certain relations. We want to express $\mathbb{C}[\mathrm{Gr}(k, \mathbb{C}^n)] =: A_{k,n}$ as a quotient of the polynomial ring $\mathbb{C}[p_J \mid J \in \binom{[n]}{k}]$ by a prime ideal encoding these relations. We define for $K \in \binom{[n]}{k-1}$ and $L \in \binom{[n]}{k+1}$ the sign $\mathrm{sgn}(j; K, L) := (-1)^{\#\{l \in L | j < l\} + \#\{k \in K | k > j\}}$. The following definition can be found for example in [53, p. 170].

Definition 23. The *Plücker relation* $R_{K,L} \in \mathbb{C}[p_J \mid J \in \binom{[n]}{k}]$ for $K \in \binom{[n]}{k-1}$ and $L \in \binom{[n]}{k+1}$ is

$$R_{K,L} := \sum_{j \in L} \mathrm{sgn}(j; K, L) p_{K \cup \{j\}} p_{L \setminus \{j\}}. \tag{3.1.1}$$

The *Plücker ideal* $I_{k,n} \subset \mathbb{C}[p_J \mid J \in \binom{[n]}{k}]$ is generated by $R_{K,L}$ for all $K \in \binom{[n]}{k-1}$ and $L \in \binom{[n]}{k+1}$ and $\mathbb{C}[\mathrm{Gr}(k, \mathbb{C}^n)] = \mathbb{C}[p_J \mid J \in \binom{[n]}{k}]/I_{k,n}$.

In the special case of $k = 2$, Plücker relations are of a particularly nice form. We simplify the notation in this case to $R_{\{i\},\{j,k,l\}} =: R_{i,j,k,l} \in \mathbb{C}[p_I \mid I \in \binom{[n]}{2}]$, where for $1 \leq i < j < k < l \leq n$ we have

$$R_{i,j,k,l} = p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk} \in I_{2,n}.$$

By setting $p_{ij} = -p_{ji}$ we see that it is enough to consider $R_{i,j,k,l}$ with $1 \leq i < j < k < l \leq n$ as generators for the ideal $I_{2,n}$ as up to sign these are all relations. We denote the polynomial ring $\mathbb{C}[p_I \mid I \in \binom{[n]}{2}]$ by $\mathbb{C}[p_{ij}]_{ij}$ for short, if it clear which n we are considering. To distinguish between the polynomial generators p_{ij} (also called *Plücker variables*) and the

Plücker coordinates in $\mathbb{C}[\mathrm{Gr}(2,\mathbb{C}^n)] = \mathbb{C}[p_{ij}]_{ij}/I_{2,n}$ we denote the Plücker coordinate by $\bar{p}_{ij} \in \mathbb{C}[\mathrm{Gr}(2,\mathbb{C}^n)] = A_{2,n}$. When there is no risk of confusion we drop this distinction.

The Grassmannian $\mathrm{Gr}(k,\mathbb{C}^n)$ can be realized as a quotient of the algebraic group SL_n over $\mathbb{C}$. Recall the basic notations from §2.1.

Consider $P_k \subset SL_n$ the parabolic subgroup of block upper traingular matrices with blocks of size $k\times k$ and $(n-k)\times(n-k)$ along the diagonal. Naturally, it contains B. Set $I_k := [n-1]\backslash k$ and consider the subgroup $W_{I_k} := \langle s_i \mid i \in I_k\rangle$ of S_n. We choose a representative w_k in the coset of w_0 in the quotient S_n/W_{I_k}. Then by the identification $S_n = N_{SL_n}(T)/T$ (here $N_{SL_n}(T)$ is the normalizer of T in SL_n) we have $P_k = \overline{Bw_kB}$. The Grassmannian is then the quotient

$$SL_n/P_k = \mathrm{Gr}(k,\mathbb{C}^n).$$

Similarly to R^+, let $R_k^+ = \{\beta \in R^+ \mid w_k(\beta) < 0\}$ be the set of positive roots for SL_n/P_k. In fact, we have $R_k^+ = \{\alpha_{i,j} \in R^+ \mid i \leq k \leq j\}$. We also have $\mathfrak{n}_k^- = \langle f_\beta \mid \beta \in R_k^+\rangle \subset \mathfrak{n}^-$ a Lie subalgebra and denote by $U_k^- \subset B^-$ the corresponding subgroup with $\mathrm{Lie}\, U_k^- = \mathfrak{n}_k^-$. It consists of lower triangular matrices with 1s on the diagonal and non-zero entries only in positions (i,j) with $k \leq i \leq n$ and $1 \leq j \leq k$.

Example 7. For $\mathrm{Gr}(2,\mathbb{C}^4)$ we have $I_2 = \{1,3\} \subset [3]$ and consider $S_n/\langle s_1, s_3\rangle$. As representative of w_0 in the quotient we can chose w_2 with reduced expression $s_2s_1s_3s_2$. Then we compute $R_2^+ = \{\alpha_2, \alpha_{1,2}, \alpha_{2,3}, \alpha_{1,3}\} = \{\epsilon_i - \epsilon_j | 1 \leq i \leq 2 < j \leq n\}$. The corresponding subgroups of SL_4 are

$$P_2 = \left\{ \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} \\ 0 & 0 & x_{3,3} & x_{3,4} \\ 0 & 0 & x_{4,3} & x_{4,4} \end{pmatrix} \right\} \text{ and } U_2^- = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ x_{3,1} & x_{3,1} & 1 & 0 \\ x_{4,1} & x_{4,2} & 0 & 1 \end{pmatrix} \right\}.$$

Note that U_k^- is open and dense in $\mathrm{Gr}(k,\mathbb{C}^n)$ and we have an isomorphism of fields of rational functions $\mathbb{C}(\mathrm{Gr}(k,\mathbb{C}^n)) \cong \mathbb{C}(U_k^-)$. We see in §4.1 that $\mathbb{C}[SL_n/B] = \bigoplus_{r\geq 1} V(r\lambda)^*$ for every $\lambda \in \Lambda^{++}$. Having $\mathrm{Gr}(k,\mathbb{C}^n) = SL_n/P_k$ similary we have for the homogeneous coordinate ring of the Grassmannian

$$\mathbb{C}[\mathrm{Gr}(k,\mathbb{C}^n)] = \bigoplus_{r\geq 1} V(r\omega_k)^*, \tag{3.1.2}$$

where $\omega_k \in \Lambda^+$ is the kth fundamental weight (see §2.1).

3.1.1 The tropical Grassmannian

In this section we recall results on the tropical Grassmannian due to Speyer and Sturmfels in [64] and [53, §4.3]. For computations in small cases we rely on *Macaulay2* [35] and *gfan* [41].

Definition 24. The *tropical Grassmannian*, denoted $\mathrm{trop}(\mathrm{Gr}(k,\mathbb{C}^n)) \subset \mathbb{R}^{\binom{n}{k}}$ is the tropical variety of the Plücker ideal $I_{k,n}$. By [64, Corollary 3.1] it is a $k(n-k)+1$-dimensional polyhedral fan whose maximal cones are all of this dimension.

By what we have seen in §2.2 $\mathrm{trop}(\mathrm{Gr}(k,\mathbb{C}^n))$ is the subfan of the Gröbner fan of $I_{k,n}$ consisting of those $\mathbf{w}$, such that $\mathrm{in}_{\mathbf{w}}(I_{k,n})$ is monomial-free. Recall that for a fixed cone C of $\mathrm{trop}(\mathrm{Gr}(k,\mathbb{C}^n))$ each two points $\mathbf{v},\mathbf{w}$ in its relative interior yield the same initial ideal, i.e. $\mathrm{in}_{\mathbf{w}}(I_{k,n}) = \mathrm{in}_{\mathbf{v}}(I_{k,n})$ and we use the notation $\mathrm{in}_C(I_{k,n})$. Recall that a maximal cone C of $\mathrm{trop}(\mathrm{Gr}(k,\mathbb{C}^n))$ by definition is prime, if $\mathrm{in}_C(I_{k,n})$ is a prime ideal.

We mainly focus on the tropicalization of $\mathrm{Gr}(2,\mathbb{C}^n)$ which has a very nice properties.

Corollary. ([64, Corollary 4.4]) Every initial ideal $\mathrm{in}_C(I_{2,n})$ associated to a maximal cone C in $\mathrm{trop}(\mathrm{Gr}(2,\mathbb{C}^n))$ is prime.

Recall that $\mathrm{trop}(V(I)) \subset \mathbb{R}^n$ for I a homogeneous ideal in $\mathbb{C}[x_1,\dots,x_n]$ contains a linear subspace L_I called *lineality space*. The elements $l \in L_I$ have the property that $\mathrm{in}_l(I) = I$. In particular, $\mathbb{R}(1,\dots,1) \subset \mathbb{R}^n$ is contained in L_I.

Theorem. ([64, Theorem 3.4]) The quotient $\mathrm{trop}(\mathrm{Gr}(2,\mathbb{C}^n))/L_{I_{2,n}} \subset \mathbb{R}^{\binom{n}{2}}/\mathbb{R}^{n-3}$ intersected with the unit sphere is, up to sign, the *space of phylogenetic trees* [6].

We explain the implications of the theorem in more detail. In particular, it implies that every maximal prime cone C can be associated with a *labelled trivalent tree* with n leaves. The set of all labels trivalent trees with n leaves is denoted by $\mathcal{T}_n$. A *trivalent tree* is a graph with internal vertices of valency three and no loops or cycles of any kind. Non-internal vertices are called *leaves* and the word *labelled* refers to labelling the leaves by $1,\dots,n$. We call an edge *internal*, if it connects two internal vertices.

We label the standard basis of $\mathbb{R}^{\binom{n}{2}}$ by pairs (i,j) with $1 \le i < j \le n$ corresponding to Plücker coordinates. The following definition shows how we can get a point in the relative interior of a maximal cone in $\mathrm{trop}(\mathrm{Gr}(2,\mathbb{C}^n))$ from a labelled trivalent tree. It follows from [64, Theorem 3.4].

Definition 25. Let T be a labelled trivalent tree with n leaves. Then the (i,j)'th entry of the weight vector $\mathbf{w}_T \in \mathrm{trop}(\mathrm{Gr}(2,\mathbb{C}^n))$ is

$$-\#\{\text{internal edges on path from leaf } i \text{ to leaf } j \text{ in } T\}.$$

For notational convenience we set $\mathrm{in}_T(I_{2,n}) := \mathrm{in}_{\mathbf{w}_T}(I_{2,n})$. The corresponding maximal cone in $\mathrm{trop}(\mathrm{Gr}(2,\mathbb{C}^n))$ is denoted C_T.

Later in §3.3 we refer to the entries of $-\mathbf{w}_T$ (note the sign change) as *tree degrees*, we denote $\deg_T p_{i,j} = (-\mathbf{w}_T)_{(i,j)}$. Combining the above, we conclude that every trivalent labelled tree induces a toric degeneration of $\mathrm{Gr}(2,\mathbb{C}^n)$ with flat family given as in (2.2.4).

The symmetric group S_n acts on $\mathcal{T}_n$ by permuting the labels of the leaves of trees. We also have a S_n-action on Plücker coordinates given by

$$\sigma(p_{ij}) = \mathrm{sgn}(\sigma) p_{\sigma^{-1}(i),\sigma^{-1}(j)} \text{ for } \sigma \in S_n.$$

This action induces a ring automorphism of $\mathbb{C}[p_{i,j}]_{ij}$ for every $\sigma \in S_n$ that sends $\mathrm{in}_T(I_{2,n})$ to $\mathrm{in}_{\sigma(T)}(I_{2,n})$ for every trivalent labelled tree T. Denote by $\mathtt{T}$ the equivalence class of $T \in \mathcal{T}_n$. It is uniquely determined by the underlying *(unlabelled) trivalent tree* with n leaves, see for example Figure 3.1. We denote the set of trivalent tree by $\mathcal{T}_n/S_n$

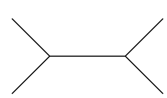


Figure 3.1: A trivalent tree with 4 leaves.

Consider a trivalent tree $\mathtt{T} \in \mathcal{T}_n/S_n$. If there are two non-internal edges connected to the same internal vertex c, then we say $\mathtt{T}$ has a *cherry* at vertex c.

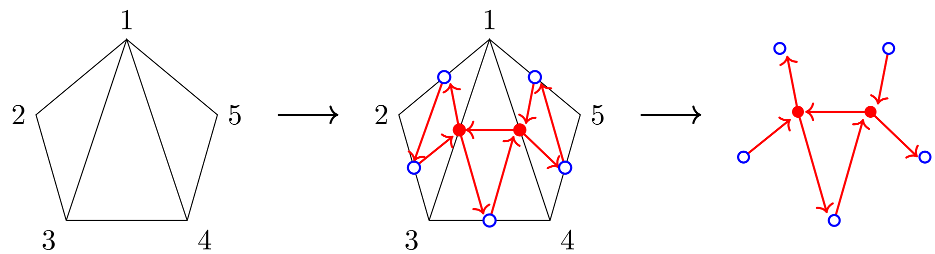


Figure 3.2: Visualizing Algorithm 2 for a triangulation of D_5.

Lemma 5. Every trivalent tree with $n \geq 4$ leaves has a cherry.

Proof. We use induction on n. For $n = 4$ Figure 3.1 displays the only trivalent tree in $\mathcal{T}_4/S_4$ and we see, it has two cherries. Now consider a trivalent tree $\mathtt{T}' \in \mathcal{T}_{n+1}/S_{n+1}$. We remove one edge connected to a leaf and obtain a tree $\mathtt{T} \in \mathcal{T}_n/S_n$. By induction, $\mathtt{T}$ has a cherry at some vertex c. Adding the removed edge back there are two possibilities: either we add it to an internal edge, then the cherry also exists in $\mathtt{T}'$. Or we add it at an edge with a leaf, hence create a new cherry. □

3.1.2 Cluster structure on $\mathbb{C}[\mathrm{Gr}(2, \mathbb{C}^n)]$

We have seen in Example 5 the cluster structure on $\mathbb{C}[\mathrm{Gr}(2, \mathbb{C}^4)]$. In this subsection we want to recall the cluster structure on $\mathbb{C}[\mathrm{Gr}(2, \mathbb{C}^n)]$ following [26] and [63].

Let D_n be a disk with n marked points on its boundary ∂D_n labelled by $[n]$ in counterclockwise order. We define an *arc* in D_n as a line connecting two marked points. A *triangulation* Δ of D_n is a maximal collection of non-crossing arcs. We call arcs that intersect $D_n^\circ := D_n \setminus \partial D_n$ *internal arc* and those along ∂D_n *boundary arc*. Note that every triangulation consists of n boundary arcs and $n-3$ internal arcs. A collection of three arcs $\{d_1, d_2, d_3\}$ in Δ is a *triangle* if pairwise they have one adjacent marked point in common.

Algorithm 2: Associating a quiver with a triangulation of D_n.

Input: A triangulation Δ of D_n.

for *every internal arc* d *in* Δ **do**
- create a mutable vertex $v_d \in Q_0$;

for *every boundary arc* b *in* Δ **do**
- create a frozen vertex $v_b \in Q_0$;

for *every triangle* $\{d_1, d_2, d_3\}$ *in* Δ **do**
- draw three arrows in Q_1 between the vertices $v_{d_1}, v_{d_2}, v_{d_3} \in Q_0$ creating a counterclockwise oriented 3-cycle in Q;

Output: The quiver $Q_\Delta := (Q_0, Q_1)$.

Definition 26. To a triangulation Δ of D_n we associate the quiver Q_Δ that is the output of Algorithm 2 and set $\mathbf{A}_\Delta := (A_{1,\Delta}, \dots, A_{n-3,\Delta}, A_{n-2}, \dots, A_{2n-3})$. Then Δ determines the cluster algebra $\mathcal{Y}_\Delta := \mathcal{Y}(\mathbf{A}_\Delta, Q_\Delta)$.

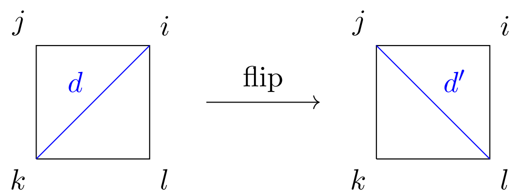


Figure 3.3: Flipping an internal arc.

Given a triangulation Δ of D_n we create a new triangulation Δ' by flipping a diagonal. More precisely, consider two adjacent triangles in Δ forming a quadrilateral with vertices the marked point i, j, k, l in circular order along ∂D_n and diagonal $d = [i, k]$. Then *flipping* d refers to replacing it with $d' = [j, l]$ (see Figure 3.3). The outcome is a new triangulation Δ' which only differs from Δ by d. Given this definition the next proposition has a straightforward proof.

Proposition 1. Let Δ and Δ' be two triangulations of D_n related to each other by flipping the (internal) arc $d \in \Delta$. Then the quivers Q_Δ and $Q_{\Delta'}$ are related to each other by quiver mutation (see Definition 16). Moreover, the cluster algebras $\mathcal{Y}_\Delta$ and $\mathcal{Y}'_\Delta$ are isomoprhic.

The main result is then the following.

Proposition. ([26], [63, Proposition 2]) For $n \geq 5$ the homogeneous coordinate ring $\mathbb{C}[\mathrm{Gr}(2, \mathbb{C}^n)]$ is isomorphic to $\mathcal{Y}_\Delta \otimes_\mathbb{Z} \mathbb{C}$ for any triangulation Δ of D_n.

Let Δ be a triangulation of D_n with extended cluster $\mathbf{A}_\Delta$. Then the cluster variables $A_{1,\Delta}, \ldots A_{n-3,\Delta}$ correspond to internal arcs of Δ, each connecting two marked points. If $A_{k,\Delta}$ corresponds to the arc connecting i and j, the isomorphism in [63, Proposition 2] identifies $A_{k,\Delta}$ with the Plücker coordinate $\bar{p}_{ij} \in \mathbb{C}[\mathrm{Gr}(2, \mathbb{C}^n)]$. The frozen variables $A_{n-2}, \ldots, A_{2n-3}$ correspond to arcs connecting successive marked points i and $i+1 \mod n$. They are identified with the corresponding Plücker coordinates $\bar{p}_{i,i+1}$.

3.2 Birational sequences for Grassmannians and $\mathrm{trop}(\mathrm{Gr}(2,\mathbb{C}^n))$

We study birational sequences due to Fang, Fourier, and Littelmann [20] for Grassmannians and introduce the class of iterated birational sequences. We show that toric degenerations of $\mathrm{Gr}(2,\mathbb{C}^n)$ constructed using the tropical Grassmannian $\mathrm{trop}(\mathrm{Gr}(2,\mathbb{C}^n))$ due to Speyer and Sturmfels [64] can also be obtained using (iterated) biratinal sequences.

3.2.1 Birational sequences

We start the section by recalling some results due to Fang, Fourier, and Littelmann in [20] regarding birational sequences and associated valuations. After proving Lemma 6,which is central in this section we define a new class of birational sequences caled *iterated* in Definition 28.

Consider a positive root $\beta \in R^+$, then the *root subgroup* corresponding to β is given by

$$U_{-\beta} := \{\exp(z f_\beta) \mid z \in \mathbb{C}\} \subset U^-.$$

Definition 27. ([20]) Let $S = (\beta_1, \dots, \beta_{k(n-k)})$ be a sequence of positive roots. Then S is called a *birational sequence* for $\mathrm{Gr}(k,\mathbb{C}^n)$ if the product map induced by multiplication is birational:

$$U_{-\beta_1} \times \cdots \times U_{-\beta_{k(n-k)}} \to U_k^-.$$

Example 8. The following are two first (and motivating) examples of birational sequences that we encounter again later in §4.3.

1. The product map $\pi : \prod_{\beta \in R_k^+} U_{-\beta} \to U_k^-$ is birational, which makes any sequence containing all roots in R_k^+ (in arbitrary order) a birational sequence called *PBW-sequence* (see [20, Example 1 and page 131]). We distinguish between PBW-sequences S and S' when the roots in both appear in different order.

2. Another example is given by a reduced decomposition $\underline{w}_k = s_{i_1} \dots s_{i_{k(n-k)}}$ of w_k, a coset representative of w_0 in S_n/W_{I_k} (see §3.1). Let $S = (\alpha_{i_1}, \dots, \alpha_{i_{k(n-k)}})$ be the corresponding sequence of simple roots. Then S is a birational sequence called *the reduced decomposition case* (see [20, Example 2]).

The second example shows that repetitions of positive roots may occur in birational sequences. Our aim is to shed some light on sequences that are *neither* PBW *nor* associated to reduced decompositions for Grassmannians. The following lemma allows us to construct such sequences for $\mathrm{Gr}(k,\mathbb{C}^{n+1})$ from sequences for $\mathrm{Gr}(k,\mathbb{C}^n)$.

Lemma 6. Let $S = (\beta_1, \dots, \beta_{k(n-k)})$ be a birational sequence for $\mathrm{Gr}(k,\mathbb{C}^n)$. Then extending it to the left by $\alpha_{i_1,n}, \dots, \alpha_{i_k,n}$ for distinct $i_1, \dots, i_k \le n$ yields the following birational sequence for $\mathrm{Gr}(k,\mathbb{C}^{n+1})$

$$S' = (\alpha_{i_1,n}, \dots, \alpha_{i_k,n}, \beta_1, \dots, \beta_{k(n-k)}). \tag{3.2.1}$$

Proof. The sequence S' yields the product of root subgroups $\mathcal{G}' = U_{-\alpha_{i_1,n}} \times \cdots \times U_{-\alpha_{i_k,n}} \times U_{-\beta_1} \times \cdots \times U_{-\beta_{k(n-k)}} \subset SL_{n+1}$, where for $y_1, \ldots, y_k, z_1, \ldots, z_{k(n-1)} \in \mathbb{C}$ elements are of form

$$\exp(y_1 f_{\alpha_{i_1,n}}) \cdots \exp(y_k f_{\alpha_{i_k,n}}) \exp(z_1 f_{\beta_1}) \cdots \exp(z_{k(n-k)} f_{\beta_{k(n-k)}})$$
$$= \begin{pmatrix} 1 & 0 & 0 & & \ldots & 0 & 0 \\ a_2^1 & 1 & 0 & & \ldots & 0 & 0 \\ a_3^1 & a_3^2 & 1 & & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & & 0 & 0 \\ a_{k+1}^1 & a_{k+1}^2 & \ldots & a_{k+1}^k & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & & \ddots & \vdots & \vdots \\ a_n^1 & a_n^2 & \ldots & a_n^k & * & \ldots & 1 & 0 \\ a_{n+1}^1 & a_{n+1}^2 & \ldots & a_{n+1}^k & * & \ldots & * & 1 \end{pmatrix} \in \mathcal{G}'.$$

Here $a_{n+1}^j = y_1 a_{i_1}^j + \cdots + y_k a_{i_k}^j$ for $1 \leq j \leq k$. Denote the i-th row of a fixed element $A \in \mathcal{G}'$ by $a_i = (a_i^1, a_i^2, \ldots, a_i^k, *, \ldots, *) \in \mathbb{C}^{n+1}$. Set $a_j^j = 1$ for $1 \leq j \leq n$ and $a_i^j = 0$ for $i < j$. The coefficient of $e_{j_1} \wedge \cdots \wedge e_{j_k}$ in $A(e_1 \wedge \cdots \wedge e_k)$ is the minor $\bar{p}_J(A) = \det \begin{pmatrix} a_{j_1} \\ \vdots \\ a_{j_k} \end{pmatrix}$ with $J = \{j_1, \ldots, j_k\} \subset [n+1]$. Now assume $J' = \{j_1, \ldots, j_{k-1}, n+1\}$. As $a_{n+1} = y_1 a_{i_1} + \cdots + y_k a_{i_k}$ then

$$p_{J'} = \det \begin{pmatrix} a_{j_1} \\ \vdots \\ a_{j_{k-1}} \\ a_{n+1} \end{pmatrix} = y_1 \det \begin{pmatrix} a_{j_1} \\ \vdots \\ a_{j_{k-1}} \\ a_{i_1} \end{pmatrix} + \cdots + y_k \det \begin{pmatrix} a_{j_1} \\ \vdots \\ a_{j_{k-1}} \\ a_{i_k} \end{pmatrix}.$$

We define the map $\varphi' : \mathbb{C}(\mathbb{A}^{k(n-k+1)}) \to \mathbb{C}(\mathrm{Gr}(k, \mathbb{C}^{n+1})) \cong \mathbb{C}(U_k^-)$ as extension of the birational map induced by S on the function fields, which we denote by $\varphi : \mathbb{C}(\mathbb{A}^{k(n-k)}) \to \mathbb{C}(\mathrm{Gr}(k, \mathbb{C}^n))$. For $I = \{i_1, \ldots, i_k\} \subset [n]$ and for $1 \leq j \leq k$ we define

$$\varphi'(y_j) := \frac{\bar{p}_{I \setminus \{i_j\} \cup \{n+1\}}}{\bar{p}_I}.$$

In order to prove that S' is birational it suffices to find a map $\psi' : \mathbb{C}(\mathrm{Gr}(k, \mathbb{C}^{n+1})) \to \mathbb{C}(\mathbb{A}^{k(n-k+1)})$ that is inverse to φ'. Let $\psi : \mathbb{C}(\mathrm{Gr}(k, \mathbb{C}^n)) \to \mathbb{C}(\mathbb{A}^{k(n-k)})$ be the inverse of φ. We define ψ' to as the extension of ψ given by

$$\psi'(p_{J'}) = y_1 \psi(\bar{p}_{J' \setminus \{n+1\} \cup \{i_1\}}) + \cdots + y_k \psi(\bar{p}_{J' \setminus \{n+1\} \cup \{i_k\}}).$$

A straightforward computation then reveals that ψ' and φ' are indeed inverse to each other. Therefore S' is a birational sequence for $\mathrm{Gr}(k, \mathbb{C}^{n+1})$. $\square$

Definition 28. For $k < n$ consider a birational sequence for $\mathrm{Gr}(k, \mathbb{C}^{k+1})$. Now extend it as in (3.2.1) to a birational sequence for $\mathrm{Gr}(k, \mathbb{C}^{k+2})$. Repeat this process until the outcome is a birational sequence for $\mathrm{Gr}(k, \mathbb{C}^n)$. Birational sequences of this form are called *iterated*.

We explain how to obtain a valuation from a fixed birational sequence $S = (\beta_1, \ldots, \beta_d)$ for $\mathrm{Gr}(k, \mathbb{C}^n)$ as constructed in [20]. Let $d := k(n-k)$ and define the *height function* $\mathrm{ht} : R^+ \to$

$\mathbb{Z}_{\geq 0}$ by sending a positive root to the number of its simple summands, i.e. $\mathrm{ht}(\alpha_{i,j}) = j-i+1$. Then the *height weighted function* $\Psi : \mathbb{Z}^d \to \mathbb{Z}$ is given by

$$\Psi(m_1,\dots,m_d) := \sum_{i=1}^{d} m_i \,\mathrm{ht}(\beta_i).$$

Let $<_{lex}$ be the lexicographic order on $\mathbb{Z}^d$. Then we define the Ψ*-weighted reverse lexicographic order* $\prec_\Psi$ on $\mathbb{Z}^d$ by setting for $\mathbf{m},\mathbf{m'} \in \mathbb{Z}^d$

$$\mathbf{m} \prec_\Psi \mathbf{m}' :\Leftrightarrow \Psi(\mathbf{m}) < \Psi(\mathbf{m}') \text{ or } \Psi(\mathbf{m}) = \Psi(\mathbf{m}') \text{ and } \mathbf{m} >_{lex} \mathbf{m}'. \tag{3.2.2}$$

Definition 29 ([20]). Let $f = \sum a_{\mathbf{u}} x^{\mathbf{u}}$ with $\mathbf{u} \in \mathbb{Z}^d_{\geq 0}$ be a non-zero polynomial in $\mathbb{C}[x_1,\dots,x_d]$. The valuation $\mathfrak{v}_S : \mathbb{C}[x_1,\dots,x_d] \setminus \{0\} \to (\mathbb{Z}^d_{\geq 0}, \prec_\Psi)$ associated to S is defined as

$$\mathfrak{v}_S(f) := \min{}_{\prec_\Psi} \{\mathbf{u} \in \mathbb{Z}^{k(n-k)}_{\geq 0} \mid a_{\mathbf{u}} \neq 0\}. \tag{3.2.3}$$

We extend $\mathfrak{v}_S$ to a valuation on $\mathbb{C}(x_1,\dots,x_d) \setminus \{0\}$ by setting for $h = \frac{f}{g}$ a rational function $\mathfrak{v}_S(h) := \mathfrak{v}_S(f) - \mathfrak{v}_S(g)$.

Valuations of form (3.2.3) are usually called *lowest term valuations*. As S is a birational sequence, for every element in $\mathbb{C}(\mathrm{Gr}(k,\mathbb{C}^n))$ there exists a unique element $f \in \mathbb{C}(x_1,\dots,x_d)$ associated to it by the isomorphism $\psi : \mathbb{C}(\mathbb{A}^d) \to \mathbb{C}(\mathrm{Gr}(k,\mathbb{C}^n))$. Hence, we have a valuation on $\mathbb{C}(\mathrm{Gr}(k,\mathbb{C}^n)) \setminus \{0\}$. Further, as $\mathbb{C}[\mathrm{Gr}(k,\mathbb{C}^n)] \setminus \{0\} \subset \mathbb{C}(\mathrm{Gr}(k,\mathbb{C}^n)) \setminus \{0\}$ we can restrict to obtain

$$\mathfrak{v}_S : \mathbb{C}[\mathrm{Gr}(k,\mathbb{C}^n)] \setminus \{0\} \to (\mathbb{Z}^d_{\geq 0}, \prec_\Psi).$$

We denote as in §2.3 by $S(\mathbb{C}[\mathrm{Gr}(k,\mathbb{C}^n)], \mathfrak{v}_S)$ the associated value semi-group and the associated graded algebra by $\mathrm{gr}_S(\mathbb{C}[\mathrm{Gr}(k,\mathbb{C}^n)])$. For the images of Plücker coordinates $\bar{p}_J \in \mathbb{C}[\mathrm{Gr}(k,\mathbb{C}^n)]$ we chose as before the notation $\overline{p_J} \in \mathrm{gr}_S(\mathbb{C}[\mathrm{Gr}(k,\mathbb{C}^n)])$ for $J \in \binom{[n]}{k}$.

We are interested in toric degenerations of $\mathrm{Gr}(k,\mathbb{C}^n)$ from the above defined valuations using the Rees algebra construction (2.3.2). We would like to apply Theorem [44] stated in §2.3 and therefore need to show that the value semi-group $S(\mathbb{C}[\mathrm{Gr}(k,\mathbb{C}^n)], \mathfrak{v}_S)$ is finitely generated. The following representation theoretic point of view on the valuation $\mathfrak{v}_S$ from [20, §8 and §9] is useful to do so for $\mathrm{Gr}(2,\mathbb{C}^n)$ in §3.2.2 below.

A birational sequence $S = (\beta_1,\dots,\beta_d)$ for $\mathrm{Gr}(k,\mathbb{C}^n)$ together with the total order $\prec_\Psi$ on $\mathbb{Z}^d$ induces a filtration on the universal enveloping algebra $U(\mathfrak{n}_k^-)$ for $0 \neq \mathbf{m} \in \mathbb{Z}^d_{\geq 0}$ by

$$U(\mathfrak{n}_k^-)_{\preceq_\Psi \mathbf{m}} := \langle \mathbf{f}^{\mathbf{k}} = f_{\beta_1}^{k_1} \cdots f_{\beta_d}^{k_d} \mid \mathbf{k} \in \mathbb{Z}_{\geq 0}, \mathbf{k} \preceq_\Psi \mathbf{m} \rangle. \tag{3.2.4}$$

We define similarly $U(\mathfrak{n}_k^-)_{\prec_\Psi \mathbf{m}}$. Recall that the highest weight module $V(\lambda)$ for $\lambda = r\omega_k$ with $r \geq 1$ is cyclically generated by a highest weight vector $v_\lambda \in V(\lambda)$ over $U(\mathfrak{n}_k^-)$. We therefore have an induced filtration for $0 \neq \mathbf{m} \in \mathbb{Z}^d_{\geq 0}$ defined by

$$V(\lambda)_{\preceq_\Psi} := U(\mathfrak{n}_k^-)_{\preceq_\Psi \mathbf{m}} \cdot v_\lambda, \tag{3.2.5}$$

Similarly we define $V(\lambda)_{\prec_\Psi}$. Then $V^{\mathrm{gr}}(\lambda) := \bigoplus_{0 \neq \mathbf{m} \in \mathbb{Z}^d_{\geq 0}} V(\lambda)_{\preceq_\Psi} / V(\lambda)_{\prec_\Psi}$ is the associated graded vector space. This leads to the following definition of *essential sets* (see [20, Definition 7])

$$\mathsf{es}_S(\lambda) := \{\mathbf{m} \in \mathbb{Z}^d_{\geq 0} \mid V(\lambda)_{\preceq_\Psi} / V(\lambda)_{\prec_\Psi} \neq 0\}. \tag{3.2.6}$$

These sets are of particular importance as $\{\mathbf{f}^{\mathbf{m}} \cdot v_\lambda \mid \mathbf{m} \in \mathsf{es}_S(\lambda)\}$ forms a basis for $V(\lambda)$ and hence $\bigcup_{r\geq 1}\{\mathbf{f}^{\mathbf{m}} \cdot v_{r\omega_k} \mid \mathbf{m} \in \mathsf{es}_S(r\omega_k)\}$ a basis for $\mathbb{C}[\mathrm{Gr}(k,\mathbb{C}^n)]$.

The connection between the valuation $\mathfrak{v}_S$ introduced above and the essential sets is the following.

Proposition. ([20, Proposition 2]) For every birational sequence S for $\mathrm{Gr}(k,\mathbb{C}^n)$ we have $\bigcup_{r\geq 1}\mathsf{es}_S(r\omega_k) = S(A_{k,n},\mathfrak{v}_S)$.

The proposition (resp. its proof) implies that $\mathsf{es}_S(\omega_k) = \{\mathfrak{v}_S(\bar p_J) \mid J \in \binom{[n]}{k}\}$ by the counting argument

$$|\,\mathsf{es}_S(\omega_k)| = \dim_{\mathbb{C}} V(\omega_k) = \binom{n}{k} = \left|\left\{\mathfrak{v}_S(\bar p_J)\middle| J \in \binom{[n]}{k}\right\}\right|.$$

Consider for $r\geq 1$ the set $r\,\mathsf{es}_S(\lambda) := \{\sum_{j=1}^r \mathbf{m}_j \mid \mathbf{m}_j \in \mathsf{es}_S(\lambda)\forall j\}$. Then by construction we have

$$r\,\mathsf{es}_S(\omega_k) \subseteq \mathsf{es}_S(r\omega_k).$$

If equality holds for all $r\geq 1$ by [20, Proposition 2] the value semi-group $S(\mathbb{C}[\mathrm{Gr}(k,\mathbb{C}^n)],\mathfrak{v}_S)$ is generated by $\{\mathfrak{v}_S(\bar p_J) \mid J \in \binom{[n]}{k}\}$. Hence, the Plücker coordinates form a Khovanskii basis for $\mathfrak{v}_S$ and we can apply Theorem [44] to get a toric degenration of $\mathrm{Gr}(k,\mathbb{C}^n)$. Our aim is to show that this is the case when S is an iterated sequences for $\mathrm{Gr}(2,\mathbb{C}^n)$.

3.2.2 Iterated sequences for $\mathrm{Gr}(2,\mathbb{C}^n)$

In this subsection we prove Theorem 11 stated in the introduction. After proving Proposition 2 it follows from Theorem 10 stated in §2.4. We focus on iterated sequences for $\mathrm{Gr}(2,\mathbb{C}^n)$ and start by making the above definitions precise.

Let $S = (\beta_1,\dots,\beta_d)$ be a birational sequence for $\mathrm{Gr}(2,\mathbb{C}^n)$. With notation as in the previous subsection we have $I_k = I_2 = [n-1]\setminus 2$ and $\ell(w_2) = 2(n-2) = d$. For $\mathfrak{n}_2^- = \mathrm{Lie}(U_2^-)$, by [20, Lemma 2] $U(\mathfrak{n}_2^-)$ is generated by monomials of form $f_{\beta_1}^{m_1}\dots f_{\beta_d}^{m_d}$. We consider the irreducible highest weight representation $V(\omega_2) = \bigwedge^2\mathbb{C}^n$ of highest weight $\omega_2\in\Lambda^+$. It is cyclically generated over $U(\mathfrak{n}_2^-)$ by a highest weight vector v_{ω_2}, which we chose to be $e_1\wedge e_2$ as in Example 2. The Plücker coordinate $\bar p_{ij}$ is the dual basis vector to $e_i\wedge e_j$ for $1\leq i<j\leq n$ in $(\bigwedge^2\mathbb{C}^n)^*$. There exists at least one monomial of form $\mathbf{f}^{\mathbf{m}} = f_{\beta_1}^{m_1}\dots f_{\beta_d}^{m_d}$ with the property $\mathbf{f}^{\mathbf{m}}(e_1\wedge e_2) = e_i\wedge e_j$ for all $i,j\in[n]$. Then by [20, Proposition 2] we have

$$\mathfrak{v}_s(\bar p_{ij}) = \min{}_{\prec_\Psi}\{\mathbf{m}\in\mathbb{Z}^d_{\geq 0} \mid \mathbf{f}^{\mathbf{m}}(e_1\wedge e_2) = e_i\wedge e_j\}. \tag{3.2.7}$$

Example 9. Consider $\mathrm{Gr}(2,\mathbb{C}^4)$ with iterated sequences $S = (\alpha_{1,3},\alpha_{2,3},\alpha_{1,2},\alpha_2)$ and $S' = (\alpha_3,\alpha_{2,3},\alpha_{1,2},\alpha_2)$. They are birational by Lemma 6, as $(\alpha_{1,2},\alpha_2)$ is of PBW type for $\mathrm{Gr}(2,\mathbb{C}^3)$. We compute the valuation $\mathfrak{v}_S$ on Plücker coordinates. There are two monomials sending $e_1\wedge e_2$ to $e_3\wedge e_4$, namely

$$\mathbf{f}^{(1,0,0,1)}\cdot e_1\wedge e_2 = \mathbf{f}^{(0,1,1,0)}\cdot e_1\wedge e_2 = e_1\wedge e_4.$$

We have $\Psi(1,0,0,1) = \Psi(0,1,1,0) = 4$, but $(1,0,0,1) >_{lex} (0,1,1,0)$. Hence, $\mathfrak{v}_S(\bar p_{34}) = (1,0,0,1)$.

For $\mathfrak{v}_{S'}$ we compute $\mathbf{f}^{(1,0,0,1)}\cdots e_1\wedge e_2 = \mathbf{f}^{(0,1,0,0)}\cdots e_1\wedge e_2 = e_1\wedge e_4$ Again, we have $\Psi(1,0,0,1) = \Psi(0,1,0,0) = 2$, but as $(1,0,0,1) >_{lex} (0,1,0,0)$ it follows $\mathfrak{v}_{S'}(\bar p_{14}) = (1,0,0,1)$. In Table 3.1 you can find the images of all Plücker coordinates under $\mathfrak{v}_S$ and $\mathfrak{v}_{S'}$.

Plücker	$\mathfrak{v}_S$	$\mathfrak{v}_{S'}$
$\bar{p}_{12}$	$(0,0,0,0)$	$(0,0,0,0)$
$\bar{p}_{13}$	$(0,0,0,1)$	$(0,0,0,1)$
$\bar{p}_{23}$	$(0,0,1,0)$	$(0,0,1,0)$
$\bar{p}_{14}$	$(0,1,0,0)$	$(1,0,0,1)$
$\bar{p}_{24}$	$(1,0,0,0)$	$(1,0,1,0)$
$\bar{p}_{34}$	$(1,0,0,1)$	$(0,1,1,0)$

Table 3.1: Images of Plücker coordinates under the valuations $\mathfrak{v}_S, \mathfrak{v}_{S'}$ associated to $S = (\alpha_{1,3}, \alpha_{2,3}, \alpha_{1,2}, \alpha_2)$ and $S' = (\alpha_3, \alpha_{2,3}, \alpha_{1,2}, \alpha_2)$ for $\mathrm{Gr}(2,\mathbb{C}^4)$.

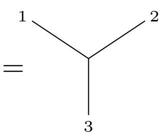
$T_3 =$ 

Figure 3.4: Labelled trivalent tree with three leaves.

From now on we consider an iterated birational sequence $S = ((i_n,n),(j_n,n),\dots,(i_3,3),(j_3,3))$ for $\mathrm{Gr}(2,\mathbb{C}^n)$, where (i_k,k) represents the positive root $\alpha_{i_k,k-1} = \epsilon_{i_k} - \epsilon_k$. We chose this notation as it easily encodes the action of $f_{\alpha_{i_k,k-1}} \in \mathfrak{n}^-$ on $\mathbb{C}^n$ (see Example 2), which we need to compute $\mathfrak{v}_S$ on Plücker coordinates. The following algorithm associates to S a trivalent tree T_S with n leaves labelled by $[n]$.

Algorithm 3: Associating a trivalent tree T_S with an iterated sequence S.

Input: An iterated birational sequence $S = ((i_n,n),(j_n,n),\dots,(i_3,3),(j_3,3))$, the trivalent tree T_3 as in Figure 3.4.

Initialization: Set $k=4$, $T_3^S := T_3$.

for k **do**

Construct a tree T_k^S from T_{k-1}^S by replacing the edge with leaf i_k in T_{k-1}^S by three edges forming a cherry with leaves labelled by i_k and k.

if $k{=}n$ **then**

Output: The tree T_n^S.

else

Replace k by $k+1$, T_{k-1}^S by T_k^S and start over.

Output: The tree $T_S := T_n^S$ and the sequence $\mathbb{T}_S := (T_n^S,\dots,T_3^S)$ of trees.

Definition 30. To an iterated sequence S we associate the trivalent tree T_S and the sequence of trees $\mathbb{T}_S = (T_n^S,\dots,T_3^S)$ that are the output of Algorithm 3. Denote by C_S the maximal cone in trop(Gr$(2,\mathbb{C}^n)$) corresponding to the tree T_S by [64, Theorem 3.4] restated in §3.1.1.

Example 10. Consider $S = ((4,6),(5,6),(2,5),(3,5),(2,4),(3,4),(1,3),(2,3))$, an iterated sequence for $\mathrm{Gr}(2,\mathbb{C}^6)$. We construct the trees $\mathbb{T}_S = (T_3^S, T_4^S, T_5^S, T_6^S)$ by Algorithm 3. Figure 3.5 shows the obtained sequence of trees.

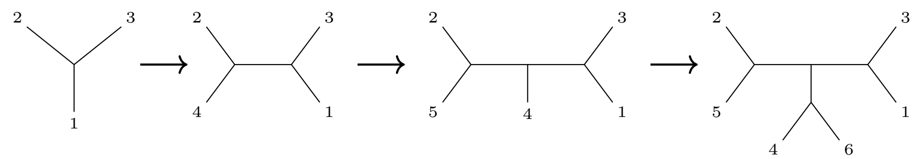


Figure 3.5: The sequence $\mathbb{T}_S$ for S as in Example 10.

Definition 31. We define the *weighting matrix* $M_S \in \mathbb{Z}^{d\times\binom{n}{2}}$ associated to S as the matrix whose columns are $\mathfrak{v}_S(\bar p_{ij})$ for $\{i,j\} \in \binom{[n]}{2}$.

Following [45, §3.1] we want to compute $\mathrm{in}_{M_S}(I_{2,n})$ to apply Theorem 10. Recall the Definition 12 from §2.4.

Proposition 2. For every interated sequence S we have $\mathrm{in}_{M_S}(I_{2,n}) = \mathrm{in}_{C_S}(I_{2,n})$.

Proof. Recall that $\mathrm{in}_{C_S}(I_{2,n}) = \langle \mathrm{in}_{C_S}(R_{i,j,k,l}) \mid i,j,k,l \in [n]\rangle$ by [64, Proof of Theorem 3.4]. This implies that it is enough to prove the following claim.

Claim: For every Plücker relation $R_{i,j,k,l}$ with $i,j,k,l \in [n]$ we have $\mathrm{in}_{C_S}(R_{i,j,k,l}) = \mathrm{in}_{M_S}(R_{i,j,k,l})$.

Let $\{e_{ij}\}_{\{i,j\}\in\binom{[n]}{2}}$ be the stansard basis for $\mathbb{R}^{\binom{n}{2}}$. Adopting the notation for monomials in the polynomial ring $\mathbb{C}[p_{ij}]_{ij}$ we have

$$R_{i,j,k,l} = p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk} = \mathbf{p}^{e_{ij}+e_{kl}} - \mathbf{p}^{e_{ik}+e_{jl}} + \mathbf{p}^{e_{il}+e_{jk}}.$$

In particular, $M_S(e_{ij}+e_{kl}) = \mathfrak{v}_S(\bar p_{ij}) + \mathfrak{v}_S(\bar p_{kl}) = \mathfrak{v}_S(\bar p_{ij}\bar p_{kl})$ and $\mathrm{in}_{M_S}(R_{i,j,k,l})$ is the sum of those monomials in $R_{i,j,k,l}$ for which the valuation $\mathfrak{v}_S$ of the corresponding monomials in $A_{2,n}$ is minimal with respect to $\prec_\Psi$.

Proof of claim: We proceed by induction. For $n=4$ let $S = ((i,4),(j,4),(i_3,3),(j_3,3))$, i.e. the tree T_S has a cherry labelled by i and 4. Consider the Plücker relation $R_{i,j,k,4} = p_{ij}p_{k4} - p_{ik}p_{j4} + p_{i4}p_{jk}$ with $\{i,j,k\} = [3]$. Then

$$\mathrm{in}_{C_S}(R_{i,j,k,4}) = p_{ij}p_{k4} - p_{ik}p_{j4}.$$

Let $S' = ((i_3,3),(j_3,3))$ be the sequence for $\mathrm{Gr}(2,\mathbb{C}^3)$ and denote by $\hat p_{rs}$ with $r,s \in [3]$ the Plücker coordinates in A_3. For $\mathbf{m} \in \mathbb{Z}^{d-2}$ and $m_d, m_{d-1} \in \mathbb{Z}$ write $(m_d, m_{d-1}, \mathbf{m}) := (m_d, m_{d-1}, m_{d-2}, \dots, m_1)$ We compute

$$\mathfrak{v}_S(\bar p_{i4}) = (0,1,\mathfrak{v}_{S'}(\hat p_{ij})), \quad \mathfrak{v}_S(\bar p_{j4}) = (1,0,\mathfrak{v}_{S'}(\hat p_{ij})), \quad \text{and} \quad \mathfrak{v}_S(\bar p_{k4}) = (1,0,\mathfrak{v}_{S'}(\hat p_{ik})).$$

This implies $\mathfrak{v}_S(\bar p_{i4}\bar p_{jk}) \succ_\Psi \mathfrak{v}_S(\bar p_{ij}\bar p_{k4}) = \mathfrak{v}_S(p_{ik}p_{j4})$, and hence $\mathrm{in}_{M_S}(R_{i,j,k,4}) = \mathrm{in}_{C_S}(R_{i,j,k,4})$.

Assume the claim is true for $n-1$ and let $S = ((i_n,n),(j_n,n),\dots,(i_3,3),(j_3,3))$ be an iterated sequence for $\mathrm{Gr}(2,\mathbb{C}^n)$. Then $S' = ((i_{n-1},n-1),(j_{n-1},n-1),\dots,(i_3,3),(j_3,3))$ is an iterated sequence for $\mathrm{Gr}(2,\mathbb{C}^{n-1})$. Denote by $\hat p_{ij}$ with $i,j \in [n-1]$ the Plücker coordinates in A_{n-1}. As $\mathfrak{v}_S(\bar{p}_{ij}) = (0,0,\mathfrak{v}_{S'}(\hat{p}_{ij}))$ for $i,j<n$ we deduce $\operatorname{in}_{C_S}(R_{i,j,k,l}) = \operatorname{in}_{M_S}(R_{i,j,k,l})$ with $i,j,k,l<n$ by induction. Consider the Plücker relation $R_{i,j,k,n}$. Then

$$\mathfrak{v}_S(\bar{p}_{rn}) = \begin{cases} (1,0,\mathfrak{v}_{S'}(\hat{p}_{ri_n})), & \text{if } r \neq i_n \\ (0,1,\mathfrak{v}_{S'}(\hat{p}_{i_nj_n})), & \text{if } r = i_n. \end{cases}$$

As (i_n,n) is a cherry in T_S we observe that the associated weight vector $\mathbf{w}_{T_S} \in C_S^\circ \subset \operatorname{trop}(\operatorname{Gr}(2,\mathbb{C}^n))$ satisfies $(\mathbf{w}_{T_S})_{rn} = (\mathbf{w}_{T_S})_{ri_n} = (\mathbf{w}_{T_{S'}})_{ri_n} - 1$. In particular, for $i,j,k\neq i_n$ we deduce by induction $\operatorname{in}_{M_S}(R_{i,j,k,n}) = \operatorname{in}_{C_S}(R_{i,j,k,n})$. The only relations left to consider are of form $R_{i_n,j,k,n}$ for $j,k\in[n-1]\setminus\{i_n\}$. For M_S we compute by the above

$$\mathfrak{v}_S(\bar{p}_{i_nj}\bar{p}_{kn}) = \mathfrak{v}_S(\bar{p}_{i_nk}\bar{p}_{jn}) \succ_\Psi \mathfrak{v}_S(\bar{p}_{i_nn}\bar{p}_{jk}).$$

Hence, $\operatorname{in}_{M_S}(R_{i_n,j,k,n}) = p_{i_nj}p_{kn} - p_{i_nk}p_{jn}$. As (i_n,n) is a cherry in T_S we obtain $\operatorname{in}_{C_S}(R_{i_n,j,k,n}) = \operatorname{in}_{M_S}(R_{i_n,j,k,n})$. □

As $\operatorname{in}_{C_S}(I_{2,n})$ is prime, Proposition 2 allows us to apply Theorem 10 from §2.4. Let us have a look at the other necessary assumptions before we formulate the statements from §2.4 in the context of valuations from iterated sequences for $\operatorname{Gr}(2,\mathbb{C}^n)$ below. For completeness we also include the proofs in this case, although this would be not necessary given that we can apply the general theorem. They just serve as an example to obtain a better understanding of the general theory.

We consider the algebra $A_{2,n}$, the homogeneous coordinate ring of $\operatorname{Gr}(2,\mathbb{C}^n)$. We have fixed the Plücker embedding, that yields a presentation $\pi:\mathbb{C}[p_{ij}]_{ij}\to A_{2,n}$ with $A_{2,n} = \mathbb{C}[p_{ij}]_{ij}/\ker(\pi)$. More precisely, $\ker(\pi) = I_{2,n}$ is the Plücker ideal. The candidate for a Khovanksii basis is therefore $\{\pi(p_{ij}) = \bar{p}_{ij}\}_{ij}\subset A_{2,n}$. As $\mathbb{C}[p_{ij}]_{ij}$ is postivily graded by $\mathbb{Z}_{\geq 0}$ (we have $\deg p_{ij} = 1$) and $I_{2,n}$ is homogeneous with respect to this grading generated by Plücker relations of degree 2, by Lemma 3 we have $\mathfrak{v}_S(\bar{p}_{ij}) = \mathfrak{v}_{M_S}(\bar{p}_{ij})$. Here $\mathfrak{v}_S: A_{2,n}\setminus\{0\}\to\mathbb{Z}^d$ is the valuation induced by the iterated sequence S and $\mathfrak{v}_{M_S}: A_{2,n}\setminus\{0\}\to\mathbb{Z}^d$ the (quasi-)valuation defined by the weighting matrix M_S of $\mathfrak{v}_S$ as above (see Definition 13).

Inspired by the proof of [45, Proposition 5.2] we obtain the next proposition. The proof is analogous to the one of Corollary 1 in §2.4.

Proposition 3. For every iterated sequence S the associated quasi-valuation $\mathfrak{v}_{M_S}$ with weighting matrix M_S satisfies $\operatorname{gr}_{M_S}(A_{2,n}) \cong \mathbb{C}[p_{ij}]_{ij}/\operatorname{in}_{C_S}(I_{2,n})$. Moreover, $\mathfrak{v}_{M_S}$ is a valuation with value semi-group $S(A_{2,n},\mathfrak{v}_{M_S})$ generated by $\mathfrak{v}_{M_S}(\bar{p}_{ij})$ for $1\leq i<j\leq n$.

Proof. As $I_{2,n}$ is homogeneous with respect to a positive grading, we have $M_S\in\operatorname{GR}^d(I_{2,n})$ and hence, can apply [45, Lemma 4.4] to get $\operatorname{gr}_{M_S}(A_{2,n})\cong\mathbb{C}[p_{ij}]_{ij}/\operatorname{in}_{M_S}(I_{2,n})$. Then the first part of the claim follows from Proposition 2.

As $\operatorname{in}_{C_S}(I_{2,n})$ is prime, $\operatorname{gr}_{M_S}(A_{2,n})\cong\mathbb{C}[p_{ij}]_{ij}/\operatorname{in}_{C_S}(I_{2,n})$ is a domain. The rest of the proof is exactly the same as the proof of Corollary 1 in §2.4. □

Corollary 3. For every iterated sequence S the Newton-Okounkov body associated with the weight valuation $\mathfrak{v}_{M_S}$ is given by

$$\Delta(A_{2,n},\mathfrak{v}_{M_S}) = \operatorname{conv}(\mathfrak{v}_S(\bar{p}_{ij}) \mid 1\leq i<j\leq n).$$

Proof. By Proposition 3 we have $\Delta(A_{2,n}, \mathfrak{v}_{M_S}) = \mathrm{conv}(\mathfrak{v}_{M_S}(\bar{p}_{ij}) \mid 1 \le i < j \le n)$. Therefore it remains to show $\mathfrak{v}_{M_S}(\bar{p}_{ij})) = \mathfrak{v}_S(\bar{p}_{ij})$ for all $i, j \in [n]$. This follows exactly by the argument in the proof of Lemma 3. □

Theorem 11. For every iterated sequence S we have $\mathrm{gr}_S(A_{2,n}) \cong \mathbb{C}[p_{ij}]_{ij}/\,\mathrm{in}_{C_S}(I_{2,n})$. Moreover, for every maximal prime cone C of $\mathrm{trop}(\mathrm{Gr}(2, \mathbb{C}^n))$ there exists a birational sequence S, such that $\mathbb{C}[p_{ij}]_{ij}/\,\mathrm{in}_C(I_{2,n}) \cong \mathrm{gr}_S(A_{2,n})$.

Proof. The first part of the claim follows from Theorem 10. We give an alternative proof here, using the essential sets to illustrate another point of view on the general theorem.

We show that $S(A_{2,n}, \mathfrak{v}_S)$ is generated by $\mathfrak{v}_S(p_{ij})$ for $1 \le i < j \le n$. This implies $S(A_{2,n}, \mathfrak{v}_S) = S(A_{2,n}, \mathfrak{v}_{M_S})$. As $\mathfrak{v}_S$ and $\mathfrak{v}_{M_S}$ are full rank and hence have one-dimensional leaves by [11, Remark 4.13] (see also §2.3) we have $\mathrm{gr}_S(A_{2,n}) \cong \mathbb{C}[S(A_{2,n}, \mathfrak{v}_S)]$ and $\mathrm{gr}_{M_s}(A_{2,n}) \cong \mathbb{C}[S(A_{2,n}, \mathfrak{v}_{M_S})]$. Therefore, $\mathrm{gr}_{M_s}(A_{2,n}) \cong \mathrm{gr}_{M_S}(A_{2,n})$. Then the first claim follows by Proposition 3. In order to do so, we use [20, Proposition 2] restated above and show $\mathsf{es}_S(k\omega_2) = k\,\mathsf{es}_S(\omega_2)$ for all $k \ge 1$.

As $\Delta(A_{2,n}, \mathfrak{v}_{M_S})$ is integral by Corollary 3, all lattice points in the kth dilation $k\Delta(A_{2,n}, \mathfrak{v}_{M_S})$ are sums of k lattice points in $\Delta(A_{2,n}, \mathfrak{v}_{M_S})$. We have $\mathsf{es}_S(\omega_2) = \{\mathfrak{v}_S(\bar{p}_{ij}) \mid 1 \le i < j \le n\}$ by [20, Proposition 2], which are the lattice points in $\Delta(A_{2,n}, \mathfrak{v}_{M_S})$. Then by Corollary 3 for $f \in A_{2,n}$ with $\deg f = k \ge 1$ there exist $\mathbf{m}_1, \dots, \mathbf{m}_k \in \mathsf{es}_S(\omega_2)$ such that $\mathfrak{v}_{M_S}(f) = \sum_{j=1}^k \mathbf{m}_j$. In particular, $\mathfrak{v}_{M_S}(f) \in k\,\mathsf{es}_S(\omega_2)$. Hence, we count

$$|k\,\mathsf{es}_S(\omega_2)| = \dim_{\mathbb{C}} V(k\omega_2) = |\,\mathsf{es}_S(k\omega_2)|$$

and the claim follows.

For the second part, note that by Algorithm 3 for every shape of tree we can find an iterated sequence, such that the output has the desired shape (ignoring the labelling for now). Therefore, for a given maximal prime cone $C \subset \mathrm{trop}(\mathrm{Gr}(2, \mathbb{C}^n))$ consider the corresponding tree T_C and its shape $\mathtt{T}_C$. Then find S with T_S of shape $\mathtt{T}_C$ (see also Corollary 4 below). The action of the symmetric group induces an isomorphism $\mathbb{C}[p_{ij}]_{ij}/\,\mathrm{in}_{T_S}(I_{2,n}) \cong \mathbb{C}[p_{ij}]_{ij}/\,\mathrm{in}_{T_C}(I_{2,n})$. The rest follows then by the first part. □

Remark 5. Note that the essential basis for $A_{2,n}$ (see [20, Remark 5]) induced by $\bigcup_{k\ge1} \mathsf{es}(k\omega_2)$ is an adapted basis for the valuation $\mathfrak{v}_S$ and therefore also for $\mathfrak{v}_{M_S}$. Having the notion of essential sets in this context allowed us to use this (more concrete) basis instead of the (more abstract) standard monomial basis for $\mathfrak{v}_{M_S}$ (that exists as M_S lies in the Gröbner region) used in the proof of Theorem 10.

For an iterated sequence S for $\mathrm{Gr}(2, \mathbb{C}^n)$ denote by $\mathtt{T}_i^S$ the (non-labelled) trivalent tree underlying the labelled trivalent tree T_i^S with i leaves in the tree sequence $\mathbb{T}_S$. The Algorithm 3 provides a tool for comparing whether two iterated sequences induce isomorphic flat toric degenerations. Construct $\mathbb{T}_{S_1}, \mathbb{T}_{S_2}$ for two such sequences S_1, S_2 and consider $\mathtt{T}_n^{S_1}$ and $\mathtt{T}_n^{S_2}$. If $\mathtt{T}_n^{S_1}$ and $\mathtt{T}_n^{S_2}$ coincide then

$$\mathrm{in}_{T_n^{S_1}}(I_{2,n}) \cong \mathrm{in}_{T_n^{S_2}}(I_{2,n}).$$

The following definition allows us to interpret iterated sequences for $\mathrm{Gr}(2, \mathbb{C}^n)$ in a combinatorial way in Corollary 4 below.

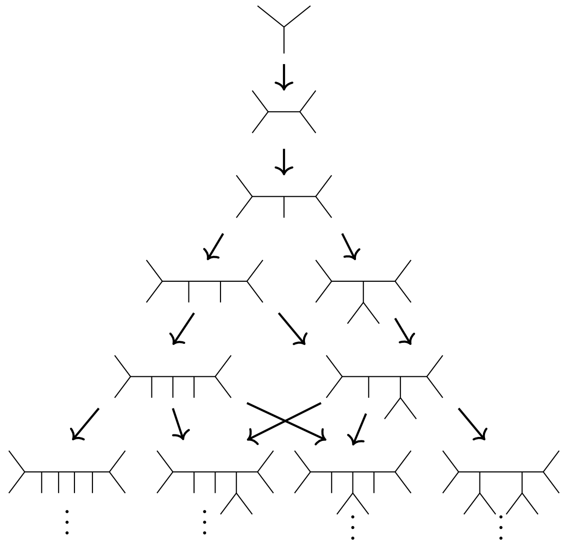


Figure 3.6: The tree graph $\mathcal{T}$ from level (#of leaves) 3 to 8.

Definition 32. The *tree graph* $\mathcal{T}$ is an infinite graph whose vertices at level $i \geq 3$ correspond to trivalent trees with i leaves. There is an arrow $\mathtt{T} \to \mathtt{T}'$, if $\mathtt{T}$ has i leaves, $\mathtt{T}'$ has $i+1$ leaves and $\mathtt{T}'$ can be obtained from $\mathtt{T}$ by attaching a new boundary edge in the middle of some edge of $\mathtt{T}$. There is a unique source $\mathtt{T}_3$ at level 3. See Figure 3.6.

Corollary 4. Every iterated sequence S for $\mathrm{Gr}(2, \mathbb{C}^n)$ corresponds to a path from $\mathtt{T}_3$ to $\mathtt{T}_n^S$ in the tree graph $\mathcal{T}$.

Proof. The underlying unlabelled trees in the sequence $\mathbb{T}_S = (T_3, T_4^S, \dots, T_n^S)$ associated to S define the path $\mathtt{T}_3 \to \mathtt{T}_4^S \to \cdots \to \mathtt{T}_n^S$ in $\mathcal{T}$. $\square$

3.3 Toric degenerations via plabic graphs

In this section we apply Theorem 10 from §2.4 to the valuation defined by Rietsch-Williams in [62] on $\mathbb{C}[\mathrm{Gr}(k,\mathbb{C}^n)]$ using the cluster structer. Theorem 12 specifies when their toric degeneration can be realized as a Gröbner toric degeneration having a Khovanskii basis in terms of Plücker coordinates. Further, Corollary 6 establishes a connection between the integrality of their associated Newton-Okounkov body and the weighting matrix of the valuation (as in Definition 14).

Moreover, we show that the weight vector defined for plabic graphs in joint work with Fang, Fourier, Hering, and Lanini in [8] is closely realted to the weighting matrix. The subsection §3.3.3 is based on this joint work, where we establish an explicit bijection between the toric degenerations of the Grassmannian $\mathrm{Gr}(2,\mathbb{C}^n)$ arising from maximal cones in tropical Grassmannians and the ones coming from plabic graphs corresponding to $\mathrm{Gr}(2,\mathbb{C}^n)$.

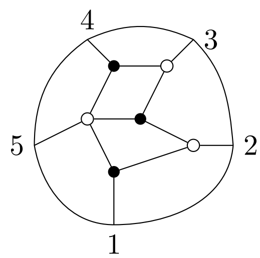


Figure 3.7: A plabic graph.

3.3.1 Plabic graphs

We review the definition of plabic graphs due to Postnikov [60]. This section is closely oriented towards Rietsch and Williams [62].

Definition 33. A *plabic graph* $\mathcal{G}$ is a planar bicolored graph embedded in a disk. It has n boundary vertices numbered $1,\dots,n$ in a counterclockwise order. Boundary vertices lie on the boundary of the disk and are not colored. Additionally there are internal vertices colored black or white. Each boundary vertex is adjacent to a single internal vertex.

For our purposes we assume that plabic graphs are connected and that every leaf of a plabic graph is a boundary vertex. We first recall the four local moves on plabic graphs.

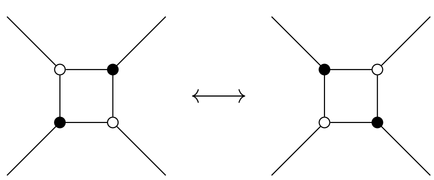


Figure 3.8: Square move (M1)

(M1) If a plabic graph contains a square of four internal vertices with alternating colors, each of which is trivalent, then the colors can be swapped. So every black vertex in the square becomes white and every white vertex becomes black (see Figure 3.3.1).

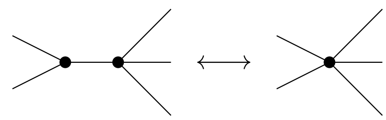


Figure 3.9: Merge vertices of same color (M2)

(M2) If two internal vertices of the same color are connected by an edge, the edge can be contracted and the two vertices can be merged. Conversely, any internal black or white vertex can be split into two adjacent vertices of the same color (see Figure 3.3.1).

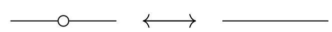


Figure 3.10: Insert/remove degree two vertex (M3)

(M3) If a plabic graph contains an internal vertex of degree 2, it can be removed. Equivalently, an internal black or white vertex can be inserted in the middle of any edge (see Figure 3.3.1).

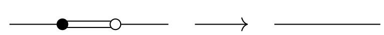


Figure 3.11: Reducing parallel edges (R)

(R) If two internal vertices of opposite color are connected by two parallel edges, they can be reduced to only one edge. This can not be done conversely (see Figure 3.3.1).

The *equivalence class* of a plabic graph $\mathcal{G}$ is defined as the set of all plabic graphs that can be obtained from $\mathcal{G}$ by applying (M1)-(M3). If in the equivalence class there is no graph to which (R) can be applied, we say $\mathcal{G}$ is *reduced*. From now on we only consider reduced plabic graphs.

Definition 34. Let $\mathcal{G}$ be a reduced plabic graph with boundary vertices $v_1, \dots, v_n$ labelled in a counterclockwise order. We define the *trip permutation* $\pi_{\mathcal{G}}$ as follows. We start at a boundary vertex v_i and form a path along the edges of $\mathcal{G}$ by turning maximally right at an internal black vertex and maximally left at an internal white vertex. We end up at a boundary vertex $v_{\pi(i)}$ and define $\pi_{\mathcal{G}} = [\pi(1), \dots, \pi(n)] \in S_n$.

It is a fact that plabic graphs in one equivalence class have the same trip permutation. Further, it was proven by Postnikov in [60, Theorem 13.4] that plabic graphs with the same trip permutation are connected by moves (M1)-(M3) and are therefore equivalent. Let $\pi_{k,n} = (n-k+1, n-k+2, \dots, n, 1, 2, \dots, n-k)$. From now on we focus on plabic graphs $\mathcal{G}$ with trip permutation $\pi_{\mathcal{G}} = \pi_{k,n}$. Each path v_i to $v_{\pi_{k,n}(i)}$ defined above, divides the disk into two regions. We label every face in the region to the left of the path by i. After repeating this for every $1 \leq i \leq n$, all faces have a labelling by an $(n-k)$-element subset of $[n]$. We denote by $\mathcal{P}_{\mathcal{G}}$ the set of all such subsets for a fixed plabic graph $\mathcal{G}$.

A face of a plabic graph is called *internal*, if it does not intersect with the boundary of the disk. Other faces are called *boundary faces*. Following [62] we define an orientation on a plabic graph. This is the first step in establishing the *flow model* introduced by Postnikov, which we use to define plabic degrees on the Plücker coordinates.

Definition 35. An orientation $\mathcal{O}$ of a plabic graph $\mathcal{G}$ is called *perfect*, if every internal white vertex has exactly one incoming arrow and every internal black vertex has exactly one outgoing arrow. The set of boundary vertices that are sources is called the *source set* and is denoted by $I_{\mathcal{O}}$.

Postnikov showed in [60] that every reduced plabic graph with trip permutation $\pi_{k,n}$ has a perfect orientation with source set of order k. See Figure 3.12 for a plabic graph with trip permutation $\pi_{2,5}$.

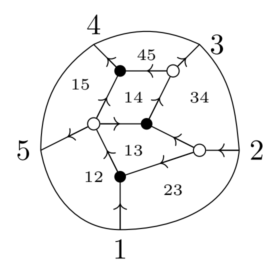


Figure 3.12: A plabic graph with a perfect orientation and source set $\{1, 2\}$.

Index the standard basis of $\mathbb{Z}^{d+1}$ by the faces of the plabic graph $\mathcal{G}$, where $d = k(n-k)$. Given a perfect orientation $\mathcal{O}$ on $\mathcal{G}$, every directed path $\mathbf{p}$ from a boundary vertex in the source set to a boundary vertex that is a sink, divides the disk in two parts. The *weight* $\mathrm{wt}(\mathbf{p}) \in \mathbb{Z}^{d+1}_{\geq 0}$ has entry 1 in the position corresponding to a face F of $\mathcal{G}$, if F is to the left of $\mathbf{p}$ with respect to the orientation. The *degree* $\deg_{\mathcal{G}}(\mathbf{p})$ is defined the number of internal faces to the left of the path.

For a set of boundary vertices J with $|J| = |I_{\mathcal{O}}|$, we define a *J-flow* as a collection of self-avoiding, vertex disjoint directed paths with sources $I_{\mathcal{O}} - (J \cap I_{\mathcal{O}})$ and sinks $J - (J \cap I_{\mathcal{O}})$. Let $I_{\mathcal{O}} - (J \cap I_{\mathcal{O}}) = \{j_1, \dots, j_r\}$ and $\mathbf{f} = \{\mathbf{p}_{j_1}, \dots, \mathbf{p}_{j_r}\}$ be a flow where each path $\mathbf{p}_{j_i}$ has sink j_i. Then the *weight of the flow* is $\mathrm{wt}(\mathbf{f}) := \mathrm{wt}(\mathbf{p}_{j_1}) + \cdots + \mathrm{wt}(\mathbf{p}_{j_r})$. Similarly, we define the *degree of the flow* as $\deg_{\mathcal{G}}(\mathbf{f}) = \deg_{\mathcal{G}}(\mathbf{p}_{j_1}) + \cdots + \deg_{\mathcal{G}}(\mathbf{p}_{j_r})$. By $\mathcal{F}_J$ we denote the set of all J-flows in $\mathcal{G}$ with respect to $\mathcal{O}$.

Valuation and plabic degree

In [62] Rietsch-Williams use the cluster structure on $\mathrm{Gr}(k,\mathbb{C}^n)$ (due to Scott, see [63]) to define a valuation on $\mathbb{C}(\mathrm{Gr}(k,\mathbb{C}^n))\setminus\{0\}$ for every seed. In fact, a plabic graph $\mathcal{G}$ defines a seed in the corresponding cluster algebra. A combinatorial algorithm associated a quiver with $\mathcal{G}$ (see e.g. [62, Definition 3.8]). The corresponding $\mathcal{A}$-cluster is given in terms of Plücker coordinates p_J, where J is a face label in $\mathcal{G}$ as described above.

As we are only interested in the values on Plücker coordinates of this valuation, we do not recall it in detail hear but refer the reader to [62, §7]. The main idea to construct the valuation is to use $\mathcal{X}$-cluster variables as coordinates for $\mathbb{C}[\mathrm{Gr}(k,\mathbb{C}^n)]$ and send a Laurent polynomial in those to its lexicographically minimal term. This is another instance of a lowest term valuation that we have encountered already in §3.2 in the context of birational sequences.

Let $A_{k,n}:=\mathbb{C}[\mathrm{Gr}(k,n)]$, then the Rietsch-Williams valuation in [62, Definition 7.1] can be restricted to a valuation $\mathfrak{v}_{\mathcal{G}}:A_{k,n}\setminus\{0\}\to(\mathbb{Z}^{d+1},\prec)$. The total order $\prec$ on $\mathbb{Z}^{d+1}$ is the lexicographic order with respect to a fixed order on the coordinates (see [62, Definition 7.1]). For $J\in\binom{[n]}{k}$ let $\mathbf{f}_J\in\mathcal{F}_J$ be such that $\deg_{\mathcal{G}}(\mathbf{f}_J)=\min\{\deg_{\mathcal{G}}(\mathbf{f})\mid \mathbf{f}\in\mathcal{F}_J\}$. Then on a Plücker coordinate $\bar p_J\in A_{k,n}$ the valuation $\mathfrak{v}_{\mathcal{G}}$ is given by

$$\mathfrak{v}_{\mathcal{G}}(\bar p_J)=\mathrm{wt}(\mathbf{f}_J).$$

Remark 6. This is for us, given the notion of degree, the most convenient way to write it and in fact equivalent to how it is described in [62].

We define closely related to the valuation the following notion of degree for Plücker variables in $\mathbb{C}[p_J]_J$ and associate a weight vector in $\mathbb{R}^{\binom{n}{k}}$.

Definition 36. For $J\in\binom{[n]}{k}$ and a plabic graph $\mathcal{G}$, the *plabic degree* of the Plücker variable p_J is defined as

$$\deg_{\mathcal{G}}(p_J)=\min\{\deg_{\mathcal{G}}(\mathbf{f})\mid \mathbf{f}\in\mathcal{F}_J\}.$$

It gives rise to a weight vector $\mathbf{w}_{\mathcal{G}}\in\mathbb{R}^{\binom{n}{k}}$ by setting $(\mathbf{w}_{\mathcal{G}})_J=\deg_{\mathcal{G}}(J)$.

By [61, Lemma 3.2] and its proof, the plabic degree is independent of the choice of the perfect orientation. We therefore fix the perfect orientation by choosing the source set $I_{\mathcal{O}}=[k]$. The following proposition guarantees that the degree (and the valuation) are well-defined. It is a reformulation of the original statement adapted to our notion degree.

Proposition. ([62, Corollary 11.4]) There is a unique J-flow in $\mathcal{G}$ with respect to $\mathcal{O}$ with degree equal to $\deg_{\mathcal{G}}(p_J)$.

Example 11. Consider the plabic graph $\mathcal{G}$ with perfect orientation from Figure 3.12 and source set is $I_{\mathcal{O}}=[2]$. We compute $\deg_{\mathcal{G}}(p_J)$ and $\mathfrak{v}_{\mathcal{G}}(\bar p_J)$ for all $J\in\binom{[5]}{2}$.

Order the faces of $\mathcal{G}$ by

$$F_{23},F_{34},F_{45},F_{15},F_{12},F_{13},F_{14}.$$

For example, consider $J=\{1,4\}$. There are two flows, $\mathbf{f}_1$ and $\mathbf{f}_2$ from $I_{\mathcal{O}}$ to $J=\{1,4\}$. Both consist of only one path from 2 to 4. One of them, say $\mathbf{f}_1$, has faces labelled by $\{2,3\}$, $\{1,5\}$

p_J	F_{23}	F_{34}	F_{45}	F_{15}	F_{12}	F_{13}	F_{14}	$\deg_{\mathcal{G}}(p_J)$	$e(M_{\mathcal{G}})$
p_{12}	0	0	0	0	0	0	0	0	0
p_{13}	1	0	1	1	1	0	0	0	13
p_{14}	1	0	0	1	1	0	0	0	10
p_{15}	1	0	0	0	1	0	0	0	6
p_{23}	0	0	1	1	1	0	0	0	12
p_{24}	0	0	0	1	1	0	0	0	9
p_{25}	0	0	0	0	1	0	0	0	5
p_{34}	1	0	1	2	2	1	1	2	22
p_{35}	1	0	1	1	2	1	1	2	18
p_{45}	1	0	0	1	2	1	1	2	15

Table 3.2: The valuation $\mathfrak{v}_{\mathcal{G}}$ for $\mathcal{G}$ as in Figure 3.12 on Plücker coordinates, the plabic degrees and an example for a weight vector $e(M_{\mathcal{G}})$ as in Proposition 4 (the multiplicities in the proof of Proposiiton 4 are chosen as $r_i = i$ and $q = 1$, the columns are ordered as below).

and $\{1,2\}$ to the left, and $\mathbf{f}_2$ has faces $\{2,3\}$, $\{1,5\}$, $\{1,2\}$ and $\{1,3\}$ to the left. Then with respect to the order of coordinates on $\mathbb{Z}^7$ we have

$$\mathrm{wt}(\mathbf{f}_1) = (1,0,0,1,1,0,0) \quad \text{and} \quad \mathrm{wt}(\mathbf{f}_2) = (1,0,0,1,1,1,0).$$

As $\deg_{\mathcal{G}}(\mathbf{f}_1) = 0$ and $\deg_{\mathcal{G}}(\mathbf{f}_2) = 1$, we have $\mathfrak{v}_{\mathcal{G}}(\bar{p}_{14}) = (1,0,0,1,1,0,0)$ and $\deg_{\mathcal{G}}(p_{14}) = 0$. All other $\mathfrak{v}_{\mathcal{G}}(\bar{p}_J)$ and $\deg_{\mathcal{G}}(p_J)$ can be found in Table 3.2.

3.3.2 The valuation $\mathfrak{v}_{\mathcal{G}}$ and the weighting matrix $M_{\mathcal{G}}$

In this section we apply Theorem 10 from §2.4 to the valuation $\mathfrak{v}_{\mathcal{G}}$ by Rietsch-Williams as seen in the last section. We show that the weight vector defined by the plabic degree on Plücker coordinates is closely related to the valuation: in fact, taking the initial ideal of the Plücker ideal with respect to the weighting matrix of $\mathfrak{v}_{\mathcal{G}}$ coincides with the initial ideal with respect to the plabic weight vector (see Proposition 4). In particular, we obtain that the associated graded algebra for $\mathfrak{v}_{\mathcal{G}}$ is the quotient of the polynomial ring in Plücker coordinates by the initial ideal, given it is prime. Moreover, in Corollary 6 we relate the property of the initial ideal being prime to integrality of the Newton-Okounkov body $\Delta(A_{k,n}, \mathfrak{v}_{\mathcal{G}})$ studied in [62, §8]. We exhibit the case of $\mathrm{Gr}(3,\mathbb{C}^6)$ in detail below and dedicate §3.3.3 to analyzing the case of $\mathrm{Gr}(2,\mathbb{C}^n)$.

Let $\mathcal{G}$ be a plabic graph for $\mathrm{Gr}(k,\mathbb{C}^n)$ with perfect orientation chosen such that $[k]$ is the source set as above. Consider the weighting matrix $M_{\mathcal{G}} := M_{\mathfrak{v}_{\mathcal{G}}}$ of $\mathfrak{v}_{\mathcal{G}}$ as in Definition 14. That is, the columns of $M_{\mathcal{G}}$ are $\mathfrak{v}_{\mathcal{G}}(p_J)$ for $J \in \binom{[n]}{k}$ and the rows $M_1,\dots,M_{d+1}$ are indexed by the faces of the plabic graph $\mathcal{G}$, where $d = k(n-k)$. Denote the boundary faces of $\mathcal{G}$ by $F_1,\dots,F_n$, where F_i is adjacent to the boundary vertices i and $i+1$. Hence, $F_k = F_{\varnothing}$ and $M_k = (0,\dots,0)$. Order the rows of $M_{\mathcal{G}}$ such that M_i is the row corresponding to the face F_i in $\mathcal{G}$.

Lemma 7. Let $r \in [n]$ and $J = \{j_1, \dots, j_k\} \in \binom{[n]}{k}$ with $j_1 < \dots j_s \leq k < j_{s+1} < \dots < j_k$. Set $[k] \setminus \{j_1, \dots, j_s\} = \{i_1, \dots, i_{k-s}\}$ with $i_1 < \dots < i_{k-s}$. Then

$$(M_r)_J = \#\{l \mid r \in [j_l, i_{k-l+1}]\},$$

where $[j_l, i_{k-l+1}]$ is the cyclic interval ($j_l > i_{k-l+1}$).

Proof. Let $\mathbf{f} = \{\mathbf{p}_{j_1}, \dots, \mathbf{p}_{j_k}\}$ be a flow from $[k]$ to J, where $\mathbf{p}_{j_i}$ denotes the path with sink j_i. The paths $\mathbf{p}_{j_r}$ for $r \leq s$ are "lazy paths", starting and ending at j_r without moving. Let $[k] \setminus \{j_1, \dots, j_s\} = \{i_1, \dots, i_{k-s}\}$ with $i_1 < \dots < i_{k-s}$. Hence, for $l > s$ the path $\mathbf{p}_{j_l}$ has source i_{k-l+1} and sink j_l. To its left are all boundary faces F_r with r in the cyclic interval $[j_l, i_{k-l+1}]$. Note that $i_{k-l+1} < k < j_l$, hence $k \notin [j_l, i_{k-l+1}]$. In particular, the claim follows □

Corollary 5. Recall that $L_{I_{k,n}} \subset \operatorname{trop}(\operatorname{Gr}(k, \mathbb{C}^n))$ is the lineality space of the Plücker ideal $I_{k,n}$. For all $r \in [n]$ we have

$$M_r \in L_{I_{k,n}}.$$

Proof. Consider a Plücker relation $R_{K,L}$ with $K \in \binom{[n]}{k-1}$ and $L \in \binom{[n]}{k+1}$ of form (3.1.1). Every term in $R_{K,L}$ equals $\pm p_J p_{J'}$ for some $J, J' \in \binom{[n]}{k}$. Let $J = \{j_1, \dots, j_k\} \in \binom{[n]}{k}$ with $j_1 < \dots j_s \leq k < j_{s+1} < \dots < j_k$ and $J' = \{j'_1, \dots, j'_k\} \in \binom{[n]}{k}$ with $j'_1 < \dots j'_{s'} \leq k < j'_{s'+1} < \dots < j'_k$. Set $[k] \setminus \{j_1, \dots, j_s\} = \{i_1, \dots, i_{k-s}\}$ with $i_1 < \dots < i_{k-s}$ and $[k] \setminus \{j'_1, \dots, j'_{s'}\} = \{i'_1, \dots, i'_{k-s'}\}$ with $i'_1 < \dots < i'_{k-s'}$. We further denote $J \cup J' \setminus ([k] \cap (J \cup J')) = \{l_1, \dots, l_m\}$. That is $\{l_1, \dots, l_m\} = \{j_{s+1}, \dots, j_k, j'_{s'+1}, \dots, j'_k\}$ and $m = 2k - s - s'$. Note that there might be repetitions among the l_i. Define for $q \in [m]$

$$i_q := \begin{cases} i_{k-l+1}, & \text{if } i_r = j_l, \\ i'_{k-l'+1}, & \text{if } i_r = j'_{l'}. \end{cases}$$

With this notation we have

$$\begin{aligned} \{j_l \in \{j_{s+1}, \dots, j_k\} \mid r \in [j_l, i_{k-l+1}]\} \quad &\cup \quad \{j'_{l'} \in \{j'_{s'+1}, \dots, j'_k\} \mid r \in [j'_{l'}, i'_{k-l'+1}]\} \\ &= \quad \{l_q \in \{l_1, \dots, l_m\} \mid r \in [l_q, i_q]\}. \end{aligned}$$

Consider $M_r \in \mathbb{R}^{\binom{n}{k}}$ as a weight vector for $\mathbb{C}[p_J]_J$. Then the M_r-weight on $\pm p_J p_{J'}$ is

$$(M_r)_J + (M_r)_{J'} \quad = \quad \#\{l_q \in \{l_1, \dots, l_m\} \mid r \in [l_q, i_q]\}.$$

As $\#\{l_q \in \{l_1, \dots, l_m\} \mid r \in [l_q, i_q]\}$ depends only on $J \cup J'$, which is equal for all monomials in $R_{K,L}$ we deduce

$$\operatorname{in}_{M_r}(R_{K,L}) = R_{K,L},$$

and the claim follows. □

Recall the plabic weight vector $\mathbf{w}_{\mathcal{G}}$ from Definition 36. The following proposition establishes the connection to what we have seen in §2.4. In terms of the weighting matrix $M_{\mathcal{G}}$, we observe

$$\mathbf{w}_{\mathcal{G}} = \sum_{j=k+1}^{d+1} M_j,$$

where the sum contains exactly those M_j corresponding to interior faces of $\mathcal{G}$. Let $e : \mathbb{Q}^{d+1} \to \mathbb{Q}$ be a linear form. Using the notation as in §2.4 we have $e(M_{\mathcal{G}}) = (e(\mathfrak{v}_{\mathcal{G}}(p_J))_{J \in \binom{[n]}{k}} \in \mathbb{Q}^{\binom{n}{k}}$.

Proposition 4. For every plabic graph $\mathcal{G}$ there exists a linear form $e:\mathbb{Q}^{d+1}\to\mathbb{Q}$ satisfying $\mathfrak{v}_{\mathcal{G}}(\bar{p}_I)\prec\mathfrak{v}_{\mathcal{G}}(\bar{p}_J)$ implies $e(\mathfrak{v}_{\mathcal{G}}(\bar{p}_I))<e(\mathfrak{v}_{\mathcal{G}}(\bar{p}_J))$ for $I,J\in\binom{[n]}{k}$, such that for the plabic weight vector $\mathbf{w}_{\mathcal{G}}$ we have

$$\operatorname{in}_{e(M_{\mathcal{G}})}(I_{k,n})=\operatorname{in}_{\mathbf{w}_{\mathcal{G}}}(I_{k,n}).$$

Proof. We use the following two observations following by definition from the fan structure we chose on $\operatorname{trop}(\operatorname{Gr}(k,\mathbb{C}^n))$. Firstly, for every $q\in\mathbb{Z}_{>0}$ we have $\operatorname{in}_{q\mathbf{w}_{\mathcal{G}}}(I_{k,n})=\operatorname{in}_{\mathbf{w}_{\mathcal{G}}}(I_{k,n})$.

Secondly, by Corollary 5, and the fan structure on $\operatorname{trop}(\operatorname{Gr}(k,\mathbb{C}^n))$ we have for $r_1,\dots,r_k\in\mathbb{Z}_{\geq0}$ that $\operatorname{in}_{\mathbf{w}_{\mathcal{G}}+r_1M_1+\dots+r_kM_k}(I_{k,n})=\operatorname{in}_{\mathbf{w}_{\mathcal{G}}}(I_{k,n})$.

In particular, these observations imply that it is enough to find $q,r_1,\dots,r_k\in\mathbb{Z}_{\geq0}$ such that $e(x_1,\dots,x_{d+1}):=\sum_{i=1}^k r_ix_i+\sum_{i=k+1}^{d+1}qx_i$ satisfies

$$\mathfrak{v}_{\mathcal{G}}(\bar{p}_I)\prec\mathfrak{v}_{\mathcal{G}}(\bar{p}_J)\Rightarrow e(\mathfrak{v}_{\mathcal{G}}(\bar{p}_I))<e(\mathfrak{v}_{\mathcal{G}}(\bar{p}_J))\text{ for }I,J\in\binom{[n]}{k}.$$

As $\mathfrak{v}_{\mathcal{G}}(p_J)\in\mathbb{Z}_{\geq0}^{d+1}$, it suffices to find $q,r_1,\dots,r_k\in\mathbb{Z}_{\geq0}$ such that all $e(\mathfrak{v}_{\mathcal{G}}(p_J))$ are distinct. If we choose $r_1,\dots,r_k\in\mathbb{Z}_{>0}$ big enough with $|r_i-r_j|$ big enough and $q\in\mathbb{Z}_{>0}$ relatively small, this is the case and the claim follows. □

Example 12. Consider the plabi graph $\mathcal{G}$ with perfect orientation as in Figure 3.12. We have seen the images of $\mathfrak{v}_{\mathcal{G}}$ in Table 3.2 above. Note that the columns corresponding to the faces of $\mathcal{G}$ are ordered as we fixed above. We have $F_{23}=F_1,F_{34}=F_2,\dots,F_{12}=F_5$. In particular, in the row for p_J the entries in columns $F_{23},\dots,F_{14}$ give $\mathfrak{v}_{\mathcal{G}}(\bar{p}_J)$. For example, for p_{14} these entires are $1,0,0,1,1,0,0$ and $\mathfrak{v}_{\mathcal{G}}(\bar{p}_{14})=(1,0,0,1,1,0)$. The matrix $M_{\mathcal{G}}$ is the transpose of the matrix with columns F_i as in the table. The last column $e(M_{\mathcal{G}})$ corresponds to the linear form $e:\mathbb{Z}^7\to\mathbb{Z}$ given by

$$e(x_1,\dots,x_7)=x_1+2x_2+3x_3+4x_4+5x_5+x_6+x_7.$$

It is an example of a linear form as in the proof of Proposition 4 with $r_i=i$ and $q=1$.

Theorem 12. If $\operatorname{in}_{\mathbf{w}_{\mathcal{G}}}(I_{k,n})$ is prime we have

$$\operatorname{gr}_{\mathfrak{v}_{\mathcal{G}}}(A_{k,n})\cong\mathbb{C}[p_J]_J/\operatorname{in}_{\mathbf{w}_{\mathcal{G}}}(I_{k,n}).$$

Moreover, $\Delta(A_{k,n},\mathfrak{v}_{\mathcal{G}}(p_J))=\operatorname{conv}(\mathfrak{v}_{\mathcal{G}}(\bar{p}_J)\mid J\in\binom{[n]}{k})$ and the Plücker coordinates form a Khovanskii basis for $(A_{k,n},\mathfrak{v}_{\mathcal{G}})$.

Proof. Consider $e:\mathbb{Q}^{d+1}\to\mathbb{Q}$ as in Proposition 4. Then by Lemma 4 we have

$$\operatorname{in}_{\mathbf{w}_{\mathcal{G}}}(I_{k,n})=\operatorname{in}_{e(M_{\mathcal{G}})}(I_{k,n})=\operatorname{in}_{M_{\mathcal{G}}}(I_{k,n}).$$

In particular, $\operatorname{in}_{M_{\mathcal{G}}}(I_{k,n})$ is prime. Moreover, as $I_{k,n}$ is homogeneous with respect to the usual grading on the polynomial ring $\mathbb{C}[p_J]_J$ and generated by elements of degree 2, we can apply Theorem 10 and obtain $\Delta(A_{k,n},\mathfrak{v}_{\mathcal{G}})=\operatorname{conv}(\mathfrak{v}_{\mathcal{G}}(p_J)\mid J\in\binom{[n]}{k})$. Recall the (quasi-)valuation $\mathfrak{v}_{M_{\mathcal{G}}}$ with weighting matrix $M_{\mathcal{G}}$. From the proof of Theorem 10 we further deduce that

$$S(A_{k,n},\mathfrak{v}_{\mathcal{G}})=S(A_{k,n},\mathfrak{v}_{M_{\mathcal{G}}}).$$

Hence, by [45, Lemma 4.4] $\operatorname{gr}_{\mathfrak{v}_{\mathcal{G}}}(A_{k,n})=\operatorname{gr}_{M_{\mathcal{G}}}(A_{k,n})\cong\mathbb{C}[p_J]_J/\operatorname{in}_{M_{\mathcal{G}}}(I_{k,n})$ and the claim follows. □

Corollary 6. If $\Delta(A_{k,n}, \mathfrak{v}_{\mathcal{G}})$ is not integral, then $\mathrm{in}_{\mathbf{w}_{\mathcal{G}}}(I_{k,n})$ is not prime.

Proof. If $\Delta(A_{k,n}, \mathfrak{v}_G)$ is not integral it is in particular not the convex hull of $\mathfrak{v}_{\mathcal{G}}(p_J) \in \mathbb{Z}^{d+1}$ for $J \in \binom{[n]}{k}$. Then the semigroup $S(A_{k,n}, \mathfrak{v}_{\mathcal{G}})$ is also not generated by $\mathfrak{v}_{\mathcal{G}}(p_J)$ for $J \in \binom{[n]}{k}$. By Theorem 12 in hence follows, that $\mathrm{in}_{\mathbf{w}_{\mathcal{G}}}(I_{k,n})$ cannot be prime. □

The case of $\mathrm{Gr}(2, \mathbb{C}^n)$

We show in §3.3.3 that $\mathrm{in}_{\mathbf{w}_{\mathcal{G}}}(I_{2,n})$ is prime for every plabic graph $\mathcal{G}$ for $\mathrm{Gr}(2, \mathbb{C}^n)$ (see Theorem 15). Further, we show that for every isomorphism class of maximal prime cones in $\mathrm{trop}(\mathrm{Gr}(2, \mathbb{C}^n))$ there exists a plabic graph $\mathcal{G}$ such that $\mathrm{in}_{\mathbf{w}_{\mathcal{G}}}$ coincides with an initial ideal from a prime cone in the equivalence class. Then Theorem 12 yields the following corollary.

Corollary 7. For every plabic graph $\mathcal{G}$ for $\mathrm{Gr}(2, \mathbb{C}^n)$ there exists a maximal prime cone $C \subset \mathrm{trop}(\mathrm{Gr}(2, \mathbb{C}^n))$ such that

$$\mathrm{gr}_{\mathfrak{v}_{\mathcal{G}}}(A_{2,n}) = \mathbb{C}[p_{ij}]_{ij} / \mathrm{in}_C(I_{2,n}).$$

In particular, the special fibre of the toric degeneration of $\mathrm{Gr}(2, \mathbb{C}^n)$ given in Rietsch-Williams by $\mathfrak{v}_{\mathcal{G}}$ occurs also as a special fibre in a Gröbner toric degeneration as in (2.2.4). Moreover, for every maximal cone $C \subset \mathrm{trop}(\mathrm{Gr}(2, \mathbb{C}^n))$ there exists a plabic graph $\mathcal{G}$ such that the toric variety $\mathrm{Proj}(\mathrm{gr}_{\mathfrak{v}_{\mathcal{G}}}(A_{2,n}))$ is isomorphic to $V(\mathrm{in}_C(I_{2,n}))$.

Note that the Corollary implies in particular, that $\mathbf{w}_{\mathcal{G}} \in \mathrm{trop}(\mathrm{Gr}(2, \mathbb{C}^n))$, which is not clear by definition. However, we observe in the next subsection that this is also the case for $\mathrm{Gr}(3, \mathbb{C}^6)$.

The case of $\mathrm{Gr}(3, \mathbb{C}^6)$[1]

For $\mathrm{Gr}(3, \mathbb{C}^6)$ there are (up to moves (M2) and (M3)) 34 plabic graphs. We compute for each of them the plabic weight vector $\mathbf{w}_{\mathcal{G}}$ and the initial ideal $\mathrm{in}_{\mathbf{w}_{\mathcal{G}}}(I_{2,n})$. Before stating the results of our computation we review what is known about $\mathrm{trop}(\mathrm{Gr}(3, \mathbb{C}^6))$ from [64]. There are 7 isomorphism classes of maximal cones in $\mathrm{trop}(\mathrm{Gr}(3, \mathbb{C}^6))$, labelled by

FFGG, EEEE, EEFF1, EEFF2, EEFG, EEEG, EEFG.

The last six are prime cones, while the initial ideals for cones in the isomorphism classes FFGG are not prime. The following Theorem summarizes our results. The weight vectors $\mathbf{w}_{\mathcal{G}}$ for all 34 plabic graphs can be found in Table A.1 in Appendix A.1.

Theorem 13. For every plabic graph $\mathcal{G}$ for $\mathrm{Gr}(3, \mathbb{C}^6)$ the plabic weight vector $\mathbf{w}_{\mathcal{G}}$ lies in $\mathrm{trop}(\mathrm{Gr}(3, \mathbb{C}^6))$. Up to isomorphism, there are six distinct initial ideals $\mathrm{in}_{\mathbf{w}_{\mathcal{G}}}(I_{3,6})$, five of which correspond to ideals from cones in the isomorphism classes

EEFF1, EEFF2, EEFG, EEEG, EEFG.

Two plabic graphs yield a weight vector lying on a ray of type GG of a maximal (non-prime) cone of type FFGG. These are in fact those plabic graphs, for which $\Delta(A_{3,6}, \mathfrak{v}_{\mathcal{G}})$ is not integral (see [62, §8]).

[1]The computations were done in joint work [8]. The code is provided by Hering in [38].

In [10] they study a combinatorial model for cluster algebras of type D_4. The 50 seeds are given by centrally symmetric pseudo-triangulations of a once punctured disk with 8 marked points. In the paper they analyze symmetries among the cluster seeds and associate each seed to an isomorphism class of maximal cones in $\mathrm{trop}(\mathrm{Gr}(3,\mathbb{C}^6))$. Although they consider all 50 cluster seeds, the outcome is similar to ours: they recover only six of the seven types of maximal cones, missing the cone of type EEEE. In Table A.1 we indicate to which seeds (using their labelling) the 34 plabic graphs correspond. We observe that our findings match theirs in the sense that our weight vectors lie in the relative interior of those cones in $\mathrm{trop}(\mathrm{Gr}(3,\mathbb{C}^6))$ they identified with the corresponding seed through symmetries.

3.3.3 Main Theorem for $\mathrm{Gr}(2,\mathbb{C}^n)$

From now on we focus on $\mathrm{Gr}(2,\mathbb{C}^n)$. The main result of this section is Theorem 15 in which we show that the initial ideal with respect to the plabic weight vector coincides with the initial ideal corresponding to a certain trivalent tree. In fact, the plabic graph and the tree are related combinatorially: they can both be obtained from the same triangulation.

Recall the cluster structure of $\mathrm{Gr}(2,\mathbb{C}^n)$ from §3.1.2. For $n \geq 4$, let D_n be a disk with n marked points on the boundary labelled by $1,\dots,n$ in the counterclockwise order (or n-gon for short). For $1 \leq i,j \leq n$, let (i,j) be the arc connecting the points i and j.

Fix $\Delta = \Delta_e \cup \Delta_d$ a triangulation of the D_n as in §3.1.2. We define

$$\Delta_d = \{(a_1,b_1),(a_2,b_2),\dots,(a_{n-3},b_{n-3})\}$$

as the set of internal arcs and

$$\Delta_e = \{(1,2),(2,3),\dots,(n-1,n),(n,1)\}$$

the set of boundary arcs connecting marked points.

A *rooted (labelled) tree* is a trivalent tree on n leaves with root 1 and the other leaves labelled counterclockwise with $2,\dots,n$. Each triangulation Δ of D_n gives such a labelled tree T_Δ by considering the dual graph to Δ. More precisely T_Δ can be constructed using the following Algorithm 4. See Figure 3.13 for an example.

Algorithm 4: Constructing a rooted tree T_Δ from a triangulation Δ of D_n.

Input: A triangulation Δ of D_n.

for *every triangle* t *in* Δ **do**
  draw a vertex v_t.

for *every two adjacent triangles* t,t' *in* Δ **do**
  connect the vertices v_t and $v_{t'}$ by an edge.

for *every boundary arc* $(i,i+1) \mod n$ *of* D_n **do**
  draw a vertex labelled $i+1$ and connect it to the unique vertex v_t for which $(i,i+1) \mod n$ is an edge of the triangle t.

Output: a rooted tree T_Δ.

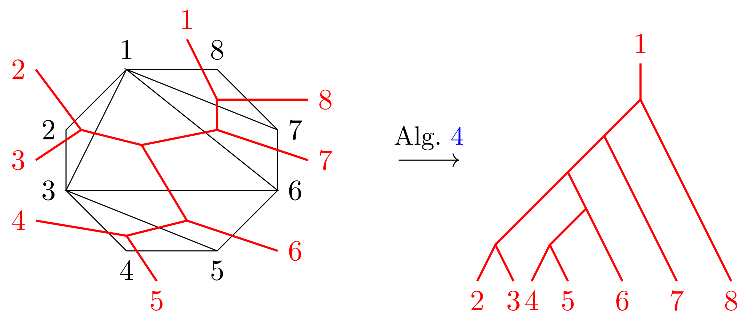


Figure 3.13: A triangulation of D_8 and the corresponding rooted tree (the output of Algorithm 4) after rescaling.

For a Plücker variable p_{ij}, recall that the *tree degree* $\deg_{T_\Delta}(p_{ij})$ is the number of internal edges between leaves i and j (see Definition 25). The tree degree admits an alternative description in terms of the corresponding triangulation as follows.

We adopt the following notations on cyclic intervals: $[i,i] = \{i\}$ and for $1 \le i < j \le n$, let $[i,j] = \{i, i+1, \ldots, j\}$ and $[j,i] = \{j, j+1, \ldots, n\} \cup \{1, 2, \ldots, i\}$. The *A-degree* on Plücker coordinates is defined for $i < j$ as

$$a_{ij} = \deg_A(p_{ij}) := \#\{(a_r, b_r) \in \Delta_d \mid \{a_r, b_r\} \cap [i, j-1] \text{ has cardinality } 1\}. \tag{3.3.1}$$

By definition the following proposition holds.

Proposition 5. For any $1 \le i < j \le n$, $\deg_{T_\Delta}(p_{ij}) = \deg_A(p_{ij})$.

Definition 37. An internal arc (a,b) in the triangulation Δ is called *connecting* $[p,q]$ and $[s,t]$, if $a \in [p,q]$ and $b \in [s,t]$ or vice versa. The number of such internal arcs are denoted by $C^{s,t}_{p,q}$ and called *connection number*.

Using this notation, the A-degree on the Plücker coordinates can be written as:

$$a_{ij} = C^{j,i-1}_{i,j-1}. \tag{3.3.2}$$

This alternative description allows us to state the following proposition.

Proposition 6. For $1 \le i < j < k < l \le n$,

(i) $a_{ij} + a_{kl} = a_{ik} + a_{jl}$ if and only if $C^{l,i-1}_{j,k-1} = 0$; when this is the case, $a_{ij} + a_{kl} > a_{il} + a_{jk}$.

(ii) $a_{il} + a_{jk} = a_{ik} + a_{jl}$ if and only if $C^{k,l-1}_{i,j-1} = 0$; when this is the case, $a_{ij} + a_{kl} < a_{il} + a_{jk}$.

To prove the proposition we need the following properties of the connection numbers that follow directly from their definition.

Lemma 8. Suppose that $1 \le p, q, s, t \le n$, the following statements hold.

(i) $C^{s,t}_{p,q} = C^{p,q}_{s,t}$.

(ii) Suppose that $[s,t] \cap [p,q] = \emptyset$. For any $r \in [s,t]$ such that $r \ne s$, $C^{s,t}_{p,q} = C^{s,r-1}_{p,q} + C^{r,t}_{p,q}$; if $r \ne t$, $C^{s,t}_{p,q} = C^{s,r}_{p,q} + C^{r+1,t}_{p,q}$.

(iii) For $t \in [s,q]$ such that $t \neq q$, $C^{s,t}_{s,q} = C^{s,t}_{s,t} + C^{s,t}_{t+1,q}$.

(iv) $C^{s,q}_{s,q} = C^{s+1,q}_{s,s} + C^{s+1,q}_{s+1,q}$.

Proof of Proposition 6. By (3.3.2) and Lemma 8(ii), we have

$$\begin{aligned}
a_{ij} + a_{kl} &= C^{j,k-1}_{i,j-1} + C^{k,l-1}_{i,j-1} + C^{l,i-1}_{i,j-1} + C^{l,i-1}_{k,l-1} + C^{i,j-1}_{k,l-1} + C^{j,k-1}_{k,l-1} \\
a_{ik} + a_{jl} &= C^{k,l-1}_{i,j-1} + C^{k,l-1}_{j,k-1} + C^{l,i-1}_{i,j-1} + C^{l,i-1}_{j,k-1} + C^{l,i-1}_{j,k-1} + C^{l,i-1}_{k,l-1} + C^{i,j-1}_{j,k-1} + C^{i,j-1}_{k,l-1} \\
a_{il} + a_{jk} &= C^{l,i-1}_{i,j-1} + C^{l,i-1}_{j,k-1} + C^{l,i-1}_{k,l-1} + C^{k,l-1}_{j,k-1} + C^{l,i-1}_{j,k-1} + C^{i,j-1}_{j,k-1}.
\end{aligned}$$

Notice that in a triangulation, $C^{l,i-1}_{j,k-1}$ and $C^{k,l-1}_{i,j-1}$ can not both be zero. The proposition follows from comparing the terms. □

We continue by defining X-degrees in terms of connection numbers that coincide with the plabic degrees. Having both, plabic and tree degrees, in terms of connection numbers allows us to directly compare them on the triangulation and prove our main theorem. Before we can do so we need a combinatorial tool relating triangulations and plabic graphs. Kodama and Williams associate to a triangulation Δ a plabic graph $\mathcal{G}_\Delta$ in [48, Algorithm 12.1]. We recall the algorithm below.

Algorithm 5: Constructing a plabic graph $\mathcal{G}_\Delta$ from a triangulation Δ of D_n.

Input: A triangulation Δ of D_n.

for *every triangle* t *in* Δ **do**
  draw a black vertex b_t and connect it to the vertices of the t.

for *every every marked point* $m \in \partial D_n$ **do**
  if *exist an arc* $(m,k) \in \Delta$ **then**
    draw a white vertex w_m for m
  else
    draw a black vertex b_m for m

Erase the arcs Δ_d and Δ_e, contract adjacent vertices of the same color.

Output: a graph D_n^Δ with n boundary vertices

for *every boundary vertex* m *of* D_n^Δ **do**
  add an edge e_m such that no two e_{m_1}, e_{m_2} intersect for m_1, m_2 boundary vertices of D_n^Δ

Embed the resulting graph in a disk such that the new vertices of edges e_m lie on the boundary.

Output: a plabic graph $\mathcal{G}_\Delta$.

We call D_n^Δ (the first output of Algorithm 5) the *plabic n-gon* associated to the triangulation Δ. Boundary vertices of D_n^Δ are colored by black or white.

Fix a perfect orientation on $\mathcal{G}_\Delta$ such that the source set is $\{1,2\}$. Recall the definition of plabic degree. The plabic degree has an alternative description in terms of the corresponding triangulation as follows. For a fixed triangulation Δ of D_n the *X-degrees* of the Plücker coordinates are defined by

$$x_{ij} = \deg_X(p_{ij}) = \begin{cases} 0, & \text{if } i=1, j=2 \\ C^{j,1}_{j,1} + C^{2,j-1}_{1,1}, & \text{if } i=1, 2<j\le n \\ C^{j,1}_{j,1}, & \text{if } i=2, 2<j\le n \\ C^{i,1}_{i,1} + C^{j,1}_{j,i-1}, & \text{otherwise.} \end{cases} \tag{3.3.3}$$

Theorem 14. For any $1 \le i < j \le n$, $\deg_{\mathcal{G}_\Delta}(p_{ij}) = \deg_X(p_{ij})$.

Example 13. We examine Theorem 14 in the case where the triangulation Δ is given by $\Delta_d = \{(2,4),(2,5),\dots,(2,n)\}$.

Claim: For every $2 < j \le n$, $\deg_{\mathcal{G}_\Delta}(P_{1j}) = \deg_{\mathcal{G}_\Delta}(P_{2j}) = 0$ and for every $2 < i < j \le n$, $\deg_{\mathcal{G}_\Delta}(P_{ij}) = n-j+1$.

Proof of Claim. The first statement follows from the fact that for $i \in \{1,2\}$ and for $2 < j \le n$ there is a path from i to j having only boundary faces of $\mathcal{G}_\Delta$ to its left (see Figure 3.14). For the second part of the claim we observe that there there is a (unique) path from 1 to j having only boundary faces of $\mathcal{G}_\Delta$ to its left. Further, there is a unique minimal path (with respect to the number of faces on its left) from 2 to i which does not intersect the one from 1 to j. We count that the second path has $n-j+1$ faces on its left, namely those coming from the internal arcs $(2,n),\dots,(2,j)$. □

By straightforward computations, Theorem 14 holds in this case.

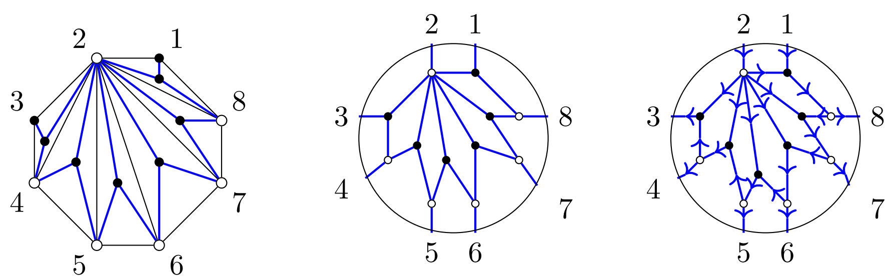


Figure 3.14: A plabic 8-gon for $\mathrm{Gr}(2,\mathbb{C}^8)$ and the corresponding plabic graph with perfect orientation.

In order to prove Theorem 14 by induction on n, we need the following properties of X-degrees.

Proposition 7. For $1 \le i < j < k < l \le n$,

(i) We have $x_{ij} + x_{kl} = x_{ik} + x_{jl}$ if and only if $C^{l,i-1}_{j,k-1} = 0$. In this is the case $x_{ij} + x_{kl} < x_{il} + x_{jk}$.

(ii) We have $x_{il} + x_{jk} = x_{ik} + x_{jl}$ if and only if $C^{k,l-1}_{i,j-1} = 0$. In this is the case $x_{ij} + x_{kl} > x_{il} + x_{jk}$.

Proof. First notice that by Lemma 8(iii), for $2 < i < j \leq n$, $x_{i,j} = C^{i,1}_{i,1} + C^{j,1}_{j,1} + C^{j,1}_{2,i-1}$. The proof is separated into four cases:

(i) When $2 < i < j < k < l \leq n$, we have:

$$x_{ij} + x_{kl} = C^{i,1}_{i,1} + C^{j,1}_{j,1} + C^{k,1}_{k,1} + C^{l,1}_{l,1} + C^{j,1}_{2,i-1} + C^{l,1}_{2,k-1}, \tag{3.3.4}$$

$$x_{ik} + x_{jl} = C^{i,1}_{i,1} + C^{j,1}_{j,1} + C^{k,1}_{k,1} + C^{l,1}_{l,1} + C^{k,1}_{2,i-1} + C^{l,1}_{2,j-1}, \tag{3.3.5}$$

$$x_{il} + x_{jk} = C^{i,1}_{i,1} + C^{j,1}_{j,1} + C^{k,1}_{k,1} + C^{l,1}_{l,1} + C^{l,1}_{2,i-1} + C^{k,1}_{2,j-1}. \tag{3.3.6}$$

By Lemma 8 (i) and (ii), subtracting (3.3.4) from (3.3.5) gives

$$C^{k,1}_{2,i-1} + C^{l,1}_{2,j-1} - C^{j,1}_{2,i-1} - C^{l,1}_{2,k-1} = -C^{j,k-1}_{2,i-1} - C^{l,1}_{j,k-1} = -C^{l,i-1}_{j,k-1}.$$

Subtracting (3.3.6) from (3.3.5) we obtain

$$C^{k,1}_{2,i-1} + C^{l,1}_{2,j-1} - C^{l,1}_{2,i-1} - C^{k,1}_{2,j-1} = -C^{k,1}_{i,j-1} + C^{l,1}_{i,j-1} = -C^{k,l-1}_{i,j-1}.$$

These computations prove the proposition in this case.

(ii) When $i = 1 < 2 < j < k < l \leq n$, we have:

$$x_{1j} + x_{kl} = C^{j,1}_{j,1} + C^{2,j-1}_{1,1} + C^{k,1}_{k,1} + C^{l,1}_{l,1} + C^{l,1}_{2,k-1}, \tag{3.3.7}$$

$$x_{1k} + x_{jl} = C^{k,1}_{k,1} + C^{2,k-1}_{1,1} + C^{j,1}_{j,1} + C^{l,1}_{l,1} + C^{l,1}_{2,j-1}, \tag{3.3.8}$$

$$x_{1l} + x_{jk} = C^{l,1}_{l,1} + C^{2,l-1}_{1,1} + C^{j,1}_{j,1} + C^{k,1}_{k,1} + C^{k,1}_{2,j-1}. \tag{3.3.9}$$

Again by Lemma 8, subtracting (3.3.7) from (3.3.8) gives

$$C^{2,k-1}_{1,1} - C^{2,j-1}_{1,1} + C^{l,1}_{2,j-1} - C^{l,1}_{2,k-1} = -C^{l,n}_{2,k-1} + C^{l,n}_{2,j-1} = -C^{l,n}_{j,k-1}.$$

Subtracting (3.3.9) from (3.3.8) gives

$$C^{2,k-1}_{1,1} - C^{2,l-1}_{1,1} + C^{l,1}_{2,j-1} - C^{k,1}_{2,j-1} = -C^{k,l-1}_{1,1} - C^{k,l-1}_{2,j-1} = -C^{k,l-1}_{1,j-1}.$$

(iii) When $i = 2 < j < k < l \leq n$, the proof is similar.

(iv) When $i = 1 < j = 2 < k < l \leq n$, we have

$$\begin{aligned} x_{12} + x_{kl} &= C^{k,1}_{k,1} + C^{l,1}_{l,1} + C^{l,1}_{2,k-1}, \\ x_{1k} + x_{2l} &= C^{k,1}_{k,1} + C^{2,k-1}_{1,1} + C^{l,1}_{l,1}, \\ x_{1l} + x_{2k} &= C^{l,1}_{l,1} + C^{2,l-1}_{1,1} + C^{k,1}_{k,1}. \end{aligned}$$

It is then easy to deduce the corresponding statement in the proposition.

∎

Proof of Theorem 14 Fix a triangulation $\Delta = \Delta_d \cup \Delta_e$ and let $\mathcal{G}_\Delta$ be the associated plabic graph obtained by applying Algorithm 5 to Δ. The proof of the theorem is executed by induction on n. The case $n = 4$ contains only two different triangulations and can be verified directly. Suppose $n \geq 5$. First notice that there exists at least two black boundary vertices in the plabic n-gon D_n^Δ and vertices 1 and 2 can not be both black vertices. In fact, all neighbors of a black vertex are white vertices. Let s be the black vertex different from 1 and 2 such that there is no black vertex in $[s+1, n]$. Then $s-1$, $s+1, \ldots, n$ are all white vertices and $(s-1, s+1) \in \Delta_d$.

Lemma 9 (Sector lemma)**.** If $(s-1, p) \in \Delta_d$ for some $s+1 < p \leq n$, then $(s-1, s+2), \ldots, (s-1, p-1) \in \Delta_d$.

Proof. Let $q \in [s+2, p-1]$ be the smallest integer such that $(s-1, q) \notin \Delta_d$. In this case, there exists an internal arc (q, r) for some $r \in [q+1, p-1]$. This is not possible since otherwise there must be at least one black vertex in $[q+1, r-1]$. □

Corollary 8. If $s = 3$, then $\Delta_d = \{(2,4), (2,5), \ldots, (2,n)\}$.

Proof. When $s = 3$, there are only two black vertices 1 and 3 in the plabic n-gon, which implies that $(2,4), (2,n) \in \Delta_d$. By the Sector Lemma, for any $r \in [5, n-1]$, $(2, r) \in \Delta_d$. □

According to the corollary, if $s = 3$, by Example 13, Theorem 14 holds. From now on suppose that $s \neq 3$. The following lemma explains the local orientation on the square containing $s-1, s, s+1$ in the plabic graph $\mathcal{G}_\Delta$, see Figure 9. In the plabic graph $\mathcal{G}_\Delta$, we use $1, 2, \ldots, n$ to denote the internal vertices connected to boundary vertices $1, 2, \ldots, n$. First note that, as s is a boundary black vertex, it has already one edge going out in the plabic graph $\mathcal{G}_\Delta$. Hence, the edges in $\mathcal{G}_\Delta$ connecting $s-1$ and $s+1$ to s have orientations pointing towards s (see e.g. Figure 3.15). Suppose that the theorem holds for any triangulation of D_{n-1}. Let $\overline{D}_n$ be the disk with $n-1$ markes points obtained from D_n by removing the makred point s. The triangulation Δ of D_n induces a triangulation $\overline{\Delta} = \overline{\Delta}_d \cup \overline{\Delta}_e$ of $\overline{D}_n$ where

$$\overline{\Delta}_d = \Delta_d \backslash \{(s-1, s+1)\} \text{ and } \overline{\Delta}_e = (\Delta_e \backslash \{(s-1, s), (s, s+1)\}) \cup \{(s-1, s+1)\}.$$

We associate to $\overline{D}_n$ and $\overline{\Delta}$ a plabic $(n-1)$-gon $\overline{D}_n^{\overline{\Delta}}$ and a plabic graph $\overline{\mathcal{G}}_{\overline{\Delta}}$. For $1 \leq i < j \leq n$ and $i, j \neq s$, we denote $\deg_{\overline{\mathcal{G}}_{\overline{\Delta}}}(\overline{p}_{ij})$ and $\deg_X(\overline{p}_{ij})$ the corresponding degrees with respect to $\overline{\mathcal{G}}_{\overline{\Delta}}$ and $\overline{\Delta}$. If one of i and j equals s, we set these degrees to be zero. The connection numbers for $\overline{\Delta}$ are denoted by $\overline{C}_{p,q}^{r,s}$. For $1 \leq i < j \leq n$, we denote

$$v_{ij} = \deg_X(p_{ij}) - \deg_X(\overline{p}_{ij}) \text{ and } w_{ij} = \deg_{\mathcal{G}_\Delta}(p_{ij}) - \deg_{\overline{\mathcal{G}}_{\overline{\Delta}}}(\overline{p}_{ij}).$$

By the induction hypothesis, to prove the theorem, it suffices to show that for any $1 \leq i < j \leq n$, $v_{ij} = w_{ij}$. We start with the following lemma.

Lemma 10. Suppose $2 < i < s$ and $s < j \leq n$. The face of $\mathcal{G}_\Delta$ corresponding to the internal arc $(s-1, s+1) \in \Delta_d$ is to the left of any directed path from 1 or 2 to i, and to the right of any directed path from 1 or 2 to j.

Proof. If there exists a directed path from 1 or 2 to j such that this face is to the left of the path, then it passes through the vertex s. This is impossible, since all arrows at s not connecting to the boundary go towards s. The proof of the statement on i is similar. □

The rest of this section is dedicated to proving that $\deg_{\mathcal{G}_\Delta}(p_{ij}) = \deg_X(p_{ij})$ for all $1 \leq i < j \leq n$. We distinguish the following 13 cases:

(i) $i = 1 < j < s$. By Lemma 10, we have $w_{1j} = 1$ and

$$v_{1j} = C_{j,1}^{j,1} + C_{1,1}^{2,j-1} - \overline{C}_{j,1}^{j,1} - \overline{C}_{1,1}^{2,j-1} = C_{j,1}^{j,1} - \overline{C}_{j,1}^{j,1} = 1.$$

(ii) $i = 1 < s < j \leq n$. By Lemma 10, we have $w_{1j} = 0$ and a similar argument as above shows that $v_{1j} = 0$.

(iii) $1 = i < j = s < n$. We need to show that $\deg_{\mathcal{G}_\Delta}(p_{1s}) = \deg_X(p_{1s})$. Notice that a directed path from 2 to s must pass through either $s-1$ or $s+1$. Since there always exists a directed path from 2 to $s+1$, by minimality, we have $\deg_{\mathcal{G}_\Delta}(p_{1s}) = \deg_{\mathcal{G}_\Delta}(p_{1,s+1})$. On the other hand, since there is no internal arc meeting s, we have $C_{s,1}^{s,1} = C_{s+1,1}^{s+1,1}$ and $C_{1,1}^{2,s-1} = C_{1,1}^{2,s}$. We compüute

$$\deg_X(p_{1s}) = C_{s,1}^{s,1} + C_{1,1}^{2,s-1} = C_{s+1,1}^{s+1,1} + C_{1,1}^{2,s} = \deg_X(p_{1,s+1}).$$

Now the claim follows from the case $s < j = s+1 \leq n$.

(iv) $1 = i < j = s = n$. In this case, a directed path from 2 to s must pass through $s - 1 = n - 1$, since it cannot pass through 1. Therefore $\deg_{\mathcal{G}_\Delta}(p_{1n}) = \deg_{\mathcal{G}_\Delta}(p_{1,n-1})$. As $(1, n-1) \in \Delta_d$, we have $C_{n,1}^{n,1} = 0$ and $C_{1,1}^{n-1,n-1} = C_{n-1,1}^{n-1,1}$. We can hence apply Lemma 8 (2) and obtain

$$\deg_X(p_{1n}) = C_{n,1}^{n,1} + C_{1,1}^{2,n-1} = C_{1,1}^{2,n-2} + C_{n-1,1}^{n-1,1} = \deg_X(p_{1,n-1}).$$

The statement follows from Case (i).

(v) $i = 2 < j \leq s$. This(ese) case(s) can be examined in a similar manner as the corresponding cases for $i = 1$.

(vi) $i = 2 < s + 1 \leq j$. The proof of this case is similar to the proof of Case (i). Nevertheless, we repeat the argument since this case is applied to prove Case (xii). By Lemma 10, we have $w_{2j} = 0$. On the other hand $v_{2j} = C_{j,1}^{j,1} - \overline{C}_{j,1}^{j,1} = 0$, since there are no internal arcs of Δ_d (or $\overline{\Delta}_d$) entirely contained in $[j, 1]$.

(vii) $2 < i < s < j$. By Lemma 10, $w_{ij} = 1$. By definition, $v_{ij} = (C_{i,1}^{i,1} - \overline{C}_{i,1}^{i,1}) + (C_{j,i-1}^{j,1} - \overline{C}_{j,i-1}^{j,1})$. Since $i \neq s$, the second bracket gives zero. The first bracket gives 1, as the internal arc $(s-1, s+1)$ is no longer in $\overline{\Delta}$.

(viii) $2 < s < i < j$. By Lemma 10, $w_{ij} = 0$. A similar argument as above shows $v_{ij} = 0$.

(ix) $2 < i < j < s$. By Lemma 10, $w_{ij} = 2$. A similar argument as above shows $v_{ij} = 2$.

(x) $2 < i = s < j = s + 1$. We consider directed paths from 1 to $s+1$ and from 2 to s. Since the vertex $s+1$ is occupied, to reach the vertex s, the path from 2 to s is forced to go through $s - 1$, which shows $\deg_{\mathcal{G}_\Delta}(p_{s,s+1}) = \deg_{\mathcal{G}_\Delta}(p_{s-1,s+1})$.

By Case (i) we have proved, $\deg_{\mathcal{G}_\Delta}(p_{s-1,s+1}) = \deg_X(p_{s-1,s+1})$. It suffices to show that $\deg_X(p_{s,s+1}) - \deg_X(p_{s-1,s+1}) = 0$. Since $s-1>2$, the left hand side equals

$$C^{s,1}_{s,1} + C^{s+1,1}_{s+1,s-1} - C^{s-1,1}_{s-1,1} - C^{s+1,1}_{s+1,s-2}.$$

By applying Lemma 8 several times, we obtain

$$\begin{aligned}
\deg_X(p_{s,s+1}) - \deg_X(p_{s-1,s+1}) &= C^{s,1}_{s,1} + C^{s+1,1}_{s+1,s-1} - C^{s-1,1}_{s-1,1} - C^{s+1,1}_{s+1,s-2}\\
&= C^{s-1,1}_{s-1,1} - C^{s,1}_{s-1,s-1} + C^{s+1,1}_{s+1,s-1} - C^{s-1,1}_{s-1,1} - C^{s+1,1}_{s+1,s-2}\\
&= -C^{s,s}_{s-1,s-1} - C^{s+1,1}_{s-1,s-1} + C^{s+1,1}_{s+1,s-1} - C^{s+1,1}_{s+1,s-2}\\
&= -C^{s,s}_{s-1,s-1} - C^{s+1,1}_{s+1,s-1} + C^{s+1,1}_{s+1,s-1}\\
&= -C^{s,s}_{s-1,s-1}.
\end{aligned}$$

The first two equalities follow from point (4) and (2) of Lemma 8 and the third one by combining (1) and (2) of Lemma 8. Since there is no internal arc touching s, the connection number $C^{s,s}_{s-1,s-1}$ is zero and the statement follows.

(xi) $i = s < s+1 < j$. We claim that $\deg_{\mathcal{G}_\Delta}(p_{sj}) = \deg_{\mathcal{G}_\Delta}(p_{s+1,j})$. To compute these degrees, we have to consider directed paths from 1 to j and from 2 to s. Note that the path of minimal degree from 1 to j is the same for both calculations.

Consider paths from 2 to s or $s+1$. As all edges in $\mathcal{G}_\Delta$ meeting $s+1$ connect to black vertices, there is a unique black vertex v such that the edge connecting v and $s+1$ goes towards $s+1$ (see e.g. Figure 3.15). Let $(p,q,s+1)$ be the triangle in Δ corresponding to v and assume that $p<q$, then $p \leq s-1$. Since v has an outgoing edge to $s+1$, the edge between p and v is directed from p to v. Hence, the plabic graph has a path $p \to v \to s+1$. As p and $s+1$ are boundary vertices, and $s+1$ can only have one incoming vertex, every path from 2 to s or $s+1$ has to pass through p. Note that the path of lowest degree must end with $p \to v \to s+1 \to s$, so the claim follows.

By Case (viii), $\deg_{\mathcal{G}_\Delta}(p_{s+1,j}) = \deg_X(p_{s+1,j})$, hence it suffices to show that $\deg_X(p_{sj}) = \deg_X(p_{s+1,j})$. Their difference is given by

$$(C^{s,1}_{s,1} - C^{s+1,1}_{s+1,1}) + (C^{j,1}_{j,s-1} - C^{j,1}_{j,s}).$$

As there is no internal arc incident to the vertex s, we have $C^{s,1}_{s,1} = 0$ and $C^{j,1}_{j,s-1} = C^{j,1}_{j,s}$. It follows from our assumptions on s that $C^{s+1,1}_{s+1,1} = 0$.

(xii) $2 < i < j = s < n$. From Case (vi), we deduce that $\deg_{\mathcal{G}_\Delta}(p_{2,s+1}) = C^{s+1,1}_{s+1,1} = 0$. This implies that we can find a path from 1 to $s+1$ of plabic degree 0. Moreover, since there is an edge between $s+1$ and s oriented towards the latter, we have just shown that there exists a path from 1 to s which does not contribute to the plabic degree of p_{is}. It follows that $\deg_{\mathcal{G}_\Delta}(p_{is}) = \deg_{\mathcal{G}_\Delta}(p_{i,s+1})$. By Case (vii), $\deg_{\mathcal{G}_\Delta}(p_{i,s+1}) = \deg_X(p_{i,s+1})$, hence it suffices to show that $\deg_X(P_{is}) = \deg_X(P_{i,s+1})$: this follows from Lemma 8 (3) and the fact that no internal arcs end at s.

(xiii) $2 < i < j = s = n$. If $s = n$, then $(1, n-1) \in \Delta_d$ and the edge between 1 and n has to be oriented towards n, since 1 is a white vertex and has already an edge going in.

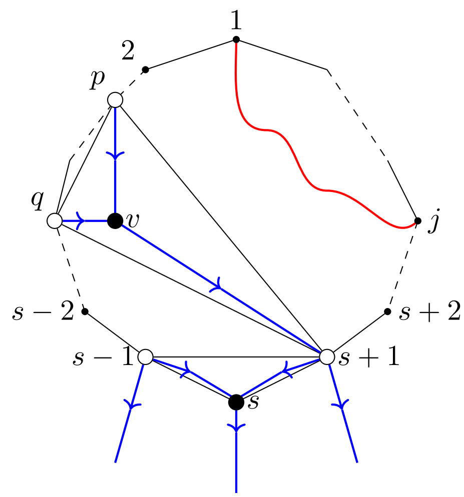


Figure 3.15: A picture for Case (xi)

This implies that only the path from 2 to i contributes to the plabic degree of p_{in} and hence $\deg_{\mathcal{G}_\Delta}(p_{in}) = \deg_{\mathcal{G}_\Delta}(p_{1i})$. On the other hand, $\deg_X(p_{1i}) = C^{i,1}_{i,1} + C^{2,i-1}_{1,1}$ and $\deg_X(p_{in}) = C^{i,1}_{i,1} + C^{n,1}_{n,i-1}$. Since there is no internal arc meeting the vertex n, the connection numbers $C^{2,i-1}_{1,1}$ and $C^{n,1}_{n,i-1} = C^{n,i-1}_{n,1}$ coincide. To conclude we hence have to show that $\deg_X(p_{1i}) = \deg_{\mathcal{G}_\Delta}(p_{1i})$, but this has been dealt with in Case (i).

Main theorem

Let $\Delta = \Delta_d \cup \Delta_e$ be a triangulation of D_n and let T_Δ be the labelled tree corresponding to Δ. Recall that the tree degrees give the weight vector $\mathbf{w}_{T_\Delta} = (-\deg_{T_\Delta}(p_{ij}))_{ij} \in \mathbb{R}^{\binom{n}{2}}$. Let $\mathrm{in}_{T_\Delta}(I_{2,n}) = \mathrm{in}_{\mathbf{w}_{T_\Delta}}(I_{2,n})$. Similarly, consider $\mathbf{w}_P = (\deg_{\mathcal{G}_\Delta}(p_{ij}))_{ij} \in \mathbb{R}^{\binom{n}{2}}$ the weight vector associated to a plabic graph $\mathcal{G}_\Delta$ from Definition 36 and let $\mathrm{in}_{\mathcal{G}_\Delta}(I_{2,n}) = \mathrm{in}_{\mathbf{w}_{\mathcal{G}_\Delta}}(I_{2,n})$.

We are now prepared to prove the main theorem as stated in the introduction. We restate it here with the notation introduced in the previous paragraphs.

Theorem 15. For a given triangulation Δ of D_n, we have $\mathrm{in}_{T_\Delta}(I_{2,n}) = \mathrm{in}_{\mathcal{G}_\Delta}(I_{2,n})$.

Proof. Recall that from Definition 25 that $\mathbf{w}_{T_\Delta} \in \mathrm{trop}(\mathrm{Gr}(2,\mathbb{C}^n))$. More precisely, $\mathbf{w}_{T_\Delta}$ lies in the relative interior of a maximal cone C_Δ of $\mathrm{trop}(\mathrm{Gr}(2,\mathbb{C}^n))$. To prove the theorem, it suffices to show that $\mathbf{w}_{\mathcal{G}_\Delta}$ lies in the relative interior of the same cone.

First note that every arc (a,b) of Δ connecting $[l,i-1]$ and $[j,k-1]$ divides the disk D_n into two parts. One of these two parts contains the marked points in $[i,j-1]$ and has empty intersection with the set of marked points $[k,l-1]$. We deduce that every internal arc connecting $[i,j-1]$ and $[k,l-1]$ intersects (a,b) and therefore $C^{l,i-1}_{j,k-1} \neq 0$ implies $C^{k,l-1}_{i,j-1} = 0$. The same argument, applied to (l,i,j,k) instead of (i,j,k,l), shows the opposite implication. We conclude that $C^{l,i-1}_{j,k-1} \neq 0$ if and only if $C^{k,l-1}_{i,j-1} = 0$.

From Proposition 7 it follows that for every $1 \leq i < j < k < l \leq n$, the minimum of the numbers $x_{ij} + x_{kl}, x_{ik} + x_{jl}, x_{il} + x_{jk}$ is attained exactly twice. In particular, by Theorem 14, this implies that $\mathrm{in}_{\mathcal{G}_\Delta}(p_{ij}p_{kl} - p_{ik}p_{ji} + p_{il}p_{jk})$ is a binomial. Further, by Propositions 5 and 7

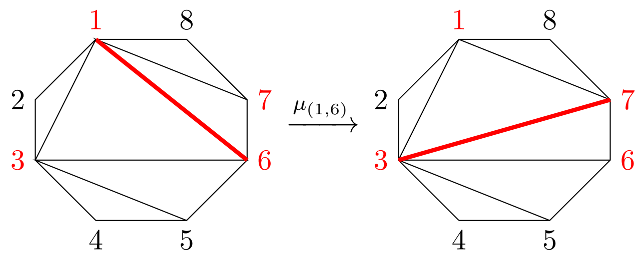


Figure 3.16: Mutation at the arc $(1, 6)$.

we deduce

$$\mathrm{in}_{\mathcal{G}_\Delta}(p_{ij}p_{kl} - p_{ik}p_{ji} + p_{il}p_{jk}) = \mathrm{in}_{T_\Delta}(p_{ij}p_{kl} - p_{ik}p_{ji} + p_{il}p_{jk}).$$

As $\mathrm{in}_{T_\Delta}(I_{2,n})$ is generated by the initial terms of the Plücker relations with respect to $\mathbf{w}_{T_\Delta}$ we conclude $\mathrm{in}_{\mathcal{G}_\Delta}(I_{2,n}) = \mathrm{in}_{T_\Delta}(I_{2,n})$. □

Corollary 9. For every plabic graph $\mathcal{G}$ there exists a maximal prime cone $C \subset \mathrm{trop}(\mathrm{Gr}(2, \mathbb{C}^n))$ with $\mathbf{w}_\mathcal{G} \in C^\circ$. In particular, $\mathcal{G}$ induces a toric degeneration of $\mathrm{Gr}(2, \mathbb{C}^n)$.

Proof. Recall that for $\mathrm{Gr}(2, \mathbb{C}^n)$ there is a bijection between seeds of the cluster algebra $\mathbb{C}[\mathrm{Gr}(2, \mathbb{C}^n)]$ and triangulations Δ of D_n by [26] and [63]. Plabic graphs for general $\mathrm{Gr}(k, \mathbb{C}^n)$ encode ($\mathcal{A}$-)seeds of $\mathbb{C}[\mathrm{Gr}(k, \mathbb{C}^n)]$ given purely in terms of Plücker coordinates (see e.g. [62, (4.1)]). In particular, for $k = 2$ there exists a triangulation Δ of D_n for every such plabic seed with $\mathcal{G}_\Delta = \mathcal{G}$. In fact, for $\mathrm{Gr}(2, \mathbb{C}^n)$ there is a bijection between seeds and plabic graphs, as all seeds consist of only Plücker coordinates. Applying Algorithm 4 we obtain T_Δ and a corresponding cone C_{T_Δ}. By Theorem 15 $\mathbf{w}_\mathcal{G} \in C^\circ_{T_\Delta}$. The toric degeneration is then given by the family described in (2.2.4). □

3.3.4 Mutation and initial ideals

Recall from §3.1.2 cluster mutation in the case of $\mathbb{C}[\mathrm{Gr}(2, \mathbb{C}^n)]$ following [63]. The aim of this section is to understand the effect cluster mutation has on the inital ideal $\mathrm{in}_\Delta(I_{2,n}) = \mathrm{in}_{T_\Delta}(I_{2,n})$. We translate mutation in terms of flipping arcs to rooted trees and analyze how this changes the associated weight vectors $\mathbf{w}_{T_\Delta}$.

Let Δ be a triangulation of D_n and T_Δ the associated rooted tree. Recall that T_Δ is a rooted tree with root corresponding to the leaf labelled by 1 and counterclockwise labelling of the leaves $1, 2, \ldots, n$. Then T_Δ can be seen as a directed graph, where an edge $\mathbf{a} - \mathbf{b}$ gets an orientation $\mathbf{a} \longrightarrow \mathbf{b}$, if the distance of $\mathbf{a}$ to 1 is less than the distance of $\mathbf{b}$ to 1.

Let $\mathbf{a}$ be an internal vertex of T_Δ and $\mathbf{a} \longrightarrow \mathbf{b}, \mathbf{a} \longrightarrow \mathbf{c}$ be the adjacent edges. We say $\mathbf{c}$ is the *left child* of $\mathbf{a}$ and $\mathbf{b}$ is the *right child* of $\mathbf{a}$ if the labels of the leaves reachable by a directed path (with respect to orientation) from $\mathbf{b}$ are smaller than those reachable from $\mathbf{c}$, having in mind that leaves are labelled counterclockwise by $1, \cdots, n$ (see Figure 3.17). The following definition formulates on the level of trees how mutation of triangulations deforms the corresponding trees. It coincides with the notation of mutation of phylogenetic trees by Buczynska and Wisniewski in [12].

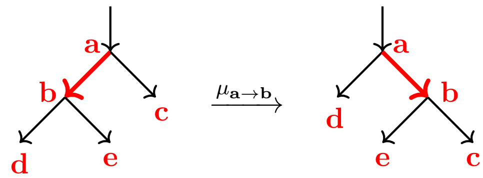


Figure 3.17: Mutation on trees

Definition 38. Let $\mathbf{a} \longrightarrow \mathbf{b}$ be an internal edge of T_Δ and $\mathbf{b}$ be the right child of $\mathbf{a}$. We further denote by $\mathbf{c}$ the left child of $\mathbf{a}$, $\mathbf{d}$ the right child of $\mathbf{b}$ and $\mathbf{e}$ the left child of $\mathbf{b}$ (see Figure 3.17). The rooted tree $\mu_{\mathbf{a}\to\mathbf{b}}(T_\Delta)$ is the tree obtained from T_Δ by defining $\mathbf{d}$ to be the right child of $\mathbf{a}$, $\mathbf{b}$ to be the left child of $\mathbf{a}$, $\mathbf{c}$ to be the left child of $\mathbf{b}$ and $\mathbf{e}$ to be the right child of $\mathbf{b}$. All other edges of the tree remain unchanged.

Remark 7. Let $(a_r, b_r) \in \Delta_d$ be the arc corresponding to $\mathbf{a} \longrightarrow \mathbf{b}$ and (a'_r, b'_r) be the arc obtained by mutating at (a_r, b_r). Set Δ' be the triangulation of the n-gon obtained through the internal arcs

$$\Delta'_d = \Delta_d \cup \{(a'_r, b'_r)\} \setminus \{(a_r, b_r)\}.$$

Then, by construction,

$$T_{\Delta'} = \mu_{\mathbf{a}\to\mathbf{b}}(T_\Delta).$$

Lemma 11. Given two arbitrary trees T_Δ and $T_{\Delta'}$, then there is a sequence of mutations on inner edges transforming T_Δ into $T_{\Delta'}$.

Proof. This is true for the triangulations Δ and Δ' on the n-gon G_n and hence by Remark 7 for the labelled trees T_Δ and $T_{\Delta'}$. □

From the direction of edges in T_Δ we obtain a partial order on the vertices. We set for two vertices $x, y \in T_\Delta$ (internal or leaves)

$$x \le y, \text{ if } \exists \text{ a directed path } x \to y \text{ in } T_\Delta.$$

We further define for a vertex x the subset of leaves $[n]_{\le x} = \{k \in [n] \mid k \le x\}$. Similarly we define $[n]_{\not\le x}$.

Let $\mathbf{a} \to \mathbf{b}$ be an internal edge of T_Δ, and we keep the same notation as in Figure 3.17 with $\mathbf{c}, \mathbf{d}, \mathbf{e}$. Observe that the set of leaves $[n]$ can be decomposed with respect to $\mathbf{a} \to \mathbf{b}$ as follows

$$[n] = [n]_{\not\le \mathbf{b}} \cup [n]_{\le \mathbf{b}} = ([n]_{\not\le \mathbf{a}} \cup [n]_{\le \mathbf{c}}) \cup ([n]_{\le \mathbf{d}} \cup [n]_{\le \mathbf{e}}).$$

After mutation the edge $\mathbf{a}' \to \mathbf{b}'$ in $T_{\Delta'} = \mu_{\mathbf{a}\to\mathbf{b}}(T_\Delta)$ separates $[n]_{\not\le \mathbf{b}'} = [n]_{\le \mathbf{d}'} \cup [n]_{\not\le \mathbf{a}'}$ from $[n]_{\le \mathbf{b}'} = [n]_{\le \mathbf{e}'} \cup [n]_{\le \mathbf{c}'}$. Note that $[n]_{\le \mathbf{x}'} = [n]_{\le \mathbf{x}}$ for a vertex $\mathbf{x} \ne \mathbf{b}$ in T_Δ resp. $T_{\Delta'}$. An example is shown in Figure 3.18.

Consider $i, j, k, l \in [n]$ pairwise distinct and the paths between each of them in T_Δ. Then there is a unique non-intersecting pair of paths, say $i \to j$ and $k \to l$. Let $\mathbf{x}$ be the first (in direction of the paths as indictaed) vertex in which the paths $i \to k$ and $j \to k$ intersect. Similary, let $\mathbf{y}$ the last vertex that lies on bath paths $i \to k$ and $i \to l$.

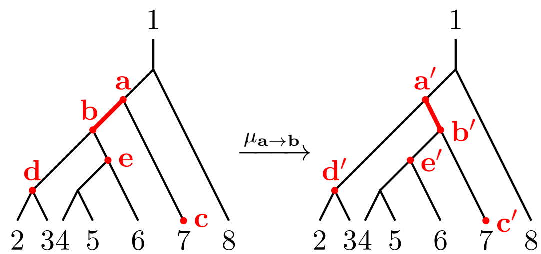


Figure 3.18: Mutation of the trees corresponding to Figure 3.16. Here we have $n = 8$ and $[8]_{\leq \mathbf{d}} = \{2, 3\}, [8]_{\leq \mathbf{e}} = \{4, 5, 6\}, [8]_{\leq \mathbf{c}} = \{7\}$ and $[8]_{\not\leq \mathbf{a}} = \{1, 8\}$.

Definition 39. We define the *contracted tree* $T_\Delta|_{i,j,k,l}$ to be the trivalent tree with four leaves i, j, k, l obtained from T_Δ by contracting all edges on the paths $i \to \mathbf{x}$, $j \to \mathbf{x}$, $\mathbf{x} \to \mathbf{y}$, $\mathbf{y} \to k$, and $\mathbf{y} \to l$ to one edge only and deleting all other edges from T_Δ. With an edge of $T_\Delta|_{i,j,k,l}$ we associate the number of internal edges of T_Δ it contracted (see Figure 3.19).

For example, the edge $i - \mathbf{x}$ in $T_\Delta|_{i,j,k,l}$ obtains the weight $d_{i\mathbf{x}} = \#\{\text{internal edges on path } i \to \mathbf{x} \text{ in } T_\Delta\}$. Note that the sums of edge weights along a path between two leaves in $T_\Delta|_{i,j,k,l}$ equals the corresponding T_Δ-degree. In particular, the tree $T_\Delta|_{i,j,k,l}$ encodes all necessary information for computing the initial form $\text{in}_\Delta(p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk})$.

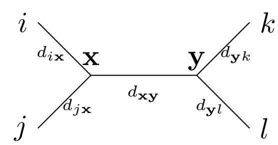


Figure 3.19: The contracted tree $T_\Delta|_{i,j,k,l}$ with edge weights.

We analyze further the tree degrees and their relation to internal edges of T_Δ. For an internal edge $\mathbf{a} \to \mathbf{b}$ in T_Δ consider $r_{\mathbf{a}\to\mathbf{b}} \in \mathbb{R}^{\binom{n}{2}}$ given by

$$(r_{\mathbf{a}\to\mathbf{b}})_{i,j} = \begin{cases} 1, & \text{if } i \in [n]_{\leq \mathbf{b}}, j \in [n]_{\not\leq \mathbf{b}} \text{ or vice versa} \\ 0, & \text{otherwise.} \end{cases} \tag{3.3.10}$$

Remark 8. The vector $-r_{\mathbf{a}\to\mathbf{b}}$ is in fact a ray generator for the maximal cone $C_T \subset \text{trop}(\text{Gr}(2, \mathbb{C}^n))$ by [64, Equation (8)] and [53, Theorem 4.3.5].

Lemma 12. For a labelled trivalent tree T we have $\mathbf{w}_T = -\sum_{\mathbf{a}\to\mathbf{b} \text{ internal edge of } T} r_{\mathbf{a}\to\mathbf{b}}$.

Proof. Consier $i < j$ in $[n]$. Then $(r_{\mathbf{a}\to\mathbf{b}})_{ij}$ is 1 if $\mathbf{a} \to \mathbf{b}$ lies on the path from i to j in T and zero otherwise. In particular, this implies

$$\sum_{\mathbf{a}\to\mathbf{b} \text{ internal edge of } T} (r_{\mathbf{a}\to\mathbf{b}})_{ij} = \deg_T(p_{ij}).$$

The claim follows as $(\mathbf{w}_T)_{ij} = -\deg_T(p_{ij})$. $\square$

The following lemma follows from Lemma 12 and the fact that by definition mutation on trees only changes one internal edge and keeps all other edges unchanged.

Lemma 13. Let Δ be a triangulation of D_n and T_Δ the corrresponding tree. Consider an internal edge $\mathbf{a} \to \mathbf{b}$ of T_Δ and let $\mu_{\mathbf{a}\to\mathbf{b}}(T_\Delta) = T_{\Delta'}$. Denote the internal edge of $T_{\Delta'}$ obtained from $\mathbf{a} \to \mathbf{b}$ by $\mathbf{a}' \to \mathbf{b}'$. Then

$$\mathbf{w}_{T_{\Delta'}} = \mathbf{w}_{T_\Delta} + r_{\mathbf{a}\to\mathbf{b}} - r_{\mathbf{a}'\to\mathbf{b}'}.$$

Theorem 16. Let Δ be a triangulation of the D_n and $\mathbf{a} \to \mathbf{b}$ be an internal edge of T_Δ and $T_{\Delta'} = \mu_{\mathbf{a}\to\mathbf{b}}(T_\Delta)$. Consider the Plücker relation $R_{i,j,k,l} = p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk} \in I_{2,n}$ for i, j, k, l cyclically ordered. Assume $\mathrm{in}_\Delta(R_{i,j,k,l}) = p_{il}p_{jk} - p_{ik}p_{jl}$, then

$$\mathrm{in}_{\Delta'}(R_{i,j,k,l}) = \begin{cases} p_{ij}p_{kl} - p_{il}p_{jk}, & \text{if } i \in [n]_{\not\leq \mathbf{a}}, j \in [n]_{\leq \mathbf{c}}, k \in [n]_{\leq \mathbf{d}}, l \in [n]_{\leq \mathbf{e}} \\ \mathrm{in}_\Delta(R_{i,j,k,l}), & \text{otherwise.} \end{cases}$$

Proof. Assume i, j, k, l are such that $T_\Delta|_{i,j,k,l}$ is of shape as in Figure 3.19, if necessary reorder them. We distinguish two cases, $\mathbf{a} \to \mathbf{b}$ contributes to the edge $\mathbf{x} - \mathbf{y}$ in $T_\Delta|_{i,j,k,l}$ or it does not.

If $\mathbf{a} \to \mathbf{b}$ does not contribute to $\mathbf{x}-\mathbf{y}$ first observe that we are in the "otherwise" case of the claim. In this case either $\mathbf{a} \to \mathbf{b}$ was deleted from T_Δ by construction $T_\Delta|_{i,j,k,l}$ (in which case the claim follows) or it contributes to one of the edges adjacent to a leaf in $T_\Delta|_{i,j,k,l}$. Assume without loss of generality that $\mathbf{a} \to \mathbf{b}$ contributes to the edge $i - \mathbf{x}$. Then by Lemma 13 the T_Δ-degrees of p_{ij}, p_{ik} and p_{il} are all changed by ± 1 while all others remain the same after mutation. In particular, this implies the claim.

If $\mathbf{a} \to \mathbf{b}$ contributes to $\mathbf{x} - \mathbf{y}$ we further distinguish depending on $q = \#(\{\mathbf{x}, \mathbf{y}\} \cap \{\mathbf{a}, \mathbf{b}\})$:

$q = 0$ In this case $\mathbf{a}' \to \mathbf{b}'$ does not contribute to the edge $\mathbf{x}' - \mathbf{y}'$ of $T_{\Delta'}|_{i,j,k,l}$. By the proof of Lemma 12 the T_Δ-degrees of $p_{ik}, p_{il}, p_{jk}, p_{jl}$ differ by -1 from the $T_{\Delta'}$-degrees while the others stay unchanged. Hence, $\mathrm{in}_\Delta(R_{i,j,k,l}) = \mathrm{in}_{\Delta'}(R_{i,j,k,l})$.

$q = 1$ We assume without loss of generality $\mathbf{x} = \mathbf{a}$ (otherwise relabel accordingly). After mutation, the edge $\mathbf{a}' \to \mathbf{b}'$ contributes to either $i - \mathbf{x}'$ or $j\mathbf{x}'$. We treat the first case, the argument for the second is the same. In this case the T_Δ-degrees of p_{ij}, p_{ik}, p_{il} equal the $T_{\Delta'}$ degrees while all others differ by -1. In particular, the degree of each monomial in $R_{i,j,k,l}$ decreases by 1 and so $\mathrm{in}_\Delta(R_{i,j,k,l}) = \mathrm{in}_{\Delta'}(R_{i,j,k,l})$.

$q = 0$ Observe that this is (up to relabelling if necessary) the case $i \in [n]_{\not\leq \mathbf{a}}, j \in [n]_{\leq \mathbf{d}}, k \in [n]_{\leq \mathbf{e}}, l \in [n]_{\leq \mathbf{c}}$. The tree $T_{\Delta'}|_{i,j,k,l}$ has cherries i, k and j, l. In particular, $\deg_{\Delta'}(p_{ik}p_{jl}) < \deg_{\Delta'}(p_{ij}p_{ik}) = \deg_{\Delta'}(p_{il}p_{jk})$ and the claim follows.

□

Chapter 4

Flag and Schubert varieties

4.1 Preliminary notions

Definition 40. A *complete flag* in the vector space $\mathbb{C}^n$ is a chain

$$\mathbb{V}:\ \{0\} = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = \mathbb{C}^n$$

of vector subspaces of $\mathbb{C}^n$ with $\dim_{\mathbb{C}}(V_i) = i$.

The set of all complete flags in $\mathbb{C}^n$ is denoted by $\mathcal{F}\ell_n$ and it has the structure of an algebraic variety. More precisely, it is a subvariety of the product of Grassmannians $\mathrm{Gr}(1,\mathbb{C}^n)\times \mathrm{Gr}(2,\mathbb{C}^n)\times\cdots\times\mathrm{Gr}(n-1,\mathbb{C}^n)$.

Composing with the Plücker embeddings of the Grassmannians, $\mathcal{F}\ell_n$ becomes a subvariety of $\mathbb{P}^{\binom{n}{1}-1}\times\mathbb{P}^{\binom{n}{2}-1}\times\cdots\times\mathbb{P}^{\binom{n}{n-1}-1}$ and so we can ask for its defining ideal I_n. Each point in the flag variety can be represented by an $n\times n$-matrix $M = (x_{i,j})$ whose first d rows generate V_d. Each V_d corresponds to a point in a Grassmannian. Moreover, they satisfy the condition $V_d \subset V_{d+1}$ for $d = 1,\dots,n-1$. In order to compute the ideal I_n defining $\mathcal{F}\ell_n$ in $\mathbb{P}^{\binom{n}{1}-1}\times\mathbb{P}^{\binom{n}{2}-1}\times\cdots\times\mathbb{P}^{\binom{n}{n-1}-1}$ we have to translate the inclusions $V_d\subset V_{d+1}$ into polynomial equations. We define the map

$$\varphi_n : \mathbb{C}[p_J \mid \varnothing \neq J \subsetneq [n]] \to \mathbb{C}[x_{i,j} \mid i,j\in[n]] \tag{4.1.1}$$

sending each Plücker variable p_J to the determinant of the submatrix of M with row indices $[|J|]$ and column indices J. The ideal I_n of $\mathcal{F}\ell_n$ is the kernel of φ_n.

Example 14. Consider $\mathcal{F}\ell_4 \hookrightarrow \mathbb{P}^3\times\mathbb{P}^5\times\mathbb{P}^3$. The Plücker variables are

$$p_1, p_2, p_3, p_4, p_{12}, p_{13}, p_{23}, p_{14}, p_{24}, p_{34}, p_{123}, p_{124}, p_{134}, p_{234}.$$

Then $\ker(\varphi_4)$ contains the Plücker relation defining $\mathrm{Gr}(2,\mathbb{C}^4)$:

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}.$$

And further relations between Plücker variables associated to different Grassmannians:

$$\begin{gathered}
p_4p_{23} - p_3p_{24} + p_2p_{34},\ p_4p_{13} - p_3p_{14} + p_1p_{34},\\
p_4p_{12} - p_2p_{14} + p_1p_{24},\ p_3p_{12} - p_2p_{13} + p_1p_{23},\\
p_{34}p_{124} - p_{24}p_{134} + p_{14}p_{234},\ p_{34}p_{123} - p_{23}p_{134} + p_{13}p_{234},\\
p_{24}p_{123} - p_{23}p_{124} + p_{12}p_{234},\ p_{14}p_{123} - p_{13}p_{124} + p_{12}p_{134},\\
p_4p_{123} - p_3p_{124} + p_2p_{134} - p_1p_{234}.
\end{gathered}$$

The above relations form a complete list of generators for the ideal I_4.

There is an action of $S_n \rtimes \mathbb{Z}_2$ on $\mathcal{F}\ell_n$. The symmetric group acts by permuting the columns of M. The action of $\mathbb{Z}_2$ maps a complete flag $\mathbb{V}$ to its complement, which is defined to be

$$\mathbb{V}^\perp: \ \{0\} = V_n^\perp \subset V_{n-1}^\perp \subset \cdots \subset V_1^\perp \subset V_0^\perp = \mathbb{C}^n.$$

We hence do computations in §4.3 up to $S_n \rtimes \mathbb{Z}_2$-symmetry.

Recall our notation for SL_n from §2.1. Representing flags by matrices corresponds to realizing the flag variety as the quotient SL_n/B. We construct line bundles on SL_n/B as follows. Consider a weight $\lambda \in \Lambda^+$, it is a character of B (i.e. a morphism of algebraic groups $\lambda : B \to \mathbb{C}^*$). We have a free action of B on $SL_n \times \mathbb{C}$, which for $b \in B, g \in SL_n$ and $t \in T$ is given by

$$b(g,t) := (gb^{-1}, \lambda(b)t).$$

Let L_λ be the fibre product $SL_n \times_B \mathbb{C} = (SL_n \times \mathbb{C})/B$. Then there is a map

$$L_\lambda \to SL_n/B, \text{ given by } (g,t)B \mapsto gB.$$

It follows that L_λ is the total space of a line bundle over SL_n/B called the *homogeneous line bundle associated to the weight* λ. These line bundles satisfy $L_{m\lambda} = L_\lambda^{\otimes m}$ for $m \geq 1$ and are ample, if $\lambda \in \Lambda^{++}$. By the *Borel-Weil-Theorem* we have the following correspondence between line bundles L_λ for $\lambda \in \Lambda^{++}$ and irreducible highest weight representations

$$H^0(SL_n/B, L_\lambda)^* \cong V(\lambda).$$

Recall that the highest weight representation is cyclically generated by a highest weight vector $v_\lambda \in V(\lambda)$. Then the above correspondence induces an embedding

$$SL_n/B \hookrightarrow \mathbb{P}(V(\lambda)), \ gB \mapsto g[v_\lambda].$$

In particular, we can realize the homogeneous coordinate ring of the flag variety as $\mathbb{C}[SL_n/B] = \bigoplus_{k\geq 0} V(k\lambda)^*$. Similarly, we obtain $\mathbb{C}[SL_n/U] = \bigoplus_{\lambda\in\Lambda^+} V(\lambda)$ which is a consequence of the *Peter-Weyl-Theorem*. The quasi-affine variety SL_n/U is sometimes also called *base affine space*.

In the next section we consider Schubert varieties. These are subvarieties of SL_n/B indexed by Weyl group elements $w \in S_n = N_{SL_n}(T)/T$. We identify $w \in S_n$ with a coset representative in the quotient $N_{SL_n}(T)/T$ and consider the *Bruhat cell* $BwB \subset SL_n$. The quotient BwB/B is called *Schubert cell*.

Definition 41. For $w \in S_n$ the *Schubert variety* $X_w \subset SL_n/B$ is defined as the Zariski closure $X_w := \overline{BwB/B}$.

Schubert varieties are normal, not necessarily smooth (but if singular having only rational singularities) subvarieties of the flag variety. Their dimension equals the length of the associated Weyl group element, i.e. $\dim X_w = \ell(w)$. The line bundles L_λ can be restricted to Schubert varieties and the Borel-Weil Theorem generalizes as follows. Fix $w \in S_n$ and $\lambda \in \Lambda^+$, then

$$H^0(X_w, L_\lambda)^* \cong V_w(\lambda),$$

where $V_w(\lambda)$ is the Demazure module (see Definition 2 in §2.1) Observe, that the Borel-Weil-Theorem is in fact a special case as we have $X_{w_0} = SL_n/B$ and $V_{w_0}(\lambda) = V(\lambda)$. Using observation one can generalize many constructions for SL_n/B that rely on Borel-Weil to Schubert varieties. An example of this incidence can be found in the following section when studying string polytopes for flag and Schubert varieties.

4.2 String cones and the Superpotential

In this section we study the combinatorics of pseudoline arrangements. We associate to each in dual ways two collections of polyhedral objects (consisting of two polyhedral cones and a $\mathbb{R}^{n-1}$-family of polytopes). We show they are unimodularly equivalent and relate the cones to the geometry of flag and Schubert varieties. We prove that one of them is the weighted string cone by Littelmann [52] (see also Berenstein-Zelevinsky [5]). It was used by Caldero [13] to degenerate flag and Schubert varieties to toric varieties. For the other we show that it is closely related to the framework of cluster varieties and mirror symmetry by Gross-Hacking-Keel-Kontsevich [37]. For the flag variety the cone is the tropicalization of their superpotential while for Schbert varieties a restriction of the superpotential is necessary. In their framework they also give a construction of toric degenerations using the superpotential. As a corollary of our combinatorial result we realize Caldero's degenerations as GHKK-degenerations using cluster theory.

The section is structured as follows: we recall pseudoline arrangements and define the two collections of polyhedral objects and unimodular equivalences among them in §4.2.1. In §4.2.2 we show that of the cones is the weighted string cone and in §4.2.3 we show how the other arises from the superpotential. Then in §4.2.4 we apply our combinatorial result and relate to toric degenrations.[1]

4.2.1 Pseudoline arrangements and Gleizer-Postnikov paths

Recall our notation for the symmetriy group S_n from §2.1. In the following section we associate for $w \in S_n$ a diagram called a pseudoline arrangement to every reduced expression $\underline{w}$. These diagrams turn out to be closely related to cluster algebras. In fact, to every pseudoline arrangement one can associate a quiver and then using the construction summarized in §2.5 define a cluster algebra. We start by introducing the combinatorial tools: to a pseudoline arrangement we associate two weighted cones and give a unimodular equivalence between them.

Definition 42. A *pseudoline arrangement* $\mathsf{pa}(\underline{w})$ associated to a reduced expression $\underline{w} = s_{i_1} \cdots s_{i_{\ell(w)}}$ is a diagram consisting of n horizontal *pseudolines* $l_1, \dots, l_n$ (or short *lines*) labelled at the left end from bottom to top, with crossings indicated by the reduced expression. A reflection s_i indicates a crossing at *level* i (see e.g. Figure 4.4).

For a given reduced expression $\underline{w} = s_{i_1} \cdots s_{i_{l(w)}}$, we associate to each s_{i_j} the positive root $\beta_{i_j} := s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$. Then $\beta_{i_j} = \alpha_{k,m-1}$ for $k, m < n$ and s_{i_j} induces the crossing of the lines l_k and l_m in $\mathsf{pa}(\underline{w})$. The crossing point is a vertex in the diagram and it is labelled (k, m). As two lines l_k, l_m cross at most once, there is at most one position with label (k, m). For a given w the pairs appearing as labels for crossing points are exactly those for which $w(\alpha_{k,m-1}) < 0$. Further, the right end of a pseudoline l_i is a vertex labelled L_i. Let $\mathsf{pa}(\underline{w})_0$ be the set of all vertices in $\mathsf{pa}(\underline{w})$.

Definition 43. [4, Definition 2.2] Let $w \in S_n$ with reduced expression $\underline{w}$. Then the quiver $Q_{\underline{w}}$ associated to $\mathsf{pa}(\underline{w})$ has *vertices* w_F associated to faces F of $\mathsf{pa}(\underline{w})$ and *arrows*:

(1) if two faces are at the same level separated by a crossing then there is an arrow from

[1] based on joint work with Ghislain Fourier.

left to right (see Figure 4.1a);

(2) if two faces are on consecutive levels separated by two crossings then there is an arrow from right to left (either upwards or downwards, see Figure 4.1b, 4.1c).

Vertices corresponding to unbounded faces are *frozen* and we disregard arrows between them. All the other vertices are called *mutable*.

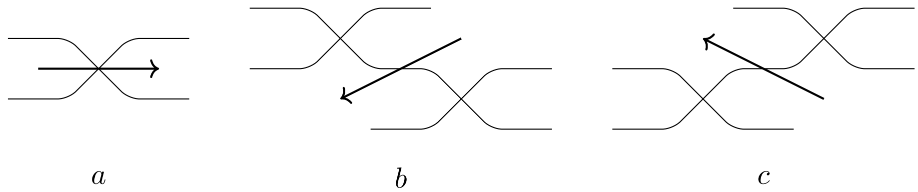


Figure 4.1: Arrows of the quiver arising from the pseudoline arrangement.

Recall the notion of of quiver mutation from Definition 16 in §2.5.

Definition 44. Let $w \in S_n$ with reduced expression $\underline{w}$. A *mutation* of $\mathsf{pa}(\underline{w})$ (resp. of $\underline{w}$) is a change of consecutive $s_r s_{r+1} s_r$ in $\underline{w}$ to $s_{r+1} s_r s_{r+1}$ (or vice versa) (see Figure 4.2). We call a face F of $\mathsf{pa}(\underline{w})$ *mutable* if it corresponds to $s_r s_{r+1} s_r$ (or $s_{r+1} s_r s_{r+1}$) and denote the corresponding mutation by μ_F. The resulting pseudoline arrangement is associated to the reduced expression $\mu_F(\underline{w})$ of w and denoted by $\mathsf{pa}(\mu_F(\underline{w}))$.

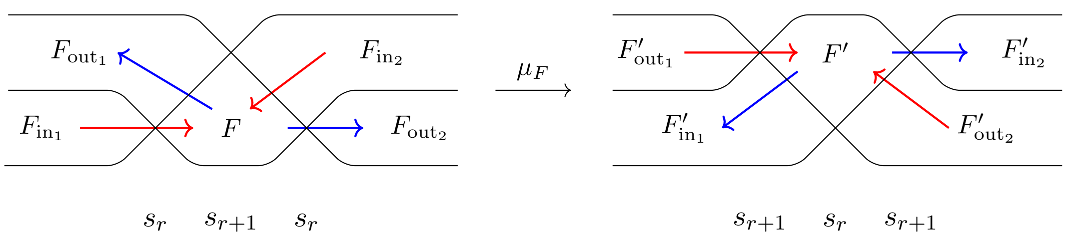


Figure 4.2: Mutation of pseudoline arrangements.

Note, that the quivers $Q_{\underline{w}}$ and $Q_{\mu_{F(\underline{w})}}$ are related by quiver mutation at the vertex w_F. However, $Q_{\underline{w}}$ has more mutable vertices than $\mathsf{pa}(\underline{w})$ has mutable faces. When mutating $Q_{\underline{w}}$ at a vertex $w_{F'}$ with F' not mutable in $\mathsf{pa}(\underline{w})$, then for $\mu_{F'}(Q_{\underline{w}})$ there is no reduced expression of w that would give rise to this quiver via a pseudoline arrangement.

Consider $\underline{w}_0 \in S_n$ with reduced expression $\underline{\hat{w}}_0 := s_1 s_2 s_1 s_3 s_2 s_1 \dots s_{n-1} s_{n-2} \dots s_3 s_2 s_1$ and the quiver $Q_{\underline{\hat{w}}_0}$. We label the vertices for faces $F_{(i,j)}$ bounded to the left by the crossing of lines l_i and l_j by $w_{(i,j)}$. In particular, the frozen vertices at the right boundary are labelled $w_{(n-1,n)}, \dots, w_{(1,n)}$ from bottom to top. Referring to their level, the frozen vertices on the left boundary are labelled by $w_1, \dots, w_{n-1}$ from bottom to top. In the following example we describe the quiver corresponding to this *initial* reduced expression $\underline{\hat{w}}_0$ for $n = 5$.

Example 15. Consider $\underline{\hat{w}}_0 = s_1 s_2 s_1 s_3 s_2 s_1 s_4 s_3 s_2 s_1 \in S_5$. The pseudoline arrangement and the corresponding quiver are depicted in Figure 4.3.

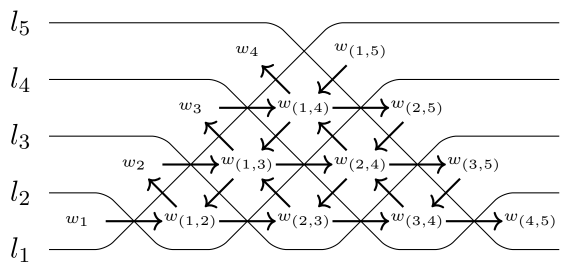


Figure 4.3: $\mathsf{pa}(\underline{\hat{w}}_0)$ and $Q_{\underline{\hat{w}}_0}$ with $\underline{\hat{w}}_0 = s_1 s_2 s_1 s_3 s_2 s_1 s_4 s_3 s_2 s_1 \in S_5$.

Orientation and paths.

For every pair (l_i, l_{i+1}) with $1 \leq i \leq n-1$ we give an orientation to a pseudoline arrangement by orienting lines $l_1, \ldots, l_i$ from right to left and lines $l_{i+1}, \ldots, l_n$ from left to right, see Figure 4.4. Consider an oriented path with three consecutive crossings $v_{k-1} \to v_k \to v_{k+1}$ belonging to the same pseudoline l_i. Then v_k is the intersection of l_i with some line l_j, i.e. $v_k = v_{(i,j)}$. If either $i < j$ and both lines are oriented to the left, or $i > j$ and both lines are oriented to the right, the path is called *non-rigorous*. Figure 4.5 shows these two situations. A path is called *rigorous* if it is not non-rigorous.

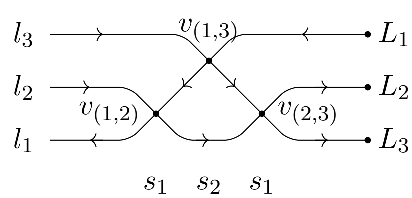


Figure 4.4: $\mathsf{pa}(\underline{w}_0)$ for $\underline{w}_0 = s_1 s_2 s_1 \in S_3$ with orientation for (l_1, l_2).

Definition 45. Let $\underline{w}$ be a fixed reduced expression of $w \in S_n$.

A *Gleizer-Postnikov path* (or short *GP-path*) is a rigorous path $\mathbf{p}$ in $\mathsf{pa}(\underline{w})$ endowed with some orientation (l_i, l_{i+1}) for $i \in [n-1]$. It has source L_p and sink L_q for $p \leq i$ and $q \geq i+1$. Further, $w(i+1) \leq w(p) \leq w(i)$ and $w(i+1) \leq w(q) \leq w(i)$. The set of all GP-paths for all orientations in the pseudoline arrangement associated to $\underline{w}$ is denoted by $\mathcal{P}_{\underline{w}}$.

Figure 4.5: The two red arrows are forbidden in rigorous paths.

Note that if $w(i) < w(i+1)$ there are no GP-paths of shape (l_i, l_{i+1}) and in case $w(p) \leq w(q)$ there are no GP-paths with source L_p and sink L_q.

Proposition 8. Let $w \in S_n$ with reduced expression $\underline{w}$. Consider $\mathbf{p} \in \mathcal{P}_{\underline{w}}$ of shape (l_i, l_{i+1}). Then $\mathbf{p}$ is either the empty path or does not cross the lines l_{i+1} and l_i. In particular, $\mathbf{p}$ does not leave the area in $\mathsf{pa}(\underline{w})$ bounded by l_i and l_{i+1} to the left.

Proof. Without loss of generality we assume $w(i) < w(i+1)$, otherwise $\mathbf{p}$ is empty and we are done. Further, let L_p be the source of $\mathbf{p}$ and L_q the sink. We assume $w(p) \leq w(q)$, otherwise, again, $\mathbf{p}$ is empty. We focus on the part of $\mathsf{pa}(\underline{w})$ to the *right* of the crossing of l_i and l_{i+1} (which exists as $w(i) < w(i+1)$). Observe the following:

- all lines crossing l_i do so oriented from top to bottom.
- all lines crossing l_{i+1} do so oriented from bottom to top.

As L_p and L_q lie in between the lines l_i and l_{i+1} this observation implies that $\mathbf{p}$ can not cross l_i and if it was to cross l_{i+1} it could not return to L_q, a contradiction. The only possibility that is left, is if $\mathbf{p}$ was to follow l_i through the crossing with l_{i+1}, but then again, it could not return to L_q. □

Cones and polytopes arising from pseudoline arrangements

We define two weighted cones, two cones, and two families of polytopes that arise from $\mathcal{P}_{\underline{w}}$ for $\underline{w}$ reduced expression of $w \in S_n$. We relate the two cones in the forthcoming sections, one to the weighted string cone (introduced by Littelmann [52] and Berenstein-Zelevinsky [5]), the other to the tropicalization of the (restriction of the) superpotential for a double Bruhat cell (see Magee [54]).

The (weighted) GP-cone For $\underline{w} = s_{i_1} \dots s_{i_{\ell(w)}}$ we label the standard basis of $\mathbb{R}^{\ell(w)}$ by crossing points in $\mathsf{pa}(\underline{w})$, i.e. $\{c_{(k,m)} \mid w(\alpha_{k,m-1}) < 0\}$. Sometimes it is also convenient to use the notation $c_{i_j} := c_{(k,m)}$, when s_{i_j} induces the crossing of l_k and l_m in $\mathsf{pa}(\underline{w})$. Consider $\mathbf{p} \in \mathcal{P}_{\underline{w}}$. It is uniquely determined by those vertices in $\mathsf{pa}(\underline{w})_0$ where $\mathbf{p}$ changes from one line to another. For some $1 \leq p \leq i < q \leq n$ we can therefore write $\mathbf{p}$ as

$$\mathbf{p} = L_p \to v_{(p,j_1)} \to v_{(j_1,j_2)} \to \dots \to v_{(j_k,q)} \to L_q.$$

Set $j_0 := p$ and $j_{k+1} := q$, then we associate to $\mathbf{p}$ the vector

$$c_{\mathbf{p}} := \sum_{s=0}^{k} c_{(j_s,j_{s+1})} \in \mathbb{R}^{\ell(w)}, \tag{4.2.1}$$

where we set $c_{(i,j)} := -c_{(j,i)}$ if $i > j$ and $c_{(i,i)} := 0$.

Definition 46. The following polyhedral cone is called *GP-cone* (due to Gleizer-Postnikov [32] who call it *principal cone*):

$$C_{\underline{w}} = \{\mathbf{x} \in \mathbb{R}^{\ell(w)} \mid (c_{\mathbf{p}})^t(\mathbf{x}) \geq 0, \forall \mathbf{p} \in \mathcal{P}_{\underline{w}}\}. \tag{4.2.2}$$

Example 16. Consider the reduced expression $\underline{w_0} = s_1 s_2 s_1 \in S_3$. We endow $\mathsf{pa}(s_1 s_2 s_1)$ with the orientation for (l_1, l_2), i.e. l_1 is oriented to the left and l_2, l_3 are oriented to the right (see Figure 4.4). There are two paths in $\mathcal{P}_{s_1 s_2 s_1}$ from L_1 to L_2,

$$\mathbf{p}_1 = L_1 \to v_{(1,3)} \to v_{(1,2)} \to v_{(2,3)} \to L_2 \text{ and } \mathbf{p}_2 = L_1 \to v_{(1,3)} \to v_{(2,3)} \to L_2.$$

They yield $c_{\mathbf{p}_1} = c_{(1,2)}$ and $c_{\mathbf{p}_2} = c_{(1,3)} - c_{(2,3)}$. Similarly for the orientation (l_2, l_3) we find a path $\mathbf{p}_3 = L_2 \to v_{(2,3)} \to L_3$ with $c_{\mathbf{p}_3} = c_{(2,3)}$. Then

$$C_{s_1 s_2 s_1} = \{(x_{(1,2)}, x_{(1,3)}, x_{(2,3)}) \in \mathbb{R}^3 \mid x_{(1,2)} \geq 0, x_{(1,3)} \geq x_{(2,3)} \geq 0\}.$$

We are interested in a weighted version of this cone to relate it to string polytopes in the next section. The weighted cone lives in $\mathbb{R}^{\ell(w)+n-1}$, where the additional basis elements are indexed $c_1, \dots, c_{n-1}$. By some abuse of notation we denote by $c_{\mathbf{p}}$ also the vector $(c_{\mathbf{p}}, 0 \dots, 0) \in \mathbb{R}^{\ell(w)} \times \{0\}^{n-1} \subset \mathbb{R}^{\ell(w)+n-1}$.

For every $i \in [n-1]$ we define the following subset of $[\ell(w)]$

$$J(i) := \{k \in [\ell(w)] \mid s_{i_k} = s_i\} \text{ with } n_i := \#J(i). \tag{4.2.3}$$

Let $J(i) = \{j_1, \dots, j_{n_i}\}$, then we set $c_{[i:0]} := c_i$ and for $1 \leq k \leq n_i$ we define

$$c_{[i:k]} := c_i - c_{i_{j_k}} - 2 \sum_{j \in J(i), j > j_k} c_{i_j} + \sum_{l \in J(i-1) \cup J(i+1), l > j_k} c_{i_l}. \tag{4.2.4}$$

These vectors are normal vectors to the faces of the following weighted cone.

Definition 47. The *weighted Gleizer-Postnikov cone* $\mathcal{C}_{\underline{w}} \subset \mathbb{R}^{\ell(w)+n-1}$ is defined as

$$\mathcal{C}_{\underline{w}} := \left\{ \mathbf{x} \in \mathbb{R}^{\ell(w)+n-1} \,\middle|\, \begin{array}{ll} (c_{\mathbf{p}})^t(\mathbf{x}) \geq 0 \,, & \forall\, \mathbf{p} \in \mathcal{P}_{\underline{w}}, \\ (c_{[i:k]})^t(\mathbf{x}) \geq 0, & \forall\, i \in [n-1], 0 \leq k \leq n_i \end{array} \right\}. \tag{4.2.5}$$

Example 17. Consider $w_0 \in S_n$ and consider the reduced expression $\underline{\hat{w}}_0$ defined above Example 15. For $i \in [n-1]$ all GP-paths in $\mathsf{pa}(\underline{w}_0)$ with orientation (l_i, l_{i+1}) are of form

$$\mathbf{p}_{i,j} := L_i \to v_{(i,n)} \to v_{(i,n-1)} \to \cdots \to v_{(i,j)} \to v_{(i+1,j)} \to \cdots \to v_{(i+1,n)} \to L_{i+1}.$$

In particular, the GP-cone $C_{\underline{w}_0}$ is described by inequalities defined by the normal vectors $c_{(i,j+1)} - c_{(i+1,j+1)}$ and $c_{(i,i+1)}$ for $i \in [n-1]$ and $j \in [i+1, n-1]$. The vectors defining weight inequalities are (for all $i < j$):

$$c_{j-i} - c_{(i,j)} - 2\sum_{k=1}^{n-j} c_{(i+k,j+k)} + \sum_{k=0}^{n-j-1} c_{(i+k,j+1+k)} + \sum_{k=0}^{n-j} c_{(i+1+k,j+k)}.$$

The (weighted) area cone We associate to the set of all GP-paths $\mathcal{P}_{\underline{w}}$ a second cone. In this setup, the standard basis of $\mathbb{R}^{\ell(w)+n-1}$ is indexed by the faces of the pseudoline arrangement $\{e_F \mid F \text{ face of } \mathsf{pa}(\underline{w})\}$. Namely, there are basis vectors associated to faces $F_{(i,j)}$ bounded to the left by a crossing (i,j), and to faces F_l unbounded to the left for every $l \in [n-1]$. Let $\mathbf{p} \in \mathcal{P}_{\underline{w}}$. We denote by $\mathsf{A}_{\mathbf{p}}$ the area to the left of $\mathbf{p}$ (with respect to the orientation), i.e. the area enclosed by $\mathbf{p}$. Note that for non-trivial $\mathbf{p}$, $\mathsf{A}_{\mathbf{p}}$ is a non-empty union of faces F in the pseudoline arrangement. We associate to $\mathbf{p}$ the vector

$$e_{\mathbf{p}} := -\sum_{F \subset \mathsf{A}_{\mathbf{p}}} e_F \in \mathbb{R}^{\ell(w)+n-1}. \tag{4.2.6}$$

With a little abuse of notation we denote by $e_{\mathbf{p}}$ also the vector in $\mathbb{R}^{\ell(w)}$ obtained by projecting onto the first $\ell(w)$ coordinates (forgetting the coordinates belonging to the faces that are unbounded to the left, which equal 0 in $e_{\mathbf{p}}$).

Definition 48. For a reduced expression $\underline{w} \in S_n$, we define the *area cone*

$$S_{\underline{w}} := \{\mathbf{x} \in \mathbb{R}^{\ell(w)} \mid (e_{\mathbf{p}})^t(\mathbf{x}) \geq 0, \forall\, \mathbf{p} \in \mathcal{P}_{\underline{w}}\}. \tag{4.2.7}$$

Again, we are interested in a weighted extension of this cone. For this, we associate to every level $i \in [n-1]$ a union of faces. Consider F_i, the face of $\mathsf{pa}(\underline{w})$ that is unbounded to the left at level i. As before for crossings we set $F_{i_j} := F_{(k,m)}$ if s_{i_j} in $\underline{w}$ induces the crossing of l_k and l_m in $\mathsf{pa}(\underline{w})$. We define $\mathsf{A}_i := F_i \cup \bigcup_{k=1}^{n_i} F_{i_k}$, then $\mathsf{A}_i \cap \mathsf{A}_{i'} = \varnothing$ if $i \neq i'$. It is called the *weight area* associated to the level i. For each k with $0 \leq k \leq n_i$, we define a vector

$$e_{[i:k]} := -e_{F_i} - \sum_{j \in J(i), j \leq j_k} e_{F_{i_j}} \in \mathbb{R}^{\ell(w)+n}. \tag{4.2.8}$$

Note that $e_{[i:0]} = -e_{F_i}$ and $e_{[i:n_i]} = -\sum_{F \subset \mathsf{A}_i} e_F$.

Definition 49. The *weighted area cone* $\mathcal{S}_{\underline{w}} \subset \mathbb{R}^{\ell(w)+n-1}$ associated to the reduced expression $\underline{w}$ of $w \in S_n$ is defined as

$$\mathcal{S}_{\underline{w}} := \left\{ \mathbf{x} \in \mathbb{R}^{\ell(w)+n-1} \;\middle|\; \begin{array}{ll} (e_{\mathbf{p}})^t(\mathbf{x}) \geq 0\,, & \forall\, \mathbf{p} \in \mathcal{P}_{\underline{w}}, \\ (e_{[i:k]})^t(\mathbf{x}) \geq 0\,, & \forall\, i \in [n-1], 0 \leq k \leq n_i \end{array} \right\}. \tag{4.2.9}$$

The additional inequalities induced by the $e_{[i:k]}$ are called *weight inequalities*.

Remark 9. In all four cases, $C_{\underline{w}}, \mathcal{C}_{\underline{w}}, S_{\underline{w}}$ and $\mathcal{S}_{\underline{w}}$, some of the inequalities might be redundant and these cones are far from being simplical in general. The vectors $e_{\mathbf{p}}, c_{\mathbf{p}}, e_{[i:k]}$ and $c_{[i:k]}$ are normal vectors to the defining hyperplanes of the cones $S_{\underline{w}}, C_{\underline{w}}, \mathcal{S}_{\underline{w}}$ and $\mathcal{C}_{\underline{w}}$ respectively. Not all of them are normal vectors to facets of these cones in general.

Example 18. Consider the reduced expression $\underline{\hat{w}}_0 \in S_5$. We have seen all GP-paths in $\mathsf{pa}(\underline{w})$ in Example 17. Take the path $\mathbf{p} = L_1 \to v_{(1,5)} \to v_{(1,4)} \to v_{(1,3)} \to v_{(1,2)} \to v_{(2,3)} \to v_{(2,4)} \to v_{(2,5)} \to L_2$. The area $\mathsf{A}_{\mathbf{p}}$ associated to this path is shaded blue in Figure 4.6. The weight area A_2 corresponding to level 2 is also shown in Figure 4.6 dotted in red.

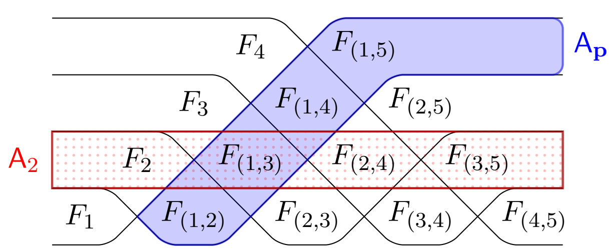


Figure 4.6: The area $\mathsf{A}_{\mathbf{p}}$ for $\mathbf{p}$ as in Example 18 shaded in blue and the weight area A_2 dotted in red.

Example 19. Consider as in Example 17 $\underline{\hat{w}}_0 \in S_n$ and recall $\mathbf{p}_{i,j} \in \mathcal{P}_{\underline{w}_0}$ with $i \in [n-1]$ and $j \in [i+1, n-1]$. The assigned area is $\mathsf{A}_{\mathbf{p}_{i,j}} = F_{(i,j)} \cup F_{(i,j+1)} \cup \cdots \cup F_{(i,n)}$ for $F_{(i,k)}$ the area bounded by $v_{(i,k)}$ to the left. Hence, the cone $S_{\underline{w}_0}$ is given by inequalities defined by

$$e_{\mathbf{p}_{i,j}} = -e_{F_{(i,j)}} - e_{F_{(i,j+1)}} - \cdots - e_{F_{(i,n)}} - e_{F_{(i,n+1)}}. \tag{4.2.10}$$

The additional weight inequalities defining the cone $\mathcal{S}_{\underline{w}_0}$ are given by the normal vectors

$$e_{[i:k]} = -e_{F_i} - e_{F_{(1,i+1)}} - e_{F_{(2,i+2)}} - \cdots - e_{F_{(k,i+k)}}, \tag{4.2.11}$$

for $i \in [n-1]$ and $0 \leq k \leq n-i$.

The polytopes Let $\pi : \mathbb{R}^{\ell(w)+n-1} \to \mathbb{R}^{n-1}$ be the projection onto the last $n-1$ coordinates, also called *weight coordinates*. We are interested in the preimage $\pi^{-1}(\lambda)$ for $\lambda \in \mathbb{R}^{n-1}$. It is the intersection of the following hyperplanes for each $i \in [n-1]$ defined by

$$(c_{[i:0]})^t(\mathbf{x}) = \lambda_i, \ \forall\, \mathbf{x} \in \mathbb{R}^{\ell(w)+n-1}. \tag{4.2.12}$$

Fix $w \in S_n$ with reduced expression $\underline{w}$. We define a second map $\tau_{\underline{w}} : \mathbb{R}^{\ell(w)+n-1} \to \mathbb{R}^{n-1}$ by $\tau_{\underline{w}}(\mathbf{x}) = ((e_{[i:n_i]})^t(\mathbf{x}))_{i=1,\dots,n-1}$. The preimage of $\lambda \in \mathbb{R}^{n-1}$ with respect to $\tau_{\underline{w}}$ is also an intersection of hyperplanes in $\mathbb{R}^{\ell(w)+n-1}$. For each $i \in [n-1]$ they are defined by

$$(e_{[i:n_i]})^t(\mathbf{x}) = \lambda_i, \ \forall\, \mathbf{x} \in \mathbb{R}^{\ell(w)+n-1}. \tag{4.2.13}$$

Definition 50. For $w \in S_n$ with reduced expression $\underline{w}$ and for $\lambda \in \mathbb{R}^{n-1}$ we define the following polytopes in $\mathbb{R}^{\ell(w)+n-1}$

$$\mathcal{S}_{\underline{w}}(\lambda) := \mathcal{S}_{\underline{w}} \cap \tau_{\underline{w}}^{-1}(\lambda) \text{ and } \mathcal{C}_{\underline{w}}(\lambda) := \mathcal{C}_{\underline{w}} \cap \pi^{-1}(\lambda). \tag{4.2.14}$$

Note that by (4.2.13) (resp. (4.2.12)) we obtain a description of $\mathcal{S}_{\underline{w}}(\lambda)$ (resp. $\mathcal{C}_{\underline{w}}(\lambda)$) in terms of defining equalities and inequalities by replacing the weight inequalities $e^t_{[i:n_i]}(\mathbf{x}) \geq 0$ in (4.2.9) (resp. $(c_i)^t(\mathbf{x}) \geq 0$ in (4.2.5)) by $(e_{[i:n_i]})^t(\mathbf{x}) = \lambda_i$ (resp. $(c_i)^t(\mathbf{x}) = \lambda_i$). In particular, the defining normal vectors for $\mathcal{S}_{\underline{w}}$ (resp. $\mathcal{C}_{\underline{w}}$) coincide with those for $\mathcal{S}_{\underline{w}}(\lambda)$ (resp. $\mathcal{C}_{\underline{w}}(\lambda)$). This observation is important in the proof of Theorem 17.

A unimodular equivalence

The above pairs of cones (resp. polytopes) $(S_{\underline{w}}, C_{\underline{w}})$ and $(\mathcal{S}_{\underline{w}}, \mathcal{C}_{\underline{w}})$ (resp. $(\mathcal{S}_{\underline{w}}(\lambda), \mathcal{C}_{\underline{w}}(\lambda))$) have in fact more in common than the combinatorics defining them. To make this statement precise we need to introduce the notion of unimodular equivalence (see e.g. [40, §2]).

Definition 51. Two polytopes $P, Q \subset \mathbb{R}^d$ (resp. polyhedral cones $C, D \subset \mathbb{R}^d$) are called *unimodularly equivalent* if there exists matrix $M \in GL_d(\mathbb{Z})$ and $w \in \mathbb{Z}^d$

$$Q = f_M(P) + w \text{ (resp. } D = f_M(C) + w),$$

where $f_M(x) = xM$ for $x \in \mathbb{R}^d$. We denote this by $Q \cong P$ (resp. $C \cong D$).

This notion of equivalence is of particular interest to us because of its implication on the associated toric varieties. Recall the construction of a projective toric variety $X_P \subset \mathbb{P}^{d-1}$ associated with a polytope $P \subset \mathbb{R}^d$ in [16, §2.1 and §2.3]. Then

$$Q \cong P \text{ implies } X_Q \cong X_P. \tag{4.2.15}$$

We want to construct a unimodular equivalence between $\mathcal{C}_{\underline{w}}$ and $\mathcal{S}_{\underline{w}}$ for all reduced expression $\underline{w}$ of $w \in S_n$. The following definition is the affine lattice transformation (f_M in Definition 51) that defines the unimodular equivalence. We give it in terms of the bases $\{e_F \mid F \text{ face of } \mathsf{pa}(\underline{w})\}$ and $\{c_{(k,m)}, c_i \mid v_{(k,m)} \in \mathsf{pa}(\underline{w})_0, i \in [n-1]\}$. Morally, we send a face F bounded to the left by a crossing to a linear combination of its adjacent crossings (see (4.2.17)). A face unbounded to the left is sent to the sum of all crossings at its level.

Definition 52. For $w \in S_n$ and $\underline{w}$ a reduced expression we define the linear map $\Psi_{\underline{w}} : \mathbb{R}^{\ell(w)+n-1} \to \mathbb{R}^{\ell(w)+n-1}$ on the basis $\{-e_F\}$ associated to faces F of $\mathsf{pa}(\underline{w})$. Let $F = F_{i_{j_k}}$ be the face bounded to the left by the crossing induced from $s_{i_{j_k}} = s_i$ and $J(i) = \{j_1, \dots, j_{n_i}\}$ (see (4.2.3)). Then

$$\Psi_{\underline{w}}(-e_{F_{i_{j_k}}}) := c_{i_{j_k}} + c_{i_{j_{k+1}}} - \sum_{\substack{j \in J(i-1)\cup J(i+1),\\ j_k < j < j_{k+1}}} c_{i_j}. \tag{4.2.16}$$

For every level $i \in [n-1]$, we define

$$\Psi_{\underline{w}}(-e_{F_i}) := c_{[i:1]}. \tag{4.2.17}$$

Example 20. Consider $\mathsf{pa}(\underline{w})$ for $\underline{w} = s_1 s_2 s_1 \in S_3$ as in Figure 4.4. The two bases for $\mathbb{R}^5$ are

$$\mathcal{B}_e = \{-e_{F_1}, -e_{F_2}, -e_{F_{(1,2)}}, -e_{F_{(1,3)}}, -e_{F_{(2,3)}}\} \text{ and } \mathcal{B}_c = \{c_1, c_2, c_{(1,2)}, c_{(1,3)}, c_{(2,3)}\}.$$

We compute the images of elements in $\mathcal{B}_e$ and express them in $\mathcal{B}_c$. The coefficients form the columns of the following matrix with the order of the bases as given above.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 \\ -2 & 1 & 1 & -1 & 1 \end{pmatrix} \in GL_5(\mathbb{Z}).$$

The observation in the example above is true in general. We obtain the following Lemma as a straightforward consequence of the definition of $\Psi_{\underline{w}}$.

Lemma 14. Let $w \in S_n$ with reduced expression $\underline{w}$. Order the bases induced by the faces of $\mathsf{pa}(\underline{w})$ resp. by the crossing points in $\mathsf{pa}(\underline{w})$ as

$$\mathcal{B}_e = \{-e_{F_1}, \dots, -e_{F_{n-1}}, -e_{F_{i_1}}, \dots - e_{F_{i_{\ell(w)}}}\}, \text{ resp. } \mathcal{B}_c = \{c_1, \dots, c_{n-1}, c_{i_1}, \dots, c_{i_{\ell(w)}}\}.$$

Then $\Psi_{\underline{w}}$ can be represented by a lower triangular matrix $M_{\underline{w}}^{e,c}$ with all diagonal entries being 1. In particular, $M_{\underline{w}}^{e,c} \in GL_{\ell(w)+n-1}(\mathbb{Z})$.

Corollary 10. With assumptions as in Lemma 14 consider $\Psi_{\underline{w}}|_{\mathbb{R}^{\ell(w)}} : \mathbb{R}^{\ell(w)} \to \mathbb{R}^{\ell(w)}$. We order as before the bases for $\mathbb{R}^{\ell(w)}$ induced by the faces resp. crossing points in $\mathsf{pa}(\underline{w})$ by

$$\overline{\mathcal{B}}_e = \{-e_{F_{i_1}}, \dots - e_{F_{i_{\ell(w)}}}\}, \text{ resp. } \overline{\mathcal{B}}_c = \{c_{i_1}, \dots, c_{i_{\ell(w)}}\}.$$

Then $\Psi_{\underline{w}}|_{\mathbb{R}^{\ell(w)}}$ can be represented by a lower triangular matrix $\overline{M}_{\underline{w}}^{e,c}$ with all diagonal entries 1. In particular, $\overline{M}_{\underline{w}}^{e,c} \in GL_{\ell(w)}(\mathbb{Z})$.

Remark 10. The map $\Psi_{\underline{w}}$ restricted to $\mathbb{R}^{\ell(w)}$ is related to the Chamber Ansatz due to Berenstein-Fomin-Zelevinsky in [3] (see also [30]).

Proposition 9. Let $w \in S_n$ with reduced expression $\underline{w}$. For every $\mathbf{p} \in \mathcal{P}_{\underline{w}}$ we have

$$\Psi_{\underline{w}}(e_{\mathbf{p}}) = c_{\mathbf{p}}.$$

In particular, $\Psi_{\underline{w}}$ sends the normal vector of a defining hyperplane of $\mathcal{S}_{\underline{w}}$ to the normal vector of a defining hyperplane of $\mathcal{C}_{\underline{w}}$.

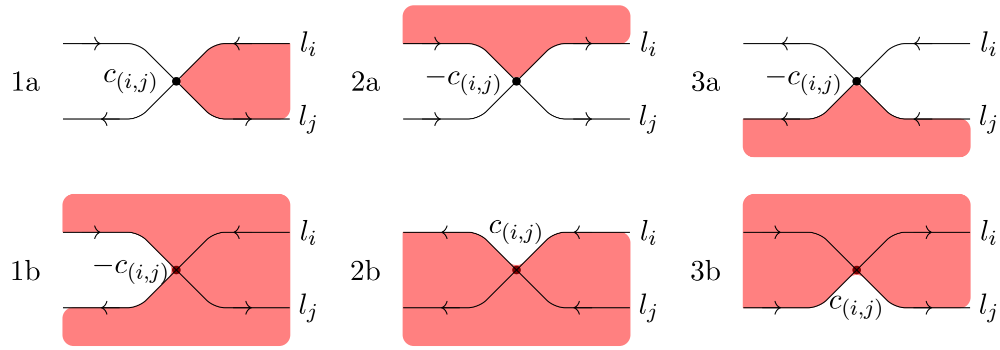


Figure 4.7: A path $\mathbf{p}$ changing the line at a crossing (i, j) and the corresponding area $\mathsf{A}_\mathbf{p}$.

Proof. We show that for every crossing point (i, j) the coefficient of $c_{(i,j)}$ coincides in $\Psi_{\underline{w}}(e_\mathbf{p})$ and $c_\mathbf{p}$. Recall that $\mathsf{A}_\mathbf{p}$ is the union of all faces to the left of $\mathbf{p}$ with respect to the given orientation. We distinguish three cases:

- If (i, j) lies in the interior of $\mathsf{A}_\mathbf{p}$, then four faces $F^r \subset \mathsf{A}_\mathbf{p}$ with $r \in [4]$ are adjacent to (i, j). For two of them in $\Psi_{\underline{w}}(-e_{F^r})$ the coefficient of $c_{(i,j)}$ is $+1$, for the other two it is -1. Hence, they cancel each other and in $\Psi_{\underline{w}}(-e_\mathbf{p})$ it is zero as it is in $c_\mathbf{p}$.

- If $\mathbf{p}$ contains (i, j) but does not change the line at (i, j), then $c_{(i,j)}$ has coefficient zero in $c_\mathbf{p}$. For $\mathsf{A}_\mathbf{p}$, this means that two faces, F^1 and F^2, are adjacent to (i, j). One of the two, say F^1, is bounded by (i, j) to the left where for F^2, (i, j) is part of the upper or lower boundary. In particular, $\Psi_{\underline{w}}(e_\mathbf{p})$ contains $c_{(i,j)}$ once with positive and once with negative sign, hence with the coefficient is zero.

- Assume $\mathbf{p}$ changes the line at the crossing (i, j). Figure 4.7 shows the three possible orientations of l_i and l_j. Each yields two possibilities for the path. If in situation 1a, there is one face F in $\mathsf{A}_\mathbf{p}$ bounded by (i, j) to the left. So $c_{(i,j)}$ has coefficient 1 in $\Psi_{\underline{w}}(e_\mathbf{p})$. As $\mathbf{p}$ changes from l_i to l_j and $i < j$, also $c_\mathbf{p}$ contains $c_{(i,j)}$ with coefficient 1.

  In cases 2a and 3a, $\mathsf{A}_\mathbf{p}$ contains only one face bounded by (i, j) below resp. above. Hence $c_{(i,j)}$ appears with coefficient -1 in $\Psi_{\underline{w}}(e_\mathbf{p})$. The same is true for $c_\mathbf{p}$: in both cases $\mathbf{p}$ changes from line l_j to l_i but $i < j$.

  Three cases remain to be checked, 1b, 2b and 3b in Figure 4.7. In all of them $\mathsf{A}_\mathbf{p}$ contains three faces F^1, F^2 and F^3 adjacent to (i, j). In case 1b, (i, j) bounds one face to the left and the other two from above, resp. below. This implies that $c_{(i,j)}$ appears with coefficient -1 in $\Psi_{\underline{w}}(e_\mathbf{p})$. As $\mathbf{p}$ changes from l_j to l_i the same is true for $c_\mathbf{p}$. For 2b and 3b we are in the opposite case: two faces in $\mathsf{A}_\mathbf{p}$ are bounded to the left, resp. right, by (i, j) and only one from above, resp. below. Hence, $\Psi_{\underline{w}}(e_\mathbf{p})$ contains $c_{(i,j)}$ with coefficient 1 and the same is true for $c_\mathbf{p}$, as $\mathbf{p}$ changes from line l_i to line l_j.

$\square$

For our application later, it remains to show that the normal vectors defining the weight inequalities are mapped onto each other by $\Psi_{\underline{w}}$. Recall the weight area $\mathsf{A}_i = F_i \cup \bigcup_{r=1}^{n_i} F_{(i_r,j_r)}$ of level $i \in [n-1]$, with n_i the number of faces bounded to the left of level i.

Proposition 10. Let $w \in S_n$ with reduced expression $\underline{w}$. Consider $i \in [n-1]$ with $J(i) = \{j_1, \dots, j_{n_i}\}$. Then for $k \in [0, n_i]$ we have

$$\Psi_{\underline{w}}(e_{[i:k]}) = c_{[i:k+1]} \text{ and } \Psi_{\underline{w}}(e_{[i:n_i]}) = c_{[i:0]}.$$

In particular, $\Psi_{\underline{w}}$ sends normal vectors of defining (weight) hyperplanes of $\mathcal{S}_{\underline{w}}$ to normal vectors of defining (weight) hyperplanes of $\mathcal{C}_{\underline{w}}$.

Proof. We prove the claim by induction on k. By definition we have $\Psi_{\underline{w}}(e_{[i:0]}) = \Psi_{\underline{w}}(-e_{F_i}) = c_{[i:1]}$. Let $1 \leq k < n_i - 1$, then using induction for the third equation, we obtain

$$\begin{aligned}
\Psi_{\underline{w}}(e_{[i:k+1]}) &\overset{(4.2.8)}{=} \Psi_{\underline{w}}(e_{[i:k]} - e_{F_{i_{j_{k+1}}}}) \\
&\overset{(4.2.16)}{=} c_{[i:k+1]} + c_{i_{j_{k+1}}} + c_{i_{j_{k+2}}} - \sum_{\substack{j \in J(i-1) \cup J(i+1), \\ j_{k+1} < j < j_{k+2}}} c_{i_j} \\
&\overset{(4.2.4)}{=} c_i - c_{i_{j_{k+2}}} - 2 \sum_{j \in J(i), j > j_{k+2}} c_{i_j} + \sum_{\substack{j \in J(i-1) \cup J(i+1), \\ j > j_{k+2}}} c_{i_j} \;=\; c_{[i:k+2]}.
\end{aligned}$$

Now consider $e_{[i:n_i]} = e_{[i:n_i-1]} - e_{F_{i_{j_{n_i}}}}$. We apply $\Psi_{\underline{w}}$ and obtain the following by induction.

$$\begin{aligned}
\Psi(e_{[i:n_i]}) &\overset{(4.2.8)}{=} \Psi_{\underline{w}}(e_{[i:n_i-1]} - e_{F_{j_{n_i}}}) \\
&\overset{(4.2.16)}{=} c_{[i:n_i]} + c_{i_{j_{n_i}}} - \sum_{\substack{j \in J(i-1) \cup J(i+1), \\ j_{n_i} < j}} c_{i_j} \overset{(4.2.4)}{=} c_i.
\end{aligned}$$

□

We can now prove the first Theorem of this section. It is a more precise formulation of Theorem 5 as stated in the Introduction.

Theorem 17. Let $w \in S_n$ and $\underline{w}$ a reduced expression. The following polyhedral objects are unimodularly equivalent

(i) $\mathcal{S}_{\underline{w}} \cong \mathcal{C}_{\underline{w}}$ via $\Psi_{\underline{w}}$,

(ii) $S_{\underline{w}} \cong C_{\underline{w}}$ via $\Psi_{\underline{w}}|_{\mathbb{R}^{\ell(w)}}$,

(iii) $\mathcal{S}_{\underline{w}}(\lambda) \cong \mathcal{C}_{\underline{w}}(\lambda)$ for all $\lambda \in \mathbb{R}^{n-1}$ via $\Psi_{\underline{w}}$.

Proof. We begin by proving (i). From Proposition 9 we know that normal vectors $e_{\mathbf{p}}$ of $\mathcal{S}_{\underline{w}}$ for $\mathbf{p} \in \mathcal{P}_{\underline{w}}$ are mapped to the normal vectors $c_{\mathbf{p}}$ of $\mathcal{C}_{\underline{w}}$, i.e. $\Psi_{\underline{w}}(e_{\mathbf{p}}) = c_{\mathbf{p}}$. Further, from Proposition 10 we know the same is true for the normal vectors $e_{[i:k]}$ of $\mathcal{S}_{\underline{w}}$ for $i \in [n-1]$: we have $\Psi_{\underline{w}}(e_{[i:k]}) = c_{[i:k+1]}$ for $k < n_i$ and $\Psi_{\underline{w}}(e_{[i:n_i]}) = c_{[i:0]}$. As the right hand side of all defining inequalities is zero, we deduce that $\Psi_{\underline{w}}(\mathcal{S}_{\underline{w}}) = \mathcal{C}_{\underline{w}}$. By Lemma 14, $\Psi_{\underline{w}}$ is given by a matrix in $GL_{\ell(w)+n-1}(\mathbb{Z})$ and hence, $\mathcal{S}_{\underline{w}} \cong \mathcal{C}_{\underline{w}}$.

To show (ii), note that by the same argument as for (i) we have $\Psi_{\underline{w}}|_{\mathbb{R}^{\ell(w)}}(S_{\underline{w}}) = C_{\underline{w}}$. By Corollary 10 we deduce $S_{\underline{w}} \cong C_{\underline{w}}$.

For (iii) recall that for $\lambda \in \mathbb{R}^{n-1}$ the polytopes $\mathcal{S}_{\underline{w}}(\lambda)$ resp. $\mathcal{C}_{\underline{w}}(\lambda)$ are defined by the same normal vectors as $\mathcal{S}_{\underline{w}}$ resp. $\mathcal{C}_{\underline{w}}$. As above, it is also true that the right hand sides of the defining (in-)equalities coincides for all normal vectors being mapped onto each other. We therefore deduce $\mathcal{S}_{\underline{w}}(\lambda) = \mathcal{C}_{\underline{w}}(\lambda)$. $\square$

4.2.2 String cones, polytopes and toric degenerations

Recall from §2.1 our notation for the representation theoretic background regarding SL_n. In this section we recall *string polytopes* and *string cones* introduced by Littelmann in [52] and Berenstein-Zelevinsky in [5] as well as the *weighted string cones* defined in [52]. We prove using a result from Gleizer-Postnikov in [32] that these are exactly $\mathcal{C}_{\underline{w}}(\lambda)$ resp. $C_{\underline{w}}$ and $\mathcal{C}_{\underline{w}}$.

Littelmann [52] introduced in the context of quantum groups and crystal bases the so called (weighted) string cones and string polytopes $Q_{\underline{w}}(\lambda)$. The motivation is to find monomial bases for the Demazure modules $V_{\underline{w}}(\lambda)$ for $w \in S_n$ and $\lambda \in \Lambda^+$. Recall that by (2.1.2) $\{f_{\alpha_{i_1}}^{m_{i_1}} \cdots f_{\alpha_{i_{\ell(w)}}}^{m_{i_{\ell(w)}}} \cdot v_\lambda \in V(\lambda) \mid m_{i_j} \geq 0\}$ is a spanning set for $V_{\underline{w}}(\lambda)$ depending on a reduced expression $\underline{w} = s_{i_1} \cdots s_{i_{\ell(w)}}$. Littelmann identifies a linearly independent subset of this spanning set by introducing the notion of *adapted string* (see [52, p. 4]) referring to a tuple $(a_1, \dots, a_{\ell(w)}) \in \mathbb{Z}_{\geq 0}^{\ell(w)}$. His basis for $V_{\underline{w}}(\lambda)$ consists of those elements $f_{i_1}^{a_1} \cdots f_{i_{\ell(w)}}^{a_{\ell(w)}} \cdot v_\lambda$ for which $(a_1, \dots, a_{\ell(w)})$ is adapted.

For a fixed reduced expression $\underline{w}$ of $w \in S_n$ and $\lambda \in \Lambda^+$ he gives a recursive definition of the the *string polytope* $Q_{\underline{w}}(\lambda) \subset \mathbb{R}^{\ell(w)}$ ([52, p. 5], see also [5]). The lattice points $Q_{\underline{w}}(\lambda) \cap \mathbb{Z}^{\ell(w)}$ are the adapted strings for $\underline{w}$ and λ. The *string cone* $Q_{\underline{w}} \subset \mathbb{R}^{\ell(w)}$ is the convex hull of all $Q_{\underline{w}}(\lambda)$ for $\lambda \in \Lambda^+$. The *weighted string cone* $\mathcal{Q}_{\underline{w}} \subset \mathbb{R}^{\ell(w)+n-1}$ is defined as

$$\mathcal{Q}_{\underline{w}} := \operatorname{conv}\left(\bigcup_{\lambda \in \Lambda^+} Q_{\underline{w}}(\lambda) \times \{\lambda\} \right) \subset \mathbb{R}^{\ell(w)+n-1}.$$

By definition, one obtains the string polytope from the weighted string cone by intersecting it with the hyperplanes given by $\pi^{-1}(\lambda)$ as in (4.2.12). The lattice points in the weighted string cone for $w = w_0$ parametrize a basis of $\mathbb{C}[SL_n/U] \cong \bigoplus_{\lambda \in \Lambda^+} V(\lambda)$.

String polytopes are of great interest to us because of Caldero's work [13] in 2002. He defines for a Schubert variety X_w a flat family over $\mathbb{A}^1$ with generic fibre X_w and special fibre a toric variety. The family is of form (2.3.2) given by a construction using Rees algebras (see §2.3). Although not defined using valuations initially, it was realized this way in [43] and [20]. His main tools are Lusztig's dual canonical basis and the string parametrization due to [5] and [52]. We summarize his results (restricted to the case of SL_n) below.

Let $\underline{w} = s_{i_1} \cdots s_{i_{\ell(w)}}$ be a reduced expression of $w \in S_n$. We extend $\underline{w}$ to the *right* to a reduced expression $\underline{w}_0 = \underline{w} s_{i_{\ell(w)+1}} \cdots s_{i_N}$ of w_0. This extension is not unique but the results are independent of the extension. Caldero realizes the string cone $Q_{\underline{w}}$ for the Demazure module $V_w(\lambda)$ as a face of the string cone $Q_{\underline{w}_0}$. He deduces the following Lemma as a consequence of [52, §1].

Lemma. (see [52], [13, Lemma 3.3]) Let $w \in S_n$ with reduced expression $\underline{w} = s_{i_1} \cdots s_{i_{\ell(w)}}$ and choose a reduced expression $\underline{w}_0 = \underline{w} s_{i_{\ell(w)+1}} \cdots s_{i_N}$. Then the weighted string cone $\mathcal{Q}_{\underline{w}}$ is obtained from the weighted string cone $\mathcal{Q}_{\underline{w}_0}$ by setting the variables corresponding to $s_{i_{\ell(w)+1}}, \dots, s_{i_N}$ equal to zero.

Caldero defines a filtration $(\mathcal{F}_{\leq m})_{m\geq 1}$ on $\mathbb{C}[SL_n/U]$ with associated graded algebra the semi-group algebra $\mathbb{C}[\mathcal{Q}_{\underline{w}_0}\cap\mathbb{Z}^{N+n-1}]$. Using the Lemma, he defines a quotient filtration $(\overline{\mathcal{F}}_{\leq m})_{m\geq 1}$ on $\mathbb{C}[SL_n/U]/I_w$, where $I_w=\bigoplus_{\lambda\in\Lambda^+}V_w(\lambda)^\perp$ (recall §2.1), i.e. (from what we have seen in §4.1)

$$\mathbb{C}[SL_n/U]/I_w=\bigoplus_{\lambda\in\Lambda^+}V(\lambda)^*/\bigoplus_{\lambda\in\Lambda^+}V_w(\lambda)^\perp.$$

The semi-group algebra $\mathbb{C}[\mathcal{Q}_{\underline{w}}\cap\mathbb{Z}^{\ell(\underline{w})+n-1}]$ is the associated graded algebra of the quotient filtration. In particular, he degenerates X_w into a toric variety Y, whose normalization is the toric variety $X_{\mathcal{Q}_{\underline{w}}(\lambda)}$ associated to the string polytope $\mathcal{Q}_{\underline{w}}(\lambda)$ for $\lambda\in\Lambda^{++}$.

Relation to the GP cones Gleizer and Postnikov develop in [32] a combinatorial model to describe string cones $Q_{\underline{w}_0}$ *non-recursively* for every reduced expression $\underline{w}_0$ of $w_0\in S_n$. They use pseudoline arrangements and GP-paths to obtain the following.

Corollary. [32, Corollary 5.8] Let $\underline{w}_0$ be a reduced expression for $w_0\in S_n$. Then $C_{\underline{w}_0}=Q_{\underline{w}_0}$.

On our way to showing that a toric variety isomorphic to $X_{\mathcal{Q}_{\underline{w}}(\lambda)}$ arises in the context of cluster varieties and mirror symmetry, we first generalize Gleizer-Postnikov's result as follows.

Theorem 18. For every $w\in S_n$ with reduced expression $\underline{w}$ and every extension $\underline{w}_0=\underline{w}s_{i_{\ell(w)+1}}\cdots s_{i_N}$ the following polyhedral objects coincide

(i) $\mathcal{C}_{\underline{w}}=\mathcal{Q}_{\underline{w}}$,

(ii) $C_{\underline{w}}=Q_{\underline{w}}$,

(iii) $\mathcal{C}_{\underline{w}}(\lambda)=\mathcal{Q}_{\underline{w}}(\lambda)$ for $\lambda\in\mathbb{R}^{n-1}$.

In order to prove Theorem 18 we show how to obtain $\mathcal{C}_{\underline{w}}$ from restricting $\mathcal{C}_{\underline{w}_0}$ for appropriate $\underline{w}_0$. The next subsection is dedicated to introducing restricted paths and concludes with the proof of Theorem 18.

Restriction of paths

We show that for $\underline{w}_0=\underline{w}s_{i_{\ell(w)+1}}\cdots s_{i_N}$ we obtain $\mathcal{C}_{\underline{w}}$ from $\mathcal{C}_{\underline{w}_0}$ by setting to zero the coordinates corresponding to crossing points $c_{i_{\ell(w)+1}},\dots,c_{i_N}$ in $\mathsf{pa}(\underline{w}_0)$.

Definition 53. Let $\underline{w}$ be a reduced expression of $w\in S_n$ and fix $\underline{w}_0=\underline{w}s_{i_{\ell(w)+1}}\cdots s_{i_N}$. Consider $\mathbf{p}_{\underline{w}_0}\in\mathcal{P}_{\underline{w}_0}$ and draw it in $\mathsf{pa}(\underline{w}_0)$. Then cut $\mathsf{pa}(\underline{w}_0)$ in two pieces along a vertical line, such that all crossing points v_{i_p} corresponding to s_{i_p} with $1\leq p\leq\ell(w)$ are on the left of the cut and all v_{i_q} corresponding to $s_{i_q},\ell(w)<q\leq N$ are on the right (see Figure 4.8). We define the *restriction* $\mathrm{res}_{\underline{w}}(\mathbf{p}_{\underline{w}_0})$ of $\mathbf{p}_{\underline{w}_0}$ to $\mathsf{pa}(\underline{w})$ as the part of $\mathbf{p}_{\underline{w}_0}$ that is to the left of the cut.

We label the intersection points of the lines l_i with the cutting line by $\hat{L}_i$. An alternative way of describing $\mathrm{res}_{\underline{w}}(\mathbf{p}_{\underline{w}_0})$ is by removing all vertices $v_{(i,j)}$ from it for which $w(\alpha_{i,j-1})>0$. Denote by $\mathrm{res}_{\underline{w}}(\mathcal{P}_{\underline{w}_0})$ the set of all paths in $\mathsf{pa}(\underline{w})$ that appear in a restriction of a path in $\mathcal{P}_{\underline{w}_0}$ (counting each path only once).

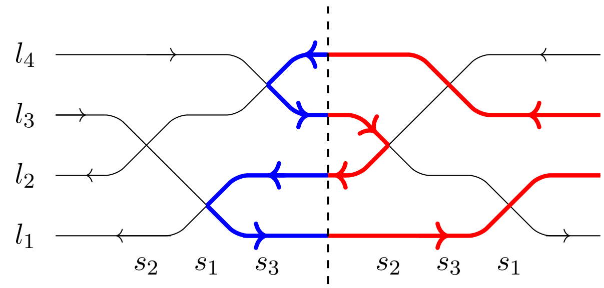


Figure 4.8: A path $\mathbf{p}_{\underline{w}_0} \in \mathcal{P}_{\underline{w}_0}$ for $\underline{w}_0 = s_2s_1s_3s_2s_3s_1$ that restricts to two paths $\mathbf{p}_{\underline{w}}, \mathbf{p}'_{\underline{w}} \in \mathcal{P}_{\underline{w}}$ (in blue to the left of the dashed cut) for $\underline{w} = s_2s_1s_3$.

Example 21. Consider $\underline{w} = s_2s_1s_3$ and extend it to $\underline{w}_0 = s_2s_1s_3s_2s_3s_1 \in S_4$. We draw $\mathsf{pa}(\underline{w})$ and endow it with the orientation for (l_2, l_3). Figure 4.8 shows a GP-path $\mathbf{p}_{\underline{w}_0}$. Its restriction $\mathrm{res}_{\underline{w}}(\mathbf{p}_{\underline{w}_0})$ consists of two GP-paths for $\underline{w}$ shown in blue to the left of the cut.

Proposition 11. Let $\underline{w}$ be a reduced expression of $w \in S_n$ and fix $\underline{w}_0 = \underline{w}s_{i_{\ell(w)+1}} \cdots s_{i_N}$. Consider $\mathbf{p}_{\underline{w}_0} \in \mathcal{P}_{\underline{w}_0}$, then $\mathrm{res}_{\underline{w}}(\mathbf{p}_{\underline{w}_0})$ is either empty or a union of paths in $\mathcal{P}_{\underline{w}}$. In particular, $\mathrm{res}_{\underline{w}}(\mathcal{P}_{\underline{w}_0}) \subset \mathcal{P}_{\underline{w}}$.

Proof. Let $\mathbf{p}_{\underline{w}_0}$ be a path for orientation (l_i, l_{i+1}), i.e. of form $\mathbf{p}_{\underline{w}_0} = L_i \to v_{(i,j_1)} \to v_{(j_1,j_2)} \to \cdots \to v_{(j_k,i+1)} \to L_{i+1}$. To simplify notation we set $i = j_0$ and $i + 1 = j_{k+1}$. First note that if $w(\alpha_{j_r,j_{r+1}-1}) > 0$ for all $0 \le r \le k$ then $\mathrm{res}_{\underline{w}}(\mathbf{p}_{\underline{w}_0}) = \varnothing$. Otherwise $\mathrm{res}_{\underline{w}}(\mathbf{p}_{\underline{w}_0})$ is a union of paths

$$\mathbf{p}_r = \hat{L}_{j_r} \to v_{(j_r,j_{r+1})} \to \cdots \to v_{(j_{r+s},j_{r+s+1})} \to \hat{L}_{j_{r+s+1}}$$

such that $w(\alpha_{j_{r+p},j_{r+p+1}-1}) < 0$ for all $0 \le p \le s$, $0 \le r \le k$ and $0 \le s \le k - r$. By definition, each $\mathbf{p}_r$ is rigorous and hence, in $\mathcal{P}_{\underline{w}}$. $\square$

We want to show the other implication, $\mathcal{P}_{\underline{w}} \subset \mathrm{res}_{\underline{w}}(\mathcal{P}_{\underline{w}_0})$. In Algorithm 6 we give a construction to obtain a path in $\mathsf{pa}(\underline{w}_0)$ from a given path in $\mathcal{P}_{\underline{w}}$. The following proposition shows that the algorithm always terminates and that the output is in fact a path in $\mathcal{P}_{\underline{w}_0}$.

Proposition 12. Algorithm 6 terminates for all $\mathbf{p}_{\underline{w}} \in \mathcal{P}_{\underline{w}}$ and $\mathrm{ind}_{\underline{w}_0}(\mathbf{p}_{\underline{w}}) \in \mathcal{P}_{\underline{w}_0}$.

Proof. By Proposition 8 $\mathbf{p}_{\underline{w}}$ lies in the region of $\mathsf{pa}(\underline{w})$ in between the lines l_i and l_{i+1}. In particular, at some point there is a p' with $l_{i+m-p-p'} = l_{i+1}$ terminating the first loop and a q' with $l_{i-l+q+q'} = l_i$ terminating the second loop.

To see that $\mathrm{ind}_{\underline{w}_0}(\mathbf{p}_{\underline{w}}) \in \mathcal{P}_{\underline{w}_0}$ observe that changing the lines as indicated by the algorithm avoids exactly the two situations from Figure 4.5 forbidden in rigorous paths. Hence, $\mathrm{ind}_{\underline{w}_0}(\mathbf{p}_{\underline{w}})$ is rigorous. $\square$

By Proposition 12 we can define the following.

Definition 54. Let $\underline{w}$ be a reduced expression of $w \in S_n$ and fix $\underline{w}_0 = \underline{w}s_{i_{\ell(w)+1}} \cdots s_{i_N}$. For $\mathbf{p}_{\underline{w}} \in \mathcal{P}_{\underline{w}}$ we define the *induced path* $\mathrm{ind}_{\underline{w}_0}(\mathbf{p}_{\underline{w}}) \in \mathcal{P}_{\underline{w}_0}$ as the output of Algorithm 6.

Example 22. Consider $\underline{w} = s_2s_1s_3$ and extend it to $\underline{w}_0 = s_2s_1s_3s_2s_3s_1 \in S_4$. We draw $\mathsf{pa}(\underline{w})$ and endow it with the orientation for (l_2, l_3). Figure 4.9 shows a GP-path $\mathbf{p}_{\underline{w}}$ in blue to the left of the cut. The extension of $\mathbf{p}_{\underline{w}}$ in red to the right of the cut completes $\mathbf{p}_{\underline{w}}$ to the induced path $\mathrm{ind}_{\underline{w}_0}(\mathbf{p}_{\underline{w}})$ that is the output of Algorithm 6.

Algorithm 6: Constructing the induced path $\mathrm{ind}_{\underline{w}_0}(\mathbf{p}_{\underline{w}})$ from $\mathbf{p}_{\underline{w}} \in \mathcal{P}_{\underline{w}}$.

Input: A path in $\mathcal{P}_{\underline{w}} \ni \mathbf{p}_{\underline{w}} = \hat{L}_{i-l} \to v_{r_1} \to \cdots \to v_{r_m} \to \hat{L}_{i+m}$ for orientation (l_i, l_{i+1}).

Initialization: extend $\underline{w}$ to $\underline{w}_0 = \underline{w} s_{i_{\ell(w)+1}} \cdots s_{i_N}$;
complete $\mathsf{pa}(\underline{w})$ to $\mathsf{pa}(\underline{w}_0)$ with orientation for (l_i, l_{i+1});
set $p = q = 0$ and $\hat{\mathbf{p}}_{\underline{w}} = \mathbf{p}_{\underline{w}}$.

for $p < m - 1$ **do**
- follow l_{i+m-p} with respect to the orientation to the next crossing with a line $l_{i+m-p-p'}$ with $p' \in [m-p-1]$,
- **if** $p' = m - p - 1$ **then**
  - **Output:** $\hat{\mathbf{p}}_{\underline{w}} \to v_{(i+m-p,i+1)} \to L_{i+1}$.
- **else**
  - replace p by $p + p'$ and $\hat{\mathbf{p}}_{\underline{w}}$ by $\hat{\mathbf{p}}_{\underline{w}} \to v_{(i+m-p,i+m-p-p')}$ and start over.

for $q < l$ **do**
- follow l_{i-l+q} against the orientation to the next crossing with a line $l_{i-l+q+q'}$ with $q' \in [l - q]$,
- **if** $q' = l - q$ **then**
  - **Output:** $L_i \to v_{(i,i-l+q+q')} \to \hat{\mathbf{p}}_{\underline{w}}$.
- **else**
  - replace q by $q + q'$ and $\hat{\mathbf{p}}_{\underline{w}}$ by $v_{(i-l+q+q',i-l+q)} \to \hat{\mathbf{p}}_{\underline{w}}$ and start over.

Output: A path
$\mathrm{ind}_{\underline{w}_0}(\mathbf{p}_{\underline{w}}) := L_i \to v_{(i,i-l+q)} \to \cdots \to \mathbf{p}_{\underline{w}} \to \cdots \to v_{(i+m-p,i+1)} \to L_{i+1}$.

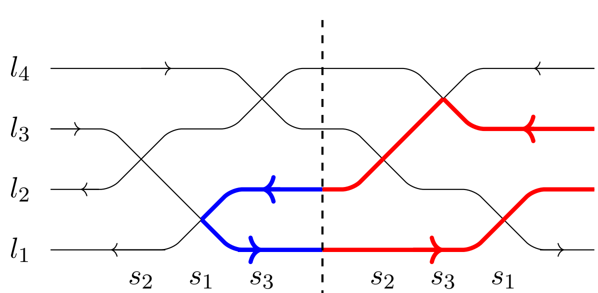


Figure 4.9: A path $\mathbf{p}_{\underline{w}} \in \mathcal{P}_{\underline{w}}$ for $\underline{w}_0 = s_2 s_1 s_3$ and the induced path $\mathrm{ind}_{\underline{w}_0}(\mathbf{p}_{\underline{w}}) \in \mathcal{P}_{\underline{w}_0}$ with $\underline{w}_0 = \underline{w} s_2 s_3 s_1$.

Proposition 13. Let $\underline{w}$ be a reduced expression of $w \in S_n$ and fix $\underline{w}_0 = \underline{w}s_{i_{\ell(w)+1}} \cdots s_{i_N}$. For every $\mathbf{p}_{\underline{w}} \in \mathcal{P}_{\underline{w}}$ there exists $\mathbf{p}_{\underline{w}_0} \in \mathcal{P}_{\underline{w}_0}$ such that, $\mathrm{res}_{\underline{w}}(\mathbf{p}_{\underline{w}_0}) = \mathbf{p}_{\underline{w}}$. In particular, we have $\mathcal{P}_{\underline{w}} \subset \mathrm{res}_{\underline{w}}(\mathcal{P}_{\underline{w}_0})$.

Proof. By construction $\mathrm{ind}_{\underline{w}_0}(\mathbf{p}_{\underline{w}}) \in \mathcal{P}_{\underline{w}_0}$ satisfies $\mathrm{res}_{\underline{w}}(\mathrm{ind}_{\underline{w}_0}(\mathbf{p}_{\underline{w}})) = \mathbf{p}_{\underline{w}}$. □

Recall for $i \in [n-1]$ the definition $J(i)$ and n_i from (4.2.3). To distinguish between the sets for $\underline{w}$ and $\underline{w}_0$, we use the notation $J(i)^{\underline{w}}$ (resp. $J(i)^{\underline{w}_0}$) and $n_i^{\underline{w}}$ (resp. $n_i^{\underline{w}_0}$). We define the following polyhedral objects from restricted paths and show they equal the (weighted) GP-cone, respectively polytope, in the subsequent key proposition for proving Theorem 18.

Definition 55. Let $\underline{w}$ be a reduced expression of $w \in S_n$ and fix $\underline{w}_0 = \underline{w}s_{i_{\ell(w)+1}} \cdots s_{i_N}$. We define the *restricted weighted GP-cone* as

$$\mathrm{res}_{\underline{w}}(\mathcal{C}_{\underline{w}_0}) := \left\{ \mathbf{x} \in \mathbb{R}^{\ell(w)+n-1} \,\middle|\, \begin{array}{ll} (c_{\mathrm{res}_{\underline{w}}(\mathbf{p}_{\underline{w}_0})})^t(\mathbf{x}) \geq 0, & \forall\, \mathbf{p}_{\underline{w}_0} \in \mathcal{P}_{\underline{w}_0}, \\ (c_{[i:k]})^t(\mathbf{x}) \geq 0, & \forall i \in [n-1], 0 \leq k \leq n_i^{\underline{w}} \end{array} \right\}. \quad (4.2.18)$$

Similarly, we define $\mathrm{res}_{\underline{w}}(C_{\underline{w}_0}) := \{\mathbf{x} \in \mathbb{R}^{\ell(w)} \mid (c_{\mathrm{res}_{\underline{w}}(\mathbf{p}_{\underline{w}_0})})^t(\mathbf{x}) \geq 0, \forall\, \mathbf{p}_{\underline{w}_0} \in \mathcal{P}_{\underline{w}_0}\}$ the *restricted GP-cone* and the polytope $\mathrm{res}_{\underline{w}}(\mathcal{C}_{\underline{w}_0}(\lambda)) := \mathrm{res}_{\underline{w}}(\mathcal{C}_{\underline{w}_0}) \cap \pi^{-1}(\lambda)$ (see (4.2.12)) for $\lambda \in \mathbb{R}^{n-1}$.

Proposition 14. For every $w \in S_n$ with reduced expression $\underline{w}$ and every extension $\underline{w}_0 = \underline{w}s_{i_{\ell(w)+1}} \cdots s_{i_N}$ the following polyhedral objects coincide

(i) $\mathcal{C}_{\underline{w}} = \mathrm{res}_{\underline{w}}(\mathcal{C}_{\underline{w}_0})$,

(ii) $C_{\underline{w}} = \mathrm{res}_{\underline{w}}(C_{\underline{w}_0})$,

(iii) $\mathcal{C}_{\underline{w}}(\lambda) = \mathrm{res}_{\underline{w}}(\mathcal{C}_{\underline{w}_0}(\lambda))$ for $\lambda \in \mathbb{R}^{n-1}$.

Proof. We start by showing (i), then (ii) and (iii) are direct implications. Note that only the inequalities induced by GP-paths differ in the definition of $\mathcal{C}_{\underline{w}}$ (4.2.5), resp. $\mathrm{res}_{\underline{w}}(\mathcal{C}_{\underline{w}_0})$ (4.2.18). By Proposition 11 we have $\mathcal{C}_{\underline{w}} \subseteq \mathrm{res}_{\underline{w}}(\mathcal{C}_{\underline{w}_0})$. By Proposition 13 we deduce $\mathrm{res}_{\underline{w}}(\mathcal{C}_{\underline{w}_0}) \subseteq \mathcal{C}_{\underline{w}}$ and hence, equality follows. □

We have now collected all ingredients necessary to provide the proof Theorem 18.

Proof of Theorem 18. We show $\mathcal{Q}_{\underline{w}} = \mathrm{res}_{\underline{w}}(\mathcal{C}_{\underline{w}_0})$ for every extension $\underline{w}_0 = \underline{w}s_{i_{\ell(w)+1}} \cdots s_{i_N}$ and then apply Proposition 14. By [13, Lemma 3.3] (restated above) we know that

$$\mathcal{Q}_{\underline{w}} = \mathcal{Q}_{\underline{w}_0} \cap \bigcap_{(i,k):\ w(\alpha_{i,k-1})>0} \{x_{(i,k)} = 0\},$$

as the $x_{(i,k)}$ appearing in the intersection of hyperplanes on the right correspond to the coordinates x_{s_p} with $\ell(w) < p \leq N$ in the extension of $\underline{w}$ to $\underline{w}_0$. Further, we observe that if $c_{\mathbf{p}_{\underline{w}_0}} = \sum_k c_{(i_k,j_k)}$ then $c_{\mathrm{res}_{\underline{w}}(\mathbf{p}_{\underline{w}_0})} = \sum_{k: w(\alpha_{i_k,j_k-1})>0} c_{(i_k,j_k)}$. Regarding the normal vectors for weight inequalities $c_{[i:k]}$ (see (4.2.4)), observe that for $k > n_i^{\underline{w}}$ we obtain c_i from $c_{[i:k]}$ when setting those $c_{(i_k,j_k)}$ to zero with $w(\alpha_{i_k,j_k-1}) > 0$. Hence, $\mathcal{Q}_{\underline{w}} = \mathrm{res}_{\underline{w}}(\mathcal{C}_{\underline{w}_0}) = \mathcal{C}_{\underline{w}}$ by Propositon 14. Then $Q_{\underline{w}} = C_{\underline{w}}$ is a direct consequence and identifying Λ^+ with $\mathbb{R}^{n-1}$ using the fundamental weights, (iii) follows. □

4.2.3 Double Bruhat cells and the superpotential

Recall the introduction to cluster varieties given in §2.5 as well as sections §2.1 and §4.1. In this section we explain the $\mathcal{A}$-cluster variety that can be associated to the quiver from a pseudoline arrangement. Based on results of Berenstein-Fomin-Zelevinsky this variety is a double Bruhat cell (see Definition 56). We apply the construction of [37] (see §2.5) and recall results of Magee in [54] regarding the superpotential.

Recall that SL_n has two cell decompositions (*the Bruhat decompositions*) in terms of Bruhat cells indexed by elements of the symmetric group

$$SL_n = \bigcup_{u\in S_n} BuB = \bigcup_{v\in S_n} B^- vB^-.$$

Definition 56. The *double Bruhat cell* associated to e and w in S_n is

$$G^{e,w} := B \cap B^- wB^- \subset SL_n.$$

The cluster structure of $G^{e,w}$ can be established as follows. Choose a reduced expression $\underline{w}$ and consider $\mathsf{pa}(\underline{w})$. Recall from Definition 43 that every face of $\mathsf{pa}(\underline{w})$ corresponds to a vertex of $Q_{\underline{w}}$. We therefore associate cluster variables to faces of $\mathsf{pa}(\underline{w})$. Let F be such a face and assume the lines $l_{j_1}, \dots, l_{j_k}$ pass below F. In particular, F is of level k. Then associate the Plücker coordinate $\bar{p}_{j_1,\dots,j_k} \in \mathbb{C}[SL_n]$ to F, i.e. the minor of the columns $[k]$ and rows $\{j_1, \dots, j_k\}$. To remember it was associated with F, we set $A_F := \bar{p}_{j_1,\dots,j_k}$.

Definition 57. Let $w \in S_n$ with reduced expression $\underline{w}$. Then the quiver $Q_{\underline{w}}$ together with the set of cluster variables $\mathbf{A}_{\underline{w}} := \{A_F \mid F \text{ a face of } \mathsf{pa}(\underline{w})\}$ forms the seed $s_{\underline{w}} := (\mathbf{A}_{\underline{w}}, Q_{\underline{w}})$.

Example 23. Recall from Example 15 and Figure 4.3 the pseudoline arrangement $\mathsf{pa}(\underline{\hat{w}}_0)$ and the quiver $Q_{\underline{\hat{w}}_0}$ for $\underline{\hat{w}}_0 \in S_5$. To a face $F_{(i,j)}$ with $i \in [n-1]$ and $j \in [i+1, n]$ we associate following the above recipe the cluster variable $A_{(i,j)} := p_{i+1,\dots,j}$. To the faces unbounded $F_i, i \in [4]$ to the left, we associate the variables $A_i := p_{5-i+1,\dots,5}$. Note that the variables associated to the frozen vertices on the left (from bottom to top) are $\bar{p}_5, \bar{p}_{45}, \bar{p}_{345}, \bar{p}_{2345}$ and those associated to the frozen vertices on the right are $\bar{p}_1, \bar{p}_{12}, \bar{p}_{123}, \bar{p}_{1234}$. These Plücker coordinates are called *consecutive minors*. The collection of all cluster variables associated to this initial seed is

$$\mathbf{A}_{\underline{\hat{w}}_0} = \{\bar{p}_1, \bar{p}_2, \bar{p}_3, \bar{p}_4, \bar{p}_5, \bar{p}_{12}, \bar{p}_{23}, \bar{p}_{34}, \bar{p}_{45}, \bar{p}_{123}, \bar{p}_{234}, \bar{p}_{345}, \bar{p}_{1234}, \bar{p}_{2345}\}.$$

Example 24. Consider $\underline{\hat{w}}_0 \in S_n$ as in Examples 17 and 19. Then the collection $\mathbf{A}_{\underline{\hat{w}}_0}$ of associated cluster variables is

$$\mathbf{A}_{\underline{\hat{w}}_0} = \{\bar{p}_{i,\dots,j} \mid i \in [n-1], j \in [i, n]\},$$

where $\bar{p}_{i,\dots,j}$ is a frozen variable if either $i = 1$ or $j = n$. Note that $\bar{p}_{1,\dots,n} = \det$, which is constant on SL_n, hence we disregard it. From now on we denote by s_0 the seed $s_0 := s_{\underline{\hat{w}}_0} = (\mathbf{A}_{\underline{\hat{w}}_0}, Q_{\underline{\hat{w}}_0})$.

Berenstein-Fomin-Zelevinsky show

Theorem. ([4, Theorem 2.10]) Let $w \in S_n$ with reduced expression $\underline{w}$. Then for the upper cluster algebra $\overline{\mathcal{Y}}(s_{\underline{w}})$ we haven an isomorphism of algebras

$$\overline{\mathcal{Y}}(s_{\underline{w}}) \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}[G^{e,w}]. \tag{4.2.19}$$

In particular, the Theorem implies the following: if $\underline{w}_1$ and $\underline{w}_2$ are two reduced expressions of $w \in S_n$ related by mutation in the sense of Definition 44, then the associated seeds $s_{\underline{w}_1}$ and $s_{\underline{w}_2}$ are related by cluster ($\mathcal{A}$-)mutation in the sense of Definition 18. This explains our abuse of notation using the same letter μ for both types of mutation.

We now focus on the $\mathcal{A}$-cluster variety G^{e,w_0} and the natural partial compactification using the frozen variables to study the superpotential as in §2.5. We partially compactify G^{e,w_0} to $\bar{G}^{e,w_0}$ by allowing the frozen variables $\bar{p}_{[i]}$ and $\bar{p}_{[n-i,n]}$ for $i \in [n-1]$ to vanish. Denote the resulting boundary divisor $D \subset \bar{G}^{e,w_0}$ and its irreducible components by

$$D_i := \{\bar{p}_{[i]} = 0\}, \quad \text{resp.} \quad D_{i,n} := \{\bar{p}_{[n-i,n]} = 0\}.$$

There is an open embedding $G^{e,w_0} \hookrightarrow SL_n/U$ given by $g \mapsto g^t U$ and up to codimension 2 the variety $\bar{G}^{e,w_0}$ agrees with SL_n/U (this follows, for example, from [54, Proposition 23]). Hence, we have an isomorphism of rings $\mathbb{C}[\bar{G}^{e,w_0}] \cong \mathbb{C}[SL_n/U]$. One of Magee's main results in [54] is the following.

Theorem. ([54, Corollary 3]) The full Fock-Goncharov conjecture holds for SL_n/U.

Moreover, Magee shows that there exists an optimized seed for every frozen vertex and therefore we can apply Algorithm 1 stated in §2.5 to compute the superpotential. This is indeed what Magee did for the intial seed s_0 (see Example 24). Let $\mathcal{X}$ denote the Fock-Goncharov dual to the $\mathcal{A}$-cluster variety G^{e,w_0} (see Definition 20). Recall that in the initial seed s_0 we have $N_{s_0} \cong \mathbb{Z}^{N+n-1}$ with basis $\{e_F \mid F \text{ face of } \mathsf{pa}(\underline{w}_0)\}$. As before we set $e_{F_{(i,j)}} =: e_{(i,j)}$ and $e_{F_k} =: e_k$. Further, recall that the superpotential $W : \mathcal{X} \to \mathbb{C}$ is given by the sum of ϑ-functions associated to frozen variables. We denote by ϑ_i (resp. $\vartheta_{(i,n)}$) the ϑ-functions associated to the frozen vertex w_i (resp. $w_{(i,n)}$) in the inital quiver Q_{s_0} (see Figure 4.3) for $i \in [n-1]$.

Proposition. ([54, Corollary 24]) Let $W : \mathcal{X} \to \mathbb{C}$ denote the superpotential. Then we have $W|_{\mathcal{X}_{s_0}} = \sum_{i=1}^{n-1} \vartheta_i|_{\mathcal{X}_{s_0}} + \vartheta_{(n-i,n)}|_{\mathcal{X}_{s_0}}$, where

$$\vartheta_i|_{\mathcal{X}_{s_0}} = \sum_{k=0}^{n-1-i} z^{-e_i - \sum_{j=1}^k e_{(j,i+j)}}, \quad \text{and} \quad \vartheta_{(i,n)}|_{\mathcal{X}_{s_0}} = \sum_{k=0}^{n-1-i} z^{-\sum_{j=0}^k e_{(i,n-j)}}, \quad \text{for} \quad i \in [n-1].$$

Example 25. Consider S_3 and the initial seed with quiver $Q_{s_1s_2s_1}$. Then

$$\begin{aligned} W|_{\mathcal{X}_{s_0}} &= \vartheta_{(1,3)} + \vartheta_{(2,3)} + \vartheta_1 + \vartheta_2 \\ &= z^{-e_{(1,3)}} + z^{-e_{(1,3)} - e_{(1,2)}} + z^{-e_{(2,3)}} + z^{-e_1} + z^{-e_1 - e_{(1,2)}} + z^{-e_2}. \end{aligned}$$

Definition 58. For $\underline{w}_0$ a reduced expression of $w_0 \in S_n$ we define the following polyhedral objects by tropicalizing a sum of ϑ-functions resp. the superpotential:

$$\begin{aligned} \Xi_{\underline{w}_0} &:= \{\mathbf{x} \in \mathbb{R}^{N+n-1} \mid W|^{\text{trop}}_{\mathcal{X}_{\underline{w}_0}}(\mathbf{x}) \geq 0\}, \\ \Xi_{\underline{w}_0} &:= \{\mathbf{x} \in \mathbb{R}^N \mid (\sum_{i=1}^{n-1} \vartheta_{(i,n)}|_{\mathcal{X}_{\underline{w}_0}})^{\text{trop}}(\mathbf{x}) \geq 0\}, \\ \Xi_{\underline{w}_0}(\lambda) &:= \Xi_{\underline{w}_0} \cap \tau^{-1}_{\underline{w}_0}(\lambda) \text{ for } \lambda \in \mathbb{R}^{n-1}. \end{aligned}$$

The $\mathcal{A}_{\mathrm{prin}}$-construction in [37] applied to our setting defines a flat family over $\mathbb{A}^{N-2n+2}$ for every choice of seed, in particular for every $\underline{w}_0$. The central fibre is by [37, Theorem 8.39] the toric variety associated to $\Xi_{\underline{w}_0}(\lambda)$ for $\lambda \in \mathbb{Z}^{n-1}_{>0}$. One generic fibre is SL_n/B, hence we have a toric degeneration of the flag variety. We do not go into the details on this construction but refer the reader to [37, §8].

Relating to the area cones Let $\underline{w}_0$ be an arbitrary reduced expression of $w_0 \in S_n$. In what follows we show how to obtain an expression of the superpotential in any seed $s_{\underline{w}_0}$ associated to $\underline{w}_0$ by "detropicalizing" the weighted cone $\mathcal{S}_{\underline{w}_0}$. We define it more generally for $\underline{w}$ a reduced expression of $w \in S_n$. Denote by $\mathcal{X}_{\underline{w}}$ the cluter torus associated to the seed $s_{\underline{w}}$.

Definition 59. Let $\underline{w}$ be an arbitrary reduced expression of $w \in S_n$. Then the *detropicalization* of the cone $\mathcal{S}_{\underline{w}}$ is defined as the function $W_{\mathcal{S}_{\underline{w}}} : \mathcal{X}_{\underline{w}} \to \mathbb{C}$ with

$$W_{\mathcal{S}_{\underline{w}}} := \sum_{\mathbf{p}\in\mathcal{P}_{\underline{w}}} z^{e_{\mathbf{p}}} + \sum_{i\in[n-1],0\leq k\leq n_i} z^{e_{[i:k]}}. \tag{4.2.20}$$

The name is self-explanatory, observe that by definition we have

$$\{\mathbf{x} \in \mathbb{R}^{\ell(w)+n-1} \mid W^{\mathrm{trop}}_{\mathcal{S}_{\underline{w}}}(\mathbf{x}) \geq 0\} = \mathcal{S}_{\underline{w}}.$$

Proposition 15. Let $\underline{w}_0 = s_1s_2s_1\cdots s_{n-1}s_{n-2}\cdots s_2s_1$ be the reduced expression associated to the initial seed s_0 as above. Then $W_{\mathcal{S}_{\underline{w}_0}} = W|_{\mathcal{X}_{s_0}}$.

Proof. Recall from Example 19 the expressions $e_{\mathbf{p}_{i,j}}$ (4.2.10) and $e_{[i:k]}$ (4.2.11) for $i \in [n-1]$ and $j, k \in [i+1, n]$. In comparison with [54, Corollary 24] (restated above) we obtain

$$\vartheta_{(i,n)}|_{\mathcal{X}_{s_0}} = \sum_{j=i+1}^{n} z^{e_{\mathbf{p}_{i,j}}}, \quad \text{and} \quad \vartheta_i|_{\mathcal{X}_{s_0}} = \sum_{k=0}^{n-1-i} z^{e_{[i:k]}}.$$

As from Example 17 we know $\mathcal{P}_{\underline{w}_0} = \{\mathbf{p}_{i,j} \mid i \in [n-1], j \in [i+1, n]\}$, the claim follows. □

Mutation of $\mathcal{S}_{\underline{w}}$

Our aim is to generalize Proposition 15 for arbitrary reduced expressions $\underline{w}_0$. We achieve this by showing that the detropicalization of $\mathcal{S}_{\underline{w}_0}$ behaves as the superpotential does when applying $\mathcal{X}$-mutation. Further, we show that if $\mu(\underline{w})$ and $\underline{w}$ are reduced expressions of $w \in S_n$, then $\mu^*(W_{\mathcal{S}_{\mu(\underline{w})}}) = W_{\mathcal{S}_{\underline{w}}}$, where $\mu^* : \mathbb{C}[\mathcal{X}_{\mu(\underline{w})}] \to \mathbb{C}[\mathcal{X}_{\underline{w}}]$ is the pull-back of the cluster mutation as in (2.5.7). This follows from Lemma 16 and Lemma 17. Recall from Definition 44 the mutation of pseudoline arrangements. The core of this subsection is the case-by-case analysis of how mutation effects GP-paths.

In Figure 4.10 we display locally around the mutable face $F = F_{(i,j)}$ (resp. F') the orientations of $\mathsf{pa}(\underline{w})$ (resp. $\mathsf{pa}(\mu_F(\underline{w}))$). The red arrows indicate which passages are forbidden in GP-paths. In Tables 4.1 to 4.4 we list in the second column all possibilities how a GP-path $\mathbf{p}$ locally looks around the face F. In the third column of each table is a complete list of how GP-paths look locally around the face F' obtained from F by mutation μ_F.

Recall the arrows for the quiver corresponding to $\mathsf{pa}(\underline{w})$ and $\mathsf{pa}(\mu_F(\underline{w}))$ from Figure 4.2. We call a face E *incoming* (resp. *outgoing*) with respect to F in $\mathsf{pa}(\underline{w})$, if there is an arrow

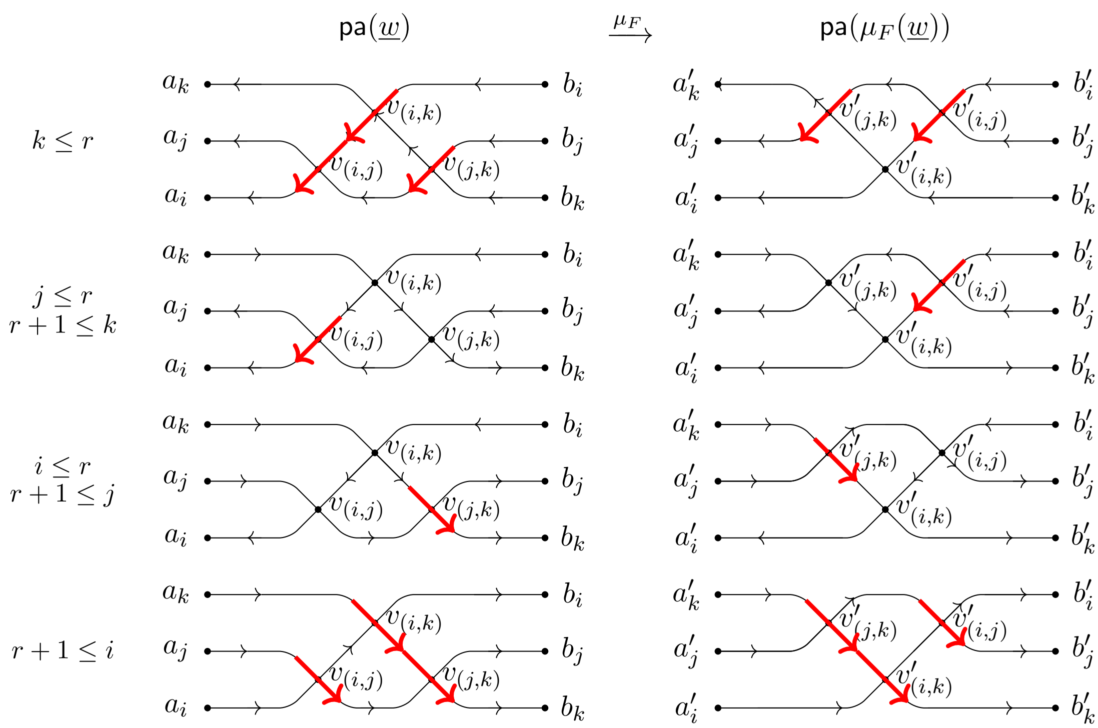


Figure 4.10: The pseudoline arrangement pa($\underline{w}$) (resp. pa($\mu_F(\underline{w})$)) locally around the face $F = F_{(i,j)}$ (resp. $F' = F'_{(j,k)}$) bounded by lines l_i, l_j, l_k with $i < j < k$ and orientations (l_r, l_{r+1}) for all possible r. The red arrows are those forbidden in GP-paths.

F-local type of $\mathbf{p}$	$\mathbf{p}$ in $\mathsf{pa}(\underline{w})$	$\mathbf{p}' = \mathrm{mut}_F(\mathbf{p})$ in $\mathsf{pa}(\mu_F(\underline{w}))$	F'-local type of $\mathbf{p}'$
$(2,1,2)$	$b_i \to v_{(i,k)} \to a_k$	$b'_i \to v'_{(i,j)} \to v'_{(j,k)} \to a'_k$	$(2,1,2)$
$(1,1,2)$	$b_j \to v_{(j,k)} \to v_{(i,k)} \to a_k$	$b'_j \to v'_{(i,j)} \to v'_{(j,k)} \to a'_k$	$(2,1,1)$
		$b'_j \to v'_{(i,j)} \to v'_{(i,k)} \to v'_{(j,k)} \to a'_k$	$(2,0,1)$
$(1,1,1)$	$b_j \to v_{(j,k)} \to v_{(i,k)} \to v_{(i,j)} \to a_j$	$b'_j \to v'_{(i,j)} \to v'_{(i,k)} \to v'_{(j,k)} \to a'_j$	$(1,0,1)$
$(1,1,1)$	$b_k \to v_{(j,k)} \to v_{(i,k)} \to a_k$	$b'_k \to v'_{(i,k)} \to v'_{(j,k)} \to a'_k$	$(1,0,1)$
$(1,1,0)$	$b_k \to v_{(j,k)} \to v_{(i,k)} \to v_{(i,j)} \to a_j$	$b'_k \to v'_{(i,k)} \to v'_{(j,k)} \to a'_j$	$(0,0,1)$
$(1,0,0)$	$b_k \to v_{(j,k)} \to v_{(i,j)} \to a_j$		
$(0,0,0)$	$b_k \to v_{(j,k)} \to v_{(i,j)} \to a_i$	$b'_k \to v'_{(i,k)} \to a'_i$	$(0,0,0)$

Table 4.1: Shapes of paths locally around F (resp. F') in $\mathcal{P}_{\underline{w}}$ (resp. $\mathcal{P}_{\mu_F(\underline{w})}$) for orientation (l_r, l_{r+1}) with $i < j < k \leq r$ (see Figure 4.10) and how they are mapped onto each other by mut_F.

in the quiver $Q_{\underline{w}}$ from (resp. to) the vertex corresponding to E to (resp. from) the vertex corresponding to F. We denote by In_F the union of all incoming faces and by Out_F the union of all outgoing faces. See for example, Figure 4.2.

Definition 60. Let $\mathbf{p} \in \mathcal{P}_{\underline{w}}$ for $\underline{w} \in S_n$ and consider a mutable face F of $\mathsf{pa}(\underline{w})$. Set $\delta_{F\subset \mathsf{A}_\mathbf{p}} := 1$ if $F \in \mathsf{A}_\mathbf{p}$ and zero otherwise. Then we define the *F-local type* of $\mathbf{p}$ as the triple

$$F(\mathbf{p}) := (i_{F,\mathbf{p}}, x_{F,\mathbf{p}}, o_{F,\mathbf{p}}) := (\#\{\mathrm{In}_F \cap \mathsf{A}_\mathbf{p}\}, \delta_{F\in\mathsf{A}_\mathbf{p}}, \#\{\mathrm{Out}_F \cap \mathsf{A}_\mathbf{p}\}).$$

For example, if $\mathsf{A}_\mathbf{p}$ in Figure 4.2 contains the faces F, F_{in_1} and F_{out_2} but not F_{in_1} and F_{out_1}, then the F-local type of $\mathbf{p}$ is $(1,1,1)$. The following lemma is a crucial observation on the F-local type of GP-paths.

Lemma 15. Let $\mathbf{p} \in \mathcal{P}_{\underline{w}}$ for $\underline{w} \in S_n$ and consider a mutable face F of $\mathsf{pa}(\underline{w})$. Then the following are all possible F-local types $\mathbf{p}$ can have:

$i_{F,\mathbf{p}} = o_{F,\mathbf{p}}$: then $F(\mathbf{p}) \in \{(0,0,0), (1,0,1), (1,1,1), (2,1,2)\}$;

$i_{F,\mathbf{p}} < o_{F,\mathbf{p}}$: then $F(\mathbf{p}) \in \{(1,1,2), (0,0,1)\}$;

$i_{F,\mathbf{p}} > o_{F,\mathbf{p}}$: then $F(\mathbf{p}) \in \{(1,0,0), (1,1,0), (2,0,1), (2,1,1)\}$.

Moreover, the F-local types of $\mathbf{p}$ with $i_{F,\mathbf{p}} > o_{F,\mathbf{p}}$ come in pairs as $((1,0,0),(1,1,0))$ or $((2,0,1),(2,1,1))$. Meaning that if a path of one type exists for a fixed orientation then so does a path of the corresponding other type for the same orientation.

Proof. The lemma follows from case-by-case consideration of all possible shapes of $\mathbf{p} \in \mathcal{P}_{\underline{w}}$ around a mutable face F of $\mathsf{pa}(\underline{w})$. First observe, that F can have two different shapes, depending on whether it is defined by simple reflections $s_m s_{m+1} s_m$ (as on the left in Figure 4.2) or by $s_{m+1} s_m s_{m+1}$ (as on the right in Figure 4.2). We endow $\mathsf{pa}(\underline{w})$ for either case of F with all possible orientations (l_r, l_{r+1}). Then locally at F, there are four cases of orientation depending

F-local type of $\mathbf{p}$	$\mathbf{p}$ in $\mathsf{pa}(\underline{w})$	$\mathbf{p}' = \mathrm{mut}_F(\mathbf{p})$ in $\mathsf{pa}(\mu_F(\underline{w}))$	F'-local type of $\mathbf{p}'$
$(1,0,1)$	$b_i \to v_{(i,k)} \to v_{(j,k)} \to v_{(i,j)} \to a_i$	$b'_i \to v'_{(i,j)} \to v'_{(j,k)} \to v'_{(i,k)} \to a'_i$	$(1,1,1)$
$(2,1,1)$ $(2,0,1)$	$b_i \to v_{(i,k)} \to v_{(i,j)} \to a_j$ $b_i \to v_{(i,k)} \to v_{(j,k)} \to v_{(i,j)} \to a_j$	$b'_i \to v'_{(i,j)} \to v'_{(j,k)} \to a'_j$	$(1,1,2)$
$(1,0,1)$	$b_i \to v_{(i,k)} \to v_{(j,k)} \to b_k$	$b'_i \to v'_{(i,j)} \to v'_{(j,k)} \to v'_{(i,k)} \to b'_k$	$(1,1,1)$
$(0,0,1)$	$b_j \to v_{(j,k)} \to v_{(i,j)} \to a_i$	$b'_j \to v'_{(i,j)} \to v'_{(j,k)} \to v'_{(i,k)} \to a'_i$ $b'_i \to v'_{(i,j)} \to v'_{(i,k)} \to a'_i$	$(1,1,0)$ $(1,0,0)$
$(1,0,1)$	$b_j \to v_{(j,k)} \to v_{(i,j)} \to a_j$	$b'_j \to v'_{(i,j)} \to v'_{(j,k)} \to a'_j$	$(1,1,1)$
$(0,0,1)$	$b_j \to v_{(j,k)} \to b_k$	$b'_j \to v'_{(i,j)} \to v'_{(j,k)} \to v'_{(i,k)} \to b'_k$ $b'_j \to v'_{(i,j)} \to v'_{(i,k)} \to b'_k$	$(1,1,0)$ $(1,0,0)$
$(1,0,1)$	$a_k \to v_{(i,k)} \to v_{(j,k)} \to v_{(i,j)} \to a_i$	$a'_k \to v'_{(j,k)} \to v'_{(i,k)} \to a'_i$	$(1,1,1)$
$(2,1,1)$ $(2,0,1)$	$a_k \to v_{(i,k)} \to v_{(i,j)} \to a_j$ $a_k \to v_{(i,k)} \to v_{(j,k)} \to v_{(i,j)} \to a_j$	$a'_k \to v'_{(j,k)} \to a'_j$	$(1,1,2)$
$(1,0,1)$	$a_k \to v_{(i,k)} \to v_{(j,k)} \to b_k$	$a'_k \to v'_{(j,k)} \to v'_{(i,k)} \to b'_k$	$(1,1,1)$

Table 4.2: Shapes of paths locally around F (resp. F') in $\mathcal{P}_{\underline{w}}$ (resp. $\mathcal{P}_{\mu_F(\underline{w})}$) for orientation (l_r, l_{r+1}) with $i < j \leq r$ and $r+1 \leq k$ (see Figure 4.10) and how they are mapped onto each other by mut_F.

on r and $r+1$ in relation to i, j, k (see Figure 4.10). We consider all possibilities for the path $\mathbf{p}$ to pass F for each case of orientation and shape of F. These are listed in Tables 4.1 to 4.4, in the second column for F as on the left of Figure 4.10 and in the third for F as on the right of Figure 4.10. In the first and last columns of these tables we indicate the corresponding F-local type. Observe, that the list in the claim of the lemma covers all occurring F-local types.

Regarding the second part of the claim, this also follows as an observation from Tables 4.1 to 4.4. □

With notation as in the lemma, if $\mathbf{p}_1, \mathbf{p}_2$ are paths with $i_{F,\mathbf{p}_j} > o_{F,\mathbf{p}_j}, j = 1,2$ such that $((i_{F,\mathbf{p}_1}, x_{F,\mathbf{p}_1}, o_{F,\mathbf{p}_1}), (i_{F,\mathbf{p}_2}, x_{F,\mathbf{p}_2}, o_{F,\mathbf{p}_2}))$ is one of the pairs, then we denote by $\mathbf{p}_1 \oplus \mathbf{p}_2$ their formal sum. If $\mathbf{p}_1$ and $\mathbf{p}_2$ are equal away from F, we denote this by $\mathbf{p}_1/F = \mathbf{p}_2/F$. Observe, that this is the case here. With this notation we define the following set of paths, respectively formal sums of paths.

$$\widehat{\mathcal{P}}_{\underline{w},F} := \left\{ \begin{array}{c|c} \mathbf{p}, & \mathbf{p} \in \mathcal{P}_{\underline{w}} \text{ with } i_{F,\mathbf{p}} = o_{F,\mathbf{p}} \text{ or } i_{F,\mathbf{p}} < o_{F,\mathbf{p}}, \\ \mathbf{p}_1 \oplus \mathbf{p}_2 & \mathbf{p}_1, \mathbf{p}_2 \in \mathcal{P}_{\underline{w}} \text{ with } i_{F,\mathbf{p}_j} > o_{F,\mathbf{p}_j} \text{ for } j = 1,2 \end{array} \right\}. \tag{4.2.21}$$

Note that for every mutable face F of $\mathsf{pa}(\underline{w})$ every path in $\mathcal{P}_{\underline{w}}$ appears in $\widehat{\mathcal{P}}_{\underline{w},F}$ either on its own or as a formal summand. This additional structure on $\mathcal{P}_{\underline{w}}$ allows us to define *mutation* on it.

Definition 61. Let $w \in S_n$ with reduced expressions $\underline{w}$ and $\mu_F(\underline{w})$, where F is a mutable face in $\mathsf{pa}(\underline{w})$. Denote by F' the corresponding face in $\mathsf{pa}(\mu_F(\underline{w}))$. We define $\mathrm{mut}_F : \widehat{\mathcal{P}}_{\underline{w},F} \to \widehat{\mathcal{P}}_{\mu_F(\underline{w}),F'}$ depending on the F-local type by

F-local type of $\mathbf{p}$	$\mathbf{p}$ in $\mathsf{pa}(\underline{w})$	$\mathbf{p}' = \mathrm{mut}_F(\mathbf{p})$ in $\mathsf{pa}(\mu_F(\underline{w}))$	F'-local type of $\mathbf{p}'$
$(1,1,1)$	$b_i \to v_{(i,k)} \to v_{(i,j)} \to a_i$	$b'_i \to v'_{(i,j)} \to v'_{(i,k)} \to a'_i$	$(1,0,1)$
$(1,0,0)$	$b_i \to v_{(i,k)} \to v_{(j,k)} \to b_j$	$b'_i \to v'_{(i,j)} \to b_k$	$(0,0,1)$
$(1,1,0)$	$b_i \to v_{(i,k)} \to v_{(i,j)} \to v_{(j,k)} \to b_j$		
$(1,1,1)$	$b_i \to v_{(i,k)} \to v_{(i,j)} \to v_{(j,k)} \to b_k$	$b'_i \to v'_{(i,j)} \to v'_{(i,k)} \to b'_k$	$(1,0,1)$
$(1,1,2)$	$a_j \to v_{(i,j)} \to a_i$	$a'_j \to v'_{(j,k)} \to v'_{(i,k)} \to a'_i$	$(2,1,1)$
		$a'_j \to v'_{(j,k)} \to v'_{(i,j)} \to v'_{(i,k)} \to a'_i$	$(2,0,1)$
$(1,1,1)$	$a_j \to v_{(i,j)} \to v_{(j,k)} \to b_j$	$a'_j \to v'_{(j,k)} \to v'_{(i,j)} \to b'_j$	$(1,0,1)$
$(1,1,2)$	$a_j \to v_{(i,j)} \to v_{(j,k)} \to b_k$	$a'_j \to v'_{(j,k)} \to v'_{(i,k)} \to b'_k$	$(2,1,1)$
		$a'_j \to v'_{(j,k)} \to v'_{(i,j)} \to v'_{(i,k)} \to b'_k$	$(2,0,1)$
$(1,1,1)$	$a_k \to v_{(i,k)} \to v_{(i,j)} \to a_i$	$a'_k \to v'_{(j,k)} \to v'_{(i,j)} \to v'_{(i,k)} \to a'_i$	$(1,0,1)$
$(1,0,0)$	$a_k \to v_{(i,k)} \to v_{(j,k)} \to b_j$	$a'_k \to v'_{(j,k)} \to v'_{(i,j)} \to b'_j$	$(0,0,1)$
$(1,1,0)$	$a_k \to v_{(i,k)} \to v_{(i,j)} \to v_{(j,k)} \to b_j$		
$(1,1,1)$	$a_k \to v_{(i,k)} \to v_{(i,j)} \to v_{(j,k)} \to b_k$	$a'_k \to v'_{(j,k)} \to v'_{(i,j)} \to v'_{(i,k)} \to b'_k$	$(1,0,1)$

Table 4.3: Shapes of paths locally around F (resp. F') in $\mathcal{P}_{\underline{w}}$ (resp. $\mathcal{P}_{\mu_F(\underline{w})}$) for orientation (l_r, l_{r+1}) with $i \leq r$ and $r+1 \leq j < k$ (see Figure 4.10) and how they are mapped onto each other by mut_F.

$i_{F,\mathbf{p}} = o_{F,\mathbf{p}}$: $\mathrm{mut}_F(\mathbf{p}) = \mathbf{p}'$ with $\mathbf{p}/F = \mathbf{p}'/F'$, where for $F(\mathbf{p}) \in \{(0,0,0),(2,1,2)\}$ we have $F(\mathbf{p}) = F'(\mathbf{p}')$, and for $F(\mathbf{p}) \in \{(1,0,1),(1,1,1)\}$ we have $F'(\mathbf{p}') = (i_{F,\mathbf{p}}, |x_{F,\mathbf{p}} - 1|, o_{F,\mathbf{p}})$;

$i_{F,\mathbf{p}} < o_{F,\mathbf{p}}$: $\mathrm{mut}_F(\mathbf{p}) = \mathbf{p}'_1 \oplus \mathbf{p}'_2$ with $\mathbf{p}/F = \mathbf{p}'_1/F' = \mathbf{p}'_2/F'$, for $F(\mathbf{p}) \in \{(0,0,1),(1,1,2)\}$ with $F'(\mathbf{p}'_1) = (o_{F,\mathbf{p}}, x_{F,\mathbf{p}}, i_{F,\mathbf{p}})$ and $F'(\mathbf{p}'_2) = (o_{F,\mathbf{p}}, |x_{F,\mathbf{p}} - 1|, i_{F,\mathbf{p}})$;

$i_{F,\mathbf{p}} > o_{F,\mathbf{p}}$: $\mathrm{mut}_F(\mathbf{p}_1 \oplus \mathbf{p}_2) = \mathbf{p}'$ with $\mathbf{p}_1/F = \mathbf{p}_2/F = \mathbf{p}'/F'$, for $(F(\mathbf{p}_1), F(\mathbf{p}_2))$ either $((1,0,0),(1,1,0))$ or $((2,1,1),(2,0,1))\}$ with $F'(\mathbf{p}') = (o_{F,\mathbf{p}_1}, x_{F,\mathbf{p}_1}, i_{F,\mathbf{p}_1})$.

Consider the torus $\mathcal{X}_{\underline{w}}$ corresponding to the seed (associated with) $\mathsf{pa}(\underline{w})$. For the lattice $N_{\underline{w}}$ we have the basis $\{e_E\}_{E \text{ face of } \mathsf{pa}(\underline{w})}$. Then $e_{\mathbf{p}} \in N$ for $\mathbf{p} \in \mathcal{P}_{\underline{w},F}$ is an expression in this basis and $z^{e_{\mathbf{p}}}$ a function on $\mathcal{X}_{\underline{w}}$. To extend our definition of $e_{\mathbf{p}}$ in (4.2.6) for $\mathbf{p} \in \mathcal{P}_{\underline{w}}$ to $\mathbf{p} \in \widehat{\mathcal{P}}_{\underline{w},F}$, we set $z^{e_{\mathbf{p}_1 \oplus \mathbf{p}_2}} := z^{e_{\mathbf{p}_1}} + z^{e_{\mathbf{p}_2}}$. Then for every mutable face F of $\mathsf{pa}(\underline{w})$ we have

$$\left\{ \mathbf{x} \in \mathbb{R}^{\ell(w)} \;\middle|\; \Big(\sum_{\mathbf{p} \in \widehat{\mathcal{P}}_{\underline{w},F}} z^{e_{\mathbf{p}}}\Big)^{\mathrm{trop}}(\mathbf{x}) \geq 0 \right\} = S_{\underline{w}}.$$

The following is the key lemma of this section.

F-local type of $\mathbf{p}$	$\mathbf{p}$ in $\mathsf{pa}(\underline{w})$	$\mathbf{p}' = \mathrm{mut}_F(\mathbf{p})$ in $\mathsf{pa}(\mu_F(\underline{w}))$	F'-local type of $\mathbf{p}'$
$(1,0,1)$	$a_i \to v_{(i,j)} \to v_{(i,k)} \to b_i$	$a'_i \to v'_{(i,k)} \to v'_{(i,j)} \to b'_i$	$(1,1,1)$
$(2,0,1)$	$a_i \to v_{(i,j)} \to v_{(i,k)} \to v_{(j,k)} \to b_j$	$a'_i \to v'_{(i,k)} \to v'_{(i,j)} \to b'_j$	$(1,1,2)$
$(2,1,1)$	$a_i \to v_{(i,j)} \to v_{(j,k)} \to b_j$		
$(2,1,2)$	$a_i \to v_{(i,j)} \to v_{(j,k)} \to b_k$	$a'_i \to v'_{(i,k)} \to b'_k$	$(2,1,2)$
$(1,0,1)$	$a_j \to v_{(i,j)} \to v_{(i,k)} \to v_{(j,k)} \to b_j$	$a'_j \to v'_{(j,k)} \to v'_{(i,k)} \to v'_{(i,j)} \to b'_j$	$(1,1,1)$
$(0,0,0)$	$a_k \to v_{(i,k)} \to b_i$	$a'_k \to v'_{(j,k)} \to v'_{(i,j)} \to b'_i$	$(0,0,0)$
$(0,0,1)$	$a_j \to v_{(i,j)} \to v_{(i,k)} \to b_i$	$a'_j \to v'_{(j,k)} \to v'_{(i,j)} \to b'_i$	$(1,0,0)$
		$a'_j \to v'_{(j,k)} \to v'_{(i,k)} \to v'_{(i,j)} \to b'_i$	$(1,1,0)$

Table 4.4: Shapes of paths locally around F (resp. F') in $\mathcal{P}_{\underline{w}}$ (resp. $\mathcal{P}_{\mu_F(\underline{w})}$) for orientation (l_r, l_{r+1}) with $r+1 \leq i < j < k$ (see Figure 4.10) and how they are mapped onto each other by mut_F.

Lemma 16. Let $w \in S_n$ with reduced expressions $\underline{w}$ and $\mu_F(\underline{w})$, where F is a mutable face of $\mathsf{pa}(\underline{w})$ and F' the corresponding face of $\mathsf{pa}(\mu_F(\underline{w}))$ (i.e. $\mu_{F'}(\mu_F(\underline{w})) = \underline{w}$). Let $\{e_E\}_E$ denote the basis for $N_{\underline{w}}$ and $\{e'_E\}_E$ the basis for $N_{\mu_F(\underline{w})}$. Then for $\mathbf{p} \in \widehat{\mathcal{P}}_{\underline{w},F}$ we have

$$\mu^*_{F'}(z^{e_{\mathbf{p}}}) = z^{e'_{\mathrm{mut}_F(\mathbf{p})}}.$$

Proof. We prove the claim case-by-case depending on the F-local type of $\mathbf{p}$ as in Lemma 15. As notation we use $n \in N_{\underline{w}}$ (resp. $n' \in N_{\mu_F(\underline{w})}$) referring to an expression of n is the basis $\{e_E\}_{E \text{ face of } \mathsf{pa}(\underline{w})}$ (resp. $\{e'_E\}_{E \text{ face of } \mathsf{pa}(\mu_F(\underline{w}))}$). Consider $\mathbf{p} \in \mathcal{P}_{\underline{w}}$, then

$$e_{\mathbf{p}} = -\sum_{E \subset \mathsf{A}_{\mathbf{p}}} e_E = -\sum_{E \subset (\mathrm{In}_F \cup \mathrm{Out}_F) \cap \mathsf{A}_{\mathbf{p}}} e_E - \sum_{E \not\subset (\mathrm{In}_F \cup \mathrm{Out}_F) \cap \mathsf{A}_{\mathbf{p}}} e_E =: n_{\mathbf{p}} + m_{\mathbf{p}}$$

As by definition mut_F effects a path only locally around F, we have $m_{\mathbf{p}} = m_{\mathrm{mut}_F(\mathbf{p})}$ (resp. $m_{\mathbf{p}} = m_{\mathbf{p}'_1} = m_{\mathbf{p}'_2}$ if $\mathrm{mut}_F(\mathbf{p}) = \mathbf{p}'_1 \oplus \mathbf{p}'_2 \in \widehat{\mathcal{P}}_{\mu_F(\underline{w}),F'}$). Both have the same expressions in bases $\{e_E\}_E$ and $\{e'_E\}_E$ as the corresponding basis elements are not effected by mutation: only basis elements corresponding to vertices (i.e. faces of $\mathsf{pa}(\underline{w})$) adjacent to F (i.e. in $\mathrm{In}_F \cup \mathrm{Out}_F$) are changed by mutation in (2.5.6). We use this fact throughout the proof. Denote basis elements associated with faces $F_{\mathrm{in}}, F_{\mathrm{in}_1}, F_{\mathrm{in}_2} \in \mathrm{In}_F$ by $e_{\mathrm{in}}, e_{\mathrm{in}_1}, e_{\mathrm{in}_2}$ and similarly for e_{out}. After mutation, e'_{in} is associated with the face $F'_{\mathrm{in}} \in \mathrm{Out}_{F'}$ in $\mathsf{pa}(\mu_F(\underline{w}))$.

We distinguish the cases as in Lemma 15.

$i_{F,\mathbf{p}} < o_{F,\mathbf{p}}$ From Lemma 15 we know that in this case $n_{\mathbf{p}} = -e_{\mathrm{in}} - e_F - e_{\mathrm{out}_1} - e_{\mathrm{out}_2}$ (resp. $n_{\mathbf{p}} = -e_{\mathrm{out}}$) and $\mathrm{mut}_F(\mathbf{p}) = \mathbf{p}'_1 \oplus \mathbf{p}'_2$ with $\mathbf{p}'_1, \mathbf{p}'_2$ as in Definition 61. Then $n'_{\mathbf{p}'_1} = -e'_{\mathrm{in}} - e'_F - e'_{\mathrm{out}_1} - e'_{\mathrm{out}_2}$ (resp. $n'_{\mathbf{p}'_1} = -e'_{\mathrm{out}} - e_F$) and $n'_{\mathbf{p}'_2} = -e'_{\mathrm{in}} - e'_{\mathrm{out}_1} - e'_{\mathrm{out}_2}$ (resp. $n'_{\mathbf{p}'_1} = -e'_{\mathrm{out}}$).

We compute using formulas (2.5.6), (2.5.7) and the observation that $m'_{\mathbf{p}'_1} = m'_{\mathbf{p}'_2}$:

$$\begin{array}{rcl}
\mu^*_{F'}(z^{n_{\mathbf{p}}+m_{\mathbf{p}}}) & = & z^{-e_{\text{in}}-e_F-e_{\text{out}_1}-e_{\text{out}_2}+m_{\mathbf{p}}}(1+z^{e_F}) \\
& = & z^{-e'_{\text{in}}-e'_{\text{out}_1}-e'_{\text{out}_2}+m'_{\mathbf{p}'}}(1+z^{-e'_F}) \\
& = & z^{n'_{\mathbf{p}'_1}+m'_{\mathbf{p}'_1}} + z^{n'_{\mathbf{p}'_2}+m'_{\mathbf{p}'_2}} = z^{e'_{\mathbf{p}'1}} + z^{e'_{\mathbf{p}'2}} \\
& \overset{\text{(by def.)}}{=} & z^{e'_{\mathbf{p}'1\oplus\mathbf{p}'2}} = z^{e'_{\text{mut}_F(\mathbf{p})}} \\
(\text{resp. } \mu^*_{F'}(z^{n_{\mathbf{p}}+m_{\mathbf{p}}}) & = & z^{-e_{\text{out}}+m_{\mathbf{p}}}(1+z^{e_F}) = z^{-e'_{\text{out}}+m'_{\mathbf{p}'}}(1+z^{-e'_F}) \\
& = & z^{n'_{\mathbf{p}'_1}+m'_{\mathbf{p}'_1}} + z^{n'_{\mathbf{p}'_2}+m'_{\mathbf{p}'_2}} = z^{e'_{\text{mut}_F(\mathbf{p})}}).
\end{array}$$

$i_{F,\mathbf{p}} = o_{F,\mathbf{p}}$ In this case $\text{mut}_F(\mathbf{p}) = \mathbf{p}' \in \widehat{\mathcal{P}}_{\mu_F(\underline{w}),F'}$ as in Definition 61. We divide into three cases: $i_{F,\mathbf{p}} \in \{0,1,2\}$. If $i_{F,\mathbf{p}} = 0$, consider $\mathsf{A}_{\mathbf{p}} = F_1 \cup \cdots \cup F_r$ then $\mathsf{A}_{\mathbf{p}'} = F'_1 \cup \cdots \cup F'_r$. Further,

$$\mu^*_{F'}(z^{e_{\mathbf{p}}}) = \mu^*_{F'}(z^{m_{\mathbf{p}}}) = z^{m_{\mathbf{p}}} = z^{m_{\mathbf{p}'}} = z^{e'_{\mathbf{p}'}} = z^{e'_{\text{mut}_F(\mathbf{p})}}.$$

If $i_{F,\mathbf{p}} = 1$ we have $n_{\mathbf{p}} = -e_F - e_{\text{in}} - e_{\text{out}}$ (resp. $n_{\mathbf{p}} = -e_{\text{in}} - e_{\text{out}}$). We have $n'_{\mathbf{p}'} = -e'_{\text{in}} - e'_{\text{out}}$ (resp. $n'_{\mathbf{p}'} = -e'_F - e'_{\text{in}} - e'_{\text{out}}$) and compute

$$\begin{array}{rcl}
\mu^*_{F'}(z^{n_{\mathbf{p}}}) & = & z^{-e_F-e_{\text{in}}-e_{\text{out}}} = z^{e'_F-(e'_{\text{in}}+e'_F)-e'_{\text{out}}} = z^{-e'_{\text{in}}-e'_{\text{out}}} = z^{n'_{\mathbf{p}'}} \\
(\text{resp. } \mu^*_{F'}(z^{n_{\mathbf{p}}}) & = & z^{-e_{\text{in}}-e_{\text{out}}} = z^{-(e'_{\text{in}}+e'_F)-e'_{\text{out}}} = z^{-e'_F-e'_{\text{in}}-e'_{\text{out}}} = z^{n'_{\mathbf{p}'}}).
\end{array}$$

If $i_{F,\mathbf{p}} = 2$ we have $n_{\mathbf{p}} = -e_{\text{in}_1} - e_{\text{in}_2} - e_F - e_{\text{out}_1} - e_{\text{out}_2}$. Now $n'_{\mathbf{p}'} = -e'_{\text{in}_1} - e'_{\text{in}_2} - e'_F - e'_{\text{out}_1} - e'_{\text{out}_2}$ and we compute

$$\begin{array}{rcl}
\mu^*_{F'}(z^{n_{\mathbf{p}}}) & = & z^{-e_{\text{in}_1}-e_{\text{in}_2}-e_F-e_{\text{out}_1}-e_{\text{out}_2}} = z^{-(e'_{\text{in}_1}+e'_F)-(e'_{\text{in}_2}+e'_F)-(-e'_F)-e'_{\text{out}_1}-e'_{\text{out}_2}} \\
& = & z^{-e'_{\text{in}_1}-e'_{\text{in}_2}-e'_F-e'_{\text{out}_1}-e'_{\text{out}_2}} = z^{n'_{\mathbf{p}'}}.
\end{array}$$

In all three cases the claim follows from the computation.

$i_{F,\mathbf{p}} > o_{F,\mathbf{p}}$ In this case by Lemma 15 there are paths $\mathbf{p}_1, \mathbf{p}_2 \in \mathcal{P}_{\underline{w}}$ with $\mathbf{p}_1 \oplus \mathbf{p}_2 \in \widehat{\mathcal{P}}_{\underline{w},F}$ and $\text{mut}_F(\mathbf{p}_1 \oplus \mathbf{p}_2) = \mathbf{p}' \in \widehat{\mathcal{P}}_{\mu_F(\underline{w}),F'}$ as in Definition 61. We have $n_{\mathbf{p}_1} = -e_{\text{in}}$ and $n_{\mathbf{p}_2} = -e_{\text{in}} - e_F$ (resp. $n_{\mathbf{p}_1} = -e_{\text{in}_1} - e_{\text{in}_2} - e_{\text{out}}$ and $n_{\mathbf{p}_2} = -e_{\text{in}_1} - e_{\text{in}_2} - e_F - e_{\text{out}}$). For $\mathbf{p}'$ we have $n'_{\mathbf{p}'} = -e'_{\text{in}}$ (resp. $n'_{\mathbf{p}'} = -e'_{\text{in}_1} - e'_{\text{in}_2} - e'_F - e'_{\text{out}}$). We compute

$$\begin{array}{rcl}
\mu^*_{F'}(z^{n_{\mathbf{p}_1}} + z^{n_{\mathbf{p}_2}}) & = & z^{-e_{\text{in}}}(1+z^{e_F})^{-1} + z^{-e_{\text{in}}-e_F}(1+z^{e_F})^{-1} \\
& = & (z^{-e'_{\text{in}}-e'_F} + z^{-e'_{\text{in}}})(1+z^{-e'_F})^{-1} = z^{-e'_{\text{in}}} = z^{n'_{\mathbf{p}'}} \\
(\text{resp. } \mu^*_{F'}(z^{n_{\mathbf{p}_1}} + z^{n_{\mathbf{p}_2}}) & = & z^{-e_{\text{in}_1}-e_{\text{in}_2}-e_{\text{out}}}(1+z^{e_F})^{-1} + z^{-e_{\text{in}_1}-e_{\text{in}_2}-e_F-e_{\text{out}}}(1+z^{e_F})^{-1} \\
& = & (z^{-e'_{\text{in}_1}-e'_{\text{in}_2}-2e'_F-e'_{\text{out}}} + z^{-e'_{\text{in}_1}-e'_{\text{in}_2}-e'_F-e'_{\text{out}}})(1+z^{-e'_F})^{-1} \\
& = & z^{-e'_{\text{in}_1}-e'_{\text{in}_2}-e'_F-e'_{\text{out}}} = z^{n'_{\mathbf{p}'}}).
\end{array}$$

In both cases the claim follows.

$\square$

Before proving a generalization of Proposition 15 we have to show that also the normal vectors associated to the weight inequalities $e_{[i:k]}$ (4.2.8) mutate as expected. We use the notation as in Lemma 16 and its proof. Recall the normal vectors of the weight inequalities for $\mathcal{S}_{\underline{w}}$ from (4.2.8). For $i \in [n-1]$ let $e_{[i:0]}, \ldots, e_{[i:n_i]}$ be those for $\underline{w}$ as expressions in $\{e_E\}_E$ and $e'_{[i:0]}, \ldots, e'_{[i,n'_i]}$ those for $\mu_F(\underline{w})$ as expressions in $\{e'_E\}_E$.

Lemma 17. With notation as above we have for every $i \in [n-1]$

$$\mu^*_{F'}\left(\sum_{k=0}^{n_i} z^{e_{[i:k]}}\right) = \sum_{k'=0}^{n'_i} z^{e'_{[i:k']}}.$$

Proof. We treat the case where F is of level l and F' of level $l+1$, with $l \in [n-2]$ (the proof of the other case is similar). Recall that $e_{[i:k]} = -e_{F_i} - e_{F_{j_1}} - \cdots - e_{F_{j_k}}$, where $k \in [0, n_i]$, $s_{j_1}, \ldots, s_{j_{n_i}} = s_i$ in $\underline{w}$, and F_{j_k} is bounded to the left by the crossing in $\mathsf{pa}(\underline{w})$ induced by s_{j_k}. Let $F'_i, F'_{j_1}, \ldots, F'_{j_{n'_i}}$ be the corresponding faces in $\mathsf{pa}(\mu_F(\underline{w}))$. In particular, if $i \notin \{l, l+1\}$ we have

$$\mu^*_{F'}(z^{-e_{F_i}-e_{F_{j_1}}-\cdots-e_{F_{j_k}}}) = z^{-e'_{F'_i}-e'_{F'_{j_1}}-\cdots-e'_{F'_{j_k}}}.$$

We therefore focus on the cases $i \in \{l, l+1\}$.

$i = l$ As F is of level l we have $F = F_{j_k}$ for one $k \in [n_l], s_{j_k} = s_l$. By (2.5.6) we have $\mu_F(e_{F_l}) = e'_{F'_l}$ and $\mu_F(e_{F_{j_r}}) = e'_{F'_{j_r}}$ for $r \in [k-1]$, hence

$$\mu^*_{F'}(z^{-e_{F_l}-e_{F_{j_1}}-\cdots-e_{F_{j_r}}}) = z^{-e'_{F'_l}-e'_{F'_{j_1}}-\cdots-e'_{F'_{j_r}}}.$$

Still by (2.5.6) we have $\mu_F(e_{F_{k-1}}) = e'_{F'_{j_{k-1}}} + e'_{F'_{j_k}}, \mu_F(e_{F_{j_k}}) = -e'_{F'_{j_k}}$ and $\mu_F(e_{F_{j_s}}) = e'_{F'_{j_s}}$ for $s \in [k+1, n_l]$. Plugging in to (2.5.7) we obtain

$$\begin{aligned}
\mu^*_{F'}(z^{e_{[l:k-1]}} + z^{e_{[l:k]}} + z^{e_{[l:k+1]}}) &= z^{-e_{F_l}-\cdots-e_{F_{j_{k-1}}}}(1+z^{e_{F_{j_k}}})^{-1} \\
&+ z^{-e_{F_l}-\cdots-e_{F_{j_k}}}(1+z^{e_{F_{j_k}}})^{-1} + z^{-e_{F_l}-\cdots-e_{F_{j_{k+1}}}} \\
&= z^{-e'_{F'_l}-\cdots-e'_{F'_{j_{k-1}}}} + z^{-e'_{F'_l}-\cdots-e'_{F'_{j_{k+1}}}} \\
&= z^{e'_{[l:k-1]}} + z^{e_{[l:k]}}.
\end{aligned}$$

Note that the index shift in the last equality comes from the fact that $\mathsf{pa}(\mu_F(\underline{w}))$ has one less face of level l than $\mathsf{pa}(\underline{w})$ as F' is of level $l+1$. So the claim follows for level l.

$i = l+1$ Let F_{j_r} be the face of level $l+1$ in Out_F and $F_{j_{r+1}}$ the one in In_F. Then we compute with notation as above

$$\begin{aligned}
\mu^*_{F'}(z^{e_{[l:r]}} + z^{e_{[l:r+1]}}) &= z^{-e_{F_{l+1}}-\cdots-e_{F_{j_r}}}(1+z^{e_F}) + z^{-e_{F_{l+1}}-\cdots-e_{F_{j_{r+1}}}} \\
&= z^{-e'_{F_{l+1}}-\cdots-e'_{F'_{j_r}}} + z^{-e'_{F'_{l+1}}-\cdots-e'_{F'_{j_r}}-e'_{F'}} + z^{-e'_{F'_{l+1}}-\cdots-e'_{F'_{j_r}}-e'_{F'}-e'_{F'_{j_{r+1}}}} \\
&= z^{e'_{[l+1:r]}} + z^{e'_{[l+1:r+1]}} + z^{e_{[l+1:r+2]}}.
\end{aligned}$$

As before the index shift occurs because $\mathsf{pa}(\mu_F(\underline{w}))$ has additionally the face F' of level $l+1$ in comparison to $\mathsf{pa}(\underline{w})$.

□

We can now prove the following theorem.

Theorem 19. Let $\underline{w}_0$ be an arbitrary reduced expression of $w_0 \in S_n$. Then the superpotential expressed in the seed given by $\mathsf{pa}(\underline{w}_0)$ satisfies $W|_{\mathcal{X}_{\underline{w}_0}} = W_{\mathcal{S}_{\underline{w}_0}}$. In particular,

$$W|_{\mathcal{X}_{\underline{w}_0}} = \sum_{\mathbf{p}\in\mathcal{P}_{\underline{w}_0}} z^{e_{\mathbf{p}}} + \sum_{i\in[n-1],0\leq k\leq n_i} z^{e_{[i:k]}}.$$

Proof. By Proposition 15 the claim is true for the seed s_0 with $\underline{w}_0 = s_1 s_2 s_1 \cdots s_{n-1} \cdots s_2 s_1$. Now Lemmata 16 and 17 imply that the claim holds for all seeds that are related to s_0 by a finite sequence of mutations. As there are only finitely many reduced expressions for w_0 and they are all related by mutation as defined in Definition 44 the claim is true for all $\underline{w}_0$. □

Corollary 11. For every reduced expression $\underline{w}_0 \in S_n$ the following polyhedral objects coincide

(i) $\mathcal{S}_{\underline{w}_0} = \Xi_{\underline{w}_0}$,

(ii) $S_{\underline{w}_0} = \Xi_{\underline{w}_0}$,

(iii) $\mathcal{S}_{\underline{w}_0}(\lambda) = \Xi_{\underline{w}_0}(\lambda)$ for $\lambda \in \mathbb{R}^{n-1}$.

Proof. The claim in (i) follows immediately from Theorem 19 by tropicalizing. Then (iii) follows by definition as we intersect both cones with the same collection of hyperplanes. To see (ii), recall from the proof of Proposition 15 that for the initial seed s_0 the ϑ-functions $\vartheta_{(i,n)}$ correspond to GP-paths. Then the claim follows by Lemma 16 and the proof of Theorem 19. □

4.2.4 Applications of Theorem 17

We have seen in the last two subsections how the cones and polytopes defined in §4.2.1 arise from a representation theoretic point of view and in the context of cluster varieties. The following theorem is the main combinatorial result of this section. We obtain it as an application of the unimodular equivalences in Theorem 17.

Theorem 20. Let $\underline{w}_0$ be an arbitrary reduced expression of $w_0 \in S_n$. Then the following polyhedral objects are unimodularly equivalent

(i) $\mathcal{Q}_{\underline{w}_0} \cong \Xi_{\underline{w}_0}$ via $\Psi_{\underline{w}_0}$,

(ii) $Q_{\underline{w}_0} \cong \Xi_{\underline{w}_0}$ via $\Psi_{\underline{w}_0}|_{\mathbb{R}^N}$,

(iii) $\mathcal{Q}_{\underline{w}_0}(\lambda) \cong \Xi_{\underline{w}_0}(\lambda)$ for $\lambda \in \mathbb{R}^{n-1}$ via $\Psi_{\underline{w}_0}$.

Proof. Combine Theorem 17 with Theorem 18 and Corollary 11. □

Remark 11. For the special case of the initial seed s_0 the theorem can also be proved by combining results of Magee and Littelmann. In [52] Littelmann shows that the string polytope $\mathcal{Q}_{\underline{w}_0}(\lambda)$ for $\underline{w}_0 = s_1s_2s_1 \cdots s_{n-1}s_{n-2} \cdots s_2s_1$ is unimodularly equivalent to the *Gelfand-Tsetlin polytope* defined in [29]. Magee shows in [54] that Ξ_{s_0} (resp. $\Xi_{s_0}(\lambda)$) is unimodularly equivalent to the Gelfand-Tsetlin cone (resp. polytope). Combining both, one obtains Theorem 20 for s_0. In fact, to understand Magee's result was driving motivation behind this project.

By the construction of toric varieties associated to polytopes as in [16, §2.1 and §2.3] and the toric degenerations of Caldero [13] and Gross-Hacking-Keel-Kontsevich [37] we obtain the following corollary from Theorem 20 relating these toric varieties. It is the main result regarding toric degenerations of flag varieties in this section and an answer to Question 4 in the introduction.

Corollary 12. Let $\underline{w}_0$ be an arbitrary reduced expression of $w_0 \in S_n$ and $\lambda \in \mathbb{Z}^{n-1}_{>0}$. We have an induced isomorphism of the following toric varieties that are degenerations (resp. normalizations of such) of SL_n/B

$$X_{\mathcal{Q}_{\underline{w}_0}}(\lambda) \cong X_{\Xi_{\underline{w}_0}}(\lambda).$$

In order to achieve a similar result for Schubert varieties, we study the restriction of the superpotential in the following subsection.

Restricted Superpotential and Schubert varieties

Caldero's degeneration works more generally for Schubert varieties. As we have seen above, he uses the degeneration for the flag variety and by a quotient construction on the level of rings he obtains a family for the Schubert variety. For the cones, taking this quotient corresponds to setting certain variables to zero, or equivalently, restricting the defining GP-paths as in Definition 53. In a similar fashion we want to proceed with the superpotential. We show how the polytopes defining toric degenerations of Schubert varieties arise in the setting of [37].

Consider $w \in S_n$ with reduced expression $\underline{w}$ and extension $\underline{w}_0 = \underline{w}s_{i_{\ell(w)+1}} \cdots s_{i_N}$. Recall that for a seed corresponding to $\underline{w}_0$ we have a basis $\{e_F \mid F \text{ face of } \mathsf{pa}(\underline{w}_0)\}$ for $N_{\underline{w}_0}$ and further $\mathbb{C}[\mathcal{X}_{\underline{w}_0}] = \mathbb{C}[z^{\pm e_F} \mid F \text{ face of } \mathsf{pa}(\underline{w}_0)]$. Then $\{e_F \mid F \text{ face of } \mathsf{pa}(\underline{w})\}$ generates a sublattice in $N_{\underline{w}_0}$, which we denote by $N_{\underline{w}}$ with dual lattice $M_{\underline{w}}$ a quotient of $M_{\underline{w}_0}$. We have the torus $\mathcal{X}_{\underline{w}} = T_{M_{\underline{w}}} = \operatorname{Spec}(\mathbb{C}[N_{\underline{w}}])$ associated with $M_{\underline{w}}$ as in (2.5.5). In particular, $\mathbb{C}[\mathcal{X}_{\underline{w}}] = \mathbb{C}[z^{\pm e_F} \mid F \text{ face of } \mathsf{pa}(\underline{w})]$ and we have a restriction morphism between the Laurent polynomial rings

$$\operatorname{res}_{\underline{w}} : \mathbb{C}[\mathcal{X}_{\underline{w}_0}] \to \mathbb{C}[\mathcal{X}_{\underline{w}}], \quad f \mapsto f|_{\mathcal{X}_{\underline{w}}}.$$

We are interested in the restrictions to $\mathcal{X}_{\underline{w}}$ of the superpotential $W|_{\mathcal{X}_{\underline{w}_0}}$ and the detropicalization $W_{\mathcal{S}_{\underline{w}_0}}$ of $\mathcal{S}_{\underline{w}_0}$ (they are equal by Theorem 19). We want to show that they coincide with the detropicalization of $\mathcal{S}_{\underline{w}}$. In analogy with Definiton 58 for w_0 we consider for arbitrary w the following polyhedral objects.

Definition 62. For $w \in S_n$ with reduced expression $\underline{w}$ and an extension $\underline{w}_0 = \underline{w}s_{i_{\ell(w)+1}} \cdots s_{i_N}$ polyhedral objects by tropicalizing the restriction of a sum of ϑ-functions resp. the superpo-

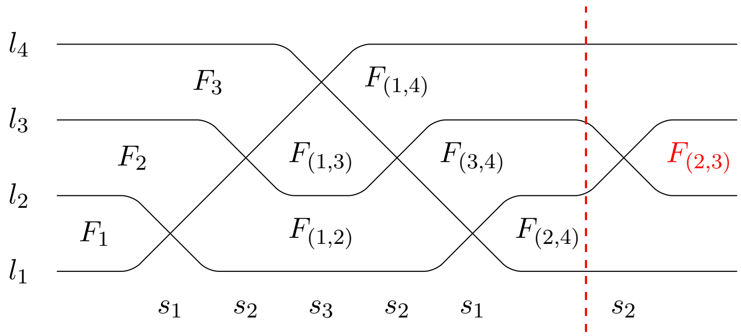


Figure 4.11: Restriction/Extension of a pseudoline arrangement.

tential:

$$\begin{aligned}
\mathrm{res}_{\underline{w}}(\Xi_{\underline{w}_0}) &:= \{\mathbf{x} \in \mathbb{R}^{\ell(w)+n-1} \mid \mathrm{res}_{\underline{w}}(W|_{\mathcal{X}_{\underline{w}_0}})^{\mathrm{trop}}(\mathbf{x}) \geq 0\},\\
\mathrm{res}_{\underline{w}}(\Xi_{\underline{w}_0}) &:= \{\mathbf{x} \in \mathbb{R}^{\ell(w)} \mid \mathrm{res}_{\underline{w}}(\sum_{i=1}^{n-1} \vartheta_{(i,n)}|_{\mathcal{X}_{\underline{w}_0}})^{\mathrm{trop}}(\mathbf{x}) \geq 0\},\\
\mathrm{res}_{\underline{w}}(\Xi_{\underline{w}_0}(\lambda)) &:= \mathrm{res}_{\underline{w}}(\Xi_{\underline{w}_0}) \cap \tau_{\underline{w}}^{-1}(\lambda) \text{ for } \lambda \in \mathbb{R}^{n-1}.
\end{aligned}$$

Example 26. Consider $\underline{w} = s_1s_2s_3s_2s_1 \in S_4$ with extension $\underline{w}_0 = \underline{w}s_2$. We compute the superpotential in $W|_{\mathcal{X}_{\underline{w}_0}} \in \mathbb{C}[\mathcal{X}_{\underline{w}_0}]$.

$$\begin{aligned}
W|_{\mathcal{X}_{\underline{w}_0}} &= \left(z^{-e_3} + z^{-e_3-e_{(1,4)}}\right) + \left(z^{-e_2} + z^{-e_2-e_{(1,3)}} + z^{-e_2-e_{(1,3)}-e_{(3,4)}}\right)\\
&+ \left(z^{-e_1} + z^{-e_1-e_{(1,2)}} + z^{-e_1-e_{(1,2)}-e_{(2,4)}}\right) + \left(z^{-e_{(2,4)}} + z^{-e_{(2,4)}-e_{(3,4)}}\right) + \left(z^{-e_{2,3}}\right)\\
&+ \left(z^{-e_{(1,4)}} + z^{-e_{(1,4)}-e_{(1,3)}} + z^{-e_{(1,4)}-e_{(1,3)}-e_{(3,4)}} + z^{-e_{(1,4)}-e_{(1,3)}-e_{(3,4)}-e_{(1,2)}}\right).
\end{aligned}$$

From Figure 4.11 we see that $F_{(2,3)}$ is a face of $\mathsf{pa}(\underline{w}_0)$, but not of $\mathsf{pa}(\underline{w})$. Hence,

$$\begin{aligned}
\mathrm{res}_{\underline{w}}(W|_{\mathcal{X}_{\underline{w}_0}}) &= \left(z^{-e_3} + z^{-e_3-e_{(1,4)}}\right) + \left(z^{-e_2} + z^{-e_2-e_{(1,3)}} + z^{-e_2-e_{(1,3)}-e_{(3,4)}}\right)\\
&+ \left(z^{-e_1} + z^{-e_1-e_{(1,2)}} + z^{-e_1-e_{(1,2)}-e_{(2,4)}}\right) + \left(z^{-e_{(2,4)}} + z^{-e_{(2,4)}-e_{(3,4)}}\right)\\
&+ \left(z^{-e_{(1,4)}} + z^{-e_{(1,4)}-e_{(1,3)}} + z^{-e_{(1,4)}-e_{(1,3)}-e_{(3,4)}} + z^{-e_{(1,4)}-e_{(1,3)}-e_{(3,4)}-e_{(1,2)}}\right).
\end{aligned}$$

Proposition 16. Let $w \in S_n$ and consider a reduced expression $\underline{w}$ with an extension to $\underline{w}_0 = \underline{w}s_{i_{\ell(w)+1}} \cdots s_{i_N}$. Then

$$\mathrm{res}_{\underline{w}}(W|_{\mathcal{X}_{\underline{w}_0}}) = W_{\mathcal{S}_{\underline{w}}}.$$

Proof. Recall the restriction of GP-paths defined in Definition 53. By Propositions 11 and 13 we have seen $\mathrm{res}_{\underline{w}}(\mathcal{P}_{\underline{w}_0}) = \mathcal{P}_{\underline{w}}$. To avoid confusion we denote as before for $i \in [n-1]$ by $n_i^{\underline{w}} := \#\{j \mid s_{i_j} = s_i \text{ in } \underline{w}\}$ and $n_i^{\underline{w}_0} := \#\{j \mid s_{i_j} = s_i \text{ in } \underline{w}_0\}$. Using Theorem 19 we compute

$$\begin{aligned}
\mathrm{res}_{\underline{w}}(W|_{\mathcal{X}_{\underline{w}_0}}) &= \sum_{\mathbf{p}\in\mathcal{P}_{\underline{w}_0}} z^{e_{\mathbf{p}}}|_{\mathcal{X}_{\underline{w}}} + \sum_{i\in[n-1],0\leq k\leq n_i^{\underline{w}_0}} z^{e_{[i:k]}}|_{\mathcal{X}_{\underline{w}}}\\
&= \sum_{\mathbf{p}\in\mathrm{res}_{\underline{w}}(\mathcal{P}_{\underline{w}_0})} z^{e_{\mathbf{p}}} + \sum_{i\in[n-1],0\leq k\leq n_i^{\underline{w}}} z^{e_{[i:k]}}\\
&= W_{\mathcal{S}_{\underline{w}}}.
\end{aligned}$$

□

The last proposition enables us to formulate a theorem similar to Theorem 20 for arbitrary $w \in S_n$.

Theorem 21. Let $w \in S_n$ and consider a reduced expression $\underline{w}$ with an extension to $\underline{w}_0 = \underline{w}s_{i_{\ell(w)+1}} \cdots s_{i_N}$. Then the following polyhedral objects are unimodularly equivalent

(i) $\mathcal{Q}_{\underline{w}} \cong \mathrm{res}_{\underline{w}}(\Xi_{\underline{w}_0})$ via $\Psi_{\underline{w}}$,

(ii) $Q_{\underline{w}} \cong \mathrm{res}_{\underline{w}}(\Xi_{\underline{w}_0})$ via $\Psi_{\underline{w}}|_{\mathbb{R}^{\ell(w)}}$,

(iii) $\mathcal{Q}_{\underline{w}}(\lambda) \cong \mathrm{res}_{\underline{w}}(\Xi_{\underline{w}_0}(\lambda))$ for $\lambda \in \mathbb{R}^{n-1}$ via $\Psi_{\underline{w}}$.

Proof. For (i) combine Proposition 16 with Theorem 17 and Theorem 18, which directly implies (iii). To see (ii), recall that by Lemma 16 and the proof of Proposition 15 we have

$$\sum_{\mathbf{p}\in\mathcal{P}_{\underline{w}_0}} z^{e_{\mathbf{p}}} = \sum_{i\in[n-1]} \vartheta_{(i,n)}|_{\mathcal{X}_{\underline{w}_0}}.$$

By the proof of Proposition 16 the same equality when replacing $\underline{w}_0$ by $\underline{w}$. Then the claim follows by Theorem 17 and Theorem 18. □

For the following corollary relating the toric degenerations of Schubert varieties by Caldero [13] to the toric degenerations of flag varieties by Gross-Hacking-Keel-Kontsevich [37] we briefly remind you about the *Orbit-Cone-Correspondence* for toric varieties (see [16, §3.2]).

For a (full-dimensional) polytope $P \subset \mathbb{R}^n$ denote by $\Sigma_P \subset \mathbb{R}^n$ its *normal fan* (see [16, Remark 2.3.3]). Every cone $\sigma \in \Sigma_P$ corresponds to a torus orbits in X_{Σ_P} of dimension $n - \dim \sigma$ ([16, Theorem 3.2.6]). The closure of each torus orbit is a toric variety. For a face Q of P let $\sigma_Q \in \Sigma_P$ be the cone in Σ_P spanned by the normal vectors of all facets of P containing Q. Then by [16, Proposition 3.2.9] the toric variety X_Q is isomorphic to the closure of the torus orbit corresponding to the cone $\sigma_Q \in \Sigma_P$.

Consider an arbitrary $w \in S_n$ with a reduced expression $\underline{w}$ and an extension $\underline{w}_0 = \underline{w}s_{i_{\ell(w)+1}} \cdots s_{i_N}$. For every $\lambda \in \Lambda^{++}$ recall that the toric variety $X_{\mathcal{Q}_{\underline{w}}(\lambda)}$ is (the normalization of) a toric degeneration of X_w by [13]. Similarly, $X_{\Xi_{\underline{w}_0}(\lambda)}$ is a flat degeneration of SL_n/B by [37]. We can now formulate the geometric version of our main result on toric degenrations of Schubert varieties.

Corollary 13. The toric variety $X_{\mathcal{Q}_{\underline{w}}(\lambda)}$ is isomorphic to a subvariety of $X_{\Xi_{\underline{w}_0}(\lambda)}$. More precisely, we have

$$X_{\mathcal{Q}_{\underline{w}}(\lambda)} \cong X_{\mathrm{res}_{\underline{w}}(\Xi_{\underline{w}_0}(\lambda))},$$

where $X_{\mathrm{res}_{\underline{w}}(\Xi_{\underline{w}_0}(\lambda))}$ is the closure of the torus orbit corresponding to the cone $\sigma_{\mathrm{res}_{\underline{w}}(\Xi_{\underline{w}_0}(\lambda))} \in \Sigma_{\Xi_{\underline{w}_0}(\lambda)}$.

Proof. By definiton $\mathrm{res}_{\underline{w}}(\Xi_{\underline{w}_0}(\lambda))$ is a union of faces of $\Xi_{\underline{w}_0}(\lambda)$. Theorem 21(iii) implies in particular, that $\mathrm{res}_{\underline{w}}(\Xi_{\underline{w}_0}(\lambda))$ is a polytope itself, hence a face of $\Xi_{\underline{w}_0}(\lambda)$. Further, the unimodular equivalence $Q_{\underline{w}}(\lambda) \cong \mathrm{res}_{\underline{w}}(\Xi_{\underline{w}_0}(\lambda))$ induces an isomorphism of toric varieties $X_{Q_{\underline{w}}(\lambda)} \cong X_{\mathrm{res}_{\underline{w}}(\Xi_{\underline{w}_0}(\lambda))}$. Then the Corollary follows by [16, Proposition 3.2.9]. □

Restriction vs. superpotential for $G^{e,w}$ We conclude with an example that shows how $\mathrm{res}_{\underline{w}}(W|_{\mathcal{X}_{\underline{w}_0}})$ is essentially different from a function one would obtain from applying Algorithm 1 to the quiver $Q_{\underline{w}}$

Example 27. Let $s = s_{\underline{w}}$ be the seed of the reduced expression $\underline{w} = s_1s_2s_3s_2s_1 \in S_4$ as in Figure 4.11. The corresponding quiver is pictured in Figure 4.12. We apply Algorithm 1 and compute optimized seeds for all frozen vertices in $Q_{\underline{w}}$. As w_3 and $w_{(2,4)}$ are sinks in $Q_{\underline{w}}$ we set $\vartheta_3|_{\mathcal{X}_s} = z^{-e_3}$ and $\vartheta_{(2,4)}|_{\mathcal{X}_s} = z^{-e_{(2,4)}}$, where $\{e_1, e_2, e_3, e_{(1,2)}, e_{(1,3)}, e_{(1,4)}, e_{(2,4)}, e_{(3,4)}\}$ is the lattice basis associated to s.

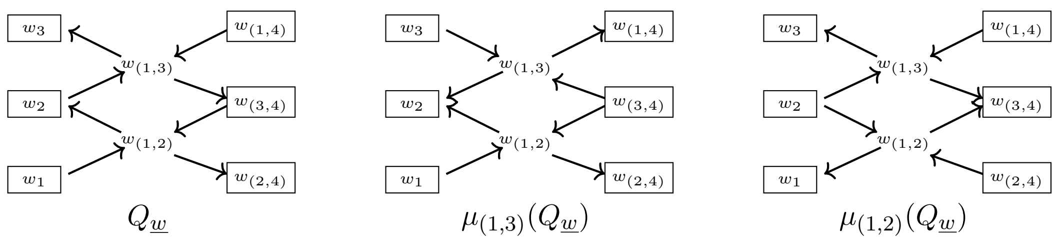


Figure 4.12: The quivers $Q_{\underline{w}}$, $\mu_{(1,3)}(Q_{\underline{w}})$ and $\mu_{(1,2)}(Q_{\underline{w}})$ for $\underline{w} = s_1s_2s_3s_2s_1$. The boxes denote frozen variables.

For the other variables we have to find a mutation sequence to an optimized seed. Mutation at $w_{(1,3)}$ (resp. $w_{(1,2)}$) yields the quiver $\mu_{(1,3)}(Q_{\underline{w}})$ (resp. $\mu_{(1,2)}(Q_{\underline{w}})$) in Figure 4.12. The seed $\mu_{(1,3)}(s)$ is optimized for $w_{(1,4)}$ and w_2, so $\vartheta_{(1,4)}|_{\mathcal{X}_{\mu_{(1,3)}(\underline{w})}} = z^{-e'_{(1,4)}}$ and $\vartheta_2|_{\mathcal{X}_{\mu_{(1,3)}(\underline{w})}} = z^{-e'_2}$. In $\mathcal{X}_{\underline{w}}$ we obtain $\vartheta_{(1,4)}|_{\mathcal{X}_s} = z^{-e_{(1,4)}} + z^{-e_{(1,4)}-e_{(1,3)}}$ and $\vartheta_2|_{\mathcal{X}_s} = z^{-e_2} + z^{-e_2-e_{(1,3)}}$. Proceeding analogously with $\mu_{(1,2)}(s)$, optimized for $w_{(3,4)}$ and w_1, we obtain a function on $\mathcal{X}_{\underline{w}}$

$$\begin{aligned} F \quad &:= \quad (z^{-e_3}) + (z^{-e_2} + z^{-e_2-e_{(1,3)}}) + (z^{-e_1} + z^{-e_1-e_{(1,2)}}) + (z^{-e_{(2,4)}}) \\ &+ \quad (z^{-e_{(3,4)}} + z^{-e_{(3,4)}-e_{(1,2)}}) + (z^{-e_{(1,4)}} + z^{-e_{(1,4)}-e_{(1,3)}}). \end{aligned}$$

Comparing to Example 26 where $\underline{w}_0 = \underline{w}s_2$ we observe that $F \neq \mathrm{res}_{\underline{w}}(W|_{\mathcal{X}_{\underline{w}_0}})$. Tropicalizing $\mathrm{res}_{\underline{w}}(W|_{\mathcal{X}_{\underline{w}_0}})$ we get the following set of inequalities defining the cone $\mathcal{S}_{\underline{w}} \subset \mathbb{R}^8$

$$\begin{gathered} -x_3 \geq 0, -x_3 - x_{(1,4)} \geq 0, \\ -x_2 \geq 0, -x_2 - x_{(1,3)} \geq 0, -x_2 - x_{(1,3)} - x_{(3,4)} \geq 0 \\ -x_1 \geq 0, -x_1 - x_{(1,2)} \geq 0, -x_1 - x_{(1,2)} - x_{(2,4)} \geq 0, \\ -x_{(2,4)} \geq 0, -x_{(2,4)} - x_{(3,4)} \geq 0, \\ -x_{(1,4)} \geq 0, -x_{(1,4)} - x_{(1,3)} \geq 0, -x_{(1,4)} - x_{(1,3)} - x_{(3,4)} \geq 0, -x_{(1,4)} - x_{(1,3)} - x_{(3,4)} - x_{(1,2)} \geq 0. \end{gathered}$$

From F^{trop} we get inequalities defining a cone $\mathcal{D}_F \subset \mathbb{R}^8$:

$$\begin{gathered} -x_3 \geq 0, \\ -x_2 \geq 0, -x_2 - x_{(1,3)} \geq 0, \\ -x_1 \geq 0, -x_1 - x_{(1,2)} \geq 0, \\ -x_{(2,4)} \geq 0, \\ -x_{(3,4)} \geq 0, -x_{(3,4)} - x_{(1,2)} \geq 0, \\ -x_{(1,4)} \geq 0, -x_{(1,4)} - x_{(1,3)} \geq 0. \end{gathered}$$

Observe that $\mathcal{D}_F \subset \mathcal{S}_{\underline{w}}$. We compute the polytopes $\mathcal{S}_{\underline{w}}(\lambda)$ and $\mathcal{D}_F \cap \tau_{\underline{w}}^{-1}(\lambda)$ for $\lambda = (1,1,1)$ and their lattice points using *polymake* [27]. The outcome is

$$|\mathcal{S}_{\underline{w}}(\lambda) \cap \mathbb{Z}^8| = 49 = \dim_{\mathbb{C}} H^0(X_w, L_\lambda) > |\mathcal{D}_F \cap \tau_{\underline{w}}^{-1}(\lambda) \cap \mathbb{Z}^8| = 30.$$

In particular, the toric variety $X_{\mathcal{D}_F \cap \tau_{\underline{w}}^{-1}(\lambda)}$ can not be a flat degeneration of the Schubert variety X_w. However, this observation is not too surprising from a geometric point of view, as the restricted superpotential and the function F correspond to different partial compactifications of the $\mathcal{A}$-cluster variety $G^{e,w}$ associated with $\mathcal{Y}(s_{\underline{w}})$.

When considering the restricted superpotential, the cluster variety we are dealing with is G^{e,w_0} and its compactification $\bar{G}^{e,w_0}$ with boundary divisors

$$\{\bar{p}_1 = 0\}, \{\bar{p}_{12} = 0\}, \{\bar{p}_{123} = 0\}, \{\bar{p}_4 = 0\}, \{\bar{p}_{34} = 0\}, \{\bar{p}_{234} = 0\}.$$

Recall that G^{e,w_0} is SL_4/U up to codimension 2. The Schubert variety of our interest is X_w with $s_1s_2s_3s_2s_1 = w$. It is given by $\{\bar{p}_{34} = 0\}$ as a subvariety SL_4/B. Note that in fact, whenever we have a reduced expression $\underline{w}$ and an extension $\underline{w}_0 = \underline{w}s_{i_{\ell(w)+1}} \cdots s_{i_N}$, then the Plücker coordinates that appear as $\mathcal{A}$-cluster variables for faces of $\mathsf{pa}(\underline{w}_0)$ that are not faces of $\mathsf{pa}(\underline{w})$ vanish identically on X_w. When restricting the superpotential, we consider the divisor of $\bar{G}^{e,w_0}$ (resp. SL_4/U) given by $\{\bar{p}_{34} = 0\}$, which is closely related to X_w.

The function F on the other hand corresponds to the $\mathcal{A}-$cluster variety $G^{e,w}$ and its partial compactification $\bar{G}^{e,w}$ with boundary divisors

$$\{\bar{p}_1 = 0\}, \{\bar{p}_{12} = 0\}, \{\bar{p}_{123} = 0\}, \{\bar{p}_4 = 0\}, \{\bar{p}_{24} = 0\}, \{\bar{p}_{234} = 0\}.$$

In this case, the defining equation for X_w in SL_4/B is not part of the boundary, so there is no reason to expect information for the Schubert variety from the potential F encoding this boundary.

4.3 Computing toric degenerations of flag varieties

In this section we compute toric degenerations arising from the tropicalization of the full flag varieties $\mathcal{F}\ell_4$ and $\mathcal{F}\ell_5$ embedded in a product of Grassmannians. For $\mathcal{F}\ell_4$ and $\mathcal{F}\ell_5$ we compare toric degenerations arising from string polytopes and the FFLV polytope with those obtained from the tropicalization of the flag varieties. We also present a general procedure to find toric degenerations in the cases where the initial ideal arising from a cone of the tropicalization of a variety is not prime.[2]

This project was initialized during the Apprenticeship Program at the Fields Institute, held 21 August–3 September 2016. The solutions to the following questions posed during the program can be found in Theorem 22 (see [66, Problem 5&6 on Grassmannians]).

5. *The complete flag variety for SL_4 is a six-dimensional subvariety of $\mathbb{P}^3 \times \mathbb{P}^4 \times \mathbb{P}^3$. Compute its ideal and determine its tropicalization.*

6. *Classify all toric ideals that arises as initial ideals for the flag variety above. For each such toric degeneration, compute the Newton-Okounkov polytope.*

This section is structured as follows. We study the tropicalization of the flag varieties $\mathcal{F}\ell_n$ for $n = 4, 5$ and the induced toric degenerations in §4.3.1.

In §4.3.2 we recall the definition of the FFLV polytope for regular dominant integral weights. We compute for $\mathcal{F}\ell_4$ and $\mathcal{F}\ell_5$ all string polytopes and the FFLV polytope for the weight $\rho \in \Lambda^{++}$, the sum of all fundamental weights. Moreover, in §4.3.2 for every string cone we construct a weight vector $\mathbf{w}_{\underline{w}_0}$ contained in the tropicalization of the flag variety in order to further explore the connection between these two different approaches. The construction is inspired by Caldero [13]. Our work is closely related to [45]. We were particularly curious about [45, Problem 1]:

> *Given a projective variety Y, find an embedding of this variety into a projective toric variety so that the resulting tropicalization contains a prime cone of maximal dimension.*

In §4.3.3 we give an algorithmic approach (see Procedure 7) to solving this problem for a subvariety X of a toric variety Y when each cone in $\mathrm{trop}(X)$ has multiplicity one. Procedure 7 aims at computing a new embedding X' of X in case $\mathrm{trop}(X)$ has some non-prime cones. Once we have such an embedding, we explain how to get new toric degenerations of X. We apply the procedure to $\mathcal{F}\ell_4$. Furthermore, we explain how to interpret the procedure in terms of finding valuations with finite Khovanskii basis on the algebra given by the homogeneous coordinate ring of X.

4.3.1 Tropicalizing $\mathcal{F}\ell_n$

In this section we study the tropicalization of $\mathcal{F}\ell_4$ and $\mathcal{F}\ell_5$. We analyze the Gröbner toric degenerations arising from $\mathrm{trop}(\mathcal{F}\ell_4)$ and $\mathrm{trop}(\mathcal{F}\ell_5)$, and we compute the polytopes associated to their normalizations. In Proposition 17 we describe the *tropical configurations* arising from the maximal cones of $\mathrm{trop}(\mathcal{F}\ell_4)$. These are configurations of a point on a tropical line in a tropical plane corresponding to the points in the relative interior of a maximal cone.

[2] Based on joint work with Sara Lamboglia, Kalina Mincheva, and Fatemeh Mohammadi in [9].

We are interested in finding distinct polytopes up to unimodular equivalence (recall Definition 51) as they give rise to non-isomorphic toric varieties. Often it is only possible to determine combinatorial equivalence (see [15, §2.2]).

Definition 63. Consider two polytopes P and Q in $\mathbb{R}^n$. Then P and Q are *combinatorially equivalent*, if there exists a bijection

$$\{\text{faces of } P\} \leftrightarrow \{\text{faces of } Q\}.$$

Note that in particular, when P and Q are unimodularly equivalent (see Definition 51) then P and Q are combinatorially equivalent. Hence, if they are not combinatorially equivalent P and Q yield non-isomorphic toric varieties. We use this fact throughout the section.

Theorem 22. The tropical variety $\mathrm{trop}(\mathcal{F}\ell_4)$ is a 6-dimensional rational fan in $\mathbb{R}^{14}/\mathbb{R}^3$ with a 3-dimensional lineality space. It consists of 78 maximal cones, 72 of which are prime. They are organized in five $S_4 \rtimes \mathbb{Z}_2$-orbits, four of which contain prime cones. The prime cones give rise to four non-isomorphic toric degenerations.

Proof. The theorem is proved by explicit computations. We developed a *Macaulay2* package called `ToricDegenerations` containing all the functions we use. The package and the data needed for this proof are available at

$$\texttt{https://github.com/ToricDegenerations}.$$

The flag variety $\mathcal{F}\ell_4$ is a subvariety of $\mathrm{Gr}(1,4) \times \mathrm{Gr}(2,4) \times \mathrm{Gr}(3,4)$, which makes it using the Plücker embedding of Grassmannians a 6-dimensional subvarity of $\mathbb{P}^3 \times \mathbb{P}^5 \times \mathbb{P}^3$. The ideal I_4 is the kernel of the map φ_4 defined in (4.1.1). It is contained in the total coordinate ring R of $\mathbb{P}^3 \times \mathbb{P}^5 \times \mathbb{P}^3$, a $\mathbb{C}$-polynomial ring in the Plücker variables

$$p_1, p_2, p_3, p_4, p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}, p_{123}, p_{124}, p_{134}, p_{234}.$$

The (multi-)grading on R is given by the matrix

$$D := \begin{pmatrix} 1&1&1&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&1&1&1&1&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix}. \tag{4.3.1}$$

The explicit form of I_4 can be found in [56, page 276]. As we have seen in §3.1.1 the tropicalization of $\mathcal{F}\ell_4$ is contained in $\mathbb{R}^{14}/H$, where H is the subspace of $\mathbb{R}^{14}$ spanned by the rows of D.

We use the *Macaulay2* [35] interface to *Gfan* [41] to compute $\mathrm{trop}(\mathcal{F}\ell_4)$. The given input is the ideal I_4 and the $S_4 \rtimes \mathbb{Z}_2$-action (see [42, §3.1.1]). The output is a subfan F of the Gröbner fan of dimension 9. We quotient it by H to get $\mathrm{trop}(\mathcal{F}\ell_4)$ as a 6-dimensional fan contained in $\mathbb{R}^{14}/H \cong \mathbb{R}^{14}/\mathbb{R}^3$.

Firstly, the function `computeWeightVectors` computes a list of vectors. There is one for every maximal cone of $\mathrm{trop}(\mathcal{F}\ell_4)$ and it is contained in the relative interior of the corresponding cone. Then `groebnerToricDegenerations` computes all the initial ideals and checks if they are binomial and prime over $\mathbb{Q}$. These are organized in a hash table, which is the output of the function.

All 78 initial ideals are binomial and all maximal cones have multiplicity one. In order to check primeness over $\mathbb{C}$, we consider for every cone C with $\text{in}_C(I_4)$ prime over $\mathbb{Q}$ the ideal $I(W_C)$ as defined in (2.2.6). We verify $\text{in}_C(I_4) = I(W_C)$ computationally using *Macaulay2* [35].

We consider the orbits of the $S_4 \ltimes \mathbb{Z}_2$-action on the set of initial ideals. These correspond to the orbits of maximal cones of F and hence of $\text{trop}(\mathcal{F}\ell_4)$. There is one orbit of non-prime initial ideals and four orbits of prime initial ideals. The varieties corresponding to initial ideals contained in the same orbit are isomorphic. Therefore, for each orbit we choose a representative of the form $\text{in}_C(I_4) = I(W_C)$ for some cone C in the orbit.

We now compute for each of the four prime orbits, the polytope of the normalization of the associated toric varieties. We use the *Macaulay2*-package *Polyhedra* [7] for the following computations.

The lattice M associated to $S/I(W_C)$ is generated over $\mathbb{Z}$ by the columns of W_C. To use *Polyhedra* we want to have a lattice with index 1 in $\mathbb{Z}^9$. If the index of M in $\mathbb{Z}^9$ is different from 1, we consider M as the lattice generated by the columns of the matrix $(\ker((\ker(W_C))^t)^t$. Here, for a matrix A we consider $\ker(A)$ to be the matrix whose columns minimally generate the kernel of the map $\mathbb{Z}^{14} \to \mathbb{Z}^9$ defined by A. We denote the set of generators of M by $\mathcal{B}_C = \{\mathbf{b}_1, \dots, \mathbf{b}_{14}\}$ so that $M = \mathbb{Z}\mathcal{B}_C$.

The toric variety $\mathbb{P}^3 \times \mathbb{P}^5 \times \mathbb{P}^3$ can be seen as $\text{Proj}(\oplus_\ell R_{\ell(1,1,1)})$ and $I(W_C)$ as an ideal in $\oplus_\ell R_{\ell(1,1,1)}$ (see [56, Chapter 10]). The associated toric variety is $\text{Proj}(\oplus_\ell \mathbb{C}[\mathbb{Z}_{\geq 0}\mathcal{B}_C]_{\ell(1,1,1)})$. The polytope P of the normalization is given as the convex hull of those lattice points in $\mathbb{Z}_{\geq 0}\mathcal{B}_C$ corresponding to degree $(1,1,1)$-monomials in $\mathbb{C}[\mathbb{Z}_{\geq 0}\mathcal{B}_C]$.

These can be found in the following way. We order the rows of the matrix $(\mathbf{b}_1, \dots, \mathbf{b}_{14})$ associated to $\mathcal{B}_C$ so that the first three rows give the matrix D from (4.3.1). Now the matrix $(\mathbf{b}_1, \dots, \mathbf{b}_{14})$ represents a map $\mathbb{Z}^{14} \to \mathbb{Z}^3 \oplus \mathbb{Z}^6$, where $\mathbb{Z}^3 \oplus \mathbb{Z}^6$ is the lattice M and the $\mathbb{Z}^3$ part gives the degree of the monomials associated to each lattice point on M. The lattice points, whose convex hull give the polytope P, are those with the first three coordinates being 1. In other words, we have obtained P by applying the reverse procedure of constructing a toric variety from a polytope (see [16, §2.1-§2.2]). Note that the difference from the procedure given in [16, §2.1-§2.2] is the $\mathbb{Z}^3$-grading and because of that we do not consider the convex hull of $\mathcal{B}_C$, but the intersection of $\mathbb{Z}_{\geq 0}\mathcal{B}_C$ with these hyperplanes.

In Table 4.5 there are the numerical invariants of the initial ideals and their corresponding polytopes. Using *polymake* [27] we first obtain that there is no combinatorial equivalence between each pair of polytopes. This means that there is no unimodular equivalence between the corresponding normal fans, hence the normalization of the toric varieties associated to these toric degenerations are not isomorphic. This implies that we obtain four non-isomorphic toric degenerations. □

Proposition 17. There are six tropical configurations up to symmetry (depicted in Figure 4.14) arising from the maximal cones of $\text{trop}(\mathcal{F}\ell_4)$. They are further organized in five $S_4 \rtimes \mathbb{Z}_2$-orbits.

Proof. The tropical variety $\text{trop}(\mathcal{F}\ell_4)$ is contained in

$$\text{trop}(\text{Gr}(1,\mathbb{C}^4)) \times \text{trop}(\text{Gr}(2,\mathbb{C}^4)) \times \text{trop}(\text{Gr}(3,\mathbb{C}^4)).$$

Each tropical Grassmannian parametrizes tropicalized linear spaces (see [53, Theorem 4.3.17]). This implies that every point p in $\text{trop}(\mathcal{F}\ell_4)$ corresponds to a chain of tropical linear subspaces

Orbit	Size	Cohen-Macaulay	Prime	#Generators	F-vector of associated polytope
1	24	Yes	Yes	10	(42, 141, 202, 153, 63, 13)
2	12	Yes	Yes	10	(40, 132, 186, 139, 57, 12)
3	12	Yes	Yes	10	(42, 141, 202, 153, 63, 13)
4	24	Yes	Yes	10	(43, 146, 212, 163, 68, 14)
5	6	Yes	No	10	Not applicable

Table 4.5: The tropical variety $\operatorname{trop}(\mathcal{F}\ell_4)$ has 78 maximal cones organized in five $S_4 \rtimes \mathbb{Z}_2$-orbits. The algebraic invariants of the initial ideals associated to these cones and the F-vectors of their associated polytopes are listed here.

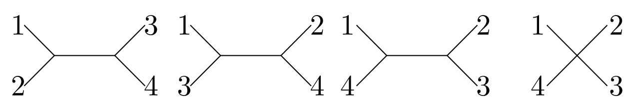


Figure 4.13: Combinatorial types of tropical lines in $\mathbb{R}^4/\mathbb{R}\mathbf{1}$.

given by a point on a tropical line contained in a tropical plane. All tropical chains are *realizable*, meaning that they are the tropicalization of the classical chains of linear spaces of $\mathbb{k}^4$ corresponding to a point q in $\mathcal{F}\ell_4(\mathbb{k})$ such that $\mathfrak{v}(q) = p$, where $\mathbb{k} = \mathbb{C}\{\{t\}\}$ is the field of Piusseux series and $\mathfrak{v}$ is the natural valuation on it (see [53, Part (3) of Theorem 3.2.3]).

In this case, there is only one combinatorial type for the tropical plane and four possible types for the lines up to symmetry (see [53, Example 4.4.9]). The plane consists of six 2-dimensional cones positively spanned by all possible pairs of vectors $(1,0,0)^t$, $(0,1,0)^t$, $(0,0,1)^t$, and $(-1,-1,-1)^t$. The combinatorial types of the tropical lines are shown in Figure 4.13. The leaves of these graphs represent the rays of the tropical line labeled 1 up to 4 corresponding to the positive hull of each of the vectors $(1,0,0)^t$, $(0,1,0)^t$, $(0,0,1)^t$, and $(-1,-1,-1)^t$.

Consider the $S_4 \rtimes \mathbb{Z}_2$-orbits of maximal cones of $\operatorname{trop}(\mathcal{F}\ell_4)$. If we compute the chain of tropical linear spaces corresponding to an element in each orbit, we get the configurations in Figure 4.14. Note that we do not include the labeling since up to symmetry we can get all possibilities. The point on the line is the black dot. In case the intersection of the line with the rays of the plane is the vertex of the plane then we denote this with a hollow dot. A vertex of the line is colored in gray if it lies on a ray of the plane. For example in orbit 2, label the rays 1 to 4 anti-clockwise starting from the top left edge. We have rays 1 and 2 in the 2-dimensional positive hull of $(1,0,0)^t$ and $(0,1,0)^t$. The vector associated to the internal edge is $(1,1,0)^t$. The gray point is the origin and the black point has coordinates $(a,1,0)^t$ for $a > 1$.

Orbits 1 and 4 in Figure 4.14 have size 24, orbits 2 and 3 have size 12 and orbit 5 has size 6. Note that orbit 5 corresponds to non-prime initial ideals. Orbit 1 contains two combinatorial types of tropical configurations and one is sent to the other by the $\mathbb{Z}_2$-action on the tropical variety. The orbits 2 and 3 differ from the fact that for each combinatorial type of line the gray dot can lie on one of the four rays of the tropical plane. These possibilities are grouped

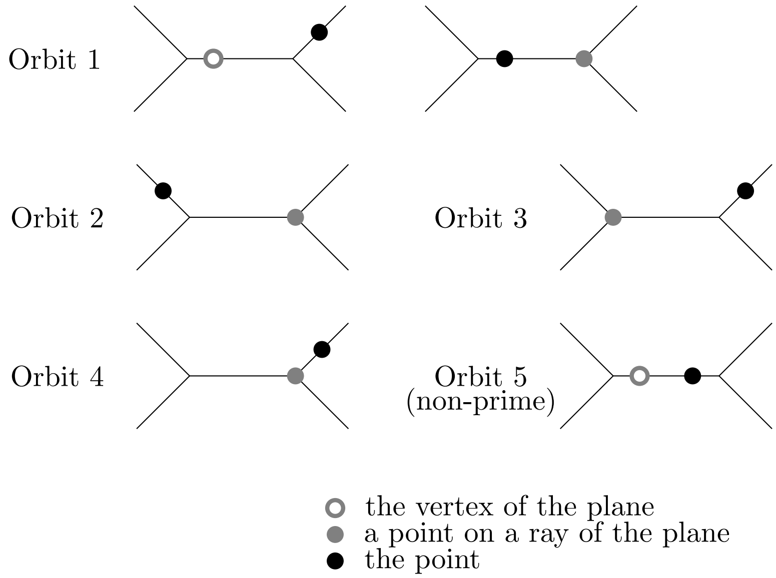


Figure 4.14: The list of all tropical configurations up to symmetry that arise in $\mathcal{F}\ell_4$. The hollow and the full gray dot denote whether that vertex of the line is the vertex of the plane or it is contained in a ray of the plane. The black dot is the position of the point on the line.

in two pairs, one is in orbit 2 and the other in orbit 3. $\square$

Theorem 23. The tropical variety $\text{trop}(\mathcal{F}\ell_5)$ is a 10-dimensional fan in $\mathbb{R}^{30}/\mathbb{R}^4$ with a 4-dimensional lineality space. It consists of 69780 maximal cones which are grouped in 536 $S_5 \rtimes \mathbb{Z}_2$-orbits. These give rise to 531 orbits of binomial initial ideals and among these 180 are prime. They correspond to 180 non-isomorphic toric degenerations.

Proof. The flag variety $\mathcal{F}\ell_5$ is a 10-dimensional variety defined by 66 quadratic polynomials in the total coordinate ring of $\mathbb{P}^4 \times \mathbb{P}^9 \times \mathbb{P}^9 \times \mathbb{P}^4$. These are of the form $\sum_{j \in J \setminus I} (-1)^{l_j} p_{I \cup \{j\}} p_{J \setminus \{j\}}$, where $J, I \subset \{1, \dots, 5\}$ and $l_j = \#\{k \in J \mid j < k\} + \#\{i \in I \mid i < j\}$.

The proof is similar to the proof of Theorem 22. The only difference is that the action of $S_5 \rtimes \mathbb{Z}_2$ on $\mathcal{F}\ell_5$ is crucial for the computations. In fact, without exploiting the symmetries the calculations to get the tropicalization would not terminate. Moreover, we only verify primeness of the initial ideals over $\mathbb{Q}$ using the *primdec* library [59] in *Singular* [17]. We compute the polytopes associated to the normalization of the 180 toric varieties in the same way as Theorem 22, only changing the matrix of the grading, which is now given by

$$D := \begin{pmatrix} 1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1 \end{pmatrix}. \tag{4.3.2}$$

Since there are no combinatorial equivalences among the normal fans to these polytopes, we deduce that the obtained toric degenerations are pairwise non-isomorphic. More information on the non-prime initial ideals is available in Table B.1 in the appendix. $\square$

$\underline{w}_0$	Normal	MP	Weight vector $-\mathbf{w}_{\underline{w}_0}$	Prime	Tropical cone
String 1:					
121321	yes	yes	$(0,32,24,7,0,16,6,48,38,30,0,4,20,52)$	yes	rays $\{10,18,19\}$, cone 71
212321	yes	yes	$(0,16,48,7,0,32,6,24,22,54,0,4,36,28)$	yes	rays $\{6,10,19\}$, cone 44
232123	yes	yes	$(0,4,36,28,0,32,24,6,22,54,0,16,48,7)$	yes	rays $\{0,3,6\}$, cone 3
323123	yes	yes	$(0,4,20,52,0,16,48,6,38,30,0,32,24,7)$	yes	rays $\{0,1,3\}$, cone 1
String 2:					
123212	yes	yes	$(0,32,18,14,0,16,12,48,44,27,0,8,24,56)$	yes	rays $\{2,10,18\}$, cone 36
321232	yes	yes	$(0,8,24,56,0,16,48,12,44,27,0,32,18,14)$	yes	rays $\{0,1,2\}$, cone 0
String 3:					
213231	yes	yes	$(0,16,48,13,0,32,12,20,28,60,0,8,40,22)$	yes	rays $\{3,6,19\}$, cone 24
String 4:					
132312	yes	no	$(0,16,12,44,0,8,40,24,56,15,0,32,10,26)$	no	rays $\{1,2,17\}$, cone 17
FFLV	yes	yes	$\mathbf{w}^{\min}=(0,2,2,1,0,1,1,2,1,2,0,1,1,1)$ $\mathbf{w}^{\mathrm{reg}}=(0,3,4,3,0,2,2,4,3,5,0,1,2,3)$	yes	rays $\{9,11,12\}$, cone 56 rays $\{9,11,12\}$, cone 56

Table 4.6: Isomorphism classes of string polytopes for $n=4$ and ρ depending on $\underline{w}_0$, normality, the weak Minkowsky property, the weight vectors $\mathbf{w}_{\underline{w}_0}$ constructed in §4.3.2, primeness of the binomial initial ideals $\mathrm{in}_{\mathbf{w}_{\underline{w}_0}}(I_4)$, and the corresponding tropical cones with their spanning rays as they appear at `http://www.mi.uni-koeln.de/~lbossing/tropflag/tropflag4.html` .

4.3.2 String&FFLV polytopes and the tropical flag variety

This section provides an introduction to FFLV polytope and explicit computations of the FFLV polytope and the string polytopes for $\mathcal{F}\ell_4$ and $\mathcal{F}\ell_5$. FFLV stands for Feigin, Fourier, and Littelmann, who defined this polytope in [23], and Vinberg who conjectured its existence in a special case. We have already seen how string polytopes can be used to construct toric degenerations of the flag variety following Caldero [13]. The same is true for the FFLV polytope. Recall the definition of string polytopes from §4.2.2 and the parametrization given in §4.2.1.

Consider the weight $\rho\in\Lambda^{++}$. The string polytope $\mathcal{Q}_{\underline{w}_0}(\rho)$ is in general *not* the Minkowski sum of string polytopes $\mathcal{Q}_{\underline{w}_0}(\omega_1),\dots,\mathcal{Q}_{\underline{w}_0}(\omega_{n-1})$, which motivates the following definition.

Definition 64. A string cone has the *weak Minkowski property* (MP), if for every lattice point $p\in\mathcal{Q}_{\underline{w}_0}(\rho)$ there exist lattice points $p_{\omega_i}\in\mathcal{Q}_{\underline{w}_0}(\omega_i)$ such that

$$p=p_{\omega_1}+p_{\omega_2}+\dots+p_{\omega_{n-1}}.$$

Remark 12. Note that the (non-weak) Minkowski property would require the above condition on lattice points to be true for arbitrary weights $\lambda\in\Lambda^{++}$. Further, note that if $\mathcal{Q}_{\underline{w}_0}(\rho)$ is the Minkowski sum of the fundamental string polytopes $\mathcal{Q}_{\underline{w}_0}(\omega_i)$, then MP is satisfied.

Proposition 18. For $\mathcal{F}\ell_4$ there are four string polytopes in $\mathbb{R}^{10}$ up to unimodular equivalence and three of them satisfy MP. For $\mathcal{F}\ell_5$ there are 28 string polytopes in $\mathbb{R}^{14}$ up to unimodular equivalence and 14 of them satisfy MP.

Proof. We first consider $\mathcal{F}\ell_4$. There are 16 reduced expressions for w_0. Simple transpositions s_i and s_j with $1 \leq i < i+1 < j < n$ commute and are also called *orthogonal*. We consider reduced expressions up to changing those, so there are eight symmetry classes. We fix the weight $\rho = \omega_1 + \omega_2 + \omega_3$ in Λ^{++}. The string polytopes are organized in four classes up to unimodular equivalence. See Table 4.6, in which 121321 denotes the reduced expression $\underline{w}_0 = s_1 s_2 s_1 s_3 s_2 s_1$. Hence, they give four different toric degenerations for the embedding $\mathcal{F}\ell_4 \hookrightarrow \mathbb{P}(V(\rho))$.

In order to verify whether the weak Minkowski property holds or not, we proceed as follows. We fix $\underline{w}_0$ to compute the string polytope $\mathcal{Q}_{\underline{w}_0}(\rho)$ using *polymake*. The number of lattice points in $\mathcal{Q}_{\underline{w}_0}(\rho)$ is $\dim(V(\rho)) = 64$. Then we compute the polytopes $\mathcal{Q}_{\underline{w}_0}(\omega_1), \mathcal{Q}_{\underline{w}_0}(\omega_2), \mathcal{Q}_{\underline{w}_0}(\omega_3)$ and set $P_{\underline{w}_0} := \mathcal{Q}_{\underline{w}_0}(\omega_1) + \mathcal{Q}_{\underline{w}_0}(\omega_2) + \mathcal{Q}_{\underline{w}_0}(\omega_3) \subset \mathbb{R}^9$. If $|P_{\underline{w}_0} \cap \mathbb{Z}^9| < 64$, then there exists a lattice point in $\mathcal{Q}_{\underline{w}_0}(\rho)$, that can not be expressed as $p_1 + p_2 + p_3$ for $p_i \in \mathcal{Q}_{\underline{w}_0}(\omega_i)$. For $\underline{w}_0 = s_1 s_3 s_2 s_3 s_1 s_2$, we compute

$$|P_{\underline{w}_0} \cap \mathbb{Z}^9| = 62 < 64.$$

Hence, polytopes in the class String 4 do not satisfy MP. For polytopes in the classes String 1, 2, and 3 equality holds and MP is satisfied.

Now consider $\mathcal{F}\ell_5$. There are 62 reduced expressions $\underline{w}_0$ up to changing orthogonal transpositions. The map $L : S_5 \to S_5$ given on simple reflections by $L(s_i) = s_{4-i+1}$ induces a symmetry on the pseudoline arrangements. Further, for a fixed $\lambda \in P^{++}$ is induces a unimodular equivalence between $\mathcal{Q}_{\underline{w}_0}(\lambda)$ and $\mathcal{Q}_{L(\underline{w}_0)}(\lambda)$. Exploiting this symmetry, we compute 31 string polytopes for ρ. They are organized in 28 unimodular equivalence classes, that arise from further symmetries of the underlying pseudoline arrangements. Table B.3 shows which reduced expressions belong to string polytopes within one unimodular equivalence class, and which string cones satisfy MP. Proceeding as for $\mathcal{F}\ell_4$, we observe that 14 out of 28 classes satisfy MP. □

We now turn to the FFLV polytope. It is defined in [23] by Feigin, Fourier, and Littelmann to describe bases of irreducible highest weight representations $V(\lambda)$. In [24] they give a construction of a flat degeneration of the flag variety into the toric variety associated to the FFLV polytope. It is also an example of the more general setup of birational sequences presented in [20]. We recall the definition and compute the FFLV polytopes for $\mathcal{F}\ell_4$ and $\mathcal{F}\ell_5$ for ρ. Recall, that $\alpha_i = \epsilon_i - \epsilon_{i+1} \in \mathbb{R}^n$ for $1 \leq i < n$ are the simple roots of $\mathfrak{sl}_n$, and $\alpha_{p,q}$ is the positive root $\alpha_p + \alpha_{p+1} + \cdots + \alpha_q$ for $1 \leq p \leq q < n$.

Definition 65. A *Dyck path* for $\mathfrak{sl}_n$ is a sequence of positive roots $\mathbf{d} = (\beta_0, \dots, \beta_k)$ with $k \geq 0$ satisfying the following conditions

1. if $k = 0$ then $\mathbf{d} = (\alpha_i)$ for $1 \leq i \leq n-1$,
2. if $k \geq 1$ then
   (a) the first and the last roots are simple, i.e. $\beta_0 = \alpha_i$, $\beta_k = \alpha_j$ for $1 \leq i < j \leq n-1$,
   (b) if $\beta_s = \alpha_{p,q}$ then β_{s+1} is either $\alpha_{p,q+1}$ or $\alpha_{p+1,q}$.

Denote by $\mathcal{D}$ the set of all Dyck paths. We choose the positive roots $\alpha > 0$ as an indexing set for a basis of $\mathbb{R}^N$.

Definition 66. The *FFLV polytope* $P(\lambda) \subset \mathbb{R}^N_{\geq 0}$ for a weight $\lambda = \sum_{i=1}^{n-1} m_i\omega_i \in \Lambda^{++}$ is defined as

$$P(\lambda) = \left\{ (r_\alpha)_{\alpha>0} \in \mathbb{R}^N_{\geq 0} \,\middle|\, \begin{array}{l} \forall \mathbf{d} \in \mathcal{D}: \text{ if } \beta_0 = \alpha_i \text{ and } \beta_k = \alpha_j \\ r_{\beta_0} + \cdots + r_{\beta_k} \leq m_i + \cdots + m_j \end{array} \right\}. \tag{4.3.3}$$

Example 28. Consider $\mathcal{F}\ell_4$. Then the Dyck paths are

$$\begin{gathered} (\alpha_1), (\alpha_2), (\alpha_3), \\ (\alpha_1, \alpha_{1,2}, \alpha_2), (\alpha_2, \alpha_{2,3}, \alpha_3), \\ (\alpha_1, \alpha_{1,2}, \alpha_2, \alpha_{2,3}, \alpha_3) \text{ and } (\alpha_1, \alpha_{1,2}, \alpha_{1,3}, \alpha_{2,3}, \alpha_3) \end{gathered}$$

For our favorite choice of weight $\lambda = \rho = \omega_1 + \omega_2 + \omega_3$ we obtain the FFLV polytope

$$P(\rho) = \left\{ (r_\alpha)_{\alpha>0} \,\middle|\, \begin{array}{c} r_{\alpha_1} \leq 1, r_{\alpha_2} \leq 1, r_{\alpha_3} \leq 1, \\ r_{\alpha_1} + r_{\alpha_{1,2}} + r_{\alpha_2} \leq 2, r_{\alpha_2} + r_{\alpha_{2,3}} + r_{\alpha_3} \leq 2, \\ r_{\alpha_1} + r_{\alpha_{1,2}} + r_{\alpha_2} + r_{\alpha_{2,3}} + r_{\alpha_3} \leq 3, \\ r_{\alpha_1} + r_{\alpha_{1,2}} + r_{\alpha_{1,3}} + r_{\alpha_{2,3}} + r_{\alpha_3} \leq 3 \end{array} \right\} \subset \mathbb{R}^6_{\geq 0}.$$

The following is a corollary of [23, Proposition 11.6], which says that a strong version of the Minkowski property is satisfied by the FFLV polytope for $\mathcal{F}\ell_n$. It can alternatively be shown for $n = 4, 5$ using the methods in the proof of Proposition 18.

Corollary 14. The FFLV polytope $P(\rho)$ satisfies the weak Minkowski property.

Remark 13. The FFLV polytope is in general not a string polytope. A computation in *polymake* shows that $P(\rho)$ for $\mathcal{F}\ell_5$ is not combinatorially equivalent to any string polytope for ρ.

String cones and points in $\operatorname{trop}(\mathcal{F}\ell_n)$

We have seen in §2.2 how to obtain toric degenerations from maximal prime cones of the tropicalization of a variety. We compare the degenerations arising from $\operatorname{trop}(\mathcal{F}\ell_n)$ with those from string polytopes and the FFLV polytope. Moreover, applying [13, Lemma 3.2] (see §2.3) we construct a weight vector from a string cone, which allows us to apply Theorem 10 from §2.4. Computational evidence for $\mathcal{F}\ell_4$ and $\mathcal{F}\ell_5$ shows that each constructed weight vector lies in the relative interior of a maximal cone in $\operatorname{trop}(\mathcal{F}\ell_n)$. A similar idea for a more general case is carried out in [45, §7]. For the FFLV polytope we compute weight vectors for $\mathcal{F}\ell_n$ with $n = 4, 5$ (see Example 31) following a construction given in [21].

We now prove the result in Theorem 9 by analyzing the polytopes associated to the different toric degenerations of $\mathcal{F}\ell_n$ for $n = 4, 5$.

Orbit	Combinatorially equivalent polytopes
1	String 2
2	String 1 (Gelfand-Tsetlin)
3	String 3 and FFLV
4	-

Table 4.7: Combinatorial equivalences among the polytopes obtained from prime cones in $\mathrm{trop}(\mathcal{F}\ell_4)$ and string polytopes resp. the FFLV polytope.

Proof of Theorem 9. In order to distinguish the different toric degenerations, we consider the normalizations of the toric varieties associated to their special fibers. Two projective normal toric varieties are isomorphic, if their corresponding polytopes are unimodularly equivalent. For this reason we first look for combinatorial equivalences between the polytopes. If they are not combinatorially equivalent then they can not be unimodularly equivalent, hence they define non-isomorphic toric varieties. We use *polymake* [27] for computations with polytopes.

From Table 4.7 one can see that for $\mathcal{F}\ell_4$ there is one toric degeneration, whose associated polytope is not combinatorially equivalent to any string polytope or the FFLV polytope for ρ. Hence, its corresponding normal toric variety is not isomorphic to any toric variety associated to these polytopes. For the toric varieties associated to the other polytopes we can not exclude isomorphisms since there might be a unimodular equivalences.

For $\mathcal{F}\ell_5$, Table B.2 in the appendix shows that there are 168 polytopes obtained from prime cones of $\mathrm{trop}(\mathcal{F}\ell_5)$ that are not combinatorially equivalent to any string polytope or the FFLV polytope for ρ. □

Remark 14. There are also string polytopes, which are not combinatorially equivalent to any polytope from prime cones in $\mathrm{trop}(\mathcal{F}\ell_n)$ for $n = 4, 5$. These are exactly those not satisfying MP, i.e. one string polytope for $\mathcal{F}\ell_4$ and 14 for $\mathcal{F}\ell_5$. See also Table B.3.

From now on, we fix a reduced expression $\underline{w}_0 = s_{i_1} \dots s_{i_N}$ and we consider the birational sequence of simple roots $S := (\alpha_{i_1}, \dots, \alpha_{i_N})$. As we have seen in Example 8.2 for Grassmannains, the same is true here: S is a birational sequence. In [20] (see also [43]) they realize string polytopes as Newton-Okounkov polytopes associated to the valuation from this birational sequence. Another necessary ingredient to obtain such a valuation on $\mathbb{C}[\mathcal{F}\ell_n]$ was the choice of total order on $\mathbb{Z}^N$. Recall therefore the definition of the Ψ-weighted reverse lexicographic order $\prec_\Psi$ from (3.2.2) and the definition of $\mathfrak{v}_S$ from (3.2.3). For our choice of S from a reduced expression $\underline{w}_0$ we denote $\mathfrak{v}_{\underline{w}_0} := \mathfrak{v}_S$.

Then $\mathfrak{v}_{\underline{w}_0}$ can be computed explicitly on Plücker coordinates. We have seen this for $\mathrm{Gr}(2, n)$ in (3.2.7), but more generally we have by [20, Proposition 2] for $\{j_1, \dots, j_k\} \subset [n]$

$$\mathfrak{v}_S(\bar{p}_{j_1,\dots,j_k}) = \min{}_{\prec_\Psi}\{\mathbf{m} \in \mathbb{Z}^N_{\geq 0} \mid \mathbf{f}^{\mathbf{m}}(e_1 \wedge \dots \wedge e_k) = e_{j_1} \wedge \dots \wedge e_{j_k}\},$$

where $\mathbf{f}^{\mathbf{m}} = f_{\alpha_{i_1}}^{m_1} \cdots f_{\alpha_{i_N}}^{m_N} \in U(\mathfrak{n}^-)$.

Example 29. For $\mathcal{F}\ell_4$ three root vectors in $\mathfrak{n}^-$ are

$$f_{\alpha_1} = \begin{pmatrix} 0&0&0&0\\ 1&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \quad f_{\alpha_2} = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&1&0&0\\ 0&0&0&0 \end{pmatrix}, \quad \text{and} \quad f_{\alpha_3} = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&1&0 \end{pmatrix}.$$

Consider $V = \bigwedge^2 \mathbb{C}^4$. As we have seen above the action of $\mathfrak{n}^-$ on $\mathbb{C}^4$ is given by $f_{\alpha_i}(e_i) = e_{i+1}$ and $f_{\alpha_i}(e_j) = 0$ for $j \neq i$. On V the $\mathfrak{n}^-$-action is given by

$$f_{\alpha_i}(e_j \wedge e_k) = f_{\alpha_i}(e_j) \wedge e_k + e_j \wedge f_{\alpha_i}(e_k).$$

The Plücker coordinate $\bar{p}_{13} \in \mathbb{C}[\mathcal{F}\ell_4]$ is of degree $(0,1,0)$, i.e. contained in $\mathbb{C}[\mathcal{F}\ell_4]_{(0,1,0)} \cong (\bigwedge^2 \mathbb{C}^4)^*$. As we have seen before $\bar{p}_{13} = (e_1 \wedge e_3)^*$, and so we consider $e_1 \wedge e_3 \in V$. Then $f_{\alpha_2}(e_1 \wedge e_2) = e_1 \wedge e_3$. We fix $\underline{\hat{w}}_0 = s_1s_2s_1s_3s_2s_1 \in S_4$, then as seen in (2.1.2) we have $U(\mathfrak{n}^-) \cdot (e_1 \wedge e_2) = \langle f_{\alpha_1}^{m_1} f_{\alpha_2}^{m_2} f_{\alpha_1}^{m_3} f_{\alpha_3}^{m_4} f_{\alpha_2}^{m_5} f_{\alpha_1}^{m_6} \cdots (e_1 \wedge e_2) \mid m_i \in \mathbb{Z}_{\geq 0} \rangle$. Hence,

$$\mathbf{f}^{(0,1,0,0,0,0)}(e_1 \wedge e_2) = \mathbf{f}^{(0,0,0,0,1,0)}(e_1 \wedge e_2) = e_1 \wedge e_3.$$

The minimal $\mathbf{m} \in (\mathbb{Z}^6, \prec_\Psi)$ satisfying $\mathbf{f}^{\mathbf{m}}(e_1 \wedge e_2) = e_1 \wedge e_3$ is $(0,1,0,0,0,0)$, so we have $\mathfrak{v}_{\underline{\hat{w}}_0}(\bar{p}_{13}) = (0,1,0,0,0,0)$.

We want to apply the results from §2.4 to the given valuations of form $\mathfrak{v}_{\underline{w}_0} : \mathbb{C}[\mathcal{F}\ell_n] \setminus \{0\} \to (\mathbb{Z}^N, \prec_\Psi)$. Let therefore $M_{\underline{w}_0} := (\mathfrak{v}_{\underline{w}_0}(\bar{p}_J))_{0 \neq J \subsetneq [n]} \in \mathbb{Z}^{N \times \binom{n}{1} + \cdots + \binom{n}{n-1}}$ be the matrix whose columns are given by the images of Plücker coordinates under $\mathfrak{v}_{\underline{w}_0}$. We define a linear form $e : \mathbb{Z}^N \to \mathbb{Z}$ by

$$-e(\mathbf{m}) := 2^{N-1}m_1 + 2^{N-2}m_2 + \ldots + 2m_{N-1} + m_N.$$

By [13, Proof of Lemma 3.2] and our choice of total order $\prec_\Psi$, it satisfies $\mathfrak{v}_{\underline{w}_0}(\bar{p}_I) \prec_\Psi \mathfrak{v}_{\underline{w}_0}(\bar{p}_J)$ implies $e(\mathfrak{v}_{\underline{w}_0}(\bar{p}_I)) < e(\mathfrak{v}_{\underline{w}_0}(\bar{p}_J))$ for $0 \neq I, J \subsetneq [n]$.

Definition 67. For a fixed reduced expression $\underline{w}_0$ the *weight* of the Plücker variable p_J is $e(\mathfrak{v}_{\underline{w}_0}(\bar{p}_J))$. We define the *weight vector* $\mathbf{w}_{\underline{w}_0}$ in $\mathbb{R}^{\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n-1}}$ by

$$\mathbf{w}_{\underline{w}_0} := e(M_{\underline{w}_0}) = (e(\mathfrak{v}_{\underline{w}_0}(\bar{p}_J)))_{0 \neq J \subsetneq [n]},$$

where we order the subsets $0 \neq J \subsetneq [n]$ lexicographically from $[1]$ to $[2, n]$.

Example 30. We continue as in Example 29 with the reduced expression $\underline{\hat{w}}_0 \in S_4$. As $\mathfrak{v}_{\underline{\hat{w}}_0}(\bar{p}_{13}) = (0,1,0,0,0,0)$ the weight of $\bar{p}_{13}$ is $e(0,1,0,0,0,0) = -(1 \cdot 2^4) = -16$. Similarly, we obtain weights for all Plücker coordinates and

$$-\mathbf{w}_{\underline{w}_0} = (0, 32, 24, 7, 0, 16, 6, 48, 38, 30, 0, 4, 20, 52).$$

Table 4.6 contains all weight vectors (up to sign) for $\mathcal{F}\ell_4$ constructed in the way just described.

Proposition 19. Consider $\mathcal{F}\ell_n$ with $n = 4, 5$. The above construction produces a weight vector $\mathbf{w}_{\underline{w}_0}$ for every string cone. This weight vector lies in the relative interior of a maximal cone of $\operatorname{trop}(\mathcal{F}\ell_n)$. If further the string cone satisfies MP, then $\mathbf{w}_{\underline{w}_0}$ lies in the relative interior of a prime cone whose associated polytope is combinatorially equivalent to $\mathcal{Q}_{\underline{w}_0}(\rho)$.

Proof. The constructed weight vectors $\mathbf{w}_{\underline{w}_0}$ can be found in Table 4.6 for $\mathcal{F}\ell_4$ and Table B.3 in the appendix for $\mathcal{F}\ell_5$. A computation in *Macaulay2* shows that all initial ideals $\operatorname{in}_{\mathbf{w}_{\underline{w}_0}}(I_n)$ for $n = 4, 5$ are binomial, hence in the relative interiors of maximal cones of $\operatorname{trop}(\mathcal{F}\ell_n)$.

Moreover, if MP is satisfied we check using *polymake* that the polytope constructed from the maximal prime cone $C^{\underline{w}_0} \subset \operatorname{trop}(\mathcal{F}\ell_n)$ with $\mathbf{w}_{\underline{w}_0}$ in its relative interior is combinatorially equivalent to the string polytope $\mathcal{Q}_{\underline{w}_0}(\rho)$. See Table 4.6 and Table B.3. $\square$

This computational outcome can actually be explained by Theorem 10. In this context in can be stated as follows.

Theorem 24. Let $\underline{w}_0$ be a reduced expression of $w_0 \in S_n$ and consider $\mathbf{w}_{\underline{w}_0} \in \mathbb{R}^{\binom{n}{1}+\dots+\binom{n}{n-1}}$. If $\text{in}_{\mathbf{w}_{\underline{w}_0}}(I_n)$ is prime, then $S(A_n, \mathfrak{v}_{\underline{w}_0})$ is generated by $\{\mathfrak{v}_{\underline{w}_0}(\bar{p}_J) \mid 0 \neq J \subsetneq [n]\}$. In particular,

$$\mathcal{Q}_{\underline{w}_0}(\rho) = \text{conv}(\mathfrak{v}_{\underline{w}_0}(\bar{p}_J) \mid 0 \neq J \subsetneq [n]),$$

and the Plücker coordinates form a Khovanskii basis for $\mathfrak{v}_{\underline{w}_0}$.

Proof. First note, that by [56, Theorem 14.6] the ideal I_n is generated by elements f satisfying $\deg f > \varepsilon_i$ for all $i \in [n-1]$. Further, by Lemma 4 $\text{in}_{\mathbf{w}_{\underline{w}_0}}(I_n) = \text{in}_{M_{\underline{w}_0}}(I_n)$ and so $\text{in}_{M_{\underline{w}_0}}(I_n)$ is prime by assumption. As $\mathfrak{v}_{\underline{w}_0}$ is of full rank, we can apply Theorem 10. If follows that $S(A_n, \mathfrak{v}_{\underline{w}_0})$ is generated by $\{\mathfrak{v}_{\underline{w}_0}(\bar{p}_J) \mid 0 \neq J \subsetneq [n]\}$ and that

$$\Delta(A_n, \mathfrak{v}_{\underline{w}_0}) = \text{conv}(\mathfrak{v}_{\underline{w}_0}(\bar{p}_J) \mid 0 \neq J \subsetneq [n]).$$

As by [20, §11] $\mathcal{Q}_{\underline{w}_0}(\rho)$ is the Newton-Okounkov body of the valuation $\mathfrak{v}_{\underline{w}_0}$ the claim follows. □

Corollary 15. With assumptions as in Theorem 24, we have:

$$\text{in}_{\mathbf{w}_{\underline{w}_0}}(I_n) \text{ is prime } \Rightarrow \mathcal{Q}_{\underline{w}_0} \text{ has the (strong) Minkowski property.}$$

Proof. By Theorem 24 the value semigroup $S(A_n, \mathfrak{v}_{\underline{w}_0})$ is generated by $\{\mathfrak{v}_{\underline{w}_0}(\bar{p}_J) \mid 0 \neq J \subsetneq [n]\}$.

Recall that $\mathbb{C}[SL_n/U] \cong \bigoplus_{\lambda \in \Lambda^+} V(\lambda)$ from §4.1. The algebra therefore has a natural multigrading given by Λ^+. Consider the valuation $\hat{\mathfrak{v}}_{\underline{w}_0} : \mathbb{C}[SL_n/U] \setminus \{0\} \to \Lambda^+ \times \mathbb{Z}^N_{\geq 0}$ given by $\hat{\mathfrak{v}}_{\underline{w}_0}(f) = (\deg f, \mathfrak{v}_{\underline{w}_0}(f))$ as in [20]. Then for every $\lambda \in \Lambda^{++}$ we have

$$\mathcal{Q}_{\underline{w}_0}(\lambda) = C(A_n, \hat{\mathfrak{v}}_{\underline{w}_0}) \cap \{\lambda\} \times \mathbb{R}^N.$$

If $\lambda = \sum_{i=1}^{n-1} a_i\omega_i$ we know (by an argument similar to [20, Proposition 2]) that the Minkowski sum of the fundamental string polytopes satisfies

$$a_1\mathcal{Q}_{\underline{w}_0}(\omega_1) + \dots + a_{n-1}\mathcal{Q}_{\underline{w}_0}(\omega_{n-1}) \subseteq \mathcal{Q}_{\underline{w}_0}(\lambda).$$

As the generators of $S(A_n, \mathfrak{v}_{w_0})$ are the union of lattice point of all fundamental string polytopes, we count

$$|(a_1\mathcal{Q}_{\underline{w}_0}(\omega_1) + \dots + a_{n-1}\mathcal{Q}_{\underline{w}_0}(\omega_{n-1})) \cap \mathbb{Z}^N| = \dim_{\mathbb{C}} V(\lambda) = |\mathcal{Q}_{\underline{w}_0}(\lambda)|.$$

Hence, the two polytopes are equal. □

Regarding the opposite implication to Corollary 15, we know that if $\mathcal{Q}_{\underline{w}_0}$ satisfies the strong Minkowski property, then $S(A_n, \mathfrak{v}_{\underline{w}_0})$ is generated by $\{\mathfrak{v}_{\underline{w}_0}(p_J) \mid 0 \neq J \subsetneq [n]\}$. Hence, $\text{gr}_{\mathfrak{v}_{\underline{w}_0}}(A_n)$ is generated by $\overline{p_J}$ for $0 \neq J \subsetneq [n]$ and we have a surjective morphism $\pi : \mathbb{C}[p_J]_J \to \text{gr}_{\mathfrak{v}_{\underline{w}_0}}(A_n)$. Then $\ker(\pi) \subset \mathbb{C}[p_J]_J$ is a prime ideal with $\text{gr}_{\mathfrak{v}_{\underline{w}_0}}(A_n) \cong \mathbb{C}[p_J]_J/\ker(\pi)$. So far, we didn't manage to prove that $\ker(\pi) = \text{in}_{\mathbf{w}_{\underline{w}_0}}(I_n)$.

Computational evidence for $\mathcal{F}\ell_4$ and $\mathcal{F}\ell_5$ leads us to the following conjecture.

Conjecture 1. Let $n \geq 3$ be an arbitrary integer. For every reduced expression $\underline{w}_0$, the weight vector $\mathbf{w}_{\underline{w}_0}$ lies in the relative interior of a maximal cone in $\operatorname{trop}(\mathcal{F}\ell_n)$.

Moreover, if the string cone satisfies MP this vector lies in the relative interior of the prime cone $C \subset \operatorname{trop}(\mathcal{F}\ell_n)$, whose associated polytope is combinatorially equivalent to the string polytope $\mathcal{Q}_{\underline{w}_0}(\rho)$.

The following example discusses a similar construction of weight vectors for the FFLV polytope.

Example 31. Consider for $\mathcal{F}\ell_4$ the birational PBW-sequence with good ordering $S := (\alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1, \alpha_2, \alpha_3)$ (similar to Example 8.1 for Grassmannians). We choose as total order on $\mathbb{Z}^N$ the *homogeneous right lexicographic order* (see [20, Example 9]), i.e.

$$\mathbf{m} \succ_{\text{rlex}} \mathbf{n} :\Leftrightarrow \sum_{i=1}^N m_i > \sum_{i=1}^N n_i, \text{ or } \sum_i m_i = \sum_i n_i \text{ and } \mathbf{m} >_{\text{rlex}} \mathbf{n},$$

where $<_{\text{rlex}}$ denotes the right lexicographic order on $\mathbb{Z}^N$. With these choices the associated Newton-Okounkov polytope to the valuation $\mathfrak{v}_S$ is the FFLV polytope (see [20, §13] and also [47]). Similar to the above we define (according to the degrees defined in [21]) linear forms $e^{\min}, e^{\text{reg}} : \mathbb{Z}^N \to \mathbb{Z}$ by

$$\begin{aligned} e^{\min}(\mathbf{m}) &:= m_1 + 2m_2 + m_3 + 2m_4 + m_5 + m_6, \\ e^{\text{reg}}(\mathbf{m}) &:= 3m_1 + 4m_2 + 2m_3 + 3m_4 + 2m_5 + m_6. \end{aligned}$$

We obtain in analogy to Definition 67 the corresponding weight vectors in $\mathbb{R}^{\binom{n}{1}+\dots+\binom{n}{n-1}}$

$$\begin{aligned} \mathbf{w}^{\min} &= (0,2,2,1,0,1,1,2,1,2,0,1,1,1), \\ \mathbf{w}^{\text{reg}} &= (0,3,4,3,0,2,2,4,3,5,0,1,2,3). \end{aligned}$$

A computation in *Macaulay2* shows that $\operatorname{in}_{\mathbf{w}^{\min}}(I_4) = \operatorname{in}_{\mathbf{w}^{\text{reg}}}(I_4)$ is a binomial prime ideal. Hence, $\mathbf{w}^{\min}$ and $\mathbf{w}^{\text{reg}}$ lie in the relative interior of the same prime cone $C \subset \operatorname{trop}(\mathcal{F}\ell_4)$. Using *polymake* [27] we verify that the polytope associated to C is combinatorially equivalent to the FFLV polytope $P(\rho)$. We did the analogue of this computation for $\mathcal{F}\ell_5$ and the outcome is the same, $\operatorname{in}_{\mathbf{w}^{\min}}(I_5) = \operatorname{in}_{\mathbf{w}^{\text{reg}}}(I_5) = \operatorname{in}_C(I_5)$ with the polytope associated to C being combinatorially equivalent to $P(\rho)$. The weight vectors $\mathbf{w}^{\min}$ and $\mathbf{w}^{\text{reg}}$ for $\mathcal{F}\ell_5$ can be found in Table B.3 in the appendix. In fact, in [19] Fang-Feigin-Fourier-Makhlin show that for arbitrary n the vectors $\mathbf{w}^{\min}$ and $\mathbf{w}^{\text{reg}}$ lie in the relative interior of a maximal prime cone of $\operatorname{trop}(\mathcal{F}\ell_n)$ and they give explicit inequalities to describe the cone.

4.3.3 Toric degenerations from non-prime cones

As we have seen in §4.3.1, not all maximal cones in the tropicalization of a variety give rise to prime initial ideals and hence to toric degenerations. In fact, there may also be tropicalizations without prime cones (see Example 32). Let $X = V(I)$ be a subvariety of a toric variety Y. In this section, we give a recursive procedure (Procedure 7) to compute a new embedding $V(I') \subset Y'$ of X in case $\operatorname{trop}(X)$ has non-prime cones. Let C be a non-prime cone. If the procedure terminates, the new tropical variety $\operatorname{trop}(X')$ has more prime cones than $\operatorname{trop}(X)$

Procedure 7: Computing new embeddings of the variety X in case $\text{trop}(X)$ contains non-prime cones

---

Input: $A = \mathbb{C}[x_1, \dots, x_n]/I$, where $\mathbb{C}[x_1, \dots, x_n]$ is the total coordinate ring of the toric variety Y and I defines the subvariety $V(I) \subset Y$, C a non-prime cone of $\text{trop}(V(I))$.

Initialization:
Compute the primary decomposition of $\text{in}_C(I)$;
$I(W_C) =$ unique prime toric component in the decomposition;
$G =$ minimal generating set of $I(W_C)$.
Compute a list of binomials $L_C = \{f_1, \dots, f_s\}$ in G, which are not in $\text{in}_C(I)$;
$A' = \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_s]/I'$ with $I' = I + \langle y_1 - f_1, \dots, y_s - f_s \rangle$;
$V(I')$ subvariety of Y' whose total coordinate ring is $\mathbb{C}[Y] := \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_s]$.
Compute $\text{trop}(V(I'))$;
for *all prime cones* $C' \in \text{trop}(V(I'))$ **do**
  if $\pi(C')$ *is contained in the relative interior of* C **then**
    Output: The algebra A' and the ideal $\text{in}_{C'}(I')$ of a toric degeneration of $V(I')$.
  else
    Apply the procedure again to A' and C'.

---

and at least one of them is projecting onto C. We apply this procedure to $\mathcal{F}\ell_4$ and compare the new toric degenerations with those obtained so far (see Proposition 20). The procedure terminates for $\mathcal{F}\ell_4$, but we are still investigating the conditions for which this is true in general.

We explain Procedure 7: consider a toric variety Y whose total coordinate ring with associated $\mathbb{Z}^k$-degree $\deg : \mathbb{Z}^n \to \mathbb{Z}^k$ is $\mathbb{C}[x_1, \dots, x_n]$. Let X be the subvariety of Y associated to an ideal $I \subset \mathbb{C}[x_1, \dots, x_n]$, where the Krull dimension of $A = \mathbb{C}[x_1, \dots, x_n]/I$ is d. Denote by $\text{trop}(V(I))$ the tropicalization of X intersected with the torus of Y. Suppose there is a non-prime cone $C \subset \text{trop}(V(I))$ with multiplicity one. By Lemma 2, we have that $I(W_C)$ is the unique toric ideal in the primary decomposition of $\text{in}_C(I)$, hence $\text{in}_C(I) \subset I(W_C)$. We compute $I(W_C)$ using the function `primaryDecomposition` in *Macaulay2*. Fix a minimal binomial generating set G of $I(W_C)$, and let $L_C = \{f_1, \dots, f_s\}$ be the set consisting of binomials in G that are not in $\text{in}_C(I)$. By *Hilbert's Basis Theorem* s is a finite number. The absence of these binomials in $\text{in}_C(I)$ is the reason why the initial ideal is not equal to $I(W_C)$, hence not prime. We introduce new variables $\{y_1, \dots, y_s\}$ and consider the algebra $A' = \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_s]/I'$, where

$$I' = I + \langle y_1 - f_1, \dots, y_s - f_s \rangle.$$

The ideal I' is a homogeneous ideal in $\mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_s]$ with respect to the grading

$$(\deg(x_1), \dots, \deg(x_n), \deg(f_1), \dots, \deg(f_s)).$$

The new variety $V(I')$ is a subvariety of the toric variety Y' with total coordinate $\mathbb{C}[Y'] := \mathbb{C}[x_0, \dots, x_n, y_1, \dots, y_s]$. For example, if $V(I)$ is a subvariety of a projective space then $V(I')$ is contained in a weighted projective space.

Since the algebras A and A' are isomorphic as graded algebras, the varieties $V(I)$ and $V(I')$ are isomorphic. We have a monomial map

$$\pi : \mathbb{C}[x_1,\dots,x_n]/I \to \mathbb{C}[x_0,\dots,x_n,y_1,\dots,y_s]/I'$$

inducing a surjective map $\mathrm{trop}(\pi) : \mathrm{trop}(V(I')) \to \mathrm{trop}(V(I))$ (see [53, Corollary 3.2.13]). The map $\mathrm{trop}(\pi)$ is the projection onto the first n coordinates. Suppose there exists a prime cone $C' \subset \mathrm{trop}(V(I'))$, whose projection has a non-empty intersection with the relative interior of C. Then by construction we have $\mathrm{in}_C(I) \subset \mathrm{in}_{C'}(I') \cap \mathbb{C}[x_0,\dots,x_n]$ and the procedure terminates. In this way we obtain a new initial ideal $\mathrm{in}_{C'}(I')$ which is toric and hence gives a new toric degeneration of the variety $V(I') \cong V(I)$. If only non-prime cones are projecting to C then run this procedure again with A' and C', where the latter is a maximal cone of $\mathrm{trop}(V(I'))$, which projects to C. We can then repeat the procedure starting from a different non-prime cone.

The function to apply Procedure 7 is `findNewToricDegenerations` and it is part of the package **ToricDegenerations**. This computes only one re-embedding for each non-prime cone. It is possible to use `mapMaximalCones` to obtain the image of $\mathrm{trop}(V(I'))$ under the map π.

Remark 15. If f_i is a polynomial in $\mathbb{C}[x_1,\dots,x_n]$ with the standard grading and $\deg(f_i) > 1$, then we need to homogenize the ideal I' before computing the tropicalization with *Gfan*. This is done by adding a new variable h. The homogenization of I' with respect to h is denoted by $I'_{proj} \subseteq \mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_s,h]$. Then by [53, Proposition 2.6.1] for every $\mathbf{w}$ in $\mathbb{R}^{n+s+2}$ the ideal $\mathrm{in}_{\mathbf{w}}(I')$ is obtained from $\mathrm{in}_{(\mathbf{w},0)}(I'_{proj})$ by setting $h = 1$.

If the cone C is prime, we can construct a valuation $\mathfrak{v}_C$ on $\Bbbk[x_1,\dots,x_n]/I$ in the following way. Consider the matrix W_C in Equation (2.2.5). For monomials $m_i = c\mathbf{x}^{\alpha_i} \in \mathbb{C}[x_1,\dots x_n]$ define

$$\mathfrak{v}(m_i) = W_C\alpha_i \quad \text{and} \quad \mathfrak{v}(\sum_i m_i) = \min_i\{\mathfrak{v}(m_i)\}, \tag{4.3.4}$$

where the minimum on the right side is taken with respect to the lexicographic order on $(\mathbb{Z}^d,+)$. This is a valuation on $\mathbb{C}[x_1,\dots,x_n]$ of rank equal to the Krull dimension of A for every cone C. Composing $\mathfrak{v}$ with the quotient morphism $p : \mathbb{C}[x_1,\dots,x_n] \to \mathbb{C}[x_1,\dots,x_n]/I$ we obtain a map $\mathfrak{v}_C$, which is a valuation if and only if the cone C is prime. Moreover, in [45] Kaveh and Manon prove that a cone C in $\mathrm{trop}(V(I))$ is prime if and only if $A = \Bbbk[x_1,\dots,x_n]/I$ has a finite *Khovanskii basis* for the valuation $\mathfrak{v}_C$ constructed from the cone C.

Procedure 7 can be interpreted as finding an extension $\mathfrak{v}_{C'}$ of $\mathfrak{v}_C$ so that A' has finite Khovanskii basis with respect to $\mathfrak{v}_{C'}$. The Khovanskii basis is given by the images of $x_1,\dots,x_n,y_1,\dots,y_s$ in A'. We illustrate the procedure in the following example.

Example 32. Consider the algebra $A = \mathbb{C}[x,y,z]/\langle xy+xz+yz\rangle$. The tropicalization of $V(\langle xy+xz+yz\rangle) \subset \mathbb{P}^2$ has three maximal cones. The corresponding initial ideals are $\langle xz+yz\rangle, \langle xy+yz\rangle$ and $\langle xy+xz\rangle$, none of which is prime. Hence they do not give rise to toric degenerations. The matrices associated to each cone are

$$W_{C_1} = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}, \quad W_{C_2} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad W_{C_3} = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

We now apply Procedure 7 to the cone C_1. The initial ideal associated to C_1 is generated by $xz + yz$. In this case $\mathrm{in}_{C_1}(I) = \langle z\rangle \cdot \langle x + y\rangle$ hence for the missing binomial $x + y$ we adjoin a new variable u to $\mathbb{C}[x, y, z]$ and the new relation $u - x - y$ to I. We have

$$I' = \langle xy + xz + yz, u - x - y\rangle \text{ and } A' = \mathbb{C}[x, y, z, u]/I'$$

with $V(I')$ a subvariety of $\mathbb{P}^3$. After computing the tropicalization of $V(I')$ we see that there exists a prime cone C' such that $\pi(C') = C$. The matrix $W_{C'}$ associated to the cone C' is

$$W' = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

The initial ideal $\mathrm{in}_{C'}(I')$ gives a toric degeneration of $V(I')$. The image of the set $\{x, y, z, u\}$ in A' is a Khovanskii basis for $S(A', \mathfrak{v}_{C'})$. Repeating this process for the cones C_2 and C_3 of $\mathrm{trop}(V(xy + xz + yz))$, we get prime cones C_2' and C_3' whose projections are C_2 and C_3 respectively. Hence, there is a valuation with finite Khovanskii basis and a corresponding toric degeneration for every maximal cone.

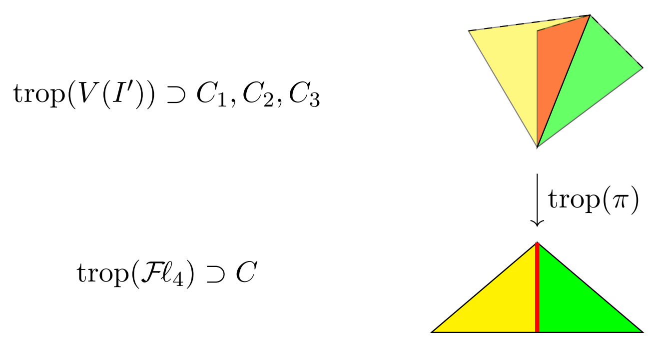


Figure 4.15: The three triangles above represent the three cones in $\mathrm{trop}(V(I'))$ which project down to the non-prime cone C in $\mathrm{trop}(\mathcal{F}\ell_4)$.

We now apply Procedure 7 to $\mathrm{trop}(\mathcal{F}\ell_4)$.

Proposition 20. Each of the non-prime cones of $\mathrm{trop}(\mathcal{F}\ell_4)$ gives rise to three toric degenerations, which are not isomorphic to any degeneration coming from the prime cones of $\mathrm{trop}(\mathcal{F}\ell_4)$. Moreover, two of the three new polytopes are combinatorially equivalent to the previously missing string polytopes for ρ in the class String 4.

Proof. By Theorem 22 we know that $\mathrm{trop}(\mathcal{F}\ell_4)$ has six non-prime cones forming one $S_4 \rtimes \mathbb{Z}_2$-orbit. Hence, we apply Procedure 7 to only one non-prime cone. The result for the other non-prime cones is the same up to symmetry. In particular, the obtained toric degenerations from one cone is isomorphic to those coming from another cone. We describe the results for the maximal cone C associated to the initial ideal $\mathrm{in}_C(I_4)$ defined by the following binomials:

$$\begin{array}{llll} p_4p_{123} - p_3p_{124}, & p_{24}p_{134} - p_{14}p_{234}, & p_{23}p_{134} - p_{13}p_{234}, & p_2p_{14} - p_1p_{24}, \\ p_2p_{13} - p_1p_{23}, & p_{24}p_{123} - p_{23}p_{124}, & p_{14}p_{123} - p_{13}p_{124}, & p_4p_{23} - p_3p_{24} \\ & p_4p_{13} - p_3p_{14}, \text{ and} & p_{14}p_{23} - p_{13}p_{24}. & \end{array}$$

We define the ideal $I' = I_4 + \langle w - p_2p_{134} + p_1p_{234}\rangle$. The grading on the variables $p_1, \dots, p_{234}$ and w is given by the matrix

$$D' := \begin{pmatrix} 1&1&1&1&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&1&1&1&1&1&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&1&1&1&1 \end{pmatrix}.$$

It extends the grading on the variables $p_1, \dots, p_{234}$ given by the matrix D in (4.3.1). The tropical variety $\mathrm{trop}(V(I'))$ has 105 maximal cones, 99 of which are prime. Among them we can find three maximal prime cones, which are mapped to C by $\mathrm{trop}(\pi)$ (see Figure 4.15). We compute the polytopes associated to the normalization of these three toric degenerations by applying the same methods as in Theorem 22. Using *polymake* we check that two of them are combinatorially equivalent to the string polytopes for ρ in the class String 4. Moreover, none of them is combinatorially equivalent to any polytope coming from prime cones of $\mathrm{trop}(\mathcal{F}\ell_4)$, hence they define different toric degenerations. $\square$

Proposition 20 suggests that for $\underline{w}_0 = s_1s_3s_2s_3s_1s_2 \in$ String 4 the weighted string cone $\mathcal{Q}_{\underline{w}_0}$ does not satisfy MP because the element

$$\mathfrak{v}_{\underline{w}_0}(\bar{p}_2\bar{p}_{134} + \bar{p}_1\bar{p}_{234}) \succeq_\Psi \min{}_{\prec_\Psi}\{\mathfrak{v}_{\underline{w}_0}(\bar{p}_2\bar{p}_{134}), \mathfrak{v}_{\underline{w}_0}(\bar{p}_1\bar{p}_{234})\}$$

is missing as a generator for $S(A_4, \mathfrak{v}_{\underline{w}_0})$. As $\mathfrak{v}_{\underline{w}_0}(\bar{p}_2\bar{p}_{134}) = \mathfrak{v}_{\underline{w}_0}(\bar{p}_1\bar{p}_{234}) = (1,0,1,1,0,0)$, we deduce $\mathfrak{v}_{\underline{w}_0}(\bar{p}_2\bar{p}_{134}+\bar{p}_1\bar{p}_{234}) \succ_\Psi (1,0,1,1,0)$. Hence, this element can not be obtained from the images of Plücker coordinates under $\mathfrak{v}_{\underline{w}_0}$ and therefore $\mathfrak{v}_{\underline{w}_0}(\bar{p}_2\bar{p}_{134} + \bar{p}_1\bar{p}_{234})$ has to be added as a generator for $S(A_4, \mathfrak{v}_{\underline{w}_0})$.

Remark 16. Procedure 7 could be applied also to $\mathcal{F}\ell_5$ but we have not been able to do so. In fact, the tropicalization for $\mathrm{trop}(V(I_5'))$ did not terminate since the computation can not be simplified by symmetries.

Appendices

Appendix A

Grassmannians

A.1 Plabic weight vectors for $\mathrm{Gr}(3, \mathbb{C}^6)$

Here are the computational findings on plabic weight vectors $\mathbf{w}_\mathcal{G}$ defined in §3.3 in more detail for $\mathrm{Gr}(3, \mathbb{C}^6)$ (based on the joint work [8]). The code can be found in [38].

There are 34 reduced plabic graphs for $\mathrm{Gr}(3, \mathbb{C}^6)$, and they give rise to the weight vectors in Table A.1. The first column indicates the unfrozen variables corresponding to a cluster and determining a plabic graph; the second column gives the corresponding weight vector in the basis indexed by

$$\{123, 124, 134, 234, 125, 135, 235, 145, 245, 345, 126, 136, 236, 146, 246, 346, 156, 236, 356, 456\}.$$

The third column gives the corresponding isomorphism class of a cone in the tropical Grassmannian as described in [64, after Lemma 5.3] and, for GG [64, after Lemma 5.1]; the fourth column gives the permutation ($\sigma = [a_1a_2a_3a_4a_5a_6]$ where $\sigma(i) = a_i$) that moves the initial ideal of the weight vector in column 2 to the initial ideal of the corresponding cone in the tropical Grassmannian using the sample vectors given in [64]. The permutations were obtained using Macaulay 2 [35], see [38] for the code. The last column refers to the enumeration of cluster seeds from [10], where a combinatorial model for cluster algebras of type D_4 is studied. The 50 seeds are given by centrally symmetric pseudo-triangulations of a once punctured dirk with 8 marked points. In the paper they analyze symmetries among the cluster seeds and associate each seed to a isomorphism class of maximal cones in $\mathrm{trop}(\mathrm{Gr}(3, \mathbb{C}^6))$. Although they consider all 50 cluster seeds, the outcome is similar to ours: they recover only six of the seven types of maximal cones, missing the cone of type $EEEE$.

mutable $\mathcal{A}$-cluster variables in seed $s_{\mathcal{G}}$	plabic weight vector $\mathbf{w}_{\mathcal{G}} \in \mathbb{R}^{\binom{6}{3}}$	type in [64]	σ	# of $s_{\mathcal{G}}$ in [10]
$p_{135}, p_{235}, p_{145}, p_{136}$	(0,0,1,1,1,1,1,1,1,4,1,1,1,1,1,1,4,4,4,5,5)	GG	123456	49
$p_{124}, p_{246}, p_{346}, p_{256}$	(0,0,0,3,0,0,3,3,4,4,3,3,4,4,4,4,4,4,4,7)	GG	134562	23
$p_{125}, p_{235}, p_{245}, p_{256}$	(0,0,1,1,0,1,1,2,2,5,3,3,3,3,3,5,3,3,5,6)	EEFF1	124563	17
$p_{235}, p_{136}, p_{236}, p_{356}$	(0,0,0,0,0,0,0,1,1,2,0,0,0,1,1,2,2,2,3,5)	EEFF1	123456	11
$p_{124}, p_{125}, p_{145}, p_{245}$	(0,0,2,3,1,2,3,2,3,6,4,4,4,4,4,6,5,5,6,6)	EEFF1	135642	15
$p_{124}, p_{125}, p_{245}, p_{256}$	(0,0,1,2,0,1,2,2,3,5,4,4,4,4,4,5,4,4,5,6)	EEFF1	134562	27
$p_{235}, p_{236}, p_{256}, p_{356}$	(0,0,0,0,0,0,0,2,2,3,1,1,1,2,2,3,2,2,3,6)	EEFF1	123564	29
$p_{136}, p_{236}, p_{346}, p_{356}$	(0,0,0,1,0,0,1,2,2,2,0,0,1,2,2,2,3,3,3,6)	EEFF1	312456	12
$p_{236}, p_{346}, p_{256}, p_{356}$	(0,0,0,1,0,0,1,3,3,3,1,1,2,3,3,3,3,3,3,7)	EEFF1	312564	28
$p_{134}, p_{136}, p_{146}, p_{346}$	(0,0,0,3,1,1,3,2,3,3,1,1,3,2,3,3,5,5,5,6)	EEFF1	356421	9
$p_{134}, p_{145}, p_{136}, p_{146}$	(0,0,1,3,2,2,3,2,3,4,2,2,3,2,3,4,6,6,6,6)	EEFF1	345612	8
$p_{124}, p_{134}, p_{145}, p_{146}$	(0,0,1,4,2,2,4,2,4,5,3,3,4,3,4,5,6,6,6,6)	EEFF1	145632	32
$p_{125}, p_{235}, p_{145}, p_{245}$	(0,0,2,2,1,2,2,2,2,6,3,3,3,3,3,6,4,4,6,6)	EEFF1	125643	14
$p_{124}, p_{134}, p_{146}, p_{346}$	(0,0,0,4,1,1,4,2,4,4,2,2,4,3,4,4,5,5,5,6)	EEFF1	156432	31
$p_{125}, p_{235}, p_{256}, p_{356}$	(0,0,0,0,0,0,0,2,2,4,2,2,2,3,3,4,3,3,4,6)	EEFF2	125346	30
$p_{124}, p_{134}, p_{125}, p_{145}$	(0,0,1,3,1,1,3,1,3,5,3,3,4,3,4,5,5,5,5,5)	EEFF2	163452	33
$p_{134}, p_{136}, p_{346}, p_{356}$	(0,0,0,2,0,0,2,2,3,3,0,0,2,2,3,3,4,4,4,6)	EEFF2	512634	10
$p_{136}, p_{236}, p_{146}, p_{346}$	(0,0,0,2,1,1,2,2,2,2,1,1,2,2,2,2,4,4,4,6)	EEFF2	612534	13
$p_{124}, p_{145}, p_{245}, p_{146}$	(0,0,2,4,2,3,4,3,4,6,4,4,4,4,4,6,6,6,7,7)	EEFF2	153462	16
$p_{235}, p_{245}, p_{236}, p_{256}$	(0,0,1,1,0,1,1,2,2,4,2,2,2,2,2,4,2,2,4,6)	EEFF2	126345	18
$p_{125}, p_{135}, p_{235}, p_{145}$	(0,0,1,1,1,1,1,1,1,5,2,2,2,2,2,5,4,4,5,5)	EFFG	123456	43
$p_{135}, p_{235}, p_{136}, p_{356}$	(0,0,0,0,0,0,0,1,1,3,0,0,0,1,1,3,3,3,4,5)	EFFG	345612	45
$p_{236}, p_{246}, p_{346}, p_{256}$	(0,0,0,2,0,0,2,3,3,3,2,2,3,3,3,3,3,3,3,7)	EFFG	612345	26
$p_{124}, p_{146}, p_{246}, p_{346}$	(0,0,0,4,1,1,4,3,4,4,3,3,4,4,4,4,5,5,5,7)	EFFG	134562	21
$p_{134}, p_{135}, p_{145}, p_{136}$	(0,0,1,2,1,1,2,1,2,4,1,1,2,1,2,4,5,5,5,5)	EFFG	561234	47
$p_{124}, p_{245}, p_{246}, p_{256}$	(0,0,1,3,0,1,3,3,4,5,4,4,4,4,4,5,4,4,5,7)	EFFG	356124	24
$p_{245}, p_{236}, p_{146}, p_{246}$	(0,0,1,3,1,2,3,3,3,4,3,3,3,3,3,4,4,4,5,7)	EEEG	265341	20
$p_{134}, p_{125}, p_{135}, p_{356}$	(0,0,0,1,0,0,1,1,2,4,1,1,2,2,3,4,4,4,4,5)	EEEG	126534	48
$p_{125}, p_{135}, p_{235}, p_{356}$	(0,0,0,0,0,0,0,1,1,4,1,1,1,2,2,4,3,3,4,5)	EEFG	342156	42
$p_{134}, p_{125}, p_{135}, p_{145}$	(0,0,1,2,1,1,2,1,2,5,2,2,3,2,3,5,5,5,5,5)	EEFG	563421	44
$p_{134}, p_{135}, p_{136}, p_{356}$	(0,0,0,1,0,0,1,1,2,3,0,0,1,1,2,3,4,4,4,5)	EEFG	215634	46
$p_{245}, p_{236}, p_{246}, p_{256}$	(0,0,1,2,0,1,2,3,3,4,3,3,3,3,3,4,3,3,4,7)	EEFG	156342	25
$p_{236}, p_{146}, p_{246}, p_{346}$	(0,0,0,3,1,1,3,3,3,3,2,2,3,3,3,3,4,4,4,7)	EEFG	634215	19
$p_{124}, p_{245}, p_{146}, p_{246}$	(0,0,1,4,1,2,4,3,4,5,4,4,4,4,4,5,5,5,6,7)	EEFG	321564	22

Table A.1: Dictionary for the 34 plabic graphs.

Appendix B

Flag varieties

In this Appendix we provide numerical evidence of our computations in §4.3. Table B.1 contains data on the non-prime maximal cones of $\mathrm{trop}(\mathcal{F}\ell_5)$. In Table B.2 there is information on the polytopes obtained from maximal prime cones of $\mathrm{trop}(\mathcal{F}\ell_5)$. This includes the F-vectors, combinatiral equivalences among the polytopes, and between those and the string polytopes, resp. FFLV polytope, for ρ. Lastly Table B.3 contains information on the string polytopes and FFLV polytope for $\mathcal{F}\ell_5$, such as the weight vectors constructed in §4.3.2, primeness of the initial ideals with respect to these vectors, and the MP property.

B.1 Algebraic and combinatorial invariants of $\mathrm{trop}(\mathcal{F}\ell_5)$

Below we collect in a table all the information about the non-prime initial ideals of $\mathcal{F}\ell_5$ up to symmetry.

Number of Orbits	#Generators
30	69
267	66
37	68
11	70
10	71
2	73

Table B.1: Data for non-prime initial ideals of $\mathcal{F}\ell_5$.

The following table shows the F-vectors of the polytopes associated to maximal prime cones of $\mathrm{trop}(\mathcal{F}\ell_5)$ for one representative in each orbit. The last column contains information on the existence of a combinatorial equivalence between these polytopes and the string polytopes resp. FFLV polytope for ρ. The initial ideals are all *Cohen-Macaulay*.

Orbit	F-vector	Combinatorial equivalences
0	475 2956 8417 14241 15690 11643 5820 1899 374 37	
1	456 2799 7843 13023 14038 10159 4938 1565 301 30	
2	425 2573 7108 11626 12333 8779 4201 1316 253 26	
3	393 2313 6200 9833 10125 7021 3297 1027 201 22	
4	433 2621 7230 11796 12473 8847 4219 1318 253 26	
5	435 2630 7246 11810 12479 8848 4219 1318 253 26	
6	425 2553 6988 11317 11888 8388 3987 1245 240 25	
7	450 2751 7677 12699 13648 9863 4800 1529 297 30	
8	435 2630 7246 11810 12479 8848 4219 1318 253 26	
9	419 2522 6922 11243 11842 8373 3985 1245 240 25	
10	453 2785 7817 12999 14027 10157 4938 1565 301 30	
11	463 2885 8237 13987 15474 11532 5788 1895 374 37	
12	463 2852 8020 13365 14459 10501 5121 1627 313 31	
13	457 2840 8078 13638 14954 10996 5413 1726 330 32	
14	454 2819 8016 13540 14870 10968 5427 1744 337 33	
15	445 2748 7770 13050 14254 10464 5161 1658 322 32	
16	441 2681 7438 12228 13056 9369 4525 1430 276 28	
17	440 2704 7602 12684 13752 10014 4897 1560 301 30	
18	471 2923 8298 13995 15369 11369 5667 1845 363 36	
19	464 2883 8200 13861 15258 11313 5651 1843 363 36	
20	467 2911 8309 14097 15574 11586 5804 1897 374 37	
21	461 2876 8225 13993 15509 11575 5814 1903 375 37	
22	397 2363 6416 10313 10755 7536 3561 1109 215 23	
23	437 2669 7447 12319 13236 9556 4642 1475 286 29	
24	425 2553 6988 11317 11888 8388 3987 1245 240 25	
25	415 2498 6861 11158 11772 8339 3976 1244 240 25	
26	470 2942 8436 14377 15944 11889 5955 1939 379 37	
27	460 2856 8109 13656 14929 10944 5374 1712 328 32	
28	449 2741 7634 12594 13487 9702 4695 1486 287 29	
29	427 2592 7181 11778 12523 8926 4270 1334 255 26	
30	425 2573 7108 11626 12333 8779 4201 1316 253 26	FFLV
31	443 2708 7557 12495 13411 9667 4686 1485 287 29	
32	397 2363 6416 10313 10755 7536 3561 1109 215 23	S22

Orbit	F-vector	Combinatorial equivalences
33	425 2553 6988 11317 11888 8388 3987 1245 240 25	
34	419 2522 6922 11243 11842 8373 3985 1245 240 25	
35	405 2407 6518 10442 10851 7578 3571 1110 215 23	
36	401 2387 6477 10398 10825 7570 3570 1110 215 23	
37	368 2154 5755 9111 9373 6497 3052 953 188 21	S21
38	379 2214 5892 9280 9494 6547 3063 954 188 21	S27, S28
39	393 2313 6200 9833 10125 7021 3297 1027 201 22	
40	358 2069 5453 8516 8653 5941 2778 870 174 20	S1, S18, S26, S29 (Gelfand-Tsetlin)
41	459 2851 8111 13720 15118 11223 5614 1834 362 36	
42	467 2913 8322 14133 15629 11636 5831 1905 375 37	
43	423 2562 7083 11596 12313 8772 4200 1316 253 26	
44	425 2573 7108 11626 12333 8779 4201 1316 253 26	S24
45	397 2363 6416 10313 10755 7536 3561 1109 215 23	S23
46	461 2876 8225 13993 15509 11575 5814 1903 375 37	
47	400 2366 6377 10175 10546 7363 3480 1089 213 23	
48	393 2313 6200 9833 10125 7021 3297 1027 201 22	
49	393 2313 6200 9833 10125 7021 3297 1027 201 22	
50	379 2214 5892 9280 9494 6547 3063 954 188 21	S2, S19
51	426 2599 7257 12034 12981 9420 4602 1470 286 29	
52	428 2594 7176 11761 12514 8947 4307 1359 263 27	
53	419 2522 6922 11243 11842 8373 3985 1245 240 25	
54	466 2917 8371 14288 15879 11870 5960 1944 380 37	
55	443 2729 7692 12867 13982 10197 4987 1585 304 30	
56	453 2787 7826 13011 14021 10122 4895 1539 293 29	
57	469 2926 8358 14188 15679 11663 5839 1906 375 37	
58	458 2825 7958 13286 14398 10472 5113 1626 313 31	
59	472 2949 8435 14335 15854 11796 5902 1923 377 37	
60	440 2704 7602 12684 13752 10014 4897 1560 301 30	
61	472 2967 8561 14720 16525 12526 6410 2144 432 43	
62	457 2842 8099 13726 15153 11266 5640 1842 363 36	
63	465 2902 8296 14096 15588 11594 5795 1884 368 36	
64	459 2851 8111 13720 15118 11223 5614 1834 362 36	
65	428 2608 7269 12028 12946 9377 4576 1462 285 29	

Orbit	F-vector	Combinatorial equivalences
66	441 2681 7438 12228 13056 9369 4525 1430 276 28	
67	418 2510 6876 11157 11753 8321 3969 1243 240 25	
68	406 2442 6713 10943 11587 8245 3950 1241 240 25	
69	373 2199 5926 9474 9849 6897 3267 1024 201 22	
70	427 2586 7144 11681 12383 8806 4209 1317 253 26	
71	451 2781 7840 13111 14243 10390 5089 1623 313 31	
72	440 2704 7602 12684 13752 10014 4897 1560 301 30	
73	406 2442 6713 10943 11587 8245 3950 1241 240 25	
74	448 2764 7800 13061 14208 10377 5087 1623 313 31	
75	462 2873 8181 13846 15258 11321 5656 1844 363 36	
76	457 2842 8099 13726 15153 11266 5640 1842 363 36	
77	469 2927 8364 14203 15699 11678 5845 1907 375 37	
78	454 2802 7903 13216 14348 10453 5110 1626 313 31	
79	451 2787 7879 13221 14419 10565 5200 1667 323 32	
80	441 2705 7584 12611 13622 9885 4823 1537 298 30	
81	454 2803 7914 13263 14455 10598 5231 1687 330 33	
82	441 2697 7532 12465 13391 9660 4685 1485 287 29	
83	445 2721 7593 12550 13461 9694 4694 1486 287 29	
84	441 2697 7532 12465 13391 9660 4685 1485 287 29	
85	445 2725 7617 12611 13546 9764 4728 1495 288 29	
86	397 2363 6416 10313 10755 7536 3561 1109 215 23	
87	368 2154 5755 9111 9373 6497 3052 953 188 21	S5, S31
88	452 2801 7946 13385 14654 10771 5309 1699 327 32	
89	430 2624 7318 12097 12974 9329 4497 1411 269 27	
90	456 2834 8071 13670 15083 11210 5612 1834 362 36	
91	432 2633 7332 12104 12975 9341 4521 1430 276 28	
92	467 2919 8359 14230 15769 11756 5892 1922 377 37	
93	456 2834 8071 13670 15083 11210 5612 1834 362 36	
94	426 2597 7244 11998 12926 9370 4575 1462 285 29	
95	440 2708 7630 12769 13898 10169 5001 1603 311 31	
96	432 2633 7332 12104 12975 9341 4521 1430 276 28	
97	412 2479 6810 11083 11707 8306 3967 1243 240 25	
98	415 2511 6945 11391 12133 8679 4174 1313 253 26	

Orbit	F-vector	Combinatorial equivalences
99	458 2845 8092 13676 15042 11132 5543 1800 353 35	
100	437 2669 7447 12319 13236 9556 4642 1475 286 29	
101	441 2703 7569 12562 13531 9780 4746 1502 289 29	
102	427 2586 7144 11681 12383 8806 4209 1317 253 26	
103	419 2522 6922 11243 11842 8373 3985 1245 240 25	
104	437 2669 7447 12319 13236 9556 4642 1475 286 29	
105	411 2470 6776 11012 11617 8235 3933 1234 239 25	
106	413 2483 6808 11043 11606 8177 3871 1201 230 24	
107	425 2553 6988 11317 11888 8388 3987 1245 240 25	
108	405 2407 6518 10442 10851 7578 3571 1110 215 23	
109	405 2427 6638 10751 11296 7969 3785 1181 228 24	S30
110	465 2904 8312 14152 15700 11734 5907 1940 384 38	
111	464 2902 8323 14204 15795 11828 5960 1956 386 38	
112	438 2690 7559 12608 13667 9952 4868 1552 300 30	
113	445 2725 7617 12611 13546 9764 4728 1495 288 29	
114	437 2669 7447 12319 13236 9556 4642 1475 286 29	
115	411 2470 6776 11012 11617 8235 3933 1234 239 25	
116	424 2574 7139 11737 12529 8983 4332 1367 264 27	
117	419 2522 6922 11243 11842 8373 3985 1245 240 25	
118	401 2387 6477 10398 10825 7570 3570 1110 215 23	
119	405 2427 6638 10751 11296 7969 3785 1181 228 24	S6
120	464 2893 8261 14019 15483 11503 5746 1869 366 36	
121	454 2806 7928 13283 14448 10543 5159 1641 315 31	
122	451 2794 7928 13370 14676 10840 5387 1746 342 34	
123	444 2736 7715 12915 14053 10273 5044 1613 312 31	
124	466 2909 8318 14138 15644 11650 5837 1906 375 37	
125	456 2815 7939 13271 14398 10480 5118 1627 313 31	
126	423 2561 7078 11586 12303 8767 4199 1316 253 26	
127	429 2580 7064 11429 11972 8402 3959 1221 232 24	
128	431 2626 7309 12058 12915 9290 4494 1422 275 28	
129	428 2602 7224 11883 12684 9087 4375 1377 265 27	
130	443 2727 7679 12831 13927 10147 4960 1577 303 30	
131	432 2637 7354 12152 13024 9356 4505 1412 269 27	

Orbit	F-vector	Combinatorial equivalences
132	451 2793 7920 13342 14620 10770 5331 1718 334 33	
133	434 2632 7273 11879 12557 8883 4210 1301 246 25	
134	452 2781 7813 13004 14042 10171 4944 1566 301 30	
135	453 2808 7969 13433 14725 10847 5366 1727 335 33	
136	451 2794 7928 13370 14676 10840 5387 1746 342 34	
137	433 2646 7390 12236 13150 9482 4589 1448 278 28	
138	442 2715 7629 12727 13808 10076 4948 1587 309 31	
139	432 2633 7332 12104 12975 9341 4521 1430 276 28	
140	423 2564 7096 11632 12368 8822 4227 1324 254 26	
141	413 2483 6808 11043 11606 8177 3871 1201 230 24	
142	427 2594 7196 11827 12614 9031 4347 1369 264 27	
143	431 2622 7281 11973 12769 9135 4390 1379 265 27	
144	431 2626 7309 12058 12915 9290 4494 1422 275 28	
145	410 2459 6725 10881 11411 8029 3802 1183 228 24	
146	428 2594 7176 11761 12514 8947 4307 1359 263 27	
147	419 2522 6922 11243 11842 8373 3985 1245 240 25	
148	451 2781 7840 13111 14243 10390 5089 1623 313 31	
149	464 2900 8310 14168 15740 11778 5933 1948 385 38	
150	446 2750 7757 12985 14123 10315 5058 1615 312 31	
151	420 2541 7021 11496 12218 8719 4184 1314 253 26	
152	441 2705 7584 12611 13622 9885 4823 1537 298 30	
153	425 2575 7119 11651 12363 8799 4208 1317 253 26	
154	448 2764 7801 13067 14223 10397 5102 1629 314 31	
155	444 2737 7724 12949 14124 10363 5115 1647 321 32	
156	452 2772 7753 12830 13755 9876 4750 1486 282 28	
157	442 2706 7565 12529 13460 9696 4684 1473 281 28	
158	441 2708 7602 12655 13676 9915 4821 1525 292 29	
159	427 2596 7207 11850 12633 9026 4324 1350 257 26	
160	452 2781 7813 13004 14042 10171 4944 1566 301 30	
161	427 2586 7144 11681 12383 8806 4209 1317 253 26	
162	400 2382 6467 10388 10820 7569 3570 1110 215 23	
163	448 2764 7800 13061 14208 10377 5087 1623 313 31	
164	470 2943 8444 14405 16000 11959 6011 1967 387 38	

Orbit	F-vector	Combinatorial equivalences
165	460 2857 8117 13684 14985 11014 5430 1740 336 33	
166	418 2530 6996 11466 12198 8712 4183 1314 253 26	
167	434 2640 7325 12025 12788 9108 4348 1353 257 26	
168	425 2577 7132 11687 12418 8849 4235 1325 254 26	
169	425 2581 7160 11772 12564 9004 4339 1368 264 27	
170	430 2614 7255 11928 12724 9109 4382 1378 265 27	
171	422 2557 7075 11597 12333 8801 4220 1323 254 26	
172	411 2470 6772 10988 11556 8150 3863 1200 230 24	S7
173	427 2586 7144 11681 12383 8806 4209 1317 253 26	
174	400 2382 6467 10388 10820 7569 3570 1110 215 23	
175	464 2898 8295 14119 15649 11673 5856 1913 376 37	
176	442 2718 7644 12754 13822 10056 4911 1562 301 30	
177	440 2698 7563 12576 13587 9864 4816 1536 298 30	
178	423 2562 7083 11596 12313 8772 4200 1316 253 26	
179	452 2781 7813 13004 14042 10171 4944 1566 301 30	

Table B.2: Orbits of maximal prime cones for $\mathcal{F}\ell_5$, the F-vectors of the corresponding polytopes, and combinatorially equivalent string polytopes resp. FFLV polytope.

B.2 Algebraic invariants of the $\mathcal{F}\ell_5$ string polytopes

The table below contains information on the $\mathcal{F}\ell_5$ string polytopes and the FFLV polytope for ρ. It shows the reduced expressions underlying the string polytopes, whether the polytopes satisfy the weak Minkowski property, the weight vectors constructed in §4.3.2, and whether the corresponding initial ideal is prime. The last column contains information on unimodular equivalences among these polytopes. If there is no information in this column this means that there is no unimodular equivalence between this polytope and any other polytope in the table.

Class	$\underline{w}_0$	Normal	MP	Weight vector $-\mathbf{w}_{\underline{w}_0}$	Prime	Uni. Eq.
S1	1213214321	yes	yes	$(0,512,384,112,0,256,96,768,608,480,$ $0,64,320,832,15,14,526,398,126,12,$ $268,108,780,620,492,0,8,72,328,840)$	yes	S18, S26, S29
S2	1213243212	yes	yes	$(0,512,384,98,0,256,96,768,608,480,$ $0,64,320,832,30,28,540,412,123,24,$ $280,120,792,632,504,0,16,80,336,848)$	yes	-
S3	1213432312	yes	no	$(0,512,384,74,0,256,72,768,584,456,$ $0,64,320,832,58,56,568,440,111,48,$ $304,108,816,620,492,0,32,96,352,864)$	no	-

Class	$\underline{w}_0$	Normal	MP	Weight vector $-\mathbf{w}_{\underline{w}_0}$	Prime	Uni. Eq.
S4	1214321432	yes	no	$(0, 512, 384, 56, 0, 256, 48, 768, 560, 432, 0, 32, 288, 800, 120, 112, 624, 496, 63, 96, 352, 54, 864, 566, 438, 0, 64, 36, 292, 804)$	no	-
S5	1232124321	yes	yes	$(0, 512, 288, 224, 0, 256, 192, 768, 704, 432, 0, 128, 384, 896, 15, 14, 526, 302, 238, 12, 268, 204, 780, 716, 444, 0, 8, 136, 392, 904)$	yes	-
S6	1232143213	yes	yes	$(0, 512, 288, 224, 0, 256, 192, 768, 704, 420, 0, 128, 384, 896, 30, 28, 540, 316, 252, 24, 280, 216, 792, 728, 437, 0, 16, 144, 400, 912)$	yes	-
S7	1232432123	yes	yes	$(0, 512, 260, 196, 0, 256, 192, 768, 704, 390, 0, 128, 384, 896, 60, 56, 568, 310, 246, 48, 304, 240, 816, 752, 423, 0, 32, 160, 416, 928)$	yes	-
S8	1234321232	yes	no	$(0, 512, 264, 152, 0, 256, 144, 768, 656, 396, 0, 128, 384, 896, 120, 112, 624, 364, 219, 96, 352, 210, 864, 722, 462, 0, 64, 192, 448, 960)$	no	-
S9	1234321323	yes	no	$(0, 512, 264, 152, 0, 256, 144, 768, 656, 394, 0, 128, 384, 896, 120, 112, 624, 362, 222, 96, 352, 212, 864, 724, 459, 0, 64, 192, 448, 960)$	no	-
S10	1243212432	yes	no	$(0, 512, 272, 112, 0, 256, 96, 768, 608, 344, 0, 64, 320, 832, 240, 224, 736, 472, 119, 192, 448, 102, 960, 614, 350, 0, 128, 68, 324, 836)$	no	-
S11	1243214323	yes	no	$(0, 512, 272, 112, 0, 256, 96, 768, 608, 338, 0, 64, 320, 832, 240, 224, 736, 466, 126, 192, 448, 108, 960, 620, 347, 0, 128, 72, 328, 840)$	no	-
S12	1321324321	yes	no	$(0, 512, 192, 448, 0, 128, 384, 640, 896, 240, 0, 256, 160, 672, 15, 14, 526, 206, 462, 12, 140, 396, 652, 908, 252, 0, 8, 264, 168, 680)$	no	-
S13	1321343231	yes	no	$(0, 512, 192, 448, 0, 128, 384, 640, 896, 228, 0, 256, 160, 672, 29, 28, 540, 220, 476, 24, 152, 408, 664, 920, 246, 0, 16, 272, 176, 688)$	no	-
S14	1321432143	yes	no	$(0, 512, 192, 448, 0, 128, 384, 640, 896, 216, 0, 256, 144, 656, 60, 56, 568, 248, 504, 48, 176, 432, 688, 944, 219, 0, 32, 288, 146, 658)$	no	-
S15	1323432123	yes	no	$(0, 512, 132, 388, 0, 128, 384, 640, 896, 198, 0, 256, 192, 704, 60, 56, 568, 182, 438, 48, 176, 432, 688, 944, 231, 0, 32, 288, 224, 736)$	no	-
S16	1324321243	yes	no	$(0, 512, 136, 392, 0, 128, 384, 640, 896, 172, 0, 256, 160, 672, 120, 112, 624, 236, 492, 96, 224, 480, 736, 992, 175, 0, 64, 320, 162, 674)$	no	-
S17	1343231243	yes	no	$(0, 512, 48, 304, 0, 32, 288, 544, 800, 60, 0, 256, 40, 552, 240, 224, 736, 188, 444, 192, 168, 424, 680, 936, 63, 0, 128, 384, 42, 554)$	no	-
S18	2123214321	yes	yes	$(0, 256, 768, 112, 0, 512, 96, 384, 352, 864, 0, 64, 576, 448, 15, 14, 270, 782, 126, 12, 524, 108, 396, 364, 876, 0, 8, 72, 584, 456)$	yes	S1, S26, S29
S19	2123243212	yes	yes	$(0, 256, 768, 98, 0, 512, 96, 384, 352, 864, 0, 64, 576, 448, 30, 28, 284, 796, 123, 24, 536, 120, 408, 376, 888, 0, 16, 80, 592, 464)$	yes	-
S20	2123432132	yes	no	$(0, 256, 768, 76, 0, 512, 72, 384, 328, 840, 0, 64, 576, 448, 60, 56, 312, 824, 111, 48, 560, 106, 432, 362, 874, 0, 32, 96, 608, 480)$	no	-

Class	$\underline{w}_0$	Normal	MP	Weight vector $-\mathbf{w}_{\underline{w}_0}$	Prime	Uni. Eq.
S21	2132134321	yes	yes	$(0, 256, 768, 224, 0, 512, 192, 320, 448, 960, 0, 128, 640, 336, 15, 14, 270, 782, 238, 12, 524, 204, 332, 460, 972, 0, 8, 136, 648, 344)$	yes	-
S22	2132143214	yes	yes	$(0, 256, 768, 224, 0, 512, 192, 320, 448, 960, 0, 128, 640, 328, 30, 28, 284, 796, 252, 24, 536, 216, 344, 472, 984, 0, 16, 144, 656, 329)$	yes	-
S23	2132343212	yes	yes	$(0, 256, 768, 194, 0, 512, 192, 320, 448, 960, 0, 128, 640, 352, 30, 28, 284, 796, 219, 24, 536, 216, 344, 472, 984, 0, 16, 144, 656, 368)$	yes	-
S24	2132432124	yes	yes	$(0, 256, 768, 196, 0, 512, 192, 320, 448, 960, 0, 128, 640, 336, 60, 56, 312, 824, 246, 48, 560, 240, 368, 496, 1008, 0, 32, 160, 672, 337)$	yes	-
S25	2134321324	yes	no	$(0, 256, 768, 152, 0, 512, 144, 272, 400, 912, 0, 128, 640, 276, 120, 112, 368, 880, 222, 96, 608, 212, 340, 468, 980, 0, 64, 192, 704, 277)$	no	-
S26	2321234321	yes	yes	$(0, 64, 576, 448, 0, 512, 384, 96, 352, 864, 0, 256, 768, 112, 15, 14, 78, 590, 462, 12, 524, 396, 108, 364, 876, 0, 8, 264, 776, 120)$	yes	S1, S18, S29
S27	2321243214	yes	yes	$(0, 64, 576, 448, 0, 512, 384, 96, 352, 864, 0, 256, 768, 104, 30, 28, 92, 604, 476, 24, 536, 408, 120, 376, 888, 0, 16, 272, 784, 105)$	yes	-
S28	2321432134	yes	yes	$(0, 64, 576, 448, 0, 512, 384, 72, 328, 840, 0, 256, 768, 74, 60, 56, 120, 632, 504, 48, 560, 432, 106, 362, 874, 0, 32, 288, 800, 75)$	yes	-
S29	2324321234	yes	yes	$(0, 8, 520, 392, 0, 512, 384, 12, 268, 780, 0, 256, 768, 14, 120, 112, 108, 620, 492, 96, 608, 480, 78, 334, 846, 0, 64, 320, 832, 15)$	yes	S1, S18, S26
S30	2343212324	yes	yes	$(0, 16, 528, 304, 0, 512, 288, 24, 280, 792, 0, 256, 768, 28, 240, 224, 216, 728, 438, 192, 704, 420, 156, 412, 924, 0, 128, 384, 896, 29)$	yes	-
S31	2343213234	yes	yes	$(0, 16, 528, 304, 0, 512, 288, 20, 276, 788, 0, 256, 768, 22, 240, 224, 212, 724, 444, 192, 704, 424, 150, 406, 918, 0, 128, 384, 896, 23)$	yes	-
FFLV	-	yes	yes	$w^{\mathrm{reg}} = (0, 4, 6, 6, 0, 3, 4, 6, 6, 9, 0, 2, 4, 6, 4, 3, 4, 7, 8, 2, 3, 5, 4, 6, 8, 0, 1, 2, 3, 4)$ $w^{\min} = (0, 3, 4, 3, 0, 2, 2, 4, 3, 5, 0, 1, 2, 3, 1, 1, 1, 3, 3, 1, 1, 2, 1, 2, 3, 0, 1, 1, 1, 1)$	yes yes	-

Table B.3: String polytopes depending on $\underline{w}_0$ and the FFLV polytope for $\mathcal{F}\ell_5$ and ρ, their normality, the weak Minkowski property, the weight vectors constructed in §4.3.2, primeness of the binomial initial ideals, and unimodular equivalences among the polytopes.

Chapter 5

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