ROBUST BEHAVIOR IN AUCTIONS

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Contents

1 Introduction

The modeling assumptions in the analysis of an economic problem highly depend on the knowledge different agents have about the particular economic setting. In particular, the assumptions made about common knowledge may strongly influence the outcome of an economic analysis. However, the assumption that specific aspects of an economic setting are common knowledge, may not always be plausible. This issue has been brought up in the Wilson Doctrine stated by Robert Wilson in 1987:

> "Game Theory has a great advantage in explicitly analyzing the consequence of trading rules that presumably are really common knowledge; it is deficient to the extent it assumes other features to be common knowledge, such as one player's probability assessment about another's preferences or information. I foresee the progress of game theory as depending on successive reductions in the base of common knowledge required to conduct useful analysis of practical problems. Only by repeated weakening of common knowledge assumptions will the theory approximate reality."

This doctrine served as an inspiration for many economic papers, in particular, it was pathbreaking for the literature on "Robustness". Typically, in this literature at least one agent faces uncertainty about the specification of one (or more) model component, i.e. the agent faces a set of possible specifications instead of knowing that a particular specification is correct. The research question is to find a strategy for this agent which ensures a certain outcome independent of which specification from the set of possible specifications is the correct one.

This thesis contains three papers which analyze such a research question in the context of auctions. In the first paper the agent facing uncertainty has to delegate the execution of an auction (or procurement auction) to an auctioneer whose particular choice of the auction mechanism is not known to the agent. While the auctioneer may have a favorite bidder, the goal of the agent is to ensure the absence of (positive or negative) discrimination. Common examples for such agents are public institutions who have to procure a good. Thus, I introduce the rule "imitation perfection" which ensures that every auction fulfilling this rule is discrimination-free. As a consequence, the agent delegating the execution of the auction can be sure that an auctioneer who is restricted to choose from imitation-perfect auctions cannot discriminate.

In the second paper bidders participating in a first-price auction face uncertainty about each other's valuation distributions. This departs from the commonly used assumption in economic literature on first-price auctions, that the valuation distributions are common knowledge. However, I assume that the support and the mean of the valuation distributions are common knowledge. In order to derive a strategy under this uncertainty, bidders form a subjective belief which is a worst-case belief. That is, under all possible beliefs about the other bidders' valuation distributions, a bidder chooses the subjective belief which minimizes her maximum possible utility.

In the third paper I consider an abstract game of incomplete information and an agent facing uncertainty about the other players' strategies. This agent seeks to ensure a certain utility independent of the other players' strategies. In other words, this player applies the maximin expected utility criterion. However, even under the given uncertainty the fact that rational players interact strategically already contains information about the other players' possible strategies. Thus, I propose a decision criterion which works in two steps. First, I assume common knowledge of rationality and restrict the set of the other players' possible strategies to the set of rationalizable strategies. Second, I apply the maximin expected utility criterion. As a result, one can derive recommendations for a player facing strategic uncertainty. I apply this decision criterion to first-price auctions where the issue of strategic uncertainty has received very little attention in economic literature so far.

In the following I provide for each paper a short summary of the motivation and the main results.

The first paper "Imitation perfection - a simple rule to prevent discrimination in procurement" is joint work Nicolas Fugger, Vitali Gretschko and Achim Wambach where I am the first author.[1] It investigates the question which rules prevent discrimination in (procurement) auctions.

In its rules for public procurement, for example, the World Trade Organization (WTO) demands that governments comply with "non-discrimination, equality of treatment, transparency". Furthermore, the WTO seeks "to avoid introducing or continuing discriminatory measures that distort open procurement"'. These regulations imply that the rules and procedures of a procurement process should treat suppliers equally.

However, Deb and Pai (2017) show that regulation requiring equal treatment of suppliers on its own poses virtually no restriction on the ability to discriminate. In particular, such symmetric auctions allow for perfect discrimination. That is, there exists a symmetric auction and an equilibrium of this auction, in which the project is always awarded to a particular bidder at the most favorable price. Hence, an auctioneer can favor a particular bidder in the most extreme way without violating existing legal hurdles. This in turn indicates that existing legal hurdles are not sufficient to prevent discrimination and that regulators should not focus on rules that imply equal treatment but need to go further to guarantee non-discriminatory outcomes. We seek to answer the question: How should a non-discriminatory outcome be defined and what rules are sufficient in order to achieve non-discriminatory outcomes?

We propose a simple rule named imitation perfection. Imitation perfection requires that for any realization of bids and the resulting allocation and payments, every bidder had the opportunity to imitate the allocation and payment of any other bidder. It ensures that in every equilibrium, bidders with the same valuation distribution and the same valuation

[1] I presented this paper at the Ruhr Graduate School Doctoral Conference 2015 in Essen, at the Conference of the Economic Design Society 2015 in Istanbul and at the Jornadas de Economía Industrial 2015 in Alicante.

earn the same expected utility. This result is robust to perturbations of homogeneity. This means, if at any point of the domain the distribution functions of two bidders differ at most by some constant, then the expected utilities of these two bidders with the same valuation in the same imitation-perfect auction differ at most by a linear expression of this constant regardless of the other bidders' distributions.

If all bidders are homogeneous (i.e. have the same valuation distributions), revenue and social surplus optimal auctions which are consistent with imitation perfection exist. For heterogeneous bidders however, imitation perfection is incompatible with revenue and social surplus optimization.

The second paper "Endogenous worst-case beliefs in first-price auctions" is joint work with Vitali Gretschko.[2] In this paper we analyze bidding behavior in a first-price auction in which the knowledge of the bidders about the distribution of their competitors' valuations is restricted to the support and the mean.

Consider a company participating in an auction or procurement auction. The knowledge of the competitors' valuation distributions is crucial for the derivation of an optimal bidding strategy. Bidders go at great lengths in order to learn their competitors' valuations. For example, companies participating in procurement auctions reverse-engineer their competitors' products in order to learn about their production costs. However, such learning has its limits and bidders may learn only some summary statistics of the underlying distribution.

We consider a bidder in a first-price auction whose only information about the valuations of her competitors is the support and the mean of their distribution. From our own experience in consulting bidders in high-stakes auctions, it is a typical approach taken by bidders to generate several scenarios with respect to the valuations of their competitors and than to tailor their strategy to the worst-case. Thus, we assume that for a given bid strategy of her competitors the bidder will tailor her bid to be optimal given that she expects to face the worst distribution of her competitors' valuations among all distributions with the same support and mean. Worst distribution, in this context, means the bidder will expect to face the distribution of valuations that minimizes her expected utility, given her bid is an optimal reaction to the bids of her competitors induced by this distribution. In other words, the worst-case belief of a bidder minimizes her maximum possible expected utility.

We assume that every bidder in the auction follows a similar logic when preparing her bid. In this case, a profile of bids is a *worst-case belief equilibrium* if each bidder chooses her optimal bid given her valuation, the bidding strategy of her competitors, and the

[2] I presented this paper in 2016 at the Seminar for Applied Microeconomics at the University of Cologne, at the Spring Meeting of Young Economists 2016 in Lisbon, at the Meeting of the Society for Social Choice and Welfare 2016 in Lund, at the Conference of the Economic Design Society 2017 in York, at the Congress of the EEA 2017 in Lisbon and at the Econometric Society European Winter Meeting 2017 in Barcelona.

worst-case belief as defined above. In particular, this implies that the worst-case belief of a bidder will crucially depend on her type (valuation) in a non-monotonic way.

Despite the absence of monotonicity in the beliefs, an ex-post efficient equilibrium exists. The intuition is that the worst-case belief of a bidder with a given valuation, just puts enough probability weight on lower valuations to induce that for this bidder it is optimal to outbid each bidder with a lower valuation. The remaining probability weight is put on the valuation of the bidder in question in order to minimize her winning probability. It follows directly that such beliefs induce bidding that leads to an efficient allocation.

In the third paper "Strategies under distributional and strategic uncertainty" I propose a decision criterion for players facing uncertainty about the other players' strategies in a game of incomplete information.[3]

In a game of incomplete information a player faces *strategic uncertainty* if the smallest set of strategies such that the player knows that the other players' true strategy is an element of this set, is not a singleton. Strategic uncertainty can occur due to many reasons, even if Nash equilibria exist. There may exist strategy profiles which formally fulfill the conditions of a (Bayes-) Nash equilibrium. However, a player may be uncertain whether her opponents employ such strategies and consequently face strategic uncertainty. As stated by Pearce (1984), "some Nash equilibria are intuitively unreasonable and not all reasonable strategy profiles are Nash equilibria". Thus, a Nash equilibrium may not be a suitable decision criterion, in particular, if multiple equilibria exist without one being focal or salient or if players cannot communicate (Bernheim (1984)). Similarly, Renou and Schlag (2010) argue that "common knowledge of conjectures, mutual knowledge of rationality and utilities, and existence of a common prior" are required in order to justify Nash equilibria as a solution concept.

Given strategic uncertainty in games of incomplete information, I propose a new decision criterion which works in two steps: First, I assume common knowledge of rationality and eliminate all actions which are not rationalizable. Afterwards, I apply the maximin expected utility criterion. Using this decision criterion, I can derive recommendations for a player facing strategic uncertainty. Furthermore, I analyze outcomes under the assumption that every player in the game uses this criterion. In an extension I discuss how the proposed decision criterion can be applied under the presence of both, distributional and strategic uncertainty.

I apply the proposed decision criterion to first-price auctions where valuations are independently and identically distributed according to a commonly known distribution function. For every type there exists a unique highest belief-free rationalizable bid. A bidder applying the decision criterion expects the other bidders to bid the highest belief-free

[3] I presented this paper in 2016 at the reading group seminar at Yale University, in 2017 at the DFG research workshop at the University of Cologne, in 2018 at the BGSE Micro Workshop at the University of Bonn and at the International Conference on Game Theory 2018 in Stony Brook.

rationalizable bid given their valuation. As a consequence, the bidder never expects to win against a bidder with an equal or higher valuation and therefore bids the highest belief-free rationalizable bid of a lower type. If all bidders apply the criterion, it turns out that due to the symmetry of beliefs about distributions and strategies, the higher the type of the bidder, the higher is the type whose highest rationalizable bid maximizes her expected utility. Therefore, every outcome under maximin strategies is efficient.

2 Imitation perfection

Procurement regulation aimed at curbing discrimination requires equal treatment of sellers. However, Deb and Pai (2017) show that such regulation imposes virtually no restrictions on the ability to discriminate. We propose a simple rule – imitation perfection – that restricts discrimination significantly. It ensures that in every equilibrium bidders with the same valuation distribution and the same valuation earn the same expected utility. If all bidders are homogeneous, revenue and social surplus optimal auctions which are consistent with imitation perfection exist. For heterogeneous bidders however, it is incompatible with revenue and social surplus optimization. Thus, a trade-off between non-discrimination and optimality exists.

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2.1 Introduction

Regulators go to great lengths to prevent discrimination in procurement. In its rules for public procurement, for example, the World Trade Organization (WTO) demands that governments comply with "non-discrimination, equality of treatment, transparency and mutual recognition". Furthermore, the WTO seeks "to avoid introducing or continuing discriminatory measures that distort open procurement."[4] The European Commission requires public buyers to reach their decision "in full accordance with the principles of equal treatment, non-discrimination and transparency."[5] These regulations imply that the *rules and procedures* of a procurement process should treat suppliers equally. That is, the rules of a procurement process must not depend on the identity of the suppliers. However, Deb and Pai (2017) show that regulation requiring equal treatment of suppliers on its own imposes virtually no restrictions on the ability to discriminate. In particular, such *symmetric auctions* allow for *perfect discrimination.* That is, there exists a symmetric auction and an equilibrium of this auction, in which the project is always awarded to a particular bidder at the most favorable price. Hence, an auctioneer can favor a particular bidder in the most extreme way without violating existing legal hurdles. This in turn, indicates that existing legal hurdles are not sufficient to prevent discrimination and that regulators should not remain satisfied with *rules* that imply equal treatment but need to go further to guarantee discrimination-free *outcomes.*

This article is complementary to Deb and Pai (2017) and provides an answer to the

[4]See the General Agreement on Tariffs and Trade (GATT) (Article 1), General Agreement on Trade in Services (GATS) (Article 2), and Agreement on Trade-Related Aspects of Intellectual Property Rights (TRIPS) (Article 4) and World Trade Organization (2012).

[5]See Directive 2004/18/EC of the European Parliament and of the Council of 31 March 2004 on the coordination of procedures for the award of public works contracts, public supply contracts and public service contracts.

question: what rules are sufficient in order to achieve discrimination-free outcomes? We propose a simple rule named *imitation perfection*. Imitation perfection requires that for any realization of bids and the resulting allocation and payments, every bidder had the opportunity to imitate the allocation and payment of any other bidder. We show that imposing imitation perfection rules out perfect discrimination. This is due to the fact that imitation perfection implies that every bidder could have won the auction at (almost) the same price as the winning bidder by slightly outbidding the winning bidder. More generally, in an imitation-perfect auction each bidder had the opportunity to come arbitrarily close to the ex-post allocation and payment of every other bidder.

We denote an equilibrium as *non-discriminatory* if among a group of (possibly heterogeneous) bidders a pair of homogeneous bidders (i.e. bidders with the same valuation distribution) with the same valuation expects the same utility. Furthermore, we denote a mechanism as *discrimination-free* if all of its equilibria are non-discriminatory. We show that each imitation-perfect auction is *discrimination-free*.

For a pair of ex-ante heterogeneous bidders there is no clear definition of a non-discriminatory equilibrium. We repurpose a measure of how two ex-ante heterogeneous bidders' distributions differ. We show that in an imitation-perfect auction the difference in the expected utility of two ex-ante heterogeneous bidders with the same valuation is limited by the measure of their heterogeneity. Thus, we show that the auction designer's ability to discriminate between (heterogeneous) bidders in an imitation-perfect auction is limited by the heterogeneity between these bidders regardless of the other bidders' distributions. In particular, this implies that the result, that a pair of ex-ante homogeneous bidders expects the same utility given their valuation, is robust with respect to small perturbations of homogeneity, even if the heterogeneity among the other bidders is arbitrarily high.

Since we want a non-corrupt auctioneer to have enough freedom to choose the appropriate auction mechanism, it is also useful to know whether an auctioneer can discriminate in favor of a bidder by choosing among different imitation-perfect auctions. We introduce a measure of the heterogeneity of all bidders and show that the expected utilities of a bidder with a given valuation in two different imitation-perfect auctions is limited by the measure of the heterogeneity of all bidders.

Usually, the beneficiary of a procurement organization (the people of a country, the CPO of a company, or its shareholders) is responsible for thousands of different procurement projects with thousands of different bidders. According to the European Commission, there are over 250,000 public authorities involved in procurement in the EU. Delegating the specific procurement project to a (potentially large) group of agents is therefore unavoidable. Most of these agents will have the buyer's best interest in mind and will use the optimal procedures. There may, however, be some agents who are corrupt and/or favor certain bidders.[6] For the buyer, it is impossible to monitor each of the procurement transactions and to check whether the implemented procedures were optimal. Thus, there is a need

[6]See Mironov and Zhuravskaya (2016) for some recent empirical evidence.

to set general procurement rules. The set of procurement regulations should have the following properties. Firstly, it should be easy to check whether these regulations have been followed. In particular, this should not require knowledge of unobservables such as subjective beliefs, or the use of complicated calculations such as equilibrium analyses. Secondly, the regulation should restrict corrupt agents in a meaningful way. Finally, honest agents should maintain enough freedom to enable them to implement the optimal procedures. Imitation perfection has all of these desirable properties. Firstly, a quick look at the rules of the particular auction is sufficient to verify whether the procurement process satisfies imitation perfection. This is due to the fact that imitation perfection is a property of the payment rule. Hence, the verification does not require information on any details of the procurement project and can also be done ex-ante or ex-post without the calculation of equilibria. Secondly, imitation perfection prevents corrupt agents from implementing perfectly discriminatory outcomes and guarantees discrimination-free outcomes. Finally, imitation perfection gives honest agents the opportunity to implement the efficient auction as well as the revenue-optimal one if bidders are homogeneous. In this respect, ensuring that the procurement mechanism is imitation-perfect comes at no costs if all bidders are ex-ante homogeneous.

If bidders are ex-ante heterogeneous, imitation perfection is neither compatible with social surplus maximization nor with revenue maximization. Efficiency requires that bidders with the same valuation place the same bids. We will show that in imitation-perfect auctions the payment of a winning bidder depends only on her own bid. This, however, implies that if bidders with the same valuation have different beliefs about the bids they are competing against, it cannot be optimal for these bidders to place the same bid. Applying a similar reasoning to virtual valuations indicates that imitation perfection is not compatible with revenue maximization in the case of ex-ante heterogeneous bidders. Thus, there is a trade-off between non-discrimination and optimality.

Common auction formats which are compatible with imitation perfection are first-price auctions and all-pay auctions with a reservation bid. A common auction format which is ruled out by imitation perfection is the second-price auction. It cannot be imitation-perfect since it has a perfect discrimination equilibrium where one bidder bids an arbitrarily high bid b and all other bidders bid zero. It is also easy to see that none of the bidders bidding zero can imitate the bidder bidding b since by bidding slightly above b, the imitating bidder would have to pay b and not zero.

Relation to the literature

Only few papers deal with the question how general procurement rules must be designed in order to achieve the goals of procurement organizations. Deb and Pai (2017) analyze the common desideratum of "non-discrimination". However, they show that even equal and anonymous treatment of all bidders does not prevent discrimination. Gretschko and Wambach (2016) analyze how far public scrutiny can help to prevent corruption and discrimination. They consider a setting in which the agent is privately informed about the preferences of the buyer regarding the specifications of the horizontally differentiated sellers. The agent colludes with one exogenously chosen seller. They show that in the optimal mechanism the agent should have no discretion with respect to the probability of the favorite seller winning, which in turn induces the agent to truthfully report the preference of the buyer whenever his favorite seller fails to win. Moreover, they demonstrate that intransparent negotiations have this feature of the optimal mechanism whenever the favorite bidder fails to win the project and thus may outperform transparent auctions. Even though we do not explicitly model an agent of the buyer, our model could easily be extended by the introduction of an agent who, in exchange for a bribe, would bend the rules of the mechanism in the most favorable way that is consistent with the procurement regulations. Contrary to Gretschko and Wambach (2016), we do not focus on the ability of the agent to manipulate the quality assessment of the buyer but rather on the ability of an agent to design procurement mechanisms. To the best of our knowledge, our article is therefore the first to investigate the design of procurement regulations in the presence of corruption and manipulation of the rules of the mechanism.[7]

In the majority of work on corruption in auctions, the ability of the agent to manipulate is defined with respect to the particular mechanism. Either the agent is able to favor one of the sellers within the rules of a particular mechanism (typically, bid-rigging in first-price auctions) or the agent is able to manipulate the quality assessment of the sellers for a particular mechanism. Examples of the first strand of literature include Arozamena and Weinschelbaum (2009), Burguet and Perry (2007), Burguet and Perry (2009), Cai et al. (2013), Compte et al. (2005), Lengwiler and Wolfstetter (2010), and Menezes and Monteiro (2006). Examples of the second strand include Burguet and Che (2004), Koessler and Lambert-Mogiliansky (2013), and Laffont and Tirole (1991).

Finally, our article is related to the literature on mechanism design with fairness concerns. As pointed out by Bolton et al. (2005) and Saito (2013) (among others), market participants care about whether the rules governing a particular market are procedurally fair. Thus, imitation perfection can be seen not only as a device to prevent favoritism and corruption, but also as a possible way of ensuring that all equilibria of a particular mechanism yield fair (discrimination-free) outcomes. Previous approaches to mechanism design with fairness concerns in auctions and other settings include Bierbrauer et al. (2017), Bierbrauer and Netzer (2016), Budish (2011), Englmaier and Wambach (2010) and Rasch et al. (2012).

[7]Previous work on mechanism design with corruption focused on the ability of the agent to manipulate the quality assessment and the principal's optimal reaction to this. In particular, the mechanism designed by the principal is tailored to the situation at hand and does not imply general procurement regulations. See Celentani and Ganuza (2002) and Burguet (2017) for details.

2.2 Model

Environment Let $\{1, \dots, n\}$ denote a set of risk-neutral bidders that compete for one indivisible item. Bidder i's valuation v_i for the item is her private information and is drawn independently from the interval $[0, \overline{v}]$ according to a continuous (i.e. atomless) differentiable distribution function F_i with corresponding continuous density f_i.[8] The functions F_i are common knowledge among the bidders. Denote by $v_{-i} \in [0, \overline{v}]^{n-1}$ the vector containing all the valuations of bidder i's competitors. [9]

Symmetric auctions We consider an auction mechanism in which all participants submit bids $b_i \in \mathbb{R}^+$ and the auction mechanism assigns the item based on these bids.[10] An *auction mechanism* is a double (x, p) of an allocation function x and a payment function p. For every number of bidders n, and for every vector of bids $b = (b_1, \dots, b_n) \in (\mathbb{R}^+)^n$ the allocation function

$$x^n : b \to (x_1, \dots, x_n) \quad \text{with } x_i \in [0, 1], \; \sum x_i \leq 1$$

determines for each participant the probability of receiving the item. For every number of bidders n and for every vector of bids $b = (b_1, \dots, b_n) \in (\mathbb{R}^+)^n$ the payment function

$$p^n : b \to (p_1, \dots, p_n) \quad \text{with } p_i \in \mathbb{R}^+$$

determines each participant's payment.[11] We require that the payment function fulfills the following minimal consistency condition.

For all bidders i and for all bid vectors $(b_1, \dots, b_i, \dots b_n)$ it holds that

$$p_i^n (b_1, \dots, b_i, \dots b_n) = p_i^{n+1} (b_1, \dots, b_i, \dots b_n, 0) .^{12}$$

In order to be able to properly account for ties throughout the paper, we introduce the term *winner with a tie*. For a given vector of bids (b_i, b_{-i}) bidder i is a winner with a tie if it holds that $b_i = \max_{j \neq i} b_j$ and there exists a bidder $k \neq i$ such that $b_k = \max_{j \neq k} b_j$.

[8] We allow for the fact that the support of F_i is a strict subset of $[0, \overline{v}]$.

[9] For a vector $(v_1, \dots, v_n)$ we denote by v_{-i} the vector $(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$.

[10] The process of a procurement auction is mostly similar to the process of a sales auction, the only difference being that the lowest bid is awarded the contract. The bidders do not have valuations for the good but costs for fulfilling the contract. Due to the existence of the correspondence between selling auctions and procurement auctions, the formal framework will be set up for selling auctions and we will use the term auctions from now on. This has the advantage that most readers are more familiar with this notation.

[11] Since the *Directive 2004/18/EC of the European Parliament and of the Council of 31 March 2004 on the coordination of procedures for the award of public works contracts, public supply contracts and public service contracts* requires the auctioneers "to post in advance all decision criteria", we define the allocation and payment functions for all possible number of bidders. That is, the auctioneer has to commit to an auction mechanism *before* observing the number of bidders. In the following all results and definitions hold for all number of bidders.

[12] If the number of bidders is not relevant, we will omit the subscript indicating the number of bidders.

A *pure strategy* of bidder i is a mapping

$$\beta_i : [0, \overline{v}] \to \mathbb{R}^+.$$

A *(mixed) strategy* of bidder i is a map from the set of valuations to the set of bid distributions on $\mathbb{R}^+$:

$$\beta_i : [0, \overline{v}] \to \Delta\mathbb{R}^+.$$

That is, for all valuations v_i and all bids b, $\beta_i(v_i)(b)$ denotes the probability that bidder i places a bid lower or equal than b. Let $supp(\beta_i(v_i))$ denote the support of the bid distribution $\beta_i(v_i)$ and $g_{v_i}^{\beta_i}$ the corresponding density.[13]

A tuple $\beta = (\beta_1, \ldots, \beta_n)$ of pure strategies constitutes an *equilibrium* of a mechanism if for all i and for all $v_i \in V$ the bid $\beta_i(v_i)$ maximizes over all bids b bidder i's expected utility

$$U_i^{\beta_{-i}}(v_i, b) = \int_{[0,\overline{v}]^{n-1}} [v_i \cdot x_i(b, \beta_{-i}(v_{-i})) - p_i(b, \beta_{-i}(v_{-i}))] f_{-i}(v_{-i})\, dv_{-i}.$$

A tuple $\beta = (\beta_1, \ldots, \beta_n)$ of (mixed) strategies constitutes an *equilibrium* of a mechanism if for all i, for all $v_i \in [0, \overline{v}]$ and for all $b_i \in supp(\beta_i(v_i))$ the bid b_i maximizes over all bids b bidder i's expected utility

$$U_i^{\beta_{-i}}(v_i, b) = \int_{[0,\overline{v}]^{n-1}} \int_{b_{-i} \in supp(\beta_{-i}(v_{-i}))} [v_i \cdot x_i(b, b_{-i}) - p_i(b, b_{-i})] \prod_{j \neq i} g_{v_j}^{\beta_j}(b_j) f_{-i}(v_{-i})\, dv_{-i}$$

where $supp(\beta_{-i}(v_{-i})) = \times_{j \neq i} supp(\beta_j(v_j))$. The expected equilibrium utility of bidder i with valuation v_i, which is given by $U_i^{\beta}(v_i, b_i)$ for $b_i \in supp(\beta_i(v_i))$, is denoted by $U^{\beta}(v_i)$.[14] In the remainder of this paper we allow for mixed strategies if we use the term strategy or equilibrium. In particular, all results hold for mixed strategies unless specified otherwise. However, we need the following assumption in order to be able to derive results:

Assumption 1. *We assume that every equilibrium of an auction mechanism consists of strategies which are continuous except a set of valuations which has measure zero.*

Current public procurement regulation aimed at preventing discrimination requires equal treatment of bidders. The restrictiveness of this requirement is analyzed by Deb and Pai (2017), who provide the following definition.

[13] A pure strategy can be interpreted as distribution of bids which puts probability weight 1 on one bid. We abuse notation since in the case of a pure strategy, $\beta_i(v_i)$ denotes an element in $\mathbb{R}^+$ while in the case of a (mixed) strategy $\beta_i(v_i)$ denotes an element in $\Delta\mathbb{R}^+$. However, in the following it will be clear whether β_i is a pure or a mixed strategy.

[14] In the following we will use the notation $U^{\beta}(v_i)$ or $U^{\beta}(v_i, b_i)$ in order to denote bidder i's equilibrium utility. We will use the notation $U^{\beta_{-i}}(v_i, b)$ in order to indicate that bidder i deviates from equilibrium to bid b.

Definition 1 (Symmetric auction)**.** *A* symmetric auction *with reservation bid r is an auction mechanism which fulfills the following two conditions:*

(i) The highest bidder wins, that is the allocation is given by

$$x_i(b_i, b_{-i}) = \begin{cases} \frac{1}{\#\{j \in \{1,\ldots,n\}: b_j = b_i\}} & \text{if } b_i \geq \max\limits_{j \neq i} \{b_j, r\} \\ 0 & \text{otherwise,} \end{cases}$$

where r is a reservation bid.[15]

(ii) The payment does not depend on the identity of the bidder and every bidder is treated equally. Formally, let π_n be a permutation of the elements $1, \ldots, n$. In a symmetric auction, it holds true for all $b = (b_1, \ldots, b_n)$ that

$$p_i\left(b_{\pi_n(1)}, \ldots, b_{\pi_n(i-1)}, b_{\pi_n(i)}, b_{\pi_n(i+1)}, \ldots, b_{\pi_n(n)}\right) = p_{\pi_n(i)}(b_i, b_{-i}).$$

In a symmetric auction, the highest bidder wins and the payment function is anonymous. Hence, a bidder's payment depends only on the bids and not on her identity. Moreover, a permutation of all bids would lead to the same permutation of payments and allocations.

In addition to the requirements of a symmetric auction, we assume that an auction mechanism fulfills some monotonicity conditions. First, we require that the payment of a bidder is non-decreasing in her own bid. Second, we require that conditional on winning or losing the payment of a bidder is non-decreasing in the other bidders' bids. Third, we require that the payment of a winning bidder is strictly increasing in at least one component of the bid vector.

Assumption 2. *We assume that the payment function of every auction mechanism is monotone. We call a payment function p* monotone *if for every bidder i and for each vector of bids (b_i, b_{-i}) the following holds:*

(i) The payment of bidder i is non-decreasing in her bid, i.e. for all b_i' with $b_i \leq b_i'$ it holds that

$$p_i(b_i, b_{-i}) \leq p_i(b_i', b_{-i}).$$

(ii) Given that a bidder is losing or winning, her payment is non-decreasing in the other bidders' bids. That is, if $b_i \neq \max\limits_{j \neq i} b_j$, then for every bid b_j' with $b_j \leq b_j'$ it holds that

$$p_i\left(b_i, b_j, b_{-(i,j)}\right) \leq p_i\left(b_i, b_j', b_{-(i,j)}\right)$$

and if $b_i = \max\limits_{j \neq i} b_j$, then for every bid b_j' with $b_j \leq b_j' \leq b_i$ it holds that

$$p_i\left(b_i, b_j, b_{-(i,j)}\right) \leq p_i\left(b_i, b_j', b_{-(i,j)}\right).$$

[15] Throughout the paper we assume that a reservation bid r is given unless specified otherwise.

(iii) If i is a winning bidder, her payment is strictly increasing in at least one component of the bid vector (b_i, b_{-i}). That is, if $b_i \geq \max_{j \neq i} \{b_j, r\}$, then there exist a bid b_j for $j \in \{1, \ldots, n\}$ such that for all b_j' with $b_j' > b_j$ and $b_i \geq b_j'$ if $j \neq i$, it holds that

$$p_i(b_j, b_{-j}) < p_i\left(b_j', b_{-j}\right).$$

We impose these conditions in order to ensure equilibrium existence. If the payment of a bidder was strictly decreasing in her own bid, she would place arbitrarily high bids. A similar reasoning applies to the second and third condition. Consider an auction with two bidders who both have a valuation of v.[16] The payment rule is defined by

$$p_i(b_i, b_j) = \begin{cases} \max\{b_i - A \cdot b_j, 0\} & \text{if } b_i > b_j \\ 0 & \text{if } b_i < b_j \end{cases}$$

for $A > 0$. An equilibrium does not exist because bidders want to place arbitrarily high bids.[17] Finally, consider an auction in which a bidder pays a constant independent of her bid, which contradicts the third condition. Again this bidder has an incentive to place arbitrarily high bids and an equilibrium does not exist. Although requiring a monotone payment function is a technical assumption, it is not restrictive in the sense that it does not rule out any of the auction formats that are popular in practice, like the first-price auction or the second-price auction.[18]

Moreover, we assume that every bidder has the possibility to achieve at least an expected utility of zero by bidding below the reservation bid or bidding zero.

Assumption 3. *We assume that for every bidder $i \in \{1, \ldots, n\}$ and every bid vector (b_i, b_{-i}) with $b_i < r$ or $b_i = 0$ it holds that*

$$p_i(b_i, b_{-i}) = 0.$$

2.2.1 Discrimination-free auctions

The main insight of Deb and Pai (2017) is that even though the *rules* of a symmetric auction treat all bidders equally, mechanisms with discriminating *outcomes* can still be implemented. In particular, they demonstrate that almost every reasonable mechanism has an implementation as a symmetric auction. Thus, requiring a symmetric auction, i.e. equal treatment, is not an effective anti-discrimination measure. To get an idea of the discrimination that is possible in symmetric auctions, consider the following example.

[16] In the following we will use the terms auction and auction mechanism interchangeably.

[17] Assume there would exist an equilibrium where bidder 1 bids b_1 and bidder 2 bids b_2. W.l.o.g. it holds that $b_2 > b_1$. It must hold that $v - b_2 + Ab_1 \geq 0$. It follows that $v - b_1 + Ab_2 > 0$. Thus, bidder 1 has an incentive to deviate to a bid above b_2.

[18] Note that Deb and Pai (2017) and Example 1 show that symmetric auctions with a monotone payment function do not prevent perfect discrimination as defined in Definition 2.

Example 1. *An agency is in charge of running an auction among n bidders with valuations in* $[0,1]$*. One of the bidders, say bidder* 1*, has close ties to the agency. Thus, the agency does not aim at maximizing revenue but instead seeks to maximize the utility of bidder* 1*. In this case, the agency can implement the following symmetric auction. If only one bidder bids a strictly positive amount, all payments are zero. If more than one bidder bids a strictly positive amount, all bidders who bid a strictly positive amount pay their own bid plus (a penalty of) one. This auction has a Bayes-Nash equilibrium in undominated strategies in which bidder* 1*, irrespective of her valuation, bids some strictly positive amount* $b_1 > 0$*. All other bidders bid zero, irrespective of their valuations. In this case, bidder* 1 *receives the object with probability one and pays nothing.*

We call an equilibrium a *perfect discrimination equilibrium* if one bidder wins the auction with probability one independent of her valuation and pays nothing.

Definition 2 (Perfect discrimination equilibrium)**.** *An equilibrium* $(\beta_1, \ldots, \beta_n)$ *of an auction mechanism* (x, p) *is called a* perfect discrimination equilibrium *if there exists a bidder i such that for any vector of valuations* $(v_1, \ldots v_n)$ *and every vector of bids* $(b_1, \ldots, b_n)$ *such that for all* $j \in \{1, \ldots, n\}$*,* $b_j \in supp\,(\beta_j\,(v_j))$*, it holds that:*

$$x_i\,(b_1, \ldots, b_n) = 1$$

$$p_i\,(b_1, \ldots, b_n) = 0.$$

Given that symmetric auctions do not prevent perfect discrimination, the aim of this article is to provide a simple extension to the existing rules that restricts discrimination in a meaningful way. A minimum requirement for the extension is that it rules out perfect discrimination equilibria.[19] In addition, we demand that in a non-discriminatory equilibrium ex-ante homogeneous bidders with the same valuation expect the same utility. We denote a symmetric auction as discrimination-free if all of its equilibria are non-discriminatory.

Definition 3 (Discrimination-free auction)**.** *An equilibrium* $(\beta_1, \ldots, \beta_n)$ *of a symmetric auction is called* non-discriminatory *if for all bidders* i, j *with* $F_i = F_j$ *it holds for all* $v \in [0, \overline{v}]$ *that*

$$U_i^\beta\,(v) = U_j^\beta\,(v)\,.$$

A symmetric auction is called discrimination-free *if all equilibria of this auction are non-discriminatory.*

[19] Note that Deb and Pai (2017) propose adjustments of symmetric auctions that may restrict the class of implementable mechanisms. In particular, they consider auction mechanisms with inactive losers, continuous payment rules, monotonic payment rules and ex-post individual rationality. However, it is easy to see that none of these adjustments prevents the existence of perfect discrimination equilibria. This is due to the fact that any of these adjustments allows for the implementation of the second-price auction. The second-price auction has perfectly-discriminating equilibria in which one of the bidders bids $b_i \geq \overline{v}$ and all other bidders bid zero.

2.3 Imitation perfection

In what follows we introduce a simple extension of the existing symmetric rules which require equal treatment. We call this extension *imitation perfection* and show that all imitation-perfect auctions are discrimination-free.

Imitation perfection requires that for any realization of bids each bidder could have achieved the same allocation and payment as any other bidder by bidding slightly higher than a bidder with a higher bid and bidding slightly lower than a bidder with a lower bid.

Definition 4 (Imitation perfection). *A symmetric auction (x, p) is imitation-perfect if for all bidders i, all bids b_i, and all $\epsilon > 0$*

(i) For all vectors of bids $\left(b_i, b_j, b_{-(i,j)}\right)$ such that bidder i is not a winner with a tie and for all $j \in \{1, \dots, n\}$ with $b_i > b_j$ there exists a bid $\bar{b} > b_i$ such that

$$\left| p_i\left(b_i, b_j, b_{-(i,j)}\right) - p_j\left(b_i, \bar{b}, b_{-(i,j)}\right) \right| < \epsilon.$$

That is, all bidders can imitate the allocation and payment of a higher bidder who is not a winner with a tie by bidding slightly higher.

(ii) For all vectors of bids $\left(b_i, b_j, b_{-(i,j)}\right)$ and for all $j \in \{1, \dots, n\}$ with $b_i < b_j$ there exists a bid $\underline{b} < b_i$ such that

$$\left| p_i\left(b_i, b_j, b_{-(i,j)}\right) - p_j\left(b_i, \underline{b}, b_{-(i,j)}\right) \right| < \epsilon.\text{[20]}$$

That is, all bidders can imitate the allocation and payment of a lower bidder by bidding slightly lower.

In an imitation-perfect auction every bidder could have imitated the (ex-post) allocation and payment of each bidder who is not a winner with a tie. By bidding slightly above a winner with a tie a bidder would become the unique winner and therefore cannot imitate the allocation and payment of a bidder with tie. We discuss the payments of winners with a tie further below.

A strength of our proposed rule is that the verification of whether an auction is imitation-perfect can be done without knowledge about the environment, such as the beliefs of the bidders or the selection of a particular equilibrium. A simple verification of the payment rule is sufficient.

In order to gain some intuition for the definition of imitation perfection, we consider the following examples.

Example 2. *Consider the mechanism proposed in Example 1. Recall that bidder 1 is the favorite bidder and if more than one bidder places a strictly positive bid, all bidders who*

[20] It is sufficient to consider only the payment function, because in a symmetric auction the allocation rule is fixed.

placed a strictly positive bid pay their bid plus a penalty of one. This mechanism is not imitation-perfect. For $b_1 > 0$ it holds that

$$p_1\left(b_1, 0, \ldots, 0\right) = 0.$$

Bidder 1 wins the auction and pays nothing. For every $b_j > b_1$ it holds that

$$p_j\left(b_1, 0, \ldots, 0, b_j, 0, \ldots 0\right) - p_1\left(b_1, 0, \ldots, 0\right) > 1,$$

which implies that bidder 1 cannot be imitated.

Example 3. *Consider a second-price auction with two bidders. If bidder 1 is bidding $b_1 = 1$ and bidder 2 is bidding $b_2 = 0$, bidder 1 will receive the object and pay a price of zero. Bidder 2 cannot imitate this outcome. By bidding above 1, bidder 2 would win the object but her payment would be 1.*

Example 4. *Consider a first-price auction with two bidders. If bidder 1 is bidding $b_1 = 1$ and bidder 2 is bidding $b_2 = 2$, bidder 2 will receive the object and pay a price of 2 while bidder 1 pays zero. By placing a bid marginally higher than 2 bidder 1 can imitate bidder 2's allocation and payment. Bidder 2 can imitate bidder 1's allocation and payment by placing a bid marginally lower than 1.*

In the following, we will present the properties of an imitation-perfect auction and of its outcomes. We start with a property of imitation perfection which we need for subsequent proofs:

Proposition 1. *In an imitation-perfect auction the payment of a bidder depends only on her own bid conditional on winning or losing. That is, for all bidders i the following holds true:*

(i) For all bid vectors $\left(b_i, b_j, b_{-(i,j)}\right)$ such that $b_i > \max_{j \neq i} b_j$ it holds for all bids b'_j with $b_i > b'_j$ that

$$p_i\left(b_i, b_j, b_{-(i,j)}\right) = p_i\left(b_i, b'_j, b_{-(i,j)}\right).$$

(ii) For all bid vectors $\left(b_i, b_j, b_{-(i,j)}\right)$ such that $b_i < \max_j b_j$ it holds for all bids b'_j with $b_i < \max\{b'_j, b_{-(i,j)}\}$ that

$$p_i\left(b_i, b_j, b_{-(i,j)}\right) = p_i\left(b_i, b'_j, b_{-(i,j)}\right).$$

The proof is relegated to Appendix 2.7.2.

As indicated above, we have to discuss the payments of winners with a tie. Proposition 1 implies that if there is a unique highest bidder i with bid b_i, then her payment depends only on her bid. That is, there exists a function p^{win} such that bidder i's payment is equal to $p^{win}(b_i)$. If bidder i is a losing bidder, then there exists a function p^{lose} such that her payment is equal to $p^{lose}(b_i)$. Since the allocation rule breaks ties randomly, an analogous property should hold for the payment rule. We can use the functions p^{win} and p^{lose} in order to define such a payment rule. This is formalized in the following assumption.

Assumption 4. *We assume that for every bidder i and for every vector of bids*

$$(b_i, b_{-i})$$

such that $b_i = \max_{j\neq i} b_j$ and $k = \#\{j \in \{1,\dots,n\} \mid b_j = b_i\}$, i.e. i is a bidder with a tie bidding b_i, the payment of bidder i is given by

$$p_i(b_i, b_{-i}) = \frac{1}{k} p^{win}(b_i) + \left(1 - \frac{1}{k}\right) p^{lose}(b_i)$$

where for every $b_i > 0$, $p^{win}(b_i)$ is defined by

$$p^{win}(b_i) = p_i(b_i, 0, \dots, 0)$$

and for every b_i, $p^{lose}(b_i)$ is defined by

$$p^{lose}(b_i) = p_i(b_i, b_j, 0 \dots, 0)$$

for any b_j with $b_j > b_i$.

It follows from Proposition 1 that the definition of the payment for a bidder who is winner with a tie and the definitions of p^{win} and p^{lose} are well-defined.

We have shown that the payment of a bidder who is not a winner with a tie does not depend on lower bids. In addition, we state the following properties of imitation perfection that will be useful in the sections to follow. They also serve as necessary and sufficient conditions for imitation perfection. First, we need the following definition.

Definition 5 (Bid-determines-payment auction)**.** *A symmetric auction is a* bid-determines-payment auction *if the payment of every bidder depends only on whether or not she wins and on her bid. Formally, an auction satisfies the bid-determines-payment rule if there exist functions $p^{win}, p^{lose} : \mathbb{R}^+ \to \mathbb{R}^+$ such that for bidder i her payment can be written as*

$$p_i(b_i, b_{-i}) = x_i(b_i, b_{-i})\, p^{win}(b_i) + [1 - x_i(b_i, b_{-i})]\; p^{lose}(b_i).$$

Note that the functions p^{win} and p^{lose} are identical for every number of bidders.

Proposition 2. *An auction is imitation-perfect if and only if the following holds true:*

(i) (Bid-determines-payment) The auction is a bid-determines-payment auction.

(ii) *(Continuity) If bidder i is not a winner with a tie, her payment is right-continuous in her bid. That is, for every bidder i, for every bid vector (b_i, b_{-i}) such that it follows from $b_i = \max_{j \neq i} b_j$ that $b_i > \max_{j \neq i} b_j$, and for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all b_i' with $b_i < b_i' < b_i + \delta$ it holds that*

$$|p_i (b_1, \ldots, b_i, \ldots, b_n) - p_i (b_1, \ldots, b_i', \ldots, b_n) | < \epsilon.$$

Moreover, if a bidder does not place the highest bid, then her payment is left-continuous in her bid. That is, for every bidder i, for every bid vector such that $b_i < \max_{j \neq i} b_j$, and for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all b_i' with $b_i - \delta < b_i' < b_i$ it holds that

$$|p_i (b_1, \ldots, b_i, \ldots, b_n) - p_i (b_1, \ldots, b_i', \ldots, b_n) | < \epsilon.$$

The proof is relegated to Appendix 2.7.3.

We continue with the results specifying the desirable properties imitation-perfect auctions have. We illustrated in Examples 2 and 3 how imitation perfection can prevent perfect discrimination equilibria. The following proposition states that imitation perfection prevents in general the existence of perfect discrimination equilibria in symmetric auctions.

Proposition 3. *If there are at least two bidders i and j who have a strictly positive valuation with a strictly positive probability, then an imitation-perfect auction does not have a perfect discrimination equilibrium.*

We illustrate a sketch of the proof for the case of pure strategies. Assume that there exists a perfect discrimination equilibrium in which bidder i wins the auction with probability 1 and pays zero. We show in Lemma 6 in Appendix 2.7.4 that every equilibrium bidding strategy is non-decreasing. Thus, the highest bid placed by bidder i is given by $\beta_i(\overline{v})$. Assume that bidder j has a strictly positive valuation v_j. Due to imitation perfection, for every $\epsilon > 0$ there exists a bid $\bar{b} > \beta_i(\overline{v})$ such that

$$|p_i \left(\beta_i (\overline{v}), \beta_j (v_j), \beta_{-(i,j)} \left(v_{-(i,j)}\right)\right) - p_j \left(\beta_i (\overline{v}), \bar{b}, \beta_{-(i,j)}\right) \left(v_{-(i,j)}\right) | < \epsilon$$

for every vector of valuations v_{-i}. Since in equilibrium bidder i always pays zero, this implies that by bidding $\bar{b}$, bidder j would win the auction with probability 1 and pay an amount which is strictly lower than her valuation. Therefore, she has an incentive to deviate. Hence, a perfect discrimination equilibrium cannot exist in an imitation-perfect auction, because every bidder $j \neq i$ would have an incentive to deviate whenever she has a strictly positive valuation for the good. The formal proof is relegated to Appendix 2.7.5.

We have established that imitation perfection fulfills the minimum requirement of preventing perfect discrimination. The following theorem states that imitation-perfect auctions are discrimination- free.

Theorem 1. *A symmetric and imitation-perfect auction is discrimination-free.*

Intuitively, Theorem 1 builds on the fundamental idea of imitation perfection that bidders can imitate the allocation and payment of the other bidders that have outbid them. Formally, we prove that homogeneous bidders follow identical strategies. This ensures that ex-ante homogeneous bidders with the same valuation have the same expected utility. In order to do so, we adapt a technique of Chawla and Hartline (2013). They show that for a given auction, if some interval $[\underline{z}, \overline{z}]$ satisfies *utility crossing*, that is, if for some bidders i and j it holds that $U_i^{\beta}(\overline{z}) \geq U_j^{\beta}(\overline{z})$ and $U_j^{\beta}(\underline{z}) \geq U_i^{\beta}(\underline{z})$ and $\beta_j(v) \geq \beta_i(v)$ for all $v \in [\underline{z}, \overline{z}]$, then the strategies of bidder i and bidder j must be identical on this interval. If there is an interval of valuations of positive measure such that the equilibrium prescribes that one bidder strictly outbids the other, we apply imitation perfection at the endpoints of this interval in order to demonstrate that this interval satisfies utility crossing. Due to imitation perfection, at the upper endpoint $\overline{z}$ a deviating bid for bidder i exists, such that bidder i can achieve the same expected utility as bidder j by bidding slightly higher than bidder j. Bidder i's utility in equilibrium cannot, therefore, be lower than bidder j's utility as bidder i would otherwise have an incentive to deviate. Similarly, bidder j can achieve the same expected utility as bidder i at the lower endpoint $\underline{z}$. The formal proof is relegated to Appendix 2.7.6.

2.4 Imitation perfection with homogeneous bidders

In this section we present further results for the case that bidders are ex-ante homogeneous. Bidders are ex-ante homogeneous if all bidders draw their valuations from the same distribution, i.e. it holds for all $i, j \in \{1, \dots n\}$ that $F_i = F_j$. We provide conditions for the existence and uniqueness of equilibria in imitation-perfect auctions. Furthermore, we show that imitation perfection is compatible with revenue and social surplus maximization.

Proposition 4. *Assume that the bid spaces of all bidders are compact intervals, i.e. every bidder is allowed to submit bids in some interval* $[0, \overline{b}]$. *Then the following holds true in an imitation-perfect auction:*

(i) There exists an equilibrium.

(ii) If bidders are ex-ante homogeneous, then there exists a unique non-decreasing equilibrium in pure strategies.

The proof is relegated to Appendix 2.7.7.

If bidders are ex-ante homogeneous, the revenue-optimal auction can be implemented as a first-price auction, which is an imitation-perfect auction, with an appropriate reservation bid (see Krishna 2009). Similarly, the efficient auction can be implemented as a first-price auction without a reservation bid. Thus, it follows from Proposition 2 that there exist imitation-perfect auctions which are revenue and social surplus optimal, as stated in the following Corollary.

Corollary 1. *If bidders are ex-ante homogeneous, the following holds true:*

(i) There exists a symmetric and discrimination-free auction that is revenue-optimal among all incentive compatible mechanisms.

(ii) There exists a symmetric and discrimination-free auction that is social surplus maximizing among all incentive compatible mechanisms.

Thus, the implementation of a discrimination-free auction is not in conflict with the aims of revenue or social surplus maximization if all bidders are ex-ante homogeneous.

2.5 Imitation perfection with heterogeneous bidders

In this section we analyze the extent to which imitation perfection limits discrimination between bidders that are ex-ante heterogeneous and examine whether imitation perfection is compatible with revenue and social surplus maximization.

If bidders are ex-ante heterogeneous it is not reasonable to require that bidders with the same valuation earn the same expected utility in equilibrium. The heterogeneity implies that different bidders face different degrees of competition even if they have the same valuation.

Nevertheless, we will show that even in settings with ex-ante heterogeneous bidders imitation perfection effectively limits the possible extent of discrimination. In order to provide a precise and tractable measure of heterogeneity, we follow Fibich et al. (2004). They show that by defining

$$H = \frac{1}{n}\sum_{i=1}^{n} F_i$$

$$\delta = \max_i \max_v |F_i(v) - H(v)|$$

$$H_i(v) = (F_i(v) - H(v))/\delta,$$

for any set of distribution functions $F_1, \ldots, F_n$ defined on some interval $[0, \overline{v}]$ and for every $i \in \{1, \ldots, n\}$ the distribution function F_i can be decomposed in the following way

$$F_i(v) = H(v) + \delta H_i(v) \quad (1)$$

where $H(0) = 0$, $H(\overline{v}) = 1$, $H_i(0) = H_i(\overline{v}) = 0$, $|H_i| \leq 1$ on $[0, \overline{v}]$ and $\delta \geq 0$. Among all H, $\{H_i\}_i$ and δ which allow such a decomposition, δ as defined above is minimal. The parameter δ formalizes the degree of heterogeneity between all bidders.

In particular, by defining

$$H = \frac{1}{2}(F_i + F_j)$$

$$\delta_{i,j} = \max_{k \in \{i,j\}} \max_{v} |F_k(v) - H(v)|$$

$$H_k(v) = (F_k(v) - H(v)) / \delta_{i,j} \quad \text{for } k \in \{i,j\}$$

for every pair of bidders i and j it holds that

$$F_i(v) = H(v) + \delta_{i,j} H_i(v), \quad F_j(v) = H(v) + \delta_{i,j} H_j(v) \tag{2}$$

where $H(0) = 0$, $H(\overline{v}) = 1$, $H_k(0) = H_k(\overline{v}) = 0$, and $|H_k| \leq 1$ on $[0, \overline{v}]$ for $k \in \{i,j\}$. Analogously, among all H, H_i, H_j and $\delta_{i,j}$ which allow such a decomposition, $\delta_{i,j}$ as defined above is minimal. The parameter $\delta_{i,j}$ formalizes the degree of heterogeneity between two specific bidders i and j.

The following proposition provides an upper bound on the difference in expected utilities of two bidders with the same valuation in an imitation-perfect auction. We will need the condition $p^{win}(b) \geq p^{lose}(b)$ for all $b \geq 0$. In order to gain some intuition for why this condition is necessary, consider an imitation-perfect auction with two bidders i and j. If both bidders adopt the same strictly increasing strategy β, then for every valuation v it holds that

$$\begin{aligned}
U_i^{\beta}(v) - U_j^{\beta}(v) &= F_j(v)\left(v - p^{win}(\beta(v))\right) - (1 - F_j(v))\, p^{lose}(\beta(v)) \\
&\quad - \Big(F_i(v)\left(v - p^{win}(\beta(v))\right) - (1 - F_i(v))\, p^{lose}(\beta(v))\Big) \\
&\leq (F_i(v) + \delta_{i,j})\left(v - p^{win}(\beta(v))\right) - (1 - (F_j(v) + \delta_{i,j}))\, p^{lose}(\beta(v)) \\
&\quad - \Big(F_i(v)\left(v - p^{win}(\beta(v))\right) - (1 - F_i(v))\, p^{lose}(\beta(v))\Big) \\
&= \delta_{i,j}\left(v - p^{win}(\beta(v)) + p^{lose}(\beta(v))\right).
\end{aligned}$$

Thus, in order to find an upper bound of the difference in the bidders' expected utility for a given valuation, one has to find an upper bound for the function $\left(-p^{win} + p^{lose}\right)(\cdot)$ which can be achieved by imposing the condition $p^{win}(b) \geq p^{lose}(b)$ for all $b \geq 0$.

Proposition 5. *In an imitation-perfect auction for every equilibrium $\beta = (\beta_1, \ldots, \beta_n)$, for every pair of bidders i, j and for every valuation v it holds that*

$$|U_i^{\beta}(v) - U_j^{\beta}(v)| \leq \delta_{i,j} + \delta_{i,j}(\overline{v} - v)$$

if it holds for all $b \in \mathbb{R}^+$ that $p^{win}(b) \geq p^{lose}(b)$ where $\delta_{i,j}$ is defined as in (2). That is, the difference in the expected utilities of two bidders with the same valuation in the same imitation-perfect auction is given by at most $\delta_{i,j} + \delta_{i,j}(\overline{v} - v)$ independent of the degree of heterogeneity of the other $n-2$ bidders.

The proof is relegated to Appendix 2.7.8.

Theorem 1 states that in an imitation-perfect auction two ex-ante homogeneous bidders with the same valuation expect the same utility even if the heterogeneity among the other bidders is arbitrarily strong. Proposition 5 implies that this finding is robust towards small pertubations of homogeneity, which is illustrated in the following example.

Example 5. *Consider a first-price auction with two bidders. The valuation of bidder 1 is uniformly distributed on the interval* $[0, 1000]$*, the valuation of bidder 2 is uniformly distributed on the interval* $[0, 1001]$*. Following Krishna (2009), one can derive the unique equilibrium and calculate the expected utilities as a function of a bidder's valuation. The difference in the expected utilities of the two bidders is maximized at* $v^* \approx 700$*. The upper bound provided in Proposition 5 is given by*

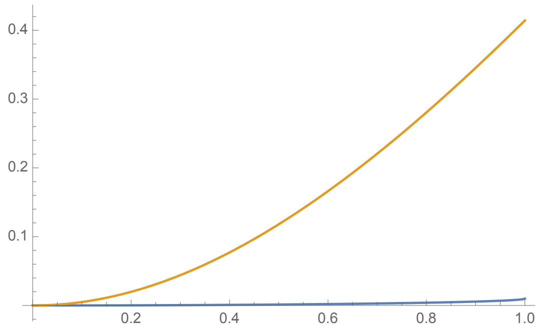
$$\delta\overline{v} + \delta\left(\overline{v} - v\right) = \frac{1001}{1001} + \frac{1001 - 700}{1001} \approx 1.302.$$

This is equal to one half percent of the utility expected by bidder i *and* j *at* v^**.*

While Proposition 5 shows that there is little room for discrimination in imitation-perfect auctions if bidders are ex-ante almost homogeneous, it is obvious that extreme heterogeneity results in outcomes that are arbitrarily close to perfect discrimination. This is illustrated in the following example.

Example 6. *Consider a first-price auction with two bidders. Bidder 1's valuation is drawn from the interval* $[0, 1]$*, whereas bidder 2's valuation is drawn from the interval* $[0, \overline{v}]$ *with* $\overline{v} >> 1$*. Following Krishna (2009), one can derive the unique equilibrium and show that for every* $v \in (0, 1]$ *it holds that* $U_2(v) > U_1(v)$*. Figure 1 illustrates the expected utilities of bidder 1 (blue line) and bidder 2 (orange line) for* $v \in [0, 1]$ *and* $\overline{v} = 100$*.*

Figure 1: $U_1(v)$, $U_2(v)$



Even if bidder 1 has a valuation of 1, bidder 2 will have a higher valuation with a probability of 0.99. In contrast to that, bidder 2 can be sure to have the higher valuation if her valuation is 1. This example highlights, that if bidders are extremely ex-ante heterogeneous, their expected utilities given the same valuation can also differ extremely.

So far, we have analyzed the auctioneer's possibility to discriminate between two heterogeneous bidders in the same imitation-perfect auction. Now we turn our attention to the auctioneer's possibility to increase a favorite bidder's expected utility by choosing among different imitation-perfect auctions. If bidders are ex-ante heterogeneous, the revenue equivalence theorem does not hold. Hence, the expected utility of a bidder with a given valuation can differ between different imitation-perfect auctions. Proposition 6 demonstrates that the possible extent of discrimination is limited by the degree of heterogeneity. If the heterogeneity between bidders is small, so is the extent to which the auctioneer can discriminate between them by choosing different auction formats.

Since we make use of the Revenue Equivalence Principle and a Lemma in Fibich et al. (2004) in the proof of the following Proposition, the following holds true for all pure strictly increasing and differentiable equilibria in imitation-perfect auctions with reservation bid zero.

Proposition 6. *Let A and B be imitation-perfect auctions with reservation bid zero and β be an equilibrium of A and β' be an equilibrium of B. If the equilibria β and β' are in pure strictly increasing and differentiable strategies, then for every bidder i and every valuation v it holds that*

$$\left|U_i\left(v\right)^{\beta}-U_i\left(v\right)^{\beta'}\right|\leq 2\left(\delta\overline{v}+\delta\left(\overline{v}-v\right)\right)+\mathcal{O}\left(\delta^2\right).$$

That is, for every bidder i with a given valuation v the difference in the expected utilities in any equilibrium of A and B is given by at most

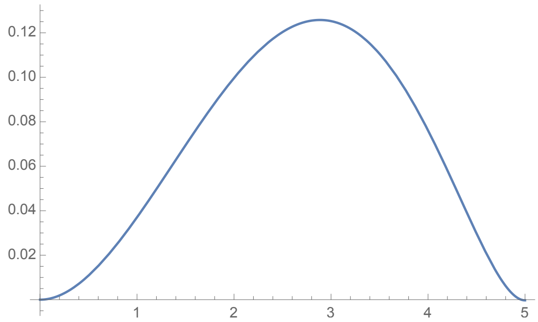
$$2\left(\delta\overline{v}+\delta\left(\overline{v}-v\right)\right)+\mathcal{O}\left(\delta^2\right).$$

The proof is relegated to Appendix 2.7.9.

If the ex-ante heterogeneity among bidders is sufficiently pronounced, an auctioneer who knows the distributions of the bidders is able to substantially influence her favorite bidder's expected utility by choosing among imitation-perfect auctions. We illustrate the auctioneer's possibility to influence her favorite bidder's expected utility with the following example.

Example 7. *Consider an auctioneer who has to conduct an imitation-perfect auction with two bidders. The valuation of bidder 1 is uniformly distributed on the interval $[0,5]$ and the valuation of bidder 2 is uniformly distributed on the interval $[0,10]$. Assume that the auctioneer can either conduct a frist-price auction or an all-pay auction. Following Krishna (2009) and Amann and Leininger (1996), we can compute the unique equilibrium bidding functions for both bidders in both auctions. If the auctioneer wants to favor bidder 1, he will conduct a first-price auction. Independent from her valuation, bidder 1 expects a weakly higher utility in a first-price auction than in an all-pay auction. Figure 2 illustrates the difference in the expected utility of bidder 1 in the first-price and the all-pay auction for all possible valuations.*

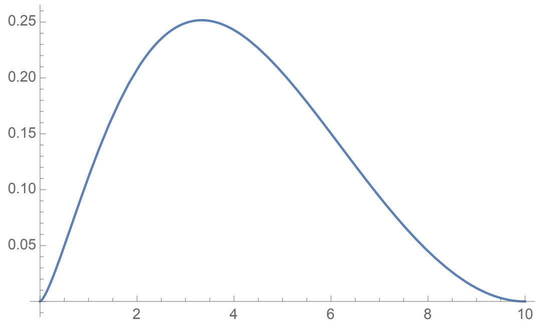
Figure 2: $U_1^{FPA}(v) - U_1^{APA}(v)$



Notes. The difference in the expected utilities $U_1^{FPA}(v) - U_1^{APA}(v)$ obtains its maximum valuation of 0.126 at $v = 2.9$. In this case, bidder 1's utility in a first-price auction is 39 percent larger than in an all-pay auction.

Vice versa, the auctioneer can favor bidder 2 by conducting an all-pay auction. Figure 3 illustrates that, independent of her valuation, bidder 2 expects a (weakly) larger utility in an all-pay auction.

Figure 3: $U_2^{APA}(v) - U_2^{FPA}(v)$



Notes. The difference in the expected utilities $U_2^{APA}(v) - U_2^{FPA}(v)$ obtains its maximum valuation of 0.252 at $v = 3.4$. In this case, bidder 2's utility in an all-pay auction is 24 percent larger than in a first-price auction.

Finally, we will show that imitation perfection is not compatible with efficiency and revenue maximization if bidders are ex-ante heterogeneous.

Proposition 7. *Assume there exists at least one pair of bidders i, j such that $\int_0^{\overline{v}} F_i(z)\,dz \neq \int_0^{\overline{v}} F_j(z)\,dz$, then there does not exist an efficient equilibrium in any imitation-perfect auction.*

The proof is relegated to Appendix 2.7.10.

In symmetric auctions efficiency requires that bidders with the same valuation place the same bid. As a consequence, ex-ante heterogeneous bidders face different bid distributions. The winner's payment in an imitation-perfect auction cannot depend on others bidders' bids. This implies that following the same bidding strategy cannot be optimal for ex-ante heterogeneous bidders. Applying a similar reasoning to virtual valuations indicates that imitation perfection is not compatible with revenue maximization in the case of ex-ante heterogeneous bidders.

Proposition 8. *For every bidder i let $V_i(v_i) = \frac{1-F_i(v_i)}{f_i(v_i)}$, that is $V_i(v_i)$ denotes the virtual valuation of bidder i with valuation v_i. Assume there exists at least one pair of bidders i, j such that $\int_0^{\overline{v}} F_i\left(V_i^{-1}(V_j(z))\right) dz \neq \int_0^{\overline{v}} F_j\left(V_j^{-1}(V_i(z))\right) dz$. In this case, all equilibria of an imitation-perfect auction are not revenue-maximizing. That is, the object is not always allocated to the bidder with the highest virtual valuation.*

The proof is relegated to Appendix 2.7.10.

2.6 Conclusion

This article demonstrates that the existing rules imposed to prevent discrimination in procurement, which require equal treatment of bidders, are not sufficient to prevent even perfect discrimination. We introduce a simple extension to the existing rules called imitation perfection. Imitation perfection requires that for any realization of bids and the resulting allocation and payments, every bidder had the opportunity to imitate the allocation and payment of every other bidder. Imitation perfection can be easily verified without specific knowledge of details of the environment and guarantees discrimination-free outcomes. If all bidders are ex-ante homogeneous, both an imitation-perfect revenue optimal auction and an imitation-perfect social surplus optimal auction exist. If bidders are heterogeneous, imitation perfection still ensures that the difference in the expected utilities of two bidders with the same valuation is limited by the heterogeneity of their valuation distributions. Moreover, the difference in the expected utilities of a bidder with a given valuation in two different imitation-perfect auctions is limited by the heterogeneity of all bidders.

2.7 Appendix

2.7.1 Definition of expected allocation and payment

If $(\beta_1, \ldots, \beta_n)$ is an equilibrium of an auction mechanism (x, p), then the expected (interim) allocation and payment of a bidder i who bids b_i are defined by

$$X_i^{\beta_{-i}}(b_i) = \int_{[0,\overline{v}]^{n-1}} x_i(b_i, \beta_{-i}(v_{-i})) f_{-i}(v_{-i})\, d(v_{-i}) \tag{3}$$

$$P_i^{\beta_{-i}}(b_i) = \int_{[0,\overline{v}]^{n-1}} p_i(b_i, \beta_{-i}(v_{-i})) f_{-i}(v_{-i})\, d(v_{-i}). \tag{4}$$

Similarly as in the notation for expected utility, we will use the notation $X_i^{\beta}(v_i)$ or $X_i^{\beta}(\beta_i(v_i))$ in order to denote the equilibrium allocation of bidder i with valuation v_i. We will use the notation $X_i^{\beta_{-i}}(b)$ in order to indicate that bidder i deviated from equilibrium to bid b. The analogous notation holds for the expected payment.

2.7.2 Proof of Proposition 1

Proof. As a preparation for this proof we will need four lemmas. The statements in these lemmas can be (informally) summarized as follows:

- Lemma 1: The payment of a bidder does not depend on lower bids.
- Lemma 2: The payment of a bidder does not depend on higher bids.
- Lemma 3: The payment of a bidder who is not a winner with a tie is right-continuous in her bid.
- Lemma 4: The payment of a bidder who is not the highest bidder is left-continuous in her bid.

We will formally state and prove the four lemmas and then continue with the proof of Proposition 1.

Lemma 1. *In an imitation-perfect auction for every bidder i and for every pair of vectors*

$$\left(b_i, b_j, b_{-(i,j)}\right) \text{ and } \left(b_i, b'_j, b_{-(i,j)}\right)$$

where $b_i > b_j$, $b_i > b'_j$ and bidder i is not a winner with a tie, it holds that

$$p_i\left(b_i, b_j, b_{-(i,j)}\right) = p_i\left(b_i, b'_j, b_{-(i,j)}\right),$$

i.e. the payment of a bidder does not depend on the bids of competitors who placed lower bids.

Proof. Let $\left(b_i, b_j, b_{-(i,j)}\right)$ and $\left(b_i, b'_j, b_{-(i,j)}\right)$ be bid vectors where bidder i is not a winner with a tie and it holds $b_i > b_j$, $b_i > b'_j$. Imitation perfection implies that for every $\epsilon > 0$ there exist bids $\bar{b}$ and $\bar{b}'$ with $\bar{b} > b_i$, $\bar{b}' > b_i$ such that

$$\left|p_i\left(b_i, b_j, b_{-(i,j)}\right) - p_j\left(b_i, \bar{b}, b_{-(i,j)}\right)\right| < \frac{\epsilon}{2} \tag{5}$$

and

$$\left|p_i\left(b_i, b'_j, b_{-(i,j)}\right) - p_j\left(b_i, \bar{b}', b_{-(i,j)}\right)\right| < \frac{\epsilon}{2}.$$

W.l.o.g. it holds that $\bar{b} \leq \bar{b}'$. Since the payment function of a bidder is non-decreasing in the other bidders' bids, it holds that

$$p_j\left(b_i, \bar{b}', b_{-(i,j)}\right) - p_i\left(b_i, b'_j, b_{-(i,j)}\right) > 0$$

and since the payment function of a bidder is non-decreasing in her own bid and $\overline{b} \leq \overline{b}'$, it holds that

$$p_j\left(b_i, \overline{b}, b_{-(i,j)}\right) - p_i\left(b_i, b_j', b_{-(i,j)}\right) < \frac{\epsilon}{2}. \tag{6}$$

Due to the triangle inequality, it follows from (5) and (6) that

$$\left|p_i\left(b_i, b_j, b_{-(i,j)}\right) - p_i\left(b_i, b_j', b_{-(i,j)}\right)\right| < \epsilon.$$

Since ϵ can be chosen arbitrarily, it holds that

$$p_i\left(b_i, b_j, b_{-(i,j)}\right) = p_i\left(b_i, b_j', b_{-(i,j)}\right),$$

□

Lemma 2. *In an imitation-perfect auction for every bidder i and for every pair of vectors*

$$\left(b_i, b_j, b_{-(i,j)}\right) \text{ and } \left(b_i, b_j', b_{-(i,j)}\right)$$

where $b_i < b_j$ and $b_i < b_j'$, it holds that

$$p_i\left(b_i, b_j, b_{-(i,j)}\right) = p_i\left(b_i, b_j', b_{-(i,j)}\right),$$

i.e. the payment of a bidder does not depend on the bids of competitors who placed higher bids.

Proof. The proof works analogously to the proof of Lemma 1. Let $\left(b_i, b_j, b_{-(i,j)}\right)$ and $\left(b_i, b_j', b_{-(i,j)}\right)$ be bid vectors such that $b_i < b_j$ and $b_i < b_j'$. Imitation perfection implies that for every $\epsilon > 0$ there exist bids $\underline{b}$ and $\underline{b}'$ with $\underline{b} < b_i, \underline{b}' < b_i$ such that

$$\left|p_i\left(b_i, b_j, b_{-(i,j)}\right) - p_j\left(b_i, \underline{b}, b_{-(i,j)}\right)\right| < \frac{\epsilon}{2}$$

and

$$\left|p_i\left(b_i, b_j', b_{-(i,j)}\right) - p_j\left(b_i, \underline{b}', b_{-(i,j)}\right)\right| < \frac{\epsilon}{2}. \tag{7}$$

W.l.o.g. it holds that $\overline{b} \leq \overline{b}'$. Since the payment function of a bidder is non-decreasing in the other bidders' bids, it holds that

$$p_i\left(b_i, b_j, b_{-(i,j)}\right) - p_j\left(b_i, \underline{b}, b_{-(i,j)}\right) > 0$$

and since the payment function of a bidder is non-decreasing in her own bid and $\overline{b} \leq \overline{b}'$, it holds that

$$p_i\left(b_i, b_j, b_{-(i,j)}\right) - p_j\left(b_i, \underline{b}', b_{-(i,j)}\right) < \frac{\epsilon}{2}. \tag{8}$$

Due to the triangle inequality, it follows from (7) and (8) that

$$\left|p_i\left(b_i, b_j, b_{-(i,j)}\right) - p_i\left(b_i, b'_j, b_{-(i,j)}\right)\right| < \epsilon.$$

Since ϵ can be chosen arbitrarily, it holds that

$$p_i\left(b_i, b_j, b_{-(i,j)}\right) = p_i\left(b_i, b'_j, b_{-(i,j)}\right).$$

$\square$

Lemma 3. *In an imitation-perfect auction for every bidder i, for all bid vectors (b_i, b_{-i}) such that it follows from $b_i = \max_{j \neq i} b_j$ that $b_i > \max_{j \neq i} b_j$, and for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all b'_i with $b_i < b'_i < b_i + \delta$ it holds that*

$$\left|p_i\left(b_1, \ldots, b_i, \ldots, b_n\right) - p_i\left(b_1, \ldots, b'_i, \ldots, b_n\right)\right| < \epsilon,$$

i.e. the payment of a bidder who is not a winner with a tie is right-continuous in her bid.

Proof. Let i be a bidder with bid b_i and $\epsilon > 0$. Let $(b_1, \ldots, b_i, \ldots, b_n)$ be a bid vector where bidder i is not a winner with a tie. It follows from imitation perfection that there exists a bid $\bar{b} > b_i$ such that

$$\left|p_i^{n+1}\left(b_1, \ldots, b_i, \ldots, b_n, 0\right) - p_{n+1}^{n+1}\left(b_1, \ldots b_i, \ldots, b_n, \bar{b}\right)\right| < \epsilon.^{21}$$

Since the auction is symmetric, it holds that

$$p_{n+1}^{n+1}\left(b_1, \ldots, b_i, \ldots, b_n, \bar{b}\right) = p_i^{n+1}\left(b_1, \ldots, \bar{b}, \ldots, b_n, b_i\right)$$

and therefore

$$\left|p_i^{n+1}\left(b_1, \ldots, b_i, \ldots, b_n, 0\right) - p_i^{n+1}\left(b_1, \ldots, \bar{b}, \ldots, b_n, b_i\right)\right| < \epsilon.$$

Since $\bar{b} > b_i \geq 0$, it follows from Lemma 1 that

$$p_i^{n+1}\left(b_1, \ldots, \bar{b}, \ldots, b_n, b_i\right) = p_i^{n+1}\left(b_1, \ldots, \bar{b}, \ldots, b_n, 0\right).$$

Therefore,

$$\left|p_i^{n+1}\left(b_1, \ldots, b_i, \ldots, b_n, 0\right) - p_i^{n+1}\left(b_1, \ldots, \bar{b}, \ldots, b_n, 0\right)\right| < \epsilon$$
$$\Leftrightarrow \left|p_i^{n}\left(b_1, \ldots, b_i, \ldots, b_n\right) - p_i^{n}\left(b_1, \ldots, \bar{b}, \ldots, b_n\right)\right| < \epsilon.$$

Define δ by $\delta := \bar{b} - b_i$. Since the payment function of bidder i is non-decreasing in her

[21] We need the construction with the $(n+1)$th bidder only to ensure that the lowest bidder in a bid vector can be also imitated by a higher bid.

own bid, it holds for every b with $b_i < b < b_i + \delta$ that

$$|p_i^n(b_1, \ldots, b_i, \ldots, b_n) - p_i^n(b_1, \ldots, b, \ldots, b_n)| < \epsilon$$

for all bid vectors $(b_1, \ldots, b_i, \ldots, b_n)$ where bidder i is not a winner with a tie. Hence, we have shown that the payment of bidder i is right-continuous if she is not a winner with a tie. □

Lemma 4. *In an imitation-perfect auction for every bidder i, for all bid vectors (b_i, b_{-i}) such that $b_i < \max_{j \neq i} b_j$, and for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all b with $b_i - \delta < b < b_i$ it holds that*

$$|p_i(b_1, \ldots, b_i, \ldots, b_n) - p_i(b_1, \ldots, b, \ldots, b_n)| < \epsilon,$$

i.e. the payment of a bidder who is not the highest bidder is left-continuous in her bid.

Proof. The proof works analogously to the proof of Lemma 3. Let i be a bidder with bid b_i and $\epsilon > 0$. Let (b_i, b_{-i}) be a bid vector such that $b_i < \max_{j \neq i} b_j$. Let b_j be a bid such that $b_j > b_i$. It follows from imitation perfection that there exists a bid $\underline{b}$ with $\underline{b} < b_i$ such that

$$|p_i(b_1, \ldots, b_i, \ldots b_j \ldots, b_n) - p_j(b_1, \ldots b_i, \ldots, \underline{b}, \ldots, b_n)| < \epsilon.$$

Since the auction is symmetric, it holds that

$$p_j(b_1, \ldots, b_i, \ldots, \underline{b}, \ldots, b_n) = p_i(b_1, \ldots, \underline{b}, \ldots, b_i, \ldots, b_n)$$

and therefore

$$|p_i(b_1, \ldots, b_i, \ldots, b_j, \ldots, b_n) - p_i(b_1, \ldots, \underline{b}, \ldots, b_i, \ldots, b_n)| < \epsilon.$$

Since $\underline{b} < b_i < b_j$, it follows from Lemma 2 that

$$p_i(b_1, \ldots, \underline{b}, \ldots, b_i, \ldots, b_n) = p_i(b_1, \ldots, \underline{b}, \ldots, b_j, \ldots, b_n)$$

from which follows that

$$|p_i(b_1, \ldots, b_i, \ldots, b_j, \ldots, b_n) - p_i(b_1, \ldots, \underline{b}, \ldots, b_j, \ldots, b_n)| < \epsilon.$$

Let δ be defined by $\delta := b_i - \underline{b}$. Since the payment function of bidder i is non-decreasing in her own bid, it holds for every b_i' with $b_i - \delta < b_i' < b_i$ that

$$|p_i(b_1, \ldots, b_i, \ldots, b_j, \ldots, b_n) - p_i(b_1, \ldots, b_i', \ldots, b_i, \ldots, b_n)| < \epsilon$$

for every vector $(b_1, \ldots, b_i, \ldots b_j \ldots, b_n)$ with $b_j > b_i$. Hence, we have shown that the payment of bidder i is left-continuous if she is not the highest bidder. □

After proving the four lemmas, we continue with the proof of Proposition 1. The first part of Proposition 1 follows from Lemma 1. However, the second part does not directly follow from Lemma 2. We have not shown yet that the payment of a losing bidder with bid b_i does not change if another bidder changes her bid from b_j to b'_j (or from b'_j to b_j) with $b'_j > b_i > b_j$ given that bidder i remains a losing bidder. First, we will show the following claim. Let (b_i, b_{-i}) be a bid vector such that i is a losing bidder, i.e. it holds $b_i < \max_{j \neq i} b_j$. Then for every b_j with $j \in \{1, \ldots, n\}$ it holds that

$$p_i\left(b_i, b_i, b_{-(i,j)}\right) = p_i\left(b_i, b_j, b_{-(i,j)}\right).$$

First, we consider the case that $b_j < b_i$. Since bidder i's payment is right-continuous in her own bid if she is not a winner with a tie, for every $\epsilon > 0$ there exists a bid $b'_i > b_i$ such that

$$\left|p_i\left(b'_i, b_i, b_{-(i,j)}\right) - p_i\left(b_i, b_i, b_{-(i,j)}\right)\right| < \frac{\epsilon}{2}$$

and

$$\left|p_i\left(b'_i, b_j, b_{-(i,j)}\right) - p_i\left(b_i, b_j, b_{-(i,j)}\right)\right| < \frac{\epsilon}{2}.$$

It holds that

$$\begin{aligned}&\left|p_i\left(b_i, b_i, b_{-(i,j)}\right) - p_i\left(b_i, b_j, b_{-(i,j)}\right)\right| \\ \leq &\left|p_i\left(b_i, b_i, b_{-(i,j)}\right) - p_i\left(b'_i, b_i, b_{-(i,j)}\right)\right| + \left|p_i\left(b_i, b_j, b_{-(i,j)}\right) - p_i\left(b'_i, b_i, b_{-(i,j)}\right)\right|.\end{aligned}$$

Since $b'_i > b_i > b_j$, it follows from Lemma 1 that this is equal to

$$\begin{aligned}&\left|p_i\left(b_i, b_i, b_{-(i,j)}\right) - p_i\left(b'_i, b_i, b_{-(i,j)}\right)\right| + \left|p_i\left(b_i, b_j, b_{-(i,j)}\right) - p_i\left(b'_i, b_j, b_{-(i,j)}\right)\right| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Now we consider the case that $b_j > b_i$. Since bidder i's payment is left-continuous in her own bid if she is not the highest bidder, for every $\epsilon > 0$ there exists a bid $b'_i < b_i$ such that

$$\left|p_i\left(b'_i, b_i, b_{-(i,j)}\right) - p_i\left(b_i, b_i, b_{-(i,j)}\right)\right| < \frac{\epsilon}{2}$$

and

$$\left|p_i\left(b'_i, b_j, b_{-(i,j)}\right) - p_i\left(b_i, b_j, b_{-(i,j)}\right)\right| < \frac{\epsilon}{2}.$$

It holds that

$$\begin{aligned}&\left|p_i\left(b_i, b_i, b_{-(i,j)}\right) - p_i\left(b_i, b_j, b_{-(i,j)}\right)\right| \\ \leq &\left|p_i\left(b_i, b_i, b_{-(i,j)}\right) - p_i\left(b'_i, b_i, b_{-(i,j)}\right)\right| + \left|p_i\left(b_i, b_j, b_{-(i,j)}\right) - p_i\left(b'_i, b_i, b_{-(i,j)}\right)\right|.\end{aligned}$$

Since $b'_i < b_i < b'_j$, it follows from Lemma 2 that this is equal to

$$\left| p_i\left(b_i, b_i, b_{-(i,j)}\right) - p_i\left(b'_i, b_i, b_{-(i,j)}\right)\right| + \left|p_i\left(b_i, b_j, b_{-(i,j)}\right) - p_i\left(b'_i, b_j, b_{-(i,j)}\right)\right|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since in both cases ϵ can be chosen arbitrarily, it holds that

$$p_i\left(b_i, b_i, b_{-(i,j)}\right) = p_i\left(b_i, b_j, b_{-(i,j)}\right)$$

for every vector of bids (b_i, b_{-i}) where i is a losing bidder.

After proving the claim, we can conclude that for two vectors of bids $\left(b_i, b_j, b_{-(i,j)}\right)$ and $\left(b_i, b'_j, b_{-(i,j)}\right)$ where bidder i is a losing bidder that

$$p_i\left(b_i, b_j, b_{-(i,j)}\right) = p_i\left(b_i, b_i, b_{-(i,j)}\right) = p_i\left(b_i, b'_j, b_{-(i,j)}\right). \tag{9}$$

This shows the second part of Proposition 1. □

2.7.3 Proof of Proposition 2

Proof. We begin by proving that imitation perfection implies the first part of Proposition 2. We begin by stating the following two statements:

1. For each bidder i and for all vectors of bids (b_i, b_{-i}) and $\left(b_i, b'_{-i}\right)$ such that $b_i > \max_{j \neq i} b_j$ and $b_i > \max_{j \neq i} b'_j$ it holds that

$$p_i\left(b_i, b_{-i}\right) = p_i\left(b_i, b'_{-i}\right).$$

2. For each bidder i and for all vectors of bids (b_i, b_{-i}) and $\left(b_i, b'_{-i}\right)$ such that $b_i < \max_{j \neq i} b_j$ and $b_i < \max_{j \neq i} b'_j$ it holds that

$$p_i\left(b_i, b_{-i}\right) = p_i\left(b_i, b'_{-i}\right).$$

The first statement follows from the repeated application of the statement in Lemma 1 and the second statement follows from the repeated application of (9).

Given these two statements, we can define the function p^{win} by:

$$p^{win}\left(b_i\right) = p_i^2\left(b_i, 0\right)$$

and define p^{lose} by

$$p^{lose}\left(b_i\right) = p_i^2\left(b_i, b_j\right)$$

for $b_j > b_i$. In order to see that this definition is consistent for all numbers of bidders and all bid vectors, consider the vector $(b_i, b_{-i}) \in (\mathbb{R}^+)^n$ such that $b_i > \max_{j \neq i} b_j$. Then due to statement 1, it holds that

$$p_i^n (b_i, b_{-i}) = p_i^2 (b_i, 0, \ldots, 0) = p_i^2 (b_i, 0).$$

Now consider the vector $(b_i, b_{-i}) \in (\mathbb{R}^+)^n$ such that $b_i < \max_{j \neq i} b_j$. Then due to statement 2, it holds for every $b_j > b_i$ that

$$p_i^n (b_i, b_{-i}) = p_i^n (b_i, b_j, 0, \ldots, 0) = p_i^2 (b_i, b_j).$$

It is left to show the following statement: For all vectors of bids (b_i, b_{-i}) and (b_i, b'_{-i}) such that

$$b_i = \max_{j \neq i} b_j, \quad b_i = \max_{j \neq i} b'_j$$

and

$$\#\{j \in \{1, \ldots, n\} \mid b_j = b_i\} = \#\{j \in \{1, \ldots, n\} \mid b'_j = b_i\}$$

it holds that

$$p_i (b_i, b_{-i}) = p_i (b_i, b'_{-i}).$$

Let

$$k = \#\{j \in \{1, \ldots, n\} \mid b_j = b_i\} = \#\{j \in \{1, \ldots, n\} \mid b'_j = b_i\}.$$

According to Assumption 4, it holds that

$$p_i (b_i, b_{-i}) = \frac{1}{k} p^{win} (b_i) + \left(1 - \frac{1}{k}\right) p^{lose} (b_i) = p_i (b_i, b'_{-i}).$$

The fact that imitation perfection implies that second part of Proposition 2, follows directly from Lemma 3 and Lemma 4.

It remains for us to show that the two conditions in Proposition 2 imply that an auction is imitation-perfect. That is, we have to show that

(i) For every vector of bids $\left(b_i, b_j, b_{-(i,j)}\right)$ such that bidder i is not a winner with a tie and for all $j \in \{1, \ldots, n\}$ with $b_i > b_j$ there exists a bid $\bar{b} > b_i$ such that

$$\left| p_i \left(b_i, b_j, b_{-(i,j)}\right) - p_j \left(b_i, \bar{b}, b_{-(i,j)}\right) \right| < \epsilon.$$

(ii) For every vector of bids $\left(b_i, b_j, b_{-(i,j)}\right)$ and for all $j \in \{1, \ldots, n\}$ with $b_i < b_j$ there exists a bid $\underline{b} < b_i$ such that

$$\left| p_i \left(b_i, b_j, b_{-(i,j)}\right) - p_j \left(b_i, \underline{b}, b_{-(i,j)}\right) \right| < \epsilon.$$

Let $\left(b_i, b_j, b_{-(i,j)}\right)$ be a bid vector such that bidder i is not a winner with a tie and $b_i > b_j$

Since bidder i's payment is right-continuous in b_i if she is not a winner with a tie, there exists a $\bar{b} > b_i$ such that

$$\left|p_i\left(b_1, \ldots, b_i, \ldots, b_j, \ldots, b_n\right) - p_i\left(b_1, \ldots, \bar{b}, \ldots, b_j, \ldots, b_n\right)\right| < \epsilon.$$

Since the payment of bidder i does not depend on lower bids and $\bar{b} > b_i > b_j$, it holds that

$$p_i\left(b_1, \ldots, \bar{b}, \ldots, b_j, \ldots, b_n\right) = p_i\left(b_1, \ldots, \bar{b}, \ldots, b_i, \ldots, b_n\right).$$

Due to the symmetry of the auction, it holds that

$$p_i\left(b_1, \ldots, \bar{b}, \ldots, b_i, \ldots, b_n\right) = p_j\left(b_1, \ldots, b_i, \ldots, \bar{b}, \ldots, b_n\right)$$

from which follows that

$$\left|p_i\left(b_1, \ldots, b_i, \ldots, b_j, \ldots, b_n\right) - p_j\left(b_1, \ldots, b_i, \ldots, \bar{b}, \ldots, b_n\right)\right| < \epsilon.$$

This completes the proof of the statement in (i). For the proof of part (ii) let $\left(b_i, b_j, b_{-(i,j)}\right)$ be a bid vector such that $b_i < b_j$. Since bidder i's payment is left-continuous in b_i if she is not the highest bidder, there exists a $\underline{b} < b_i$ such that

$$\left|p_i\left(b_1, \ldots, b_i, \ldots, b_j, \ldots, b_n\right) - p_i\left(b_1, \ldots, \underline{b}, \ldots, b_j, \ldots, b_n\right)\right| < \epsilon.$$

Since the payment of bidder i does not depend on higher bids and $b_j > b_i > \underline{b}$, it holds that

$$p_i\left(b_1, \ldots, \underline{b}, \ldots, b_j, \ldots, b_n\right) = p_i\left(b_1, \ldots, \underline{b}, \ldots, b_i, \ldots, b_n\right).$$

Due to the symmetry of the auction, it holds that

$$p_i\left(b_1, \ldots, \underline{b}, \ldots, b_i, \ldots, b_n\right) = p_j\left(b_1, \ldots, b_i, \ldots, \underline{b}, \ldots, b_n\right)$$

from which follows that

$$\left|p_i\left(b_1, \ldots, b_i, \ldots, b_j, \ldots, b_n\right) - p_j\left(b_1, \ldots, b_i, \ldots, \underline{b}, \ldots, b_n\right)\right| < \epsilon.$$

This completes the proof of part (ii). □

2.7.4 Lemmas

After proving Propositions 1 and 2, which provided statements regarding the payment rules in imitation-perfect auctions, we use these results in order to prove the following Lemmas. They provide statements about possible equilibria in imitation-perfect auctions and will be used throughout most of the proofs.

Lemma 5. *In an imitation-perfect auction and for every equilibrium $\beta = (\beta_1, \ldots, \beta_n)$ the expected payment as defined in (4) is strictly increasing above the reservation bid. That is, for all bids b_i, b_i' with $r \leq b_i < b_i'$ it holds that*

$$P_i^{\beta}(b_i) < P_i^{\beta}(b_i').$$

Proof. Due to Assumption 2, it holds that the payment of a winning bidder is strictly increasing in at least one component of the bid vector. Since we have shown in Proposition 1 that the payment of a winning bidder in an imitation-perfect auction does not depend on other bids, we conclude that the payment of a winning bidder is strictly increasing in her own bid.

Assume that the lemma is not true and there exists an equilibrium β, a bidder i and bids b_i, b_i' with $r \leq b_i < b_i'$ such that

$$P_i^{\beta}(b_i) \geq P_i^{\beta}(b_i').$$

This implies that bidder i with bid b_i' wins the auction with probability zero given the equilibrium strategies of the other bidders β_{-i}. It follows that there exists an interval $[0, v]$ such that for all bidders $j \neq i$, for all $z \in [0, v]$ and for all $b \in supp(\beta_j(z))$ it holds that $b > b_i'$ (except a measure zero set of valuations in [0,v]). Therefore, there exists a valuation v_ϵ, a bid $b > b_i'$ and a bidder $j \neq i$ such that $v_\epsilon < p^{win}(b_i')$ and $b \in supp(\beta_j(v_\epsilon))$. Since this cannot be optimal, this leads to a contradiction to the assumption

$$P_i^{\beta}(b_i) \geq P_i^{\beta}(b_i').$$

□

Lemma 6. *In an imitation-perfect auction every equilibrium is non-decreasing above the reservation bid. That is, for every equilibrium $\beta(\beta_1, \ldots, \beta_n)$, for every bidder $i \in \{1, \ldots, n\}$, for every pair of valuations $v', v \in [0, \overline{v}]$ such that $v' > v$ and for every pair of bids b', b with $b \geq r, b' \geq r$ such that $b' \in supp(\beta_i(v'))$ and $b \in supp(\beta_i(v))$ it holds that $b' \geq b$.*

Proof. The proof works analogously to the proof of Lemma 3.9 in Chawla and Hartline (2013). Assume the Lemma is not true and there exists an equilibrium β of an imitation-perfect auction which is decreasing. Then there exists a bidder i, valuations $v', v \in [0, \overline{v}]$ with $v' > v$ and bids b', b with $b' \in supp(\beta_i(v'))$ and $b \in supp(\beta_i(v))$ such that $b' < b$. It holds that

$$U_i^{\beta}(v', b) = U_i^{\beta}(v, b) + (v' - v)\, X_i^{\beta}(b).$$

$$U_i^{\beta}(v', b') = U_i^{\beta}(v, b') + (v' - v)\, X_i^{\beta}(b')$$

Since $b' \in supp\left(\beta_i\left(v'\right)\right)$ and $b \in supp\left(\beta_i\left(v\right)\right)$, it holds that

$$U_i^{\beta}\left(v', b\right) \leq U_i^{\beta}\left(v', b'\right).$$

and

$$U_i^{\beta}\left(v, b\right) \geq U_i^{\beta}\left(v, b'\right).$$

Therefore, it must hold that

$$\left(v' - v\right) X_i^{\beta}\left(b'\right) \geq \left(v' - v\right) X_i^{\beta}\left(b\right).$$

It follows from $b' < b$ that $X_i^{\beta}\left(b'\right) \leq X_i^{\beta}\left(b\right)$. Hence, it holds that $X_i^{\beta}\left(b'\right) = X_i^{\beta}\left(b\right)$ and $U_i^{\beta}\left(v', b'\right) = U_i^{\beta}\left(v', b\right)$. This implies that

$$X_i^{\beta}\left(b'\right) v' - P_i^{\beta}\left(b'\right) = X_i^{\beta}\left(b\right) v' - P_i^{\beta}\left(b\right)$$

and

$$P_i^{\beta}\left(b'\right) = P_i^{\beta}\left(b\right)$$

which is a contradiction to Lemma 5. □

In order to state the next lemma, we need the following definition

Definition 6. *For an equilibrium $\beta = \left(\beta_1, \ldots, \beta_n\right)$ and a bidder $i \in \{1, \ldots, n\}$ we denote the endpoints of an interval of valuations over which $\beta_i\left(v\right) = b$ by $\underline{v}_i\left(b\right)$ and $\overline{v}_i\left(b\right)$. Formally, we define*

$$\underline{v}_i\left(b\right) = \inf\{v \in [0, \overline{v}] \mid \beta_i\left(v\right) = b\}$$

and

$$\overline{v}_i\left(b\right) = \sup\{v \in [0, \overline{v}] \mid \beta_i\left(v\right) = b\}.$$

Lemma 7. *In an imitation-perfect auction for every pure strategy equilibrium $\beta = \left(\beta_1, \ldots, \beta_n\right)$, for every bidder $i \in \{1, \ldots, n\}$ and for every pair of valuations v, v' such that*

$$r \leq b = \beta_i\left(v\right) < b' = \beta_i\left(v'\right)$$

it holds that

$$\overline{v}_i\left(b\right) \leq \underline{v}_i\left(b'\right).$$

Proof. Assume there exists a pure strategy equilibrium $\beta = \left(\beta_1, \ldots, \beta_n\right)$, a bidder i and valuations v, v' with $r \leq b = \beta_i\left(v\right) < b' = \beta_i\left(v'\right)$ such that

$$\overline{v}_i\left(b\right) > \underline{v}_i\left(b'\right).$$

Then there exist $\hat{v}$ and $\hat{v}'$ such that

$$\beta_i(\hat{v}) = b,\ \beta_i(\hat{v}') = b' \text{ and } \hat{v}' < \hat{v}.$$

This is a contradiction to Lemma 6. Thus, we conclude that it must hold

$$\overline{v}_i(b) \leq \underline{v}_i(b').$$

□

In several proofs we will show the particular statement for pure strategy equilibria and use the following Lemma in order to derive the statement for general strategies.

Lemma 8. *Let $\beta = (\beta_1, \dots, \beta_n)$ be an equilibrium of an imitation-perfect auction. Then there exists a pure strategy equilibrium $\beta' = (\beta_1', \dots, \beta_n')$ such that it holds for all $i \in \{1, \dots, n\}$ and for all $v \in [0, \overline{v}]$ that*

$$X_i^{\beta}(v) = X_i^{\beta'}(v)$$

except a set of valuations which has measure zero.

Proof. Since we have shown that an imitation-perfect auction is a bid-determines-payment auction, we can follow the same steps as the proof of Lemma 3.10 in Chawla and Hartline (2013). □

2.7.5 Proof of Proposition 3

Proof. Let bidder i be the bidder who wins the auction with probability 1 and pays zero. As indicated in the sketch of the proof, we will show that in a perfect discrimination equilibrium a bidder with a strictly positive valuation can deviate to a bid which is strictly higher than any bid placed in equilibrium and come arbitrarily close to bidder i's payment which is zero.

Let bidder j be a bidder who has a strictly positive valuation v_j and let ϵ be such that $0 < \epsilon < v_j$. For every vector of valuations (v_i, v_{-i}) and for every vector of bids $(b_i(v_i), b_{-i}(v_{-i}))$, where $b_k(v_k) \in supp(\beta_k(v_k))$ for $k \in \{1, \dots, n\}$, bidder i is the unique winner. Thus, due to imitation perfection, for every $b_i(v_i) \in supp(\beta_i(v_i))$ there exists a bid $\overline{b}_j(b_i(v_i))$ such that

$$|p_j\left(b_i(v_i), \overline{b}_j(b_i(v_i)), b_{-(i,j)}\left(v_{-(i,j)}\right)\right) - p_i\left(b_i(v_i), b_j(v_j), b_{-(i,j)}\left(v_{-(i,j)}\right)\right)| < \frac{\epsilon}{2} \quad (10)$$

for every vector of valuations v_{-i} and for every $b_{-i} \in supp(\beta_{-i}(v_{-i}))$. Let

$$\hat{b}_j = \sup\left\{\overline{b}_j(b_i(v_i)) \quad | \ b_i(v_i) \in supp(\beta_i(v_i)),\ v_i \in [0, \overline{v}]\right\}.$$

It is not a priori clear that the statement in (10) holds for the supremum. In the following we will show that the statement in (10) also holds for the supremum.

Since for every vector of valuations v_{-j} and for every $b_{-j} \in supp\left(\beta_{-j}\left(v_{-j}\right)\right)$ in the vector

$$\left(b_i\left(v_i\right), \overline{b}_j\left(b_i\left(v_i\right)\right), b_{-(i,j)}\left(v_{-(i,j)}\right)\right)$$

bidder j is the unique winner, it follows from Proposition 2 that her payment function is right-continuous in her bid and there exists a $\delta > 0$ such that for all b_j with $\overline{b}_j\left(b_i\left(v_i\right)\right) < b_j < \overline{b}_j\left(b_i\left(v_i\right)\right) + \delta$ it holds that

$$\left|p_j\left(b_i\left(v_i\right), \overline{b}_j\left(b_i\left(v_i\right)\right), b_{-(i,j)}\left(v_{-(i,j)}\right)\right) - p_j\left(b_i\left(v_i\right), b_j, b_{-(i,j)}\left(v_{-(i,j)}\right)\right)\right| < \frac{\epsilon}{2}. \tag{11}$$

Fix a δ for which (11) holds. There exists a valuation $v_i^* \in [0, \overline{v}]$ and a bid $b_i\left(v_i^*\right) \in supp\left(\beta_i\left(v_i^*\right)\right)$ such that

$$\overline{b}_j\left(b_i\left(v_i^*\right)\right) < \hat{b}_j < \overline{b}_j\left(b_i\left(v_i^*\right)\right) + \delta.$$

Otherwise, it would hold for all $v_i \in [0, \overline{v}]$ and for all $b_i\left(v_i\right) \in supp\left(\beta_i\left(v_i\right)\right)$ that $\hat{b}_j \geq \overline{b}_j\left(b_i\left(v_i\right)\right) + \delta$. This is a contradiction to the fact that $\hat{b}_j$ is the smallest upper bound of the set

$$\left\{\overline{b}_j\left(b_i\left(v_i\right)\right) \quad | \; b_i\left(v_i\right) \in supp\left(\beta_i\left(v_i\right)\right), \; v_i \in [0, \overline{v}]\right\}.$$

Let $v_i^* \in [0, \overline{v}]$ and $b_i\left(v_i^*\right) \in supp\left(\beta_i\left(v_i^*\right)\right)$ be such that

$$\overline{b}_j\left(b_i\left(v_i^*\right)\right) < \hat{b}_j < \overline{b}_j\left(b_i\left(v_i^*\right)\right) + \delta.$$

Then for every vector of valuations v_{-i} and for every $b_{-i} \in supp\left(\beta_{-i}\left(v_{-i}\right)\right)$ it follows from (10) and (11) that

$$\left|p_j\left(b_i\left(v_i^*\right), \hat{b}_j, b_{-(i,j)}\left(v_{-(i,j)}\right)\right) - p_i\left(b_i\left(v_i^*\right), b_j\left(v_j\right), b_{-(i,j)}\left(v_{-(i,j)}\right)\right)\right|$$

$$\begin{aligned}
\leq & \left|p_j\left(b_i\left(v_i^*\right), \hat{b}_j, b_{-(i,j)}\left(v_{-(i,j)}\right)\right) - p_j\left(b_i\left(v_i^*\right), \overline{b}_j\left(b_i\left(v_i^*\right)\right), b_{-(i,j)}\left(v_{-(i,j)}\right)\right)\right| \\
& + \left|p_j\left(b_i\left(v_i^*\right), \overline{b}_j\left(b_i\left(v_i^*\right)\right), b_{-(i,j)}\left(v_{-(i,j)}\right)\right) - p_i\left(b_i\left(v_i^*\right), b_j\left(v_j\right), b_{-(i,j)}\left(v_{-(i,j)}\right)\right)\right| \\
& < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

For every vector of valuations v_{-i} and for every $b_{-i} \in supp\left(\beta_{-i}\left(v_{-i}\right)\right)$ it follows from

$$p_i\left(b_i\left(v_i^*\right), b_j\left(v_j\right), b_{-(i,j)}\left(v_{-(i,j)}\right)\right) = 0,$$

that

$$p_j\left(b_i\left(v_i^*\right), \hat{b}_j, b_{-(i,j)}\left(v_{-(i,j)}\right)\right) < \epsilon.$$

Since $\hat{b}_j$ is higher than any other bid placed by any bidder in equilibrium, bidder j would win the auction when bidding $\hat{b}_j$. According to Proposition 2 (or Lemma 1), the payment of a bidder does not depend on lower bids from which follows for every vector of valuations v_{-j} and for every $b_{-j} \in supp\,(\beta_{-i}\,(v_{-i}))$ that

$$p_j\left(b_i\,(v_i)\,, \hat{b}_j, b_{-(i,j)}\left(v_{-(i,j)}\right)\right) = p_j\left(b_i\,(v_i^*)\,, \hat{b}_j, b_{-(i,j)}\left(v_{-(i,j)}\right)\right) < \epsilon < v_j.$$

That is, by bidding $\hat{b}_j$, bidder j would pay an amount which is strictly smaller than her valuation. Hence, a perfect discrimination equilibrium cannot exist in an imitation-perfect symmetric auction, because each bidder $j \neq i$ would have an incentive to deviate whenever she has a strictly positive valuation for the good. □

2.7.6 Proof of Theorem 1

Proof. Although this theorem directly follows from Proposition 5, we provide a separate proof for Theorem 1 since this proof is less technical and may help to understand the intuition behind the results in Theorem 1 and Proposition 5. We prove that the auction is discrimination-free by demonstrating that in every equilibrium two bidders with the same distribution function follow identical strategies above the reservation bid except a set of valuations which has measure zero. In order to do so, we adapt a proof by Chawla and Hartline (2013).

First, we show the theorem for the case of equilibria in pure strategies and afterwards use Lemma 8 in order to derive the result for mixed strategy equilibria. We begin the proof by showing the following two lemmas, Lemma 9 and Lemma 10.

Lemma 9. *Let $\beta = (\beta_1, \ldots, \beta_n)$ be a pure strategy equilibrium of an imitation-perfect auction.*

(i) Let i and j be two bidders with the same distribution function and v a valuation such that

$$r \leq \beta_i\,(v) < \beta_j\,(v)\,.$$

Then it holds that

$$X_j^\beta\,(v) > X_i^\beta\,(v)$$

where $X_j^\beta\,(v)$ and $X_i^\beta\,(v)$ are defined as in (3) in Appendix 2.7.1.

(ii) Let i and j be two bidders with the same distribution function and v a valuation such that

$$\beta_i\,(v) = \beta_j\,(v) = b \text{ and } \underline{v}_j\,(b) \leq \underline{v}_i\,(b)\,,\ \overline{v}_j\,(b) \leq \overline{v}_i\,(b)$$

where $\underline{v}_j\,(b)\,, \underline{v}_i\,(b)\,, \overline{v}_j\,(b)\,, \overline{v}_i\,(b)$ are defined as in Definition 6 in Appendix 2.7.4. Then it holds that

$$X_j^\beta\,(v) \geq X_i^\beta\,(v)\,.$$

Proof. Part *(i)*: Let $v \in [0, \overline{v}]$ be a valuation such that $b_j > b_i \geq r$ where $b_j = \beta_j(v)$ and $b_i = \beta_i(v)$. The allocation probability of bidder i is equal or lower than her allocation probability if she wins every tie with probability one. In this case her winning probability is determined by the case that she bids equal or higher than any other bidder. Formally, we define a new allocation rule $\tilde{x}$ which is identical to the allocation rule x except for all bid vectors where bidder i is a winning bidder with a tie. For such a bid vector it holds that $\tilde{x}_i(b_i, b_{-i}) = 1$. It holds that

$$X_i^{\beta_{-i}}(b_i) \leq \tilde{X}_i^{\beta_{-i}}(b_i)$$

$$= \prod_{k \neq i} Pr\left[\text{bidder } k \text{ bids lower than } b_i + \text{bidder } k \text{ bids } b_i\right]$$

$$= \prod_{k \neq i} \left[F\left(\underline{v}_k(b_i)\right) + \left(F\left(\overline{v}_k(b_i)\right) - F\left(\underline{v}_k(b_i)\right)\right)\right] = \prod_{k \neq i} F\left(\overline{v}_k(b_i)\right)$$

$$= F\left(\overline{v}_j(b_i)\right) \prod_{k \neq i,j} F\left(\overline{v}_k(b_i)\right)$$

where F is defined by $F := F_i = F_j$. Due to Lemma 7, it holds that $\overline{v}_k(b_i) \leq \underline{v}_k(b_j)$ for all bidders $k \in \{1, \ldots, n\}$. Therefore, it holds that

$$\overline{v}_j(b_i) \leq \underline{v}_j(b_j) \leq v \leq \overline{v}_i(b_i) \leq \underline{v}_i(b_j).$$

It follows that

$$F\left(\overline{v}_j(b_i)\right) \prod_{k \neq i,j} F\left(\overline{v}_k(b_i)\right) \leq F\left(\underline{v}_i(b_j)\right) \prod_{k \neq i,j} F\left(\underline{v}_k(b_j)\right) = \prod_{k \neq i} F\left(\underline{v}_k(b_j)\right)$$

$$\leq \prod_{k \neq i} F\left(\underline{v}_k(b_j)\right) + \sum_{k=1}^{n-1} \frac{1}{k+1} Pr\left(\text{k bidders bid } b_i \text{ and none higher}\right) = X_j^{\beta_{-j}}(b_j).$$

According to Lemma 5, the expected payment of a bidder is strictly increasing in her own bid above the reservation bid. Thus, it cannot hold that $X_i^{\beta_{-i}}(b_i) = X_j^{\beta_{-j}}(b_j)$ in equilibrium. Otherwise, bidder j could deviate to a bid b_j' with $b_i < b_j' < b_j$. With the analogous reasoning as above, one can show that $X_j^{\beta_{-j}}\left(b_j'\right) \geq X_i^{\beta_{-i}}(b_i) = X_j^{\beta_{-j}}(b_j)$. Due to Lemma 5, it holds that $P_j^{\beta_{-j}}\left(b_j'\right) < P_j^{\beta_{-j}}(b_j)$. Hence, deviating to b_j' would increase bidder j's expected utility. Therefore, $b_j > b_i$ implies

$$X_i^{\beta_{-i}}(b_i) < X_j^{\beta_{-j}}(b_j).$$

Proof of part *(ii)*: If $b < r$, then the allocation probability for both bidders is zero and therefore the same. If $b \geq r$, it holds that

$$X_i^\beta(b) = F\left(\underline{v}_j(b)\right) E_{v_{-i}}[x_i(b, \beta_{-i}(v_{-i})) \mid b > \beta_j(v_j)] \\ + \left[F(\overline{v}_j(b)) - F\left(\underline{v}_j(b)\right)\right] E_{v_{-i}}[x_i(b, \beta_{-i}(v_{-i})) \mid b = \beta_j(v_j)].$$

Since $\underline{v}_j(b) \leq \underline{v}_i(b)$ and $\overline{v}_j(b) \leq \overline{v}_i(b)$, this is smaller or equal than

$$F(\underline{v}_i(b)) E_{v_{-i}}[x_i(b, \beta_{-i}(v_{-i})) \mid b > \beta_j(v_j)] \\ + [F(\overline{v}_i(b)) - F(\underline{v}_i(b))] E_{v_{-i}}[x_i(b, \beta_{-i}(v_{-i})) \mid b = \beta_j(v_j)].$$

The term $E_{v_{-i}}[x_i(b, \beta_{-i}(v_{-i})) \mid b > \beta_j(v_j)]$ denotes the probability that bidder i wins against all other bidders besides bidder j if she bids b and given that she wins against bidder j. This is equal to the probability that bidder j wins against all other bidders besides bidder i if she bids b given that she wins against bidder i, which is denoted by the term $E_{v_{-j}}[x_j(b, \beta_{-j}(v_{-j})) \mid b > \beta_i(v_i)]$. An analogous reasoning applies to the term $E_{v_{-i}}[x_i(b, \beta_{-i}(v_{-i})) \mid b = \beta_j(v_j)]$. Therefore, the expression above is equal to

$$= F(\underline{v}_i(b)) E_{v_{-j}}[x_j(b, \beta_{-j}(v_{-j})) \mid b > \beta_i(v_i)] \\ + [F(\overline{v}_i(b)) - F(\underline{v}_i(b))] E_{v_{-j}}[x_j(b, \beta_{-j}(v_{-j})) \mid b = \beta_i(v_i)] = X_j^\beta(b).$$

□

Lemma 10. *If there exists a pure strategy equilibrium $\beta = (\beta_1, \ldots, \beta_n)$ of an imitation-perfect auction, a bidder k and an interval $[\underline{v}_k, \overline{v}_k]$ with $F_k(\overline{v}_k) - F_k(\underline{v}_k) > 0$ such that $\beta_k(v) = b_k \geq r$ for all $v \in [\underline{v}_k, \overline{v}_k]$, then there does not exist another bidder j and a valuation v_j such that $\beta_j(v_j) = b_k$ and*

$$Pr(b_k > \beta_i(v_i) \text{ for all } i \neq k, j) \left(v_j - p^{win}(b_k)\right) > 0.$$

That is, if bidder k's bidding strategy is constant over some interval, then due to the continuity of the payment function, every other bidder would never bid the same amount but has an incentive to slightly overbid bidder k.

Proof. Assume there exists a bidder j and a valuation v_j such that $\beta_j(v_j) = b_k$ and $Pr(b_k > \beta_i(v_i) \text{ for all } i \neq k, j)(v_j - p^{win}(b_k)) > 0$. First, we consider the case that bidder k is the only bidder such that there exists an interval of valuations with measure larger than zero over which the bidder bids b_k. Let $\left(\hat{\underline{v}}_k, \hat{\overline{v}}_k\right)$ be the maximal interval over which it holds that $\beta_k(v) = b_k$ for all $v \in \left(\hat{\underline{v}}_k, \hat{\overline{v}}_k\right)$, i.e.

$$\hat{\underline{v}}_k = \inf_{v \in [0, \overline{v}]} \beta_k(v) = b$$

$$\hat{\overline{v}}_k = \sup_{v\in[0,\overline{v}]} \beta_k(v) = b.$$

For a shorter notation let P be defined by

$$P = Pr\left(b_k > \beta_i(v_i) \text{ for all } i \neq k, j\right).$$

Let $\epsilon > 0$ be such that

$$\frac{1}{2}[F_k\left(\hat{\overline{v}}_k\right) - F_k\left(\hat{\underline{v}}_k\right)][v_j - p^{win}(b_k)]P - \epsilon > 0.$$

Due to the right-continuity of the payment function, for all $\epsilon > 0$ there exists a bid $b' > b_k$ such that

$$p_j\left(b', b_{-j}\right) - p_j\left(b_k, b_{-j}\right) < \frac{\epsilon}{2}$$

for all vectors (b_k, b_{-j}) where bidder j is not a winning bidder with a tie.

If bidder j deviates to b', then her winning probability increases from

$$E_{v_{-j}}[x_j\left(b_k, \beta_{-j}(v_{-j})\right)] = \frac{1}{2}[F_k\left(\hat{\overline{v}}_k\right) - F_k\left(\hat{\underline{v}}_k\right)]P$$

to

$$E_{v_{-j}}[x_j\left(b', \beta_{-j}(v_{-j})\right)] = [F_k\left(\hat{\overline{v}}_k\right) - F_k\left(\hat{\underline{v}}_k\right)]Pr\left(b' > \beta_i(v_i) \text{ for all } i \neq k, j\right).$$

This implies that the winning probability increases by at least

$$\frac{1}{2}[F_k\left(\hat{\overline{v}}_k\right) - F_k\left(\hat{\underline{v}}_k\right)]P.$$

Her expected payment increases by at most

$$\frac{1}{2}\left(p^{win}(b')\right)[F_k\left(\hat{\overline{v}}_k\right) - F_k\left(\hat{\underline{v}}_k\right)]P + \int_{[0,\overline{v}]^{n-1}} \frac{\epsilon}{2} f(v_{-j})\, dv_{-j}$$

$$\leq \frac{1}{2}\left(p^{win}(\beta_j(v_j)) + \frac{\epsilon}{2}\right)[F_k\left(\hat{\overline{v}}_k\right) - F_k\left(\hat{\underline{v}}_k\right)]P + \frac{\epsilon}{2}$$

$$\leq \frac{1}{2}\left(p^{win}(\beta_j(v_j))\right)[F_k\left(\hat{\overline{v}}_k\right) - F_k\left(\hat{\underline{v}}_k\right)]P + \frac{\epsilon}{4}[F_k\left(\hat{\overline{v}}_k\right) - F_k\left(\hat{\underline{v}}_k\right)]P + \frac{\epsilon}{2}$$

$$\leq \frac{1}{2}\left(p^{win}(\beta_j(v_j))\right)[F_k\left(\hat{\overline{v}}_k\right) - F_k\left(\hat{\underline{v}}_k\right)]P + \epsilon.$$

Therefore, the expected utility gain for bidder j from deviating to b' is given by at least

$$\frac{1}{2}[F_k\left(\hat{\overline{v}}_k\right) - F_k\left(\hat{\underline{v}}_k\right)]Pv_j - \frac{1}{2}\left(p^{win}(\beta_j(v_j))\right)[F_k\left(\hat{\overline{v}}_k\right) - F_k\left(\hat{\underline{v}}_k\right)]P - \epsilon$$

$$= \frac{1}{2}[F_k\left(\hat{\overline{v}}_k\right) - F_k\left(\hat{\underline{v}}_k\right)][v_j - p^{win}(\beta_j(v_j))]P - \epsilon > 0.$$

We conclude that there does not exist a bidder j and a valuation v_j such that $\beta_j(v_j) = b_k$ because otherwise bidder j would have an incentive to deviate.

The case where more than one bidder bids b_k over an interval of valuations with measure greater than zero, can be excluded analogously. $\square$

We continue with the proof of Theorem 1. Let $\beta = (\beta_1, \ldots, \beta_n)$ be an equilibrium in pure strategies. In order to show that bidders with the same distribution function follow identical strategies, we consider two arbitrary bidders i and j who draw their valuations from the same distribution. Assume that the strategies of bidder i and bidder j differ above the reservation bid over some set of valuations with strictly positive measure. Since strategies are continuous except a set of valuations which has measure zero, there exists an interval of valuations with strictly positive measure over which the strategies differ. Consider the lowest valuation at which the strategies differ above the reservation bid over an interval of valuations with strictly positive measure. Formally, let

$$\underline{z} = \inf \left\{ v' \mid \exists\, z'' > v' \text{ s.t. } \beta_j(v) \neq \beta_i(v) \text{ and } \beta_i(v) \geq r, \beta_j(v) \geq r \text{ for all } v \in [v', z''] \right\}.$$

Since strategies are continuous besides a set of valuations which has measure zero, there exists a valuation v'' such that w.l.o.g. it holds for all $z \in (\underline{z}, v'')$ that $\beta_j(z) > \beta_i(z)$.

In order to show that this leads to a contradiction, we use the following definition.

Definition 7 (Utility crossing)**.** *For a given equilibrium β an interval $(\underline{z}, \overline{z})$ satisfies* utility crossing *if $X_j^\beta(v) \geq X_i^\beta(v)$ for all $v \in (\underline{z}, \overline{z})$ and $U_j^\beta(\underline{z}) \geq U_i^\beta(\underline{z})$ and $U_i^\beta(\overline{z}) \geq U_j^\beta(\overline{z})$.*

We will show that strategies have to be equal on an interval which satisfies utility crossing.

Lemma 11. *Let β be an equilibrium of an imitation-perfect auction and $(\underline{z}, \overline{z})$ be an interval satisfying utility crossing. Then it holds that $\beta_i(v) = \beta_j(v)$ for all $v \in (\underline{z}, \overline{z})$ except a set of measure zero.*

Proof. Suppose that $\beta_j \neq \beta_i$ over some subset of $(\underline{z}, \overline{z})$ with strictly positive measure. Since strategies are discontinuous on a measure zero set of valuations, there exists a set with strictly positive measure such that either $\beta_j > \beta_i$ or $\beta_i > \beta_j$ for all valuations of this set. Since $X_j^\beta(v) \geq X_i^\beta(v)$ for all $v \in (\underline{z}, \overline{z})$, it follows from Lemma 9 that $\beta_j > \beta_i$ over some subset of $(\underline{z}, \overline{z})$ with strictly positive measure. Due to Lemma 9, it holds that $X_j^\beta(v) > X_i^\beta(v)$ for all v with $\beta_j(v) > \beta_i(v)$. According to Myerson (1981), in every auction mechanism the expected utility of a bidder can be written as a function of the winning probability. Formally, it holds for every k and every v_k that

$$U_k^\beta(v_k) = U_k^\beta(0) + \int_0^{v_k} X_k^\beta(z)\, dz. \tag{12}$$

Since the payment function of a winning bidder is strictly increasing in her bid [22], a

[22] Recall the argument provided in the proof of Lemma 5: Due to Assumption 2, the payment of a bidder has to be strictly increasing in at least one component of the bid vector. We have shown in Proposition 1 (and Lemma 1) that the payment of a winning bidder depends only on her own bid. Therefore, the

bidder with valuation zero does not win with positive probability. Because payments are non-negative, the expected payment of a bidder with valuation zero is zero. Therefore, the expected utility of a bidder with valuation zero is zero.

Thus, applying equation (12) to $\overline{z}$ and $\underline{z}$ and rearranging it accordingly gives

$$U_i^{\beta}(\overline{z}) - U_i^{\beta}(\underline{z}) = \int_{\underline{z}}^{\overline{z}} X_i(z)\, dz$$

and

$$U_j^{\beta}(\overline{z}) - U_j^{\beta}(\underline{z}) = \int_{\underline{z}}^{\overline{z}} X_j^{\beta}(z)\, dz.$$

Since $X_j^{\beta} > X_i^{\beta}$ over a subset of $(\underline{z}, \overline{z})$ with strictly positive measure, it holds that

$$U_j^{\beta}(\overline{z}) - U_j^{\beta}(\underline{z}) > U_i^{\beta}(\overline{z}) - U_i^{\beta}(\underline{z}),$$

which contradicts utility crossing. It therefore holds that $\beta_i(v) = \beta_j(v)$ for all v in $(\underline{z}, \overline{z})$ except a set of measure zero. □

We will show that $(\underline{z}, v'')$ lies in an interval satisfying utility crossing. Hence, the strategy of bidder i and j cannot differ on the interval $(\underline{z}, v'')$. We will show that the interval which satisfies utility crossing is given by $(\underline{z}, \overline{z})$ where $\overline{z}$ is defined by

$$\overline{z} = \inf\{v > \underline{z} | \beta_i(v) \geq \beta_j(v)\}.$$

If the infimum does not exist, we redefine $\overline{z} = \overline{v}$. It follows from Lemma 9 that $X_j^{\beta}(v) > X_i^{\beta}(v)$ for all $v \in (\underline{z}, \overline{z})$. Thus, it is left to show that

$$U_i(\overline{z}) \geq U_j(\overline{z}) \text{ and } U_j(\underline{z}) \geq U_i(\underline{z}).$$

Let $b < \beta_j(\underline{z})$ be such that there exists a valuation $z \in [0, \underline{z}]$ with $\beta_i(z) = b$. Since $z < \underline{z}$ and $\underline{z}$ is the infimum of valuations at which the strategies of bidder i and bidder j differ, it holds that

$$\beta_i(z) = \beta_j(z) = b$$

and

$$\underline{v}_i(b) = \underline{v}_j(b).$$

It also holds that $\overline{v}_j(b) \leq \overline{v}_i(b)$. Assume this is not true. Then it holds that $\overline{v}_j(b) > \overline{v}_i(b)$. Since the equilibrium is non-decreasing, this implies that there exists an interval $(\overline{v}_i(b), \hat{v})$ such that $\beta_i(v) > \beta_j(v)$ for all $v \in (\overline{v}_i(b), \hat{v})$. This is a contradiction to the assumption that $\underline{z}$ is the infimum of valuations at which the strategies of bidder i and j differ and that bidder j bids higher than bidder i on some interval $(\underline{z}, v'')$.

payment of a winning bidder has to be strictly increasing in her own bid.

Thus, it holds that $\overline{v}_j(b) \leq \overline{v}_i(b)$ and it follows from Lemma 9 that $X_i^{\beta}(b) = X_j^{\beta}(b)$.[23] If $\beta_j(\underline{z}) > \beta_i(\underline{z})$, it follows from part (ii) of Lemma 9 that $X_i^{\beta}(b) < X_j^{\beta}(b)$.

If $\beta_j(\underline{z}) = \beta_i(\underline{z})$, it holds that $\underline{v}_i(\beta_i(\underline{z})) = \underline{v}_j(\beta_i(\underline{z}))$ because strategies are equal below $\underline{z}$. Moreover, it holds that $\overline{v}_i(\beta_i(\underline{z})) \geq \overline{v}_j(\beta_i(\underline{z}))$ because otherwise bidder i's strategy would be decreasing. Thus, it follows from Lemma 9 that $X_i^{\beta}(z) \leq X_j^{\beta}(z)$ for all $z \in [\underline{v}_i(\beta_i(\underline{z})), \underline{z}]$.

We conclude that for all $z \in [0, \underline{z}]$, it holds that $X_j^{\beta}(z) \geq X_i^{\beta}(z)$. Hence, it follows from Myerson (1981) that

$$U_j^{\beta}(\underline{z}) = \int_0^{\underline{z}} X_j^{\beta}(z)\, dz \geq \int_0^{\underline{z}} X_i^{\beta}(z)\, dz = U_i^{\beta}(\underline{z}).$$

It is left to show that

$$U_i(\overline{z}) \geq U_j(\overline{z}).$$

In order to do so, we show that for every $\epsilon > 0$ there exists a bid for bidder i with which she could achieve an expected utility of at least $U_j^{\beta}(\overline{z}) - \epsilon$ if she has valuation $\overline{z}$. Therefore, the expected utility of bidder i's equilibrium bid has to induce at least an expected utility of $U_j^{\beta}(\overline{z})$ and it holds that

$$U_i^{\beta}(\overline{z}) \geq U_j^{\beta}(\overline{z}).$$

It follows from Proposition 2 that the expected payment in equilibrium of bidder j at $\overline{z}$ is given by

$$P_j^{\beta}(\beta_j(\overline{z})) = X_j^{\beta}(\beta_j(\overline{z}))\, p^{win}(\beta_j(\overline{z})) + \left(1 - X_j^{\beta}(\beta_j(\overline{z}))\right)\, p^{lose}(\beta_j(\overline{z})).$$

Let ϵ be greater than zero. Due to the right-continuity of the functions p^{win} and p^{lose}, there exists a bid $b > \beta_j(\overline{z})$ such that

$$p^{win}(b) - p^{win}(\beta_j(\overline{z})) < \epsilon$$

and

$$p^{lose}(b) - p^{lose}(\beta_j(\overline{z})) < \epsilon.$$

We can assume that $Pr(\beta_j(\overline{z}) > \beta_k(v_k) \text{ for all } k \neq i, j)(\overline{z} - \beta_j(\overline{z})) > 0$ because otherwise it immediately follows that $U_i^{\beta}(\overline{z}) \geq U_j^{\beta}(\overline{z}) = 0$. Thus, by Lemma 10, the event that bidder j is a winner with a tie bidding $\beta_j(\overline{z})$ has probability zero. In particular, the interval $[\underline{v}_i(\beta_j(\overline{z})), \overline{v}_i(\beta_j(\overline{z}))]$ has measure zero.

[23] Recall that es stated in (3) in Appendix 2.7.1, by $X_i^{\beta}(b)$ we denote the allocation probability of a bidder i who submits bid b given the equilibrium β.

In equilibrium the expected utility of bidder j bidding $\beta_j(\overline{z})$ is given by

$$
\begin{aligned}
&U_j^{\beta}(\overline{z}, \beta_j(\overline{z})) \\
&\quad = F(\underline{v}_i(\beta_j(\overline{z})))\, E_{v_{-j}}[x_j(\beta_j(\overline{z}), \beta_{-j}(v_{-j})) \mid \beta_j(\overline{z}) > \beta_i(v_i)] \left(\overline{z} - p^{win}(\beta_j(\overline{z}))\right) \\
&\qquad + \Big(1 - F(\underline{v}_i(\beta_j(\overline{z})))\, E_{v_{-j}}[x_j(\beta_j(\overline{z}), \beta_{-j}(v_{-j})) \mid \beta_j(\overline{z}) > \beta_i(v_i)]\Big) p^{lose}(\beta_j(\overline{z})).
\end{aligned}
$$

Since the event that bidder j is a winner with a tie has measure zero, this equation does not account for the possibility of ties. The expected utility of bidder i deviating to bid b at $\overline{z}$ is given by

$$
\begin{aligned}
U_i^{\beta_{-i}}(\overline{z}, b) &= F\left(\underline{v}_j(b)\right) E_{v_{-i}}[x_i(b, \beta_{-i}(v_{-i})) \mid b > \beta_j(v_j)] \left(\overline{z} - p^{win}(b)\right) \\
&\quad + [F(\overline{v}_j(b)) - F\left(\underline{v}_j(b)\right)] E_{v_{-i}}[x_i(b, \beta_{-i}(v_{-i})) \mid b = \beta_j(v_j)] \left(\overline{z} - p^{win}(b)\right) \\
&\qquad - \Big(1 - F\left(\underline{v}_j(b)\right) E_{v_{-i}}[x_i(b, \beta_{-i}(v_{-i})) \mid b > \beta_j(\overline{z})] \\
&\qquad - [F(\overline{v}_j(b)) - F\left(\underline{v}_j(b)\right)] E_{v_{-i}}[x_i(b, \beta_{-i}(v_{-i})) \mid b = \beta_j(v_j)]\Big) p^{lose}(b)
\end{aligned}
$$

$$
\begin{aligned}
&\geq F\left(\underline{v}_j(b)\right) E_{v_{-i}}[x_i(b, \beta_{-i}(v_{-i})) \mid b > \beta_j(v_j)] \left(\overline{z} - p^{win}(b)\right) \\
&\qquad - \Big(1 - F\left(\underline{v}_j(b)\right) E_{v_{-i}}[x_i(b, \beta_{-i}(v_{-i})) \mid b > \beta_j(\overline{z})]\Big) p^{lose}(b). \qquad (13)
\end{aligned}
$$

Since $b > \beta_j(\overline{z})$, it follows from Lemma 7 that $\overline{v}_j(b) \geq \underline{v}_j(b) \geq \overline{v}_j(\beta_j(\overline{z})) \geq \overline{z}$. Since the interval $[\underline{v}_i(\beta_j(\overline{z})), \overline{v}_i(\beta_j(\overline{z}))]$ has measure zero, it holds that $\beta_i(v) > \beta_i(\overline{z})$ for all $v > \overline{z}$ except a set of valuations which has measure zero. It follows that $\overline{v}_i(\beta_j(\overline{z})) \leq \overline{z}$. Hence, it holds that

$$
\underline{v}_i(\beta_j(\overline{z})) \leq \overline{v}_i(\beta_j(\overline{z})) \leq \overline{z} \leq \overline{v}_j(\beta_j(\overline{z})) \leq \underline{v}_j(b). \qquad (14)
$$

It follows from (14) that the term in (13) is greater or equal than

$$
\begin{aligned}
&F(\underline{v}_i(\beta_j(\overline{z})))\, E_{v_{-i}}[x_i(\beta_j(\overline{z}), \beta_{-i}(v_{-i})) \mid \beta_j(\overline{z}) > \beta_j(v_j)] \left(\overline{z} - p^{win}(b)\right) \\
&\qquad - \Big(1 - F(\underline{v}_i(\beta_j(\overline{z})))\, E_{v_{-i}}[x_i(\beta_j(\overline{z}), \beta_{-i}(v_{-i})) \mid \beta_j(\overline{z}) > \beta_j(v_j)]\Big) p^{lose}(b)
\end{aligned}
$$

$$
\begin{aligned}
&> F(\underline{v}_i(\beta_j(\overline{z})))\, E_{v_{-i}}[x_i(\beta_j(\overline{z}), \beta_{-i}(v_{-i})) \mid \beta_j(\overline{z}) > \beta_j(v_j)] \left(\overline{z} - \left(p^{win}(\beta_j(\overline{z})) + \epsilon\right)\right) \\
&- \Big(1 - F(\underline{v}_i(\beta_j(\overline{z})))\, E_{v_{-i}}[x_i(\beta_j(\overline{z}), \beta_{-i}(v_{-i})) \mid \beta_j(\overline{z}) > \beta_j(v_j)]\Big) \left(p^{lose}(\beta_j(\overline{z})) + \epsilon\right)
\end{aligned}
$$

$$= F\left(\underline{v}_i\left(\beta_j\left(\overline{z}\right)\right)\right) E_{v_{-j}}[x_j\left(\beta_j\left(\overline{z}\right), \beta_{-j}\left(v_{-j}\right)\right) \mid \beta_j\left(\overline{z}\right) > \beta_i\left(v_i\right)] \left(\overline{z} - \left(p^{win}\left(\beta_j\left(\overline{z}\right)\right) + \epsilon\right)\right)$$
$$- \left(1 - F\left(\underline{v}_i\left(\beta_j\left(\overline{z}\right)\right)\right) E_{v_{-j}}[x_j\left(\beta_j\left(\overline{z}\right), \beta_{-j}\left(v_{-j}\right)\right) \mid \beta_j\left(\overline{z}\right) > \beta_i\left(v_i\right)]\right) \left(p^{lose}\left(\beta_j\left(\overline{z}\right)\right) + \epsilon\right)$$

$$= F\left(\underline{v}_i\left(\beta_j\left(\overline{z}\right)\right)\right) E_{v_{-j}}[x_j\left(\beta_j\left(\overline{z}\right), \beta_{-j}\left(v_{-j}\right)\right) \mid \beta_j\left(\overline{z}\right) > \beta_i\left(v_i\right)] \left(\overline{z} - p^{win}\left(\beta_j\left(\overline{z}\right)\right)\right)$$
$$- \left(1 - F\left(\underline{v}_i\left(\beta_j\left(\overline{z}\right)\right)\right) E_{v_{-j}}[x_j\left(\beta_j\left(\overline{z}\right), \beta_{-j}\left(v_{-j}\right)\right) \mid \beta_j\left(\overline{z}\right) > \beta_i\left(v_i\right)]\right) p^{lose}\left(\beta_j\left(\overline{z}\right)\right) - \epsilon.$$

It follows that

$$U_j^{\beta}\left(\overline{z}, \beta_j\left(\overline{z}\right)\right) - U_i^{\beta_{-i}}\left(\overline{z}, b\right) < \epsilon.$$

Hence, we have shown that for every $\epsilon > 0$ there exists a deviating bid b such that bidder i can achieve an expected utility of at least $U_j^{\beta}\left(\overline{z}\right) - \epsilon$ from which follows that

$$U_i^{\beta}\left(\overline{z}\right) \geq U_j^{\beta}\left(\overline{z}\right).$$

We conclude that the interval $\left(\underline{z}, \overline{z}\right)$ fulfills utility crossing. Therefore, it hods that $\beta_i\left(v\right) = \beta_j\left(v\right)$ for all $v \in \left(\underline{z}, \overline{z}\right)$ except a measure zero set of valuations. Thus, the assumption, that there exists a measurable interval over which the bidding strategies of two bidders differ above the reservation bid, leads to a contradiction.

It is left to consider the case of mixed equilibria. Let $\beta = \left(\beta_1, \ldots, \beta_n\right)$ be a (possibly mixed equilibrium). According to Lemma 8, there exists a pure strategy equilibrium $\beta' = \left(\beta_1', \ldots, \beta_n'\right)$ such that it holds for all $i \in \{1, \ldots, n\}$ and for all $v \in \{1, \ldots, n\}$ that

$$X_i^{\beta}\left(v\right) = X_i^{\beta'}\left(v\right)$$

except for a set of valuations with measure zero. Since we have shown that in a pure strategy equilibrium all bidders adopt identical strategies above the reservation bid, for every pair of bidders i and j and for every $v \in [0, \overline{v}]$ it holds that

$$X_i^{\beta'}\left(v\right) = X_j^{\beta'}\left(v\right).$$

Thus, for every $v \in [0, \overline{v}]$ it holds that

$$\left|U_i^{\beta}\left(v\right) - U_j^{\beta}\left(v\right)\right| = \left|\int_0^v X_j^{\beta}\left(z\right) dz - \int_0^v X_i^{\beta}\left(z\right) dz\right| = \left|\int_0^v X_j^{\beta'}\left(z\right) dz - \int_0^v X_i^{\beta'}\left(z\right) dz\right| = 0.$$

Thus, it hods that $U_i^{\beta}\left(v\right) = U_j^{\beta}\left(v\right)$ which completes the proof.

$\square$

2.7.7 Proof of Proposition 4

Proof of part (i). Since we apply a result from Reny (1999), we begin the proof by explaining his framework.

A *game* consists of a set of players $\{1, \ldots, n\}$, a set of pure strategies X_i which is a subset of a topological vector space for every player i and a utility function

$$u_i : X \to \mathbb{R}$$

for every player i where $X = \times_{i=1}^{n} X_i$. The function u denotes the vector of all utility functions $(u_1, \ldots, u_n)$. Note that in a setting with incomplete information like our auction setting, the pure strategy set for a bidder i corresponds to the set of pure bidding strategies for this bidder and the utility function u_i corresponds to bidder i's ex-ante utility given the valuation distributions of all bidders. That is, in our setting in the notation used in Reny (1999) the function u_i maps a profile of bidding strategies to an ex-ante utility for bidder i. Given the notation used in this paper, it holds that

$$X_i = \{\beta_i : \Theta_i \to A_i\}$$

and

$$u_i\left(x_1, \ldots, x_n\right) = \int_{[0,\overline{v}]} U_i^x\left(v_i\right) dF_i\left(v_i\right).$$

where u_i is the utility function as defined in Reny (1999) and U_i is defined as in this paper in the model section.

A game is a *compact Hausdorff game* if for every $i \in \{1, \ldots, n\}$ it holds that X_i is a compact Hausdorff space and u_i is bounded. If $f : A \to B$ is a function and A, B are topological spaces, then a tuple (a, b) with $a \in A$ and $b \in B$ is in the *closure of the graph* of f if there exists a sequence $\{a^m\}_{m=1}^{\infty}$ converging to a such that $b = \lim_{m\to\infty} f\left(a^m\right)$.

Player i can *secure* a utility $p \in \mathbb{R}$ at $x \in X$ if there exists $\hat{x}_i \in X_i$ such that $u_i\left(\hat{x}_i, x'_{-i}\right) \geq p$ for all x'_{-i} in some open neighborhood of x_{-i}. A game is *better-reply secure* if whenever (x^*, u^*) is in the closure of the graph of u and x^* is not an equilibrium, there exists a player i who can secure a utility strictly above u_i^* at x^*.

The existence of a Bayes-Nash equilibrium (in possibly mixed strategies) follows from Corollary 5.2 in Reny (1999). It states that if the mixed extension of a compact Hausdorff Game is better-reply secure, then an equilibrium exists. We apply this result to our setting by adopting the proof for Example 5.2 in Reny (1999) which applies Corollary 5.2 to first-price auctions. As stated in Example 5.2, the space of pure bidding strategies for a bidder is a compact Hausdorff space with respect to the topology of pointwise convergence.[24] Thus, it is left to show that an imitation-perfect auction (where also mixed strategies are

[24]Recall that for a set X and a space of functions from X to R the topology of pointwise convergence is generated by the subbase $U_{x,a,b}$ for $x \in X, a, b \in \mathbb{R}, a < b$ with $U, x, b = \{f : X \to \mathbb{R} \mid \alpha < f(x) < b\}$.

allowed) is better-reply secure.

Assume that $x^* \in X$ is not an equilibrium and let (x^*, u^*) be in the closure of the graph of u. By definition, there exists a sequence of strategies $\{x^m\}_{m=1}^{\infty}$ converging to x^* such that $u^* = \lim_{m\to\infty} u(x^m)$.

First, we consider the case that ties occur with probability zero at x^*. Then u is continuous at x^*. Let $u_i^s(x_{-i}^*)$ denote bidder i's supremum utility at x_{-i}^*, i.e. for all $i \in \{1, \ldots, n\}$ it holds that

$$u_i^s(x_{-i}^*) = \sup_{x_i \in X_i} u_i(x_i, x_{-i}^*).$$

Since by assumption it holds that x^* is not an equilibrium, there exists a bidder i such that

$$u_i^s(x_{-i}^*) > u_i(x^*).$$

Therefore, for every $\epsilon > 0$ there exists a strategy for bidder i, x_i^ϵ, such that

$$\left|u_i(x_i^\epsilon, x_{-i}^*) - u_i^s(x_{-i}^*)\right| < \epsilon.$$

For a sufficiently small ϵ it holds that

$$u_i(x_i^\epsilon, x_{-i}^*) > u_i(x^*).$$

Since u is continuous at x^*, there exists an open neighborhood of $x_{-i}{}^*$ such that

$$u_i(x_i^\epsilon, x'_{-i}) > u_i(x^*)$$

for all x'_{-i} in this neighborhood. Thus, if ties occur with probability zero, the game is better-reply secure. Now we consider the case that ties occur with positive probability at x^*. Let i and j be two bidders who tie with positive probability, that is, there exists an interval $[\underline{v}_t, \overline{v}_t]$ on which they submit the same bid b_t. For every strategy profile x^m either bidder i or bidder j loses with positive (ex-ante) probability over the interval $[\underline{v}_t, \overline{v}_t]$. This implies that there exists a bidder l for $l \in \{i, j\}$ and a subsequence $\{x^{m_k}\}$ such that bidder l loses with positive (ex-ante) probability on the interval $[\underline{v}_t, \overline{v}_t]$ for every x^{m_k}. Then this subsequence converges to x^*. W.l.o.g. we assume that $l = i$.

It must hold that $p^{win}(b_t) \leq \underline{v}_t < \overline{v}_t$. Therefore, bidder i would strictly increase her ex-ante utility if her ex-ante winning probability increases on a subset of $[\underline{v}_t, \overline{v}_t]$ with positive measure. For every $k > 0$ it holds that at x^{m_k} bidder i can strictly increase her ex-ante winning probability on a subset of $[\underline{v}_t, \overline{v}_t]$ with positive measure.

Let $u_i^s\left(x_{-i}^{m_k}\right)$ denote bidder i's supremum utility at $x_{-i}^{m_k}$. Since the functions p^{win} and p^{lose} are right-continuous, there exists $K > 0$ such that for all $k > K$ it holds that

$$u_i^s\left(x_{-i}^{m_k}\right) > u_i(x^{m_k}).$$

That is, bidder i's strategy cannot be optimal since by slightly increasing her bid, she could increase her ex-ante utility. Since this holds for all $k > K$, this also holds at the limit. That is,

$$u_i^s\left(x_{-i}^*\right) > u_i\left(x^*\right).$$

Hence, for every $\epsilon > 0$ there exists a strategy x_i^ϵ such that

$$\left|u_i\left(x_i^\epsilon, x_{-i}^*\right) - u_i^s\left(x_{-i}^*\right)\right| < \epsilon.$$

Again, for a sufficiently small ϵ it holds that

$$u_i\left(x_i^\epsilon, x_{-i}^*\right) > u_i\left(x^*\right).$$

Since at $\left(x_i^\epsilon, x_{-i}^*\right)$ bidder i does not tie with another bidder, u_i is continuous at x_{-i}^*. Therefore, there exists an open neighborhood of x_{-i}^* such that

$$u_i\left(x_i^\epsilon, x'_{-i}\right) > u_i\left(x^*\right)$$

for all x'_{-i} in this neighborhood.

We conclude that an imitation-perfect auction with compact intervals as bid spaces is better-reply secure and therefore an equilibrium exists. □

Proof of part (ii). It follows from Theorem 4.5 in Chawla and Hartline (2013) that in a bid determines-payment auction with homogeneous bidders there exists only one symmetric equilibrium, that is, an equilibrium where all bidders adopt identical strategies. Due to Proposition 2, all imitation-perfect auctions are bid-determines-payment auctions. As shown in the proof of Theorem 1, in an imitation-perfect auction all equilibria are symmetric. Therefore, an imitation-perfect auction with homogeneous bidders has a unique equilibrium. It follows from Lemma 8 that if a mixed strategy equilibrium exists, then a pure strategy equilibrium also exists. Since the equilibrium is unique, it has to be a pure strategy equilibrium. It follows from Lemma 6 that the equilibrium is non-decreasing.

□

2.7.8 Proof of Proposition 5

Proof. As in the proof of Theorem 1, we will first show the statement in Proposition 5 for pure strategy equilibria and then apply Lemma 8 in order to show the statement for mixed strategy equilibria. We will prove Proposition 5 for pure strategy equilibria by showing the following claim: For every equilibrium $\beta = (\beta_1, \ldots, \beta_n)$, every valuation v and every pair of bidders i and j it holds that

$$\left|\int_0^v X_j^\beta(z) - X_i^\beta(z)\, dz\right| \leq \delta_{i,j}\overline{v} + \delta_{i,j}\left(\overline{v} - v\right). \tag{15}$$

Given this claim, Proposition 5 directly follows from Myerson (1981) since

$$\left|U_j^\beta(v) - U_i^\beta(v)\right| = \left|\int_0^v X_j^\beta(z) - X_i^\beta(z)\, dz\right|.$$

The proof of the claim works similarly to the proof of Theorem 1. We start by proving the following Lemma which provides an analogous result to Lemma 9 for heterogeneous bidders.

Lemma 12. *Let $\beta = (\beta_1, \ldots, \beta_n)$ be a pure strategy equilibrium equilibrium of an imitation-perfect auction.*

(i) Let i and j be two bidders and v a valuation such that

$$\beta_i(v) < \beta_j(v).$$

Then it holds that

$$X_j^\beta(v) + \delta_{i,j} \geq X_i^\beta(v)$$

where $\delta_{i,j}$ is defined as in (2).

(ii) Let i and j be two bidders and v a valuation such that

$$\beta_i(v) = \beta_j(v) = b \text{ and } \underline{v}_j(b) \leq \underline{v}_i(b),\ \overline{v}_j(b) \leq \overline{v}_i(b).$$

Then it holds that

$$X_j^\beta(v) + \delta_{i,j} \geq X_i^\beta(v)$$

where $\delta_{i,j}$ is defined as in (2).

Proof. Part *(i)*: Let v be a valuation and b_i, b_j be defined by $\beta_i(v) = b_i$ and $\beta_j(v_j) = b_j$. If $b_i < r$, it holds that $X_i^\beta(b_i) = 0$ and the statement follows directly. Otherwise, similarly to the proof of Lemma 9 we have:

$$X_i^\beta(b_i) \leq \tilde{X}_i^\beta(b_i), \text{ with}$$

$$\begin{aligned}
\tilde{X}_i^\beta(b_i) &= \prod_{k \neq i} Pr\left[\text{bidder } k \text{ bids lower than } b_i + \text{bidder } k \text{ bids } b_i\right] \\
&= \prod_{k \neq i}\left[F_k\left(\underline{v}_j(b_i)\right) + \left(F_k(\overline{v}_k(b_i)) - F_k(\underline{v}_k(b_i))\right)\right] = \prod_{k \neq i} F_k(\overline{v}_k(b_i)) \\
&= F_j(\overline{v}_j(b_i)) \prod_{k \neq i,j} F_k(\overline{v}_k(b_i)) \leq F_j(\underline{v}_i(b_j)) \prod_{k \neq i,j} F(\underline{v}_k(b_j))
\end{aligned}$$

Since for every $v \in [0, \overline{v}]$ it holds that $F_j(v) \leq F_i(v) + \delta$, this is smaller or equal than

$$\left(F_i(\underline{v}_i(b_j)) + \delta_{i,j}\right) \prod_{k \neq i,j} F(\underline{v}_k(b_j)) = \prod_{k \neq i} F(\underline{v}_k(b_j)) + \delta_{i,j} \prod_{k \neq i,j} F(\underline{v}_k(b_j))$$

$$\leq \prod_{k\neq i} F\left(\underline{v}_k\left(b_j\right)\right)+\delta_{i,j} \leq \prod_{k\neq i} F\left(\underline{v}_k\left(b_j\right)\right)+\delta_{i,j}+\sum_{k=1}^{n-1} \frac{1}{k+1} Pr\left(\text{k bidders bid } b_i \text{ and none higher}\right)$$

$$= X_j^{\beta}\left(b_j\right)+\delta_{i,j}.$$

Proof of part *(ii)*: If $b < r$, the allocation probability for both bidders is zero and therefore the same. If $b \geq r$, it holds that

$$X_i^{\beta}\left(b\right) = F_j\left(\underline{v}_j\left(b\right)\right) E_{v_{-i}}[x_i\left(b, \beta_{-i}\left(v_{-i}\right)\right) \mid b > \beta_j\left(v_j\right)] \\ + \left[F_j\left(\overline{v}_j\left(b\right)\right) - F_j\left(\underline{v}_j\left(b\right)\right)\right] E_{v_{-i}}[x_i\left(b, \beta_{-i}\left(v_{-i}\right)\right) \mid b = \beta_j\left(v_j\right)]$$

$$= F_j\left(\underline{v}_j\left(b\right)\right)\left(E_{v_{-i}}[x_i\left(b, \beta_{-i}\left(v_{-i}\right)\right) \mid b > \beta_j\left(v_j\right)] - E_{v_{-i}}[x_i\left(b, \beta_{-i}\left(v_{-i}\right)\right) \mid b = \beta_j\left(v_j\right)]\right) \\ + F_j\left(\overline{v}_j\left(b\right)\right) E_{v_{-i}}[x_i\left(b, \beta_{-i}\left(v_{-i}\right)\right) \mid b = \beta_j\left(v_j\right)]$$

$$\leq \left(F_i\left(\underline{v}_i\left(b\right)\right)+\delta_{i,j}\right)\left(E_{v_{-i}}[x_i\left(b, \beta_{-i}\left(v_{-i}\right)\right) \mid b > \beta_j\left(v_j\right)] \\ - E_{v_{-i}}[x_i\left(b, \beta_{-i}\left(v_{-i}\right)\right) \mid b = \beta_j\left(v_j\right)]\right) \\ + \left(F_i\left(\overline{v}_i\left(b\right)\right)+\delta_{i,j}\right) E_{v_{-i}}[x_i\left(b, \beta_{-i}\left(v_{-i}\right)\right) \mid b = \beta_j\left(v_j\right)]$$

$$\leq F_i\left(\underline{v}_i\left(b\right)\right)\left(E_{v_{-i}}[x_i\left(b, \beta_{-i}\left(v_{-i}\right)\right) \mid b > \beta_j\left(v_j\right)] - E_{v_{-i}}[x_i\left(b, \beta_{-i}\left(v_{-i}\right)\right) \mid b = \beta_j\left(v_j\right)]\right) \\ F_i\left(\overline{v}_i\left(b\right)\right) E_{v_{-i}}[x_i\left(b, \beta_{-i}\left(v_{-i}\right)\right) \mid b = \beta_j\left(v_j\right)] + \delta_{i,j}$$

$$= F_i\left(\underline{v}_i\left(b\right)\right) E_{v_{-j}}[x_j\left(b, \beta_{-j}\left(v_{-j}\right)\right) \mid b > \beta_i\left(v_i\right)] \\ + \left[F_i\left(\overline{v}_i\left(b\right)\right) - F_i\left(\underline{v}_i\left(b\right)\right)\right] E_{v_{-j}}[x_j\left(b, \beta_{-j}\left(v_{-j}\right)\right) \mid b = \beta_i\left(v_i\right)] + \delta_{i,j} = X_j^{\beta}\left(b\right)+\delta_{i,j}$$

where the first inequality follows from the assumption that $\underline{v}_j\left(b\right) \leq \underline{v}_i\left(b\right)$ and $\overline{v}_j\left(b\right) \leq \overline{v}_i\left(b\right)$. □

Note that Lemma 10 holds independent of the bidders' homogeneity. We will show the claim in (15) given by

$$\left|\int_0^v X_j^{\beta}\left(z\right) - X_i^{\beta}\left(z\right) dz\right| \leq \delta_{i,j}\overline{v} + \delta_{i,j}\left(\overline{v} - v\right)$$

by contradiction. Assume that there exist bidders i and j and a valuation v such that

$$\left|\int_0^v X_j^{\beta}\left(z\right) - X_i^{\beta}\left(z\right) dz\right| > \delta_{i,j}\overline{v} + \delta_{i,j}\left(\overline{v} - v\right).$$

W.l.o.g. we can assume that it holds

$$\int_0^v X_j^\beta(z) - X_i^\beta(z)\, dz > \delta_{i,j}\overline{v} + \delta_{i,j}(\overline{v} - v).$$

We will need the following Lemma:

Lemma 13. *Let β be an equilibrium of an imitation-perfect auction and $\overline{z}$ a valuation. Then the following holds true:*

(i) If $\beta_i(\overline{z}) \geq \beta_j(\overline{z})$, then it holds that

$$U_i^\beta(\overline{z}) + \delta_{i,j}\overline{v} \geq U_j^\beta(\overline{z}).$$

(ii) If there exists a valuation $\hat{v} > \overline{z}$ such that for all $z \in (\overline{z}, \hat{v})$ it holds that $\beta_i(z) \geq \beta_j(z)$, then it holds that

$$U_i^\beta(\overline{z}) + \delta_{i,j}\overline{v} \geq U_j^\beta(\overline{z}).$$

(iii) If $\overline{z} = \overline{v}$, then it holds that

$$U_i^\beta(\overline{z}) = U_j^\beta(\overline{z}).$$

.

Proof. For all three parts we can make the following statements. If $\beta_j(\overline{z}) < r$, it holds that $U_j^\beta(\overline{z}) = 0$ and the statement follows directly. If $\beta_j(\overline{z}) \geq r$, let ϵ be greater than zero.

Due to Proposition 2, it holds that the expected payment of bidder j at $\overline{z}$ is given by

$$P_j^\beta(\beta_j(\overline{z})) = X_j^\beta(\beta_j(\overline{z}))\, p^{win}(\beta_j(\overline{z})) + \left(1 - X_j^\beta(\beta_j(\overline{z}))\right)\, p^{lose}(\beta_j(\overline{z})).$$

We continue with the proof for part *(i)* and *(ii)*. Analogously as in the proof of Theorem 1, we will prove that

$$U_i^\beta(\overline{z}) + \delta_{i,j}\overline{v} \geq U_j^\beta(\overline{z})$$

by showing that for every $\epsilon > 0$ there exists a deviating bid b for bidder i at valuation $\overline{z}$ with which she could achieve at least a utility of $U_j^{\beta_i}(\overline{z}) - \delta_{i,j}\overline{v} - \epsilon$.

In equilibrium the expected utility of bidder j bidding $\beta_j(\overline{z})$ is given by

$$\begin{aligned}
&U_j^\beta(\overline{z}, \beta_j(\overline{z})) \\
&\quad = F_i(\underline{v}_i(\beta_j(\overline{z})))\, E_{v_{-j}}[x_j(\beta_j(\overline{z}), \beta_{-j}(v_{-j})) \mid \beta_j(\overline{z}) > \beta_i(v_i)]\left(\overline{z} - p^{win}(\beta_j(\overline{z}))\right) \\
&\qquad - \left(1 - F_i(\underline{v}_i(\beta_j(\overline{z})))\, E_{v_{-j}}[x_j(\beta_j(\overline{z}), \beta_{-j}(v_{-j})) \mid \beta_j(\overline{z}) > \beta_i(v_i)]\right) p^{lose}(\beta_j(\overline{z})).
\end{aligned}$$

We can assume that $Pr(\beta_j(\overline{z}) > \beta_k(v_k)$ for all $k \neq i, j) > 0$ because otherwise it directly follows that $U_i^\beta(\overline{z}) \geq U_j^\beta(\overline{z}) = 0$. Thus, according to Lemma 10, the event that bidder j is a winner with a tie bidding $\beta_j(\overline{z})$, has probability zero. In particular, the interval

$[\underline{v}_i(\beta_j(\overline{z})), \overline{v}_i(\beta_j(\overline{z}))]$ has measure zero. Thus, the above equation does not account for the possibility of ties. Due to the right-continuity of the functions p^{win} and p^{lose}, there exists a bid $b > \beta_j(\overline{z})$ such that

$$p^{win}(b) - p^{win}(\beta_j(\overline{z})) < \epsilon$$

and

$$p^{lose}(b) - p^{lose}(\beta_j(\overline{z})) < \epsilon.$$

The expected utility of bidder i deviating to bid b at $\overline{z}$ is given by

$$\begin{aligned}
U_i^{\beta_{-i}}(\overline{z}, b) = F_j\left(\underline{v}_j(b)\right) E_{v_{-i}}[x_i(b, \beta_{-i}(v_{-i})) \mid b > \beta_j(v_j)]\left(\overline{z} - p^{win}(b)\right) \\
+ [F_j(\overline{v}_j(b)) - F_j\left(\underline{v}_j(b)\right)] E_{v_{-i}}[x_i(b, \beta_{-i}(v_{-i})) \mid b = \beta_j(v_j)]\left(\overline{z} - p^{win}(b)\right) \\
- \Big(1 - F_j\left(\underline{v}_j(b)\right) E_{v_{-i}}[x_i(b, \beta_{-i}(v_{-i})) \mid b > \beta_j(\overline{z})] \\
- [F_j(\overline{v}_j(b)) - F_j\left(\underline{v}_j(b)\right)] E_{v_{-i}}[x_i(b, \beta_{-i}(v_{-i})) \mid b = \beta_j(v_j)]\Big) p^{lose}(b).
\end{aligned}$$

$$\begin{aligned}
\geq F_j\left(\underline{v}_j(b)\right) E_{v_{-i}}[x_i(b, \beta_{-i}(v_{-i})) \mid b > \beta_j(v_j)]\left(\overline{z} - p^{win}(b)\right) \\
- \Big(1 - F_j\left(\underline{v}_j(b)\right) E_{v_{-i}}[x_i(b, \beta_{-i}(v_{-i})) \mid b > \beta_j(\overline{z})]\Big) p^{lose}(b). \quad (16)
\end{aligned}$$

Since $b > \beta_j(\overline{z})$, it follows from Lemma 7 that $\overline{v}_j(b) \geq \underline{v}_j(b) \geq \overline{v}_j(\beta_j(\overline{z})) \geq \overline{z}$. Since the interval $[\underline{v}_i(\beta_j(\overline{z})), \overline{v}_i(\beta_j(\overline{z}))]$ has measure zero, it holds that $\beta_i(v) > \beta_i(\overline{z})$ for all $v > \overline{z}$ except a set of valuations which has measure zero. It follows that $\overline{v}_i(\beta_j(\overline{z})) \leq \overline{z}$. Hence, it holds that

$$\underline{v}_i(\beta_j(\overline{z})) \leq \overline{v}_i(\beta_j(\overline{z})) \leq \overline{z} \leq \overline{v}_j(\beta_j(\overline{z})) \leq \underline{v}_j(b). \quad (17)$$

It follows from (17) that the expression in (16) is greater or equal than

$$\begin{aligned}
F_j(\underline{v}_i(\beta_j(\overline{z}))) E_{v_{-i}}[x_i(\beta_j(\overline{z}), \beta_{-i}(v_{-i})) \mid \beta_j(\overline{z}) > \beta_j(v_j)]\left(\overline{z} - p^{win}(b)\right) \\
- \Big(1 - F_j(\underline{v}_i(\beta_j)(\overline{z})) E_{v_{-i}}[x_i(\beta_j(\overline{z}), \beta_{-i}(v_{-i})) \mid \beta_j(\overline{z}) > \beta_j(v_j)]\Big) p^{lose}(b).
\end{aligned}$$

$$\begin{aligned}
> F_j(\underline{v}_i(\beta_j(\overline{z}))) E_{v_{-i}}[x_i(\beta_j(\overline{z}), \beta_{-i}(v_{-i})) \mid \beta_j(\overline{z}) > \beta_j(v_j)]\left(\overline{z} - \left(p^{win}(\beta_j(\overline{z})) + \epsilon\right)\right) \\
- \Big(1 - F_j(\underline{v}_i(\beta_j(\overline{z}))) E_{v_{-i}}[x_i(\beta_j(\overline{z}), \beta_{-i}(v_{-i})) \mid \beta_j(\overline{z}) > \beta_j(v_j)]\Big)\left(p^{lose}(\beta_j(\overline{z})) + \epsilon\right)
\end{aligned}$$

$$\geq \left(F_i\left(\underline{v}_i\left(\beta_j\right)(\overline{z})\right)-\delta_{i,j}\right) E_{v_{-i}}[x_i\left(\beta_j\left(\overline{z}\right), \beta_{-i}\left(v_{-i}\right)\right) \mid \beta_j\left(\overline{z}\right)>\beta_j\left(v_j\right)]\left(\overline{z}-p^{win}\left(\beta_j\left(\overline{z}\right)\right)\right)$$
$$-\Big(1-(F_i\left(\underline{v}_i\left(\beta_j\left(\overline{z}\right)\right)-\delta_{i,j}\right)) E_{v_{-i}}[x_i\left(\beta_j\left(\overline{z}\right), \beta_{-i}\left(v_{-i}\right)\right) \mid \beta_j\left(\overline{z}\right)>\beta_j\left(v_j\right)]\Big)\left(p^{lose}\left(\beta_j\left(\overline{z}\right)\right)\right)-\epsilon$$

$$\geq F_i\left(\underline{v}_i\left(\beta_j\left(\overline{z}\right)\right)\right) E_{v_{-i}}[x_i\left(\beta_j\left(\overline{z}\right), \beta_{-i}\left(v_{-i}\right)\right) \mid \beta_j\left(\overline{z}\right)>\beta_j\left(v_j\right)]\left(\overline{z}-p^{win}\left(\beta_j\left(\overline{z}\right)\right)\right)$$
$$-\Big(1-F_i\left(\underline{v}_i\left(\beta_j\left(\overline{z}\right)\right)\right) E_{v_{-i}}[x_i\left(\beta_j\left(\overline{z}\right), \beta_{-i}\left(v_{-i}\right)\right) \mid \beta_j\left(\overline{z}\right)>\beta_j\left(v_j\right)]\Big) p^{lose}\left(\beta_j\left(\overline{z}\right)\right)$$
$$-\delta_{i,j}\left(\overline{z}-p^{win}\left(\beta_j\left(\overline{z}\right)\right)+p^{lose}\left(\beta_j\left(\overline{z}\right)\right)\right)-\epsilon$$
$$\geq F_i\left(\underline{v}_i\left(\beta_j\left(\overline{z}\right)\right)\right) E_{v_{-i}}[x_i\left(\beta_j\left(\overline{z}\right), \beta_{-i}\left(v_{-i}\right)\right) \mid \beta_j\left(\overline{z}\right)>\beta_j\left(v_j\right)]\left(\overline{z}-p^{win}\left(\beta_j\left(\overline{z}\right)\right)\right)$$
$$-\Big(1-(F_i\left(\underline{v}_i\left(\beta_j\left(\overline{z}\right)\right)-\delta_{i,j}\right)) E_{v_{-i}}[x_i\left(\beta_j\left(\overline{z}\right), \beta_{-i}\left(v_{-i}\right)\right) \mid \beta_j\left(\overline{z}\right)>\beta_j\left(v_j\right)]\Big) p^{lose}\left(\beta_j\left(\overline{z}\right)\right)$$
$$-\delta_{i,j}\overline{z}-\epsilon.$$
$$=F_i\left(\underline{v}_i\left(\beta_j\left(\overline{z}\right)\right)\right) E_{v_{-j}}[x_j\left(\beta_j\left(\overline{z}\right), \beta_{-j}\left(v_{-j}\right)\right) \mid \beta_j\left(\overline{z}\right)>\beta_i\left(v_i\right)]\left(\overline{z}-p^{win}\left(\beta_j\left(\overline{z}\right)\right)\right)$$
$$+\Big(1-F_i\left(\underline{v}_i\left(\beta_j\left(\overline{z}\right)\right)\right) E_{v_{-j}}[x_j\left(\beta_j\left(\overline{z}\right), \beta_{-j}\left(v_{-j}\right)\right) \mid \beta_j\left(\overline{z}\right)>\beta_i\left(v_i\right)]\Big) p^{lose}\left(\beta_j\left(\overline{z}\right)\right)$$
$$-\delta_{i,j}\overline{z}-\epsilon.$$

It follows that

$$U_j^{\beta}\left(\overline{z}, \beta_j\left(\overline{z}\right)\right)-U_i^{\beta_{-i}}\left(\overline{z}, b\right)<\delta_{i,j}\overline{z}+\epsilon.$$

Hence, we have shown that for every $\epsilon>0$ there exists a deviating bid b such that bidder i can achieve an expected utility of at least $U_j^{\beta}\left(\overline{z}\right)-\delta_{i,j}\overline{z}-\epsilon$ from which follows that

$$U_i^{\beta}\left(\overline{z}\right)+\delta_{i,j}\left(\overline{z}\right) \geq U_j^{\beta}\left(\overline{z}\right).$$

Now we provide a proof for part *(iii)*. It is sufficient to show that $U_i^{\beta}\left(\overline{z}\right) \geq U_j^{\beta}\left(\overline{z}\right)$, since then $U_j^{\beta}\left(\overline{z}\right) \geq U_i^{\beta}\left(\overline{z}\right)$ follows by symmetry. As in part *(i)* and *(ii)*, the expected utility of bidder j bidding $\beta_j\left(\overline{z}\right)$ is given by

$$\begin{aligned}
&U_j^{\beta}\left(\overline{z}, \beta_j\left(\overline{z}\right)\right)\\
&\quad=F_i\left(\underline{v}_i\left(\beta_j\left(\overline{z}\right)\right)\right) E_{v_{-j}}[x_j\left(\beta_j\left(\overline{z}\right), \beta_{-j}\left(v_{-j}\right)\right) \mid \beta_j\left(\overline{z}\right)>\beta_i\left(v_i\right)]\left(\overline{z}-p^{win}\left(\beta_j\left(\overline{z}\right)\right)\right)\\
&\qquad-\Big(1-F_i\left(\underline{v}_i\left(\beta_j\left(\overline{z}\right)\right)\right) E_{v_{-j}}[x_j\left(\beta_j\left(\overline{z}\right), \beta_{-j}\left(v_{-j}\right)\right) \mid \beta_j\left(\overline{z}\right)>\beta_i\left(v_i\right)]\Big) p^{lose}\left(\beta_j\left(\overline{z}\right)\right).
\end{aligned}$$

Since $\overline{z}=\overline{v}$, by bidding b bidder i will bid higher than bidder j with probability one. Thus, the expected utility of bidder i bidding b at $\overline{z}$ is given by

$$\begin{aligned}
U_i^{\beta_{-i}}\left(\overline{z}, b\right)=E_{v_{-i}}[x_i\left(b, \beta_{-i}\left(v_{-i}\right)\right) \mid b>\beta_j\left(v_j\right)]\left(\overline{z}-p^{win}\left(b\right)\right)&\\
-\left(1-E_{v_{-i}}[x_i\left(b, \beta_{-i}\left(v_{-i}\right)\right) \mid b>\beta_j\left(v_j\right)]\right) p^{lose}\left(b\right)&
\end{aligned}$$

$$\geq F_i\left(\underline{v}_i\left(\beta_j\left(\overline{z}\right)\right)\right) E_{v_{-i}}[x_i\left(b,\beta_{-i}\left(v_{-i}\right)\right) \ \mid b>\beta_j\left(v_j\right)] \left(\overline{z}-p^{win}\left(b\right)\right)$$
$$-\left(1-F_i\left(\underline{v}_i\left(\beta_j\left(\overline{z}\right)\right)\right) E_{v_{-i}}[x_i\left(b,\beta_{-i}\left(v_{-i}\right)\right) \ \mid b>\beta_j\left(v_j\right)]\right)p^{lose}\left(b\right)$$

$$> F_i\left(\underline{v}_i\left(\beta_j\left(\overline{z}\right)\right)\right) E_{v_{-i}}[x_i\left(b,\beta_{-i}\left(v_{-i}\right)\right) \ \mid b>\beta_j\left(v_j\right)] \left(\overline{z}-\left(p^{win}\left(\beta_j\left(\overline{z}\right)\right)+\epsilon\right)\right)$$
$$-\left(1-F_i\left(\underline{v}_i\left(\beta_j\left(\overline{z}\right)\right)\right) E_{v_{-i}}[x_i\left(b,\beta_{-i}\left(v_{-i}\right)\right) \ \mid b>\beta_j\left(v_j\right)]\right) \left(p^{lose}\left(\beta_j\left(\overline{z}\right)\right)+\epsilon\right)$$

$$= F_i\left(\underline{v}_i\left(\beta_j\left(\overline{z}\right)\right)\right) E_{v_{-i}}[x_i\left(b,\beta_{-i}\left(v_{-i}\right)\right) \ \mid b>\beta_j\left(v_j\right)] \left(\overline{z}-p^{win}\left(\beta_j\left(\overline{z}\right)\right)\right)$$
$$-\left(1-F_i\left(\underline{v}_i\left(\beta_j\left(\overline{z}\right)\right)\right) E_{v_{-i}}[x_i\left(b,\beta_{-i}\left(v_{-i}\right)\right) \ \mid b>\beta_j\left(v_j\right)]\right)p^{lose}\beta_j\left(\overline{z}\right)-\epsilon$$

$$\geq F_i\left(\underline{v}_i\left(\beta_j\left(\overline{z}\right)\right)\right) E_{v_{-j}}[x_j\left(\beta_j\left(\overline{z}\right),\beta_{-j}\left(v_{-j}\right)\right) \ \mid \beta_j\left(\overline{z}\right)>\beta_i\left(v_i\right)] \left(\overline{z}-p^{win}\left(\beta_j\left(\overline{z}\right)\right)\right)$$
$$-\left(1-F_i\left(\underline{v}_i\left(\beta_j\left(\overline{z}\right)\right)\right) E_{v_{-j}}[x_j\left(\beta_j\left(\overline{z}\right),\beta_{-j}\left(v_{-j}\right)\right) \ \mid \beta_j\left(\overline{z}\right)>\beta_i\left(v_i\right)]\right)p^{lose}\beta_j\left(\overline{z}\right)-\epsilon.$$

Hence, we have shown that for every $\epsilon > 0$ there exists a deviating bid b such that bidder i can achieve an expected utility of at least $U_j^{\beta}\left(\overline{z}\right)-\epsilon$ from which follows that

$$U_i^{\beta}\left(\overline{z}\right) \geq U_j^{\beta}\left(\overline{z}\right).$$

□

We continue with the proof of Proposition 5 by considering the following three cases.

Case 1: First, we will consider the case that there exists an interval (v', v) such that $\beta_j\left(z\right) > \beta_i\left(z\right)$ for all $z \in (v', v)$. In this case let

$$\overline{z} = \inf\{z > v \mid \beta_i\left(z\right) \geq \beta_j\left(v\right)\}.$$

If the infimum does not exist, we redefine $\overline{z} = \overline{v}$.

In both cases it follows from Lemma 13 that

$$\int_0^{\overline{z}} X_j^{\beta}\left(z\right) - X_i^{\beta}\left(z\right) dz = U_j^{\beta}\left(\overline{z}\right) - U_i^{\beta}\left(\overline{z}\right) \leq \delta_{i,j}\overline{z}.$$

It holds that

$$\int_0^{v} X_j^{\beta}\left(z\right) - X_i^{\beta}\left(z\right) dz = \int_0^{\overline{z}} X_j^{\beta}\left(z\right) - X_j^{\beta}\left(z\right) dz - \int_v^{\overline{z}} X_j^{\beta}\left(z\right) - X_i^{\beta}\left(z\right) dz$$
$$= \int_0^{\overline{z}} X_j^{\beta}\left(z\right) - X_i^{\beta}\left(z\right) dz + \int_v^{\overline{z}} X_i^{\beta}\left(z\right) - X_j^{\beta}\left(z\right) dz$$
$$\leq \delta_{i,j}\overline{z} + \int_v^{\overline{z}} X_i^{\beta}\left(z\right) - X_j^{\beta}\left(z\right) dz$$

$$\leq \delta_{i,j}\overline{z} + \delta_{i,j}\left(\overline{z} - v\right).$$

Due to $\beta_j(z) > \beta_i(z)$, for all $z \in (v, \overline{z})$, the last inequality follows from Lemma 12.

We conclude that the assumption that

$$\int_0^v X_j^\beta(z) - X_i^\beta(z)\, dz > \delta_{i,j}\overline{v} + \delta_{i,j}\left(\overline{v} - v\right)$$

leads to a contradiction.

Case 2: Second, we consider the case that there exists an interval (v', v) such that $\beta_i(z) > \beta_j(z)$ for all $z \in (v', v)$. It follows from Lemma 13 that

$$U_i^\beta(v') + \delta_{i,j}v' \geq U_j^\beta(v'). \tag{18}$$

Therefore, it holds that

$$\int_0^{v'} X_j^\beta(z) - X_i^\beta(z)\, dz \leq \delta_{i,j}v'.$$

It follows from the fact that $\beta_i(z) > \beta_j(z)$ for all $z \in (v', v)$ and from Lemma 12 that

$$\int_{v'}^v X_j^\beta(z) - X_i^\beta(z)\, dz \leq \delta_{i,j}\left(v - v'\right).$$

Therefore, we can conclude that

$$\int_0^v X_j^\beta(z) - X_i^\beta(z)\, dz = \int_0^{v'} X_j^\beta(z) - X_i^\beta(z)\, dz + \int_{v'}^v X_j^\beta(z) - X_i^\beta(z)\, dz$$

$$\leq \delta_{i,j}v' + \delta_{i,j}\left(v - v'\right) \leq \delta_{i,j}\overline{v} + \delta_{i,j}\left(\overline{v} - v\right)$$

which leads to a contradiction.

Case 3: Finally, we consider the case that there exists an interval (v', v) such that $\beta_i(z) = \beta_j(z)$ for all $z \in (v', v)$. Since the bidding functions of bidders i and j are continuous except a measure zero set of valuations, case 1, 2 and 3 constitute all possible cases. It follows from Lemma 13 that

$$U_i^\beta(v) + \delta_{i,j}v \geq U_j^\beta(v). \tag{19}$$

Thus, it holds that

$$\int_0^v X_j^\beta(z) - X_i^\beta(z)\, dz \leq \delta_{i,j}v \leq \delta_{i,j}\overline{v} + \delta_{i,j}\left(\overline{v} - v\right)$$

which leads to a contradiction.

We conclude that for every possible case the assumption that

$$\int_0^v X_j^\beta(z) - X_i^\beta(z)\, dz > \delta_{i,j}\overline{v} + \delta_{i,j}\left(\overline{v} - v\right)$$

leads to a contradiction which completes the proof of the claim in (15) and hence the proof of Proposition 5 for pure strategy equilibria.

It is left to consider the case of mixed equilibria. Let $\beta = (\beta_1, \ldots, \beta_n)$ be a (possibly mixed equilibrium). According to Lemma 8, there exists a pure strategy equilibrium $\beta' = (\beta'_1, \ldots, \beta'_n)$ such that it holds for all $i \in \{1, \ldots, n\}$ and for all $v \in \{1, \ldots, n\}$ that

$$X_i^{\beta}(v) = X_i^{\beta'}(v)$$

except a set of valuations which has measure zero. Since we have shown Proposition 5 for pure strategy equilibria, it holds that

$$\left| \int_0^v X_j^{\beta'}(z)\, dz - \int_0^v X_i^{\beta'}(z)\, dz \right| \leq \delta_{i,j}\overline{v} + \delta_{i,j}(\overline{v} - v).$$

Thus, for every $v \in [0, \overline{v}]$ it holds that

$$|U_i^{\beta}(v) - U_j^{\beta}(v)| = \left| \int_0^v X_j^{\beta}(z)\, dz - \int_0^v X_i^{\beta}(z)\, dz \right|$$

$$= \left| \int_0^v X_j^{\beta'}(z)\, dz - \int_0^v X_i^{\beta'}(z)\, dz \right| \leq \delta_{i,j}\overline{v} + \delta_{i,j}(\overline{v} - v).$$

This completes the proof.

□

2.7.9 Proof of and Proposition 6

Proof. Let A be a mechanism and i a bidder with valuation v. Let β^{δ} denote an equilibrium of mechanism A for a given valuation of δ and let $U_i\left(v, \beta^{\delta}\right)$ denote the expected utility of bidder i with valuation v in the equilibrium β^{δ}. If δ equals to zero, then we can deduce from the Revenue Equivalence Theorem (e.g. as stated in Krishna (2009)) that for every mechanism A and every strictly increasing equilibrium β of A the expected utility of bidder i with valuation v is given by

$$U_i^{\beta}(v) = vX_i^{\beta}(v) - P_i^{\beta}(v) = vG(v) - \int_0^v zg(z)\, dz$$

where $G(v) = H^{n-1}(v)$ and $g(v)$ denotes the corresponding density. That is, for $\delta = 0$ the expression $U_i\left(v, \beta^{\delta}\right)$ neither depends on the mechanism nor on the equilibrium (if the equilibrium is strictly increasing which we assume) and therefore can be denoted by $U_i(v, 0)$.

It holds that

$$\sum_{i=1}^{n} \frac{d}{d\delta} U_i\left(v, \beta^{\delta}\right)\bigg|_{\delta=0} = \sum_{i=1}^{n} \frac{d}{d\delta} \int_0^v X_i\left(z, \beta^{\delta}\right) dz\bigg|_{\delta=0}.$$

According to Lemma 1 in Fibich et al. (2004) this is equal to

$$\int_0^v (n-1)\, H^{n-2}(z) \sum_{i=1}^n H_i(z)\, dz.$$

As in the proof of Theorem 1 in Fibich et al. (2004), we use the Taylor series in order to conclude that for a given equilibrium β and a given δ it holds that

$$\sum_{i=1}^n U_i^\beta(v) = \sum_{i=1}^n U_i\left(v, \beta^\delta\right) = \sum_{i=1}^n U_i\left(v, \beta^0\right) + \delta \frac{d}{d\delta} \sum_{i=1}^n U_i\left(v, \beta^\delta\right) \Big|_{\delta=0} + \mathcal{O}\left(\delta^2\right)$$

$$= \sum_{i=1}^n U_i(v, 0) + \delta \int_0^v (n-1)\, H^{n-2}(z) \sum_{i=1}^n H_i(z)\, dz + \mathcal{O}\left(\delta^2\right)$$

where the term $\mathcal{O}\left(\delta^2\right)$ may depend on the particular mechanism while

$$\mathcal{U}(v) := \sum_{i=1}^n U_i(v, 0) + \delta \int_0^v (n-1)\, H^{n-2}(z) \sum_{i=1}^n H_i(z)\, dz$$

does not. It follows from Proposition 5 that for every $j \neq i$ it holds that

$$\left| U_j^\beta(v) - U_i^\beta(v) \right| \leq \delta_{i,j} + \delta_{i,j}(\overline{v} - v) \leq \delta + \delta(\overline{v} - v)$$

from which follows for every bidder i that

$$\left| \sum_{i=1}^n U_i^\beta(v) - nU_i^\beta(v) \right| \leq n\delta + n\delta(\overline{v} - v).$$

Hence it holds that

$$\left| \mathcal{U}(v) + \mathcal{O}\left(\delta^2\right) - nU_i^\beta(v) \right| \leq n\delta + n\delta(\overline{v} - v).$$

$$\Leftrightarrow nU_i^\beta(v) - n\left(\delta\overline{v} + \delta(\overline{v} - v)\right) \leq \mathcal{U}(v) + \mathcal{O}\left(\delta^2\right) \leq nU_i^\beta(v) + n\left(\delta\overline{v} + \delta(\overline{v} - v)\right)$$

$$\Leftrightarrow \frac{1}{n}\left(\mathcal{U}(v) + \mathcal{O}\left(\delta^2\right) - n\left(\delta\overline{v} + \delta(\overline{v} - v)\right)\right) \leq U_i^\beta(v) \leq \frac{1}{n}\left(\mathcal{U}(v) + \mathcal{O}\left(\delta^2\right) + n\left(\delta\overline{v} + \delta(\overline{v} - v)\right)\right).$$

Let B be a mechanism with equilibrium β'. Since the same statement holds for equilibrium β', it follows that

$$\left| U_i(v)^\beta - U_i(v)^{\beta'} \right| \leq 2\left(\delta\overline{v} + \delta(\overline{v} - v)\right) + \mathcal{O}\left(\delta^2\right).$$

□

2.7.10 Proof of Propositions 7 and 8

Proof. First, we will show Proposition 7 for pure strategy equilibria and afterwards apply Lemma 8 in order to derive the result for mixed strategy equilibria. We will show Proposition 7 for pure strategy equilibria by contradiction. Let $\beta = (\beta_1, \dots, \beta_n)$ be an efficient equilibrium of an imitation-perfect auction. Let bidders i and j be such that

$$\int_{[0,\overline{v}]^{n-1}} F_{-j}(z)\, dz > \int_{[0,\overline{v}]^{n-1}} F_{-i}(z)\, dz.$$

Since the equilibrium is efficient, it follows from Myerson (1981) that

$$U_j^{\beta}(\overline{v}) = \int_0^{\overline{v}} X_j(z)\, dz = \int_0^{\overline{v}} F_{-j}(z)\, dz > \int_0^{\overline{v}} F_{-i}(z)\, dz = \int_0^{\overline{v}} X_i(z)\, dz = U_i^{\beta}(\overline{v}). \quad (20)$$

According to Lemma 13, it holds that $U_j^{\beta}(\overline{v}) = U_i^{\beta}(\overline{v})$ which leads to a contradiction. This completes the proof for pure strategy equilibria. It is left to consider the case of mixed strategy equilibria. Let $\beta = (\beta_1, \dots, \beta_n)$ be a (possibly mixed equilibrium). According to Lemma 8, there exists a pure strategy equilibrium $\beta' = (\beta'_1, \dots, \beta'_n)$ such that it holds for all $i \in \{1, \dots, n\}$ and for all $v \in \{1, \dots, n\}$ that

$$X_i^{\beta}(v) = X_i^{\beta'}(v)$$

except a set of valuations which has measure zero. Since the equilibrium β' is efficient, for every pair of bidders i and j and for every pair of valuations v_i and v_j such that $v_j > v_i$ it holds that $X_i^{\beta'}(v_i) < X_j^{\beta'}(v_j)$. Therefore, it holds that $X_i^{\beta}(v_i) < X_j^{\beta}(v_j)$ except a measure zero set of valuations. Conclusively, given equilibrium β, the bidder with the highest valuation wins with probability one and the same reasoning as above applies.

The proof of Proposition 8 works in the same way with the only difference being that valuations are replaced with the corresponding virtual valuations. Assume there exists a pure strategy equilibrium $\beta = (\beta_1, \dots, \beta_n)$ of an imitation-perfect auction such that the bidder with the highest virtual valuation wins with probability 1. Let bidders i and j be such that

$$\int_0^{\overline{v}} F_i\left(V_i^{-1}(V_j(z))\right) dz > \int_0^{\overline{v}} F_j\left(V_j^{-1}(V_i(z))\right) dz.$$

Since the bidder with the highest virtual valuation wins with probability 1, it follows from Myerson (1981) that

$$\begin{aligned} U_j^{\beta}(\overline{v}) = \int_0^{\overline{v}} X_j(z)\, dz &= \int_0^{\overline{v}} \prod_{k \neq j} F_k\left(V_k^{-1}(V_j(z))\right)(z)\, dz \\ &> \int_0^{\overline{v}} \prod_{k \neq i} F_k\left(V_k^{-1}(V_i(z))\right) dz = \int_0^{\overline{v}} X_i(z)\, dz = U_i^{\beta}(\overline{v}). \end{aligned}$$

As before, one can show that this leads to a contradiction since the expected utilities of bidders i and j at $\overline{v}$ are equal. The same reasoning as above applies to mixed strategy equilibria.

$\square$

3 Endogenous worst-case beliefs in first-price auctions

Bidding in first-price auctions crucially depends on the beliefs of the bidders about their competitors' willingness to pay. We analyze bidding behavior in a first-price auction in which the knowledge of the bidders about the distribution of their competitors' valuations is restricted to the support and the mean. To model this situation, we assume that under such uncertainty a bidder will expect to face the distribution of valuations that minimizes her expected utility, given her bid is an optimal reaction to the bids of her competitors induced by this distribution. This introduces a novel way to endogenize beliefs in games of incomplete information. We find that for a bidder with a given valuation her worst-case belief just puts sufficient probability weight on lower valuations of her competitors to induce a high bid. At the same time the worst-case belief puts as much as possible probability weight on the same valuation in order to minimize the bidder's winning probability. This implies that even though the worst-case beliefs are type dependent in a non-monotonic way, an efficient equilibrium of the first-price auction exists.

JEL classification: D44, D81, D82

Keywords: Auctions, mechanism design, beliefs, uncertainty

3.1 Introduction

Consider a company preparing a bid for a first-price procurement auction. Their optimal bidding strategy will crucially depend on their belief about the costs of their competitors. Typically, this company would spend a considerable amount of resources to reverse-engineer the products of their competitors and learn about their cost structure. However, such learning has its limits. For example, reverse-engineering may inform the company about the used components and the general complexities in producing this part. But it cannot inform about the production processes and the used equipment of its competitors. Thus, it is reasonable to assume that learning about the distribution of the competitors' costs is not perfect and just specifies some summary statistic of the underlying distribution like the support and the mean. How to weigh the probabilities of certain costs within this support is subjective and hard to objectify. Thus, in order to submit a bid in the auction, the company has to form a subjective belief.

In this paper we consider the problem of a bidder in a first-price auction whose only information about the valuations of her competitors is the support and the mean of their distribution. Given such a large uncertainty, it seems natural for this bidder to prepare for the worst case.[25] Thus, we assume that for a given bidding strategy of her competitors the

[25]From our own experience in consulting bidders in high-stakes (procurement) auctions, it is a typical approach taken by bidders to generate several scenarios with respect to the valuations (costs) of their competitors and than to tailor their strategy to the worst-case.

bidder will tailor her bid to be optimal given that she expects to face the worst distribution of of her competitors' valuations among all distributions with the same support and mean. Worst distribution, in this context, means the bidder will expect to face the distribution of valuations that minimizes her expected utility, given her bid is an optimal reaction to the bids of her competitors induced by this distribution. In other words, the worst-case belief of a bidder minimizes her maximum possible expected utility. We assume that every bidder in the auction follows a similar logic when preparing her bid. In this case, a profile of bids is an equilibrium if each bidder chooses her optimal bid given her valuation (type), the bidding strategy of her competitors, and the worst-case belief as defined above. In particular, this implies that the worst-case belief of a bidder will crucially depend on her type (valuation) in a non-monotonic way.

Our contribution is threefold. Our first contribution is to introduce a novel way to model endogenous beliefs in a first-price auction. Endogenous, in this case, means that a bidder's beliefs about the valuations of the competitors are not assumed as a primitive of the environment but arise naturally as worst-case beliefs from the game induced by the rules of the first-price auction.[26] This can be viewed as a relaxation of the paradigm of symmetric independent private value (IPV) auctions that each bidder's valuation for the object is drawn from a distribution that is common knowledge among all bidders. Moreover, our solution concept constitutes a novel way to analyze games with asymmetric information and can be straightforwardly extended to any kind of such game.

Our second contribution is to show that even though the endogenous beliefs that arise from our solution concept are type dependent in a non-monotonic way, an ex-post efficient equilibrium exists. That is, even though the worst-case beliefs of bidders with a higher valuation do not imply that they believe to face a stronger competition in the auction than bidders with a lower valuation, in equilibrium the object is allocated with probability one to the bidder with the highest valuation.

Our third contribution is to introduce a novel proof method that we use in order to derive the worst-case strategies and beliefs in the efficient equilibrium. The method encompasses an elegant way to compare the solutions of an infinite set of minimization problems. To fix ideas and to gain some intuition for our results, consider the case that the valuation of a bidder can take one of three valuations 0, θ and 1. Suppose furthermore that it is common knowledge among the bidders that the mean of the distribution of valuations is μ with $\mu < \theta$. In this case, the efficient equilibrium takes the following form: all bidders with valuation 0 bid 0, all bidders with valuation θ mix between 0 and some $\bar{b}_\theta$, and all bidders with a valuation of 1 mix between $\bar{b}_\theta$ and some $\bar{b}_1$. The beliefs of a bidder with valuation 0 are arbitrary as she will always bid 0 and expect a utility of 0. A bidder with valuation θ believes that she is facing only bidders with valuations 0 and θ with probabilities such that the mean of her belief is μ. A bidder with valuation 1 believes that she is facing bidders with valuations 0, θ, and 1 with probabilities such that she is indifferent between

[26] A different auction format would generate different worst-case beliefs.

mixing in $\left[\bar{b}_\theta, \bar{b}_1\right]$ and bidding 0 and such that the mean of her belief is μ. Given their beliefs, all bidders best reply to the bidding strategies of their competitors. Given the bidding strategies, the beliefs make each bidder worst off given her type. It may appear counterintuitive that, given her bid, the worst-case scenario for a bidder with valuation θ is that she is the strongest bidder. However, given the bidding strategies in the efficient equilibrium, the utility of a bidder with a valuation of θ depends only on the probability that she is facing bidders with a valuation of 0. Given that the mean of the belief is fixed, this probability is minimized if the probability of facing bidders with a valuation of 1 is zero. In other words, for a bidder with a valuation of θ it is the worst-case to that the probability that she will face only bidders with a valuation of 0, against whom she will win for sure, is minimized.

For bidders with a valuation of 1, the worst-case is determined by minimizing her winning probability while keeping the incentives intact to bid above $\bar{b}_\theta$. Thus, the belief of a bidder with a valuation of 1 puts just enough probability weight on 0 and θ such that she will bid above the highest bid of a bidder with a valuation of θ and then as much probability as possible on 1.

The intuitions from the case with three types carry over to the general model. In particular, the worst-case belief of a bidder with a given valuation just puts enough probability weight on lower valuations to induce that for this bidder it is optimal to outbid each bidder with a lower valuation. The remaining probability weight is put on the valuation of the bidder in question in order to minimize her winning probability. It follows directly that such beliefs induce bidding that leads to an efficient allocation.

In order to show that the proposed strategies indeed constitute an equilibrium with worst-case beliefs, it remains to show that there is no other belief that would induce a bid that would make a bidder worse off than in the proposed equilibrium. For this we introduce a novel proof method. The underlying idea of the proof is to show that we can switch from comparing different beliefs and their induced utilities to comparing different bids and their induced utilities. This is due to the fact that a given best reply b can be induced by a multitude of beliefs (given the bidder's valuation and the other bidders' strategies). It follows that every bid b can be identified by a minimization problem: among all distribution functions with mean μ which induce bid b as a best reply it suffices to consider the belief which leads to the minimum utility. Using this concept, we can map every bid to a belief and a corresponding utility. Therefore, checking whether the utility induced by b is lower than the utility induced by some other b' establishes a transitive total order on the set of bids.

We use three different tools with which we can compare different bids with respect to the introduced transitive order. The first tool is to show that for certain types there exists only one distribution which induces a particular bid. This allows to directly compute the minimum expected utility which can be induced by this bid for these types. The second tool constitutes a connection between binding constraints in the minimization problem corresponding to a bid b and bids which are lower than b with respect to our order. Third, we show that for a given type there exist bids which can never be a best reply independent of the subjective belief. Hence, these bids cannot be possible deviations from the proposed worst-case equilibrium for the given type. Using these three tools, we construct a chain where all bids are arranged with respect to our order and the efficient equilibrium bids are the lowest. Due to the transitivity of our relation, this excludes exclude all other bids as possible deviations from the proposed equilibrium strategy.

Besides specifying an efficient worst-case belief equilibrium, we provide a comparison of expected revenues of a second-price auction and a first-price auction under endogenous worst-case beliefs for the case where bidders can have three discrete valuations 0, θ and 1. We show that for certain parameter constellations of θ and μ the first-price or the second-price auction perform better in terms of expected revenue independent of the true valuation distribution. There also exist parameters θ and μ such that the revenue-maximizing choice of the auction format depends on the true valuation distribution.

The remainder of the paper is organized as follows. We conclude the introduction with an overview over the related literature. The second section contains the formal model including the formal description of our solution concept, the worst-case belief equilibrium. In the third section we show the existence of an efficient worst-case belief-equilibrium and derive the corresponding beliefs and strategies for the special case of two bidders and three types. We consider this special case in order to focus on the intuition of the results and to illustrate the techniques of our proof. In the fourth section we conduct the revenue comparison between the first-price and the second-price auction under endogenous worst-case beliefs for the case of two bidders and three valuations. The fifth section contains the formal model and an outline of the proof for the general case with an arbitrary number of bidders and discrete valuations. We conclude in section six and section seven provides an overview over the most used notation and definitions. The appendix contains the proofs not provided in previous sections. We provide all proofs in the Appendix for the case of two bidders and three valuations and the general case separately. The proofs for the special case are provided in order to give an intuition for the general case. However, the model for the general case in the fifth section as well as the proofs for the general case can be also understood without reading the special case first.

Relation to the literature

Our paper complements two strands of literature: the literature on robust auction design and the literature on first-price auctions with non-standard priors. Both strands of literature relax the typically strict assumptions that are placed on the beliefs of the designer and the participants of an auction.

Contrary to the literature on robust auction design that focuses on the problem of the designer who does not have precise beliefs about the bidders, we focus on the problem of the bidder who does not have precise beliefs about her competitors. Departing from the ideas posed in this literature, we propose that not only the designer may be uninformed about the environment but also the bidders, if they do not interact frequently, may have some uncertainty. We then use modeling techniques developed in this literature and develop a novel solution concept to analyze this problem. For example, Bergemann and Schlag (2008) consider optimal monopoly pricing under uncertainty about demand distribution with a seller who either maximizes worst-case expected utilities or minimizes the maximal regret. They find that the optimal pricing policy hedges against uncertainty by randomizing over a range of prices. Buyers with low valuations cannot generate substantial regret and are priced out of the market. Bergemann and Schlag (2011) consider a robust version of the classic problem of optimal monopoly pricing with incomplete information. The seller faces model uncertainty and only knows that the true demand distribution is in the neighborhood of a given model distribution. They find that the equilibrium price under either criterion is lower then in the absence of uncertainty. The concern for robustness leads the seller to concede a larger information rent to all buyers with valuations below the optimal price without uncertainty. Carrasco et al. (2018) analyze the optimal selling mechanism if the seller maximizes worst-case expected profits and is only informed about one moment of the distribution of the buyer's valuations. They show that the optimal mechanism entails distortions at the intensive margin, e.g., except for the highest valuation buyer, sales will take place with probability strictly smaller than one. The seller can implement such allocation by committing to post prices drawn from a non-degenerate distribution, so that randomizing over prices is an optimal robust selling mechanism. Brooks (2016) considers the mechanism design problem of a seller who is uninformed about demand, while potential buyers are well-informed. The seller's goal is to maximize the minimum ratio between expected revenue and the expected efficient utility. He characterizes simple mechanisms that maximize the minimum extraction ratio. In these mechanisms, the seller runs a second-price auction and simultaneously surveys the beliefs of buyers about other's valuations. Carroll (2015) considers a moral hazard problem where the principal is uncertain what the agent can and cannot do: She knows some actions available to the agent, but other, unknown actions may also exist. The principal demands robustness, evaluating possible contracts by their worst-case performance, over unknown actions the agent might potentially take. He finds that the optimal contract from the point of view of the principal is linear.

The literature on first-price auctions with non-standard priors relaxes the assumptions placed on the priors of the bidders by the standard IPV model. For example, Fang and Morris (2006) consider parametric examples of symmetric two-bidder private valuation auctions in which each bidder observes her own private valuation as well as noisy signals about her opponent's private valuation. They show that in such environments the revenue equivalence between the first and second-price auction (SPA) breaks down and there is no definite revenue ranking; while the SPA always allocates efficiently, the first price auction (FPA) may be inefficient; equilibria may fail to exist for the FPA. Kim and Che (2004) study auctions in which bidders may know the types of some rival bidders but not others. They show that the first-price auction results in an inefficient allocation and that this inefficient allocation translates into a poor revenue performance. Bergemann et al. (2017) characterize the set of all possible outcomes that may arise in a first-price auction under any given information structure among the bidders. They find that revenue is maximized when buyers know who has the highest valuation, but the highest valuation buyer has partial information about others' valuations. Revenue is minimized when buyers are uncertain about whether they will win or lose and incentive constraints are binding for all upward bid deviations. Contrary to this literature, we do not assume an exogenously given prior but rather introduce a novel way to model endogenous beliefs that will depend on the specific game structure. We find, in contrast to most findings in this literature, that the first-price auction allocates efficiently.

3.2 Model

3.2.1 Setup

There are n risk-neutral bidders competing in a first-price sealed-bid auction for one indivisible object. Before the auction starts, each bidder $i \in \{1, \dots, n\}$ privately observes her valuation (type) $\theta_i \in \Theta = \{0 = \theta^1, \theta^2, \dots, \theta^{m-1}, 1 = \theta^m\}$. The valuation distributions are unknown to the bidders. However, it is common knowledge among the bidders that the mean of this distribution is μ. Hence, every bidder knows that the probability mass function of the other bidders' valuations is an element from

$$\mathscr{F}_\mu^{n-1} = \left\{ f_1 \times \cdots \times f_{n-1} : \Theta^{n-1} \to [0,1] \;\middle|\; \sum_{j=1}^{m} \theta^j f_i\left(\theta^j\right) = \mu \text{ for all } i \in \{1, \dots, n-1\} \right\},$$

where for every $i \in \{1, \dots, n-1\}$ and every $j \in \{1, \dots, m\}$, $f_i(\theta^j)$ denotes the probability with which valuation θ^j occurs according to the probability mass function f_i. In other words, this is the set of all probability mass functions of independently drawn valuations from the set Θ for $n-1$ bidders with mean μ. For a shorter notation we will use the term probability function instead of probability mass function.

In the auction the bidders submit bids, the bidder with the highest bid wins the object and pays her bid. In addition, we assume an efficient tie-breaking rule[27]. Thus, the utility of bidder i with valuation θ_i and bid b_i given that the other bids are b_{-i} is denoted by [28]

[27] We assume an efficient tie-breaking rule since it simplifies notation. With a random tie-breaking rule one would need to assume a discrete bid grid (which may be arbitrarily fine) in order to ensure equilibrium existence. However, the equilibrium strategies under both tie breaking rules would differ by at most one bid step in the bid grid.

[28] For a vector $(v_1, \dots, v_n)$ we denote by v_{-i} the vector $(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$.

$$u_i(\theta_i, b_i, b_{-i}) = \begin{cases} \theta_i - b_i & \text{if } b_i > \max\limits_{j\neq i} b_j \\ \theta_i - b_i & \text{if } b_i = \max\limits_{j\neq i} b_j \text{ and } \theta_i > \max\limits_{j\neq i}\{\theta_j \mid b_j = b_i\} \\ 0 & \text{if } b_i = \max\limits_{j\neq i} b_j \text{ and } \theta_i < \max\limits_{j\neq i}\{\theta_j \mid b_j = b_i\} \\ \frac{1}{k}(\theta_i - b_i) & \text{if } b_i = \max\limits_{j\neq i} b_j \text{ and } \theta_i = \max\limits_{j\neq i}\{\theta_j \mid b_j = b_i\} \\ 0 & \text{if } b_i < \max\limits_{j\neq i} b_j \end{cases}$$

where θ_j denotes the valuation of bidder j with bid b_j for $j \in \{1, ..., n\}$ and $k = \#\{\max\{\theta_j | b_j = b_i\}\}$.

A *(mixed) strategy* β_i of a bidder i maps the valuation (type) of a bidder to a distribution of bids:

$$\beta_i : \Theta \to \Delta\mathbb{R}^+$$

$$\theta_i \mapsto \beta_i(\theta_i)$$

where $\Delta\mathbb{R}^+$ is the set of all probability distributions on $\mathbb{R}^+$. For bidder i with valuation θ_i it is a cumulative distribution function of bids, denoted by $G^{\beta_i}_{\theta_i}$ with corresponding density $g^{\beta_i}_{\theta_i}$ and support *supp* $(\beta_i(\theta_i))$. A *pure strategy* of bidder i with valuation θ_i is a mapping

$$\beta_i : \Theta \to \mathbb{R}^+$$

$$\theta_i \mapsto \beta_i(\theta_i),$$

i.e. this is a mapping from the set of valuations to the set of bids.[29] The expected utility of a bidder i with valuation θ_i, belief $f_{-i} \in \mathscr{F}^{n-1}_{\mu}$ and bid b_i given that her competitors employ bidding strategies β_{-i} can be written as

$$U_i(\theta_i, f_{-i}, b_i, \beta_{-i}) = \int_{\theta_{-i}} \int_{b_{-i}} u_i(\theta_i, b_i, b_{-i}) \prod_{j\neq i} g^{\beta_j}_{\theta j}(b_j) d\theta_{-j} f_{-i}(\theta_{-i}) d\theta_{-i}.$$

3.2.2 Solution Concept

We are interested in the bidding behavior of a bidder who apart from the support and mean has no information about the distribution of the valuations of her competitors. Thus, in order to derive a bid, the bidder has to form a subjective belief. We assume that this bidder will prepare for the worst case. Prepare means that the bidder will choose her optimal bid given she expects to face the worst-case distribution of valuations. That is, the bidder will expect to face the distribution of valuations that minimizes her expected

[29] A pure strategy can be interpreted as distribution of bids which puts probability weight 1 on one bid. We abuse notation since in the case of a pure strategy, $\beta_i(\theta_i)$ denotes an element in $\mathbb{R}^+$ while in the case of a (mixed) strategy $\beta_i(\theta_i)$ denotes an element in $\Delta\mathbb{R}^+$. However, in the following it will be clear whether β_i is a pure or a mixed strategy. In addition, we will also use the notation $G^{\beta_i}_{\theta_i}$ instead of $\beta_i(\theta_i)$ in case of mixed strategies.

utility, given her bid is an optimal reaction to the bids of her competitors induced by this distribution and their bidding strategy. We will introduce the concept in several steps. First, we define the best reply of bidder i to a given belief f_{-i} and a given bidding strategy of the competitors β_{-i}. Second, we introduce the worst-case belief for a given bidding strategy of the competitors β_{-i}. That is, we derive the belief that minimizes the expected utility of bidder i given her best reply to this belief and the bidding strategy of her competitors. Third, we will define the worst-case belief equilibrium in which each type of each bidder bids the optimal bid given her worst-case belief and the bidding strategy of her competitors.

Best reply to a belief and the competitors' strategies For bidder i with valuation θ_i and for each belief f_{-i} about the other bidders' valuations and bidding strategies β_{-i}, the set of *best replies* of bidder i is given by

$$B_i^r\left(\theta_i, f_{-i}, \beta_{-i}\right) = \underset{b_i}{\operatorname{argmax}}\ U_i\left(\theta_i, f_{-i}, b_i, \beta_{-i}\right).$$

Bidder i's best reply induces an expected utility of

$$U\left(\theta_i, f_{-i}, b_i^r\left(\theta_i, f_{-i}, \beta_{-i}\right), \beta_{-i}\right)$$

for $b_i^r\left(\theta_i, f_{-i}, \beta_{-i}\right) \in B_i^r\left(\theta_i, f_{-i}, \beta_{-i}\right)$.

Worst-case belief given a best reply and the competitors' strategies As argued before, we will assume that a bidder prepares for the worst case, i.e. she will assume that the distribution of her competitors' valuations induces the worst utility given her best reply and the bidding strategy of her competitors. Since after forming a belief, a bidder will choose an optimal bid given this belief, a distribution induces the worst outcome for a bidder if it minimizes the expected utility of a bidder given her optimal bid. That is, the worst-case belief minimizes the maximum expected utility of the bidder. Formally, a worst-case belief $f_{-i}^{\theta_i}$ of bidder i with valuation θ_i is given by

$$\begin{aligned} f_{-i}^{\theta_i} &= \underset{f_{-i}\in\mathscr{F}_\mu^{n-1}}{\operatorname{argmin}}\ U_i\left(\theta_i, f_{-i}, b^r\left(\theta_i, f_{-i}, \beta_{-i}\right), \beta_{-i}\right) \\ &= \underset{f_{-i}\in\mathscr{F}_\mu^{n-1}}{\operatorname{argmin}}\ \max_{b_i}\ U_i\left(\theta_i, f_{-i}, b_i, \beta_{-i}\right). \end{aligned}$$

Given the other bidders' strategies β_{-i}, a bidder i with type θ_i calculates her best reply to each belief in $\mathscr{F}_\mu^{n-1}$ and the corresponding utility. The worst-case belief of bidder i is the one inducing the lowest utility. In other words, the worst-case belief minimizes the maximum possible expected utility of a bidder given her valuation and the other bidders' strategies. Note that a worst-case belief is not necessarily unique but every worst-case belief yields the same utility.

Worst-case belief equilibrium In equilibrium, after forming a worst-case belief as described above, each bidder will choose an optimal bid given her valuation, her worst-case belief and the other bidders' strategies. That is, in equilibrium it has to hold for every valuation of every bidder that

(i) Given her valuation, her belief, and the other bidders' strategies the bid of a bidder maximizes her expected utility.

(ii) For every bidder there does not exists another belief such that a best reply to this belief induces a lower expected utility.

This leads to the following definition.

Definition 8 (Worst-case belief equilibrium)**.** *A profile of bidding strategies* $(\beta_1, \ldots, \beta_n)$ *together with a profile of beliefs* $\left([f_{-1}^{\theta^1}, \ldots, f_{-1}^{\theta_{m-1}}, f_{-1}^{\theta^m}], \ldots, [f_{-n}^{\theta^1}, \ldots, f_{-n}^{\theta_{m-1}} f_{-n}^{\theta^m}]\right) \in \left(\mathscr{F}_\mu^{n-1}\right)^m$ *form a* worst-case belief equilibrium *if for all* $i \in \{1, ..., n\}$*, all* $\theta_i \in \Theta$*, all* $f_{-i} \in \mathscr{F}_\mu^{n-1}$ *and all* $b_i \in supp\,(\beta_i(\theta_i))$ *it holds that*

$$b_i \in B_i^r\left(\theta_i, f_{-i}^{\theta_i}, \beta_{-i}\right) \tag{21}$$

and

$$U_i\left(\theta_i, f_{-i}^{\theta_i}, b_i, \beta_{-i}\right) \leq U_i\left(\theta_i, f_{-i}, b^r\left(\theta_i, f_{-i}, \beta_{-i}\right), \beta_{-i}\right). \tag{22}$$

In the following we will refer to the first condition as the best-reply condition *and to the second condition as the* worst-case belief condition*.*

3.3 Worst-case belief equilibria: two bidders, three valuations

This section focuses on our main result which states that an efficient worst-case belief equilibrium exists. We characterize the beliefs and strategies in the worst-case belief equilibrium and illustrate the techniques of our proof. We start our analysis with the case of two bidders, A and B and and three possible valuations $0, \theta$ and 1. This allows us to focus on the main features of the concept without complex notation. The general case with n bidders and m types is analyzed in section 3.5.

3.3.1 Efficient worst-case belief equilibrium

Theorem 2. *In a first-price auction there exists an efficient worst-case belief equilibrium.*

In order to prove the existence of an efficient worst-case equilibrium, we specify a profile of increasing strategies and beliefs and show that they constitute a worst-case belief equilibrium. The underlying idea of the proof is to show that we can switch from comparing different beliefs and their induced utilities to comparing different bids and their induced utilities. This is due to the fact that a given best reply b can be induced by a multitude of beliefs (given the bidder's valuation and the other bidders' strategies). It follows that every bid b can be identified by a minimization problem: among all distribution functions with mean μ which induce bid b as a best reply it suffices to consider the belief which leads to the minimum utility. Using this concept, we can map every bid to a belief and a corresponding utility. Therefore, checking whether the utility induced by b is lower than the utility induced by some other b' establishes a transitive total order on the set of bids.

We use three different tools with which we can compare different bids with respect to the introduced transitive order. The first tool is to show that for certain types there exists only one distribution which induces a particular bid. This allows to directly compute the minimum expected utility which can be induced by this bid for these types. The second tool constitutes a connection between binding constraints in the minimization problem corresponding to a bid b and bids which are lower than b with respect to our order. Third, we show that for a given type there exist bids which can never be a best reply independent of the belief. Hence, these bids cannot be possible deviations from the proposed worst-case equilibrium for the given type. Using these three tools, we construct a chain where all bids are arranged with respect to our order and the efficient equilibrium bids are the lowest. Due to the transitivity of our relation, this excludes all other bids as possible deviations from the proposed worst-case strategy.

We start with the formal description of the strategies and beliefs we claim to constitute a worst-case belief equilibrium. We will consider two possible cases: $\theta \leq \mu$ and $\theta > \mu$.

3.3.2 Characterization of the efficient worst-case belief equilibrium for $\theta \leq \mu$

We start with the simpler case $\theta \leq \mu$ and claim that the following strategies and beliefs constitute a worst-case belief equilibrium. The proof of this claim is provided in section 3.3.3. Since both bidders will have symmetric beliefs and strategies, we omit the identity of the bidder in the notation of beliefs and strategies.

We denote the strategy which we claim to be played in a worst-case belief equilibrium by β^*. We define

$$\beta^*(0) = 0, \quad \beta^*(\theta) = \theta, \quad \beta^*(1) = G_1^*. \tag{23}$$

That is, a bidder with valuation zero bids zero, a bidder with valuation θ bids θ and a bidder with valuation 1 plays a mixed strategy on the interval $[\theta, \bar{b}_1]$ according to a continuous bid distribution G_1^*. We will calculate G_1^* and the exact valuation of $\bar{b}_1$ further below. One can immediately see that these strategies constitute an efficient equilibrium, that is, the bidder with the highest valuation wins the auction with probability 1.

We will denote the belief which we claim to constitute a worst-case belief equilibrium together with the strategies specified above, by $f^{\hat{\theta},*} = \left(f_0^{\hat{\theta},*}, f_\theta^{\hat{\theta},*}, f_1^{\hat{\theta},*}\right)$ for $\hat{\theta} \in \{0, \theta, 1\}$.[30] That is, $f_0^{\hat{\theta},*}$ denotes the probability with which bidder A with valuation $\hat{\theta}$ believes that

[30]In the following we will refer to β^* and $f^{\hat{\theta}^*}$ as the worst-case strategy and the worst-case beliefs.

bidder B has valuation zero (and analogously for other valuations and bidder B).

The subjective worst-case beliefs are defined as follows. Type zero can have any belief from the set $\mathscr{F}_\mu^{n-1}$. A bidder with valuation θ has the subjective worst-case belief that the probability weight in the other bidder's probability function is solely distributed between valuations θ and 1. Since probabilities have to add up to one and the mean has to be preserved, it must hold that

$$f_0^{\theta,*} + f_\theta^{\theta,*} + f_1^{\theta,*} = 1$$

and

$$f_0^{\theta,*}0 + f_\theta^{\theta,*}\theta + f_1^{\theta,*}1 = \mu.$$

In the following we will refer to these two constraints as the first and second *probability constraint.* If it holds that $f_0^{\theta,*} = 0$, it follows from these constraints that

$$f_0^{\theta,*} = 0,\ f_\theta^{\theta,*} = \frac{1-\mu}{1-\theta},\ f_1^{\theta,*} = \frac{\mu-\theta}{1-\theta}. \tag{24}$$

We define the subjective worst-case belief of a bidder with valuation 1 to be the solution of the following minimization problem which we denote by $M_{<\theta}^1$:

$$\min_{(f_0,f_\theta,f_1)} \ (f_0 + f_\theta)$$

$$s.t.\ f_{\hat\theta} \geq 0 \text{ for all } \hat\theta \in \{0,\theta,1\}$$

$$f_0 + f_\theta + f_1 = 1$$

$$f_\theta\theta + f_1 = \mu$$

$$(f_0 + f_\theta)(1-\theta) \geq f_0.$$[31]

The second and third constraints are the above described probability constraints. The last constraint ensures that bidding θ is weakly better for a bidder with valuation 1 than bidding any lower bid given the other bidder's strategy.[32] That is, there is just enough probability weight on lower types in order to induce a bid of at least θ for type 1. It is sufficient to consider only a possible deviation to bid 0 because all bids in the interval $(0,\theta)$ are placed with zero probability and therefore are never best replies. Note that the feasible set of this minimization problem is not empty since the worst-case belief of type θ is an element of the feasible set.

In the case with three types such that $\theta \leq \mu$ the solution of minimization problem $M_{<\theta}^1$

[31] We use the expression "the solution" instead of "a solution" since we will show that this minimization problem has a unique solution. Also in the remainder of the paper we will use the term "the solution" in order to indicate that we will show that the particular minimization problem has a unique solution.

[32] In the following we will use the notation with subscript "$<$" like in $M_{<\theta}^1$ in order to indicate that a minimization problem does not contain all possible constraints but only the constraints which ensure that bidding a given bid is weakly better than bidding any lower bid.

can be obtained directly. Consider the minimization problem

$$\min_{(f_0, f_\theta, f_1)} (f_0 + f_\theta)$$

$$s.t.\ f_{\hat{\theta}} \geq 0 \text{ for all } \hat{\theta} \in \{0, \theta, 1\}$$

$$f_0 + f_\theta + f_1 = 1$$

$$f_\theta \theta + f_1 = \mu.$$

The solution of this minimization problem puts zero probability weight on type θ. Such a solution does not fulfill the constraint

$$(f_0 + f_\theta)(1 - \theta) \geq f_0.$$

Since this is the only constraint besides the probability constraints, this constraint has to be binding in minimization problem $M^1_{<\theta}$. Therefore, the solution of minimization problem $M^1_{<\theta}$ is the unique solution of a system of three linear equations with three unknowns. It holds that

$$f_0^{1,*} = \frac{1-\mu}{1+\theta},\ f_\theta^{1,*} = \frac{\theta(1-\mu)}{1-\theta^2},\ f_1^{1,*} = \frac{\mu - \theta^2}{1-\theta^2}. \tag{25}$$

Given the subjective worst-case belief of a bidder with valuation 1, one can compute the upper endpoint of her bidding interval, denoted by $\bar{b}_1$, and the bid distribution, denoted by G_1.[33] The upper endpoint of this bidding interval is defined by

$$\left(f_0^{1,*} + f_\theta^{1,*}\right)(1-\theta) = 1 - \bar{b}_1.$$

$$\Leftrightarrow \bar{b}_1 = f_1^{1,*} + \theta\left(f_0^{1,*} + f_\theta^{1,*}\right) = \frac{\mu - \theta^2 + \theta - \mu\theta}{1-\theta^2} = \frac{\theta + \mu}{1+\theta}.$$

The bid distribution is defined such that bidders A and B with valuation 1 make each other indifferent between any bid in their bidding interval. For every $s \in [\theta, \bar{b}_1]$ it holds that

$$\left(f_0^{1,*} + f_\theta^{1,*} + f_1^{1,*} G_1(s)\right)(1-s) = \left(f_0^{1,*} + f_\theta^{1,*}\right)(1-\theta). \tag{26}$$

3.3.3 Proving the best-reply and the worst-case belief condition for $\theta \leq \mu$

After specifying the worst-case beliefs and strategy, we have to show that these indeed constitute a worst-case belief equilibrium. That is, we have to show that the best-reply and the worst-case belief condition are fulfilled.

[33] Since according to the worst-case strategy the support of the bid distribution for every type is an interval (which may consists only of one point), we use the term "bidding interval" for the support of the bid distribution prescribed by the worst-case strategy for a given type.

Proposition 9. *Given the worst-case strategy as defined in (23) and the worst-case beliefs as defined in (24) and (25), it holds for all* $\hat{\theta} \in \{0, \theta, 1\}$ *that*

(i) *The best-reply condition given by*

$$b_{\hat{\theta}} \in B^r\left(\hat{\theta}, f^{\hat{\theta},*}, \beta^*\right) \text{ for all } b_{\hat{\theta}} \in supp\left(\beta^*\left(\hat{\theta}\right)\right)$$

is fulfilled, i.e. every bidder plays a best reply given her valuation, her worst-case belief and the other bidder's worst-case strategy.

(ii) *The worst-case belief condition is fulfilled, i.e. for all* $b_{\hat{\theta}} \in supp\left(\beta^*\left(\hat{\theta}\right)\right)$ *it holds that*

$$U\left(\hat{\theta}, f^{\hat{\theta},*}, b_{\hat{\theta}}, \beta^*\right) \leq U\left(\hat{\theta}, f, b^r\left(\hat{\theta}, f, \beta^*\right), \beta^*\right) \text{ for all } f \in \mathscr{F}_\mu.^{34}$$

That is, there does not exist another belief such that a best reply to this belief induces a lower expected utility than the worst-case belief.

Proof. Part *(i)*: Due to the symmetry of beliefs and strategies, it is sufficient to show the best-reply condition for bidder A. The result is obvious for a bidder with valuation zero. Given the subjective worst-case belief of bidder A with valuation θ as defined in (24) and bidder B's strategy, bidder A with valuation θ considers θ to be the lowest bid placed by bidder B. Therefore, she expects a utility of zero and bidding θ is a best reply. It follows from the definition of the worst-case belief of bidder A with valuation 1 as defined in (25) that she does not earn a higher expected utility by bidding any bid lower than θ. Bids in the interval $(0, \theta)$ are never played according to β^* and therefore cannot be a best reply. The constraint

$$(f_0 + f_\theta)(1 - \theta) \geq f_0$$

in minimization problem $M^1_{<\theta}$ ensures that bidding zero does not induce a higher expected utility than bidding θ. Since bidder B does not place bids above $\bar{b}_1$, it cannot be a best reply for bidder A to bid above $\bar{b}_1$. The bid distribution G_1 is constructed in a way which makes bidder A with valuation 1 indifferent between any bid in $[\theta, \bar{b}_1]$ which completes the proof of part *(i)*. □

The remainder of this section is dedicated to proving the worst-case belief condition. That is, for every type we have to consider all probability functions over the valuations 0, θ and 1 with mean μ and have to show that none of these probability functions induces a lower expected utility than the worst-case beliefs of the given type. Before we can complete the proof of part (ii), we need to introduce several proof techniques.

As a first step, we will introduce the concept of *minimizing probability functions* which enables us to switch from comparing the induced utility of probability functions to comparing

[34] Since utility functions are symmetric among bidders, we will omit the identitiy of the bidder in the notation of utility functions.

the induced utility of bids. Afterwards, we will introduce different tools with which we can compare the utility induced by different bids and therefore exclude bids as possible deviations from the proposed worst-case strategy.

Minimizing probability functions Consider the list of possible subjective beliefs from which bidder A chooses. Given the type of bidder A and bidder B's strategy, every probability function induces a best reply for bidder A. The best reply induces an expected utility:

$$\begin{array}{lll}
\text{probability function } f_B \rightarrow & \text{best reply } b^r\left(\theta_A, f_B, \beta_B\right) \rightarrow & \text{expected utility } U\left(\theta_A, f_B, b^r, \beta_B\right) \\
f_B^a & b^a\left(\theta_A, f_B^a, \beta_B\right) & U\left(\theta_A, f_B^a, b^a, \beta_B\right) \\
f_B^b & b^b\left(\theta_A, f_B^b, \beta_B\right) & U\left(\theta_A, f_B^b, b^b, \beta_B\right) \\
\vdots & \vdots & \vdots
\end{array}$$

Here $\theta_A \in \{0, \theta, 1\}$ denotes a valuation of bidder A and $f_B^a, f_B^b, \ldots$ denotes a list of probability functions of bidder B's valuations among which bidder A chooses her subjective worst-case belief. Note that different probability functions can induce the same best reply. Therefore, the list can be rearranged by grouping those probability functions together which induce the same best reply:

$$\begin{array}{lll}
\text{probability function } f_B \rightarrow & \text{best reply } b^r\left(\theta_A, f_B, \beta_B\right) \rightarrow & \text{expected utility } U\left(\theta_A, f_B, b^r, \beta_B\right) \\
f_B^a & b^a\left(\theta_A, f_B^a, \beta_B\right) & U\left(\theta_A, f_B^a, b^a, \beta_B\right) \\
f_B^{a'} & b^a\left(\theta_A, f_B^{a'}, \beta_B\right) & U\left(\theta_A, f_B^{a'}, b^a, \beta_B\right) \\
\vdots & \vdots & \vdots \\
 & & \\
f_B^b & b^b\left(\theta_A, f_B^b, \beta_B\right) & U\left(\theta_A, f_B^b, b^b, \beta_B\right) \\
f_B^{b'} & b^b\left(\theta_A, f_B^{b'}, \beta_B\right) & U\left(\theta_A, f_B^{b'}, b^{b'}, \beta_B\right) \\
\vdots & \vdots & \vdots
\end{array}$$

Among the probability functions which induce the same bid, it is sufficient to consider the probability functions which induce the minimum expected utility. That is, it is sufficient to select the probability functions inducing the minimum expected utility from each group and compare the induced utilities. Hence, we can switch from comparing probability functions to comparing bids. This is formalized in the following definition and observation which we provide for bidder A to simplify notation.

Definition 9. *For bidder A with valuation $\theta_A \in \{0, \theta, 1\}$, a bid b_A and the competitor's strategy β_B, the set of probability functions $\mathscr{F}^{min}\left(\theta_A, b_A, \beta_B\right)$ given by*

$$\mathscr{F}^{min}\left(\theta_A, b_A, \beta_B\right) = \underset{f_B \in \mathscr{F}_\mu}{\operatorname{argmin}} \left\{U\left(\theta_A, f_B, b_A, \beta_B\right) \mid b_A \in B^r\left(\theta_A, f_B, \beta_B\right)\right\}$$

is called the set of minimizing probability functions *of bid b_A for a bidder with valuation θ_A given the other bidder's strategy β_B. Among all probability functions which induce bid b_A as a best reply, a minimizing probability function is a probability function which induces the minimum utility.*

Observation 1. *Let β_K be a strategy of bidder K for $K \in \{A, B\}$ and $\left(f_B^0, f_B^\theta, f_B^1\right)$ be a profile of beliefs bidder A has about bidder B's valuation. For a valuation $\theta_A \in \{0, \theta, 1\}$ of bidder A and a bid $b_A \in supp\left(\beta_A(\theta_A)\right)$ the worst-case belief condition for bid b_A, given by*

$$U\left(\theta_A, f_B^{\theta_A}, b_A, \beta_B\right) \leq U\left(\theta_A, f_B, b^r\left(\theta_A, f_B, \beta_B\right), \beta_B\right)$$

for all $f_B \in \mathscr{F}_\mu$, is equivalent to the following two conditions:

(i) The belief $f_B^{\theta_A}$ is an element in $\mathscr{F}^{min}\left(\theta_A, b_A, \beta_B\right)$, i.e. a minimizing probability function of bid b_A for a bidder with valuation θ_A given B's strategy β_B.

(ii) Let b'_A be a bid and f_B be an element in $\mathscr{F}^{min}\left(\theta_A, b'_A, \beta_B\right)$, i.e. a minimizing probability function of bid b'_A for a bidder with valuation θ_A given β_B. Then it holds

$$U\left(\theta_A, f_B^{\theta_A}, b_A, \beta_B\right) \leq U\left(\theta_A, f_B, b'_A, \beta_B\right).$$

Clearly, a belief cannot be a worst-case belief of a given type if this belief induces a bid as a best reply for this type but there exists another belief which induces the same bid but with a lower expected utility. Therefore, a worst-case belief has to be a minimizing probability function for all bids in the support of bidder A's bidding strategy, as stated in the first condition of the observation. Moreover, for every type and every bid in the support of the given type there cannot exist another bid which induces a lower expected utility together with a minimizing probability function for this type and this bid, as stated in the second condition. In other words, if we group together all probability functions which induce the same bid and consider the minimizing probability function in every group, we can compare the expected utility induced by bids instead the expected utility induced by beliefs.

That is, it is sufficient to compare bids if we compare them with respect to the expected utility they induce together with their minimizing probability function. In order to apply this technique, we need the following definitions.

Definition 10. *For a bidder with valuation 1 minimization problem M_b^1 of a bid $b \in [\theta, \bar{b}_1]$ is the minimization problem corresponding to its minimizing probability functions, i.e. all solutions of minimization problem M_b^1 are minimizing probability functions of b for a bidder with valuation 1 given the other bidder's worst-case strategy β^*. Formally, minimization problem M_b^1 is given by*

$$\min_{(f_0, f_\theta, f_1)} \left(f_0 + f_\theta + f_1 G_1(b)\right)(1 - b)$$

$$s.t.\ f_{\hat{\theta}} \geq 0 \ for\ all\ \hat{\theta} \in \{0, \theta, 1\}$$

$$f_0 + f_\theta + f_1 = 1$$

$$f_\theta \theta + f_1 = \mu$$

$$(f_0 + f_\theta + f_1 G_1(b))(1-b) \geq (f_0 + f_\theta G_\theta(s))(1-s) \ \ for\ all\ s \in [0, \bar{b}_\theta]$$

$$(f_0 + f_\theta + f_1 G_1(b))(1-b) \geq (f_0 + f_\theta + f_1 G_1(s))(1-s) \ \ for\ all\ s \in [\bar{b}_\theta, \bar{b}_1].$$

In other words, among all probability functions which induce bid b for type 1 as a best reply, the solutions of minimization problem M_b^1 induce the minimum expected utility. Note that since bids above $\bar{b}_1$ are never a best reply, it is not necessary to include constraints which ensure that bidding b induces at least the same expected utility as bids above $\bar{b}_1$.

Definition 11. *Apart from the constraints*

$$f_{\hat{\theta}} \geq 0 \ for\ all\ \hat{\theta} \in \{0, \theta, 1\}$$

$$f_0 + f_\theta + f_1 = 1$$

$$f_\theta \theta + f_1 = \mu,$$

every constraint in minimization problem M_b^1 compares the utility of bidding b to the utility of bidding some other bid b', which is formalized by

$$U(1, f, b, \beta^*) \geq U(1, f, b', \beta^*).$$

We call such a constraint an incentive constraint corresponding to bid b'.

Definition 12. *For a type $\hat{\theta} \in \{0, \theta, 1\}$ and bids b, b' we use the notation $b \leq^{\hat{\theta}} b'$ if for the $\hat{\theta}$-type bid b' does not induce a strictly lower expected utility than bid b together with their minimizing probability functions given the other bidder's worst-case strategy β^*. Formally, let $f^{min}\left(\hat{\theta}, b, \beta^*\right) \in \mathscr{F}^{min}\left(\hat{\theta}, b, \beta^*\right)$ and $f^{min}\left(\hat{\theta}, b', \beta^*\right) \in \mathscr{F}^{min}\left(\hat{\theta}, b', \beta^*\right)$. Then it holds that*

$$U\left(\hat{\theta}, f^{min}\left(\hat{\theta}, b, \beta^*\right), b, \beta^*\right) \leq U\left(\hat{\theta}, f^{min}\left(\hat{\theta}, b', \beta^*\right), b', \beta^*\right) \ \Rightarrow b \leq^{\hat{\theta}} b',$$

$$U\left(\hat{\theta}, f^{min}\left(\hat{\theta}, b, \beta^*\right), b, \beta^*\right) < U\left(\hat{\theta}, f^{min}\left(\hat{\theta}, b', \beta^*\right), b', \beta^*\right) \ \Rightarrow b <^{\hat{\theta}} b'$$

and

$$U\left(\hat{\theta}, f^{min}\left(\hat{\theta}, b, \beta^*\right), b, \beta^*\right) = U\left(\hat{\theta}, f^{min}\left(\hat{\theta}, b', \beta^*\right), b', \beta^*\right) \ \Rightarrow b =^{\hat{\theta}} b'.$$

We also use the notation $b <^{\hat{\theta}} b'$ if b' does not have a minimizing probability function given $\hat{\theta}$ because it is never a best reply for a bidder with valuation $\hat{\theta}$, but b does have a minimizing probability function. We use the notation $b =^{\hat{\theta}} b'$ if neither b, nor b' have a minimizing probability function.

Given the notation provided in this definition and Observation 1, we can state a condition which is equivalent to the worst-case belief condition but is more tractable:

Observation 2. *The worst-case belief condition for a bidder with valuation $\hat{\theta} \in \{0, \theta, 1\}$, bid $b_{\hat{\theta}} \in supp\left(\beta^*\left(\hat{\theta}\right)\right)$ and the other bidder's strategy β^* given by*

$$U\left(\hat{\theta}, f^{\hat{\theta},*}, b_{\hat{\theta}}, \beta^*\right) \leq U\left(\hat{\theta}, f, b^r\left(\hat{\theta}, f, \beta^*\right), \beta^*\right) \text{ for all } f \in \mathscr{F}_\mu$$

is equivalent to

(i) $$f^{\hat{\theta},*} \in \mathscr{F}^{min}\left(\hat{\theta}, b_{\hat{\theta}}, \beta^*\right)$$

(ii) $$b_{\hat{\theta}} \leq^{\hat{\theta}} b' \text{ for all } b' \in [0, \overline{b}_1].$$

In order to apply this observation, we will make use of the fact that the relation $\leq^{\hat{\theta}}$ constitutes a transitive order which allows us to build chains of the form

$$b_{\hat{\theta}} \leq^{\hat{\theta}} b_1 \cdots \leq^{\hat{\theta}} b_k$$

and exclude all bids $b_1, \ldots, b_k$ as bids which could induce a lower expected utility.

After reframing the worst-case belief condition, we prove two lemmas which correspond to two different tools with which we can compare the utility induced by different bids and therefore exclude bids as possible deviations from the proposed worst-case strategy.[35] The first tool is to show that for every bid in the interval $(\theta, \overline{b}_1)$ there exists only one probability function which induces this bid as a best reply for the 1-type. As a consequence, one can directly compute the minimum utility which can be induced for a bid in the interval $[\theta, \overline{b}_1]$ and show that the minimum utility is equal for all bids in the interval $[\theta, \overline{b}_1]$. This is formalized in the following Lemma and Corollary.

Lemma 14. *Let $b \in (\theta, \overline{b}_1)$ be such that b is an element in $B^r\left(1, f^{1,b}, \beta^*\right)$ for $f^{1,b} \in \mathscr{F}_\mu$. Then $f^{1,b}$ equals to $f^{1,*} = \left(f_0^{1,*}, f_\theta^{1,*}, f_1^{1,*}\right)$, the worst-case belief of a bidder with valuation 1.*

The intuition behind this is that the worst-case belief of a 1-type together with the strategy of the other 1-type makes her indifferent between any bid in the interval $[\theta, \overline{b}_1]$. Any change of the worst-case belief makes either a deviation to θ or to $\overline{b}_1$ more profitable. Hence, a bid $b \in \left(\theta, \overline{b}_1\right)$ cannot be induced by a belief different from the worst-case belief of the 1-type. The formal proof is relegated to Appendix 3.8.1.

Corollary 2. *For every $b \in [\theta, \overline{b}_1]$ it holds that $\theta =^1 b$.*

That is, every bid in the interval $[\theta, \overline{b}_1]$ induces the same expected utility together with a minimizing probability function.

[35] We will need a third tool in the case $\theta > \mu$.

Proof. As defined in (25), the worst-case belief of a bidder with valuation 1, denoted by $f^{1,*}$, is the solution of minimization problem $M^1_{<\theta}$. Since we have shown that the best-reply condition is fulfilled for type 1, it holds that $f^{1,*}$ is an element of the feasible set of minimization problem M^1_θ. Since the constraints of minimization problem $M^1_{<\theta}$ are a subset of the constraints of minimization problem M^1_θ, it follows that $f^{1,*}$ is a solution of M^1_θ. It follows from Lemma 14 and the definition of the worst-case belief of the 1-type that every bid in $[\theta, \bar{b}_1)$ together with its unique minimizing probability function induces the same expected utility given by

$$\left(f_0^{1,*} + f_\theta^{1,*}\right)(1-\theta).$$

Independent of the probability function the expected utility of bidding $\bar{b}_1$ is equal to $1 - \bar{b}_1$ which is equal to $\left(f_0^{1,*} + f_\theta^{1,*}\right)(1-\theta)$. Therefore, it holds for all $b \in [\theta, \bar{b}_1]$ that

$$\theta =^1 b.$$

$\square$

The second tool constitutes a connection between binding incentive constraints in minimization problem M^1_b and bids which are lower than b with respect to the introduced transitive order $\leq^1$.

Lemma 15. *Let b be a bid and $f^{1,b}$ a solution of minimization problem M^1_b. If there exists a binding incentive constraint with corresponding bid $\hat{b}$, i.e.*

$$U\left(1, f^{1,b}, b, \beta^*\right) = U\left(1, f^{1,b}, \hat{b}, \beta^*\right),$$

then it holds that $\hat{b} \leq^1 b$.

Proof. Let L^1_b and $L^1_{\hat{b}}$ be the feasible sets, $f^{1,b} = \left(f_0^{1,b}, f_\theta^{1,b}, f_1^{1,b}\right)$ and $f^{1,\hat{b}} = \left(f_0^{1,\hat{b}}, f_\theta^{1,\hat{b}}, f_1^{1,\hat{b}}\right)$ solutions and $U\left(1, f^{1,b}, b, \beta^*\right)$ and $U\left(1, f^{1,\hat{b}}, \hat{b}, \beta^*\right)$ the values of the objective functions of minimization problem M^1_b and $M^1_{\hat{b}}$ respectively. In minimization problem $M^1_{\hat{b}}$ for every $s \in [\theta, \bar{b}_1]$ the incentive constraint corresponding to bid s given by

$$U\left(1, f, \hat{b}, \beta^*\right) \geq U\left(1, f, s, \beta^*\right)$$

is fulfilled for $f = f^{1,b}$ because it holds that

$$U\left(1, f^{1,b}, \hat{b}, \beta^*\right) = U\left(1, f^{1,b}, b, \beta^*\right) \geq U\left(1, f^{1,b}, s, \beta^*\right).$$

The equality follows from the fact that the incentive constraint corresponding to $\hat{b}$ is binding in minimization problem M_b^1. The inequality

$$U\left(1, f^{1,b}, b, \beta^*\right) \geq U\left(1, f^{1,b}, s, \beta^*\right)$$

holds because $f^{1,b}$ is a solution of minimization problem M_b^1. Since every constraint in $M_{\hat{b}}^1$ is fulfilled by $f^{1,b}$, it holds that $f^{1,b}$ is an element of $L_{\hat{b}}^1$. This also shows that the feasible set of minimization problem $M_{\hat{b}}^1$ is not empty. Therefore, in $M_{\hat{b}}^1$ the solution of minimization problem $M_{\hat{b}}^1$ has to induce a lower or equal utility than the solution of minimization problem M_b^1 and it follows that

$$U\left(1, f^{1,\hat{b}}, \hat{b}, \beta^*\right) \leq U\left(1, f^{1,b}, \hat{b}, \beta^*\right) = U\left(1, f^{1,b}, b, \beta^*\right).$$

We conclude that bid b together with a minimizing probability function does not induce a lower expected utility than bid $\hat{b}$ together with a minimizing probability function and it therefore holds that $\hat{b} \leq^1 b$. □

After introducing two tools with which we can compare bids with respect to the introduced transitive order, we can prove the second part of Proposition 9.

Proof. Since by bidding zero a bidder with valuation zero expects a utility of zero and this is the lowest possible utility, the worst-case belief condition is fulfilled for type zero. The expected utility of a bidder with valuation θ induced by her worst-case belief and the other bidder's strategy is zero and therefore, the worst-case belief condition is fulfilled for type θ. It is left to show the worst-case belief condition for type 1. As stated in Observation 2, the worst-case belief condition for type 1 is equivalent to

(i) $$f^{1,*} \in \mathscr{F}^{min}(1, b, \beta^*)$$

(ii) $$b \leq^1 b' \text{ for all } b' \in [0, \bar{b}_1]$$

for all $b \in [\theta, \bar{b}_1]$. Analogously as in the proof of Corollary 2, one can show that the worst-case belief of type 1 is a solution of minimization problem M_b^1 for all $b \in [\bar{b}_\theta, \bar{b}_1]$. It follows from Lemma 14 that condition (i) is fulfilled for all bids in $[0, \bar{b}_\theta)$. By definition of the worst-case belief of type 1, this belief induces $\bar{b}_1$ as best reply for a bidder with valuation 1. Since any probability function which induces $\bar{b}_1$ as a best reply for the 1-type yields an expected utility of $1 - \bar{b}_1$, any probability function with this property is a minimizing probability function. Therefore, condition (i) is also fulfilled for bid $\bar{b}_1$. Given the result in Corollary 2, condition (ii) reduces to

$$\theta \leq^1 b \text{ for all } b \in [0, \theta). \tag{27}$$

The only candidate for a bid in the interval $[0, \theta)$ which could induce a lower expected utility than bid θ is 0 since all other bids cannot be a best reply independently of the belief.

A minimizing probability function of zero is a solution of minimization problem M_0^1:

$$\min_{(f_0,f_\theta,f_1)} \ f_0$$

$$s.t.\ f_{\hat{\theta}} \geq 0 \text{ for all } \hat{\theta} \in \{0,\theta,1\}$$

$$f_0 + f_\theta + f_1 = 1$$

$$f_\theta\theta + f_1 = 1 = \mu$$

$$f_0 \geq (f_0 + f_\theta + f_1 G_1(s))(1-s) \ \text{ for all } s \in [\theta, \bar{b}_1].$$

Note that it is not necessary to include incentive constraints with corresponding bid in the interval $(0,\theta)$ since such a bid is never a best reply. If only the constraints

$$s.t.\ f_{\hat{\theta}} \geq 0 \text{ for all } \hat{\theta} \in \{0,\theta,1\}$$

$$f_0 + f_\theta + f_1 = 1$$

$$f_\theta\theta + f_1 = 1 = \mu$$

would be considered, it would hold for the solution of M_0^1 that $f_1 = 0$. But then the constraint

$$f_0 \geq (f_0 + f_\theta)(1-\theta)$$

would be violated. Therefore, one of the incentive constraints in M_0^1 has to be binding. Let $\hat{b}$ be the bid such that the corresponding incentive constraint is binding. It follows from Lemma 15 that $\hat{b} \leq^1 0$. Since bids in the interval $(0,\theta)$ are never a best reply, it must hold that $\hat{b} \in [\theta, \bar{b}_1]$. Using the transitivity of the relation $\leq^1$, we conclude that

$$0 \leq^1 \hat{b} =^1 \theta.$$

Thus, we have shown (27) which completes the proof. □

After proving the best-reply and the worst-case belief condition, we conclude that the strategies and beliefs specified in 3.3.2 indeed constitute a worst-case belief equilibrium. This completes the example with two bidders and three types such that $\theta \leq \mu$ and now we turn to the case where $\theta > \mu$. As before, we first specify the worst-case strategy and beliefs.

3.3.4 Characterization of the efficient worst-case belief equilibrium for $\theta > \mu$

Again, we denote the worst-case strategy by β^* and define

$$\beta^*(0) = 0,\ \beta^*(\theta) = G_\theta,\ \beta^*(1) = G_1. \tag{28}$$

That is, type zero bids zero, type θ plays a mixed strategy on the interval $[0, \bar{b}_\theta]$ and type 1 plays a mixed strategy on the interval $[\bar{b}_\theta, \bar{b}_1]$. As before, one can immediately see that this constitutes an efficient equilibrium. We denote the worst-case belief of a bidder with valuation $\hat{\theta}$ by $\left(f_0^{\hat{\theta},*}, f_\theta^{\hat{\theta},*}, f_1^{\hat{\theta},*}\right)$ for $\hat{\theta} \in \{0, \theta, 1\}$. Type zero can have any belief. The worst-case belief of type θ is the solution of the following minimization problem, denoted by $M_{<0}^\theta$:

$$\min_{(f_0, f_\theta, f_1)} \; f_0$$

$$s.t.\; f_{\hat{\theta}} \geq 0 \text{ for all } \hat{\theta} \in \{0, \theta, 1\}$$

$$f_0 + f_\theta + f_1 = 1$$

$$f_\theta \theta + f_1 = \mu.$$

Recall that in the case $\theta \leq \mu$, by definition, the worst-case belief of a bidder with a given type contained all incentive constraints with corresponding bids which are lower than the lower endpoint of the type's bidding interval. This also holds for the case $\theta > \mu$. Since type θ plays a mixed strategy on an interval beginning with zero, there are no incentive constraints in this minimization problem. Any solution of minimization problem M_0^θ has to fulfill the two probability constraints:

$$f_0 + f_\theta + f_1 = 1$$

$$f_\theta \theta + f_1 = \mu.$$

Rearranging the second probability constraint w.r.t. to f_θ and plugging in into the first probability constraint gives:

$$f_0 + \frac{\mu - f_1}{\theta} + f_1 = 1$$

$$\Leftrightarrow f_0 = \frac{\theta - \mu}{\theta} + \frac{f_1 (1 - \theta)}{\theta}.$$

Thus, the minimum value for f_0 is given by $\frac{\theta - \mu}{\theta}$ and the solution of minimization problem $M_{<0}^\theta$ is given by

$$f_0^{\theta,*} = \frac{\theta - \mu}{\theta}, \; f_\theta^{\theta,*} = \frac{\mu}{\theta}, \; f_1^{\theta,*} = 0. \tag{29}$$

The upper endpoint of the bidding interval of a bidder with valuation θ is obtained by the equation

$$f_0^{\theta,*} \theta = \left(f_0^{\theta,*} + f_\theta^{\theta,*}\right) \left(\theta - \bar{b}_\theta\right) \tag{30}$$

$$\Leftrightarrow \bar{b}_\theta = f_\theta^{\theta,*} \theta = \mu.$$

The bid distribution of bidders A and B with valuation θ makes them indifferent between any bid in their bidding interval. That is, it for every $s \in [0, \bar{b}_\theta]$ it holds that

$$f_0^{\theta,*}\theta = \left(f_0^{\theta,*} + f_\theta^{\theta,*}G_\theta(s)\right)(\theta - s). \tag{31}$$

The subjective worst-case belief of a bidder with valuation 1 is the solution of the following minimization problem, denoted by $M^1_{<\bar{b}_\theta}$:

$$\min_{(f_0, f_\theta, f_1)} \quad f_0 + f_\theta$$

$$s.t.\ f_{\hat{\theta}} \geq 0 \text{ for all } \hat{\theta} \in \{0, \theta, 1\}$$

$$f_0 + f_\theta + f_1 = 1$$

$$f_\theta \theta + f_1 = \mu$$

$$(f_0 + f_\theta)\left(1 - \bar{b}_\theta\right) \geq (f_0 + f_\theta G_\theta(s))(1 - s) \text{ for all } s \in [0, \bar{b}_\theta].$$

As before, the minimization problem contains all incentive constraints with corresponding bids which are lower than the lower endpoint of type 1's bidding interval. This implies that there is just enough probability weight on types zero and θ in order to incentivize the 1-type to play a mixed strategy on an interval beginning with $\bar{b}_\theta$. The upper endpoint of type 1's bidding interval is obtained by the equation

$$\left(f_0^{1,*} + f_\theta^{1,*}\right)\left(1 - \bar{b}_\theta\right) = \left(f_0^{1,*} + f_\theta^{1,*} + f_1^{1,*}\right)\left(1 - \bar{b}_1\right).$$

The bid distribution of bidders A and B with valuation 1 makes them indifferent between any bid int heir bidding interval. That is, for every $s \in [\bar{b}_\theta, \bar{b}_1]$ it holds that

$$\left(f_0^{1,*} + f_\theta^{1,*}\right)\left(1 - \bar{b}_\theta\right) = \left(f_0^{1,*} + f_\theta^{1,*} + f_1^{1,*}G_1(s)\right)(1 - s). \tag{32}$$

Note that in contrast to previous minimization problems we cannot derive the solution of minimization problem $M^1_{<\bar{b}_\theta}$ directly since we have to consider an uncountable number of incentive constraints . For now, we proceed with the given definition of the worst-case belief of a bidder with valuation 1 and provide the explicit solution of the minimization problem later on. However, it is easy to see that the feasible set of minimization problem $M^1_{<\bar{b}_\theta}$ is not empty since the worst-case belief of type θ is an element of this set.

3.3.5 Proving the best-reply and the worst-case belief condition for $\theta > \mu$

After specifying the worst-case strategy and beliefs, we have to show that these indeed constitute a worst-case belief equilibrium. That is, we have to show the optimality and the worst-case belief condition.

Proposition 10. *Given the worst-case strategy as defined in (28) and the worst-case beliefs as defined in (29) and (32), it holds for all $\hat{\theta} \in \{0, \theta, 1\}$ that* *(i) The best-reply condition given by*

$$b_{\hat{\theta}} \in B^r\left(\hat{\theta}, f^{\hat{\theta},*}, \beta^*\right) \text{ for all } b_{\hat{\theta}} \in supp\left(\beta^*\left(\hat{\theta}\right)\right)$$

is fulfilled, i.e. every bidder plays a best reply given her valuation, her worst-case belief and the other bidder's worst-case strategy.

(ii) The worst-case belief condition is fulfilled, i.e. for all $b_{\hat{\theta}} \in supp\left(\beta^*\left(\hat{\theta}\right)\right)$ it holds that

$$U\left(\hat{\theta}, f^{\hat{\theta},*}, b_{\hat{\theta}}, \beta^*\right) \leq U\left(\hat{\theta}, f, b^r\left(\hat{\theta}, f, \beta^*\right), \beta^*\right) \text{ for all } f \in \mathscr{F}_\mu.$$

That is, there does not exist another belief such that a best reply to this belief induces a lower expected utility than the worst-case belief.

Proof. Part *(i)*: The result is obvious for a bidder with valuation zero. The worst-case belief of bidder A with valuation θ is that bidder B has valuation 1 with probability zero. Hence, bidder A expects $\bar{b}_\theta = \mu$ to be the highest bid placed by bidder B. The bid distribution of type θ makes bidder A with valuation θ indifferent between any bid in the interval $[0, \bar{b}_\theta]$. Therefore, she has no incentive to deviate. It follows from the definition of the worst-case belief of bidder A with valuation 1 that she does not earn a higher expected utility by bidding any bid lower than $\bar{b}_\theta$. Since bidder B does not play a bid above $\bar{b}_1$, it cannot be a best reply for bidder A to bid above $\bar{b}_1$. The bid distribution G_1 is constructed in a way which makes bidder A with valuation 1 indifferent between any bid in $[\bar{b}_\theta, \bar{b}_1]$ which completes the proof. □

The remainder of the section is dedicated to proving the worst-case belief condition. Since type zero expects the lowest possible utility of zero by bidding zero, the worst-case belief condition is fulfilled for type zero. We will prove the worst-case belief condition for types θ and 1 separately, i.e. we divide part *(ii)* of Proposition 10 into two different parts

(ii.1) The worst-case belief condition is fulfilled for type θ, i.e. for all $b \in [0, \bar{b}_\theta]$ it holds that

$$U\left(\theta, f^{\theta,*}, b, \beta^*\right) \leq U\left(\theta, f, b^r\left(\theta, f, \beta^*\right), \beta^*\right) \text{ for all } f \in \mathscr{F}_\mu.$$

(ii.2) The worst-case belief condition is fulfilled for type 1, i.e. for all $b \in [\bar{b}_\theta, \bar{b}_1]$ it holds that

$$U\left(1, f^{\theta,*}, b, \beta^*\right) \leq U\left(1, f, b^r\left(1, f, \beta^*\right), \beta^*\right) \text{ for all } f \in \mathscr{F}_\mu.$$

We begin with part *(ii.1)*. Similarly, as in the case $\theta \leq \mu$, we prove three lemmas which correspond to three different tools with which we can compare the utility induced by different bids.[36]

[36]In contrast to the case $\theta \leq \mu$, in the case $\theta > \mu$ we will make use of three tools.

The first lemma provides a similar result as Lemma 14 and Corollary 2. That is, we show that for every bid in the interval $(0, \bar{b}_\theta)$ there exists only one probability function which induces this bid as a best reply for the θ-type. As a consequence, one can directly compute the minimum utility which can be induced for a bid in the interval $[0, \bar{b}_\theta]$ and show that the minimum utility is equal for all bids in the interval $[0, \bar{b}_\theta]$.

Lemma 16. *Let $b \in (0, \bar{b}_\theta)$ be such that b is an element in $B^r\left(\theta, f^{\theta,b}, \beta^*\right)$ for $f^{\theta,b} \in \mathscr{F}_\mu$. Then $f^{\theta,b}$ equals to $f^{\theta,*} = \left(f_0^{\theta,*}, f_\theta^{\theta,*}, f_1^{\theta,*}\right)$, the worst-case belief of a bidder with valuation θ.*

We omit the formal proof since it works similarly to the proof of Lemma 14 and is also covered by the general case.

Corollary 3. *For every $b \in [0, \bar{b}_\theta]$ it holds that $0 =^\theta b$.*

That is, every bid in the interval $[0, \bar{b}_\theta]$ induces the same expected utility together with a minimizing probability function.

Proof. Analogously as in the proof of Corollary 2, one can conclude that every bid in $[0, \bar{b}_\theta)$ together with its unique minimizing probability function induces the same expected utility given by $f_0^{\theta,*}\theta$.

It is left to show that $0 =^\theta \bar{b}_\theta$. Any probability function (f_0, f_θ, f_1) which induces bid $\bar{b}_\theta = \mu$ as a best reply for type θ has to fulfill

$$(f_0 + f_\theta)(\theta - \mu) \geq f_0\theta. \tag{33}$$

Since due to the probability constraints the smallest possible value for f_0 is given by $\frac{\theta-\mu}{\theta}$, it must hold that

$$(f_0 + f_\theta)(\theta - \mu) \geq \theta - \mu$$

from which follows that $f_0 + f_\theta = 1$. Hence, f_0 and f_θ are uniquely determined by the two probability constraints. Any probability function which fulfills the probability constraints and inequality (33) coincides with the worst-case belief of type θ.

Therefore, the worst-case belief of type θ is the only probability function which induces $\bar{b}_\theta = \mu$ as a best reply for type θ. Hence, the worst-case belief is the unique minimizing probability function for bid $\bar{b}_\theta$ and it follows from the definition of the worst-case belief that bids 0 and $\bar{b}_\theta$ induce the same expected utility together with a minimizing probability function given by

$$\left(f_0^{\theta,*} + f_0^{\theta,*}\right)(\theta - \mu) = f_0^{\theta,*}\theta.$$

Therefore, it holds for all $b \in [0, \bar{b}_\theta]$ that

$$0 =^1 b.$$

□

The second lemma corresponds to Lemma 15. That is, it establishes a connection between binding incentive constraints in minimization problem M_b^θ and bids which are lower than b with respect to the order $\leq^\theta$

Lemma 17. *Let b be a bid and $f^{\theta,b}$ a solution of minimization problem M_b^θ. If there exists a binding incentive constraint with corresponding bid $\hat{b}$, i.e.*

$$U\left(\theta, f^{\theta,b}, b, \beta^*\right) = U\left(\theta, f^{\theta,b}, \hat{b}, \beta^*\right),$$

then it holds that $\hat{b} \leq^\theta b$.

The same proof as for Lemma 15 applies. For the third tool, we show that for a given type there exist bids which can never be a best reply independent of the subjective belief. Hence, these bids cannot be possible deviations from the proposed worst-case equilibrium for the given type.

Lemma 18. *The feasible set of minimization problem M_b^θ for all $b \in (\bar{b}_\theta, \bar{b}_1]$ is empty.*

Assume there exists a bid b in the interval $(\bar{b}_\theta, \bar{b}_1]$ such that the feasible set of minimization problem M_b^θ is not empty. Then in a solution of minimization problem M_b^θ, denoted by $\left(f_0^{\theta,b}, f_\theta^{\theta,b}, f_1^{\theta,b}\right)$, there must be strictly positive probability weight on the 1-type because otherwise there would be no incentive to bid higher than $\bar{b}_\theta$. In contrast, the worst-case equilibrium belief of the θ-type has no probability weight on the 1-type. Hence, in order to preserve the mean, the probability weight on the zero-type or the θ-type in the solution of minimization problem M_b^θ must be higher than in the worst-case belief. Given the worst-case belief, the θ-type is indifferent among all bids in the interval $[0, \bar{b}_\theta]$. If the probability weight of the zero-type is increased, it is optimal for the θ-type to bid zero. Therefore, the probability weight on the 0-type cannot be increased. Similarly, if the probability weight on the θ-type is increased, it is optimal for the 1-type to bid $\bar{b}_\theta$ or lower. Therefore, the belief $\left(f_0^{\theta,b}, f_\theta^{\theta,b}, f_1^{\theta,b}\right)$ cannot induce a bid above $\bar{b}_\theta$ for the θ-type. The formal proof is relegated to Appendix 3.8.2.

After introducing the three tools, we can start with the proof of part *(ii.1)*.

Proof. As stated in Observation 2, the worst-case belief condition for type θ is equivalent to

(i) $$f^{\theta,*} \in \mathscr{F}^{min}(\theta, b, \beta^*)$$

(ii) $$b \leq^\theta b' \text{ for all } b' \in [0, \bar{b}_1]$$

for all $b \in [0, \bar{b}_\theta]$.[37] Analogously as in the proof of Corollary 2, one can show that the worst-case belief of type θ is a solution of minimization problem M_b^θ for all $b \in [0, \bar{b}_\theta]$. It follows from Lemma 16 that condition (i) is fulfilled for all bids in $[0, \bar{b}_\theta)$. As shown in the

[37]We use the notation provided in Definitions 10-12 also for the case $\theta > \mu$ but use β^* as defined in 3.3.4.

proof of Corollary 3, the worst-case belief of the θ-type is the only probability function which induces $\overline{b}_\theta$ as a best reply. Therefore, condition (i) is fulfilled. Given the result in Corollary 3, condition (ii) reduces to

$$0 \leq^\theta b \text{ for all } b \in (\overline{b}_\theta, \overline{b}_1].$$

It follows from Lemma 18 that for all $b \in (\overline{b}_\theta, \overline{b}_1]$ it holds that $0 <^\theta b$ which completes the proof of part *(ii.1)*. □

It is left to show part *(ii.2)* of Proposition 10, i.e. the worst-case belief condition for type 1. Again, we prove three lemmas which correspond to the three tools presented above.

The first lemma provides a similar result as Lemma 16 and Corollary 3 (and as Lemma 14 and Corollary 2 in the case $\theta \leq \mu$).

Lemma 19. *Let $b \in (\overline{b}_\theta, \overline{b}_1)$ be such that b is an element in $B^r\left(1, f^{1,b}, \beta^*\right)$ for $f^{1,b} \in \mathscr{F}_\mu$. Then $f^{1,b}$ equals to $f^{1,*} = \left(f_0^{1,*}, f_\theta^{1,*}, f_1^{1,*}\right)$, the worst-case belief of a bidder with valuation 1.*

Corollary 4. *For every $b \in [\overline{b}_\theta, \overline{b}_1]$ it holds that $\overline{b}_\theta =^1 b$.*

We omit the proofs of the Lemma and the Claim since they work with the same arguments as before and are covered by the proof of the general case. The following lemma provides the second tool and corresponds to Lemma 15 and Lemma 17.

Lemma 20. *Let b be a bid and $f^{1,b}$ a solution of minimization problem M_b^1. If there exists a binding incentive constraint with corresponding bid $\hat{b}$, i.e.*

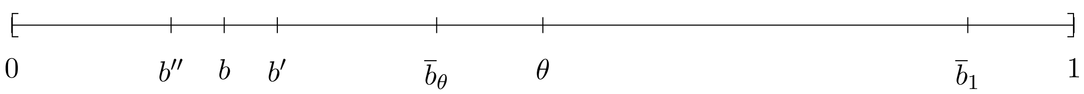
$$U\left(1, f^{1,b}, b, \beta^*\right) = U\left(1, f^{1,b}, \hat{b}, \beta^*\right),$$

then it holds that $\hat{b} \leq^1 b$.

The same proof as for Lemma 15 applies. The third tool in the proof of the worst-case belief condition for the 1-type is similar to the third tool (Lemma 18) in the proof of the worst-case belief condition for type θ. That is, we show that for a given type there exist bids which can never be a best reply independent of the subjective belief. Hence, these bids cannot be possible deviations from the proposed worst-case equilibrium for the given type.

Lemma 21. *The feasible set of minimization problem M_b^1 for all $b \in \left(0, \overline{b}_\theta\right)$ is empty.*

Assume there exists a bid $b \in \left(0, \overline{b}_\theta\right)$ such that the minimization problem M_b^1 has a solution which we denote by $\left(f_0^{1,b}, f_\theta^{1,b}, f_1^{1,b}\right)$. Consider two bids b', b'' with $0 \leq b'' < b < b' \leq \overline{b}_\theta$.



Given the belief $\left(f_0^{1,b}, f_\theta^{1,b}, f_1^{1,b}\right)$, the utility for the 1-type of bidding b must be at least as high as the utilities of bidding b'' or b'. The higher $f_0^{1,b}$, the lower is the optimal bid for type 1. Therefore, the incentive constraint corresponding to bid b'' sets a lower bound on the value of $f_0^{1,b}$ while the incentive constraint corresponding to bid b' sets an upper bound. We will show that the conditions resulting from these two bounds contradict each other. Intuitively, a bidder bidding in the interval $[0, \overline{b}_\theta]$ faces the bid distribution G_θ of the θ-type which is constructed in order to make the other θ-type indifferent. Thus, only for the θ-type the upper and the lower bound are compatible. The formal proof is relegated to Appendix 3.8.3 .

Given the three tools, we can show part *(ii.2)*.

Proof. As stated in Observation 2, the worst-case belief condition for type 1 is equivalent to

$$\text{(i)} \qquad f^{1,*} \in \mathscr{F}^{min}(1, b, \beta^*)$$

$$\text{(ii)} \qquad b \leq^1 b' \text{ for all } b' \in [0, \overline{b}_1]$$

for all $b \in [\theta, \overline{b}_1]$. Condition (i) can be proven analogously as in the proof of Proposition 9 and due to Corollary 4, the second condition reduces to

$$\overline{b}_\theta \leq^1 b' \text{ for all } b' \in [0, \overline{b}_\theta).$$

It follows from Lemma 21 that for all $b \in \left(0, \overline{b}_\theta\right)$ it holds that $\overline{b}_\theta <^1 b$. Therefore, in order to show the worst-case belief condition for type 1, it is left to show that

$$\overline{b}_\theta \leq^\theta 0. \tag{34}$$

As a next step, we use Lemma 21, in order to calculate the worst-case belief of a bidder with valuation 1. This belief is the solution of minimization problem $M^1_{<\overline{b}_\theta}$:

$$\min_{(f_0, f_\theta, f_1)} \; f_0 + f_\theta$$

$$s.t. \; f_{\hat{\theta}} \geq 0 \text{ for all } \hat{\theta} \in \{0, \theta, 1\}$$

$$f_0 + f_\theta + f_1 = 1$$

$$f_\theta \theta + f_1 = \mu$$

$$(f_0 + f_\theta)\left(1 - \overline{b}_\theta\right) \geq (f_0 + f_\theta G_\theta(s))(1 - s) \;\text{ for all } s \in [0, \overline{b}_\theta].$$

The solution of the reduced minimization problem which contains only the constraints

$$f_{\hat{\theta}} \geq 0 \text{ for all } \hat{\theta} \in \{0, \theta, 1\}$$

$$f_0 + f_\theta + f_1 = 1$$

$$f_\theta \theta + f_1 = \mu$$

would distribute the probability weight solely between type zero and one which would violate the incentive constraint corresponding to bid zero. Hence, at least one of the incentive constraints has to be binding. Let $\hat{b}$ be a bid such that the corresponding incentive constraint is binding. Since we have shown that the best-reply condition is fulfilled for type 1, it holds that $f^{1,*}$ is an element of the feasible set of minimization problem $M^1_{\bar{b}_\theta}$. Since the constraints of minimization problem $M^1_{<\bar{b}_\theta}$ are a subset of the constraints of minimization problem $M^1_{\bar{b}_\theta}$, it follows that $f^{1,*}$ is a solution of $M^1_{\bar{b}_\theta}$. Therefore, it follows from Lemma 20 that $\hat{b} \leq^1 \bar{b}_\theta$. Due to Lemma 21, it holds that $\bar{b}_\theta <^1 b$ for all $b \in \left(0, \bar{b}_\theta\right)$ from which follows that $\hat{b} = 0$. Therefore, the worst-case belief of type 1 is the unique solution of a system of three linear equation with three unknowns given by

$$f_0^{1,*} + f_\theta^{1,*} + f_1^{1,*} = 1$$

$$f_\theta^{1,*}\theta + f_1^{1,*} = \mu$$

$$\left(f_0^{1,*} + f_\theta^{1,*}\right)\left(1 - \bar{b}_\theta\right) = f_0^{1,*}$$

The solution is given by

$$f_0^{1,*} = \frac{(1-\mu)^2}{1-\mu\theta},\ f_\theta^{1,*} = \frac{\mu(1-\mu)}{1-\mu\theta},\ f_1^{1,*} = \frac{\mu(1-\theta)}{1-\mu\theta}.$$

Consider minimization problem M^1_0:

$$\min_{(f_0, f_\theta, f_1)} \ f_0$$

$$s.t.\ f_{\hat{\theta}} \geq 0 \text{ for all } \hat{\theta} \in \{0, \theta, 1\}$$

$$f_0 + f_\theta + f_1 = 1$$

$$f_\theta \theta + f_1 = 1 = \mu$$

$$f_0 \geq (f_0 + f_\theta G_\theta(s))(1-s) \ \text{ for all } s \in [0, \bar{b}_\theta].$$

$$f_0 \geq (f_0 + f_\theta + f_1 G_1(s))(1-s) \ \text{ for all } s \in [\theta, \bar{b}_1].$$

The solution of the reduced minimization problem which contains only the constraints

$$s.t.\ f_{\hat{\theta}} \geq 0 \text{ for all } \hat{\theta} \in \{0, \theta, 1\}$$

$$f_0 + f_\theta + f_1 = 1$$

$$f_\theta \theta + f_1 = 1 = \mu$$

would distribute the probability weight solely between types zero and θ which would violate the incentive constraint corresponding to bid θ. Therefore, one of the incentive constraints with corresponding bid different from zero has to be binding. Let $\hat{b}'$ be the bid corresponding to the binding incentive constraint. It follows from Lemma 20 that $\hat{b}' \leq^1 0$.

As argued above, the worst-case belief of type 1 is an element of the feasible set of minimization problem $M^1_{\bar{b}_\theta}$. Since the incentive constraint corresponding to bid zero is binding in this minimization problem, it follows that the worst-case belief of type 1 is an element of the feasible set of minimization problem M^1_0. This implies that the feasible set of minimization problem M^1_0 is not empty. As stated in Lemma 21, the feasible set of minimization problem M^1_b is empty for all $b \in (0, \bar{b}_\theta)$. Hence, it holds that $0 <^1 b$ for all $b \in (0, \bar{b}_\theta)$. Therefore, it holds that $\hat{b}' \in [\bar{b}_\theta, \bar{b}_1]$. It follows from Corollary 4 that $\bar{b}_\theta =^1 \hat{b}'$. Thus, we can construct the transitive chain

$$\bar{b}_\theta =^1 \hat{b}' \leq^1 0.$$

We have shown that (34) holds which we established as a sufficient condition for the worst-case belief condition for type 1. $\square$

Since we have shown that the best-reply and the worst-case belief condition hold for all types, we conclude that the beliefs and strategies specified in 3.3.4 indeed constitute a worst-case belief equilibrium.

3.4 Revenue comparison of the first-price and second-price auction

We want to compare the revenue of a first-price and a second-price auction in a setting where bidders do not know the distribution of their competitors' valuations. As described in the model, we assume that the number of bidders, the set possible valuations Θ and the exogenously given mean μ of valuations is common knowledge. In a second-price auction bidding the own valuation is a weakly dominant strategy and thus independent of the belief about the other bidders' valuations. Therefore, we assume that in a second-price auction bidders bid their valuation. For the first-price auction we assume that bidders play the efficient worst-case belief equilibrium.

Since the computation of revenue of the first-price auction involves the computation of the worst-case beliefs and strategy which is computationally complex, we provide the formal revenue comparison for the simplified case of two bidders with three possible types 0,θ and 1. As we will see, it highly depends on the valuation distribution which auction leads to the higher revenue. Hence, we cannot state any general theorems. The revenue comparison for a given valuation distribution and a given number of bidders requires a computational solution.

3.4.1 Revenue of the second-price auction

In order to compute the revenue of the first-price or the second-price auction, we need to know the true valuation distribution which we denote by (f_0, f_θ, f_1). Given that in a second-price auction all bidders bid their valuation, the revenue of the second-price auction is obtained as follows. The expected revenue from type zero is zero. The expected revenue from type θ is determined by the probability that the θ-type meets another θ-type against whom she wins with probability $\frac{1}{2}$ and pays θ which gives an expected revenue of $\frac{1}{2} f_\theta \theta$. The expected revenue from a 1-type is determined by the probability that she meets a θ-type, in this case the 1-type wins with probability 1 and pays θ, and by the probability that she meets a 1-type, in this case the 1-type wins with probability $\frac{1}{2}$ and pays 1. This results in an expected revenue of $f_\theta \theta + \frac{1}{2} f_1$. The total expected revenue of a second-price auction from one bidder is given by

$$\frac{1}{2} f_\theta^2 \theta + f_1 \left(f_\theta \theta + \frac{1}{2} f_1 \right). \tag{35}$$

Due to the probability constraints given by

$$f_0 + f_\theta + f_1 = 1$$

$$f_\theta \theta + f_1 = \mu,$$

there is only one degree of freedom in left in the choice of the probability function (f_0, f_θ, f_1). The probability constraints can be rewritten as

$$f_0 = 1 - f_\theta - f_1$$

$$f_1 = \mu - f_\theta \theta$$

which gives

$$f_0 = 1 - (1 - \theta) f_\theta - \mu$$

$$f_1 = \mu - f_\theta \theta.$$

Substituting the expression for f_1 in (35) gives a revenue of

$$\frac{1}{2} f_\theta^2 \theta + (\mu - f_\theta \theta) \left(f_\theta \theta + \frac{1}{2} (\mu - f_\theta \theta) \right)$$

$$= \frac{1}{2} \left(f_\theta^2 \theta - f_\theta^2 \theta^2 + \mu^2 \right).$$

3.4.2 Revenue of the first-price auction

For the revenue calculation of the first-price auction with worst-case beliefs we have to differentiate between the case $\mu \geq \theta$ and $\mu < \theta$. We start with the case $\mu \geq \theta$. In this case the θ-type bids θ. The winning probability of the θ-type is $f_0 + \frac{1}{2}f_\theta$ which gives an expected revenue of

$$\theta\left(f_0 + \frac{1}{2}f_\theta\right) = \theta\left(1 - (1-\theta)\, f_\theta - \mu + \frac{1}{2}f_\theta\right) = \theta\left(1 - \mu + f_\theta\left(\theta - \frac{1}{2}\right)\right).$$

As shown in in section 3.3.2, the worst-case belief of the 1-type, which we denote by $\left(f_0^{1,*}, f_\theta^{1,*}, f_1^{1,*}\right)$, is the unique solution of the following system of linear equations:

$$f_0^{1,*} + f_\theta^{1,*} + f_1^{1,*} = 1$$

$$f_\theta^{1,*}\theta + f_1^{1,*} = \mu$$

$$\left(f_0^{1,*} + f_\theta^{1,*}\right)(1-\theta) = f_0^{1,*}$$

which leads to

$$f_0^{1,*} = \frac{1-\mu}{1+\theta},\ f_\theta^{1,*} = \frac{\theta\,(1-\mu)}{1-\theta^2},\ f_1^{1,*}\frac{\mu-\theta^2}{1-\theta^2}.$$

The bid distribution of the 1-type, denoted by G_1, is determined by the equation

$$\left(f_0^{1,*} + f_\theta^{1,*}\right)(1-\theta) = \left(f_0^{1,*} + f_\theta^{1,*} + f_1^{1,*}G_1(s)\right)(1-s)$$

for $s \in [\theta, \overline{b}_1]$ where $\overline{b}_1$ is defined by

$$\left(f_0^{1,*} + f_\theta^{1,*}\right)(1-\theta) = 1 - \overline{b}_1,$$

i.e. $G_1\left(\overline{b}_1\right) = 1$. This is equivalent to

$$\overline{b}_1 = 1 - \left(f_0^{1,*} + f_\theta^{1,*}\right)(1-\theta)$$

$$\Leftrightarrow \overline{b}_1 = 1 - \frac{1-\mu}{1-\theta^2}(1-\theta) = \frac{\theta+\mu}{1+\theta}.$$

After plugging in the values for $f_0^{1,*}, f_\theta^{1,*}, f_1^{1,*}$ into

$$G_1(s) = \frac{\left(f_0^{1,*} + f_\theta^{1,*}\right)(s-\theta)}{(1-s)\, f_1^{1,*}}$$

we get

$$G_1(s) = \frac{(1-\mu)(s-\theta)}{(1-s)(\mu-\theta^2)}$$

and

$$\frac{dG_1(s)}{ds} = \frac{(1-\mu)(1-s)(\mu-\theta^2) + (1-\mu)(s-\theta)(\mu-\theta^2)}{(1-s)^2(\mu-\theta^2)^2} = \frac{(1-\mu)(1-\theta)}{(1-s)^2(\mu-\theta^2)}.$$

The expected revenue from a 1-type is given by

$$\int_{\theta}^{\bar{b}_1} \left(f_0 + f_\theta + f_1 G_1(s)\right) s dG_1(s)\, ds$$

$$= \int_{\theta}^{\frac{\theta+\mu}{1+\theta}} \left(1-\mu+\theta f_\theta + (\mu - f_\theta\theta)\frac{(1-\mu)(s-\theta)}{(1-s)(\mu-\theta^2)}\right) s \frac{(1-\mu)(1-\theta)}{(1-s)^2(\mu-\theta^2)} ds.$$

The total expected revenue from one bidder in a first-price auction with $\theta \le \mu$ is given by

$$f_\theta\theta\left(f_0 + \frac{1}{2}f_\theta\right) + f_1 \int_{\theta}^{\bar{b}_1} \left(f_0 + f_\theta + f_1 G_1(s)\right) s dG_1(s)\, ds$$

$$\begin{aligned} &= f_\theta\theta\left(1-\mu + f_\theta\left(\theta - \frac{1}{2}\right)\right) \\ &\quad + (\mu - f_\theta\theta)\int_{\theta}^{\frac{\theta+\mu}{1+\theta}} \left(1-\mu+\theta f_\theta + (\mu - f_\theta\theta)\frac{(1-\mu)(s-\theta)}{(1-s)(\mu-\theta^2)}\right) s \frac{(1-\mu)(1-\theta)}{(1-s)^2(\mu-\theta^2)} ds. \end{aligned}$$

Now we will calculate the revenue of a first-price auction if $\theta > \mu$. Let $\left(f_0^{\theta,*}, f_\theta^{\theta,*}, f_1^{\theta,*}\right)$ denote the worst-case equilibrium belief of type θ. As shown in section 3.3.4, the θ-type believes that there is no 1-type and therefore it follows from the probability constraints that

$$f_0^{\theta,*} = \frac{\theta-\mu}{\theta},\; f_\theta^{\theta,*} = \frac{\mu}{\theta},\; f_1^{1,*} = 0.$$

The θ-type plays a mixed strategy on the interval $[0, \bar{b}_\theta]$ where $\bar{b}_\theta$ is defined by

$$\theta - \bar{b}_\theta = f_0^{\theta,*}\theta$$

$$\Leftrightarrow \bar{b}_\theta = \theta - \frac{\theta-\mu}{\theta}\theta = \mu.$$

For all $s \in [0, \bar{b}_\theta]$ the bid distribution G_θ is defined by

$$f_0^{\theta,*}\theta = \left(f_0^{\theta,*} + f_\theta^{\theta,*} G_\theta(s)\right)(\theta - s)$$

$$\Leftrightarrow G_\theta(s) = \frac{s f_0^{\theta,*}}{f_\theta^{\theta,*}(\theta-s)} = \frac{s(\theta-\mu)}{\mu(\theta-s)}.$$

It follows that

$$\frac{dG_\theta(s)}{ds} = \frac{(\theta-\mu)\mu(\theta-s) + s(\theta-\mu)\mu}{\mu^2(\theta-s)^2} = \frac{(\theta-\mu)\theta}{\mu(\theta-s)^2}.$$

The expected revenue from a bidder with valuation θ is given by

$$\int_0^{\mu} \left(f_0 + f_\theta G_\theta(s)\right) s dG_\theta(s)\, ds$$

$$= \int_0^{\mu} \left(1 - \mu - (1-\theta) f_\theta + f_\theta \frac{s(\theta-\mu)}{\mu(\theta-s)}\right) s \frac{(\theta-\mu)\theta}{\mu(\theta-s)^2} ds.$$

The belief of the 1-type, denoted by $\left(f_0^{1,*}, f_\theta^{1,*}, f_1^{1,*}\right)$, is the unique solution of the following system of linear equations

$$f_0^{1,*} + f_\theta^{1,*} + f_1^{1,*} = 1$$

$$f_\theta^{1,*}\theta + f_1^{1,*} = \mu$$

$$\left(f_0^{1,*} + f_\theta^{1,*}\right)(1-\mu) = f_0^{1,*}$$

which leads to

$$f_0^{1,*} = \frac{(1-\mu)^2}{1-\mu\theta}, \quad f_\theta^{1,*} = \frac{\mu(1-\mu)}{1-\mu\theta}, \quad f_1^{1,*} = \frac{\mu(1-\theta)}{1-\mu\theta}.$$

The 1-type plays a mixed strategy on the interval $[\bar{b}_\theta, \bar{b}_1]$ where $\bar{b}_1$ is determined by

$$1 - \bar{b}_1 = \left(f_0^{1,*} + f_\theta^{1,*}\right)(1-\mu)$$

$$\Leftrightarrow \bar{b}_1 = 1 - \left(f_0^{1,*} + f_\theta^{1,*}\right)(1-\mu) = 1 - \frac{(1-\mu)^2}{1-\mu\theta} = \frac{2\mu - \mu - \mu\theta}{1-\mu\theta}.$$

For all $s \in [\bar{b}_\theta, \bar{b}_1]$ the bid distribution G_1 is determined by

$$\left(f_0^{1,*} + f_\theta^{1,*}\right)(1-\mu) = \left(f_0^{1,*} + f_\theta^{1,*} + f_1^{1,*} G_1(s)\right)(1-s)$$

$$\Leftrightarrow G_1(s) = \frac{\left(f_0^{1,*} + f_\theta^{1,*}\right)(s-\mu)}{f_1^{1,*}(1-s)} = \frac{(1-\mu)(s-\mu)}{\mu(1-\theta)(1-s)}$$

from which follows that

$$\frac{dG_1(s)}{ds} = \frac{(1-\mu)(1-b)\mu(1-\theta) + (1-\mu)(b-\mu)\mu(1-\theta)}{\mu^2(1-\theta)^2(1-s)^2} = \frac{(1-\mu)^2}{\mu(1-\theta)(1-s)^2}.$$

The expected utility from a bidder with type 1 is given by

$$\int_{\mu}^{\frac{2\mu-\mu-\mu\theta}{1-\mu\theta}} \left(f_0 + f_\theta + f_1 G_1(s)\right) s dG_1(s) ds$$

$$= \int_{\mu}^{\frac{2\mu-\mu-\mu\theta}{1-\mu\theta}} \left(1 - \mu + \theta f_\theta + (\mu - f_\theta\theta) \frac{(1-\mu)(s-\mu)}{\mu(1-\theta)(1-s)}\right) s \frac{(1-\mu)^2}{\mu(1-\theta)(1-s)^2} ds.$$

The total expected revenue from a bidder is given by

$$f_\theta \int_0^{\mu} \left(f_0 + f_\theta G_\theta(s)\right) s dG_\theta(s) ds + f_1 \int_{\mu}^{\frac{2\mu-\mu-\mu\theta}{1-\mu\theta}} \left(f_0 + f_\theta + f_1 G_1(s)\right) s dG_1(s) ds$$

$$= f_\theta \int_0^\mu \left(1-\mu-(1-\theta) f_\theta + f_\theta \frac{s(\theta-\mu)}{\mu(\theta-s)}\right) s \frac{(\theta-\mu)\theta}{\mu(\theta-s)^2} ds$$

$$+ (\mu - f_\theta \theta) \int_\mu^{\frac{2\mu-\mu-\mu\theta}{1-\mu\theta}} \left(1-\mu+\theta f_\theta + (\mu - f_\theta\theta) \frac{(1-\mu)(s-\mu)}{\mu(1-\theta)(1-s)}\right) s \frac{(1-\mu)^2}{\mu(1-\theta)(1-s)^2} ds.$$

3.4.3 Revenue comparison

After calculating the expected revenue of the first-price and the second-price auction we can compare the revenue for a given θ and μ in dependence of the valuation of f_θ. The minimum possible valuation for f_θ is zero. In case $\theta \leq \mu$ the maximum possible valuation of f_θ is obtained if $f_0 = 0$ and is equal to $\frac{1-\mu}{1-\theta}$. In case $\theta > \mu$, the maximum possible valuation of f_θ is obtained if $f_1 = 0$ and is equal to $\frac{\mu}{\theta}$.

The following graph illustrates the revenue comparison for the first-price auction (blue line) and the second-price auction (red line) for the parameters $\theta = 0.4$ and $\mu = 0.5$.

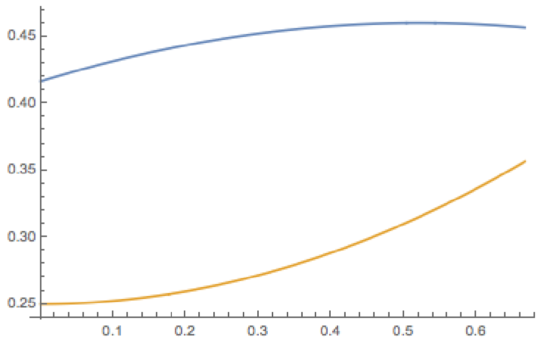


Figure 4: Revenue of the first-price auction (blue line) and second-price auction (red line) plotted against f_θ for $\theta = 0.4$ and $\mu = 0.5$

In this case the auctioneer would choose the first-price auction independent of the true valuation distributions. However, there exist valuations for θ and μ where the revenue functions cross, i.e. it depends on the true valuation distribution which auction leads to the higher revenue.

The following graph illustrates the revenue comparison for the first-price auction (blue line) and the second-price auction (red line) for the parameters $\theta = 0.6$ and $\mu = 0.5$.

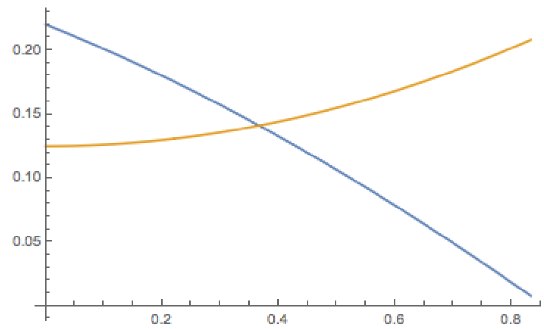


Figure 5: Revenue of the first-price auction (blue line) and second-price auction (red line) plotted against f_θ for $\theta = 0.6$ and $\mu = 0.5$

We conclude that the revenue comparison highly depends on the parameters θ and μ and depending on the valuation of θ and μ, it can depend on the true valuation distribution.

3.5 General Case: n bidders with m valuations

In this section we provide all definitions and results required for the general case with n bidders and m types. As before, the main result is that there exists an efficient worst-case belief equilibrium.

Theorem 3. *In a first-price auction there exists an efficient worst-case belief equilibrium.*

3.5.1 Characterization of the efficient worst-case belief equilibrium

As in the case of two bidders and three types we begin with the characterization of the strategies and beliefs which we claim to constitute a worst-case belief equilibrium.[38] We denote the worst-case strategy by β^*. The support of the bid distribution of a bidder with valuation θ^k is denoted by $[\bar{b}_{\theta^{k-1}}, \bar{b}_{\theta^k}]$. As every bidder adopts the same worst-case belief-equilibrium, we omit the identity of the bidder in the notation. Every bidder has the same worst-case belief and moreover, in the worst-case belief of a bidder every other bidder has the same valuation distribution. Thus, we can denote the worst-case belief of a bidder with valuation $\theta^k \in \Theta$ by $\left(f_{\theta^1}^{\theta^k,*}, \ldots, f_{\theta^m}^{\theta^k,*}\right)$, i.e. for $l \in \{1, \ldots, m\}$ let $f_{\theta^l}^{\theta^k,*}$ be the probability with which one of the other $n-1$ bidders has the θ^l-type in the belief of a bidder with valuation θ^k.

For a bidder with valuation $\theta^k \leq \mu$ we define the bidder's strategy to be a pure strategy with $\beta^* \left(\theta^k\right) = \theta^k$. Let θ^z be the lowest type which is strictly greater than μ. The belief of a bidder with valuation $\theta^k \leq \mu$ is the probability function which puts strictly positive weight only on $f_{\theta^k}^{\theta^k}$ and $f_{\theta^z}^{\theta^k}$. Therefore, the probability weight is determined by the equations

$$f_{\theta^k}^{\theta^k,*} + f_{\theta^z}^{\theta^k,*} = 1$$

[38] As before, we call the strategy and beliefs we claim to constitute a worst-case belief equilibrium worst-case strategy and worst-case beliefs.

$$f_{\theta^k}^{\theta^k,*}\theta^k + f_{\theta^z}^{\theta^k,*}\theta^z = \mu.$$

The unique solution of this system of linear equations is given by

$$f_{\theta^k}^{\theta^k,*} = \frac{\theta^z - \mu}{\theta^z - \theta^k}, \quad f_{\theta^k}^{\theta^z,*} = \frac{\mu - \theta^k}{\theta^z - \theta^k}.$$

Given this belief, it is a best reply for a bidder with valuation θ^k to bid θ^k since the lowest bid which such a bidder believes is played by another bidder is given by θ^k. This induces the lowest possible expected utility of zero and therefore, the strategies and beliefs specified for types $\theta^k \leq \mu$ fulfill the best-reply condition and the worst-case belief condition.

Now we define the bidding strategy and beliefs for a bidder with valuation θ^k with $\theta^k > \mu$. A bidder with type $\theta^k > \mu$ plays a mixed strategy on the interval $[\overline{b}_{\theta^{k-1}}, \overline{b}_{\theta^k}]$ where the upper limit of type θ^k's bidding interval is the lower limit of type θ^{k+1}'s bidding interval. We will derive the boundaries of this bidding interval inductively starting with the boundaries of the bidding interval of the θ^z-type, which we defined above as the lowest type strictly greater than μ. The θ^z-type plays a mixed strategy on the interval $[\overline{b}_{\theta^{z-1}}, \overline{b}_{\theta^z}]$ with $\overline{b}_{\theta^{z-1}} = \theta^{z-1}$. We define the worst-case belief of a bidder with valuation θ^z to be the solution of the following minimization problem which we denote by $M^{\theta^z}_{<\theta^{z-1}}$:

$$\min_{(f_{\theta^1},\dots,f_{\theta^m})} \; (f_{\theta^1} + \cdots + f_{\theta^{z-1}})^{n-1}\left(\theta^z - \theta^{z-1}\right)$$

$$s.t. \; f_{\theta^j} \geq 0 \text{ for all } 1 \leq j \leq m$$

$$\sum_{l=1}^{m} f_{\theta^l} = 1$$

$$\sum_{l=1}^{m} f_{\theta^l}\theta^l = \mu$$

$$(f_{\theta^1} + \cdots + f_{\theta^{z-1}})^{n-1}\left(\theta^z - \theta^{z-1}\right) \geq (f_{\theta^1})^{n-1}\theta^z$$

$$(f_{\theta^1} + \cdots + f_{\theta^{j-1}})^{n-1}\left(\theta^z - \theta^{z-1}\right) \geq (f_{\theta^1} + f_{\theta^2})^{n-1}\left(\theta^z - \theta^2\right)$$

$$\vdots$$

$$(f_{\theta^1} + \cdots + f_{\theta^{z-1}})^{n-1}\left(\theta^z - \theta^{z-1}\right) \geq (f_{\theta^1} + \cdots + f_{\theta^{z-2}})^{n-1}\left(\theta^z - \theta^{z-2}\right),$$

i.e. among all distributions with mean μ such that for a bidder with valuation θ^z it is weakly better to bid θ^{z-1} than any lower bid given the other bidders' strategies, it is the distribution inducing the minimum utility. We do not have to include the incentive constraints with corresponding bid b for $b \in (\theta^{h-1}, \theta^h)$ for $1 < h < z-1$ since these bids are never played according to the worst-case strategy and thus are never a best reply. Note that the feasible set of this minimization problem is non-empty since a distribution which puts strictly positive probability weight only on the θ^{z-1}- and the θ^z-type preserving the mean μ is an element of the feasible set. The upper endpoint of the bidding interval of the θ^z-type is obtained by the equation

$$\left(f_{\theta^1}^{\theta^z,*} + \cdots + f_{\theta^{z-1}}^{\theta^z,*}\right)^{n-1}\left(\theta^z - \theta^{z-1}\right) = (\theta^z - \bar{b}_{\theta^z}).$$

The bid distribution G_{θ^z} is defined such that every bidder with valuation θ^z is indifferent between every bid in her bidding interval given her belief and the other bidders' strategies, i.e. for every $s \in [\bar{b}_{\theta^{z-1}}, \bar{b}_{\theta^z}]$ where $\bar{b}_{\theta^{z-1}} = \theta^{z-1}$ it holds

$$\left(f_{\theta^1}^{\theta^z,*} + \cdots + f_{\theta^{z-1}}^{\theta^z,*} + f_{\theta^z}^{\theta^z,*} G_{\theta^z}(s)\right)^{n-1}(\theta^z - s) = \left(f_{\theta^1}^{\theta^z,*} + \cdots + f_{\theta^{z-1}}^{\theta^z,*}\right)^{n-1}\left(\theta^z - \theta^{z-1}\right).$$

After we have specified the strategies and beliefs for the θ^z-type, we can proceed inductively. Assume, that strategies and beliefs have been specified for types $1, \ldots, k-1$ with $z \leq k-1 < m$, then strategies and beliefs for type k are defined as follows. A bidder with valuation θ^k plays a mixed strategy on the interval $[\bar{b}_{\theta^{k-1}}, \bar{b}_{\theta^k}]$ where $\bar{b}_{\theta^{k-1}}$ is the upper bound of the bidding interval of the θ^{k-1}-type. We define the worst-case belief of type θ^k to be the solution of the following minimization problem which we denote by $M^{\theta^k}_{<\bar{b}_{\theta^{k-1}}}$:

$$\begin{aligned}
\min_{(f_{\theta^1}, \ldots, f_{\theta^m})} \quad & (f_{\theta^1} + \cdots + f_{\theta^{k-1}})^{n-1}\left(\theta^k - \bar{b}_{\theta^{k-1}}\right) \\
s.t.\ & f_{\theta^j} \geq 0 \text{ for all } 1 \leq j \leq m \\
& \sum_{j=1}^{m} f_{\theta^j} = 1 \\
& \sum_{j=1}^{m} f_{\theta^j}\theta^j = \mu \\
(f_{\theta^1} + \cdots + f_{\theta^{k-1}})^{n-1}\left(\theta^k - \bar{b}_{\theta^{k-1}}\right) & \geq \left(\left(\sum_{j=1}^{h-1} f_{\theta^j}\right) + f_{\theta^h} G_{\theta^h}(s)\right)^{n-1}\left(\theta^k - s\right) \\
& \text{for all } h \in \{1, \ldots, k-1\} \text{ and all } s \in [\bar{b}_{\theta^{h-1}}, \bar{b}_{\theta^h}],
\end{aligned}$$

i.e. among all distributions with mean μ such that for a bidder with valuation θ^k it is weakly better to bid $\bar{b}_{\theta^{k-1}}$ than any lower bid given the other bidders' strategies, it is the distribution inducing the minimum utility. The bid distribution G_{θ^k} and $\bar{b}_{\theta^k}$ are determined such that given this belief every bidder with valuation θ^k is indifferent between all bids in $[\bar{b}_{\theta^{k-1}}, \bar{b}_{\theta^k}]$. Formally, for every $s \in [\bar{b}_{\theta^{k-1}}, \bar{b}_{\theta^k}]$, $G_{\theta^k}(s)$ is defined by

$$\left(\sum_{j=1}^{k-1} f_{\theta^j}^{\theta^k,*}\right)^{n-1}\left(\theta - \bar{b}_{\theta^{k-1}}\right) = \left(\sum_{j=1}^{k-1} f_{\theta^j}^{\theta^k,*} + f_{\theta^k}^{\theta^k,*} G_{\theta^k}(s)\right)^{n-1}(\theta - s).$$

Obviously, the worst-case strategy is efficient. We will show in the next section that the feasible set of minimization problem $M^{\theta^k}_{<\bar{b}_{\theta^{k-1}}}$ is not empty. Moreover, in Lemma 28 in Appendix 3.8.5 we derive the unique solution of this minimization problem. We show that for the worst-case belief of a bidder with valuation θ^k it holds that $f_j^{\theta^k,*} = 0$ for $j > k$ and that the vector $\left(f_{\theta^1}^{\theta^k,*}, \ldots, f_{\theta^k}^{\theta^k,*}\right)$ is the unique solution of the system of k linear equations which includes the two probability constraints and the binding incentive constraints with corresponding bid $\bar{b}_{\theta^j}$ for $1 \leq j \leq k-2$.

3.5.2 Proving the best-reply and the worst-case belief condition

After specifying the worst-case strategy and beliefs, we have to show that they indeed constitute a worst-case belief equilibrium. That is, we have to show the best-reply and the worst-case belief condition.

Proposition 11. *Given the worst-case strategy and the worst-case beliefs as defined in 3.5.1, it holds for all $\hat{\theta} \in \Theta$ that*

(i) The best-reply condition given by

$$b_{\hat{\theta}} \in B^r\left(\hat{\theta}, f^{\hat{\theta},*}, \beta^*\right) \text{ for all } b_{\hat{\theta}} \in supp\left(\beta^*\left(\hat{\theta}\right)\right)$$

is fulfilled, i.e. every bidder plays a best reply given her valuation, her worst-case belief and the other bidders' worst-case strategy.

(ii) The worst-case belief condition is fulfilled, i.e. for all $b_{\hat{\theta}} \in supp\left(\beta^*\left(\hat{\theta}\right)\right)$ it holds that

$$U\left(\hat{\theta}, f^{\hat{\theta},*}, b_{\hat{\theta}}, \beta^*\right) \leq U\left(\hat{\theta}, f, b^r\left(\hat{\theta}, f, \beta^*\right), \beta^*\right) \text{ for all } f \in \mathscr{F}_\mu^{n-1}.^{39}$$

That is, there does not exist another belief such that a best reply to this belief induces a lower expected utility than the worst-case belief.

Proof. Part *(i)*: It follows directly from the definition of the worst-case beliefs, that for a bidder with valuation θ^k it is weakly better to bid $\bar{b}_{\theta^{k-1}}$ than any lower bid. By construction, a bidder with valuation θ^k is indifferent between any bid in $[\bar{b}_{\theta^{k-1}}, \bar{b}_{\theta^k}]$. Hence, it is left to show that it is weakly better to bid $\bar{b}_{\theta^{k-1}}$ than any bid higher than $\bar{b}_{\theta^k}$. In order to do so, we will compare the solutions of the following two minimization problems. Let $M^{\theta^k}_{<\bar{b}_{\theta_{k-1}}}$ be the minimization problem which corresponds to the worst-case belief of the bidder as defined above:

$$\min_{(f_{\theta^m}, \ldots, f_{\theta^m})} \left(f_{\theta^1} + \cdots + f_{\theta^{k-1}}\right)^{n-1}\left(\theta^k - \bar{b}_{\theta^{k-1}}\right)$$

[39] Since utility functions are symmetric among bidders, we will omit the identitiy of the bidder in utility function. Moreover, if there exists an asymmetric belief about the other bidders' valuations which violates the worst-case belief condition then due to the symmetry of the worst-case strategy, there exists also a symmetric belief. Therefore, it is sufficient to focus only on symmetric beliefs as possible deviations.

$$s.t.\ f_{\theta^j} \geq 0 \text{ for all } 1 \leq j \leq m$$

$$\sum_{j=1}^{m} f_{\theta^j} = 1$$

$$\sum_{j=1}^{m} f_{\theta^j} \theta^j = \mu$$

$$\left(f_{\theta^1} + \cdots + f_{\theta^{k-1}}\right)^{n-1} \left(\theta^k - \overline{b}_{\theta^{k-1}}\right) \geq \left(\sum_{j=1}^{h-1} f_{\theta^j} + f_{\theta^h} G_{\theta^h}(s)\right)^{n-1} \left(\theta^k - s\right)$$

$$\text{for all } h \in \{1, \ldots, k-1\} \text{ and all } s \in [\overline{b}_{\theta^{h-1}}, \overline{b}_{\theta^h}],$$

i.e. among all distributions with mean μ such that for a bidder with valuation θ^k it is weakly better to bid $\overline{b}_{\theta^{k-1}}$ than any *lower* bid given the other bidders' strategies, it is the distribution inducing the minimum utility. Now we consider minimization problem $M^{\theta^k}_{\overline{b}_{\theta^{k-1}}}$ which determines the distribution inducing the minimum utility among all distributions with mean μ such that for a bidder with valuation θ^k it is weakly better to bid $\overline{b}_{\theta^{k-1}}$ than any *other* bid given the other bidders' strategies:

$$\min_{(f_{\theta^1}, \ldots, f_{\theta^m})} \left(f_{\theta^1} + \cdots + f_{\theta^{k-1}}\right)^{n-1} \left(\theta^k - \overline{b}_{\theta^{k-1}}\right)$$

$$s.t.\ f_{\theta^j} \geq 0 \text{ for all } 1 \leq j \leq m$$

$$\sum_{j=1}^{m} f_{\theta^j} = 1$$

$$\sum_{j=1}^{m} f_{\theta^j} \theta^j = \mu$$

$$\left(f_{\theta^1} + \cdots + f_{\theta^{k-1}}\right)^{n-1} \left(\theta^k - \overline{b}_{\theta^{k-1}}\right) \geq \left(\left(\sum_{j=1}^{h} f_{\theta^j}\right) + f_{\theta^h} G_{\theta^h}(s)\right)^{n-1} \left(\theta^k - s\right)$$

$$\text{for all } h \in \{1, \ldots, m\} \text{ and all } s \in [\overline{b}_{\theta^{h-1}}, \overline{b}_{\theta^h}].$$

Let $f^{\theta^k,*}$ and f^{θ^k} be solutions of $M^{\theta^k}_{<\overline{b}_{\theta^{k-1}}}$ and $M^{\theta^k}_{\overline{b}_{\theta^{k-1}}}$ respectively. The constraints of $M^{\theta^k}_{<\overline{b}_{\theta^{k-1}}}$ are a subset of the constraints of $M^{\theta^k}_{<\overline{b}_{\theta^{k-1}}}$. Therefore, it is sufficient to show that $f^{\theta^k,*}$ is an element of the feasible set of $M^{\theta^k}_{\overline{b}_{\theta^{k-1}}}$.

In $f^{\theta^k,*}$ there is no probability weight on types above θ^k because this would require more probability weight on types below μ and hence increase the value of the objective function. If we plug in $f^{\theta^k,*}$ into $M^{\theta^k}_{\overline{b}_{\theta^{k-1}}}$, then all constraints which correspond to a bid above $\overline{b}_{\theta^k}$ are fulfilled because there is no probability weight on types above θ^k. As argued above, all constraints with corresponding bid in the interval $[0, \overline{b}_{\theta^k}]$ are fulfilled. Therefore, $f^{\theta^k,*}$ is an element of the feasible set of $M^{\theta^k}_{\overline{b}_{\theta^{k-1}}}$. $\square$

Computing the worst-case belief of a bidder is equivalent to computing the distribution inducing the minimum utility of a bidder given the other bidders' strategies. Thus, one has to solve the trade-off between putting probability weight on lower types in order to induce a high bid and putting probability weight on higher types in order to reduce the winning probability.

This proof shows that this trade-off is solved such that the worst-case belief of a bidder with valuation θ^k puts just enough probability weight on lower types in order to induce the bid $\bar{b}_{\theta^{k-1}}$ and puts as much as possible probability weight on type θ^k in order to reduce the bidder's winning probability.

One can use this proof in order to show that the worst-case belief of the θ^{k-1}-type is an element of the feasible set of minimization problem $M^{\theta^k}_{<\bar{b}_{\theta^{k-1}}}$. Hence, one can show by induction that for all $1 \leq k \leq m$ the feasible set of minimization problem $M^{\theta^k}_{<\bar{b}_{\theta^{k-1}}}$ is not empty.

By definition, the worst-case belief of the θ^{k-1}-type is a solution of minimization problem $M^{\theta^{k-1}}_{<\bar{b}_{\theta^{k-2}}}$. Moreover, it holds by definition of $\bar{b}_{\theta^{k-1}}$ that

$$\left(\sum_{j=1}^{k-2} f^{\theta^{k-1},*}_{\theta^j}\right)\left(\theta^{k-1} - \bar{b}_{\theta^{k-2}}\right) = \theta^{k-1} - \bar{b}_{\theta^{k-1}}.$$

Therefore, for all s with $s \in [\bar{b}_{\theta^{h-1}}, \bar{b}_{\theta^h}]$ with $h \leq k-1$ it holds that

$$\theta^{k-1} - \bar{b}_{\theta^{k-1}} = \left(\sum_{j=1}^{k-2} f^{\theta^{k-1},*}_{\theta^j}\right)^{n-1}\left(\theta^{k-1} - \bar{b}_{\theta^{k-2}}\right) \geq \left(\sum_{j=1}^{h-1} f^{\theta^{k-1},*}_{\theta^j} + G_{\theta^h}(s)\right)^{n-1}\left(\theta^{k-1} - s\right).$$

It follows that for all s with $s < \bar{b}_{\theta^{k-1}}$ the incentive constraint corresponding to s is fulfilled if plugging in $f^{\theta^{k-1},*}$ into $M^{\theta^k}_{\bar{b}_{\theta^{k-1}}}$ because adding the inequalities

$$\theta^{k-1} - \bar{b}_{\theta^{k-1}} \geq \left(\sum_{j=1}^{h-1} f^{\theta^{k-1},*}_{\theta^j} + G_{\theta^h}(s)\right)^{n-1}\left(\theta^{k-1} - s\right)$$

and

$$\theta^k - \theta^{k-1} \geq \left(\sum_{j=1}^{h-1} f^{\theta^{k-1},*}_{\theta^j} + G_{\theta^h}(s)\right)^{n-1}\left(\theta^k - \theta^{k-1}\right)$$

yields

$$\theta^k - \bar{b}_{\theta^{k-1}} \geq \left(\sum_{j=1}^{h-1} f^{\theta^{k-1},*}_{\theta^j} + G_{\theta^h}(s)\right)^{n-1}\left(\theta^k - s\right).$$

Conclusively, $f^{\theta^{k-1},*}$ is an element of the feasible set of minimization problem $M^{\theta^k}_{<\bar{b}_{\theta^{k-1}}}$.

We have already shown the worst-case belief condition for all types $\theta^k \leq \mu$. In order to show the worst-case belief condition for higher types, as in the case of three types and two bidders, we introduce the concept of minimizing probability functions and show that we can switch from comparing the induced utility of distributions to comparing the induced utility of bids. This is formalized in the following definition and observation.

Definition 13. *For a bidder with valuation θ_i, a bid b_i and a strategy β_{-i} of the other bidders, the set of probability functions $\mathscr{F}_{n-1}^{min}(\theta_i, b_i, \beta_{-i})$ given by*

$$\mathscr{F}_{n-1}^{min}(\theta_i, b_i, \beta_{-i}) = \underset{f_{-i}\in\mathscr{F}_{\mu}^{n-1}}{\operatorname{argmin}} \{U(\theta_i, f_{-i}, b_i, \beta_{-i}) \mid b_i \in B^r(\theta_i, f_{-i}, \beta_{-i})\}$$

is called the set of minimizing probability functions *of bid b_i for a bidder with valuation θ_i given the other bidders' strategies β_{-i}. Among all probability functions which induce bid b_i as a best reply, a minimizing probability function is a probability function which induces the minimum utility.*

Observation 3. *Let $(\beta_1, \ldots, \beta_n)$ be a profile of strategies and $\left(f_{-i}^{\theta^1}, \ldots, f_{-i}^{\theta^m}\right)$ be a profile of beliefs bidder i has about the other bidders' valuations. For a valuation θ_i of bidder i and a bid $b_i \in supp(\beta_i(\theta_i))$ the worst-case belief condition for bid b_i, given by*

$$U\left(\theta_i, f_{-i}^{\theta_i}, b_i, \beta_{-i}\right) \leq U\left(\theta_i, f_{-i}^{\theta_i}, b^r(\theta_i, f_{-i}, \beta_i), \beta_{-i}\right)$$

for all $f_{-i} \in \mathscr{F}_{\mu}^{n-1}$, is equivalent to the following two conditions:

(i) The belief $f_{-i}^{\theta_i}$ is an element in $\mathscr{F}_{n-1}^{min}(\theta_i, b_i, \beta_{-i})$, i.e. a minimizing probability function of bid b_i for bidder i with valuation θ_i given the other bidders' strategies β_{-i}.

(ii) Let b_i' be a bid and f_{-i} be an element in $\mathscr{F}_{n-1}^{min}(\theta_i, b_i', \beta_{-i})$, i.e. a minimizing probability function of bid b_i' for bidder i with valuation θ_i. Then it holds

$$U\left(\theta_i, f_{-i}^{\theta_i}, b_i, \beta_{-i}\right) \leq U(\theta_i, f_{-i}, b_i', \beta_{-i}).$$

That is, it is sufficient to compare bids if we compare them with respect to the expected utility they induce together with a minimizing probability function. In order to apply this technique, we need the following definitions.

Definition 14. *For a bidder with valuation θ minimization problem M_b^{θ} of a bid $b \in [\bar{b}_{\theta^{l-1}}, \bar{b}_{\theta^l}]$ is the minimization problem corresponding to its minimizing probability functions, i.e. all solutions of minimization problem M_b^{θ} are minimizing probability function of b for a bidder with valuation θ given the other bidders' worst-case strategy β^*. Formally, minimization problem M_b^{θ} is given by*

$$\min_{(f_{\theta^1}, \ldots, f_{\theta^m})} \left(\sum_{j=1}^{l-1} f_{\theta^j} + f_{\theta^l} G_l(b)\right)^{n-1} (\theta - b)$$

$$s.t.\ f_{\theta^j} \geq 0 \ for\ all\ 1 \leq j \leq m$$

$$\sum_{j=1}^{m} f_{\theta^j} = 1$$

$$\sum_{j=1}^{m} f_{\theta^j} \theta^j = \mu$$

$$\left(f_{\theta^1} + \cdots + f_{\theta^{l-1}}\right)^{n-1} (\theta - b) \geq \left(\sum_{j=1}^{h} f_{\theta^j} + f_{\theta^h} G_{\theta^h}(s)\right)^{n-1} (\theta - s)$$

$$for\ all\ h \in \{1, \ldots, m\}\ and\ all\ s \in [\bar{b}_{\theta^{h-1}}, \bar{b}_{\theta^h}].$$

Definition 15. *Apart from the constraints*

$$f_{\theta^j} \geq 0 \ for\ all\ 1 \leq j \leq m$$

$$\sum_{j=1}^{m} f_{\theta^j} = 1$$

$$\sum_{j=1}^{m} f_{\theta^j} \theta^j = \mu,$$

every constraint in minimization problem M_b^θ compares the utility of bidding b to the utility of bidding some other bid b', which is formalized by

$$U(\theta, f, b, \beta^*) \geq U(\theta, f, b', \beta^*).$$

We call such a constraint an incentive constraint corresponding to bid b'.

Definition 16. *For a type θ and bids b, b' we use the notation $b \leq^\theta b'$ if for the θ-type bid b' does not induce a strictly lower expected utility than bid b together with their minimizing probability functions given the other bidders' worst-case strategy β^*. Formally, let $f^{min}(\theta, b, \beta^*) \in \mathscr{F}_{n-1}^{min}(\theta, b, \beta^*)$ and $f^{min}(\theta, b', \beta^*) \in \mathscr{F}_{n-1}^{min}(\theta, b', \beta^*)$. Then it holds that*

$$U\left(\theta, f^{min}(\theta, b, \beta^*), b, \beta^*\right) \leq U\left(\theta, f^{min}(\theta, b', \beta^*), b', \beta^*\right) \Rightarrow b \leq^\theta b',$$

$$U\left(\theta, f^{min}(\theta, b, \beta^*), b, \beta^*\right) < U\left(\theta, f^{min}(\theta, b', \beta^*), b', \beta^*\right) \Rightarrow b <^\theta b'$$

and

$$U\left(\theta, f^{min}(\theta, b, \beta^*), b, \beta^*\right) = U\left(\theta, f^{min}(\theta, b', \beta^*), b', \beta^*\right) \Rightarrow b =^\theta b'.$$

We also use the notation $b <^\theta b'$ if b' does not have a minimizing probability function given θ because it is never a best reply for a bidder with valuation θ, but b does have a minimizing probability function. We use the notation $b =^\theta b'$ if neither b, nor b' have a minimizing probability function.

Given the notation provided in this Definition, we can state a condition which is equivalent to the worst-case belief condition but is more tractable:

Observation 4. *The worst-case belief condition for bidder i with valuation $\hat{\theta}$ and bid $b_{\hat{\theta}} \in supp\left(\beta^*\left(\hat{\theta}\right)\right)$ and bidder B's strategy β^* given by*

$$U\left(\hat{\theta}, f^{\hat{\theta},*}, b_{\hat{\theta}}, \beta^*\right) \leq U\left(\hat{\theta}, f, b^r\left(\hat{\theta}, f, \beta^*\right), \beta^*\right) \text{ for all } f \in \mathscr{F}_{\mu}^{n-1}$$

is equivalent to

(i) $$f^{\hat{\theta},*} \in \mathscr{F}_{n-1}^{min}\left(\hat{\theta}, b_{\hat{\theta}}, \beta^*\right)$$

(ii) $$b_{\hat{\theta}} \leq^{\hat{\theta}} b' \text{ for all } b' \in [0, \bar{b}_{\theta^m}].$$

As in the case with two bidders and three valuations, we prove three lemmas which correspond to three different tools with which we can compare the utility induced by different bids and therefore exclude bids as possible deviations from the proposed worst-case strategy. The first tool is to show that for every valuation $\theta^k \geq \theta^z$ for every bid in the interval $(\bar{b}_{\theta^{k-1}}, \bar{b}_{\theta^k}]$ there exists only one probability function which induces this bid as a best reply for the θ^k-type.[40] As a consequence, one can directly compute the minimum utility which can be induced for a bid in the interval $[\bar{b}_{\theta^{k-1}}, \bar{b}_{\theta^k}]$ and show that the minimum utility is equal for all bids in the interval $[\bar{b}_{\theta^{k-1}}, \bar{b}_{\theta^k}]$. This is formalized in the following Lemma and Corollary.

Lemma 22. *For a valuation θ^k with $\theta^z \leq \theta^k \leq \theta^m$ and $b \in [\bar{b}_{\theta^{k-1}}, \bar{b}_{\theta^k}]$ let $f^{min}\left(\theta^k, b, \beta^*\right)$ be an element in $\mathscr{F}_{n-1}^{min}\left(\theta^k, b, \beta^*\right)$. Then $f^{min}\left(\theta^k, b, \beta^*\right)$ equals to $f^{\theta^k,*}$, the worst-case belief of a bidder with valuation θ.*

The intuition behind this result works similarly as for the result for two bidders and three types in Lemma 14. The formal proof is relegated to Appendix 3.8.9.

Corollary 5. *For every valuation θ^k with $\theta^k \geq \theta^z$ and for every $b \in (\bar{b}_{\theta^{k-1}}, \bar{b}_{\theta^k}]$ it holds that $b =^{\theta^k} \bar{b}_{\theta^{k-1}}$.*

That is, every bid in the interval $[\bar{b}_{\theta^{k-1}}, \bar{b}_{\theta^k}]$ induces the same expected utility together with a minimizing probability function.

Proof. We have shown in the first part of Proposition 11 that the best-reply condition is fulfilled for all types. Hence, it holds that the worst-case belief of a bidder with valuation $\theta^k \geq \theta^z$, which is the solution of minimization problem $M^{\theta^k}_{<\bar{b}_{\theta^{k-1}}}$, is an element of the feasible set of minimization problem $M^{\theta^k}_{\bar{b}_{\theta^k}}$. Since the constraints of $M^{\theta^k}_{<\bar{b}_{\theta^{k-1}}}$ are a subset of $M^{\theta^k}_{\bar{b}_{\theta^{k-1}}}$, it holds that $f^{\theta,*}$ is a solution of $M^{\theta^k}_{<\bar{b}_{\theta^{k-1}}}$. It follows from Lemma 22 and the

[40] Recall that we defined θ^z to be the smallest valuation which is strictly greater than μ.

definition of the worst-case belief of a bidder with valuation θ^k that every bid in $(\overline{b}_{\theta^{k-1}}, \overline{b}_{\theta^k}]$ together with its unique minimizing probability function induces the same expected utility given by

$$\left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^{k-1}}^{\theta^k,*}\right)\left(\theta^k - \overline{b}_{\theta^{k-1}}\right).$$

Thus, it holds for every $b \in (\overline{b}_{\theta^{k-1}}, \overline{b}_{\theta^k}]$ that

$$b =^{\theta^k} \overline{b}_{\theta^{k-1}}.$$

$\square$

The second tool constitutes a connection between binding incentive constraints in the minimization problem corresponding to a bid b and bids which are lower than b with respect to our order. It corresponds to Lemmas 15,17 and 20 in the case of two bidders and three types.

Lemma 23. *Let θ be a valuation, b a bid and $f^{\theta,b}$ a solution of minimization problem M_b^θ. If there exists a binding incentive constraint with corresponding bid $\hat{b}$, i.e.*

$$U\left(\theta, f^{\theta,b}, b, \beta^*\right) = U\left(\theta, f^{\theta,b}, \hat{b}, \beta^*\right),$$

then it holds that $\hat{b} \leq^\theta b$.

Proof. Let L_b^θ and $L_{\hat{b}}^\theta$ be the set of feasible solutions , $f^{\theta,b} = \left(f_{\theta^1}^{\theta,b}, \ldots, f_{\theta^m}^{\theta,b}\right)$ and $f^{\theta,\hat{b}} = \left(f_0^{\theta,\hat{b}}, \ldots, f_{\theta^m}^{\theta,\hat{b}}\right)$ solutions and $U\left(\theta, f^{\theta,b}, b, \beta^*\right)$ and $U\left(\theta, f^{\theta,\hat{b}}, \hat{b}, \beta^*\right)$ the values of the objective functions of minimization problem M_b^θ and $M_{\hat{b}}^\theta$ respectively. In minimization problem $M_{\hat{b}}^\theta$ for every bid s the incentive constraint corresponding to s given by

$$U\left(\theta, f, \hat{b}, \beta^*\right) \geq U\left(\theta, f, s, \beta^*\right)$$

is fulfilled for $f = f^{\theta,b}$ because it holds that

$$U\left(\theta, f^{\theta,b}, \hat{b}, \beta^*\right) = U\left(\theta, f^{\theta,b}, b, \beta^*\right) \geq U\left(\theta, f^{\theta,b}, s, \beta^*\right).$$

The equality follows from the fact that the incentive constraint corresponding to $\hat{b}$ is binding in minimization problem M_b^θ. The inequality

$$U\left(\theta, f^{\theta,b}, b, \beta^*\right) \geq U\left(\theta, f^{\theta,b}, s, \beta^*\right)$$

holds because f_b^θ is a solution of minimization problem M_b^θ. Since every constraint in $M_{\hat{b}}^\theta$ is fulfilled by $f^{\theta,b}$, it holds that $f^{\theta,b}$ is an element of $L_{\hat{b}}^\theta$. Therefore in $M_{\hat{b}}^\theta$, the solution of minimization problem $M_{\hat{b}}^\theta$ has to induce a lower or equal utility than the solution of minimization problem M_b^θ and it follows that

$$U\left(\theta, f^{\theta,\hat{b}}, \hat{b}, \beta^*\right) \leq U\left(\theta, f^{\theta,b}, \hat{b}, \beta^*\right) = U\left(\theta, f^{\theta,b}, b, \beta^*\right).$$

We conclude that bid b together with a minimizing probability function does not induce a lower expected utility than bid $\hat{b}$ together with a minimizing probability function and therefore it holds that $\hat{b} \leq^\theta b$. □

For the third tool, we show that for a given type there exist bids which can never be a best reply independent of the subjective belief. Hence, these bids cannot be possible deviations from the proposed worst-case equilibrium for the given type.

Lemma 24. *For every pair of valuations θ^l and θ^k such that $\theta^z \leq \theta^l \leq \theta^{k-1}$ and every b with $\overline{b}_{\theta^{l-1}} < b < \overline{b}_{\theta^l}$ the feasible set of minimization problem $M_b^{\theta^k}$ is empty.*

Lemma 25. *For every pair of valuations θ^l and θ^k such that $\theta^z \leq \theta^{k+1} \leq \theta^l$ and every b with $\overline{b}_{\theta^{l-1}} < b \leq \overline{b}_{\theta^l}$ the feasible set of minimization problem $M_{\overline{b}_{\theta^l}}^{\theta^k}$ is empty.*

The formal proof is relegated to Appendices 3.8.4 and 3.8.6. The intuition for Lemma 24 is similar to Lemma 21 and the intuition for Lemma 25 is similar to Lemma 18, i.e. as in the case of two bidders and three valuations. We provide a detailed intuition for both results at the end of this section.

After introducing these three tools, we can provide the proof of part *(ii)* of Proposition 11. That is, we prove that the worst-case belief condition is fulfilled for all types. In this proof we construct a chain where all bids are arranged with respect to our order and the efficient equilibrium bid is the lowest. Due to the transitivity of our relation, this excludes all other distributions than the efficient worst-case beliefs as a potential deviation.

Proof. Analogously as in the proof of Corollary 5, one can show that for all $\theta^k \geq \theta^z$ it holds that $f^{\theta,*}$ is a solution of $M_b^{\theta^k}$ for all $b \in [\overline{b}_{\theta^{k-1}}, \overline{b}_{\theta^k}]$. It follows from Lemma 22 that

$$f^{\theta^k,*} \in \mathscr{F}^{min}\left(\theta^k, b, \beta^*\right)$$

for all $b \in [\overline{b}_{\theta^{k-1}}, \overline{b}_{\theta^k}]$ Thus, we can conclude from Observation 4 and Corollary 5 that in order to show the worst-case belief condition, it is left to show that for all $\theta^k \geq \theta^z$ it holds that

$$\overline{b}_{\theta^{k-1}} \leq^{\theta^k} b \text{ for all } b \in [0, \overline{b}_{\theta^m}] \backslash [\overline{b}_{\theta^{k-1}}, \overline{b}_{\theta^k}]. \tag{36}$$

Lemma 24 shows that if $b \in [\overline{b}_{\theta^{l-1}}, \overline{b}_{\theta^l}]$ was to induce a lower expected utility than $\overline{b}_{\theta^{k-1}}$ and $l < k$, then b needs to be either $\overline{b}_{\theta^{l-1}}$ or $\overline{b}_{\theta^l}$. Since every lower bound of a bidding interval is the upper bound of some interval, it is w.l.o.g. to assume that b is equal to $\overline{b}_{\theta^l}$ for an appropriate l. Lemma 25 shows that a lower expected utility can be achieved by inducing a bid only in the bidding interval of a lower type. Lemma 24 and 25 combined state that if $b \in [\overline{b}_{\theta^{l-1}}, \overline{b}_{\theta^l}]$ was to induce a lower expected utility than $\overline{b}_{\theta^{k-1}}$, then $b = \overline{b}_{\theta^l}$

for $0 \le l \le k-2$. In order to show that all bids $\overline{b}_{\theta^l}$ with $0 \le l \le k-2$ do not induce a lower expected utility than $\overline{b}_{\theta^{k-1}}$, we need to show the following two Lemmas.

Lemma 26. *For every pair of valuations θ^l and θ^k such that $\theta^z \le \theta^l \le \theta^{k-1}$ the unique solution of minimization problem $M^{\theta^k}_{\overline{b}_{\theta^l}}$ denoted by $\left(f^{\theta^k,\overline{b}_{\theta^l}}_{\theta^1}, \ldots, f^{\theta^k,\overline{b}_{\theta^l}}_{\theta^m}\right)$ is obtained as follows. Choose the minimum $p \in \{1,\ldots,m\}$ such that the probability vector $(f_{\theta^1},\ldots,f_{\theta^m})$ is an element of the feasible set of minimization problem $M^{\theta^k}_{\overline{b}_{\theta^l}}$, i.e. it satisfies*

$$\sum_{j=1}^{m} f_{\theta^j} = 1$$

$$\sum_{j=1}^{m} f_{\theta^j}\theta^j = \mu$$

$$(f_{\theta^1}+\cdots+f_{\theta^l})^{n-1}\left(\theta^k - \overline{b}_{\theta^l}\right) \ge \left(\sum_{j=1}^{h} f_{\theta^j}\right)^{n-1}\left(\theta^k - \overline{b}_{\theta^h}\right) \quad \textit{for all } h \in \{1,\ldots,m\}.$$

$$f_{\theta^j} \ge 0 \quad \textit{for all } j \in \{1,\ldots,m\}$$

where $(f_{\theta^1},\ldots,f_{\theta^{p+2}})$ is the unique solution of the system of linear equations with $p+2$ equations (or p+1 equations if $p \ge l$) given by

$$\sum_{j=1}^{p+2} f_{\theta^j} = 1$$

$$\sum_{j=1}^{p+2} f_{\theta^j}\theta^j = \mu$$

$$(f_{\theta^1}+\cdots+f_{\theta^p})^{n-1}\left(\theta^k - \overline{b}_{\theta^l}\right) = \left(\sum_{j=1}^{h} f_{\theta^j}\right)^{n-1}\left(\theta^k - \overline{b}_{\theta^h}\right) \quad \textit{for all } h \in \{1,\ldots,p\}$$

and $f_{\theta^j} = 0$ for all $j > p+2$ (or all $j > p+1$ if $p \ge l$). [41] *Let p^* be the minimum p. Then $\left(f^{\theta^k,\overline{b}_{\theta^l}}_{\theta^1}, \ldots, f^{\theta^k,\overline{b}_{\theta^l}}_{\theta^{p+2}}\right)$ is the unique solution of the to system of equations*

$$\sum_{j=1}^{p^*+2} f_{\theta^j} = 1$$

$$\sum_{j=1}^{p^*+2} f_{\theta^j}\theta^j = \mu$$

[41] If $p \ge l$, then the number of equations equals to $p+1$ since the equation which is the binding incentive constraint corresponding to bid $\overline{b}_{\theta^l}$ is redundant.

$$\left(f_{\theta^1}+\cdots+f_{\theta^l}\right)^{n-1}\left(\theta^k-\bar{b}_{\theta^l}\right)=\left(\sum_{j=1}^{h} f_{\theta^j}\right)^{n-1}\left(\theta^k-\bar{b}_{\theta^h}\right) \quad \text{for all } h\in\{1,\dots,p^*\}$$

and it holds that $f_{\theta^j}^{\theta^k,\bar{b}_{\theta^l}}=0$ *for all* $m\geq j>p^*+2$ *if* $p^*<l$*. If* $p^*\geq l$*, then there are* $p+1$ *equations and it holds that* $f_{\theta^j}^{\theta^k,\bar{b}_{\theta^l}}=0$ *for all* $p^*+1<j\leq m$*.*

The construction in this Lemma works as follows. We start with the equalities

$$\sum_{j=1}^{2} f_{\theta^j}=1$$

$$\sum_{j=1}^{2} f_{\theta^j}\theta^j=\mu.$$

This is a linear system of two equations which gives a unique f_{θ^1} and f_{θ^2}. If with the probability vector $(f_{\theta^1},f_{\theta^2},0\dots,0)$ we obtain an element of the feasible set of minimization problem $M_{\bar{b}_{\theta^l}}^{\theta^k}$, then we stop. Otherwise we add the equation which is identical to the binding incentive constraint with corresponding bid 0, i.e.

$$\left(f_{\theta^1}+f_{\theta^2}+f_{\theta^3}\right)^{n-1}\left(\theta^k-\bar{b}_{\theta^l}\right)=f_{\theta^1}^{n-1}\theta^k$$

and obtain a unique solution for $(f_{\theta^1},f_{\theta^2},f_{\theta^3})$ and check whether the vector $(f_{\theta^1},f_{\theta^2},f_{\theta^3},0\dots,0)$ is an element of the feasible set and so forth until we find an element of the feasible set of minimization problem $M_{\bar{b}_{\theta^l}}^{\theta^k}$. Let $\bar{b}_{\theta^{p^*}}$ be the bid corresponding to this final binding incentive constraint. The solution of minimization problem $M_{\bar{b}_{\theta^l}}^{\theta^k}$ is given by $\left(f_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}},\dots,f_{\theta^{p^*+2}}^{\theta^k,\bar{b}_{\theta^l}},0,\dots,0\right)$ where $\left(f_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}},\dots,f_{\theta^{p^*+2}}^{\theta^k,\bar{b}_{\theta^l}}\right)$ is the unique solution of the system of equations given by the probability constraints and all added incentive constraints if $p^*<l-1$. In case $p^*\geq l-1$, the solution has p^*+1 variables which are greater than zero.

Lemma 27. *For every pair of valuations* θ^l *and* θ^k *such that* $\theta^z\leq\theta^l\leq\theta^{k-2}$ *the minimum* p *for minimization problem* $M_{\bar{b}_{\theta^l}}^{\theta^k}$ *is greater or equal then* $l+1$*.*

Finally, Lemma 27 states that the construction in Lemma 26 leads to a minimum p which is greater than l. This implies that the binding incentive constraint with corresponding bid $l+1$, i.e.

$$\left(f_{\theta^1}+\cdots+f_{\theta^l}\right)^{n-1}\left(\theta^k-\bar{b}_{\theta^l}\right)=\left(f_{\theta^1}+\cdots+f_{\theta^{l+1}}\right)^{n-1}\left(\theta^k-\bar{b}_{\theta^{l+1}}\right)$$

is added to the system of equations. Therefore, in minimization problem $M_{\bar{b}_{\theta^l}}^{\theta^k}$ the constraint corresponding to $\bar{b}_{\theta^{l+1}}$ is binding and it follows from Lemma 23 that $\bar{b}_{\theta^l}\geq^{\theta^k}\bar{b}_{\theta^{l+1}}$. With the same reasoning in minimization problem $M_{\bar{b}_{\theta^{l+1}}}^{\theta^k}$ the constraint corresponding to $\bar{b}_{\theta^{l+2}}$

is binding and it follows from Lemma 23 that $\overline{b}_{\theta^{l+1}} \geq^{\theta^k} \overline{b}_{\theta^{l+2}}$ so on. Therefore, we can construct the following transitive chain

$$\overline{b}_{\theta^l} \geq^{\theta^k} \overline{b}_{\theta^{l+1}} \geq^{\theta^k} \cdots \geq^{\theta^k} \overline{b}_{\theta^{k-1}}.$$

We conclude that there does not exist a bid which induces a lower expected utility than $\overline{b}_{\theta^{k-1}}$ which shows the statement in (36). This completes the proof of part *(ii)* of Proposition 11 which states that the worst-case belief condition is fulfilled for all types. □

We relegate the formal proofs of Lemma 26 and 27 to Appendices 3.8.7 and 3.8.8 and provide an intuition for Lemma 24-27.

Intuition for Lemma 24-27 The intuition for Lemma 24 works similarly as for Lemma 21: Assume there exists a solution of minimization problem $M_b^{\theta^k}$ such that $\overline{b}_{\theta^{l-1}} < b < \overline{b}_{\theta^l}$ and $\theta^z \leq \theta^l \leq \theta^{k-1}$, denoted by $\left(f_{\theta^1}^{\theta^k,b}, \ldots, f_{\theta^m}^{\theta^k,b}\right)$. Consider two bids $b', b'' \in [\overline{b}_{\theta^{l-1}}, \overline{b}_{\theta^l}]$ with $b'' < b < b'$. The utility for the θ^k-type of bidding b must be at least as high as the utilities of bidding b'' or b'. The higher $f_{\theta^{l-1}}^{\theta^k,b}$, the lower is the optimal bid for type θ^k (if we allow only for bids in the interval $[\overline{b}_{\theta^{l-1}}, \overline{b}_{\theta^l}]$). Therefore, the incentive constraint corresponding to bid b' sets an upper bound on the valuation of $f_{\theta^{l-1}}^{\theta^k,b}$ while the incentive constraint corresponding to bid b'' sets a lower bound. We will show that the conditions resulting from these two bounds contradict each other. Intuitively, a bidder bidding in the interval $[\overline{b}_{\theta^{l-1}}, \overline{b}_{\theta^l}]$ faces the bid distribution G_{θ^l} of the θ^l-type which is constructed in order to make her indifferent between any bid in the interval $[\overline{b}_{\theta^{l-1}}, \overline{b}_{\theta^l}]$. Thus, only for the θ^l-type the upper and the lower bound imposed by the incentive constraints corresponding to bids b'' and b' are compatible.

In order to explain to intuition for Lemma 25 and Lemma 26, we illustrate how to construct a solution of minimization problem $M_{\overline{b}_{\theta^l}}^{\theta^k}$. Given some belief $(f_{\theta^1}, \ldots, f_{\theta^m})$, the expected utility of bidder i with valuation θ^k and bid $\overline{b}_{\theta^l}$ is given by

$$(f_{\theta^1} + \cdots + f_{\theta^l})^{n-1} \left(\theta^k - \overline{b}_{\theta^l}\right).$$

Choosing a probability function which minimizes the expected utility is equivalent to choosing a distribution which minimizes the sum $f_{\theta^1} + \cdots + f_{\theta^l}$. If we would look for a probability function which minimizes the sum $f_{\theta^1} + \cdots + f_{\theta^l}$ considering only the first probability constraint, we would set $f_{\theta^1} + \cdots + f_{\theta^l}$ to zero and put all the probability weight on types above θ^l. If we add the constraint that the probability function must have mean μ, this is not longer possible because the mean would be too high. Therefore, one would select types on which to put a strictly positive probability weight in a way such that the mean of the probabilities of types equal or lower than θ^l is minimized. Then one would put as much as possible probability weight on types above θ^l without violating the constraint that the mean has to be μ. In other words, independently of the valuation of μ

one would put strictly positive probability weight only on types 0 and θ^{l+1} because this choice minimizes the mean of the probabilities of types equal or lower than θ^l. Then we would choose f_{θ^1} and $f_{\theta^{l+1}}$ such that the mean is μ. If we add the incentive constraints, one would shift only so much probability weight on types above 0 as it is necessary to fulfill the incentive constraints. In particular, one would put probability weight on some type θ^j only if the probability weight on lower types cannot be increased without violating a constraint.

The statement in Lemma 26 reflects exactly this reasoning. Consider the system of equations given by the probability constraints and the equations which are identical to the binding incentive constraints with corresponding bids $\bar{b}_{\theta^1}, \ldots, \bar{b}_{\theta^{p-1}}$, i.e.

$$\sum_{j=1}^{p+1} f_{\theta^j} = 1$$

$$\sum_{j=1}^{p+1} f_{\theta^j} \theta^j = \mu$$

$$(f_{\theta^1} + \cdots + f_{\theta^l})^{n-1} \left(\theta^k - \bar{b}_{\theta^l}\right) = \left(\sum_{j=1}^{h} f_{\theta^j}\right)^{n-1} \left(\theta^k - \bar{b}_{\theta^h}\right) \quad \text{for all } h \in \{1, \ldots, p-1\}.$$

Assume that for the solution $f_{\theta^1}, \ldots, f_{\theta^{p+1}}$ of this system of equations (or $f_{\theta^1}, \ldots, f_{\theta^p}$ if $p-1 \geq l$) it does not hold that $(f_{\theta^1}, \ldots, f_{\theta^{p+1}}, 0, \ldots 0)$ is an element of the feasible set of minimization problem $M^{\theta^k}_{\bar{b}_{\theta^l}}$. If we now add the equation with corresponding bid $\bar{b}_{\theta_p}$, i.e.

$$(f_{\theta^1} + \cdots + f_{\theta^l})^{n-1} \left(\theta^k - \bar{b}_{\theta^l}\right) = (f_{\theta^1} + \cdots + f_p)^{n-1} \left(\theta^k - \bar{b}_{\theta^p}\right),$$

then in the solution of the extended system of equations it holds that $f_{p+2} > 0$ (or $f_{\theta^{p+1}} > 0$ if $p \geq l$). We have to check whether the vector $(f_{\theta^1}, \ldots, f_{\theta^{p+2}}, 0, \ldots 0)$ is an element of the feasible set of minimization problem $M^{\theta^k}_{\bar{b}_{\theta^l}}$. Since the new vector has positive probability weight on $f_{\theta^{p+2}}$, it must hold that there is less probability weight on types below f_{p+2} than in the old vector $(f_{\theta^1}, \ldots, f_{\theta^{p+2}}, 0, \ldots 0)$ (and analogously for the case $p \geq l$). Therefore, the construction in Lemma 26 ensures that probability weight on a higher type is shifted only if a constraint in minimization problem $M^{\theta^k}_{\bar{b}_{\theta^l}}$ is not fulfilled and shifting weight on lower types is not possible because all constraints corresponding to lower types already hold with equality.

This reasoning also explains the intuition of Lemma 25. It states that for every $k \in \{1, \ldots, m\}$ and $l > k$ the feasible set of minimization problem $M^{\theta^k}_b$ for $b \in (\bar{b}_{\theta^{l-1}} \bar{b}_{\theta^l}]$ is empty. The belief of type θ^k is constructed such that there is just enough probability weight on types below θ^k in order to induce a mixed strategy in the bidding interval of the θ^k-type. As argued above, the choice of types on which there is strictly positive probability weight minimizes the mean of the probabilities of types below θ^k. If one would try to induce a bid $\overline{b}_{\theta^l}$ above $\overline{b}_{\theta^k}$, the probability weight on the θ^l-type has to be increased. In order to preserve the mean, this would imply a decrease of the probability weight on lower types. This is not possible without violating a constraint since the belief of type θ^k had already the lowest possible mean of the probabilities of types below θ^k.

In order to understand Lemma 27, consider minimization problem $M^{\theta^l}_{\overline{b}_{\theta^{l-1}}}$ for $1 \leq l \leq m$. As shown in the proof of part *(i)* of Proposition 11, the solution of this minimization problem is the worst-case belief of type θ^l denoted by $\left(f^{\theta^l,*}_{\theta^1}, \ldots, f^{\theta^l,*}_{\theta^m}\right)$. Since in this proof we have also shown that in the worst-case belief of the θ^l-type there is no probability weight on types above θ^l, one can also write $\left(f^{\theta^l,*}_{\theta^1}, \ldots, f^{\theta^l,*}_{\theta^l}, 0, \ldots 0\right)$. In Appendix 3.8.5 we prove Lemma 28 which states that the solution of minimization problem $M^{\theta^l}_{\overline{b}_{\theta^{l-1}}}$ is the solution of the system of l equations given by the two probability constraints and the $l-2$ incentive constraints given by the bids $\overline{b}_{\theta^j}$ for $1 \leq j \leq l-2$. Hence, for this minimization problem the minimum p equals to $l-2$. Now consider minimization problem $M^{\theta^k}_{\overline{b}_{\theta^{l-1}}}$ for $1 \leq l \leq k-2$. Let $\left(\tilde{f}^{\theta^k}_{\theta^1}, \ldots, \tilde{f}^{\theta^k}_{\theta^m}\right)$ be the solution of the system of l equations given by the two probability constraints and the binding incentive constraints with corresponding bid $\overline{b}_{\theta^j}$ for $1 \leq j \leq l-2$. How does the vector $\left(\tilde{f}^{\theta^k}_{0}, \ldots, \tilde{f}^{\theta^k}_{\theta^l}\right)$ differ from $\left(f^{\theta^l,*}_{\theta^1}, \ldots, f^{\theta^l,*}_{\theta^l}\right)$? In minimization problem $M^{\theta^l}_{\overline{b}_{\theta^{l-1}}}$ a constraint with corresponding bid $\overline{b}_{\theta^j}$ given by

$$\left(f_{\theta^1} + \ldots + f_{\theta^{l-1}}\right)\left(\theta^l - \overline{b}_{\theta^{l-1}}\right) \geq \left(f_{\theta^1} + \ldots + f_{\theta^j}\right)\left(\theta^l - \overline{b}_{\theta^j}\right)$$

is equivalent to

$$\left(f_{j+1} + \ldots + f_{\theta^{l-1}}\right)\left(\theta^l - \overline{b}_{\theta^{l-1}}\right) - \left(f_{\theta^1} + \ldots + f_{\theta^j}\right)\left(\overline{b}_{\theta^l} - \overline{b}_{\theta^j}\right) \geq 0.$$

In minimization problem $M^{\theta^k}_{\overline{b}_{\theta^{l-1}}}$ the same incentive constraint is equivalent to

$$\left(f_{j+1} + \ldots + f_{\theta^{l-1}}\right)\left(\theta^k - \overline{b}_{\theta^l}\right) - \left(f_{\theta^1} + \ldots + f_{\theta^j}\right)\left(\overline{b}_{\theta^l} - \overline{b}_{\theta^j}\right) \geq 0.$$

This shows that in minimization problem $M^{\theta^k}_{\overline{b}_{\theta^{l-1}}}$ it is possible to put more probability weight on lower types. Thus, the value of the objective function is lower under $\left(\tilde{f}^{\theta^k}_{\theta^1}, \ldots, \tilde{f}^{\theta^k}_{\theta^l}\right)$ than under $\left(f^{\theta^l,*}_{\theta^1}, \ldots, f^{\theta^l,*}_{\theta^l}\right)$. But then the constraint corresponding to bid $\overline{b}_{\theta^l}$ is not fulfilled under $\left(\tilde{f}^{\theta^k}_{\theta^1}, \ldots, \tilde{f}^{\theta^k}_{\theta^l}\right)$. Hence, one has to add an additional constraint. Since the constraint corresponding to bid $\overline{b}_{\theta^{l-1}}$ is redundant, one has to add the constraint corresponding to bid $\overline{b}_{\theta^l}$. Thus, the minimum p in minimization problem $M^{\theta^k}_{\overline{b}_{\theta^{l-1}}}$ is greater than $l-1$.

After proving the best-reply and the worst-case belief condition, we conclude that the strategies and belief specified in 3.5.1 indeed constitute a worst-case belief equilibrium.

3.6 Conclusion

We provide a novel approach to endogenize beliefs in games of incomplete information and apply this approach to bidding in first-price auctions. Our model is based on the assumption that bidders in a first-price auction who, apart from the mean of the distribution, have little information about the valuations of their competitors prepare for the worst case. Preparing for the worst-case means that the bidders assume that given the bidding strategies of their competitors they will face ex-ante the worst distribution of valuations. Given that all bidders prepare in the same way a worst-case belief equilibrium arises whenever all bidders best-reply to the bidding strategies of their competitors and their corresponding worst-case beliefs. In particular, this implies there is no other belief such that the best reply to this belief will yield a higher pay-off than in equilibrium. The resulting beliefs are type-dependent and due to the assumption of a constant mean of the distribution the beliefs cannot be strictly ordered by first-order stochastic dominance. In particular this implies that bidders with higher valuations not necessarily face higher competition. Nevertheless, we show that a worst-case equilibrium exists that allocates the object to the bidder with the highest valuation with probability one.

Our concept of the worst-case belief equilibrium can be easily extended to any game of incomplete information and provides a very intuitive way to endogenize beliefs. This is in particular helpful when modeling situations in which players only interact infrequently and thus may not be able to form reasonable objective beliefs.

3.7 Notation

- $U_i\left(\theta_i, f_{-i}, b_i, \beta_{-i}\right)$ denotes the expected utility of a bidder i with valuation θ_i, belief about the other bidders' valuations f_{-i}, bid b_i given the other bidders' strategies β_{-i}.

- For bidder i with valuation θ_i and for each belief f_{-i} about the other bidders' valuations and bidding strategies β_{-i}, the set of *best replies* of bidder i is given by

$$B_i^r\left(\theta_i, f_{-i}, \beta_{-i}\right) = \underset{b_i}{\text{argmax}}\ U_i\left(\theta_i, f_{-i}, b_i, \beta_{-i}\right).$$

- For $\theta^k \in \Theta$, $\beta^*\left(\theta^k\right)$ denotes the worst-case strategy of a bidder with valuation θ^k and

$$\left(f_{\theta^1}^{\theta^k,*}, \ldots, f_{\theta^m}^{\theta^k,*}\right)$$

  denotes the worst-case belief of a bidder with valuation θ^k. The bid distribution of a bidder with valuation θ^k which is prescribed by the worst-case strategy is denoted by G_{θ^k}, i.e. $\beta^*\left(\theta^k\right) = G_{\theta^k}$. The support of this bid distribution is given by $[\bar{b}_{\theta^{k-1}}, \bar{b}_{\theta^k}]$.

- The worst-case belief of a bidder with valuation θ^k is the solution of minimization problem $M_{<\bar{b}_{\theta^{k-1}}}$ which is defined by

$$\min_{(f_{\theta^1},\ldots,f_{\theta^m})} \left(\sum_{j=1}^{k-1} f_{\theta^j}\right)^{n-1} (\theta^k - \bar{b}_{\theta^{k-1}})$$

$$s.t.\ f_{\theta^j} \geq 0 \text{ for all } 1 \leq j \leq m$$

$$\sum_{j=1}^{m} f_{\theta^j} = 1$$

$$\sum_{j=1}^{m} f_{\theta^j} \theta^j = \mu$$

$$(f_{\theta^1} + \cdots + f_{\theta^{l-1}})^{n-1} (\theta^k - \bar{b}_{\theta^{k-1}}) \geq \left(\sum_{j=1}^{h-1} f_{\theta^j} + f_{\theta^h} G_{\theta^h}(s)\right)^{n-1} \left(\theta^k - s\right)$$

$$\text{for all } h \in \{1, \ldots, k-1\} \text{ and all } s \in [\bar{b}_{\theta^{h-1}}, \bar{b}_{\theta^h}].$$

That is, minimization problem $M_{<\bar{b}_{\theta^{k-1}}}$ ensures that bidding $\bar{b}_{\theta^{k-1}}$ induces at least the expected utility than bidding any *lower* bid given the other bidders' worst-case strategy β^*.

- For a bidder with valuation θ_i, a bid b_i and a strategy β_{-i} of the other bidders, the set of probability functions $\mathscr{F}_{n-1}^{min}(\theta_i, b_i, \beta_{-i})$ given by

$$\mathscr{F}_{n-1}^{min}(\theta_i, b_i, \beta_{-i}) = \underset{f_{-i} \in \mathscr{F}_{\mu}^{n-1}}{\operatorname{argmin}} \left\{ U(\theta_i, f_{-i}, b_A, \beta_{-i}) \mid b_A \in B^r(\theta_i, f_{-i}, \beta_{-i}) \right\}$$

  is called the *set of minimizing probability functions* of bid b_i for a bidder with valuation θ_i given the other bidders' strategies β_{-i}.

- For a bidder with valuation θ minimization problem M_b^θ of a bid $b \in [\bar{b}_{\theta^{l-1}}, \bar{b}_{\theta^l}]$ is the minimization problem corresponding to its minimizing probability function, i.e. all solutions of minimization problem M_b^θ are minimizing probability function of b for bidder with valuation θ given the other bidders' worst-case strategy β^*. Formally, minimization problem M_b^θ is given by

$$\min_{(f_{\theta^1},\ldots,f_{\theta^m})} \left(\sum_{j=1}^{l-1} f_{\theta^j} + f_{\theta^l} G_l(b)\right)^{n-1} (\theta - b)$$

$$s.t.\ f_{\theta^j} \geq 0 \text{ for all } 1 \leq j \leq m$$

$$\sum_{j=1}^{m} f_{\theta^j} = 1$$

$$\sum_{j=1}^{m} f_{\theta^j}\theta^j = \mu$$

$$\left(f_{\theta^1} + \cdots + f_{\theta^{l-1}}\right)^{n-1}(\theta - b) \geq \left(\sum_{j=1}^{h} f_{\theta^j} + f_{\theta^h} G_{\theta^h}(s)\right)^{n-1}(\theta - s)$$

$$\text{for all } h \in \{1, \ldots, m\} \text{ and all } s \in [\bar{b}_{\theta^{h-1}}, \bar{b}_{\theta^h}],$$

- We denote the solution of minimization problem M_b^θ by $f^{\theta,b} = \left(f_{\theta^1}^{\theta,b}, \ldots, f_{\theta^m}^{\theta,b}\right)$.
- Apart from the constraints

$$f_{\theta^j} \geq 0 \text{ for all } 1 \leq j \leq m$$

$$\sum_{j=1}^{m} f_{\theta^j} = 1$$

$$\sum_{j=1}^{m} f_{\theta^j}\theta^j = \mu,$$

every constraint in minimization problem M_b^θ compares the utility of bidding b to the utility of bidding some other bid b', which is formalized by

$$U(\theta, f, b, \beta^*) \geq U(\theta, f, b', \beta^*).$$

We call such a constraint an *incentive constraint corresponding to bid b'.*

- For a type θ and bids b, b' we use the notation $b \leq^\theta b'$ if for the θ-type bid b' does not induce a strictly lower expected utility than bid b together with their minimizing probability functions given the other bidders' worst-case strategy β^*. Formally, let $f^{min}(\theta, b, \beta^*) \in \mathscr{F}^{min}(\theta, b, \beta^*)$ and $f^{min}(\theta, b', \beta^*) \in \mathscr{F}^{min}(\theta, b', \beta^*)$. Then it holds that

$$U\left(\theta, f^{min}(\theta, b, \beta^*), b, \beta^*\right) \leq U\left(\theta, f^{min}(\theta, b', \beta^*), b', \beta^*\right) \Rightarrow b \leq^\theta b',$$

$$U\left(\theta, f^{min}(\theta, b, \beta^*), b, \beta^*\right) < U\left(\theta, f^{min}(\theta, b', \beta^*), b', \beta^*\right) \Rightarrow b <^\theta b'$$

and

$$U\left(\theta, f^{min}(\theta, b, \beta^*), b, \beta^*\right) = U\left(\theta, f^{min}(\theta, b', \beta^*), b', \beta^*\right) \Rightarrow b =^\theta b'.$$

We also use the notation $b <^\theta b'$ if b' does not have a minimizing probability function given θ because it is never a best reply for a bidder with valuation θ, but b does have a minimizing probability function. We use the notation $b =^\theta b'$ if neither b, nor b' have a minimizing probability function.

3.8 Appendix

3.8.1 Proof of Lemma 14

Assume there exists a belief $f^{1,b} = \left(f_0^{1,b}, f_\theta^{1,b}, f_1^{1,b}\right) \in \mathscr{F}_\mu$ and a bid $b \in \left(\theta, \bar{b}_1\right)$ such that b is a best reply to $f^{1,b}$ for a bidder with valuation 1 but $f^{1,b}$ differs from the worst-case belief of the 1-type given by $f^{1,*}$. Let $\delta_0, \delta_\theta, \delta_1$ be real numbers such that

$$\left(f_0^{1,b}, f_\theta^{1,b}, f_1^{1,b}\right) = \left(f_0^{1,*} + \delta_0, f_\theta^{1,*} + \delta_\theta, f_1^{1,*} + \delta_1\right).$$

Since $f^{1,b}$ has to fulfill the two probability constraints, it must hold that

$$\delta_0 + \delta_\theta + \delta_1 = 0 \tag{37}$$

$$\delta_\theta \theta + \delta_1 = 0. \tag{38}$$

Due to (37), (38) and $f^{1,b} \neq f^{1,*}$, it must hold that either $\delta_0 < 0$ or $\delta_0 > 0$. First, we consider the case $\delta_0 < 0$. Subtracting (38) from (37) gives

$$\delta_0 + \delta_\theta \left(1 - \theta\right) = 0$$

$$\Leftrightarrow \delta_\theta = -\frac{\delta_0}{1 - \theta} \tag{39}$$

from which follows that $\delta_\theta > 0$. Due to (38), it follows that $\delta_1 < 0$. By definition of the bid distribution of the 1-type in (26), it holds that

$$\left(f_0^{1,*} + f_\theta^{1,*} + f_1^{1,*} G_1\left(b\right)\right)\left(1 - b\right) = \left(f_0^{1,*} + f_\theta^{1,*}\right)\left(1 - \theta\right).$$

Since b is a best reply given $f^{1,b}$, it holds that

$$\left(f_0^{1,b} + f_\theta^{1,b} + f_1^{1,b} G_1\left(b\right)\right)\left(1 - b\right) \geq \left(f_0^{1,b} + f_\theta^{1,b}\right)\left(1 - \theta\right)$$

$$\Leftrightarrow \left(f_0^{1,*} + \delta_0 + f_\theta^{1,*} + \delta_\theta + (f_1^{1,*} + \delta_1) G_1\left(b\right)\right)\left(1 - b\right) \geq \left(f_0^{1,*} + \delta_0 + f_\theta^{1,*} + \delta_\theta\right)\left(1 - \theta\right)$$

from which follows that

$$\left(\delta_0 - \frac{\delta_0}{1 - \theta} + \delta_1\right)\left(1 - b\right) \geq \left(\delta_0 - \frac{\delta_0}{1 - \theta}\right)\left(1 - \theta\right)$$

$$\Leftrightarrow -\theta \delta_0 \left(b - \theta\right) - \delta_1 (1 - \theta)\left(1 - b\right) \leq 0.$$

Since $b > \theta$ and δ_0 and δ_1 are smaller than zero, this leads to a contradiction.

Now we consider the case $\delta_0 > 0$. It follows from (39) that $\delta_\theta < 0$. Due to (38), it follows that $\delta_1 > 0$. By definition of the bid distribution of the 1-type in (26), it holds that

$$\left(f_0^{1,*} + f_\theta^{1,*} + f_1^{1,*} G_1\left(b\right)\right)\left(1 - b\right) = \left(f_0^{1,*} + f_\theta^{1,*} + f_1^{1,*}\right)\left(1 - \bar{b}_1\right).$$

Since b is a best reply given $f^{1,b}$, it holds that

$$\left(f_0^{1,b} + f_\theta^{1,b} + f_1^{1,b} G_1(b)\right)(1-b) \geq \left(f_0^{1,b} + f_\theta^{1,b} + f_1^{1,b}\right)\left(1 - \bar{b}_1\right)$$

$$\Leftrightarrow \left(f_0^{1,*} + \delta_0 + f_\theta^{1,*} + \delta_\theta + \left(f_1^{1,*} + \delta_1\right) G_1(b)\right)(1-b)$$
$$\geq \left(f_0^{1,*} + \delta_0 + f_\theta^{1,*} + \delta_\theta + f_1^{1,*} + \delta_1\right)\left(1 - \bar{b}_1\right)$$

from which follows that

$$\left(\delta_0 - \frac{\delta_0}{1-\theta} + \delta_1 G_1(b)\right)(1-b) \geq \left(\delta_0 - \frac{\delta_0}{1-\theta} + \delta_1\right)\left(1 - \bar{b}_1\right)$$

$$\Leftrightarrow -\theta\delta_0\left(\bar{b}_1 - b\right) - \delta_1(1-\theta)\left(1 - \bar{b}_1 - G_1(b)(1-b)\right) \geq 0.$$

Since $1 - \bar{b}_1 > 1 - b > G_1(b)(1-b)$, $\bar{b}_1 > b$ and δ_0 and δ_1 are greater than zero, this leads to a contradiction. We conclude that if a bid $b \in \left(\theta, \bar{b}_1\right)$ is a best reply to a belief for a bidder with valuation 1, then this belief coincides with the worst-case belief of the 1-type.

3.8.2 Proof of Lemma 18

Proof. Assume that the feasible set of minimization problem M_b^θ with $b \in (\bar{b}_\theta, \bar{b}_1]$ is not empty. Since $b > \theta$ is never a best reply, we can assume that $b \leq \theta$. Let $\left(f_0^{\theta,b}, f_\theta^{\theta,b}, f_1^{\theta,b}\right)$ denote a solution of this minimization problem. Let $\delta_0, \delta_\theta, \delta_1$ be real numbers such that

$$\left(f_0^{\theta,b}, f_\theta^{\theta,b}, f_1^{\theta,b}\right) = \left(f_0^{\theta,*} + \delta_0, f_\theta^{\theta,*} + \delta_\theta, f_1^{\theta,*} + \delta_1\right).$$

It holds that $f_1^{\theta,b} > 0$ because otherwise bidding above $\bar{b}_\theta$ is not a best reply. Since $f_1^{\theta,*} = 0$, it follows that $\delta_1 > 0$. Due to the probability constraints, it must hold

$$\delta_0 + \delta_\theta + \delta_1 = 0 \tag{40}$$

$$\delta_\theta\theta + \delta_1 = 0. \tag{41}$$

Hence, it must hold that $\delta_\theta < 0$ because otherwise (41) cannot be fulfilled. Subtracting (41) from (40) gives

$$\delta_0 + \delta_\theta - \delta_\theta\theta = 0.$$

Since $\delta_\theta - \delta_\theta\theta < 0$, it follows that $\delta_0 > 0$. Because the expected utility from bidding b must be as least as high as the expected utility from bidding any other bid, it holds that

$$\left(f_0^{\theta,b} + f_\theta^{\theta,b} + f_1^{\theta,b} G_1(b)\right)(\theta - b) \geq \left(f_0^{\theta,b}\right)\theta.$$

$$\Leftrightarrow \left(f_0^{\theta,*} + \delta_0 + f_\theta^{\theta,*} + \delta_\theta + \left(f_1^{\theta,*} + \delta_1\right) G_1(b)\right)(\theta - b) \geq \left(f_0^{\theta,*} + \delta_0\right)\theta.$$

Since $f_1^{\theta,*} = 0$ it holds that

$$\left(f_0^{\theta,*} + f_\theta^{\theta,*} + f_1^{\theta,*} G_1(b)\right)(\theta - b) = \left(f_0^{\theta,*} + f_\theta^{\theta,*}\right)(\theta - b) < \left(f_0^{\theta,*} + f_\theta^{\theta,*}\right)\left(\theta - \bar{b}_\theta\right) = f_0^{\theta,*}\theta$$

where the last equality follows from the definition of $\bar{b}_\theta$ in (30). It follows that

$$(\delta_0 + \delta_\theta + \delta_1 G_1(b))(\theta - b) > \delta_0 \theta > 0.$$

Because $b \leq \theta$, it must hold that

$$\delta_0 + \delta_\theta + \delta_1 G_1(b) > 0.$$

Since $\delta_1 > 0$ and $G_1(b) \leq 1$, it holds that

$$0 < \delta_0 + \delta_\theta + \delta_1 G_1(b) \leq \delta_0 + \delta_\theta + \delta_1 G_1(b) + \delta_1 (1 - G_1(b)) = \delta_0 + \delta_\theta + \delta_1 = 0.$$

We conclude that the assumption that the feasible set of minimization problem M_b^θ with $b \in (\bar{b}_\theta, \bar{b}_1]$ is not empty, leads to a contradiction. $\square$

3.8.3 Proof of Lemma 21

Proof. The formal proof works by contradiction. Assume that there exists a $b \in \left(0, \bar{b}_\theta\right)$ such that there exists an element of the feasible set of minimization problem M_b^1 denoted by $\left(f_0^{1,b}, f_\theta^{1,b}, f_1^{1,b}\right)$. Then for every $s', s'' \in [0, \bar{b}_\theta]$ it holds

$$\left(f_0^{1,b} + f_\theta^{1,b} G_\theta(b)\right)(1 - b) \geq \left(f_0^{1,b} + f_\theta^{1,b} G_\theta(s')\right)(1 - s') \tag{42}$$

$$\left(f_0^{b1,b} + f_\theta^{1,b} G_\theta(b)\right)(1 - b) \geq \left(f_0^{1,b} + f_\theta^{1,b} G_\theta(s'')\right)(1 - s''). \tag{43}$$

Let $s'' < b < s'$ be such that

$$s' - b = b - s'' = \alpha \tag{44}$$

for some appropriate $\alpha > 0$. Rearranging of (42) gives

$$\Leftrightarrow f_0^{1,b} \geq \frac{f_\theta^{1,b} G_\theta(s')(1 - s') - f_\theta^{1,b} G_\theta(b)(1 - b)}{s' - b}. \tag{45}$$

Rearranging of (43) gives

$$\Leftrightarrow f_0^{1,b} \leq \frac{f_\theta^{1,b} G_\theta(b)(1 - b) - f_\theta^{1,b} G_\theta(s'')(1 - s'')}{b - s''}. \tag{46}$$

If we show that

$$\frac{f_\theta^{1,b} G_\theta(b)(1 - b) - f_\theta^{1,b} G_\theta(s'')(1 - s'')}{b - s''} < \frac{f_\theta^{1,b} G_\theta(s')(1 - s') - f_\theta^{1,b} G_\theta(b)(1 - b)}{s' - b}, \tag{47}$$

we find a contradiction between inequalities (45) and (46). Due to (44), inequality (47) is equivalent to

$$f_\theta^{1,b} G_\theta(b)(1-b) - f_\theta^{1,b} G_\theta(s'')(1-s'') < f_\theta^{1,b} G_\theta(s')(1-s') - f_\theta^{1,b} G_\theta(b)(1-b).$$

If b is a best reply to $f^{1,b}$, it must hold that $f_\theta^{1,b} > 0$ because otherwise bidding zero or above $\bar{b}_\theta$ would be strictly better. Therefore, the inequality is equivalent to

$$-2G_\theta(b)(1-b) + G_\theta(s'')(1-s'') + G_\theta(s')(1-s') > 0.$$

Due to (44), this is equivalent to

$$-2G_\theta(b)(1-s'+\alpha) + G_\theta(s'')(1-s'+2\alpha) + G_\theta(s')(1-s') > 0$$

$$\Leftrightarrow (1-s')\left[-2G_\theta(b) + G_\theta(s'') + G_\theta(s')\right] + \alpha[-2G_\theta(b) + 2G_\theta(s'')] > 0. \tag{48}$$

As defined in (31), for all $s \in [0, \bar{b}_\theta]$ the distribution G_θ is given by the equation

$$f_0^{\theta,*}\theta = \left(f_0^{\theta,*} + f_\theta^{\theta,*} G_\theta(s)\right)(\theta - s)$$

$$\Leftrightarrow G_\theta(s) = \frac{f_0^{\theta,*} s}{f_\theta^{\theta,*}(\theta - s)}.$$

If $b \le \frac{\mu}{2}$, we choose $s'' = 0$ and it holds that $s' = 2b \le \mu = \bar{b}_\theta$.

Then inequality (48) is equivalent to

$$(1-s')\left(\frac{-2f_0^{\theta,*}b}{f_\theta^{\theta,*}(\theta-b)} + \frac{f_0^{\theta,*}s'}{f_\theta^{\theta,*}(\theta-s')}\right) - \frac{2\alpha f_0^{\theta,*}b}{f_\theta^{\theta,*}(\theta-b)} > 0. \tag{49}$$

It holds that

$$\theta - b - (\theta - 2b) > 0$$

$$\Leftrightarrow -2b(\theta - 2b) + 2b(\theta - b) > 0.$$

Due to (44), this is equivalent to

$$-2b(\theta - s') + s'(\theta - b) > 0$$

$$\Leftrightarrow \frac{-2f_0^{\theta,*}b}{f_\theta^{\theta,*}(\theta-b)} + \frac{f_0^{\theta,*}s'}{f_\theta^{\theta,*}(\theta-s')} > 0.$$

It follows that in order to show (49), it is sufficient to show that

$$(\theta-s')\left(\frac{-2f_0^{\theta,*}b}{f_\theta^{\theta,*}(\theta-b)} + \frac{f_0^{\theta,*}s'}{f_\theta^{\theta,*}(\theta-s')}\right) - \frac{2\alpha f_0^{\theta,*}b}{f_\theta^{\theta,*}(\theta-b)} \ge 0.$$

Multiplying the inequality with $(\theta - b)$ and plugging in $\alpha = (s' - b)$ reduces the problem to

$$-2b(\theta - s') + s'(\theta - b) - 2b(s' - b) \geq 0. \tag{50}$$

It holds that

$$s' \geq 2b$$
$$\Leftrightarrow -2b(\theta - b) + s'(\theta - b) \geq 0$$
$$\Leftrightarrow -2b\theta + s'\theta - s'b + 2b^2 \geq 0$$
$$-2b\theta + 2bs' + s'\theta - s'b - 2bs' + 2b^2 \geq 0$$
$$-2b(\theta - s') + s'(\theta - b) - 2b(s' - b) \geq 0.$$

Thus, we have shown inequality (50) from which follows that inequality (47) holds. This shows that inequalities (42) and (43) lead to a contradiction in case $b \leq \frac{\mu}{2}$.

By definition of $\bar{b}_\theta = \mu$, it holds for all $s \in [0, \mu]$ that

$$\left(f_0^{\theta,*} + f_\theta^{\theta,*}\right)(\theta - \mu) = \left(f_0^{\theta,*} + f_\theta^{\theta,*} G_\theta(s)\right)(\theta - s)$$

$$\Leftrightarrow G_\theta(s) = \frac{-f_0^{\theta,*}(\mu - s) + f_\theta^{\theta,*}(\theta - \mu)}{f_\theta^{\theta,*}(\theta - s)}.$$

If $b > \frac{\mu}{2}$, then we set $s' = \mu$ and it holds that $s'' = 2b - \mu > 0$. Then inequality (48) is equivalent to

$$\begin{aligned}
&(1-\mu)\left(\frac{2f_0^{\theta,*}(\mu - b) - 2f_\theta^{\theta,*}(\theta - \mu)}{f_\theta^{\theta,*}(\theta - b)} + \frac{-f_0^{\theta,*}(\mu - (2b - \mu)) + f_\theta^{\theta,*}(\theta - \mu)}{f_\theta^{\theta,*}(\theta - (2b - \mu))} + 1\right) \\
&\quad + \alpha\left(\frac{2f_0^{\theta,*}(\mu - b) - 2f_\theta^{\theta,*}(\theta - \mu)}{f_\theta^{\theta,*}(\theta - b)} + \frac{-2f_0^{\theta,*}(\mu - (2b - \mu)) + 2f_\theta^{\theta,*}(\theta - \mu)}{f_\theta^{\theta,*}(\theta - (2b - \mu))}\right) > 0
\end{aligned}$$

$$\begin{aligned}
\Leftrightarrow\ &(1-\mu)\left(\frac{2f_0^{\theta,*}(\mu - b) - 2f_\theta^{\theta,*}(\theta - \mu)}{f_\theta^{\theta,*}(\theta - b)} + \frac{-2f_0^{\theta,*}(\mu - b) + f_\theta^{\theta,*}(\theta - \mu)}{f_\theta^{\theta,*}(\theta - 2b + \mu)} + 1\right) \\
&\quad + \alpha\left(\frac{2f_0^{\theta,*}(\mu - b) - 2f_\theta^{\theta,*}(\theta - \mu)}{f_\theta^{\theta,*}(\theta - b)} + \frac{-4f_0^{\theta,*}(\mu - b) + 2f_\theta^{\theta,*}(\theta - \mu)}{f_\theta^{\theta,*}(\theta - 2b + \mu)}\right) > 0
\end{aligned}$$

$$\begin{aligned}
\Leftrightarrow\ &2f_0^{\theta,*}(\mu - b)(\theta - 2b + \mu)(1 - \mu + \alpha) - 2f_0^{\theta,*}(\mu - b)(\theta - b)(1 - \mu + 2\alpha) \\
&- 2f_\theta^{\theta,*}(\theta - \mu)(\theta - 2b + \mu)(1 - \mu + \alpha) + f_\theta^{\theta,*}(\theta - \mu)(\theta - b)(1 - \mu + 2\alpha) \\
&\qquad + f_\theta^{\theta,*}(1 - \mu)(\theta - 2b + \mu)(\theta - b) > 0
\end{aligned}$$

$$\Leftrightarrow 2f_0^{\theta,*}(\mu-b)[\theta-\theta\mu+\theta\alpha-2b+2b\mu-2b\alpha+\mu-\mu+\mu\alpha-(\theta-\theta\mu+2\theta\alpha-b+b\mu+2b\alpha)]$$
$$-2f_\theta^{\theta,*}(\theta-\mu)(\theta-2b+\mu)(1-\mu+\alpha)+f_\theta^{\theta,*}(\theta-\mu)(\theta-b)(1-\mu+2\alpha)$$
$$+f_\theta^{\theta,*}(1-\mu)(\theta-2b+\mu)(\theta-b)>0.$$

$$\Leftrightarrow 2f_0^{\theta,*}(\mu-b)[-\alpha\theta-b+\mu b+\mu-\mu+\mu\alpha]$$
$$-2f_\theta^{\theta,*}(\theta-\mu)(\theta-2b+\mu)(1-\mu+\alpha)+f_\theta^{\theta,*}(\theta-\mu)(\theta-b)(1-\mu+2\alpha)$$
$$+f_\theta^{\theta,*}(1-\mu)(\theta-2b+\mu)(\theta-b)>0.$$

By definition of α in (44), this is equivalent to

$$\Leftrightarrow 2f_0^{\theta,*}(\mu-b)[-(\mu-b)\theta-b+\mu b+\mu-\mu+\mu(\mu-b)]$$
$$-2f_\theta^{\theta,*}(\theta-\mu)(\theta-2b+\mu)(1-\mu+\alpha)+f_\theta^{\theta,*}(\theta-\mu)(\theta-b)(1-\mu+2\alpha)$$
$$+f_\theta^{\theta,*}(1-\mu)(\theta-2b+\mu)(\theta-b)>0.$$

$$\Leftrightarrow 2f_0^{\theta,*}(\mu-b)[\mu-b-\theta(\mu-b)]$$
$$-2f_\theta^{\theta,*}(\theta-\mu)(\theta-2b+\mu)(1-\mu+\alpha)+f_\theta^{\theta,*}(\theta-\mu)(\theta-b)(1-\mu+2\alpha)$$
$$+f_\theta^{\theta,*}(1-\mu)(\theta-2b+\mu)(\theta-b)\geq 0.$$

Since $2f_0^{\theta,*}(\mu-b)[\mu-b-\theta(\mu-b)]>0$, it is sufficient to show that

$$-2f_\theta^{\theta,*}(\theta-\mu)(\theta-2b+\mu)(1-\mu+\alpha)$$
$$+f_\theta^{\theta,*}(\theta-\mu)(\theta-b)(1-\mu+2\alpha)+f_\theta^{\theta,*}(1-\mu)(\theta-2b+\mu)(\theta-b)>0. \quad (51)$$

It holds that

$$\mu>b$$
$$\Leftrightarrow(\mu-b)(1-\theta)(-\theta+\mu+\theta-2b+\mu)>0$$
$$\Leftrightarrow-(\theta-\mu)(\mu-b)(1-\theta)+(\theta-2b+\mu)(\mu-b)(1-\theta)>0$$
$$\Leftrightarrow(\theta-\mu)[-\theta b+b-\mu+\mu\theta]+(\theta-2b+\mu)[\theta b+\mu-b-\mu\theta]>0$$

$$\Leftrightarrow(\theta-\mu)[-\theta+\theta b+2b-2b^2-\mu+\mu b+\theta-2b\theta+\theta\mu-b+2b^2-b\mu]$$
$$+(\theta-2b+\mu)[-\theta+\theta b+\mu-\mu b+\theta-b-\mu\theta+\mu b]>0$$

$$\Leftrightarrow (\theta-\mu)\left[-(\theta-2b+\mu)(1-b)+(\theta-b)(1-2b+\mu)\right] \\ +(\theta-2b+\mu)\left[-(\theta-\mu)(1-b)+(1-\mu)(\theta-b)\right]>0$$

Since $f_\theta^{\theta,*}>0$, this is equivalent to

$$-2f_\theta^{\theta,*}(\theta-\mu)(\theta-2b+\mu)(1-b)+f_\theta^{\theta,*}(\theta-\mu)(\theta-b)(1-2b+\mu) \\ +f_\theta^{\theta,*}(1-\mu)(\theta-2b+\mu)(\theta-b)>0.$$

Since $-b=-\mu+\alpha$ and $-2b+\mu=-\mu+2\alpha$, this is equivalent to

$$-2f_\theta^{\theta,*}(\theta-\mu)(\theta-2b+\mu)(1-\mu+\alpha)+f_\theta^{\theta,*}(\theta-\mu)(\theta-b)(1-\mu+2\alpha) \\ +f_\theta^{\theta,*}(1-\mu)(\theta-2b+\mu)(\theta-b)>0.$$

Thus, we have shown inequality (51) from which follows that inequality (47) holds. This shows that inequalities (42) and (43) lead to a contradiction in case $b>\frac{\mu}{2}$. We conclude that in any possible case the assumption that the feasible set of minimization problem M_b^1 with $b\in(0,\overline{b}_\theta)$ is not empty, leads to a contradiction. □

3.8.4 Proof of Lemma 24

Proof. We have to show that for every pair of valuations θ^l and θ^k such that $\theta^z\leq\theta^l\leq\theta^{k-1}$ and every b with $\overline{b}_{\theta^{l-1}}<b<\overline{b}_{\theta^l}$ the feasible set of minimization problem $M_b^{\theta^k}$ is empty. Assume that there exist l and b with $\overline{b}_{\theta^{l-1}}<b<\overline{b}_{\theta^l}$ such that there exists an element of the feasible set of minimization problem $M_b^{\theta^k}$ denoted by $\left(f_{\theta^1}^{\theta^k,b},\ldots,f_{\theta^m}^{\theta^k,b}\right)$. Then for every $s',s''\in[\overline{b}_{\theta^{l-1}},\overline{b}_{\theta^l}]$ it holds

$$\left(f_{\theta^1}^{\theta^k,b}+\ldots+f_{\theta^{l-1}}^{\theta^k,b}+f_{\theta^l}^{\theta^k,b}G_{\theta^l}(b)\right)^{n-1}\left(\theta^k-b\right) \\ \geq\left(f_{\theta^1}^{\theta^k,b}+\ldots+f_{\theta^{l-1}}^{\theta^k,b}+f_{\theta^l}^{\theta^k,b}G_{\theta^l}(s')\right)^{n-1}\left(\theta^k-s'\right) \quad (52)$$

$$\left(f_{\theta^1}^{\theta^k,b}+\ldots+f_{\theta^{l-1}}^{\theta^k,b}+f_{\theta^l}^{\theta^k,b}G_{\theta^l}(b)\right)^{n-1}\left(\theta^k-b\right) \\ \geq\left(f_{\theta^1}^{\theta^k,b}+\ldots+f_{\theta^{l-1}}^{\theta^k,b}+f_{\theta^l}^{\theta^k,b}G_{\theta^l}(s'')\right)^{n-1}\left(\theta^k-s''\right). \quad (53)$$

Let $s''<b<s'$ be such that

$$\sqrt[n-1]{\theta^k-s''}-\sqrt[n-1]{\theta^k-b}=\sqrt[n-1]{\theta^k-b}-\sqrt[n-1]{\theta^k-s'}=\alpha \quad (54)$$

$$\Leftrightarrow\sqrt[n-1]{\theta^k-s'}+\alpha=\sqrt[n-1]{\theta^k-b}\ ,\ \sqrt[n-1]{\theta^k-s'}+2\alpha=\sqrt[n-1]{\theta^k-s''}$$

for some appropriate $\alpha > 0$. Rearranging of (52) gives

$$\begin{aligned}
&\left(f_{\theta^1}^{\theta^k,b} + \ldots + f_{\theta^{l-1}}^{\theta^k,b} + f_{\theta^l}^{\theta^k,b} G_{\theta^l}(b)\right) \sqrt[n-1]{\theta^k - b} \\
&\qquad\qquad \geq \left(f_{\theta^1}^{\theta^k,b} + \ldots + f_{\theta^{l-1}}^{\theta^k,b} + f_{\theta^l}^{\theta^k,b} G_{\theta^l}(s')\right) \sqrt[n-1]{\theta^k - s'}
\end{aligned}$$

$$\begin{aligned}
\Leftrightarrow &\left(f_{\theta^1}^{\theta^k,b} + \ldots + f_{\theta^{l-1}}^{\theta^k,b}\right)\left(\sqrt[n-1]{\theta^k - b} - \sqrt[n-1]{\theta^k - s'}\right) \\
&\qquad\qquad \geq f_{\theta^l}^{\theta^k,b} G_{\theta^l}(s') \sqrt[n-1]{\theta^k - s'} - f_{\theta^l}^{\theta^k,b} G_{\theta^l}(b) \sqrt[n-1]{\theta^k - b}
\end{aligned}$$

$$\Leftrightarrow f_{\theta^1}^{\theta^k,b} + \ldots + f_{\theta^{l-1}}^{\theta^k,b} \geq \frac{f_{\theta^l}^{\theta^k,b} G_{\theta^l}(s') \sqrt[n-1]{\theta^k - s'} - f_{\theta^l}^{\theta^k,b} G_{\theta^l}(b) \sqrt[n-1]{\theta^k - b}}{\sqrt[n-1]{\theta^k - b} - \sqrt[n-1]{\theta^k - s'}}. \qquad (55)$$

Rearranging of (53) gives

$$\begin{aligned}
&\left(f_{\theta^1}^{\theta^k,b} + \ldots + f_{\theta^{l-1}}^{\theta^k,b} + f_{\theta^l}^{\theta^k,b} G_{\theta^l}(b)\right) \sqrt[n-1]{\theta^k - b} \\
&\qquad\qquad \geq \left(f_{\theta^1}^{\theta^k,b} + \ldots + f_{\theta^{l-1}}^{\theta^k,b} + f_{\theta^l}^{\theta^k,b} G_{\theta^l}(s'')\right) \sqrt[n-1]{\theta^k - s''}
\end{aligned}$$

$$\begin{aligned}
\Leftrightarrow &\left(f_{\theta^1}^{\theta^k,b} + \ldots + f_{\theta^{l-1}}^{\theta^k,b}\right)\left(\sqrt[n-1]{\theta^k - s''} - \sqrt[n-1]{\theta^k - b}\right) \\
&\qquad\qquad \leq f_{\theta^l}^{\theta^k,b} G_{\theta^l}(b) \sqrt[n-1]{\theta^k - b} - f_{\theta^l}^{\theta^k,b} G_{\theta^l}(s'') \sqrt[n-1]{\theta^k - s''}
\end{aligned}$$

$$\Leftrightarrow f_{\theta^1}^{\theta^k,b} + \ldots + f_{\theta^{l-1}}^{\theta^k,b} \leq \frac{f_{\theta^l}^{\theta^k,b} G_{\theta^l}(b) \sqrt[n-1]{\theta^k - b} - f_{\theta^l}^{\theta^k,b} G_{\theta^l}(s'') \sqrt[n-1]{\theta^k - s''}}{\sqrt[n-1]{\theta^k - s''} - \sqrt[n-1]{\theta^k - b}}. \qquad (56)$$

If we show that

$$\begin{aligned}
&\frac{f_{\theta^l}^{\theta^k,b} G_{\theta^l}(b) \sqrt[n-1]{\theta^k - b} - f_{\theta^l}^{\theta^k,b} G_{\theta^l}(s'') \sqrt[n-1]{\theta^k - s''}}{\sqrt[n-1]{\theta^k - s''} - \sqrt[n-1]{\theta^k - b}} \\
&\qquad\qquad < \frac{f_{\theta^l}^{\theta^k,b} G_{\theta^l}(s') \sqrt[n-1]{\theta^k - s'} - f_{\theta^l}^{\theta^k,b} G_{\theta^l}(b) \sqrt[n-1]{\theta^k - b}}{\sqrt[n-1]{\theta^k - b} - \sqrt[n-1]{\theta^k - s'}}, \qquad (57)
\end{aligned}$$

we find a contradiction between inequalities (55) and (56). Inequality (57) is equivalent to

$$\begin{aligned}
&\frac{f_{\theta^l}^{\theta^k,b} G_{\theta^l}(b) \sqrt[n-1]{\theta^k - b} - f_{\theta^l}^{\theta^k,b} G_{\theta^l}(s'') \sqrt[n-1]{\theta^k - s''}}{\alpha} \\
&\qquad\qquad < \frac{f_{\theta^l}^{\theta^k,b} G_{\theta^l}(s') \sqrt[n-1]{\theta^k - s'} - f_{\theta^l}^{\theta^k,b} G_{\theta^l}(b) \sqrt[n-1]{\theta^k - b}}{\alpha}
\end{aligned}$$

$$\Leftrightarrow -2 f_{\theta^l}^{\theta^k,b} G_{\theta^l}(b) \sqrt[n-1]{\theta^k - b} + f_{\theta^l}^{\theta^k,b} G_{\theta^l}(s'') \sqrt[n-1]{\theta^k - s''} + f_{\theta^l}^{\theta^k,b} G_{\theta^l}(s') \sqrt[n-1]{\theta^k - s'} > 0.$$

If bid b is a best reply, it must hold that $f_{\theta^l}^{\theta^k,b} > 0$ and therefore the inequality is equivalent to

$$-2G_{\theta^l}\left(b\right)\sqrt[n-1]{\theta^k-b}+G_{\theta^l}\left(s''\right)\sqrt[n-1]{\theta^k-s''}+G_{\theta^l}\left(s'\right)\sqrt[n-1]{\theta^k-s'}>0.$$

$$\Leftrightarrow -2G_{\theta^l}\left(b\right)\left(\sqrt[n-1]{\theta^k-s'}+\alpha\right)+G_{\theta^l}\left(s''\right)\left(\sqrt[n-1]{\theta^k-s'}+2\alpha\right)+G_{\theta^l}\left(s'\right)\sqrt[n-1]{\theta^k-s'}>0$$

$$\Leftrightarrow \sqrt[n-1]{\theta^k-s'}\left(-2G_{\theta^l}\left(b\right)+G_{\theta^l}\left(s''\right)+G_{\theta^l}\left(s'\right)\right)+\alpha\left(-2G_{\theta^l}\left(b\right)+2G_{\theta^l}\left(s''\right)\right)>0$$

$$\Leftrightarrow \sqrt[n-1]{\theta^k-s'}\left(-2G_{\theta^l}\left(b\right)+G_{\theta^l}\left(s''\right)+G_{\theta^l}\left(s'\right)\right)>\alpha\left(2G_{\theta^l}\left(b\right)-2G_{\theta^l}\left(s''\right)\right). \tag{58}$$

For all $s\in[\overline{b}_{\theta^{l-1}},\overline{b}_{\theta^l}]$ the distribution G_{θ^l} is defined by the equation

$$\left(f_{\theta^1}^{\theta^l,*}+\ldots+f_{\theta^{l-1}}^{\theta^l,*}\right)^{n-1}\left(\theta^l-\overline{b}_{\theta^{l-1}}\right)=\left(f_{\theta^1}^{\theta^l,*}+\ldots+f_{\theta^{l-1}}^{\theta^l,*}+f_{\theta^l}^{\theta^l,*}G_{\theta^l}\left(s\right)\right)^{n-1}\left(\theta^l-s\right)$$

$$\Leftrightarrow G_{\theta^l}\left(s\right)=\frac{\left(f_{\theta^1}^{\theta^l,*}+\ldots+f_{\theta^{l-1}}^{\theta^l,*}\right)\left(\sqrt[n-1]{\theta^l-\overline{b}_{\theta^{l-1}}}-\sqrt[n-1]{\theta^l-s}\right)}{f_{\theta^l}^{\theta^l,*}\sqrt[n-1]{\theta^l-s}}$$

where $\left(f_{\theta^1}^{\theta^l,*}+\ldots+f_{m}^{\theta^l,*}\right)$ denotes the worst-case belief of the θ^l-type.

Let b^* be defined by

$$\sqrt[n-1]{\theta^k-\overline{b}_{\theta^{l-1}}}-\sqrt[n-1]{\theta^k-b^*}=\sqrt[n-1]{\theta^k-b^*}-\sqrt[n-1]{\theta^k-\overline{b}_{\theta^l}}.$$

If $b\le b^*$, we choose $s''=\overline{b}_{\theta^{l-1}}$ and it holds that

$$\begin{aligned}\sqrt[n-1]{\theta^k-b}-\sqrt[n-1]{\theta^k-s'}&=\sqrt[n-1]{\theta^k-\overline{b}_{\theta^{l-1}}}-\sqrt[n-1]{\theta^k-b}\\&\le\sqrt[n-1]{\theta^k-\overline{b}_{\theta^{l-1}}}-\sqrt[n-1]{\theta^k-b^*}=\sqrt[n-1]{\theta^k-b^*}-\sqrt[n-1]{\theta^k-\overline{b}_{\theta^l}}\end{aligned}$$

$$\Rightarrow \sqrt[n-1]{\theta^k-s'}\ge\sqrt[n-1]{\theta^k-b}-\sqrt[n-1]{\theta^k-b^*}+\sqrt[n-1]{\theta^k-\overline{b}_{\theta^l}}\ge\sqrt[n-1]{\theta^k-\overline{b}_{\theta^l}}$$

$$\Rightarrow s'\le\overline{b}_{\theta^l}.$$

Moreover, we define

$$\alpha_1^{s'}:=\sqrt[n-1]{\theta^l-\overline{b}_{\theta^{l-1}}}-\sqrt[n-1]{\theta^l-b}\quad\text{and }\alpha_2^{s'}:=\sqrt[n-1]{\theta^l-b}-\sqrt[n-1]{\theta^l-s'}.$$

Then inequality (58) is equivalent to

$$\begin{aligned}\sqrt[n-1]{\theta^k-s'}\frac{\left(f_{\theta^1}^{\theta^l,*}+\ldots+f_{\theta^{l-1}}^{\theta^l,*}\right)^{n-1}}{f_{\theta^l}^{\theta^l,*}}&\left(\frac{-2\alpha_1^{s'}}{\sqrt[n-1]{\theta^l-b}}+\frac{\alpha_1^{s'}+\alpha_2^{s'}}{\sqrt[n-1]{\theta^l-s'}}\right)\\&>\frac{\left(f_{\theta^1}^{\theta^l,*}+\ldots+f_{\theta^{l-1}}^{\theta^l,*}\right)^{n-1}}{f_{\theta^l}^{\theta^l,*}}\alpha\frac{2\alpha_1^{s'}}{\sqrt[n-1]{\theta^l-b}}.\end{aligned}$$

Since $1>f_{\theta^l}^{\theta^l,*}>0$ and $\sum_{j=1}^{l}f_{\theta^j}^{\theta^l,*}=1$, it holds that $\sum_{j=1}^{l-1}f_{\theta^j}^{\theta^l,*}>0$ and therefore the inequality is equivalent to

$$\sqrt[n-1]{\theta^k - s'}\left(\frac{-2\alpha_1^{s'}}{\sqrt[n-1]{\theta^l - b}} + \frac{\alpha_1^{s'} + \alpha_2^{s'}}{\sqrt[n-1]{\theta^l - s'}}\right) > \alpha \frac{2\alpha_1^{s'}}{\sqrt[n-1]{\theta^l - b}}.$$

Since $\sqrt[n-1]{}$ is concave, it holds that $\alpha_1^{s'} \leq \alpha_2^{s'}$ from which follows that

$$\frac{-2\alpha_1^{s'}}{\sqrt[n-1]{\theta^l - b}} + \frac{\alpha_1^{s'} + \alpha_2^{s'}}{\sqrt[n-1]{\theta^l - s'}} > 0.$$

Hence, if $\theta^k > \theta^l$, it is sufficient to show that

$$-2\alpha_1^{s'} \sqrt[n-1]{\theta^l - s'} + \left(\alpha_1^{s'} + \alpha_2^{s'}\right) \sqrt[n-1]{\theta^l - b} \geq 2\alpha\alpha_1^{s'}.$$

Since $\alpha_1^{s'} \leq \alpha_2^{s'}$, it is sufficient to show that

$$-2\alpha_1^{s'} \sqrt[n-1]{\theta^l - s'} + 2\alpha_1^{s'} \sqrt[n-1]{\theta^l - b} \geq 2\alpha\alpha_1^{s'}$$

$$\Leftrightarrow - \sqrt[n-1]{\theta^l - s'} + \sqrt[n-1]{\theta^l - b} \geq \alpha$$

which is true since $\sqrt[n-1]{}$ is concave. Thus, we have shown inequality (57) and conclude that in the case $b \leq b^*$ the assumption that the feasible set of minimization problem $M_b^{\theta^k}$ is not empty, leads to a contradiction.

If $b > b^*$, then we choose $s' = \overline{b}_{\theta^l}$ and it holds that

$$\begin{aligned}\sqrt[n-1]{\theta^k - s''} - \sqrt[n-1]{\theta^k - b} &= \sqrt[n-1]{\theta^k - b} - \sqrt[n-1]{\theta^k - \overline{b}_{\theta^l}} \\ &\leq \sqrt[n-1]{\theta^k - b^*} - \sqrt[n-1]{\theta^k - \overline{b}_{\theta^l}} = \sqrt[n-1]{\theta^k - \overline{b}_{\theta^{l-1}}} - \sqrt[n-1]{\theta^k - b^*}\end{aligned}$$

$$\Rightarrow \sqrt[n-1]{\theta^k - s''} \leq \sqrt[n-1]{\theta^k - \overline{b}_{\theta^{l-1}}} - \sqrt[n-1]{\theta^k - b^*} + \sqrt[n-1]{\theta^k - b} \leq \sqrt[n-1]{\theta^k - \overline{b}_{\theta^{l-1}}}$$

$$\Rightarrow s'' \geq \overline{b}_{\theta^{l-1}}.$$

Moreover, we define

$$\alpha_1^{s''} := \sqrt[n-1]{\theta^l - b} - \sqrt[n-1]{\theta^l - \overline{b}_{\theta^l}} \qquad \alpha_2^{s''} := \sqrt[n-1]{\theta^l - s''} - \sqrt[n-1]{\theta^l - b}. \tag{59}$$

By definition of $\overline{b}_{\theta^l}$, it holds that

$$\left(f_{\theta^1}^{\theta^l,*} + \ldots + f_{\theta^l}^{\theta^l,*}\right)^{n-1}\left(\theta^l - \overline{b}_{\theta^l}\right) = \left(f_{\theta^1}^{\theta^l,*} + \ldots + f_{\theta^{l-1}}^{\theta^l,*} + f_{\theta^l}^{\theta^l,*} G_{\theta^l}(s)\right)^{n-1}\left(\theta^l - s\right)$$

$$\Leftrightarrow G_{\theta^l}(s) = \frac{-\left(f_{\theta^1}^{\theta^l,*} + \ldots + f_{\theta^{l-1}}^{\theta^l,*}\right)\left(\sqrt[n-1]{\theta^l - s} - \sqrt[n-1]{\theta^l - \overline{b}_{\theta^l}}\right) + f_{\theta^l}^{\theta^l,*} \sqrt[n-1]{\theta^l - \overline{b}_{\theta^l}}}{f_{\theta^l}^{\theta^l,*} \sqrt[n-1]{\theta^l - s}}.$$

Then inequality (58) is equivalent to

$$\sqrt[n-1]{\theta^k - \overline{b}_{\theta^l}} \left(\frac{2\left(f_{\theta^1}^{\theta^l,*} + \cdots + f_{\theta^{l-1}}^{\theta^l,*}\right)\alpha_1^{s''} - 2 f_{\theta^l}^{\theta^l,*} \sqrt[n-1]{\theta^l - \overline{b}_{\theta^l}}}{f_{\theta^l}^{\theta^l,*} \sqrt[n-1]{\theta^l - b}} + 1 \right.$$
$$\left. + \frac{-\left(f_{\theta^1}^{\theta^l,*} + \cdots + f_{\theta^{l-1}}^{\theta^l,*}\right)\left(\alpha_1^{s''} + \alpha_2^{s''}\right) + f_{\theta^l}^{\theta^l,*} \sqrt[n-1]{\theta^l - \overline{b}_{\theta^l}}}{f_{\theta^l}^{\theta^l,*} \sqrt[n-1]{\theta^l - s''}} \right)$$
$$> \alpha \left(\frac{-2\left(f_{\theta^1}^{\theta^l,*} + \cdots + f_{\theta^{l-1}}^{\theta^l,*}\right)\alpha_1^{s''}}{f_{\theta^l}^{\theta^l,*} \sqrt[n-1]{\theta^l - b}} \right.$$
$$\left. + \frac{2\left(f_{\theta^1}^{\theta^l,*} + \cdots + f_{\theta^{l-1}}^{\theta^l,*}\right)\left(\alpha_1^{s''} + \alpha_2^{s''}\right) - 2 f_{\theta^l}^{\theta^l,*} \sqrt[n-1]{\theta^l - \overline{b}_{\theta^l}}}{f_{\theta^l}^{\theta^l,*} \sqrt[n-1]{\theta^l - s''}} \right)$$

$$\Leftrightarrow 2\left(f_{\theta^1}^{\theta^l,*} + \cdots + f_{\theta^{l-1}}^{\theta^l,*}\right)\alpha_1^{s''}\left(\sqrt[n-1]{\theta^k - \overline{b}_{\theta^l}} + \alpha\right)\sqrt[n-1]{\theta^l - s''}$$
$$-2 f_{\theta^l}^{\theta^l,*} \sqrt[n-1]{\theta^l - \overline{b}_{\theta^l}}\left(\sqrt[n-1]{\theta^k - \overline{b}_{\theta^l}} + \alpha\right)\sqrt[n-1]{\theta^l - s''} + f_{\theta^l}^{\theta^l,*} \sqrt[n-1]{\theta^l - s''}\sqrt[n-1]{\theta^l - b}\sqrt[n-1]{\theta^k - \overline{b}_{\theta^l}}$$
$$-\left(f_{\theta^1}^{\theta^l,*} + \cdots + f_{\theta^{l-1}}^{\theta^l,*}\right)\left(\alpha_1^{s''} + \alpha_2^{s''}\right)\left(\sqrt[n-1]{\theta^k - \overline{b}_{\theta^l}} + 2\alpha\right)\sqrt[n-1]{\theta^l - b}$$
$$+ f_{\theta^l}^{\theta^l,*} \sqrt[n-1]{\theta^l - \overline{b}_{\theta^l}}\left(\sqrt[n-1]{\theta^k - \overline{b}_{\theta^l}} + 2\alpha\right)\sqrt[n-1]{\theta^l - b} > 0.$$

Since $\sqrt[n-1]{\ }$ is concave, it holds that $\alpha_1^{s''} \geq \alpha_2^{s''}$ and therefore, it is sufficient to show that

$$\left(f_{\theta^1}^{\theta^l,*} + \cdots + f_{\theta^{l-1}}^{\theta^l,*}\right)\left(\alpha_1^{s''} + \alpha_2^{s''}\right)\left(\sqrt[n-1]{\theta^k - \overline{b}_{\theta^l}} + \alpha\right)\alpha_2^{s''}$$
$$-\left(f_{\theta^1}^{\theta^l,*} + \cdots + f_{\theta^{l-1}}^{\theta^l,*}\right)\left(\alpha_1^{s''} + \alpha_2^{s''}\right)\alpha \sqrt[n-1]{\theta^l - b}$$
$$-2 f_{\theta^l}^{\theta^l,*} \sqrt[n-1]{\theta^l - \overline{b}_{\theta^l}}\left(\sqrt[n-1]{\theta^k - \overline{b}_{\theta^l}} + \alpha\right)\sqrt[n-1]{\theta^l - s''}$$
$$+ f_{\theta^l}^{\theta^l,*} \sqrt[n-1]{\theta^l - s''}\sqrt[n-1]{\theta^l - b}\sqrt[n-1]{\theta^k - \overline{b}_{\theta^l}}$$
$$+ f_{\theta^l}^{\theta^l,*} \sqrt[n-1]{\theta^l - \overline{b}_{\theta^l}}\left(\sqrt[n-1]{\theta^k - \overline{b}_{\theta^l}} + 2\alpha\right)\sqrt[n-1]{\theta^l - b} > 0.$$

If $\theta^k > \theta^l$, it holds that $\alpha \sqrt[n-1]{\theta^l - b} < \alpha \sqrt[n-1]{\theta^k - b} = \alpha\left(\sqrt[n-1]{\theta^k - \overline{b}_{\theta^l}} + \alpha\right)$ and therefore it is sufficient to show that

$$\begin{aligned}
&\left(f_{\theta^1}^{\theta^l,*}+\cdots+f_{\theta^{l-1}}^{\theta^l,*}\right)\left(\alpha_1^{s''}+\alpha_2^{s''}\right)\left(\sqrt[n-1]{\theta^k-\overline{b}_{\theta^l}}+\alpha\right)\alpha_2^{s''}\\
&\qquad-\left(f_{\theta^1}^{\theta^l,*}+\cdots+f_{\theta^{l-1}}^{\theta^l,*}\right)\left(\alpha_1^{s''}+\alpha_2^{s''}\right)\left(\sqrt[n-1]{\theta^k-\overline{b}_{\theta^l}}+\alpha\right)\alpha\\
&\qquad-2f_{\theta^l}^{\theta^l,*}\sqrt[n-1]{\theta^l-\overline{b}_{\theta^l}}\left(\sqrt[n-1]{\theta^k-\overline{b}_{\theta^l}}+\alpha\right)\sqrt[n-1]{\theta^l-s''}\\
&\qquad+f_{\theta^l}^{\theta^l,*}\sqrt[n-1]{\theta^l-s''}\sqrt[n-1]{\theta^l-b}\sqrt[n-1]{\theta^k-\overline{b}_{\theta^l}}\\
&\qquad+f_{\theta^l}^{\theta^l,*}\sqrt[n-1]{\theta^l-\overline{b}_{\theta^l}}\left(\sqrt[n-1]{\theta^k-\overline{b}_{\theta^l}}+2\alpha\right)\sqrt[n-1]{\theta^l-b}\geq 0.
\end{aligned}$$

Since $\sqrt[n-1]{\ }$ is concave, it holds that $\alpha_2^{s''}\geq\alpha$ and it is sufficient to show that

$$\begin{aligned}
&-2f_{\theta^l}^{\theta^l,*}\sqrt[n-1]{\theta^l-\overline{b}_{\theta^l}}\left(\sqrt[n-1]{\theta^k-\overline{b}_{\theta^l}}+\alpha\right)\sqrt[n-1]{\theta^l-s''}\\
&\qquad+f_{\theta^l}^{\theta^l,*}\sqrt[n-1]{\theta^l-s''}\sqrt[n-1]{\theta^l-b}\sqrt[n-1]{\theta^k-\overline{b}_{\theta^l}}\\
&\qquad+f_{\theta^l}^{\theta^l,*}\sqrt[n-1]{\theta^l-\overline{b}_{\theta^l}}\left(\sqrt[n-1]{\theta^k-\overline{b}_{\theta^l}}+2\alpha\right)\sqrt[n-1]{\theta^l-b}\geq 0.
\end{aligned}$$

By definition of α in (54), it holds that $\sqrt[n-1]{\theta^k-\overline{b}_{\theta^l}}+\alpha=\sqrt[n-1]{\theta^k-b}$ and hence, we have to show that

$$\begin{aligned}
\Leftrightarrow\ &-2\sqrt[n-1]{\theta^l-\overline{b}_{\theta^l}}\sqrt[n-1]{\theta^k-b}\sqrt[n-1]{\theta^l-s''}+\sqrt[n-1]{\theta^l-s''}\sqrt[n-1]{\theta^l-b}\sqrt[n-1]{\theta^k-\overline{b}_{\theta^l}}\\
&\qquad+\sqrt[n-1]{\theta^l-\overline{b}_{\theta^l}}\sqrt[n-1]{\theta^k-s''}\sqrt[n-1]{\theta^l-b}\geq 0
\end{aligned}$$

$$\begin{aligned}
\Leftrightarrow\ &\sqrt[n-1]{\theta^l-s''}\left(\sqrt[n-1]{\theta^l-b}\sqrt[n-1]{\theta^k-\overline{b}_{\theta^l}}-\sqrt[n-1]{\theta^l-\overline{b}_{\theta^l}}\sqrt[n-1]{\theta^k-b}\right)\\
&\qquad-\sqrt[n-1]{\theta^l-\overline{b}_{\theta^l}}\left(\sqrt[n-1]{\theta^k-b}\sqrt[n-1]{\theta^l-s''}-\sqrt[n-1]{\theta^k-s''}\sqrt[n-1]{\theta^l-b}\right)\geq 0.
\end{aligned}$$

Since

$$\begin{aligned}
&\sqrt[n-1]{\theta^l-b}\sqrt[n-1]{\theta^k-\overline{b}_{\theta^l}}-\sqrt[n-1]{\theta^l-\overline{b}_{\theta^l}}\sqrt[n-1]{\theta^k-b}>0,\\
&\sqrt[n-1]{\theta^k-b}\sqrt[n-1]{\theta^l-s''}-\sqrt[n-1]{\theta^k-s''}\sqrt[n-1]{\theta^l-b}>0
\end{aligned}$$

and

$$\sqrt[n-1]{\theta^l-s''}\geq\sqrt[n-1]{\theta^l-\overline{b}_{\theta^l}},$$

it is sufficient to show that

$$\begin{aligned}
\Leftrightarrow\ &\left(\sqrt[n-1]{\theta^l-b}\sqrt[n-1]{\theta^k-\overline{b}_{\theta^l}}-\sqrt[n-1]{\theta^l-\overline{b}_{\theta^l}}\sqrt[n-1]{\theta^k-b}\right)\\
&\qquad-\left(\sqrt[n-1]{\theta^k-b}\sqrt[n-1]{\theta^l-s''}-\sqrt[n-1]{\theta^k-s''}\sqrt[n-1]{\theta^l-b}\right)\geq 0.
\end{aligned}$$

It holds that

$$\left(\sqrt[n-1]{\theta^l - b}\, \sqrt[n-1]{\theta^k - \bar{b}_{\theta^l}} - \sqrt[n-1]{\theta^l - \bar{b}_{\theta^l}}\, \sqrt[n-1]{\theta^k - b} \right)$$
$$- \left(\sqrt[n-1]{\theta^k - b}\, \sqrt[n-1]{\theta^l - s''} - \sqrt[n-1]{\theta^k - s''}\, \sqrt[n-1]{\theta^l - b} \right)$$
$$= \sqrt[n-1]{\theta^l - b} \left(\sqrt[n-1]{\theta^k - \bar{b}_{\theta^l}} + \sqrt[n-1]{\theta^k - s''} \right) - \sqrt[n-1]{\theta^k - b} \left(\sqrt[n-1]{\theta^l - \bar{b}_{\theta^l}} + \sqrt[n-1]{\theta^l - s''} \right).$$

By definition of α in (54), this is equal to

$$\sqrt[n-1]{\theta^l - b} \left(\sqrt[n-1]{\theta^k - b} - \alpha + \sqrt[n-1]{\theta^k - b} + \alpha \right) - \sqrt[n-1]{\theta^k - b} \left(\sqrt[n-1]{\theta^l - \bar{b}_{\theta^l}} + \sqrt[n-1]{\theta^l - s''} \right).$$

By definition of α_1 and α_2 in (59), this is equal to

$$= \sqrt[n-1]{\theta^l - b} \left(2 \sqrt[n-1]{\theta^k - b} \right) - \sqrt[n-1]{\theta^k - b} \left(\sqrt[n-1]{\theta^l - b} - \alpha_1 + \sqrt[n-1]{\theta^l - b} + \alpha_2 \right)$$
$$\geq \sqrt[n-1]{\theta^l - b} \left(2 \sqrt[n-1]{\theta^k - b} \right) - \sqrt[n-1]{\theta^k - b} \left(2 \sqrt[n-1]{\theta^l - b} \right) = 0.$$

Thus, we have shown inequality (57) and conclude that also in the case $b > b^*$ the assumption that the feasible set of minimization problem $M_b^{\theta^k}$ for $\theta^k > \theta^l$ and $\bar{b}_{\theta^{l-1}} < b < \bar{b}_{\theta^l}$ is not empty, leads to a contradiction.

□

For the proofs to follow we will need the following Lemma.

3.8.5 Lemma 28

Lemma 28. *For every valuation $\theta^k \geq \theta^z$ the worst-case belief $f^{\theta^k,*}$ is the solution of the following system of equations:*

$$\sum_{j=1}^{m} f_{\theta^j} = 1$$

$$\sum_{j=1}^{m} f_{\theta^j} \theta^j = \mu$$

$$(f_{\theta^1} + \cdots + f_{\theta^{k-1}})^{n-1} \left(\theta^k - \bar{b}_{\theta^{k-1}} \right) = \left(\sum_{j=1}^{h} f_{\theta^j} \right)^{n-1} \left(\theta^k - \bar{b}_{\theta^h} \right) \quad \textit{for all } h \in \{1, \ldots, k-2\}.$$

Proof. As defined in 3.5.1, the worst-case belief of type θ^k is the solution of the minimization problem with objective function

$$(f_{\theta^1} + \cdots + f_{\theta^{k-1}})^{n-1} \left(\theta^k - \bar{b}_{\theta^{k-1}} \right)$$

which consists of the two probability constraints and all incentive constraints with corresponding bid lower than $\bar{b}_{\theta^{k-1}}$. We denote this minimization problem by $M^{\theta^k}_{<\bar{b}_{\theta^{k-1}}}$.

The constraints of this minimization problem can be summarized as

$$\sum_{j=1}^{m} f_{\theta^j} = 1$$

$$\sum_{j=1}^{m} f_{\theta^j}\theta^j = \mu$$

$$(f_{\theta^1} + \cdots + f_{\theta^{k-1}})^{n-1}\left(\theta^k - \bar{b}_{\theta^{k-1}}\right) \geq \left(\left(\sum_{j=1}^{h-1} f_{\theta^j}\right) + f_{\theta^h} G_{\theta^h}(s)\right)\left(\theta^k - s\right)$$

for all $h \in \{1, \ldots, k-1\}$ and all $s \in [\bar{b}_{\theta^{h-1}}, \bar{b}_{\theta^h}]$.

According to part *(i)* of Proposition 11, the solution of minimization problem $M^{\theta^k}_{<\bar{b}_{\theta^{k-1}}}$ is also a solution of minimization problem $M^{\theta^k}_{\bar{b}_{\theta^{k-1}}}$. Hence, an incentive constraint corresponding to a bid b with $\bar{b}_{\theta^{j-1}} < b < \bar{b}_{\theta^j}$ with $1 < j < k-1$ cannot be binding because otherwise it would follow from Lemma 23 that the solution of minimization problem $M^{\theta^k}_{<\bar{b}_{\theta^{k-1}}}$ would be an element of the feasible set of minimization problem $M^{\theta^k}_{b}$. But this would be a contradiction to Lemma 24.

Hence, the set of possible binding incentive constraints is a subset of the incentive constraints with corresponding bids $\bar{b}_{\theta^j}$ with $j \in \{1, \ldots, k-2\}$. It is left to show that every incentive constraint with corresponding bid $\bar{b}_{\theta^j}$ with $j \in \{1, \ldots, k-2\}$ is binding. As shown in the proof of part *(i)* of Proposition 11, in the worst-case belief of type θ^k there is no probability weight on types above θ^k and therefore we can write the worst-case belief of type θ^k as $\left(f^{\theta^k,*}_{\theta^1}, \ldots, f^{\theta^k,*}_{\theta^k}, 0, \ldots, 0\right)$. Assume that under $\left(f^{\theta^k,*}_{\theta^1}, \ldots, f^{\theta^k,*}_{\theta^k}, 0, \ldots, 0\right)$ an incentive constraint with corresponding bid $\bar{b}_{\theta^j}$ for some $j \in \{1, \ldots, k-2\}$ is not binding in minimization problem $M^{\theta^k}_{<\bar{b}_{\theta^{k-1}}}$. Then we will construct a feasible solution of minimization problem $M^{\theta^k}_{<\bar{b}_{\theta^{k-1}}}$ which leads to a lower value of the objective function than $\left(f^{\theta^k,*}_{\theta^1}, \ldots, f^{\theta^k,*}_{\theta^k}, 0, \ldots, 0\right)$ given by

$$\left(f^{\theta^k,*}_{\theta^1} + \cdots + f^{\theta^k,*}_{\theta^{k-1}}\right)^{n-1}\left(\theta^k - \bar{b}_{\theta^{k-1}}\right).$$

Given the intuition provided above for Lemma 26, this should not come as a surprise. We stated that in the solution of a minimization problem $M^{\theta^k}_{\bar{b}_{\theta^k-1}}$ the probability weight on lower types should be as high as possible without violating a constraint because this allows to put probability weight on high types without violating the second probability constraint. More precisely, if a constraint with corresponding bid $\bar{b}_{\theta^j}$ is not binding, this implies that one can reduce the probability weight on $f_{\theta^{j-1}}$ and increase probability weight on f_{θ^j} without violating an incentive constraint. This reduces the mean and therefore one can increase the probability weight on f_{θ^k}. This results in a lower value of the objective function. The rest of the proof formalizes this idea.

Case 1: $j = 1$. Let

$$l = \min_{\hat{l}>1} \left\{ \hat{l} \mid f^{\theta^k,*}_{\theta^{\hat{l}}} > 0 \right\},$$

i.e. let θ^l be the smallest valuation such that $f^{\theta^k,*}_{\theta^l}$ is strictly greater than zero. We claim that the vector

$$f^{\theta^k}_{\epsilon} = \left(f^{\theta^k,*}_{\theta^1} + \epsilon_{\theta^1}, f^{\theta^k,*}_{\theta^2}, \ldots, f^{\theta^k,*}_{\theta^l} - \epsilon_{\theta^l}, f^{\theta^k,*}_{\theta^{l+1}}, \ldots, f^{\theta^k,*}_{\theta^k} - \epsilon_{\theta^k}, f^{\theta^k,*}_{\theta^{k+1}}, \ldots, f^{\theta^k,*}_{\theta^m} \right)$$

fulfills all constraints of $M^{\theta^k}_{<\bar{b}_{\theta^{k-1}}}$ but leads to a lower value of the objective function. Here $\epsilon_{\theta^1}, \epsilon_{\theta^l}, \epsilon_{\theta^k}$ are strictly positive real numbers such that it holds

$$\epsilon_{\theta^1} - \epsilon_{\theta^l} + \epsilon_{\theta^k} = 0 \tag{60}$$

$$-\epsilon_{\theta^l}\theta^l + \epsilon_{\theta^k}\theta^k = 0. \tag{61}$$

First, we will show that such $\epsilon_{\theta^1}, \epsilon_{\theta^l}, \epsilon_{\theta^k}$ exist, then we will show that the proposed vector is an element of the feasible set of minimization problem $M^{\theta^k}_{<\bar{b}_{\theta^{k-1}}}$. Since it follows from (60) that $\epsilon_{\theta^1} - \epsilon_{\theta^l} < 0$, it follows directly that the constructed vector leads to a lower value of the objective function than $f^{\theta^k,*}$.

Equations (60) and (61) are solved by any choice of $\epsilon_{\theta^1}, \epsilon_{\theta^l}, \epsilon_{\theta^k}$ that fulfill

$$\epsilon_{\theta^1} - \epsilon_{\theta^l} + \epsilon_{\theta^k} = 0, \quad \epsilon_{\theta^k} = \frac{\epsilon_{\theta^l}\theta^l}{\theta^k}$$

$$\Leftrightarrow \epsilon_{\theta^1} - \epsilon_{\theta^l} + \frac{\epsilon_{\theta^l}\theta^l}{\theta^k} = 0, \quad \epsilon_{\theta^k} = \frac{\epsilon_{\theta^l}\theta^l}{\theta^k}$$

$$\Leftrightarrow \epsilon_{\theta^1} = \frac{\epsilon_{\theta^l}\left(\theta^k - \theta^l\right)}{\theta^k}, \quad \epsilon_{\theta^k} = \frac{\epsilon_{\theta^l}\theta^l}{\theta^k}.$$

This shows that $\epsilon_{\theta^1}, \epsilon_{\theta^l}, \epsilon_{\theta^k}$ can be chosen as strictly positive real numbers. Moreover, it holds that the smaller the valuation of ϵ_{θ^1}, the smaller the valuation of ϵ_{θ^l}. Therefore, $\epsilon_{\theta^1}, \epsilon_{\theta^l}, \epsilon_{\theta^k}$ can be chosen such that the incentive constraint corresponding to bid zero given by

$$\left(f^{\theta^k,*}_{\theta^1} + \epsilon_{\theta^1} + \cdots + f^{\theta^k,*}_{\theta^l} - \epsilon_{\theta^l} + \cdots + f^{\theta^k,*}_{\theta^{k-1}} \right)^{n-1} \left(\theta^k - \bar{b}_{\theta^{k-1}} \right) \geq \left(f^{\theta^k,*}_{\theta^1} + \epsilon_{\theta^1} \right)^{n-1} \theta^k$$

is fulfilled. The probability constraints are fulfilled by construction. Since all incentive constraints with corresponding bid b with $\bar{b}_{\theta^{h-1}} < b < \bar{b}_{\theta^h}$ and $h < k-1$ are not binding under $\left(f^{\theta^k,*}_{\theta^1}, \ldots, f^{\theta^k,*}_{\theta^k}, 0, \ldots, 0 \right)$, they will be fulfilled under $f^{\theta^k}_{\epsilon}$ if ϵ_{θ^1} and ϵ_{θ^l} are sufficiently small. Since $f^{\theta^k,*}_{\theta^j} = 0$ for all $1 < j < l$, all incentive constraints with corresponding bid $\bar{b}_{\theta^h}$ with $1 < h < l$ are fulfilled if ϵ_{θ^l} is sufficiently small. Every other incentive constraint given by

$$\left(f_{\theta^1}^{\theta^k,*} + \epsilon_{\theta^1} + f_{\theta^2}^{\theta^k,*} - \epsilon_{\theta^2} + \cdots + f_{\theta^{k-1}}^{\theta^k,*}\right)^{n-1} \left(\theta^k - \bar{b}_{\theta^{k-1}}\right)$$
$$\geq \left(f_{\theta^1}^{\theta^k,*} + \epsilon_{\theta^1} + f_{\theta^2}^{\theta^k} - \epsilon_{\theta^2} + \ldots + f_{\theta^h}^{\theta^k,*}\right)^{n-1} \left(\theta^k - \bar{b}_{\theta^h}\right)$$

for $l \leq h \leq k-2$ is fulfilled since it holds that

$$\left(f_{\theta^1}^{\theta^k,*} + f_{\theta^2}^{\theta^k,*} - + \cdots + f_{\theta^{k-1}}^{\theta^k,*}\right)^{n-1} \left(\theta^k - \bar{b}_{\theta^{k-1}}\right) \geq \left(f_{\theta^1}^{\theta^k,*} + f_{\theta^2}^{\theta^k} + \ldots + f_{\theta^h}^{\theta^k,*}\right)^{n-1} \left(\theta^k - \bar{b}_{\theta^h}\right)$$

and $\epsilon_{\theta^1} - \epsilon_{\theta^2} < 0$. Hence, we have found a vector of probabilities which fulfills all probability and all incentive constraints while inducing a lower value of the objective function. Since $f_{\theta^l}^{\theta^k,*} > 0$, the constraint that all probabilities have to be non-negative is also fulfilled if ϵ_{θ^l} is sufficiently small. We conclude that the assumption that the incentive constraint with corresponding bid 0 is not binding in the worst-case belief of type θ^k, leads to a contradiction.

Case 2: $j > 1$. If the non-binding incentive constraint is an incentive constraint with corresponding bid $\bar{b}_{\theta^j}$ with $j > 1$, we proceed similarly, by constructing a vector which is an element of the feasible set of minimization problem $M^{\theta^k}_{<\bar{b}_{\theta^{k-1}}}$ but leads to a lower value of the objective function. Let

$$l' = \min_{\hat{l} > j} \left\{ \hat{l} \mid f_{\theta^{\hat{l}}}^{\theta^k,*} > 0 \right\},$$

then it must hold that $l' \leq k-1$ because otherwise bidding $\bar{b}_{\theta^{k-1}}$ would never be a best reply. We claim that the desired vector is given by

$$f_\epsilon^{\theta^k} = \left(f_{\theta^1}^{\theta^k,*} - \epsilon_{\theta^1}, f_{\theta^2}^{\theta^k,*}, \ldots, f_{\theta^j}^{\theta^k,*} + \epsilon_{\theta^j}, f_{\theta^{j+1}}^{\theta^k,*}, \ldots, f_{\theta^{l'}}^{\theta^k,*} - \epsilon_{\theta^{l'}}, f_{\theta^{l'+1}}^{\theta^k,*}, \ldots, f_{\theta^k}^{\theta^k,*} + \epsilon_{\theta^k}, f_{\theta^{k+1}}^{\theta^k,*}, \ldots, f_{\theta^m}^{\theta^k,*}\right)$$

where $\epsilon_{\theta^1}, \epsilon_{\theta^j}, \epsilon_{\theta^{l'}}, \epsilon_{\theta^k}$ are strictly positive real numbers such that it holds

$$-\epsilon_{\theta^1} + \epsilon_{\theta^j} - \epsilon_{\theta^{l'}} + \epsilon_{\theta^k} = 0 \tag{62}$$

$$\epsilon_{\theta^j}\theta^j - \epsilon_{\theta^{l'}}\theta^{l'} + \epsilon_{\theta^k}\theta^k = 0. \tag{63}$$

Since it follows from (62) that $-\epsilon_{\theta^1} + \epsilon_{\theta^j} - \epsilon_{\theta^l} < 0$, it follows directly that the constructed vector leads to a lower value of the objective function than $f^{\theta^k,*}$. In addition, we choose $\epsilon_{\theta^{l'}}$ sufficiently small such that the non-binding incentive constraint

$$\left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^{k-1}}^{\theta^k,*}\right)^{n-1} \left(\theta^k - \bar{b}_{\theta^{k-1}}\right) > \left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^j}^{\theta^k,*}\right)^{n-1} \left(\theta^k - \bar{b}_{\theta^j}\right)$$

is still fulfilled, i.e. it holds that

$$\left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^{k-1}}^{\theta^k,*} - \epsilon_{\theta^1} + \epsilon_{\theta^j} - \epsilon_{\theta^{l'}}\right)^{n-1} \left(\theta^k - \bar{b}_{\theta^{k-1}}\right)$$
$$\geq \left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^j}^{\theta^k,*} - \epsilon_{\theta^1} + \epsilon_{\theta^j}\right)^{n-1} \left(\theta^k - \bar{b}_{\theta^j}\right). \quad (64)$$

Again, we will first show that such $\epsilon_{\theta^1}, \epsilon_{\theta^j}, \epsilon_{\theta^j}, \epsilon_{\theta^k}$ exist, then we will show that the proposed vector is an element of the feasible set of minimization problem $M^{\theta^k}_{<\bar{b}_{\theta^{k-1}}}$.

Equations (62) and (63) are solved by any choice of $\epsilon_{\theta^1}, \epsilon_{\theta^j}, \epsilon_{\theta^j}, \epsilon_{\theta^k}$ which fulfills

$$-\epsilon_{\theta^1} + \epsilon_{\theta^j} - \epsilon_{\theta^{l'}} + \epsilon_{\theta^k} = 0, \quad \epsilon_{\theta^k} = \frac{-\epsilon_{\theta^j}\theta^j + \epsilon_{\theta^{l'}}\theta^{l'}}{\theta^k}$$

$$\Leftrightarrow -\epsilon_{\theta^1} + \epsilon_{\theta^j} - \epsilon_{\theta^{l'}} + \frac{-\epsilon_{\theta^j}\theta^j + \epsilon_{\theta^{l'}}\theta^{l'}}{\theta^k} = 0, \quad \epsilon_{\theta^k} = \frac{-\epsilon_{\theta^j}\theta^j + \epsilon_{\theta^{l'}}\theta^{l'}}{\theta^k}$$

$$\Leftrightarrow -\epsilon_{\theta^1}\theta^k + \epsilon_{\theta^j}\theta^k - \epsilon_{\theta^{l'}}\theta^k - \epsilon_{\theta^j}\theta^j + \epsilon_{\theta^{l'}}\theta^{l'} = 0, \quad \epsilon_{\theta^k} = \frac{-\epsilon_{\theta^j}\theta^j + \epsilon_{\theta^{l'}}\theta^{l'}}{\theta^k}$$

$$\Leftrightarrow \epsilon_{\theta^j}\left(\theta^k - \theta^j\right) = \epsilon_{\theta^{l'}}\left(\theta^k - \theta^j\right) + \epsilon_{\theta^1}\theta^k, \quad \epsilon_{\theta^k} = \frac{-\epsilon_{\theta^j}\theta^j + \epsilon_{\theta^{l'}}\theta^{l'}}{\theta^k}$$

$$\Leftrightarrow \epsilon_{\theta^j} = \epsilon_{\theta^{l'}} + \frac{\epsilon_{\theta^1}\theta^k}{\theta^k - \theta^j}, \quad \epsilon_{\theta^k} = \frac{-\epsilon_{\theta^j}\theta^j + \epsilon_{\theta^{l'}}\theta^{l'}}{\theta^k}$$

$$\Leftrightarrow \epsilon_{\theta^j} = \epsilon_{\theta^{l'}} + \frac{\epsilon_{\theta^1}\theta^k}{\theta^k - \theta^j}, \quad \epsilon_{\theta^k} = \frac{-\left(\epsilon_{\theta^{l'}} + \frac{\epsilon_{\theta^1}\theta^k}{\theta^k - \theta^j}\right)\theta^j + \epsilon_{\theta^{l'}}\theta^{l'}}{\theta^k}$$

$$\Leftrightarrow \epsilon_{\theta^j} = \epsilon_{\theta^{l'}} + \frac{\epsilon_{\theta^1}\theta^k}{\theta^k - \theta^j}, \quad \epsilon_{\theta^k} = \frac{-\epsilon_{\theta^{l'}}\left(\theta^k - \theta^j\right) + \epsilon_{\theta^1}\theta^k\theta^j + \epsilon_{\theta^{l'}}\theta^{l'}}{\theta^k\left(\theta^k - \theta^j\right)}$$

$$\Leftrightarrow \epsilon_{\theta^j} = \epsilon_{\theta^{l'}} + \frac{\epsilon_{\theta^1}\theta^k}{\theta^k - \theta^j}, \quad \epsilon_{\theta^k} = \frac{\epsilon_{\theta^{l'}}\left(\theta^{l'} - \left(\theta^k - \theta^j\right)\theta^j\right) + \epsilon_{\theta^1}\theta^k\theta^j}{\theta^k\left(\theta^k - \theta^j\right)}.$$

This shows that $\epsilon_{\theta^1}, \epsilon_{\theta^j}, \epsilon_{\theta^{l'}}, \epsilon_{\theta^k}$ can be chosen as strictly positive real numbers. Moreover, it holds that the smaller the valuation of ϵ_{θ^1} and $\epsilon_{\theta^{l'}}$, the smaller the valuation of ϵ_{θ^j} and ϵ_{θ^k}. Therefore, $\epsilon_{\theta^1}, \epsilon_{\theta^j}$ and $\epsilon_{\theta^{l'}}$ can be both chosen sufficiently small such that the incentive constraint (64) is fulfilled.

The probability constraints are fulfilled by construction. Since all incentive constraints with corresponding bid b with $\bar{b}_{\theta^{h-1}} < b < \bar{b}_{\theta^h}$ for $h < k-1$ are not binding under $f^{\theta^k,*}$, they will be fulfilled under $f^{\theta^k}_{\epsilon}$ if $\epsilon_{\theta^1}, \epsilon_{\theta^j}, \epsilon_{\theta^{l'}}, \epsilon_{\theta^k}$ are sufficiently small. Any incentive constraint with corresponding bid $\bar{b}_{\theta^h}$ for $1 \leq h \leq j-1$ given by

$$\left(f_{\theta^1}^{\theta^k,*} + f_{\theta^2}^{\theta^k,*} + \cdots + f_{\theta^{k-1}}^{\theta^k,*} - \epsilon_{\theta^1} + \epsilon_{\theta^j} - \epsilon_{\theta^{l'}}\right)^{n-1} \left(\theta^k - \bar{b}_{\theta^{k-1}}\right) \\ \geq \left(f_{\theta^1}^{\theta^k,*} + f_{\theta^2}^{\theta^k,*} + \ldots + f_{\theta^h}^{\theta^k,*} - \epsilon_{\theta^1}\right)^{n-1} \left(\theta^k - \bar{b}_{\theta^h}\right)$$

is fulfilled since it holds that

$$\left(f_{\theta^1}^{\theta^k,*} + f_{\theta^2}^{\theta^k,*} + \cdots + f_{\theta^{k-1}}^{\theta^k,*}\right)^{n-1} \left(\theta^k - \bar{b}_{\theta^{k-1}}\right) \geq \left(f_{\theta^1}^{\theta^k,*} + f_{\theta^2}^{\theta^k,*} + \ldots + f_{\theta^h}^{\theta^k,*}\right)^{n-1} \left(\theta^k - \bar{b}_{\theta^h}\right)$$

and $0 > -\epsilon_{\theta^1} + \epsilon_{\theta^j} - \epsilon_{\theta^{l'}} > -\epsilon_{\theta^1}$. The incentive constraint with corresponding bid $\bar{b}_{\theta^j}$ is fulfilled by construction. Since $f_{\theta^h}^{\theta^k,*} = 0$ for $j < h < l'$, it holds that all incentive constraints with corresponding bid $\bar{b}_{\theta^h}$ with $j < h < l'$ are fulfilled if $\epsilon_{\theta^{l'}}$ is sufficiently small.

An incentive constraint with corresponding bid $\bar{b}_{\theta^h}$ for $l' \leq h \leq k-2$ given by

$$\left(f_{\theta^1}^{\theta^k} + f_{\theta^2}^{\theta^k,*} + \cdots + f_{\theta^{k-1}}^{\theta^k,*} - \epsilon_{\theta^1} + \epsilon_{\theta^j} - \epsilon_{\theta^{l'}}\right)^{n-1} \left(\theta^k - \bar{b}_{\theta^{k-1}}\right) \\ \geq \left(f_{\theta^1}^{\theta^k,*} + f_{\theta^2}^{\theta^k,*} + \ldots + f_{\theta^h}^{\theta^k,*} - \epsilon_{\theta^1} + \epsilon_{\theta^j} - \epsilon_{\theta^{l'}}\right)^{n-1} \left(\theta^k - \bar{b}_{\theta^h}\right)$$

is fulfilled since it holds that

$$\left(f_{\theta^1}^{\theta^k,*} + f_{\theta^2}^{\theta^k,*} + \cdots + f_{\theta^{k-1}}^{\theta^k,*}\right)^{n-1} \left(\theta^k - \bar{b}_{\theta^{k-1}}\right) \geq \left(f_{\theta^1}^{\theta^k,*} + f_{\theta^2}^{\theta^k,*} + \ldots + f_{\theta^h}^{\theta^k,*}\right)^{n-1} \left(\theta^k - \bar{b}_{\theta^h}\right)$$

and $-\epsilon_{\theta^1} + \epsilon_{\theta^j} - \epsilon_{\theta^{l'}} < 0$. Hence, we have found a vector of probabilities, $f_{\epsilon}^{\theta^k}$, which fulfills all probability and all incentive constraints while inducing a lower value of the objective function. We can assume that $f_{\theta^1}^{\theta^k} > 0$ because otherwise, the incentive constraint corresponding to bid $0 = \bar{b}_{\theta^1}$ is not binding and the first case applies. Since $f_{\theta^{l'}}^{\theta^k} > 0$, the constraint that probabilities are non-negative is also fulfilled if $\epsilon_{\theta^{l'}}$ is sufficiently small. We conclude that the assumption that an incentive constraint with corresponding bid $\bar{b}_{\theta^j}$ for $1 \leq j \leq k-2$ is not binding, leads to a contradiction.

□

3.8.6 Proof of Lemma 25

Proof. We have to show that for every pair of valuations θ^l and θ^k such that $\theta^z \leq \theta^{k+1} \leq \theta^l$ and every b with $\bar{b}_{\theta^{l-1}} < b \leq \bar{b}_{\theta^l}$ the feasible set of minimization problem $M_{\bar{b}_{\theta^l}}^{\theta^k}$ is empty.

Assume that the feasible set of minimization problem $M_b^{\theta^k}$ for some $b \in (\bar{b}_{\theta^{l-1}}, \bar{b}_{\theta^l}]$ with $l > k$ is not empty. Let $\left(f_{\theta^1}^{\theta^k,b}, \ldots, f_{\theta^m}^{\theta^k,b}\right)$ denote a solution. We can write

$$\left(f_{\theta^1}^{\theta^k,b}, \ldots, f_{\theta^m}^{\theta^k,b}\right) = \left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1}, \ldots, f_{\theta^m}^{\theta^k,*} + \delta_{\theta^m}\right)$$

for some appropriate be real numbers $\delta_{\theta^1}, \ldots, \delta_{\theta^m}$. We will prove the claim in four steps:

(1) For every j with $k+1 \leq j \leq m$ it holds $\delta_{\theta^j} \geq 0$.

(2) There exist strictly positive real numbers α and β such that

$$\sum_{j=1}^{k} \delta_{\theta^j} = -\alpha \quad \text{and} \quad \sum_{j=k+1}^{m} \delta_{\theta^j} = \beta$$

(3) Let $\left(\hat{\delta}_{\theta^1}, \ldots, \hat{\delta}_{\theta^m}\right) =$

$$\operatorname{argmin} \left\{ \sum_{j=1}^{k} \tilde{\delta}_{\theta^j} \theta^j \;\middle|\; \left(f_{\theta^1}^{\theta^k,*} + \tilde{\delta}_{\theta^1}, \ldots, f_{\theta^m}^{\theta^k,*} + \tilde{\delta}_{\theta^m}\right) \text{ is element of feasible set of } M_b^{\theta^k}, \sum_{j=1}^{k} \tilde{\delta}_{\theta^j} = -\alpha \right\},$$

then it holds that $\hat{\delta}_{\theta^j} \leq 0$ for all $1 \leq j \leq k-1$.

(4) We use steps (1)-(3) in order to show that the assumption that $\left(f_{\theta^1}^{\theta^k,b}, \ldots, f_{\theta^m}^{\theta^k,b}\right)$ is a solution of minimization problem $M_b^{\theta^k}$, leads to a contradiction.

Proof of step (1)
As shown in the proof of part *(i)* of Proposition 11, it holds that $f_{\theta^j}^{\theta^k,*} = 0$ for all $j > k$. Since probabilities cannot be negative, it follows that $\delta_{\theta^j} \geq 0$ for all $k+1 \leq j \leq m$.

Proof of step (2)
Since $k < l$ and $f_{\theta^j}^{\theta^k,*} = 0$ for all $j > k$, it holds that

$$\left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^k}^{\theta^k,*}\right)^{n-1} \left(\theta^k - \overline{b}_{\theta^k}\right) > \left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^l}^{\theta^l,*} G_{\theta^l}(b)\right)^{n-1} \left(\theta^k - b\right).$$

As $\left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1}, \ldots, f_{\theta^k}^{\theta^k,*} + \delta_{\theta^k}\right)$ is an element of the feasible set of minimization problem $M_b^{\theta^k}$, it must hold that

$$\begin{aligned}\left(f_{\theta^1}^{\theta^k} + \delta_{\theta^1}, \ldots, f_{\theta^k}^{\theta^k,*} + \delta_{\theta^k} + \cdots + \left(f_{\theta^l}^{\theta^k,*} + \delta_{\theta^l}\right) G_{\theta^l}(b)\right)^{n-1} \left(\theta^k - b\right) \\ \geq \left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1}, \ldots, f_{\theta^k}^{\theta^k,*} + \delta_{\theta^k}\right)^{n-1} \left(\theta^k - \overline{b}_{\theta^k}\right).\end{aligned}$$

It follows that either $\sum_{j=1}^{k} \delta_{\theta^j} < 0$ or $\sum_{j=k+1}^{l} \delta_{\theta^j} > 0$. Due to the first probability constraint, it holds $\sum_{j=1}^{m} \delta_{\theta^j} = 0$. Assume that $\sum_{j=1}^{k} \delta_{\theta^j} \geq 0$. Then it must hold $\sum_{j=k+1}^{l} \delta_{\theta^j} > 0$. Since due to step (1) it holds that $\delta_{\theta^j} \geq 0$ for all $k+1 \leq j \leq m$, it follows that $\sum_{j=1}^{m} \delta_{\theta^j} > 0$ which leads to a contradiction. Hence, it must hold that $\sum_{j=1}^{k} \delta_{\theta^j} < 0$. Therefore, there exist strictly positive be real numbers α and β such that $\sum_{j=1}^{k} \delta_{\theta^j} = -\alpha$ and $\sum_{j=k+1}^{m} \delta_{\theta^j} = \beta$.

Proof of step (3)

We start the proof of step (3) by showing the following claim: Let

$$\left(\hat{\delta}_{\theta^1}, \ldots, \hat{\delta}_{\theta^m}\right) =$$

$$\operatorname{argmin} \left\{ \sum_{j=1}^{k} \tilde{\delta}_{\theta^j} \theta^j \;\middle|\; \left(f_{\theta^1}^{\theta^k,*} + \tilde{\delta}_{\theta^1}, \ldots, f_{\theta^m}^{\theta^k,*} + \tilde{\delta}_{\theta^m}\right) \text{ is element of feasible set of } M_b^{\theta^k}, \sum_{j=1}^{k} \tilde{\delta}_{\theta^j} = -\alpha \right\}.$$

Then it holds that under $\left(f_{\theta^1}^{\theta^k} + \hat{\delta}_{\theta^1}, \ldots, f_{\theta^m}^{\theta^k} + \hat{\delta}_{\theta^m}\right)$ in minimization problem $M_b^{\theta^k}$ all constraints with corresponding bid $\overline{b}_{\theta^t}$ with $t \leq k-1$ are binding i.e.

$$\left(f_{\theta^1}^{\theta^k,*} + \hat{\delta}_{\theta^1}, \ldots, f_{\theta^k}^{\theta^k,*} + \hat{\delta}_{\theta^k} + \cdots + \left(f_{\theta^l}^{\theta^l,*} + \hat{\delta}_{\theta^l}\right) G_{\theta^l}(b)\right)^{n-1} \left(\theta^k - b\right)$$
$$= \left(f_{\theta^1}^{\theta^k,*} + \hat{\delta}_{\theta^1} + \cdots + f_{\theta^t}^{\theta^k,*} + \hat{\delta}_{\theta^t}\right)^{n-1} \left(\theta^k - \overline{b}_{\theta^t}\right) \quad (65)$$

for all $t \leq k-1$.

In order to show this claim, consider all real numbers $\tilde{\delta}_{\theta^1}, \ldots, \tilde{\delta}_{\theta^m}$ such that $\sum_{j=1}^{k} \tilde{\delta}_{\theta^j} = -\alpha$ and $\left(f_{\theta^1}^{\theta^k,*} + \tilde{\delta}_{\theta^1}, \ldots, f_{\theta^m}^{\theta^k,*} + \tilde{\delta}_{\theta^m}\right)$ is an element of feasible set of minimization problem $M_b^{\theta^k}$. If we would consider only the constraint $\sum_{j=1}^{k} \tilde{\delta}_{\theta^j} = -\alpha$, then one could achieve arbitrarily small values of the term $\sum_{j=1}^{k} \tilde{\delta}_{\theta^j} \theta^j$ by choosing high values of $\delta_{\theta^1}, \ldots, \delta_{k-1}$ which results in a low value of δ_{θ^k}. Adding the constraint that $\left(f_{\theta^1}^{\theta^k,*} + \tilde{\delta}_{\theta^1}, \ldots, f_{\theta^m}^{\theta^k,*} + \tilde{\delta}_{\theta^m}\right)$ is an element of feasible set of minimization problem $M_b^{\theta^k}$, the value of the term $\sum_{j=1}^{k} \tilde{\delta}_{\theta^j} \theta^j$ is minimized if the values of all $\tilde{\delta}_{\theta^j}$ with $1 \leq j \leq k-1$ are as high as possible and the value of δ_{θ^k} is as low as possible without violating any incentive constraint.

An incentive constraint with corresponding bid $\overline{b}_{\theta^{t-1}} < b' < \overline{b}_{\theta^t}$ with $t < k$ cannot be binding because then due to Lemma 23, $\left(f_{\theta^1}^{\theta^k,b}, \ldots, f_{\theta^m}^{\theta^k,b}\right)$ would be an element of the feasible set of minimization problem $M_{b'}^{\theta^k}$ which would be a contradiction to Lemma 24. It follows that if all constraints with corresponding bid $\overline{b}_{\theta^t}$ with $t < k$ are binding, $\tilde{\delta}_{\theta^1}, \ldots, \tilde{\delta}_{\theta^{k-1}}$ cannot be increased without violating an incentive constraint in minimization problem $M_b^{\theta^k}$. A decrease of $\tilde{\delta}_{\theta^t}$ with $t \leq k-1$ would imply a higher $\tilde{\delta}_{\theta^k}$ which would lead to a higher value of the term $\sum_{j=1}^{k} \delta_{\theta^j} \theta^j$. We conclude that the values of all δ_{θ^j} with $1 \leq j \leq k-1$ are as high as possible if all constraints with corresponding bid $\overline{b}_{\theta^t}$ with $t \leq k-1$ are binding i.e.

$$\left(f_{\theta^1}^{\theta^k,*} + \hat{\delta}_{\theta^1}, \ldots, f_{\theta^k}^{\theta^k,*} + \hat{\delta}_{\theta^k} + \cdots + \left(f_{\theta^l}^{\theta^l,*} + \hat{\delta}_{\theta^l,}\right) G_{\theta^l}(b)\right)^{n-1} \left(\theta^k - b\right)$$
$$= \left(f_{\theta^1}^{\theta^k,*} + \hat{\delta}_{\theta^1} + \cdots + f_{\theta^t}^{\theta^k,*} + \hat{\delta}_{\theta^t}\right)^{n-1} \left(\theta^k - \overline{b}_{\theta^t}\right)$$

for $t \leq k-1$.

We will use this claim in order to show inductively that all $\hat{\delta}_{\theta^1}, \ldots, \hat{\delta}_{\theta^{k-1}}$ are non-positive. We start the induction by showing that $\hat{\delta}_{\theta^1} \leq 0$. Due to the first probability constraint, it holds that

$$\sum_{j=1}^{k} \hat{\delta}_{\theta^j} + \sum_{j=k+1}^{l} \hat{\delta}_{\theta^j} + \sum_{j=l+1}^{m} \hat{\delta}_{\theta^j} = 0$$

and since $l > k$, we can conclude with the same reasoning as in step (1) that

$$\sum_{j=l+1}^{m} \hat{\delta}_{\theta^j} \geq 0.$$

It follows that

$$\sum_{j=1}^{k} \hat{\delta}_{\theta^j} + \sum_{j=k+1}^{l} \hat{\delta}_{\theta^j} \leq 0.$$

Since $\hat{\delta}_{\theta^l} \geq 0$, it holds that

$$\hat{\delta}_{\theta^1} + \ldots + \hat{\delta}_{\theta^l} G_{\theta^l}(b) \leq 0.$$

Moreover, it follows from $f_{\theta^j}^{\theta^k,*} = 0$ for $j > k$, that

$$\begin{aligned} &\left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^l}^{\theta^k,*} G_{\theta^l}(b)\right)^{n-1} \left(\theta^k - b\right) \\ &\qquad < \left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^k}^{\theta^k,*}\right)^{n-1} \left(\theta^k - \overline{b}_{\theta^k}\right) = \left(f_{\theta^1}^{\theta^k,*}\right)^{n-1} \theta^k \end{aligned}$$

where the equality follows from Lemma 28. It also holds that

$$\left(f_{\theta^1}^{\theta^k,*} + \hat{\delta}_{\theta^1} + \cdots + \left(f_{\theta^l}^{\theta^k,*} + \hat{\delta}_{\theta^l}\right) G_{\theta^l}(b)\right)^{n-1} \left(\theta^k - b\right) \geq \left(f_{\theta^1}^{\theta^k} + \hat{\delta}_{\theta^1}\right)^{n-1} \theta^k$$

from which it follows that

$$\hat{\delta}_{\theta^1} \leq \frac{\left(\hat{\delta}_{\theta^1} + \cdots + \hat{\delta}_{\theta^l} G_{\theta^l}(b)\right) \sqrt[n-1]{\theta^k - b}}{\sqrt[n-1]{\theta^k}} \leq 0.$$

We now turn our attention to the inductive step. Assume it is already shown that $\hat{\delta}_t \leq 0$ for all $1 \leq t < k-2$. It follows from Lemma 28 that

$$\left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^t}^{\theta^k,*}\right)^{n-1} \left(\theta^k - \overline{b}_{\theta^t}\right) = \left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^{t+1}}^{\theta^k,*}\right)^{n-1} \left(\theta^k - \overline{b}_{\theta^{t+1}}\right)$$

and due to the construction of $\hat{\delta}_{\theta^1}, \ldots \hat{\delta}_{\theta^m}$ it holds that

$$\begin{aligned} &\left(f_{\theta^1}^{\theta^k,*} + \hat{\delta}_{\theta^1} + \cdots + f_{\theta^t}^{\theta^k,*} + \hat{\delta}_{\theta^t}\right)^{n-1} \left(\theta^k - \overline{b}_{\theta^t}\right) \\ &\qquad = \left(f_{\theta^1}^{\theta^k,*} + \hat{\delta}_{\theta^1} + \cdots + f_{\theta^{t+1}}^{\theta^k,*} + \hat{\delta}_{\theta^{t+1}}\right)^{n-1} \left(\theta^k - \overline{b}_{\theta^{t+1}}\right) \end{aligned}$$

from which follows that

$$\hat{\delta}_{\theta^{t+1}} = \frac{\left(\hat{\delta}_{\theta^1} + \cdots + \hat{\delta}_{\theta^t}\right)\left(\sqrt[n-1]{\theta^k - \bar{b}_{\theta^t}} - \sqrt[n-1]{\theta^k - \bar{b}_{\theta^{t+1}}}\right)}{\sqrt[n-1]{\theta^k - \bar{b}_{\theta^{t+1}}}} \leq 0.$$

We conclude that for all $1 \leq j \leq k-1$ it holds $\hat{\delta}_{\theta^j} \leq 0$.

Proof of step (4)
Recall that we defined $\delta_{\theta^1}, \ldots, \delta_{\theta^m}$ by

$$\left(f_{\theta^1}^{\theta^k,b}, \ldots, f_{\theta^m}^{\theta^k,b}\right) = \left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1}, \ldots, f_{\theta^m}^{\theta^k,*} + \delta_{\theta^m}\right).$$

According to step (3) it holds $\sum_{j=1}^{k} \hat{\delta}_{\theta^j}\theta^j \geq \sum_{j=1}^{k} \hat{\delta}_{\theta^j}\theta^k = -\alpha\theta^k$. Hence, the maximal possible valuation for the term $-\sum_{j=1}^{k} \delta_{\theta^j}\theta^j$ equals to $\alpha\theta^k$. Since due to step (1), $\delta_{\theta^j} \geq 0$ for all $k+1 \leq j \leq m$, it follows that $\sum_{j=k+1}^{m} \delta_{\theta^j}\theta^j \geq \sum_{j=k+1}^{m} \delta_{\theta^j}\theta^{k+1} = \beta\theta^{k+1}$. Hence, the maximal possible valuation for the term $-\sum_{j=k+1}^{m} \delta_{\theta^j}\theta^j$ equals to $-\beta\theta^{k+1}$. It follows from the probability constraints that

$$-\alpha + \beta = 0$$

$$\sum_{j=1}^{k} \delta_{\theta^j}\theta^j + \sum_{j=k+1}^{m} \delta_{\theta^j}\theta^j = 0.$$

Subtracting the second equation from the first gives

$$-\alpha - \sum_{j=1}^{k} \delta_{\theta^j}\theta^j + \beta - \sum_{j=k+1}^{m} \delta_{\theta^j}\theta^j = 0.$$

It holds

$$-\alpha - \sum_{j=1}^{k} \delta_{\theta^j}\theta^j + \beta - \sum_{j=k+1}^{m} \delta_{\theta^j}\theta^j \leq -\alpha + \alpha\theta^k + \beta - \beta\theta^{k+1}.$$

Since $\alpha = \beta$ it holds

$$-\alpha + \alpha\theta^k + \beta - \beta\theta^{k+1} < 0.$$

Hence, the assumption that the feasible set of minimization problem $M_b^{\theta^k}$ is not empty, leads to a contradiction. □

3.8.7 Proof of Lemma 26

Let θ^l and θ^k be a pair of valuations such that $\theta^z \leq \theta^l \leq \theta^{k-2}$ and let p^* be the minimum p in the construction in Lemma 26. Such a minimum p exists since the worst-case belief of the θ^k-type is an element of the feasible set of minimization problem $M_{\bar{b}_{\theta^l}}^{\theta^k}$ and due to Lemma 28, in $M_{\bar{b}_{\theta^l}}^{\theta^k}$ only the incentive constraints with corresponding bid $\bar{b}_{\theta^j}$ for $1 \leq j \leq k-1$ are binding if plugging in the worst-case belief. That is, the construction in Lemma 26 stops at the latest after adding the binding incentive constraint with corresponding bid $\bar{b}_{\theta^{k-1}}$. Let $\left(\tilde{f}_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}}, \ldots, \tilde{f}_{\theta^m}^{\theta^k,\bar{b}_{\theta^l}}\right)$ denote the solution of minimization problem $M_{\bar{b}_{\theta^l}}^{\theta^k}$ as constructed in Lemma 26, i.e. if $p^* \geq l$ for all $j > p^* + 1$ (and for all $j > p^* + 2$ if $p^* < l$) it holds that $\tilde{f}_0^{\theta^k,\bar{b}_{\theta^l}} = 0$ and the vector $\left(\tilde{f}_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}}, \ldots, \tilde{f}_{\theta^{p^*+1}}^{\theta^k,\bar{b}_{\theta^l}}\right)$ is the unique solution of the system of equations given by

$$\sum_{j=1}^{p^*+1} f_{\theta^j} = 1$$

$$\sum_{j=1}^{p^*+1} f_{\theta^j}\theta^j = \mu$$

$$(f_{\theta^1} + \cdots + f_{\theta^l})^{n-1}\left(\theta^k - \bar{b}_{\theta^l}\right) = \left(\sum_{j=1}^{h} f_{\theta^j}\right)^{n-1}\left(\theta^k - \bar{b}_{\theta^h}\right) \quad \text{for all } h \in \{1, \ldots, p^*\}.$$

We will prove this Lemma using the following steps:

(1) Let $f^{\theta^k,\bar{b}_{\theta^l}} = \left(f_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}}, \ldots, f_{\theta^m}^{\theta^k,\bar{b}_{\theta^l}}\right)$ be a solution of minimization problem $M_{\bar{b}_{\theta^l}}^{\theta^k}$. Then it holds that $f_{\theta^j}^{\theta^k,\bar{b}_{\theta^l}} = 0$ for all $j > p^* + 1$ if $p^* \geq l$ and for all $j > p^* + 2$ if $p^* < l$.

(2) It holds for $f^{\theta^k,\bar{b}_{\theta^l}}$ that all constraints in $M_{\bar{b}_{\theta^l}}^{\theta^k}$ with corresponding bid $\bar{b}_{\theta^j}$ with $j \leq p^*$ have to be binding.

(3) It holds that
$$\left(\tilde{f}_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}}, \ldots, \tilde{f}_{\theta^m}^{\theta^k,\bar{b}_{\theta^l}}\right) = \left(f_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}}, \ldots, f_{\theta^m}^{\theta^k,\bar{b}_{\theta^l}}\right).$$

Proof of step (1)
If $p^* > k$, then the equation which is the binding incentive constraint corresponding to bid $\bar{b}_{\theta^l}$ is obviously redundant and therefore, the system of equations in Lemma 26 consist of two probability constraints and $p^* - 1$ binding incentive constraints. This gives a system of $p^* + 1$ equations for $p^* + 1$ variables. We will provide the proof for the case $p^* \geq l$ since the case $p^* < l$ works analogously and we will show in Lemma 27 that it indeed holds that $p^* \geq l$.

Assume that there exists at least one h with $p^* + 1 < h \leq m$ such that $f_{\theta^h}^{\bar{b}_{\theta^l}} > 0$. Let $\delta_{\theta^1}, \ldots, \delta_{\theta^m}$ be such that

$$\left(\tilde{f}_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}} + \delta_{\theta^1}, \ldots, \tilde{f}_{\theta^m}^{\theta^k,\bar{b}_{\theta^l}} + \delta_{\theta^m}\right) = \left(f_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}}, \ldots, f_{\theta^m}^{\theta^k,\bar{b}_{\theta^l}}\right).$$

It holds that $\tilde{f}_j^{\theta^k,\bar{b}_{\theta^l}} = 0$ for all j with $j > p^* + 1$. Therefore, it holds that $\delta_{\theta^j} \geq 0$ for all $p^* + 1 < j \leq m$ and there exists at least one j with $p^* + 1 < j \leq m$ such that $\delta_{\theta^j} > 0$.

Before we proceed with the proof, we introduce the concept of *δ-sequences*. We define a δ-*sequence* as a vector $\left(\delta_{\theta^{l^{min}}}, \dots, \delta_{\theta^{l^{max}}}, \delta_{\theta^{k^{min}}}, \dots, \delta_{\theta^{k^{max}}}\right)$ with $\delta_{k^{min}} = \delta_{l^{max}+1}$ such that for all j with $l^{min} \leq j \leq l^{max}$ it holds $\delta_{\theta^j} < 0$ and for all $k^{min} \leq j \leq k^{max}$ it holds $\delta_{\theta^j} \geq 0$. If at least one δ_{θ^j} is not equal to zero, it holds

$$\sum_{j=l^{min}}^{l^{max}} \delta_{\theta^j}\theta^j + \sum_{j=k^{min}}^{k^{max}} \delta_{\theta^j}\theta^j > \sum_{j=l^{min}}^{l^{max}} \delta_{\theta^j}\theta^{l^{max}} + \sum_{j=k^{min}}^{k^{max}} \delta_{\theta^j}\theta^{l^{max}} = \sum_{j=l^{min}}^{k^{max}} \delta_{\theta^j}\theta^{l^{max}}. \tag{66}$$

Every given vector $(\delta_{\theta^1}, \dots, \delta_{\theta^m})$ can be decomposed into δ-sequences. Let m' be the number of δ-sequences and $\left(\delta_{\theta^{j,l^{min}}}, \dots, \delta_{\theta^{j,l^{max}}}, \delta_{\theta^{j,k^{min}}}, \dots, \delta_{\theta^{j,k^{max}}}\right)$ be the $j-th$ δ-sequence. Let $\delta'_{\theta^j} := \sum_{s=j,l^{min}}^{j,k^{max}} \delta_{\theta^s}$ and $\theta^{j'} := \theta^{j,l^{max}}$. Then it holds

$$\sum_{j=1}^{m} \delta_{\theta^j}\theta^j > \sum_{j=1}^{m'} \delta'_{\theta^j}\theta^{j,l^{max}} = \sum_{j=1}^{m'} \delta'_{\theta^j}\theta^{j'}.$$

The vector $(\delta'_{\theta^1}, \dots, \delta'_{\theta^m})$ can again be decomposed into δ-sequences. Let m'' be the number of δ-sequences in the vector $(\delta'_{\theta^m}, \dots, \delta'_{\theta^m})$ and $\left(\delta'_{\theta^{j,l^{min}}}, \dots, \delta'_{\theta^{j,l^{max}}}, \delta'_{\theta^{j,k^{min}}}, \dots, \delta'_{\theta^{j,k^{max}}}\right)$ be the $j-th$ δ-sequence. Let $\delta''_{\theta^j} := \sum_{s=j,l^{min}}^{j,k^{max}} \delta'_{\theta^s}$ and $\theta^{j''} := \theta^{j,l^{max}}$. As in (66), we conclude that

$$\sum_{j=1}^{m} \delta_{\theta^j}\theta^j > \sum_{j=1}^{m'} \delta'_{\theta^j}\theta^{j,l^{max}} \geq \sum_{j=1}^{m''} \delta''_{\theta^j}\theta^{j',l^{max}}.$$

If there does not exist a t with $1 \leq t \leq m$ such that $\sum_{j=1}^{t} \delta_{\theta^j} > 0$, the process of decomposing into δ-sequences ends with a δ-sequence of length 2, i.e. with some vector $\left(\delta_1^{final}, \delta_2^{final}\right)$ with $\delta_1^{final} < 0$ and $\delta_2^{final} > 0$. Since $\sum_{j=1}^{m} \delta_{\theta^j} = 0$, it holds that $\delta_1 = -\delta_2$ and there exists some θ^{final} such that

$$\sum_{j=1}^{m} \delta_{\theta^j}\theta^j > \left(\delta_1^{final} + \delta_2^{final}\right)\theta^{final} = 0.$$

We illustrate the concept of δ-sequences with the following example.

Example 8. *Let*

$$(\delta_{\theta^1}, \dots, \delta_{\theta^m}) = \left(-\frac{1}{12}, -\frac{1}{6}, \frac{1}{12}, -\frac{1}{8}, \frac{1}{4}, \frac{1}{8}\right).$$

The vector has two relevant properties. It holds that $\sum_{j=1}^{m} \delta_{\theta^j} = 0$ *and there does not exist a* t *with* $1 \leq t \leq m$ *such that* $\sum_{j=1}^{t} \delta_{\theta^j} = 0$. *This vector can be decomposed into two* δ*-sequences given by* $\left(-\frac{1}{12}, -\frac{1}{6}, \frac{1}{12}\right)$ *and* $\left(-\frac{1}{8}, \frac{1}{4}, \frac{1}{8}\right)$. *It holds that*

$$-\frac{1}{12}\theta^1 - \frac{1}{6}\theta^2 + \frac{1}{12}\theta^3 > -\frac{1}{12}\theta^2 - \frac{1}{6}\theta^2 + \frac{1}{12}\theta^2 = \sum_{1}^{3} \delta_{\theta^j}\theta^2$$

and

$$-\frac{1}{8}\theta^4 + \frac{1}{4}\theta^5 + \frac{1}{8}\theta^6 > -\frac{1}{8}\theta^4 + \frac{1}{4}\theta^4 + \frac{1}{8}\theta^4 = \sum_{4}^{6} \delta_{\theta^j}\theta^4.$$

We define $\delta'_1 = \sum_1^3 \delta_{\theta^j} = -\frac{1}{4}$ *and* $\delta'_2 = \sum_4^6 \delta_{\theta^j} = \frac{1}{4}$. *It holds*

$$\sum_{j=1}^{m} \delta_{\theta^j}\theta^j > \sum_{j=1}^{3} \delta_{\theta^j}\theta^2 + \sum_{j=4}^{6} \delta_{\theta^j}\theta^4 = \delta'_1\theta^2 + \delta'_2\theta^4.$$

The new vector $(\delta'_1, \delta'_2) = \left(-\frac{1}{4}, \frac{1}{4}\right)$ *is a* δ*-sequence and it holds*

$$\delta'_1\theta^3 + \delta'_2\theta^4 = -\frac{1}{4}\theta^2 + \frac{1}{4}\theta^4 > -\frac{1}{4}\theta^4 + \frac{1}{4}\theta^4 = 0.$$

Hence, it holds that

$$\sum_{j=1}^{m} \delta_{\theta^j}\theta^j > \delta'_1\theta^2 + \delta'_2\theta^4 > 0.$$

Now we proceed with the proof of step (1) of Lemma 26. Recall that $\left(f_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}}, \ldots, f_{\theta^m}^{\theta^k,\bar{b}_{\theta^l}}\right)$ denotes a solution of minimization problem $M_{\bar{b}_{\theta^k}}^{\theta^l}$ and $\left(\tilde{f}_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}}, \ldots, \tilde{f}_{\theta^m}^{\theta^k,\bar{b}_{\theta^l}}\right)$ denotes the solution of minimization problem $M_{\bar{b}_{\theta^l}}^{\theta^k}$ as constructed in Lemma 26. Let the vector $(\delta_{\theta^1}, \ldots, \delta_{\theta^m})$ be defined by

$$\left(\tilde{f}_0^{\theta^k,\bar{b}_{\theta^l}} + \delta_{\theta^1}, \ldots, \tilde{f}_1^{\theta^k,\bar{b}_{\theta^l}} + \delta_{\theta^m}\right) = \left(f_0^{\theta^k,\bar{b}_{\theta^l}}, \ldots, f_1^{\theta^k,\bar{b}_{\theta^l}}\right).$$

We can decompose the vector $(\delta_{\theta^1}, \ldots, \delta_{\theta^m})$ into δ-sequences. Due to the two probability constraints it must hold that

$$\sum_{j=1}^{m} \delta_{\theta^j} = 0$$

and

$$\sum_{j=1}^{m} \delta_{\theta^j}\theta^j = 0.$$

Assume that the process of decomposing into δ-sequences ends with some vector $\left(\delta_1^{final}, \delta_2^{final}\right)$ with $\delta_1^{final} > 0$ and $\delta_2^{final} < 0$. Then there exists some $1 \leq t \leq m$ such that $\sum_{j=1}^{t} \delta_{\theta^j} > 0$.

First, we consider the case that $t > p^*$. It holds that $\tilde{f}_j^{\theta^k,\bar{b}_{\theta^l}} = 0$ for all $j > p^* + 1$. Thus, it holds that $\delta_{\theta^j} \geq 0$ for all $j > p^* + 1$ from which follows that $\sum_{j=t+1}^{m} \delta_{\theta^j} \geq 0$. Since $\sum_{j=1}^{t} \delta_{\theta^j} > 0$, it holds that $\sum_{j=1}^{m} \delta_{\theta^j} > 0$ which leads to a contradiction to the first probability constraint.

Second, we consider the case that $t \leq p^*$. Since the vector $\left(\tilde{f}_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}}, \ldots, \tilde{f}_{\theta^m}^{\theta^k,\bar{b}_{\theta^l}}\right)$ cannot induce a lower value of the objective function than the solution of the minimization problem, it must hold that

$$\sum_{j=1}^{l} \delta_{\theta^j} \leq 0. \tag{67}$$

Since the solution of the minimization problem $M_{\bar{b}_l}^{\theta^k}$, $\left(f_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}}, \ldots, f_{\theta^m}^{\theta^k,\bar{b}_{\theta^l}}\right)$, is an element of the feasible set of minimization problem $M_{\overline{b}_l}^{\theta^k}$ and we defined the real numbers δ_{θ^j} for $1 \leq j \leq m$ by

$$\left(\tilde{f}_{\theta^1}^{\theta^k,\overline{b}_{\theta^l}} + \delta_{\theta^1}, \ldots, \tilde{f}_{\theta^m}^{\theta^k,\overline{b}_{\theta^l}} + \delta_{\theta^m}\right) = \left(f_{\theta^1}^{\theta^k,\overline{b}_{\theta^l}}, \ldots, f_{\theta^m}^{\theta^k,\overline{b}_{\theta^l}}\right).$$

it holds that

$$\begin{aligned}\left(\tilde{f}_{\theta^1}^{\theta^k,\overline{b}_{\theta^l}} + \delta_{\theta^1} + \cdots + \tilde{f}_{\theta^l}^{\theta^k,\overline{b}_{\theta^l}} + \delta_{\theta^l}\right)^{n-1} &\left(\theta^k - \overline{b}_{\theta^l}\right) \\ &\geq \left(\tilde{f}_{\theta^1}^{\theta^k,\overline{b}_{\theta^l}} + \delta_{\theta^1} + \cdots + \tilde{f}_{\theta^t}^{\theta^k,\overline{b}_{\theta^l}} + \delta_{\theta^t}\right)^{n-1} \left(\theta^k - \overline{b}_{\theta^t}\right).\end{aligned}$$

By construction of the vector $\left(\tilde{f}_{\theta^1}^{\theta^k,\overline{b}_{\theta^l}}, \ldots, \tilde{f}_{\theta^m}^{\theta^k,\overline{b}_{\theta^l}}\right)$, it holds that

$$\left(\tilde{f}_{\theta^1}^{\theta^k,\overline{b}_{\theta^l}} + \cdots + \tilde{f}_{\theta^l}^{\theta^k,\overline{b}_{\theta^l}}\right)^{n-1} \left(\theta^k - \overline{b}_{\theta^l}\right) = \left(\tilde{f}_{\theta^1}^{\theta^k,\overline{b}_{\theta^l}} + \cdots + \tilde{f}_{\theta^t}^{\theta^k,\overline{b}_{\theta^l}}\right)^{n-1} \left(\theta^k - \overline{b}_{\theta^t}\right)$$

from which follows that

$$\begin{gathered}\left(\delta_{\theta^1} + \ldots + \delta_{\theta^l}\right) \sqrt[n-1]{\theta^k - \overline{b}_{\theta^l}} \geq \left(\delta_{\theta^1} + \ldots + \delta_{\theta^t}\right) \sqrt[n-1]{\theta^k - \overline{b}_{\theta^t}} \\ \Leftrightarrow \left(\delta_{\theta^1} + \ldots + \delta_{\theta^l}\right) \geq \frac{\left(\delta_{\theta^1} + \ldots + \delta_{\theta^t}\right) \sqrt[n-1]{\theta^k - \overline{b}_{\theta^t}}}{\sqrt[n-1]{\theta^k - \overline{b}_{\theta^l}}} > 0\end{gathered}$$

which leads to a contradiction to (67). Therefore, the existence of $\delta_1^{final} > 0$ and $\delta_1^{final} < 0$ leads to a contradiction. Hence, there exists some θ^{final} such that

$$\sum_{j=1}^{m} \delta_{\theta^j} \theta^j \geq \sum_{j=1}^{m} \delta_{\theta^j} \theta^{final}.$$

Since there exists a δ_{θ^h} for $p^* + 1 < h \leq m$ with $\delta_{\theta^h} > 0$, this inequality is strict and it holds

$$\sum_{j=1}^{m} \delta_{\theta^j} \theta^j > \sum_{j=1}^{m} \delta_{\theta^j} \theta^{final} = 0.$$

Since this is a contradiction to the second probability constraint, it follows that the assumption that there exists some h with $p^* + 1 < h \leq m$ such that $f_{\theta^h}^{\theta^k,\overline{b}_{\theta^l}} > 0$ leads to contradiction.

Proof of step (2)
It follows from Lemma 28 that

$$\left(f_{\theta^1}^{\theta^k,*} + \ldots + f_{\theta^{k-1}}^{\theta^k,*}\right)^{n-1} \left(\theta^k - \overline{b}_{\theta^{k-1}}\right) = \left(f_{\theta^1}^{\theta^k,*} + \ldots + f_{\theta^l}^{\theta^k,*}\right)^{n-1} \left(\theta^k - \overline{b}_{\theta^l}\right).$$

The worst-case belief of type θ^k is an element of the feasible set of minimization problem $M_{\bar{b}_{\theta^{k-1}}}$. Thus, for every $j \in \{1, \ldots, m\}$ and every $b \in [\bar{b}_{\theta^{j-1}}, \bar{b}_{\theta^j}]$ it holds that

$$\begin{aligned}\left(f_{\theta^1}^{\theta^k,*} + \ldots + f_{\theta^l}^{\theta^k,*}\right)^{n-1}\left(\theta^k - \bar{b}_{\theta^l}\right) &= \left(f_{\theta^1}^{\theta^k,*} + \ldots + f_{\theta^{k-1}}^{\theta^k,*}\right)^{n-1}\left(\theta^k - \bar{b}_{\theta^{k-1}}\right) \\ &\geq \left(f_{\theta^1}^{\theta^k,*} + \ldots + f_{\theta^j}^{\theta^k,*} G_{\theta^j}(b)\right)^{n-1}\left(\theta^k - b\right).\end{aligned}$$

Hence, the worst-case belief equilibrium of type θ^k is an element of the feasible set of minimization problem $M_{\bar{b}_{\theta^l}}^{\theta^k}$. Assume that the construction in Lemma 26 has reached the step where the constraint with corresponding bid $\bar{b}_{\theta^{k-1}}$ was added, i.e. all constraints with corresponding bid $\bar{b}_{\theta^j}$ for $1 \leq j \leq k-1$ were added and are binding. Consider the solution vector in this step i.e. the solution of the system of linear equations consisting of the two probability constraints and the binding incentive constraints with corresponding bid $\bar{b}_{\theta^j}$ for $1 \leq j \leq k-1$. According to Lemma 28, this solution vector coincides with the worst-case belief equilibrium of type θ^k. As argued above, this is an element of the feasible set of minimization problem $M_{\bar{b}_{\theta^l}}^{\theta^k}$ and therefore the construction in Lemma 26 would stop. We conclude that it holds $p^* \leq k-1$.

It follows from step (1) that $f_{\theta^j}^{\theta^k,\bar{b}_{\theta^l}} = 0$ for all $j > k$. Assume that there exists an incentive constraint with corresponding bid $\bar{b}_{\theta^h}$ with $1 \leq h \leq p^*$ which is not binding. Let $(\delta_{\theta^1}, \ldots, \delta_{\theta^m})$ be defined such that

$$\left(f_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}}, \ldots, f_{\theta^m}^{\theta^k,\bar{b}_{\theta^l}}\right) = \left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1}, \ldots, f_{\theta^m}^{\theta^k,*} + \delta_{\theta^m}\right).$$

Then there exists j with $1 \leq j \leq m$ such that $\delta_j \neq 0$.

We consider the following two cases:

- Case 1: It holds that

$$\left(f_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}} + \cdots + f_{\theta^l}^{\theta^k,\bar{b}_{\theta^l}}\right)^{n-1}\left(\theta^k - \bar{b}_{\theta^l}\right) = \left(f_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}} + \cdots + f_{\theta^k}^{\theta^k,\bar{b}_{\theta^l}}\right)^{n-1}\left(\theta^k - \bar{b}_{\theta^k}\right).$$

- Case 2: It holds that

$$\left(f_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}} + \cdots + f_{\theta^l}^{\theta^k,\bar{b}_{\theta^l}}\right)^{n-1}\left(\theta^k - \bar{b}_{\theta^l}\right) > \left(f_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}} + \cdots + f_{\theta^k}^{\theta^k,\bar{b}_{\theta^l}}\right)^{n-1}\left(\theta^k - \bar{b}_{\theta^k}\right).$$

Since by definition of $\left(f_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}}, \ldots, f_{\theta^m}^{\theta^k,\bar{b}_{\theta^l}}\right)$, this vector is an element of the feasible set of minimization problem $M_{\bar{b}_{\theta^l}}^{\theta^k}$, these two cases constitute all possible cases.

Case 1:

As before, we decompose the vector $(\delta_{\theta^1}, \ldots, \delta_{\theta^m})$ into δ-sequences. If we can show that there does not exist a t with $1 \leq t \leq m$ such that

$$\sum_{j=1}^{t} \delta_{\theta^j} > 0, \tag{68}$$

the process of decomposing ends with some δ-sequence $\left(\delta_1^{final}, \delta_2^{final}\right)$ with $\delta_1^{final} < 0$ and $\delta_2^{final} > 0$. Assume there exists a t with $1 \leq t \leq m$ such that $\sum_{j=1}^{t} \delta_{\theta^j} > 0$. Since $p^* \leq k-1$, it follows from step (1) that $f_{\theta^j}^{\theta^k, \bar{b}_{\theta l}} = 0$ for all $j > k$. Because $f_{\theta^j}^{\theta^k,*} = 0$ for all $j > k$, it follows that $\delta_{\theta^j} = 0$ for all $j > k$. Due to the first probability constraint, it holds that $\sum_{j=1}^{m} \delta_{\theta^j} = 0$ and therefore it must hold that $\sum_{j=1}^{k} \delta_{\theta^j} = 0$. Hence, t must be smaller than k. It follows from Lemma 28 that

$$\left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^k}^{\theta^k,*}\right)^{n-1}\left(\theta^k - \bar{b}_{\theta^k}\right) = \left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^t}^{\theta^k,*}\right)^{n-1}\left(\theta^k - \bar{b}_{\theta^t}\right).$$

Since $\left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1}, \ldots, f_{\theta^m}^{\theta^k,*} + \delta_{\theta^m}\right)$ is an element of the feasible set of minimization problem $M_{\bar{b}_{\theta l}}^{\theta^k}$ and the incentive constraint with corresponding bid $\bar{b}_{\theta^k}$ is binding, it holds that

$$\begin{aligned}\left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1}, \ldots, f_{\theta^k}^{\theta^k,*} + \delta_{\theta^k}\right)^{n-1}\left(\theta^k - \bar{b}_{\theta^k}\right) &= \left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1}, \ldots, f_{\theta^l}^{\theta^k,*} + \delta_{\theta^l}\right)^{n-1}\left(\theta^k - \bar{b}_{\theta^l}\right) \\ &\geq \left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1}, \ldots, f_{\theta^t}^{\theta^k,*} + \delta_{\theta^t}\right)^{n-1}\left(\theta^k - \bar{b}_{\theta^t}\right).\end{aligned}$$

It follows that $\sum_{j=1}^{t} \delta_{\theta^j} \leq 0$ which is a contradiction to (68). Thus, the process of decomposing into δ-sequences ends with some δ-sequence $\left(\delta_{\theta^m}^{final}, \delta_2^{final}\right)$ with $\delta_{\theta^m}^{final} < 0$ and $\delta_2^{final} > 0$. Hence, it holds that

$$\sum_{j=1}^{m} \delta_{\theta^j}\theta^j > \sum_{j=1}^{m} \delta_{\theta^j}\theta^{final} = 0.$$

But then the solution of minimization problem $M_{\bar{b}_{\theta l}}^{\theta^k}$, given by

$$\left(f_{\theta^1}^{\theta^k,\bar{b}_{\theta l}}, \ldots, f_{\theta^m}^{\theta^k,\bar{b}_{\theta l}}\right) = \left(\tilde{f}_0^{\theta^k,\bar{b}_{\theta l}} + \delta_{\theta^1}, \ldots, \tilde{f}_1^{\theta^k,\bar{b}_{\theta l}} + \delta_{\theta^m}\right).$$

violates the second probability constraint.

Case 2:
As in the first case, if follows from the first probability constraint that $\sum_{j=1}^{k} \delta_{\theta^j} = 0$. Since $\left(f_{\theta^1}^{\theta^k,*}, \ldots, f_{\theta^m}^{\theta^k,*}\right)$ is an element of the feasible set of minimization problem $M_{\bar{b}_{\theta l}}^{\theta^k}$, it must hold that the value of the objective function if plugging in $\left(f_{\theta^1}^{\theta^k,*}, \ldots, f_{\theta^m}^{\theta^k,*}\right)$ is not greater than if plugging in $\left(\tilde{f}_{\theta^1}^{\theta^k,\bar{b}_{\theta l}}, \ldots, \tilde{f}_{\theta^m}^{\theta^k,\bar{b}_{\theta l}}\right)$. Therefore, it must hold $\sum_{j=1}^{l} \delta_{\theta^j} \leq 0$. By

assumption, it holds that

$$\left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1} + \cdots + f_{\theta^l}^{\theta^k,*} + \delta_{\theta^l}\right)^{n-1} \left(\theta^k - \overline{b}_{\theta^l}\right) > \left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1} + \cdots + f_{\theta^k}^{\theta^k,*} + \delta_{\theta^k}\right)^{n-1} \left(\theta^k - \overline{b}_{\theta^k}\right)$$

and due to Lemma 28, it holds that

$$\left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^l}^{\theta^k,*}\right)^{n-1} \left(\theta^k - \overline{b}_{\theta^l}\right) = \left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1} + \cdots + f_{\theta^k}^{\theta^k,*} + \delta_{\theta^k}\right)^{n-1} \left(\theta^k - \overline{b}_{\theta^k}\right)$$

from which follows that $\sum_{j=1}^{l} \delta_{\theta^j} > \sum_{j=1}^{k} \delta_{\theta^j} = 0$ which leads to a contradiction.

We conclude that in both cases the assumption that there exists a h with $1 \leq h \leq p^*$ such that the constraint with corresponding bid $\overline{b}_{\theta^h}$ is not binding in the solution of minimization problem $M_{\overline{b}_{\theta^l}}^{\theta^k}$, leads to a contradiction.

Proof of step (3):
According to the first step, it holds that $f_j^{\theta^k,\overline{b}_{\theta^l}} > 0$ only for $1 \leq j \leq p^* + 1$ are greater than zero. According to step (2), this vector has to fulfill $p^* + 1$ equations given by

$$\sum_{j=1}^{p^*+1} f_{\theta^j} = 1$$

$$\sum_{j=1}^{p^*+1} f_{\theta^j} \theta^j = \mu$$

$$(f_{\theta^1} + \cdots + f_{\theta^l})^{n-1} \left(\theta^k - \overline{b}_{\theta^l}\right) = \left(\sum_{j=1}^{h} f_{\theta^j}\right)^{n-1} \left(\theta^k - \overline{b}_{\theta^h}\right) \quad \text{for all } h \in \{1, \ldots, p^*\}.$$

If we consider only roots which are real positive numbers, this is equivalent to

$$\sum_{j=1}^{p^*+1} f_{\theta^j} = 1$$

$$\sum_{j=1}^{p^*+1} f_{\theta^j} \theta^j = \mu$$

$$(f_{\theta^1} + \cdots + f_{\theta^l}) \sqrt[n-1]{\theta^k - \overline{b}_{\theta^l}} = \left(\sum_{j=1}^{h} f_{\theta^j}\right) \sqrt[n-1]{\theta^k - \overline{b}_{\theta^h}} \quad \text{for all } h \in \{1, \ldots, p^*\}.$$

We will show that this system of linear equations has a unique solution. In order to do so, we will show that the matrix corresponding to the system of equations has rank $p^* + 1$ by applying the Gauss elimination method and obtaining a row echelon form. The incentive constraints can be also summarized as

$$(f_{\theta^1}+\cdots+f_{\theta^h})\left(\sqrt[n-1]{\theta^k-\overline{b}_{\theta^h}}-\sqrt[n-1]{\theta^k-\overline{b}_{\theta^{h+1}}}\right)-f_{\theta^{h+1}}\sqrt[n-1]{\theta^k-\overline{b}_{\theta^{h+1}}}=0$$

for all $h\in\{1,\ldots,p^*-1\}$. In order to obtain an upper triangular matrix, we will successively eliminate the variables $f_{\theta^{p^*+1}}, f_{\theta^{p^*}},\ldots,f_{\theta^2}$. We eliminate the variable $f_{\theta^{p^*+1}}$ by multiplying the equation

$$\sum_{j=1}^{p^*+1} f_{\theta^j}=1$$

by $-\theta^{p^*+1}$ and adding it to

$$\sum_{j=1}^{p^*+1} f_{\theta^j}\theta^j=\mu.$$

Multiplying the resulting equation by (-1) gives

$$\sum_{j=1}^{p^*} f_{\theta^j}\left(\theta^{p^*+1}-\theta^j\right)=\theta^{p^*+1}-\mu$$

which eliminates the variable $f_{\theta^{p^*+1}}$. Moreover, the coefficient $\left(\theta^{p^*+1}-\theta^j\right)$ is strictly positive. Now we subsequently use the transformed incentive constraints given by

$$(f_{\theta^1}+\cdots+f_{\theta^h})\left(\sqrt[n-1]{\theta^k-\overline{b}_{\theta^h}}-\sqrt[n-1]{\theta^k-\overline{b}_{\theta^{h+1}}}\right)-f_{\theta^{h+1}}\sqrt[n-1]{\theta^k-\overline{b}_{\theta^{h+1}}}=0$$

for all $h\in\{1,\ldots,p^*-1\}$ in order to eliminate the variables $f_{\theta^{p^*}}, f_{\theta^{p^*-1}},\ldots f_{\theta^2}$. We show by induction that in every elimination step all coefficients are strictly positive. In particular, this implies that none of the coefficients is equal to zero and hence, we obtain an upper triangular matrix after applying the Gauss elimination method. We start the induction by showing that in the equation which is obtained after eliminating $f_{\theta^{p^*}}$ all coefficients are strictly positive. The variable $f_{\theta^{p^*}}$ is eliminated by multiplying the incentive constraint given by

$$(f_{\theta^1}+\cdots+f_{\theta^{p^*-1}})\left(\sqrt[n-1]{\theta^k-\overline{b}_{\theta^{p^*-1}}}-\sqrt[n-1]{\theta^k-\overline{b}_{\theta^{p^*}}}\right)-f_{\theta^{p^*}}\sqrt[n-1]{\theta^k-\overline{b}_{\theta^{p^*}}}=0.$$

by the factor

$$\frac{\theta^{p^*+1}-\theta^{p^*}}{\sqrt[n-1]{\theta^k-\overline{b}_{\theta^{p^*}}}}$$

and adding it to the equation

$$\sum_{j=1}^{p^*} f_{\theta^j}\left(\theta^{p^*+1}-\theta^j\right)=\theta^{p^*+1}-\mu.$$

This gives the equation

$$\sum_{j=1}^{p^*-1} f_{\theta^j} \left(\theta^{p^*+1} - \theta^j + \frac{\left(\theta^{p^*+1} - \theta^{p^*}\right) \left(\sqrt[n-1]{\theta^k - \overline{b}_{\theta^{p^*-1}}} - \sqrt[n-1]{\theta^k - \overline{b}_{\theta^{p^*}}} \right)}{\sqrt[n-1]{\theta^k - \overline{b}_{\theta^{p^*}}}} \right) = \theta^{p^*+1} - \mu$$

where all coefficients are strictly positive. Now we turn our attention to the induction step and assume that the variables $f_{\theta^{p^*}}, f_{\theta^{p^*-1}}, \ldots, f_{\theta^{h+1}}$ have been eliminated and in the resulting equation

$$\sum_{j=1}^{h} c_j f_{\theta^j} = c$$

all coefficients c and c_j for $1 \leq j \leq h$ are strictly positive. Now we have to eliminate the variable f_{θ^h} using the incentive constraint

$$(f_{\theta^1} + \cdots + f_{\theta^{h-1}}) \left(\sqrt[n-1]{\theta^k - \overline{b}_{\theta^{h-1}}} - \sqrt[n-1]{\theta^k - \overline{b}_{\theta^h}} \right) - f_{\theta^h} \sqrt[n-1]{\theta^k - \overline{b}_{\theta^h}} = 0.$$

We multiply this equation by the factor

$$\frac{c_h}{\sqrt[n-1]{\theta^k - \overline{b}_{\theta^h}}}$$

and add it to the equation

$$\sum_{j=1}^{h} c_j f_{\theta^j} = c.$$

This gives the equation

$$\sum_{j=1}^{h-1} f_{\theta^j} \left(c_j + \frac{c_h \left(\sqrt[n-1]{\theta^k - \overline{b}_{\theta^{h-1}}} - \sqrt[n-1]{\theta^k - \overline{b}_{\theta^h}} \right)}{\sqrt[n-1]{\theta^k - \overline{b}_{\theta^h}}} \right) = c$$

in which all coefficients are strictly positive. We conclude that the system of equations given by

$$\sum_{j=1}^{p^*+1} f_{\theta^j} = 1$$

$$\sum_{j=1}^{p^*+1} f_{\theta^j} \theta^j = \mu$$

$$(f_{\theta^1} + \cdots + f_{\theta^l})^{n-1} \left(\theta^k - \overline{b}_{\theta^l}\right) = \left(\sum_{j=1}^{h} f_{\theta^j} \right)^{n-1} \left(\theta^k - \overline{b}_{\theta^h}\right) \quad \text{for all } h \in \{1, \ldots, p^*\}$$

can be rearranged to a system of linear equations such that the resulting matrix has rank $p^* + 1$ and therefore this system of equations has a unique solution.

Since the vector $\left(\tilde{f}_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}}, \ldots, \tilde{f}_{\theta^{p^*+1}}^{\theta^k,\bar{b}_{\theta^l}}\right)$ fulfills the same p^*+1 equations and the solution of the linear system of equations with p^*+1 equations and p^*+1 unknowns is unique, it holds that

$$\left(\tilde{f}_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}}, \ldots, \tilde{f}_{\theta^m}^{\theta^k,\bar{b}_{\theta^l}}\right) = \left(f_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}}, \ldots, f_{\theta^m}^{\theta^k,\bar{b}_{\theta^l}}\right).$$

3.8.8 Proof of Lemma 27

We have to show that for every pair of valuations θ^l and θ^k such that $\theta^z \leq \theta^l \leq \theta^{k-2}$ the minimum p for minimization problem $M_{\bar{b}_{\theta^l}}^{\theta^k}$ is greater or equal then $l+1$. We will prove the claim by contradiction. Let $\left(f_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}}, \ldots, f_{\theta^m}^{\theta^k,\bar{b}_{\theta^l}}\right)$ denote the solution of minimization problem $M_{\bar{b}_{\theta^l}}^{\theta^k}$ as constructed in Lemma 26. Assume that the minimum p is strictly smaller than $l+1$. Under this assumption, we will show the following steps:

(1) The minimum p is equal to $l-1$.

(2) Let $\delta_{\theta^1}, \ldots, \delta_{\theta^m}$ be real numbers such that

$$\left(f_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}}, \ldots, f_{\theta^m}^{\theta^k,\bar{b}_{\theta^l}}\right) = \left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1}, \ldots, f_{\theta^m}^{\theta^k,*} + \delta_{\theta^m}\right).$$

Then for all $1 \leq j \leq l$ it holds $\delta_{\theta^j} > 0$, for all $l+2 \leq j \leq k$ it holds that $\delta_{\theta^j} < 0$ and for all $k+1 \leq j \leq m$ it holds that $\delta_{\theta^j}=0$.

(3) We use step (2) in order to show that the fact that $\left(f_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}}, \ldots, f_{\theta^m}^{\theta^k,\bar{b}_{\theta^l}}\right)$ is the solution of minimization problem $M_{\bar{b}_{\theta^l}}^{\theta^k}$ leads to a contradiction to the assumption $p^* < l+1$

Proof of step (1):
If the minimum p, denoted by $p*$, is strictly smaller than $l+1$, then the last equation added in the construction of Lemma 26 has a corresponding bid which is lower or equal than $\bar{b}_{\theta^{l-1}}$ because the incentive constraint corresponding to $\bar{b}_{\theta^l}$ given by

$$\left(f_{\theta^1}^{\bar{b}_{\theta^l}} + \cdots + f_{\theta^l}^{\bar{b}_{\theta^l}}\right)^{n-1}\left(\theta^k - \bar{b}_{\theta^l}\right) \geq \left(f_{\theta^1}^{\bar{b}_{\theta^l}} + \cdots + f_{\theta^l}^{\bar{b}_{\theta^l}}\right)^{n-1}\left(\theta^k - \bar{b}_{\theta^l}\right)$$

is fulfilled trivially. Therefore, it holds $p^* < l$. It cannot hold that $p^* < l-1$ because then according to Lemma 26 there would be no probability weight on types above θ^l. This would imply that $f_{\theta^1} + \cdots + f_{\theta^l}$ equals to 1 and therefore, the valuation of the objective function

$$\left(f_{\theta^1}^{\bar{b}_{\theta^l}} + \cdots + f_{\theta^l}^{\bar{b}_{\theta^l}}\right)^{n-1}\left(\theta^k - \bar{b}_{\theta^l}\right)$$

is maximized. This cannot be optimal because the worst-case belief of the θ^k-type is an element of the feasible set of minimization problem $M_{\bar{b}_{\theta^l}}^{\theta^k}$ and has a lower value of the objective function. We conclude that $p^* = l-1$.

Proof of step (2):
Let $\delta_{\theta^1}, \ldots, \delta_{\theta^m}$ be real numbers such that it holds

$$\left(f_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}}, \ldots, f_{\theta^m}^{\theta^k,\bar{b}_{\theta^l}}\right) = \left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1}, \ldots, f_{\theta^m}^{\theta^k,*} + \delta_{\theta^m}\right).$$

Since the minimum p equals to $l-1$, it follows from Lemma 26 that in the solution of minimization problem $M_{\bar{b}_{\theta^l}}^{\theta^k}$ there is no probability weight on types above θ^{l+1}. In the worst-case belief of the θ^k-type there is probability weight on types θ^j for $1 \leq j \leq k$ and there is no probability weight on types θ^j for $k+1 \leq j \leq m$. Therefore, for all j with $l+2 \leq j \leq k$ it holds that $\delta_{\theta^j} < 0$ and for $k+1 \leq j \leq m$ it holds that $\delta_{\theta^j} = 0$. Note that the set $\{j \mid l+2 \leq j \leq k\}$ is not empty because $l \leq k-2$. Since $\sum_{j=1}^{m} \delta_{\theta^j}$ has to be zero, it follows that $\sum_{j=1}^{l+1} \delta_{\theta^j} > 0$.

According to Lemma 26, if plugging in the solution $\left(f_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}}, \ldots, f_{\theta^m}^{\theta^k,\bar{b}_{\theta^l}}\right)$ into minimization problem $M_{\bar{b}_{\theta^l}}^{\theta^k}$, all constraints with corresponding bid below $\bar{b}_{\theta^l}$ have to binding. We use this in order to show by induction that $\delta_{\theta^1}, \ldots, \delta_{\theta^l}$ have to be strictly positive.

According to Lemma 28 it holds that

$$\begin{aligned}
&\left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^k}^{\theta^k,*}\right)^{n-1}\left(\theta^k - \bar{b}_{\theta^k}\right) \\
&\qquad = \left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^l}^{\theta^k,*}\right)^{n-1}\left(\theta^k - \bar{b}_{\theta^l}\right) = \left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^{l+1}}^{\theta^k,*}\right)^{n-1}\left(\theta^k - \bar{b}_{\theta^{l+1}}\right) \\
&\Leftrightarrow \left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^l}^{\theta^k,*}\right)\sqrt[n-1]{\left(\theta^k - \bar{b}_{\theta^l}\right)} = \left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^{l+1}}^{\theta^k,*}\right)\sqrt[n-1]{\left(\theta^k - \bar{b}_{\theta^{l+1}}\right)}. \qquad (69)
\end{aligned}$$

It also holds that

$$\begin{aligned}
&\left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1} + \cdots + f_{\theta^l}^{\theta^k,*} + \delta_{\theta^l}\right)^{n-1}\left(\theta^k - \bar{b}_{\theta^l}\right) \\
&\qquad\qquad \geq \left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1} + \cdots + f_{\theta^{l+1}}^{\theta^k,*} + \delta_{\theta^{l+1}}\right)^{n-1}\left(\theta^k - \bar{b}_{\theta^{l+1}}\right)
\end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow \left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1} + \cdots + f_{\theta^l}^{\theta^l,*} + \delta_{\theta^l}\right)\sqrt[n-1]{\left(\theta^k - \bar{b}_{\theta^l}\right)} \\
&\qquad\qquad \geq \left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1} + \cdots + f_{\theta^{l+1}}^{\theta^k,*} + \delta_{\theta^{l+1}}\right)\sqrt[n-1]{\left(\theta^k - \bar{b}_{\theta^{l+1}}\right)}. \qquad (70)
\end{aligned}$$

Subtracting (69) from (70) gives

$$\delta_{\theta^1} + \cdots + \delta_{\theta^l} \geq \frac{\left(\delta_{\theta^1} + \cdots + \delta_{\theta^{l+1}}\right)\sqrt[n-1]{\left(\theta^k - \bar{b}_{\theta^{l+1}}\right)}}{\sqrt[n-1]{\left(\theta^k - \bar{b}_{\theta^l}\right)}} > 0.$$

We start the inductive proof by showing that δ_{θ^1} is strictly positive. According to Lemma 26, it holds that

$$\left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1} + \cdots + f_{\theta^l}^{\theta^k,*} + \delta_{\theta^l}\right)^{n-1} \left(\theta^k - \overline{b}_{\theta^l}\right) = \left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1}\right)^{n-1} \theta^k$$

$$\Leftrightarrow \left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1} + \cdots + f_{\theta^l}^{\theta^k,*} + \delta_{\theta^l}\right) \sqrt[n-1]{\left(\theta^k - \overline{b}_{\theta^l}\right)} = \left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1}\right) \sqrt[n-1]{\theta^k}. \tag{71}$$

According to Lemma 26 it also holds that

$$\left(f_{\theta^1}^{\theta^k,*}\delta_{\theta^1} + \cdots + f_{\theta^k}^{\theta^k,*} + \delta_{\theta^k}\right)^{n-1} \left(\theta^k - \overline{b}_{\theta^k}\right) = \left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1} + \cdots + f_{\theta^l}^{\theta^k,*} + \delta_{\theta^l}\right)^{n-1} \left(\theta^k - \overline{b}_{\theta^l}\right)$$

$$\Leftrightarrow \left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1} + \cdots + f_{\theta^k}^{\theta^k,*} + \delta_{\theta^k}\right) \sqrt[n-1]{\left(\theta^k - \overline{b}_{\theta^l}\right)}$$
$$= \left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1} + \cdots + f_{\theta^l}^{\theta^k,*} + \delta_{\theta^l}\right) \sqrt[n-1]{\left(\theta^k - \overline{b}_{\theta^l}\right)}. \tag{72}$$

Subtracting (71) from (72) gives

$$\sqrt[n-1]{\left(\theta^k - \overline{b}_{\theta^l}\right)} \left(\delta_{\theta^1} + \cdots + \delta_{\theta^l}\right) = \sqrt[n-1]{\theta^k}\delta_{\theta^1}$$

$$\Leftrightarrow \delta_{\theta^1} = \frac{\sqrt[n-1]{\left(\theta^k - \overline{b}_{\theta^l}\right)} \left(\delta_{\theta^1} + \cdots + \delta_{\theta^l}\right)}{\sqrt[n-1]{\theta^k}} > 0.$$

Assume that we have shown that $\delta_{\theta^j} > 0$ for all $1 \leq j < h$ for some $1 < h < l$. Then we can show that $\delta_{\theta^{h+1}} > 0$. According to Lemma 26 it holds that

$$\left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1} + \cdots + f_{\theta^{h+1}}^{\theta^k,*} + \delta_{\theta^{h+1}}\right)^{n-1} \left(\theta^k - \overline{b}_{\theta^{h+1}}\right)$$
$$= \left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1} + \cdots + f_{\theta^h}^{\theta^k,*} + \delta_{\theta^h}\right)^{n-1} \left(\theta^k - \overline{b}_{\theta^h}\right)$$

and according to Lemma 28 it holds that

$$\left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^{h+1}}^{\theta^k,*}\right)^{n-1} \left(\theta^k - \overline{b}_{\theta^{h+1}}\right) = \left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^h}^{\theta^k,*}\right)^{n-1} \left(\theta^k - \overline{b}_{\theta^h}\right)$$

from which follows that

$$\left(\delta_{\theta^1} + \cdots + \delta_{\theta^h}\right) \sqrt[n-1]{\theta^k - \overline{b}_{\theta^h}} = \left(\delta_{\theta^1} + \cdots + \delta_{\theta^h} + \delta_{\theta^{h+1}}\right) \sqrt[n-1]{\theta^k - \overline{b}_{\theta^{h+1}}}$$

$$\Leftrightarrow \delta_{\theta^{h+1}} = \frac{\left(\delta_{\theta^1} + \cdots + \delta_{\theta^h}\right) \left(\sqrt[n-1]{\theta^k - \overline{b}_{\theta^h}} - \sqrt[n-1]{\theta^k - \overline{b}_{\theta^{h+1}}}\right)}{\sqrt[n-1]{\theta^k - \overline{b}_{\theta^{h+1}}}} > 0.$$

We conclude that for all j with $1 \leq j \leq l$ it holds $\delta_{\theta^j} > 0$.

Proof of step (3):
Let α and β be strictly positive be real numbers such that $\sum_{j=1}^{l+1} \delta_{\theta^j} = \alpha$ and $\sum_{j=l+2}^{k} \delta_{\theta^j} = -\beta$. Due to the two probability constraints it must hold that

$$\alpha - \beta = 0 \tag{73}$$

$$\sum_{j=1}^{l+1} \delta_{\theta^j} \theta^j + \sum_{j=l+2}^{k} \delta_{\theta^j} \theta^j = 0. \tag{74}$$

Since for $l+2 \leq j \leq k$ it holds that $\delta_{\theta^j} < 0$, it holds that $\sum_{j=l+2}^{k} \delta_{\theta^j} \theta^j < \sum_{j=l+2}^{k} \delta_{\theta^j} \theta^{l+2} = -\beta\theta^{l+2}$. It follows from step (2) that $\sum_{j=1}^{l+1} \delta_{\theta^j} \theta^j < \sum_{j=1}^{l+1} \delta_{\theta^j} \theta^{l+1} = \alpha\theta^{l+1}$. According to (73), it holds that $\alpha = \beta$ and it follows that

$$\sum_{j=1}^{l+1} \delta_{\theta^j} \theta^j + \sum_{j=l+2}^{k} \delta_{\theta^j} \theta^j < \alpha\theta^{l+1} - \beta\theta^{l+2} = \beta\theta^{l+1} - \beta\theta^{l+2} < 0$$

which is a contradiction to (74). Hence, we have found a contradiction to the assumption that the minimum p is strictly smaller than $l+1$.

3.8.9 Proof of Lemma 22

Proof. For $b \in [\bar{b}_{\theta^{k-1}}, \bar{b}_{\theta^k}]$ let $\left(f_{\theta^1}^{\theta^k,b}, \ldots, f_{\theta^m}^{\theta^k,b}\right)$ denote a solution of minimization problem $M_b^{\theta^k}$. Let $(\delta_{\theta^1}, \ldots, \delta_{\theta^m})$ be real numbers such that

$$\left(f_{\theta^1}^{\theta^k,*}, \ldots, f_{\theta^m}^{\theta^k,*}\right) = \left(f_{\theta^1}^{\theta^k,b} + \delta_1, \ldots, f_{\theta^m}^{\theta^k,b} + \delta_m\right).$$

Assume that $f^{\theta^k,b} \neq f^{\theta^k,*}$. Then there exists $1 \leq j \leq m$ such that $\delta_j \neq 0$. Therefore, one can decompose the vector $(\delta_{\theta^1}, \ldots, \delta_{\theta^m})$ into δ-sequences and if there does not exist a $1 \leq t \leq m$ with $\sum_{j=1}^{t} \delta_{\theta^j} > 0$, the process of decomposing into δ-sequences end with a δ-sequence of length 2, i.e. with some vector $\left(\delta_1^{final}, \delta_2^{final}\right)$ with $\delta_1^{final} < 0$ and $\delta_2^{final} > 0$.

Assume there exists a $1 \leq t \leq m$ with $\sum_{j=1}^{t} \delta_{\theta^j} > 0$. We consider two cases: $t \leq k$ and $t > k$.

Case 1: $t \leq k$.
Following the steps in the proof of Lemma 24, one can show that it either holds

$$\left(f_{\theta^1}^{\theta^k,b} + \cdots + f_{\theta^{k-1}}^{\theta^k,b} + f_{\theta^k}^{\theta^k,b} G_{\theta^k}(b)\right)^{n-1} \left(\theta^k - b\right) = \left(f_{\theta^1}^{\theta^k,b} + \cdots + f_{\theta^k}^{\theta^k,b}\right)^{n-1} \left(\theta^k - \bar{b}_{\theta^k}\right) \tag{75}$$

or

$$\left(f_{\theta^1}^{\theta^k,b} + \cdots + f_{\theta^{k-1}}^{\theta^k,b} + f_{\theta^k}^{\theta^k,b} G_{\theta^k}(b)\right)^{n-1} \left(\theta^k - b\right)$$
$$= \left(f_{\theta^1}^{\theta^k,b} + \cdots + f_{\theta^{k-1}}^{\theta^{k-1},b}\right)^{n-1} \left(\theta^k - \bar{b}_{\theta^{k-1}}\right). \quad (76)$$

Thus, we consider two subcases.

Case 1.1: (75) holds. It follows from the definition of $\bar{b}_{\theta^k}$ and from Lemma 28 that

$$\left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^k}^{\theta^k,*}\right)^{n-1} (\theta^k - \bar{b}_{\theta^k}) = \left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^t}^{\theta^k,*}\right)^{n-1} (\theta^k - \bar{b}_{\theta^t}). \quad (77)$$

Since $f^{\theta^k,b}$ is an element of the feasible set of minimization problem $M_b^{\theta^k}$, it holds that

$$\left(f_{\theta^1}^{\theta^k,b} + \cdots + f_{\theta^{k-1}}^{\theta^k,b} + f_{\theta^k}^{\theta^k,b} G_{\theta^k}(b)\right)^{n-1} (\theta^k - b) \geq \left(f_{\theta^1}^{\theta^k,b} + \cdots + f_{\theta^t}^{\theta^k,b}\right)^{n-1} (\theta^k - \bar{b}_{\theta^t})$$

It follows from (75) that

$$\left(f_{\theta^1}^{\theta^k,b} + \cdots + f_{\theta^k}^{\theta^k,b}\right)^{n-1} \left(\theta^k - \bar{b}_{\theta^k}\right) \geq \left(f_{\theta^1}^{\theta^k,b} + \cdots + f_{\theta^t}^{\theta^k,b}\right)^{n-1} (\theta^k - \bar{b}_{\theta^t}).$$

Subtracting equation (77) gives

$$\left(\sum_{j=1}^{k} \delta_{\theta^j}\right) \sqrt[n-1]{\theta^k - \bar{b}_{\theta^k}} \geq \left(\sum_{j=1}^{t} \delta_{\theta^j}\right) \sqrt[n-1]{\theta^k - \bar{b}_{\theta^t}}.$$

Thus, it holds that $\sum_{j=1}^{k} \delta_{\theta^j} > 0$. Due to the first probability constraint, it follows that $\sum_{j=k+1}^{m} \delta_{\theta^j} < 0$. Since $f_{\theta^j}^{\theta^k,*} = 0$ for all $j > k$, this leads to a contradiction to the constraint $f_{\theta^j}^{\theta^k,b} \geq 0$ for all $1 \leq j \leq m$.

Case 1.2: (76) holds. It follows from Lemma 28 that

$$\left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^{k-1}}^{\theta^k,*}\right)^{n-1} (\theta^k - \bar{b}_{\theta^{k-1}}) = \left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^t}^{\theta^k,*}\right)^{n-1} (\theta^k - \bar{b}_{\theta^t}). \quad (78)$$

Since $f^{\theta^k,b}$ is an element of the feasible set of minimization problem $M_b^{\theta^k}$, it holds that

$$\left(f_{\theta^1}^{\theta^k,b} + \cdots + f_{\theta^{k-1}}^{\theta^k,b} + f_{\theta^k}^{\theta^k,b} G_{\theta^k}(b)\right)^{n-1} (\theta^k - b) \geq \left(f_{\theta^1}^{\theta^k,b} + \cdots + f_{\theta^t}^{\theta^k,b}\right)^{n-1} (\theta^k - \bar{b}_{\theta^t}).$$

It follows from (76) that

$$\left(f_{\theta^1}^{\theta^k,b} + \cdots + f_{\theta^{k-1}}^{\theta^k,b}\right)^{n-1} \left(\theta^k - \bar{b}_{\theta^{k-1}}\right) \geq \left(f_{\theta^1}^{\theta^k,b} + \cdots + f_{\theta^t}^{\theta^k,b}\right)^{n-1} (\theta^k - \bar{b}_{\theta^t}).$$

Subtracting equation (78) gives

$$\left(\sum_{j=1}^{k-1} \delta_{\theta^j}\right) \sqrt[n-1]{\theta^k - \bar{b}_{\theta^{k-1}}} \geq \left(\sum_{j=1}^{t} \delta_{\theta^j}\right) \sqrt[n-1]{\theta^k - \bar{b}_{\theta^t}}.$$

Thus, it holds that $\sum_{j=1}^{k-1} \delta_{\theta^j} > 0$. Since $f_{\theta^j}^{\theta^k,*} = 0$ for all $j > k$, it holds due to the constraint

$$f_{\theta^j}^{\theta^k,b} \geq 0 \text{ for all } 1 \leq j \leq m$$

that $\delta_{\theta^j} \geq 0$ for all $k+1 \leq j \leq m$. Since $\sum_{j=1}^{m} \delta_{\theta^j} = 0$, it follows that $\delta_k < 0$ which is a contradiction to (76).

Case 2: $t > k$.
Due to the first probability constraint, it follows from $\sum_{j=1}^{t} \delta_{\theta^j} > 0$ that $\sum_{j=t+1}^{m} \delta_{\theta^j} < 0$. Since $f_{\theta^j}^{\theta^k,*} = 0$ for all $j > k$, this leads to a contradiction to the constraint $f_{\theta^j}^{\theta^k,b} \geq 0$ for all $1 \leq j \leq m$.

We conclude that in both cases the process of decomposing into δ-sequences ends with some vector $\left(\delta_1^{final}, \delta_2^{final}\right)$ with $\delta_1^{final} < 0$ and $\delta_2^{final} > 0$ and there exists a θ^{final} such that

$$\sum_{j=1}^{m} \delta_{\theta^j} \theta^j > \sum_{j=1}^{m} \delta_{\theta^j} \theta^{final} = 0.$$

Since this is a contradiction to the fact that the vector $f^{\theta^k,b}$ fulfills the constraint

$$\sum_{j=1}^{m} f_{\theta^j}^{\theta^k,b} \theta^j = \mu,$$

the assumption that $f^{\theta^k,b} \neq f^{\theta^k,*}$, leads to a contradiction.

$\square$

4 Strategies under strategic uncertainty

I investigate the decision problem of a player in a game of incomplete information who faces uncertainty about the other players' strategies. I propose a new decision criterion which works in two steps. First, I assume common knowledge of rationality and eliminate all strategies which are not rationalizable. Second, I apply the maximin expected utility criterion. Using this decision criterion, one can derive predictions about outcomes and recommendations for players facing strategic uncertainty. A bidder following this decision criterion in a first-price auction expects all other bidders to bid their highest rationalizable bid given their valuation. As a consequence, the bidder never expects to win against an equal or higher type and resorts to win against lower types with certainty.

JEL classification: C72, D81, D82, D83

Keywords: Auctions, Incomplete Information, Informational Robustness, Rationalizability

4.1 Introduction

I investigate the decision problem of a player in a game of incomplete information who faces strategic uncertainty. Formally, a player faces strategic uncertainty if the smallest set of strategies such that the player knows that the other players' true strategy is an element of this set, is not a singleton. I propose a new decision criterion which works in two steps: First, I assume common knowledge of rationality and eliminate all actions which are not best replies. That is, the set of the other players' possible strategies is restricted to the set of rationalizable strategies. Afterwards, I apply the maximin expected utility criterion. Using this decision criterion, I can derive recommendations for a player facing strategic uncertainty. Furthermore, I analyze outcomes under the assumption that every player in the game uses this decision criterion. In sections 4.2-4.4 I consider a game with incomplete information under strategic uncertainty with common knowledge of type distributions. In an extension in section 4.5 I discuss how the proposed decision criterion can be applied under the presence of both, distributional and strategic uncertainty.

Before I explain the decision criterion in more detail, I argue why strategic uncertainty can occur in games (of complete or incomplete information). Consider a game and a player who has to decide about her strategy. There may exist strategy profiles which formally fulfill the conditions of a (Bayes-) Nash equilibrium. However, a player may be uncertain whether her opponents employ such strategies and consequently face strategic uncertainty. As stated by Pearce (1984), "some Nash equilibria are intuitively unreasonable and not all reasonable strategy profiles are Nash equilibria". He argues that if players cannot communicate, then a player will best reply to Nash equilibrium strategies only if she is able to deduce these equilibrium strategies. However, a player may consider more than one strategy of the other players' as possible. For example, this can occur under the existence of multiple Nash equilibria without one being focal or salient (Bernheim (1984)). Thus, a Nash equilibrium may not be a suitable decision criterion if a player does not observe or does not deduce a unique conjecture about the other players' strategies. Similarly, Renou and Schlag (2010) argue that "common knowledge of conjectures, mutual knowledge of rationality and utilities, and existence of a common prior" are required in order to justify Nash equilibria as a decision criterion.

So far, I argued that a player may not know which strategies are played by the other players. But a player may not consider all strategies of the other players as possible. The fact that rational players interact strategically given some commonly known rules of a game (e.g. the rules of a first-price auction), already contains information about the set of possible strategies. Therefore, in the first step of the decision criterion I propose to consider strategies which a player can deduce only from common knowledge of rationality. Under strategic uncertainty a player is rational if her action is a best reply given her type, the commonly known type distribution and a conjecture about the other players' strategies. A strategy which a player assumes to be played by another rational player has to be rational as well, i.e. the action prescribed by a strategy for a given type has to be a best reply given the type, the commonly known type distribution and a conjecture about the other players' strategies. This reasoning continues ad infinitum. Pearce (1984) and Bernheim (1984) (and Battigalli and Siniscalchi (2003b) for games of incomplete information) show that common knowledge of rationality is equivalent to bidders playing *rationalizable strategies* [42]. These are strategies which survive the iterated elimination of actions which are not best replies to some strategy which consists of actions which have not been eliminated in previous elimination rounds.

In the second step I apply the maximin expected utility criterion due to Gilboa and Schmeidler (1989). A player applying this criterion chooses the action which maximizes her minimum expected utility given her type. The application of the maximin expected utility criterion can be modeled as a simultaneous zero-sum game against an adverse nature whose action space consists of the other players' rationalizable strategies. Given the strategy of the adverse nature, the player applying the maximin criterion chooses the action which maximizes her expected utility. The adverse nature's utility is the player's expected utility multiplied by -1.

In other words, under the proposed decision criterion a player facing strategic uncertainty forms a subjective belief about the other players' strategies and acts optimally given this subjective belief. The first step of the decision criterion determines the set from which a player chooses her subjective belief. The second step determines how the subjective belief

[42] For games of incomplete information where also the type distribution is not known, i.e. only the type spaces and action spaces are common knowledge, Battigalli and Siniscalchi (2003b) use the term *belief-free rationalizable strategies*. If additional information about possible strategies or distributions is common knowledge, i.e. more than the type spaces and the action spaces is common knowledge, they use the term Δ-*rationalizable strategies*. If the type distribution is common knowledge but nothing besides the actions spaces is known about strategies, they use the term *rationalizable strategies*. I will use the term rationalizable strategies throughout the paper.

is chosen. The subjective belief is given by the adverse nature's equilibrium strategy, in the following called *subjective maximin belief.* In order to distinguish the Nash equilibrium in the simultaneous game between a player and the adverse nature and the Bayes-Nash equilibrium which may exist in a given game of incomplete information, I will refer to the Nash equilibrium in the former case as a *maximin equilibrium.*

By assuming common knowledge of rationality and applying the maximin expected utility criterion, I am able to derive recommendations for players facing distributional and strategic uncertainty. Moreover, I characterize outcomes under the assumption that every player follows the proposed decision criterion.

The following two examples illustrate two different reasons for why strategic uncertainty can occur and how the proposed decision criterion applies under strategic uncertainty. In the first example there exist multiple Nash equilibria without one being salient. In the second example a salient Nash equilibrium exists but is not the unique rationalizable action. In particular, the salient Nash equilibrium is not compatible with actions derived from the maximin utility or minimax regret criterion. Afterwards, I will summarize the results for first-price auctions under strategic uncertainty and provide the results for the extension of the decision criterion to both, distributional and strategic uncertainty.

For the first example consider a sender who has to deposit a package either in places A, B or C. A receiver has to decide to which places she sends one or two drivers in order to pick up the package. If the package is picked up, sender and receiver earn each a utility of P and zero otherwise. In addition, the receiver faces a cost of c if a driver travels to place A or B and a cost of $\tilde{c}$ if a driver travels to place C. The game is summarized in the following utility table:

	A	B	C	AB	AC	BC
A	$P; P-c$	$0; -c$	$0; -\tilde{c}$	$P; P-2c$	$P; P-c-\tilde{c}$	$0; -c-\tilde{c}$
B	$0; -c$	$P; P-c$	$0; -\tilde{c}$	$P; P-2c$	$0; -c-\tilde{c}$	$P; P-c-\tilde{c}$
C	$0; -c$	$0; -c$	$P; P-\tilde{c}$	$0; -2c$	$0; P-c-\tilde{c}$	$P; P-c-\tilde{c}$

Assume it is common knowledge that it holds $P-\tilde{c} < -c$ and $P-2c > -c$. The Nash equilibria in this game are $(A; A)$, $(B; B)$ and both players mixing between A and B with probability $\frac{1}{2}$. Although Nash equilibria exist, the players may be uncertain about each other's strategy since there does not exist a particularly salient one. The application of the maximin criterion leaves both players indifferent between actions A and B. The maximin criterion does not yield to action AB for the receiver since by choosing AB she would face the risk that the sender deposits the package in C, leaving the receiver with the costs of two drivers $-2c$. However, the result of the maximin criterion changes after assuming common knowledge of rationality. Excluding actions which are not best replies leads to the elimination of strategies C, AC and BC for the receiver, leading to the elimination of action C for the sender:

	A	B	AB
A	$P; P-c$	$0; -c$	$P; P-2c$
B	$0; -c$	$P; P-c$	$P; P-2c$

Now the maximin criterion leads to action AB for the receiver. In other words, if the receiver anticipates that the sender anticipates that she will never send a driver to C, the application of the maximin criterion leads to action AB. In this case, the receiver earns a utility of $P-2c$ with certainty. If she would follow a Nash equilibrium strategy or apply the maximin criterion directly, she would face the risk of getting a utility of $-c$.

As a second example consider the following utility table. It illustrates the decision problem of a player who is uncertain about which of the possible rationalizable actions her opponent will choose:

	X	Y	Z
A	10;10	0;9	0,0
B	5;1	5;9	0,0
C	4;1	4;9	4;0
D	1;10	6;9	0;0

The unique Nash equilibrium in pure strategies, (A, X), is focal in the sense that it is the social optimum and leads to the highest possible utility for both players. However, a rational column player can also choose Y instead of X. Action Y is rationalizable and moreover, the application of the maximin or the minimax regret criterion would lead to action Y for the column player. In other words, the column player may prefer to get a utility of 9 with certainty instead of aiming for the utility of 10 and risking to get a utility of 1. Given this uncertainty about the column player's strategy, the row player may resort to the application of the maximin criterion. This leads to action C which ensures a utility of 4 for the row player. However, the row player can anticipate that action Z is strictly dominated for the column player. After the elimination of this action, C becomes strictly dominated for the row player. The iterated elimination of actions which are not best replies, i.e. the elimination of actions Z and C, leads to the following utility table:

	X	Y
A	10;10	0;9
B	5;1	5;9
D	1;10	6;9

Now the application of the maximin criterion leads to action B for the row player. That is, after anticipating that the column player will never play Z, the row player can ensure a utility of 5 instead a utility of 4.

These examples show how the proposed decision criterion provides recommendations under strategic uncertainty. Moreover, they show why players may not expect their opponents to play Nash equilibria and why the application of the maximin utility criterion alone may cause forgone profits. After discussing the two examples, I provide an intuition and a summary of the results for first-price auctions where bidders' valuations are identically and independently distributed according to a commonly known distribution function. Consider the simple example of a first-price auction with two bidders who can have either a valuation of zero with probability p or a valuation of 1 with probability $1-p$. For simplicity, assume an efficient tie-breaking rule. We have to compute the highest rationalizable bids of each type. The highest rationalizable bid of a bidder with valuation zero is zero. If a bidder with valuation 1 bids zero, she gets an expected utility of p. Hence, bidding too close to the own valuation (or even above) cannot be rational for a 1-type.

The highest rationalizable bid of a bidder with valuation 1 makes her indifferent between winning against the 0-type by bidding zero and winning with probability one. That is, it is obtained by the equation[43]

$$1 - \bar{b}_1 = p \;\Leftrightarrow\; \bar{b}_1 = 1 - p.$$

A bidder with valuation 1 who applies the proposed decision criterion has the subjective maximin belief that the other bidder with valuation 1 bids $\bar{b}_1$. Therefore, her best reply is to win against the 0-type of the other bidder with certainty by bidding zero.

For the general case with an arbitrary number of bidders and valuations, for every type there exists a unique highest rationalizable bid. A bidder applying the proposed decision criterion assumes that every other bidder places the highest rationalizable bid given her type. As a consequence, the bidder never expects to win against a bidder with an equal or higher type and therefore bids the highest rationalizable bid of a lower type in order to

[43]For the case with two possible valuations the highest rationalizable bid of a bidder with the higher valuation coincides with the highest bid played in the unique Bayes-Nash equilibrium. With more than two valuations the highest rationalizable bid of a type is strictly higher than the highest bid played in the unique Bayes-Nash equilibrium.

win against the lower type with certainty. If every bidder applies this decision criterion, then every bidder has the same beliefs about distributions and strategies. Every bidder calculates which highest rationalizable bid of a lower type maximizes her expected utility. It turns out that due to the symmetry of beliefs about valuation distributions and strategies, the higher the type of the bidder, the higher is the type whose highest rationalizable bid maximizes her expected utility. Therefore, the outcome is efficient, i.e. the bidder with the highest valuation wins the auction with probability one.

In an extension I analyze both, distributional and strategic uncertainty. In this case the strategy space of the adverse nature consists of all rationalizable strategies and all possible valuation distributions. For a restriction of the set of possible distributions I assume common knowledge of an exogenously given mean μ of bidders' valuations.[44] Although in reality bidders go at great lengths in order to learn about their competitors' valuations, such learning has its limits and bidders may be able to learn only the support and the mean of the valuation distribution.

Under strategic uncertainty with common knowledge of rationality and distributional uncertainty with common knowledge of an exogenously given mean, as before, for every type there exists a unique highest rationalizable bid. A bidder applying the proposed decision criterion assumes that every other bidder places the highest rationalizable bid given her type. Let θ_μ be the lowest valuation which is higher than the mean. The highest rationalizable bid of a bidder with a valuation lower than θ_μ is her valuation. The subjective maximin belief of such a bidder about the other bidders' valuation distributions is that the probability weight is distributed between her own valuation and θ_μ. As a consequence, a bidder with a valuation lower than θ_μ expects a utility of zero and is indifferent between any bid between zero and her valuation. Every bidder with valuation θ such that $\theta \geq \theta_\mu$ never expects to win against a bidder with the same valuation. Hence, the subjective maximin belief of such a bidder about the other bidders' valuation distribution maximizes the probability weight on θ and makes the bidder indifferent between any highest rationalizable bid of lower types. As a consequence, the bidder mixes among all highest rationalizable bids of lower types. Therefore, the outcome is not efficient.

The remainder of the paper is organized as follows. I conclude the introduction with an overview over the related literature. The second section contains the formal description of the proposed decision criterion. In the third section I collect sufficient conditions for actions to be rationalizable which will be useful for the derivation of subjective maximin beliefs and outcomes under maximin strategies. Moreover, I provide sufficient conditions for the existence of such outcomes. In the fourth section I apply the decision criterion to first-price auctions under strategic uncertainty. The fifth section contains the formal description of the decision criterion under distributional and strategic uncertainty and its application to first-price auctions. The appendix contains the proofs not provided in

[44] The assumption of common knowledge of an exogenously given mean under distributional uncertainty has been used before. See for example Montiero (2009).

previous sections.

Relation to the literature

This paper relates to two strands of literature - the literature on decision criteria under uncertainty and robustness and the literature on rationalizability. Two widely used decision criteria under uncertainty are the maximin utility and the minimax regret criterion. The axiomatization of the maximin expected utility criterion is provided in Gilboa and Schmeidler (1989), the axiomatization of the minimax regret criterion is provided in Stoye (2011). In Bergemann and Schlag (2008) both criteria are applied to a monopoly pricing problem where a seller faces uncertainty about the buyer's valuation distribution. Since the seller knows that the buyer will obtain the good if the price is equal or lower than her valuation, the seller does not face strategic uncertainty.

The maximin expected utility criterion has been applied to first-price auctions under distributional uncertainty. Lo (1998) derives Bayes-Nash equilibrium bidding strategies in a first-price auction under the maximin expected utility criterion where it is common knowledge that the true valuation distribution is an element of a given set of distributions. Salo and Weber (1995) assume that only the set of possible valuations is common knowledge and that ambiguity averse bidders use a convex transformation of the uniform distribution as a prior. They find, that the more ambiguity averse a bidder is, the higher is the bid. Chen et al. (2007) analyze first- and second-price auctions where bidders face one of two possible distributions which can be ordered with respect to first-order stochastic dominance. Thus, an ambiguity-averse bidder would assume the stochastically dominating distribution. In their experimental findings they reject the hypothesis that bidders are ambiguity-averse. These three papers use Bayes-Nash equilibria as a solution concept, that is, the issue of strategic uncertainty is not addressed.

Bose et al. (2006) derive the optimal auction in a setting where seller and bidders may face different degrees of ambiguity, that is, they may face different sets of possible valuation distributions. Carrasco et al. (2018) consider a seller facing a single buyer. The set of distributions the seller considers to be possible is determined by a given support and mean. In these two papers strategic uncertainty is not an issue since the seller chooses a strategy proof direct mechanism.

Renou and Schlag (2010) analyze strategic uncertainty using the minimax regret criterion. Besides Kasberger and Schlag (2017), I am the only one addressing distributional *and* strategic uncertainty. They use the minimax regret criterion and allow for the possibility that a bidder can impose bounds on the other bidders' bids or valuation distributions. For example, they consider the case where a bidder can impose a lower bound on the highest bid.

In their literature on robust mechanism design Dirk Bergemann and Stephen Morris consider the problem of a social planner facing uncertainty about the players' actions. In Bergemann and Morris (2005) a social planner can circumvent uncertainty about the players' strategies by choosing ex-post implementable mechanisms. Bergemann and Morris (2013) provide predictions in games independent of the specification of the information structure. In order to do so, they characterize the set of Bayes correlated equilibria. An application of this concept to first-price auctions is carried out in Bergemann et al. (2017). In Carroll (2016) two agents accept or reject a proposed deal where the valuation for each agent depends on an unknown state. The main result provides an upper bound of welfare loss among all information structures.

The concept of rationalizable strategies has been first introduced by Bernheim (1984) and Pearce (1984) for games with complete information. Battigalli and Siniscalchi (2003b) extend rationalizability to games of incomplete information. An application to first-price auctions has been carried out by Dekel and Wolinsky (2001). They apply rationalizable strategies to a first-price auction with discrete private valuations and discrete bids. They present a condition on the distribution of types under which the only rationalizable action is to bid the highest bid below valuation. Battigalli and Siniscalchi (2003a) assume that valuation distributions in a first-price auction are common knowledge but not the strategies of the bidders. They characterize the set of rationalizable actions under the assumption of *strategic sophistication*, which implies common knowledge of rationality and of the fact that bidders with positive bids win with positive probability. They find that for a bidder with a given valuation θ all bids in an interval $(0, b^{max}(\theta))$ are rationalizable where $b^{max}(\theta)$ is higher than the Bayes-Nash equilibrium bid. Using this result, one can immediately tell that under common knowledge of rationality a bidder applying the maximin expected utility criterion has the subjective maximin belief that every other bidder with valuation θ bids $b^{max}(\theta)$. I replicate this result in section 4.4 for first-price auctions with discrete valuations.

To the best of my knowledge I am the first one applying the maximin expected utility criterion to strategic uncertainty and the first one combining rationalizable strategies with a decision criterion under uncertainty.

4.2 Model

Underlying game of incomplete information The starting point of the model is a game of incomplete information which is denoted by $\left(\{1,\ldots,I\}, \Theta, A, \{u_i\}_{i\in\{1,\ldots,I\}}\right)$ where $\{1,\ldots,I\}$ is the set of players and for every $i \in \{1,\ldots,I\}$, $A_i \subseteq \mathbb{R}$ is the set of possible actions and $\Theta_i \subseteq \mathbb{R}$ is the set of possible privately known types of player i. A and Θ are defined by $A = A_1 \times \ldots \times A_I$ and $\Theta = \Theta_1 \times \ldots \times \Theta_I$. A *pure strategy* of player i is a mapping

$$\beta_i : \Theta_i \to A_i$$

$$\theta_i \mapsto a_i.$$

The set S_i is the set of all pure strategies of player i. A *strategy* of player i is a mapping

$$\beta_i : \Theta_i \to \Delta A_i$$

$$\theta_i \mapsto a_i$$

where ΔA_i is the set of probability distributions on A_i. In the following $g_{\theta_i}^{\beta_i}$ will denote the density of the bid distribution $\beta_i(\theta_i)$ and *supp* $(\beta_i(\theta_i))$ its support.[45] Let

$$u_i : A \times \Theta_i \to \mathbb{R}$$

$$(a_1, \ldots, a_I, \theta_i) \mapsto u_i(a_1, \ldots, a_I, \theta_i)$$

denote the utility function of player i. That is, I consider a setting with private valuations.

For a given profile of strategies $(\beta_1, \ldots, \beta_n)$ and a given type distribution

$$F : \Theta \to [0,1]$$

the expected utility of a player i is given by

$$\begin{aligned} &U_i(\theta_i, \beta_i(\theta_i), \beta_{-i}, F_{-i}) \\ &\qquad = \int_{\theta_{-i}} \int_{a_{-i}} u_i(a_1, \ldots, a_i, \ldots, a_I, \theta_i) \prod_{j \neq i} g_{\theta^j}^{\beta_j}(a_j)\, d\theta_{-j} dF_{-i}(\theta_{-i})\, d\theta_{-i} \end{aligned} \tag{79}$$

where the function u_i stems from the underlying game of incomplete information and where F_{-i} is defined by $F_{-i}(\theta_{-i}) = F(\theta_{-i}, \theta_i)$.[46]

Action space of adverse nature In order to formalize the maximin expected utility criterion, a new player, denoted by n, is introduced, representing the adverse nature a player i applying the maximin expected utility criterion faces. Players i and n play a simultaneous zero-sum game where utilities are induced by the underlying game of incomplete information. The first step of a formal description of this game is the definition of the adverse nature's action space. It accounts for the residual uncertainty of player i. In sections 4.2-4.4 I study only strategic uncertainty and assume common knowledge of a type distribution given by

$$F : \Theta \to [0,1].$$

That is, the adverse nature's action space is the set of all other players' strategies which player i considers to be possible which is the set of rationalizable strategies.

[45] A pure strategy can be interpreted as distribution of bids which puts probability weight 1 on one bid. I abuse notation since in the case of a pure strategy, $\beta_i(\theta_i)$ denotes an element in A_i while in the case of a (mixed) strategy $\beta_i(\theta_i)$ denotes an element in ΔA_i. However, in the following it will be clear whether β_i is a pure or a mixed strategy.

[46] For a vector $(v_1, \ldots, v_I)$ I denote by v_{-i} the vector $(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_I)$.

Rationalizable strategies As argued in the introduction, in many economic settings players may face uncertainty about the other players' strategies. Even if a (Bayes-) Nash equilibrium exists, a player may consider also other strategies of her opponents to be possible. For example, multiple Nash equilibria can exist or the Nash equilibrium strategies are not aligned with preferences the other players may have, e.g. maximin or minimax regret preferences. In order to determine the set of strategies a player can expect from rational opponents, I assume common knowledge of rationality. That is, it is common knowledge that every player i maximizes her expected utility given her type, the commonly known type distribution F and a conjecture about the other players' strategies.

The assumption of common knowledge of rationality leads to the following reasoning. Every player i maximizes her expected utility given her type, the type distribution F and a conjecture about the other players' strategies. The strategy which player i assumes is played by some player $j \neq i$ has also to be compatible with common knowledge of rationality. Therefore, for every possible type of player j, the action prescribed by the strategy assumed by player i maximizes j's expected utility given her type, the type distribution F and a conjecture about the other players' strategies. Again, player j's conjecture has to be compatible with common knowledge of rationality. This reasoning continues ad infinitum. [47]

Given the type of a player, an action which is compatible with common knowledge of rationality is called *rationalizable.* Battigalli and Siniscalchi (2003b) have shown that it is equivalent to define a rationalizable action as follows.

Definition 17.

(i) Let $i \in \{1, \dots, I\}$ be a player and $\theta_i \in \Theta_i$ be a type of player i. The set of rationalizable actions *for player i is defined as follows. Set $RS_i^1(\theta_i) := A_i$. Assume that for $k \in \mathbb{N}$ the set $RS_i^k(\theta_i)$ is already defined. Then the set $RS_i^{k+1}(\theta_i)$ is defined as the set of all elements a_i in A_i for which there exists a strategy profile β_{-i} of the other players such that it holds*

(i) $a_j \in supp(\beta_j(\theta_j))$ for $\theta_j \in \Theta_j \ \Rightarrow \ a_j \in RS_j^k(\theta_j)$ for all $j \neq i$

(ii) $a_i \in \underset{a_i' \in A_i}{\operatorname{argmax}} \ U_i(\theta_i, a_i', \beta_{-i}, F_{-i})$

and $RS_i(\theta_i)$ is given by

$$RS_i(\theta_i) = \bigcap_{k \geq 1} RS_i^k(\theta_i).$$

(ii) A strategy β_i of a player i is rationalizable *if for every $\theta_i \in \Theta_i$ every action a_i with $a_i \in supp(\beta_i(\theta_i))$ is rationalizable, i.e. an element of $RS_i(\theta_i)$.*

[47] As stated above, under strategic uncertainty a rational player acts optimally given a conjecture about the other players' strategies (and a conjecture about the other players' type distributions if also distributional uncertainty is present). Instead of "conjecture" other terms have been used in economic literature, e.g. belief, subjective prior, assumption, assessment ect. I use the term conjecture as proposed in Bernheim (1984).

(iii) For a player i let RS_{-i} be the set of rationalizable strategies of the other $I-1$ players.

The intuition behind this definition is that an action for a player which is consistent with common knowledge of rationality, i.e. a rationalizable action, is an action which survives the iterated elimination of actions which are not best replies. An action is a best reply if it maximizes the player's expected utility given her type, the commonly known type distribution F and a conjecture about the other players' strategies which prescribe actions that have not been eliminated yet.

The definition of rationalizable strategies allows for a formal definition of the adverse nature's action space and therefore for a formal definition of the simultaneous game against the adverse nature.

Simultaneous game against adverse nature The following definition summarizes all components describing a game under strategic uncertainty.

Definition 18. *A* game under strategic uncertainty *consists of an underlying game of incomplete information, denoted by $(\{1,\dots,I\},\Theta,A,\{u_i\}_i \in \{1,\dots,I\})$, a subset of players $\{i_1,\dots i_k\} \subseteq \{1,\dots,I\}$ applying the maximin expected utility criterion, and a player n. For every $i \in \{i_1,\dots,i_k\}$ player i chooses a strategy*

$$\beta_i : \Theta_i \to \Delta A_i.$$

A strategy of n is a mapping which for every player $i \in \{i_1,\dots i_k\}$ and for every possible type of player i assigns a strategy of the other players:

$$\beta_n = (\beta^{n_{i_1}},\dots,\beta^{n_{i_k}}) : \Theta_{i_1} \times \dots \times \Theta_{i_k} \to RS_{-i_1} \times \dots \times RS_{-i_k}.$$

$$(\theta_{i_1},\dots,\theta_{i_k}) \to \left(\beta_{-i_1}^{n_{i_1},\theta_{i_1}},\dots,\beta_{-i_k}^{n_{i_k},\theta_{i_k}}\right).$$

Here the superscript n_{i_j},θ_{i_j} for $j \in \{1,\dots,k\}$ indicates that the other players' strategies $\beta_{-i_j}^{n_{i_j},\theta_{i_j}}$ are chosen by the adverse nature faced by player i_j and depend on the player's type θ_{i_j}. The utility of a player $i \in \{i_1,\dots,i_k\}$ is given by

$$U_i\left(\theta_i,\beta_i(\theta_i),\beta_{-i}^{n_i,\theta_i},F_{-i}\right)$$

which is defined as in (79) and depends on the utility function of player i in the underlying game of incomplete information, denoted by u_i:

$$u_i : A \times \Theta_i \to \mathbb{R}.$$

$$(a_1,\dots,a_I,\theta_i) \mapsto u_i(a_1,\dots,a_I,\theta_i).$$

The utility of player nature is given by

$$-\sum_{j=1}^{k} U_i \left(\theta_i, \beta_i\left(\theta_i\right), \beta_{-i}^{n_i,\theta_i}, F_{-i}\right).$$

Throughout the remainder of the paper it will be assumed that a game under strategic uncertainty is given without explicitly stating all its ingredients.

Since the other players' strategies the adverse nature chooses for a player $i \in \{i_1, \ldots, i_k\}$, are not observed by a player $j \neq i$, $j \in \{i_1, \ldots, i_k\}$, the adverse nature faces an independent minimization problem for every player applying the expected maximin utility criterion.[48]

Note that after specifying the subset of players who apply the maximin expected utility criterion, a given game of incomplete information uniquely defines a game under strategic uncertainty.

Now it is possible to define a maximin strategy in a game under strategic uncertainty which can be seen as a recommendation for a player facing strategic uncertainty.

Definition 19. *In a game under strategic uncertainty for a player i a strategy*

$$\beta_i : \Theta_i \to \Delta A_i$$

is a maximin strategy *if there exists a Nash equilibrium in the simultaneous game between nature and player i such that β_i is player i's equilibrium strategy.*

The Nash equilibrium in the simultaneous game between nature and player i is called maximin equilibrium*.*

As described above, such a maximin strategy has two properties. First, if a player would not choose an action according to a maximin strategy, then there would exist a rationalizable strategy of the other players under which the player's expected utility is lower than under the action prescribed by a maximin strategy. Second, the strategy chosen by nature can be interpreted as the player's subjective belief about the state of the world against which she maximizes her expected utility given her type. The second property is formalized in the following definition.

Definition 20. *In a game under strategic uncertainty let β^{n_i} be the adverse nature's maximin equilibrium strategy projected on the i'th component. A* subjective maximin belief *of player i with valuation θ_i is defined as*

$$\beta^{n_i}\left(\theta_i\right) = \beta_{-i}^{n_i,\theta_i},$$

that is, the adverse nature's maximin equilibrium strategy evaluated at θ_i.

[48] Equivalently, one could introduce an additional adverse nature for every player applying the minimax expected utility criterion.

Note that the subjective maximin belief of player is not necessarily unique. However, every best reply of a player i to any subjective maximin belief induces the same expected utility for player i.

4.3 Outcomes under strategic uncertainty

So far, I have characterized the set of strategies of a player which are obtained if this particular player applies the maximin expected utility criterion. In addition to the derivation of maximin strategies for particular players, one can analyze what happens if *all* players adopt maximin strategies. Since under strategic uncertainty players do not observe each other's strategies, I do not use the term *equilibrium*, but the term *outcome*.

Definition 21. *In a game under strategic uncertainty an* outcome under maximin strategies *is a strategy profile $(\beta_1, \ldots, \beta_I, \beta_n)$ such that for every $i \in \{1, \ldots, I\}$ it holds that player i's strategy is a maximin strategy given the adverse nature's strategy β_n.*

In other words, for every player $i \in \{1, \ldots, I\}$ it holds that $(\beta_1, \ldots, \beta_I, \beta_n)$ constitutes a maximin equilibrium in the simultaneous game between *all* players and the adverse nature. The following Proposition follows from the definition of rationalizable strategies and of an outcome under maximin strategies.

Proposition 12. *In a game under strategic uncertainty let $(\beta_1, \ldots, \beta_I, \beta_n)$ be an outcome under maximin strategies. Then for every $i \in \{1, \ldots, I\}$ it holds that β_i is a rationalizable strategy for player i.*

One can prove this Proposition by showing per induction that for every $k \in \mathbb{N}$, for every $\theta_i \in \Theta_i$ and for every $a_i \in supp\,(\beta_i(\theta_i))$ it holds that a_i is an element in $RS_i^k\,(\theta_i)$. The formal proof is relegated to Appendix 4.7.1.

The following conclusions can be derived from this proposition. First, this proposition shows that the maximin expected utility criterion is consistent with common knowledge of rationality. That is, every action resulting from the application of the maximin utility criterion is rationalizable. Second, it provides a sufficient condition for a strategy to be rationalizable which will be useful in subsequent proofs. Third, the same proof as for Proposition 12 can be used in order to show that an action which is a best reply to a rationalizable strategy is again rationalizable. The last statement is formalized in the following Corollary.

Corollary 6. *In a game under strategic uncertainty let $i \in \{1, \ldots, I\}$ be a player with valuation θ_i and for every $j \in \{1, \ldots, I\} \backslash \{i\}$ let β_j be a rationalizable strategy for player j. Let $a_i \in A_i$ be a best reply to β_{-i}, i.e. it holds that*

$$a_i \in \underset{a_i' \in A_i}{\operatorname{argmax}}\; U_i\left(\theta_i, a_i', \beta_{-i}, F_{-i}\right),$$

then $a_i \in RS_i\,(\theta_i)$, that is, a_i is a rationalizable action for player i with valuation θ_i.

Another sufficient condition for an action to be rationalizable is that it is played in a Bayes-Nash equilibrium. It follows from Corollary 6 that a best reply to strategies played in a Bayes-Nash equilibrium is rationalizable. This constitutes another sufficient condition for an action to be rationalizable. These two conditions are formalized in the following definition and proposition.

Definition 22. *In a game of incomplete information a strategy profile* $(\beta_1, \ldots \beta_I)$ *together with a profile of type distributions* $\left(\hat{F}_1, \ldots, \hat{F}_I\right)$ *is a* Bayes-Nash equilibrium with a common prior *if for every* $i \in \{1, \ldots, I\}$*, every* $\theta_i \in \Theta_i$ *and every* $a_i \in A_i$ *such that* $a_i \in supp\left(\beta_i\left(\theta_i\right)\right)$ *it holds that*

$$a_i \in \underset{a_i' \in A_i}{\operatorname{argmax}} \; U_i\left(\theta_i, a_i', \beta_{-i}, \hat{F}_{-i}\right).$$

That is, every player maximizes her expected utility given the other players' strategies and the other players' commonly known type distributions.

Proposition 13. *Let the profile of strategies* $(\beta_1, \ldots, \beta_I)$ *together with the profile of type distributions* $\left(\hat{F}_1, \ldots, \hat{F}_I\right)$ *constitute a Bayes-Nash equilibrium with a common prior of a game of incomplete information. Then the following holds true:*

(i) For every $i \in \{1, \ldots, I\}$ *the strategy* β_i *is rationalizable.*

(ii) Let $i \in \{1, \ldots I\}$ *be a player with valuation* θ_i *and let* $a_i \in A_i$ *be a best reply to* β_{-i} *and some distribution of the other players' types* ${F_{-i}}' \in \Delta_{\Theta_{-i}}$*, i.e. it holds that*

$$a_i \in \underset{a_i' \in A_i}{\operatorname{argmax}} \; U_i\left(\theta_i, a_i', \beta_{-i}, F_{-i}'\right),$$

then $a_i \in RS_i\left(\theta_i\right)$*, that is,* a_i *is a rationalizable action for player* i *with valuation* θ_i*.*

The formal proof is relegated to Appendix 4.7.2.

As mentioned above, for every player applying the maximin expected utility criterion, the adverse nature faces an independent optimization problem. Thus, if $\{i_1, \ldots, i_k\}$ is the set of players applying the maximin expected utility criterion, then the game against an adverse nature can be seen as k independent two-player zero-sum games. This allows for the application of all results for two-player zero-sum games including the existence result for Nash equilibria.

4.4 First-Price Auctions under Strategic uncertainty

In this section I apply the proposed decision criterion to first-price auctions. The first subsection specifies the model for first-price auctions. The second subsection gives a rather informal preview of the results. The third subsection provides a list of the necessary notation and definitions. The fourth and fifth subsection contain a detailed description and derivation of the results for first price auctions under strategic uncertainty with common knowledge of valuations and common knowledge of the valuation distribution, respectively.

4.4.1 Model

Underlying game of incomplete information As in the general model, the model description starts with the specification of the underlying game of incomplete information. There are I risk-neutral bidders competing in a first-price sealed-bid auction for one indivisible good. Before the auction starts, every bidder $i \in \{1, ..., I\}$ privately observes her valuation (type) $\theta_i \in \Theta = \{0 = \theta^1, \theta^2, \ldots, \theta^{m-1}, 1 = \theta^m\}$. A *pure strategy* of bidder i is a mapping

$$\beta_i : \Theta \to \mathscr{B}$$

$$\theta_i \mapsto \beta_i(\theta_i)$$

where $\mathscr{B}$ is a finite (arbitrarily fine) grid of bids on an interval $[0, B]$ with $\Theta \subseteq \mathscr{B}$.[49] A *strategy* of a bidder i is a mapping

$$\beta_i : \Theta \to \Delta\mathscr{B}$$

$$\theta_i \mapsto \beta_i(\theta_i)$$

where $\Delta\mathscr{B}$ is the set of bid distributions on $\mathscr{B}$. For every $b \in \mathscr{B}$ with $b > 0$ there exists a predecessor in $\mathscr{B}$ denoted by

$$b^- = \max_{b' \in \mathscr{B}} b' < b$$

and for every $b \in \mathscr{B}$ with $b < B$ there exists a successor in $\mathscr{B}$ denoted by

$$b^+ = \min_{b' \in \mathscr{B}} b' > b.$$

In a first-price auction the bidders submit bids, the bidder with the highest bid wins the object and pays her bid. In addition, it holds an efficient tie-breaking rule.[50] Thus, the utility of bidder i with valuation θ_i and bid b_i given that the other bids are b_{-i} is denoted by

[49] A finite grid is used for the set of all possible bids instead of the interval $[0, B]$ because of the following reason. Consider two bidders 1 and 2 with the same valuation θ. If bidder 1 bids some amount $b < \theta$, one has to identify the smallest bid which is strictly higher than b since this would be the unique best reply of bidder 2. This allows a more formal analysis than using expressions like "bidding an arbitrarily small amount more than b". The grid is assumed to be finite in order to ensure that any subset of the bid grid is compact. Since the grid can be arbitrarily fine, I assume for simplicity that $\Theta \subseteq \mathscr{B}$.

[50] The core statements in the results do not depend on the choice of the tie-breaking rule, i.e. under a random tie-breaking rule for every bidder and every valuation the bid prescribed by the maximin strategy would change by at most one step on the bid grid.

$$u_i\left(\theta_i, b_i, b_{-i}\right) = \begin{cases} \theta_i - b_i & \text{if } b_i > \max\limits_{j \neq i} b_j \\ \theta_i - b_i & \text{if } b_i = \max\limits_{j \neq i} b_j \text{ and } \theta_i > \max\limits_{j \neq i}\{\theta_j \mid b_j = b_i\} \\ 0 & \text{if } b_i = \max\limits_{j \neq i} b_j \text{ and } \theta_i < \max\limits_{j \neq i}\{\theta_j \mid b_j = b_i\} \\ \frac{1}{k}\left(\theta_i - b_i\right) & \text{if } b_i = \max\limits_{j \neq i} b_j \text{ and } \theta_i = \max\limits_{j \neq i}\{\theta_j \mid b_j = b_i\} \\ 0 & \text{if } b_i < \max\limits_{j \neq i} b_j \end{cases}$$

where θ_j denotes the valuation of bidder j with bid b_j for $j \in \{1, ..., n\}$ and $k = \#\{\max\{\theta_j | b_j = b_i\}\}$.

The bidders' valuations are identically and independently distributed according to a distribution function

$$F : \Theta \to [0, 1].$$

It is assumed that all components of the underlying game of incomplete information as well as rationality are common knowledge among all bidders.

As mentioned above, the above defined game of incomplete information uniquely defines a game under strategic uncertainty (after specifying the players applying the maximin expected utility criterion). In the following I will call this game *first-price auction under strategic uncertainty.* Moreover, I also consider the case where the valuations of the bidders are common knowledge. In this case I use the term *first-price auction under strategic uncertainty and common knowledge of valuations.*

4.4.2 Preview of results

Common knowledge of valuations If in a first-price auction under strategic uncertainty and common knowledge of valuations there exists a unique bidder with the highest valuation, this bidder's maximin strategy is to bid the second-highest valuation and every other bidder's maximin strategy prescribes to be indifferent between any bid between zero and her valuation. If at least two bidders have the highest valuation, then every bidder's maximin strategy prescribes to be indifferent between zero and her valuation.

Common knowledge of the valuation distribution In a first-price auction under strategic uncertainty the bidders' strategies are equal in every outcome under maximin strategies and every outcome is efficient.

For every type there exists a unique highest rationalizable bid. For every bidder and every type the adverse nature chooses as the strategy of the other bidders that every bidder places the highest rationalizable bid given her type. As a consequence, a bidder applying the maximin expected utility criterion never expects to win against a bidder with an equal or higher type. The bidder calculates which highest rationalizable bid of a lower type maximizes her expected utility. It turns out that due to the symmetry of beliefs about distributions and strategies, the higher the type of the bidder, the higher is the type whose highest rationalizable bid maximizes her expected utility. Therefore, every outcome under maximin strategies is efficient.

4.4.3 Notation and definitions

For the formal analysis it is useful to have an overview over the notation and definitions which will be used in the remainder of this paper.

- For $\theta^k \in \Theta$ let $\overline{b}^{\theta^k}$ be the highest rationalizable bid of a bidder with valuation θ^k.
- For $\theta^k, \theta^l \in \Theta$, $f^{i,\theta^k}_{j,\theta^l}$ denotes the probability with which type θ^l of bidder j occurs in the subjective maximin belief of a bidder i with valuation θ^k.
- If $f^{i,\theta^k}_{j,\theta^l}$ does not depend on the identities of bidder i and j, I use the notation $f^{\theta^k}_{\theta^l}$.

Definition 23. *An* auction mechanism *is a double (x,p) of an allocation function x and a payment function p. The allocation function*

$$x : (\mathscr{B})^I \to [0,1]^I$$

$$x : (b_1, \dots, b_I) \to (x_1, \dots, x_I) \quad \textit{with } x_i \in [0,1],\ \sum x_i \le 1$$

determines for each participant the probability of receiving the item and the payment function

$$p : (\mathscr{B})^I \to \left(\mathbb{R}^+\right)^I$$

$$p : (b_1, \dots, b_I) \to (p_1, \dots, p_I) \quad \textit{with } p_i \in \mathbb{R}^+$$

determines each participant's payment.

Definition 24. *In a first-price auction a bidder i with valuation θ_i and strategy β_i* overbids *a bidder j with valuation θ_j and strategy β_j if for every b, b' such that $b \in supp\,(\beta_i(\theta_i))$ and $b' \in supp(\beta_j\,(\theta_j))$ it holds that $b \ge b'$ if $\theta_i > \theta_j$ and $b > b'$ if $\theta_i \le \theta_j$.*

Note that due to the efficient tie-breaking rule, a bidder who overbids every other bidder wins with probability 1 in any auction mechanism where the highest bid wins.

In order to evaluate outcomes in terms of social surplus, I introduce the following definition.

Definition 25. *Let $(\beta_1, \dots, \beta_I, \beta_n)$ be an outcome under maximin strategies of an auction mechanism. The outcome $(\beta_1, \dots, \beta_I, \beta_n)$ is* efficient *if for all bid vectors $(b_1, \dots, b_I)$, such that for every $i \in \{1, \dots I\}$ there exists a valuation θ_i with $b_i \in supp(\beta_i(\theta_i))$, it holds that*

$$x_i\,(b_1, \dots, b_I) > 0 \;\Rightarrow\; \theta_i = \max_{j \ne i} \theta_j.$$

That is, the good is allocated with probability one to a group of bidders who have the highest valuation.

4.4.4 Common knowledge of valuations

Proposition 14. *Consider a first-price auction under strategic uncertainty and common knowledge of valuations. Then there exists an outcome under maximin strategies and the following holds true:*

(i) If $\theta_k > \max_{j \neq i} \theta_j$, i.e. there exists a unique bidder k with the highest valuation, then bidder k bids $\theta_{k'} = \max_{\theta_j \in \Theta \setminus \{\theta_k\}} \theta_j$, i.e. the bidder with the highest valuation bids the second-highest valuation and every bidder $i \neq k$ is indifferent between any bid between zero and her valuation.

(ii) If it holds that $\theta_k = \theta_l = \max_{j \in \{1,\dots,I\}} \theta_j$, i.e. there exist at least two bidders k and l with the highest valuation, then every bidder is indifferent between any bid between zero and her valuation.

The formal proof is relegated to Appendix 4.7.3.

The intuition behind part (i) is that one can show that the second-highest valuation $\theta_{k'}$ is the highest rationalizable bid of bidder k with the highest valuation θ_k. If the adverse nature chooses for all other bidders the subjective maximin belief that bidder k bids $\theta_{k'}$, this induces a utility of zero for any other bidder. Hence, any strategy of the adverse nature has to induce an expected utility of at most zero for all bidders besides k. That is, the subjective maximin belief of a bidder $i \neq k$ with valuation θ_i is that at least one other bidder bids an amount which is equal or greater than θ_i. As a consequence, all bidders are indifferent between zero and their valuation. The adverse nature chooses the subjective maximin belief for bidder k such that the bidder with the second-highest valuation bids her valuation $\theta_{k'}$. Hence, it is a best reply for bidder k to bid $\theta_{k'}$. Similar arguments apply to part (ii).

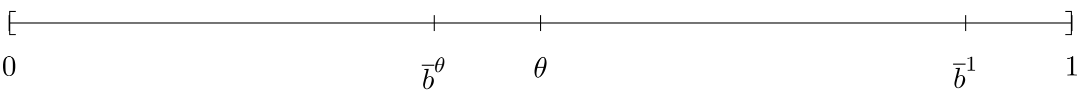
This Proposition describes the bidding behavior in a first-price auction under strategic uncertainty under the assumption that every bidder applies the proposed decision criterion. Since the bidders do not best reply to each other's strategies, the Proposition also provides the maximin strategy for every bidder. That is, the strategy prescribed in the outcome also serves as a recommendation for a bidder facing strategic uncertainty in a first-price auction.

Note that while the unique Nash equilibrium in this setting is rationalizable, there are much more rationalizable actions than played in the Nash equilibrium. In particular, in the case of two bidders who have the same valuation v all actions in the interval $[0, v]$ are rationalizable. This leaves room for more outcomes than the unique Nash-equilibrium which is weakly dominated.

4.4.5 Common knowledge of the valuation distribution

Now I consider the case where not the bidders' valuations but the distribution of the valuations is common knowledge. In this case in an outcome under maximin strategies for every type there exists a unique highest rationalizable bid. For every bidder and every type the adverse nature chooses as the strategy of the other bidders that every other bidder will bid the highest rationalizable bid given her type. As a consequence, it is never a best reply for a bidder to overbid bidders with the same type. Hence, every bidder overbids only lower types and it depends on the commonly known valuation distribution which types are overbid. Since the strategy chosen by the adverse nature is the same for every bidder and every type, this results in an efficient outcome. This is illustrated by the following example.

Example 9. *Consider a first-price auction under strategic uncertainty with two bidders 1 and 2 and three possible valuations* $0, \theta$ *and* 1 *which are identically and independently distributed according to a commonly known distribution function* $F \in \Delta\{0, \theta, 1\}$*. For every type* $\theta^k \in \{0, \theta, 1\}$ *there exists a highest rationalizable bid* $\overline{b}^{\theta^k}$*. For every bidder and every type the adverse nature chooses a strategy of the other bidder to bid the highest rationalizable bid given her type. That is, every bidder with every type has the subjective maximin belief that the 0-type bids zero, the* θ*-type bids* $\overline{b}^{\theta}$ *and the 1-type bids* $\overline{b}^{1}$*.*



Hence, bidder 1 with type θ *never expects to win against bidder 2 with type* θ *and therefore bids 0. Bidder 1 with type 1 never expects to win against bidder 2 with type 1 and has to decide between bidding* 0 *and bidding* $\overline{b}^{\theta}$*. Since the same reasoning holds for bidder 2, in any case the outcome is efficient.*

The insights from this example are formalized in the following Proposition.

Proposition 15. *In a first-price auction under strategic uncertainty there exists an outcome under maximin strategies. Every outcome is efficient.*

The formal proof is relegated to Appendix 4.7.4.

I will show the existence of an efficient outcome under maximin strategies by construction. Then I will show that every strategy of the adverse nature in an outcome under maximin strategies induces the same bidding strategies as in the constructed outcome and therefore every outcome has to be efficient. The proof by construction has the advantage that it determines the maximin strategies for every bidder, i.e. it provides a recommendation for a bidder facing strategic uncertainty.

For the construction of the efficient outcome it is crucial to calculate the highest rationalizable bid for every type. The following three steps serve as a preparation for this calculation.

(I) Show that for every type $\theta^k \in \Theta$ there exists a unique highest rationalizable bid $\overline{b}^{\theta^k}$.

(II) Show that for every type zero is a rationalizable bid.

(III) Show that for every type $\theta^k \in \Theta$ every bid in the interval $[0, \overline{b}^{\theta^k}]$ is rationalizable.

The first step follows from the fact that $\mathscr{B}$ is compact and well-ordered with respect to $\leq$. For a proof sketch of step (II) consider a proof by induction with respect to the valuations in Θ. Assume it has been shown that for every bidder with valuation θ^j such that $j < k+1$ bidding zero is a rationalizable action. Assume that a bidder with valuation θ^{k+1} conjectures that all lower types bid zero. Due to step (I), for every type there exists a highest rationalizable bid. Assume further, that the bidder with valuation θ^{k+1} conjectures that all equal or higher types bid their highest rationalizable bid, then it is a best reply of this bidder to bid zero. As stated in Corollary 6, a best reply to a rationalizable strategy profile is rationalizable and therefore zero is a rationalizable action for a bidder with valuation θ^{k+1}.

For an intuition of step (III) consider the bid 0^+. Since bidding zero is a rationalizable action for every bidder and every type, it is straight-forward that for a sufficiently fine bid grid bidding 0^+ is a rationalizable action for every bidder and every type besides zero. Because if a bidder conjectures that all bidders bid zero, than she could win the auction with probability 1 by bidding 0^+. The same holds for $(0^+)^+$ and so on. This process reaches some bid b such that for type θ^2 it is more profitable to bid zero and win against the zero-type than to bid b^+ even if all other bidders with a type higher than zero bid b. Then b is the highest rationalizable bid for type θ^2 and all bids in the interval $[0, b]$ are rationalizable for a bidder with valuation θ^2. The analogous reasoning applies to every higher type. Since the bids in $\mathscr{B}$ are well-ordered with respect to $\leq$, one can show the result by double induction with respect to the types and the bids.

Given these steps, one can calculate the highest rationalizable bid for every type. The highest rationalizable bid $\overline{b}^{\theta^k}$ for a bidder i with valuation θ^k is induced by the belief about the other bidders' strategies such that

(i) All bidders with a lower type bid their highest rationalizable bid.

(ii) All bidders with an equal or higher type bid $\left(\overline{b}^{\theta^k}\right)^-$.

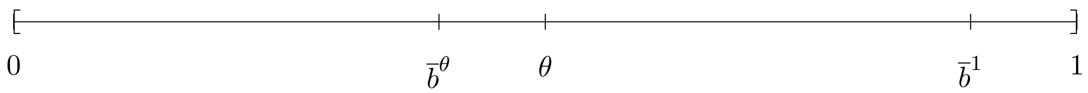
That is, $\overline{b}^{\theta^k}$ is a best reply to the belief which maximizes the expected utility of bidding $\overline{b}^{\theta^k}$. The strategies in (i) are rationalizable by definition and it follows from step (III) that the strategies specified in (ii) are rationalizable. Hence, the highest rationalizable bid $\overline{b}^{\theta^k}$ of type θ^k makes this type indifferent between winning with probability 1 by bidding $\overline{b}^{\theta^k}$

and the most profitable overbidding of a lower type given that all lower types bid their highest rationalizable bid.

In an outcome under maximin strategies for every bidder and every type the adverse nature chooses as the strategy of the other bidders that every other bidder places the highest rationalizable bid given her type. As a consequence, a bidder never expects to win against an equal or higher type. The maximin strategy of a bidder is determined by the most profitable overbidding of a lower type given that every lower type places her highest rationalizable bid.

The following example continues with Example 9 and illustrates the calculation of the highest rationalizable bids and the maximin strategies.

Example 10. *Consider again the case with two bidders 1 and 2 and three possible valuations 0,θ and 1 which are identically and independently distributed according to a commonly known distribution function $F \in \Delta\{0, \theta, 1\}$.*



The highest rationalizable bid for type zero is zero. The highest rationalizable bid for type θ is given by the bid $\overline{b}^{\theta}$ which makes her indifferent between winning with probability 1 by bidding $\overline{b}^{\theta}$ and just overbidding type zero:

$$\theta - \overline{b}^{\theta} = F(0)(\theta - 0)$$

$$\Leftrightarrow \overline{b}^{\theta} = \theta(1 - F(0)) + F(0).$$

The highest rationalizable bid for type 1 is given by the bid $\overline{b}^{1}$ which makes her indifferent between winning with probability 1 by bidding $\overline{b}^{1}$ and the most profitable overbidding of a lower type. That is, type 1 has to be indifferent between bidding $\overline{b}^{1}$ and the maximum utility of bidding either $0 = \overline{b}^{0}$ or $\overline{b}^{\theta}$:

$$1 - \overline{b}^{1} = \max\left\{F(0)(1-0), F(\theta)\left(1 - \overline{b}^{\theta}\right)\right\}.$$

For a numerical example consider the parameters $\theta = \frac{1}{2}$, $F(0) = \frac{1}{3}$, $F(\theta) = \frac{2}{3}$ and $F(1) = 1$. Then it holds that

$$\overline{b}^{\theta} = \frac{1}{2}\left(1 - \frac{1}{3}\right) = \frac{1}{3}$$

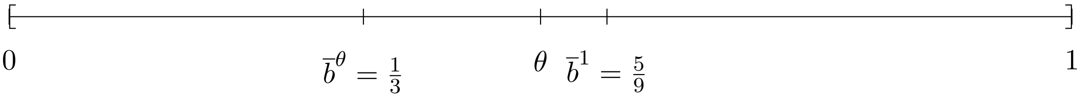
and

$$\max\left\{F(0), F(\theta)\left(1 - \overline{b}^{\theta}\right)\right\} = \max\left\{\frac{1}{3}, \frac{2}{3}\left(1 - \frac{1}{3}\right)\right\} = \frac{4}{9} = F(\theta)\left(1 - \overline{b}^{\theta}\right)$$

from which follows that

$$\bar{b}^1 = 1 - \frac{4}{9} = \frac{5}{9}$$

which is illustrated below:



After computing the highest rationalizable bids, one can compute bidding strategies in an outcome under maximin strategies. Type zero bids zero. Note that the highest rationalizable bid of type θ is determined by the case that type θ wins with probability 1. Since type θ of bidder 1 has the subjective maximin belief that type θ of bidder 2 bids $\bar{b}^\theta$, bidder 1 would win only with probability $\frac{1}{2}$ by bidding $\bar{b}^\theta$. Hence, given the subjective maximin of bidder 1 with type θ, it is not a best reply to bid equal or higher than $\bar{b}^\theta$. Therefore, she does not expect to win against type θ of bidder 2 and bids zero. Similarly, type 1 of bidder 1 does not expect to win against type 1 of bidder 2 and has to decide whether to overbid type 0 or type θ of bidder 2, i.e. whether to bid 0 or $\bar{b}^\theta$. In any case the outcome is efficient.[51] *Bidding zero gives an expected utility of*

$$F(0) = \frac{1}{3}$$

and bidding $\bar{b}^\theta = \frac{1}{3}$ gives an expected utility of

$$F(\theta)\left(1 - \bar{b}^\theta\right) = \frac{2}{3}\left(1 - \frac{1}{3}\right) = \frac{4}{9}.$$

Hence, type 1 of bidder 1 will bid $\bar{b}^\theta$ (and analogously for type θ of bidder 2).

Applying the same procedure, one can compute the highest rationalizable bids for every number of types and every choice of parameters and then compute the bids under maximin strategies. The following two graphs show the highest rationalizable bids for m equidistant types with a uniform distribution for $m = 10$ and $m = 20$.

[51] Due to the efficient tie-breaking rule the outcome is efficient even if different types submit equal bids. However, efficiency does not depend on thy choice of the tie-breaking rule. Under a random tie-breaking rule type 1 would just decide between the bids 0^+ and $\left(\bar{b}^\theta\right)^+$.

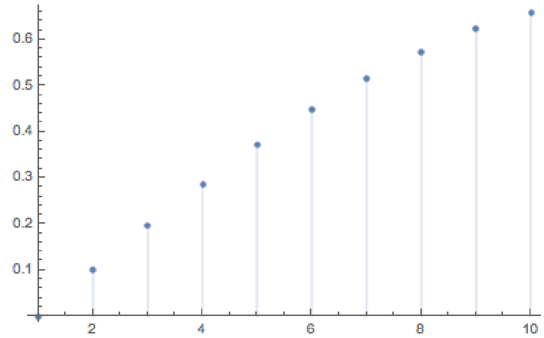


Figure 6: Highest rationalizable bids for $m = 10$

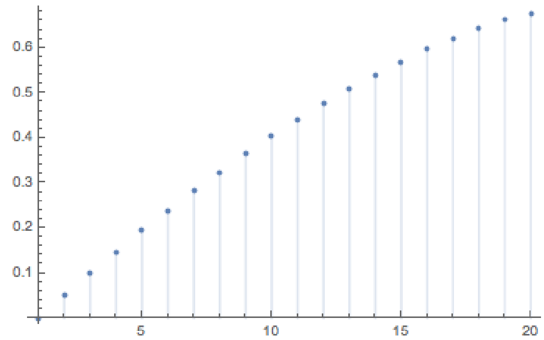


Figure 7: Highest rationalizable bids for $m = 20$

The following two graphs show the bids in an outcome under maximin strategies for m equidistant types with a uniform distribution for $m = 10$ and $m = 20$.

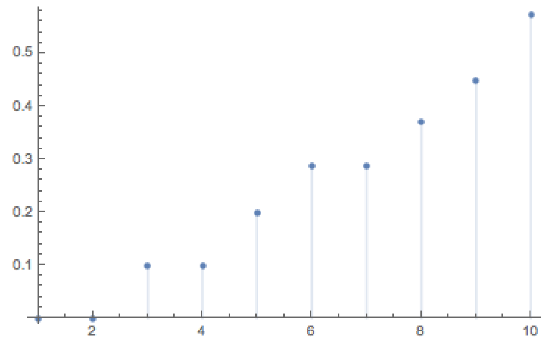


Figure 8: Bids in an outcome under maximin strategies for $m = 10$

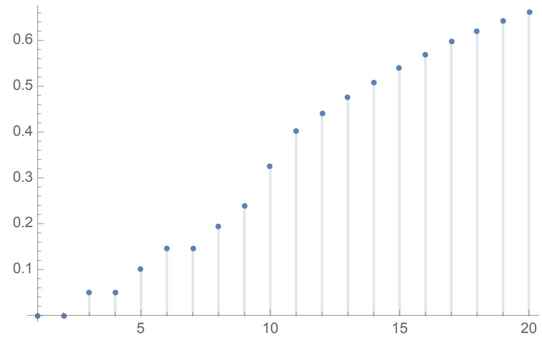


Figure 9: Bids in an outcome under maximin strategies for $m = 20$

Figures 3 and 4 show that the outcome under maximin strategies is efficient since the bidder with the highest valuation wins the auction with probability 1. However, it is possible that different types submit equal bids. Whenever a bidder is not indifferent between two bids, her bidding strategy is unique which is the case in figures 3 and 4.

The strategy of the adverse nature is not necessarily unique. Assume that it is a best reply of a bidder i with valuation $\theta^k \in \Theta$ to bid $\overline{b}^{\theta^l}$ for $l < k$. Then it is possible that the adverse nature decreases the bid of some bidder $j \neq i$ with type $\theta^{l'}$ for $l \neq l'$ without changing the best reply of bidder i and hence without changing her expected utility. Since all possible strategies of the adverse nature induce the same bidding strategies, the non-uniqueness of the adverse nature's strategy does not affect efficiency.

The recursive computation of the highest rationalizable bids for all types is formalized in the following Proposition.

Proposition 16. *In a first-price auction under strategic uncertainty the highest rationalizable bids can be defined by the following recursion. The highest rationalizable bid of a bidder with valuation zero is zero. Assume that for every type θ^j with $j < k$ the highest rationalizable bid $\overline{b}^{\theta^j}$ has been already defined. Then the highest rationalizable bid of a bidder with valuation θ^k is determined by the equality*

$$\theta^k - \overline{b}^{\theta^k} = \max_{\theta^j < \theta^k} F^{I-1}\left(\theta^j\right)\left(\theta^k - \overline{b}^{\theta^j}\right). \tag{80}$$

The formal proof is relegated to Appendix 4.7.4.[52]

Proposition 16 states that the highest rationalizable bid of a bidder with valuation θ^k makes this bidder indifferent between winning the auction with probability 1 by bidding $\overline{b}^{\theta^k}$ and the most profitable overbidding of some lower type given that all lower types bid their highest rationalizable bid.

[52]For a simpler notation I assume that the highest rationalizable bids $\overline{b}^{\theta^j}$ for $1 \leq j \leq m$ lie on the bid grid. Otherwise, the highest rationalizable bid of a bidder with valuation θ^k would be defined by $\max\{b \in \mathscr{B} \mid b < b^{\theta^k}\}$ where the bid b^{θ^k} is defined by $\overline{b}^{\theta^k} = \theta^k - b^{\theta^k} = \max_{\theta^j < \theta^k} F^{I-1}\left(\theta^j\right)\left(\theta^k - \overline{b}^{\theta^j}\right)$.

After calculating the highest rationalizable bid for every type, one can specify the strategies played in an outcome under maximin strategies.

Proposition 17. *In a first-price auction under strategic uncertainty it holds for every outcome under maximin strategies and for every bidder i and every valuation θ^k that*

$$b \in supp\left(\beta_i\left(\theta^k\right)\right) \Rightarrow b \in \underset{\overline{b}^{\theta j}}{\operatorname{argmax}}\left\{F^{I-1}\left(\theta^j\right)\left(\theta^k - \overline{b}^{\theta^j}\right) \mid \theta^j < \theta^k\right\}.$$

This proposition states that every bidder chooses the most profitable overbidding of a lower type. The intuition for this result is as follows. I show in the proof of Proposition 15 that there exists an outcome under maximin strategies such that for every bidder the adverse nature chooses as the other bidders' strategy that every bidder places the highest rationalizable bid given her type. The strategy specified in Proposition 17 is a best reply to this strategy of the adverse nature. Moreover, I show in the proof of Proposition 15 that in every outcome under maximin strategies of a first-price auction under strategic uncertainty the bidders' strategies are equal. Therefore, in every outcome the bidders' strategies are as specified in Proposition 17. The formal proof is relegated to Appendix 4.7.4. Note that as before, this Proposition also provides the maximin strategy, i.e. a recommendation, for a bidder facing strategic uncertainty in a first-price auction.

Proposition 17 shows that a bidder with a given type does not need to know the exact value or distribution of higher types but only of lower types. This stems from the fact that a bidder with a given type does not expect to win against bidders with the same or a higher type. This Proposition also provides an intuition for the fact that every outcome is efficient. The higher the valuation of a bidder, the higher is the type such that bidding the highest rationalizable bid of this type maximizes the bidder's expected utility.

Similarly as in the case where bidders' valuations are known, there are more rationalizable actions than actions played in the Bayes-Nash equilibrium as formalized in the following Proposition.

Proposition 18. *Consider a first-price auction under strategic uncertainty. Let $\overline{b}_*^{\theta^k} = \max supp\left(\beta^*\left(\theta^k\right)\right)$ in the unique Bayes-Nash equilibrium β^*.[53] If $m \geq 3$, it holds for all $k \neq 1$ that $\overline{b}_*^{\theta^k} < \overline{b}^{\theta^k}$.*

Proof. The formal proof is relegated to Appendix 4.7.5. □

Proposition 16 provides the explanation for this result. Since the Bayes-Nash equilibrium is efficient, the highest bid in the Bayes-Nash equilibrium is induced if a bidder overbids all bidders with an equal or lower type. In contrast, the highest rationalizable bid is induced if a bidder overbids all other bidders.

[53] It follows from Montiero (2009) that a unique Bayes-Nash equilibrium exists.

4.5 Distributional and strategic uncertainty

In this section the formal model and the application to first-price auctions allow not only for strategic but also for distributional uncertainty. The first subsection contains the formal model, the second subsection collects all results for the general model. The third subsection specifies the formal model for first-price auctions and the fourth subsection provides the results.

4.5.1 Model

This subsection provides the formal model for a game under both, distributional and strategic uncertainty. The underlying game of incomplete information is the same as in section 4.2. As before, a player applying the maximin expected utility criterion plays a simultaneous game against an adverse nature.

Action space of adverse nature Under distributional uncertainty the adverse nature's action space does not only consist of rationalizable strategies but also of the set of distributions which the player considers to be possible. I allow for the possibility that a player does not know the exact type distribution but has more knowledge than just the other players' type spaces. This is formalized in the following definition.

Definition 26. *Let $\Delta\Theta_{-i}$ be the set of all probability distributions on Θ_{-i}. The set $\Delta_{\Theta_{-i}}$ is the smallest subset of $\Delta\Theta_{-i}$ such that player i knows that the true type distribution is an element in $\Delta_{\Theta_{-i}}$.*

Analogously as for strategic uncertainty, it holds that distributional uncertainty is present if the set $\Delta_{\Theta_{-i}}$ is not a singleton. Note that the assumption that a player knows that the true type distribution (or the true strategy) is an element of some set is w.l.o.g. since it covers any possible knowledge structure. For example, if a bidder i knows only the type spaces of the other bidders but nothing else about the type distribution, then $\Delta_{\Theta_{-i}}$ is equal to $\Delta\Theta_{-i}$, the set of all type distributions on Θ_{-i}. In contrast, if bidder i faces no distributional uncertainty and knows that the distribution of the other bidders' types is given by a function F_{-i}, then the set $\Delta_{\Theta_{-i}}$ is equal to $\{F_{-i}\}$.

Throughout the paper I use the axiomatization of the knowledge operator where the statement that a player knows something implies that it is true. Therefore, for every player i it holds that the true type distribution of the other player is indeed an element in $\Delta_{\Theta_{-i}}$.

As in section 4.2, the strategies which a player considers to be possible are the set of rationalizable strategies.

Rationalizable strategies As before, I assume common knowledge of rationality which implies that the adverse nature has to choose from the set of rationalizable strategies. If distributional uncertainty is added to strategic uncertainty, the definition of a rationalizable action changes. Under distributional and strategic uncertainty a player is rational if her action is a best reply given her type, a conjecture about the other players' strategies and a conjecture about the other players' type distributions which is an element in $\Delta_{\Theta_{-i}}$.

Definition 27.

(i) *Let $i \in \{1,\ldots,I\}$ be a player and $\theta_i \in \Theta_i$ be a type of player i. The set of* rationalizable actions *for player i is defined as follows. Set $RS_i^1(\theta_i) := A_i$. Assume that for $k \in \mathbb{N}$ the set $RS_i^k(\theta_i)$ is already defined. Then the set $RS_i^{k+1}(\theta_i)$ is defined as the set of all elements a_i in A_i for which there exists a type distribution $F_{-i} \in \Delta_{\Theta_{-i}}$ and a strategy profile β_{-i} of the other players such that it holds*

   (i) *$a_j \in supp\left(\beta_j\left(\theta_j\right)\right)$ for $\theta_j \in \Theta \Rightarrow$ $a_j \in RS_j^k(\theta_j)$ for all $j \neq i$*

   (ii) *$a_i \in \underset{a_i' \in A_i}{\operatorname{argmax}}\ U_i\left(\theta_i, a_i', \beta_{-i}, F_{-i}\right)$*

   and $RS_i(\theta_i)$ is given by

$$RS_i(\theta_i) = \bigcap_{k \geq 1} RS_i^k(\theta_i).$$

(ii) *A strategy β_i of a player i is* rationalizable *if for every $\theta_i \in \Theta_i$ every action a_i with $a_i \in supp\left(\beta_i\left(\theta_i\right)\right) > 0$ is rationalizable, i.e. an element of $RS_i(\theta_i)$.*

(iii) *For a player i let RS_{-i} be the set of rationalizable strategies of the other $I-1$ players.*

The definition of the possible distributions and strategies and of rationalizable strategies allows for a formal definition of the adverse nature's action space and therefore for a formal definition of the simultaneous game against an adverse nature.

Simultaneous game against adverse nature The following definition summarizes all components describing a game under distributional and strategic uncertainty.

Definition 28. *A* game under distributional and strategic uncertainty *consists of a game of incomplete information, denoted by $(\{1,\ldots,I\}, \Theta, A, \{u_i\}_i \in \{1,\ldots,I\})$, a subset of players $\{i_1,\ldots i_k\} \subseteq \{1,\ldots,I\}$ applying the maximin expected utility criterion, and a player n. For every $i \in \{i_1,\ldots,i_k\}$ player i chooses a strategy*

$$\beta_i : \Theta_i \rightarrow \Delta A_i.$$

A strategy of n is a mapping which for every player $i \in \{i_1,\ldots i_k\}$ and for every possible type of player i assigns a distribution of the other players' valuations in a convex set $\Delta_{\Theta_{-i}}$ and a strategy of the other players in RS_{-i}:

$$\left(\beta^{n_{i_1}},\ldots,\beta^{n_{i_k}}\right) : \Theta_{i_1} \times \ldots \times \Theta_{i_k} \rightarrow \left(RS_{-i_1} \times \Delta_{\Theta_{-i_1}}\right) \times \cdots \times \left(RS_{-i_k} \times \Delta_{\Theta_{-i_k}}\right).$$

$$\left(\theta_{i_1},\ldots,\theta_{i_k}\right) \rightarrow \left(\left(\beta_{-i_1}^{n_{i_1},\theta_{i_1}}, F_{-i_1}^{n_{i_1},\theta_{i_1}}\right),\ldots,\left(\beta_{-i_k}^{n_{i_k},\theta_{i_k}}, F_{-i_k}^{n_{i_k},\theta_{i_k}}\right)\right).$$

The utility of a player $i \in \{i_1, \dots, i_k\}$ is given by

$$U_i\left(\theta_i, \beta_i(\theta_i), \beta_{-i}^{n_i,\theta_i}, F_{-i}^{n_i,\theta_i}\right)$$

which is defined as in (79) and depends on the utility function of player i in the underlying game of incomplete information, denoted by u_i:

$$u_i : A \times \Theta_i \to \mathbb{R}.$$

$$(a_1, \dots, a_I, \theta_i) \mapsto u_i(a_1, \dots, a_I, \theta_i)$$

The utility of player nature is given by

$$-\sum_{j=1}^{k} U_i\left(\theta_i, \beta_i(\theta_i), \beta_{-i}^{n_i,\theta_i}, F_{-i}^{n_i,\theta_i}\right).$$

The term uncertainty can include distributional uncertainty or strategic uncertainty or both. If only one type of uncertainty is present, I will refer to this case as *pure distributional* or *pure distributional uncertainty.* For instance, a game under pure strategic uncertainty as defined in section 4.2, is a special case of a game under distributional and strategic uncertainty. If the type distribution F is common knowledge as defined in section 4.2, then it holds for all players i that $\Delta_{\Theta_{-i}} = \{F_{-i}\}$.

Now it is possible to define a maximin strategy in a game under distributional and strategic uncertainty.

Definition 29. *In a game under distributional and strategic uncertainty for a player i a strategy*

$$\beta_i : \Theta_i \to \Delta A_i$$

is a maximin strategy *if there exists a maximin equilibrium in the simultaneous game between nature and player i such that β_i is player i's equilibrium strategy.*

The Nash equilibrium in the simultaneous game between nature and player i is called maximin equilibrium*.*

Analogously to section 4.2, one can define the subjective maximin belief of a bidder.

Definition 30. *In a game under uncertainty let β^{n_i} be the adverse nature's maximin equilibrium strategy projected on the i'th component. A* subjective maximin belief *of player i with valuation θ_i is defined as*

$$\beta^{n_i}(\theta_i) = \left(\beta_{-i}^{n_i,\theta_i}, F_{-i}^{n_i,\theta_i}\right),$$

that is, the adverse nature's maximin equilibrium strategy evaluated at θ_i.

4.5.2 Outcomes under distributional and strategic uncertainty

The definition of an outcome in a game under distributional and strategic uncertainty is analogous to the definition in a game under distributional and strategic uncertainty.

Definition 31. *In a game under distributional and strategic uncertainty an* outcome under maximin strategies *is a strategy profile* $(\beta_1, \ldots, \beta_I, \beta_n)$ *such that for every* $i \in \{1, \ldots, I\}$ *it holds that player i's strategy is a maximin strategy given the adverse nature's strategy* β_n*.*

Analogously to Proposition 12 and Corollary 6 the following Proposition and Corollary hold true which state that a strategy played in an outcome under maximin strategies is rationalizable and that every action which is a best reply to a profile of rationalizable strategies is also rationalizable.

Proposition 19. *In a game under distributional and strategic uncertainty let* $(\beta_1, \ldots, \beta_I, \beta_n)$ *be an outcome under maximin strategies. Then for every* $i \in \{1, \ldots, I, \beta_n\}$ *it holds that* β_i *is a rationalizable strategy for player* i*.*

The proof is relegated to Appendix 4.7.1.

Corollary 7. *In a game under distributional and strategic uncertainty let* $i \in \{1, \ldots, I\}$ *be a player with valuation* θ_i *and for every* $j \in \{1, \ldots, I\} \backslash \{i\}$ *let* β_j *be a rationalizable strategy for player* j*. Let* $a_i \in A_i$ *be a best reply to* β_{-i}*, i.e. it holds that*

$$a_i \in \underset{a_i' \in A_i}{\operatorname{argmax}} \; U_i\left(\theta_i, a_i', \beta_{-i}, F_{-i}\right),$$

then $a_i \in RS_i(\theta_i)$*, that is,* a_i *is a rationalizable action for player* i *with valuation* θ_i*.*

I will now provide another simple condition which is sufficient for an action to be rationalizable and therefore facilitates the derivation of maximin strategies. In order to do so, the following definition is needed.

Definition 32. *For a game under distributional and strategic uncertainty a profile of strategies* $(\beta_1, \ldots, \beta_I) \in \Delta A_1 \times \cdots \times \Delta A_I$ *together with a profile of subjective beliefs about the other players' type distributions* $\left(F_{-1}^1, \ldots, F_{-I}^I\right) \in \Delta_{\Theta_{-1}} \times \cdots \times \Delta_{\Theta_{-I}}$ *is called* subjective-belief equilibrium with given strategies *if every player acts optimally given her belief and the other players' strategies, i.e. for every* $i \in \{1, \ldots, I\}$*, every* $\theta_i \in \Theta_i$ *and every* $a_i \in supp\left(\beta_i(\theta_i)\right)$ *it holds that*

$$a_i \in \underset{a_i' \in A_i}{\operatorname{argmax}} \; U_i\left(\theta_i, a_i', \beta_{-i}, F_{-i}^i\right).$$

That is, in a subjective-belief equilibrium players best reply to each other's strategies but do not know each other's type distributions. Every player forms a subjective belief about the other players' type distributions and acts optimally given this subjective belief and the other players' strategies. An example for a subjective-belief equilibrium is a Bayes-Nash equilibrium with a common prior.

Example 11. *Let the strategy profile $(\beta_1, \dots \beta_I)$ together with the profile of beliefs $\left(\hat{F}_{-1}, \dots, \hat{F}_{-I}\right)$ be a Bayes-Nash equilibrium with a common prior. Then $(\beta_1, \dots \beta_I)$ together with $\left(\hat{F}_{-1}, \dots, \hat{F}_{-I}\right)$ constitutes a subjective-belief equilibrium.*

The following proposition states that a strategy which is played in a subjective-belief equilibrium is rationalizable.

Proposition 20. *In a game under distributional and strategic uncertainty an action $a_i \in A_i$ is rationalizable for a player i with valuation θ_i if there exists a subjective-belief equilibrium with strategies $(\beta_1, \dots, \beta_I)$ such that $a_i \in supp\left(\beta_i\left(\theta_i\right)\right)$.*

Proof. Let $(\beta_1, \dots, \beta_I)$ together with $\left(F^1_{-1}, \dots, F^I_{-I}\right)$ be a subjective-belief equilibrium. Let i be a player with valuation θ_i and a_i be an action such that $a_i \in supp\left(\beta_i\left(\theta_i\right)\right)$. It is to show that $a_i \in RS_i(\theta_i)$. I show by induction with respect to k that for every $j \in \{1, \dots, I\}$, for every $k \geq 1$ and for all $\theta_j \in \Theta_j$ it holds that

$$a_j \in supp\left(\beta_j\left(\theta_j\right)\right) \Rightarrow a_j \in RS^k_j\left(\theta_j\right).$$

Then it follows that $a_j \in RS_j\left(\theta_j\right)$ and one can conclude that $a_i \in RS_i\left(\theta_i\right)$ because $a_i \in supp\left(\beta_i\left(\theta_i\right)\right)$. It holds for all $j \in \{1, \dots, I\}$ that

$$a_j \in supp\left(\beta_j\left(\theta_j\right)\right) \Rightarrow a_j \in RS^1_j\left(\theta_j\right) \quad \text{for all } \theta_j \in \Theta_j$$

since $RS^1_j\left(\theta_j\right) = A_j$ by definition. Assume it is already shown that for all $j \in \{1, \dots, I\}$ it holds that

$$a_i \in supp\left(\beta_j\left(\theta_j\right)\right) \Rightarrow a_j \in RS^k_{\theta_j} \quad \text{for all } \theta_j \in \Theta_j.$$

Let j be some player with type θ_j and subjective belief $F^j_{-j} = \left(F^j_1, \dots, F^j_{j-1}, F^j_{j+1}, \dots, F^j_I\right)$. Then F^j_{-j} and β_{-j} fulfill the properties

(i) $a_l \in supp\left(\beta_l\left(\theta_l\right)\right) \Rightarrow a_l \in {RS_l}^k\left(\theta_l\right)$ for all $l \neq j$

(ii) $a_j \in supp\left(\beta_j\left(\theta_j\right)\right) \Rightarrow a_j \in \underset{a'_j \in A_j}{\operatorname{argmax}}\ U_j\left(\theta_j, a'_j, \beta_{-j}, F^j_{-j}\right)$.

The first property follows from the induction hypothesis and the second property follows from the definition of a subjective-belief equilibrium with given strategies. By definition of a rationalizable action, it follows that $\beta_j\left(\theta_j\right) \in RS^{k+1}_j$. Hence, it is shown that $a_j \in supp\left(\beta_j\left(\theta_j\right)\right) \Rightarrow a_j \in RS_j\left(\theta_j\right)$. ☐

The analogous result as in Proposition 13 holds also for games under distributional and strategic uncertainty. That is, every strategy played in a Bayes-Nash equilibrium is rationalizable and every action which is a best reply to a Bayes-Nash equilibrium is rationalizable.

Proposition 21. *Let the profile of strategies* $(\beta_1, \ldots, \beta_I)$ *together with the profile of type distributions* $(F_1, \ldots, F_I)$ *constitute a Bayes-Nash equilibrium with a common prior of a game of incomplete information. Then the following holds true:*

(i) For every $i \in \{1, \ldots, I\}$ *the strategy* β_i *is rationalizable.*

(ii) Let $i \in \{1, \ldots I\}$ *be a player with valuation* θ_i *and let* $a_i \in A_i$ *be a best reply to* β_{-i} *and some distribution of the other players' types* $F'_{-i} \in \Delta_{\Theta_{-i}}$*, i.e. it holds that*

$$a_i \in \underset{a'_i \in A_i}{\operatorname{argmax}} \, U_i\left(\theta_i, a'_i, \beta_{-i}, F'_{-i}\right),$$

then $a_i \in RS_i(\theta_i)$*, that is,* a_i *is a rationalizable action for player* i *with valuation* θ_i*.*

It can be proved as a direct result of Proposition 20.

Proof. As stated in Example 11, every Bayes-Nash equilibrium is a subjective-belief equilibrium. Due to Proposition 20, every strategy played in a subjective-belief equilibrium is rationalizable. Hence, every strategy played in a Bayes-Nash equilibrium is rationalizable which proves the first part. Corollary 7 states that best replies to rationalizable strategies are rationalizable. Therefore, a best reply to a strategy which is played in a Bayes-Nash equilibrium is rationalizable which shows the second part. □

If $\{i_1, \ldots, i_k\}$ is the set of players applying the maximin expected utility criterion, then as under pure strategic uncertainty, the game against an adverse nature can be seen as k independent two-player zero-sum games. This allows for the application of all results for two-player zero-sum games including the existence result for Nash equilibria.

After presenting the formal model, I turn to the application to first-price auctions under distributional and strategic uncertainty.

4.5.3 First-price auctions under distributional and strategic uncertainty: Model

The underlying game of incomplete information is the same as for first-price auctions under strategic uncertainty in subsection 4.4.1. What differs is the set of distributions a bidder applying the maximin expected utility criterion considers to be possible. Before, the valuation distribution was common knowledge. Now I assume that the set of possible valuations, i.e. the support of the valuation distribution, and the mean of the valuation distribution is common knowledge.

Possible distributions It is common knowledge that every bidder's valuation is drawn from the set $\Theta = \left\{0 = \theta^1, \theta^2, \ldots, \theta^{m-1}, 1 = \theta^m\right\}$ according to a distribution with an exogenously given mean μ. Formally, let

$$\mathcal{F}_{\mu}^{I-1} = \left\{F_1 \times \cdots \times F_{I-1} \in \Delta^{I-1}(\Theta) \;\middle|\; \sum_{i=1}^{m} \theta_i Pr\left(\theta^i\right) = \mu\right\},$$

the set of all distributions of independently drawn valuations for $I-1$ bidders with mean μ. Then it holds for every $i \in \{1, \ldots, I\}$ that

$$\Delta_{\Theta_{-i}} = \mathcal{F}_{\mu}^{I-1}.$$

As argued before, the above defined game of incomplete information uniquely defines a game under distributional strategic uncertainty (after specifying the player applying the maximin expected utility criterion). In the following I will call this game *first-price auction under distributional and strategic uncertainty.*

4.5.4 First-price auctions under distributional and strategic uncertainty: Results

Preview or results If in a first-price auction under distributional and strategic uncertainty there exist types $\theta^k, \theta^{k'}, \theta^{k''} \in \Theta$ such that $0 < \theta^k \leq \mu < \theta^{k'} < \theta^{k''}$, then every outcome is inefficient.

For every type there exists a unique highest rationalizable bid. For every bidder and every type the adverse nature chooses as the strategy of the other bidders that every bidder places the highest rationalizable bid given her type.

Let θ_μ be the lowest valuation which is higher than the mean. The highest rationalizable bid of a bidder with a valuation lower than θ_μ is her valuation. The subjective maximin belief of a bidder with valuation lower than θ_μ about the other bidders' valuation distributions is that the probability weight is distributed between her own valuation and θ_μ. As a consequence a bidder with a valuation lower than μ expects a utility of zero and is indifferent between any bid between zero and her valuation.

A bidder applying the maximin expected utility criterion with a valuation θ^k such that $\theta^k \geq \theta_\mu$ never expects to win against a bidder with the same valuation. Hence, the subjective maximin belief of the bidder about the other bidders' valuation distribution maximizes the probability weight on θ^k and makes the bidder indifferent between any highest rationalizable bid of lower types. As a consequence, the bidder mixes among all highest rationalizable bids of lower types. Therefore, if types $\theta^k, \theta^{k'}, \theta^{k''} \in \Theta$ such that $0 < \theta^k \leq \mu < \theta^{k'} < \theta^{k''}$ exist, then with positive probability type $\theta^{k''}$ bids zero and type $\theta^{k'}$ bids the highest rationalizable bid of type θ^k which is θ^k. Conclusively, the outcome is not efficient.

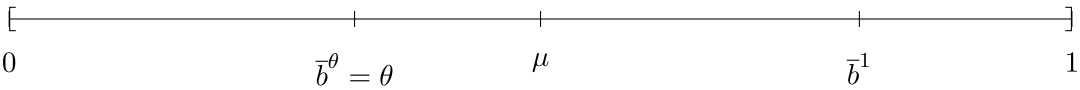
Example 12. *Consider a first-price auction under distributional and strategic uncertainty with two bidders 1 and 2 and three possible valuations $0, \theta$ and 1 which are identically and independently distributed with a commonly known mean μ. Assume that it holds $\theta < \mu$. The first step is to calculate the highest rationalizable bid for every valuation.*

The highest rationalizable bid of a bidder with valuation zero is zero. Assume that bidder 1 and bidder 2 have the subjective belief that the other bidder's valuation distribution distributes the probability weight between types θ and 1, i.e. there is zero probability weight on type 0. Given that bidder 1 and bidder 2 have this subjective belief, the following strategies constitute a Bayes-Nash equilibrium:

(i) Type θ of bidder 1 and bidder 2 bids θ.

(ii) Type 1 of bidder 1 and bidder 2 plays a mixed strategy on the interval $[\theta, b^1]$ for $\theta < b^1 < 1$.

Thus, it is part of a subjective-belief equilibrium that a bidder with valuation θ bids θ. It follows from Proposition 20 that bidding θ is a rationalizable action for a bidder with valuation θ. Since bidding above valuation cannot be rationalizable, the highest rationalizable action of a bidder with valuation θ is to bid θ.



Let $\overline{b}^1$ denote the highest rationalizable bid of a bidder with valuation 1. In order to compute $\overline{b}^1$, consider the conjecture of a bidder with valuation 1 that the strategy of the other bidder is such that

(iii) Type zero bids zero,

(iv) Type θ bids θ,

(vi) Type 1 bids $\left(\overline{b}^1\right)^-$.

It has been already shown that (iv) is rationalizable and similarly as in the case of pure strategic uncertainty, one can show that (vi) is rationalizable. It follows from Corollary 7 that a best reply to the strategy described in $(iii) - (vi)$ is rationalizable. Thus, this is the rationalizable strategy which maximizes the expected utility of bidding $\overline{b}^1$ and therefore induces the highest rationalizable bid of a bidder with valuation 1, i.e. bidding $\overline{b}^1$ is a best reply to this strategy.

A rationalizable bid is a best reply to a strategy of the other bidders and to a distribution of their valuations. Hence, in addition to the strategy inducing $\overline{b}^1$, one has to derive the *valuation distribution inducing* $\overline{b}^1$*. Let* $\left(f_0^1, f_\theta^1, f_1^1\right)$ *denote the corresponding probability mass function. It must hold that*

$$1 - \overline{b}^1 \geq f_0^1$$

$$1 - \overline{b}^1 \geq \left(f_0^1 + f_\theta^1\right)(1-\theta)$$

which is equivalent to

$$1 - \overline{b}^1 \geq \max\left\{f_0^1, \left(f_0^1 + f_\theta^1\right)(1-\theta)\right\}.$$

Since $\overline{b}^1$ *is the highest bid for which this condition is fulfilled, it holds that*

$$\overline{b}^1 = 1 - \min\max\left\{f_0^1, \left(f_0^1 + f_\theta^1\right)(1-\theta)\right\}$$

which is equivalent to

$$\overline{b}^1 = 1 - f_0^1 = 1 - \left(f_0^1 + f_\theta^1\right)(1-\theta). \tag{81}$$

Since probabilities have to add up to zero and the mean has to be preserved, the vector $\left(f_0^1, f_\theta^1, f_1^1\right)$ *is the unique solution to the following system of linear equations*

$$f_0^1 + f_\theta^1 + f_1^1 = 1$$

$$f_0^1 0 + f_\theta^1 \theta + f_1^1 1 = \mu$$

$$f_1^0 = \left(f_1^0 + f_1^\theta\right)(1-\theta).$$

After obtaining the solution

$$f_0^1 = \frac{1-\mu}{1+\theta}, \quad f_\theta^1 = \frac{\theta(1-\mu)}{1-\theta^2}, \quad f_1^1 = \frac{\mu - \theta^2}{1-\theta^2},$$

one can compute $\overline{b}^1$ *using equation (81), i.e. it holds*

$$\overline{b}^1 = 1 - f_1^0 = 1 - \left(f_1^0 + f_1^\theta\right)(1-\theta) = \frac{\mu}{1+\theta}.$$

After deriving the highest rationalizable bids for every type, the second step is to derive the adverse nature's strategy. In the setting of strategic and distributional uncertainty the adverse nature's strategy determines for every bidder and every type a strategy and a valuation distribution of the other bidder. As in the case of pure strategic uncertainty, for every bidder and every type the adverse nature chooses as the strategy of the other bidder to place the highest rationalizable bid given her valuation.[54]

[54] As in the case of pure strategic uncertainty, the strategy of the adverse nature is not necessarily unique in a maximin equilibrium but in every equilibrium the strategies of the bidders coincide with the best reply to the strategy of the adverse nature as described.

The subjective maximin belief of a bidder with valuation zero is irrelevant since such a bidder always earns a utility of zero. For a bidder with valuation θ the adverse nature chooses a distribution of the other bidder's valuations which puts zero probability weight on type zero. Since type θ bids θ, this induces an expected utility of zero for a bidder with valuation θ. A bidder with valuation 1 never expects to win against a bidder with valuation 1. Therefore, a bidder with valuation 1 has to decide between bidding zero and bidding θ. Hence, the adverse nature has to choose a valuation distribution $\left(\tilde{f}_1^0, \tilde{f}_1^\theta, \tilde{f}_1^1\right)$ such that it holds

$$\min \max \left\{ \tilde{f}_1^0, \left(\tilde{f}_1^0 + \tilde{f}_1^\theta\right)(1-\theta) \right\}.$$

Since probabilities have to add up to one and the mean has to be preserved, the vector $\left(\tilde{f}_1^0, \tilde{f}_1^\theta, \tilde{f}_1^1\right)$ is the unique solution of the same system of linear equations as the vector $\left(f_1^0, f_1^\theta, f_1^1\right)$. Therefore, it holds that

$$\left(\tilde{f}_1^0, \tilde{f}_1^\theta, \tilde{f}_1^1\right) = \left(f_1^0, f_1^\theta, f_1^1\right).$$

In the final step, for every bidder and every type one has to find the set of best replies to the adverse nature's strategy. Moreover, one has to identify the best replies such that the adverse nature does not have an incentive to deviate from her strategy derived in the second step. Since the expected utility of a bidder does not decrease if one of the other bidders places a lower bid, the adverse nature does not have an incentive to deviate from the strategy where for every bidder and every type she prescribes the highest rationalizable bid.[55] *Hence, it is sufficient to check whether the adverse nature has an incentive to deviate from the chosen distributions.*

A bidder with valuation zero bids zero. A bidder with valuation θ expects a utility of zero and is indifferent between any bid in the interval $[0, \theta]$. Hence, the adverse nature does not have an incentive to deviate. A bidder with valuation 1 is indifferent between bidding 0 and θ. In a maximin equilibrium in the game against the adverse nature, a bidder with valuation 1 mixes between 0 and θ in a way such that the adverse nature is indifferent among any valuation distribution which fulfills the constraints that probabilities add up to one and the mean μ is preserved. Therefore, the adverse nature does not have an incentive to deviate.

Note that the distribution of the other bidder's valuations which the adverse nature chooses for a type is the same distribution which induces the highest rationalizable bid for this type. That is, a bidder i with a given type assumes that her opponent j has the same assumption about i's valuation distribution as i's assumption about j's valuation distribution. But bidder i assumes that j has a different belief about i's strategy than i's

[55] An exception is that if one bidder bids above her valuation, it would be a best reply of the adverse nature to choose as the strategy of the other bidders that every other bidder bids zero. This would induce a strictly negative utility for the bidder bidding above her valuation. However, one can exclude this exception in a maximin equilibrium.

belief about j's strategy.

The insights from the example about bidders' strategies are generalized in the following Proposition.

Proposition 22. *Consider a first-price auction under distributional and strategic uncertainty. There exists an outcome under maximin strategies. If there exist types $\theta^k, \theta^{k'}, \theta^{k''} \in \Theta$ such that $0 < \theta^k \leq \mu < \theta^{k'} < \theta^{k''}$, then there does not exist an efficient outcome.*

The proof is relegated to Appendix 4.7.6.

The inefficiency stems from the fact that every type above μ mixes between all highest rationalizable bids of all lower types. With positive probability type $\theta^{k''}$ bids zero and type $\theta^{k'}$ bids the highest rationalizable bid of type θ^k which is θ^k. Conclusively, the outcome is not efficient.

Similarly as under pure strategic uncertainty, I will show the existence of an outcome under maximin strategies by construction. The following three steps serve as a preparation for the calculation of the highest rationalizable bids.

(I) Show that for every type $\theta^k \in \Theta$ there exists a unique highest rationalizable bid $\overline{b}^{\theta^k}$.

(II) Show that for every type zero is a rationalizable bid.

(III) Show that for every type $\theta^k \in \Theta$ every bid in the interval $[0, \overline{b}^{\theta^k}]$ is rationalizable.

The explanation for steps (I)-(III) works analogously as for steps (I)-(III) in the case of pure strategic uncertainty. For the calculation of the highest rationalizable bids, first, consider valuations equal or below μ. Analogously as in the example, on can show that the highest rationalizable bid of a bidder with valuation θ^k such that $\theta^k \leq \mu$ is θ^k. This bid is induced by the subjective belief equilibrium where the probability weight is distributed between types θ^k and θ_μ and all bidders with valuation θ^k bid θ^k, where θ_μ is the smallest valuation strictly higher than μ.

The calculation of the highest rationalizable bids for higher types works by recursion. Assume that for a bidder i with valuation $\theta^k \geq \theta_\mu$ and that for all $j < k$ the highest rationalizable bids has been already computed. The highest rationalizable bid $\overline{b}^{\theta^k}$ of a bidder with valuation θ^k is a best reply to a conjecture about the other bidders' strategies and distributions.

The strategies which induce $\overline{b}^{\theta^k}$ are given by

(i) Every bidder with valuation θ^j such that $\theta^j < \theta^k$ bids her highest rationalizable bid.

(ii) Every bidder with valuation θ^k bids $\left(\overline{b}^{\theta^k}\right)^-$.

The valuation distribution of the other bidders which induces $\overline{b}^{\theta^k}$ has to minimize the incentive to bid another bid. In addition, probabilities have to add up to zero and the mean μ has to be preserved. Let $\left(f_{\theta^1}^{\theta^k}, \ldots, f_{\theta^m}^{\theta^k}\right)$ be a vector of probabilities such that according to the valuation distribution inducing $\overline{b}^{\theta^k}$, type θ^l of some bidder $j \neq i$ occurs with probability $f_{\theta^l}^{\theta^k}$.

Hence, the vector $\left(f_{\theta^1}^{\theta^k}, \ldots, f_{\theta^m}^{\theta^k}\right)$ is the solution to the following minimization problem

$$\min \max \left\{ \left(f_{\theta^1}^{\theta^k}\right)^{I-1} \theta^k, \left(f_{\theta^1}^{\theta^k} + f_{\theta^2}^{\theta^k}\right)^{I-1} \left(\theta^k - \overline{b}^{\theta^2}\right), ..., \left(f_{\theta^1}^{\theta^k} + \cdots + f_{\theta^{k-1}}^{\theta^k}\right)^{I-1} \left(\theta^k - \overline{b}^{\theta^{k-1}}\right) \right\}$$

$$s.t. \;\; f_{\theta^1}^{\theta^k} + \cdots + f_{\theta^m}^{\theta^k} = 1$$

$$f_{\theta^1}^{\theta^k} \theta^1 + \cdots + f_{\theta^m}^{\theta^k} \theta^m = \mu.$$

As proved in the Appendix, in the solution of this minimization problem all terms of the form

$$\left(\sum_{i=1}^{j} f_{\theta^i}^{\theta^k}\right)^{I-1} \left(\theta^k - \overline{b}^{\theta^j}\right) \quad \text{for } 1 \leq j < k$$

have to be equal.

The recursive calculation of the highest rationalizable bids and the distributions inducing them, is formalized in the following Proposition.

Proposition 23. *Consider a first-price auction under distributional and strategic uncertainty. For $\theta^k \leq \mu$ the highest rationalizable bid $\overline{b}^{\theta}$ is equal to θ^k.*

Assume that for all $j < k$, the highest rationalizable bid $\overline{b}^{\theta^j}$ has been already defined and it holds $\theta^k > \mu$. Then for the vector $f^{\theta^k} = \left(f_{\theta^1}^{\theta^k}, \ldots, f_{\theta^m}^{\theta^k}\right)$ it holds that $f_{\theta^j}^{\theta^k} = 0$ for $j > k$ and the vector $\left(f_{\theta^1}^{\theta^k}, \ldots, f_{\theta^k}^{\theta^k}\right)$ is the unique solution of the following system of k linear equations given by

$$\sum_{i=1}^{k} f_{\theta^i}^{\theta^k} = 1$$

$$\sum_{i=1}^{k} f_{\theta^i}^{\theta^k} \theta^i = \mu$$

$$\left(f_{\theta^1}^{\theta^k}\right)^{I-1} \theta^k = \left(\sum_{i=1}^{j} f_{\theta^i}^{\theta^k}\right)^{I-1} \left(\theta^k - \overline{b}^{\theta^j}\right) \quad \textit{for } 1 < j < k.$$

The highest rationalizable bid $\overline{b}^{\theta^k}$ is obtained by the equation

$$\overline{b}^{\theta^k} = \theta^k - \left(f_{\theta^1}^{\theta^k}\right)^{I-1} \theta^k.$$

The proof is relegated to Appendix 4.7.6. After calculating the highest rationalizable bid for every type, one can specify the strategies played in an outcome under maximin strategies.

Proposition 24. *In a first-price auction under distributional and strategic uncertainty it holds for every outcome under maximin strategies and for every bidder i and every valuation θ^k that*

(i) Every bidder with valuation θ^k such that $\theta^k \leq \mu$ is indifferent between any bid in the interval $[0, \theta^k]$.

(ii) Every bidder with valuation θ^k such that $\theta^k > \mu$ mixes among the bids $\{\bar{b}^{\theta^j} \mid j < k\}$, that is, among the set of all highest rationalizable bids of lower types.

The proof is relegated to Appendix 4.7.6.

4.6 Conclusion

I conclude by first providing a short summary and afterwards discussing the assumptions made in this paper as well as possible extensions.

4.6.1 Summary

I propose a new decision criterion for players who face strategic uncertainty in games of incomplete information. The decision criterion works in two steps. First, I assume common knowledge of rationality and eliminate all strategies which do not survive the iterated elimination of strategies which are not best replies. Second, I apply the maximin expected utility criterion. With this decision criterion one can derive recommendations for a player facing strategic uncertainty and analyze outcomes under the assumption that every player follows this decision criterion. Moreover, I provide an extension of the model to distributional and strategic uncertainty.

I apply this decision criterion to first-price auctions under pure strategic uncertainty and under both, distributional and strategic uncertainty. In both cases every bidder has the subjective belief that every other bidder places the highest rationalizable bid given her type. Therefore, a bidder applying the proposed decision criterion resorts to win against lower types with certainty by bidding highest rationalizable bids of lower types. Besides providing recommendations for bidders facing strategic oder distributional uncertainty in first-price auctions, I characterize all outcomes under the assumption that every bidder applies this criterion. Under pure strategic uncertainty every outcome is efficient. Under distributional and strategic uncertainty every outcome is inefficient (under a mild condition on the number and distribution of possible valuations).

4.6.2 Discussion

Choice of decision criterion

The decision criterion under uncertainty used in this paper is the maximin expected utility criterion. The analogous analysis could be conducted with other criteria such as the minimax expected regret criterion.

Possible distributions and strategies

In this paper I restricted the set of possible strategies by assuming common knowledge of rationality and the set of possible distributions by assuming common knowledge of a mean. This restriction is crucial for the application of the maximin expected utility criterion. Otherwise, in first-price auctions there would exist a distribution or strategy inducing an expected utility of zero for a player independent of her action.

For example, Bergemann and Schlag (2008) apply the maximin expected utility criterion to a monopoly pricing problem where a seller faces uncertainty about the buyer's valuation distribution. Without a restriction of the set of possible distributions the adverse nature would choose a distribution which puts the whole probability weight on valuation zero. Thus, they assume that the seller knows that the buyer's true valuation distribution is in the neighborhood of a model distribution. Other papers applying the maximin expected utility criterion to distributional uncertainty also assume an exogenously given restriction of the set of possible distributions.

Due to a similar reasoning, a restriction of the set of possible strategies is required if the maximin expected utility criterion is applied to strategic uncertainty. For example, if one would apply the maximin expected utility criterion to a first-price auction where a bidder faces uncertainty about the other bidders' strategies, then without a restriction of the set of possible strategies the adverse nature would choose a strategy of the other bidders such that all bidders place arbitrarily high bids. First, such strategies do not seem plausible. Second, the maximin expected utility criterion does not provide a useful recommendation. In order to solve these issues, one could also exogenously restrict the set of possible strategies. For example, Kasberger and Schlag (2017) apply the minimax regret criterion to first-price auctions and assume common knowledge of an exogenously given restriction of the players' bidding strategies, for instance, in form of a lower bound of the highest bid.

However, the fact that rational agents interact strategically in a given economic setting already contains information about the possible strategies. Thus, it is possible to use an endogenous restriction of the set of possible strategies - which is given by the set of rationalizable strategies - in order to apply the maximin expected utility criterion. The model can be easily extended in a way which allows for additional (exogenously given) knowledge about possible strategies.

Under distributional uncertainty an exogenously given restriction of the set of possible distributions is still necessary. Besides fixing the mean, there exist other possibilities to restrict the set of possible distributions and strategies. For instance, one could investigate outcomes under distributional uncertainty under the assumption that further moment conditions of the type distribution are common knowledge.

Cognitive complexity

Formally, the derivation of the set of rationalizable actions for an agent with a given type requires an infinite intersection of sets. However, the proofs use a finite number of recursion steps. In the model under strategic uncertainty and in the model under distributional uncertainty the bid of a bidder with type θ^k is obtained after at most k recursion steps. One could argue that a sufficiently rational player can conduct the necessary calculations. But one could also argue that for some players these calculations may be too difficult. Therefore, similarly as in level-k models, one could define the concept of k-rationalizability. That is, a player i could know that her opponent can compute the set RS_j^k for all players j and for $k \in \mathbb{N}$, but cannot compute the sets $RS_j^{k'}$ for $k' > k$ (see Bernheim (1984)). Depending on the parameters, this knowledge can influence player i's maximin strategy.

Robustness

In addition to the maximin expected utility criterion, one could introduce an additional robustness criterion in the following sense: Does the maximin strategy of an agent change if the adverse nature deviates from her strategy to another strategy in an ϵ-neighborhood? If there is a change, does the strategy and the resulting expected utility change continuously?

As an example, consider a first-price auction under pure strategic uncertainty with a commonly known distribution function, two bidders and three valuations 0, θ and 1. Bidder 1 with valuation 1 has the subjective maximin belief that bidder 2 with valuation 1 bids $\bar{b}^1$. Hence, bidder 1 with valuation 1 bids either $\bar{b}_\theta$ or zero. However, all bids in the interval $[0, \bar{b}^1]$ are rationalizable for a bidder with valuation 1. Hence, (if the bid grid is sufficiently fince) an ϵ-neighborhood of $\bar{b}^1$ and its intersection with the set of rationalizable actions contains bids lower than $\bar{b}^1$. If bidder 1 with valuation 1 has the subjective belief that bidder 2 with valuation 1 bids lower than $\bar{b}^1$, e.g. $\left(\bar{b}^1\right)^-$, then $\bar{b}^1$ becomes a best reply for bidder 1 with valuation 1. This constitutes a discontinuity in her best reply.

As a second example, consider a first-price auction under pure strategic uncertainty with two bidders and a commonly known common valuation v. To bid v is the highest rationalizable action for both bidders. Therefore, bidder 1 has the subjective maximin belief that bidder 2 bids v. As a consequence, bidder 1 is indifferent between any bid in $[0, v]$. Assume that bidder 1 chooses the action v (or v^-). As any other bid, this leads to a utility of zero given the subjective maximin belief that bidder 2 bids v. An ϵ-neighborhood of v and its intersection with the set of rationalizable actions contains only bids below v, e.g. it contains the bids v,v^- and $(v^-)^-$. The best replies to these bids are in an ϵ-neighborhood of v (or v^-) and the induced utilities are in an ϵ-neighborhood of zero. Hence, bidding v (or v^-) fulfills the robustness property that an ϵ-deviation of the subjective maximin belief induces an ϵ-deviation of the best replies and expected utility.

4.7 Appendix

4.7.1 Proof of Proposition 12 and 19

Proof. Proposition 12 is a special case of Proposition 19 such that for every player i it holds that $\Delta_{\Theta_{-i}} = F_{-i}$ where F is the commonly known valuation distribution as assumed in Proposition 12. Therefore, it is sufficient to prove Proposition 19. Every player maximizes her expected utility given a distribution of the other players' types and a rationalizable strategy of the other players chosen by nature. Let $(\beta_1, \dots, \beta_I, \beta_n)$ be an outcome under maximin strategies. It is to show that for every player i and for every type θ_i an action a_i which is in the support of $\beta_i(\theta_i)$ is an element in $RS_i^k(\theta_i)$ for every $k \geq 1$. The proof works by induction. It is true that for every $i \in \{1, \dots, I\}$ every action $a_i \in A_i$ is an element in $RS_i^1(\theta_i)$ since it holds by definition that $RS_i^1(\theta_i) = A_i$. Assume it is already shown for every $i \in \{1, \dots, I\}$ and every $\theta_i \in \Theta_i$ that every action a_i with $a_i \in supp(\beta_i(\theta_i))$ is an element in $RS_i^k(\theta_i)$. Let i bid a bidder with valuation θ_i. Since n can choose only among rationalizable strategies, it holds for every $j \neq i$ that $\beta_j^{n_i,\theta_i}$ is a rationalizable strategy. By definition, this implies that for every $\theta_j \in \Theta_j$ and every action a_j with $a_j \in supp\left(\beta_j^{n_i,\theta_i}(\theta_j)\right)$ it holds that $a_j \in \bigcap_{k\geq 1} RS_j^k(\theta_j)$. It follows that

(i) $a_j \in supp\left(\beta_j^{n_i,\theta_i}(\theta_j)\right)$ for $\theta_j \in \Theta_j \Rightarrow a_j \in {RS_j}^k(\theta_j)$ for all $j \neq i$.

By definition of an outcome under maximin strategies, it holds for every action a_i with $a_i \in supp(\beta_i(\theta_i))$ that a_i is a best reply given the adverse nature's strategy, i.e. a best reply to the other bidders' valuation distribution and strategies chosen by the adverse nature. Therefore, it holds that

(ii) $a_i \in supp(\beta_i(\theta_i)) \Rightarrow a_i \in \underset{a_i' \in A_i}{\operatorname{argmax}} U_i\left(\theta_i, a_i', \beta_{-i}^{n_i,\theta_i}, F_{-i}^{n_i,\theta_i}\right)$.

By definition of the set $RS_i^{k+1}(\theta_i)$, it follows from (i) and (ii) that for every a_i with $a_i \in supp(\beta_i(\theta_i))$ is an element in $RS_i^{k+1}(\theta_i)$ and it follows by induction that a_i is an element in $RS_i^k(\theta_i)$ for every $k \geq 1$. □

4.7.2 Proof of Proposition 13

Since Proposition 13 is a special case of Proposition 21 such that for every player i it holds that $\Delta_{\Theta_{-i}} = F_{-i}$, where F is the commonly known valuation distribution as assumed in Proposition 13, the proof follows from the proof of Proposition 21.

4.7.3 Proof of Proposition 14

Proof. (i) At first, I consider the case where there exists a unique bidder k who has the highest valuation and show that her highest rationalizable bid is the second-highest valuation, denoted by θ_k'. In order to do so, I will show by induction that for every bidder $i \neq k$ the bids in the interval $(\theta_{k'}, 1]$ are not rationalizable. Let i be an arbitrary bidder which is not bidder k. Hence, bidder i's valuation is strictly lower than 1. The induction steps are descending and start with 1. Since 1 is the highest possible bid, bidder i wins with strictly positive probability if she bids 1 which cannot be rationalizable since she would earn a negative utility with positive probability. For the induction step assume that it has been shown that all bids equal or higher than b with $b \in (\theta_{k'}, 1]$ are not rationalizable for all bidders $i \neq k$. It is to show that for an arbitrary bidder $i \neq k$ the bid b^- is not rationalizable if $b^- > \theta_{k'}$. Since all bids strictly higher than b^- are not rationalizable for all bidders besides bidder k, it is also never a best reply for bidder k to bid strictly higher than b^-. Therefore, bidder i wins with strictly positive probability if she bids b^-. Since b^- is strictly higher than her valuation, this cannot be optimal. This completes the induction step from which follows that for all bidders $i \neq k$ the bids in the interval $(\theta_{k'}, 1]$ are not rationalizable. It follows that for bidder k all bids in the interval $(\theta_{k'}, 1]$ are not rationalizable. In every Nash equilibrium the highest bidder bids the second-highest valuation $\theta_{k'}$. Since according to part (i) of Proposition 13 a strategy played in a Bayes-Nash equilibrium is rationalizable, the bid $\theta_{k'}$ is rationalizable for bidder k. It follows that $\theta_{k'}$ is the highest rationalizable bid of bidder k.

If the adverse nature chooses for all bidders $i \neq k$ as the action of bidder k to bid θ'_k , i.e. $\beta_k^{n_i,\theta_i}(\theta_k) = \theta_{k'}$, every bidder $i \neq k$ with valuation θ_i expects a utility of zero independent of her action. Therefore, any other strategy of the adverse nature which is played in a maximin equilibrium, has to induce an expected utility of zero for every bidder $i \neq k$. That is, the subjective maximin belief of a bidder $i \neq k$ with valuation θ_i is that at least one other bidder bids equal or higher than θ_i. As a result, every bidder $i \neq k$ is indifferent between all bids in the interval $[0, \theta_i]$. It is left to show that a bidder $i \neq k$ does not bid above her valuation. Assume there exists a bidder i with valuation θ_i who bids $b > \theta_i$. Since for all bidders $j \neq k$ bidding zero is rationalizable, it is rationalizable for bidder k to bid zero. Given that bidder i bids b, the adverse nature chooses as the strategy of the other bidders to bid zero, i.e. for every $j \neq i$ it holds that $\beta_j^{n_i,\theta_i}(\theta_j) = 0$. As a result, bidder i wins with probability 1 and expects a negative utility which cannot be part of a maximin equilibrium. Hence, none of the bidders places bids strictly higher than her valuation in a maximin equilibrium.

In order to minimize the expected utility of bidder k, the adverse nature chooses as the strategy of the second-highest bidder, i.e. bidder k' with valuation $\theta_{k'}$, to bid her valuation i.e. $\beta_{k'}^{n_k,\theta_k}(\theta_{k'}) = \theta_{k'}$. This is the highest rationalizable bid which can be placed by a bidder who is not bidder k. As a consequence, bidder k bids $\theta_{k'}$.

(ii) Finally, I consider the case where at least two bidders have the highest valuation θ_k. Analogously as before, one can show by induction that for every bidder the bids in the interval $(\theta_k, 1]$ are not rationalizable. In every Nash equilibrium every highest bidder bids her valuation θ_k. Therefore, it holds due to Corollary 13 that the bid θ_k is rationalizable for every highest bidder. It follows that θ_k is the highest rationalizable bid and therefore is the action which the adverse nature chooses as the action of a highest bidder k for a bidder $i \neq k$, i.e. $\beta_k^{n_i,\theta_i}(\theta_k) = \theta_k$. This implies that every bidder does not expect to earn a positive utility and therefore is indifferent between any bid between zero and her valuation. Bids strictly higher than the own valuation can be excluded analogously as above. $\square$

4.7.4 Proof of Propositions 15, 16 and 17

In order to prove Propositions 15 and 16, I will show the following Lemmas which formalize steps (I) -(III).

Lemma 29. *For every bidder i and every valuation $\theta^k \in \Theta$ there exists a unique highest rationalizable bid $\bar{b}^{\theta^k}$. This bid does not depend on the identity of bidder i.*

Proof. For every bidder i and every valuation θ_i the set of rationalizable actions $RS_i(\theta_i)$ is a finite set in a metric space and therefore compact. Since every compact set contains its supremum, there exists a maximum element of the set $RS_i(\theta_i)$. Since this is a subset of $\mathscr{B}$ and by definition, $\mathscr{B}$ is well-ordered with respect to the relation $\leq$, the maximum element of $RS_i(\theta_i)$ has to be unique. Due to the symmetry of the bidders, the highest rationalizable bid does not depend on the identity of the bidder. $\square$

Lemma 30. *For every type $\theta^k \in \Theta$ zero is a rationalizable bid.*

Proof. The proof works by induction with respect to the types in Θ. The induction starts with $\theta^1 = 0$. Montiero (2009) shows that with a given commonly known distribution there exists a Bayes-Nash equilibrium in the first-price auction with discrete valuations where type zero bids zero. It follows from part (i) of Corollary 13 that zero is a rationalizable action for type zero.

For the induction step assume that it has been already shown for all types θ^j with $j \leq k$ that zero is a rationalizable action for type θ^j. Consider a bidder i with valuation θ^{k+1} who conjectures that all other bidders with type θ^j such that $j \leq k$ bid zero which is rationalizable by assumption. According to Lemma 29, for every bidder and every type there exists a highest rationalizable bid. Let the conjecture of bidder i with valuation θ^{k+1} be such that every other bidder with type θ^j such that $j > k$ bids her highest rationalizable bid. Since all types with valuation θ^j such that $j > k$ bid at least the highest rationalizable bid of type θ^{k+1} and all other types bid zero, it is a best reply of bidder i with valuation θ^{k+1} to bid zero. As stated in Corollary 6, a best reply to a rationalizable strategy profile is rationalizable and therefore zero is a rationalizable action for bidder i with type θ^{k+1}. This completes the induction step and hence one can conclude that for every bidder and every type zero is a rationalizable action. $\square$

Lemma 31. *For every type $\theta^k \in \Theta$ every bid in $[0, \bar{b}^{\theta^k}]$ is rationalizable.*

Proof. The proof works by showing a formally stronger statement by induction with respect to the types in Θ. The statement is that for every type $\theta^k \in \Theta$ it holds that every bid in the interval $[0, \bar{b}^{\theta^k}]$ is rationalizable for every type θ^j such that $j \geq k$.

The induction starts with $k = 1$, i.e. with $\theta^1 = 0$. The highest rationalizable bid for type θ^1 is zero and it follows from Lemma 30 that zero is a rationalizable bid for every type $\theta^j \geq \theta^1$.

For the induction step assume that it has been already shown that for all $l \leq k$ it holds that every bid in the interval $[0, \overline{b}^{\theta^l}]$ is rationalizable for every type θ^j such that $j \geq l$. By using induction with respect to the bids, I will show that the same statement holds for type θ^{k+1}. The induction starts with the bid zero. It follows from Lemma 30 that zero is rationalizable for every type. For the induction step assume that it has been already shown that every bid in the interval $[0, b]$ with $b < \overline{b}^{\theta^{k+1}}$ is rationalizable for every type θ^j with $j \geq k+1$. In order to show that b^+ is rationalizable for every type θ^j with $j \geq k+1$, consider a bidder i with valuation θ^j with $j \geq k+1$ and strategies of the other bidders such that for every other bidder it holds that

(i) Every type θ^h with $h < j$ bids her highest rationalizable bid.

(ii) Every type θ^h with $h \geq j$ bids b.

The strategies in (i) are rationalizable by definition and the strategies in (ii) are rationalizable by the assumption in the induction step (in the second induction with respect to the bids in the interval $[0, \overline{b}^{\theta^k}]$). Given this conjecture about the other bidders' strategies it is a best reply for bidder i with valuation θ^j to bid b^+. Any change in part (i) would imply that there exists a bidder with valuation θ^l such that $l < k+1$ who bids some bid $b^{\theta^l} < \overline{b}^{\theta^l}$ instead of $\overline{b}^{\theta^l}$ which does not increase the expected utility of bidding b^+. Any deviation from (ii) implies that there exists at least one bidder and a valuation θ^h with $h \geq j$ such that this bidder places either a lower or a higher bid than b. If the bid is lower, then the same reasoning as above applies. If a bidder with valuation θ^h deviates to a higher bid, then by bidding b^+ bidder i with valuation θ^j does not overbid type θ^l of the deviating bidder anymore which decreases bidder i's winning probability.

Conclusively, any conjecture deviating from the strategies in (i) and (ii) does not increase the expected utility of bidding b^+. Therefore, if bidding b^+ is not a best reply to the beliefs in (i) and (ii), then b is the highest rationalizable bid for type θ^j which is a contradiction to the assumption $b < \overline{b}^{\theta^k}$. This completes the induction step of the second induction. It follows that any bid in the interval $[0, \overline{b}^{\theta^{k+1}}]$ is rationalizable for every type θ^j with $j \geq k+1$. This completes the induction step of the first induction. Therefore, it has been shown that for every type $\theta^k \in \Theta$ it holds that every bid in the interval $[0, \overline{b}^{\theta^k}]$ is rationalizable for every type θ^j such that $j \geq k$. $\square$

After proving Lemmas 29-31, I continue with the proof of Proposition 16.

<u>Proof of Proposition 16</u>

Proof. Consider a bidder with valuation θ^k. As shown in the proof of Lemma 31, for every type the conjecture given by

(i) Every type θ^l with $l < k$ bids her highest rationalizable bid .

(ii) Every type θ^j with $j \geq k$ bids $\left(\overline{b}^{\theta^k}\right)^-$.

induces the highest rationalizable bid of a bidder with valuation θ^k , that is, the highest rationalizable bid $\overline{b}^{\theta^k}$ of a bidder with valuation θ^k is a best reply to the conjecture that all other bidders employ this strategy. Given this conjecture, the expected utility of a bidder with type $\theta^k \in \Theta$ who bids $\overline{b}^{\theta^k}$ is given by

$$\theta^k - \overline{b}^{\theta^k}.$$

This utility has to be higher than the utility induced by any other bid. A bid can be a best reply for a bidder if she just overbids some other bidder. Formally, a bid b can be best reply for a bidder with valuation θ^k only if there exists a bidder $j \neq i$ and a valuation $\theta^l < \theta^k$ such that bidder j with valuation θ^l bids b (or there exists a bidder j with valuation $\theta^l \geq \theta^k$ such that bidder j with valuation θ^l bids b^- or b). Hence, the only potential candidates for best replies besides $\overline{b}^{\theta^k}$ are bids $\overline{b}^{\theta^j}$ with $j < k$. Thus, equation (80) ensures that bidding $\overline{b}^{\theta^k}$ induces at least the same expected utility than any other bid which can be a best reply. □

Proof of Proposition 15

Proof. I show the existence of an efficient outcome under maximin strategies by construction and then show that bidders' strategies are equal in every outcome. Conclusively, there exists an outcome under maximin strategies and every outcome is efficient. The construction of an efficient outcome works as follows. According to Lemma 29 for every type there exists a unique highest rationalizable bid. For every type and every player the adverse nature chooses the other bidders' strategies such that every bidder places the highest rationalizable bid given her type, i.e. for pair of bidders i, j and for every pair of valuations θ_i, θ_j it holds that

$$\beta_j^{n_i,\theta_i}(\theta_j) = \overline{b}^{\theta_j}. \tag{82}$$

Let β_n denote this adverse nature's strategy. Independent of the bidders' strategies there does not exist another strategy of the adverse nature which induces a lower expected utility for any of the bidders.[56] Thus, the adverse nature does not have an incentive to deviate from this strategy. Every bidder plays a best reply given her type and the adverse nature's strategy. Due to the compactness of $\mathscr{B}$, such a best reply always exists. I will show that the outcome defined by these best replies is efficient.

[56] An exception is that if one bidder bids above her valuation, it would be a best reply of the adverse nature to choose as the strategy of the other bidders that every other bidder bids zero. This would induce a strictly negative utility for the bidder bidding above her valuation. However, one can exclude this exception in a maximin equilibrium.

For a bidder with valuation θ^k the best reply is given by the most profitable overbidding of a lower type, that is, by

$$\underset{\overline{b}^{\theta^j}<\theta^k}{\operatorname{argmax}}\ F\left(\theta^j\right)\left(\theta^k-\overline{b}^{\theta^j}\right).$$

Let $\overline{b}^{\theta^l}$ be a best reply of a bidder with valuation θ^k. Then it holds for all $j \in \{1,\ldots,k-1\}$ that

$$F\left(\theta^l\right)\left(\theta^k-\overline{b}^{\theta^l}\right) \geq F\left(\theta^j\right)\left(\theta^k-\overline{b}^{\theta^j}\right) \tag{83}$$

$$\Leftrightarrow \theta^k\left(F\left(\theta^l\right)-F\left(\theta^j\right)\right)-F\left(\theta^l\right)\overline{b}^{\theta^l}+F\left(\theta^j\right)\overline{b}^{\theta^j} \geq 0. \tag{84}$$

Since $F\left(\theta^l\right)-F\left(\theta^j\right) \geq 0$ for $1 \leq j < l$, it follows from (84) that for all $\theta^{k'}$ such that $\theta^{k'} > \theta^k$ and for all $1 \leq j < l$ it holds that

$$\theta^{k'}\left(F\left(\theta^l\right)-F\left(\theta^j\right)\right)-F\left(\theta^l\right)\overline{b}^{\theta^l}+F\left(\theta^j\right)\overline{b}^{\theta^j} \geq 0$$

$$\Leftrightarrow F\left(\theta^l\right)\left(\theta^{k'}-\overline{b}^{\theta^l}\right) \geq F\left(\theta^j\right)\left(\theta^{k'}-\overline{b}^{\theta^j}\right). \tag{85}$$

First, consider the case where for every $j \in \{1,\ldots,k-1\}$ the inequality in (83) is strict. Then for a bidder with valuation θ^k there exists a unique best reply, denoted by $\overline{b}^{\theta^l}$. Hence, in order to show efficiency, it is sufficient to show that every best reply of a bidder with valuation $\theta^{k'}$ with $\theta^{k'} > \theta^k$ is equal or greater than $\overline{b}^{\theta^l}$.

It holds for every $1 \leq j < l$ that the inequality in (85) is strict. It follows that none of the bids $\overline{b}^{\theta^j}$ for $1 \leq j < l$ can be a best reply for a bidder with valuation $\theta^{k'}$. Thus, a best reply of a bidder with valuation $\theta^{k'}$ with $\theta^{k'} > \theta^k$ is equal or greater than $\overline{b}^{\theta^l}$.

Second, consider the case where for at least one $j \in \{1,\ldots,k-1\}$ the expression in (83) holds with equality. Let $j_1,\ldots j_h$ be all indices for which it holds that the expression in (83) holds with equality where $\theta^{j_h} = \max\limits_{j\in\{j_1,\ldots,j_h\}} \theta^j$. That is, $\overline{b}_{\theta^h}$ is the highest best reply of a bidder with valuation θ^k. Then for all $j \in \{j_1,\ldots,j_h\}\backslash\{j_h\}$ it must hold that $F\left(\theta^{j_h}\right) > F\left(\theta^j\right)$. Thus, for all $j < j_h$ it holds that

$$\theta^{k'}\left(F\left(\theta^{j_h}\right)-F\left(\theta^j\right)\right)-F\left(\theta^{j_h}\right)\overline{b}^{\theta^{j_h}}+F\left(\theta^j\right)\overline{b}^{\theta^j} > 0$$

$$\Leftrightarrow F\left(\theta^{j_h}\right)\left(\theta^{k'}-\overline{b}^{\theta^{j_h}}\right) > F\left(\theta^j\right)\left(\theta^{k'}-\overline{b}^{\theta^j}\right).$$

For all j such that $j < j_h$ and $j \notin \{j_1,\ldots,j_h\}$ as in the first case, it holds that the inquality in (83) is strict and therefore also the inequality in (85). Conclusively, the best reply of a bidder with valuation $\theta^{k'}$ is at least as high as the highest best reply of a bidder with valuation θ^k. Therefore, the outcome is efficient.

So far, I have shown by construction that an outcome under maximin strategies exists and that this outcome is efficient. More precisely, I have shown that any combination of best replies to the adverse nature's strategy β_n as defined in (82) is efficient. Formally, let B^{θ^j} be the set of best replies for a bidder with valuation θ^j given the adverse nature's strategy β_n, i.e. B^{θ^j} is defined by

$$B^{\theta^j} = \underset{b' \in \mathscr{B}}{\operatorname{argmax}} \, U\left(\theta^j, b', \beta_n, F\right).$$

Let $(b_1, \ldots, b_m)$ be a vector of bids such that $b_j \in B^{\theta^j}$ for all $j \in \{1, \ldots, m\}$. Then for every i, j with $j > i$, it holds that $b_j \geq b_i$.

It remains to show that every outcome is efficient. In order to show that every outcome is efficient, I will show that if $\left(\hat{\beta}_1, \ldots \hat{\beta}_I, \hat{\beta}_n\right)$ is an outcome under maximin strategies, then it holds for every bidder j and every valuation θ_j that

$$\hat{B}^{\theta_j} \subseteq B^{\theta_j}$$

where $\hat{B}^{\theta^j}$ is defined by

$$\hat{B}^{\theta^j} = \underset{b' \in \mathscr{B}}{\operatorname{argmax}} \, U\left(\theta^j, b', \hat{\beta}_n, F\right).$$

Assume there exists an outcome under maximin strategies, denoted by $\left(\hat{\beta}_1, \ldots \hat{\beta}_I, \hat{\beta}_n\right)$, such that there exists a bidder i with valuation θ_i and a bid b such that $b \in supp\left(\hat{\beta}_i\left(\theta_i\right)\right)$ and $b \notin B^{\theta_i}$. This implies that there exists a bidder $j \neq i$ and a valuation θ^l such that $\hat{\beta}_j^{n_i,\theta_i}(\theta^l) \neq \beta^{n_i,\theta_i}(\theta^l)$ and

$$b \in supp\left(\beta_j^{n_i,\theta_i}\left(\theta^l\right)\right) \text{ or } b^- \in supp\left(\beta_j^{n_i,\theta_i}\left(\theta^l\right)\right)$$

(depending on whether $\theta_i > \theta^l$ or $\theta_i \leq \theta^l$). In other words, since the outcome is not efficient, there exists a bidder i who bids differently than in the efficient outcome by bidding b. This in turn implies that there exists another bidder j such that in the subjective maximin belief of bidder i with valuation θ_i bidder j's strategy differs from the strategy prescribed by β_n in a way which makes the bid b a best reply for bidder i.

Since in the subjective maximin belief of bidder i bidder j with valuation θ^l deviates from bidding her highest rationalizable bid and cannot bid higher than the highest rationalizable bid of type θ^l, it holds that $b < \overline{b}^{\theta^l}$. Therefore, the adverse nature could strictly decrease the winning probability of bidder i with valuation θ_i by deviating to the strategy which prescribes to bid $\overline{b}_{\theta^l}$ for bidder j with valuation θ^l. Thus, $\left(\hat{\beta}_1, \ldots \hat{\beta}_I, \hat{\beta}_n\right)$ cannot constitute an outcome under maximin strategies. Conclusively, every outcome under maximin strategies has to be efficient. □

Proof of Proposition 17

Proof. With the same reasoning as in the proof of Proposition 15, one can show that for every outcome $\left(\hat{\beta}_1, \ldots \hat{\beta}_I, \hat{\beta}_n\right)$ under maximin strategies it holds for every bidder i and every valuation θ_i that

$$B^{\theta_i} \subseteq \hat{B}^{\theta_i}$$

where $\hat{B}^{\theta_i}$ and B^{θ_i} are defined by

$$B^{\theta^j} = \underset{b' \in \mathscr{B}}{\operatorname{argmax}}\, U\left(\theta^j, b', \beta_n, F\right), \quad \hat{B}^{\theta^j} = \underset{b' \in \mathscr{B}}{\operatorname{argmax}}\, U\left(\theta^j, b', \hat{\beta_n}, F\right)$$

and β_n is defined as in (82). This is the strategy of the adverse nature which chooses as the subjective maximin belief for every bidder and every valuation that every other bidder places the highest rationalizable bid given her valuation. It follows that

$$\hat{B}^{\theta_i} = B^{\theta_i}$$

That is, in every outcome under maximin strategies, bidders play best replies to the adverse nature's strategy β_n. Therefore, in every outcome the bidders' strategies are as specified in Proposition 17.

□

4.7.5 Proof of Proposition 18

Proof. The proof works by induction with respect to the type. Since $\overline{b}^{\theta^1} = \overline{b}_*^{\theta^1}$, the induction starts with θ^2. The highest rationalizable bid for type θ^2 is obtained by the equation

$$\theta^2 - \overline{b}^{\theta^2} = F^{I-1}(0)\,\theta^2$$
$$\Leftrightarrow \overline{b}^{\theta^2} = \theta^2\left(1 - F^{I-1}(0)\right).$$

The highest bid which is placed by a bidder with valuation θ^2 in a Bayes-Nash equilibrium is obtained by the equation

$$F^{I-1}\left(\theta^2\right)\left(\theta^2 - \overline{b}_*^{\theta^2}\right) = F^{I-1}(0)\,\theta^2$$

$$\Leftrightarrow \overline{b}_*^{\theta^2} = \frac{\theta^2\left(F^{I-1}(\theta^2) - F^{I-1}(0)\right)}{F^{I-1}(\theta^2)}.$$

Since $m \geq 3$, it holds that $F^{I-1}\left(\theta^2\right) < 1$ from which follows that

$$F^{I-1}\left(\theta^2\right)F^{I-1}(0) < F^{I-1}(0)$$

$$\Leftrightarrow F^{I-1}\left(\theta^2\right) - F^{I-1}\left(\theta^2\right)F^{I-1}(0) > F^{I-1}\left(\theta^2\right) - F^{I-1}(0)$$

$$\Leftrightarrow 1 - F^{I-1}(0) > \frac{F^{I-1}(\theta^2) - F^{I-1}(0)}{F^{I-1}(\theta^2)}$$

$$\Leftrightarrow \overline{b}^{\theta^2} > \overline{b}_*^{\theta^2}.$$

For the induction step assume that it has been already shown that $\overline{b}^{\theta^j} > \overline{b}_*^{\theta^j}$ for all $j \leq k$. It has to be shown that

$$\overline{b}^{\theta^{k+1}} > \overline{b}_*^{\theta^{k+1}}.$$

As stated in Proposition 16, it holds that

$$\theta^{k+1} - \overline{b}^{\theta^{k+1}} = \max_{\theta^j < \theta^{k+1}} F^{I-1}\left(\theta^j\right)\left(\theta^{k+1} - \overline{b}^{\theta^j}\right).$$

Let

$$F^{I-1}\left(\theta^l\right)\left(\theta^{k+1} - \overline{b}^{\theta^l}\right) = \max_{\theta^j < \theta^{k+1}} F^{I-1}\left(\theta^j\right)\left(\theta^{k+1} - \overline{b}^{\theta^j}\right).$$

Since $\overline{b}_*^{\theta^{k+1}}$ is a best reply, it must induce an expected utility which is greater or equal than the expected utility induced by any other bid, given that every other bidder plays equilibrium strategies. Hence, it holds that

$$F^{I-1}\left(\theta^{k+1}\right)\left(\theta^{k+1} - \overline{b}_*^{\theta^{k+1}}\right) \geq F^{I-1}\left(\theta^l\right)\left(\theta^{k+1} - \overline{b}_*^{\theta^l}\right).$$

Due to the induction assumption, it holds that $\overline{b}_*^{\theta^l} < \overline{b}^{\theta^l}$ from which follows that

$$\begin{aligned}\theta^{k+1} - \overline{b}^{\theta^{k+1}} = F^{I-1}\left(\theta^l\right)\left(\theta^{k+1} - \overline{b}^{\theta^l}\right) &< F^{I-1}\left(\theta^l\right)\left(\theta^{k+1} - \overline{b}_*^{\theta^l}\right)\\ &\leq F^{I-1}\left(\theta^{k+1}\right)\left(\theta^{k+1} - \overline{b}_*^{\theta^{k+1}}\right)\end{aligned}$$

and therefore it holds that

$$\theta^{k+1} - \overline{b}^{\theta^{k+1}} < F^{I-1}\left(\theta^{k+1}\right)\left(\theta^{k+1} - \overline{b}_*^{\theta^{k+1}}\right).$$

$$\Leftrightarrow \overline{b}_*^{\theta^{k+1}} < \frac{\overline{b}^{\theta^{k+1}} - \theta^{k+1}\left(1 - F^{I-1}\left(\theta^{k+1}\right)\right)}{F^{I-1}\left(\theta^{k+1}\right)}. \tag{86}$$

It holds that

$$\begin{aligned}\theta^{k+1} - \overline{b}^{\theta^{k+1}} &\geq 0\\ \Leftrightarrow \theta^{k+1}\left(1 - F^{I-1}\left(\theta^{k+1}\right)\right) - \overline{b}^{\theta^{k+1}}\left(1 - F^{I-1}\left(\theta^{k+1}\right)\right) &\geq 0\\ \Leftrightarrow \overline{b}^{\theta^{k+1}} - \theta^{k+1}\left(1 - F^{I-1}\left(\theta^{k+1}\right)\right) &\leq F^{I-1}\left(\theta^{k+1}\right)\overline{b}^{\theta^{k+1}}\\ \Leftrightarrow \frac{\overline{b}^{\theta^{k+1}} - \theta^{k+1}\left(1 - F^{I-1}\left(\theta^{k+1}\right)\right)}{F^{I-1}\left(\theta^{k+1}\right)} &\leq \overline{b}^{\theta^{k+1}}.\end{aligned}$$

Due to equation (86), it follows that

$$\overline{b}^{\theta^{k+1}} > \overline{b}_*^{\theta^{k+1}}.$$

This completes the induction step and the proof.

□

4.7.6 Proof of Propositions 22, 23 and 24

First, I prove Proposition 23 which formalizes the recursive calculation of the highest rationalizable bids for every type. This calculation is crucial for the proofs of Propositions 22 and 24. In order to prove Proposition 23, I state the following three lemmas which formalize steps (I)-(III) in section 4.5. The proofs work analogously as for Lemmas 29, 30 and 31 in section 4.4 and are therefore omitted.

Lemma 32. *For every bidder i and every valuation $\theta^k \in \Theta$ there exists a unique highest rationalizable bid $\overline{b}_i^{\theta_i}$.*

Lemma 33. *For every type zero is a rationalizable bid.*

Lemma 34. *For every type $\theta^k \in \Theta$ it holds that every bid in $[0, \overline{b}^{\theta^k}]$ is rationalizable.*

Proof of Proposition 23

Proof. First, I examine the highest rationalizable bids of a bidder with valuation θ^k such that θ^k is lower or equal than μ. Consider a subjective belief equilibrium where every bidder has the subjective belief that the other bidders' valuation distribution distributes the probability weight between types θ^k and θ^μ where $\theta_\mu = \min\{\theta^k \in \Theta \mid \theta^k > \mu\}$. Formally, the distribution of the other bidders' valuation is defined by the vector $f^{\theta^k} = \left(f^{\theta^k}_{\theta^1}, \dots, f^{\theta^k}_{\theta^m}\right)$ where for all $j \in \{1, \dots, m\}$ it holds that $f^{\theta^k}_{\theta^j}$ denotes the probability with which type θ^j occurs. This vector is defined by

$$f^{\theta^k}_{\theta^k} = \frac{\theta_\mu - \mu}{\theta_\mu - \theta^k}, \quad f^{\theta^k}_{\theta_\mu} = \frac{\mu - \theta^k}{\theta_\mu - \theta^k} \quad \text{and} \quad f^{\theta^k}_{\theta^j} = 0 \quad \text{for } \theta^j \neq \theta^k, \theta_\mu.$$

Given this subjective belief, in every subjective-belief equilibrium every bidder with valuation θ^k bids θ^k. It follows from Proposition 20 that bidding θ^k is a rationalizable action for a bidder with valuation θ^k. Since it is not rationalizable to bid above valuation, θ^k is the highest rationalizable bid of a bidder with valuation θ^k.

Now I examine the highest rationalizable bids of a bidder with valuation θ^k such that θ^k is strictly greater than μ. Analogously as in the proof Proposition 16, the highest rationalizable bid of a bidder with valuation θ^k is induced by the strategy of the other bidders' such that

(i) All bidders with a lower type bid their highest rationalizable bid.

(ii) All bidders with an equal or higher type bid $\left(\overline{b}^{\theta^k}\right)^-$.

The strategies in (i) are rationalizable by definition and the strategies in (ii) are rationalizable due to Lemma 34. It follows from Corollary 7 that a best reply to these strategies is rationalizable. The highest rationalizable bid of a bidder with valuation θ^k is a best reply to the strategies in (i) and (ii) and to a distribution of the other bidders' valuations. Let the vector $f^{\theta^k} = \left(f^{\theta^k}_{\theta^1}, \dots, f^{\theta^k}_{\theta^m}\right)$ be defined by $f_{\theta^j} = 0$ for $j > k$ and let $\left(f^{\theta^k}_{\theta^1}, \dots, f^{\theta^k}_{\theta^k}\right)$ be the unique solution of the system of k linear equations given by

$$\sum_{i=1}^{k} f^{\theta^k}_{\theta^i} = 1$$

$$\sum_{i=1}^{k} f^{\theta^k}_{\theta^i} \theta^i = \mu$$

$$\left(f^{\theta^k}_{\theta^1}\right)^{I-1} \theta^k = \left(\sum_{i=1}^{j} f^{\theta^k}_{\theta^i}\right)^{I-1} \left(\theta^k - \overline{b}^{\theta^j}\right) \quad \text{for } 1 < j < k.$$

It is to show that this is the unique solution of minimization problem

$$\min \max_{l<k} \left\{ \left(\sum_{i=1}^{l} f_{\theta^i}\right)^{I-1} \left(\theta^k - \overline{b}^{\theta^l}\right) \right\}$$

$$s.t. \;\; f_{\theta^j} \geq 0 \text{ for all } 1 \leq j \leq m$$

$$f_{\theta^1} + \dots + f_{\theta^m} = 1$$

$$f_{\theta^1}\theta^1 + \dots + f_{\theta^m}\theta^m = \mu.$$

Assume, this is not true. Then let $\tilde{f}^{\theta^k} = \left(\tilde{f}^{\theta^k}_{\theta^1}, \dots, \tilde{f}^{\theta^k}_{\theta^m}\right)$ denote the solution vector of this minimization problem, which I will denote by M^{θ^k}. Let $(\delta_{\theta^1}, \dots, \delta_{\theta^m})$ be numbers such that

$$\left(\tilde{f}^{\theta^k}_{\theta^1}, \dots, \tilde{f}^{\theta^k}_{\theta^m}\right) = \left(f^{\theta^k}_{\theta^1} + \delta_{\theta^1}, \dots, f^{\theta^k}_{\theta^m} + \delta_{\theta^m}\right).$$

Since $\tilde{f}^{\theta^k} \neq f^{\theta^k}$, it holds that at least one δ_{θ^j} for $1 \leq j \leq m$ is unequal to zero. Therefore, as in the proof of Lemma 26 in section 3.8.7, one can decompose the vector $(\delta_{\theta^1}, \dots, \delta_{\theta^m})$ into δ-sequences and if there does not exist a $1 \leq t \leq m$ with $\sum_{j=1}^{t} \delta_{\theta^j} > 0$, the process of decomposing into δ-sequences end with a δ-sequence of length 2, i.e. with some vector $\left(\delta^{final}_1, \delta^{final}_2\right)$ with $\delta^{final}_1 < 0$ and $\delta^{final}_2 > 0$.

Assume there exists a $1 \leq t \leq m$ with $\sum_{j=1}^{t} \delta_{\theta^j} > 0$. Since a bidder with valuation θ^k never expects to win against an equal type and the mean μ has to be preserved, it is not optimal for the adverse nature to put positive probability weight on types above θ^k. If there would be positive probability weight on types above θ^k, one could shift probability weight from types above θ^k and types below θ^k to type θ^k in a way which preserves the mean. Since this reduces the winning probability of a bidder with valuation θ^k, it cannot be optimal for the adverse nature to put positive probability weight on types above θ^k. Therefore, it holds that $\delta_{\theta^j} > 0$ for $j > k$. Since it must hold that $\sum_{j=1}^{m} \delta_{\theta^j} = 0$, it holds that $t < k$. Let

$$h \in \underset{l<k}{\operatorname{argmax}} \left\{ \left(\sum_{j=1}^{l} \tilde{f}_{\theta^j} \right)^{I-1} \left(\theta^k - \overline{b}^{\theta^l} \right) \right\}.$$

This implies that

$$\left(\sum_{j=1}^{h} \tilde{f}_{\theta^j} \right)^{I-1} \left(\theta^k - \overline{b}^{\theta^h} \right)$$

is the minimum value of the objective function of minimization problem M^{θ^k}. Since f^{θ^k} is an element of the feasible set of minimization problem M^{θ^k}, the vector f^{θ^k} cannot induce a lower value of the objective function than $\tilde{f}^{\theta^k}$. Therefore, it holds that

$$\left(\sum_{j=1}^{h} \tilde{f}_{\theta^j} \right)^{I-1} \left(\theta^k - \overline{b}^{\theta^h} \right) \leq \left(\sum_{j=1}^{h} f_{\theta^j} \right)^{I-1} \left(\theta^k - \overline{b}^{\theta^h} \right)$$

$$\Leftrightarrow \left(\sum_{j=1}^{h} f_{\theta^j} + \delta_{\theta^j} \right) \sqrt[I-1]{\left(\theta^k - \overline{b}^{\theta^h} \right)} \leq \left(\sum_{j=1}^{h} f_{\theta^j} \right) \sqrt[I-1]{\left(\theta^k - \overline{b}^{\theta^h} \right)}$$

$$\sum_{j=1}^{h} \delta_{\theta^j} \leq 0. \tag{87}$$

By definition of the vector f^{θ^k}, it holds that

$$\left(\sum_{j=1}^{h} f_{\theta^j} \right)^{I-1} \left(\theta^k - \overline{b}^{\theta^h} \right) = \left(\sum_{j=1}^{t} f_{\theta^j} \right)^{I-1} \left(\theta^k - \overline{b}^{\theta^t} \right).$$

By definition of h, it holds that

$$\left(\sum_{j=1}^{h} \tilde{f}_{\theta^j} \right)^{I-1} \left(\theta^k - \overline{b}^{\theta^h} \right) \geq \left(\sum_{j=1}^{t} \tilde{f}_{\theta^j} \right)^{I-1} \left(\theta^k - \overline{b}^{\theta^t} \right)$$

from which follows that

$$\left(\sum_{j=1}^{h} \delta_{\theta^j} \right) \sqrt[I-1]{\theta^k - \overline{b}^{\theta^h}} \geq \left(\sum_{j=1}^{t} \delta_{\theta^j} \right) \sqrt[I-1]{\theta^k - \overline{b}^{\theta^t}}.$$

Since $\sum_{j=1}^{t} \delta_{\theta^j} > 0$, it follows that $\sum_{j=1}^{h} \delta_{\theta^j} > 0$ which is a contradiction to (87). Therefore, the process of decomposing into δ-sequences ends with some vector $\left(\delta_1^{final}, \delta_2^{final}\right)$ with

$\delta_1^{final} < 0$ and $\delta_2^{final} > 0$ and there exists a θ^{final} such that

$$\sum_{j=1}^{m} \delta_{\theta^j} \theta^j > \sum_{j=1}^{m} \delta_{\theta^j} \theta^{final} = 0.$$

Since this is a contradiction to the fact that the vector $\tilde{f}^{\theta^k}$ fulfills the constraint

$$\sum_{j=1}^{m} \tilde{f}_j^{\theta^k} \theta^j = \mu,$$

one can conclude that the assumption that the solution of minimization problem M^{θ^k} does not coincide with the unique solution of the system of k linear equations as specified in Proposition 22, leads to a contradiction. Therefore, the solution of this system of linear equations is the unique distribution inducing the highest rationalizable bid of a bidder with valuation θ^k. □

Proof of Proposition 22 and Proposition 24

Proof. Given the distribution of the other bidders' valuations, the adverse nature chooses for a bidder with valuation $\theta^k \leq \mu$, the bidder expects the lowest possible utility of zero. Thus, the adverse nature does not have an incentive to deviate from this strategy. In order to choose for a bidder with valuation $\theta^k > \mu$ a distribution of the other bidders' valuations, the adverse nature has to solve the following minimization problem:

$$\min \max_{l<k} \left\{ \left(\sum_{i=1}^{l} f_{\theta^i}^{\theta^k} \right)^{I-1} \left(\theta^k - \overline{b}^{\theta^l} \right) \right\}$$
$$s.t. \quad f_{\theta^j} \geq 0 \text{ for all } 1 \leq j \leq m$$
$$f_{\theta^1}^{\theta^k} + \cdots + f_{\theta^m}^{\theta^k} = 1$$
$$f_{\theta^1}^{\theta^k} \theta^1 + \cdots + f_{\theta^m}^{\theta^k} \theta^m = \mu.$$

Since this minimization coincides with the minimization problem in the proof of Proposition 23 and the minimization problem has a unique solution, the distribution of the other bidders' valuations chosen by the adverse nature for a bidder with valuation $\theta^k > \mu$ coincides with the vector $\left(f_{\theta^1}^{\theta^k}, \ldots, f_{\theta^m}^{\theta^k} \right)$ as specified in Proposition 23. A bidder with valuation θ^k best replies to the adverse nature's strategy. If $\theta^k \leq \mu$, the bidder expects a utility of zero and is indifferent between any bid between zero and her valuation. If $\theta^k > \mu$, the bidder is indifferent between any highest rationalizable bid of a lower type. Thus, mixing among all highest rationalizable bids of lower types is a best reply to the adverse nature's strategy. In a maximin equilibrium every player with valuation $\theta > \mu$ will mix in a way such that the adverse nature does not have an incentive to deviate from the proposed equilibrium. Thus, the strategies proposed in Propositions 22 and 24 indeed constitute a maximin equilibrium in the game with I players and an adverse nature. Analogously as in the proof of Proposition 15, one can show that bidders' strategies are equal in every outcome under maximin strategies. $\square$

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