Essays in Public Economics

Inauguraldissertation
zur
Erlangung des Doktorgrades
der
Wirtschafts- und Sozialwissenschaftlichen Fakultät
der
Universität zu Köln
2018

vorgelegt
von

Diplom-Mathematikerin Laura Kohlleppel

aus

Bochum - Wattenscheid

Referent:	Prof. Dr. Felix Bierbrauer
Korreferent:	Prof. Dr. Martin Barbie
Tag der Promotion:	07.09.2018

Acknowledgments

First of all, I am very grateful to Felix Bierbrauer, my first supervisor, for advising me during my dissertation and teaching me how to ask the relevant questions. His enthusiasm towards economic research has been a constant source of learning and inspiration. The first chapter is joint work with him.

Particular thanks must also be given to Martin Barbie, my second supervisor, for diligently reading my papers and giving valuable feedback and further to John Weymark for having me twice at Vanderbilt University providing an environment that allowed me to make great progress on this thesis.

Representative for all the fellow students who supported me while working on this thesis, I would like to express my deepest gratitude towards my co-authors Désirée Rückert and Frederik Thenée. It was by far the most joyful experience during my dissertation to work together on our common projects. I am very grateful to them for not getting tired of discussing, explaining, commenting and motivating. I have learned so much and feel honored to call them not only co-authors but also friends. The first and the third chapter, respectively, are joint work.

Finally and, most importantly, I owe big thanks to my family. To my parents who believe in me even when I do not – for their unconditional support. To my sisters who are the most loyal companions in life – for always fighting on my side whichever it is. To Clemens who has been by my side, day by day, and went all the way with me – for holding me and never loosening the grip.

Laura Kohlleppel

Contents

List of Figures

List of Tables

Introduction

The given thesis includes three independent chapters which all contribute to the theory of public economics. They are linked, theoretically, by the fact that an information asymmetry is included in the presented environment. In the first two chapters agents are assumed to hold private information on their preferences. Making use of a mechanism design approach, applications such as a bilateral trade problem, the provision of a public good and the design of an optimal income tax schedule are studied. More precisely, Chapter 1 relates discrete and continuous formulations of the independent private values model and derives conditions under which these two behave qualitatively the same. Chapter 2 studies how a Mirrleesian income tax model is affected by the assumption of arbitrary directions of redistribution and addresses the question how the corresponding optimal income tax schedules should be designed. In the third chapter individuals are incompletely informed about the characteristics of a public good. Only incomplete information on the good's productivity and governmental contributions are available such that salience becomes a driving factor for individual behavior and hence, the provision of the good.

A common aim of all three chapters is to disentangle and evaluate the effects of information asymmetry in the respective setting. In an environment as considered in the first two chapters, individuals being privately informed about preferences yields a social cost. Thus, only a second-best situation can be reached where welfare is lower compared to a full information environment. However, this is not true in a situation as studied in Chapter 3. Here, the information asymmetry can be exploited by a benevolent government to improve welfare compared to a situation with complete information. Conclusively, the effects of information asymmetry on welfare and optimal allocations are non-trivial and depend on the considered environment.

Chapter 1 The first chapter is based on joint work with Felix Bierbrauer and Désirée Rückert. It studies two representative applications – the bilateral trade problem and the provision of a public good – in different formulations of the independent private values model. In this workhorse model of mechanism design, agents are privately informed about their preferences. In our setup, consumers are privately informed about the marginal benefit of consumption, while producers are privately informed about the marginal cost of production. Typically, in the independent private values model, those "types" are drawn from a continuum with a corresponding atomless distribution. We contrast this continuous formulation with a discrete specification and compare the implementability of efficient outcomes.

At first, necessary and sufficient conditions for the implementability of a social choice function are derived for the discrete specification of the model. Secondly, we study how the implementability of a social choice function varies in the parameters of the model, e.g. the number of types or the number of individuals. For this purpose, we define a measure which quantifies how costly it is to implement efficient outcomes.

Regarding the two applications, contrary observations are made with respect to the question whether the implementability of efficient outcome is the same in both the continuous and the discrete formulation. While the impossibility result of Mailath and Postlewaite (1990) on the provision of public goods preserves in any model with discrete types, this is not true for a private good setting. The impossibility result by Myerson and Satterthwaite (1983) for the bilateral trade problem with a continuum of types does not extend to a setting with discrete types. We deliver parameter constellations that allow to efficiently trade a private good. The final section in this chapter provides conditions under which discrete and continuous formulations of the independent private values model behave approximately the same.

Chapter 2 In the second chapter, a discrete version of the Mirrleesian income tax model is used to study welfare maximizing tax schedules under the assumption of arbitrary redistributive preferences of the planner. An economy with three productivity types is studied and the welfare function is a weighted Utilitarian. Redistributive preferences are reflected by the welfare weights assigned to the three productivity types, respectively. The model does not put any assumptions on these welfare weights and, hence, covers every possible redistributive preference that can be expressed by a Utilitarian welfare function. Typically, models building on Mirrlees (1971) consider welfare weights that decrease in productivity. This yields optimal redistribution towards the poor, formally given by downward binding incentive constraints.

For weights that are non-monotonic, the set of binding incentive constraints is a priori unknown and has to be identified. A perturbation argument is used to assign binding incentive constraints to given welfare weights. The results are aggregated in two formal conditions which partition the set of welfare weights into four subsets (and its boundaries) according to the pattern of binding incentive constraints.

Having identified the binding constraints for each subset, allows to state a reduced form of the original welfare maximization problem. Optimal allocations are derived by a first order approach. The corresponding optimal marginal tax rates are such that it is never optimal to overly encourage low-skilled individuals to work or to discourage high-skilled individuals from work which is equivalent to non-negative marginal tax rates for low-skilled and non-positive tax rates for high-skilled. The median type's optimal tax rate can have either sign depending on the direction of redistribution. Furthermore, depending on the specific form of preferences bunching can occur. Though, given that preferences fulfill the single-crossing condition, some types of bunching can be ruled out for further specified welfare weights.

The chapter concludes by discussing the relation of welfare maximizing and second-best Pareto efficient allocations.

Chapter 3 The third chapter is based on joint work with Frederik Thenée. It investigates how the provision of a public good is affected by salience. Two types of salience are studied: firstly, individuals are assumed to be incompletely informed about the quality of the public good and, hence, responsive to salience. More precisely, individuals are assumed to underestimate the marginal benefit of a contribution to the public good. Secondly, the salience of taxes has an impact on the individuals' behavior as they are assumed to misperceive governmental contributions to the public good. The analysis focuses on incompletely salient taxes where individuals underestimate the amount of public contributions. In a model similar to Bergstrom et al. (1986) or Andreoni (1990) the interplay of public and private contributions to charity is analyzed, in particular with regard to salience as a driving force of behavior.

Chapter 3 can be subdivided in two main parts. The first is dedicated to the investigation of individual equilibrium behavior as a function of various factors such as public contributions, salience – in both of it dimensions – and the quality of the public good. The second part evaluates these results and delivers a normative approach to public good scenarios where not only the provision level of the public good but also the induced welfare is considered.

As real world observations would suggest, private equilibrium contributions increase in the perceived quality of the public good and decrease with the level of perceived public contributions. Furthermore, as public contributions are only an imperfect substitute for private contributions, the aggregate provision level increases in public contributions. This increase in the provision level alleviates the underprovision of the public good resulting from free-riding.

However, an increase in the aggregate provision is not equivalent to an increase in welfare. Opposed to voluntary contributions, taxes do not create a warm glow. So, crowding out private ones, public contributions come at the cost of a destroyed warm glow. To evaluate the overall effect on welfare, the negative effect of a destroyed glow has to be contrasted with the positive effects of reduced free-riding and increased aggregate contributions. It can be shown that the positive effect never outweighs the negative when taxes are perfectly salient. Only by their non-salience public contributions to a public good are potentially beneficial. To identify conditions when this is actually the case, crowding-in as defined by the concomitant increase of public and private contributions is introduced as an indicator. Under the assumption of a welfare maximizing government, it can be shown that it depends on the trigger of crowding-in whether the increase in taxes raises welfare or not: if an increase in the good's salience is the reason for higher equilibrium contributions, increased taxes are never optimal. Contrary, if crowding-in is caused by an increase in the good's quality, higher taxes are potentially welfare increasing. Raising taxes is actually beneficial if the problem of free-riding became more severe by the quality shock.

For exposition, we study the public good "charity".

1

On the independent private values model – A unified approach

1.1 Introduction

The independent private values model is an important workhorse model for the theory of mechanism design. In this model, economic agents are privately informed about their characteristics, typically preferences or costs, and, moreover, the characteristics of different agents are modeled as the realizations of independent random variables. In addition, an individual's payoff does not depend on the types of other individuals. This framework has been applied to study a wide range of allocation problems. These include the allocation of indivisible private goods (auctions), the provision of pure or excludable public goods, the regulation of externalities, the problem of partnership dissolution, or redistributive income taxation.

The seminal papers in this literature are based on the assumption that, for each agent, there is a continuum of possible types and that the corresponding probability distribution has no mass points and a monotone hazard rate. Moreover, the typical approach is to use the envelope theorem for a characterization of incentive compatible social choice functions. In this paper, we develop an alternative characterization of implementable social choice functions based on the assumption that the set of possible types is discrete. More specifically, our analysis proceeds as follows.

We first provide necessary and sufficient conditions for the implementability of a social choice function. For our characterization, we introduce the notion of a *minimal subsidy*. It is defined as the difference between the maximal payment that one can extract from individuals in the presence of incentive and participation constraints, and the payment that would be required in order to ensure budget balance. That is to say, the minimal subsidy is the amount of money an external party would have to provide so as to make a given social choice function compatible with the requirements of incentive compatibility, voluntary participation, and budget balance. However, we do not assume that such an external party is actually available. Consequently, a social choice function can be implemented if and only if the minimal subsidy is negative. We then apply our characterization to clarify conditions under which the famous impossibility results by

Myerson and Satterthwaite (1983) and Mailath and Postlewaite (1990) extend to a model with a discrete set of types. Second, we provide a comparative statics analysis of how a change in exogenous parameters – such as the number of individuals, or the number of possible types per individual – affect the minimal subsidy. This allows us, for instance, to check whether a change in the economic environment makes it more or less difficult to implement an efficient provision rule for public goods. A final contribution of our paper is to spell out the conditions under which a model with a large but discrete number of types behaves approximately in the same way as a model with a continuum of types.

These results are derived in a model in which many consumers, who have private information about their preferences, benefit from the provision of a private or public good. Their payoffs are quasi-linear in the transfers they need to pay for the good. Further, many firms, which have private information about their costs, profit from the production of goods. Firm profits are quasi-linear in the revenues they receive for producing the good. Consumers' consumption is bounded by the total output that is made available by the firms. To derive the minimal subsidy, we proceed as follows: The social choice function can be divided into a transfer and consumption rule for consumers and a revenue and production rule for firms. First, we hold the consumption rule for consumers fixed and derive the maximal transfers that consumers are able to make if incentive compatibility constraints and participation constraints need to be respected simultaneously. Similarly, we hold the production rule fixed for firms and derive the minimal revenue that firms are willing to accept if again incentive compatibility constraints and participation constraints need to be respected. The differences between the maximal consumer transfers and the minimal firm revenues is the minimal subsidy. If the minimal subsidy is positive, i.e., the mechanism runs a deficit, then the specified consumption and production rules are not implementable. Contrary, if the minimal subsidy is negative, the implementation of the social choice function is possible. For the characterization of maximal consumer transfers and minimal firm revenues we use techniques developed in the non-linear pricing literature (e.g. Mussa and Rosen (1978)).

Our analysis proceeds as follows: We first derive necessary and sufficient conditions for the implementation of a social choice function. For the characterization of the first condition, we consider the problem of maximizing consumers' transfers and the problem of minimizing firms' revenues, taking only a subset of incentive compatibility and participation constraints into consideration. Specifically, we consider the participation constraints of the consumer with the lowest valuation for consumption and the incentive constraints that prevent consumers to communicate lower preferences. Similarly, we take into consideration the participation constraint of the firm with the highest costs of production and the incentive constraints that prevent firms from exaggerating their costs. The expression that arises from this *relaxed* problem, where only a subset of constraints is considered, provides a lower bound on the minimal subsidy. Thus, a necessary condition for the implementation is that the minimal subsidy of this relaxed problem is negative. Second, we derive a sufficient condition, which assures that the lower bound of the minimal subsidy can be reached. This condition is stated for monotone consumption and allocation rules so that the consumption rule and the production rule are monotone, so that consumers with higher willingness to pay for the good consume more than consumers with a lower willingness to pay; and similarly, firms with lower costs produce more output than firms with higher costs.

These conditions have the following implications: first-best consumption and provision rules are monotone. Therefore, first-best implementation is possible if and only if the minimal subsidy is negative. When the first-best provision rule is not implementable, monotonicity of the consumption and provision rules can be achieved when the distribution of agents' types satisfies a monotone hazard rate assumption. Hence, consumption and production plans that maximize a social surplus function subject to the constraint that the minimal subsidy is negative, are monotone and therefore implementable. To derive the necessary condition, the monotonicity of hazard rates does not play a role. We present a version of the impossibility results of Mailath and Postlewaite (1990), when consumers have a discrete number of types. Our specification uses only the necessary condition to derive this result. We do not require the assumption of a monotone hazard rate that was imposed by Mailath and Postlewaite (1990) in order to attain the impossibility result. Hence, our result holds under less restrictive assumptions.

We provide comparative static results that show how the minimal subsidy varies with the number of types and the number of agents. In particular, we can compare the comparative static properties of the minimal subsidy in a private good setting with a public good setting. A change in the number of agents affects the minimal subsidy in both settings differently. In a public good setting, an increase in the number of consumers leads to a positive minimal subsidy, so that it is impossible to efficiently provide the public good. Contrary, when a private good setting is considered, an increase in the number of buyers and sellers leads to a negative minimal subsidy. An increase in the number of types, on the other hand, increases the minimal subsidy, so that in the private good setting, as well as the public good setting, impossibility results occur when the number of types grows large.

In order to understand how parameter changes affect the minimal subsidy, we decompose the effect of a change in parameters in the surplus measure and the measure for information rents separately. We show that when each agent has a binary type set, then parameters can be found such that efficient bilateral trade is possible. Further, if only two consumers are considered, parameters can be found such that the public good can be provided efficiently. We show that the ability to reach possibility results hinges on the observation that parameters need to be chosen in such a way that the surplus measure is bigger than the information rents that need to be guaranteed. This raises the question how (i) the agent's type parameters, (ii) the probability weights on types (iii) the number of types and (iv) the number of agents influence the minimal subsidy. In particular, we show that the possibility results that are derived with a binary type set 'approach' the impossibility results of Myerson and Satterthwaite (1983) and Mailath and Postlewaite (1990) if the number of types increases and if the new types are introduced in such a way that the finite type set lies *dense* in the infinite type set, i.e., every point of the infinite type set can be approximated by a point of the finite subset of types. We demonstrate that the minimal subsidy in the discrete setting converges to the minimal subsidy in the continuous setting if the environments are aligned.

Based on these observations, we study general convergence results. We specify what we mean by one environment approaching another environment, so that results in a continuous setting and a discrete setting coincide. Therefore, we define an environment that allows us to compare and relate different economies, e.g. the discrete and the continuous bilateral trade economy. Each of the economies is characterized by four decisive factors for implementability: the number of agents, the number of types, the probability distribution and the parameter constellation. Formally, we can approximate 'similar' economies by adjusting the single components; i.e., as we increase the number of types and adjust the probability distribution, we transfer the discrete bilateral trade setting into the setting of Myerson and Satterthwaite (1983). If the components are adjusted 'appropriately', we say that one economy will converge to the other economy. To relate and analyze implementability results of different settings, we calculate the minimal subsidy and study what drives the possibility to attain efficient implementation in each economy. We give general insights on how different applications of the independent private values can be linked.

The reminder is organized as follows. The next section contains a more detailed discussion of the literature. Section 1.3 provides counterparts to Myerson and Satterthwaite (1983) and Mailath and Postlewaite (1990) impossibility results. Section 1.4 introduces the model. To motivate our more general analysis in subsequent sections, Section 1.5 presents necessary and sufficient conditions for the implementation of social choice functions and characterizes first-best and second-best provision rules. Section 1.6 discusses comparative statics properties of the bilateral trade problem and the public good provision problem. Further, this section studies the impact of increasing the number of agents on the possibility result. Section 1.7 then shows how the discrete type setting converges to the continuous type setting and analyzes the convergence in our examples. The last section contains concluding remarks. Preliminary proofs are in Part 1.A of the Appendix. Part 1.C introduces further applications.

1.2 Related literature

The independent private values model has been applied to study a wide variety of allocation problems, from the allocation of indivisible private goods (auctions), to the bilateral trade problem, the provision of pure or excludable public goods, the regulation of externalities, the regulation of a monopolist, or the problem of partnership dissolution.

In auction theory, the independent private values model is central. The seminal paper that introduced the second-price auction and the revenue equivalence theorem is Vickrey (1961). Optimal auctions for risk neutral bidders with independent types are derived in Myerson (1981), Riley and W. (1981), Harris and Raviv (1981) (see McAfee and McMillan (1987), for further references). Che and Gale (2006) show that the revenue equivalence theorem does not need to apply when the number of buyer types is finite.

Further, for the example of bilateral trade, Myerson and Satterthwaite (1983) have shown that if the buyer's preferences and the seller's costs are private information and voluntary participation needs to be assured, efficient trade is not possible. They introduce the notion of the minimal subsidy and thereby provide a measure of how severe the impossibility result is. We will show that this impossibility result does translate into a model with many finite types. For few finite types, however, parameters can be found such that efficient bilateral trade is possible; i.e., the minimal subsidy is negative. A special case of our setup is the paper of Matsuo (1989), who provides conditions under which efficiency in the bilateral trade example can be reached for discrete distributions with two types.

The possibility to achieve efficient public good provision as a Bayes-Nash equilibrium in an independent private values model has first been established by D'Aspremont and Gerard-Varet (1979) and Arrow (1979). This literature has not taken voluntary participation into account. Güth and Hellwig (1986), Rob (1989) and Mailath and Postlewaite (1990) have shown that if preferences for public goods are private information, so that incentive compatibility constraints need to be considered and if at the same time voluntary participation need to be guaranteed, then first-best efficient public good provision cannot be achieved. We will show that this impossibility result does not rely on the assumption that preferences for public goods are continuously distributed by showing that the impossibility extends to a setup with an arbitrary discrete type set. The provision of a non-rival, but excludable good is studied in Güth and Hellwig (1986), Hellwig (2003), Schmitz (1997) and Norman (2004).

The independent private values model has also been applied to study the dissolution of partnerships, see Cramton et al. (1987). They look at situations where each of several agents possesses a fraction of a good and assume a continuous symmetric distribution of agents' valuations. They show that an efficient reassignment of shares is possible if initial shares are sufficiently equal distributed. However, when a single agent possesses all shares of the partnership, then the same arguments as in Myerson and Satterthwaite (1983) apply and an efficient dissolution is impossible. Hence, whether the partnership can be dissolved efficiently relies on the initial shares of the partnership. As a corollary of our analysis of Myerson and Satterthwaite (1983) with discrete types, we show that if the number of types is finite, then parameters can be found such that the partnership can always be dissolved efficiently, even when shares are unevenly distributed.

Hellwig (2007) provides separate characterizations of optimal income taxes for a model with a discrete set of types and for a model with a continuum of types. He argues that for all steps in the proof for the continuous type set there exists an analogous step for the discrete type set. The strategy of our paper is different in that we analyze the implementation of social choice functions for an arbitrary number of discrete types. We investigate the implication of this modeling choice by approximating the continuous type set.

Kos and Messner (2013) provide a general characterization of implementable allocation rules. They describe bounds on the set of transfers that implement an allocation rule. They do refrain from any assumption on the agent's type set and utility function. The work of Kos and Messner (2013) is related to this paper in that it makes use of minimal subsidies to evaluate whether implementation of social choice functions is possible. Opposed to them, we make more specific assumptions that allow us to elaborate more clearly on necessary and sufficient conditions for implementation. Further, it enables us to do comparative statics analysis.

The specification of the independent private values model with a finite number of types is well suited for directly testing mechanisms in the laboratory. Bierbrauer et al. (2015) use this specification to test whether mechanism that are robust to agents' probabilistic beliefs (see Bergemann and Morris (2005)) fail when agents have social preferences.

1.3 Motivating examples

This section contains motivating examples, which illustrate the difficulty of extending the impossibility results by Myerson and Satterthwaite (1983) and Mailath and Postlewaite (1990) to models with a discrete set of types. Throughout, we will use these two examples to illustrate conceptual issues that arise.

1.3.1 The bilateral trade problem

In the private good setting, there is one buyer and one seller. The seller produces $y \in [0,1]$ units of a good. The buyer can purchase $q \in [0,1]$ units of the good. The buyer's utility is given by $u(\theta, q, t) = \theta q - t$, so that θ is the buyer's valuation for the good and t is the transfer the buyer has to pay for the good. The seller's profit is given by $\pi(\delta, y, r) = r - \delta y$, so that δ is the cost of producing the good and r is the revenue the seller receives for providing the good. The quantity that is consumed by the buyer is equal to the quantity produced by the seller, so that for all $(\theta, \delta) \in \Theta \times \Delta$, $q(\theta, \delta) = y(\theta, \delta) \in [0,1]$. Further, it is assumed that trade is voluntary and, in the absence of trade, both parties realize a utility, respectively a profit of 0. We define for the buyer a function $Q : \Theta \mapsto [0,1]$, where $Q(\theta^k) = \mathbb{E}_{(\delta)}[q(\theta^k, \delta)|\theta^l]$. This gives the conditional expectation over the probability that the buyer gets the good, in case that he announces type θ^k but having a true type θ^l. The conditional expected value of the transfers $T(\theta^k)$, and the conditional expected values of revenues $R(\delta^l)$ and produced quantity $Y(\delta^l)$ for the seller are defined analogously. The seminal analysis of the bilateral trade problem by Myerson and Satterthwaite (1983) has focused on the question whether there exists a Pareto efficient or surplus-maximizing social choice function that is incentive-compatible for the buyer,

$$\theta^l Q(\theta^l) - T(\theta^l) \geq \theta^l Q(\theta^k) - T(\theta^k) \;, \qquad \forall\, \theta^l, \theta^k \in \Theta \;, \tag{1.1}$$

incentive-compatible for the seller,

$$R(\delta^l) - \delta^l Y(\delta^l) \geq R(\delta^k) - \delta^l Y(\delta^k) \;, \qquad \forall\, \delta^l, \delta^k \in \Delta \;, \tag{1.2}$$

and compatible with the budget requirement,

$$\mathbb{E}_{(\theta,\delta)}\left[t(\theta,\delta)\right] \geq \mathbb{E}_{(\theta,\delta)}\left[r(\theta,\delta)\right]. \tag{1.3}$$

Surplus-maximization requires that the function $q : \Theta \times \Delta \to [0,1]$ is chosen so as to maximize

$$\mathbb{E}_{(\theta,\delta)}\left[(\theta - \delta) q(\theta,\delta))\right] \;.$$

Hence, surplus-maximization requires that

$$q(\theta,\delta) = \begin{cases} 0, & \text{if} \quad \theta < \delta \;, \\ 1, & \text{if} \quad \theta > \delta \;. \end{cases}$$

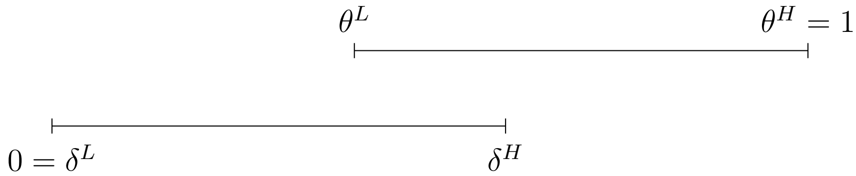
Myerson and Satterthwaite (1983) analyzed the bilateral trade problem under the assumption of an atomless distribution functions with a monotone hazard rate[1], and established the following impossibility result.

Proposition 1.1. ***Myerson and Satterthwaite (1983):*** *If the buyer's valuation for the good is independently drawn from the intervals $[\theta^L, \theta^H]$ and the seller's costs for the good are drawn from the interval $[\delta^L, \delta^H]$ with strictly positive densities, such that the intervals $[\theta^L, \theta^H]$ and $[\delta^L, \delta^H]$ are not disjunct, then there is no Bayesian incentive compatible social choice function that is ex post efficient and gives every buyer type and every seller type non-negative expected gains from trade.*

We change the assumption of atomless type distributions and show: when the buyer's and the seller's type set is discrete, then efficient trade is possible for some parameter constellations.

[1] The hazard rate for the buyer is defined as $\frac{f(\theta)}{1-F(\theta)}$, where the cumulative distribution function of the random variable is denoted by F and f is the density; for the seller, the hazard rate is defined as $\frac{p(\delta)}{P(\delta)}$, respectively. The *monotone hazard rate assumption* assumes these to be non-decreasing.

Figure 1.1: Binary type set



Assume that each agent's type occurs with equal probability. For certain parameter constellations, e.g.

$$\underline{\theta}_s = 0 \ , \qquad \underline{\theta}_b = \frac{1}{8} \ , \qquad \bar{\theta}_s = \frac{7}{8} \qquad \text{and} \qquad \bar{\theta}_b = 1 \ ,$$

the Myerson and Satterthwaite (1983) impossibility result is obtained. Consider the following *relaxed problem*: The mechanism designer is interested in maximizing expected surplus, subject to the incentive compatibility constraints in (1.1) and (1.2) and subject to the constraints that gains from trade have to be non-negative. The following Table gives a solution to this relaxed problem.

Table 1.1: Positive minimal subsidy

(q, r, t)	$\underline{\theta}_s$	$\bar{\theta}_s$
$\underline{\theta}_b$	$\left(1, \frac{3}{8}, \frac{1}{8}\right)$	$(0, 0, 0)$
$\bar{\theta}_b$	$\left(1, \frac{4}{8}, \frac{4}{8}\right)$	$\left(1, \frac{7}{8}, \frac{5}{8}\right)$

The maximal expected transfer that the buyer is willing to make is $\mathbb{E}_{(\theta,\delta)}\left[t(\theta,\delta)\right] = \frac{5}{16}$. The minimal expected revenue the seller is willing to accept is $\mathbb{E}_{(\theta,\delta)}\left[r(\theta,\delta)\right] = \frac{7}{16}$. Hence, the solution to the relaxed problem violates the budget constraint in (1.3). The minimal subsidy that is necessary for efficient bilateral trade is $\mathbb{E}_{(\theta,\delta)}\left[r(\theta,\delta)\right] - \mathbb{E}_{(\theta,\delta)}\left[t(\theta,\delta)\right] = \frac{1}{8}$.

Contrary, for the following parameter constellation

$$\underline{\theta}_s = 0 \ , \qquad \underline{\theta}_b = \frac{1}{3} \ , \qquad \bar{\theta}_s = \frac{2}{3} \qquad \text{and} \qquad \bar{\theta}_b = 1 \ ,$$

the social choice function, which specifies (q, t, r) for all possible type combinations, in Table 1.2 leads to efficient bilateral trade and satisfies the conditions in (1.1), (1.2), (1.3) and assures non-negative payoffs.[2] Whenever the buyer's marginal valuation for the good is higher than the seller's marginal costs, the good is exchanged. The maximal

[2]See Observation 1.1 below, for a proof.

expected transfer that the buyer is willing to make is $\mathbb{E}_{(\theta,\delta)}\left[t(\theta,\delta)\right] = \frac{1}{2}$. The minimal expected revenue the seller is willing to accept is $\mathbb{E}_{(\theta,\delta)}\left[r(\theta,\delta)\right] = \frac{1}{3}$. Hence, the budget constraint in (1.3) is satisfied. The minimal subsidy that is necessary for efficient bilateral trade is $\mathbb{E}_{(\theta,\delta)}\left[r(\theta,\delta)\right] - \mathbb{E}_{(\theta,\delta)}\left[t(\theta,\delta)\right] = -\frac{1}{6}$.

Table 1.2: Negative minimal subsidy

(q,r,t)	$\underline{\theta}_s$	$\bar{\theta}_s$
$\underline{\theta}_b$	$\left(1,\frac{1}{3},\frac{2}{3}\right)$	$(0,0,0)$
$\bar{\theta}_b$	$\left(1,\frac{1}{3},\frac{2}{3}\right)$	$\left(1,\frac{2}{3},\frac{2}{3}\right)$

1.3.2 Public good provision

An indivisible public good is either provided or not. There are $I = \{1,\dots,n\}$ consumers and one producer. The utility function is taken to be linear so that $u(\theta_i, q, t) = \theta_i\, q - t$, where $q = 1$ if the public good is provided, and $q = 0$, otherwise. The producer's cost function is taken to be publicly known. If the public good is produced, the costs are equal to nc, where c is the per capita cost of public-goods provision. Since the cost function is known, the producer's incentive compatibility constraints are irrelevant, and a state of the economy is exclusively defined by the vector of preference parameters θ. When the public good is not provided, all consumers realize a utility of zero.

The analysis of public good provision by Mailath and Postlewaite (1990) has focused on the question whether there exists a Pareto efficient social choice function that is incentive compatible, so that the incentive constraints in (1.1) are satisfied for all i, and that satisfies the resource requirement

$$\mathbb{E}_{(\theta)}\left[\sum_{i=1}^{n} t_i(\theta)\right] \geq nc\, \mathbb{E}_{(\theta)}[q(\theta)] \;, \tag{1.4}$$

where n denotes the number of consumers.

Surplus maximization requires that the function $q : \Theta^n \mapsto [0,1]$ is chosen so as to maximize

$$\mathbb{E}_{(\theta)}\left[\left(\sum_{i=1}^{n} \theta_i - nc\right) q(\theta)\right] \;.$$

Hence, surplus maximization requires that

$$q(\theta) = \begin{cases} 0, & \text{if } \frac{1}{n}\sum_{j=1}^{n} \theta_i < c \;, \\ 1, & \text{if } \frac{1}{n}\sum_{j=1}^{n} \theta_i > c \;. \end{cases}$$

If the average valuation of a consumer exceeds the per capita costs, $\mathbb{E}_{(\theta)}[\theta_i] > c$, then the public good should be provided. Mailath and Postlewaite (1990) analyze the public good provision under the assumption of atomless distribution functions with a monotone hazard rate. They show that if the number of consumers grows without limit, public good provision is zero under any social choice function that is incentive compatible and respects participation constraints.

Proposition 1.2. ***Mailath and Postlewaite (1990):*** *If the consumers' valuation for the public good are independently drawn from the intervals $[\theta^L, \theta^H]$ with strictly positive densities and the per capita costs are such that $\theta^L < c < \theta^H$, then $lim_{n\to\infty} prob\,(q(\theta)^n > 0) = 0$, for any mechanism in the sequence of mechanism satisfying incentive compatibility constraints, voluntary participation and expected budget balance.*

With many consumers, if the average valuation is higher than the marginal per capita costs, then the efficient amount of public good provision will be almost surely equal to 1, yet the amount that is going to be implemented will be almost surely equal to 0, under any mechanism that respects consumers' voluntary participation.

Consider now the case where consumers have a binary type set and assume that each consumer's type occurs with equal probability. For all parameter constellations $\theta_i^L < c < \theta_i^H$, there is no social choice function, which maximizes surplus and satisfies incentive compatibility constraints, ensures non-negative utility for all consumers and fulfills the budget requirement (see Proposition 1.5 below). With private information on public good preferences, consumer i's transfers have to be chosen such that the incentive compatibility constraints in (1.1) are satisfied. However, when the number of consumers grows large, a consumer's impact on the public good provision becomes insignificant. If no consumer has an impact on the provision, then incentive compatibility constraints imply that the transfers have to be similar. Thus, the maximal transfers per capita is θ_i^L, which is smaller than the per capita costs of public good provision.

1.3.3 Comparison of private and public good.

Assume that there are two consumers and that each consumer type occurs with equal probability. For the following parameters, $\underline{\theta}_b = 1,\ c = 3$ and $\bar{\theta}_b = 10$, there is a social choice function, which maximizes surplus, satisfies incentive compatibility constraints in (1.1), assure voluntary participation and fulfills the budget requirement in (1.4). By contrast, for $\underline{\theta}_b = 1,\ c = 3$ and $\bar{\theta}_b = 6$, there is no social choice function, which maximizes surplus and fulfills all constraints.[3] It depends on the parameters whether we have an impossibility result or not. With many individuals, the Mailath and Postlewaite (1990) result extends to a model with a discrete type set. As has been shown by Gresik and Satterhwaite (1989), the Myerson and Satterthwaite (1983) result does not extend

[3]See Observation 1.4 below, for a proof.

to a model with a large number of buyers and sellers. This raises the following more general questions: what impact does the number of agents have on the impossibility results, and what impact does the assumption on the type set have? To address these questions we will develop a general framework in the subsequent section.

1.4 The model

Consumers. There is a finite set of consumers, $I = \{1, ..., n\}$. The preferences of consumer i are represented by the utility function

$$u_i(\theta_i, q_i, t_i) = v(\theta_i, q_i) - t_i \ ,$$

where q_i denotes i's consumption of a public or private good and the function v gives the utility of consumption. It depends on a preference parameter θ_i that belongs to a finite ordered set of possible preference parameters $\Theta_i = \{\theta_i^0, \theta_i^1, ..., \theta_i^s\}$, with $\theta_i^0 < \theta_i^1$, etc. for every $i \in I$. The monetary payment of consumer i is denoted by t_i.

The function v is assumed to have the following properties. Zero consumption gives zero utility: for all $\theta_i \in \Theta$, $v(\theta_i, 0) = 0$. The lowest type does not benefit from consumption: for all q_i, $v(\theta_i^0, q_i) = 0, \forall\ i \in I$. For all other types, the marginal benefit from increased consumption is positive and decreasing, so that for all $\theta_i > \theta_i^0$ and all q_i, $v_2(\theta_i, q_i) > 0$[4] and $v_{22}(\theta_i, q_i) \leq 0$. The marginal benefit of consumption is increasing in the individual's type, so that $\theta_i' \geq \theta_i$ implies that $v_2(\theta_i', q_i) \geq v_2(\theta_i, q_i)$.

The consumer privately observes θ_i. From the perspective of all other agents it is a random variable with support Θ_i and probability distribution $f_i = (f_i^0, ..., f_i^s)$ where $f_i(\theta_i)$ takes the value f_i^l if $\theta_i = \theta_i^l$. The cumulative distribution is given by $F_i(\theta_i)$, such that $F_i(\theta_i)$ takes the value $\sum_{k=0}^{l} f_i^k$, if $\theta_i = \theta_i^l$. The random variables $(\theta_i)_{i \in I}$ are independently and identically distributed *(i.i.d.)*. We write $\theta = (\theta_1, ..., \theta_n)$ for a vector of all consumers' taste parameters and θ_{-i} for a vector that lists all taste parameters except θ_i.

Producers. There is a set of producers, $J = \{1, ..., m\}$. Each producer contributes to the supply of a public or private good. The contribution of producer j is denoted by y_j and comes with production costs $k(\delta_j, y_j)$, where δ_j is a cost characteristic of firm j that belongs to the finite ordered set $\Delta_j = \{\delta_j^1, ..., \delta_j^r\}$ of possible technology parameters. We assume that $\delta_j^1 < \delta_j^2$ etc. $\forall\ j \in J$. The profit of producer j is given by

$$\pi_j(\delta_j, r_j, y_j) = r_j - k(\delta_j, y_j) \ ,$$

where r_j is producer j's revenue, or, equivalently, a monetary payment to producer j.

[4]The index 2 denotes the partial derivative with respect to the second argument; $v_2(\theta_i, q_i) = \frac{\partial v(\theta_i, q_i)}{\partial q_i}$.

The function k is assumed to have the following properties. Zero production is costless: for all $\delta_j \in \Delta_j$, $k(\delta_j, 0) = 0$. The marginal costs from increased production is positive and increasing, so that for all δ_j and all y_j, $k_2(\delta_j, y_j) > 0$ and $k_{22}(\delta_j, y_j) \geq 0$. The marginal cost of production is increasing in the firm's type, so that $\delta_j' \geq \delta_j$ implies that $k_2(\delta_j', y_j) \geq k_2(\delta_j, y_j)$.

The technology parameter δ_j is privately observed by producer j. From the perspective of all other agents, it is a random variable with support Δ_j and probability distribution $p_j = (p_j^1, ..., p_j^r)$ where $p_j(\delta_j)$ that takes the value p_j^l if $\delta_j = \delta_j^l$. The cumulative distribution is given by $P_j(\delta_j)$, such that $P_j(\delta_j)$ takes the value $\sum_{k=1}^{l-1} p_j^k$ if $\delta_j = \delta_j^l$. The random variables $(\delta_j)_{j \in J}$ are *i.i.d.* We write $\delta = (\delta_1, ..., \delta_m)$ for a vector of technology parameters and δ_{-j} for a vector that lists all technology parameters except δ_j.

The consumers' preference parameters and the firms' cost parameters are taken to to be independent random variables. We will also refer to a vector (θ, δ) that lists all taste and cost parameters as a state of the economy. The set of all states is given by $(\Theta_i^n)_{i \in I} \times (\Delta_j^m)_{j \in J}$.

Social choice functions/ Direct Mechanisms. A social choice function or direct mechanism consists of a consumption and a payment rule for each consumer i and a production and revenue rule for each producer j. The consumption rule is a function $q_i : \Theta^n \times \Delta^m \mapsto \mathbb{R}_+$, that assigns to each state of the economy a consumption level for consumer i. Analogously, $t_i : \Theta^n \times \Delta^m \mapsto \mathbb{R}$ specifies i's payment as a function of the state of the economy. The production and revenue rule for producer j are, respectively, given by $y_j : \Theta^n \times \Delta^m \mapsto \mathbb{R}_+$ and $r_j : \Theta^n \times \Delta^m \mapsto \mathbb{R}$.

A social choice function is implementable as a Bayes-Nash equilibrium if there is a game with Bayes-Nash equilibrium, so that the equilibrium allocation of this game coincides in each state of the economy with the allocation stipulated by the social choice function. For the given setup, the revelation principle holds, so that we can without loss of generality limit attention to the implementation of a social choice function via a direct mechanism that induces a game in which truth-telling is a Bayes-Nash equilibrium. Thus, we say that a social choice function is incentive-compatible if truth-telling is a Bayes-Nash equilibrium of the corresponding direct mechanism.

Incentive-compatibility. Incentive-compatibility for consumer i holds, provided that for each $\theta_i^l \in \Theta_i$ and for all $\theta_i^k \in \Theta_i$,

$$V(\theta_i^l \mid \theta_i^l, q_i(\theta_{-i}, \theta_i^l)) - T(\theta_i^l) \geq V(\theta_i^k \mid \theta_i^l, q_i(\theta_{-i}, \theta_i^k)) - T(\theta_i^k) \;, \qquad (IC_C)$$

where $V(\theta_i^k \mid \theta_i^l, q_i(\theta_{-i}, \theta_i^k)) := \mathbb{E}_{(\theta_{-i},\delta)}\left[v(\theta_i^l, q_i(\theta_{-i}, \theta_i^k, \delta))\right]$ is the expected consumption utility for type θ_i^l of consumer i in case of announcing θ_i^k to the mechanism designer, given that all other consumers and producers reveal their preferences and technologies. Analogously, $T(\theta_i^k) := \mathbb{E}_{(\theta_{-i},\delta)}\left[t_i(\theta_{-i}, \theta_i^k, \delta)\right]$ is i's expected payment in case of reporting a preference parameter θ_i^k. The expectations operator $\mathbb{E}_{(\theta_{-i},\delta)}$ indicates that expectations are computed with respect to the random variable (θ_{-i}, δ). By contrast, the realization of θ_i is known when computing this expectation.

Likewise, incentive-compatibility for firm j requires that for all $\delta_j^l \in \Delta_j$ and for all $\delta_j^k \in \Delta_j$,

$$R(\delta_j^l) - K(\delta_j^l \mid \delta_j^l, y_j(\delta_{-j}, \delta_j^l)) \geq R(\delta_j^k) - K(\delta_j^k \mid \delta_j^l, y_j(\delta_{-j}, \delta_j^k)) \ , \qquad (IC_F)$$

where $R(\delta_j^k) := \mathbb{E}_{(\theta,\delta_{-j})}\left[r_j(\theta, \delta_{-j}, \delta_j^k)\right]$ is j's expected revenue in case of reporting a cost parameter δ_j^k, and $K(\delta_j^k \mid \delta_j^l, y_j(\delta_{-j}, \delta_j^k)) := \mathbb{E}_{(\theta,\delta_{-j})}[k(\delta^l, y_j(\theta, \delta_{-j}, \delta_j^k))]$ is the expected cost for type δ_j^l of firm j in case of announcing δ_j^k to the mechanism designer.

Participation Constraints. We will assume that social choice functions have to respect a lower bound on the utility that consumers realize and the profits that are realized by producers. Formally, we require that for all i and for all $\theta_i^l \in \Theta_i$,

$$V(\theta_i^l \mid \theta_i^l, q_i(\theta_{-i}, \theta_i^l)) - T(\theta_i^l) \geq \underline{u}_i \ , \qquad (PC_C)$$

where $\underline{u}_i$ denotes a lower bound for the expected utility of consumer i. Likewise, for all j and $\delta_j^l \in \Delta_j$,

$$R(\delta_j^l) - K(\delta_j^l \mid \delta_j^l, y_j(\delta_{-j}, \delta_j^l)) \geq \underline{\pi}_j \ , \qquad (PC_F)$$

where $\underline{\pi}_j$ is a lower bound for the expected profit of firm j.

The interpretation of these participation constraints depends on the application at hand. For instance, we may think that the implementation of the given social choice function replaces a status quo outcome and moreover requires a unanimous consent of all consumers and producers. In this case, $\underline{u}_i$ and $\underline{\pi}_j$ would, respectively, be interpreted as consumer i's and producer j's payoff in the status quo. Alternatively, in a model in which a government has coercive power, such a consent may not be needed but producers may have the possibility to shut down, so that a social choice function has to provide them at least with the level of profits that they would realize in this case. By choosing $\underline{u}_i$ and $\underline{\pi}_j$ arbitrarily small, we can also capture situations for which participation constraints are irrelevant.

Physical constraints. For many applications we assume that the consumers' consumption is bounded by the total output that is made available by the producers. Denote total output by $Y(\theta, \delta) = \sum_{j=1}^m y_j(\theta, \delta)$. If we consider an allocation problem involving private goods, then it has to be the case that, for all (δ, θ), $\sum_{i=1}^n q_i(\theta, \delta) \leq Y(\theta, \delta)$. If the good is non-rival and non-excludable then, for all i and all (δ, θ), $q_i(\delta, \theta) = Y(\delta, \theta)$. If the good is non-rival, but excludable, then, for all i and all (δ, θ), $0 \leq q_i(\delta, \theta) \leq Y(\delta, \theta)$.

We capture all these cases by postulating that, for all (θ, δ),

$$(q_i(\theta, \delta))_{i \in I} \in \Lambda(Y(\delta, \theta)) \ , \tag{1.5}$$

where $\Lambda(Y(\delta, \theta))$ is an abstract consumption set. Its structure depends on whether the goods in question are public or private.

Budget balance. We often assume that a social choice function has to satisfy a budget constraint, which requires that the consumers' expected payments suffice to cover the producers' expected revenues,

$$\mathbb{E}_{(\theta,\delta)}\left[\sum_{i=1}^{n} t_i(\theta, \delta)\right] \geq \mathbb{E}_{(\theta,\delta)}\left[\sum_{j=1}^{m} r_j(\theta, \delta)\right] \ . \tag{1.6}$$

An alternative that we will also consider is that the consumer's expected payments have to be sufficient to cover the producer's expected costs, i.e.,

$$\mathbb{E}_{(\theta,\delta)}\left[\sum_{i=1}^{n} t_i(\theta, \delta)\right] \geq \mathbb{E}_{(\theta,\delta)}\left[\sum_{j=1}^{m} k(\delta_j, y_j(\theta, \delta))\right] \ . \tag{1.7}$$

The budget condition in (1.6) is relevant in models in which producers have private information. The budget condition in (1.7) is employed in models in which the producers' cost functions are assumed to be publicly known information and in which profits in the hands of producers are considered undesirable. Since there is no private information there is also no impediment to reaching an outcome with zero expected profits, i.e., with

$$\mathbb{E}_{(\theta,\delta)}\left[\sum_{j=1}^{m} r_j(\theta, \delta)\right] = \mathbb{E}_{(\theta,\delta)}\left[\sum_{j=1}^{m} k(\delta_j, y_j(\theta, \delta))\right] \ .$$

However, as we will see below, such an outcome is out of reach if producers have private information and if their participation in the system is voluntary so that $\pi_j(\delta_j, r_j, y_j) \geq \underline{\pi}_j$, for all j.

These budget conditions allow for the possibility that there are deficits in some states of the economy and surpluses in others, provided that, in expectation, the surpluses are at least as large as the deficits. Thus, it is more permissive than having a separate budget balance condition for each state of the economy. There are various justifications for working with this permissive notion of budget balance. First, for many applications of the independent private values model, the following proposition holds true: if there is a social choice function that is incentive-compatible, respects the relevant participation constraints and budget balance in expectation, there is an 'equivalent' social choice function that satisfies in addition a state-wise requirement of budget balance, see Börgers and Norman (2009). Second, a requirement of budget balance in expectation may be justified with an appeal to the Law of Large Numbers.[5] If the numbers of consumers and producers is large, the discrepancy between budget balance in expectation and budget balance for each state separately becomes small, see Bierbrauer (2011). Finally, many analyses of the independent private values model have established impossibility results, see Myerson and Satterthwaite (1983) or Mailath and Postlewaite (1990). If there is no social choice function that satisfies budget balance in expectation, then there is also no social choice function that gives rise to budget balance in each state separately. Thus, for the purpose of establishing an impossibility result, working with the requirement of budget balance in expectation can be a useful modeling device.

Surplus measures. The total expected surplus that is generated by a social choice function is given by

$$S((q_i)_{i\in I},(y_j)_{j\in J}) = \mathbb{E}_{(\theta,\delta)}\left[\sum_{i=1}^{n} v(\theta_i, q_i(\theta,\delta)) - \sum_{j=1}^{m} k(\delta_j, y_j(\theta,\delta))\right] .$$

In a model with quasi-linear preferences, a social choice function is Pareto efficient if and only if the relevant budget constraint holds as an equality, the participation constraints in (PC_C) and (PC_F) are satisfied and $(q_i)_{i\in I}$, and $(y_j)_{j\in J}$ are chosen so as to maximize total surplus $S((q_i)_{i\in I},(y_j)_{j\in J})$ subject to the constraint of physical feasibility in (1.5). Note that there are typically many different Pareto efficient social choice functions. While the criterion of surplus-maximization pins down the functions $(q_i)_{i\in I}$ and $(y_j)_{j\in J}$, alternative specifications of the payment and revenue rules $(t_i)_{i\in I}$ and $(r_j)_{j\in J}$ give rise to different distributions of the surplus among consumers and producers.

1.5 Implementable provision rules

Before we turn to the question under which conditions efficient outcomes can be obtained, we will provide, as a preliminary step, a characterization of the set of implementable social choice functions, i.e., social choice functions with the property that there exists a direct mechanism that is incentive compatible, satisfies participation constraints, and is budgetary and physically feasible.

We begin by deriving a necessary and a sufficient condition for a social choice function to be implementable as Bayes-Nash equilibrium.

1.5.1 Necessary condition

The following proposition states a necessary condition for the possibility to implement a social choice function. More specifically, it states an inequality constraint so that, if this

[5]See e.g. Judd (1985) and Feldman and Gilles (1985).

inequality is violated, we know that there is no mechanism that satisfies the incentive compatibility constraints in (IC_C) and (IC_F), participation constraints in (PC_C) and (PC_F), and the expected budget constraint in (1.6).

Proposition 1.3. $\left\{(q_i)_{i=1}^n, (y_j)_{j=1}^m\right\}$ *is part of an implementable social choice function only if*[6]

$$\mathbb{E}_{(\theta,\delta)}\left[\sum_{i=1}^{n}\sum_{l=1}^{s}\left(v(\theta_i,q_i(\theta,\delta)) - f(\theta_i^l)\{v(\theta_i^{l+1},q_i(\theta,\delta)) - v(\theta_i^l,q_i(\theta,\delta))\}\frac{1-F(\theta_i)}{f(\theta_i)}\right)\right] - \sum_{i=1}^{n}\underline{u}_i \geq$$
$$\mathbb{E}_{(\theta,\delta)}\left[\sum_{j=1}^{m}\sum_{l=1}^{r-1}\left(k(\delta_j,y_j(\theta,\delta)) + p(\delta_j^l)\{k(\delta_j^l,y_j(\theta,\delta)) - k(\delta_j^{l-1},y_j(\theta,\delta))\}\frac{P(\delta_j)}{p(\delta_j)}\right)\right] + \sum_{j=1}^{m}\underline{\pi}_j\ .$$

A proof of the Proposition is in part 1.B of the Appendix. The social choice function can be split into a transfer and consumption rule for consumers and a revenue and production rule for producers. We begin by holding fixed consumers' consumption rule and derive the maximal transfers that consumers are able to make if incentive compatibility and participation constraints need to be both respected at the same time. Similarly, we hold fixed firms' production rule and derive the minimal revenues that firms are willing to accept if again incentive compatibility and participation constraints need to be respected. We then check whether consumers maximal transfers cover firms' minimal revenues. If the mechanism is unable to generate consumer payments that are high enough to cover the minimal revenue of provision, then there is no mechanism that reaches efficiency.

We make use of techniques used in the non-linear pricing literature (see e.g. Mussa and Rosen (1978)). We consider the *relaxed problem* of maximizing consumers' transfers subject to the *local downward* incentive compatibility constraints and the participation constraints for the lowest preference type. The local downward incentive compatibility constraints prevent type θ_i^l of consumer i to announce the next lower type to the mechanism designer. Hence, only a subset of consumers' incentive compatibility and participation constraints are taken into account. Thus, the maximal transfers that can be obtained at the solution to the relaxed problem, is an upper bound on the transfers that can be obtained if all incentive compatibility constraints are taken into account. We then show that the maximal transfers are given by the left-hand-side of the inequality constraint in Proposition 1.3 above.

Similarly, we solve a relaxed problem on the production side, to define the minimal revenues firms need to receive, where the *local upward* incentive compatibility constraints and the participation constraints of the worst technology type δ^r are taken into

[6] Note that the following expressing entails components that are not defined, i.e. θ_i^{s+1} δ_j^0. This is not problematic as they are weighted with zero.

consideration. As under consumer transfer maximization, only a subset of firms' incentive compatibility constraints is considered, so that the minimal revenues that need to be generated at the solution to the relaxed problem, is a lower bound on the revenues that can be generated if all incentive compatibility constraints are taken into account. If we plug the maximal transfers and the minimal revenues in the requirement of the budget constraint in (1.6), we obtain the inequality in Proposition 1.3 above.

For the derivation of the necessary condition in Proposition 1.3, we did not assume that the utility function u_i and the profit function π_j are differentiable. Neither did we need to assume, that q_i and y_j are monotonic. In contrast to a model that assumes the set of possible types for all agents to be infinite, we can avoid such assumptions on endogenous objects.

1.5.2 Sufficient condition

The next proposition establishes a sufficient condition for the implementability of a social choice function. For this purpose, we consider only a subset of all provision and production rules, namely those that satisfy the following monotonicity conditions: For every i,

$$q(\theta_i^l, \theta_{-i}, \delta) \geq q(\theta_i^k, \theta_{-i}, \delta) \qquad \text{for} \qquad \theta_i^l > \theta_i^k,\ \theta_i \in \Theta_i\ , \tag{1.8}$$

i.e., consumption must be monotonically increasing in θ_i for every consumer i. For every j,

$$y(\delta_j^l, \delta_{-j}, \theta) \geq y(\delta_j^k, \delta_{-j}, \theta) \qquad \text{for} \qquad \delta_j^l < \delta_j^k,\ \delta_j \in \Delta_j\ , \tag{1.9}$$

i.e., firm j's contribution to production must be monotonically decreasing in δ_j.

Efficient mechanisms satisfy these monotonicity conditions. Thus, as long as we limit ourselves to surplus-maximizing provision rules, the assumptions on monotonicity are not restrictive.

Proposition 1.4. *Let the monotonicity constraints in (1.8) and (1.9) be satisfied, then we can implement the social choice function if*[7]

$$\mathbb{E}_{(\theta,\delta)}\left[\sum_{i=1}^{n}\sum_{l=1}^{s}\left(v(\theta_i, q_i(\theta,\delta)) - f(\theta_i^l)\{v(\theta_i^{l+1}, q_i(\theta,\delta)) - v(\theta_i^l, q_i(\theta,\delta))\}\frac{1-F(\theta_i)}{f(\theta_i)}\right)\right] - \sum_{i=1}^{n}\underline{u}_i \geq$$
$$\mathbb{E}_{(\theta,\delta)}\left[\sum_{j=1}^{m}\sum_{l=1}^{r-1}\left(k(\delta_j, y_j(\theta,\delta)) + p(\delta_j^l)\{k(\delta_j^l, y_j(\theta,\delta)) - k(\delta_j^{l-1}, y_j(\theta,\delta))\}\frac{P(\delta_j)}{p(\delta_j)}\right)\right] + \sum_{j=1}^{m}\underline{\pi}_j\ .$$

Suppose that the condition in Proposition 1.4 holds. We need to show that we can construct a payment scheme that satisfies all relevant constraints. We can choose our transfer and revenue scheme such that they solve the relaxed problems that we studied in the proof of Proposition 1.3. After that, we need to verify that the transfer and revenue

[7] This is also true if the less demanding weak monotonicity condition holds instead of condition (1.8) and (1.9). See equation (WM) and (wm) in the appendix for a definition.

schemes, which solve the relaxed problems, satisfy not only the local downward incentive compatibility constraints of consumers and the local upward incentive compatibility constraints of firms, but all consumers' and firms' incentive compatibility constraints. We prove that the fact that all local downward incentive compatibility constraints of consumers are binding, together with the monotonicity constraints of consumption, implies that all incentive compatibility constraints are satisfied. With that, consumption is efficient for the consumer of highest type and distorted downwards for all other consumers. And similarly, production is efficient for the firm of lowest costs and distorted upwards for all other firms.

Definition 1.1. *We define the minimal subsidy* $(MS(\cdot))$ *as*

$$\left.\begin{aligned} MS((q_i)_{i\in I},(y_j)_{j\in J}) :=& \sum_{i=1}^{n} \underline{u}_i - \sum_{j=1}^{m} \overline{\pi}_j \\ &- \mathbb{E}_{(\theta,\delta)}\left[\sum_{i=1}^{n} v(\theta_i, q_i(\theta,\delta)) - \sum_{j=1}^{m} k(\delta_j, y_j(\theta,\delta))\right] \end{aligned}\right\} S(\cdot)$$

$$\left.\begin{aligned} &+ \mathbb{E}_{(\theta,\delta)}\Bigg[\sum_{i=1}^{n}\sum_{l=1}^{s} f(\theta_i^l)\{v(\theta_i^{l+1}, q_i(\theta,\delta)) - v(\theta_i^l, q_i(\theta,\delta))\}\frac{1-F(\theta_i)}{f(\theta_i)} \\ &\qquad -\sum_{j=1}^{m}\sum_{l=1}^{r-1} p(\delta_j^l)\{k(\delta_j^l, y_j(\theta,\delta)) - k(\delta_j^{l-1}, y_j(\theta,\delta))\}\frac{P(\delta_j)}{p(\delta_j)}\Bigg] \end{aligned}\right\} IR(\cdot)$$

Under the monotonicity constraints (1.8) and (1.9), the minimal subsidy gives the amount of money that is required from an outside party to satisfy Proposition 1.4.[8] Apart from the reservation utilities and profits, it consists of two components: the first component is surplus $S(\cdot)$, given by the difference of consumers' valuations and firms' costs. The second component $IR(\cdot)$ is the difference of consumers' and firms' information rents. In an environment with complete information, the maximal transfer a consumer i is is willing to pay is his true valuation. With private information about preferences, however, the transfers are decreased by the hazard rate expression $\sum_{l=1}^{s} f(\theta_i^l)\{v(\theta_i^{l+1}, q_i(\theta,\delta)) - v(\theta_i^l, q_i(\theta,\delta))\}\frac{1-F(\theta_i)}{f(\theta_i)}$. That is why these hazard rate expressions are often interpreted as information rents. Similarly, for firms, the virtual costs account for the private information. The revenues, firms would accept in an environment with complete information, are increased by an information rent.

1.5.3 Efficiency

In the previous section, we have been concerned with deriving conditions such that social choice functions could be implemented in an environment where agents' have

[8] This notion follows Myerson and Satterthwaite (1983), who call the amount that would be required from an outside party to overcome the impossibility result *minimal lump-sum subsidy*.

private information about their preferences and costs, respectively. Now, we focus on the welfare evaluation of implementable social choice functions.

Corollary 1.1. *A mechanism that maximizes surplus $S(\theta, \delta)$, in the following denoted by $\left\{(q_i^*)_{i=1}^n, (y_j^*)_{j=1}^m\right\}$, is implementable if and only if*

$$\mathbb{E}_{(\theta,\delta)}\left[\sum_{i=1}^n\sum_{l=1}^s\left(v(\theta_i, q_i^*(\theta,\delta)) - f(\theta_i^l)\{v(\theta_i^{l+1}, q_i^*(\theta,\delta)) - v(\theta_i^l, q_i^*(\theta,\delta))\}\frac{1-F(\theta_i)}{f(\theta_i)}\right)\right] - \sum_{i=1}^n \underline{u}_i \geq$$
$$\mathbb{E}_{(\theta,\delta)}\left[\sum_{j=1}^m\sum_{l=1}^{r-1}\left(k(\delta_j, y_j^*(\theta,\delta)) + p(\delta_j^l)\{k(\delta_j^l, y_j^*(\theta,\delta)) - k(\delta_j^{l-1}, y_j^*(\theta,\delta))\}\frac{P(\delta_j)}{p(\delta_j)}\right)\right] + \sum_{j=1}^m \underline{\pi}_j\ .$$

A necessary condition for the implementability of $((q_i^*)_{i=1}^n, (y_j^*)_{j=1}^m)$ is that the maximal transfers that can be extracted from consumers suffice to cover the minimal revenues that the production sector needs to obtain, which means that the minimal subsidy is negative.

Sufficiency can be shown by constructing the transfers of consumers such that all local downward incentive compatibility constraints are binding, and constructing the revenues of producers such that all local upward incentive compatibility constraints are binding. Finally, we choose the transfer of the consumer with the lowest preference parameter such that budget balance is satisfied and the minimal utilities and resource requirement is taken into consideration. Then, we obtain a mechanism that achieves $((q_i^*)_{i=1}^n, (y_j^*)_{j=1}^m)$ and satisfies all relevant constraints stated in Corollary 1.1.

From Corollary 1.1, it follows immediately that when participation constraints can be violated, a social choice function can be implemented efficiently. The minimal subsidy can be decreased without limit. This is an alternative proof of the possibility results obtained by D'Aspremont and Gerard-Varet (1979) and Arrow (1979).

The following Corollary shows how the second-best mechanism can be derived.

Corollary 1.2. *The second-best mechanism, in the following denoted by $((q_i^{**})_{i=1}^n, (y_j^{**})_{j=1}^m)$, can be found by maximizing $S(\theta, \delta)$ subject to*

$$\mathbb{E}_{(\theta,\delta)}\left[\sum_{i=1}^n\sum_{l=1}^s\left(v(\theta_i, q_i(\theta,\delta)) - f(\theta_i^l)\{v(\theta_i^{l+1}, q_i(\theta,\delta)) - v(\theta_i^l, q_i(\theta,\delta))\}\frac{1-F(\theta_i)}{f(\theta_i)}\right)\right] - \sum_{i=1}^n \underline{u}_i \geq$$
$$\mathbb{E}_{(\theta,\delta)}\left[\sum_{j=1}^m\sum_{l=1}^{r-1}\left(k(\delta_j, y_j(\theta,\delta)) + p(\delta_j^l)\{k(\delta_j^l, y_j(\theta,\delta)) - k(\delta_j^{l-1}, y_j(\theta,\delta))\}\frac{P(\delta_j)}{p(\delta_j)}\right)\right] + \sum_{j=1}^m \underline{\pi}_j$$

and the monotonicity constraints in (1.8) and (1.9).

In the solution to the second-best problem, the budget condition has to be binding. Otherwise, expected consumer transfers can be reduced without violating any of the incentive compatibility and participation constraints. The provision rule that solves the second-best problem satisfies the monotonicity constraints for all consumers and all firms. This follows because the optimal provision level is given by the first-order condition where the sum of virtual marginal valuation equals the sum of virtual marginal costs. The assumption that hazard rates are non-decreasing for consumers implies that virtual valuation is increasing in consumer i's type, and the assumption that the monotone hazard rate is non-increasing for firms implies that virtual costs are increasing in the cost type.

Continuous types. To study whether a model with a large but discrete number of types behaves approximately in the same way as a model with continuum types, we state the expression for the minimal subsidy for continuous type distributions. We know from the literature (compare e.g. Mas-Colell et al., 1995) that for a surplus maximizing mechanism $\left\{(q_i^*)_{i=1}^n, (y_j^*)_{j=1}^m\right\}$ in a continuous environment, implementability is possible if the constraint in the following Remark is satisfied. It is based on the assumption that functions are continuously differentiable.

Remark 1. *Suppose that q_i and y_j are continuously differentiable functions. Then a mechanism that maximizes $S(\theta, \delta)$, is implementable if and only if*

$$\begin{aligned}
&\mathbb{E}_{(\theta,\delta)}\left[\sum_{i=1}^n\left(v(\theta_i, q_i^*(\theta,\delta)) - \int_{\theta_i^0}^{\theta_i^s} f(\theta_i)v_1(\theta_i, q_i^*(\theta,\delta))\frac{1-F(\theta_i)}{f(\theta_i)}d\theta_i\right)\right] - \sum_{i=1}^n \underline{u}_i \geq \\
&\mathbb{E}_{(\theta,\delta)}\left[\sum_{j=1}^m\left(k(\delta_j, y_j^*(\theta,\delta)) + \int_{\delta_j^1}^{\delta_j^r} p(\delta_j)k_1(\delta_j, y_j^*(\theta,\delta))\frac{P(\delta_j)}{p(\delta_j)}d\delta_j\right)\right] + \sum_{j=1}^m \underline{\pi}_j.
\end{aligned} \tag{1.10}$$

The differences, in comparison to Corollary 1.1, are that $\{v(\theta_i^{l+1}, q_i^*(\theta,\delta)) - v(\theta_i^l, q_i^*(\theta,\delta))\}$ and $\{k(\delta_j^l, y_j^*(\theta,\delta)) - k(\delta_j^{l-1}, y_j^*(\theta,\delta))\}$ are replaced by the derivatives $v_1(\theta_i, q_i^*(\theta,\delta))$ and $k_1(\delta_j, y_j^*(\theta,\delta))$, respectively. Therefore, the differentiability assumption is crucial with a continuum of types.

1.6 Comparative statics I: From few to many agents

We use our results in Proposition 1.3 and 1.4 to obtain a more systematic understanding of how a change in the parameters of the model affects the possibility to implement efficient social choice functions. Again, we will check whether our results depend on whether the allocation problem involves private or public goods.

1.6.1 Possibility results when the type set is binary

Bilateral trade. We have shown on Section 1.3.1 that parameter constellations can be found such that efficient bilateral trade is possible. In order to investigate what drives this possibility result, we make some simplifying assumptions.

Assumption 1.1. *All agents have a binary type set.*[9]

Under Assumption 1.1, the buyer's marginal valuation can be high or low, $\Theta = \{\theta^L, \theta^H\}$, and the seller's marginal costs can take two values, $\Delta = \{\delta^L, \delta^H\}$. We denote the respective probabilities by $Prob(\theta = \theta_L) = f^L$, $Prob(\theta = \theta^H) = f^H = 1 - f^L$ and $Prob(\delta = \delta^L) = p^L$, $Prob(\delta = \delta^H) = p^H = 1 - p^L$.

Assumption 1.2. ***(Symmetry)*** *The distance between the low type and the high type is the same for buyer and seller, $\theta^H - \theta^L = \delta^H - \delta^L$.*[10] *The probability of a high valuation buyer is equal to the probability of a low cost seller $f^H = p^L$ and therefore $f^L = p^H$.*

Analogously to the continuous environment of Myerson and Satterthwaite (1983), where $[\theta^L, \theta^H] \cap [\delta^L, \delta^H] \neq \emptyset$ and $[\theta^L, \theta^H] \neq [\delta^L, \delta^H]$, we assume that the parameter constellation is such that $\delta^L < \theta^L < \delta^H < \theta^H$, with the normalization $\delta^L = 0$ and $\theta^H = 1$. The social choice function f is efficient if and only if $q(\theta, \delta) = y(\theta, \delta)$, $q(\theta, \delta)$ satisfies

$$q^f(\theta, \delta) = \begin{cases} 0, & \text{if} \quad \theta < \delta \ , \\ \in \{0, 1\} & \text{if} \quad \theta = \delta \ , \\ 1, & \text{if} \quad \theta > \delta \ , \end{cases}$$

and $\mathbb{E}_{(\theta,\delta)}[t^f(\theta, \delta)] = \mathbb{E}_{(\theta,\delta)}[r^f(\theta, \delta)]$.

Observation 1.1. *Suppose Assumptions 1.1 and 1.2 are satisfied and consider the direct mechanism for f. There is a social choice function, which is efficient, implementable as Bayes-Nash equilibrium, and yields a non-negative material payoff for every type of agent if and only if*

$$f^L(\theta^H - \delta^L) + (\theta^L - \delta^H) > 0 \quad \Leftrightarrow \quad f^L > \delta^H - \theta^L \ .$$

The proof of this Observation and all following Observations can be found in Appendix 1.E. It provides a sufficient condition for the possibility to reach efficient bilateral trade. Specifically, the condition states that the probability of the low valuation buyer, respectively on the high cost seller, needs to be bigger than the ratio that measures the overlap of buyer's and seller's type sets ($\delta^H - \theta^L$), relative to the whole type set that we normalized to 1 ($\theta^H - \delta^L = 1$). We call this ratio in the following d,

$$d := \frac{\delta^H - \theta^L}{\theta^H - \delta^L} = \delta^H - \theta^L \ .$$

The bigger the ratio d, the bigger is the overlap of buyer's and seller's type sets. If $d = 0$, the type sets of buyer and seller are disjunct, and efficient trade is always possible,

[9] Without loss of generality, we specify the binary type set, in the following, as consisting of a low and high type, where the high type exceeds the low type.

[10] We assume that $\Theta \subseteq \mathbb{R}$ and $\Delta \subseteq \mathbb{R}$, such that 'distance' is well-defined by the Euclidean metric.

regardless of the probability distribution. On the other hand, the maximum of d is given by 1, where the type sets of the buyer and the seller are congruent.

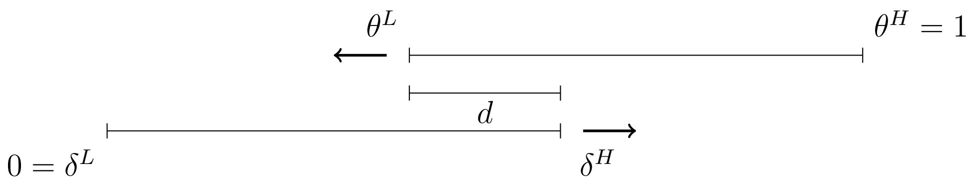
To understand what drives the possibility result above, it is instructive to look at the proof of Observation 1.1. By Proposition 1.4, we know that efficient bilateral trade is possible if the minimal subsidy is negative. Given that trade is efficient for all states of the economy but when the low valuation buyer faces the high cost seller, we have three states of the economy that need to be evaluated, and weighted with the probability of occurrence.

$$\text{MS} = -\left(\begin{array}{c} \\ + \\ + \end{array} \underbrace{\begin{array}{c} f^H f^L(\theta^H - \delta^H) \\ f^H f^H(\theta^H - \delta^L) \\ f^L f^H(\theta^L - \delta^L) \end{array}}_{S(\cdot)} \begin{array}{c} - \\ \\ - \end{array} \underbrace{\begin{array}{c} f^H f^L(\delta^H - \delta^L)\frac{f^H}{f^L} \\ \\ f^L f^H(\theta^H - \theta^L)\frac{f^H}{f^L} \end{array}}_{IR(\cdot)}\right)$$

Figure 1.2: Minimal subsidy for a binary type set

In the following we do comparative statics for different components of the model. As a first exercise, we fix the probability of the low valuation buyer and the high cost seller f^L and vary the ratio d. Without loss of generality, we keep the normalization of the whole type set, so that $\theta^H - \delta^L = 1$. In order to change d, the position of θ^L and δ^H can be varied. By Assumption 1.2, $\theta^H - \theta^L = \delta^H - \delta^L$, so that a shift from the low valuation buyer to the left goes hand in hand with a move of the high cost seller to the right, and vice versa, as the following Figure illustrates.

Figure 1.3: Changing parameters in the bilateral trade setting



Observation 1.2. *Suppose we move the types as in Figure 1.3 above, such that d is increased. This affects the minimal subsidy via the expected information rents and the expected surplus, where*

$$\frac{\partial IR(\cdot)}{\partial d} > 0 \,, \qquad \text{and} \qquad \frac{\partial S(\cdot)}{\partial d} < 0 \,.$$

For types outside the overlap, efficient trade takes place – independent of the opponent's type, i.e., the private information of the other party does not matter for the decision whether trade takes place. This means, that the two-sided private information only matters in the overlap. Thus, d can be understood as a measure for the importance of two-sided private information. The first effect depends on the distance of types for each player. If d increases, higher expected information rents need to be paid under the new parameter constellation, since the distance of high and low types increases for both players. This has a negative effect on the possibility to achieve efficient trade, and therefore yields a negative information rent effect. Second, if d increases, the expected surplus decreases, i.e., we have a negative surplus effect. Further, the average valuation for the good goes down relative to the average costs if d is increased. Therefore, expected surplus decreases, which has a negative effect on the possibility to achieve efficient trade. Thus, an increase of d decreases expected surplus and increases expected information rents. It is hence *more costly* to achieve efficiency.[11]

Next, consider the situation where for a given d and f^L implementation is possible. We fix the ratio that measures the overlap d and analyze how varying the probability f^L affects implementability.

Observation 1.3. *Suppose d is fixed and the probability f^L is increased. This affects the minimal subsidy via the expected information rents and the expected surplus, where*

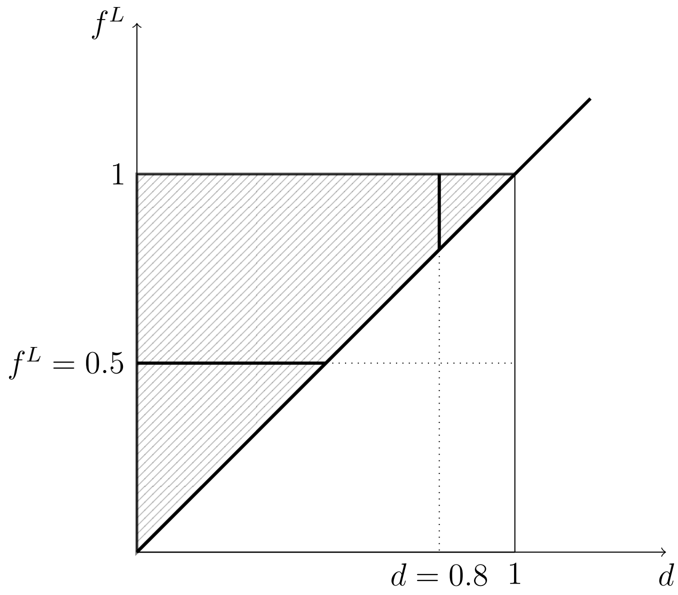
$$\frac{\partial IR(\cdot)}{\partial f^L} < 0 \,, \qquad \text{and} \qquad \frac{\partial S(\cdot)}{\partial f^L} < 0 \,.$$

Incentive compatibility constraints are binding for the high valuation buyer and the low cost seller, and slack for the other types. Hence, if f^L increases, we will have to pay less individuals information rents. This has a positive effect on the possibility to achieve efficient trade. Second, we find that the surplus effect is negative if $\theta^H - \delta^H = \theta^L - \delta^L < \frac{1}{2}$. Since we start our analysis in a situation where implementation is possible, we know that this conditions always holds. Thus, the increase of f^L has an unambiguous negative effect on implementability since it lowers the expected surplus. Intuitively, the

[11] Whenever we write efficient implementation gets more costly, we mean the following: if we compare two sets that represent the tuples of f^L and d, for which efficient trade is possible, the set for which efficiency is more costly is a subset of the other. Analogously for the case where efficiency is less costly to achieve.

expected surplus decreases since we expect more low valuation buyers, respectively high cost sellers. Overall, an increases in f^L increases the minimal subsidy, and thus has a negative effect on the possibility to reach efficient trade: by Observation 1.1, we know that an increase of f^L makes it less costly to achieve efficient trade. Therefore, when f^L is increased, the reduction in expected information rents is bigger than the reduction in expected surplus, so that an increase of f^L makes it less costly to achieve efficiency. The intuition for this result is that there is more mass on the types that do not receive an information rent. Overall, the effect of an increase in f^L on the minimal subsidy is negative such that implementation gets less costly.

Figure 1.4: Possibility of efficient trade



By assumption, f^L and d are bounded by 0 and 1. If f^L is for example fixed at 0.5, then the distance between the high cost seller and the low valuation buyer needs to be lower than 0.5 in order to achieve efficient trade. And if the distance is for example fixed at $d = 0.8$, then the probability of the low type buyer, respectively the high cost seller, needs to be higher than 0.8 to get a possibility result.

Figure 1.3 illustrates the interplay of the expected surplus effect and the expected information rent effect on the possibility of efficient bilateral trade. The grey shaded area T_0 gives every combination of f^L and d, such that, efficient trade is possible,

$$T_0 = \left\{ (f^L, d) : f^L > d = \frac{\delta^H - \theta^L}{\theta^H - \delta^L} \right\} .$$

Public Good. Given Assumption 1.1 holds, a social choice function f' specifies whether the public good is provided $q^{f'}(\theta_1, \theta_2)$, and the accompanying transfers $t_1^{f'}(\theta_1, \theta_2)$ and $t_2^{f'}(\theta_1, \theta_2)$. Pareto efficient public good provision requires

$$q^{f'}(\theta_1, \theta_2) = \begin{cases} 0, & \text{if } \frac{1}{n}\sum_{i=1}^{n} \theta_i < c \ , \\ \in \{0, 1\} & \text{if } \frac{1}{n}\sum_{i=1}^{n} \theta_i = c \ , \\ 1, & \text{if } \frac{1}{n}\sum_{i=1}^{n} \theta_i > c \ . \end{cases}$$

and $\mathbb{E}_{(\theta)}[t_1^{f'}(\theta_1, \theta_2) + t_2^{f'}(\theta_1, \theta_2)] = 2c\, \mathbb{E}_{(\theta)}[q^{f'}(\theta)]$ for all $(\theta_1, \theta_2) \in \Theta$.

The following Observation highlights that if there are just two consumers, then efficient implementation of the public good will be possible for some parameter constellations.

Observation 1.4. *Suppose Assumption 1.1 is satisfied, the per capita costs are such that $\theta^L < c < \theta^H$, and consider the direct mechanism for f'. There is a social choice function which is efficient, implementable as Bayes Nash equilibrium, and it yields non-negative utility for every type of consumer if and only if*

$$f^L - \frac{c - \theta^L}{\theta^H - c} > 0 \ .$$

Observation 1.4 provides a sufficient condition for efficient public good provision. Following the proof of Observation 1.4, the public good can be provided efficiently, if and only if consumers' expected transfers cover the costs of public good provision, so that the minimal subsidy is negative

$$MS = -\left(\underbrace{\left[\theta^H - c - f_L(c - \theta^L)\right]}_{S(\cdot)} - \underbrace{(1 - f^L)(\theta^H - \theta^L)}_{IR(\cdot)} \right) < 0 \ .$$

The first term denotes expected surplus, depending on the cost c and the type distribution, whereas the second term describes expected information rents.

In the following, we analyze what drives the possibility to reach efficient public good provision. Again, we analyze the effect of a change in the type distribution f^L and a change in the cost parameters c and split the resulting effects into the expected surplus and the expected information rent effect.

Observation 1.5. *Suppose Assumptions 1.1 and 1.2 hold.*

i) Suppose further that f^L is fixed, then $\frac{\partial S(\cdot)}{\partial c} < 0$.

ii) Suppose further that c is fixed, then

$$\frac{\partial IR(\cdot)}{\partial f^L} < 0 \ , \qquad \textit{and} \qquad \frac{\partial S(\cdot)}{\partial f^L} < 0 \ .$$

When costs are increased, this does not affect the information rents of the agents, but lowers the expected surplus. Higher per capita costs imply that in expectation, the share of benefiting types reduces compared to the share of suffering types. This means that for higher c, the average valuation for the public good decreases compared to the fixed per capita costs.

If c is fixed, a change in f^L affects the minimal subsidy in two ways. First, the expected surplus decreases if f^L increases. This has a negative effect on the possibility of efficient public good provision. And second, for the expected information rents, we find, that if f^L increases, we have to pay in expectation less agents an information rent. This means, the higher the probability for low valuation consumers, the lower the expected information rent term and the higher the possibility for efficient public good provision, ceteris paribus. As we can see in Observation 1.4, the reduced expected information rent effect dominates the negative expected surplus effect, such that an increase in f^L makes it less costly to provide the public good efficiently, i.e., the minimal subsidy decreases.

Comparison. When a binary type set and two agents are considered, efficient bilateral trade and efficient public good provision are possible when parameters are chosen appropriately. Further, when the average valuation for the private or public good is increased, it gets cheaper to achieve efficiency, in the sense that the minimal subsidy goes down. To understand the difference between private and public goods, we study how the increase in the number of the agents affects the minimal subsidy.

1.6.2 Many agents

The classical bilateral trade setting has one buyer and one seller. To that extent Observation 1.1 is not restrictive. Contrary, the impossibility result in the public good setting is studied for a large economy, with many individuals. In the following, we want to analyze these impossibility results under the assumption that the type set of individuals remains binary.

Public good. Mailath and Postlewaite (1990) show that efficient public good provision is impossible when the number of individuals goes to infinity. The following Proposition shows that the assumption of continuous type sets has no impact on this impossibility result. Even for a binary type set, efficient public good provision is impossible if the number of individuals grows without limit.

Proposition 1.5. *Suppose Assumption 1.1 holds. Consider the public good example and an economy with n individuals. For any sequence of incentive-compatible mechanisms $(q, t_1^n, \ldots, t_n^n)$,*

$$lim_{n\mapsto\infty}\mathbb{E}_{(\theta)}\left[\frac{1}{n}\sum_{i=1}^{n} t_i^n(\theta)\right] = 0\ .$$

As $n \to \infty$, the probability of public good provision converges to 0 if $c > \theta^L$, and converges to 1 otherwise.

According to Proposition 1.5, the per capita revenue from individuals' contribution goes to zero. That is, with many individuals, any social choice function, which is attainable when voluntary participation needs to be guaranteed, prescribes a public good provision level that is equal to 0. Even if the average valuation for the public good is larger than the per capita costs, the amount that is implemented is almost equal to 0, although the efficient amount is almost equal to 1.

The reason for the impossibility result is that, for $n \to \infty$, any individual's impact on the public good provision becomes negligible. The free-rider problem in public good provision becomes extreme as the number of individuals becomes large. Since no individual is pivotal for the production of the public good, incentive compatibility implies that all individuals have to make the same lump-sum transfer that does not depend on their announced preference intensity. Participation constraints imply that this lump-sum transfer must not exceed θ^L. Thus, the aggregate of all individuals' transfers is as if every individual has a low valuation for the public good. This makes it impossible to cover the costs of public good provision. Hence, for the impossibility result, the assumption that the type set is discrete has no impact. Remember that Proposition 1.2 states that public good provision is as well not possible under the assumption of a continuous type set when the number of individuals goes to infinity.

Private good. Consider a finite economy with $I = \{1, \ldots n\}$ buyers and $J = \{1, \ldots n\}$ sellers. Buyers and sellers have equivalent binary type sets with the corresponding probability distributions (f_i^L, f_i^H) and (p_j^L, p_j^H).[12] The agents face a price $\rho \in [\theta_i^L, \delta_i^H]$. As we have seen in previous sections, there exist parameter constellations and probability distributions, where trade does not take place. Now, we assume that the number of agents grows without limits, i.e., $n \longrightarrow \infty$. The Law of Large Numbers applies, and probabilities can be interpreted as cross-sectional distribution of types. In particular, this means that for large economies one knows with probability 1, that there exist high type buyers with a share of f_i^H, and low type sellers, with a share of $p_j^L = f_i^H$. For these agents, trade always takes place. This proves the existence of trade in large economies for prices $\rho \in [\theta_i^L, \delta_i^H]$ if agents have binary type sets.[13]

Gresik and Satterhwaite (1989) additionally show that trade in large economies is efficient – ex-ante and ex-post – holding the ratio of buyer and seller fixed. Further, they provide results on the rate of convergence to the competitive equilibrium.

Comparison. When private and public goods are compared for economies with many agents, we see that efficient public good provision is impossible when the number of con-

[12] $f_i^H, f_i^L, p_j^H, p_j^L > 0, \forall\, i, j$.

[13] This argument holds independently of whether the type set is finite or infinite.

sumers is large. Contrary, for the private good model efficient bilateral trade is possible if there are many agents. The impossibility result of Mailath and Postlewaite (1990) is thus *stronger* than the impossibility result of Myerson and Satterthwaite (1983) in the sense that it extends to any model with a discrete set of types.

1.7 Comparative Statics II: From few to many types

This section shows that the minimal subsidy for a discrete type set with a large number of types converges to the minimal subsidy for a continuous type set environment. We proceed as follows: we first show how one additional type affects the implementation rule for private and public goods and add a third type for each agent. To study the changes of an increased number of types separately from changes in the number of agents, we hold the number of agents fixed in this section. Second, we consider the situation where the number of types gets larger and larger and study general convergence. We provide conditions that need to be met for the convergence result and discuss how violating these conditions affects our analysis.

1.7.1 Introducing a third type.

Bilateral trade. In the following, we consider what happens if we add step-by-step new types in *rounds* to agents' type sets of the pre-round. Thereby, we do not move the existing types, but add new types 'between' them.[14] If the procedure of adding types fulfills two conditions we call it 'uniform extension'.

Definition 1.2. (Uniform Extension) *Consider the agent sets I and J, fully ordered finite type sets $\Theta_{0i} = \{\theta_{0i}^L, \dots, \theta_{0i}^H\}$, $\#\Theta_{0i} < \infty, \forall\, i \in I$, $\Delta_{0j} = \{\delta_{0j}^L, \dots, \delta_{0j}^H\}$, $\#\Delta_{0j} < \infty, \forall\, j \in J$ and the respective probabilities, such that $f_{0i}^L + \dots + f_{0i}^H = 1$ and $p_{0j}^L + \dots + p_{0j}^H = 1$. We add finitely many new types in every round k to every agent's type set, such that for every new type θ_{ki} and δ_{kj}, it holds that $\theta_{0i}^L < \theta_{ki} < \theta_{0i}^H$ and $\delta_{0j}^L, < \delta_{kj} < \delta_{0j}^H$, respectively. We call this procedure uniform extension if there exists a round $K < \infty$[15] such that for all $i \in I$ and $j \in J$*

- $\overline{\Theta_{Ki}} = [\theta_{0i}^L, \theta_{0i}^H]$ *and* $\overline{\Delta_{Kj}} = [\delta_{0j}^L, \delta_{0j}^H]$.[16]
- *For every $\theta_{ki} \in \Theta_{ki}$ and $\delta_{kj} \in \Delta_{kj}$, it holds that $f_i(\theta_{ki})$ and $p(\delta_{kj})$ are monotonically decreasing from round to round.*

The first property assures, that there is round K from which on, there are no neighbored types that have a distance of more than an ϵ, which can be arbitrary small. That

[14] Since we consider fully ordered type sets, 'between' is well-defined.

[15] This does not have to be the same K for producers and consumers or each individual. In such a case, round K as used here, is the maximum of all those.

[16] The *closure* of a set A is defined as $\overline{A} = A \cup \{\lim_n a_n, a_n \in A\}$.

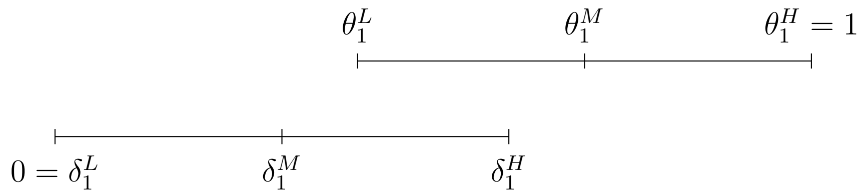
this is crucial to achieve implementability in the limit can be seen in the section counterexamples later on.
We start in a situation with a binary type set, see Section 1.3, and call this setting 'round 0'. Successively, we add new types and index the new situations by round numbers $k = 1, 2, \ldots$. Exemplary we calculate the new efficient trade possibility condition for 'round 1', i.e., there are three types for the buyer and the seller. Hence, the buyer has the extended type set $\Theta_1 = \{\theta_1^L, \theta_1^M, \theta_1^H\}$, and the seller has the extended type set $\Delta_1 = \{\delta_1^L, \delta_1^M, \delta_1^H\}$.[17] Without loss of generality, we add the new types according to the following procedure and call it *uniform extension type I.*

Definition 1.3. (Uniform Extension Type I) *Consider given agent sets I_0 and J_0, for which every agent $i \in I_0$ has the same finite type set Θ_{0i} and for which every agent $j \in J_0$ has the same finite type set Δ_{0j}. The probabilities assigned to the possible types of any agent sum up to 1. This situation is called round 0. In every round $k = 1, 2, \ldots$ finitely many new types are added to each agents' type set. Thereby, new types θ_{ki} and δ_{kj} are positioned at the center between adjacent types from round $k = 0, \ldots, k-1$.*

In the following, we start with round 0, which is given by $I_0 = 1$ and $J_0 = 1$ and the type sets $\Theta_0 = \{\theta^L, \theta^H\}$ and $\Delta_0 = \{\delta^L, \delta^H\}$ and the respective probabilities, such that $f_0^L + f_0^H = 1$.[18]

Continuing, new types are introduced,in every 'round k', in the middle of the subintervals of type sets that existed already by 'round $k-1$'. With this procedure, the continuous type interval is approached, such that the first condition in Definition 1.2 is fulfilled. Since the probability of an existing type cannot increase, both conditions of Definition 1.2 are fulfilled. We consider the case where the introduced third type is positioned such that $\theta_1^M > \delta_1^H$.[19]

Figure 1.5: Three buyer and seller types



We can now derive which condition needs to be satisfied in order achieve efficient bilateral trade when the buyer and the seller have an extended type set with three types. Applying Proposition 1.4, the following Observation shows a condition for reaching efficient bilateral trade.

[17]Since $I = 1$ and $J = 1$, we drop the indices that label the agent for ease of notation.
[18]Remember that by Assumption 1.2, $f^L = p^H$ and $f^H = p^L$ and analogously for every round k.
[19]The case where $\theta_1^M < \delta_1^H$ can be found in Appendix 1.E.

Observation 1.6. *Without loss of generality, suppose parameters are as in Figure 1.5. For the 'Uniform Extension Type I' in 'round 1', there is a social choice function which is efficient, implementable as Bayes-Nash equilibrium, and yields non-negative material payoff for every type of player if and only if*

$$f_1^L > \frac{\delta_1^H - \theta_1^L}{\theta_1^M - \delta_1^M} \qquad \Leftrightarrow \qquad \frac{\frac{1}{2} f_1^L}{1 + \frac{1}{2} f_1^L} > d \; .$$

d still denotes $\frac{\delta^H - \theta^L}{\theta^H - \delta^L}$ and did not change from 'round 0' to 'round 1'. For the purpose of illustration, assume that every type has equal probability: In 'round 0' with a binary type set, the condition in Observation 1.1 states that $\frac{1}{2} > d$. After adding a third type, Observation 1.6 states that efficient bilateral trade is possible if $\frac{1}{7} > d$. So, even for only one additional type, the possibility condition for efficient trade gets more restrictive.

As for the binary type set, the minimal subsidy is important to evaluate whether efficient trade is possible. It is decisive, for which states of the economy trade takes place, i.e., for which combinations of buyer and seller types trade should take place, so that $q^*(\theta, \delta) = 1$. From Observation 1.6 we know that the high type buyer and the low type seller are not relevant. Intuitively, this is because even when a buyer with a lower valuation, i.e., θ_1^M, can trade with every possible type of seller, then this is as well true for high type buyers. By the same logic, the seller with the lowest valuation does not enter the implementability condition. Mathematically, for buyer's high type (and seller's low type), terms of the expected surplus effect and the expected information rent effect coincide and thus cancel out. Instead, the middle type (which is the lowest types that can trade with every possible seller) needs to be considered. When subsequent rounds are considered, we generalize this observation and analyze for which types the expected surplus effect and the expected information rent effect coincide. We define the buyer types and the seller types, for which the effects do not cancel out and thus are relevant for the evaluation of possibility.

When the type sets Θ_1 and Δ_1 are considered, expected surplus is given by

$$S(\cdot) = (\theta_1^M - \delta_1^M) f_1^M - d \; f_1^L (f_1^M + f_1^H) \; .$$

Expected information rents are given by

$$IR(\cdot) = (f_1^M + f_1^H)^2 (\theta_1^M - \delta_1^M + d) - f_1^H (\theta_1^M - \delta_1^M) \; .$$

Expected surplus is bigger than expected information rents if $\frac{\frac{1}{2} f_1^L}{1 + \frac{1}{2} f_1^L} > d$. This is the inequality we know from Observation 1.6.

The following Figure illustrates that implementation is more costly, i.e., the minimal subsidy goes up, if we add types to the setting. The lighter shaded area T_0, known from Figure 1.4, gives the set of tuples (f^L, d), for which trade with two types is possible. If we add a third type, the set T_1 (the darker shaded area) gives the combinations for which efficient trade is possible, where

$$T_1 = \left\{ (f^L, d) : f_1^L > \frac{\delta_1^H - \theta_1^L}{\theta_1^M - \delta_1^M} = \frac{2d}{1-d} \right\} ,$$

so that $T_1 \subseteq T_0$.

Given Definition 1.3, we can express the new type θ_1^M in terms of the types θ_1^L and θ_1^H. Therefore we can compare the sets T_0 and T_1 in a coordinate system, with axes labelled d and f^L.

Figure 1.6: Bilateral trade – Comparison between 'round 0' and 'round 1'

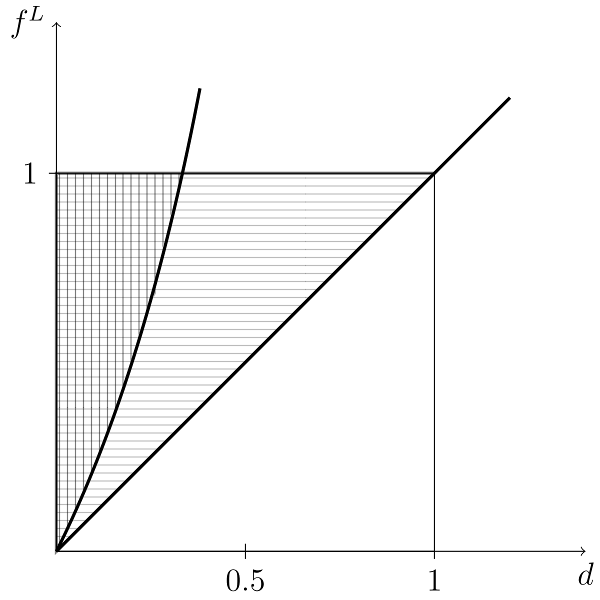


Figure 1.6 implies that if efficient trade is possible in round 1, then it is as well possible in 'round 0'. The opposite is, however, not true. Imagine that the ratio of overlapping is 0.5 in 'round 0'. Then, there is no value f^L can take, such that efficient bilateral trade is still possible.

Counterexamples. In order to illustrate the importance of Definition 1.2 and 1.3, we present examples that violate the 'Uniform Extension' procedure.

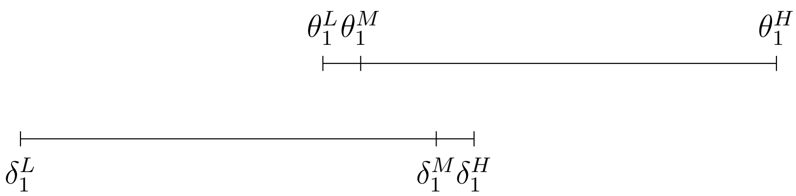
Observation 1.7. *Suppose parameters are as in Figure 1.5. Implementation in 'round 1' is less costly then in 'round 0' if*

$$f_1^L > \frac{2f_0^L}{1+f_0^L} .$$

For the purpose of illustration, assume that $f_0^L = \frac{1}{5}$. In 'round 0' with a binary type set, Observation 1.1 states that if $\frac{1}{5} > d$, then efficient bilateral trade is possible. Suppose now that $f_1^L = \frac{1}{2}$, according to Observation 1.7 efficient bilateral is possible if $\frac{1}{2} > d$. Hence, $T_1 \supseteq T_0$. If the probability of the low type increases over rounds, so that the second condition of Definition 1.2 is violated, then efficient bilateral trade gets less costly. For this result, the monotone hazard rate assumption plays no role. The probabilities on types f_1^H and f_1^M can be chosen such that the monotone hazard rate either holds or is violated.

Consider next the case, where the introduced third type is positioned such that $\theta_1^M < \delta_1^M$.

Figure 1.7: Violating the 'Uniform Extension' procedure



Applying Proposition 1.4, the following Observation shows a condition for reaching efficiency.

Observation 1.8. *Suppose parameters are as in Figure 1.7. There is a social choice function which is efficient, implementable as Bayes-Nash equilibrium, and yields non-negative material payoff for every type of agent if and only if*

$$f_1^L + f_1^M > d \,.$$

Comparing Observation 1.6 with Observation 1.8 illustrates that violating the first condition in Definition 1.2 makes it less costly to achieve efficient bilateral trade.

1.7.2 Introducing many types

To understand what leads to the Myerson and Satterthwaite (1983) impossibility result when the number of types is getting large, we must analyze the following characteristics of their model: type sets are closed and connected sets.[20] Intuitively, this means that there are no gaps in the type set. When we add new types, we have to respect this characteristic of the model and achieve it in the limit.

[20] A set B is closed if it contains the limit points of every possible sequence $x_n \in B$, with $x_n \to x$, for $n \to \infty$. A set B is connected if it cannot be represented as the union of two or more disjoint non-empty open subsets.

Definition 1.4. (Uniform Extension Type II) *Consider the 'Uniform Extension Type I' Definition. Additionally, equal probability mass is put on all types, in every round.*

Making use of Definition 1.4, we can show that a discrete specification of the model approaches in the limit the continuous specification of the model.

Proposition 1.6. *Suppose Assumption 1.1 holds. Consider the 'Uniform Extension Type II' procedure. There is no social choice function which is efficient, implementable as Bayes-Nash equilibrium and yields non-negative material payoff for every type of agent if* $k \to \infty$.

The uniform extension procedure is one way to assure that the discrete specification of the bilateral trade problem approaches the continuous specification of Myerson and Satterthwaite (1983). In the proof we show that an increase in the number of types leads to an increase of the minimal subsidy. If sufficiently many types are introduced, then there exists a round $k < \infty$, where the minimal subsidy is positive, i.e., it is impossible to reach efficient bilateral trade.

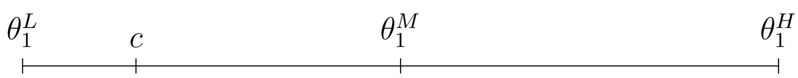
Public Good provision. Similar to the bilateral trade problem, the condition relates the type distribution to the relative position of the per capita costs for the public good. Therefore, we define

$$e = \frac{c - \theta^L}{\theta^H - c} \, .$$

Since $\theta^L < c < \theta^H$, the per capita costs c split the type set into two sub-intervals: the first, where the costs exceed the low valuation, and the second, where the costs lie below the high valuation. Thus, e is a measure for the net gains of free-riding. If consumer i has high preferences for the public good, he faces the following trade-off: if he announces his type truthfully, the public good will be provided for sure but he will have to pay higher transfers than if he is lying. However, if he understates his preferences, he will have to pay lower transfers but will risk that the public good will not be provided.

As for the bilateral trade setting, we use Definition 1.3 to show that implementation is getting more costly when more types are introduced. Analogously to the bilateral trade example, we consider the case where the third type is positioned such that $\frac{1}{2}\left(\theta_1^L + \theta_1^M\right) > c$.

Figure 1.8: Three consumer types



In order to efficiently provide the public good, when there are three consumer types, the inequality in the following Observation needs to be satisfied.

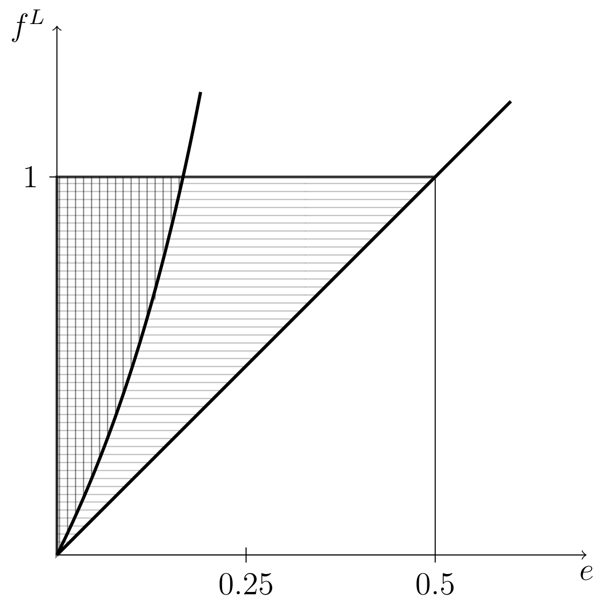
Observation 1.9. *Suppose parameters are as in Figure 1.8. There is a social choice function which is efficient, implementable as Bayes-Nash equilibrium and yields non-negative utility for every type of consumer, if and only if*

$$f_1^L > \frac{c - \theta_1^L}{\theta_1^M - c} \ .$$

As for the private good example, the expected surplus effect for the high valuation consumer equals the expected information rent effect. Thus, the parameter for the high-valuation buyer does not show up in Observation 1.9.

The comparison of Observations 1.4 and 1.9 shows that implementation is getting more costly with three consumer types. Even for one additional type, the possibility condition for efficient public good provision gets more restrictive. Graphically, a comparison of 'round 0' and 'round 1' resembles the graphic in the bilateral trade example.

Figure 1.9: Public good – Comparison between 'round 0' and 'round 1'



The gray shaded area D_0 in Figure 1.9 gives every combination of f^L and e, such that the public good is efficiently provided, i.e., Observation 1.9 holds,

$$D_0 = \left\{ (f^L, e) : f^L > e = \frac{c - \theta_0^L}{\theta_0^H - c} \right\} \ .$$

Analogously, D_1 gives the tuples (f^L, e), for which efficiency is achieved, if we have three possible types,

$$D_1 = \left\{ (f^L, e) : f^L > \frac{c - \theta_1^L}{\theta_1^M - c} = \frac{2e}{1-e} \right\} .$$

Hence, $D_1 \subseteq D_0$.

If we introduce a continuous type set to the public good setting with only two consumers, we can show that efficient public good provision is impossible.

Proposition 1.7. *If consumers' valuation for the public good are independently drawn from the interval $[\theta^L, \theta^H]$, then there is no Bayesian incentive compatible social choice function that is efficient and yields non-negative utility for every type of agent.*

The proof is an adaption of Myerson and Satterthwaite (1983) to the public good provision setting. Comparing Propositions 1.6 and 1.7, we can see that the assumption of a continuous type set makes both efficient bilateral trade as well as public good provision impossible. For the latter, the assumption of a large economy is hence not necessary for the impossibility to reach efficient public good provision.

1.7.3 General convergence

As we have seen in the previous subsections, there exist qualitatively different results concerning implementability in discrete and continuous environments. We analyze now how the results for discrete settings relate to the results in continuous settings. To answer this question, we reconstruct our given environment, such that we are able to link discrete and continuous environments technically and can specify how the respective implementation conditions are related. This reconstructed environment enables us to generalize our results concerning the effects of changing type sets or number of agents on implementability.

The economic environment in our paper contains a set of agents, a type set for every agent, and density functions. To combine the decisive factors for implementability, we bundle them in the definition of an *economy*. This generates a set of different economies whose elements can differ in the number of types, the number of agents, the density functions, or the allocation functions. On this set, we define sequences and the meaning of convergence. We frame requirements for this environment under which it is possible to align the implementability conditions in different economies. Thereby we find that under some assumptions concerning the *similarity* of these two environments (type sets, density functions and allocation functions), the implementability result in discrete settings approaches the result in the continuous setting. We are able to approach the Myerson and Satterthwaite (1983) result, even if we start with a parametrization, for which efficient trade is possible. We show conditions that need to be met in order to guarantee that we end up in an environment, that equals the setting of Myerson and Satterthwaite

(1983). This means, that it is not a sufficient condition to have infinitely many types in the bilateral trade setting to get the impossibility result. The position of the types in the type set and the probabilities matter. A first intuitive evidence for this relation is given by Observation 1.1. To show that even in case of infinitely many types efficient trade can take place, we will give some counterexamples to justify the conditions that we introduce afterwards. Thereafter we formulate the environment of economies for which we define the needed convergence and phrase the conditions, under which the infinite number of types is a sufficient condition for the impossibility of implementation. Even if the analysis of the counterexamples is done for the bilateral trade setting, the derived results apply to every independent private value setting. There will be a general result, when and under which conditions it is possible to link discrete and continuous (with respect to the type set) settings.

Consumption Economy.

Definition 1.5. *We define a consumption economy (I, Θ, f, q) as the tuple consisting of a consumer set $I = \{1, \ldots, n\}$, the Cartesian product of type sets (one type set for every consumer i) $\Theta = \times_{i \in I} \Theta_i$ with the corresponding density functions $f_i : \Theta_i \to [0, 1]$, where $f := (f_0, \ldots, f_n)$, and the allocation rule $q : \Theta \to \mathbb{R}_+$, which maps types into allocations.*

The economies can differ in the number of agents, the type set, the density function, or the allocation rule. To relate different economies, we have to introduce convergence to our set of economies. Especially, we want to link the finite and the infinite economy, where we consider finite and infinite type sets. For this, we define a sequence of consumption economies as given by $(I_k, \Theta_k, f_k, q_k)_{k \in \mathbb{N}}$, where I_k is a sequence of consumer sets, Θ_k a sequence of type set, f_k a sequence of density-functions, and q_k a sequence of allocation functions. We denote the infinite setting by $(I_\infty, \Theta_\infty, f_\infty, q_\infty)$, where we have an infinite number of agents and types. f_∞, q_∞ are the corresponding density or allocations functions. As our economy contains different formal objects, namely sets and functions, we have to define convergence component-wise.

Definition 1.6. *A sequence of consumption economies $(I_k, \Theta_k, f_k, q_k)_{k \in \mathbb{N}}$ converges to a limit economy (I, Θ, f, q), if the sequences of consumers, type sets, density functions and allocation functions converge to the respective limit given by the limit consumption economy.*

When we look at the convergence of an economy, we require that every element of the economy converges. In the following we define in detail the components of an economy sequence and the corresponding concept of convergence.

Consumers. Let $I_k = \{1, \ldots, n_k\}$ be the set of consumers in round k. We say, that $(I_k)_{k \in \mathbb{N}} \to I_\infty$ if the number of consumers is increasing for $k \to \infty$.

I_k affects the dimension of the other components of an economy. Since we need a type set for every agent, the type set component in the economy consists of a Cartesian product over the type sets for every single agent. The domain of the density and the allocation function change accordingly.

Type sets. Fix a consumer set $I = \{1, \dots, n\}$. An element Θ_k of the sequence $(\Theta_k)_{k\in\mathbb{N}}$ is the Cartesian product over the consumers' type sets in round k. Since we assume symmetry, every consumer $i \in I$ in round k has the same type set. We call this type set Θ_{ki}. The component Θ_k of an economy in a sequence is hence given by $\Theta_k = (\Theta_{ki})^n$.

Since the type set component of the economy is a product over many type sets, we define the convergence of this Cartesian product element-wise, hence consumer-wise. This means, that for every consumer $i \in I$, the sequence of his type sets has to converge. We use the topological concept of density to define convergence.[21]

Definition 1.7. *Fix a consumer set I. If we write $(\Theta_k)_{k\in\mathbb{N}} \to \Theta$, this means $(\Theta_{ki})_{k\in\mathbb{N}} \to \Theta_i$, $k \to \infty$ for every $i \in I$. At that, Θ_i gives the limit of the sequence of type sets that is assigned to consumer $i \in I$. Thereby, it holds, that Θ_i is the limit of the sequence $(\Theta_{ki})_{k\in\mathbb{N}}$, if and only if, there exists a K, such that for all $k \geq K$, Θ_{ki} lies dense in $\Theta_i, \forall\, i \in I$.*

The sequence of the Cartesian products over type sets is generated by adding new types to every single type set of every consumer in a round k.[22] We add the same finite number of types to every one's type set. As mentioned above, whenever the number of consumers or the type sets changes, the density function and the allocation rule has to change as well. In every round k and for every consumer $i \in I_k$, the sum over all probabilities of types is 1. Thus, we face the following issue: whenever we add new types to type sets, we do not only need to assign a probability to them but also have to change probabilities of existing types, i.e., take probability mass of the old types and redistribute it to the new types. Thereby we make the assumption that for every single type, the probability cannot increase over rounds. This means, that the collected probabilities of the old types are shared exclusively among the newly added types.

Density Functions. Fix a consumer set $I = \{1, \dots, n\}$. The sequence $(f_k)_{k\in\mathbb{N}}$ consists of $f_k = (f_{k0}, \dots, f_{kn})$, where f_{ki} is the density function of consumer $i \in I$ in round k over his type set Θ_{ki}. The change in the type sets means, that we face a change in the domains of the density functions, such that the standard concept of convergence for function sequences cannot be used. Thus, for our purpose we require that for every type in the limit type set, that is also contained in one of the finite type sets, the difference between the probability assigned by the limit density function and the sequence of density functions has to become arbitrarily small if the number of rounds becomes large.

[21] A set $A \subseteq B$ lies dense in the set B, if $\overline{A} = B$.

[22] Mathematically this means: $\Theta_{ki} = \Theta_{(k-1)i} \cup \{\theta_m | m = 1, \dots, M; M < \infty\}$.

Definition 1.8. *Fix a consumer set* $I = \{1, \ldots, n\}$. *We say* $(f_k)_{k\in\mathbb{N}} \to f, f = (f_1, \ldots, f_n) : \Theta \to [0,1]^n$ *if for any* $\theta \in \Theta$*, that lies also in* Θ_k*, for some* $k < \infty$*, there exists a* $K < \infty$*, such that for all* $k \geq K$ *it holds that* $|f(\theta) - f_k(\theta)| < \epsilon$*. This has to hold component-wise. Hence, for every consumer* κ*:* $|f_i(\theta) - f_{ki}(\theta)| < \epsilon, i = 1, \ldots, n, k \geq K$.

For the allocation functions, we do not have to change the outcomes in round k for the types that are contained in $\Theta_k \cap \Theta_{k-1}$. Since we consider first-best environments, we only have to define the outcomes assigned to new types. Since the domain changes for allocation functions, we require analogously for the density functions that for every type in the limit type set, that is also contained in one of the finite type sets, the difference between the outcome assigned by the limit allocation function and sequence of allocation functions has to become arbitrarily small if the number of rounds becomes large.

Allocation Functions. Fix a consumer set $I = \{1, \ldots, n\}$. The sequence of allocation functions is given by $(q_k)_{k\in\mathbb{N}} : (\Theta_k)_{k\in\mathbb{N}} \to (z_k)_{k\in\mathbb{N}}, z_k \in \mathbb{R}$, where $q_k : \Theta_k = (\Theta_{ki})^n \to \mathbb{R}_+$.

Definition 1.9. *We say* $(q_k)_{k\in\mathbb{N}} \to q : \Theta \to \mathbb{R}_+$ *if for any* $\theta \in \Theta$*, that lies also in* Θ_k*, for some* $k < \infty$*, there exists a* $K < \infty$*, such that for all* $k \geq K$ *it holds that* $|q(\theta) - q_k(\theta)| < \epsilon$.

For the public good example, we have already seen in Proposition 1.5 that a finite but sufficiently large number of consumers yields the same result as the Theorem by Mailath and Postlewaite (1990) for infinitely many consumers: the public good will not be provided. To apply these concepts to the private good example, we need the analogous definitions for the producer side. The respective definitions and concepts of convergence follow the same logic as for the consumer side and can be found in Appendix 1.D.

1.7.4 Convergence of type set

We are now able to bring together the implementation condition in discrete and continuous settings. The terms 'discrete' and 'continuous' refer to the set of types – for the private good application. We found that the implementability conditions in discrete environments approached the conditions in continuous environments such that the qualitative results concerning efficient implementation converge. Now, we can be more precise on what we mean by 'approach the results'.

As we know from Section 1.5, efficient implementation is possible when consumers' expected transfers exceed firms' expected revenues. The expected transfers and revenues are calculated, taking the respective incentive compatibility and participation constraints into account. This logic also applies for continuous settings. To compare the implementability conditions for private goods in discrete and continuous environments, we increase the number of types, while holding the set of agents fixed. We find, that with the given definitions of economies and convergence, the expected payments for a finite type set converge to the expected payments in the continuous economy. The payments that we get for a mechanism (q, t) in a continuous environment that fulfills incentive compatibility and participation constraints can be found in equation (1.10). With the same arguments, we get the convergence of the revenues for the producer side, see Appendix 1.D.

Proposition 1.8. *Let $(I, \Theta_k, f_k, q_k)_{k\in\mathbb{N}} \to (I, \Theta_\infty, f_\infty, q_\infty)$, for $k \to \infty$. Then it holds that for every consumer $i \in I$: For every $\epsilon > 0 \quad \exists K : \forall\ k \geq K$*

$$\Bigg| \mathbb{E}_{(\theta_{ki})} \left[\left(v(\theta_{ki}, q_{ki}(\theta, \delta)) - \frac{1 - F_k(\theta_{ki})}{f_k(\theta_{ki})} \sum_{l=1}^{s} f(\theta_{ki}^l)\{v(\theta_{ki}^{l+1}, q_{ki}(\theta, \delta)) - v(\theta_{ki}^l, q_{ki}(\theta, \delta))\} \right) \right] - \mathbb{E}_{(\theta_{\infty i})} \left[v(\theta_{\infty i}, q_{\infty i}(\theta, \delta)) - \int_{\theta^0_{\infty i}}^{\theta^s_{\infty i}} f(\theta_{\infty i}) v_1(\theta_{\infty i}, q_{\infty i}(\theta)) \frac{1 - F_\infty(\theta_{\infty i})}{f_\infty(\theta_{\infty i})} d\theta_{\infty i} \right] \Bigg| < \epsilon.$$

Thus, for the private good application, we get equivalent conditions for efficient implementation of finite but sufficiently large type sets and the infinite type set. Note, that this result does not require a monotonic change in the implementability from round to round: the convergence $(I, \Theta_k, f_k, q_k)_{k\in\mathbb{N}} \to (I, \Theta_\infty, f_\infty, q_\infty)$ does not rule out that there exists a finite number of rounds where the implementability develops in the opposite direction of the limit case. With respect to our applications, this means, that even for the counterexample 1.7 where implementation gets less costly in round 1, the impossibility result will hold in the limit if the aforementioned convergence is fulfilled. Generally spoken, the specific extension procedure is irrelevant for the result in proposition 1.8.

For public goods, we know from Mailath and Postlewaite (1990) that for an infinite number of consumer and a finite type set, the public good will never be provided if incentive compatibility, participation, and the resource constraints have to be fulfilled. As shown in Proposition 1.6, we find that for a finite but sufficiently large set of consumers, the same result is true: the public good will not be provided. While increasing the number of agents, the number of types remains unchanged.

Summarizing, we changed the element of the economies in the private and the public good example that had infinite dimension. For the bilateral trade application this is given by the type set, whereas it is the set of agents in the public good example. If we approach these infinite components of the economies (and the resulting changes in the density and allocation functions) by a sequence of finite but increasing elements, we can generate the same implementation result for the finite environments as for the original infinite cases.

The stated convergence result does also apply to every possible application of the independent private values model that fulfills the conditions. Thus, this result enables us to link results for discrete and infinite settings. Thereby, either type sets or agent sets (or both) can be varied from a finite number of elements to an infinite set.

1.8 Concluding remarks

In this paper, we investigate the independent private values model when types are discrete. We use this model for the analysis of how impossibility results by Myerson and Satterthwaite (1983) and Mailath and Postlewaite (1990) are affected by the specification of the number of individuals and the specification of the type set. The existing literature on this topic neglects the question of how the discrete specification and the continuous specification of the independent private values model relate to each other. Our analysis provides a framework to study the convergence.

Our analysis yields the following key insights: First, the impossibility results for efficient bilateral trade and efficient public good provision vanish when a binary type set is considered. Moreover, we find that the Mailath and Postlewaite (1990) result extends to any model with a discrete set of types. The Myerson and Satterthwaite (1983) result, by contrast, extends only to a model with a large but finite number of types.

Second, the discrete version of the independent private values model leads to the same outcomes as the continuous version of the model if many types are introduced in the right way. We discuss various factors that have an influence on the convergence. This analysis does not support the presumption that the increase in the type set alone leads to the convergence of both model specifications.

The analysis was made possible by a combination of insights from the non-linear pricing literature and mechanism design. We believe that the applicability of a large class of problems that have been studied in the empirical economics and behavioral economics literature can be simplified by using the discrete specification of the independent private values model.

Appendix 1.A Preliminaries

Lemma 1.1. *For all i, the incentive constraints in (IC_C) hold if the following local incentive constraints are satisfied: For any $l < s$,*

$$V(\theta_i^l \mid \theta_i^l, q_i(\theta_{-i}, \theta_i^l)) - T(\theta_i^l) \geq V(\theta_i^{l+1} \mid \theta_i^l, q_i(\theta_{-i}, \theta_i^{l+1})) - T(\theta_i^{l+1}) \ , \tag{1.11}$$

and for all $l > 0$,

$$V(\theta_i^l \mid \theta_i^l, q_i(\theta_{-i}, \theta_i^l)) - T(\theta_i^l) \geq V(\theta_i^{l-1} \mid \theta_i^l, q_i(\theta_{-i}, \theta_i^{l-1})) - T(\theta_i^{l-1}) \ . \tag{1.12}$$

Moreover, the local incentive constraints imply that, for all i, a weak monotonicity condition as defined by Müller et al. (2007) holds.

Proof: We first show that the weak monotonicity condition of Müller et al. (2007) holds for each i and l. Equation (1.12) as stated in the Lemma for $\theta_i = \theta_i^{l+1}$:

$$V(\theta_i^{l+1} \mid \theta_i^{l+1}, q_i(\theta_{-i}, \theta_i^{l+1})) - T(\theta_i^{l+1}) \geq V(\theta_i^l \mid \theta_i^{l+1}, q_i(\theta_{-i}, \theta_i^l)) - T(\theta_i^l) \ .$$

Adding equation (1.11) as stated above yields:

$$\begin{aligned} V(\theta_i^l \mid \theta_i^l, q_i(\theta_{-i}, \theta_i^l)) + V(\theta_i^{l+1} \mid \theta_i^{l+1}, q_i(\theta_{-i}, \theta_i^{l+1})) - T(\theta_i^l) - T(\theta_i^{l+1}) \geq \\ V(\theta_i^{l+1} \mid \theta_i^l, q_i(\theta_{-i}, \theta_i^{l+1})) + V(\theta_i^l \mid \theta_i^{l+1}, q_i(\theta_{-i}, \theta_i^l)) - T(\theta_i^{l+1}) - T(\theta_i^l) \end{aligned}$$

$$\begin{aligned} \Leftrightarrow V(\theta_i^l \mid \theta_i^l, q_i(\theta_{-i}, \theta_i^l)) + V(\theta_i^{l+1} \mid \theta_i^{l+1}, q_i(\theta_{-i}, \theta_i^{l+1})) \geq \\ V(\theta_i^{l+1} \mid \theta_i^l, q_i(\theta_{-i}, \theta_i^{l+1})) + V(\theta_i^l \mid \theta_i^{l+1}, q_i(\theta_{-i}, \theta_i^l)) \end{aligned}$$

$$\begin{aligned} \Leftrightarrow V(\theta_i^{l+1} \mid \theta_i^{l+1}, q_i(\theta_{-i}, \theta_i^{l+1})) - V(\theta_i^l \mid \theta_i^{l+1}, q_i(\theta_{-i}, \theta_i^l)) \geq \\ V(\theta_i^{l+1} \mid \theta_i^l, q_i(\theta_{-i}, \theta_i^{l+1})) - V(\theta_i^l \mid \theta_i^l, q_i(\theta_{-i}, \theta_i^l)) \end{aligned} \tag{WM}$$

According to Müller et al. (2007) the condition in equation (WM) is called weak monotonicity. If preferences are linear in the type, incentive compatibility implies not only weak monotonicity but also monotonicity.
We show that equation (1.11) implies that

$$V(\theta_i^l \mid \theta_i^l, q_i(\theta_{-i}, \theta_i^l)) - T(\theta_i^l) \geq V(\theta_i^{l+2} \mid \theta_i^l, q_i(\theta_{-i}, \theta_i^{l+2})) - T(\theta_i^{l+2}) \ .$$

To see this, rewrite equation (1.11) as

$$\begin{aligned}V(\theta_i^l \mid \theta_i^l, q_i(\theta_{-i}, \theta_i^l)) - T(\theta_i^l) \geq \\ & V(\theta_i^{l+1} \mid \theta_i^{l+1}, q_i(\theta_{-i}, \theta_i^{l+1})) - T(\theta_i^{l+1}) \\ & - \left[V(\theta_i^{l+1} \mid \theta_i^{l+1}, q_i(\theta_{-i}, \theta_i^{l+1})) - V(\theta_i^{l+1} \mid \theta_i^l, q_i(\theta_{-i}, \theta_i^{l+1}))\right] \ .\end{aligned}$$

Since (WM) holds for all l, we have

$$\begin{aligned}V(\theta_i^l \mid \theta_i^l, q_i(\theta_{-i}, \theta_i^l)) - T(\theta_i^l) \geq \\ & V(\theta_i^{l+1} \mid \theta_i^{l+1}, q_i(\theta_{-i}, \theta_i^{l+1})) - T(\theta_i^{l+1}) \\ & - \left[V(\theta_i^{l+2} \mid \theta_i^{l+1}, q_i(\theta_{-i}, \theta_i^{l+2})) - V(\theta_i^{l+2} \mid \theta_i^l, q_i(\theta_{-i}, \theta_i^{l+2}))\right] \ .\end{aligned}$$

Moreover, condition (1.11) for $\theta_i = \theta_i^{l+1}$ is

$$V(\theta_i^{l+1} \mid \theta_i^{l+1}, q_i(\theta_{-i}, \theta_i^{l+1})) - T(\theta_i^{l+1}) \geq V(\theta_i^{l+2} \mid \theta_i^{l+1}, q_i(\theta_{-i}, \theta_i^{l+2})) - T(\theta_i^{l+1}) \ .$$

Adding the last two inequalities yields

$$V(\theta_i^l \mid \theta_i^l, q_i(\theta_{-i}, \theta_i^l)) - T(\theta_i^l) \geq V(\theta_i^{l+2} \mid \theta_i^l, q_i(\theta_{-i}, \theta_i^{l+2})) - T(\theta_i^{l+1}) \ .$$

Hence, an individual with preference parameter θ_i^l does not benefit from announcing θ_i^{l+2}. Iterating this argument once more establishes that this individual does neither benefit from announcing θ_i^{l+3}, etc. The proof that an individual with preference parameter θ_i^l does not benefit from announcing θ_i^{l-j}, for any $j \geq 1$ is analogous and left to the reader.

Lemma 1.2. *Suppose that, for some i, all local downward incentive compatibility constraints are binding and that the weak monotonicity condition (WM) holds. Then all incentive constraints of i are satisfied.*

Proof: If all local downward incentive constraints are binding for individual i, this implies that, for all $l \geq 1$,

$$T(\theta_i^l) = \sum_{k=1}^{l} \{V(\theta_i^k \mid \theta_i^k, q_i(\theta_{-i}, \theta_i^k)) - V(\theta_i^{k-1} \mid \theta_i^k, q_i(\theta_{-i}, \theta_i^{k-1}))\} + T(\theta_i^0) \ .$$

For all $l > 0$, the equation can be equivalently written as

$$\begin{aligned}T(\theta_i^l) =& V(\theta_i^l \mid \theta_i^l, q_i(\theta_{-i}, \theta_i^l)) \\ & - \sum_{k=0}^{l-1} \{V(\theta_i^k \mid \theta_i^{k+1}, q_i(\theta_{-i}, \theta_i^k)) - V(\theta_i^k \mid \theta_i^k, q_i(\theta_{-i}, \theta_i^k))\} + T(\theta_i^0) \ .\end{aligned}$$

To establish incentive compatibility, Lemma 1.1 implies that it suffices to show that all local upward incentive constraints are satisfied, i.e., for all l,

$$V(\theta_i^l \mid \theta_i^l, q_i(\theta_{-i}, \theta_i^l)) - T(\theta_i^l) \geq V(\theta_i^{l+1} \mid \theta_i^l, q_i(\theta_{-i}, \theta_i^{l+1})) - T(\theta_i^{l+1}) \ ,$$

or equivalently,

$$\begin{aligned} V(\theta_i^l \mid \theta_i^l, q_i(\theta_{-i}, \theta_i^l)) - T(\theta_i^l) \geq & \\ & V(\theta_i^{l+1} \mid \theta_i^{l+1}, q_i(\theta_{-i}, \theta_i^{l+1})) - T(\theta_i^{l+1}) \\ & - \left[V(\theta_i^{l+1} \mid \theta_i^{l+1}, q_i(\theta_{-i}, \theta_i^{l+1})) - V(\theta_i^{l+1} \mid \theta_i^l, q_i(\theta_{-i}, \theta_i^{l+1}))\right] \ . \end{aligned}$$

By $T(\theta_i^l) = V(\theta_i^l \mid \theta_i^l, q_i(\theta_{-i}, \theta_i^l)) - \sum_{k=0}^{l-1}\{V(\theta_i^k \mid \theta_i^{k+1}, q_i(\theta_{-i}, \theta_i^k)) - V(\theta_i^k \mid \theta_i^k, q_i(\theta_{-i}, \theta_i^k))\}$, this inequality can be written as

$$\begin{aligned} \sum_{k=0}^{l-1}\{V(\theta_i^k \mid \theta_i^{k+1}, q_i(\theta_{-i}, \theta_i^k)) - V(\theta_i^k \mid \theta_i^k, q_i(\theta_{-i}, \theta_i^k))\} \geq & \\ & \sum_{k=0}^{l}\{V(\theta_i^k \mid \theta_i^{k+1}, q_i(\theta_{-i}, \theta_i^k)) - V(\theta_i^k \mid \theta_i^k, q_i(\theta_{-i}, \theta_i^k))\} \\ & - \left[V(\theta_i^{l+1} \mid \theta_i^{l+1}, q_i(\theta_{-i}, \theta_i^{l+1})) - V(\theta_i^{l+1} \mid \theta_i^l, q_i(\theta_{-i}, \theta_i^{l+1}))\right] \ , \end{aligned}$$

or

$$\begin{aligned} V(\theta_i^{l+1} \mid \theta_i^{l+1}, q_i(\theta_{-i}, \theta_i^{l+1})) - V(\theta_i^{l+1} \mid \theta_i^l, q_i(\theta_{-i}, \theta_i^{l+1})) \geq & \\ V(\theta_i^l \mid \theta_i^{l+1}, q_i(\theta_{-i}, \theta_i^l)) - V(\theta_i^l \mid \theta_i^l, q_i(\theta_{-i}, \theta_i^l)), & \end{aligned}$$

which is equivalent to weak monotonicity which is satisfied by assumption.

Lemma 1.3. *If for individual i, all local downward incentive compatibility constraints are binding, then the expected utility of individual i from ex ante perspective is given by*

$$\mathbb{E}_{(\theta)}[v(\theta_i, q_i(\theta)) - t_i(\theta)] = \mathbb{E}_{(\theta)}\left[\sum_{l=1}^{s} f(\theta_i^l)\{v(\theta_i^{l+1}, q_i(\theta)) - v(\theta_i^l, q_i(\theta))\}\frac{1 - F(\theta_i)}{f(\theta_i)}\right] - T(\theta_i^0) \ .$$

Proof: Equation

$$\begin{aligned} T(\theta_i^l) = & V(\theta_i^l \mid \theta_i^l, q_i(\theta_{-i}, \theta_i^l)) \\ & - \sum_{k=0}^{l-1}\{V(\theta_i^k \mid \theta_i^{k+1}, q_i(\theta_{-i}, \theta_i^k)) - V(\theta_i^k \mid \theta_i^k, q_i(\theta_{-i}, \theta_i^k))\} + T(\theta_i^0) \ , \end{aligned}$$

in the proof of Lemma 1.2 and the law of iterated expectations imply that,

$$
\begin{aligned}
\mathbb{E}_{(\theta)}[t_i(\theta)] &= \sum_{j=0}^{s} f^j T(\theta_i^j) \\
&= \sum_{j=0}^{s} f^j \Big[\mathbb{E}_{(\theta_{-i})}[v(\theta_i^j, q_i(\theta_{-i}, \theta_i^j))] \\
&\quad - \sum_{k=0}^{j-1} \{ V(\theta_i^k \mid \theta_i^{k+1}, q_i(\theta_{-i}, \theta_i^k)) - V(\theta_i^k \mid \theta_i^k, q_i(\theta_{-i}, \theta_i^k)) \} + T(\theta_i^0) \Big] \\
&= \mathbb{E}_{(\theta)}[v(\theta_i, q_i(\theta))] + T(\theta_i^0) \\
&\quad - \sum_{j=1}^{s} f^j \sum_{k=0}^{j-1} \{ V(\theta_i^k \mid \theta_i^{k+1}, q_i(\theta_{-i}, \theta_i^k)) - V(\theta_i^k \mid \theta_i^k, q_i(\theta_{-i}, \theta_i^k)) \} \\
&= \mathbb{E}_{(\theta)}[v(\theta_i, q_i(\theta))] + T(\theta_i^0) \\
&\quad - \sum_{j=1}^{s} \left(1 - \sum_{k=0}^{j} f^k \right) \{ V(\theta_i^j \mid \theta_i^{j+1}, q_i(\theta_{-i}, \theta_i^j)) - V(\theta_i^j \mid \theta_i^j, q_i(\theta_{-i}, \theta_i^j)) \} \\
&= \mathbb{E}_{(\theta)}[v(\theta_i, q_i(\theta))] + T(\theta_i^0) \\
&\quad - \sum_{j=1}^{s} f^j \{ V(\theta_i^{j+1} \mid \theta_i^j, q_i(\theta_{-i}, \theta_i^{j+1})) - V(\theta_i^j \mid \theta_i^j, q_i(\theta_{-i}, \theta_i^j)) \} \frac{1 - \sum_{k=0}^{j} f^k}{f^j} \\
&= \mathbb{E}_{(\theta)}[v(\theta_i, q_i(\theta))] - \mathbb{E}_{(\theta)} \left[\sum_{l=0}^{s} f(\theta_i^l) \{ v(\theta_i^{l+1}, q_i(\theta)) - v(\theta_i^l, q_i(\theta)) \} \frac{1 - F(\theta_i)}{f(\theta_i)} \right] + T(\theta_i^0)
\end{aligned}
$$

Lemma 1.4. *For all i, if the (PC_C) is satisfied for $\theta_i = \theta_i^0$, then it is satisfied as well for $\theta_i \neq \theta_i^0$.*

Proof: Let $\theta_i \neq \theta_i^0$. Then by (IC_C)

$$V(\theta_i^l \mid \theta_i^l, q_i(\theta_{-i}, \theta_i^l)) - T(\theta_i^l) \geq V(\theta_i^0 \mid \theta_i^l, q_i(\theta_{-i}, \theta_i^0)) - T(\theta_i^0) \ .$$

Moreover, $\theta_i > \theta_i^0$ implies that the right-hand side of this inequality exceeds

$$V(\theta_i^0 \mid \theta_i^0, q_i(\theta_{-i}, \theta_i^0)) - T(\theta_i^0) \ ,$$

which is non-negative by (PC_C) for $\theta_i = \theta_i^0$. This proves that (PC_C) is not binding for $\theta_i \neq \theta_i^0$.

Lemma 1.5. *Let q_i be an arbitrary given provision rule. Consider the problem of choosing a mechanism $(t_1, ..., t_n)$ in order to maximize total transfers*

$$\mathbb{E}_{(\theta)} \left[\sum_{i=1}^{n} t_i(\theta) \right] \ ,$$

subject to the incentive compatibility constraints in (IC_C) and the interim participation constraints in (PC_C). At a solution to this problem, the participation constraint in (PC_C) is binding for $\theta_i = \theta_i^0$ and slack otherwise.

Proof: By Lemma 1.4 we only need to show that it is binding for $\theta_i = \theta_i^0$. We show that it is possible to increase the expected payments of individual i in an incentive compatible way if, for some i, the participation constraint for $\theta_i = \theta_i^0$ does not hold as an equality. It is instructive to rewrite the incentive compatibility constraints in (IC_C) as follows: For each i, for each $\theta_i^l \in \Theta_i$, and for each $\hat{\theta}_i \in \Theta_i$,

$$V(\theta_i^l \mid \theta_i^l, q_i(\theta_{-i}, \theta_i^l)) - V(\hat{\theta}_i^l \mid \theta_i^l, q_i(\theta_{-i}, \hat{\theta}_i^l)) \geq T(\theta_i^l) - T(\hat{\theta}_i) \ .$$

Consider a new payment rule for individual i such that for each $\theta_i \in \Theta_i$, $T(\theta_i^l)$ increases by some $\epsilon > 0$, this implies that the right-hand side of the incentive constraints states above remains constant, i.e., the increase of i's expected payments does not violate the incentive compatibility. Since revenue increases in the expected payments of individual i, the revenue maximizing mechanism must be such that a biding participation constraint for $\theta_i = \theta_i^0$ prevents a further increase of individual i's payments.

Lemma 1.6. *Let q_i be an arbitrary given provision rule. Consider the "relaxed problem" of choosing a mechanism $(t_1, ..., t_n)$ in order to maximize total transfers*

$$\mathbb{E}_{(\theta)}\left[\sum_{i=1}^{n} t_i(\theta)\right] \ ,$$

subject to the downward incentive compatibility constraints in (IC_C) and the ex interim participation constraints in (PC_C). At a solution to this problem, all downward incentive constraints are binding, and the participation constraint in (PC_C) is binding for $\theta_i = \theta_i^0$ and slack otherwise.

Proof: It is straightforward to verify that, for all i, all downward incentive constraints are binding. Otherwise the expected payments of some individual could be increased without violating any one of the constraints of the relaxed problem. It remains to be shown that, for all i, the participation constraint in (PC_C) is binding for $\theta_i = \theta_i^0$ and is slack otherwise. By Lemma 1.4 we only need to show that, for all i, the participation constraint in (PC_C) is binding for $\theta_i = \theta_i^0$. Suppose otherwise, then is was possible to increase $T(\theta_i^0)$ without violating any constraint.

Lemma 1.7. *For all j, the incentive constraints in (IC_F) hold if the following local incentive constraints are satisfied: For any $l < r$,*

$$R(\delta_j^l) - K(\delta_j^l \mid \delta_j^l, y_j(\delta_{-j}, \delta_j^l)) \geq R(\delta_j^{l+1}) - K(\delta_j^{l+1} \mid \delta_j^l, y_j(\delta_{-j}, \delta_j^{l+1})) \ , \tag{1.13}$$

and for all $l > 1$,

$$R(\delta_j^l) - K(\delta_j^l \mid \delta_j^l, y_j(\delta_{-j}, \delta_j^l)) \geq R(\delta_j^{l-1}) - K(\delta_j^{l-1} \mid \delta_j^l, y_j(\delta_{-j}, \delta_j^{l-1})) \ . \tag{1.14}$$

Moreover, the local incentive constraints imply that, for all j, a weak monotonicity condition as defined by Müller et al. (2007) holds.

Proof: We first show that the weak monotonicity condition of Müller et al. (2007) holds for each j and each l. Equation (1.14) as stated in the Lemma for $\delta_j = \delta_j^{l+1}$:

$$R(\delta_j^{l+1}) - K(\delta_j^{l+1} \mid \delta_j^{l+1}, y_j(\delta_{-j}, \delta_j^{l+1})) \geq R(\delta_j^l) - K(\delta_j^l \mid \delta_j^{l+1}, y_j(\delta_{-j}, \delta_j^l)) \ .$$

Adding equation (1.13) as stated above yields:

$$\begin{aligned} K(\delta_j^l \mid \delta_j^{l+1}, y_j(\delta_{-j}, \delta_j^l)) - K(\delta_j^{l+1} \mid \delta_j^{l+1}, y_j(\delta_{-j}, \delta_j^{l+1})) \geq \\ K(\delta_j^l \mid \delta_j^l, y_j(\delta_{-j}, \delta_j^l)) - K(\delta_j^{l+1} \mid \delta_j^l, y_j(\delta_{-j}, \delta_j^{l+1})). \end{aligned} \tag{wm}$$

This is exactly the weak monotonicity condition of Müller et al. (2007).

We show that equation (1.13) implies that

$$R(\delta_j^l) - K(\delta_j^l \mid \delta_j^l, y_j(\delta_{-j}, \delta_j^l)) \geq R(\delta_j^{l+2}) - K(\delta_j^{l+2} \mid \delta_j^l, y_j(\delta_{-j}, \delta_j^{l+2})) \ .$$

To see this rewrite equation (1.13) as

$$\begin{aligned} R(\delta_j^l) - & K(\delta_j^l \mid \delta_j^l, y_j(\delta_{-j}, \delta_j^l)) \geq \\ & R(\delta_j^{l+1}) - K(\delta_j^{l+1} \mid \delta_j^{l+1}, y_j(\delta_{-j}, \delta_j^{l+1})) \\ & + [K(\delta_j^{l+1} \mid \delta_j^{l+1}, y_j(\delta_{-j}, \delta_j^{l+1})) - K(\delta_j^{l+1} \mid \delta_j^l, y_j(\delta_{-j}, \delta_j^{l+1}))] \ . \end{aligned}$$

Since condition (wm) holds, we have

$$\begin{aligned} R(\delta_j^l) - & K(\delta_j^l \mid \delta_j^l, y_j(\delta_{-j}, \delta_j^l)) \geq \\ & R(\delta_j^{l+1}) - K(\delta_j^{l+1} \mid \delta_j^{l+1}, y_j(\delta_{-j}, \delta_j^{l+1})) \\ & + K(\delta_j^{l+2} \mid \delta_j^{l+1}, y_j(\delta_{-j}, \delta_j^{l+2})) - K(\delta_j^{l+2} \mid \delta_j^l, y_j(\delta_{-j}, \delta_j^{l+2})) \ . \end{aligned}$$

Moreover, condition (1.13) for $\delta_j = \delta_j^{l+1}$ is

$$R(\delta_j^{l+1}) - K(\delta_j^{l+1} \mid \delta_j^{l+1}, y_j(\delta_{-j}, \delta_j^{l+1})) \geq R(\delta_j^{l+2}) - K(\delta_j^{l+2} \mid \delta_j^{l+1}, y_j(\delta_{-j}, \delta_j^{l+2})) \ .$$

Adding the two last inequalities yields

$$R(\delta_j^l) - K(\delta_j^l \mid \delta_j^l, y_j(\delta_{-j}, \delta_j^l)) \geq R(\delta_j^{l+2}) - K(\delta_j^{l+2} \mid \delta_j^l, y_j(\delta_{-j}, \delta_j^{l+2})) \ .$$

Hence, a firm with technology parameter δ_j^l does not benefit from announcing δ_j^{l+2}. Iterating this argument once more establishes that this individual does neither benefit from announcing δ_j^{l+3}, etc.

The proof that a firm with technology parameter δ_j^l does not benefit from announcing δ_j^{l-j}, for any $j \geq 1$ is analogous and left to the reader.

Lemma 1.8. *Suppose that, for some firm j, all local upward incentive constraints are binding and that the weak monotonicity condition (wm) holds, for all $l > 1$. Then all incentive compatibility constraints are satisfied.*

Proof: If all local upward incentive constraints are binding for firm j, this implies that, for all $l \geq 2$,

$$R(\delta_j^l) = \sum_{k=l}^{r-1} \{K(\delta_j^k \mid \delta_j^k, y_j(\delta_{-j}, \delta_j^k)) - K(\delta_j^k \mid \delta_j^{k+1}, y_j(\delta_{-j}, \delta_j^{k+1}))\} - R(\delta_j^r) \ .$$

For all $l > 1$, the equation can equivalently be written as

$$\begin{aligned} R(\delta_j^l) =& K(\delta_j^l \mid \delta_j^l, y_j(\delta_{-j}, \delta_j^l)) \\ &+ \sum_{k=l+1}^{r} \{K(\delta_j^l \mid \delta_j^l, y_j(\delta_{-j}, \delta_j^l)) - K(\delta_j^l \mid \delta_j^{l-1}, y_j(\delta_{-j}, \delta_j^l))\} - R(\delta_j^r) \ . \end{aligned}$$

To establish incentive compatibility, Lemma 1.7 implies that it suffices to show that all local downward incentive compatibility constraints are satisfied, i.e., for all l,

$$R(\delta_j^l) - K(\delta_j^l \mid \delta_j^l, y_j(\delta_{-j}, \delta_j^l)) \geq R(\delta_j^{l-1}) - K(\delta_j^{l-1} \mid \delta_j^l, y_j(\delta_{-j}, \delta_j^{l-1})) \ ,$$

or equivalently,

$$\begin{aligned} R(\delta_j^l) - & K(\delta_j^l \mid \delta_j^l, y_j(\delta_{-j}, \delta_j^l)) \geq \\ & R(\delta_j^{l-1}) - K(\delta_j^{l-1} \mid \delta_j^{l-1}, y_j(\delta_{-j}, \delta_j^{l-1})) \\ & + \left[K(\delta_j^{l-1} \mid \delta_j^{l-1}, y_j(\delta_{-j}, \delta_j^{l-1})) - K(\delta_j^{l-1} \mid \delta_j^l, y_j(\delta_{-j}, \delta_j^{l-1}))\right] \ . \end{aligned}$$

By equation

$$\begin{aligned} R(\delta_j^l) =& K(\delta_j^l \mid \delta_j^l, y_j(\delta_{-j}, \delta_j^l)) \\ &+ \sum_{k=l+1}^{r} \{K(\delta_j^k \mid \delta_j^k, y_j(\delta_{-j}, \delta_j^k)) - K(\delta_j^k \mid \delta_j^{k-1}, y_j(\delta_{-j}, \delta_j^k))\} + \underline{\pi}_j \ , \end{aligned}$$

this inequality can be written as:

$$\sum_{k=l+1}^{r} \{K(\delta_j^k \mid \delta_j^k, y_j(\delta_{-j}, \delta_j^k)) - K(\delta_j^k \mid \delta_j^{k-1}, y_j(\delta_{-j}, \delta_j^k))\} \geq \sum_{k=l}^{r}\{K(\delta_j^k \mid \delta_j^k, y_j(\delta_{-j}, \delta_j^k)) - K(\delta_j^k \mid \delta_j^{k-1}, y_j(\delta_{-j}, \delta_j^k))\} + \left[K(\delta_j^{l-1} \mid \delta_j^{l-1}, y_j(\delta_{-j}, \delta_j^{l-1})) - K(\delta_j^{l-1} \mid \delta_j^{l}, y_j(\delta_{-j}, \delta_j^{l-1}))\right],$$

or

$$K(\delta_j^{l-1} \mid \delta_j^{l}, y_j(\delta_{-j}, \delta_j^{l-1})) - K(\delta_j^{l-1} \mid \delta_j^{l-1}, y_j(\delta_{-j}, \delta_j^{l-1})) \geq K(\delta_j^{l} \mid \delta_j^{l}, y_j(\delta_{-j}, \delta_j^{l})) - K(\delta_j^{l} \mid \delta_j^{l-1}, y_j(\delta_{-j}, \delta_j^{l})) .$$

This inequality is identical to the weak monotonicity condition which holds by assumption.

$$y_j(\delta_{-j}, \delta_j^{l-1}) \geq y_j(\delta_{-j}, \delta_j^{l}) .$$

These monotonicity constraints are fulfilled by assumption.

Lemma 1.9. *If for firm j, all local downward incentive constraints are biding, then the expected profit of firm j from ex ante perspective is given by*

$$\mathbb{E}_{(\delta)}[r(\delta) - k(\delta_j, y(\delta))] = \mathbb{E}_{(\delta)}\left[\sum_{l=1}^{r-1}\{k(\delta_j^l, y_j(\delta)) - k(\delta_j^{l-1}, y_j(\theta, \delta))\}\frac{P(\delta_j)}{p(\delta_j)}\right] - R(\delta_j^r) .$$

Proof: Equation

$$R(\delta_j^l) = K(\delta_j^l \mid \delta_j^l, y_j(\delta_{-j}, \delta_j^l)) + \sum_{k=l+1}^{r} \{K(\delta_j^l \mid \delta_j^l, y_j(\delta_{-j}, \delta_j^l)) - K(\delta_j^l \mid \delta_j^{l-1}, y_j(\delta_{-j}, \delta_j^l))\} + \underline{\pi}_j ,$$

in the proof of Lemma 1.8 and the law of iterated expectation imply that

$$
\begin{aligned}
\mathbb{E}_{(\delta)}[r(\delta)] &= \sum_{i=1}^{r} R(\delta_j^i) \\
&= \sum_{i=1}^{r} p^i \Big[\mathbb{E}_{(\delta_{-j})}[k(\delta_j^i, y_j(\delta_{-j}, \delta_j^i))] \\
&\quad + \sum_{k=i+1}^{r} \{K(\delta_j^k \mid \delta_j^k, y_j(\delta_{-j}, \delta_j^k)) - K(\delta_j^k \mid \delta_j^{k-1}, y_j(\delta_{-j}, \delta_j^k))\} - R(\delta_j^r)\Big] \\
&= \quad \mathbb{E}_{(\delta)}[k(\delta_j, y_j(\delta))] + \sum_{i=1}^{r-1} p^i \sum_{k=i+1}^{r} \{K(\delta_j^k \mid \delta_j^k, y_j(\delta_{-j}, \delta_j^k)) \\
&\quad - K(\delta_j^k \mid \delta_j^{k-1}, y_j(\delta_{-j}, \delta_j^k))\} - R(\delta_j^r) \\
&= \quad \mathbb{E}_{(\delta)}[k(\delta_j, y_j(\delta))] + \sum_{i=1}^{r-1} \sum_{k=i}^{r} p^{k-1} \{K(\delta_j^i \mid \delta_j^i, y_j(\delta_{-j}, \delta_j^i)) \\
&\quad - K(\delta_j^i \mid \delta_j^{i-1}, y_j(\delta_{-j}, \delta_j^i))\} - R(\delta_j^r) \\
&= \quad \mathbb{E}_{(\delta)}[k(\delta_j, y_j(\delta))] + \sum_{i=1}^{r-1} p^i \{K(\delta_j^i \mid \delta_j^i, y_j(\delta_{-j}, \delta_j^i)) \\
&\quad - K(\delta_j^i \mid \delta_j^{i-1}, y_j(\delta_{-j}, \delta_j^i))\} \frac{\sum_{k=i}^{r} p^{k-1}}{p^i} - R(\delta_j^r) \\
&= \quad \mathbb{E}_{(\delta)}[k(\delta_j, y_j(\delta))] + \mathbb{E}_{(\delta)}\left[\sum_{l=1}^{r-1} p(\delta_j^l)\{k(\delta_j^l, y_j(\delta)) - k(\delta_j^{l-1}, y_j(\theta, \delta))\} \frac{P(\delta_j)}{p(\delta_j)}\right] - R(\delta_j^r)
\end{aligned}
$$

Lemma 1.10. *For all j, if the (PC_F) is satisfied for $\delta_j = \delta_j^r$, then it is satisfied as well for $\delta_j \neq \delta_j^r$.*

Proof: Let $\delta_j \neq \delta_j^r$. Then by (IC_F)

$$R(\delta_j^l) - K(\delta_j^l \mid \delta_j^l, y_j(\delta_{-j}, \delta_j^l)) \geq R(\delta_j^r) - K(\delta_j^r \mid \delta_j^l, y_j(\delta_{-j}, \delta_j^r)) \ .$$

Moreover, $\delta_j < \delta_j^r$ implies that the right-hand side of this inequality exceeds

$$R(\delta_j^r) - K(\delta_j^r \mid \delta_j^r, y_j(\delta_{-j}, \delta_j^r)) \ ,$$

which is non-negative by (PC_F) for $\delta_j = \delta_j^r$. This proves that (PC_F) is not binding for $\delta_j \neq \delta_j^r$.

Lemma 1.11. *Let y be an arbitrary production rule. Consider the problem of choosing a mechanism $(r_1, ..., r_m)$ in order to minimize revenue*

$$\mathbb{E}_{(\delta)}\left[\sum_{j=1}^{m} r(\delta)\right] \ ,$$

subject to the incentive compatibility constraints in (IC_F) and the interim participation constraints (PC_F). At a solution to this problem, the participation constraint in (PC_F) is binding for $\delta_j = \delta_j^r$ and slack otherwise.

Proof: By Lemma 1.10 we only need to show that it is binding for $\delta_j = \delta_j^r$. We show that it is possible to decrease expected revenues of firm j in an incentive compatible way if, for some j, the participation constraints for $\delta_j = \delta_j^r$ does not hold as an equality. It is instructive to rewrite the incentive compatibility constraints in (IC_F) as follows: For each j, for each $\delta_j^l \in \Delta_j$, and for each $\hat{\delta}_j \in \Delta_j$,

$$R(\delta_j^l) - R(\hat{\delta}_j) \geq K(\delta_j^l \mid \delta_j^l, y_j(\delta_{-j}, \delta_j^l)) - K(\hat{\delta}_j \mid \delta_j^l, y_j(\delta_{-j}, \hat{\delta}_j)) \ .$$

Consider a new revenue rule for firm j such that for every $\delta_j \in \Delta_j$, $R(\delta_j^l)$ decreases by some $\epsilon > 0$. This implies that the left-hand side of the incentive constraint remains constant, i.e., the decrease of j's expected revenue does not violate the incentive compatibility. Since revenue increase in the expected transfers to firm j, the revenue minimizing mechanism must be such that a binding participation constraint for $\delta_j = \delta_j^r$ prevents a further increase of firms j's revenues.

Lemma 1.12. *Let y be an arbitrary given provision rule. Consider the "relaxed problem" of choosing a mechanism $(r_1, ..., r_m)$ in order to minimize total revenue*

$$\mathbb{E}_{(\delta)}\left[\sum_{j=1}^{m} r(\delta)\right] \ ,$$

subject to the upward incentive compatibility constraints in (IC_F) and the interim participation constraints (PC_F). At a solution to this problem, all upward incentive constraints are biding, and the participation constraint in (PC_F) is binding for $\delta_j = \delta_j^r$ and slack otherwise.

Proof: It is straightforward to verify that, for all j, all upward incentive constraints are binding. Otherwise, the expected revenue of some firm j could be decreased without violating any one of the constraints of the relaxed problem. It remains to be shown that, for all j, the participation constraint in (PC_F) is binding for $\delta_j = \delta_j^r$. Suppose otherwise, then it was possible to decrease $R(\delta_j^r)$ without violating any constraint.

Appendix 1.B Proofs of Propositions and Corollaries

Proof of Proposition 1.3.
Consider the 'first relaxed problem' of maximizing expected transfers subject to local downward incentive constraints in (1.12) and interim participation constraints in (PC_C). The arguments in the proofs of Lemma 1.4, 1.5 and 1.6 imply that, for all i, all local downward incentive constraints as well as the interim participation constraints are binding for $\theta_i = \theta^0$. In these Lemmata the provision rule for the publicly provided good is not taken as given. However, this does not affect the logic of the argument.

Given that all local incentive compatibility constraints are biding, Lemma 1.2 implies that, for all i,

$$\mathbb{E}_{(\theta,\delta)}[t_i(\theta,\delta)] =$$
$$\mathbb{E}_{(\theta,\delta)}\left[\sum_{l=1}^{s}\left(v(\theta_i, q_i(\theta,\delta)) - f(\theta_i^l)\{v(\theta_i^{l+1}, q_i(\theta,\delta)) - v(\theta_i^l, q_i(\theta,\delta))\}\frac{1-F(\theta_i)}{f(\theta_i)}\right)\right] + T(\theta_i^0)\ .$$

Since the participation constraints are binding, for all i, whenever $\theta_i = \theta_i^0$, we have $T(\theta_i^0) = -\underline{u}_i$, for all i, and hence,

$$\mathbb{E}_{(\theta,\delta)}[t_i(\theta,\delta)] =$$
$$\mathbb{E}_{(\theta,\delta)}\left[\sum_{l=1}^{s}\left(v(\theta_i, q_i(\theta,\delta)) - f(\theta_i^l)\{v(\theta_i^{l+1}, q_i(\theta,\delta)) - v(\theta_i^l, q_i(\theta,\delta))\}\frac{1-F(\theta_i)}{f(\theta_i)}\right)\right] - \underline{u}_i\ .$$

Second, consider the 'second relaxed problem' of minimizing revenues subject to the local downward incentive constraints in (1.14) and the interim participation constraints in (PC_F). The arguments in the proofs of Lemmata 1.8, 1.10 and 1.11 imply that, for all j, all local upward incentive constraints as well as the interim participation constraints are binding for $\delta_j = \delta^r$. In these Lemmata the production rule for the good is not taken as given. However, this does not affect the logic of the argument.

Given all local incentive compatibility constraints are binding, Lemma 1.8 implies, for all j,

$$\mathbb{E}_{(\theta,\delta)}[r_j(\theta,\delta)] =$$
$$\mathbb{E}_{(\theta,\delta)}\left[\sum_{l=1}^{r-1}\left(k(\delta_j, y_j(\theta,\delta)) + p(\delta_j^l)\{k(\delta_j^l, y_j(\theta,\delta)) - k(\delta_j^{l-1}, y_j(\theta,\delta))\}\frac{P(\delta_j)}{p(\delta_j)}\right)\right] + R(\delta_j^r)\ .$$

Since the participation constraints are binding, for all j, whenever $\delta_j = \delta^r$, we have $R(\delta_j^r) = \underline{\pi}_j$, for all j, and hence,

$$\mathbb{E}_{(\theta,\delta)}\left[\sum_{l=1}^{r-1}\left(k(\delta_j, y_j(\theta,\delta)) + p(\delta_j^l)\{k(\delta_j^l, y_j(\theta,\delta)) - k(\delta_j^{l-1}, y_j(\theta,\delta))\}\frac{P(\delta_j)}{p(\delta_j)}\right)\right] + \underline{\pi}_j\ .$$

Consequently, a necessary condition for the implementability of $\{(q_i)_{i=1}^n, (y)_{j=1}^m\}$ is that

$$\mathbb{E}_{(\theta,\delta)}\left[\sum_{i=1}^{n}\sum_{l=1}^{s}\left(v(\theta_i, q_i(\theta,\delta)) - f(\theta_i^l)\{v(\theta_i^{l+1}, q_i(\theta,\delta)) - v(\theta_i^l, q_i(\theta,\delta))\}\frac{1-F(\theta_i)}{f(\theta_i)} - \underline{u}_i\right)\right] \geq$$
$$\mathbb{E}_{(\theta,\delta)}\left[\sum_{j=1}^{m}\sum_{l=1}^{r-1}\left(k(\delta_j, y_j(\theta,\delta)) + p(\delta_j^l)\{k(\delta_j^l, y_j(\theta,\delta)) - k(\delta_j^{l-1}, y_j(\theta,\delta))\}\frac{P(\delta_j)}{p(\delta_j)} + \underline{\pi}_j\right)\right] .$$

Proof of Proposition 1.4.

Suppose that the condition in Proposition 1.4 holds. We need to show that we can construct a payment scheme satisfying all relevant constraints. Suppose first that the condition in Proposition 1.4 holds as an equality. Then we can choose a payment scheme that solves the relaxed problem, that we studied in the Proof of Proposition 1.3. To show this, we need to verify that the payment scheme, which solves the relaxed problem in Proposition 1.3 satisfies not only the local downward incentive constraints, but all incentive compatibility constraints. This is a consequence of the Lemmata 1.2 and 1.8.

Since the given provision rule q_i satisfies the monotonicity constraints $q_i(\theta_{-i}, \theta_i^l) \geq q_i(\theta_{-i}, \theta_i^{l-1})$ for all i and all l, Lemma 1.8 implies that all incentive compatibility constraints are binding. Lemma 1.3 implies that, for all i,

$$\mathbb{E}_{(\theta,\delta)}[t_i(\theta,\delta)] =$$
$$\mathbb{E}_{(\theta,\delta)}\left[\sum_{l=1}^{s}\left(v(\theta_i, q_i(\theta,\delta)) - f(\theta_i^l)\{v(\theta_i^{l+1}, q_i(\theta,\delta)) - v(\theta_i^l, q_i(\theta,\delta))\}\frac{1-F(\theta_i)}{f(\theta_i)}\right)\right] + T(\theta_i^0) .$$

Now choose

$$T(\theta_i^0) = \frac{1}{n}\mathbb{E}_{(\theta,\delta)}\left[\sum_{j=1}^{m}\sum_{l=1}^{r-1}\left(k(\delta_j, y_j(\theta,\delta)) + p(\delta_j^l)\{k(\delta_j^l, y_j(\theta,\delta)) - k(\delta_j^{l-1}, y_j(\theta,\delta))\}\frac{P(\delta_j)}{p(\delta_j)}\right) + \underline{\pi}_j\right] -$$
$$\frac{1}{n}\mathbb{E}_{(\theta,\delta)}\left[\sum_{i=1}^{n}\sum_{l=1}^{s}\left(v(\theta_i, q_i(\theta,\delta)) - f(\theta_i^l)\{v(\theta_i^{l+1}, q_i(\theta,\delta)) - v(\theta_i^l, q_i(\theta,\delta))\}\frac{1-F(\theta_i)}{f(\theta_i)}\right) - \underline{u}_i\right] ,$$

for all i. By assumption this is smaller or equal to zero, so that the interim participation constraints are satisfied, for all i. It remains to be shown that budget balance holds. This follows since, by construction,

$$\begin{aligned}
& \mathbb{E}_{(\theta,\delta)}\left[\sum_{i=1}^{n} t_i(\theta,\delta)\right] \\
= \quad & \mathbb{E}_{(\theta,\delta)}\left[\sum_{i=1}^{n}\sum_{l=1}^{s}\left(v(\theta_i, q_i(\theta,\delta)) - f(\theta_i^l)\{v(\theta_i^{l+1}, q_i(\theta,\delta)) - v(\theta_i^l, q_i(\theta,\delta))\}\frac{1-F(\theta_i)}{f(\theta_i)}\right)\right] + T(\theta_i^0) \\
= \quad & \mathbb{E}_{(\theta,\delta)}\left[\sum_{i=1}^{n}\sum_{l=1}^{s}\left(v(\theta_i, q_i(\theta,\delta)) - f(\theta_i^l)\{v(\theta_i^{l+1}, q_i(\theta,\delta)) - v(\theta_i^l, q_i(\theta,\delta))\}\frac{1-F(\theta_i)}{f(\theta_i)}\right)\right] \\
& \mathbb{E}_{(\theta,\delta)}\left[\sum_{j=1}^{m}\sum_{l=1}^{r-1}\left(k(\delta_j, y_j(\theta,\delta)) + p(\delta_j^l)\{k(\delta_j^l, y_j(\theta,\delta)) - k(\delta_j^{l-1}, y_j(\theta,\delta))\}\frac{P(\delta_j)}{p(\delta_j)} + \underline{\pi}_j\right)\right] \\
& -\mathbb{E}_{(\theta,\delta)}\left[\sum_{i=1}^{n}\sum_{l=1}^{s}\left((v(\theta_i, q_i(\theta,\delta)) - f(\theta_i^l)\{v(\theta_i^{l+1}, q_i(\theta,\delta)) - v(\theta_i^l, q_i(\theta,\delta))\}\frac{1-F(\theta_i)}{f(\theta_i)} - \underline{u}_i\right)\right] \ .
\end{aligned}$$

Proof of Proposition 1.5.

Recall that for an economy with n individuals at a solution to consumer transfer maximization

$$T_i(\theta^L) = \theta^L Q_i^n(\theta^L) \ ,$$

and

$$T_i(\theta^H) = \theta^H(Q_i^n(\theta^H) - Q_i^n(\theta^L)) + \theta^L Q_i^n(\theta^L) \ ,$$

which implies that

$$R^n = \frac{1}{n}\sum_{j=1}^{n}\{f(\theta^H)\theta^H(Q_i^n(\theta^H) - Q_i^n(\theta^L))\} + \theta^L Q_i^n(\theta^L) \ .$$

We can view the transfer maximization problem as consisting of $\theta^L Q_i^n(\theta^L)$ that every consumer has to pay and an incremental transfer if $f(\theta^H)\theta^H(Q_i^n(\theta^H) - Q_i^n(\theta^L))$ that applies only for consumers with a high valuation for the public good. Proposition 1.5 states that the revenue due to these incremental payments goes to zero as the number of consumers becomes large.

To proof this, we proceed in two steps:

Step 1. We first show that $lim_{n\to\infty} V^n = 0$, where

$$V^n := max_{q^n()}\frac{1}{n}\sum_{i=1}^{n}\left(Q_i^n(\theta^H) - Q_i^n(\theta^L)\right) \ .$$

we start by analyzing the stated sum and decompose the expected values $Q_i^n(\theta^k),\ k \in \{L, H\}$ in its components q^n.

Fix θ'. The contribution of $q_n(\theta')$ to $\sum_{i=1}^n (Q_i^n(\theta^H) - Q_i^n(\theta^L))$ is given by

$$w^n(\theta') := m(\theta')f(\theta^H)^{m(\theta')-1}(1-f(\theta^H))^{n-m(\theta')} - (n-m(\theta'))f(\theta^H)^{m(\theta')}(1-f(\theta^H))^{n-1-m(\theta')} ,$$

where $m(\theta')$ is the number of individuals with $\theta'_i = \theta^H$.

Consequently, $q^n(\theta')$ is chosen equal to 1 if $w^n(\theta) \geq 0$ and equal to 0 otherwise. Equivalently,

$$q^n(\theta') = \begin{cases} 1, & \text{if } \frac{m(\theta')}{n} \geq f(\theta^H) , \\ 0 & \text{if } \frac{m(\theta')}{n} < f(\theta^H) . \end{cases}$$

Substituting these expression into V^n implies

$$V^n = \frac{1}{n} \sum_{x=f(\hat{\theta^H})n}^{n} \binom{n}{x} f(\theta^H)^x (1-f(\theta^H))^{n-x} - (n-x)f(\theta^H)^x(1-f(\theta^H))^{n-1-x} ,$$

where $f(\hat{\theta^H})n$ is the smallest integer larger than $nf(\theta^H)$. Equivalently,

$$\begin{aligned} V^n &= \frac{1}{f(\theta^H)(1-f(\theta^H))} \sum_{x=f(\hat{\theta^H})n}^{n} \binom{n}{x} f(\theta^H)^x(1-f(\theta^H))^{n-x} \left(\frac{x}{n} - f(\theta^H)\right) \\ &= \frac{1}{f(\theta^H)(1-f(\theta^H))} \sum_{x=f(\hat{\theta^H})n}^{n} prob\left(\frac{m(\theta)}{n} = x\right)\left(\frac{x}{n} - f(\theta^H)\right) . \end{aligned}$$

Note that $\frac{m(\theta)}{n} = \frac{1}{n}\sum_{i=1}^n z_i$, where $(z_i)_{i=1}^n$ is a collection function of *i.i.d.* random variables such that $z_i = 1$ if $\theta_i = \theta^H$ and $z_i = 0$ otherwise. By the weak Law of Large Numbers, for any $\epsilon > 0$,

$$lim_{n\to\infty} prob\left(\left|\frac{m(\theta)}{n} - f(\theta^H)\right| > \epsilon\right) = 0 .$$

Hence, $lim_{n\to\infty} V^n = 0$.

Step 2. Under any incentive compatible mechanism, $\frac{1}{n}\sum_{i=1}^n (Q_i^n(\theta^H) - Q_i^n(\theta^L))$ converges to zero. It follows from the incentive compatibility constraints that $Q_i^n(\theta^H) - Q_i^n(\theta^L) \geq 0$ since utility is defined linearly in type. This implies that $\frac{1}{n}\sum_{i=1}^n (Q_i^n(\theta^H) - Q_i^n(\theta^L)) \geq 0$. By *Step 1*, the upper bound on $\frac{1}{n}\sum_{i=1}^n (Q_i^n(\theta^H) - Q_i^n(\theta^L))$ converges to 0. Hence, $\frac{1}{n}\sum_{i=1}^n (Q_i^n(\theta^H) - Q_i^n(\theta^L))$ also converges to 0.

Proof of Proposition 1.6.

The proof is omitted as it is a special case of proposition 1.8.

Proof of Proposition 1.7.

Consider the Mailath and Postlewaite (1990) setup for $I = \{1, 2\}, \Theta = [\theta^L, \theta^H]$. We assume, that $\theta^L + \theta^H > 2c$. Then the efficient allocation rule is given by

$$q(\theta) = \begin{cases} 0, & \text{if} \quad \theta_1 + \theta_2 < 2c \ , \\ 1, & \text{if} \quad \theta_1 + \theta_2 > 2c \ . \end{cases}$$

Then g can be implemented, if and only if

$$\frac{1}{2} \sum_{i=1,2} \mathbb{E}_{(\theta)}[v(\theta_i)q(\theta)] \geq c\mathbb{E}_{(\theta)}[q(\theta)]$$

Or equivalently,

$$\begin{aligned} &\int_{\theta_1}\int_{\theta_2} \frac{1}{2}\left(\theta_1 - \frac{1-F_1(\theta_1)}{f_1(\theta_1)} + \theta_2 - \frac{1-F_2(\theta_2)}{f_2(\theta_2)}\right) q(\theta_1,\theta_2) f_1(\theta_1) f_2(\theta_2) d\theta_1 d\theta_2 \\ &\qquad \geq c \int_{\theta_1}\int_{\theta_2} q(\theta_1,\theta_2) f_1(\theta_1) f_2(\theta_2) d\theta_1 d\theta_2 \end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned} &\int_{\theta_1}\int_{max\{\theta_2^L, 2c-\theta_1\}}^{\theta_2^H} \frac{1}{2}\left(\theta_1 - \frac{1-F_1(\theta_1)}{f_1(\theta_1)} + \theta_2 - \frac{1-F_2(\theta_2)}{f_2(\theta_2)}\right) f_1(\theta_1) f_2(\theta_2) d\theta_1 d\theta_2 \\ &\qquad \geq c \int_{\theta_1}\int_{max\{\theta_2^L, 2c-\theta_1\}}^{\theta_2^H} 1 f_1(\theta_1) f_2(\theta_2) d\theta_1 d\theta_2 \end{aligned}$$

The left-hand side of the inequality can be written as

$$\begin{aligned} &\int_{\theta_1}\int_{max\{\theta_2^L, 2c-\theta_1\}}^{\theta_2^H} \frac{1}{2}\left[\left(\theta_1 - \frac{1-F_1(\theta_1)}{f_1(\theta_1)} + \theta_2\right) f(\theta_2) - 1 + F_2(\theta_2)\right] f_1(\theta_1) d\theta_1 d\theta_2 \\ &= \int_{\theta_1}\left[\frac{1}{2}\left(\theta_1 - \frac{1-F_1(\theta_1)}{f_1(\theta_1)}\right) F_2(\theta_2)\Big|_{max\{\theta_2^L, 2c-\theta_1\}}^{\theta_2^H} + \theta_2^H \right. \\ &\qquad - max\{\theta_2^L, 2c-\theta_1\} F_2(max\{\theta_2^L, 2c-\theta_1\}) - \int_{max\{\theta_2^L, 2c-\theta_1\}}^{\theta_2^H} F_2(\theta_2) d\theta_2 \\ &\qquad \left. - \theta_2^H + max\{\theta_2^L, 2c-\theta_1\} + \int_{max\{\theta_2^L, 2c-\theta_1\}}^{\theta_2^H} F_2(\theta_2) d\theta_2 \right] f_1(\theta_1) d\theta_1 \\ &= \int_{\theta_1} \frac{1}{2}\left((\theta_1 - \frac{1-F_1(\theta_1)}{f_1(\theta_1)} + max\{\theta_2^L, 2c-\theta_1\}\right)\left(1 - F_2(max\{\theta_2^L, 2c-\theta_1\})\right) f_1(\theta_1) d\theta_1 \end{aligned}$$

Case 1: $max\{\theta_2^L, 2c-\theta_1\} = 2c - \theta_1$.

$$\int_{\theta_2^L}^{2c-\theta_1^H} \frac{1}{2}\left(\theta_1 - \frac{1-F_1(\theta_1)}{f_1(\theta_1)} + 2c - \theta_1\right)\left(1 - F_2(2c-\theta_1^H)\right) f_1(\theta_1) d\theta_1$$
$$= \frac{1}{2}\int_{\theta_2^L}^{2c-\theta_1^H} \left(-(1-F_1(\theta_1))(1-F_2(2c-\theta_1^H)) + 2cf_1(\theta_1)(1-F_2(2c-\theta_1^H))\right) d\theta_1$$
$$= \frac{1}{2}\int_{\theta_2^L}^{2c-\theta_1^H} F_2(2c-\theta_1^H)\left[1 - 2cf(\theta_1) - F_1(\theta_1)\right] d\theta_1$$
$$+ \int_{\theta_2^L}^{2c-\theta_1^H} cf(\theta_1) d\theta_1 - \frac{1}{2}\int_{\theta_2^L}^{2c-\theta_1^H} (1-F_1(\theta_1)) d\theta_1$$
$$= \frac{1}{2}\int_{\theta_2^L}^{2c-\theta_1^H} \left[F_2(2c-\theta_1^H) - 1\right](1-F_1(\theta_1)) d\theta_1 + cF_2(2c-\theta_1^H)$$
$$- \frac{1}{2}\int_{\theta_2^L}^{2c-\theta_1^H} F_2(2c-\theta_1^H) 2cf(\theta_1) d\theta_1$$

Case 2: $max\{\theta_2^L, 2c-\theta_1\} = \theta_2^L$.

$$\int_{2c-\theta_1^H}^{\theta_2^H} \frac{1}{2}\left(\theta_1 - \frac{1-F_1(\theta_1)}{f_1(\theta_1)} + \theta_2^L\right) f_1(\theta_1) d\theta_1$$
$$= \frac{1}{2}\int_{2c-\theta_1^H}^{\theta_2^H} \left(\theta_1 f_1(\theta_1) - (1-F_1(\theta_1)) + \theta_2^L f_1(\theta_1)\right) d\theta_1$$
$$= \frac{1}{2}\left(\theta_1 F_1(\theta_1)\Big|_{2c-\theta_1^H}^{\theta_2^H} - \int_{2c-\theta_1^H}^{\theta_2^H} F_1(\theta_1) d\theta_1\right.$$
$$\left. - \int_{2c-\theta_1^H}^{\theta_2^H} (1-F_1(\theta_1)) d\theta_1 + \theta_2^L F(\theta_1)\Big|_{2c-\theta_1^H}^{\theta_2^H}\right)$$
$$= \frac{1}{2}\left(\theta_2^H - (2c-\theta_1^H)F_1(2c-\theta_1^H)\right) - \int_{2c-\theta_1^H}^{\theta_2^H} 1 d\theta_1 + \theta_2^L(1 - F_1(2c-\theta_1^H))$$
$$= \frac{1}{2}\left(\theta_2^H - (2c-\theta_1^H)F_1(2c-\theta_1^H) - \theta_2^H + (2c-\theta_1^H) + \theta_2^L(1-F_1(2c-\theta_1^H))\right)$$
$$= \frac{1}{2}(\theta_2^L - \theta_1^H)(1 - F_1(2c-\theta_1^H)) + c - cF_1(2c-\theta_1^H)$$

Combining *Case 1* and *Case2* the left-hand side of the inequality is given by

$$\int_{\theta_1} \frac{1}{2}\left((\theta_1 - \frac{1-F_1(\theta_1)}{f_1(\theta_1)} + max\{\theta_2^L, 2c-\theta_1\}\right)\left(1 - F_2(max\{\theta_2^L, 2c-\theta_1\})\right) f_1(\theta_1) d\theta_1$$
$$= \frac{1}{2}\int_{\theta_2^L}^{2c-\theta_1^H} \left[F_2(2c-\theta_1^H) - 1\right](1-F_1(\theta_1)) d\theta_1 + cF_2(2c-\theta_1^H)$$

$$
\begin{aligned}
&-\frac{1}{2}\int_{\theta_2^L}^{2c-\theta_1^H} F_2(2c-\theta_1^H)2cf(\theta_1)d\theta_1\\
&+\frac{1}{2}(\theta_2^L-\theta_1^H)(1-F_1(2c-\theta_1^H))+c-cF_1(2c-\theta_1^H)\\
=&\frac{1}{2}\int_{\theta_2^L}^{2c-\theta_1^H}\left[F_2(2c-\theta_1^H)-1\right](1-F_1(\theta_1))d\theta_1+c\\
&-\frac{1}{2}\int_{\theta_2^L}^{2c-\theta_1^H} F_2(2c-\theta_1^H)2cf(\theta_1)d\theta_1+\frac{1}{2}(\theta_2^L-\theta_1^H)(1-F_1(2c-\theta_1^H))
\end{aligned}
$$

The right-hand side of the inequality can be written as

$$
\begin{aligned}
&c\int_{\theta_1}\int_{max\{\theta_2^L,2c-\theta_1\}}^{\theta_2^H} 1f_1(\theta_1)f_2(\theta_2)d\theta_1 d\theta_2\\
&=\int_{\theta_1}\left(1-F_2(max\{\theta_2^L,2c-\theta_1^H\})\right)f_1(\theta_1)d\theta_1\\
&=c\int_{\theta_2^L}^{2c-\theta_1^H}\left(1-F_2(2c-\theta_1^H)\right)f(\theta_1)d\theta_1+\int_{2c-\theta_1^H}^{\theta_2^H}1f(\theta_1)d\theta_1\\
&=c\left(\int_{\theta_2^L}^{\theta_2^H}1f_1(\theta_1)d\theta_1-\int_{\theta_2^L}^{2c-\theta_1^H}F_2(2c-\theta_1^H)f(\theta_1)d\theta_1\right)\\
&=c\left(1-\int_{\theta_2^L}^{2c-\theta_1^H}F_2(2c-\theta_1^H)f(\theta_1)d\theta_1\right)
\end{aligned}
$$

The inequality can hence be written as

$$
\begin{aligned}
&\frac{1}{2}\int_{\theta_2^L}^{2c-\theta_1^H}\left[F_2(2c-\theta_1^H)-1\right](1-F_1(\theta_1))d\theta_1+c\\
&\quad-\frac{1}{2}\int_{\theta_2^L}^{2c-\theta_1^H} F_2(2c-\theta_1^H)2cf(\theta_1)d\theta_1+\frac{1}{2}(\theta_2^L-\theta_1^H)(1-F_1(2c-\theta_1^H))\\
&\geq c\left(1-\int_{\theta_2^L}^{2c-\theta_1^H}F_2(2c-\theta_1^H)f(\theta_1)d\theta_1\right)
\end{aligned}
$$

$\Leftrightarrow$

$$
\begin{aligned}
&\frac{1}{2}\int_{\theta_2^L}^{2c-\theta_1^H}\underbrace{\left[F_2(2c-\theta_1^H)-1\right]}_{<0}(1-F_1(\theta_1))d\theta_1+\frac{1}{2}\underbrace{(\theta_2^L-\theta_1^H)}_{<0}(1-F_1(2c-\theta_1^H))\\
&\geq 0
\end{aligned}
$$

The inequality cannot be fulfilled. The public good can therefore not be provided efficiently.

Proof of Proposition 1.8.

Reminder:f_D, F_D are the density-, distribution function in the discrete environment. For every individual, they are defined as follows:

$$f_D(\theta) = \int_{\frac{\theta^l+\theta^{l-1}}{2}}^{\frac{\theta^{l+1}+\theta^l}{2}} f(\theta^l)d\theta^l \ , \qquad \theta \in (\theta^0, \theta^s)$$

$$f_D(\theta^0) = \int_0^{\frac{\theta^1+\theta^0}{2}} f(\theta^l)d\theta^l$$

$$f_D(\theta^s) = \int_{\frac{\theta^s+\theta^{s-1}}{2}}^{\theta^s} f(\theta^l)d\theta^l$$

Without loss of generality (for $I < \infty$), we consider a situation with one individual on each side we take δ as given and calculate the ex ante expected payment. All arguments also apply in case of more than one agent and taking expectation over (θ, δ). The ex ante expected payment is given by the following equation

$$\mathbb{E}_\theta \left[\left(v(\theta^l, q^*(\theta,\delta)) - \sum_{l=1}^{s} f_D(\theta^l)\{v(\theta^{l+1}, q^*(\theta,\delta)) - v(\theta^l, q^*(\theta,\delta))\} \frac{1-F_D(\theta^l)}{f_D(\theta^l)} \right) \right] .$$

We neglect the first term for the moment and consider the second:

$$\begin{aligned} &\sum_l f_D(\theta^l)\{v(\theta^{l+1}, q^*(\theta,\delta)) - v(\theta^l, q^*(\theta,\delta))\} \frac{1-F_D(\theta^l)}{f_D(\theta^l)} \\ &= \sum_l \{v(\theta^{l+1}, q^*(\theta,\delta)) - v(\theta^l, q^*(\theta,\delta))\}(1-F_D(\theta^l)) \\ &= \sum_l \frac{v(\theta^{l+1}, q^*(\theta,\delta)) - v(\theta^l, q^*(\theta,\delta))}{\theta^{l+1}-\theta_l}(\theta^{l+1}-\theta^l)(1-F_D(\theta^l)) \ . \end{aligned}$$

Adding types in every round k, yields that $|\theta^{l+1} - \theta^l| \to 0$. Thus the difference quotient of the value function converges to the partial derivative of v with respect to θ_l.

$$\begin{aligned} &\lim_{|\theta^{l+1}-\theta^l|\to 0} \sum_l \frac{v(\theta^{l+1}, q^*(\theta,\delta)) - v(\theta^l, q^*(\theta,\delta))}{\theta^{l+1}-\theta^l}(\theta^{l+1}-\theta^l)(1-F_D(\theta^l)) \\ &= \sum_l \frac{\partial v}{\partial \theta^l}\Big|_{\theta^l}(1-F_D(\theta^l)) \ . \end{aligned}$$

Since v is monotone we can change summation and take the infinum/supremum. The limit of the upper- and the lower-sum of the above function coincide and $F_D(\theta^l) \to$

$F(\theta^l)$, thus the integral exists and is given by[23]

$$\lim_{|\theta^{l+1}-\theta^l|\to 0} \sum_l \inf_z \left(\frac{\partial v}{\partial \theta^l}\Big|_z (1 - F_D(z)) \right) = \lim_{|\theta^{l+1}-\theta^l|\to 0} \sum_l \sup_z \left(\frac{\partial v}{\partial \theta^l}\Big|_z (1 - F_D(z)) \right)$$
$$= \int_{\theta_0}^{\theta^s} \frac{\partial v}{\partial \theta^l}\Big|_z (1 - F(z))dz \ .$$

Using integration by parts and Fubini's Theorem (to change the order of integration), we get

$$\begin{aligned}
\int_{\theta^0}^{\theta^s} \frac{\partial v}{\partial \theta^l}\Big|_s (1 - F(s))dr &= \int_{\theta^0}^{\theta^s} \frac{\partial v}{\partial \theta^i}\Big|_x (1 - F(x))dx \\
&= \int_{\theta^0}^{\theta^s} \frac{\partial v}{\partial \theta^l}\Big|_x \int_x^{\theta^s} f(z)dzdx \\
&= \int_{\theta^0}^{\theta^s} \int_x^{\theta^s} \frac{\partial v}{\partial \theta^l}\Big|_x f(z)dzdx \\
&= \int_{\theta^0}^{\theta^s} \int_{\theta^s}^{z} \frac{\partial v}{\partial \theta^l}\Big|_x f(z)dxdz \\
&= \int_{\theta^0}^{\theta^s} \int_{\theta^s}^{z} \frac{\partial v}{\partial \theta^l}\Big|_x dxf(z)dz \ .
\end{aligned}$$

For the neglected term $\mathbb{E}_\theta \left[v(\theta^l, q^*(\theta, \delta)) \right] = \sum_l f_D(\theta^l) v(\theta^l, q^*(\theta, \delta))$, we consider the limes inferior and the limes superior and find, that it converges:

$$\lim_{|\theta^{l+1}-\theta^l|\to 0} \sum_l \inf_{\theta^l} \left(v(\theta^l, q^*(\theta^l, \delta)) \right) = \lim_{|\theta^{l+1}-\theta^l|\to 0} \sum_l \sup_{\theta^l} \left(v(\theta^l, q^*(\theta^l, \delta)) \right)$$
$$= \int_{\theta^0}^{\theta^s} v(z, q^*(z, \delta)) f(z)dz \ .$$

Combining the converged terms, this yields $\int_{\theta^0}^{\theta^s} \left(v(z, q^*(z, \delta)) - \int_{\theta^s}^{z} \frac{\partial v}{\partial \theta^l}\Big|_x dx \right) f(z)dz = \mathbb{E}_\theta[t(\theta, \delta)]$, which gives the claimed result.

Proof of Corollary 1.1.
Let us assume for a moment that the mechanism that maximizes expected surplus subject to (IC_F) and (PC_F) satisfies the monotonicity constraint $y_j(\delta_{-j}, \delta_j^l) \geq y_j(\delta_{-j}, \delta_j^{l+1})$, for all l. This will be verified below.

A necessary condition for the maximization of S is that consumers' transfers need to be higher than firms' revenues. The budget constraint (1.6) needs to hold with equality, otherwise it was possible to decrease transfers without violating any constraints of the

[23]Reminder: We summarize over all possible types $l = 0, \ldots, s$. This is not a summation over agents.

surplus maximization problem. Hence,

$$S(\theta,\delta) := \mathbb{E}_{(\theta,\delta)}\left[\sum_{i=1}^{n}(v(\theta_i, q_i(\theta,\delta)) - \sum_{j=1}^{m} k(\delta_j, y_j(\theta,\delta))\right] .$$

Now suppose that the solution to this problem involves overproduction,

$$\sum_{i=1}^{n} q_i(\theta,\delta) \leq \sum_{j=1}^{m} y_j(\theta,\delta) .$$

Then increasing $\sum_{i=1}^{n} q_i(\theta,\delta)$ involves no costs, i.e., firm profits remain unaffected, but increases consumer surplus as $\mathbb{E}_{(\theta,\delta)}[\sum_{i=1}^{n} v(\theta_i, q_i(\theta,\delta))]$ goes up. This is a contradiction to the assumption that the optimum involves underproduction. Hence, we need that $\sum_{i=1}^{n} q_i(\theta,\delta) = \sum_{j=1}^{m} y_j(\theta,\delta)$.

We can therefore once more rewrite the problem of choosing an optimal provision rule: Choose $(q_i)_{i=1}^{n}, (y_j)_{j=1}^{m}$ in order to maximize

$$S(\theta,\delta) := \mathbb{E}_{(\theta,\delta)}\left[\sum_{i=1}^{n} v(\theta_i, q_i(\theta,\delta)) - \sum_{j=1}^{m} k(\delta_j, y_j(\theta,\delta))\right] ,$$

subject to $\sum_{i=1}^{n} q_i(\theta,\delta) = \sum_{j=1}^{m} y_j(\theta,\delta)$. The solution to that problem is such that the following first order condition is satisfied:

$$v_2(\theta_i, q_i^*(\theta,\delta)) = k_2(\delta_j, y_j^*(\theta,\delta)) .$$

The first order condition implies that for every δ, $q_i(\theta_{-i}, \theta_i^l) \geq q_i(\theta_{-i}, \theta_i^{l-1})$ and for every $y_j(\delta_{-j}, \delta_j^l) \geq y_j(\delta_{-j}, \delta_j^{l+1})$. This implies that the monotonicity conditions, for all l, are satisfied. As the monotonicity of the allocation rule is satisfied, which in turn yields weak monotonicity, Proposition 1.4 yields the given statement. See below in more detail, how maximal payments are constructed such that all local downward incentive compatibility constraints are binding.

We choose $T(\theta_i^0)_{i=1}^{n}$ such that

$$\begin{aligned}\sum_{i=1}^{n} T(\theta_i^0) = \mathbb{E}_{(\theta,\delta)}\Big[\sum_{i=1}^{n}\sum_{l=1}^{s}\Big(& v(\theta_i, q_i^*(\theta,\delta)) - f(\theta_i^l)\{v(\theta_i^{l+1}, q_i^*(\theta,\delta)) - v(\theta_i^l, q_i^*(\theta,\delta))\}\frac{1-F(\theta_i)}{f(\theta_i)} - \underline{u}_i\Big)\\ &- \sum_{j=1}^{m}\sum_{l=1}^{r-1}\Big(k(\delta_j, y_j^*(\theta,\delta)) + p(\delta_j^l)\{k(\delta_j^l, y_j^*(\theta,\delta)) - k(\delta_j^{l-1}, y_j^*(\theta,\delta))\}\frac{P(\delta_j)}{p(\delta_j)} + \underline{\pi}_j\Big)\Big] .\end{aligned}$$

It follows from Lemma 1.2 that for all consumers incentive compatibility constraints are satisfied. Also it follows from Proposition 1.4, that the expected transfers are given by

$$\mathbb{E}_{(\theta,\delta)}\left[\sum_{i=1}^{n} t_i(\theta,\delta)\right] =$$
$$\mathbb{E}_{(\theta,\delta)}\left[\sum_{j=1}^{m}\sum_{l=1}^{r-1}\Big(k(\delta_j, y_j^*(\theta,\delta)) + p(\delta_j^l)\{k(\delta_j^l, y_j^*(\theta,\delta)) - k(\delta_j^{l-1}, y_j^*(\theta,\delta))\}\frac{P(\delta_j)}{p(\delta_j)} + \underline{\pi}_j\Big)\right] - \sum_{i=1}^{n}\underline{u}_i\ .$$

It follows as well from Proposition 1.4 that maximal revenue that can be extracted from consumers are equal to

$$\begin{aligned}\mathbb{E}_{(\theta,\delta)}\Big[\sum_{i=1}^{n}\sum_{l=1}^{s}\Big(&v(\theta_i, q^*(\theta,\delta))\\ &- f(\theta_i^l)\{v(\theta_i^{l+1}, q_i^*(\theta,\delta)) - v(\theta_i^l, q_i^*(\theta,\delta))\}\frac{1-F(\theta_i)}{f(\theta_i)} - \underline{u}_i\Big)\Big]\ .\end{aligned}$$

Consequently, a necessary condition for the implementability of $(q_i^*)_{i=1}^n, (y_j^*)_{j=1}^m$ is that

$$\begin{aligned}\mathbb{E}_{(\theta,\delta)}\left[\sum_{i=1}^{n}\sum_{l=1}^{s}\Big(v(\theta_i, q_i^*(\theta,\delta)) - f(\theta_i^l)\{v(\theta_i^{l+1}, q_i^*(\theta,\delta)) - v(\theta_i^l, q_i^*(\theta,\delta))\}\frac{1-F(\theta_i)}{f(\theta_i)}\Big)\right] - \sum_{i=1}^{n}\underline{u}_i \geq \\ \mathbb{E}_{(\theta,\delta)}\left[\sum_{j=1}^{m}\sum_{l=1}^{r-1}\Big(k(\delta_j, y_j^*(\theta,\delta)) + p(\delta_j^l)\{k(\delta_j^l, y_j^*(\theta,\delta)) - k(\delta_j^{l-1}, y_j^*(\theta,\delta))\}\frac{P(\delta_j)}{p(\delta_j)}\Big)\right] + \sum_{j=1}^{m}\underline{\pi}_j\ .\end{aligned}$$

Sufficiency of this condition can be shown by using once more, the construction in the Proof of Proposition 1.4. If the condition above holds and we let for all i,

$$\begin{aligned}T(\theta_i^0) = -\Big(E\Big[&\sum_{i=1}^{n}\sum_{l=1}^{s}\Big(v(\theta_i, q_i^*(\theta,\delta)) - f(\theta_i^l)\{v(\theta_i^{l+1}, q_i^*(\theta,\delta)) - v(\theta_i^l, q_i^*(\theta,\delta))\}\frac{1-F(\theta_i)}{f(\theta_i)} - \underline{u}_i\Big)\\ &-\sum_{j=1}^{m}\sum_{l=1}^{r-1}\Big(k(\delta_j, y_j^*(\theta,\delta)) + p(\delta_j^l)\{k(\delta_j^l, y_j^*(\theta,\delta)) - k(\delta_j^{l-1}, y_j^*(\theta,\delta))\}\frac{P(\delta_j)}{p(\delta_j)} + \underline{\pi}_j\Big)\Big]\Big) \geq 0\ ,\end{aligned}$$

we obtain a mechanism that achieves $(q_i^*)_{i=1}^n, (y_j^*)_{j=1}^m$, satisfies all relevant constraints.

Proof of Corollary 1.2.

Suppose that the constraint in Corollary 1.1 is violated, then the inequality constraint in Proposition 1.2 is binding, and the optimal provision rule maximizes the following

Lagrangian

$$\mathcal{L} = \mathbb{E}_{(\theta,\delta)}\left[\sum_{i=1}^{n}(v(\theta_i, q(\theta,\delta)) - k(\delta_j, y_j(\theta,\delta)))\right]$$
$$+ \lambda\Bigg(\mathbb{E}_{(\theta,\delta)}\left[\sum_{i=1}^{n}\sum_{l=1}^{s}\left(v(\theta_i, q_i(\theta,\delta)) - f(\theta_i^l)\{v(\theta_i^{l+1}, q_i(\theta,\delta)) - v(\theta_i^l, q_i(\theta,\delta))\}\frac{1-F(\theta_i)}{f(\theta_i)} - \underline{u}_i\right)\right]$$
$$- \mathbb{E}_{(\theta,\delta)}\left[\sum_{j=1}^{m}\sum_{l=1}^{r-1}\left(k(\delta_j, y_j(\theta,\delta)) + p(\delta_j^l)\{k(\delta_j^l, y_j(\theta,\delta)) - k(\delta_j^{l-1}, y_j(\theta,\delta))\}\frac{P(\delta_j)}{p(\delta_j)} + \underline{\pi}_j\right)\right]\Bigg) ,$$

where λ is the Lagrangian multiplier, which, at a solution to this maximization problem, has to be strictly positive, $\lambda > 0$.

To complete the proof it remains to be shown that a solution to the maximization problem satisfies the monotonicity constraints.

To see that $q_i(\theta_{-i}, \theta_i^l) \geq q_i(\theta_{-i}, \theta_i^{l-1})$, for all i and all l holds, note that the monotone hazard rate assumption implies that

$$\mathbb{E}_{(\theta,\delta)}\left[\sum_{l=1}^{s}\left(v(\theta_i, q_i(\theta,\delta)) - f(\theta_i^l)\{v(\theta_i^{l+1}, q_i(\theta,\delta)) - v(\theta_i^l, q_i(\theta,\delta))\}\frac{1-F(\theta_i)}{f(\theta_i)}\right)\right] \geq 0 ,$$

is an increasing function. Consequently, the solution to the maximization problem is such that

$$q_i(\theta_{-i}, \theta_i^l) \geq q_i(\theta_{-i}, \theta_i^{l-1}) ,$$

for all i, l, θ_{-i} and δ.

To see that $y_j(\delta_{-j}, \delta_j^l) \geq y_j(\delta_{-j}, \delta_j^{l+1})$, for all j and all l, note that

$$\mathbb{E}_{(\theta,\delta)}\left[\sum_{l=1}^{r-1}\left(k(\delta_j, y_j^*(\theta,\delta)) + p(\delta_j^l)\{k(\delta_j^l, y_j(\theta,\delta)) - k(\delta_{j-1}, y_j(\theta,\delta))\}\frac{P(\delta_j)}{p(\delta_j)}\right)\right] \geq 0 ,$$

also is an increasing function. This implies that, for all j, l, δ_{-j} and θ, $y_j(\delta_{-j}, \delta_j^l) \geq y_j(\delta_{-j}, \delta_j^{l+1})$.

Appendix 1.C Applications

Here, we present further applications that highlight the difference between discrete and continuous type settings. We show that the revenue equivalence theorem for first-price and second-price auctions (see i.e., Vickrey (1961), Harris and Raviv (1981), Riley and W. (1981); and for further references McAfee and McMillan (1987)) does not need to apply for finitely many types in Myerson (1981) single-unit auction. Further, for the partnership dissolution framework of Cramton et al. (1987), we demonstrate that irrespectively of the original distribution of shares in the partnership, this partnership can be dissolved efficiently for examples with discrete types.

This list of applications is not complete. For instance, Rob (1989) considers a model with one producer and n consumers. The producer considers to locate in a particular area. His location would come with emissions, which are a public bad from the perspective of consumers who live in the vicinity of the new production site. Rob characterizes the social choice function, which maximizes the producer's expected profits subject to incentive and participation constraints for the consumers.

A single-unit auction, Myerson (1981). There are n buyers and one producer, also referred to as the auctioneer, so that $I = \{1, \ldots, n\}$, and $J = \{1\}$. Payoff functions of consumers are assumed to be linear in the quantity traded so that $v(\theta_i, q_i) = \theta_i \; q_i$. The seller in-elastically supplies one unit of an indivisible private good, and does not incur production costs. A state of the economy is therefore defined by a vector of valuations θ. The set of physically feasible consumption rules is such that

$$\sum_{i=1}^{n} q_i(\theta) = 1 \quad \text{and} \quad q_i(\theta) \in \{0,1\} \ , \quad \text{for all } i \ . \tag{1.15}$$

It is assumed that trade is voluntary and, that, in the absence of trade, everyone realizes a utility of 0. Thus, $\underline{u}_i = 0$, for all i, and $\underline{\pi}_1 = 0$. Surplus-maximization requires to choose the functions $q_i : \Theta \to \{0,1\}$ so as to maximize $\mathbb{E}_\theta \left[\sum_{i=1}^{n} \theta_i q_i(\theta)\right]$ subject to the constraints in (1.15). The auctioneer receives the payments of the buyers, so that, for all θ, $r_1(\theta) = \sum_{i=1}^{n} t_i(\theta)$.

The famous revenue equivalence theorem (see i.e., Vickrey (1961), Harris and Raviv (1981), Riley and W. (1981); and for further references McAfee and McMillan (1987)) states that all auction formats that give rise to the same consumption rules $(q_i)_{i \in I}$ yield the same expected revenue for the auctioneer. For instance, if these functions, are chosen in a surplus-maximizing way, or, equivalently, so that the object is allocated to the consumer with the highest valuation, then a first price auction and a second price auction, generate the same expected revenue. This theorem is proven under the assumption that all distributions are atomless. Below, we will show that a model with a discrete set of types provides additional degrees of freedom in the specification of payment rules, so that the revenue equivalence theorem no longer holds.

Proposition 1.9. *Consider a first price auction and a second price auction with n risk-neutral buyers, in which buyer i's valuation is drawn from Θ_i. Then the first price auction and the second price auction do not generate the same expected revenue for the seller.*

In a second-price auction, it is a weakly dominant strategy for each buyer i to bid his true valuation. This argument does not rely on the assumption that buyers' values were continuously distributed. However, in the first-price auction, buyers have an incentive to understate their valuation. A buyer that bids an equal amount to his value has a payoff of 0. If he bids less than his value, he reduces the probability of winning but at the same time increases his payoff from winning. Thus, it is a weakly dominant strategy for buyer i to understate his valuation. If these values are distributed discretely, the expected transfers under a first-price auction and a second-price auction are not the same. Hence, the expected revenues do not coincide under the two auctions.

Partnership dissolution, Cramton et al. (1987). There are $I = \{1, 2\}$ consumers, also referred to as partners, and no producers. They form a partnership in which any one agent i initially holds a share $e_i \in [0, 1]$, with $e_1 + e_2 = 1$. The allocation problem is to change the ownership structure. Let s_i be agent i's share in the partnership after the reassignment of shares. Let t_i be the monetary payment of i, which is positive if i has to compensate others for receiving their shares and is negative if i sells some of her shares to other partners. Partner i evaluates this outcome according to the utility function $\theta_i\ s_i - t_i$. We can relate this setup to the general framework developed in Section 1.4 by defining $q_i = s_i - e_i$ as the change of the shares held by agent i. Partner i's utility gain from the change of the ownership structure can then be written as $\theta_i\ q_i - t_i$. A social choice function consists of a collection of consumption functions $q_i : \Theta^n \to \mathbb{R}$, $i \in I$, so that, for all θ,

$$q_1(\theta) + q_1(\theta) = 0 \quad \text{and,} \quad \text{for all } i,\ -e_i \leq q_i(\theta) \leq 1 - e_i\ . \tag{1.16}$$

The partners have to agree unanimously on the new ownership structure so that $\underline{u}_i = 0$, for all i. The surplus that is generated by the change of the ownership structure is, again, given by $\mathbb{E}_\theta\,[\theta_1 q_1(\theta) + \theta_2 q_2(\theta)]$.

A key insight by Cramton et al. (1987) is that the specification of the initial ownership structure, i.e., the choice of $e = (e_1, e_2)$, has an influence on the possibility to dissolve a partnership in a surplus-maximizing way. Below, we will show that our discrete type specification allows us to communicate this insight in a very simply way, without having to invoke all the calculus that an analysis with an atomless distribution would require.

In the following we assume that there are two types per individual, i.e., $\Theta_1 = \{\theta_1^L, \theta_1^H\}$ and $\Theta_2 = \{\theta_2^L, \theta_2^H\}$. We denote the probability of the event $\theta_i = \theta_i^L$ by f_i^L and $f_i^H :=$

$1 - f_i^L$. For ease of notation we define the interim expected change of share in partnership as:

$$Q_1(\theta_1^L) := f_1^L q_1(\theta_1^L, \theta_2^L) + f_1^H q_1(\theta_1^L, \theta_2^H) \ ,$$

and analogously $Q_1(\theta_1^H)$, $Q_2(\theta_2^L)$ and $Q_2(\theta_2^H)$

Proposition 1.10. *Suppose that q_1 and q_2 are such that*

$$Q_1(\theta_1^L) \leq Q_1(\theta_1^H) \leq 0, \quad \textit{and} \quad Q_2(\theta_2^L) \leq Q_2(\theta_2^H) \leq 0 \ ,$$

then there exists (t_1, t_2) so that (q_1, q_2, t_1, t_2) satisfies incentive compatibility, voluntary participation and expected budget balance if and only if

$$\mathbb{E}_\theta[\theta_1 q_1(\theta_1, \theta_2) + \theta_2 q_2(\theta_1, \theta_2)] - f_1^L(\theta_1^H - \theta_1^L)Q_1(\theta_1^H) - f_2^H(\theta_2^H - \theta_2^L)Q_2(\theta_2^L) \geq 0 \ .$$

In a discrete type version of Cramton et al. (1987) an efficient dissolution of partnership is possible, even though the initial ownership is such that one party owns everything and the other party nothing. This is different under a continuous distribution of types. The possibility result relates to Observation 1.1. If initial shares are distributed more equally between two parties, efficient dissolution of partnership is possible with discrete and continuous distribution of types, as the following Proposition highlights.

Proposition 1.11. *Suppose that q_1 and q_2 are such that*

$$Q_1(\theta_1^L) \leq 0 \leq Q_1(\theta_1^H), \quad \textit{and} \quad Q_2(\theta_2^L) \geq 0 \geq Q_2(\theta_2^H) \ ,$$

then there exists (t_1, t_2) so that (q_1, q_2, t_1, t_2) satisfies incentive compatibility, voluntary participation and expected budget balance if and only if

$$\mathbb{E}_\theta[\theta_1 q_1(\theta_1, \theta_2) + \theta_2 q_2(\theta_1, \theta_2)] \geq 0 \ .$$

Proofs of Applications.

Proof of Proposition 1.9
Assume that there are n bidders. The bidder's valuation for the good can take two values $\Theta_i = \{\theta_i^L, \theta_i^H\}$.

Second price auction. In a second-price auction, each bidder submits a sealed bid of b_i. This gives rise to the following utilities:

$$U_i = \begin{cases} 0, & \text{if} \quad b_1 < max_{j \neq i} b_j \ , \\ \theta_i - max_{j \neq i} b_j, & \text{if} \quad b_1 > max_{j \neq i} b_j \ . \end{cases}$$

If $b_1 = max_{j \neq i} b_j$, the object goes to each winning bidder with the same probability. It's a weakly dominant strategy for bidder i to bid his own valuation $b_i(\theta_i) = \theta_i$ (see Vickrey (1961) for a proof).

We fix a bidder i and let the random variable $X_i \equiv X_i^{n-1}$ denote the highest value among the $n-1$ remaining bidders. Since each bidder will bid their value, the transfer of each bidder in a second-price auction (SP) is

$$\begin{aligned} t_i^{SP}(\theta_i) &= Prob(Win) \times \mathbb{E}_{(\theta)} \left[\text{2nd highest bid} | \ \theta_i \text{ is highest bid} \right] \\ &= F(\theta_i) \times \mathbb{E}_{(\theta)}[X_1 | X_1 < \theta_i] \ . \end{aligned}$$

First price auction. Bidder i's expected payoff, as a function of his bid b_i and valuation θ_i is

$$U_i = \begin{cases} 0, & \text{if} \quad b_1 < max_{j \neq i} b_j \ , \\ \theta_i - b_i, & \text{if} \quad b_1 > max_{j \neq i} b_j \ . \end{cases}$$

The bidder chooses b_i in order to maximize

$$max_{b_i} (\theta_i - b_i) Pr[b_j = b(\Theta_j) \leq b_i, \quad \forall \ j \neq i] \ .$$

It is a weakly dominant strategy to understate one's valuation, as this leads to positive expected profits. If instead the bidder bids truthfully, his expected payoff will be zero. Thus, the transfer from the first-price auction (FP) is

$$t_i^{FP}(\theta_i) = \theta_i^L \ .$$

Hence, the revenues under the first-price auction and the second-price auction do not coincide when types are discretely distributed.

Proof of Proposition 1.10.

For the given allocation function q the incentive and participation constraints for individual i are given by

$$\theta_i^L Q_i(\theta_i^L) - T_i(\theta_i^L) \geq \theta_i^L Q_i(\theta_i^H) - T_i(\theta_i^H) \tag{1.17}$$

$$\theta_i^H Q_i(\theta_i^H) - T_i(\theta_i^H) \geq \theta_i^H Q_i(\theta_i^L) - T_i(\theta_i^L) \tag{1.18}$$

$$\theta_i^L Q_i(\theta_i^L) - T_i(\theta_i^L) \geq 0 \tag{1.19}$$

$$\theta_i^H Q_i(\theta_i^H) - T_i(\theta_i^H) \geq 0 \ . \tag{1.20}$$

Additional we have the expected budget balance condition:

$$\mathbb{E}_{(\theta)} \left[t_1(\theta_1, \theta_2) + t_2(\theta_1, \theta_2) \right] = 0$$

For standard arguments, we assume, that the participation constraint for the worst-off type and the incentive constraint for the best-off are binding and solve the relaxed problem under these conditions. Subsequently we show, that the found solution fulfills the neglected conditions. Since $Q_1(\theta_{1j}) \leq 0, j \in L, H$ we refer to individual 1 as the seller and individual 2 as the buyer. Thus, equations (2.3) and (2.4) are binding for individual 2 and equations (2.2) and (2.5) are binding for individual 1.

Rearranging these conditions yields

$$
\begin{aligned}
T_2(\theta_2^L) &= \theta_2^L Q_2(\theta_2^L) \\
T_1(\theta_1^H) &= \theta_1^H Q_1(\theta_1^H) \\
T_2(\theta_2^H) &= \theta_2^H Q_2(\theta_2^H) - (\theta_2^H - \theta_2^L) Q_2(\theta_2^L) \\
T_1(\theta_1^L) &= \theta_1^L Q_1(\theta_1^L) - (\theta_1^L - \theta_1^H) Q_1(\theta_1^H)
\end{aligned}
$$

Plugging these transfers into the neglected conditions:

$$0 \geq \theta_2^L Q_2(\theta_2^H) - \left(\theta_2^H Q_2(\theta_2^H) - (\theta_2^H - \theta_2^L) Q_2(\theta_2^L)\right) \Leftrightarrow Q_2(\theta_2^H) \geq Q_2(\theta_2^L) \tag{1.21}$$

$$\theta_2^H Q_2(\theta_2^H) - \theta_2^H Q_2(\theta_2^H) - (\theta_2^H - \theta_2^L) Q_2(\theta_2^L) \geq 0 \; . \tag{1.22}$$

Equation (1.22) always holds. The constraints for player 1 are analogous. We check now the budget balance condition

$$
\begin{aligned}
&f_1^H \left(\theta_1^H Q_1(\theta_1^H)\right) + f_1^L \left(\theta_1^L Q_1(\theta_1^L) - (\theta_1^L - \theta_1^H) Q_1(\theta_1^H)\right) \\
&= - \left(f_2^L (\theta_2^L Q_2(\theta_2^L)) + f_2^H (\theta_2^H Q_2(\theta_2^H) - (\theta_2^H - \theta_2^L) Q_2(\theta_2^L))\right) \; .
\end{aligned}
$$

Thus, for $Q_i(\theta_i^H) \geq Q_i(\theta_i^L)$ we found the desired transfers, such that $t_1(\theta_1, \theta_2)$, $t_2(\theta_1, \theta_2)$, $q_1(\theta_1, \theta_2)$, $q_2(\theta_1, \theta_2)$ fulfill incentive constraints, participation constraints, if and only if

$$\mathbb{E}_\theta[\theta_1 q_1(\theta_1, \theta_2) + \theta_2 q_2(\theta_1, \theta_2)] - f_1^L (\theta_1^H - \theta_1^L) Q_1(\theta_1^H) - f_2^H (\theta_2^H - \theta_2^L) Q_2(\theta_2^L) \geq 0 \; .$$

Proof of Proposition 1.11.

We solve the relaxed problem and check, whether the solution fulfill the neglected conditions. Incentive and participation constraints are given by equations (2.2) - (2.5).

We assume that the participation constraints are binding. Thus,

$$
\begin{aligned}
T_i(\theta_i^L) &= \theta_i^L Q_i(\theta_i^L) \\
T_i(\theta_1^H) &= \theta_i^H Q_i(\theta_i^H)
\end{aligned}
$$

Check for the incentive constraints:

$$0 \geq \theta_i^L Q_i(\theta_i^H) - \theta_i^H Q_i(\theta_i^H)$$

$$0 \geq \theta_i^H Q_i(\theta_i^L) - \theta_i^L Q_i(\theta_i^L) \ .$$

Since $Q_i(\theta_i^H) \geq 0$ and $Q_i(\theta_i^L) \leq 0$, both incentive constraints hold. To check budget balance, we plug the transfers into the budget balance condition and get the desired result.

Appendix 1.D From discrete to continuous for the firm side

Production Economy. We define a production economy (J, Δ, p, y) as the vector consisting of producers $j \in J$, the Cartesian product of type sets (one type set for every firm j) $\Delta = \Delta^1 \times \cdots \times \Delta^m$ with the corresponding density functions $p^j : \Delta_j \to [0,1]$, where $p := (p_0, \ldots, p_m)$, and the allocation rule $y : \Delta \to \mathbb{R}_+$, that maps reported types into allocations. Thus, a sequence of production economies is given by $(J_k, \Delta_k, p_k, y_k)_{k \in \mathbb{N}}$, where J_k is a sequence of producer sets, Δ_k a sequence of type set-products, p_k a sequence of density-function vectors and y_k a sequence of allocation functions. We introduce some additional definitions and notations:

Producer. We keep the set $J = 1, \ldots, m_k$ be the set of consumers in round k. For ease of notation, we write J instead of J_k and $J_{\infty k}$ in every element of the sequence of production economies.

Type sets. An element Δ_l of the sequence $(\Delta_k)_{k \in \mathbb{N}}$ is given by the Cartesian product over the type sets of the firms in round k. Since we assume symmetric firms, every firm $\kappa \in J$ has the same type set $\Delta_{k\kappa}$. Thus, $\Delta_k = (\Delta_{k\kappa})^{m_k}$

Density Functions. The sequence $(p_k)_{k \in \mathbb{N}}$ consists of $p_k = (p_{k1}, \ldots, p_{km_k})$, where p_{kj} is the density function over a finite type set, namely Δ_{kj}, for all $j < m_k < \infty, j \in J$.

*Allocation Functions.*The sequence of allocations functions is given by $(y_k)_{k \in \mathbb{N}} : (\Delta_k)_{k \in \mathbb{N}} \to (x_k)_{k \in \mathbb{N}}, x_k \in \mathbb{R}$, where $y_k : \Delta_k \to \mathbb{R}$. *Remark*: The definitions of convergence are analogously to the consumption side. The revenues that we receive for a mechanism (y, r) in a continuous environment, that fulfills incentive compatibility constraints and participation constraints can be found in equation (1.10)

Proposition 1.12. *Let $(J, \Delta_k, p_k, y_k)_{k \in \mathbb{N}} \to (J, \Delta_\infty, p_\infty, y_\infty)$, for $k \to \infty$. Then it holds, that for every producer $j \in J$: For every $\epsilon > 0 \; \exists \quad K : \forall \; k \geq K$*

$$\Bigg| \mathbb{E}_{(\delta_{kj})} \left[\sum_{k=1}^{r-1} \left(k(\delta_{kj}, y_{kj}(\theta, \delta)) + p(\delta_j^l)\{k(\delta_{kj}^k, y_{kj}(\theta, \delta)) - k(\delta_{k(kj)}^{k-1}, y_{kj}(\theta, \delta))\} \frac{P_k(\delta_{kj})}{p_k(\delta_{kj})} \right) \right] - \\ \mathbb{E}_{(\delta_{\infty j})} \left[k(\delta_{\infty j}, y_{\infty j}(\theta, \delta)) + \int_{\delta_{\infty j}^1}^{\delta_{\infty j}^r} p_\infty(\delta_{\infty j}) k_1(\delta_{\infty j}, y_{\infty j}(\theta, \delta)) \frac{P_\infty(\delta_{\infty j})}{p_\infty(\delta_{\infty j})} d\delta_{\infty j} \right] \Bigg| < \epsilon \; .$$

Proof of Proposition 1.12.
The arguments are similar to the ones used in Proposition 1.8 and therefore left to the reader.

Appendix 1.E Proof of Observations

Proof of Observation 1.1.
In the direct mechanism, there are 4 states of the economy, namely: (θ^H, δ^H), (θ^H, δ^L), (θ^L, δ^H) and (θ^L, δ^L). Since $q(\theta^L, \delta^H) = 0$, straightforward computations yield

$$\begin{aligned}
&(\theta_1^H, \delta_1^H) : f^H f^L \left[\theta^H - \delta^H + (\delta^H - \delta^L)\frac{f^H}{f^L}\right] \\
&(\theta_1^H, \delta_1^L) : +f^H f^H \left[\theta^H - \delta^L\right] \\
&(\theta_1^L, \delta_1^L) : +f^L f^H \left[\theta^L - (\theta^H - \theta^L)\frac{f^H}{f^L} - \delta^L\right] \geq 0
\end{aligned}$$

Hence, implementation of the social choice function is possible if and only if

$$f^L \geq \frac{\delta^H - \theta^L}{\theta^H - \delta^L} \ .$$

Proof of Observation 1.2.
The expected surplus is

$$\begin{aligned}
S() =& f^H f^L \left[\theta^H - \delta^H\right] + f^H f^H \left[\theta^H - \delta^L\right] + f^L f^H \left[\theta^L - \delta^L\right] \\
=& f^H f^L \left[\theta^H - \delta^H + \theta^L - \delta^L\right] + f^H f^H \left[\theta^H - \delta^L\right] \\
=& f^H f^L \left[1 - (\delta^H - \theta^L)\right] + f^H f^H 1 \\
=& f^H f^L \left[1 - d\right] + f^H f^H
\end{aligned}$$

Hence $\frac{\partial S()}{\partial d} < 0$.
The expected information rents are

$$\begin{aligned}
IR() =& f^H f^L \left[\delta^H - \delta^L + \theta^H - \theta^L\right] \\
=& f^H f^L \left[1 + \delta^H - \theta^L\right] \\
=& f^H f^L \left[1 + d\right]
\end{aligned}$$

Hence $\frac{\partial IR()}{\partial d} > 0$.

Proof of Observation 1.3.
From the proof of Observation 1.2, we know that

$$S() = (1 - f^L) f^L \left[1 - d\right] + (1 - f^L)^2 \ .$$

Therefore, whenever $\theta^H - \delta^H < \frac{1}{2}$,
$\frac{\partial S()}{\partial f^L} < 0$.

The expected information rents are

$$IR() = f^H f^L [1 - d] \ .$$

So that $\frac{\partial IR()}{\partial f^L} < 0$.

Proof of Observation 1.4
Consider the Mailath and Postlewaite (1990) setup for $I = \{1, 2\}, \Theta_i = \Theta = \{\theta^L, \theta^H\}$. We assume, that $0.5\left(\theta^L + \theta^H\right) > c$. (For ease of notation: $f(\theta^L) = f^L$) Then g can be implemented efficiently, if and only if

$$\frac{1}{2} \sum_{i=1,2} \mathbb{E}_{(\theta)}[v(\theta_i)g(\theta)] \geq c\mathbb{E}_{(\theta)}[g(\theta)] \ .$$

Since trade takes place, if at least one consumer has a high valuation, and g is either 0 or 1, the right-hand side of the equation equals

$$\begin{aligned} c\mathbb{E}_{(\theta)}[g(\theta)] =& c(1 - Prob(\text{both have a low valuation})) \\ &= c\left(1 - (f^L)^2\right) = c\left((1 + f^L)(1 - f^L)\right) = c\left((1 + f^L)f^H\right) \ . \end{aligned}$$

The left-hand side is given by

$$\begin{aligned} &\frac{1}{2} \sum_{i=1,2} \mathbb{E}_{(\theta)}[v(\theta_i)g(\theta)] \\ &\quad = \frac{1}{2}(f^H)^2 \left[2\theta^H\right] + \frac{1}{2} 2 f^H \, f^L \left[\theta^L - \frac{1 - F(\theta^L)}{f^L}(\theta^H - \theta^L) + \theta^H\right] \\ &\quad = f^H \left(\theta^L + f^L \theta^H\right) \ . \end{aligned}$$

This yields that efficient implementation is possible, if and only if

$$\theta^L + f^L \theta^H \geq (1 - f^L)c \Leftrightarrow f^L \geq \frac{c - \theta^L}{\theta^H - c}$$

Proof of Observation 1.5.

$$MS = -\left(\underbrace{\left[\theta^H - c - f_L(c - \theta^L)\right]}_{S(\cdot)} - \underbrace{(1 - f^L)(\theta^H - \theta^L)}_{IR(\cdot)}\right) \leq 0 \ .$$

i) If f^L is fixed, then $\frac{\partial S()}{\partial c} < 0$.
ii) If c is fixed, then $\frac{\partial S()}{\partial c} < 0$, and $\frac{\partial IR()}{\partial c} < 0$.

Proof of Observation 1.6.

Consider the mechanism, in which the third type θ^M and δ^M for each player is added by cutting the existing intervals $[\theta^H, \theta^L]$ and $[\delta^H, \delta^L]$ into halves and set

$$\delta^M = \frac{1}{2}\left(\delta^H - \theta^L + \frac{1-(\delta^H - \theta^L)}{2}\right) ,$$

and

$$\theta^M = \frac{1}{2}\left(\delta^H - \theta^L + \frac{1-(\delta^H - \theta^L)}{2}\right) + \frac{1-(\delta^H - \theta^L)}{2} .$$

The mechanism works like the mechanism with outcome function f. We assume, that the new middle type has probability, f_1^M.
Efficient implementation is possible, whenever Corollary 1.1 applies.
If an additional type is introduced for the buyer and the seller, there are 9 states of the economy. Trade is inefficient when the low valuation buyer faces the high cost seller.

$$\begin{aligned}
(\theta^H, \delta^H) &: f_1^H f_1^L \left[\theta^H - \left(\delta^H + (\delta^H - \delta^M)\frac{f_1^H + f_1^M}{f_1^L}\right)\right]\\
(\theta^H, \delta^M) &: f_1^H f_1^M \left[\theta^H - \left(\delta^M + (\delta^M - \delta^L)\frac{f_1^H}{f_1^M}\right)\right]\\
(\theta^H, \delta^L) &: f_1^H f_1^H \left[\theta^H - \delta^L\right]\\
(\theta^M, \delta^H) &: f_1^M f_1^H \left[\theta^M - (\theta^H - \theta^M)\frac{f_1^H}{f_1^M} - \left(\delta^H + (\delta^H - \delta^M)\frac{f_1^H + f_1^M}{f_1^L}\right)\right]\\
(\theta^M, \delta^M) &: f_1^M f_1^M \left[\theta^M - (\theta^H - \theta^M)\frac{f_1^H}{f_1^M} - \left(\delta^M + (\delta^H - \delta^M)\frac{f_1^H}{f_1^M}\right)\right]\\
(\theta^M, \delta^L) &: f_1^M f_1^L \left[\theta^M - (\theta^H - \theta^M)\frac{f_1^H}{f_1^M} - \delta^L\right]\\
(\theta^L, \delta^M) &: f_1^L f_1^M \left[\theta^L - (\theta^M - \theta^L)\frac{f_1^H + f_1^M}{f_1^L} - \left(\delta^M + (\delta^M - \delta^L)\frac{f_1^H}{f_1^M}\right)\right]\\
(\theta^L, \delta^L) &: f_1^L f_1^H \left[\theta^L - (\theta^M - \theta^L)\frac{f_1^H + f_1^M}{f_1^L} - \delta^L\right]
\end{aligned}$$

Hence, by using the definition for θ_1^1 and δ_1^1 implementation of the social choice function is possible if and only if

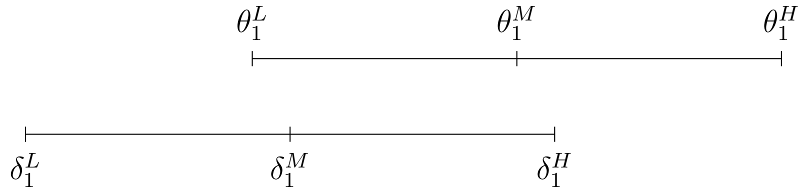
$$f_1^L(\theta_1^M - \delta_1^M) \geq \delta_1^H - \theta_1^L .$$

If we introduce a third type for each player, so that $\theta_1^M < \delta_1^H$
Applying Proposition 1.4, efficient bilateral trade is possible if and only if

$$f_1^L(\theta_1^H - \delta_1^L) \geq \delta_1^H - \theta_1^L .$$

Proof of Observation 1.7.

Figure 1.10: 'Round 1', Case b



By Observation 1.1, we know that implementation in 'round 0' is possible if

$$f_0^L > d \, .$$

By Observation 1.9, we know that implementation in 'round 1' is possible if

$$f_1^L > \frac{\delta_1^H - \theta_1^L}{\theta_1^M - \delta_1^M} \, .$$

Making use of Definition 1.2, implementation in 'round 1' is less costly than in 'round 0', i.e., the minimal subsidy decreases, if

$$f_1^L > \frac{2f_0^L}{1 + f_0^L} \, .$$

The monotone hazard rate assumption is satisfied if

$$0 \leq \frac{f_1^H}{f_1^M} \leq \frac{f_1^H + f_1^M}{f_1^L} \, .$$

When f_1^M is sufficiently small, the monotone hazard rate assumption is violated and implementation can still be possible in 'round 1'.

Proof of Observation 1.8.
Consider the mechanism, in which the third type θ^M and δ^M for each player is added such that $\theta_1^M < \delta_1^M$. Efficient implementation is possible, whenever Corollary 1.1 applies. If an additional type is introduced for the buyer and the seller, there are 9 states of the

economy. Trade is efficient only in 5 of them.

$$(\theta^H,\delta^H): f_1^H f_1^L\left[\theta^H-\left(\delta^H+(\delta^H-\delta^M)\frac{f_1^H+f_1^M}{f_1^L}\right)\right]$$
$$(\theta^H,\delta^M): f_1^H f_1^M\left[\theta^H-\left(\delta^M+(\delta^M-\delta^L)\frac{f_1^H}{f_1^M}\right)\right]$$
$$(\theta^H,\delta^L): f_1^H f_1^H\left[\theta^H-\delta^L\right]$$
$$(\theta^M,\delta^L): f_1^M f_1^L\left[\theta^M-(\theta^H-\theta^M)\frac{f_1^H}{f_1^M}-\delta^L\right]$$
$$(\theta^L,\delta^L): f_1^L f_1^H\left[\theta^L-(\theta^M-\theta^L)\frac{f_1^H+f_1^M}{f_1^L}-\delta^L\right]$$

Hence, implementation of the social choice function is possible if and only if

$$f_1^L \geq \frac{\delta^H-\theta^L}{\theta^H-\delta^L}\ .$$

Proof of Observation 1.9.
Consider the Mailath and Postlewaite (1990) setup for $I=\{1,2\}$, $\Theta_i=\Theta=\{\theta_1^L,\theta_1^M,\theta_1^H\}$. We assume, that $0.5\left(\theta_1^L+\theta_1^M\right)>c$. Then g can be implemented efficiently, if and only if

$$\frac{1}{2}\sum_{i=1,2}\mathbb{E}_{(\theta)}[v(\theta_i)g(\theta)] \geq c\mathbb{E}_{(\theta)}[g(\theta)]$$

Since trade takes place, if at least one consumer has a middle valuation, and g is either 0 or 1, the right-hand side of the equation equals

$$\begin{aligned}c\mathbb{E}_{(\theta)}[g(\theta)] =& c(1-Prob(\text{both have a low valuation}))\\ &= c\left(1-(f_1^L)^2\right)=c\left((1+f_1^L)(1-f_1^L)\right)=c(1+f_1^L)(f^H+f_1^M)\end{aligned}$$

The left-hand side is given by:

$$\begin{aligned}&\frac{1}{2}\sum_{i=1,2}\mathbb{E}_{(\theta)}[v(\theta_i)g(\theta)]\\ &\qquad=\theta_1^M\left[f_1^L\left(f_1^H+f_1^M\right)\right]+\theta_1^L\left(f_1^H+f_1^M\right)\\ &\qquad=\theta_1^M f_1^L\left(f_1^H+f_1^M\right)+\theta_1^L\left(f_1^H+f_1^M\right)\end{aligned}$$

This yields that efficient implementation is possible, if and only if

$$f_1^L > \frac{c-\theta_1^L}{\theta_1^M-c}$$

2

Optimal non-linear income taxation for arbitrary welfare weights

2.1 Introduction

The given paper addresses the question how optimal income tax schedules should be designed under the assumption that the direction of redistribution can vary arbitrarily. A model building on Mirrlees (1971) is used where the welfare function is assumed to be weighted Utilitarian. The welfare weights assigned to productivity types reflect the redistributive preference of the planner. Typically, the existing literature in tradition of Mirrlees (1971) focuses on redistribution towards the poor. In discrete type models this translates to welfare weights that decrease in the individuals' productivity as e.g. in Weymark (1986a) or Simula (2010). For models with continuous type sets, optimal redistribution towards the poor is a consequence of a concave welfare function, see e.g. Mirrlees (1971) or Ebert (1992), Brunner (1993, 1995) and more recently Hellwig (2007). The present paper considers a broader set of welfare objectives, given by weighted Utilitarian welfare functions with arbitrary weights, and analyzes the formal implications for the corresponding optimization problem and its solutions. It, thereby, provides theoretical contributions to the literature on mechanism design theory and, in particular, to the field of optimal tax theory.

Extending the analysis to a broader set of welfare weights is in line with various papers in recent years. These papers question the monotonicity assumption Mirrleesian models apply on welfare weights and study alternative formulations. A comprehensive investigation of "generalized" welfare weights in a taxation framework is provided by Saez and Stantcheva (2016). They introduce a formulation of welfare weights that allows to study various redistributive preferences; among other features, those weights include fairness considerations as these distinguish between individual characteristics that require compensation via a tax and those that do not. Furthermore, there is a branch in the literature taking an inverse perspective on optimal tax schemes and, thereby, shedding more light on the importance of analyzing various welfare weights and deriving the corresponding income tax schedules. Zoutman et al. (2014, 2017), Bourguignon and Spadaro (2012) as well as Lockwood and Weinzierl (2016) use income tax schemes that are implemented to infer marginal social welfare weights. By this "inverse-optimum" method, Bourguignon and Spadaro (2012) find that marginal welfare weights are negative for high earners which implies that the tax scheme is not Paretian.[1] For the Netherlands, Zoutman et al. (2014, 2017) infer a non-monotonic pattern of welfare weight. i.e the weight that is put on middle incomes is much higher than the one on small incomes or non-working people. This contradicts strict redistribution towards the poor as considered by Mirrlees (1971) and subsequent work. Conclusively, these findings motivate to deepen the examination of welfare weights as determining factor of optimal tax schemes.[2]

Slightly changing the perspective, the given paper can also be read from a political economy perspective such that welfare weights or, equivalently, the corresponding tax rates, reflect a political position. Citizen candidate models as firstly considered by Osborne and Slivinski (1996) or Besley and Coate (1997) can be used to identify politically feasible policies. Among others, Röell (2012) and Brett and Weymark (2017) use this framework to study voting over tax schedules and show that the median's most preferred tax schedule is politically feasible, more precisely, wins in a majority voting procedure. They focus on selfishly optimal taxes formalized by welfare weights that are zero for all but one productivity type and, thus, yield redistribution towards the type with the non-zero welfare weight. In game-theoretical terms these binary weights can be interpreted as pure strategies in the policy space of Utilitarian welfare objectives. Analogously, a vector of welfare weights that consists of elements between zero and one can be seen as a mixed strategy that results in various directions of optimal redistribution. In that sense, the given paper is a first step towards a broader analysis on politically feasible income tax schedules and, in turn, on the link of taxation theory and political economy. Note that the interpretation of welfare weights as mixed strategies also refers to a strand in the political science literature that studies optimal party positioning in multidimensional policy spaces. The phenomenon of choosing a position that is a mixture of several dimensions is called "blurring", see Rovny (2012) and Elias et al. (2015).

Concluding, it seems to be useful from different perspectives to widen the focus and consider the complete set of welfare weights, i.e. to expand the existing and well understood results on optimal income taxation in the fashion of Mirrlees (1971) to a broader set of social preferences. To do so, it is decisive to understand how the underlying optimization problem is affected by that. The following section provides more detailed information about the model, the technical details and gives an overview of the main findings.

Main Results The model in the present paper is a Mirrleesian income tax model with a continuum of individuals and three different productivity levels which are distributed

[1] Exemptions are situations including a very small labor supply elasticity.

[2] With regard to the support of Mirrleesian redistribution pattern in reality, Weinzierl (2014) provides empirical evidence that people disagree with the unweighted Utilitarian approach of Mirrlees (1971) but rather prefer an equal sacrifice objective.

uniformly. Individuals hold private information about their productivity such that the optimization problem includes, in addition to the resource constraint, also incentive constraints. The welfare objective is a weighted Utilitarian where the sum of weights is w.l.o.g. normalized to one.

In the first part of the paper, the incentive structure of the optimization problem is analyzed. For every possible vector of welfare weights, binding incentive constraints are identified. As these reflect the optimal directions of redistribution, binding constraints vary in the weights assigned to the productivity types. The analysis considers each pair of adjacent types and derives the respectively binding incentive constraint. Disentangling incentives this way is only possible as types are assumed to be discrete. In models with a continuous type set, the concept of adjacent types and, hence, local incentive constraints is not welldefined such that incentive effects of various welfare objectives are disguised.

There are two prevailing pattern of binding constraints in the literature: for the canonical case of monotonically decreasing welfare weights, all local downward incentive constraints bind, see Weymark (1986a) or Simula (2010).[3] For the case of selfishly optimal welfare weights, Röell (2012) and Brett and Weymark (2017) show that the incentive constraints pointing towards the favored type bind at the optimum.[4]

Yet, the existing work lacks not only an examination of additional pattern or, equivalently, welfare weights but does also not address the question whether those pattern are obtained injectively in the sense that the considered welfare weights are the only ones inducing the given pattern of binding constraints.[5]

With regard to these observations, the present paper tackles the following two points: firstly, it extends the subject of investigation towards taxation problems with non monotonic welfare weights. Secondly, it refines the existing analysis by identifying the set of welfare weights that induce the established pattern of binding constraints. It is noteworthy that the given setup includes three productivity types which increases the complexity of the problem compared to a two-type model. Not only the number of combinations of potentially binding constraints multiplies but also qualitatively different pattern of binding constraints, e.g. bunching, can occur at the optimum. This is never true for two-type models as shown by Bierbrauer and Boyer (2014).

The analysis in the first part of the presented paper uses a perturbation argument to deliver a characterization of the incentive structure of the given problem. Defining utility levels as variables, the binding incentive constraints can be inferred from the welfare effect of the considered utility perturbation. The findings are summarized in

[3] See Mirrlees (1971), Ebert (1992), Brunner (1993, 1995) or Hellwig (2007) for the continuous equivalent.

[4] For results on binding incentive constraints in an investigation of second-best Pareto efficient tax schedules, see Stiglitz (1982) and Bierbrauer and Boyer (2014).

[5] With regard to this, Weymark (1986a) includes a sidenote saying that monotonically decreasing welfare weights are not necessary for local downward incentive constraints to bind. However, there is no further specification of the corresponding weights.

two formal conditions that assign a set of binding incentive constraints to a vector of welfare weights. For each pair of adjacent productivity types, the conditions determine the respectively binding constraint. It becomes clear that relative weights are decisive for an incentive constraint to bind.

Furthermore, the conditions imply that depending on the welfare weight, none, one or two incentive constraints are determined to bind. If there is no desire for redistribution between adjacent types, the corresponding vector of welfare weights pins down no or only one binding constraint. For scenarios with two binding incentive constraints, it holds that these are not mutually binding between adjacent types but belong to two different pairs of types.

To illustrate the resulting correspondence between welfare weights and incentive constraints, a geometrical presentation is introduced: as welfare weights sum up to one, the set of weights can be presented by a simplex. The triangle of welfare weights can be partitioned according to the binding constraints. The separating lines are defined by the aforementioned conditions and represent welfare weights that pin down only one binding incentive constraint. At the intersection of those lines, no incentive constraint binds which describes the laissez faire situation.[6]

The second part of the paper is aimed at solving the optimization problem and deriving optimal marginal tax rates. A mechanism design approach is used to compute optimal income-consumption bundles for each productivity type. The identification of binding constraints and the assumption of quasilinear preferences allow to simplify the analysis: a reduced form problem can be stated for every possible vector of welfare weights. It is obtained by rearranging binding constraints such that a new welfare function can be formulated which incorporates all binding constraints and is a function of income only. This halves the dimensionality of the problem as welfare is no longer a function of both income and consumption. The procedure was firstly used in a non-taxation context by Mussa and Rosen (1978) who study a monopoly pricing problem, and later by Myerson (1981) and Guesnerie and Laffont (1984) in an auction setting and a general principal-agent-model, respectively. The first applying it to a tax problem were Lollivier and Rochet (1983) in a continuous and Weymark (1986a) in a discrete type setting. More recently, Simula (2010) uses this technique to characterize optimal tax schemes in a setting with finitely many types. He focuses on strictly decreasing welfare weights.

Studying the restated welfare function reveals further insight in the incentive effects of a given vector of welfare weights. Beside the budgetary restriction of the optimization, binding incentive constraints show up in the new welfare function by information rents. These have to be paid to prevent individuals from announcing another than their true productivity. Depending on the directions of redistribution, the sign and the argument of information rents differ. Whenever a local incentive constraint binds downwards

[6]The simplex representation is similar to the one in Laslier and Picard (2002) where the redistribution of one unit of a homogeneous good among n individuals is studied in a political competition framework.

(upwards), the corresponding information rent enters welfare negatively (positively). From slack incentive constraints, as given e.g. in the laissez faire economy, no social cost arise and no information rent has to be paid. The argument of an information rent reveals which type the receiving individuals have an incentive to mimic. As a result, all previously mentioned combinations of binding constraints yield a different reduced form, i.e. welfare function. For a formulation of the canonical case on redistributing downwards see Weymark (1986a) or Simula (2010).

The subsequent analysis derives optimal marginal tax rates based on a first order approach. First order conditions are computed based on the restated welfare function. Assuming these to be sufficient, those conditions characterize the optimal income level. Due to the quasilinearity of preferences, optimal income unambiguously determines optimal consumption. Since the restated welfare function is formulated only based on binding constraints, one further step is necessary to prove that the derived allocation is optimal. Only if the computed allocation is monotonically increasing in the productivity type, it characterizes an optimum. Hellwig (2007) shows that when all local downward incentive constraints bind, upward incentive constraints can be replaced by a monotonicity condition. The given paper expands this result to the newly derived pattern of binding incentive constraints. For non-monotonic solutions, the optimal allocation includes bunching. Utilizing the first order conditions and the single-crossing characteristic of preferences, some types of bunching can be ruled out: if optimal redistribution is directed towards the boundary types, bunching cannot be optimal. There is no bunching around the highly productive type if redistribution towards the poor is optimal. This was firstly noted by Guesnerie and Seade (1982) for a non-linear pricing context, and later studied e.g. by Weymark (1986b). Vice versa, redistributing towards the rich does not allow for bunching around the least skilled type. For weights that induce redistribution towards the median-skilled type bunching cannot be ruled out. Detailed work on bunching for scenarios with decreasing welfare weights or selfishly optimal welfare weights is done by Weymark (1986b) and Brett and Weymark (2017), respectively.

Focusing on fully separating allocations, the subsequent analysis delivers the following results on optimal tax schedules: it is never optimal to have negative marginal tax rates for low productive or positive tax rates for highly productive individuals. Individuals with a median productivity can face positive or negative marginal tax rates at the optimum depending on the welfare weights. Generally, tax rates of a productivity are zero if his local incentive constraints bind. This is in line with the existing literature. For scenarios with redistribution towards the poor, it holds that optimal tax rates are positive for everyone except the most productive individuals who face marginal taxes of zero at the optimum, see Seade (1977) who extends the results of Mirrlees (1971). For vectors of degenerate welfare weights, where all except one weight are zero, Röell (2012) and Brett and Weymark (2017) find that optimal tax rates below the favored type are negative and above the favored type are positive.

Lastly, comparative static properties of optimal tax rates with respect to the welfare weights are stated. Regardless of the sign, marginal tax rates are non-decreasing (non-increasing) in the welfare weight of the low (high) productivity type. For the Mirrleesian redistribution pattern, Weymark (1987) and, more recently, Simula (2010) deliver results on comparative static properties of optimal non-linear income tax rates also with respect to welfare weights. Further results on how welfare weights affect optimal income tax rates can be found in Bierbrauer and Boyer (2014) who describe the complete second-best Pareto frontier in a two-type model.[7]

The remainder is organized as follows: Section 2.2 describes the formal environment. It is followed by the introduction of the simplex of welfare weights and the formal identification of binding incentive constraints in Section 2.3. Section 2.4 starts with the formulation of the reduced form problem. Subsequently, optimal tax rates and comparative static properties are derived. Part 2.5.1 closes the analysis with a discussion on the relation between the delivered results and (second-best) Pareto efficiency. The last section gives an outlook on future steps.

2.2 Environment

The following analysis uses a simpler version of the Mirrleesian income tax model, see Mirrlees (1971), where preferences are quasilinear and given by the following utility function $U(c^i, h^i) = c^i - v(h^i)$. Person $i's$ consumption level of a private good is denoted by c^i and h^i denotes her hours worked. The utility function is increasing in consumption and decreasing in labor, $v'(\cdot) > 0$, $v''(\cdot) > 0$ and $\lim_{h\to\infty} v'(\cdot) = \infty$ and $\lim_{h\to 0} v'(\cdot) = 0$.

The set of individuals is given by $I = [0, 1]$. They differ in their productive ability where the set of all possible productivity levels is $\Omega = \{w_L, w_M, w_H\}$. It holds that $0 < w_L < w_M < w_H$. In the economy, equally many individuals of each type exist.[8] The distribution of types is common knowledge.

The production possibilities in the economy are such that if an individual with productivity w_k works for one hour she produces w_k units of the consumption good. The output she produces is therefore given by $y^i_k = w_k h^i_k$. For a given productivity w_k, we can rewrite person i's utility function as

$$U^i_k(c^i_k, h^i_k) \;=\; c^i_k - v\left(\frac{y^i_k}{w_k}\right) \;=: u_k.$$

Individuals of the same productive ability w_k are treated equally in the sense that they consume the same amount of the private good c_k and work for h_k hours, i.e. produce the same output y_k. Hence, the index i distinguishing between single individuals can

[7] Comparative static results for linear tax schedules are derived by Hellwig (1986).

[8] This assumption simplifies the analysis but is in fact only a normalization and thus w.l.o.g..

be omitted such that an allocation for the described economy is given by a collection of consumption and output levels for each productivity level $(c_k, y_k)_{k=L,M,H} \in \mathbb{R}^+ \times \mathbb{R}$.

In this setting, the Spence-Mirrlees single-crossing condition is satisfied where preferences are such that indifference curves intersect only once,

$$\frac{\frac{\partial U_k}{\partial y_k}}{\frac{\partial U_k}{\partial c_k}} > \frac{\frac{\partial U_l}{\partial y_l}}{\frac{\partial U_l}{\partial c_l}} \quad \Leftrightarrow \quad \frac{1}{w_k} v'\left(\frac{y_k}{w_k}\right) > \frac{1}{w_l} v'\left(\frac{y_l}{w_l}\right), \quad \text{if} \quad w_l > w_k. \tag{SC}$$

An allocation is *resource feasible* if aggregate consumption does not exceed aggregate production,[9]

$$\sum_{k=L,M,H} c_k \leq \sum_{k=L,M,H} y_k,$$

Individuals have private information about their productive abilities w_k and about hours worked h_k. By contrast, consumption c_k and output $y_k = w_k h_k$ are assumed to be observable.

For this information structure, we call an allocation *incentive compatible* if it satisfies the following incentive compatibility constraints: for every pair of productivity level w_k and w_l,

$$U_k(c_k, y_k) = c_k - v(\tfrac{y_k}{w_k}) \geq c_l - v(\tfrac{y_l}{w_k}) = U_k(c_l, y_l).$$

So, incentive compatibility ensures that an individual of type w_k prefers consuming c_k and working $\frac{y_k}{w_k}$ hours over consuming c_l and working for $\frac{y_l}{w_k}$ hours.

An allocation (c, y) is *incentive feasible* if it is resource feasible and incentive compatible.

To study the Mirrleesian income tax problem a mechanism design approach is used: instead of choosing a tax schedule $T(y)$ as a function of income, a set of allocations $(c_k, y_k)_{k=L,M,H}$ with a designated allocation for each skill type is stated. The allocations are designed such that they allow for a decentralization via a tax system: by the taxation principle due to Hammond (1979) and Guesnerie (1995), an allocation $(c_k, y_k)_{k=L,M,H}$ can be reached by an income tax system $T : y \mapsto T(y)$ if and only if it is incentive feasible.

Implicit marginal tax rates are defined by the wedge between the marginal rate of substitution and the marginal rate of transformation:

$$\tau_k := T'(y_k) = 1 - \tfrac{1}{\omega_k} v'(\tfrac{y_k}{\omega_k}). \tag{2.1}$$

As preferences are quasilinear in consumption, marginal tax rates do not depend on consumption. When marginal tax rates are zero, labor supply is undistorted. In con-

[9] One could account for positive governmental expenditures, but they do not affect results qualitatively.

trast, whenever incentive constraints bind, a distortion in the respective labor supply is induced.
The following chapter identifies the binding incentive constraints as a function of welfare weights.

2.3 Simplex of welfare weights

The following analysis examines how the mathematical formulation of the welfare maximization problem is affected the assumption of arbitrary welfare weights.

The analysis is aimed at simplifying the optimization problem in the sense that binding constraints are determined for every possible pattern of redistributive preferences, formally, every possible vector of welfare weights.

The welfare maximization problem $P(g)$ is given by:
for a given vector of welfare weights, choose an allocation $(c_k, y_k)_{k=L,M,H}$ to maximize the corresponding weighted Utilitarian welfare function $W(\cdot; g)$ subject to incentive constraints and a resource constraint. Formally,

$$\max_{(c,y)} W(c, y; g) = g_L U_L(c_L, \tfrac{y_L}{w_L}) + g_M U_M(c_M, \tfrac{y_M}{w_M}) + g_H U_H(c_H, \tfrac{y_H}{w_H})$$

$$c_L - v(\tfrac{y_L}{w_L}) \geq c_M - v(\tfrac{y_M}{w_L}), \qquad (IC_L)$$

$$c_M - v(\tfrac{y_M}{w_M}) \geq c_L - v(\tfrac{y_L}{w_M}), \qquad (IC_{ML})$$

$$c_M - v(\tfrac{y_M}{w_M}) \geq c_H - v(\tfrac{y_H}{w_M}), \qquad (IC_{MH})$$

$$c_H - v(\tfrac{y_H}{w_H}) \geq c_M - v(\tfrac{y_M}{w_H}), \qquad (IC_H)$$

$$y_L + y_M + y_H \geq c_L + c_M + c_H. \qquad (RC)$$

where $g = (g_L, g_M, g_H) \in G$ is the vector of welfare weights. It expresses the redistributive preferences of the planner and is assumed to be arbitrary. Using a normalized presentation, the complete set of possible welfare weights is $G = \{g = (g_L, g_M, g_H) : g_L + g_M + g_H = 1\}$. As the solution of the optimization problem depends on g, the optimal allocation is denoted by $(c^*(g), y^*(g)) = (c_k^*(g), y_k^*(g))_{k=L,M,H}$.

To compute the optimal allocation for a given weight $g \in G$, it is useful to know the binding constraints as it simplifies the analysis. Those constraints depend on the desired directions of redistribution. In particular, if g entails conflicting directions of redistribution, which means that welfare weights are not monotonic in the type, it is a priori unclear which constraint binds. The following analysis delivers conditions that unambiguously determine binding constraints as a function of g.

If welfare weights are egalitarian, such that $g_L = g_M = g_H = \frac{1}{3}$, there is no desire to redistribute between different productivity types since preferences are quasilinear. Hence, in this case, no incentive constraint binds. Whenever $g_j \neq g_k \neq \frac{1}{3}$, redistribution is desirable such that incentive constraints are no longer slack. To specify the binding incentive constraints for each vector $g \in G$, it is useful to put some structure on the set G by distinguishing weights g with respect to the ordinal ranking of single welfare weights (g_L, g_M, g_H). There are six different ways to rank weights strictly given by the list below. Additionally, there are elements $g \in G$ that entail two (or more) identical weights.

$$g_L > g_M > g_H \tag{i}$$
$$g_L > g_H > g_M \tag{ii}$$
$$g_M > g_L > g_H \tag{iii}$$
$$g_M > g_H > g_L \tag{iv}$$
$$g_H > g_L > g_M \tag{v}$$
$$g_H > g_M > g_L \tag{vi}$$

The normalized welfare weights can be illustrated by a simplex where the welfare weight of the median type is expressed in terms of the others' welfare weights, such that $g_M = 1 - g_L - g_H$, see Figure 2.1.

Some welfare weights, or more generally, sets of welfare weights, are well studied. The Mirrleesian case of redistributing towards the poor is represented in the given framework by non-increasing welfare weights which is region (i) of the simplex. For every element in this subset of G, all local downward incentive constraints bind. Analogously, welfare weights that are non-decreasing in the skill type as in $((vi))$ induce all local upward incentive constraints to bind. The degenerate cases of selfishly optimal weights where $g_k = 1$ for some k, yield incentive constraints to bind towards type k as her utility is maximized.[10]

The following analysis replicates these findings, but, additionally delivers insights for welfare weights that are neither monotonic nor selfishly optimal illustrated by regions $ii) - v)$.

[10]For details on the results see Weymark (1986a), Simula (2010), Röell (2012) or Brett and Weymark (2017), respectively.

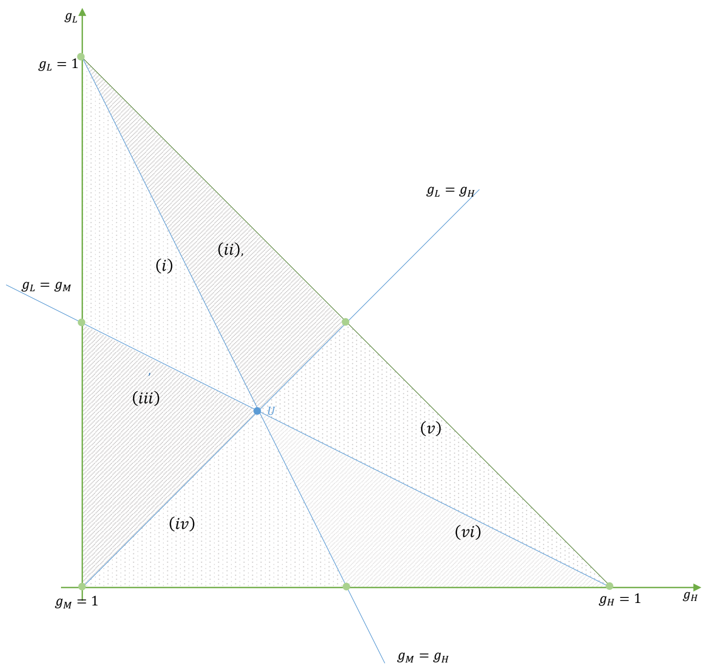


Figure 2.1: Simplex of welfare weights

To figure out what pattern of binding constraints applies for a given welfare weight g, redistributive preferences have to be disentangled. If welfare weights are non-monotonic and, so, include various directions of redistribution, the dominating ones have to be identified as they determine the binding incentive constraints.

The procedure starts at an agnostic point, where all four incentive constraints are assumed to be slack. In the next step, this point is left as small perturbations in utility levels are considered and the resulting welfare effect is computed. If the change leads to an increase in welfare without violating any incentive or the resource constraint, the redistribution expressed by the perturbation is desirable in terms of welfare. Hence, redistributing in this direction should be carried on until one of the incentive constraints hinders further redistribution and binds. Vice versa, if welfare has dropped due to the change, redistributing in the opposite direction is desirable. It becomes clear that depending on relative weights, the pattern of binding incentive constraints varies.

Proposition 2.1. *Consider a given welfare weight $g \in G$ and the corresponding optimization problem $P(g)$.*

1) If the following condition holds, IC_{ML} binds at the optimum

$$g_L \quad > \quad \frac{g_M \ + \ g_H}{2}. \tag{A}$$

If the opposite is true, IC_L binds at the optimum.

If the condition holds with equality, neither IC_{ML} nor IC_L has to bind at the optimum.

2) *If the following condition holds strictly, IC_H binds at the optimum*

$$g_H \quad < \quad \frac{g_L \; + \; g_M}{2}. \tag{B}$$

If the opposite is true, IC_{MH} binds at the optimum.
If the condition holds with equality, neither IC_{MH} nor IC_H has to bind at the optimum.

Intuitively, if the welfare weight of the low-skilled type is sufficiently high, she needs to be made off as good as possible to maximize welfare. This creates an incentive for the median-skilled type to mimic this type, such that IC_{ML} binds. More precisely, if low-skilled individuals are assigned a welfare weight that is as least as high as the weighted sum of the others' welfare weights, the local downward incentive constraint of the median skill type binds – and vice versa. Analogously, if the welfare weight of the highly productive type falls below the weighted sum of the others' welfare weights, her local downward incentive constraint binds – and vice versa.

Hence, optimal redistribution depends solely on the welfare weights.[11] Note that the desire to redistribute between adjacent productivity types depends not only on their welfare weights but also on the weight of the third productivity type. E.g. even if $g_L > g_M$, it depends on the welfare weight of the high-skilled type whether the local incentive constraint towards the low-skilled binds. For instance, if we consider area $v)$, where $g_H > g_L > g_M$, it is not a priori clear whether the desire to redistribute towards the high-skilled outweighs the desire to redistribute towards the low-skilled or vice versa. Only if welfare weights are monotonic no other information than the ordinal ranking of singular weights is needed to identify the binding constraints. This is the case for area $i)$ and $iv)$. In the remaining four areas where weights are non-monotonic, condition (A) and (B) have to be checked to 'compare' the conflicting desire to redistribute upwards or downwards to make out the binding incentive constraints.
Replacing $g_M = 1 - g_L - g_H$ in condition (A) and (B), the expressions become even more simple and can be added by lines to Figure 2.1. Condition (A) and (B), respectively, become

$$g_L \; > \; \frac{1}{3}\,, \tag{A}$$

$$g_H \; < \; \frac{1}{3}\,. \tag{B}$$

[11] The population shares enters the given conditions as well. As we assumed w.l.o.g. that the type distribution is uniform, they do not show up explicitly in the conditions.

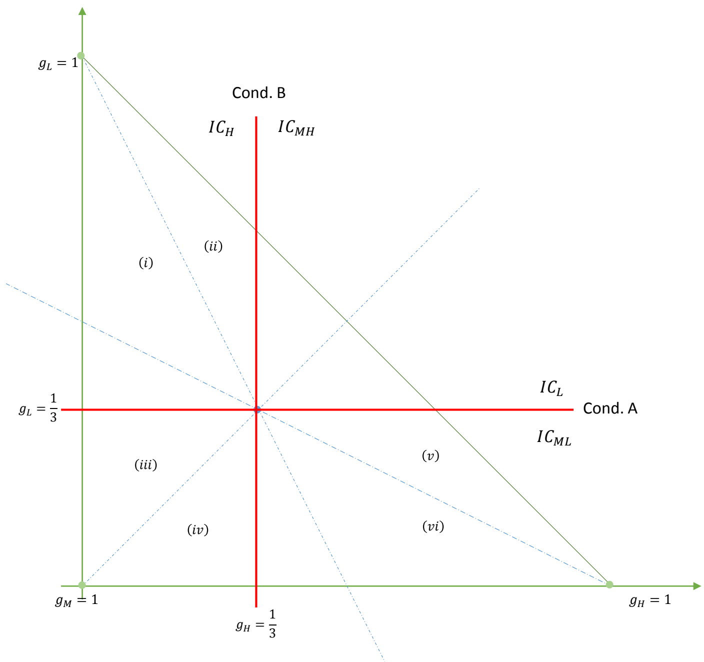


Figure 2.2: Simplex of welfare weights with partitioning

The lines given by $g_L \equiv \frac{1}{3}$ and $g_H \equiv \frac{1}{3}$ in Figure 2.2 represent welfare weights such that condition (A) and (B) hold with equality, respectively. According to proposition 2.1 there is no desire to redistribute between the two respective adjacent skill types. Hence, welfare weights g where either the low- or the high-skilled individuals' welfare weight equals the average of the others weights pin down only one binding incentive constraint. At the intersection of these lines, no incentive constraint binds which describes the laissez faire situation.

In cases where neither condition (A) nor condition (B) hold with equality formally $g_k \neq \frac{1}{3},\ \forall k$, two binding incentive constraints are determined by the conditions. The following lemma summarizes these findings.

Lemma 2.1. *The set of normalized welfare weights G consists of elements that determine no, one or two binding incentive constraints for the solution of problem $P(g)$.*

Local upward (downward) constraints bind if welfare weights are monotonically increasing (decreasing) in the productivity type. No other information than the ordinal ranking of single weights is necessary to make out the two binding constraints. This shows up in the graphic by area $i)$ and $vi)$ not being split by the lines representing (A) and (B), respectively. Contrary, area $ii) - v)$ are cut by those lines. Intuitively, there are conflicting directions of redistribution for welfare weights that lie in those areas. The strength of the respective redistribution desire decides on the final pattern of binding incentive constraints. Constraints bind towards the median skill type if his weight is sufficiently high compared to the other two weights. And, vice versa, if the median's type weight is sufficiently low, constraints pointing towards the boundary types of the skill distribution bind.

It is worth noting, that the assumption of a uniform skill distribution is not restrictive. Depending on the specific distribution, the functional form of the lines that correspond to condition (A) and (B), respectively, differ. However, the qualitative results are independent of that: the line linked to condition (A) splits section $iii)$ and $v)$ and the line linked to condition (B) splits section $ii)$ and $iv)$. So, whenever a group of median-skilled individuals exists, all before mentioned combinations of binding incentive constraints are induced by some welfare weight. The upcoming analysis builds on the pattern of binding incentive constraints, so that the assumption on the skill distribution is w.l.o.g. for the results.

2.4 Welfare maxima and optimal marginal taxes

2.4.1 The reduced-form problem

The findings of the preceding chapter allow to restate the optimization problem $P(g)$ in a way that simplifies the analysis drastically. The binding constraints can be utilized to state a reduced-form problem for which the welfare function incorporates already the binding incentive constraints. Slack constraints are replaced by a monontonicity constraint.[12]

The solution of the original problem $P(g)$ and the reduced-form problem $P'(g)$ are identical.[13]

The analysis proceeds in two steps: in a first step, the reduced-form problem $P'(g)$ is derived. Contrary to the original problem $P(g)$ which involves two control variables, consumption and income, the reduced form entails only one, income. Secondly, first order conditions on optimal income are provided which implicitly describe the optimal allocation as a solution of the reduced-form problem and are henceforth, assumed to be sufficient.[14] Finally, the question whether and for which welfare weights the optimal allocation might include bunching is addressed.

Beside the incentive constraints, the following Corollary proves the resource constraint to bind in the optimal allocation.

[12]Lollivier and Rochet (1983) applied this method to taxation problems with a continuum of individuals and Weymark (1986a) uses this approach in the context of optimal taxation problems when finitely many individuals live in the economy.

[13]For a proof see Proposition 2.3.

[14]Whether those conditions are sufficient depends on the functional form of preferences, i.e. on $v(\cdot)$, see Corollary 2.3.

Corollary 2.1. *If an allocation is welfare maximizing, the resource constraint (RC) binds.*

Given the set of binding constraints, the reduced-form problem $P'(g)$ and its solution can be obtained as follows: rearranging the binding resource and incentive constraints allows to state expressions of consumption $c^*(y)$ that are functions of income and respect all binding constraints:[15]

$$\begin{aligned} c_L^*(y) &= \tfrac{1}{3} \sum_{i=L,M,H} y_i + \sum_{k=M}^{H} \big(1 - \sum_{j=L}^{k-1} f_j\big)\big(v(\tfrac{y_{k-1}}{\omega^k}) - v(\tfrac{y_k}{\omega^k})\big) \\ c_j^*(y() &= c_L^*(y) - \sum_{k=M}^{j} \big(v(\tfrac{y_{k-1}}{\omega^k}) - v(\tfrac{y_k}{\omega^k})\big), \quad j = M, H \end{aligned}$$

with $\omega^M := \begin{cases} \omega_L & \text{if } IC_L \text{ binds,} \\ \omega_M & \text{if } IC_{ML} \text{ binds,} \end{cases}$ and $\omega^H := \begin{cases} \omega_H & \text{if } IC_H \text{ binds,} \\ \omega_M & \text{if } IC_{MH} \text{ binds.} \end{cases}$

Given that, welfare can be restated such that it is a function of income only and incorporates all binding constraints, denoted by $W^*(c^*(y), y; g)$. Maximizing the restated welfare function subject to a monotonicity constraint yields an income vector $y^*(g)$ that respects all constraints and solves the problem. According to Hellwig (2007) non-binding incentive constraints can be replaced by a monotonicity constraint if all local downward incentive constraints bind. The next Corollary extends the result to other pattern of binding incentive constraints.

Corollary 2.2. *Consider the given economy. There are two pairs of adjacent incentive constraints. An allocation is incentive feasible if it is monotonic and one constraint out of each pair of adjacent incentive constraints binds.*

Hence, a monotonic solution of the reduced-form optimization is incentive feasible regardless of the pair of binding incentive constraints. The optimal income level can be obtained by inserting the derived income level in the respective consumption function $c^*\left(y(g)\right)$ such that the final solution $\left(c^*\left(y^*(g)\right), y^*(g)\right)$ is obtained.
A necessary prerequisite for this procedure is the knowledge of the binding incentive constraints. See Simula (2010) applying this method to a similar setting when redistributing downwards is desirable.

Proposition 2.2. *For a given $g \in G$, the reduced form problem $P'(g)$ of problem $P(g)$ is given by: choose an income vector $y = (y_L, y_M, y_H)$ that maximizes the following welfare function $W^*(\cdot)$ subject to the monotonicity and non-negativity constraints*

[15] See the proof of Proposition 2.2 for details on the following expressions.

$$\max_y W^*(c^*(y), y; g) \;=\; \sum_{k=L,M,H} \big(y_k \;-\; v(\tfrac{y_k}{w_k})\big) \;+\; \sum_{j=l,h} \alpha_j\, R_j(y_j)$$
$$s.t. \quad 0 \le y_L \le y_M \le y_H.$$

where

$$y_l \in \{y_L, y_M\} \qquad \textit{and} \qquad y_h \in \{y_M, y_H\},$$

$$\alpha_l \;=\; 1 \;-\; 3\,g_L \qquad \textit{and} \qquad \alpha_h \;=-(1 \;-\; 3\,g_H).$$

and

$$R_l(\cdot) \;=\; \begin{cases} v(\frac{y_M}{w_L}) \;-\; v(\frac{y_M}{w_M}), & \textit{if}\, g_L \;<\; \frac{1}{3}, \\ v(\frac{y_L}{w_L}) \;-\; v(\frac{y_L}{w_M}), & \textit{if}\, g_L \;\ge\; \frac{1}{3}, \text{[16]} \end{cases} \tag{R_l}$$

and

$$R_h(\cdot) \;=\; \begin{cases} v(\frac{y_M}{w_M}) \;-\; v(\frac{y_M}{w_H}), & \textit{if}\, g_H \;\le\; \frac{1}{3}, \text{[17]} \\ v(\frac{y_H}{w_M}) \;-\; v(\frac{y_H}{w_H}), & \textit{if}\, g_H \;>\; \frac{1}{3}. \end{cases} \tag{R_h}$$

$R_j(\cdot),\;\; j \;\in\; \{l, h\}$ is the information rent associated with a binding incentive constraint: $R_l(\cdot)$ is the rent arising from either IC_L or IC_{ML} being binding and, analogously, $R_h(\cdot)$ is the rent arising from either IC_H or IC_{MH} being binding. Those rents have to be paid due to the unobservability of productivity types, so to prevent individuals from mimicking another than their true type. The argument of $R_j(\cdot)$ depends on the direction of redistribution. Information rent $R_j(y_k)$ is paid to an individual who has an incentive to report type w_k.
The factor $\alpha_j \in \{\alpha_l,\, \alpha_h\}$ mirrors condition (A) and (B), respectively. Its sign is determined by the direction of the binding incentive constraint, formally

$$\alpha_l \;=\; \begin{cases} >\; 0\;, & \text{if } IC_L \text{ binds,} \\ <\; 0\;, & \text{if } IC_{ML} \text{ binds,} \\ =\; 0\;, & \text{if neither } IC_{ML} \text{ nor } IC_L \text{ binds.} \end{cases}$$

and

[16] This definition of the information rent entails the case $g_L = \frac{1}{3}$ when no information rent has to be paid. As the corresponding weight α_j is zero, the given definition is w.l.o.g..

[17] This definition of the information rent entails the case $g_H = \frac{1}{3}$ when no information rent has to be paid. As the corresponding weight α_j is zero, the given definition is w.l.o.g..

$$\alpha_h \ = \ \begin{cases} < \ 0 \ , & \text{if } IC_H \text{ binds,} \\ > \ 0 \ , & \text{if } IC_{MH} \text{ binds,} \\ = \ 0 \ , & \text{if neither } IC_{MH} \text{ nor } IC_H \text{ binds.} \end{cases}$$

The weight α_k reflects the redistributive preferences: while its sign mirrors the direction of redistribution, its size quantifies the strength of the desire to redistribute in this direction. More precisely, α_l captures the net welfare effect of increasing the consumption of low-skilled individuals by one unit without taking incentive effects into account. Similarly, α_h mirrors the net welfare effect when one more unit of consumption is given to low- and median-skilled types, respectively. Thereby, welfare increases by the sum of their welfare weights. To restore the resource constraint, everyone's consumption is reduced by the same amount. This yields the stated expressions.

2.4.2 Optimal marginal taxes and bunching

In the following, a first order approach is used to study the reduced form problem $P'(g)$. First order conditions are computed for every type, respectively, and the corresponding optimal tax rates are derived. Thereby, the second order conditions are assumed to be satisfied. Whether the given first order conditions are actually sufficient to determine a maximum depends on the preferences' characteristics, i.e. on $v(\cdot)$. See Appendix, Corollary 2.3 for details on the conditions on $v(\cdot)$ that need to be fulfilled for second order conditions to hold. Note that the conditions vary across the simplex as they depend on binding constraints and, therefore, are not introduced with the general setting in Section 2.2. Based on the first order conditions comparative static properties of optimal income and tax rates with respect to welfare weights are derived.

The analysis starts by solving a relaxed version of problem $P'(g)$ where the monotonicity and the non-negativity constraint is neglected. The condition is checked ex-post. If the solution of the relaxed problem fulfills the monotonicity and the non-negativity constraint, a solution of $P'(g)$ is found. Otherwise, the optimal solution includes bunching, which will be addressed in Section 2.4.2.

The relaxed problem $P^r(g)$ is given by: maximize $W^*(c^*(y), y; g)$ over $\mathbb{R}$. Solving this unconstrained optimization problem yields optimal income $y_k^r(g), k = L, M, H$ that is implicitly given by the following first order conditions,[18] respectively

$$\frac{\partial W^*(c^*(y), y; g)}{\partial y_k} \ = \ 1 - \frac{1}{w_k} v'(\frac{y_k}{w_k}) + \sum_{j=l,h} \alpha_j \frac{\partial R_j(\cdot)}{\partial y_k} \ = \ 0, \qquad k = L, M, H. \quad \text{(FOC)}$$

[18] The proof of the following expression is entailed in the proof of Proposition 2.2.

If the implicitly given solution $y^r(g)$ of the relaxed problem $P^r(g)$ is fully separating, formally $y^r_k(g) \neq y^r_j(g),\ j \neq k$, it is in fact a solution of problem $P'(g)$. In this case, the first order conditions FOC reveal further insights in the underlying mechanisms.

The derivative of $\frac{\partial W^*(\cdot)}{\partial y_k}$ captures the effect of an increase in the income of type k on welfare. The first component of the first order condition refers to an increase in person k's income by one unit without accounting for the incentive constraints. An increase in y_k allows to consume an additional amount of $1 - \frac{1}{w_k} v'(\frac{y_k}{w_k})$. This is always positive regardless of the welfare weights. Contrary, if incentive constraints towards type k bind, those need to be restored after the change in income. As an information rent reflects the adjustment that has to be made in the utility of a person that has an incentive to mimic type ω_k, it varies in the respective income, mirrored by $\frac{\partial R_j(\cdot)}{\partial y_k}$. Whether this adjustment affects welfare positively or negatively depends on the direction of redistribution which, in turn, is determined by α_j.[19] If a downward incentive constraint binds towards type k, the increase in person k's utility requires an increase in the utility of types above k. Consequently, restoring incentives due to binding downward incentive constraints, yields a negative welfare effect. Vice versa, if an upward constraint binds towards type k, consumption of less skilled types can be reclaimed (and evenly distributed among individuals), such that restoring incentives has a positive effect on welfare. However, α_j not only reflects the direction of redistribution but also quantifies the desire to redistribute. Formally, it weights the change in information rents and, hence, measures the social costs arising from reestablishing binding incentive constraints.

Before studying the corresponding optimal marginal taxes, the following section approaches how optimal income is defined if the solution of the relaxed problem is decreasing for some k and, hence, not fully separating but includes bunching.

Bunching

Bunching occurs if the solution $y^r(g)$ of the relaxed problem violates the monotonicity or the non-negativity constraint. So it is either $y^r_L(g) > y^r_M(g)$ or $y^r_M(g) > y^r_H(g)$ or $y^r_L(g) < 0$. Depending on the welfare regime, i.e. the binding constraints, some types of bunching can be ruled out as they contradict the first order conditions.

Proposition 2.3. *Consider the relaxed problem $P^r(g)$. For the solution $y^r(g)$ implicitly given by the first order conditions FOC, it holds that*

- *there is no bunching, if IC_{ML} and IC_{MH} bind. Formally, $y^r_L(g) < y^r_M(g) < y^r_H(g)$.*
- *bunching cannot be ruled out, if IC_L and IC_H bind.*

[19] The adjustment in the information rent is always positive such that the sign of $\alpha_k \frac{\partial R_j(y_k)}{\partial y_k}$ is determined by α_k.

- *there is no bunching at the top, if IC_{ML} and IC_H bind. Formally, $y^r_M(g) < y^r_H(g)$.*
- *there is no bunching at the bottom, if IC_L and IC_{MH} bind. Formally $y^r_L(g) < y^r_M(g)$.*

Bunching can be ruled out in some cases by utilizing the single-crossing condition (SC). It ensures that for every point in a $c-y-$diagram, the indifference curve of low-skilled individuals is steeper than that of median-skilled individuals, which, again is steeper than the one of highly productive individuals. This relation is formally described by the first inequality in condition (SC). This condition allows to draw conclusions on the ordering of income levels from comparing the respective marginal rates of substitution. As the first order conditions (FOC) provide information on the marginal rates of substitution for optimal income, they can be used to rule out some types of bunching. It depends on g whether and which type of bunching is refused by the first order conditions.

E.g. if incentive constraints bind downwards at the optimum, there is no bunching at the top:[20] the marginal rate of substitution of the high-skilled income level is one, while it is smaller than one for the median- and low-skilled individuals' income. Hence, $y^r_M(g) < y^r_H(g)$ and $y^r_L(g) < y^r_H(g)$, which means that there is no bunching at the top but no conclusion on the ranking of optimal median- and low-skilled income is possible.

By similar arguments, at least one type of bunching can be ruled out in two other regions. Only in the region where IC_L and IC_H bind, bunching can not be ruled out at all. See Brett and Weymark (2017) who use a continuous type model and find that for the analogue pattern of binding constraints, a bunching region around the median type exists.

Assume now that the optimal allocation includes bunching. Then, the next proposition provides the first order conditions that implicitly determine the income level $\bar{y}$ at which the respective productivity types are bunched. By definition, two adjacent types generate the same income in a bunching allocation, so that the respective monotonicity constraint of $P'(g)$ binds.

The corresponding Lagrangian of the reduced-form problem is considered where λ_L,λ_M, λ_H are the Kuhn-Tucker multipliers of the (non-negativity and monotonicity) constraints, respectively.

$$\mathcal{L}(\cdot) = W^*(c^*(y)y; g) \ + \ \lambda_L y_L \ + \ \lambda_M(y_M - y_L) \ + \ \lambda_H(y_H - y_M) \tag{2.2}$$

yielding the following first order conditions

[20]See Guesnerie and Seade (1982) who firstly observed this for a non-linear pricing context.

$$
\begin{aligned}
\frac{\partial \mathcal{L}(\cdot)}{\partial y_k} &= \frac{\partial W^*(\cdot)}{\partial y_k} + \lambda_k - \lambda_{k+1} = 0, \quad k = L, M, && (2.3)\\
\frac{\partial \mathcal{L}(\cdot)}{\partial y_H} &= \frac{\partial W^*(\cdot)}{\partial y_H} + \lambda_H = 0, && (2.4)\\
\lambda_L &\geq 0 \;\; (= 0, \text{ if } y_L > 0), && (2.5)\\
\lambda_k &\geq 0 \;\; (= 0, \text{ if } y_k > y_{k-1}), \;\; k = M, H. && (2.6)
\end{aligned}
$$

Formally, the two types of bunching are described by $i)$ $\lambda_L > 0$ (and $\lambda_k = 0, k \neq L$) and $ii)$ $\lambda_k > 0,\ k = M, H$ and ($\lambda_j = 0, j \neq k$). This formal characterization of bunching yields an implicit definition of the income level where individuals are bunched at.

The following proposition sums up the findings on the optimal bunching allocation.[21]

Proposition 2.4. *a) If there is no bunching, the solution of the relaxed problem $P^r(g)$ is identical to the solution of reduced-form problem $P'(g)$.*

b) Bunching of all three productivity types is never optimal.

c) If the non-negativity constraint is violated, it holds that

$$\sum_{k=L,M} \frac{\partial W^*(\cdot)}{\partial y_k}\bigg|_{y_L=0} < 0.$$

d) If type j and $j+1$ are bunched at $\bar{y} \neq 0$, it holds that

$$\sum_{k=j}^{j+1} \frac{\partial W^*(\cdot)}{\partial y_k}\bigg|_{\bar{y}} = 0.$$

Those first order conditions provide an implicit characterization of the optimal bunching level $\bar{y}$ if two adjacent types are bunched together.

In the following, we focus on fully separating allocations and study optimal marginal tax rates. Second order conditions are still assumed to be satisfied.

Optimal distortions and the sign of optimal marginal tax rates

Under the given assumptions, the formulation of the first order conditions (FOC) yields an instant representation of optimal marginal tax rates. Further, it enables to directly learn the tax rate's sign.

[21] A special case of this is given by Proposition 3 in Simula (2010) who focuses on redistribution downwards.

As mentioned before, the implicit marginal tax rate of type k at income level y_k is given by the wedge between the marginal rate of transformation and the marginal rate of substitution, formally

$$\tau_k \;=\; 1 - \tfrac{1}{w_k}\, v'(\tfrac{y_k}{w_k}).$$

If redistribution is desirable, there is a tradeoff between equity and efficiency such that marginal taxes are unequal to zero at the optimal allocation. In this case, optimal labor supply is distorted compared to the first-best allocation. If taxes are positive, individuals are discouraged from work at the margin, i.e. there is a downward distortion. Vice versa, if taxes are negative, individuals are overly encouraged to work which yields an upward distortion at the margin.
It follows directly from the first order conditions (FOC) that optimal implicit marginal tax rates in the given setup are

$$\tau_k^*(g) \;:=\; \tau_k(y_k^*(g)) \;=\; \sum_{j=l,h} -\,\alpha_j\, \tfrac{\partial R_j(\cdot)}{\partial y_k}\,. \tag{2.7}$$

Except for the region where IC_L and IC_H bind,[22] the argument of $R_l(\cdot)$ differs from the argument of $R_h(\cdot)$ such that the above expression simplifies to

$$\tau_k^*(g) \;=\; -\,\alpha_j\, \frac{\partial R_j(y_k^*(g))}{\partial y_k}\,. \tag{2.8}$$

Since $v(\cdot)$ has increasing differences in w_k,[23] the derivative of $R_j(\cdot)$ w.r.t. y_k is positive. Hence, the weight $-\alpha_j$ determines the sign of optimal marginal tax rates, or equivalently, the direction of the distortion.
For three of the four areas, i.e. whenever tax rates take the form of equation (2.8), the direction of the distortion is unambiguously determined by the respective weight α_j. For the fourth area, where IC_L and IC_H bind, marginal tax rates are of the form of equation (2.7). Since α_l is positive and α_h is negative the resulting sign of the marginal tax rate is a priori ambiguous. The specific welfare weights are decisive to identify the direction of the distortion. Revisiting the respective first order condition allows to derive a formal condition that distinguishes between cases where $y_M^*(g)$ is distorted up- or downwards, respectively. Stated for later reference, the following expression defines welfare regimes $g \in \{g \in G : g_L, g_H \leq \frac{1}{3}\}$[24] for which $y_M^*(g)$ is distorted upwards:[25]

[22]When IC_L and IC_H bind, both, $R_l(\cdot)$ and $R_h(\cdot)$ are functions of y_M.
[23]$v(y_i, w_k)$ has increasing differences since: for $w_l < w_k$ it holds that $v(y_2, w_l) - v(y_2, w_k) > v(y_1, w_l) - v(y_1, w_k)(> 0),\ \forall y_2 > y_1$.
[24]This describes the region where IC_L and IC_H bind.
[25]For an illustrative representation of the separating line ($\star$), see Figure 2.3 in the proof of proposition 2.5.

$$\begin{aligned} g_L \quad &< \quad \underbrace{\tfrac{1}{3}\left(1-(1-3g_H)\frac{\frac{1}{\omega_M}v'(\frac{y_M^*}{\omega_M})-\frac{1}{\omega_H}v'(\frac{y_M^*}{\omega_H})}{\frac{1}{\omega_L}v'(\frac{y_M^*}{\omega_L})-\frac{1}{\omega_M}v'(\frac{y_M^*}{\omega_M})}\right)}_{=:(\star)} \\ \Leftrightarrow \quad \alpha_l\frac{\partial R_l(y_M^*)}{\partial y_M} \quad &> \quad -\alpha_h\frac{\partial R_h(y_M^*)}{\partial y_M} \qquad (>0). \end{aligned}$$

If the opposite holds, $y_M^*(g)$ is distorted downwards, or, if $g_L = (\star)$, the optimal income of the median type is undistorted. Note that $g_H = \frac{1}{3}$ implies $g_L = \frac{1}{3}$ and for $g_H = 0$ it holds that $g_L \in [0, \frac{1}{3}]$ so that the expression is welldefined in the sense that it actually separates only subregions of the region where IC_L and IC_H bind.
This region is the only one where the sign of a distortion changes for some type k. Within all other regions, the sign of distortions and, hence, of marginal tax rates, is constant.

Proposition 2.5. *The sign of optimal marginal tax rates $\tau_k^*(g),\ k = L, M, H$ are, respectively*

$$\begin{aligned}
\tau_L^*(g) &> 0 && \textit{if} \quad g_L > \tfrac{1}{3}, && (\tau_L)\\
\tau_L^*(g) &= 0 && \textit{if} \quad g_L \leq \tfrac{1}{3}, &&\\
\tau_H^*(g) &< 0 && \textit{if} \quad g_H > \tfrac{1}{3}, && (\tau_M)\\
\tau_H^*(g) &= 0 && \textit{if} \quad g_H \leq \tfrac{1}{3}, &&\\
\tau_M^*(g) &< 0 && \textit{if} \quad g_L < \tfrac{1}{3} \quad \textit{and} \quad g_L < (\star), && (\tau_H)\\
\tau_M^*(g) &> 0 && \textit{if} \quad g_H < \tfrac{1}{3} \quad \textit{and} \quad g_L > (\star), &&\\
\tau_M^*(g) &= 0 && \textit{if} \quad g_H \geq \tfrac{1}{3} \quad \textit{and} \quad g_L \geq \tfrac{1}{3}, \quad \textit{or if} \quad g_L = (\star). &&
\end{aligned}$$

It is never be optimal to force low-skilled individuals to work more than in the undistorted first-best allocation, i.e. optimal labor supply is distorted downwards or undistorted at the margin. Equivalently, the optimal marginal tax rate of low-skilled individuals is non-negative. Vice versa, high-skilled individuals face a non-positive marginal tax rate, such that optimal labor supply is undistorted or distorted upwards at the margin for highly productive individuals. The intuitive reason for the opposite directions of distortions lies in the effort costs: for low productivity levels generating more output is relatively costly, so it can never be optimal. Contrary, highly productive individuals can produce more output at a relatively low costs such that their labor supply is distorted upwards or undistorted at the optimum. For the median productivity types, optimal labor supply can be either upward, downward or undistorted.[26]

Comparative statics of optimal marginal tax rates

The following analysis studies the effect of changes in the welfare weights on optimal income. Since welfare weights sum up to one, changing g_k for some k goes hand in hand with a compensating change in at least one other welfare weight. Specific changes are introduced that leave one weight fixed and, graphically, represent movements parallel to the axes in the simplex. Considering those is without loss of generality as we can study every possible variation of g by combining those basic movements.

Definition 2.1. Small variations in the welfare weights g that leave either g_L or g_H unaffected are called *basic changes*. There exist two basic changes: a basic-L-change given by a marginal increase in g_L and a compensating decrease of g_M and a basic-H-change given by a marginal increase in g_H and a compensating decrease of g_M.

Comparative static results with respect to welfare weights are delivered by utilizing the implicit function theorem. Although basic changes encompass two welfare weights to change, this is actually welldefined: a basic change leaves either g_L or g_H fixed but includes a change in the remaining two weights. As optimal income is determined by the first order conditions (FOC) which are separable in the welfare weights and do not depend on g_M, using the implicit function theorem is not problematic. The following observations can be made regarding the effect of a basic change on optimal income.[27]

Lemma 2.2. *Consider for a given $g \in G$ the optimal income vector $y^*(g)$ as implicitly given by the respective first order condition (FOC). Then, optimal income decreases or remains the same after a basic-L change and increases or remains the same after a basic-H change.*

Depending on the binding incentive constraints the productivity type whose optimal income is a function of g_L (g_H), respectively, varies. However, regardless of the pattern of binding constraints, a basic-L-change induces optimal income to drop (or does not affect it) while a basic-H-change has the opposite effect.

Those findings are directly transferable to optimal marginal tax rates. As preferences are quasilinear, the previous results on income are sufficient to derive how optimal marginal tax rates behave under basic changes in g.

Lemma 2.3. *For every welfare regime $g \in G$ it holds that optimal marginal tax rates are non-decreasing in the welfare weight of low-skilled individuals and non-increasing in the welfare weight of high-skilled individuals.*

[26] Brett and Weymark (2017) provide a similar result for selfishly optimal welfare weights. Non-boundary types are distorted up- or downwards depending on the type who's utility is maximized.

[27] See also Weymark (1987) and Simula (2010) for comparative static results with respect to paramteres beside the welfare weights. And for comparative static results of linear taxes see Hellwig (1986).

Within each area it holds, that the undistorted income level and, hence, the corresponding tax rate, remains constant under a basic change. Mathematically, this is a consequence of the quasilinear utility functions which imply that the no-distortion-conditions completely pins down the corresponding income levels: optimal marginal tax rates of zero for low- and high-skilled individuals are preserved under a basic change if IC_L or IC_H remain binding, respectively, after the variation in g. The same is true for the median type, which requires IC_{ML} and IC_{MH} to be binding before and after a change in the welfare weights.

Otherwise, changes in the welfare weights always yield a corresponding change in the tax rate. One can make the following observations: optimal marginal taxes increase in the welfare weight of low-skilled individuals and decrease in the welfare weight of the high-skilled – or remain the same, respectively. So, independently of the binding incentive constraints, the more we care about the poor, lowering marginal taxes is never optimal. Vice versa, if we focus on the well-being of the rich, who have relatively low effort costs, we want to encourage individuals to work more, i.e. taxes should not go up but rather decrease or remain the same.

The second observation is, that even though the direction of a tax change (due to basic changes in g) is the same for all areas of the simplex, the change in the corresponding distortions is not. Depending on the sign of the marginal tax rate, decreasing (increasing) tax rates can lead to bigger or smaller distortions. If a change in the welfare weights points into the direction of redistribution, i.e. in the same direction as the binding incentive constraint, distortions get bigger. The reason is that the change emphasizes the desire to redistribute towards the respective skill type. If the welfare change points to the opposite direction of the direction of redistribution, it alleviates the equity- efficiency trade-off and reduces distortions.

2.5 Further steps to go and conclusion

2.5.1 Notes on the second-best Pareto frontier

The given paper takes a normative perspective and considers weighted Utilitarian welfare as the objective of the optimization problem. A natural question to follow is how the resulting set of welfare maximizing allocations relates to the set of (second-best) Pareto efficient allocations.

The second-best Pareto frontier is given by the set of incentive feasible allocations (c, y) such that there exists no other incentive feasible allocation $(c', y') \neq (c, y)$ for which the following conditions holds.[28]

$$U_k(c'_k, y'_k) \geq U_k(c_k, y_k), \quad \forall k \in \{L, M, H\} \quad \text{and} \quad U_k(c'_k, y'_k) > U_k(c_k, y_k) \text{ for at least one } k.$$

[28] For a detailed analysis of the second-best Pareto frontier in a taxation model with two types see Bierbrauer and Boyer (2014).

Or to phrase it differently: the second-best Pareto frontier encompasses all allocations that solve the following optimization problem $PP(v_M, v_H)$:

$$\max_{(c,y)\in\mathcal{A}(v_m,v_H)} U_L(c_L, y_L) \qquad \text{s.t.} \quad U_M(c_M, y_M) = v_M$$
$$U_H(c_H, y_H) = v_H,$$

where $\mathcal{A}(v_M, v_H)$ is the set of incentive feasible allocations for a pair of given utility levels (v_M, v_H) which are assigned to median- and high-skilled individuals, respectively.

From the first welfare theorem we know that every welfare maximum is Pareto efficient. Whether the reverse is true for the given economy can not be said, yet. The following section sketches one way to approach this question by utilizing some results of the given analysis. However, it is left for future work to actually prove the equivalence of the solution sets of $PP(v_M, v_H)$ and $P(g)$. The first step described below is similar to the one of the welfare maximization problem and is aimed at solving problem $PP(v_M, v_H)$. The second step makes use of a geometric argument to link the two solution sets.

Firstly, the domain of problem $PP(v_M, v_H)$, i.e. the incentive feasible allocations $\mathcal{A}(v_M, v_H)$, has to be characterized. Depending on the exogenously given utility levels v_M and v_H, the binding incentive constraints vary, and, hence, the set of incentive feasible allocations. Analogue to Section 2.3, potential pattern of binding constraints need to be identified.

With regard to this, the given analysis of the welfare maximization problem yields the following lemma.

Lemma 2.4. *For the given economy, the (second-best) Pareto frontier encompasses regions where none, one or two incentive constraints bind - or, if preferences are such that the welfare maximum includes bunching, three.*

This follows directly from the fact that every welfare maximum is also Pareto efficient. As shown in Proposition 2.4, bunching of all three types can not occur along the frontier as it is Pareto dominated by the laissez faire allocation. But there exists no combination of binding incentive constraints, that is not covered be those pattern. That said, Lemma 2.4 lists all potential combinations of binding constraints. Further, we know from Corollary 2.1 that the resource constraint has to bind at a Pareto efficient allocation.

These findings allow to compute Pareto efficient allocations by solving the Lagrangian of problem $PP(v_M, v_H)$ for each pattern of binding constraints, respectively. Analogue to the approach in the welfare maximization problem in Section 2.4.1, incentive compatibility constraints that are slack can be replaced by a monotonicity constraint and checked ex-post, while binding incentive constraints are included explicitly in the Lagrangian. This delivers the set of second-best Pareto efficient allocations.

However, even knowing this set does not allow to draw conclusions of the relation to the welfare maximizing allocations. To prove the equivalence, it has to be ruled out that the set of welfare maximizing allocations is only a subset of the set of Pareto efficient allocations. To show that there is a one-to-one mapping between the set of second-best Pareto efficient allocations and the set of welfare maximizing allocations, the (second-best) Pareto frontier has to be everywhere convex. By a separating-hyperplane-argument, the convexity ensures that there is a bijective mapping between the two sets of allocations. For a setup without information asymmetries, Negishi (1960) uses this argument to show that every competitive equilibrium is also a maximum of a welfare function that is a weighted sum of utilities. As mentioned before, this is left for future research.

Finally, the following gives an idea why this is of interest – not only from a formal, but also from an applied perspective. Suppose there was a bijective mapping between the mentioned sets, then this allows to link a rather theoretical optimal income tax problem to applications in the field of political economy.[29]

Formally, points on the (second-best) Pareto frontier are defined by utility levels $(v_M v_H)$ whereas welfare maxima are computed based on the vector of welfare weights g. Both, v_M and v_H and g can be interpreted as political positions with regard to redistribution. However, there is a remarkable weakness of studying welfare weights compared to points on the Pareto frontier.

With the given analysis on three productivity types, it is not possible to say how utility changes with respect to welfare weights at an optimum. Hence, marginal tax schemes can not be compared in terms of utility based on the analysis of welfare maximizing allocations – which is possible along the second-best Pareto frontier.

That said, the preceding analysis does e.g. not allow to address questions on voting over tax schedules. To infer voting behavior of utility maximizing individuals, it is necessary to know how utility is affected by different tax systems which requires a one-to-one relation between the second-best Pareto frontier and welfare weights. Therefore, it is valuable to tackle this problem in future work.

2.5.2 Conclusion

The given paper provides a characterization of the incentive structure of a welfare maximization problem in a Mirrleesian income tax model with three productivity levels when welfare weights can vary arbitrarily. More precisely, conditions are stated that map welfare weights to binding incentive constraints.

Secondly, reduced form problems in the fashion of Weymark (1986a) are formulated for each pattern of binding constraints and solved by a first order approach. This yields

[29] See Bierbrauer and Boyer (2013), who exploit the congruence of these sets in a two type setting and examine welfare implications of political competition.

formulations of marginal tax rates for every productivity type which are such that low-skilled (high-skilled) individuals never face a negative (positive) marginal tax rate at the optimum, while the median type can be distorted up- or downwards depending of the direction of redistribution. Bunching can not be ruled out completely, however, some bunching pattern contradict first order conditions and, hence, do not occur for the respective welfare weight.

The final, but important note relates the stated results on welfare maximization to (second-best) Pareto efficiency.

Appendix 2.A Proof of Corollaries

Corollary 2.1

Proof. As preferences are quasilinear in consumption, a non-binding resource constraint can not be optimal: each person's consumption could be increased by a small amount without violating incentive constraints. □

Corollary 2.2

Proof. We have $\Omega = \{\omega_L, \omega_M, \omega_H\}$. The analysis proceeds locally in the sense that each pair of adjacent incentive constraints is studied separately.
We consider the case where IC_L and IC_H bind and show that those imply IC_{ML} and IC_{MH} to bind, respectively, if income is monotonic. The remaining cases go through by the same arguments.

Binding IC_L and IC_H are given by

$$c_L - v\left(\frac{y_L}{\omega_L}\right) = c_M - v\left(\frac{y_M}{\omega_L}\right) \tag{IC_L}$$

$$c_H - v\left(\frac{y_H}{\omega_H}\right) = c_M - v\left(\frac{y_M}{\omega_H}\right) \tag{IC_H}$$

Then, (IC_L) can be rearranged to $c_M - c_L = v\left(\frac{y_M}{\omega_L}\right) - v\left(\frac{y_L}{\omega_L}\right)$. For IC_{ML} to be satisfied, it has to hold

$$c_M - c_L \overset{!}{>} v\left(\frac{y_M}{\omega_M}\right) - v\left(\frac{y_L}{\omega_M}\right)$$

Inserting the rearranged condition IC_L yields

$$v\left(\frac{y_M}{\omega_L}\right) - v\left(\frac{y_L}{\omega_L}\right) \overset{!}{>} v\left(\frac{y_M}{\omega_M}\right) - v\left(\frac{y_L}{\omega_M}\right).$$

But this holds due to the characteristics of $v(\cdot)$ if $y_M > y_L$. Thus, IC_L + Monotonicity implies IC_{ML}. Analogously, it can be proven that IC_H + Monotonicity imply IC_{MH}, IC_{MH} + Monotonicity imply IC_H and IC_{ML} + Monotonicity imply IC_L. This proves the given statement.
Note that the arguments used in the proof are independent of the quasilinearity of the utility function but use the separability of consumption and income. Furthermore it can extended to models with n types since the analysis proceeds locally in the sense that each pair of local constraints is studied separately. □

Corollary 2.3

Corollary 2.3. *The given specification of $v(\cdot)$ is not sufficient to ensure that second order conditions for a maximum hold. The sign of $\frac{\partial^2 R_j(\cdot)}{\partial y_k^2}\Big|_{y^*}$ is decisive to determine the sign of the second order conditions. Depending on the welfare regimes, the requirements change.*

Proof. To prove that the first order conditions are necessary and sufficient to determine a welfare maximizing allocation (for problem $P'(g)$), it has to be shown that the respective Hessian matrix is negative definite. A direct implication of the first order condition FOC is that the cross-derivatives are zero, formally $\frac{\partial^2 W^*(\cdot)}{\partial y_k \partial y_j} = 0$. Hence, proving negative definiteness simplifies to checking $\frac{\partial^2 W^*(\cdot)}{\partial y_k^2} \overset{?}{<} 0$. The following conditions provide the respective results and can be understood as sufficient conditions on $v(\cdot)$ that have to hold to ensure that derived solution is in fact a maximum.

Based on the first order condition FOC, the respective second order conditions are

- IC_L and IC_{MH}

  Take the derivative w.r.t. $y_k,\ k = L, M, H$.

$$\frac{\partial^2 W^*(\cdot)}{\partial y_L^2} = -\frac{1}{\omega_L^2} v''(\frac{y_L}{\omega_L}) < 0.$$

$$\frac{\partial^2 W^*(\cdot)}{\partial y_M^2} = -\frac{1}{\omega_M^2} v''(\frac{y_M}{\omega_M}) \underbrace{+(1-3g_L)}_{>0} \underbrace{(\frac{1}{\omega_L^2} v''(\frac{y_M}{\omega_L}) - \frac{1}{\omega_H^2} v''(\frac{y_M}{\omega_M}))}_{\overset{!}{<}0} \overset{?}{<} 0.$$

$$\frac{\partial^2 W^*(\cdot)}{\partial y_H^2} = -\frac{1}{\omega_H^2} v''(\frac{y_H}{\omega_H}) \underbrace{-(1-3g_H)}_{>0} \underbrace{(\frac{1}{\omega_M^2} v''(\frac{y_H}{\omega_M}) - \frac{1}{\omega_H^2} v''(\frac{y_H}{\omega_H}))}_{\overset{!}{<}0} \overset{?}{<} 0.$$

- IC_L and IC_H

  Take the derivative w.r.t. $y_k,\ k = L, M, H$.

$$\begin{aligned}
\frac{\partial^2 W^*(\cdot)}{\partial y_M^2} &= -\tfrac{1}{\omega_M^2}v''(\tfrac{y_M}{\omega_M}) \underbrace{+(1-3g_L)}_{>0}\underbrace{(\frac{1}{\omega_L^2}v''(\frac{y_M}{\omega_L}) - \frac{1}{\omega_M^2}v''(\frac{y_M}{w_M}))}_{\overset{!}{<}0} \\
&\quad \underbrace{-(1-3g_H)}_{<0}\underbrace{(\frac{1}{\omega_M^2}v''(\frac{y_M}{\omega_M}) - \frac{1}{\omega_H}v''(\frac{y_M}{w_H}))}_{\overset{!}{>}0} \overset{?}{<} 0. \\
\frac{\partial^2 W^*(\cdot)}{\partial y_H^2} &= -\tfrac{1}{\omega_H^2}v''(\tfrac{y_H}{\omega_H}) < 0. \\
\frac{\partial^2 W^*(\cdot)}{\partial y_L^2} &= -\tfrac{1}{\omega_L^2}v''(\tfrac{y_L}{\omega_L}) < 0.
\end{aligned}$$

- IC_{ML} and IC_{MH}

  Take the derivative w.r.t. $y_k,\ k = L, M, H$.

$$\begin{aligned}
\frac{\partial^2 W^*(\cdot)}{\partial y_L^2} &= -\tfrac{1}{\omega_L^2}v''(\tfrac{y_L}{\omega_L}) \underbrace{+(1-3g_L)}_{<0}\underbrace{(\frac{1}{\omega_L^2}v'(\frac{y_L}{\omega_L}) - \frac{1}{\omega_M^2}v'(\frac{y_L}{w_M}))}_{\overset{!}{>}0} \overset{?}{<} 0. \\
\frac{\partial^2 W^*(\cdot)}{\partial y_M^2} &= -\tfrac{1}{\omega_M^2}v''(\tfrac{y_M}{\omega_M}) < 0. \\
\frac{\partial^2 W^*(\cdot)}{\partial y_H^2} &= -\tfrac{1}{\omega_H^2}v''(\tfrac{y_H}{\omega_H})) \underbrace{-(1-3g_H)}_{>0}\underbrace{(\frac{1}{\omega_M^2}v''(\frac{y_H}{\omega_M}) - \frac{1}{\omega_H^2}v''(\frac{y_H}{\omega_H}))}_{\overset{!}{<}0} \overset{?}{<} 0.
\end{aligned}$$

- IC_{ML} and IC_H

  Take the derivative w.r.t. $y_k,\ k = L, M, H$

$$\begin{aligned}
\frac{\partial^2 W^*(\cdot)}{\partial y_L^2} &= -\tfrac{1}{\omega_L^2}v''(\tfrac{y_L}{\omega_L})) \underbrace{+(1-3g_L)}_{<0}\underbrace{(\frac{1}{\omega_L^2}v''(\frac{y_L}{\omega_L}) - \frac{1}{\omega_M^2}v''(\frac{y_L}{w_M}))}_{\overset{!}{>}0} \overset{?}{<} 0. \\
\frac{\partial^2 W^*(\cdot)}{\partial y_M^2} &= -\tfrac{1}{\omega_H}v'(\tfrac{y_H}{\omega_H})) \underbrace{-(1-3g_H)}_{<0}\underbrace{(\frac{1}{\omega_M^2}v''(\frac{y_M}{\omega_M}) - \frac{1}{\omega_H^2}v''(\frac{y_M}{w_H}))}_{>0} \overset{?}{<} 0. \\
\frac{\partial^2 W^*(\cdot)}{\partial y_H^2} &= -\tfrac{1}{\omega_H^2}v''(\tfrac{y_H}{\omega_H}) < 0.
\end{aligned}$$

Since $\frac{1}{\omega_k^2}\upsilon''(\cdot) > 0$, first order conditions defining undistorted labor supply are sufficient for a maximum. However, when distortions are present, the claimed characteristics of $\upsilon(\cdot)$ in section 2.2 are not sufficient to yield $\frac{\partial^2 W^*(\cdot)}{\partial y_k^2} < 0$. That said, the second derivatives of the information rents are decisive for the sign of the derivatives, respectively. Sufficient conditions to ensure that second order conditions are fulfilled are given by the desired sign of the $\frac{\partial^2 R_j(y_k)}{\partial y_k^2}$ in the stated second order conditions respectively. $\square$

Appendix 2.B Proof of Propositions

Proposition 2.1

Proof. The constraints of the given welfare maximization problem $P(g)$ can be expressed in terms of utility (u_L, w_M, u_H) and income y. The resource constraint and the incentive constraints become respectively

$$u_L + u_M + u_H \leq s(y) \ := \sum_{k=L,M,H} (y_k - v(\tfrac{y_k}{w_k})) \ ,$$

and

$$\begin{aligned} u_L &\geq u_M - \Big(v(\tfrac{y_M}{\omega_L}) - v(\tfrac{y_M}{\omega_M})\Big) \ , \\ u_M &\geq u_L + \Big(v(\tfrac{y_L}{\omega_L}) - v(\tfrac{y_L}{\omega_M})\Big) \ , \\ u_M &\geq u_H - \Big(v(\tfrac{y_H}{\omega_M}) - v(\tfrac{y_H}{\omega_H})\Big) \ , \\ u_H &\geq u_M + \Big(v(\tfrac{y_M}{\omega_M}) - v(\tfrac{y_M}{\omega_H})\Big) \ . \end{aligned}$$

At the optimum, the resource constraint binds, see Corollary 2.1 such that utility of high-skilled individuals can be expressed on terms of u_M and u_L, formally $u_H = s(y) - u_M - u_L$. Inserting into problem $P(g)$ yields the following version of the problem

$$\max_{u_L, u_M, y} W = (g_L - g_H)u_L \ + \ (g_M - g_H)\, u_M \ + g_H s(y)$$

s.t.

$$v(\tfrac{y_M}{\omega_L}) - v(\tfrac{y_M}{\omega_M}) \ \geq \ u_M \ - \ u_L \qquad (IC_L)$$

$$u_M \ - \ u_L \ \geq \ v(\tfrac{y_L}{\omega_L}) - v(\tfrac{y_L}{\omega_M}) \qquad (IC_{ML})$$

$$2u_M \ + u_L \ \geq \ s(y) \ - \ \Big(v(\tfrac{y_H}{\omega_M}) - v(\tfrac{y_H}{\omega_H})\Big) \qquad (IC'_{MH})$$

$$s(y) \ - \ \Big(v(\tfrac{y_M}{\omega_M}) - v(\tfrac{y_M}{\omega_H})\Big) \ \geq \ 2u_M \ + \ u_L \qquad (IC'_H)$$

Each pair of local incentive constraints is studied separately. Small changes in utility are used to figure out which incentive constraint binds: if the considered changes yields welfare to go up without violating any constraints, a desired direction of redistribution is found and the binding incentive constraints can be inferred. To guarantee that no constraint is violated, an auxiliary function $h(\cdot)$ is introduced which prescribes how to vary utility levels without violating incentive constraints.

1) To figure out whether IC_L or IC_{ML} bind at the optimum the following auxiliary function $h_1(u_L, u_M)$ is defined. It represents the lefthand side and righthand side of IC'_{MH} and IC'_H, respectively.

$$h_1(u_L, u_M) = 2u_M + u_L$$

$$dh_1(\cdot) = 0 \Leftrightarrow 2du_M + du_L = 0 \Leftrightarrow \frac{du_L}{du_M}\Big|_{dh_2=0} = -2$$

If we change u_L and u_M by a an appropriate ε, i.e. such that $dh_1(\cdot) = 0$, the lefthand side of equation (IC'_{MH}) and the righthand side of equation (IC'_H) remain unaffected: if those constraints are satisfied before the variation, they will be as well afterwards.

$$u'_L = u_L + \varepsilon$$

$$u'_M = u_M - \frac{1}{2}\varepsilon$$

$$h_1(u'_L, u'_M) = 2u'_M + u'_L = 2u_M + u_L - \varepsilon + \varepsilon = h_1(u_L, u_M)$$

W.l.o.G. we have increased u_L. To check, whether this change leads to an increase in welfare, we calculate welfare for u'_L and u'_M and compare it with the original welfare level.

$$\begin{aligned} W(u'_L, u'_M) &= g_H s(y) + (g_L - g_H)u'_L + (g_M - g_H)u'_M \\ &= \underbrace{g_H s(y) + (g_L - g_H)u_L + (g_M - g_H)u_M}_{=W(u_L,u_M)} + \underbrace{\left(g_L - g_H - g_M\frac{1}{2} + \frac{1}{2}g_H\right)}_{=\Delta W}\varepsilon \end{aligned}$$

If $\Delta W > 0$, the welfare increases by the variation in utility. This is the case if condition (A) holds, formally

$$g_L - g_H - g_M\frac{1}{2} + \frac{1}{2}g_H > 0 \Leftrightarrow g_L > \frac{g_M + g_H}{2}.$$

The increase of u_L is restricted by the lefthand side of IC_{ML}. Thus, if ΔW is positive, IC_{ML} will bind at the optimum. Contrary, if it is negative, IC_L will bind. In this case we want to make u_M as big as possible. Otherwise, the variation has no effect on welfare such that redistribution is not desirable.

2) Analogously to part 1): to figure out whether IC_H or IC_{MH} bind, consider the following auxiliary function which represents the lefthand side of IC_{ML} and the righthand side of IC_H, respectively.

$$h_2(u_L, u_M) = u_M - u_L$$

$$dh_2(\cdot) = 0 \Leftrightarrow \tfrac{du_M}{du_L} = 1, \text{ if } u'_M = u_m + \varepsilon \text{ and } u'_L = u_L + \varepsilon$$

$$h_2(u'_L, u'_M) = u_M - u_L + \varepsilon - \varepsilon = h_2(u_L, u_M)$$

W.l.o.G. consider $u_M + \varepsilon$. To check, whether this change leads to an increase in welfare, we calculate welfare for u'_L and u'_M and compare it with the original welfare level.

$$\begin{aligned} W(u'_L, u'_M) &= g_H s(y) + (g_L - g_H)u'_L + (g_M - g_H)u'_M \\ &= \underbrace{g_H s(y) + (g_L - g_H)u_L + (g_M - g_H)u_M}_{=W(u_L,u_M)} + \underbrace{\Big(g_L - g_H + g_M - g_H\Big)\,\varepsilon}_{=\Delta W} \end{aligned}$$

This means, an increase in u_L and u_M increases the welfare, if and only if $\Delta W > 0$. This is equivalent to

$$g_H < \frac{g_L + g_M}{2} \ .$$

This is identical to condition (B). If it holds, u_M is increased until the respective incentive constraint IC'_H binds. If the opposite is true, IC'_{MH} binds or, if $\Delta W = 0$, neither of both binds.

□

Proposition 2.2

Proof. A given welfare weight $g \in G$ pins down none, one or two binding incentive constraints, see Proposition 2.1. If no constraint binds, more than one optimal allocation exist. If one or two constraints, the following procedure delivers a reduced-form problem which is solvable with simple calculus.

Let g be such that two incentive constraints (and the resource constraint) bind. Then, rearranging those yield the following expressions of utility.

$$u_L^*(\cdot) = \tfrac{1}{3} \sum_{k=L,M,H} \big(y_k - v(\tfrac{y_k}{w_k})\big) \ - \ \tfrac{1}{3} R_h(\cdot) \ - \ \tfrac{2}{3} R_l(\cdot) \tag{UTI_L}$$

$$u_M^*(\cdot) = \tfrac{1}{3} \sum_{k=L,M,H} \big(y_k - v(\tfrac{y_k}{w_k})\big) \ - \ \tfrac{1}{3} R_h(\cdot) \ + \ \tfrac{1}{3} R_l(\cdot) \tag{UTI_M}$$

$$u_H^*(\cdot) = \tfrac{1}{3} \sum_{k=L,M,H} \big(y_k - v(\tfrac{y_k}{w_k})\big) \ + \ \tfrac{2}{3} R_h(\cdot) \ + \ \tfrac{1}{3} R_l(\cdot), \tag{UTI_H}$$

Summing up those utilities $u_k^*(\cdot)$ weighted with the associated welfare weight g_k yields a welfare function $W^*(c^*(y), y; g)$ that incorporates all binding constraints. To respect non-binding incentive constraints, a monotonicity constraint has to be added, see Corollary 2.2. This completes the derivation of the reduced form problem $P'(g)$. See Weymark (1986a) or Simula (2010) for analogue formulations.

To derive the above utility expressions, two incentive constraints are required to bind. The following arguments explain why the stated problem $P'(g)$ covers as well welfare weights g for which only one incentive constraint binds.

The derived welfare function $W^*(\cdot)$ consists of two components where the first reflects the resource restriction of the problem and the second the informational restrictions due to binding incentive constraints. Consider now a welfare weight that pins down one binding incentive constraint. Welfare weights that pin down only one binding incentive constraint are of the form $g = (g_L, g_M, g_H)$ where $g_k = \frac{1}{3}$ for one type k. Pretending that a second constraint binds, allows to state the reduced form problem $P'(g)$. Not that the incentive constraints, that is in fact not binding, needs to belong to the pair constraints that does not include the truels binding constraint.

Both, the actually and the allegedly binding, and binding incentive constraint shows up in the welfare function $W^*(\cdot)$ by the respective information rents. However, only the one resulting from the actually binding incentive constraints is weighted with a weight $\alpha_j \neq 0$. The other incentive constraint induces a weight of $\alpha_j = 0$ due to $g_k = \frac{1}{3}$. Hence, the resulting social costs from the second (actually not) binding constraint are weighted with zero and hence, have no effect on welfare and the optimal solution. Equivalently, the Lagrangian could be computed for situations with one binding and three slack incentive constraints. This yields the same results.

In the following the four relevant cases – as there are four different pairs of binding incentive constraints – and the respective results are listed in more detail. Note that, equivalently, consumption $c_k^*(y)$ instead of utility $u_k^*(y)$ level can be studied. It holds that $u_k^*(c^*(y), y) = c_k^*(y) - v\left(\frac{y_k}{\omega_k}\right)$. For each of the four cases, those consumption levels are computed for every productivity type.

- IC_L and IC_{MH}

Rearranging the binding incentive constraints and making use of the binding resource constraint, we get the following utility functions

$$u_L^*(\cdot) = \frac{1}{3}s(y) - \frac{1}{3}\big(v(\tfrac{y_H}{w_M}) - v(\tfrac{y_H}{w_H})\big) - \frac{2}{3}\big(v(\tfrac{y_M}{w_L}) - v(\tfrac{y_M}{w_M})\big), \qquad \text{(ICLMH)}$$

$$u_M^*(\cdot) = \frac{1}{3}s(y) - \frac{1}{3}\big(v(\tfrac{y_H}{w_M}) - v(\tfrac{y_H}{w_H})\big) + \frac{1}{3}\big(v(\tfrac{y_M}{w_L}) - v(\tfrac{y_M}{w_M})\big),$$

$$u_H^*(\cdot) = \frac{1}{3}s(y) + \frac{2}{3}\big(v(\tfrac{y_H}{w_M}) - v(\tfrac{y_H}{w_H})\big) + \frac{1}{3}\big(v(\tfrac{y_M}{w_L}) - v(\tfrac{y_M}{w_M})\big).$$

And the corresponding consumption level are given by

$$\begin{aligned}
c_L^*(y) &= \tfrac{1}{3}\textstyle\sum_{i=l,m,h} y_i + \tfrac{1}{3}\big(v(\tfrac{y_M}{w_M}) - v(\tfrac{y_H}{w_M})\big) + \tfrac{2}{3}\big(v(\tfrac{y_L}{w_L}) - v(\tfrac{y_M}{w_L})\big),\\
c_M^*(y) &= \tfrac{1}{3}\textstyle\sum_{i=l,m,h} y_i + \tfrac{1}{3}\big(v(\tfrac{y_M}{w_M}) - v(\tfrac{y_H}{w_M})\big) - \tfrac{1}{3}\big(v(\tfrac{y_L}{w_L}) - v(\tfrac{y_M}{w_L})\big),\\
c_H^*(y) &= \tfrac{1}{3}\textstyle\sum_{i=l,m,h} y_i - \tfrac{2}{3}\big(v(\tfrac{y_M}{w_M}) - v(\tfrac{y_H}{w_M})\big) - \tfrac{1}{3}\big(v(\tfrac{y_L}{w_L}) - v(\tfrac{y_M}{w_L})\big).
\end{aligned}$$

Plugging in the utility functions in the welfare function yields the following expression

$$W^*(c^*(y), y; g) = s(y) - \big(v(\tfrac{y_H}{w_M}) - v(\tfrac{y_H}{w_H})\big)(1-3g_h) + \big(v(\tfrac{y_M}{w_L}) - v(\tfrac{y_M}{w_M})\big)(1-3g_L).$$

To derive the optimal income level, we take the derivative w.r.t. $y_k,\ k = L, M, H.$

$$\begin{aligned}
\frac{\partial W^*(\cdot)}{\partial y_L} &= 1 - \tfrac{1}{\omega_L}v'(\tfrac{y_L}{\omega_L}) && = 0.\\
\frac{\partial W^*(\cdot)}{\partial y_M} &= (1 - \tfrac{1}{\omega_M}v'(\tfrac{y_M}{\omega_M})) + (1-3g_L)(\tfrac{1}{\omega_L}v'(\tfrac{y_M}{\omega_L}) - \tfrac{1}{\omega_H}v'(\tfrac{y_M}{\omega_M})) && = 0.\\
\frac{\partial W^*(\cdot)}{\partial y_H} &= (1 - \tfrac{1}{\omega_H}v'(\tfrac{y_H}{\omega_H})) - (1-3g_H)(\tfrac{1}{\omega_M}v'(\tfrac{y_H}{\omega_M}) - \tfrac{1}{\omega_H}v'(\tfrac{y_H}{\omega_H})) && = 0.
\end{aligned}$$

- IC_L and IC_H
  Rearranging the binding incentive constraints and making use of the binding re-

source constraint, we get the following utility functions

$$u_L(\cdot) = \frac{1}{3}s(y) - \frac{1}{3}\big(v(\tfrac{y_M}{w_M}) - v(\tfrac{y_M}{w_H})\big) - \frac{2}{3}\big(v(\tfrac{y_M}{w_L}) - v(\tfrac{y_M}{w_M})\big). \tag{ICLH}$$

$$u_M(\cdot) = \frac{1}{3}s(y) - \frac{1}{3}\big(v(\tfrac{y_M}{w_M}) - v(\tfrac{y_M}{w_H})\big) + \frac{1}{3}\big(v(\tfrac{y_M}{w_L}) - v(\tfrac{y_M}{w_M})\big).$$

$$u_H(\cdot) = \frac{1}{3}s(y) + \frac{2}{3}\big(v(\tfrac{y_M}{w_M}) - v(\tfrac{y_M}{w_H})\big) + \frac{1}{3}\big(v(\tfrac{y_M}{w_L}) - v(\tfrac{y_M}{w_M})\big).$$

And the corresponding consumption level are given by

$$\begin{aligned}
c_L^*(y) &= \tfrac{1}{3}\textstyle\sum_{i=l,m,h} y_i + \tfrac{1}{3}\big(v(\tfrac{y_M}{w_H}) - v(\tfrac{y_H}{w_H})\big) + \tfrac{2}{3}\big(v(\tfrac{y_L}{w_L}) - v(\tfrac{y_M}{w_L})\big),\\
c_M^*(y) &= \tfrac{1}{3}\textstyle\sum_{i=l,m,h} y_i + \tfrac{1}{3}\big(v(\tfrac{y_M}{w_H}) - v(\tfrac{y_H}{w_H})\big) - \tfrac{1}{3}\big(v(\tfrac{y_L}{w_L}) - v(\tfrac{y_M}{w_L})\big),\\
c_H^*(y) &= \tfrac{1}{3}\textstyle\sum_{i=l,m,h} y_i - \tfrac{2}{3}\big(v(\tfrac{y_M}{w_H}) - v(\tfrac{y_H}{w_H})\big) - \tfrac{1}{3}\big(v(\tfrac{y_L}{w_L}) - v(\tfrac{y_M}{w_L})\big).
\end{aligned}$$

Plugging in the utility functions in the welfare function yields the following expression

$$W^*(c^*(y), y; g) = s(y) - \big(v(\tfrac{y_M}{w_M}) - v(\tfrac{y_M}{w_H})\big)(1-3g_h) + \big(v(\tfrac{y_M}{w_L}) - v(\tfrac{y_M}{w_M})\big)(1-3g_L).$$

To derive the optimal income level, we take the derivative w.r.t. $y_k,\ k = L, M, H$.

$$\begin{aligned}
\tfrac{\partial W^*(\cdot)}{\partial y_M} &= (1 - \tfrac{1}{\omega_M}v'(\tfrac{y_M}{\omega_M})) + (1-3g_L)(\tfrac{1}{\omega_L}v'(\tfrac{y_M}{\omega_L}) - \tfrac{1}{\omega_M}v'(\tfrac{y_M}{\omega_M}))\\
&\qquad -(1-3g_H)(\tfrac{1}{\omega_M}v'(\tfrac{y_M}{\omega_M}) - \tfrac{1}{\omega_H}v'(\tfrac{y_M}{\omega_H})) = 0.\\
\tfrac{\partial W^*(\cdot)}{\partial y_H} &= 1 - \tfrac{1}{\omega_H}v'(\tfrac{y_H}{\omega_H}) = 0.\\
\tfrac{\partial W^*(\cdot)}{\partial y_L} &= 1 - \tfrac{1}{\omega_L}v'(\tfrac{y_L}{\omega_L}) = 0.
\end{aligned}$$

- IC_{ML} and IC_{MH}
  Rearranging the binding incentive constraints and making use of the binding re-

source constraint, we get the following utility functions

$$u_L(\cdot) = \frac{1}{3}s(y) - \frac{1}{3}\big(v(\tfrac{y_H}{w_M}) - v(\tfrac{y_H}{w_H})\big) - \frac{2}{3}\big(v(\tfrac{y_L}{w_L}) - v(\tfrac{y_L}{w_M})\big). \tag{ICMLMH}$$

$$u_M(\cdot) = \frac{1}{3}s(y) - \frac{1}{3}\big(v(\tfrac{y_H}{w_M}) - v(\tfrac{y_H}{w_H})\big) + \frac{1}{3}\big(v(\tfrac{y_L}{w_L}) - v(\tfrac{y_L}{w_M})\big).$$

$$u_H(\cdot) = \frac{1}{3}s(y) + \frac{2}{3}\big(v(\tfrac{y_H}{w_M}) - v(\tfrac{y_H}{w_H})\big) + \frac{1}{3}\big(v(\tfrac{y_L}{w_L}) - v(\tfrac{y_L}{w_M})\big).$$

And the corresponding consumption level are given by

$$\begin{aligned}
c_L^*(y) &= \tfrac{1}{3}\textstyle\sum_{i=l,m,h} y_i + \tfrac{1}{3}\big(v(\tfrac{y_M}{w_M}) - v(\tfrac{y_H}{w_M})\big) + \tfrac{2}{3}\big(v(\tfrac{y_L}{w_M}) - v(\tfrac{y_M}{w_M})\big),\\
c_M^*(y) &= \tfrac{1}{3}\textstyle\sum_{i=l,m,h} y_i + \tfrac{1}{3}\big(v(\tfrac{y_M}{w_M}) - v(\tfrac{y_H}{w_M})\big) - \tfrac{1}{3}\big(v(\tfrac{y_L}{w_M}) - v(\tfrac{y_M}{w_M})\big),\\
c_H^*(y) &= \tfrac{1}{3}\textstyle\sum_{i=l,m,h} y_i - \tfrac{2}{3}\big(v(\tfrac{y_M}{w_M}) - v(\tfrac{y_H}{w_M})\big) - \tfrac{1}{3}\big(v(\tfrac{y_L}{w_M}) - v(\tfrac{y_M}{w_M})\big).
\end{aligned}$$

Plugging in the utility functions in the welfare function yields the following expression

$$W^*(c^*(y), y; g) = s(y) - \big(v(\tfrac{y_H}{w_M}) - v(\tfrac{y_H}{w_H})\big)(1-3g_h) + \big(v(\tfrac{y_L}{w_L}) - v(\tfrac{y_L}{w_M})\big)(1-3g_L).$$

To derive the optimal income level, we take the derivative w.r.t. $y_k,\ k = L, M, H$.

$$\begin{aligned}
\tfrac{\partial W^*(\cdot)}{\partial y_L} &= (1 - \tfrac{1}{\omega_L}v'(\tfrac{y_L}{\omega_L})) + (1-3g_L)(\tfrac{1}{\omega_L}v'(\tfrac{y_L}{\omega_L}) - \tfrac{1}{\omega_M}v'(\tfrac{y_L}{w_M})) &&= 0.\\
\tfrac{\partial W^*(\cdot)}{\partial y_M} &= 1 - \tfrac{1}{\omega_M}v'(\tfrac{y_M}{\omega_M}) &&= 0.\\
\tfrac{\partial W^*(\cdot)}{\partial y_H} &= (1 - \tfrac{1}{\omega_H}v'(\tfrac{y_H}{\omega_H})) - (1-3g_H)(\tfrac{1}{\omega_M}v'(\tfrac{y_H}{\omega_M}) - \tfrac{1}{\omega_H}v'(\tfrac{y_H}{w_H})) &&= 0.
\end{aligned}$$

- IC_{ML} and IC_H
  Rearranging the binding incentive constraints and making use of the binding re-

source constraint, we get the following utility functions

$$u_L(\cdot) = \frac{1}{3}s(y) - \frac{1}{3}\big(v(\tfrac{y_M}{w_M}) - v(\tfrac{y_M}{w_H})\big) - \frac{2}{3}\big(v(\tfrac{y_L}{w_L}) - v(\tfrac{y_L}{w_M})\big). \tag{ICMLH}$$

$$u_M(\cdot) = \frac{1}{3}s(y) - \frac{1}{3}\big(v(\tfrac{y_M}{w_M}) - v(\tfrac{y_M}{w_H})\big) + \frac{1}{3}\big(v(\tfrac{y_L}{w_L}) - v(\tfrac{y_L}{w_M})\big).$$

$$u_H(\cdot) = \frac{1}{3}s(y) + \frac{2}{3}\big(v(\tfrac{y_M}{w_M}) - v(\tfrac{y_M}{w_H})\big) + \frac{1}{3}\big(v(\tfrac{y_L}{w_L}) - v(\tfrac{y_L}{w_M})\big).$$

And the corresponding consumption level are given by

$$\begin{aligned}
c_L^*(y) \quad &= \tfrac{1}{3}\textstyle\sum_{i=l,m,h} y_i + \tfrac{1}{3}\big(v(\tfrac{y_M}{w_H}) - v(\tfrac{y_H}{w_H})\big) + \tfrac{2}{3}\big(v(\tfrac{y_L}{w_M}) - v(\tfrac{y_M}{w_M})\big),\\
c_M^*(y) \quad &= \tfrac{1}{3}\textstyle\sum_{i=l,m,h} y_i + \tfrac{1}{3}\big(v(\tfrac{y_M}{w_H}) - v(\tfrac{y_H}{w_H})\big) - \tfrac{1}{3}\big(v(\tfrac{y_L}{w_M}) - v(\tfrac{y_M}{w_M})\big),\\
c_H^*(y) \quad &= \tfrac{1}{3}\textstyle\sum_{i=l,m,h} y_i - \tfrac{2}{3}\big(v(\tfrac{y_M}{w_H}) - v(\tfrac{y_H}{w_H})\big) - \tfrac{1}{3}\big(v(\tfrac{y_L}{w_M}) - v(\tfrac{y_M}{w_M})\big).
\end{aligned}$$

Plugging in the utility functions in the welfare function yields the following expression

$$W^*(c^*(y), y; g) = s(y) - \big(v(\tfrac{y_M}{w_M}) - v(\tfrac{y_M}{w_H})\big)(1 - 3g_h) + \big(v(\tfrac{y_L}{w_L}) - v(\tfrac{y_L}{w_M})\big)(1 - 3g_L).$$

To derive the optimal income level, we take the derivative w.r.t. $y_k,\ k = L, M, H$

$$\begin{aligned}
\frac{\partial W^*(\cdot)}{\partial y_L} \quad &= \quad (1 - \tfrac{1}{\omega_L}v'(\tfrac{y_L}{\omega_L})) + (1 - 3g_L)(\tfrac{1}{\omega_L}v'(\tfrac{y_L}{\omega_L}) - \tfrac{1}{\omega_M}v'(\tfrac{y_L}{w_M})) \quad &&= \quad 0.\\
\frac{\partial W^*(\cdot)}{\partial y_M} \quad &= \quad (1 - \tfrac{1}{\omega_H}v'(\tfrac{y_H}{\omega_H})) - (1 - 3g_H)(\tfrac{1}{\omega_M}v'(\tfrac{y_M}{\omega_M}) - \tfrac{1}{\omega_H}v'(\tfrac{y_M}{w_H})) \quad &&= \quad 0.\\
\frac{\partial W^*(\cdot)}{\partial y_H} \quad &= \quad 1 - \tfrac{1}{\omega_H}v'(\tfrac{y_H}{\omega_H}) \quad &&= \quad 0.
\end{aligned}$$

□

Proposition 2.3

Proof. To study bunching for the four different areas, respectively, we, technically, make use of the distortions as described by the first order conditions (FOC). We know that if the income of type k is undistorted it holds that $MRS_{y_k} = 1$, see equation (2.1). Undistorted income is independent of g and denoted by y_k^{FB}. It holds that $y_L^{FB} < y_M^{FB} < y_H^{FB}$.

If the income of type k is distorted upwards (downwards) it holds that $MRS_{y_k} > 1$ (<1). Hence, it exceeds (falls below) the undistorted income level y_k^{FB}.

- IC_{ML}, IC_{MH} binding
  The stated first order conditions imply that $y_L^r(g)$ is distorted downwards, $y_M^r(g)$ is undistorted, and $y_H^r(g)$ is distorted upwards. This implies

  $$y_L^r(g) < y_L^{FB} < y_M^{FB} = y_M^r(g) < y_H^{FB} < y_H^r(g).$$

  So, there is no bunching.

- IC_L, IC_H binding
  The stated first order conditions imply that y_L^r is undistorted, y_M^r is distorted (up- or downwards), and $y_H^r(g)$ is undistorted. Hence, $y_L^r(g) = y_L^{FB} < y_H^{FB} = y_H^r(g)$. But $y_M^r(g) > y_H^r(g)$ and $y_L^r(g) > y_M^r(g)$ can't be ruled out by the given information.

- IC_{ML}, IC_H binding
  The stated first order conditions imply that y_L^r is distorted downwards, y_M^r is distorted downwards, and y_H^r is undistorted. Hence, $y_L^r < y_L^{FB} < y_H^{FB} = y_H^r$ and $y_M^r < y_M^{FB} < y_H^{FB} = y_H^r$. Hence, there is no bunching at the top. ($y_L^r > y_M^r$ can not be ruled out.)

- IC_L, IC_{MH} binding
  The stated first order conditions imply that $y_L^r(g)$ is undistorted, $y_M^r(g)$ is distorted upwards, and $y_H^r(g)$ is distorted upwards. Hence, $y_L^r(g) = y_L^{FB} < y_H^{FB} < y_H^r(g)$ and $y_L^r = y_L^{FB} < y_M^{FB} < y_M^r(g)$. Hence, there is no bunching at the bottom. ($y_M^r(g) > y_H^r(g)$ can not be ruled out.)

□

Proposition 2.4

Proof. a) If there is no bunching, all multipliers are equal to zero. Hence, every income y_k that satisfies equation 2.3 satisfies the first order conditions FOC as well. As those are assumed to be necessary and sufficient, this completes the proof of part a).

b) If all three types are bunched together, the allocation must lie on the $45°$ line in the $c - y-$diagram due to the binding resource constraint. However, it is Pareto dominated by the laissez-faire allocation and, thus, can not be optimal.

c) If the non-negativity constraint binds $\lambda_L > 0, \lambda_M = \lambda_H = 0$. Summing up equation 2.3 for $k = L, M$ yields

$$\sum_{k=L}^{M} \frac{\partial W^*(\cdot)}{\partial y_k}\Big|_{\bar{y}} + \lambda_L - \lambda_M + \lambda_M - \lambda_H = 0.$$

Since $\lambda_L > 0, \lambda_M = \lambda_H = 0$, this proves the given statement.

d) If the non-negativity constraint does not bind but the low and the median type are bunched together, it holds $\lambda_L = 0, \lambda_M > 0, \lambda_H = 0$. Summing up equation 2.3 for $k = L, M$ yields

$$\sum_{k=L}^{M} \frac{\partial W^*(\cdot)}{\partial y_k}\Big|_{\bar{y}} + \lambda_L - \lambda_M + \lambda_M - \lambda_H = 0.$$

Since $\lambda_L = 0, \lambda_H = 0$, this proves the given statement.
Analogously, the case where median- and high-skilled individuals are bunched together can be shown.

□

Proposition 2.5

Proof. The sign of marginal tax rates is a direct implication from the first order conditions FOC. For a given g those first order conditions vary, such that optimal income and, hence, the optimal marginal tax rates vary in g. To keep the notation simple, we omit g as an argument of income but denote $\tau_k(g)$.

It holds that

$$\tau_k^*(g) \;=\; -\,\alpha_j \frac{\partial R_j(y_k^*)}{\partial y_k}$$

Thus, for g_L and g_H respectively

$$\tau_k^*(g_L) \;=\; \underbrace{-\,(1-3g_L)}_{=-\alpha_l} \frac{\partial R_l(y_k^*(g))}{\partial y_k},$$

$$\tau_k^*(g_H) \;=\; \underbrace{(1-3g_H)}_{=-\alpha_h} \frac{\partial R_h(y_k^*(g))}{\partial y_k}.$$

Since $\frac{\partial R_j(y_k^*(g))}{\partial y_k} > 0$, the sign of α_j determines the sign of the tax rate.

In the area where IC_L and IC_H bind, we get that $\tau_L(g) = \tau_H(g) = 0$. Marginal tax rates of the median type are given by

$$\tau_M(g) = -\sum_{j=l,h} \alpha_j \frac{\partial R_j(y_M^*(g))}{\partial y_M} = \underbrace{(1-3g_H)}_{>0} \frac{\partial R_h(y_M^*(g))}{\partial y_M} \underbrace{-(1-3g_L)}_{<0} \frac{\partial R_l(y_M^*(g))}{\partial y_M} \gtrless 0.$$

Depending on the specific welfare weight, $\tau_M^*(g)$ is positive or negative. Rearranging the previous equation yields condition τ_L.

For illustration of partitioned simplex including line $(\star)$ see the following Figure

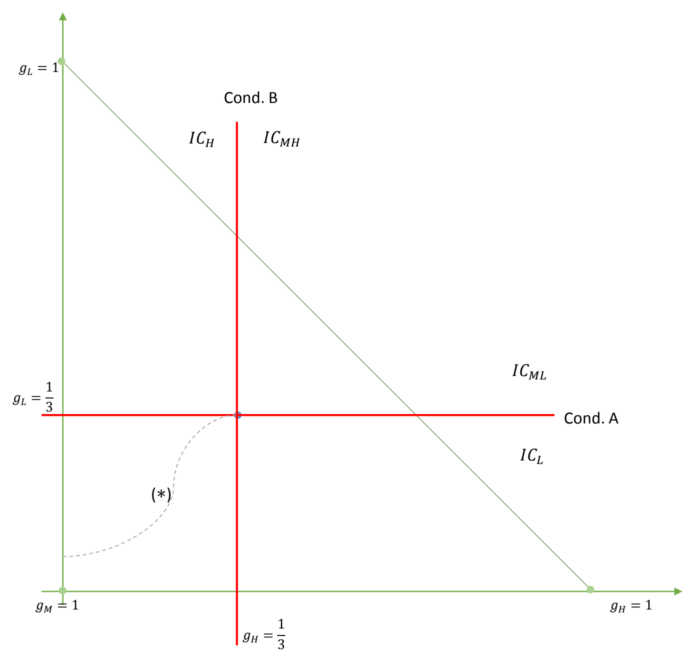


Figure 2.3: Simplex of welfare weights with separating line $(\star)$

□

Appendix 2.C Proof of Lemmata

Lemma 2.1

Proof. This is a direct consequence of Proposition 2.1. If $g_k \neq g_j \neq \frac{1}{3},\ j \neq k$, the welfare weight pins down two binding constraints. If one weight equals the others' average weight, i.e. $g_k = \frac{1}{3}$, Proposition 2.1 tells us that due to the lacking desire to redistribute none of the two associated incentive constraints has to bind at the optimum. Hence, at most one incentive constraint (out of the two non-associated) binds for those g. Finally, at the laissez faire point $g_L = g_M = g_H = \frac{1}{3}$, no incentive constraint binds at the optimum. □

Lemma 2.2

Proof. As we investigate how optimal income changes in the welfare weight, the following proof uses the extensive notation $y^*(g)$.

- Step 1: Prove that implicitly differentiating is welldefined for the considered situation:
  For the comparative static results on the optimal income $y^*(g)$ with respect to welfare weights, the first order conditions FOC are used and basic-changes are considered. Those changes affect two welfare weights, g_M and one other weight. As optimal income is independent of g_M it holds for basic-L-changes

$$dy_k^*(g) = \frac{\partial y_k^*(g)}{\partial g_L} dg_L + \underbrace{\frac{\partial y_k^*(g)}{\partial g_M}}_{=0} dg_M + \frac{\partial y_k^*(g)}{\partial g_H} \underbrace{dg_H}_{=0} \quad \Rightarrow \quad dy_k^*(g) = \frac{\partial y_k^*(g)}{\partial g_L} dg_L.$$

  Since a basic-L-change leaves g_H fixed, the respective term in the differential is zero. Hence using the implicit function theorem for basic-L-changes is well-defined. The same is true for basic-H-changes. Note that, due to $v(\cdot)$ being continuous optimal income is a continuous function of g.

  The next step specifies the actual change in $y^*(g)$ after basic changes.

- Step 2: Applying the implicit function theorem to equation (FOC) yields:

  For $F(g, y^*(g)) := \frac{\partial W^*(\cdot)}{\partial y_k} \equiv 0$ as given in FOC, we get

$$\frac{dy_k^*(g)}{dg_j} = -\frac{\frac{\partial F(\cdot)}{\partial g_j}}{\frac{\partial F(\cdot)}{\partial y_k}} = -\underbrace{\frac{1}{\frac{\partial^2 W^*(\cdot)}{\partial y_k^2}}}_{>0} \frac{\partial^2 W^*(\cdot)}{\partial y_k \partial g_j}$$

As we assume the second order conditions (see Corollary 2.3) to be satisfied, it holds that $-\frac{\partial^2 W^*(\cdot)}{\partial y_k^2} > 0$ such that $\frac{\partial^2 W^*(\cdot)}{\partial y_k \partial g_j}$ determines the sign of $\frac{dy_k^*(\cdot)}{dg_j}$.
With the stated first order condition FOC we have

$$\frac{\partial^2 W^*(\cdot)}{\partial y_k \partial g_L} \;=\; -3R_l(\cdot) \;<\; 0,$$
$$\frac{\partial^2 W^*(\cdot)}{\partial y_k \partial g_H} \;=\; 3R_h(\cdot) \;>\; 0,$$

As information rents are positive, we have that optimal income – if it is distorted – decreases in g_L and increases in g_H. This proofs the stated lemma.

□

Lemma 2.3

Proof. Optimal marginal tax rates are by definition given by the following expression

$$\tau_k^*(g) \;=\; 1 - \tfrac{1}{\omega_k} v'(\tfrac{y_k^*(g)}{\omega_k})$$

With the comparative static properties on optimal income for basic changes in g as studied in Lemma 2.2, the changes in $\tau_k^*(g)$ are a direct implication of $v''(\cdot) > 0$. □

3
Public Goods and Salience

3.1 Introduction

Providing financial support to refugees and people in crisis regions gained more and more importance in the last decade; not least since thousands upon thousands of refugees left Syria in 2015. But, do you know how much you actually contribute to specific Humanitarian aid projects taking contributions by paying taxes into account? And, do you know the improvement in the needy people's living conditions generated by your donation?

Regarding the first question, the German Federal Ministry for Economic Cooperation and Development assigned in 2017 approximately 102 Euro per citizen, i.e. about 8.5 billion Euro in total, to development cooperation projects. So, neglecting or being unaware of these mandatory contributions is likely to have an impact on individuals' privately made donations choices.

Our paper studies the effect of individuals misperceiving the amount of taxes dedicated to specific crisis regions on aggregate—public and private—donation levels. We consider the case where the misperception of taxes results in underestimating or neglecting the entailed contribution to charity and analyze how this affects privately made donations. Intuitively one might think of a situation where the donation made via a tax is included in an aggregate tax payment and hard to be quantified exactly.

Regarding the second motivational question, it has to be noted that it in fact matters how beneficial a dollar donated is as altruistic individuals are interested in the induced improvement and so their donation's effectiveness, see Warren and Walker (1991) or more recently Krasteva and Yildrim (2013). Here, a donation's effectiveness or productivity reflects the improvement a 1\$-donation generates in the living conditions of people in the crisis region. However, individuals are not capable of completely figuring out what the exact improvement associated with their donation is: they hold a perception that is based on accessible information regarding the considered crisis, the aid organization and the projects. In our notion, perceived productivity is a function of both actual productivity and its salience. While actual productivity depends on objective factors such as the infrastructural situation in the crisis region, the aid organization's effectiveness or costs for medicine and equipment in the specific country, salience of a project mirrors how accessible this information is. Hence, actual productivity reflects relevant information about a crisis region or an aid organization in a condensed form and individuals build a perception of a donation's productivity based on available information such as newspaper articles, photos etc. that does not coincide with actual productivity.

Remembering the photo of the drowned boy Alan Kurdi, the importance of a crisis' *salience* becomes striking: donations to the Swedish Red Cross were more than 50 times larger in the week after than in the week before the picture's publication Slovic et al. (2017). The painful death of the young boy made refugees' need more salient but had hardly changed the productivity of a donation to the Red Cross (compared to the days before the picture's publication).

However, even after this peak in the donation flow the requirements of the crisis region Syria were not fully met (see the end-of-year-report of UNHCR 2015). We assume that individuals underestimate the actual improvement that is induced by a donation and, in turn, donate less compared to a full information benchmark case. By that, the underestimation of productivity is one source of underfunding. It is of particular interest, especially in the charity context, to investigate causes of underfunding to be able to tackle those and provide sufficient financial support.

This instantly begs the question to what extent the underprovision of financial help is due to the individuals' misperceptions as opposed to an implication of utility maximizing behavior, i.e. free-riding. Further, is the response to a given governmental behavior affected by the misperception of productivity and how does it interfere with the misperception of taxes?

One interesting observation can be made with regard to the composition of public and private donations: between 2012 and 2016, not only public but also private contributions have gone up.[1] This simultaneous increase is remarkable as it seems to be at odds with the well established crowding-out hypothesis of Bergstrom et al. (1986). Hence, an explanation is needed that rationalizes the observed concomitant increase.

We approach these questions in our paper and study the aforementioned factors as being crucial for understanding what drives individual donation choices in the presence of public donors. One aim of our paper is to disentangle the mechanisms driving the interplay of public and private donors and to make out what determines an individual's donation choice for a given tax schedule. In that respect, we study in particular the role misperceptions—of taxes as well as a donation's effectiveness—play.

The first part of our paper focuses on the positive analysis of charity as a public good and studies the effect of the previously described factors on individual behavior. The second part of our paper is dedicated to the question how different allocations, i.e. different compositions of donations regarding the source of funding (public or private), can be

[1]While the German Federal Ministry for Economic Cooperation and Development increased its contributions to development cooperation projects from 6.3 billion Euro in 2012 to 7.8 billion Euro in 2016 (German Federal Ministry for Economic Cooperation and Development, 2018), private donations in Germany increased between 2012 and 2016 from 4.1 to 5.3 billion Euro (Deutscher Spendenrat, 2018).

evaluated in terms of welfare. We study how the answer to this question depends on the factors productivity and salience (in both of its dimensions). In particular, we introduce a welfare measure that accounts for the individuals' misperceptions and analyze how welfare is affected by governmental intervention.

The following section describes the model we use, how it relates to the existing literature and the main results that can be derived for the given setup.

Main Results and Model To answer the stated questions, we develop a theoretical model studying the voluntary provision to the public good help in tradition of Warr (1982), Roberts (1984) and Bergstrom et al. (1986). We refine their workhorse model by introducing parameters that characterize the public good and describe the informational circumstances: first, tax salience reflects the fact that individuals are prone to fiscal illusion, i.e. are not fully aware of the amount of government contributions to the charity project. Second, a donation's efficacy or, more technically, a contribution's productivity can vary across projects. Third, a charity project can be more or less salient. The more salient a charity project is, the more productive individuals perceive their donation to be. Note that the complete analysis we provide is independent of the application to charity and holds for any public good context. We use charity as an illustrative example.

Chetty et al. (2009), Finkelstein (2009) and Farhi and Gabaix (2015) recently analyzed the effect of non-salient taxes in various contexts. However, none of them studies the effects of tax salience in the context of the voluntary provision of public goods.[2]

The second feature of the public good, perceived efficacy, that we introduce is approached from many economic fields: however, most of it focuses on the informational aspect by studying the effect of quality signals, seed money or sequential donation procedures; see among others Vesterlund (2003), Andreoni (2006), Potters et al. (2007), Heutel (2014). Contrary, we study a static theoretical model and do not adress the question how to overcome the lack of information and its effects on the contribution level to the public good but instead focus on the investigation of its interplay with tax salience with regard to the resulting allocation.

In our model individuals decide on how to split a given income between private consumption and donation. As individuals are interested in the well-being of one another, they are willing to donate to make people in a crisis region better off. Besides the pure altruistic motive, we follow among others Andreoni (1989) and Andreoni (1990), assuming that individuals act out of an egoistic motive as well receiving a warm glow when contributing voluntarily to the public good charity.[3] When making their decisions, indi-

[2]See Eckel et al. (2005) or Hickey et al. (2015) for experimental and field evidence on the role of tax awareness or tax salience on charitable giving, Goldin and Listokin (2014) for the limited awareness of charitable deduction availability in the US, Chetty et al. (2009), Finkelstein (2009) or Chetty and Saez (2013) for work on the salience of taxes in non-charity contexts and, among others, Wagner (1976), Buchanan (1960), Buchanan (2014), Baekgaard et al. (2016) for contributions on fiscal illusion.

[3]While Harbaugh et al. (2007) shows impressive evidence for positive neural responses to voluntary (as

viduals take the donation's productivity into account. Yet, they only hold a perception of productivity which is a function of the project's salience and its actual productivity. In such a framework, individuals raise their donations if they consider these as being more beneficial either due to higher productivity or due to higher salience of the project. This first implication of the model fits well with the extensive empirical evidence for salience as a central influence factor for the provision of help.[4] Regarding perceived taxes, individuals raise their contributions if they consider taxes to be lower — either due to actually reduced public contributions or due to a drop in their salience. This finding is in line with the standard crowding-out result of Bergstrom et al. (1986) and foots in the substitutability of public and private contributions.

Given the importance of salience for individual contribution behavior, the gap in theoretical literature studying the effect of salience in the public good context is surprising. The only paper that studies government interventions in the context of varying tax' salience theoretically is Eckel et al. (2005). Although we are in line with the theoretical motivation for their laboratory experiment, our assumptions on how tax salience affects behavior differ from theirs. As mentioned before, we further extend the framework assuming that behavior is driven by a second misperception, the salience of productivity. And, most importantly, we provide a welfare analysis that delivers an evaluation of individuals' rational behavior.

To be able to address normative questions, we firstly derive the aggregate equilibrium donation and study how its composition regarding the source of funding changes if circumstances vary, i.e. if salience or the respective true parameter change:

In the presence of misperceptions, investigating the interplay between government contributions to the public good—being financed via mandatory lump sum taxes—and the individually rational donation behavior, we make the following observations. Public contributions crowd-out private contributions. Crowding-out is perfect only if taxes are perfectly salient and individuals do not experience a warm glow (Warr (1982), Roberts (1984), and Bergstrom et al. (1986)), and it is partial in all other cases. Here, the degree of (partial) crowding-out depends negatively on the substitutability between private and public contributions and positively on the salience of taxes. So, the higher the experienced warm glow and the lower the salience of taxes are, the larger the increase in

opposed to mandatory) donations, there exists an extensive evidence on the relevance of warm glow in the laboratory and field. See e.g. Andreoni and Payne (2013) for a survey in the context of charitable giving.

[4]While the before mentioned paper of Slovic et al. (2017) on the media presence of the image of the young Syrian boy studies an extreme example, Eisensee and Strömberg (2007) sensitize for the effect of salience on governmental relief using Olympic Games as an exogenous source lowering the news coverage of simultaneously occurring disasters. While disasters are five percent less likely to receive reliefs during Olympic Games, it is, in particular, reliefs to marginally newsworthy crises that are strongest affected by the reduced salience. Furthermore, focusing on Canadian donations to reliefs on the Haiti earthquake, Hickey et al. (2015) point out that the effect of tax reductions associated with giving is positively affected by the salience of the announcement of the policy.

aggregate contributions to the public good is if taxes increase ceteris paribus. As a direct consequence of the previously described individual behavior, it holds that the sum of public and private contributions increases in productivity and its salience.

Although outlining how the sum of public and private contributions to the public good is affected by changes in salience and taxes, the first part of the analysis does not allow us to draw conclusions on welfare. The main reason is that increased overall contributions are by definition accompanied by reductions in private consumption which potentially reduces welfare. Furthermore, both productivity and taxes are not perfectly salient when individuals make their decision implying a difference between decision and experienced utility (Kahneman et al. (1997)). For the normative analysis we define an unweighted Utalitarian welfare function based on experienced utility. The results of the comparative static analysis are interesting themselves:

Regarding perceived productivity, we find that independently of what causes the raise, welfare goes up if perceived productivity increases. Though, the underlying mechanism differs: A positive shock on the salience of productivity improves the informational structure and so yields smaller mistakes in the individuals donation choice, which is beneficial. If true productivity goes up, the marginal benefit of every dollar donated is higher, which raises utility and, in turn, welfare. Considering changes in the perceived tax system, things are less clear a priori.

First, if taxes are fully salient, equilibrium welfare decreases in public contributions. This implication is not obvious given that the result also holds when the public good is underprovided and the overall provision of the public good increases due to the tax increase. The main driving channel is the imperfect substitutability between private and public contributions. The tax increase reduces warm glow incorporated in private contributions being crowded-out by public ones. Second, if taxes and, in turn, tax increases are not fully salient, the effect of a tax increase on equilibrium welfare is ambiguous. While again warm glow is crowded-out, the degree of crowding-out is smaller than under full salience of taxes. Dependent on the size of the free-riding problem—which itself decreases in the salience of a donation's productivity—an increase in taxes can be both welfare increasing and welfare harming.

We find that this indeterminacy can be tackled by considering tax increases that are paralleled by increases in private contributions due to raised perceived productivity which is exactly the empirical pattern from our motivation. In a first step, we formally introduce this phenomenon called *crowding-in* and rationalize it in our model.[5] Secondly, we address the question what can be learned from crowding-in regarding the evalua-

[5]Note that our notion from crowding-in differs from the before mentioned known stamp of approval mechanism where government's contributions induce higher private contributions by signaling an organization's quality to uninformed individuals. Moreover, it also abstracts from associated changes in the prices of help associated with tax deductions for donations, see among others Auten et al. (2002) or, more generally, with matching gifts Karlan and List (2007), Huck and Rasul (2011), Huck et al. (2015), and Adena and Huck (2017).

tion of tax changes if we assume that the government behaves in a welfare maximizing manner. We show that the welfare implications of crowding-in strongly depend on the respective trigger: crowding-in triggered by a salience of productivity shock indicates a non-optimal, i.e. welfare decreasing, tax change. However, if crowding-in is triggered by a shock in true productivity, the implications of crowding-in are determined by whether the shock has led to a higher level of underprovision of the public good, ceteris paribus. If underprovision has become more severe after the shock, not observing crowding-in points to a non-optimal government behavior as the government should have intervened to alleviate underprovision. However, whether underprovision gets more severe after a productivity shock is determined by the individuals' preferences and a priori ambiguous.

With respect to practical applicability, these findings are highly relevant as they build a first step towards identifying changes in welfare and its direction based on observed behavior. E.g. consider again a salience shock as given by the photo of Alan Kurdi: observing crowding-in in such a situation of pure salience shock is an indicator for a decrease in welfare if governments behaved optimally before. Thereby, our analysis delivers a first approach to link the rather theoretical concept of welfare to the real world and allows under specific circumstances to deduce evaluations based on observables.

The remainder is organized as follows. In Section 3.2, we introduce the theoretical framework (Subsection 3.2.1) and analyze comparative statics of the equilibrium contributions (Subsection 3.2.2). In section 3.3, the efficiency of the Nash-equilibrium outcome and its reaction to changes in public contributions, productivity, and salience are investigated (Subsection 3.3.1) and implications of crowding-in are discussed (Subsection 3.3.2). Section 3.4 summarizes and concludes.

3.2 Model description

3.2.1 Theoretical framework

We consider a set of individuals $I = \{1, 2, \ldots, N\}$ living in a safe country while there exist as well a crisis region where living conditions are unstable. This crisis region is in need of financial help. W.l.o.g. we focus on the case of one crisis region.

As individuals in the safe country are altruistic and interested in the well-being of people in the crisis region they wish to improve the living conditions there via financial support. What matters for their decision as potential donors is the productivity of financial help. In the model, we denote the productivity of a donation to the region by a. It subsumes all potential influence factors driving how beneficial a dollar spent to the crisis is. In the real world productivity is, for example, determined by the need level, the infrastructural situation in the crisis region or the organizational structure of the aid organization.

Beside financial help provided privately by individuals there also is financial support by the government of the safe country financed via mandatory tax payments. In contrast to public contributions, private donations are made voluntarily by splitting a given income e_i between private consumption c_i and donation d_i to the crisis region. As standard in the literature, we assume that giving voluntarily generates a warm glow. It describes the preference for giving per se apart from the unselfish motive of improving the living conditions in the crisis regions. In this way, a donation d_i consists of a public good component help and a private good component called 'warm glow'.

Denoting the sum of individual private donations by $D := \sum_i d_i$, and the aggregate amount of individual tax payments dedicated to the region in need of financial help by $T := \sum_i t_i$, the utility an individual i experiences from donating d_i is described by the following function:

$$U_i(\cdot) = c_i + v(D + T + g \cdot d_i, a) \ . \tag{3.1}$$

Here, $v(\cdot)$ is a region-specific sub-utility function depending on the total amount of financial support provided, $D+T$, as well as its productivity, a and the warm glow associated with contributing voluntarily. Here, g reflects the strength of warm glow.[6] W.l.o.g. we assume that $g \in [0, 1]$.[7] We impose standard assumptions on $v(\cdot)$, assuming that $v(\cdot)$ is concavely increasing in the total support for the crisis, i.e. $v_1 > 0, v_{11} < 0$.[8] Moreover, as in other public good contexts, we focus on cases where contributions to the public good are desired by the safe country's population assuming that $\lim_{D\to 0} v_1 = \infty$ and $\lim_{D\to\infty} v_1 = 0$. Finally, we assume that $v(\cdot)$ has a positive cross-derivative, i.e. $v_{12} > 0$. The intuitive interpretation is that the utility increase associated with an additional dollar spent is larger the larger the productivity of financial help and, in turn, the associated improvement in the crisis region is.

The utility function given in equation (3.1) describes a situation with perfectly informed individuals. However, this hardly fits empirical observations. Neither do individuals estimate the actual productivity of financial contributions correctly nor are they really informed about the amount of taxes that go to the crisis region. For that reason we incorporate salience of both the crisis and taxes in the model. In-salience reflects the two most important types of misperception and is a driver of individual donations..

First, while being (impure) altruists potential donors can only build a belief of the productivity of their donation that differs from actual productivity level. This perception is a function of the salience of the crisis and its actual productivity. Salience represents the level of media representation or, more generally, the accessibility of crisis-specific information. We denote the salience of the region by $\sigma_a \in [0, 1]$ and assume that the perceived productivity $\tilde{a}(a, \sigma_a) = \sigma_a a$ is a function that is strictly increasing in both of its arguments productivity a and salience σ_a.[9]

[6] Note that being mandatory the tax payment does not create a warm glow.

[7] This ensures that we actually face a public good problem as the weight of the egoistic component is restricted.

[8] Here, v_k denotes the partial derivative of v w.r.t. the kth argument.

[9] The specification of the salience function is w.l.o.g. To maintain our results qualitatively, it is crucial

Second, we assume that individuals are not fully aware of the amount of taxes that are collected in order to support charity projects. Rather individuals seem to be unattentive towards these tax payments and do not fully account for these when deciding on their private donations. Similar to the notation before, we assume that taxes t_i are not fully salient and misperceived by a factor $\sigma_T \in [0,1]$, such that $\tilde{t}_i = \sigma_T t_i$ and analogously $\tilde{T} = \sigma_T \sum_i t_i$, similar to Eckel et al. (2005). Taxes are assumed to be exogeneously given and fixed.

To sum up, individual i's utility is a function of private consumption c_i, aggregate donation D, private donation d_i, tax level T, and the parameter for productivity a and warm glow g. Thereby, the perception of productivity and taxes is biased by its salience, respectively. Individuals build their donation choice based on their *decision utility* function [10] which is given by

$$\tilde{U}(\cdot) = c_i + v(D + \tilde{T} + g \cdot d_i, \tilde{a}) \ . \tag{3.2}$$

For ease of notation, we introduce $\tilde{v}(\cdot) := v(D + \tilde{T} + g \cdot d_i, \tilde{a})$ and $v(\cdot) := v(D + T + g \cdot d_i, a)$. The stated utility function enlightens the meaning of salience in the given context. Salience or, to be more precise, the lack of salience leads to an underestimation of two crucial factors of the model: taxes and productivity. Due to inattention towards taxes and the lack of information about the crisis, individuals make a mistake: Instead of using their actual *experienced utility* defined in equation (3.1), they decide upon their donations based on the decision utility formalized in equation (3.2).

To condense the notation, we summarize the circumstances under which individuals make their decisions by the state of the world $\omega := ((a, T), (\sigma_a, \sigma_T))$. It describes the informational structure (σ_a, σ_T) and the two focal attributes of the considered crisis, (a, T).

In the following chapter, we study individual equilibrium behavior and how it varies in the given components of ω, i.e. productivity and taxes as well as productivity salience and tax salience. Especially, we focus on the effect of previously described misperception has on individual behavior.

3.2.2 Equilibrium analysis and comparative statics

In this section we analyze individually optimal donation behavior and derive the Nash equilibrium for the public good help. We start by characterizing individually optimal behavior as a function of the tax schedule and held perceptions. Individuals are responsive to productivity, taxes and salience in both of its dimensions. Regarding the tax schedule, we focus on taxes that do not change the set of contributors. We derive the individually optimal donation behavior under the given assumptions.

that the perception is an increasing function in both of its arguments.

[10] See Kahneman et al. (1997) who was the first distinguishing experienced and decision utility.

Individually rational donation behavior

For a given state of the world ω, individual i chooses private good consumption c_i and the donation level d_i to solve the following maximization problem:

$$\max_{c_i,d_i} \quad c_i + v((1+g)d_i + D_{-i} + \tilde{T}, \tilde{a}) \tag{3.3}$$

$$\text{s.t.} \qquad c_i + d_i \leq e_i - \tilde{t}_i \tag{3.4}$$

$$c_i \geq 0 \tag{3.5}$$

$$d_i \geq 0 \tag{3.6}$$

Here, $D_{-i} := \sum_{k \neq i} d_k$ denotes the sum of donations excluding those of person i. Equation (3.4) reflects the (perceived) budget constraint[11] and equations (3.5) and (3.6) are standard non-negativity constraints. Under efficient use of resources equation (3.4) binds such that a solution of the preceding problem is characterized by the expression in the following lemma.

Lemma 3.1. *Consider a fixed state of the world ω. Then, person i's donation d_i is a best response to the others' contribution D_{-i} if individual i's decision utility $\tilde{U}(\cdot)$ is maximized. Formally, if the following first-order condition is satisfied:*

$$d_i^* = \arg_{d_i}\{1 = (1+g)v_1\big((1+g)d_i + D_{-i} + \tilde{T}, \tilde{a}\big)\} \tag{FOC*}$$

For a given state of the world ω, individual i's optimal donation is a function of D_{-i} only. Hence, the preceding first-order condition implicitly defines a Nash equilibrium by the intersection of those best response functions:

Definition 3.1 (Nash-equilibrium)**.** The Nash-equilibrium for a given state of the world ω, is a vector of contributions $(d_i^*)_{i=1}^n$ fulfilling condition (FOC*) such that for all i: $d_i^* = d_i(D_{-i}^*)$.
As preferences are identical for all individuals, it holds that

$$d_i^*(D_{-i}^*) \;=\; d_j^*(D_{-j}^*) \quad ,\forall\, i \neq j \in I$$

This symmetric Nash-equilibrium is unique in pure strategies. For this reason we use the symmetric equilibrium as the benchmark case. The aggregate equilibrium level of private donations is given by

$$D^*(\tilde{a}, \tilde{T}) = N \cdot d_i^*(\tilde{a}, \tilde{T}) \; .$$

[11] As shown by Baekgaard et al. (2016), fiscal illusion foots in unawareness towards taxes but not the lack of information. Thereby, it's reasonable to assume that individuals misperceive their available budget. As we consider quasi linear utility function, this is w.l.o.g..

Although sufficiently large public contributions induce individuals not to donate at all, we are interested in how the equilibrium private contributions change in response to changes in the state of the world. For that reason the following analysis focuses on Nash-equilibria in which individuals contribute a positive amount to the region, $d_i^* > 0$.

The public good character of donations—following from individuals being (impure) altruists—implies that individual contributions are substitutes. More precisely, individual i's donation decreases in everyone else's contribution to the same region. However, since a voluntary donation creates a warm glow, D_{-i} is only an imperfect substitute for d_i. This idea is formalized in the following lemma.

Lemma 3.2. *For a given state of the world ω, the best response function $d_i^*(D_{-i})$ is decreasing in D_{-i}. It holds that $\frac{dd_i^*}{dD_{-i}} = -\frac{1}{1+g}$, i.e. d_i and D_{-i} are imperfect substitutes if $g > 0$.*

The partial derivative of d_i with respect to D_{-i} reflects the degree of substitutability between contributions and describes the factor by which d_i can be replaced by D_{-i} such that the first order condition still holds. As the derivative is a function of g, this degree of substitutability varies in the strength of warm glow. Individual i's contribution to the region can be replaced one-by-one by others' contributions if there is no warm glow, i.e. if $g = 0$. In this case, individuals are pure altruists as they are only interested in the aggregate amount of help provided in the crisis region. Thus, without the egoistic motive to donate, d_i and D_{-i} are perfectly substitutable. Contrary, the higher the experienced warm glow is, the lower is the degree of substitutability between d_i and D_{-i}. As the private good character of d_i becomes more important relative to the public good character if g is higher, d_i has to be replaced by a far larger amount of D_{-i} to keep utility constant.

Lemma 3.2 allows another noteworthy conclusion as the partial derivative of d_i^* with respect to D_{-i} is a function of g but is independent of the tax schedule or salience. The degree of substitutability is a constant and does not change if the state of the world varies. In particular, the degree by which d_i can be replaced by D_{-i} is independent of what the government does. Yet, the amount of public contributions, productivity and salience of both determine the actual amount of private contributions in equilibrium.

The next paragraph clarifies how the state of the world affect individually rational behavior and, thereby, the equilibrium donation level D^*.

Comparative static analysis of the individually rational donation

One aim of our analysis is to get a better understanding of how the funding of charity projects relates to salience and the tax regime. For this purpose, it is insightful to distinguish the effect of changes in the state of the world on private donations from its effect on aggregate contributions, i.e. on the sum of public and private contributions.

The analysis in this chapter delivers insights in what drives changes in donation behavior and a charity project's funding. Specifically, we learn how individuals adapt their behavior to changes in ω and how this affects the aggregate support level. This positive analysis sets the stage for answering normative questions regarding the generated welfare in the next chapter.

Lemma 3.1 and the characterization of the symmetric Nash-equilibrium in Definition 3.1 imply that the (perceived) public good level in equilibrium $D^* + \tilde{T}$ satisfies the following optimality condition

$$\frac{1}{1+g} = \tilde{v}_1\big(D^* \tfrac{N+g}{N} + \tilde{T}, \tilde{a}\big).$$

As a dollar spent is considered as more productive if the crisis' salience or the donation's productivity go up, the equilibrium level of aggregate support increases in a donation's perceived effectiveness. For later reference, we denote this observation in the following corollary.

Corollary 3.1. *The true equilibrium support $D^* + T$ is increasing in productivity a and salience σ_a, formally $\frac{d(D^*+T)}{d\tilde{a}} \frac{d\tilde{a}}{da} > 0$ and $\frac{d(D^*+T)}{d\tilde{a}} \frac{d\tilde{a}}{d\sigma_a} > 0$.*

As taxes are by assumption exogenously given, this result is driven by the change in private contributions with respect to perceived productivity, i.e. with respect to productivity and its salience.[12] Formally,

$$\frac{dD^*}{d\sigma_a} = -\tfrac{\tilde{v}_{12}(\cdot)}{\tilde{v}_{11}(\cdot)} \tfrac{N}{N+g} a \quad > 0, \tag{3.7}$$

$$\frac{dD^*}{da} = -\tfrac{\tilde{v}_{12}(\cdot)}{\tilde{v}_{11}(\cdot)} \tfrac{N}{N+g} \sigma_a \quad > 0. \tag{3.8}$$

The preceding corollary shows that whatever raises an individual's perception of productivity, increased productivity or increased salience of the crisis, leads to a higher level of provided help in equilibrium. The higher individuals expect a donation's productivity to be, the more money they are willing to spend and the higher the aggregate financial support $D^* + T$ in equilibrium is. Technically, the result is driven by the positive cross-derivative $\tilde{v}_{12}(\cdot) > 0$.

A direct consequence of this observation is that under the given assumption $\tilde{a} \leq a$, the individuals' perception induces a level of private financial support that lies below the equilibrium level under perfect information, i.e.

$$D^*(\tilde{a}, T) \leq D^*(a, T) \quad \forall\, \omega.$$

[12] By that argument, it holds as well that perceived equilibrium contributions increase in perceived productivity, i.e. $\frac{d(D^*+\tilde{T})}{d\tilde{a}} > 0$.

Hence, the underestimation of productivity is a source of underprovision that is independent of free-riding—a second source of the public good's underprovision.

Apart from productivity and salience shocks, changes in the tax schedule can affect the aggregate level of support. As mentioned before, we study tax changes that do not change the set of contributors, ensuring the existence of an interior Nash equilibrium solution. The following Lemma 3.3 states how these changes in the tax policy alter the (aggregate) Nash-equilibrium support for the region in need.

Lemma 3.3. *The equilibrium support $D^* + T$ increases in T.*

$$\frac{d\left(D^* + T\right)}{dT} \;=\; 1 - \frac{N}{N+g}\sigma_T \;\geq\; 0.$$

An increase in public contributions to the region has a direct and an indirect effect on aggregate contributions. The two summands in the stated lemma represent these effects. An increase in public contributions of one dollar raises the sum of private and public contributions by the same amount (*direct effect*). This increase, in turn, induces individuals to reduce their individually rational contributions (*indirect effect*) as reflected in the second summand. Hence, higher public contributions crowd-out private contributions.

The degree of crowding-out crucially depends on the substitutability between private and public contributions and the individuals' awareness of the tax schedule: first, taxes are mandatory and do not induce a warm glow. Hence, these are an imperfect substitute for private donations inducing individuals to a partial response only (even if individuals were fully informed about taxes). Thereby, the bigger the private good component of donations, mirrored by g, is, the less private contributions are crowded-out by public ones. Second, as taxes are not fully salient, the effect is biased by its salience σ_T. The size of the crowding-out effect is lowered due to the limited awareness of government interventions—here, the increase in public contributions. The overall effect on the aggregate equilibrium support $D^* + T$ is determined via both channels and is non-negative.

Complete crowding-out as outlined by Bergstrom et al. (1986), i.e. the situation where the reduction in D^* balances the increase in T, occurs if only if individuals do not experience a warm glow ($g = 0$) and are additionally perfectly aware of taxes ($\sigma_T = 1$). In all other cases, crowding-out is partial and, thus, the aggregate amount of financial help sent to the crisis region increases in T. This partial crowding-out is in line with the findings in the existing warm glow literature, following Andreoni (1989). Here, the government intervention alleviates the problem of free riding in the sense that the aggregate amount of financial help increases in T.[13] This holds—opposed to the results of Andreoni (1989)—even for a model without warm glow as individuals do not take taxes

[13]Due to free-riding, public goods are underprovided in a Nash equilibrium compared to the efficient provision level.

at the full extent into account when deciding on their donation. So, $\sigma_T < 1$ mitigates the underprovision of the public good in Nash equilibrium.

In contrast, if $g = 0$ and $\sigma_T = 1$, public and private contributions are perfect substitutes such that the equilibrium amount of the public good remains unaffected by small government interventions.[14]

The considerations above as well as the existing literature focus on the overall provision of the public good as its underprovision is one crucial issue in both economic theory and practice. Though, they neglect the private good component in the individuals' utility functions. Private consumption is affected if the contributions to the public good changes. This has to be accounted for when the quality of an allocation is assessed, i.e. when normative questions are addressed such that the existing approach of evaluating public good equilibria by their amount of reduced free riding is unsatisfactory.

This, in turn, points to our next step where we abstract from considering the specific provision level (of the public good) only but are interested in the generated utility resulting from the respective contribution allocation, i.e. in welfare.

In the following chapter, we introduce a welfare measure and analyze how welfare is affected by changes in ω. Thereby, we not only account for changes in private good consumption, but also incorporate the evaluation of allocations in terms of the generated utility. This widens the focus of the analysis of public goods: Instead of considering the effect of a tax change on the provision level of a public good only, we study the effects on the individuals' utility and, hence, on welfare.

3.3 Welfare

So far, we took a positive perspective and studied how equilibrium donations are affected by changes in the state of the world. In the following section, we introduce a normative analysis of the public good help in case of uninformed individuals: we tackle the question how to evaluate a given allocation in terms of welfare. In a first step, we introduce a welfare measure and analyze if and how changes in perceived productivity or the perceived tax system affect welfare. Secondly, we study the phenomenon crowding-in, i.e. the concomitant increase in public and private contributions, and ask: Is there anything we can learn from observing this development in contributions regarding welfare or the evaluation of a given tax system? As we will see, under certain circumstances, crowding-in points to a non-optimal tax policy.

[14] Under these circumstances, free-riding can only be reduced if public contributions are so high that they induce private donors to stop from voluntarily contributing at all.

3.3.1 Unweighted Utilitarian welfare and comparative statics

The welfare measure we use takes into account that individuals make mistakes when choosing their donation as they face nonsalient taxes and nonsalient productivity. This causes a discrepancy between the utility individuals experience $U_i(\cdot)$, see equation (3.1), and the utility they base their decisions on $\tilde{U}_i(\cdot)$, see equation (3.2). In line with Kahneman et al. (1997) and Chetty et al. (2009) we define welfare based on experienced utility $U_i(\cdot)$. For a given state of the world ω, we identify welfare associated with the private contribution level D by the following unweighted Utilitarian welfare function:

$$W(D,\omega) = \sum_{i=1}^{N} U_i(D,\omega) = Ne_i - D - T + Nv\left(D\tfrac{N+g}{N} + T,\ a\right). \tag{3.9}$$

The welfare associated with the Nash-equilibrium for a given state of the world ω is accordingly denoted by $W^* := W(D^*,\omega)$. By considering the equilibrium welfare we account for individuals' response to changes in the state of the world.[15]

The following normative analysis focuses on the investigation of shocks on perceived productivity and changes in the perceived tax system. The effects on welfare strongly depend on whether the public good is underprovided for a given state of the world or not: due to the imperfect salience of taxes, it is no implication of the model's assumptions.[16]

We proceed as follows: in a first step, we formally identify underprovision. To do so, we characterize the welfare maximizing contribution for each state of the world. Based on this, we study in a second step how underprovision itself as well as the associated welfare loss are affected by the mentioned shocks.

Welfare maximizing private contributions and the characterization of underprovision

The next lemma determines the efficient, i.e. welfare maximizing, private contribution level D^e which gives a reference point for the identification of underprovision.

Lemma 3.4. *For a given state of the world ω, the efficient private contribution $D^e := \arg\max_D\{W(D,\omega)\}$ is implicitly given by*

$$\frac{1}{N+g} = v_1(D^e\tfrac{N+g}{N} + T, a). \tag{3.10}$$

The preceding lemma enables us to phrase a formal definition of *underprovision*, i.e.

[15] While the equilibrium contribution level D^* is a function of perceived productivity and perceived taxes, welfare is based on true productivity and taxes. Hence, in contrast to productivity and taxes, the salience of both affects welfare only indirectly via its effect on individual contribution levels.

[16] Although its practical relevance might be questionable, consider the extreme case where government's behavior is completely insalient, i.e. $\sigma_T = 0$. If the government contributes an amount T, positive contributions of individuals who do not take the government's contribution into account, can result in overprovision. In the following section we provide a formal definition of overprovision.

of a level of private contributions that is too small compared to the welfare maximizing level.

Definition 3.2 (Underprovision)**.** For a given state of the world ω, the public good is underprovided if and only if $D^* < D^e$ or equivalently if

$$(N+g)v_1(D^* \tfrac{N+g}{N} + T, a) \; - \; 1 \; > \; 0.$$

Note that—as typical for the voluntary provision of public goods under individually rational behavior—underprovision of the public good is the relevant case. Though, for the sake of completeness, the results presented in the following distinguish according to the provision level. This distinction will be decisive for the evaluation of shocks in terms of welfare. Intuitively, individuals respond to shocks by adjusting their donations. The evaluation of this adjustment depends on whether higher (or lower) overall contributions are desirable from a welfare perspective or not.

Note that in the considered setting, underprovision results as in the standard models from free-riding but, in addition, from the imperfect salience of productivity.

The effect of changes in the state of the world on equilibrium welfare

In this section we analyze how changes in perceived productivity or perceived taxes affect equilibrium welfare. Stating differently, we compute the overall effect of positive shocks on productivity and its salience as well as the tax schedule and tax salience on utility. Section 3.2.2 described the effects of shocks on private donations. Now, we derive the induced welfare changes.

We start with shocks on perceived productivity. These can be triggered either by shocks on productivity or its salience. A positive shock on salience occurs if e.g. the crisis gets in the focus of media or the accessibility of information becomes better such that individuals' degree of underestimated productivity decreases. Technically, an increase in the productivity's salience reduces the discrepancy between experienced and decision utility such that the mistake individuals make when choosing their donation gets smaller. As a result, individuals donate more such that—if there is underprovision—underprovision due to misperception is lowered.[17] With regard to welfare, this implies that salience shocks affect welfare positively:

$$\frac{dW^*}{d\sigma_a} \quad = \quad \underbrace{\frac{dD^*}{d\sigma_a}}_{>0} \cdot \underbrace{((N+g)v_1(\cdot) - 1)}_{\substack{>0 \\ \Leftrightarrow \text{ underprovision}}} > 0 \ \ .$$

The second factor is positive only if there is underprovision. It is negative, if the

[17]Note that we distinguish underprovision due to misperception or imperfect productivity salience from underprovision due to free-riding.

efficient aggregate contribution level is already exceeded such that the increase in D^* induced by the salience shock is welfare harming.[18]

For productivity shocks, the same arguments apply as these also induce higher private contributions which, in turn, affect welfare positively if there is underprovision. However, compared to a crisis' salience, productivity shocks affect welfare via a second channel, reflected in the second summand of the following equation:

$$\frac{dW^*}{da} \quad = \quad \underbrace{\frac{dD^*}{da}}_{>0} \cdot \underbrace{((N+g)v_1(\cdot)-1)}_{\substack{>0 \\ \Leftrightarrow \text{ underprovision}}} + \ Nv_2(\cdot) > 0.$$

The first component refers to the net effect of reduced underprovision: it is positive when higher donations are desirable. The second term mirrors the increased productivity of every dollar donated. It has a positive effect on welfare itself so that the overall welfare effect of a productivity shock can be positive even in the theoretical case of overprovision.

Proposition 3.1. *Consider a state of the world ω such that there is underprovision. Then positive shocks on productivity a or its salience σ_a raise welfare, i.e.*

$$\frac{dW^*}{d\tilde{a}} \ > \ 0 \quad .$$

We found that whatever raises the individuals perception of productivity leads to higher welfare when underprovision is an issue. Moreover, both shocks lead to an increase in aggregate contributions to the public good. In this sense, the discussed positive welfare effects stated above result (partly) from reduced free-riding.

In contrast to the effects of shocks on perceived productivity, the effects of changes in the perceived tax schedule are less clear and depend on the triggering shock: perceived taxes can change either due to variations in the tax or its salience. We firstly focus on the welfare effect of a tax salience shock.

Intuitively, a positive shock on tax salience makes individuals more aware of public contributions. This induces private donors to reduce their voluntary donations such that the aggregate contribution level decreases. If this aggregate contribution level is too low from a welfare perspective already before the tax shock, the decrease worsens the situation in the crisis region and lowers welfare as shown in the following expression:

$$\frac{dW^*}{d\sigma_T} \quad = \quad \underbrace{\frac{dD^*}{d\sigma_T}}_{<0} \cdot \underbrace{((N+g)v_1(\cdot)-1)}_{\substack{>0 \\ \Leftrightarrow \text{ underprovision}}} < 0 \quad .$$

[18]Note that the stated expression reflecting the mitigation of underprovision is a net effect and accounts for the accompanied reduction in private consumption that is linked to higher donations.

Note again the relevance of underprovision as a determinant for the direction of the effect. Given underprovision, a reduction of tax salience is welfare enhancing which is summarized in the following proposition:

Proposition 3.2. *Consider a state of the world ω such that there is underprovision. Then a positive shock on the tax salience σ_T lowers welfare, i.e.*

$$\frac{dW^*}{d\sigma_T} < 0 \ .$$

A similar result regarding the welfare effect of (nonsalient) taxes is derived by Chetty et al. (2009) for private goods and non-salient sales taxes: If taxes are not fully taken into account by individuals when making a consumption choice, the arising distortion due to the tax is lower the less salient the sales tax is. This, in turn, induces the dead-weight loss to be smaller under less salient sales taxes.

Having derived the welfare effect of a shock on tax salience, we can turn to the investigation of the welfare implications of changes in the tax schedule:

$$\frac{dW^*(\cdot)}{dT} = \underbrace{\frac{dD^* + T}{dT}}_{>0} \cdot \underbrace{((N+g)v_1(\cdot) - 1)}_{\substack{>0 \\ \Leftrightarrow \text{ underprovision}}} \underbrace{- g v_1(\cdot)}_{<0} \ . \tag{3.11}$$

As in all before mentioned cases and illustrated in the first summand, underprovision plays a crucial role. The first summand of equation (3.11) reflects the welfare effect of mitigated free-riding in case of underprovision.[19] Yet, there is a second effect determining the overall effect. If taxes go up, private consumption decreases as higher taxes do not create a warm glow and, hence, are an imperfect substitute. This is mirrored by the second summand: it reflects the lost warm glow if a dollar is contributed obligatorily via a tax and not voluntarily via private donations.[20] This affects welfare always negatively.

So, the aggregate welfare effect of a tax increase is not determined unambiguously by the occurrence of underprovision as in all before mentioned cases but also by a second opposing effect. The benefits of mitigated free-riding and the harm of a lost warm glow which yields reductions in private consumption have to be contrasted. The relative size of these two effects strongly depends on the state of the world such that the overall effect of a tax increase is in general ambiguous, even if there is underprovision. We make note of this point the following proposition and will discuss in more depth which component of ω plays a crucial role for determining the effect.

Proposition 3.3. *Consider a state of the world ω such that there is underprovision. Then tax increases can be both welfare increasing or harming.*

[19] In case of overprovision, a tax increase is never beneficial.

[20] Remember that we consider tax changes that do not change the set of contributors.

To better understand how the welfare effect of tax changes differs from other shocks' effect it proves helpful to rewrite equation (3.11):[21]

$$\frac{dW^*(\cdot)}{dT} = (1-\sigma_T)\frac{N}{N+g}\left((N+g)v_1(\cdot)-1\right) - \frac{g}{N+g} \ . \tag{3.12}$$

As before, the two main forces remain recognizable: the lost warm glow by which private consumption is reduced as reflected in the second summand and, further, the reduction of free-riding as represented by the first summand. Regarding the latter, the rephrased equation allows to make out a crucial determinant of the size of this effect: the tax salience.

If taxes are not fully salient, i.e. $\sigma_T < 1$, individuals underestimate the increase in taxes. This reduces the scope of crowding-out and allows the beneficial mitigation of free-riding if the public good is underprovided. The mechanism gets clear when considering the extreme case of perfectly insalient taxes, i.e. $\sigma_T = 0$. Here, individuals choose a donation in anticipation of a laissez-faire economy. So, they do not respond to changes in taxes. As a result, the aggregate contribution level $D^* + T$ increases by exactly the amount of a tax increase so that underprovision can be fully eliminated by an (appropriately chosen) tax such that the welfare maximum (see Lemma 3.4) is reached.

More generally, tax increases are more likely to raise welfare if the public good is underprovided and the tax salience is low. Then, the positive effect of reduced free-riding outweighs the lost warm glow. To get a better understanding, we close this section with a discussion on the other extreme case of full tax salience which is the classic setting Bergstrom et al. (1986) and Andreoni (1989), Andreoni (1990) study in their paper.

Full Tax Salience Under the assumption of full tax salience, $\sigma_T = 1$, individuals perfectly account for taxes when making their donation decision. Hence, the utility that is generated by the public good is constant[22] such that taxes are never beneficial as their only welfare effect is a lost warm glow. This yields the following proposition:

Proposition 3.4. *Consider a state of the world ω with fully salient taxes, i.e. $\sigma_T = 1$. Then taxes are welfare harming: for all tax schedules T_1 and T_2 with $T_2 > T_1$ it holds—ceteris paribus—that $W^*(D^*, T_1) > W^*(D^*, T_2)$.*

The stated proposition is a direct consequence of equation (3.12) for the case of $\sigma_T = 1$. It holds independently of a donation's productivity or its salience. Moreover, equation (3.12) implies that in the standard situation of Bergstrom et al. (1986) where $g = 0$ taxes do not affect welfare at all. Then, there is one-to-one crowding-out such that individuals completely balance the tax increase in their donation choice so that taxes do not impact the aggregate equilibrium contribution.

[21] Here, we make use of the comparative static results from section 3.2.2.

[22] The utility generated by the public good is pinned down by the first order condition (FOC*) if preferences are quasi-linear.

Contrary, if there is warm glow, as in the considered situation in proposition 3.4, governmental intervention incompletely crowds-out private donations. The resulting rise in the aggregate equilibrium support $D^* + T$ (see Lemma 3.3) comes in terms of welfare at the cost of reduced private consumption.

The latter observation underlines that the provision level of a public good is an insufficient measure for the quality of an allocation in terms of welfare. Even in the case of underprovision, an increase in overall public good provision level can be accompanied by a decrease in welfare, as for $\sigma_T = 1, g > 0$. So, the reduction of free-riding is not sufficient for a positive overall welfare effect.

Up to now, we studied welfare from a theoretical perspective. Though, we are actually interested in linking it to real world observations. The preceding analysis has emphasized that the observation of a tax increase is not informative itself as welfare can be boosted or lowered by it. However, if the tax increase is paralleled by an increase in private contributions, there exist cases where the direction of the welfare effect can be unambiguously identified based on concomitant changes in D^* and T.

3.3.2 Optimal government behavior and crowding-in

We only were able to determine the sign of the welfare effect of tax changes in case $\sigma_T = 1$. The aim of this section is to go one step further and to identify cases where a tax increase is beneficial or not if its salience is incomplete, $\sigma_T < 1$. We will link this theory based question to a real world situation where tax increases are observed: there are cases when individuals raise their private donations after shocks on perceived productivity even though public contributions have gone up as well. We call this phenomenon crowding-in and rationalize it in our model. Based on this, we are able to specify situations in which crowding-in is informative regarding our initial question whether a tax increase is beneficial.

Optimal Government Behavior

Up to now, we put no further assumption on the specification of the tax system. In the following we derive the optimal tax as a function of productivity, its salience and tax salience. This is an auxiliary step as it is not possible to state in general whether tax increases are beneficial or not (see Proposition 3.3). However, if we analyze the welfare maximizing tax, we have an anchor point for the evaluation of tax increases that allows us to identify situations where tax changes can be evaluated in terms of welfare.

For a given level of perceived productivity and tax salience, the optimal tax T^o conditional on rational individual behavior is characterized as follows.

Definition 3.3 (Optimal tax)**.** For a given level productivity a, productivity salience σ_a, and tax salience σ_T, the optimal tax T^o maximizes equilibrium welfare. It is characterized by the following equation:

$$T^o = \max_T \left\{0, \arg_T\{\tfrac{dW^*}{dT} = 0\}\right\}$$

Although not outlined explicitly, the optimal tax T^o is a function of perceived productivity and tax salience as it nests the equilibrium donation. In the following we analyze how optimal government behavior adjusts in response to changes in the state of the world.[23] The intuitions behind the formal results are then directly linked to the welfare analysis in Section 3.3.1 which already incorporates reactions in individual behavior to changes in the state of the world and its implications for welfare. Yet, the following analysis goes one step further as individuals respond to changes in two dimensions of the state of the world: the change in productivity or its salience as well as the change in the tax system.

Proposition 3.5. *Consider a given state of the world $\omega = (a, T^o, \sigma_a, \sigma_T)$ such that $T^o > 0$.*

(a) Assume there is a positive shock on productivity salience σ_a. Then, the optimal tax decreases:

$$\frac{dT^o}{d\sigma_a} < 0.$$

(b) Assume there is a positive shock on productivity a. Then, the optimal tax can decrease or increase dependent on ω and $v(\cdot)$:

$$\frac{dT^o}{da} \gtrless 0.$$

Let us shortly comment on the different parts of Proposition 3.5.

Proposition 3.5(a) investigates optimal taxes after a shock on productivity salience. If salience of productivity is imperfect, individuals underestimate the positive effect of their donations and donate too little. This underprovision is alleviated by mandatory and not fully salient taxes. If now the underprovision shrinks due to a positive shock on the salience of productivity, the problem that is tackled by a tax has become smaller by the shock; by that the value of a tax in terms of welfare has been reduced. On the other hand, the downside of taxes as given by destroyed warm glow still exists and gained relatively more weight in terms of welfare. In other words, if public contributions increase (or stay constant) concomitantly with the positive salience shock, the government counteracts more a less relevant problem reducing welfare compared to welfare under governmental inactivity. Hence, the optimal tax can neither stay constant nor increase if productivity gets more salient.

[23] Obviously, the interesting part for the following comparative static analysis requires government intervention itself not to be welfare harming. For that reason—even if not always mentioning it—we focus on cases where $T^o > 0$.

Proposition 3.5(b) deals with the change in the optimal tax in response to an increase in productivity. Again, to make out, whether the welfare maximizing tax decreases or increases after a shock on true productivity, the effect on the existing underprovision, i.e. on $((N+g)v_1(\cdot)-1)$, has to be evaluated. As in part (a) of this proposition, individuals donate more after the shock than before it which reduces underprovision. Though, a shock on true productivity a has an effect on the basis of the evaluation of this as the sub-utility function $v(\cdot, a)$ is affected as well. Thereby, the aggregate effect is a priori ambiguous as both arguments of $v(\cdot)$ change and tear $v(\cdot)$ in opposite directions. Intuitively, this reflects the contrasting effects of reduced free-riding and stolen glow in case of raised marginal benefits of donations.

These results set the stage for the next section which studies what we can conclude from the observation of crowding-in.

What we can learn from observing crowding-in

Before we start with introducing the phenomenon of *crowding-in* in the following definition, note that only by considering the reference case of optimal taxes, we are able to unambiguously identify how to evaluate tax increases. This would not be possible in case of arbitrary tax schedules, see Proposition 3.3.

Definition 3.4 (Crowding-in)**.** Consider a change in the state of the world from $\omega_1 = (a_1, T_1, \sigma_{a1}, \sigma_T)$ to $\omega_2 = (a_2, T_2, \sigma_{a2}, \sigma_T)$ with $T_1 < T_2$ and $\tilde{a}_1 < \tilde{a}_2$. Then, there is crowding-in if $D_1^* < D_2^*$.

There is crowding-in if private donations go up after a simultaneous increase in perceived productivity and public contributions. Intuitively, there is crowding-in if the positive effect of the former on private donations outweighs the negative effect of the latter resulting from crowded-out private donations.

From the individuals' perspective, crowding-in indicates an insufficient government response to the considered shock on $\tilde{a}$: the change in ω induces individuals to consider higher aggregate contributions to the charity to be optimal. The tax increase at least partly covers the raised requirements in contributions. Though, if crowding-in is observed, the tax increase was too small to completely cover the increased demand for the public good inducing individuals to step in and donate more to reach the new equilibrium. In that sense, we can interpret crowding-in as pointing to an insufficient government response to a shock on perceived productivity from the individuals' (distorted) perspective.

As we will see in the following, this identification of non-optimality of governmental behavior due to crowding-in per se does not always coincide with the optimality discussed in the last chapter, i.e. the stated welfare maximizing government behavior. To indicate non-optimality in the sense of the previous chapter the respective trigger of crowding-in is decisive.

Proposition 3.6. *Consider an initial state of the world* $\omega = (a, T^o, \sigma_a, \sigma_T)$ *with* $T^o > 0$ *and a positive shock on productivity salience* σ_a*. Then, observing crowding-in is sufficient to identify non-optimal government behavior.*

As the optimal, i.e. the welfare maximizing, tax decreases if the salience of the crisis goes up (see Proposition 3.5(a)), crowding-in points to non-optimal government behavior in the given setting. In particular, this implies that the induced welfare level falls below the one that could have been reached and even under the one that resulted from government inactivity.

With respect to the already mentioned real world situation when the photo of the drowned boy was published, this proposition tells us that it would have been better if the government had not raised their contributions after the photo was published—at least under the assumption of a benevolent government before the picture's publication.

Up to now, we found that in case of salience shocks we can unambiguously make out non-optimal government behavior by observing crowding-in. Hence, the problem of an ambiguous evaluation of tax increases as given in equation (3.12) is solved if the tax increase is paralleled by a salience shock.

This is not true if we consider shocks on true productivity a.

Proposition 3.7. *Consider an initial state of the world* $\omega = (a, T^o, \sigma_a, \sigma_T)$ *with* $T^o > 0$ *and a positive shock on productivity* a*. Observing crowding-in is neither necessary nor sufficient to indicate optimal government behavior.*

As the direction of the change in optimal taxes itself is a priori unclear after a productivity shock, the observation of crowding-in itself not informative in this case. The reason is that the general assumptions on ω and $v(\cdot)$ do not allow to draw conclusions on whether underprovision under individually rational behavior as proxied by $((N + g)v_1(\cdot) - 1)$ increases or decreases after a productivity shock. However, this is crucial to identify optimal government responses. If individuals 'underesponse' to the productivity—reflected in an increased underprovision of the public good—crowding-in is necessary for optimal government behavior. If, in contrast, underprovision increased in the absence of government responses to the need shock, not observing crowding-in indicates a non-optimal government behavior.

We can show for Cobb-Douglas preferences of the form $\tilde{U}_i(\cdot) = c_i + \tilde{a}^\beta \left(d_i(1+g) + D_{-i} + \tilde{T}\right)^\gamma$, $\beta, \gamma < 1$ that for these the optimal tax increases in the crisis' productivity. So in this case, not observing crowding-in always points to a tax schedule that is not welfare maximizing for the reasons given above.

To recapitulate: the implications of the observation of crowding-in strongly depend on the trigger. Identifying crowding-in triggered by a productivity salience shock identifies non-optimal government behavior while the absence of crowding-in after a pure productivity shock, in contrast, identifies non-optimal government behavior if underprovision (see Definition 3.2) increased ceteris paribus.

3.4 Conclusion

Our paper is the first tackling the yawning gap between the famous literature on public goods provision and the more recent branch of literature on salience. We provide an approach towards the question how the provision of public goods, in particular of charity, is affected by the fact that individuals hold biased beliefs: due to imperfect salience, they misperceive the level of public contributions and a donations' effectiveness. Consequently, the resulting equilibrium contribution remarkably differs from the one in the full salience benchmark case.

We provide an extensive analysis on how the aggregate equilibrium donation varies in response to changes in the circumstances, i.e. changes in perceived productivity and in perceived taxes, and derive the respective changes in welfare. Regarding welfare which is based on experienced utility, we find that it is crucial for the evaluation of a given allocation whether the public good is underprovided (compared to the welfare maximizing level). Although being the relevant case both theoretically and empirically, that the public good is underprovided is not a technical implication of our model.

Shocks on perceived productivity raise aggregate contributions since individuals respond to the shock such that those shocks are welfare enhancing if and only if the good was underprovided before the shock.

Moreover, even if underprovision is an issue, an increase in the perceived tax schedule that causes aggregate contributions go up can drag welfare in either direction. We solve this ambiguity by considering a benevolent government as a reference point and utilize the phenomenon crowding-in as an auxiliary tool to assess whether a tax change was beneficial or not. Based on the observation of crowding-in, we can identify a tax change as being welfare enhancing if it counteracts a worsening underprovision due to a shock. As this is never true if crowding-in was triggered by a salience shock, in this case crowding-in indicates non-optimal governmental behavior. If instead a tax increase is paralleled by a shock on true productivity such that individuals raise their contributions as well, this can lead to higher or lower welfare.

The presented normative analysis allows to use observable changes in the state of the world to draw conclusions about whether welfare has increased or decreased. It is left for future research to test our results empirically to check whether the conclusions drawn from the theoretical investigation can be proven true in practice.

Appendix 3.A Proof of Propositions

Proposition 3.1

Proof. Equilibrium welfare is given by

$$W^* = Ne_i - D^* - T + Nv\left(D^* \tfrac{N+g}{N} + T, a\right)$$

The first order conditions with respect to actual productivity or its salience are given by

$$\begin{aligned}\frac{dW^*}{d\sigma_a} &= -\tfrac{dD^*}{d\sigma_a} + Nv_1(\cdot)\tfrac{dD^*}{d\sigma_a}\tfrac{N+g}{N} \\ &= \underbrace{\tfrac{dD^*}{d\sigma_a}}_{>0 \text{ see (3.7)}} ((N+g)v_1(\cdot) - 1)\end{aligned}$$

if and only if there is underprovision according to Definition 3.2.

$$\begin{aligned}\frac{dW^*}{da} &= -\tfrac{dD^*}{da} + Nv_1(\cdot)\tfrac{dD^*}{da}\tfrac{N+g}{N} + Nv_2(\cdot) \\ &= \underbrace{\tfrac{dD^*}{da}}_{>0 \text{ see (3.8)}} ((N+g)v_1(\cdot) - 1) + Nv_2(\cdot) > 0\end{aligned}$$

if there is underprovision according to Definition 3.2.

$$\Rightarrow \frac{dW^*}{d\tilde{a}} > 0 \text{ if there is underprovision.}$$

□

Proposition 3.2

Proof. Equilibrium welfare is given by

$$W^* = Ne_i - D^* - T + Nv\left(D^* \tfrac{N+g}{N} + T, a\right)$$

The first order condition with respect to tax salience is given by

$$\frac{dW^*}{d\sigma_T} = -\tfrac{dD^*}{d\sigma_T} + Nv_1(\cdot)\tfrac{dD^*}{d\sigma_T}\tfrac{N+g}{N}$$

$$= \underbrace{\tfrac{dD^*}{d\sigma_T}}_{<0} \left((N+g)v_1(\cdot) - 1\right) \; < \; 0$$

if and only if there is underprovision according to Definition 3.2. $\square$

Proposition 3.3

Proof. Equilibrium welfare is given by

$$W^* = Ne_i - D^* - T + Nv\left(D^* \tfrac{N+g}{N} + T, a\right)$$

The first order condition with respect to the tax is given by

$$\begin{aligned}
\frac{dW^*}{dT} &= -\tfrac{dD^*}{dT} \; - \; 1 \; + \; Nv_1(\cdot) \underbrace{\left(\tfrac{dD^*}{dT}\tfrac{N+g}{N} + 1\right)}_{=(1-\sigma_T)} \\
&= \frac{N}{N+g}\sigma_T \; - \; 1 \; + \; Nv_1(\cdot)(1-\sigma_T) \\
&= N(1-\sigma_T)\underbrace{\left(v_1(\cdot) - \tfrac{1}{N+g}\right)}_{\substack{>0 \\ \Leftrightarrow \text{ underprovision}}} \; \underbrace{- \; \frac{g}{N+g}}_{<0}
\end{aligned}$$

Depending on the level of tax salience σ_T and the specific preferences $v(\cdot)$, this expression can be positive or negative:

If $\sigma_T = 1$, then $\frac{dW^*}{dT} = -\frac{g}{N+g} < 0$ so that $T^o = 0$.

If $\sigma_T = 0$, then $D^* := \arg_D\{v_1(D\frac{N+g}{N}, \tilde{a}) = \frac{1}{1+g}\} < \bar{D} := \arg_D\{v_1(D\frac{N+g}{N}, a) = \frac{1}{1+g}\}$ as by assumption $v_{12}(\cdot) > 0$, $v_{11}(\cdot) < 0$, and $\tilde{a} < a$.[24] Moreover, as $v_1(\cdot)$ is decreasing, it holds that $v_1(D^*\frac{N+g}{N}, a) > v_1(\bar{D}\frac{N+g}{N}, a) = \frac{1}{1+g}$ where the last step follows from the definition of $\bar{D}$.

So, if $\sigma_T = 0$, then $\frac{dW^*}{dT}|_{T=0} = Nv_1(D^*\frac{N+g}{N}, a) - 1 > \frac{N}{1+g} - 1 \geq 0$.

The last statement implies that $T^o > 0$ if $g \leq N-1$ which holds as $g \in [0,1]$. $\square$

Proposition 3.4

This proposition is a direct implication of the previous proposition.

[24]Note that (i) $v_{12}(\cdot) > 0$ implies that $v_1(D, \tilde{a}) < v_1(D, a)$ for all $\tilde{a} < a$. Moreover, (ii) $v_{11}(\cdot) < 0$ implies that $v_1(D, a) > v_1(\bar{D}, a)$ for all $D < \bar{D}$. Combining the two implications results in the stated inequality, i.e. $\bar{D} > D^*$:

$$\begin{aligned}
D^* := \arg_D\{v_1(D\tfrac{N+g}{N}, \tilde{a}) = \tfrac{1}{1+g}\} \;&\overset{(i)}{\Rightarrow}\; v_1(D^*\tfrac{N+g}{N}, a) > \tfrac{1}{1+g} \\
&\overset{(ii)}{\Rightarrow}\; \left(v_1(\bar{D}\tfrac{N+g}{N}, a) = \tfrac{1}{1+g} \Rightarrow \bar{D} > D^*\right).
\end{aligned}$$

Proposition 3.5

Proof. The optimal tax T^o is implicitly given by the following first order condition

$$\begin{aligned}\frac{dW^*}{dT} &= -\tfrac{N}{N+g}\sigma_T \;-\; 1 \;+\; Nv_1(D^*(T)\tfrac{N+g}{N} + T, a)(1-\sigma_T) \;=\; 0 \\ &\Leftrightarrow: \;\; F(\sigma_a, T(\sigma_a)) = 0.\end{aligned}$$

(a) By the implicit function theorem applied to the previously stated funtion $F(\cdot)$, we have

$$\frac{dT^o}{d\sigma_a} = -\frac{\frac{dF}{d\sigma_a}}{\frac{dF}{dT}}.$$

$$\begin{aligned}\frac{dF}{dT} &= N(1-\sigma_T)v_{11}(\cdot)\left(\tfrac{N+g}{N}\tfrac{dD^*}{dT}+1\right) \\ &= N(1-\sigma_T)^2 v_{11}(\cdot) \;<\; 0 \\ &\Rightarrow \; sgn\left(\frac{dT^o}{d\sigma_a}\right) = sgn\left(\frac{dF}{d\sigma_a}\right).\end{aligned}$$

$$\frac{dF}{d\sigma_T} = N\underbrace{v_{11}(\cdot)}_{<0}(1-\sigma_T)\frac{N+g}{N}\underbrace{\frac{dD^*}{d\sigma_a}}_{>0 \text{ see (3.7)}} < 0$$

This implies the given statement, $\frac{dT^o}{d\sigma_a} < 0$.

(b) By the implicit function theorem applied to the previously stated funtion $F(\cdot)$, we have

$$\frac{dT^o}{da} = -\frac{\frac{dF}{da}}{\frac{dF}{dT}}.$$

By the same arguments as before, we have

$$\left(\frac{dT^o}{da}\right) = sgn\left(\frac{dF}{da}\right)$$

$$\frac{dF}{da} = N(1-\sigma_T)\left(\underbrace{v_{11}(\cdot)\tfrac{N+g}{N}\tfrac{dD^*}{da}}_{<0} + \underbrace{v_{12}(\cdot)}_{>0}\right) \gtrless 0$$

Hence, the sign of $\frac{dT^o}{da}$ is ambiguous.

Moreover, it holds that

$$\frac{d}{da}\left[(N+g)v_1(\cdot)-1\right] = (N+g)\cdot\left[v_{11}(\cdot)\tfrac{dD^*}{da}\tfrac{N+g}{N} + v_{12}(\cdot)\right]$$

so that

$$\frac{dT^o}{da} \gtrless 0 \iff \frac{d}{da}\left[(N+g)v_1(\cdot)-1\right] \gtrless 0.$$

□

Proposition 3.6

This is a direct consequence of Proposition 3.5(a).

Proposition 3.7

This is a direct consequence of Proposition 3.5(b).

Appendix 3.B Proof of Lemmata

Lemma 3.1

Proof. d_i^* maximizes individual i's decision utility if it maximizes

$$\tilde{U}(\cdot)_i = e_i - d_i - t_i + v(D_{-i} + d_i(1+g) + \tilde{T}, \tilde{a})$$

This yields the following first order condition

$$\frac{d\tilde{U}}{dd_i} = -1 + (1+g)v_1(\cdot) = 0 \quad \forall i$$

By the concavity of $v(\cdot)$ the solution is in fact a maximum. $\square$

Lemma 3.2

Proof. The first order condition stated in lemm 3.1 determines implicitly, how d_i^* changes in D_{-i}. Rearranging the first order condition yields

$$1/(1+g) = v_1(d_i^*(D_{-i})(1+g) + D_{-i} + \tilde{T}, \tilde{a}).$$

Implicitly differentiating yields

$$0 = (1+g)v_{11}(\cdot)\left(\frac{dd_i^*}{dD_{-i}}(1+g) + 1\right) \Rightarrow \frac{dd_i^*}{dD_{-i}} = -\frac{1}{1+g}.$$

$\square$

Lemma 3.3

Proof. Applying the implicit function theorem to (FOC*) yields

$$\frac{dD^*(\tilde{a},\tilde{T})}{dT} = -\frac{(1+g)v_{11}(\cdot)\sigma_T}{(1+g)\frac{n+g}{N}v_{11}(\cdot)} = -\frac{N}{N+g}\sigma_T$$

so that

$$\frac{d(D^*(\tilde{a},\tilde{T}) + T)}{dT} = \frac{dD^*(\tilde{a},\tilde{T})}{dT} + 1 = 1 - \frac{N}{N+g}\sigma_T > 0.$$

$\square$

Lemma 3.4

Proof. With the given definition of welfare and quasi linear preferences, the efficient private donation has to solve

$$\max_{d_1,\ldots,d_N} W(a) \;=\; \sum_i (e_i - d_i) + \sum_i v\Big((1+g)d_i + \sum_{j\neq i} d_j, a\Big).$$

This yields the following N identical first-order conditions:

$$0 = -1 + (1+g)v_1(\cdot) + (N-1)v_1(\cdot)\ .$$

Using symmetry the FOCs simplify to

$$1 = (N+g)v_1(\cdot)\ ,$$

which is equivalent to the stated expression. □

Appendix 3.C Proof of Corollaries

Corollary 3.1

Proof. The following first-order condition determines the optimal aggregate support level $S^*(\cdot) = D^* + T$, see Lemma 3.1.

$$0 = -1 + (1+g)v_1(S^*(\cdot) + gd_i, \tilde{a})$$

Implicitly differentiating yields

$$\frac{dS^*(\cdot)}{d\tilde{a}} = -\frac{v_{12}(\cdot)(1 + \overbrace{\tfrac{dd_i^*}{d\tilde{a}}}^{>0})}{v_{11}(\cdot)}$$

With $\frac{d\tilde{a}}{da} > 0$ and $\frac{d\tilde{a}}{d\sigma_a} > 0$ the corollary is a direct implication.
As T is exogenously given and fixed, it holds that

$$\frac{dS^*(\cdot)}{d\tilde{a}} = \frac{dD^*(\cdot)}{d\tilde{a}} \text{ with } \frac{d\tilde{a}}{da} = \sigma_a, \quad \frac{d\tilde{a}}{d\sigma_a} = a.$$

□

Bibliography

Adena, M. and S. Huck (2017). "Matching donations without crowding out? Some theoretical considerations, a field, and a lab experiment". *Journal of Public Economics* 148, 32 –42.

Andreoni, J. (1989). "Giving with Impure Altruism: Applications to Charity and Ricardian Equivalence". *Journal of Political Economy* 97, 1447–1458.

——— (1990). "Impure Altruism and Donations to Public Goods: A Theory of Warm-Glow Giving". *Economic Journal* 100, 464–477.

——— (2006). "Leadership Giving in Charitable Fund-Raising". *Journal of Public Economic Theory* 8, 1–22.

Andreoni, J. and A. A. Payne (2013). "Charitable giving". In: *Handbook of public economics.* Vol. 5. Elsevier, 1–50.

Arrow, K. (1979). "The Property Rights Doctrine and Demand Revelation under Incomplete Information". In: *Economics and Human Welfare.* Ed. by M. J. Boskin. New York: Academic Press.

Auten, G. E. et al. (2002). "Charitable Giving, Income, and Taxes: An Analysis of Panel Data". *American Economic Review* 92.

Baekgaard, M. et al. (2016). "Causes of fiscal illusion: Lack of information or lack of attention?" *Public Budgeting & Finance* 36, 26–44.

Bergemann, D. and S. Morris (2005). "Robust Mechanism Design". *Econometrica* 73, 1771–1813.

Bergstrom, T. et al. (1986). "On the private provision of public goods". *Journal of Public Economics* 29, 25 –49.

Besley, T. and S. Coate (1997). "An economic model of representative democracy". *The Quarterly Journal of Economics* 112, 85–114.

Bierbrauer, F. (2011). "On the Optimality of Optimal Income Taxation". *Journal of Economic Theory* 146, 2105–2116.

Bierbrauer, F. and P. Boyer (2013). "Political competition and Mirrleesian income taxation: A first pass". *Journal of Public Economics* 103, 1–14.

——— (2014). "The Pareto-frontier in a simple Mirrleesian model of income taxation". *Annals of Economics and Statistics/Annales d'Économie et de Statistique*, 185–206.

Bierbrauer, F. et al. (2015). "Robust Mechanism Design and Social Preferences". CES ifo Working Paper No. 4713.

Börgers, T. and P. Norman (2009). "A Note on Budget Balance under Interim Participation Constraints: the Case of Independent Types". *Economic Theory* 39, 477–489.

Bourguignon, F. and A. Spadaro (2012). "Tax-benefit revealed social preferences". *Journal of Economic Inequality* 10, 75–108.

Brett, C. and J. A. Weymark (2017). "Voting over selfishly optimal nonlinear income tax schedules". *Games and Economic Behavior* 101, 172–188.

Brunner, J. K. (1993). "A note on the optimum income tax". *Journal of Public Economics* 50, 445–451.

——— (1995). "A theorem on utilitarian redistribution". *Social Choice and Welfare* 12, 175–179.

Buchanan, J. M. (1960). *The theory of public finance.*

——— (2014). *Public finance in democratic process: Fiscal institutions and individual choice.* UNC Press Books.

Che, Y. and I. Gale (2006). "Revenue Comparison for Auctions when Bidders have Arbitrary Types". *Theoretical Economics* 1, 95–118.

Chetty, R. and E. Saez (2013). "Teaching the Tax Code: Earnings Responses to an Experiment with EITC Recipients". *American Economic Journal: Applied Economics* 5.

Chetty, R. et al. (2009). "Salience and Taxation: Theory and Evidence". *American Economic Review* 99.

Cramton, P. et al. (1987). "Dissolving a Partnership Efficiently". *Econometrica* 55, 615–632.

D'Aspremont, C. and L.-A. Gerard-Varet (1979). "Incentives and Incomplete Information". *Journal of Public Economics* 11, 25–45.

Ebert, U. (1992). "A reexamination of the optimal nonlinear income tax". *Journal of Public Economics* 49, 47–73.

Eckel, C. C. et al. (2005). "An experimental test of the crowding out hypothesis". *Journal of Public Economics* 89, 1543–1560.

Eisensee, T. and D. Strömberg (2007). "News Droughts, News Floods, and U. S. Disaster Relief". *Quarterly Journal of Economics* 122, 693–728.

Elias, A. et al. (2015). "Position, selective emphasis and framing: How parties deal with a second dimension in competition". *Party Politics* 21, 839–850.

Farhi, E. and X. Gabaix (2015). "Optimal Taxation with Behavioral Agents". NBER Working Paper 21524.

Feldman, M. and C. Gilles (1985). "An Expository Note on Individual Risk Without Aggregate Uncertainty". *Journal of Economic Theory* 35, 26–32.

Finkelstein, A. (2009). "E-ztax: Tax salience and tax rates". *Quarterly Journal of Economics* 124, 969–1010.

Goldin, J. and Y. Listokin (2014). "Tax Expenditure Salience". *American Law and Economics Review* 16, 144–176.

Gresik, T. and M. Satterhwaite (1989). "The Rate at which a Simple Market Converges to Efficiency as the Number of Traders Increases: An Asymptotic Result for Optimal Trading Mechanisms". *Journal of Economic Theory* 48, 304–332.

Guesnerie, R. (1995). *A Contribution to the Pure Theory of Taxation.* Cambridge University Press.

Guesnerie, R. and J.-J. Laffont (1984). “A complete solution to a class of principal-agent problems with an application to the control of a self-managed firm”. *Journal of public Economics* 25, 329–369.

Guesnerie, R. and J. Seade (1982). “Nonlinear pricing in a finite economy”. *Journal of public Economics* 17, 157–179.

Güth, W. and M. Hellwig (1986). “The Private Supply of a Public Good”. *Journal of Economics* Supplement 5, 121–159.

Hammond, P. J. (1979). “Straightforward individual incentive compatibility in large economies”. *The Review of Economic Studies* 46, 263–282.

Harbaugh, W. T. et al. (2007). “Neural Responses to Taxation and Voluntary Giving Reveal Motives for Charitable Donations”. *Science* 316, 1622–1625.

Harris, M. and A. Raviv (1981). “Allocation Mechanisms and the Design of Auctions”. *Econometrica* 49, 1477–1499.

Hellwig, M. (1986). “The Optimal Linear Income Tax Revisited”. *Journal of Public Economics* 31, 163–179.

——— (2003). “Public-Good Provision with Many Participants”. *Review of Economic Studies* 70, 589–614.

——— (2007). “A Contribution to the Theory of Optimal Utilitarian Income Taxation”. *Journal of Public Economics* 91, 1449–1477.

Heutel, G. (2014). “Crowding out and crowding in of private donations and government grants”. *Public Finance Review* 42, 143–175.

Hickey, R. et al. (2015). “The Sensitivity of Charitable Giving to the Timing and Salience of Tax Credits”. mimeo.

Huck, S. and I. Rasul (2011). “Matched fundraising: Evidence from a natural field experiment”. *Journal of Public Economics* 95. Charitable Giving and Fundraising Special Issue, 351 –362.

Huck, S. et al. (2015). “Comparing Charitable Fundraising Schemes: Evidence from a Natural Field Experiment and a Structural Model”. *American Economic Journal: Economic Policy* 7.

Judd, K. (1985). “The Law of Large Numbers with a Continuum of I.I.D. Random Variables”. *Journal of Economic Theory* 35, 19–25.

Kahneman, D. et al. (1997). “Back to Bentham? Explorations of Experienced Utility”. *Quarterly Journal of Economics* 112, 375–405.

Karlan, D. and J. A. List (2007). “Does Price Matter in Charitable Giving? Evidence from a Large-Scale Natural Field Experiment”. *American Economic Review* 97.

Kos, N. and M. Messner (2013). “Extremeal Incentive Compatible Transfers”. *Journal of Economic Theory* 148, 134–164.

Krasteva, S. and H. Yildrim (2013). “(Un)Informed charitable giving”. *Journal of Public Economics* 106.

Laslier, J.-F. and N. Picard (2002). "Distributive politics and electoral competition". *Journal of Economic Theory* 103, 106–130.

Lockwood, B. B. and M. Weinzierl (2016). "Positive and normative judgments implicit in U.S. tax policy, and the costs of unequal growth and recessions". *Journal of Monetary Economics* 77, 30–47.

Lollivier, S. and J.-C. Rochet (1983). "Bunching and second-order conditions: A note on optimal tax theory". *Journal of Economic Theory* 31, 392–400.

Mailath, G. and A. Postlewaite (1990). "Asymmetric Bargaining Procedures With Many Agents". *Review of Economic Studies* 57, 351–367.

Mas-Colell, A. et al. (1995). *Microeconomic Theory.* USA: Oxford University Press.

Matsuo, T. (1989). "On Incentive Compatible, Individually Rational and Ex Post Efficient Mechanisms for Bilateral Trading". *Journal of Economic Theory* 49, 189–194.

McAfee, P. and J. McMillan (1987). "Auctions and Biddings". *Journal of Economic Literature* 25, 699–738.

Mirrlees, J. (1971). "An Exploration in the Theory of Optimum Income Taxation". *Review of Economic Studies* 38, 175–208.

Müller, R. et al. (2007). "Weak monotonicity and Bayes–Nash incentive compatibility". *Games and Economic Behavior* 61, 344–358.

Mussa, M. and S. Rosen (1978). "Monopoly and Product Quality". *Journal of Economic Theory* 18, 301–317.

Myerson, R. (1981). "Optimal Auction Design". *Mathematics of Operations Research* 6, 58–73.

Myerson, R. and M. Satterthwaite (1983). "Efficient Mechanisms for Bilateral Trading". *Journal of Economic Theory* 28, 265–281.

Negishi, T. (1960). "Welfare economics and existence of an equilibrium for a competitive economy". *Metroeconomica* 12, 92–97.

Norman, P. (2004). "Efficient Mechanisms for Public Goods with Use Exclusion". *Review of Economic Studies* 71, 1163–1188.

Osborne, M. J. and A. Slivinski (1996). "A model of political competition with citizen-candidates". *Quarterly Journal of Economics* 111, 65–96.

Potters, J. et al. (2007). "Leading-by-example and signaling in voluntary contribution games: an experimental study". *Economic Theory* 33, 169–182.

Riley, J. and S. W. (1981). "Optimal Auctions". *American Economic Review* 71, 381–392.

Rob, R. (1989). "Pollution Claim Settlements under Private Information". *Journal of Economic Theory* 47, 307–333.

Roberts, R. D. (1984). "A Positive Model of Private Charity and Public Transfers". *Journal of Political Economy* 92, 136–148.

Röell, A. (2012). "Voting over nonlinear income tax schedules". *Columbia University, unpublished manuscript.*

Rovny, J. (2012). "Who emphasizes and who blurs? Party strategies in multidimensional competition". *European Union Politics* 13, 262–292.
Saez, E. and S. Stantcheva (2016). "Generalized social marginal welfare weights for optimal tax theory". *American Economic Review* 106, 24–45.
Schmitz, P. (1997). "Monopolistic Provision of Excludable Public Goods under Private Information". *Public Finance/ Finance Publiques* 52, 89–101.
Seade, J. K. (1977). "On the shape of optimal tax schedules". *Journal of public Economics* 7, 203–235.
Simula, L. (2010). "Optimal nonlinear income tax and nonlinear pricing; optimality conditions and comparative static properties". *Social Choice and Welfare* 35, 199–220.
Slovic, P. et al. (2017). "Iconic photographs and the ebb and flow of empathic response to humanitarian disasters". *Proceedings of the National Academy of Sciences* 114, 640–644.
Stiglitz, J. (1982). "Self-Selection and Pareto-Efficient Taxation". *Journal of Public Economics* 17, 213–240.
Vesterlund, L. (2003). "The informational value of sequential fundraising". *Journal of Public Economics* 87, 627 –657.
Vickrey, W. (1961). "Counterspeculation, Auctions and Competitive Sealed Tenders". *Journal of Finance* 16, 8–39.
Wagner, R. E. (1976). "Revenue structure, fiscal illusion, and budgetary choice". *Public choice* 25, 45–61.
Warr, P. G. (1982). "Pareto optimal redistribution and private charity". *Journal of Public Economics* 19, 131 –138.
Warren, P. E. and I. Walker (1991). "Empathy, effectiveness and donations to charity: Social psychology's contribution". *British Journal of Social Psychology* 30, 325–337.
Weinzierl, M. (2014). "The promise of positive optimal taxation: normative diversity and a role for equal sacrifice". *Journal of Public Economics* 118, 128–142.
Weymark, J. (1987). "Comparative Static Properties of Optimal Nonlinear Income Taxes". *Econometrica* 55, 1165–1185.
Weymark, J. A. (1986a). "A reduced-form optimal nonlinear income tax problem". *Journal of Public Economics* 30, 199–217.
——— (1986b). "Bunching properties of optimal nonlinear income taxes". *Social Choice and Welfare* 3, 213–232.
Zoutman, F. T. et al. (2014). "Optimal redistributive taxes and redistributive preferences in the Netherlands". *Working Paper.*
——— (2017). "Revealed social preferences of Dutch political parties". *Journal of Public Economics* 156, 81–100.

Curriculum Vitae Laura Kohlleppel

Persönliche Daten

Geburtsdatum:	06.11.1986
Geburtsort:	Bochum, Deutschland
Staatsangehörigkeit:	deutsch

Universitätsausbildung

seit 11/2012	Promotion im Fach Volkswirtschaftslehre, Universität zu Köln
06/2009-06/2012	Mathematik Diplom, Technische Universität Darmstadt
10/2008-05/2009	Betriebswirtschaftslehre Vordiplom, Technische Universität Darmstadt
04/2006-10/2008	Mathematik Vordiplom, Universität Frankfurt

Akademische Positionen

seit 04/2014	Wissenschaftliche Mitarbeiterin am Center for Macroeconomic Research, Universität zu Köln
11/2012-03/2014	Wissenschaftliche Mitarbeiterin am Lehrstuhl von Professor Achim Wambach, PhD, Universität zu Köln