Long-time asymptotics for the massive Thirring model

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Kurzzusammenfassung

Das *massive Thirring Modell* (MTM) wurde 1958 vom österreichischen Physiker Walter Thirring im Kontext der relativistischen Quantenfeldtheorie eingeführt. Es dient zur mathematischen Beschreibung von wechselwirkenden Fermionen mit Spin 1/2 (also zum Beispiel Elektronen) in einer Raumdimension. Aus analytischer Sicht ist dieses System aus nichtlinearen partiellen Differentialgleichungen von besonderem Interesse, da es eine Darstellung als Lax-Paar mit zwei linearen Operatoren L und A besitzt und das MTM somit ein *integrables* System ist. Dies ermöglicht es, das MTM mit Hilfe der *inversen Streutransformation* (IST) exakt zu lösen. Da die Abhängigkeit der Operatoren L und A vom spektralen Paramter λ allerdings Singularitäten im Ursprung und bei Unendlich aufweist, kann die IST nicht ohne Weiteres für Anfangsdaten mit geringer Regularität definiert werden. In der vorliegenden Dissertation werden durch entsprechend gewählte Transformationen zwei äquivalente Lax-Paare gefunden, mit deren Hilfe die IST für einen optimalen L^2-basierten Sobolev-Raum konstruiert werden kann. Die Rücktransformation ist anschließend als Riemann–Hilbert-Problem formuliert, dessen Lösbarkeit entsprechend bewiesen wird.
Wie viele andere dispersive Gleichungen besitzt auch das MTM sogenannte *Solitone* als Lösungen. Diese speziellen in ihrer Form unveränderlich bleibenden und sich mit konstanter Geschwindigkeit fortbewegenden Wellen können nur auf Grund der vorhandenen Nichtlinearität des MTM existieren. Anhand ihrer Streudaten lassen sie sich auf einfache Weise charakterisieren und mit Hilfe geeigneter Riemann–Hilbert Techniken ist es möglich, zu berechnen, wie zwei (oder mehrere) Solitone wechselwirken. Desweiteren kann präzise gezeigt werden, dass sich alle Solitone nach einer gewissen Zeit innerhalb des Lichtkegels $\{|t| > |x|\}$ befinden. Mit Hilfe der sogenannten $\overline{\partial}$–Methode kann sogar gezeigt werden, dass alle Lösungen (also nicht nur Solitone) außerhalb des Lichtkegels mit einer Rate von $|t|^{-3/4}$ gegen Null konvergieren. Innerhalb des Lichtkegels gibt es zwei mögliche Szenarien, welche in der folgenden Arbeit beide rigoros untersucht werden. Falls die Anfangsdaten frei von Solitonen sind, lässt sich - wiederum mit der $\overline{\partial}$–Methode und bekannten Modell-Riemann–Hilbert-Problemen - zeigen, dass die Lösung des MTM in die Nähe einer Lösung der linearen Dirac-Gleichung (modulo Phasen-Korrektur) kommt. Diese Lösung kann sogar explizit aus den Streudaten errechnet werden und ihre Amplitude selbst fällt mit einer Rate von $\sim |t|^{-1/2}$. Die zweite Möglichkeit ist, dass die Anfangsdaten endlich viele Solitone enthalten. Hier findet dann das Hauptresultat der vorliegenden Arbeit Anwendung. Dieses Resultat besagt, dass sich jede Lösung des MTM für sogenannte generische Anfangswerte auf lange Zeit in endlich viele einzelne Solitone zerlegt, die sich mit unterschiedlichen Geschwindigkeiten auseinander bewegen. Der Restterm verschwindet dabei mit einer Rate von $\sim |t|^{-1/2}$.

Damit liefert die vorliegende Dissertation einen kompletten analytischen Beweis der sogenannten Soliton-Zerlegungs-Vermutung (*soliton resolution conjecture*) für das MTM. Außerdem kann das Ergebnis auch als *asymptotische Stabilität von Solitonen* gedeutet werden.

Abstract

The *massive Thirring model* (MTM) was introduced in 1958 by the Austrian physicist Walter Thirring in the context of relativistic quantum field theory. It describes the self-interaction of a Dirac field in one space dimension. From the analytical point of view, this system of non-linear partial differential equations is of special interest, because it has a representation in terms of a Lax pair, consisting of two linear operators L and A. Thanks to the Lax pair, the MTM admits an exact solution by the inverse scattering transform (IST). Since the dependence of L and A on the spectral parameter λ is singular at the origin and at infinity, the IST cannot be defined for initial data of low regularity as straightforward as it is done for other equations, the NLS equation for instance. One key ingredient of the present thesis is to transform the known Lax pair to two equivalent Lax pairs: one is suitable for the spectral parameter at the origin and the other one is suitable at infinity. Using the equivalent operators the direct scattering transform is developed for an optimal L^2-based Sobolev space. The inverse scattering map is then formulated in terms of two Riemann–Hilbert problems whose solvability is proven.
As it is also known from other nonlinear dispersive equations one can create *solitons* for the MTM. These special solutions are waves that move at constant speed and do not change in shape. They can refuse to disperse only because of the presence of the nonlinearity in the equation. It is relatively simple to characterize solitons, based on their scattering data. Using suitable Riemann-Hilbert techniques it is possible to analyse the interaction of two (or more) solitons. Furthermore, it can be shown precisely that each soliton will eventually enter the light cone $\{|t| > |x|\}$. Using the so-called $\overline{\partial}$–method (nonlinear steepest descent) we show that outside the light cone any solution (not only solitons) converges to zero with a rate of $\sim |t|^{-3/4}$. Inside the light-cone there are basically two different possibilities. Assuming that the initial data is free of solitons we use the $\overline{\partial}$–method and some well-known model Riemann–Hilbert–problems to show that the solution of the MTM scatters to a linear solution modulo phase correction. This linear solution can be computed explicitly from the scattering data and its amplitude decays with a rate of $\sim |t|^{-1/2}$. The second possibility is that the initial data contains finitely many solitons. Then, as the main result of the thesis, we prove that any solution breaks up into finitely many single solitons that travel at different speeds and thus, diverge. The remainder term is $\mathcal{O}(|t|^{-1/2})$.

Summarizing, the present thesis provides an analytical proof of the soliton resolution conjecture for the MTM. This result also implies the *asymptotic stability* of solitons.

Contents

Chapter 1

Introduction

1.1 The massive Thirring model

This dissertation is concerned with solutions of the *massive Thirring model* (MTM). This model was introduced by Thirring [Thi58] in the context of general relativity in the form

$$(i\partial_\mu \gamma^\mu + m)\psi + g^2 \gamma^\mu \psi(\overline{\psi}\gamma_\mu \psi) = 0,$$

where

$$\gamma^0 = \gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = -\gamma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \overline{\psi} = \psi^* \gamma^0.$$

The unknown ψ is a function on $\mathbb{R} \times \mathbb{R}$ with values in $\mathbb{C} \times \mathbb{C}$. The number g is the so-called coupling constant and m is the mass parameter. The MTM describes the vector-vector self interaction of a Dirac field in (1+1) dimensions and is a simplification of the Dirac–Maxwell system [Gro66]. Another relativistic Dirac equation is the massive Soler model [Sol70] with scalar-scalar self interaction. By setting $m = 1$ and $g = 1/\sqrt{2}$ we write the MTM system in laboratory coordinates as:

$$\begin{cases} i(u_t + u_x) + v + u|v|^2 = 0, \\ i(v_t - v_x) + u + |u|^2 v = 0. \end{cases} \tag{1.1.1}$$

Here, u and v are functions of $t \in \mathbb{R}$ (time) and $x \in \mathbb{R}$ (space) with values in $\mathbb{C}$. Subscripts denote partial derivatives. The relativistic invariance of the massive Thirring model can be stated as follows. Let $(u(t,x), v(t,x))$ be a solution of (1.1.1) and $\nu \in (-1,1)$. Then,

$$\begin{cases} \widetilde{u}(t,x) := \delta^{-1} u(t',x'), \\ \widetilde{v}(t,x) := \delta v(t',x'), \end{cases}$$

with

$$x' = \gamma(x - \nu t), \qquad t' = \gamma(t - \nu x), \qquad \gamma = \frac{1}{\sqrt{1-\nu^2}}, \qquad \delta = \sqrt{\frac{1-\nu^2}{1+\nu^2}},$$

is a new solution of (1.1.1). The transformation $(t,x) \mapsto (t',x')$ is a Lorentz transformation that maps the original coordinate frame in spacetime to another frame that moves at constant velocity ν.

In this thesis we consider the Cauchy problem for the MTM system. That is, we look at solutions of (1.1.1) with $u(0,x) = u_0(x)$ and $v(0,x) = v_0(x)$ for given initial data (u_0, v_0). However, we are not concerned with the well-posedness of the problem. Indeed, the local and global existence of solutions to the Cauchy problem for the MTM system (1.1.1) in the L^2-based Sobolev spaces $H^m(\mathbb{R})$, $m \in \mathbb{N}$ can be proven with the standard contraction and energy methods, see the review of the literature in [Pel11]. Low regularity solutions in $L^2(\mathbb{R})$ were already obtained for the MTM system by Selberg and Tesfahun [ST10], Candy [Can11], Huh and Moon [Huh11, Huh13, HM15], and Zhang [Zha13, ZQ15]. The well-posedness results can be formulated as follows.

Theorem 1.1.1. *[Can11, HM15] For every $u_0, v_0 \in H^m(\mathbb{R})$, $m \in \mathbb{N}$, there exists a unique global solution $(u,v) \in C(\mathbb{R}, H^m(\mathbb{R}) \times H^m(\mathbb{R}))$ of (1.1.1) such that $(u,v)|_{t=0} = (u_0, v_0)$ and the solution (u,v) depends continuously on the initial data (u_0, v_0). Moreover, for every $u_0, v_0 \in L^2(\mathbb{R})$, there exists a global solution $(u,v) \in C(\mathbb{R}, L^2(\mathbb{R}) \times L^2(\mathbb{R}))$ such that $(u,v)|_{t=0} = (u_0, v_0)$.*

For $u_0, v_0 \in L^2(\mathbb{R})$ it is also shown in [Can11, HM15] that the solution (u,v) is unique in a certain subspace of $C(\mathbb{R}, L^2(\mathbb{R}) \times L^2(\mathbb{R}))$ and it depends continuously on the initial data (u_0, v_0). But the authors do not provide specific details on restrictions of this 'certain subspace' of L^2.

Applying $(\partial_t - \partial_x)$ to the first line and $(\partial_t + \partial_x)$ to the second line of (1.1.1) we obtain a nonlinear Klein–Gordon equation of the form

$$\begin{cases} u_{tt} - u_{xx} + u = -|u|^2 v + i(\partial_t - \partial_x)(u|v|^2), \\ v_{tt} - v_{xx} + v = -|v|^2 u + i(\partial_t + \partial_x)(|u|^2 v). \end{cases} \tag{1.1.2}$$

Large-time asymptotics for the Klein–Gordon equation with different kind of nonlinearities can be obtained, see for instance [Sun05]. But these results cannot be applied to (1.1.1) because the nonlinear terms in (1.1.2) do not satisfy the requirements of [Sun05]. The only work concerned with the pointwise behavior of the equation (1.1.1) is [CL18]. Therein the authors show that the solution scatters at infinity to a linear solution modulo phase correction if the initial data satisfy at least

$$\|\langle x\rangle^{7/2} u_0(x)\|_{H^5_x(\mathbb{R})} + \|\langle x\rangle^{7/2} v_0(x)\|_{H^5_x(\mathbb{R})} \leq \epsilon \tag{1.1.3}$$

for $\langle x\rangle := \sqrt{1+x^2}$ and some sufficiently small $\epsilon > 0$. The proof is based on arguments of Lindblad-Soffer [LS05a, LS05b, LS15] and uses energy estimates and ODE theory. In particular, the result in [CL18] also implies that $\|u(t,\cdot)\|_{L^\infty(\mathbb{R})} + \|v(t,\cdot)\|_{L^\infty(\mathbb{R})}$ decays at the rate $\mathcal{O}(|t|^{-1/2})$. This kind of asymptotic long-time behavior is in agreement with the general theory on linear dispersive equations [LP14]. However, as it is known for many other dispersive equations such as NLS, DNLS, KdV or sine–Gordon equations, once nonlinear effects are included, there may exists solutions that refuse to disperse and travel at constant speed without changing their shape. These waves are called *solitons* and in the case of the MTM system they are explicitly given by

$$\begin{cases} u_{sol}(t,x;\{\lambda_1, C_1\}) = |\lambda_1|^{-1} \sin(2\arg\lambda_1)\,\mathrm{sech}\left(E(x-\nu t - x_0) + i\arg\lambda_1\right) e^{-i\beta(t-\nu x)+i\phi_1}, \\ v_{sol}(t,x;\{\lambda_1, C_1\}) = -|\lambda_1| \sin(2\arg\lambda_1)\,\mathrm{sech}\left(E(x-\nu t - x_0) - i\arg\lambda_1\right) e^{-i\beta(t-\nu x)+i\phi_1}, \end{cases}$$

for parameters $\lambda_1, C_1 \in \mathbb{C}$ which also determine the other real parameters E, ν, β, x_0 and ϕ_1, see Chapter 4. Not only their very existence is a surprising phenomenon, but so is their behaviour after colliding with each other. For example, let us consider initial data

$$\begin{aligned} u_0(x) &= |\lambda_1|^{-1}\,\mathrm{sech}\left(E_1(x-x_{0,1}) + i\frac{\pi}{4}\right) e^{i\beta_1\nu_1 x} + |\lambda_2|^{-1}\,\mathrm{sech}\left(E_2(x-x_{0,2}) + i\frac{\pi}{4}\right) e^{i\beta_2\nu_2 x}, \\ v_0(x) &= -|\lambda_1|\,\mathrm{sech}\left(E_1(x-x_{0,1}) + i\frac{\pi}{4}\right) e^{i\beta_1\nu_1 x} - |\lambda_2|\,\mathrm{sech}\left(E_2(x-x_{0,2}) + i\frac{\pi}{4}\right) e^{i\beta_2\nu_2 x}, \end{aligned}$$

with the soliton centers $x_{0,1}$ and $x_{0,2}$ far away from each other, say $x_{0,1} \ll 0 \ll x_{0,2}$. If the corresponding soliton velocities satisfy $-1 < \nu_2 < 0 < \nu_1 < 1$ we initially have two solitons moving towards each other. It turns out that for $t \to \infty$, the solution of (1.1.1) takes the form

$$\begin{cases} u(t,x) \approx |\lambda_1|^{-1}\,\mathrm{sech}\left(E_1(x-\nu_1 t - x_{0,1} - \Delta x_{0,1}) + i\frac{\pi}{4}\right) e^{-i\beta_1(t-\nu_1 x)+i\Delta\phi_1} \\ \qquad\qquad + |\lambda_2|^{-1}\,\mathrm{sech}\left(E_2(x-\nu_2 t - x_{0,2} - \Delta x_{0,2}) + i\frac{\pi}{4}\right) e^{-i\beta_2(t-\nu_2 x)+i\Delta\phi_2}, \\ v(t,x) \approx -|\lambda_1|\,\mathrm{sech}\left(E_1(x-\nu_1 t - x_{0,1} - \Delta x_{0,1}) + i\frac{\pi}{4}\right) e^{-i\beta_1(t-\nu_1 x)+i\Delta\phi_1} \\ \qquad\qquad - |\lambda_2|\,\mathrm{sech}\left(E_2(x-\nu_2 t - x_{0,2} - \Delta x_{0,2}) + i\frac{\pi}{4}\right) e^{-i\beta_2(t-\nu_2 x)+i\Delta\phi_2}, \end{cases}$$

and we observe two facts:

1. As time evolves, the solitons collide, but emerge almost unchanged.
2. The change of the first (second) soltion consists of a small spatial shift $\Delta x_{0,1}$ ($\Delta x_{0,2}$) and a phase shift $\Delta\phi_1$ ($\Delta\phi_2$).

These observations are remarkable in two respects: on one hand, if the MTM system was linear, the two solitons would collide without any interaction. On the other hand, it is surprising that the sum of two solutions of a nonlinear equation is also a solution (at least approximatively). In the present thesis we devote the entire Chapter 4 to the phenomenon of solitary waves.

1.2 The inverse scattering approach

We now briefly give an overview over the technique of inverse scattering, which allows us to solve the MTM system in the following way. Assume that $(u(t,x),v(t,x))$ is a solution of (1.1.1) and associate the following matrix to $(u,v)=(u(t,x),v(t,x))$:

$$L(t,x;\lambda)=\frac{i}{4}(|u|^2-|v|^2)\sigma_3-\frac{i\lambda}{2}\begin{bmatrix}0 & \overline{v}\\ v & 0\end{bmatrix}+\frac{i}{2\lambda}\begin{bmatrix}0 & \overline{u}\\ u & 0\end{bmatrix}+\frac{i}{4}\left(\lambda^2-\frac{1}{\lambda^2}\right)\sigma_3, \tag{1.2.1}$$

where

$$\sigma_3=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

and $\lambda\in\mathbb{C}$. If u and v are decaying at $x=\pm\infty$ for every t, then we see that (vector-valued) solutions to the spectral problem

$$\psi_x=L\psi$$

must take the form

$$\psi(t,x;\lambda)\sim\alpha_\pm(t;\lambda)\begin{pmatrix}1\\0\end{pmatrix}e^{\frac{i}{4}(\lambda^2-\lambda^{-2})x}+\beta_\pm(t;\lambda)\begin{pmatrix}0\\1\end{pmatrix}e^{-\frac{i}{4}(\lambda^2-\lambda^{-2})x} \tag{1.2.2}$$

as $x\to\pm\infty$. If we fix the boundary conditions $\alpha_-(t;\lambda)=1$ and $\beta_-(t;\lambda)=0$ this determines $\alpha_+(t;\lambda)$ and $\beta_+(t;\lambda)$ uniquely and defines a map

$$p(t;\lambda):=\frac{\beta_+(t;\lambda)}{\alpha_+(t;\lambda)},$$

called the *reflection coefficient*. So far we have not made use of the fact, that (u,v) solves (1.1.1). To use this fact, we introduce another matrix A, namely,

$$A(t,x;\lambda)=-\frac{i}{4}(|u|^2+|v|^2)\sigma_3-\frac{i\lambda}{2}\begin{bmatrix}0 & \overline{v}\\ v & 0\end{bmatrix}-\frac{i}{2\lambda}\begin{bmatrix}0 & \overline{u}\\ u & 0\end{bmatrix}+\frac{i}{4}\left(\lambda^2+\frac{1}{\lambda^2}\right)\sigma_3,$$

and find $[\partial_x-L,\partial_t-A]=0$ if and only if (u,v) solves (1.1.1). Using this rather miraculous algebraic identity, we are then led to the time evolution of p given by

$$p(t;\lambda)=p(0;\lambda)e^{-it(\lambda^2+\lambda^{-2})/2}.$$

We see that we can linearize the MTM system by the scattering map $\mathcal{S}:(u,v)\mapsto p$ which is quite remarkable. The reflection coefficient p is generally defined for all $\lambda\in\mathbb{R}\cup i\,\mathbb{R}$. However, it turns out that different potentials, say (u,v) and $(\widetilde{u},\widetilde{v})$, can have identical reflection coefficients, say $p=\widetilde{p}$. Indeed, the zero-solution has a vanishing reflection coefficient $p=0$, but also all solitons admit a vanishing reflection coefficient. In order to make the scattering map one-to-one, we also need to examine the eigenvalues of L: assume that a solution of $\psi_x=L\psi$ with $\mathrm{Im}(\lambda^2)<0$ in the form (1.2.2) with $\alpha_-(t;\lambda)=1$ and $\beta_-(t;\lambda)=0$ does not blow up in one of the directions $x\to+\infty$ or $x\to-\infty$. Then, $\alpha_+(t;\lambda)=0$ must hold. Hence, zeroes λ_j of $\alpha_+(t;\cdot)$ have to be considered as a component of the *scattering data* and it turns out that each eigenvalue corresponds to one soliton contained in the initial data (u_0, v_0). Its parameters are furthermore determined by the so-called *norming constants* $C_j := \beta_+(t;\lambda_j)/\alpha'_+(t;\lambda_j)$. They also have to be added to the scattering data and are well-defined in all (generic) cases where $\alpha'_+(t;\lambda_j) \neq 0$. As time evolves, the spectrum $\{\lambda_1, ..., \lambda_N\}$ stays constant and the norming constants are controlled by $C_j(t) = C_j(0)e^{-it(\lambda_j^2+\lambda_j^{-2})/2}$. This suggests the following solution procedure for the massive Thirring model:

Step 1: For generic initial data (u_0, v_0), compute the scattering data $\mathcal{S}(u_0, v_0) = (p, \{\lambda_j, C_j\}_{j=1}^N)$.

Step 2: At a time $t \neq 0$, obtain the evolved scattering data by

$$\mathcal{S}(u(t), v(t)) = (p(\lambda)e^{-it(\lambda^2+\lambda^{-2})/2}, \{\lambda_j, C_j e^{-it(\lambda_j^2+\lambda_j^{-2})/2}\}_{j=1}^N).$$

Step 3: At $t \neq 0$, recover (u, v) from the evolved scattering data.

The operator pair (L, A) for (1.1.1) was already presented and studied in [Mik76, KN77, KM77]. Even though the inverse scattering machinery is a powerful tool in many nonlinear wave equations, so far it has not been applied to the question of pointwise asymptotic behavior of (1.1.1). One reason for overlooking the possibilities of the method lies perhaps in the fact that there is a difficulty in the above outlined technique, namely, the third step. According to [BC84, BC85, BDT88] a linear operator L can be recovered from its scattering data by means of a Riemann–Hilbert problem . However, due to its rational dependence of the operator L given in (1.2.1) on the spectral parameter λ, it is not straightforward to setup the right Riemann–Hilbert problem . Also in the existing literature such as [Vil91], the treatment of the inverse problem in terms of Riemann–Hilbert problems is somehow sketchy. There also exist abstract conditions for solvability through Riemann–Hilbert problems if the operator has rational [Zho95] or even arbitrary [Zho89] spectral dependence. Although the MTM system (1.1.1) does not appear in the list of examples in [Zho95], one can show that the abstract method of Zhou is also applicable to the MTM system. However, based on [PS18a], Chapters 2 and 3 of the present dissertation solve the inverse scattering problem for the MTM relying on recent progress in the inverse scattering transform method for the derivative NLS equation [PS18b, SSP17]. The key element of this technique is a transformation of the spectral plane λ for the operator L in (1.2.1) to the spectral planes $w = \lambda^{-2}$ and $z = \lambda^2$ for two equivalent spectral problems $\Psi_x = \mathcal{L}\Psi$ and $\widehat{\Psi}_x = \widehat{\mathcal{L}}\widehat{\Psi}$. Therefore, two Riemann–Hilbert problem s are derived: one recovers the component u, the other one recovers v. Due to this transformation, we also find two new reflection coefficients r and $\widehat{r}$ which are both now functions on $\mathbb{R}$. Analogously, eigenvalues and norming constants are transformed as well. Hence, the schematic solution procedure for the MTM system is given separately for the components u and v by

$$\begin{tikzcd}
(u_0, v_0) \arrow[r, mapsto, "\text{Scattering transform}"] \arrow[d, mapsto, "\text{MTM}"] & \left(r(w), \{w_j, c_j\}_{j=1}^N\right) \arrow[d, mapsto] \\
u(t) & \left(r(w)e^{-it(w+w^{-1})/2}, \{w_j, c_j e^{-it(w_j+w_j^{-1})/2}\}_{j=1}^N\right) \arrow[l, mapsto, "\text{RHP}"']
\end{tikzcd}$$

$$\begin{tikzcd}
(u_0, v_0) \arrow[r, mapsto, "\text{Scattering transform}"] \arrow[d, mapsto, "\text{MTM}"] & \left(\widehat{r}(z), \{z_j, \widehat{c}_j\}_{j=1}^N\right) \arrow[d, mapsto] \\
v(t) & \left(\widehat{r}(z)e^{-it(z+z^{-1})/2}, \{z_j, \widehat{c}_j e^{-it(z_j+z_j^{-1})/2}\}_{j=1}^N\right) \arrow[l, mapsto, "\text{RHP}"']
\end{tikzcd}$$

In order to formulate the main result of the inverse scattering we need to specify some function spaces. The transformations $(u,v)\mapsto r$ and $(u,v)\mapsto \widehat{r}$ turn out to be controllable if $(u,v)\in X_{2,1}$, where

$$X_{2,1} := (H^2(\mathbb{R})\cap H^{1,1}(\mathbb{R}))\times(H^2(\mathbb{R})\cap H^{1,1}(\mathbb{R})).$$

Here, H^k is the standard Sobolev space and $H^{1,1}$ is defined by

$$H^{1,1}(\mathbb{R}) = \left\{ f\in L^{2,1}(\mathbb{R}), \quad f_x u \in L^{2,1}(\mathbb{R})\right\},$$

with $\|f\|^p_{L^{p,s}} := \int (1+x^2)^{ps/2}|f(x)|^p dx$. On the other hand, for the reflection coefficients we consider the space $X^{2,1}_{-2,1}$, where in general the space $X^{k,l}_{g,h}$ is defined by the following norm

$$\|f\|^2_{X^{k,l}_{g,h}} := \int_{-1}^{1} |x|^{2g}|f(x)|^2 + |x|^{2h}|f'(x)|^2 dx + \int_{\mathbb{R}\setminus[-1,1]} |x|^{2k}|f(x)|^2 + |x|^{2l}|f'(x)|^2 dx.$$

The following theorem represents the main result for the inverse scattering.

Theorem 1.2.1. *For every $N\in\mathbb{N}$ and for every $(u_0,v_0)\in X_{2,1}$ admitting no resonances in the sense of Definition 2.6.1 and N simple eigenvalues in the sense of Definition 2.7.1, there exist two direct scattering transforms*

$$\mathcal{S}_w(u_0,v_0) = (r, \{w_j, c_j\}_{j=1}^N)$$

and

$$\mathcal{S}_z(u_0,v_0) = (\widehat{r}, \{z_j, \widehat{c}_j\}_{j=1}^N)$$

with the reflection coefficients r and $\widehat{r}$ defined in $X^{2,1}_{-2,1}$. The unique solution $(u,v)\in C(\mathbb{R}, X_{2,1})$ to the MTM system (1.1.1) can be recovered by means of the inverse scattering transform for every $t\in\mathbb{R}$.

The set of all $(u_0,v_0)\in X_{2,1}$ admitting no resonances and N simple eigenvalues is denoted by $\mathcal{G}_N$. As it is shown in [BC84, Theorem A], the union $\bigcup_{N\in\mathbb{N}}\mathcal{G}_N$ is a dense and open subset of $X_{2,1}$. Hence, the assumption of Theorem 1.2.1 is *generic* in some sense. Moreover, it can be shown that the maps

$$\mathcal{S}_w : \mathcal{G}_N \to X^{1,0}_{-2,1}\times(\mathbb{C}^+)^N\times(\mathbb{C}^*)^N$$

and

$$\mathcal{S}_z : \mathcal{G}_N \to X^{1,0}_{-2,1}\times(\mathbb{C}^-)^N\times(\mathbb{C}^*)^N$$

are Lipschitz continuous for each $N\in\mathbb{N}$. Here we use the notation

$$\mathbb{C}^{\pm} = \{z\in\mathbb{C}\,|\,\pm\Im(z)>0\}, \quad \mathbb{C}^* = \mathbb{C}\setminus\{0\}.$$

The proof of the Lipschitz continuity is not worked out in the present thesis but follows from analogous arguments as in [PS18b]. In the asymptotic analysis it turns out that $r,\widehat{r}\in X^{2,1}_{-2,1}$ is not a sufficient condition and one needs to demand that $r,\widehat{r}\in X^{2,2}_{-2,0}$ instead. From general properties of scattering maps [Zho98], it is reasonable to expect that the requirement $r,\widehat{r}\in X^{2,2}_{-2,0}$ complies with the assumption $u,v\in H^{2,1}(\mathbb{R})$, where

$$H^{2,1}(\mathbb{R}) = \left\{ u\in L^{2,1}(\mathbb{R}), \quad \partial_x^2 u\in L^{2,1}(\mathbb{R})\right\}.$$

Again, this property of the scattering maps is not worked out in this thesis.

1.3 Main results

Solutions of (1.1.1) associated to $(u_0,v_0)\in\mathcal{G}_0$ are soliton free or pure radiation solutions, as they do not contain any solitons. For pure radiation solutions, the inverse map is given by the following Riemann–Hilbert problem : Find $M=M(t,x;w)$ a 2×2 matrix, satisfying the following conditions:

$$\begin{cases} M(t,x;\cdot) \text{ is analytic on } \mathbb{C}\setminus\mathbb{R}. \\ M_+(t,x;w) = M_-(t,x;w)(1+R(t,x;w)) \text{ for } w\in\mathbb{R} \text{ with } R \text{ specified below.} \\ M = 1+\mathcal{O}(w^{-1}) \text{ as } w\to\infty. \end{cases}$$

The jump matrix R is defined on $\mathbb{R}$ as follows:

$$R(t,x;w) = \begin{bmatrix} w|r(w)|^2 & \overline{r(w)}e^{-\frac{i}{2}x(w-w^{-1})+\frac{i}{2}t(w+w^{-1})} \\ wr(w)e^{\frac{i}{2}x(w-w^{-1})-\frac{i}{2}t(w+w^{-1})} & 0 \end{bmatrix}.$$

Once such a function M is found, the component u of the solution of (1.1.1) is given by

$$u(t,x) = [M(t,x;0)]_{11}\overline{\lim_{|w|\to\infty} w[M(t,x;w)]_{12}}.$$

Thus, the point-wise analysis of $u(t,x)$ involves the detailed analysis of the solution M of the above Riemann–Hilbert problem . It turns out that, to first order in r, we have

$$|u(t,x)| \sim \frac{1}{2\pi}\left|\int_{\mathbb{R}} \overline{r(w)}e^{-\frac{i}{2}x(w-w^{-1})+\frac{i}{2}t(w+w^{-1})}dw\right|,$$

which indicates that at least up to a first approximation, u can treated as a linear solution by the method of classical steepest descent. However, Deift and Zhou [DZ93] were the first who were able to present a nonlinear steepest method for oscillatory Riemann–Hilbert problem s. As an application, they restricted themselves to the modified Korteweg–de Vries equation. But since then, the method has been applied to several other equations as the sine-Gordon [CVZ99], NLS [DZ94, DZ03, DM08, CP14, Saa17a, BJM16] or DNLS equation [LPS16, LPS18, JLPS18a, JLPS18b]. Since the method of nonlinear steepest descent is a method and not a theorem, it has to be tailored in a particular way to the problem at hand. The present thesis treats in detail this method for the first time in the context of the MTM system (1.1.1). One of the main results is presented in the following statement, which even allows the initial data to contain finitely many solitons. It describes the decay to zero in the exterior region, where $|x| > |t|$:

Theorem 1.3.1. *Suppose that $(u_0, v_0) \in \mathcal{G}_N$ and consider the solution (u, v) of (1.1.1) with initial data (u_0, v_0). Additionally, assume that the transformed reflection coefficients satisfy $r, \widehat{r} \in X^{2,2}_{-2,0}$. Then, there exists a positive number T_0 depending on (u_0, v_0) and a positive number C not depending on (u_0, v_0) such that*

$$|u(t,x)| \le C\min\left\{|t-x|^{-1}, |t+x|^{-3/4}\right\}\|r\|_{X^{2,2}_{-2,0}}, \qquad |x| > \max\{|t|, T_0\}$$

and

$$|v(t,x)| \le C\min\left\{|t+x|^{-1}, |t-x|^{-3/4}\right\}\|\widehat{r}\|_{X^{2,2}_{-2,0}}, \qquad |x| > \max\{|t|, T_0\}.$$

Our second main result describes the asymptotic behavior of pure radiation solutions in the interior region:

Theorem 1.3.2. *Let $(u_0, v_0) \in \mathcal{G}_0 \cap X_{2,1}$ and assume additionally that the transformed reflection coefficients satisfy $r, \widehat{r} \in X^{2,2}_{-2,0}$. Then there exist positive constants $C = C(u_0, v_0)$ and $\tau_0 = \tau_0(u_0, v_0)$ and bounded functions $f_\pm : (-1, 1) \to \mathbb{C}$ such that*

$$\begin{aligned} \left|u(t,x) - \frac{1}{\sqrt{t-x}}\left(e^{i\tau+i|f_-(\frac{x}{t})|^2\ln(\tau)}f_-\left(\frac{x}{t}\right) + e^{-i\tau+i|f_+(\frac{x}{t})|^2\ln(\tau)}f_+\left(\frac{x}{t}\right)\right)\right| &\le C\tau^{-3/4}, \\ \left|v(t,x) - \frac{1}{\sqrt{t+x}}\left(e^{i\tau+i|f_-(\frac{x}{t})|^2\ln(\tau)}f_-\left(\frac{x}{t}\right) - e^{-i\tau+i|f_+(\frac{x}{t})|^2\ln(\tau)}f_+\left(\frac{x}{t}\right)\right)\right| &\le C\tau^{-3/4} \end{aligned} \tag{1.3.1}$$

for all $t > |x|$ and $\tau := \sqrt{t^2 - x^2} > \tau_0$.

For negative times $t < -|x|$, a similar expansion can be found. From the details of the proof in Chapter 6 and from the remarks following Theorem 1.2.1 above it is seen that there exists a $\lambda_0 > 0$ such that for all initial data satisfying

$$\lambda := \|u_0\|_{H^{2,1}(\mathbb{R})} + \|v_0\|_{H^{2,1}(\mathbb{R})} \le \lambda_0,$$

the assumptions of Theorem 1.3.2 are fulfilled and, moreover, the constant C in (1.3.1) can be chosen as $C = c\lambda$ where the constant c does not depend on u_0 and v_0 anymore. We need to mention that the same asymptotic behavior as stated in Theorem 1.3.2 is already derived in [CL18]. However, our result has two main features:

(i) Compared to [CL18], the assumptions on the initial data $(u_0, v_0) \in \mathcal{G}_0 \cap X_{2,1}$ are improved.

(ii) While in [CL18] the functions $f_\pm$ are given implicitly, we provide an explicit derivation in terms of the reflection coefficient, see (6.1.31) and (6.1.32).

The full description of the long-time behavior of the massive Thirring model is finally given by the following theorem:

Theorem 1.3.3. *Suppose that $(u_0, v_0) \in \mathcal{G}_N$ with scattering data $\mathcal{S}(u_0, v_0) = (p; \{\lambda_k, C_k\}_{k=1}^N)$, where $|\lambda_j| \neq |\lambda_k|$ for $j \neq k$. In addition, assume that $r, \widehat{r} \in X_{-2,0}^{2,2}$ and consider the solution (u, v) of (1.1.1) with initial data (u_0, v_0). Then there exist norming constants $\widetilde{C}_1^\pm, ..., \widetilde{C}_N^\pm$ such that*

$$\left| u(t,x) - \sum_{k=1}^N u_{sol}(t,x;\{\lambda_k, \widetilde{C}_k^\pm\}) \right| \leq c|t|^{-1/2},$$
$$\left| v(t,x) - \sum_{k=1}^N v_{sol}(t,x;\{\lambda_k, \widetilde{C}_k^\pm\}) \right| \leq c|t|^{-1/2}.$$

It is believed that for many other dispersive equations, solutions with generic initial data should eventually resolve into finite many solitons. This conjecture is known as the *soliton resolution conjecture* and is unsolved for most of the equations. It could be proven rigorously only for equations that admit the method of inverse scattering and this thesis adds another example to the list of equations for which the conjecture is proven. See [Tao08] for a survey on the stability of solitons.

The thesis is organized as follows: We begin with a detailed construction of the scattering and inverse scattering transform in Chapters 2 and 3. Chapter 4 is devoted to the analysis of pure soliton solutions. Long-time asymptotics in the exterior region including the proof of Theorem 1.3.1 can be found in Chapter 5. Chapter 6 contains all the details for the steepest descent needed for the proof of Theorem 1.3.2. Finally, the proof of the soliton resolution is provided in Chapter 7.

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Chapter 2

Direct Scattering

2.1 Operator pair and Jost functions

As noted in the introductory chapter, the inverse scattering transform (IST) can be developed for a nonlinear evolution equation, if there is an association of the evolution equation with a pair of linear operators. In the case of the massive Thirring model, this pair is given by the following 2×2-matrices L and A, namely,

$$L = \frac{i}{4}(|u|^2 - |v|^2)\sigma_3 - \frac{i\lambda}{2}\begin{pmatrix} 0 & \overline{v} \\ v & 0 \end{pmatrix} + \frac{i}{2\lambda}\begin{pmatrix} 0 & \overline{u} \\ u & 0 \end{pmatrix} + \frac{i}{4}\left(\lambda^2 - \frac{1}{\lambda^2}\right)\sigma_3, \tag{2.1.1}$$

and

$$A = -\frac{i}{4}(|u|^2 + |v|^2)\sigma_3 - \frac{i\lambda}{2}\begin{pmatrix} 0 & \overline{v} \\ v & 0 \end{pmatrix} - \frac{i}{2\lambda}\begin{pmatrix} 0 & \overline{u} \\ u & 0 \end{pmatrix} + \frac{i}{4}\left(\lambda^2 + \frac{1}{\lambda^2}\right)\sigma_3. \tag{2.1.2}$$

In these definitions, $u = u(t,x)$ and $v = v(t,x)$ are functions $\mathbb{R} \times \mathbb{R} \to \mathbb{C}$ and λ is a spectral parameter, sometimes also referred to as the *scattering parameter*, which is independent of x and t. The association of the operator pair (A, L) with the massive Thirring model relies on the following fact that can be easily checked by a straight forward calculation: we have

$$[\partial_x - L, \partial_t - A] = 0, \tag{2.1.3}$$

for all $\lambda \in \mathbb{C} \setminus \{0\}$ if and only if u and v satisfy the MTM system (1.1.1). Alternatively, we may formulate the same statement in the following way: for a matrix-valued (or two-component vector) solution $\psi(\lambda; t, x)$ satisfying the (over-determined) system

$$\left\{ \begin{array}{l} \psi_x = L\psi \\ \psi_t = A\psi \end{array} \right. , \tag{2.1.4}$$

the equality of the mixed derivatives for all $\lambda \in \mathbb{C} \setminus \{0\}$, that is, $\psi_{xt} = \psi_{tx}$, is equivalent to the statement that u and v satisfy the MTM system (1.1.1). For this reason, we can understand the massive Thirring model (1.1.1) as the compatibility condition of equations (2.1.4). Commonly, (nonlinear) equations admitting a pair of operators in the sense as above described are said to be *integrable*. The pair (L, A) as given in (2.1.1)–(2.1.2) and thus the integrability of the massive Thirring model, was discovered in 1970s, see [Mik76, KM77, OW75].

Now, we want to define the so-called *Jost functions*. Therefore, we freeze the time variable t and drop it from the list of arguments. If $|u(x)| + |v(x)| \to 0$ as $|x| \to \infty$, then it is reasonable to assume that there exist 2×2-matrix-valued solutions $\psi^{(+)}$ and $\psi^{(-)}$ to the spectral problem

$$\psi_x = L\psi \tag{2.1.5}$$

with the following asymptotic behavior:

$$\psi^{(\pm)}(x; \lambda) \sim e^{ix(\lambda^2 - \lambda^{-2})\sigma_3/4} := \begin{bmatrix} e^{ix(\lambda^2-\lambda^{-2})/4} & 0 \\ 0 & e^{-ix(\lambda^2-\lambda^{-2})/4} \end{bmatrix}, \quad \text{as } x \to \pm\infty. \tag{2.1.6}$$

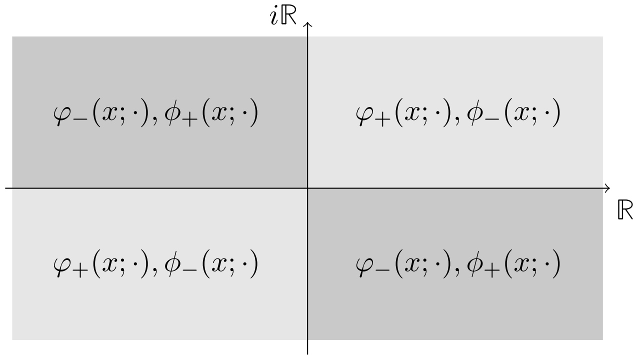


Figure 2.1: *Domains of analyticity of the normalized Jost functions.*

Writing $\psi^{(\pm)} = [\psi_1^{(\pm)}|\psi_2^{(\pm)}]$ with column vectors $\psi_j^{(\pm)}$, we can define the *normalized Jost functions* by

$$\varphi_\pm(x;\lambda) = \psi_1^{(\pm)}(x;\lambda)e^{-ix(\lambda^2-\lambda^{-2})/4}, \qquad \phi_\pm(x;\lambda) = \psi_2^{(\pm)}(x;\lambda)e^{ix(\lambda^2-\lambda^{-2})/4}. \tag{2.1.7}$$

They satisfy constant boundary conditions at infinity, that is,

$$\lim_{x\to\pm\infty} \varphi_\pm(x;\lambda) = e_1 \quad \text{and} \quad \lim_{x\to\pm\infty} \phi_\pm(x;\lambda) = e_2, \tag{2.1.8}$$

where $e_1 = (1,0)^T$ and $e_2 = (0,1)^T$. Moreover, the normalized Jost functions are solutions to the following Volterra's integral equations:

$$\begin{aligned} \varphi_\pm(x;\lambda) &= e_1 + \int_{\pm\infty}^x \begin{bmatrix} 1 & 0 \\ 0 & e^{-\frac{i}{2}(\lambda^2-\lambda^{-2})(x-y)} \end{bmatrix} Q(\lambda;u(y),v(y))\, \varphi_\pm(y;\lambda)dy, \\ \phi_\pm(x;\lambda) &= e_2 + \int_{\pm\infty}^x \begin{bmatrix} e^{\frac{i}{2}(\lambda^2-\lambda^{-2})(x-y)} & 0 \\ 0 & 1 \end{bmatrix} Q(\lambda;u(y),v(y))\, \phi_\pm(y;\lambda)dy. \end{aligned} \tag{2.1.9}$$

Here we are using the definition

$$Q(\lambda;u,v) := \frac{i}{4}(|u|^2-|v|^2)\sigma_3 - \frac{i\lambda}{2}\begin{pmatrix} 0 & \overline{v} \\ v & 0 \end{pmatrix} + \frac{i}{2\lambda}\begin{pmatrix} 0 & \overline{u} \\ u & 0 \end{pmatrix}, \tag{2.1.10}$$

which also makes it possible to rewrite the linear operator L in (2.1.1) in the form $L = Q(\lambda;u,v) + i(\lambda^2 - \lambda^{-2})\sigma_3/4$. The following lemma shows that we can associate Jost functions even to functions u and v for which we do not necessarily know that $|u(x)| + |v(x)| \to 0$ as $|x| \to \infty$.

Lemma 2.1.1. *Let $(u,v) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. For every $\lambda \in (\mathbb{R} \cup i\,\mathbb{R}) \setminus \{0\}$, there exist unique solutions $\varphi_\pm(\cdot;\lambda) \in L^\infty(\mathbb{R})$ and $\phi_\pm(\cdot;\lambda) \in L^\infty(\mathbb{R})$ satisfying the integral equations (2.1.9). Moreover, for every $x \in \mathbb{R}$, $\varphi_\pm(x;\cdot)$ and $\phi_\mp(x;\cdot)$ can be continued analytically in $\{\lambda \in \mathbb{C} : \lambda^2 \in \mathbb{C}^\pm\}$ and continuously in $\{\lambda \in \mathbb{C} : \lambda^2 \in \mathbb{C}^\pm\} \cup (\mathbb{R} \cup i\,\mathbb{R}) \setminus \{0\}$.*

Proof. The proof is standard and we repeat the main argument for the convenience of the reader. It suffices to prove the statement for the Jost function φ_-. By assumption we have $Q(\lambda;u(\cdot),v(\cdot)) \in L^1(\mathbb{R})$ and thus the operator K defined as

$$K[f](x) := \int_{-\infty}^x \begin{bmatrix} 1 & 0 \\ 0 & e^{-\frac{i}{2}(\lambda^2-\lambda^{-2})(x-y)} \end{bmatrix} Q(\lambda;u(y),v(y))\, f(y)dy$$

is a bounded operator from $L^\infty(\mathbb{R})$ to $L^\infty(\mathbb{R})$ for any fixed λ such that $\mathrm{Im}(\lambda^2) \le 0$. Moreover, we can find finitely many $-\infty = x_0 < x_1 < ... < x_{n-1} < x_n = +\infty$ such that K_j as

$$K_j[f](x) := \int_{x_{j-1}}^x \begin{bmatrix} 1 & 0 \\ 0 & e^{-\frac{i}{2}(\lambda^2-\lambda^{-2})(x-y)} \end{bmatrix} Q(\lambda;u(y),v(y))\, f(y)dy$$

is a contraction from $L^\infty(x_{j-1}, x_j)$ to $L^\infty(x_{j-1}, x_j)$ for every $j = 1, ..., n$. Set $f_0 \equiv e_1$. By the Banach Fixed Point Theorem, for every j we can find a function $f_j \in L^\infty(x_{j-1}, x_j)$ solving the equation $f_j(x) = f_{j-1}(x_j) + K_j[f_j](x)$ on the interval (x_{j-1}, x_j). These functions $f_1, ..., f_n$ can now be glued together and we find a continuous function in $L^\infty(\mathbb{R})$ satisfying Volterra's integral equation for φ_-, (2.1.9). The analyticity properties of the Jost function φ_- are proven by considering the above introduced operator K as an operator acting on functions $f(x; \lambda) \in L^\infty(\mathbb{R} \times B)$ where B is some compact subset of $\{\lambda \in \mathbb{C} \setminus \{0\} : \mathrm{Im}(\lambda^2) \leq 0\}$. There exists a positive constant c depending on B such that the inequality

$$|K^j[f](x; \lambda)| \leq \frac{\|f\|_{L^\infty(\mathbb{R}\times B)}}{j!} \left(c \int_{-\infty}^{x} |Q(\lambda; u(y), v(y))| dy \right)^j \tag{2.1.11}$$

holds for all $f(x; \lambda) \in L^\infty(\mathbb{R} \times B)$, all $(x; \lambda) \in \mathbb{R} \times B$ and all $j \in \mathbb{N}$. This inequality can be proven by induction as follows:

$$\begin{aligned} |K^{j+1}[f](x; \lambda)| &\leq c \frac{\|f\|_{L^\infty(\mathbb{R}\times B)}}{j!} \int_{-\infty}^{x} |Q(\lambda; u(y), v(y))| \left(\int_{-\infty}^{y} c|Q(\lambda; u(y'), v(y'))| dy' \right)^j dy \\ &= c \frac{\|f\|_{L^\infty(\mathbb{R}\times B)}}{(j+1)!} \int_{-\infty}^{x} \frac{d}{dy} \left(c \int_{-\infty}^{y} |Q(\lambda; u(y'), v(y'))| dy' \right)^{j+1} dy \\ &= \frac{\|f\|_{L^\infty(\mathbb{R}\times B)}}{(j+1)!} \left(c \int_{-\infty}^{x} |Q(\lambda; u(y'), v(y'))| dy' \right)^{j+1} . \end{aligned}$$

It follows from (2.1.11) that

$$\|K^j\|_{L^\infty(\mathbb{R}\times B) \to L^\infty(\mathbb{R}\times B)} \leq c \frac{\sup_{\lambda \in B} \|Q(\lambda; u(\cdot), v(\cdot))\|^j_{L^1(\mathbb{R})}}{j!},$$

and thus the Neumann series for the equation $f = e_1 + K[f]$ converges absolutely and uniformly for every $x \in \mathbb{R}$ and $\lambda \in B$. Therefore, $\varphi_-(x; \cdot)$ is analytic in B for every $x \in \mathbb{R}$. Since B is an arbitrary bounded open subset of $\{\lambda \in \mathbb{C} : \lambda^2 \in \mathbb{C}^-\}$, we find that φ_- can be continued analytically into the second and forth quadrant of the complex λ-plane. □

The assumptions of Lemma 2.1.1 guarantee that $Q(\lambda; u(\cdot), v(\cdot)) \in L^1(\mathbb{R})$ for every $\lambda \neq 0$. However, we are not able to control this L^1-norm uniformly in λ as $\lambda \to 0$ or $|\lambda| \to \infty$. In particular, the constant c in (2.1.11) does indeed depend on the choice of the subset B. We are not able to find a constant C that does not depend on λ such that

$$\|\varphi_\pm(\cdot; \lambda)\|_{L^\infty(\mathbb{R})} + \|\phi_\mp(\cdot; \lambda)\|_{L^\infty(\mathbb{R})} \leq C, \quad \lambda^2 \in \mathbb{C}^\pm .$$

This causes difficulties in studying the behaviour of $\varphi_\pm(\cdot; \lambda)$ and $\phi_\pm(\cdot; \lambda)$ as $\lambda \to 0$ and $|\lambda| \to \infty$ and thus we need to transform the spectral problem (2.1.5) into two equivalent forms. These transformations are performed in the subsequent section. We end the present section with two remarks on symmetry properties of the Jost functions.

Remark 2.1.2. We see that $L(-\lambda) = \sigma_3 L(\lambda) \sigma_3$ directly from the definition of L (2.1.1). It follows that the Jost functions $\psi^{(\pm)}$ defined by the (non-constant) boundary conditions (2.1.6) satisfy the same symmetry $\psi^{(\pm)}(x; -\lambda) = \sigma_3 \psi^{(\pm)}(x; \lambda) \sigma_3$ for all $x \in \mathbb{R}$. Thus, for the normalized Jost functions defined in (2.1.7) we conclude the following:

$$\varphi_\pm(x; -\lambda) = \sigma_3 \varphi_\pm(x; \lambda), \qquad \phi_\pm(x; -\lambda) = -\sigma_3 \phi_\pm(x; \lambda).$$

Alternatively, the same can be obtained from the integral equations (2.1.9).

Remark 2.1.3. Directly from the definition (2.1.1) of L we can see that

$$L(\overline{\lambda}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \overline{L(\lambda)} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

It follows that the Jost functions $\psi^\pm$ defined by the (non-constant) boundary conditions (2.1.6) satisfy the same symmetry

$$\psi^\pm(x;\overline{\lambda}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \overline{\psi^\pm(x;\lambda)} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

for all $x \in \mathbb{R}$. Thus, the normalized Jost functions defined in (2.1.7) admit the symmetry.

$$\phi_\pm(x;\overline{\lambda}) = \begin{bmatrix} 0 & \mp 1 \\ \pm 1 & 0 \end{bmatrix} \overline{\varphi_\pm(x;\lambda)}.$$

Alternatively, the same can be obtained from the integral equations (2.1.9).

2.2 Transformations of the Jost functions

In order to control their behavior at $\lambda = 0$ and $\lambda = \infty$, we have to apply certain transformations of the Jost functions. These transformations form the key element of the technique presented in the present thesis.

Let us define the following two new spectral parameters, given by

$$w := \lambda^{-2}, \qquad z := \lambda^2. \tag{2.2.1}$$

Assume $u, v \in L^\infty(\mathbb{R})$, $\lambda \neq 0$ and suppose that $\psi(x;\lambda)$ is a solution of the spectral problem (2.1.5). Now, define two new functions in the following way:

$$\Psi(x;w) = \begin{pmatrix} 1 & 0 \\ u(x) & \lambda^{-1} \end{pmatrix} \psi(x;\lambda), \qquad \widehat{\Psi}(x;z) = \begin{pmatrix} 1 & 0 \\ v(x) & \lambda \end{pmatrix} \psi(x;\lambda). \tag{2.2.2}$$

Here is a short motivation why these definitions make sense. Indeed, for given w or z there are always two possible choices of λ that satisfy (2.2.1), but thanks to the symmetries discussed in Remark 2.1.2 the right hand sides of both equations in (2.2.2) are even expressions with respect to λ and therefore, $\Psi(x;w)$ and $\widehat{\Psi}(x;z)$ are well-defined by these equations.

It is shown by direct computations that the transformations (2.2.2) make (2.1.5) equivalent to spectral problems

$$\Psi_x(x;w) = \mathcal{L}(x;w)\Psi(x;w), \qquad \widehat{\Psi}_x(x;z) = \widehat{\mathcal{L}}(x;z)\widehat{\Psi}(x;z), \tag{2.2.3}$$

with new linear operators $\mathcal{L}$ and $\widehat{\mathcal{L}}$ that are explicitly given by

$$\begin{aligned} \mathcal{L} &= \mathcal{Q}_1(u,v) + \lambda^2 \mathcal{Q}_2(u,v) + \frac{i}{4}\left(\lambda^2 - \frac{1}{\lambda^2}\right)\sigma_3 \\ &= \mathcal{Q}_1(u,v) + \frac{1}{w}\mathcal{Q}_2(u,v) - \frac{i}{4}\left(w - \frac{1}{w}\right)\sigma_3 \end{aligned} \tag{2.2.4}$$

where

$$\mathcal{Q}_1(u,v) = \begin{pmatrix} -\frac{i}{4}(|u|^2+|v|^2) & \frac{i}{2}\overline{u} \\ u_x - \frac{i}{2}u|v|^2 - \frac{i}{2}v & \frac{i}{4}(|u|^2+|v|^2) \end{pmatrix}, \quad \mathcal{Q}_2(u,v) = \frac{i}{2}\begin{pmatrix} u\overline{v} & -\overline{v} \\ u + u^2\overline{v} & -u\overline{v} \end{pmatrix},$$

and

$$\begin{aligned} \widehat{\mathcal{L}} &= \widehat{\mathcal{Q}}_1(u,v) + \frac{1}{\lambda^2}\widehat{\mathcal{Q}}_2(u,v) + \frac{i}{4}\left(\lambda^2 - \frac{1}{\lambda^2}\right)\sigma_3 \\ &= \widehat{\mathcal{Q}}_1(u,v) + \frac{1}{z}\widehat{\mathcal{Q}}_2(u,v) + \frac{i}{4}\left(z - \frac{1}{z}\right)\sigma_3 \end{aligned} \tag{2.2.5}$$

where

$$\widehat{\mathcal{Q}}_1(u,v) = \begin{pmatrix} \frac{i}{4}(|u|^2+|v|^2) & -\frac{i}{2}\overline{v} \\ v_x + \frac{i}{2}|u|^2 v + \frac{i}{2}u & -\frac{i}{4}(|u|^2+|v|^2) \end{pmatrix}, \quad \widehat{\mathcal{Q}}_2(u,v) = -\frac{i}{2}\begin{pmatrix} \overline{u}v & -\overline{u} \\ v + \overline{u}v^2 & -\overline{u}v \end{pmatrix}.$$

The normalized Jost functions associated to the spectral problems (2.2.3) denoted by $\{m_\pm, n_\pm\}$ and $\{\widehat{m}_\pm, \widehat{n}_\pm\}$, respectively, can be obtained from the original Jost functions $\{\varphi_\pm, \psi_\pm\}$ by the transformation formulas:

$$\begin{aligned} m_\pm(x;w) &= \begin{pmatrix} 1 & 0 \\ u(x) & \lambda^{-1} \end{pmatrix} \varphi_\pm(x;\lambda), & n_\pm(x;w) &= \begin{pmatrix} \lambda & 0 \\ \lambda\, u(x) & 1 \end{pmatrix} \phi_\pm(x;\lambda), \\ \widehat{m}_\pm(x;z) &= \begin{pmatrix} 1 & 0 \\ v(x) & \lambda \end{pmatrix} \varphi_\pm(x;\lambda), & \widehat{n}_\pm(x;z) &= \begin{pmatrix} \lambda^{-1} & 0 \\ \lambda^{-1} v(x) & 1 \end{pmatrix} \phi_\pm(x;\lambda), \end{aligned} \tag{2.2.6}$$

They satisfy the following constant boundary conditions at infinity:

$$\begin{aligned} \lim_{x\to\pm\infty} m_\pm(x;w) = e_1 \quad &\text{and} \quad \lim_{x\to\pm\infty} n_\pm(x;w) = e_2, \\ \lim_{x\to\pm\infty} \widehat{m}_\pm(x;z) = e_1 \quad &\text{and} \quad \lim_{x\to\pm\infty} \widehat{n}_\pm(x;z) = e_2. \end{aligned} \tag{2.2.7}$$

In what follows we give the Volterra's integral equations for the transformed normalized Jost functions $m_\pm$ and $n_\pm$:

$$\begin{aligned} m_\pm(x;w) &= e_1 + \int_{\pm\infty}^x \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{i}{2}(w-w^{-1})(x-y)} \end{pmatrix} \mathcal{Q}(w;u(y),v(y)) m_\pm(y;w)dy, \\ n_\pm(x;w) &= e_2 + \int_{\pm\infty}^x \begin{pmatrix} e^{-\frac{i}{2}(w-w^{-1})(x-y)} & 0 \\ 0 & 1 \end{pmatrix} \mathcal{Q}(w;u(y),v(y)) n_\pm(y;w)dy, \end{aligned} \tag{2.2.8}$$

with

$$\mathcal{Q}(w;u,v) := \mathcal{Q}_1(u,v) + \frac{1}{w}\mathcal{Q}_2(u,v).$$

Since the purpose of the transformation $\{\varphi_\pm, \phi_\pm\} \mapsto \{m_\pm, n_\pm\}$ was to gain control of the Jost functions at $\lambda = 0$ which corresponds to $w = \infty$, we have to compare the integral equations (2.1.9) for $\{\varphi_\pm, \phi_\pm\}$ with those for $\{m_\pm, n_\pm\}$, (2.2.8). We make the following observations:

- The assumptions of Lemma 2.1.1 are not sufficient for $\mathcal{Q}(w;u(\cdot),v(\cdot))$ to be in $L^1(\mathbb{R})$.
- $\mathcal{Q}(w;u(\cdot),v(\cdot)) \in L^1(\mathbb{R})$ requires additional assumptions on u and v. But assuming $\mathcal{Q}(w;u(\cdot),v(\cdot)) \in L^1(\mathbb{R})$, then the L^1-norm is also controlled uniformly in w if $w \geq r$ for some fixed positive r.

Correspondingly to these observations we have the following rigorous result.

Lemma 2.2.1. *Let $(u,v) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $u_x \in L^1(\mathbb{R})$. For every $w \in \mathbb{R}\backslash\{0\}$, there exist unique solutions $m_\pm(\cdot;w) \in L^\infty(\mathbb{R})$ and $n_\pm(\cdot;w) \in L^\infty(\mathbb{R})$, satisfying the integral equations (2.2.8). Moreover, for every $x \in \mathbb{R}$, $m_\mp(x;\cdot)$ and $n_\pm(x;\cdot)$ can be continued analytically in $\mathbb{C}^\pm$ and continuously in $\mathbb{C}^\pm \cup \mathbb{R}\backslash\{0\}$. Also, for any fixed $r > 0$, there exists a positive constant C depending on r such that*

$$\|m_\mp(\cdot;w)\|_{L^\infty(\mathbb{R})} + \|n_\pm(\cdot;w)\|_{L^\infty(\mathbb{R})} \leq C, \quad w \in \mathbb{C}^\pm, |w| \geq r. \tag{2.2.9}$$

Proof. The assumptions on u and v imply that $\mathcal{Q}(w;u(\cdot),v(\cdot)) \in L^1(\mathbb{R})$ for all $w \in \mathbb{C}\backslash\{0\}$. Thus, the existence of the Jost functions for $w \in \mathbb{R}\backslash\{0\}$ and their analyticity properties can be obtained by the same analysis as in the proof of Lemma 2.1.1. The global bound (2.2.9) follows from the fact that $\|\mathcal{Q}(w;u(\cdot),v(\cdot))\|_{L^1(\mathbb{R})}$ is controlled uniformly in w as long w is away from the origin. □

The next lemma outlines the usefulness of the transformations (2.2.2). Whereas the original integral equations (2.1.9) were not controllable at $\lambda = 0$, we will derive very explicit expansions for $m_\pm$ and $n_\pm$ at $w = \infty$ in terms of the functions u and v. These expansions create the basis for the inverse scattering problem.

Lemma 2.2.2. *Under the conditions of Lemma 2.2.1, for every $x \in \mathbb{R}$ the normalized Jost functions $m_\pm$ and $n_\pm$ satisfy the following limits as $|\operatorname{Im}(w)| \to \infty$ along a contour in the domains of their analyticity:*

$$\lim_{|w|\to\infty} m_\pm(x;w) = \begin{pmatrix} m_\pm^\infty(x) \\ 0 \end{pmatrix}, \quad \lim_{|w|\to\infty} n_\pm(x;w) = \begin{pmatrix} 0 \\ n_\pm^\infty(x) \end{pmatrix}, \tag{2.2.10}$$

where

$$m_\pm^\infty(x) = e^{-\frac{i}{4}\int_{\pm\infty}^x (|u|^2+|v|^2)dy}, \quad n_\pm^\infty(x) = e^{\frac{i}{4}\int_{\pm\infty}^x (|u|^2+|v|^2)dy}. \tag{2.2.11}$$

If in addition $u \in C^1(\mathbb{R})$ and $v \in C^0(\mathbb{R})$, then

$$\begin{aligned} \lim_{|w|\to\infty} w\left[m_\pm(x;w) - \begin{pmatrix} m_\pm^\infty(x) \\ 0 \end{pmatrix}\right] &= \begin{pmatrix} \mathfrak{q}_\pm^{(1)}(x) \\ \mathfrak{q}_\pm^{(2)}(x) \end{pmatrix}, \\ \lim_{|w|\to\infty} w\left[n_\pm(x;w) - \begin{pmatrix} 0 \\ n_\pm^\infty(x) \end{pmatrix}\right] &= \begin{pmatrix} \mathfrak{r}_\pm^{(1)}(x) \\ \mathfrak{r}_\pm^{(2)}(x) \end{pmatrix}, \end{aligned} \tag{2.2.12}$$

where

$$\begin{aligned} \mathfrak{q}_\pm^{(1)}(x) &:= -m_\pm^\infty(x)\int_{\pm\infty}^x \left[\overline{u}\left(u_x - \frac{i}{2}u|v|^2 - \frac{i}{2}v\right) - \frac{i}{2}u\overline{v}\right]dy, \\ \mathfrak{q}_\pm^{(2)}(x) &:= m_\pm^\infty(x)(2iu_x(x) + u(x)|v(x)|^2 + v(x)), \\ \mathfrak{r}_\pm^{(1)}(x) &:= n_\pm^\infty(x)\overline{u}(x), \\ \mathfrak{r}_\pm^{(2)}(x) &:= n_\pm^\infty(x)\int_{\pm\infty}^x \left[\overline{u}\left(u_x - \frac{i}{2}u|v|^2 - \frac{i}{2}v\right) - \frac{i}{2}u\overline{v}\right]dy. \end{aligned} \tag{2.2.13}$$

Proof. We prove the assertion for m_-. Let us denote the two-component vector m_- by $(m_-^{(1)}, m_-^{(2)})^t$. Rewriting the second component of the first integral equation in (2.2.8) yields

$$m_-^{(2)}(x;w) = \int_{-\infty}^x e^{\frac{i}{2}(w-w^{-1})(x-y)}\left[\mathcal{Q}(w;u(y),v(y))m_-(y;w)\right]_2 dy.$$

Due to the assumptions on u and v and by the global bound (2.2.9), the integrand is bounded for every $w \in \mathbb{C}^+, |w| \geq r$ by an w-independent L^1-function. Additionally, for $w \in \mathbb{C}^+$ and $|\operatorname{Im}(w)| \to \infty$ the exponential factor converges to zero. Thus, Lebesgue's Dominated Convergence Theorem applies and we conclude that $\lim_{|w|\to\infty} m_-^{(2)}(x;w) = 0$. On the other hand, by taking the limit $|w| \to \infty$ in the first component of the right hand side of (2.2.8), we find that $m_-^\infty(x) := \lim_{|w|\to\infty} m_-^{(1)}(x;w)$ satisfies the following integral equation:

$$m_-^\infty(x) = 1 + \int_{-\infty}^x \left[\mathcal{Q}_1(u(y),v(y))\right]_{11} m_-^\infty(y)dy. \tag{2.2.14}$$

Substituting the definition of $\mathcal{Q}_1$ and solving the integral equation explicitly we are able to obtain the formula given in (2.2.11).
Now, let us define

$$H(x;w) := \left[\mathcal{Q}(w;u(x),v(x))\right]_{21} m_-^{(1)}(x;w) + \left[\mathcal{Q}(w;u(x),v(x))\right]_{22} m_-^{(2)}(x;w),$$

such that $m_-^{(2)}(x;w) = I + II + III$ with

$$\begin{aligned} I &:= \int_{-\infty}^{x-\delta} e^{\frac{i}{2}(w-w^{-1})(x-y)}H(y;w)dy, \\ II &:= H(x;w)\int_{x-\delta}^x e^{\frac{i}{2}(w-w^{-1})(x-y)}dy, \\ III &:= \int_{x-\delta}^x e^{\frac{i}{2}(w-w^{-1})(x-y)}[H(y;w) - H(x;w)]dy, \end{aligned}$$

for $\delta = (\mathrm{Im}(w))^{-1/2}$. Since $H(\cdot; w) \in L^1(\mathbb{R})$, the summand I is decaying exponentially as $\mathrm{Im}(w) \to \infty$. For the second we have the exact value

$$II = H(x;w)\frac{2i}{w-w^{-1}}\left(1 - e^{\frac{i}{2}\frac{w-w^{-1}}{(\mathrm{Im}(w))^{1/2}}}\right).$$

In order to estimate III we assume $u \in C^1(\mathbb{R})$, such that $H(\cdot; w) \in C^0(\mathbb{R})$. From this continuity it follows that $\|H(\cdot; w) - H(x; w)\|_{L^\infty(x-\delta,x)} \to 0$ as $\delta \to 0$. Additionally, by evaluating the exponential factor we find that

$$|III| \le \frac{c}{|w|}\|H(\cdot;w) - H(x;w)\|_{L^\infty(x-\delta,x)}.$$

Altogether, we learn that only summand II contributes to the limit of $w \cdot m_-^{(2)}(x; w)$ as $\mathrm{Im}(w) \to \infty$. We obtain

$$\mathfrak{q}_-^{(2)}(x) = \lim_{|w|\to\infty} w\, m_-^{(2)}(x;w) = 2i \lim_{|w|\to\infty} H(x;w) = 2i\,[\mathcal{Q}_1(u(x),v(x))]_{21}\, m_-^\infty(x). \tag{2.2.15}$$

For the expansion of $m_-^{(1)}(x; w)$ we can use the integral equation (2.2.8) to obtain

$$\partial_x(m_-^{(1)}(x;w)) = [\mathcal{Q}(w;u(x),v(x))m_-(x;w)]_1 \tag{2.2.16}$$

and the integral equation (2.2.14) to get

$$\partial_x(m_-^\infty(x)) = [\mathcal{Q}_1(u(x),v(x))]_{11}\, m_-^\infty(x). \tag{2.2.17}$$

From these formal expressions for the derivatives it follows that

$$\partial_x\left(\frac{m_-^{(1)}(x;w)}{m_-^\infty(x)}\right) = \frac{[\mathcal{Q}(w;u(x),v(x))m_-(x;w)]_1 - [\mathcal{Q}_1(w;u(x),v(x))]_{11}\, m_-^{(1)}(x;w)}{m_-^\infty(x)}.$$

Converting this back to an integral representation and multiplying by $m_-^\infty(x)$ we get

$$\begin{aligned} &m_-^{(1)}(x;w) - m_-^\infty(x) \\ &\quad = m_-^\infty(x)\int_{-\infty}^x \frac{[\mathcal{Q}(w;u(y),v(y))m_-(y;w)]_1 - [\mathcal{Q}_1(w;u(y),v(y))]_{11}\, m_-^{(1)}(y;w)}{m_-^\infty(y)}dy \\ &\quad = m_-^\infty(x)\int_{-\infty}^x \frac{[\mathcal{Q}_1(u(y),v(y))]_{12}\, m_-^{(2)}(y;w) + \frac{1}{w}\,[\mathcal{Q}_2(u(y),v(y))m_-(y;w)]_1}{m_-^\infty(y)}dy. \end{aligned} \tag{2.2.18}$$

This explicit formula for $m_-^{(1)} - m_-^\infty$ and the known limits (2.2.10) and (2.2.15) enable us to compute

$$\begin{aligned} \mathfrak{q}_-^{(1)}(x) &= \lim_{|w|\to\infty} w\,(m_-^{(1)}(x;w) - m_-^\infty(x)) \\ &= m_-^\infty(x)\int_{-\infty}^x 2i\,[\mathcal{Q}_1(u(y),v(y))]_{12}\,[\mathcal{Q}_1(u(y),v(y))]_{21} + [\mathcal{Q}_2(u(y),v(y))]_{11}\,dy. \end{aligned} \tag{2.2.19}$$

Substituting the definition of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ yields the limit (2.2.12) for $m_-^{(1)}$. The rest of the Jost functions are analyzed in an analogous way and this concludes the proof of the lemma. $\square$

The results of Lemmas 2.2.1 and 2.2.2 can be restated in terms of the second set of transformed Jost functions $\{\widehat{m}_\pm, \widehat{n}_\pm\}$. These Jost functions satisfy the following integral equations:

$$\begin{aligned} \widehat{m}_\pm(x;z) &= e_1 + \int_{\pm\infty}^x \begin{pmatrix} 1 & 0 \\ 0 & e^{-\frac{i}{2}(z-z^{-1})(x-y)} \end{pmatrix} \widehat{\mathcal{Q}}(z;u(y),v(y))\widehat{m}_\pm(y;z)dy, \\ \widehat{n}_\pm(x;z) &= e_2 + \int_{\pm\infty}^x \begin{pmatrix} e^{\frac{i}{2}(z-z^{-1})(x-y)} & 0 \\ 0 & 1 \end{pmatrix} \widehat{\mathcal{Q}}(z;u(y),v(y))\widehat{n}_\pm(y;z)dy, \end{aligned} \tag{2.2.20}$$

with

$$\widehat{\mathcal{Q}}(z;u,v) := \widehat{\mathcal{Q}}_1(u,v) + \frac{1}{z}\widehat{\mathcal{Q}}_2(u,v).$$

The equivalent formulation in terms of the transformed Jost functions are given below without proofs, as they are analogous to the previous ones.

Lemma 2.2.3. *Let $u, v \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $v_x \in L^1(\mathbb{R})$. For every $z \in \mathbb{R}\backslash\{0\}$, there exist unique solutions $\widehat{m}_\pm(\cdot;z) \in L^\infty(\mathbb{R})$ and $\widehat{n}_\pm(\cdot;z) \in L^\infty(\mathbb{R})$ satisfying the integral equations (2.2.20). Moreover, for every $x \in \mathbb{R}$, $\widehat{m}_\pm(x;\cdot)$ and $\widehat{n}_\mp(x;\cdot)$ are continued analytically in $\mathbb{C}^\pm$ and continuously in $\mathbb{C}^\pm \cup \mathbb{R}\backslash\{0\}$. Also, for any fixed $r > 0$, there exists a positive constant C depending on r such that*

$$\|\widehat{m}_\pm(\cdot;z)\|_{L^\infty(\mathbb{R})} + \|\widehat{n}_\mp(\cdot;z)\|_{L^\infty(\mathbb{R})} \le C, \quad z \in \mathbb{C}^\pm, |z| \ge r. \tag{2.2.21}$$

Lemma 2.2.4. *Under the conditions of Lemma 2.2.3, for every $x \in \mathbb{R}$, the normalized Jost functions $\widehat{m}_\pm$ and $\widehat{n}_\pm$ satisfy the following limits as $|\operatorname{Im}(z)| \to \infty$ along a contour in the domains of their analyticity:*

$$\lim_{|z|\to\infty} \widehat{m}_\pm(x;z) = \begin{pmatrix} \widehat{m}_\pm^\infty(x) \\ 0 \end{pmatrix}, \quad \lim_{|w|\to\infty} \widehat{n}_\pm(x;w) = \begin{pmatrix} 0 \\ \widehat{n}_\pm^\infty(x) \end{pmatrix}, \tag{2.2.22}$$

where

$$\widehat{m}_\pm^\infty(x) = e^{\frac{i}{4}\int_{\pm\infty}^x (|u|^2+|v|^2)dy}, \quad \widehat{n}_\pm^\infty(x) = e^{-\frac{i}{4}\int_{\pm\infty}^x (|u|^2+|v|^2)dy}. \tag{2.2.23}$$

If in addition $u \in C^0(\mathbb{R})$ and $v \in C^1(\mathbb{R})$, then

$$\begin{aligned} &\lim_{|z|\to\infty} z\left[\widehat{m}_\pm(x;z) - \begin{pmatrix} \widehat{m}_\pm^\infty(x) \\ 0 \end{pmatrix}\right] = \begin{pmatrix} \widehat{\mathfrak{q}}_\pm^{(1)}(x) \\ \widehat{\mathfrak{q}}_\pm^{(2)}(x) \end{pmatrix}, \\ &\lim_{|z|\to\infty} z\left[\widehat{n}_\pm(x;z) - \begin{pmatrix} 0 \\ \widehat{n}_\pm^\infty(x) \end{pmatrix}\right] = \begin{pmatrix} \widehat{\mathfrak{r}}_\pm^{(1)}(x) \\ \widehat{\mathfrak{r}}_\pm^{(2)}(x) \end{pmatrix}, \end{aligned} \tag{2.2.24}$$

where

$$\begin{aligned} \widehat{\mathfrak{q}}_\pm^{(1)}(x) &:= -\widehat{m}_\pm^\infty(x)\int_{\pm\infty}^x \left[\overline{v}\left(v_x + \frac{i}{2}|u|^2 v + \frac{i}{2}u\right) + \frac{i}{2}\overline{u}v\right] dy, \\ \widehat{\mathfrak{q}}_\pm^{(1)}(x) &:= \widehat{m}_\pm^\infty(x)(-2iv_x(x) + |u(x)|^2 v(x) + u(x), \\ \widehat{\mathfrak{r}}_\pm^{(1)}(x) &:= \widehat{n}_\pm^\infty(x)\overline{v}(x), \\ \widehat{\mathfrak{r}}_\pm^{(2)}(x) &:= \widehat{n}_\pm^\infty(x)\int_{\pm\infty}^x \left[\overline{v}\left(v_x + \frac{i}{2}|u|^2 v + \frac{i}{2}u\right) + \frac{i}{2}\overline{u}v\right] dy. \end{aligned} \tag{2.2.25}$$

2.3 Continuation of the transformed Jost functions to zero

In Lemma 2.2.1 we showed the existence of the transformed Jost functions $\{m_\pm(\cdot;w), n_\pm(\cdot;w)\}$ for $w \neq 0$ and $\operatorname{Im}(w) \ge 0$ or $\operatorname{Im}(w) \le 0$ depending on the Jost function. Analogously, in Lemma 2.2.3 we showed the existence of the transformed Jost functions $\{\widehat{m}_\pm(\cdot;z), \widehat{n}_\pm(\cdot;z)\}$ for $z \neq 0$. But we did not show the existence of the limits

$$\lim_{w\to 0} m_\pm(x;w), \quad \lim_{w\to 0} n_\pm(x;w), \quad \lim_{z\to 0} \widehat{m}_\pm(x;z), \quad \lim_{z\to 0} \widehat{n}_\pm(x;z).$$

Because both sets of the transformed Jost functions are connected to the set $\{\varphi_\pm, \phi_\pm\}$ of the original Jost functions by the transformation formulas (2.2.6), we find the following connection formulas for every $w \neq 0$:

$$\begin{aligned} m_\pm(x;w) &= \begin{bmatrix} 1 & 0 \\ u(x) - w\,v(x) & w \end{bmatrix} \widehat{m}_\pm(x;w^{-1}), \\ n_\pm(x;w) &= \begin{bmatrix} w^{-1} & 0 \\ u(x)w^{-1} - v(x) & 1 \end{bmatrix} \widehat{n}_\pm(x;w^{-1}), \end{aligned} \tag{2.3.1}$$

Figure 2.2: *Domains of analyticity of the transformed Jost functions. Because the new spectral parameters are connected by $w = z^{-1}$ the domains of analyticity of, for instance, $m_\pm(x;\cdot)$ coincide with those of $\widehat{m}_\mp(x;\cdot)$.*

or in the opposite direction for $z \neq 0$,

$$\begin{aligned} \widehat{m}_\pm(x;z) &= \begin{bmatrix} 1 & 0 \\ v(x) - zu(x) & z \end{bmatrix} m_\pm(x;z), \\ \widehat{n}_\pm(x;z) &= \begin{bmatrix} z^{-1} & 0 \\ v(x)z^{-1} - u(x) & 1 \end{bmatrix} n_\pm(x;z). \end{aligned} \tag{2.3.2}$$

By Lemmas 2.2.3 and 2.2.4, the right-hand sides of (2.3.1) yield the following limits as $w \to 0$ along a contour in the domains of their analyticity:

$$\lim_{w\to 0} m_\pm(x;w) = \begin{pmatrix} \widehat{m}_\pm^\infty(x) \\ \widehat{m}_\pm^\infty(x)u(x) \end{pmatrix}, \quad \lim_{w\to 0} n_\pm(x;w) = \begin{pmatrix} \widehat{n}_\pm^\infty(x)\bar{v}(x) \\ \widehat{n}_\pm^\infty(x)(1 + u(x)\bar{v}(x)) \end{pmatrix}. \tag{2.3.3}$$

Analogously, by Lemmas 2.2.1 and 2.2.2, the right-hand sides of (2.3.2), yield the following limits as $z \to 0$ along a contour in the domains of their analyticity:

$$\lim_{z\to 0} \widehat{m}_\pm(x;z) = \begin{pmatrix} m_\pm^\infty(x) \\ m_\pm^\infty(x)v(x) \end{pmatrix}, \quad \lim_{z\to 0} \widehat{n}_\pm(x;z) = \begin{pmatrix} n_\pm^\infty(x)\bar{u}(x) \\ n_\pm^\infty(x)(1 + \bar{u}(x)v(x)) \end{pmatrix}. \tag{2.3.4}$$

By Lemmas 2.2.1, 2.2.3 and the continuation formulas (2.3.1), (2.3.2), we obtain the following result.

Lemma 2.3.1. *Under the conditions of Lemmas 2.2.1 and 2.2.3, for every $x \in \mathbb{R}$, the Jost functions defined by the integral equations (2.2.8) and (2.2.20) can be continued such that $m_\mp(x;\cdot)$, $n_\pm(x;\cdot)$, $\widehat{m}_\pm(x;\cdot)$, and $\widehat{n}_\mp(x;\cdot)$ are analytic in $\mathbb{C}^\pm$ and continuous in $\mathbb{C}^\pm \cup \mathbb{R}$ with bounded limits as $w, z \to 0$ and $|w|, |z| \to \infty$ given by (2.2.10), (2.2.22), (2.3.3), (2.3.4).*

Remark 2.3.2. By Sobolev's embedding of $H^1(\mathbb{R})$ into the space of continuous, bounded, and decaying at infinity functions, if $u, v \in H^1(\mathbb{R})$, then $u, v \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $u(x), v(x) \to 0$ as $|x| \to \infty$. By the embedding of $L^{2,1}(\mathbb{R})$ into $L^1(\mathbb{R})$, if $u, v \in H^{1,1}(\mathbb{R})$, then $u, v \in L^1(\mathbb{R})$ and $u_x, v_x \in L^1(\mathbb{R})$. Thus, requirements of Lemmas 2.2.1 and 2.2.3 are satisfied if $(u, v) \in H^{1,1}(\mathbb{R})$. The additional requirement $u, v \in C^1(\mathbb{R})$ of Lemmas 2.2.2 and 2.2.4 is satisfied if $u, v \in H^2(\mathbb{R})$. Hence,

$$X_{2,1} := H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R}) \tag{2.3.5}$$

is an optimal L^2-based Sobolev space for direct scattering of the MTM system (1.1.1).

2.4 Properties of the transformed Jost functions on the real axis

Once we want to study properties of the scattering coefficients in suitable function spaces, the following lemma will become important. We note that if $u, v \in H^{1,1}(\mathbb{R})$, then the conditions of Lemmas 2.2.1 and 2.2.3 are satisfied such that we do not need to worry about the existence of the Jost functions. In the second part of the following lemma we assume $u, v \in H^2(\mathbb{R})$, such that also the additional conditions of Lemmas 2.2.2 and 2.2.4 are satisfied.

Lemma 2.4.1. *Let $(u,v) \in H^{1,1}(\mathbb{R})$. Then for every $x \in \mathbb{R}^{\pm}$, we have*

$$m_{\pm}(x;w) - m_{\pm}^{\infty}(x)e_1 \in H^1_w(\mathbb{R}\setminus[-1,1]), \quad n_{\pm}(x;w) - n_{\pm}^{\infty}(x)e_2 \in H^1_w(\mathbb{R}\setminus[-1,1]) \tag{2.4.1}$$

and

$$\widehat{m}_{\pm}(x;z) - \widehat{m}_{\pm}^{\infty}(x)e_1 \in H^1_z(\mathbb{R}\setminus[-1,1]), \quad \widehat{n}_{\pm}(x;z) - \widehat{n}_{\pm}^{\infty}(x)e_2 \in H^1_z(\mathbb{R}\setminus[-1,1]). \tag{2.4.2}$$

If in addition $(u,v) \in H^2(\mathbb{R})$, then for every $x \in \mathbb{R}^{\pm}$, we have

$$\begin{aligned} &w\left[m_{\pm}(x;w) - \begin{pmatrix} m_{\pm}^{\infty}(x) \\ 0 \end{pmatrix}\right] - \begin{pmatrix} \mathfrak{q}_{\pm}^{(1)}(x) \\ \mathfrak{q}_{\pm}^{(2)}(x) \end{pmatrix} \in L^2_w(\mathbb{R}\setminus[-1,1]), \\ &w\left[n_{\pm}(x;w) - \begin{pmatrix} 0 \\ n_{\pm}^{\infty}(x) \end{pmatrix}\right] - \begin{pmatrix} \mathfrak{r}_{\pm}^{(1)}(x) \\ \mathfrak{r}_{\pm}^{(2)}(x) \end{pmatrix} \in L^2_w(\mathbb{R}\setminus[-1,1]), \end{aligned} \tag{2.4.3}$$

and

$$\begin{aligned} &z\left[\widehat{m}_{\pm}(x;z) - \begin{pmatrix} \widehat{m}_{\pm}^{\infty}(x) \\ 0 \end{pmatrix}\right] - \begin{pmatrix} \widehat{\mathfrak{q}}_{\pm}^{(1)}(x) \\ \widehat{\mathfrak{q}}_{\pm}^{(2)}(x) \end{pmatrix} \in L^2_z(\mathbb{R}\setminus[-1,1]), \\ &z\left[\widehat{n}_{\pm}(x;z) - \begin{pmatrix} 0 \\ \widehat{n}_{\pm}^{\infty}(x) \end{pmatrix}\right] - \begin{pmatrix} \widehat{\mathfrak{r}}_{\pm}^{(1)}(x) \\ \widehat{\mathfrak{r}}_{\pm}^{(2)}(x) \end{pmatrix} \in L^2_z(\mathbb{R}\setminus[-1,1]). \end{aligned} \tag{2.4.4}$$

Proof. Again, we prove the statement for the Jost function m_-. The proof for the other Jost functions is similar. Analogously to the proof of Lemma 2.1.1 we consider the operator

$$\mathcal{K}[f](x;w) := \int_{-\infty}^{x} \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i}{2}(w-w^{-1})(x-y)} \end{bmatrix} \mathcal{Q}(w;u(y),v(y))\, f(y;w)dy. \tag{2.4.5}$$

By similar estimates as in the proof of (2.1.11), we can find a constant c such that for all functions $f(x;w) \in L^\infty_x(\mathbb{R}, L^2_w(\mathbb{R}\setminus[-1,1]))$, all $x \in \mathbb{R}$ and all $j \in \mathbb{N}$, we have

$$\begin{aligned} &\|\mathcal{K}^j[f](x;\cdot)\|_{L^2(\mathbb{R}\setminus[-1,1])} \le \\ &\qquad c\frac{\sup_{y\le x}\|f(y;\cdot)\|_{L^2(\mathbb{R}\setminus[-1,1])}}{j!}\left(\|\mathcal{Q}_1(u(\cdot),v(\cdot))\|_{L^1(-\infty,x)} + \|\mathcal{Q}_2(u(\cdot),v(\cdot))\|_{L^1(-\infty,x)}\right)^j. \end{aligned} \tag{2.4.6}$$

Moreover, for the same constant, for all functions f with $\langle x\rangle f(x;w) \in L^\infty_x(\mathbb{R}, L^2_w(\mathbb{R}\setminus[-1,1]))$, all $x \le 0$ and all $j \in \mathbb{N}$ we have the estimate

$$\begin{aligned} &\langle x\rangle\|\mathcal{K}^j[f](x;\cdot)\|_{L^2(\mathbb{R}\setminus[-1,1])} \le \\ &\qquad c\frac{\sup_{y\le x}\langle y\rangle\|f(y;\cdot)\|_{L^2(\mathbb{R}\setminus[-1,1])}}{j!}\left(\|\mathcal{Q}_1(u(\cdot),v(\cdot))\|_{L^1(-\infty,x)} + \|\mathcal{Q}_2(u(\cdot),v(\cdot))\|_{L^1(-\infty,x)}\right)^j. \end{aligned} \tag{2.4.7}$$

For $x_0 \in \mathbb{R}\cup\{\infty\}$, let us use the notation

$$\begin{aligned} X_0(x_0) &:= L^\infty_x((-\infty,x_0), L^2_w(\mathbb{R}\setminus[-1,1])), \\ X_1(x_0) &:= \left\{ f \in X_0(x_0) : \sup_{y\le x_0}\langle y\rangle\|f(y;\cdot)\|_{L^2(\mathbb{R}\setminus[-1,1])} < \infty \right\}. \end{aligned}$$

By (2.4.6) we can consider $\mathcal{K}$ as an operator $X_0(x_0) \to X_0(x_0)$ such that $(1-\mathcal{K})$ is invertible with the bound

$$\|(1-\mathcal{K})^{-1}\|_{X_0(x_0)\to X_0(x_0)} \le c\exp\left(\|\mathcal{Q}_1(u(\cdot),v(\cdot))\|_{L^1(-\infty,x_0)} + \|\mathcal{Q}_2(u(\cdot),v(\cdot))\|_{L^1(-\infty,x_0)}\right) \tag{2.4.8}$$

for every $x_0 \in \mathbb{R}\cup\{\infty\}$. Moreover, for every $x_0 \le 0$, by (2.4.7) we can consider $\mathcal{K}$ as an operator $X_1(x_0) \to X_1(x_0)$ such that $(1-\mathcal{K})$ is invertible with the bound

$$\|(1-\mathcal{K})^{-1}\|_{X_1(x_0)\to X_1(x_0)} \le c\exp\left(\|\mathcal{Q}_1(u(\cdot),v(\cdot))\|_{L^1(-\infty,x_0)} + \|\mathcal{Q}_2(u(\cdot),v(\cdot))\|_{L^1(-\infty,x_0)}\right). \tag{2.4.9}$$

The integral equation for the Jost function m_- is given in terms of the operator $\mathcal{K}$ by

$$m_- = e_1 + \mathcal{K}[m_-],$$

which in turn is equivalent to

$$(1-\mathcal{K})[m_- - m_-^\infty e_1] = h$$

with $h := e_1 - m_-^\infty e_1 + \mathcal{K}[m_-^\infty e_1]$. Thanks to the bound (2.4.8) it suffices to show that $h \in X_0(x_0)$ in order to conclude $(m_- - m_-^\infty e_1) \in X_0(x_0)$. An explicit calculation with the integral equation (2.2.14) for $m_-^\infty(x)$ yields

$$\begin{aligned} h(x;w) = {} & e_1 \frac{1}{w}\int_{-\infty}^x [\mathcal{Q}_2(u(y),v(y))]_{11} m_-^\infty(y)dy \\ & + e_2 \frac{1}{w}\int_{-\infty}^x e^{\frac{i}{2}(w-w^{-1})(x-y)} [\mathcal{Q}_2(u(y),v(y))]_{21} m_-^\infty(y)dy \\ & + e_2 \int_{-\infty}^x e^{\frac{i}{2}(w-w^{-1})(x-y)} [\mathcal{Q}_1(u(y),v(y))]_{21} m_-^\infty(y)dy \end{aligned}$$

Clearly, the first summand is in $X_0(x_0)$ if $[\mathcal{Q}_2(u(\cdot),v(\cdot))]_{11} \in L^1(\mathbb{R})$ which is guaranteed by the assumptions of the lemma. Also the first summand is in $X_1(x_0)$, $x_0 \le 0$, if $[\mathcal{Q}_2(u(\cdot),v(\cdot))]_{11} \in L^{1,1}(\mathbb{R})$. Using the new variable $s := \frac{w-w^{-1}}{4\pi}$, we can estimate the $L^2_w(\mathbb{R}\setminus[-1,1])$-norm of the second and third summand by the $L^2_s(\mathbb{R})$-norm of

$$\int_{-\infty}^x e^{2\pi i s(x-y)}\,\mathfrak{f}(y)dy, \quad \mathfrak{f}(y) = \begin{cases} [\mathcal{Q}_2(u(y),v(y))]_{21} m_-^\infty(y), & \text{second summand;} \\ [\mathcal{Q}_1(u(y),v(y))]_{21} m_-^\infty(y), & \text{third summand.} \end{cases}$$

From standard Fourier theory we know that

$$\left\| \int_{-\infty}^x e^{2\pi i s(x-y)}\,\mathfrak{f}(y)dy \right\|^2_{L^2_s(\mathbb{R})} = \int_{-\infty}^x |\mathfrak{f}(y)|^2 dy \le \frac{1}{\langle x\rangle^2}\int_{-\infty}^x \langle y\rangle^2 |\mathfrak{f}(y)|^2 dy \tag{2.4.10}$$

and thus, altogether:

$$x_0 \in \mathbb{R}\cup\{\infty\}: \quad \|h\|_{X_0(x_0)} \le c\left\{\|[\mathcal{Q}_2]_{11}\|_{L^1(\mathbb{R})} + \|[\mathcal{Q}_2]_{21}\|_{L^2(\mathbb{R})} + \|[\mathcal{Q}_1]_{21}\|_{L^2(\mathbb{R})}\right\}, \tag{2.4.11}$$

such that $(m_-(x;\cdot) - m_-^\infty(x)e_1) \in L^2(\mathbb{R}\setminus[-1,1])$ for all $x \in \mathbb{R}$. On the other hand we also have the following stronger result that will be used later:

$$x_0 \le 0: \quad \|h\|_{X_1(x_0)} \le c\left\{\|[\mathcal{Q}_2]_{11}\|_{L^{1,1}(\mathbb{R})} + \|[\mathcal{Q}_2]_{21}\|_{L^{2,1}(\mathbb{R})} + \|[\mathcal{Q}_1]_{21}\|_{L^{2,1}(\mathbb{R})}\right\}. \tag{2.4.12}$$

For the proof of $\partial_w m_- \in X_0(x_0)$ for all $x_0 \le 0$ we differentiate the integral equation (2.2.8) in w and obtain

$$\begin{aligned} \partial_w m_-(x;w) = {} & \mathcal{K}[\partial_w m_-](x;w) + \frac{i}{2}(1+w^{-2})x \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathcal{K}[m_-](x;w) \\ & - \frac{i(1+w^{-2})}{2}\int_{-\infty}^x y \begin{bmatrix} 0 & 0 \\ 0 & e^{\frac{i}{2}(w-w^{-1})(x-y)} \end{bmatrix} \mathcal{Q}(w;u(y),v(y))\, m_-(y;w)dy \\ & - \frac{1}{w^2}\int_{-\infty}^x \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i}{2}(w-w^{-1})(x-y)} \end{bmatrix} \mathcal{Q}_2(u(y),v(y))\, m_-(y;w)dy. \end{aligned}$$

Using $\mathcal{K}[m_-] = m_- - e_1$ and introducing the function

$$\mathfrak{m}_-(x;w) := \frac{i}{2}(1+w^{-2})x \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} m_-(x;w),$$

we can slightly simplify our equation for $\partial_w m_-$, namely,

$$\begin{aligned}\partial_w m_-(x;w) &= \mathcal{K}[\partial_w m_-](x;w) + \mathfrak{m}_-(x;w) - \mathcal{K}[\mathfrak{m}_-](x;w)\\ &\quad - \frac{i(1+w^{-2})}{2}\int_{-\infty}^{x} y \begin{pmatrix} -[\mathcal{Q}(w;u(y),v(y))]_{12} m_-^{(2)}(y;w) \\ e^{\frac{i}{2}(w-w^{-1})(x-y)}[\mathcal{Q}(w;u(y),v(y))]_{21} m_-^{(1)}(y;w) \end{pmatrix} dy\\ &\quad - \frac{1}{w^2}\int_{-\infty}^{x} \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i}{2}(w-w^{-1})(x-y)} \end{bmatrix} \mathcal{Q}_2(u(y),v(y))\, m_-(y;w)dy.\end{aligned}$$

This representation can be rewritten in the following abstract form

$$(1-\mathcal{K})[\partial_w m_- - \mathfrak{m}_-] = \widetilde{h}, \tag{2.4.13}$$

with $\widetilde{h} = \widetilde{h}_1 + ... + \widetilde{h}_4$, where

$$\begin{aligned}\widetilde{h}_1(x;w) &= -e_1 \frac{i}{2}\int_{-\infty}^{x}(1+w^{-2})y\mathcal{Q}(w;u(y),v(y))]_{12} m_-^{(2)}(y;w)dy,\\ \widetilde{h}_2(x;w) &= -\frac{i}{2}\int_{-\infty}^{x}(1+w^{-2})ye^{\frac{i}{2}(w-w^{-1})(x-y)}[\mathcal{Q}(w;u(y),v(y))]_{21}(m_-^{(1)}(y;w) - m_-^{\infty}(y))dy,\\ \widetilde{h}_3(x;w) &= -\frac{i}{2}\int_{-\infty}^{x}(1+w^{-2})ye^{\frac{i}{2}(w-w^{-1})(x-y)}[\mathcal{Q}(w;u(y),v(y))]_{21} m_-^{\infty}(y)dy,\\ \widetilde{h}_4(x;w) &= -\frac{1}{w^2}\int_{-\infty}^{x} \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i}{2}(w-w^{-1})(x-y)} \end{bmatrix} \mathcal{Q}_2(u(y),v(y))\, m_-(y;w)dy.\end{aligned}$$

Using $\mathcal{Q} \in L^1(\mathbb{R})$ and $(m_- - e_1 m_-^{\infty}) \in X_1(x_0)$ which follow from (2.4.12) and (2.4.8), we obtain $\widetilde{h}_1, \widetilde{h}_2 \in X_0(x_0)$ for all $x_0 \le 0$. Using the same Plancherel formula as in (2.4.10) we can bound the $X_0(x_0)$-norm of $\widetilde{h}_3$ by the $L^{2,1}$-norms of $[\mathcal{Q}_1]_{21}$ and $[\mathcal{Q}_2]_{21}$. Finally, the $X_0(x_0)$-norm of $\widetilde{h}_4$ is simply bounded by $\|\mathcal{Q}_2\|_{L^1}$. Thus, making use of (2.4.8) and (2.4.13) we get that $(\partial_w m_- - \mathfrak{m}_-) \in X_0(x_0)$. Since we are interested in showing that $\partial_w m_- \in X_0(x_0)$ we also have to show that $\mathfrak{m}_- \in X_0(x_0)$. But the latter follows from the definition of $\mathfrak{m}_-$ and $(m_- - e_1 m_-^{\infty}) \in X_1(x_0)$.

Let us now turn to the final step of the proof which is the verification of (2.4.3). We define the following operator:

$$\mathfrak{K}[f](x;w) := \int_{-\infty}^{x} e^{\frac{i}{2}(w-w^{-1})(x-y)}[\mathcal{Q}(w;u(y),v(y))]_{22}\ f(y;w)dy.$$

In contrast to $\mathcal{K}$ defined in (2.4.5) this operator $\mathfrak{K}$ acts on scalar-valued functions and not on vector-valued functions. But similarly to $\mathcal{K}$ we can show that

$$\|(1-\mathfrak{K})^{-1}\|_{X_0(x_0)\to X_0(x_0)} \le c\exp\left(\|[\mathcal{Q}_1]_{22}\|_{L^1(-\infty,x_0)} + \|[\mathcal{Q}_2]_{22}\|_{L^1(-\infty,x_0)}\right). \tag{2.4.14}$$

We define $j(x;w) := wm_-^{(2)}(x;w) - \mathfrak{q}_-^{(2)}(x)$ and recall that

$$\begin{aligned} wm_-^{(2)}(x;w) - \mathfrak{q}_-^{(2)}(x) &= w\int_{-\infty}^{x} e^{\frac{i}{2}(w-w^{-1})(x-y)}[\mathcal{Q}(w;u(y),v(y))m_-(y;w)]_2 dy\\ &\quad -2i\left[\mathcal{Q}_1(u(x),v(x))\right]_{21} m_-^{\infty}(x).\end{aligned}$$

For this reason we have

$$(1-\mathfrak{K})[j] = L, \tag{2.4.15}$$

where

$$\begin{aligned} L(x;w) &= w\int_{-\infty}^{x} e^{\frac{i}{2}(w-w^{-1})(x-y)}[\mathcal{Q}(w;u(y),v(y))]_{21} m_-^{(1)}(y;w)dy\\ &\quad -2i\left[\mathcal{Q}_1(u(x),v(x))\right]_{21} m_-^{\infty}(x)\\ &\quad + \int_{-\infty}^{x} e^{\frac{i}{2}(w-w^{-1})(x-y)}[\mathcal{Q}(w;u(y),v(y))]_{22}\ \mathfrak{q}_-^{(2)}(y)dy.\end{aligned}$$

Let us show that $L \in X_0(x_0)$ by using a decomposition $L = L_1 + ... + L_6$ with

$$\begin{aligned}
L_1(x;w) &= (w-w^{-1})\int_{-\infty}^{x} e^{\frac{i}{2}(w-w^{-1})(x-y)}[\mathcal{Q}_1(u(y),v(y))]_{21}(m_-^{(1)}(y;w)-m_-^{\infty}(y))dy \\
&= -2i\int_{-\infty}^{x} e^{\frac{i}{2}(w-w^{-1})(x-y)}\partial_y\left([\mathcal{Q}_1(u(y),v(y))]_{21}(m_-^{(1)}(y;w)-m_-^{\infty}(y))\right)dy \\
&\quad +2i[\mathcal{Q}_1(u(x),v(x))]_{21}(m_-^{(1)}(x;w)-m_-^{\infty}(x)), \\
L_2(x;w) &= (w-w^{-1})\int_{-\infty}^{x} e^{\frac{i}{2}(w-w^{-1})(x-y)}\,[\mathcal{Q}_1(u(y),v(y))]_{21}\,m_-^{\infty}(y)dy \\
&\quad -2i\,[\mathcal{Q}_1(u(x),v(x))]_{21}\,m_-^{\infty}(x) \\
&= -2i\int_{-\infty}^{x} e^{\frac{i}{2}(w-w^{-1})(x-y)}\partial_y\left([\mathcal{Q}_1(u(y),v(y))]_{21}\,m_-^{\infty}(y)\right)dy, \\
L_3(x;w) &= \frac{1}{w}\int_{-\infty}^{x} e^{\frac{i}{2}(w-w^{-1})(x-y)}\,[\mathcal{Q}_1(u(y),v(y))]_{21}\,m_-^{(1)}(y;w)dy, \\
L_4(x;w) &= \int_{-\infty}^{x} e^{\frac{i}{2}(w-w^{-1})(x-y)}\,[\mathcal{Q}_2(u(y),v(y))]_{21}\,(m_-^{(1)}(y;w)-m_-^{\infty}(y))dy, \\
L_5(x;w) &= \int_{-\infty}^{x} e^{\frac{i}{2}(w-w^{-1})(x-y)}\,[\mathcal{Q}_2(u(y),v(y))]_{21}\,m_-^{\infty}(y)dy, \\
L_6(x;w) &= \int_{-\infty}^{x} e^{\frac{i}{2}(w-w^{-1})(x-y)}[\mathcal{Q}(w;u(y),v(y))]_{22}\,\mathfrak{q}_-^{(2)}(y)dy.
\end{aligned}$$

Using the idea of (2.4.10) we have the bound

$$\|L_2\|_{X_0(x_0)}^2 \leq 2\int_{-\infty}^{\infty}\left|\partial_y\left([\mathcal{Q}_1(u(y),v(y))]_{21}\,m_-^{\infty}(y)\right)\right|^2 dy,$$

which is finite by the assumptions. Clearly, we also have $L_3 \in X_0(x_0)$. L_4 can be bounded by the L^1-norm of $\mathcal{Q}_2$ and the $X_0(x_0)$-norm of $(m_-^{(1)}(y;w) - m_-^{\infty}(y))$. For L_5 and L_6 we can use again (2.4.10). Thus, it remains to show that $L_1 \in X_0(x_0)$. By (2.2.16) and (2.2.17) we are able to find the following:

$$\begin{aligned}
&\partial_x\left([\mathcal{Q}_1(u(x),v(x))]_{21}(m_-^{(1)}(x;w)-m_-^{\infty}(x))\right) \\
&\qquad = \partial_x[\mathcal{Q}_1(u(x),v(x))]_{21}(m_-^{(1)}(x;w)-m_-^{\infty}(x)) \\
&\qquad\quad + [\mathcal{Q}_1(u(x),v(x))]_{21}[\mathcal{Q}_1(u(x),v(x))(m_-(x;w)-m_-^{\infty}(x)e_1)]_1 \\
&\qquad\quad + \frac{1}{w}[\mathcal{Q}_1(u(x),v(x))]_{21}[\mathcal{Q}_2(u(x),v(x))m_-(x;w)]_1 \\
&\qquad = l_1(x;w)+l_2(x;w)+l_3(x;w).
\end{aligned}$$

This decomposition in turn gives rise to a decomposition $L_1 = \widetilde{L}_1 + ... + \widetilde{L}_4$:

$$\begin{aligned}
\widetilde{L}_j(x;w) &= -2i\int_{-\infty}^{x} e^{\frac{i}{2}(w-w^{-1})(x-y)}l_j(y;w)dy, \qquad j=1,2,3, \\
\widetilde{L}_4(x;w) &= 2i[\mathcal{Q}_1(u(x),v(x))]_{21}(m_-^{(1)}(x;w)-m_-^{\infty}(x)).
\end{aligned}$$

Using the Hölder inequality, for $x_0 \leq 0$, we have

$$\begin{aligned}
\|\widetilde{L}_1\|_{X_0(x_0)} &\leq c_1\int_{-\infty}^{x_0}\langle y\rangle^{-1}\left|\partial_y[\mathcal{Q}_1(u(y),v(y))]_{21}\right|\langle y\rangle\|m_-^{(1)}(\cdot;y)-m_-^{\infty}(y)\|_{L^2(\mathbb{R}\setminus[-1,1])}dy \\
&\leq c_2\|[\mathcal{Q}_1(u(\cdot),v(\cdot))]_{21}\|_{H^1(\mathbb{R})}\|m_-^{(1)}-m_-^{\infty}\|_{X_1(x_0)}.
\end{aligned}$$

The other summands $\widetilde{L}_2, ..., \widetilde{L}_4$ are estimated easily in the $X(x_0)$-norm if we use $(m_-(y;w)-m_-^{\infty}(y)e_1) \in X_0(x_0)$. Altogether, we conclude by (2.4.14) and (2.4.15) that $j \in X(x_0)$ for all $x_0 \leq 0$. The remaining part of the lemma's proof is to show that $w(m_-^{(1)}(x;w) - m_-^\infty(x)) - \mathfrak{q}_-^{(1)}(x) \in X(x_0)$. We recall (2.2.18) and replace $2i\,[\mathcal{Q}_1(u(x),v(x))]_{21}$ with $\mathfrak{q}_-^{(2)}(x)/m_-^\infty(x)$ in the definition of $\mathfrak{q}_-^{(1)}$, (2.2.19). Then, we directly compute

$$\begin{aligned} w(m_-^{(1)}(x;w) - m_-^\infty(x)) - \mathfrak{q}_-^{(1)}(x) &= m_-^\infty(x)\int_{-\infty}^x \frac{[\mathcal{Q}_1(u(y),v(y))]_{12}\,(wm_-^{(2)}(y;w) - \mathfrak{q}_-^{(2)}(x))}{m_-^\infty(y)}dy \\ &\quad + m_-^\infty(x)\int_{-\infty}^x \frac{\big[\mathcal{Q}_2(u(y),v(y))(m_-(y;w) - m_-^\infty(y)e_1)\big]_1}{m_-^\infty(y)}dy. \end{aligned}$$

Here, the first summand is in $X_0(x_0)$ because of the above result $j \in X_0(x_0)$ and $[\mathcal{Q}_1(u(\cdot),v(\cdot))]_{12} \in L^1(\mathbb{R})$. The second summand is estimated in $X_0(x_0)$ by the L^1-norms of $[\mathcal{Q}_2(u(\cdot),v(\cdot))]$ and the $X_0(x_0)$-norm of $m_- - m_-^\infty e_1$. This completes the proof of the lemma. □

2.5 Scattering coefficients

In order to define the scattering coefficients between the transformed Jost functions $\{m_\pm, n_\pm\}$ and $\{\widehat{m}_\pm, \widehat{n}_\pm\}$, we go back to the original Jost functions $\{\varphi_\pm, \phi_\pm\}$ and the non-normalized matrix-valued Jost functions $\psi^{(\pm)}$. We recall that $\psi^{(+)}$ and $\psi^{(-)}$ are defined by $\psi_x^{(\pm)} = L\psi^{(\pm)}$ and the asymptotics (2.1.6). It follows from this asymptotic behavior that

$$\lim_{x\to\pm\infty} \det \psi^{(\pm)}(x;\lambda) = 1.$$

Since the matrix operator L in (2.1.1) has zero trace, we conclude that

$$\det \psi^{(\pm)}(x;\lambda) = 1, \quad \lambda \in (\mathbb{R}\cup i\,\mathbb{R}) \setminus \{0\}, \quad x \in \mathbb{R}. \tag{2.5.1}$$

It follows that there exists a unique matrix $T(x;\lambda)$ such that $\psi^{(-)}(x;\lambda) = \psi^{(+)}(x;\lambda)T(x;\lambda)$. Differentiating this in x and using $\psi_x = L\psi$, we find $L\psi^{(-)} = L\psi^{(+)}T + \psi^{(+)}\partial_x T$. Inserting again the relation $\psi^{(-)} = \psi^{(+)}T$ and subtracting $L\psi^{(+)}T$ from both sides of the equation we find $0 = \psi^{(+)}\partial_x T$. Hence, by (2.5.1) we have $\partial_x T = 0$. We summarize as follows: there exists a matrix $T(\lambda)$, not depending on x, such that

$$\psi^{(-)}(x;\lambda) = \psi^{(+)}(x;\lambda)T(\lambda), \quad \lambda \in (\mathbb{R}\cup i\,\mathbb{R}) \setminus \{0\}, \quad x \in \mathbb{R}. \tag{2.5.2}$$

Additionally, from (2.5.1) we also know that

$$\det T(\lambda) = 1, \qquad \lambda \in (\mathbb{R}\cup i\,\mathbb{R}) \setminus \{0\}. \tag{2.5.3}$$

Moreover, using the statements of Remarks 2.1.2 and 2.1.3 we find the following symmetries for $T(\lambda)$:

$$T(-\lambda) = \sigma_3 T(\lambda)\sigma_3, \qquad T(\overline{\lambda}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \overline{T(\lambda)} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus, T takes the form

$$T(\lambda) = \begin{bmatrix} \alpha(\lambda) & -\overline{\beta(\overline{\lambda})} \\ \beta(\lambda) & \overline{\alpha(\overline{\lambda})} \end{bmatrix} \tag{2.5.4}$$

with functions α and β defined on $(\mathbb{R}\cup i\,\mathbb{R}) \setminus \{0\}$ satisfying

$$\alpha(\lambda) = \alpha(-\lambda), \qquad \beta(\lambda) = -\beta(-\lambda). \tag{2.5.5}$$

Furthermore, thanks to (2.5.3), we have the following constraints:

$$\begin{cases} |\alpha(\lambda)|^2 + |\beta(\lambda)|^2 = 1, & \lambda \in \mathbb{R}\backslash\{0\}, \\ |\alpha(\lambda)|^2 - |\beta(\lambda)|^2 = 1, & \lambda \in i\mathbb{R}\backslash\{0\}. \end{cases} \tag{2.5.6}$$

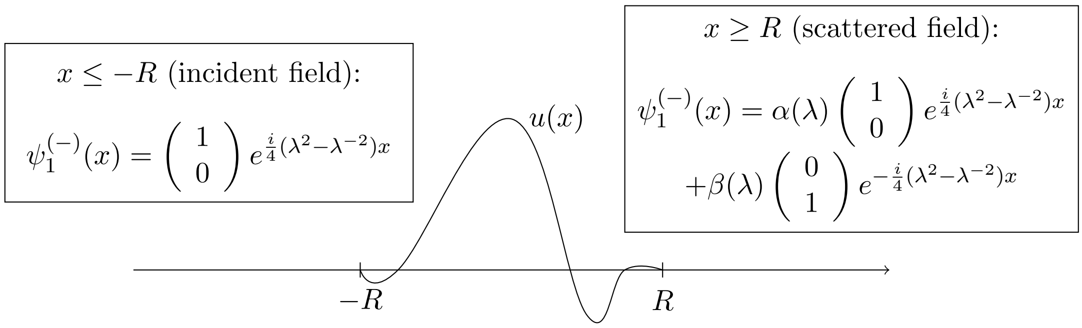


Figure 2.3: *Assuming that u and v are supported in an interval $[-R, R]$ we have an explicit expression for the Jost function $\psi_1^{(-)}$ on the left hand side of the support. In general we do not expect that for $x \ge R$, $\psi_1^{(-)}$ is of the same form. Instead, this function will be scattered into a linear combination of the Jost functions $\psi_1^{(+)}$ and $\psi_2^{(+)}$.*

The matrix T is called *scattering matrix* and we will refer to (2.5.2) as the *scattering relation*. The reason for this terminology is the following. By definition, we know precisely the behavior of, for instance, $\psi_1^{(-)}(x;\lambda)$ as $x \to -\infty$:

$$\psi_1^{(-)}(x;\lambda) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{ix(\lambda^2-\lambda^{-2})/4}, \qquad x \to -\infty.$$

Let us consider this as our incident field. After interacting with the operator L by the differential equation, we expect to observe a scattered field as $x \to \infty$. Indeed, by the relation (2.5.2), we get

$$\begin{aligned} \psi_1^{(-)}(x;\lambda) &= \alpha(\lambda)\psi_1^{(+)}(x;\lambda) + \beta(\lambda)\psi_2^{(+)}(x;\lambda) \\ &\sim \alpha(\lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{ix(\lambda^2-\lambda^{-2})/4} + \beta(\lambda) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-ix(\lambda^2-\lambda^{-2})/4}, \qquad x \to +\infty. \end{aligned}$$

Thus, the coefficient $\alpha(\lambda)$ records the portion of the incident field $e_1 e^{ix(\lambda^2-\lambda^{-2})/4}$ that was transmitted by L. For that reason, α is sometimes also called the *transmission* coefficient. On the other hand, β measures how much of $e_1 e^{ix(\lambda^2-\lambda^{-2})/4}$ was turned into $e_2 e^{-ix(\lambda^2-\lambda^{-2})/4}$, which, additionally to the change $e_1 \to e_2$, can be understood as a switch from x to $-x$. Therefore, in some sense, the coefficient β is measuring the *reflection* of the incident field. The constraints in (2.5.6) can be interpreted as a conservation law for the scattering process. See Figure 2.3 for an illustration of the direct scattering in the special case of $u, v \in C_c^\infty(\mathbb{R})$.

For future reference we want to mention that an equivalent form of (2.5.2) is given by

$$\begin{aligned} \varphi_-(x;\lambda)e^{ix(\lambda^2-\lambda^{-2})/4} &= \alpha(\lambda)\, \varphi_+(x;\lambda)e^{ix(\lambda^2-\lambda^{-2})/4} + \beta(\lambda)\, \phi_+(x;\lambda)e^{-ix(\lambda^2-\lambda^{-2})/4}, \\ \phi_-(x;\lambda)e^{-ix(\lambda^2-\lambda^{-2})/4} &= -\,\overline{\beta(\overline{\lambda})}\, \varphi_+(x;\lambda)e^{ix(\lambda^2-\lambda^{-2})/4} + \overline{\alpha(\overline{\lambda})}\, \phi_+(x;\lambda)e^{-ix(\lambda^2-\lambda^{-2})/4}. \end{aligned} \tag{2.5.7}$$

Let us now continue with investigating the scattering matrix $T(\lambda)$. Since T is determined by the linear system (2.5.2) we can also write $T(\lambda) = [\psi^{(+)}(x;\lambda)]^{-1}\psi^{(-)}(x;\lambda)$. A component-wise evaluation of this representation with the help of (2.5.1) yields

$$\alpha(\lambda) = \det[\psi_1^{(-)}(x;\lambda)|\psi_2^{(+)}(x;\lambda)], \qquad \beta(\lambda) = \det[\psi_1^{(+)}(x;\lambda)|\psi_1^{(-)}(x;\lambda)],$$

where $x \in \mathbb{R}$ is arbitrary. Using the relations (2.1.7) at $x = 0$, we can rewrite these formulas as

$$\begin{aligned} \alpha(\lambda) &= \det[\varphi_-(0;\lambda)|\phi_+(0;\lambda)], \\ \beta(\lambda) &= \det[\varphi_+(0;\lambda)|\varphi_-(0;\lambda)]. \end{aligned} \tag{2.5.8}$$

Applying Lemma 2.1.1 to the formula for α we find an analytic continuation of α into the set $\{\lambda \in \mathbb{C} : \lambda^2 \in \mathbb{C}^-\}$. However, we encounter the same issue as in the analysis of the functions $\varphi_\pm$ and $\phi_\pm$ in Section 2.1: we do not know anything about the limits of $\alpha(\lambda)$ as $\lambda \to 0$ or $|\lambda| \to \infty$. Regularity of the function $\alpha(\cdot)$ is not available, either. This observation gives rise to the following idea: we introduce *transformed* scattering coefficients that are associated to the transformed Jost functions defined and studied in the preceding Sections 2.2–2.4. We recall $w = \lambda^{-2}$ and, motivated by (2.5.8), we set

$$a(w) := \det[m_-(0;w)|n_+(0;w)], \qquad b(w) := \det[m_+(0;w)|m_-(0;w)]. \tag{2.5.9}$$

By the explicit transformations (2.2.6) and by (2.5.8), it is seen that these new functions are related to the original functions in a trivial way:

$$a(w) = \alpha(\lambda), \qquad b(w) = \frac{\beta(\lambda)}{\lambda}. \tag{2.5.10}$$

The same functions appear in the following transformed scattering relation which is obtained from (2.2.6) and (2.5.7):

$$\begin{aligned} m_-(x;w)e^{-ix(w-w^{-1})/4} &= a(w)\, m_+(x;w)e^{-ix(w-w^{-1})/4} + b(w)\, n_+(x;w)e^{ix(w-w^{-1})/4}, \\ n_-(x;w)e^{ix(w-w^{-1})/4} &= -\frac{\overline{b(w)}}{w}\, m_+(x;w)e^{-ix(w-w^{-1})/4} + \overline{a(w)}\, n_+(x;w)e^{ix(w-w^{-1})/4}. \end{aligned} \tag{2.5.11}$$

For that reason, the functions defined in (2.5.9) can be understood as the transformed scattering coefficients. Additionally, we have the following result:

Lemma 2.5.1.

(i) Under the same conditions on u and v as in Lemma 2.3.1, the scattering coefficient $a(w)$ can be continued analytically into $\mathbb{C}^+$ with the limits

$$a_\infty := \lim_{|w|\to\infty} a(w) = e^{-\frac{i}{4}\left(\|u\|^2_{L^2(\mathbb{R})}+\|v\|^2_{L^2(\mathbb{R})}\right)}, \quad a_0 := \lim_{w\to 0} a(w) = e^{\frac{i}{4}\left(\|u\|^2_{L^2(\mathbb{R})}+\|v\|^2_{L^2(\mathbb{R})}\right)}.$$

(ii) Under the assumptions of Lemma 2.4.1, we have

$$(a(w) - a_\infty) \in H^1_w(\mathbb{R}\backslash[-1,1]), \quad b(w) \in H^1_w(\mathbb{R}\backslash[-1,1]) \cap L^{2,1}_w(\mathbb{R}\backslash[-1,1]). \tag{2.5.12}$$

Proof. (i) Using the definition of a and the analyticity properties of the Jost functions m_- and n_+ stated in Lemma 2.3.1 we find that $a(w)$ can be continued analytically into $\mathbb{C}^+$. The limit at infinity follows from (2.2.10):

$$a_\infty = \det\begin{pmatrix} m_-^\infty(x) & 0 \\ 0 & n_+^\infty(x) \end{pmatrix} = m_-^\infty(x)n_+^\infty(x) = e^{-\frac{i}{4}\left(\|u\|^2_{L^2(\mathbb{R})}+\|v\|^2_{L^2(\mathbb{R})}\right)}.$$

Similarly, the limit at zero follows from (2.3.3):

$$a_0 = \det\begin{pmatrix} \widehat{m}_-^\infty(x) & \widehat{n}_+^\infty(x)\bar{v}(x) \\ \widehat{m}_-^\infty(x)u(x) & \widehat{n}_+^\infty(x)(1+u(x)\bar{v}(x)) \end{pmatrix} = \widehat{m}_-^\infty(x)\widehat{n}_+^\infty(x) = e^{\frac{i}{4}\left(\|u\|^2_{L^2(\mathbb{R})}+\|v\|^2_{L^2(\mathbb{R})}\right)}.$$

(ii) The definition (2.5.9) of a can be rewritten in the following way.

$$\begin{aligned} a(w) - a_\infty = {} & \left[m_-^{(1)}(0;w) - m_-^\infty(0)\right]\left[n_+^{(2)}(0;w) - n_+^\infty(0)\right] \\ & + n_+^\infty(0)\left[m_-^{(1)}(0;w) - m_-^\infty(0)\right] + m_-^\infty(0)\left[n_+^{(2)}(0;w) - n_+^\infty(0)\right] \\ & - m_-^{(2)}(0;w)n_+^{(1)}(0;w). \end{aligned}$$

Since, by Lemma 2.4.1, in this formulation every summand is in $H^1_w(\mathbb{R}\setminus[-1,1])$, it follows that $(a(w)-a_\infty)\in H^1_w(\mathbb{R}\setminus[-1,1])$. Writing explicitly

$$b(w)=m_+^{(1)}(0;w)m_-^{(2)}(0;w)-m_-^{(1)}(0;w)m_+^{(2)}(0;w),$$

we directly see that, again by Lemma 2.4.1, $b(w)\in H^1_w(\mathbb{R}\setminus[-1,1])$. The proof of $b(w)\in L^{2,1}_w(\mathbb{R}\setminus[-1,1])$ is based on the following decomposition,

$$\begin{aligned} wb(w) \;=\;& m_+^{(1)}(0;w)\left[wm_-^{(2)}(0;w)-\mathfrak{q}_-^{(2)}(0)\right]-m_-^{(1)}(0;w)\left[wm_+^{(2)}(0;w)-\mathfrak{q}_+^{(2)}(0)\right]\\ &+\mathfrak{q}_-^{(2)}(0)\left[m_+^{(1)}(0;w)-m_+^\infty(0)\right]-\mathfrak{q}_+^{(2)}(0)\left[m_-^{(1)}(0;w)-m_-^\infty(0)\right]\\ &+\mathfrak{q}_-^{(2)}(0)m_+^\infty(0)-\mathfrak{q}_+^{(2)}(0)m_-^\infty(0), \end{aligned}$$

where it is important to notice that the last line vanishes by the definitions (2.2.11) and (2.2.13) of $\mathfrak{q}_\pm^{(2)}(x)$ and $m_\pm^\infty(x)$. The remaining terms are in $L^2(\mathbb{R}\setminus[-1,1])$ by Lemma 2.4.1 and thus, the proof of the lemma is completed. □

Now we define the analogues to (2.5.9) in the z-variable:

$$\widehat{a}(z):=\det[\widehat{m}_-(z;0)|\widehat{n}_+(z;0)],\qquad \widehat{b}(z):=\det[\widehat{m}_+(z;0)|\widehat{m}_-(z;0)]. \tag{2.5.13}$$

These transformed scattering coefficients are related to the original coefficients by

$$\widehat{a}(z)=\alpha(\lambda),\qquad \widehat{b}(z)=\lambda\beta(\lambda) \tag{2.5.14}$$

and connect the transformed Jost functions $\{\widehat{m}_\pm,\widehat{n}_\pm\}$ in the following way:

$$\begin{aligned} \widehat{m}_-(x;z)e^{ix(z-z^{-1})/4} &= \widehat{a}(z)\,\widehat{m}_+(x;z)e^{ix(z-z^{-1})/4}+\widehat{b}(z)\,\widehat{n}_+(x;z)e^{-ix(z-z^{-1})/4},\\ \widehat{n}_-(x;z)e^{-ix(z-z^{-1})/4} &= -\frac{\overline{\widehat{b}(z)}}{z}\,\widehat{m}_+(x;z)e^{ix(z-z^{-1})/4}+\overline{\widehat{a}(z)}\,\widehat{n}_+(x;z)e^{-ix(z-z^{-1})/4}. \end{aligned} \tag{2.5.15}$$

Lemma 2.5.2.

(i) Under the same conditions on u and v as in Lemma 2.3.1, the scattering coefficient $\widehat{a}(z)$ can be continued analytically into $\mathbb{C}^-$ with the limits

$$\widehat{a}_\infty:=\lim_{|z|\to\infty}\widehat{a}(z)=e^{\frac{i}{4}\left(\|u\|^2_{L^2(\mathbb{R})}+\|v\|^2_{L^2(\mathbb{R})}\right)},\quad \widehat{a}_0:=\lim_{z\to 0}\widehat{a}(z)=e^{-\frac{i}{4}\left(\|u\|^2_{L^2(\mathbb{R})}+\|v\|^2_{L^2(\mathbb{R})}\right)}.$$

(ii) Under the assumptions of Lemma 2.4.1 we have

$$(\widehat{a}(z)-\widehat{a}_\infty)\in H^1_z(\mathbb{R}\setminus[-1,1]),\quad \widehat{b}(z)\in H^1_z(\mathbb{R}\setminus[-1,1])\cap L^{2,1}_z(\mathbb{R}\setminus[-1,1]). \tag{2.5.16}$$

2.6 The reflection coefficient

As discussed in the preceding section, the entry $\alpha(\lambda)$ of the scattering matrix T can be understood as a coefficient that measures transmission. On the other hand, $\beta(\lambda)$ measures reflection. The ratio

$$p(\lambda):=\frac{\beta(\lambda)}{\alpha(\lambda)} \tag{2.6.1}$$

is commonly called the *reflection coefficient*, even though it is the ratio of the reflection and transmission parameters. However, from now on we refer to $p(\lambda)$ as the reflection coefficient. Obviously, we need $\alpha(\lambda)\neq 0$ in order to define the reflection coefficient.

Definition 2.6.1. We say that the functions u and v admit a *resonance* at $\lambda_0\in(\mathbb{R}\cup i\,\mathbb{R})\setminus\{0\}$ if $\alpha(\lambda_0)=0$.

Remark 2.6.2. Due to the relation $\alpha(\lambda) = a(w) = \widehat{a}(z)$ with $w = z^{-1} = \lambda^{-2}$ we know that a and $\widehat{a}$ have zeroes on $\mathbb{R}$ if and only if u and v admit resonances. Moreover, by Lemmas 2.5.1 and 2.5.2 we conclude that a and $\widehat{a}$ are continuous functions on $\mathbb{R}$ with non-zero limits for $w, z \to \pm\infty$. This implies that, if u and v do not admit resonances, then there exists a constant $c_0 > 0$ such that

$$|a(w)| \geq c_0, \quad w \in \mathbb{R}, \qquad |\widehat{a}(z)| \geq c_0, \quad z \in \mathbb{R}. \tag{2.6.2}$$

This bound is an essential fact for the present work, even though the constant c_0 depends on u and v.

In the following we always assume that (u, v) does not admit resonances. Now, let us define the matrices $P_\pm(x; w) \in \mathbb{C}^{2\times 2}$ for every $x \in \mathbb{R}$ and $w \in \mathbb{R}$ by

$$P_+(x; w) := \left[\frac{m_-(x; w)}{a(w)}, n_+(x; w)\right], \quad P_-(x; w) := \left[m_+(x; w), \frac{n_-(x; w)}{\overline{a(w)}}\right]. \tag{2.6.3}$$

The purpose of these matrices is to rewrite the scattering relation (2.5.11) as a jump condition for a Riemann–Hilbert problem:

$$P_+(x; w) = P_-(x; w) \begin{bmatrix} 1 + \dfrac{|b(w)|^2}{w|a(w)|^2} & \dfrac{\overline{b(w)}}{w\overline{a(w)}} e^{-\frac{i}{2}(w - w^{-1})x} \\ \dfrac{b(w)}{a(w)} e^{\frac{i}{2}(w - w^{-1})x} & 1 \end{bmatrix}, \quad w \in \mathbb{R}\backslash\{0\}. \tag{2.6.4}$$

Analogously, we define for every $x \in \mathbb{R}$ and $z \in \mathbb{R}$

$$\widehat{P}_+(x; z) := \left[\widehat{m}_+(x; z), \frac{\widehat{n}_-(x; z)}{\overline{\widehat{a}(z)}}\right], \quad \widehat{P}_-(x; z) := \left[\frac{\widehat{m}_-(x; z)}{\widehat{a}(z)}, \widehat{n}_+(x; z)\right], \tag{2.6.5}$$

such that by (2.5.15), we have

$$\widehat{P}_+(x; z) = \widehat{P}_-(x; z) \begin{bmatrix} 1 & \dfrac{-\overline{\widehat{b}(z)}}{z\overline{\widehat{a}(z)}} e^{\frac{i}{2}(z - z^{-1})x} \\ -\dfrac{\widehat{b}(z)}{\widehat{a}(z)} e^{-\frac{i}{2}(z - z^{-1})x} & 1 + \dfrac{|\widehat{b}(z)|^2}{z|\widehat{a}(z)|^2} \end{bmatrix}, \quad z \in \mathbb{R}\backslash\{0\}.$$

We realize that the matrices connecting P_+ with P_- and $\widehat{P}_+$ with $\widehat{P}_-$ do not depend on $a(w)$ and $b(w)$ or $\widehat{a}(z)$ and $\widehat{b}(z)$ separately, but only on the ratios $a(w)/b(w)$ and $\widehat{a}(z)/\widehat{b}(z)$. This gives rise to the following definition:

Definition 2.6.3. For functions u and v not admitting resonances, the two *transformed reflection* coefficients on $\mathbb{R}$ are defined by

$$r(w) := \frac{b(w)}{wa(w)}, \quad w \in \mathbb{R}\backslash\{0\}, \qquad \widehat{r}(z) := \frac{\widehat{b}(z)}{z\widehat{a}(z)}, \quad z \in \mathbb{R}\backslash\{0\}, \tag{2.6.6}$$

and by $r(0) := 0$ and $\widehat{r}(0) := 0$.

It is clear from this definition and relations (2.5.10) and (2.5.14) that r and $\widehat{r}$ can be computed from the function p defined in (2.6.1) by

$$r(w) = \lambda p(\lambda), \qquad \widehat{r}(z) = \frac{p(\lambda)}{\lambda}. \tag{2.6.7}$$

Lemma 2.6.4. *Let $u, v \in X_{2,1}$, where $X_{2,1}$ is given in (2.3.5), and assume, that u and v do not admit resonances in the sense of Definition 2.6.1. Then, the reflection coefficients r and $\widehat{r}$ satisfy the following properties:*

(i) $wr(w) \in H^1_w(\mathbb{R}\backslash[-1,1]) \cap L^{2,1}_w(\mathbb{R}\backslash[-1,1])$ *and* $z\widehat{r}(z) \in H^1_z(\mathbb{R}\backslash[-1,1]) \cap L^{2,1}_z(\mathbb{R}\backslash[-1,1])$.

(ii) The two reflection coefficients are connected via

$$\begin{aligned} r(w) &= w^{-1}\widehat{r}(w^{-1}), \quad w \in \mathbb{R}\backslash\{0\}, \\ \widehat{r}(z) &= z^{-1}r(z^{-1}), \qquad z \in \mathbb{R}\backslash\{0\}. \end{aligned} \tag{2.6.8}$$

(iii) r *and* $\widehat{r}$ *are bounded continuous functions on* $\mathbb{R}$.

(iv) There is a constant $c_1 > 0$ *such that for all* $w = z^{-1} \in \mathbb{R}\backslash\{0\}$,

$$1 + w|r(w)|^2 = 1 + z|\widehat{r}(z)|^2 > c_1. \tag{2.6.9}$$

Proof. (i) is a direct consequence of Lemmas 2.5.1 and 2.5.2 and the bound (2.6.2) as we have

$$w^2|r(w)| \leq \frac{|w|}{c_0}|b(w)|, \quad |\partial_w(wr(w))| \leq \frac{|a(w)b'(w) - a'(w)b(w)|}{c_0^2}.$$

The relations stated in (ii) are verified by relations (2.5.10) and (2.5.14). (iii) follows from (i) on the set $\mathbb{R}\backslash[-1,1]$. For continuity on the interval $[-1,1]$ we can use (2.6.8) and (i) again. In order to prove (iv) we first remark that the equality $1 + w|r(w)|^2 = 1 + z|\widehat{r}(z)|^2$ follows from (2.6.8). Additionally, we can use (2.5.10) to show that $|\beta(\lambda)|^2 = -w|b(w)|^2$ for $\lambda \in i\,\mathbb{R}$ which corresponds to $w \in \mathbb{R}_-$. Thus, using the second line of (2.5.6), for $w < 0$, we get

$$1 + w|r(w)|^2 = 1 - \frac{|\beta(\lambda)|^2}{|\alpha(\lambda)|^2} = \frac{1}{|\alpha(\lambda)|^2} = \frac{1}{|a(w)|^2} > \left(\sup_{w\in\mathbb{R}_-} |a(w)|^2\right)^{-1} > 0.$$

For $w \geq 0$, we clearly have $1 + w|r(w)|^2 \geq 1$. Thus, the proof of the Lemma is completed. □

We shall now ask if the regularities of the reflection coefficients given by Lemma 2.6.4 (i) can be extended to the full real line. Let us make some basic observations: Assume $f \in H^1(1,\infty) \cap L^{2,1}(1,\infty)$ and set $\tilde{f}(y) := f(y^{-1})$. We find

$$\int_1^\infty x^2|f(x)|^2 dx = \int_0^1 \frac{1}{y^4}|\tilde{f}(y)|^2 dy$$

and conclude $\tilde{f} \in L^{2,-2}(0,1)$. Moreover, using the chain rule $f'(x) = -x^{-2}\tilde{f}'(x^{-1})$ we obtain

$$\int_1^\infty |f'(x)|^2 dx = \int_0^1 y^2|\tilde{f}'(y)|^2 dy,$$

which does not imply that $\tilde{f} \in H^1(0,1)$. In particular, statement (i) of Lemma 2.6.4 does not remain true if we replace $\mathbb{R}\backslash[-1,1]$ by $\mathbb{R}$. Instead, we have to consider another space of functions. For $g,h,k,l \in \mathbb{Z}$, let us denote by $X^{k,l}_{g,h}$ the closure of $\mathcal{S}(\mathbb{R})$ with respect to the norm

$$\|f\|^2_{X^{k,l}_{g,h}} := \int_{-1}^1 |x|^{2g}|f(x)|^2 + |x|^{2h}|f'(x)|^2 dx + \int_{\mathbb{R}\backslash[-1,1]} |x|^{2k}|f(x)|^2 + |x|^{2l}|f'(x)|^2 dx. \tag{2.6.10}$$

For example, we then have $X^{0,0}_{0,0} = H^1(\mathbb{R})$ and $X^{1,1}_{0,0} = H^{1,1}(\mathbb{R})$. Using the chain rule as in the above examples and the relations 2.6.8 we find the following.

Corollary 2.6.5. *Under the same assumptions as Lemma 2.6.4, we have*

$$\begin{aligned} wr(w) &\in X^{1,0}_{-3,0}, \qquad z\widehat{r}(z) \in X^{1,0}_{-3,0}, \\ r(w) &\in X^{2,1}_{-2,1}, \qquad \widehat{r}(z) \in X^{2,1}_{-2,1}. \end{aligned} \tag{2.6.11}$$

In particular, the space which all of the four functions are contained in, is given by $X^{1,0}_{-3,0} \cup X^{2,1}_{-2,1} = X^{1,0}_{-2,1}$.

Remark 2.6.6. It may appear strange at a first glance that the direct and inverse scattering transforms for the MTM system (1.1.1) connect potentials $u, v \in X_{1,2} = H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$ and reflection coefficients

$$r, \widehat{r} \in X^{2,1}_{-2,1}$$

in different spaces, whereas the Fourier transform provides an isomorphism in the space $X_{1,2}$. However, the appearance of $X_{1,2}$ spaces for the potential (u, v) is not surprising due to the transformation of the linear operator L to the equivalent forms (2.2.4) and (2.2.5). The condition $u, v \in X_{1,2}$ ensures that $Q_{1,2}, \hat{Q}_{1,2} \in H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R})$, hence, the direct and inverse scattering transform for the MTM system (1.1.1) provides a transformation between $Q_{1,2}, \hat{Q}_{1,2} \in H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R})$ and $r, \widehat{r} \in X^{2,1}_{-2,1}$, which is a natural transformation under the Fourier transform with oscillatory phase $e^{ix(\omega-\omega^{-1})}$.

Remark 2.6.7. By the explanation of the above remark, it is reasonable to expect that

$$r, \widehat{r} \in X^{2,2}_{-2,0}$$

if $Q_{1,2}, \hat{Q}_{1,2} \in H^{1,1}(\mathbb{R})$. The condition $Q_{1,2}, \hat{Q}_{1,2} \in H^{1,1}(\mathbb{R})$ in turn is equivalent to $u, v \in H^{2,1}(\mathbb{R})$. However, even if $r, \widehat{r} \in X^{2,2}_{-2,0}$ will turn out to be a necessary condition in the analysis of the long-time behavior of $(u(t,x), v(t,x))$, we do not prove rigourously in this thesis that $u, v \in H^{2,1}(\mathbb{R})$ is indeed a sufficient assumption. Therefore, in the majority of the cases we will use the formulation *"let $u, v \in X_{1,2}$ and assume additionally that $r, \widehat{r} \in X^{2,2}_{-2,0}$"*. But the reader may think of $u, v \in H^{2,1}(\mathbb{R})$.

2.7 Eigenvalues and norming constants

Definition 2.7.1. We say that the pair (u, v) admits a *simple eigenvalue* at λ_1 with $\mathrm{Im}(\lambda_1^2) < 0$ if $\alpha(\lambda_1) = 0$ and $\alpha'(\lambda_1) \neq 0$. In particular, due to the symmetry $\alpha(\lambda) = \alpha(-\lambda)$, if λ_1 is a simple eigenvalue, then $-\lambda_1$ is also a simple eigenvalue. By $\mathcal{G}_N$ with $N \in \mathbb{N}_0$ we denote all pairs of functions $(u, v) \in X_{2,1} \times X_{2,1}$ that admit exactly N simple eigenvalues in the second quadrant, i.e. $\mathrm{Im}(\lambda) < 0$ and $\mathrm{Re}(\lambda) > 0$, and no resonances in the sense of Definition 2.6.1. Furthermore, we set

$$\mathcal{G} := \bigcup_{N=0}^{\infty} \mathcal{G}_N$$

and call elements of $\mathcal{G}$ *generic* potentials.

Remark 2.7.2. Let $\{\lambda_1, ..., \lambda_N\}$ be the set of simple eigenvalues in the second quadrant and set

$$w_j = \lambda_j^{-2}, \qquad z_j = \lambda_j^2, \qquad j = 1, ..., N. \tag{2.7.1}$$

Then, by (2.5.10) and (2.5.14), $\{w_1, ..., w_N\} \subset \mathbb{C}^+$ is exactly the set of zeroes of $a(w)$ in the upper half plane and, analogously, $\{z_1, ..., z_N\} \subset \mathbb{C}^-$ forms the set of zeroes of $\widehat{a}$.

Let us find out what is special about the eigenvalues and why this terminology is chosen. Using

$$\alpha(\lambda) = \det[\psi_1^{(-)}(\lambda; x) | \psi_2^{(+)}(x; \lambda)]$$

it follows that for each eigenvalues λ_j in the sense of Definition 2.7.1, the two two-component functions $\psi_1^{(-)}(x; \lambda_j)$ and $\psi_2^{(+)}(x; \lambda_j)$ are linearly dependent. Hence, we can find a constant $\gamma_j(x)$ such that

$$\psi_1^{(-)}(x; \lambda_j) = \gamma_j(x) \psi_2^{(+)}(x; \lambda_j).$$

Differentiating this equation in x and using $\psi_x^{(\pm)} = L\psi^{(\pm)}$ we find that γ_j is independent of x. Applying (2.1.7) we find

$$\varphi_-(x; \lambda_j) = \gamma_j \phi_+(x; \lambda_j) e^{-ix(\lambda_j^2 - \lambda_j^{-2})/2}. \tag{2.7.2}$$

Using the relations (2.2.6), for w_j and z_j as in (2.7.1), we obtain:

$$\begin{aligned} m_-(x;w_j) &= \frac{\gamma_j}{\lambda_j} n_+(x;w_j) e^{ix(w_j - w_j^{-1})/2}, \\ \widehat{m}_-(x;z_j) &= \lambda_j \gamma_j \widehat{n}_+(x;z_j) e^{-ix(z_j - z_j^{-1})/2}. \end{aligned} \tag{2.7.3}$$

By definition we have $\alpha'(\lambda_j) \neq 0$. The chain rule yields $\alpha'(\lambda_j) = \frac{-2}{\lambda_j^3} a'(\lambda_j^{-2})$ and $\alpha'(\lambda_j) = 2\lambda_j \widehat{a}'(\lambda_j^2)$ such that $a'(w_j) \neq 0$ and $\widehat{a}(z_j) \neq 0$. Thus, we can define the following numbers

$$C_j := \frac{\gamma_j}{\alpha'(\lambda_j)}, \qquad c_j := \frac{\lambda_j \gamma_j}{a'(w_j)}, \qquad \widehat{c}_j := \frac{\gamma_j}{\lambda_j \widehat{a}'(z_j)}, \tag{2.7.4}$$

which are called *norming constants*. These three constants are related via

$$c_j = \frac{-2C_j}{\lambda_j^2}, \qquad \widehat{c}_j = 2C_j, \qquad c_j = -w_j \widehat{c}_j, \qquad \widehat{c}_j = -z_j c_j. \tag{2.7.5}$$

Additionally, in terms of these numbers and by (2.7.2)–(2.7.3), we find:

$$\begin{aligned} \operatorname*{Res}_{\lambda=\lambda_j} \left(\frac{\varphi_-(x;\lambda)}{\alpha(\lambda)} \right) &= C_j \phi_+(x;\lambda_j) e^{-ix(\lambda_j^2 - \lambda_j^{-2})/2}, \\ \operatorname*{Res}_{w=w_j} \left(\frac{m_-(x;w)}{a(w)} \right) &= w_j c_j n_+(x;w_j) e^{ix(w_j - w_j^{-1})/2}, \\ \operatorname*{Res}_{z=z_j} \left(\frac{\widehat{m}_-(x;z)}{\widehat{a}(z)} \right) &= z_j \widehat{c}_j \widehat{n}_+(x;z_j) e^{-ix(z_j - z_j^{-1})/2}. \end{aligned} \tag{2.7.6}$$

The second and the third residue calculations can be used to compute the residues of the matrices $P_+(x;w)$ and $\widehat{P}_-(x;z)$. We recall the definitions (2.6.3) and (2.6.5) and get

$$\begin{aligned} \operatorname*{Res}_{w=w_j} P_+(x;w) &= \lim_{w \to w_j} P_+(x;w) \begin{bmatrix} 0 & 0 \\ w_j c_j e^{ix(w_j - w_j^{-1})/2} & 0 \end{bmatrix}, \\ \operatorname*{Res}_{z=z_j} \widehat{P}_-(x;z) &= \lim_{z \to z_j} \widehat{P}_-(x;z) \begin{bmatrix} 0 & 0 \\ z_j \widehat{c}_j e^{-ix(z_j - z_j^{-1})/2} & 0 \end{bmatrix}. \end{aligned} \tag{2.7.7}$$

Using the symmetries of the Jost functions as stated in Remarks 2.1.2 and 2.1.3 and using relations (2.2.6), we derive in a similar fashion the following residue conditions:

$$\begin{aligned} \operatorname*{Res}_{w=\overline{w}_j} P_-(x;w) &= \lim_{w \to \overline{w}_j} P_-(x;w) \begin{bmatrix} 0 & -\overline{c}_j e^{-ix(\overline{w}_j - \overline{w}_j^{-1})/2} \\ 0 & 0 \end{bmatrix}, \\ \operatorname*{Res}_{z=z_j} \widehat{P}_+(x;z) &= \lim_{z \to z_j} \widehat{P}_+(x;z) \begin{bmatrix} 0 & -\overline{\widehat{c}}_j e^{ix(\overline{z}_j - \overline{z}_j^{-1})/2} \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{2.7.8}$$

We are now ready to define the scattering data of the given functions $u, v \in X_{2,1}$.

Definition 2.7.3. For $N \in \mathbb{N}$, let $(u,v) \in \mathcal{G}_N$ and denote by p the function $p : \mathbb{R} \cup i\,\mathbb{R} \to \mathbb{C}$ as defined in (2.6.1). Furthermore, let $\lambda_1, ..., \lambda_N$ be the pairwise distinct simple eigenvalues of (u,v) in the second quadrant and $C_1, ..., C_N$ the norming constants defined in (2.7.4). Then, we call

$$\mathcal{S}(u,v) = (p; \{\lambda_j, C_j\}_{j=1}^N) \tag{2.7.9}$$

the *scattering data* of (u,v). Additionally, for r and $\widehat{r}$ as defined in (2.6.6), for $w_1, ..., w_n, z_1, ..., z_N$ as in (2.7.1) and for the norming constants $c_1, ..., c_N, \widehat{c}_1, ..., \widehat{c}_N$ defined in (2.7.4), we call

$$\mathcal{S}_w(u,v) = (r; \{w_j, c_j\}_{j=1}^N), \qquad \mathcal{S}_z(u,v) = (\widehat{r}; \{z_j, \widehat{c}_j\}_{j=1}^N) \tag{2.7.10}$$

the *transformed scattering data* of (u,v).

Remark 2.7.4. We will call $\mathcal{S}(u,v)$ the *original* scattering data because those were introduced in the aforementioned pioneering work [KM77]. On the one hand, the transformed scattering data $\mathcal{S}_w(u,v)$ and $\mathcal{S}_z(u,v)$ are completely determined by $\mathcal{S}(u,v)$. On the other hand, the main purpose for introducing the transformed data is that the regularity of r and $\widehat{r}$ as functions $\mathbb{R} \to \mathbb{C}$ is easier expressed as the regularity of p as a function $\mathbb{R} \cup i\,\mathbb{R} \to \mathbb{C}$.

Using the same techniques as in [PS18b, Corollary 4] we can also prove Lipschitz continuity of the scattering map:

Theorem 2.7.5. *For each $N \in \mathbb{N}_0$, the maps*

$$\mathcal{S}_w : \mathcal{G}_N \to X^{1,0}_{-2,1} \times (\mathbb{C}^+)^N \times (\mathbb{C}^*)^N$$

and

$$\mathcal{S}_z : \mathcal{G}_N \to X^{1,0}_{-2,1} \times (\mathbb{C}^-)^N \times (\mathbb{C}^*)^N$$

are Lipschitz continuous.

2.8 Formulation of the Riemann-Hilbert problems

The inverse scattering problem consists of the construction of a map $(p; \{\lambda_j, C_j\}_{j=1}^N) \mapsto (u,v)$. In fact we work with the transformed scattering data (2.7.10). The potential u is reconstructed from $\mathcal{S}_w(u,v)$ and v is reconstructed from $\mathcal{S}_z(u,v)$. The inverse scattering map is based on the analyticity properties of the Jost functions. More precisely, for generic u and v in the sense of Definition 2.7.1, the matrices $P_\pm(x;w)$ defined in (2.6.3) admit a meromorphic continuation in $\mathbb{C}^\pm$. The poles are located at $w_1, ..., w_N, \overline{w}_1, ..., \overline{w}_N$ and the residues at these poles fulfill the conditions (2.7.7) and (2.7.8). For $w \in \mathbb{R}$ the matrices are connected by $P_+ = P_-(1+R)$, where the matrix R is obtained from (2.6.4) in terms of the reflection coefficient $r(w)$. Denoting the meromorphic continuations by the same letters, we obtain by (2.2.10) the following behavior for large $|w|$:

$$P_\pm(x;w) \to \begin{bmatrix} m_+^\infty(x) & 0 \\ 0 & (m_+^\infty(x))^{-1} \end{bmatrix} = \left[m_+^\infty(x)\right]^{\sigma_3}, \quad \text{as } |w| \to \infty, \quad w \in \mathbb{C}^\pm .$$

Since we prefer to work with x-independent boundary conditions, we define

$$M(x;w) := \begin{cases} \left[m_+^\infty(x)\right]^{-\sigma_3} P_+(x;w), & w \in \mathbb{C}^+, \\ \left[m_+^\infty(x)\right]^{-\sigma_3} P_-(x;w), & w \in \mathbb{C}^- . \end{cases} \tag{2.8.1}$$

Since M is obtained from $P_\pm$ by multiplication from the left, the residue conditions and also the discontinuity condition on $\mathbb{R}$ remain unchanged. Thus, we obtain the following Riemann-Hilbert problem for the function $M(x;\cdot)$.

Riemann-Hilbert problem 2.8.1. For given scattering data $(r, \{w_j, c_j\}_{j=1}^N)$ and $x \in \mathbb{R}$, find a 2×2-matrix-valued function $\mathbb{C} \setminus \mathbb{R} \ni w \mapsto M(x; w)$ satisfying

1. $M(x; \cdot)$ is meromorphic in $\mathbb{C} \setminus \mathbb{R}$.
2. $M(x; w) = 1 + \mathcal{O}\left(\frac{1}{w}\right)$ as $|w| \to \infty$.
3. The non-tangential boundary values $M_\pm(x; w)$ exist for $w \in \mathbb{R}$ and satisfy the jump relation
$$M_+ = M_-(1 + R),$$
where $R = R(x; w)$ is defined by $R(x; 0) = 0$ and
$$R(x; w) = \begin{bmatrix} w|r(w)|^2 & \overline{r(w)}e^{-\frac{i}{2}(w-w^{-1})x} \\ wr(w)e^{\frac{i}{2}(w-w^{-1})x} & 0 \end{bmatrix} \tag{2.8.2}$$
for $w \in \mathbb{R} \setminus \{0\}$.
4. $M(x; \cdot)$ has simple poles at $w_1, ..., w_N, \overline{w}_1, ..., \overline{w}_N$ with
$$\begin{aligned} \operatorname*{Res}_{w=w_j} M(x; w) &= \lim_{w \to w_j} M(x; w) \begin{bmatrix} 0 & 0 \\ w_j c_j e^{\frac{i}{2}(w_j - w_j^{-1})x} & 0 \end{bmatrix}, \\ \operatorname*{Res}_{w=\overline{w}_j} M(x; w) &= \lim_{w \to \overline{w}_j} M(x; w) \begin{bmatrix} 0 & -\overline{c}_j e^{-\frac{i}{2}(\overline{w}_j - \overline{w}_j^{-1})x} \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{2.8.3}$$

The expansions of the Jost functions as $|w| \to \infty$ yield the following reconstruction formulas

$$\left(u_x(x) - \frac{i}{2}u(x)|v(x)|^2 - \frac{i}{2}v(x)\right) e^{\frac{i}{2}\int_x^{+\infty} |u|^2 + |v|^2 dy} = \frac{1}{2i} \lim_{|w| \to \infty} w \cdot [M(x; w)]_{21}, \tag{2.8.4}$$

$$\overline{u}(x)e^{-\frac{i}{2}\int_x^{+\infty} |u|^2 + |v|^2 dy} = \lim_{|w| \to \infty} w \cdot [M(x; w)]_{12}\,. \tag{2.8.5}$$

In both equations (2.8.4)–(2.8.5) we denote by $[\cdot]_{ij}$ the i-j-entry of the matrix inside the brackets. Assuming that the Riemann-Hilbert problem is uniquely solvable, we can use the solution $M(x; w)$ to recover $u(x)$ by means of (2.8.4) and (2.8.5) as explained below. Thus, in total we have constructed a map $(r; \{w_j, c_j\}_{j=1}^N) \mapsto u$. This is virtually half of the inverse map for the direct scattering map $(u, v) \mapsto \mathcal{S}_w(u, v)$. The remaining part $\mathcal{S}_z(u, v) \mapsto v$ is found via a second RHP. Repeating all the arguments from above, we define

$$\widehat{M}(x; z) := \begin{cases} \left[\widehat{m}_+^\infty(x)\right]^{-\sigma_3} \widehat{P}_+(x; z), & z \in \mathbb{C}^+, \\ \left[\widehat{m}_+^\infty(x)\right]^{-\sigma_3} \widehat{P}_-(x; z), & z \in \mathbb{C}^-\,. \end{cases} \tag{2.8.6}$$

Hence, we obtain the following Riemann-Hilbert problem for the function $\widehat{M}(x; \cdot)$.

Riemann-Hilbert problem 2.8.2. For given scattering data $(\widehat{r}, \{z_j, \widehat{c}_j\}_{j=1}^N)$ and $x \in \mathbb{R}$, find a 2×2-matrix-valued function $\mathbb{C} \setminus \mathbb{R} \ni z \mapsto \widehat{M}(x; z)$ satisfying

1. $\widehat{M}(x; \cdot)$ is meromorphic in $\mathbb{C} \setminus \mathbb{R}$.
2. $\widehat{M}(x; z) = 1 + \mathcal{O}\left(\frac{1}{z}\right)$ as $|z| \to \infty$.
3. The non-tangential boundary values $\widehat{M}_\pm(x; z)$ exist for $z \in \mathbb{R}$ and satisfy the jump relation
$$\widehat{M}_+ = \widehat{M}_-(1 + \widehat{R}),$$
where $\widehat{R} = \widehat{R}(x; z)$ is defined by $\widehat{R}(x; 0) = 0$ and
$$\widehat{R}(x; z) = \begin{bmatrix} 0 & -\overline{\widehat{r}(z)} e^{\frac{i}{2}(z - z^{-1})x} \\ -z\widehat{r}(z) e^{-\frac{i}{2}(z - z^{-1})x} & z|\widehat{r}(z)|^2 \end{bmatrix} \tag{2.8.7}$$
for $z \in \mathbb{R} \setminus \{0\}$.
4. $\widehat{M}(x; \cdot)$ has simple poles at $z_1, ..., z_N, \overline{z}_1, ..., \overline{z}_N$ with
$$\begin{aligned} \operatorname*{Res}_{z=z_j} \widehat{M}(x; z) &= \lim_{z \to z_j} \widehat{M}(x; w) \begin{bmatrix} 0 & 0 \\ z_j \widehat{c}_j e^{-\frac{i}{2}(z_j - z_j^{-1})x} & 0 \end{bmatrix}, \\ \operatorname*{Res}_{z=\overline{z}_j} \widehat{M}(t, x; z) &= \lim_{z \to \overline{z}_j} \widehat{M}(t, x; z) \begin{bmatrix} 0 & -\overline{\widehat{c}}_j e^{+\frac{i}{2}(\overline{z}_j - \overline{z}_j^{-1})x} \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{2.8.8}$$

Analogously to (2.8.4)–(2.8.5), the following reconstruction formulae are available.

$$\left(v_x(x) + \frac{i}{2}|u(x)|^2 v(x) + \frac{i}{2} u(x) \right) e^{-\frac{i}{2}\int_x^{+\infty} |u|^2 + |v|^2 dy} = \frac{i}{2} \lim_{|z| \to \infty} z \cdot \left[\widehat{M}(x; z) \right]_{21}, \tag{2.8.9}$$

$$\overline{v}(x) e^{\frac{i}{2}\int_x^{+\infty} |u|^2 + |v|^2 dy} = \lim_{|z| \to \infty} z \cdot \left[\widehat{M}(x; z) \right]_{12}. \tag{2.8.10}$$

Again, unique solvability of RHP 2.8.2 would ensure the existence of the map $\mathcal{S}_z(u, v) \mapsto v$. In Chapter 3 we will work out the solvability of the two Riemann-Hilbert problems and prove estimates on their solutions.

Remark 2.8.3. It follows from $R(x; 0) = \widehat{R}(x; 0) = 0$ that $M_+(x; 0) = M_-(x; 0)$ and $\widehat{M}_+(x; 0) = \widehat{M}_-(x; 0)$. More precisely, using (2.3.3), (2.3.4) and the definitions (2.6.3) and (2.6.5) of $P_\pm(x; w)$ and $\widehat{P}_\pm(x; z)$, respectively, we find

$$M(x; 0) = \begin{bmatrix} m_+^\infty(x) & 0 \\ 0 & n_+^\infty(x) \end{bmatrix}^{-1} \begin{bmatrix} 1 & \overline{v}(x) \\ u(x) & 1 + u(x)\overline{v}(x) \end{bmatrix} \begin{bmatrix} \widehat{m}_+^\infty(x) & 0 \\ 0 & \widehat{n}_+^\infty(x) \end{bmatrix}$$

and

$$\widehat{M}(x; 0) = \begin{bmatrix} \widehat{m}_+^\infty(x) & 0 \\ 0 & \widehat{n}_+^\infty(x) \end{bmatrix}^{-1} \begin{bmatrix} 1 & \overline{u}(x) \\ v(x) & 1 + \overline{u}(x)v(x) \end{bmatrix} \begin{bmatrix} m_+^\infty(x) & 0 \\ 0 & n_+^\infty(x) \end{bmatrix}.$$

In particular, the following holds:

$$[M(x; 0)]_{11} = \frac{\widehat{m}_+^\infty(x)}{m_+^\infty(x)} = e^{-\frac{i}{2}\int_x^{+\infty} (|u|^2 + |v|^2) dy}, \qquad [\widehat{M}(x; 0)]_{11} = \frac{m_+^\infty(x)}{\widehat{m}_+^\infty(x)} = e^{\frac{i}{2}\int_x^{+\infty} (|u|^2 + |v|^2) dy}.$$

In these formulas, we regain the same exponential factors as those in the reconstruction formulas (2.8.5) and (2.8.10). Hence, by substitution we obtain the following two decoupled reconstruction formulas:

$$u(x) = [M(x;0)]_{11}\overline{\lim_{|w|\to\infty} w[M(x;w)]_{12}}, \qquad v(x) = [\widehat{M}(x;0)]_{11}\overline{\lim_{|z|\to\infty} z[\widehat{M}(x;z)]_{12}}. \tag{2.8.11}$$

While equations (2.8.4), (2.8.5), (2.8.9) and (2.8.10) are suitable for studying the inverse map of the scattering transformation in the sense of Theorem 3.1.1, the equivalent formulas (2.8.11) will become useful in the analysis of the asymptotic behavior of $u(x)$ and $v(x)$ as $|x| \to \infty$.

Remark 2.8.4. From the construction of $M(x;w)$ it follows that for

$$\Omega(x) := \lim_{|w|\to\infty} w[M(x;w)]_{12},$$

M admits the following symmetry

$$M(x;w) = \frac{1}{w}\begin{bmatrix} -\Omega(x) & \frac{1}{\Omega(x)} \\ -w - |\Omega(x)|^2 & \end{bmatrix}\overline{M(x;\overline{w})}\begin{bmatrix} 0 & -1 \\ w & 0 \end{bmatrix}. \tag{2.8.12}$$

This is in contrast, for example, to the NLS equation, where we have

$$M(x;w) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\overline{M(x;\overline{w})}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The symmetry (2.8.12) will be useful for the Bäcklund transformation, see Section 3.6.

2.9 Time evolution of the scattering data

In this section we compute the time evolution of the scattering data. In other words, we assume that (u, v) is a solution of the MTM system (1.1.1). Thus, u and v are both functions of t and x. We can compute the scattering data of Definition 2.7.3 at every time t and may ask the question if at some given time t the scattering data can be computed from the scattering data of the initial data (u_0, v_0). The answer is quite surprising, due to the linearity of the evolution of the scattering data. This remarkable fact demonstrates the significance of the direct scattering transform.

Theorem 2.9.1. *Assume that $(u(t), v(t))$ solves (1.1.1) for initial data (u_0, v_0) with scattering data $\mathcal{S}(u_0, v_0) = (p(\lambda; 0); \{\lambda_j(0), C_j(0)\}_{j=1}^N)$. Then, at any time t, the scattering data $\mathcal{S}(u(t), v(t))$ of the solution $(u(t), v(t))$ is calculated from $\mathcal{S}(u_0, v_0)$ by:*

$$\mathcal{S}(u(t), v(t)) = (p(0;\lambda)e^{-it(\lambda^2+\lambda^{-2})/2}; \{\lambda_j(0), C_j(0)e^{-it(\lambda_j^2+\lambda_j^{-2})/2}\}_{j=1}^N). \tag{2.9.1}$$

In particular, the number of eigenvalues does not vary in time and sets $\mathcal{G}_N$ are invariant under the MTM flow.

Proof. By $\det \psi^{(\pm)}(x,t;\lambda) = 1$ we know that there exists a matrix $C(x,t;\lambda)$ such that

$$\psi^{(\pm)}C = (\partial_t - A)\psi^{(\pm)}. \tag{2.9.2}$$

By the definition of $\psi^{(\pm)}$, it follows that $(\partial_x - L)[\psi^{(\pm)}C] = \psi^{(\pm)}C_x$. Substituting this into (2.9.2) we get that $\psi^{(\pm)}C_x = (\partial_x - L)(\partial_t - A)\psi^{(\pm)}$. But due to (2.1.3), the right hand side is equal to $-(\partial_t - A)(\partial_x - L)\psi^{(\pm)}$, which in turn vanishes because of $(\partial_x - L)\psi^{(\pm)} = 0$. It follows that $C_x = 0$ and thus, C does not depend on x. Now let us assume that $|u(x,t)| + |v(t,x)| \to 0$ as $x \to \pm\infty$. Taking then limits $x \to \pm\infty$ of (2.9.2) we find

$$e^{ix(\lambda^2-\lambda^{-2})\sigma_3/4}C(t;\lambda) = -\frac{i}{4}\left(\lambda^2 + \frac{1}{\lambda^2}\right)\sigma_3 e^{ix(\lambda^2-\lambda^{-2})\sigma_3/4}$$

which is equivalent to

$$C(t;\lambda) = -\frac{i}{4}\left(\lambda^2 + \frac{1}{\lambda^2}\right)\sigma_3.$$

Thus,

$$\psi_t^{(\pm)} = A\psi^{(\pm)} - \frac{i}{4}\left(\lambda^2 + \frac{1}{\lambda^2}\right)\psi^{(\pm)}\sigma_3.$$

Furthermore, we compute

$$\begin{aligned}
\partial_t \alpha(\lambda) &= \det[(\psi_1^{(-)})_t, \psi_2^{(+)}] + \det[\psi_1^{(-)}, (\psi_2^{(+)})_t] \\
&= \det[A\psi_1^{(-)}, \psi_2^{(+)}] - \frac{i}{4}\left(\lambda^2 + \frac{1}{\lambda^2}\right)\det[\psi_1^{(-)}, \psi_2^{(+)}] \\
&\quad + \det[\psi_1^{(-)}, A\psi_2^{(+)}] + \frac{i}{4}\left(\lambda^2 + \frac{1}{\lambda^2}\right)\det[\psi_1^{(-)}, \psi_2^{(+)}] \\
&= \mathrm{tr}(A)\det[\psi_1^{(-)}, \psi_2^{(+)}] \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
\partial_t \beta(\lambda) &= \det[(\psi_1^{(-)})_t, \psi_1^{(+)}] + \det[\psi_1^{(-)}, (\psi_1^{(+)})_t] \\
&= \det[A\psi_1^{(-)}, \psi_1^{(+)}] - \frac{i}{4}\left(\lambda^2 + \frac{1}{\lambda^2}\right)\det[\psi_1^{(-)}, \psi_1^{(+)}] \\
&\quad + \det[\psi_1^{(-)}, A\psi_1^{(+)}] - \frac{i}{4}\left(\lambda^2 + \frac{1}{\lambda^2}\right)\det[\psi_1^{(-)}, \psi_1^{(+)}] \\
&= \mathrm{tr}(A)\det[\psi_1^{(-)}, \psi_1^{(+)}] - \frac{i}{2}\left(\lambda^2 + \frac{1}{\lambda^2}\right)\det[\psi_1^{(-)}, \psi_1^{(+)}] \\
&= -\frac{i}{2}\left(\lambda^2 + \frac{1}{\lambda^2}\right)\beta(\lambda).
\end{aligned}$$

From these ordinary differential equations it follows that the reflection coefficient $p(\lambda) = \beta(\lambda)/\alpha(\lambda)$ admits the following time evolution:

$$p(t;\lambda) = e^{-i(\lambda^2+\lambda^{-2})t/2}p(0;\lambda)$$

The time evolution of the norming constants can be proved as follows. We recall the equation $\psi_1^{(-)}(x,t;\lambda_j) = \gamma_j(t)\psi_2^{(+)}(x,t;\lambda_j)$ and differentiate it in t. We obtain

$$A\psi_1^{(-)} - \frac{i}{4}\left(\lambda^2 + \frac{1}{\lambda^2}\right)\psi_1^{(-)} = \partial_t\gamma_j\psi_2^{(+)} + \gamma_j\left(A\psi_2^{(+)} + \frac{i}{4}\left(\lambda^2 + \frac{1}{\lambda^2}\right)\psi_2^{(+)}\right).$$

Using $\psi_1^{(-)} = \gamma_j\psi_2^{(+)}$ again, several terms cancel out and we get

$$-\frac{i}{2}\left(\lambda^2 + \frac{1}{\lambda^2}\right)\gamma_j\psi_2^{(+)} = \partial_t\gamma_j\psi_2^{(+)},$$

which finally yields

$$-\frac{i}{2}\left(\lambda^2 + \frac{1}{\lambda^2}\right)\gamma_j = \partial_t\gamma_j.$$

□

Using the relations between the original scattering data and the transformed scattering data, we derive the following Corollary:

Corollary 2.9.2. *Assume that $(u(t), v(t)) \in \mathcal{G}_N$ solves the MTM system (1.1.1) and at time $t = 0$ the transformed scattering data are given as in (2.7.10). Then, at any other time t we have*

$$\mathcal{S}_w(u(t), v(t)) = (r(w)e^{-it(w+w^{-1})/2}; \{w_j, c_j e^{-it(w_j+w_j^{-1})/2}\}_{j=1}^N),$$
$$\mathcal{S}_z(u(t), v(t)) = (\widehat{r}(z)e^{-it(z+z^{-1})/2}; \{z_j, \widehat{c}_j e^{-it(z_j+z_j^{-1})/2}\}_{j=1}^N).$$

An important property of the time evolution is that the reflection coefficients remain in the spaces in which they are contained at time $t = 0$. Taking the derivative in w of $r(w)e^{-it(w+w^{-1})/2}$ directly yields the following Corollary.

Corollary 2.9.3. *Assume that $(u(t), v(t)) \in \mathcal{G}_N$ solves the MTM system (1.1.1) and at time $t = 0$ the reflection coefficients $r(0; \cdot)$ and $\widehat{r}(0; \cdot)$ are contained in $X^{1,0}_{-2,1}$. Then, for every time t we have $r(t; \cdot) \in X^{1,0}_{-2,1}$ and $\widehat{r}(t; \cdot) \in X^{1,0}_{-2,1}$.*

Remark 2.9.4. Assume that M solves RHP 2.8.1 for transformed scattering data $(r; \{w_j, c_j\}_{j=1}^N)$. Moreover, assume that $\widetilde{M}$ is a solution for RHP 2.8.1 with scattering data $(e^{i\alpha}r; \{w_j e^{i\alpha} c_j\}_{j=1}^N)$, where α is a fixed real number. Then a direct computation shows that

$$\widetilde{M}(t, x; w) = \begin{bmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{bmatrix} M(t, x; w) \begin{bmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{bmatrix}$$

for all $w \in \mathbb{C} \setminus \mathbb{R}$. Thus we may conclude that $[M(t, x; 0)]_{11} = [\widetilde{M}(t, x; 0)]_{11}$ and

$$\lim_{|w| \to \infty} w \cdot \left[\widetilde{M}(t, x; w)\right]_{12} = e^{-i\alpha} \lim_{|w| \to \infty} w \cdot [M(t, x; w)]_{12}.$$

Taking into account the reconstruction formulas (2.8.11) we see that if a function u belongs to scattering data $(r; \{w_j, c_j\}_{j=1}^N)$, then $e^{i\alpha}u$ belongs to scattering data $(e^{i\alpha}r; \{w_j, e^{i\alpha}c_j\}_{j=1}^N)$. Repeating the same argument for RHP 2.8.2 we may summarize the statements as follows. If $\mathcal{S}(u, v) = (p; \{\lambda_j, C_j\}_{j=1}^N)$, then $\mathcal{S}(e^{i\alpha}u, e^{i\alpha}v) = (e^{i\alpha}p; \{\lambda_j, e^{i\alpha}C_j\}_{j=1}^N)$. This property of the scattering map will be useful in the proof of Theorem 7.1.1 and represents the invariance of (1.1.1) under phase shifts $(u, v) \mapsto (e^{i\alpha}u, e^{i\alpha}v)$.

Remark 2.9.5. Assume that $(u(t, x), v(t, x))$ solves the MTM system subject to initial data $(u_0(x), v_0(x))$. Defining new initial data by

$$\tilde{u}_0(x) := \overline{v_0(x)}, \qquad \tilde{v}_0(x) := \overline{u_0(x)},$$

then we know that the solution $(\tilde{u}(t, x), \tilde{v}(t, x))$ is given by

$$\tilde{u}(t, x) := \overline{v(-t, x)}, \qquad \tilde{v}(t, x) := \overline{u(-t, x)}.$$

Associating to (u, v) the linear operator $L(\lambda)$ as in (2.1.1) and, correspondingly, associating to $(\tilde{u}, \tilde{v})$ a linear operator $\tilde{L}(\lambda)$, then we find

$$\tilde{L}(\lambda) = \overline{L(\lambda^{-1})}.$$

Denoting by p and $\tilde{p}$ the reflection coefficients of (u, v) and $(\tilde{u}, \tilde{v})$, respectively, this implies

$$\tilde{p}(\lambda) = \overline{p(1/\lambda)}.$$

The respective transformed reflection coefficients $r(w) = \lambda p(\lambda)$ and $\tilde{r}(w) = \lambda \tilde{p}(\lambda)$ are related by

$$\tilde{r}(w) = w^{-1}\overline{r(w^{-1})}$$

which can also be expressed as $\tilde{r}(w) = \overline{\widehat{r}(w)}$ by (2.6.8). We conclude that conjugating and changing u and v at time $t = 0$ entails a time reversion and also a conjugation and a change of r and $\widehat{r}$. Moreover, possible eigenvalues and norming constants $\left\{\tilde{\lambda}_j, \tilde{C}_j\right\}$ can be found as

$$\tilde{\lambda}_j = 1/\overline{\lambda}_j, \qquad \tilde{C}_j = -\tilde{\lambda}_j^{-2}\overline{C}_j.$$

By (2.7.5) we see again, that the transformed norming constants c_j and $\widehat{c}_j$ are conjugated and change theirs places by the new initial data.

These symmetries are useful in the following sense. Assuming that the long-time behavior of $(u(t,x), v(t,x))$ as $t \to +\infty$ is known in terms of its scattering data, then by the above considerations one can easily compute the long-time behavior as $t \to -\infty$.

Chapter 3

Inverse Scattering

3.1 Overview

The inverse scattering problem is the construction of a map $(p; \{\lambda_j, C_j\}_{j=1}^N) \mapsto (u, v)$. In fact we work with the transformed scattering data (2.7.10). The potential u will be reconstructed from $\mathcal{S}_w(u, v)$ and v is reconstructed from $\mathcal{S}_z(u, v)$. We start with the case where the scattering data consist of only the reflection coefficient. Thus, for a given function $r \in X_{-2,1}^{2,1}$ with the property

$$1 + w|r(w)|^2 \geq c_1 > 0, \qquad w \in \mathbb{R}, \tag{3.1.1}$$

we want to find a solution $M(x; w)$ of Riemann–Hilbert problem 2.8.1. At the same time we would also like to find a solution $\widehat{M}(x; z)$ of Riemann–Hilbert problem 2.8.2. Then by (2.8.11) or the other reconstruction formulas, one would recover the functions u and v. The following summarizes the main result of this chapter.

Theorem 3.1.1. *For any $N \in \mathbb{N}$ and given scattering data $\mathcal{S}_w(u, v) = (r; \{w_j, c_j\}_{j=1}^N)$ with $r \in X_{-2,1}^{2,1}$ satisfying (3.1.1), Riemann–Hilbert problem 2.8.1 admits a unique solution $M(x; w)$ with*

$$\lim_{|w|\to\infty} w \cdot [M(x; w)]_{12} \in H_x^1(\mathbb{R}) \cap L_x^{2,1}(\mathbb{R}), \qquad \lim_{|w|\to\infty} w \cdot [M(x; w)]_{21} \in H_x^1(\mathbb{R}) \cap L_x^{2,1}(\mathbb{R}). \tag{3.1.2}$$

Additionally, for the corresponding data $\mathcal{S}_z(u, v) = (\widehat{r}; \{z_j, \widehat{c}_j\}_{j=1}^N)$, Riemann–Hilbert problem 2.8.2 admits a unique solution $\widehat{M}(x; z)$ satisfying

$$\lim_{|z|\to\infty} z \cdot \left[\widehat{M}(x; z)\right]_{12} \in H_x^1(\mathbb{R}) \cap L_x^{2,1}(\mathbb{R}), \qquad \lim_{|z|\to\infty} z \cdot \left[\widehat{M}(x; z)\right]_{21} \in H_x^1(\mathbb{R}) \cap L_x^{2,1}(\mathbb{R}). \tag{3.1.3}$$

The two Riemann-Hilbert problems 2.8.1 and 2.8.2 are structurally equivalent. After all, this allows us to skip the proof of (3.1.3) because it is proven analogously to (3.1.2). The complete proof of (3.1.2) is given throughout the following sections in several steps:

(i) Solvability of Riemann–Hilbert problem 2.8.1 for $N = 0$ is shown in Section 3.2. We use the method of Beals and Coifman and an extension to non-symmetric jump matrices developed in [PS18b].

(ii) Estimates in $H_x^1(\mathbb{R}_+) \cap L_x^{2,1}(\mathbb{R}_+)$ for $N = 0$ can be obtained by the analysis of the Beals-Coifman integral equation, see Section 3.4. Technical results which are needed for this analysis can be found in the intermediate Section 3.3

(iii) Estimates in $H_x^1(\mathbb{R}_-) \cap L_x^{2,1}(\mathbb{R}_-)$ for $N = 0$ are obtained in Section 3.5. Therefore, we rewrite Riemann–Hilbert problem 2.8.1 in an equivalent form, which is useful for $x \leq 0$.

(iv) Solvability of Riemann–Hilbert problem 2.8.1 for $N = 1$ and estimates in $H_x^1(\mathbb{R}) \cap L_x^{2,1}(\mathbb{R})$ are worked out in Section 3.6. Therein an auto-Bäcklund transformation is used in order to add eigenvalues w_1 and $\overline{w}_1$ to a pure-radiation RHP.

(v) An inductive argument shows that Theorem 3.1.1 for $N \geq 2$ is obtained by an N-fold application of the Bäcklund transformation.

The meaning of Theorem 3.1.1 is made clear in terms of the following Corollary:

Corollary 3.1.2. *Under the assumptions of Theorem 3.1.1, the potentials u and v, recovered by means of reconstruction formulas (2.8.4), (2.8.5),(2.8.9) and (2.8.10) satisfy*

$$u, v \in H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R}).$$

Proof. By Theorem 3.1.1 the right-hand sides of (2.8.5) and (2.8.10) are controlled in the space $H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R})$. Thus, $\tilde{u}, \tilde{v} \in H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R})$, where

$$\tilde{u}(x) = u(x) e^{\frac{i}{2}\int_x^{+\infty}(|u|^2+|v|^2)dy}, \quad \tilde{v}(x) = v(x) e^{-\frac{i}{2}\int_x^{+\infty}(|u|^2+|v|^2)dy}.$$

Since $|\tilde{u}(x)| = |u(x)|$ and $|\tilde{v}(x)| = |v(x)|$, the gauge factors can be immediately inverted, and since $H^1(\mathbb{R})$ is continuously embedded into $L^p(\mathbb{R})$ for any $p \geq 2$, we then have $u, v \in H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R})$. Since, again by Theorem 3.1.1, the right-hand sides of (2.8.4) and (2.8.9) are also controlled in $H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R})$, similar arguments give $u', v' \in H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R})$, that is, $u, v \in H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$. □

Remark 3.1.3. Without further theory we can observe that if Riemann–Hilbert problem 2.8.1 is solvable, then the solution is unique. In order to show the uniqueness of solutions, we firstly find the following (trivial) Riemann–Hilbert problem for the map $w \mapsto \det M(x; w)$:

$$\begin{cases} \det M(x; w) \text{ is an entire function with respect to the parameter } w, \\ \det M(x; w) \to 1, \text{ as } |w| \to \infty. \end{cases}$$

By Liouville's theorem we conclude that

$$\det M(x; w) \equiv 1, \text{ for all } x \in \mathbb{R} \text{ and } w \in \mathbb{C}. \tag{3.1.4}$$

Hence, for a possible solution M of Riemann–Hilbert problem 2.8.1, $[M(x; w)]^{-1}$ exists for all $x \in \mathbb{R}$ and $w \in \mathbb{C}$. If we have a second solution $\widetilde{M}(x; w)$, the ratio $\widetilde{M}(x; w)[M(x; w)]^{-1}$ satisfies

$$\begin{cases} \widetilde{M}(x; w)[M(x; w)]^{-1} \text{ is an entire function with respect to the parameter } w, \\ \widetilde{M}(x; w)[M(x; w)]^{-1} \to 1, \text{ as } |w| \to \infty, \end{cases}$$

such that $\widetilde{M}(x; w)[M(x; w)]^{-1} \equiv 1$.

Remark 3.1.4. Without requiring relations (2.6.8), (2.7.1) and (2.7.5) we can still obtain a pair of functions $(u(t, \cdot), v(t, \cdot)) \in X_{2,1}$ by Theorem 3.1.1. But in general this is not necessarily a solution of (1.1.1).

3.2 The method of Beals and Coifman

For any function $f \in L^p(\mathbb{R})$ with $1 \leq p < \infty$, the Cauchy operator denoted by $\mathcal{C}$ is given by

$$\mathcal{C}[f](z) := \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(s)}{s - z} ds, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

When z approaches a point on the real line transversely from the upper and lower half planes, the Cauchy operator becomes the following projection operators:

$$\mathcal{P}^{\pm}[f](z) := \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(s)}{s - (z \pm i\,\varepsilon)} ds, \quad z \in \mathbb{R}. \tag{3.2.1}$$

The following proposition summarizes all properties of the Cauchy and projection operators which are needed to establish the inverse scattering map.

Proposition 3.2.1.

(i) *For every $f \in L^p(\mathbb{R})$, $1 \le p < \infty$, the Cauchy operator $\mathcal{C}[f]$ is analytic off the real line.*

(ii) *For $f \in L^1(\mathbb{R})$, $\mathcal{C}[f](z)$ decays to zero as $|z| \to \infty$ and admits the asymptotic*

$$\lim_{|z|\to\infty} z\mathcal{C}[f](z) = -\frac{1}{2\pi i}\int_{\mathbb{R}} f(s)ds, \tag{3.2.2}$$

where the limit is taken either in $\mathbb{C}^+$ or $\mathbb{C}^-$.

(iii) *The projection operators $\mathcal{P}^\pm$ are linear bounded operators $L^p(\mathbb{R}) \to L^p(\mathbb{R})$ for each $p \in (1,\infty)$. For $p = 2$ we have $\|\mathcal{P}^\pm\|_{L^2\to L^2} = 1$.*

(iv) *(Sokhotski-Plemelj theorem) The following two identities hold:*

$$\begin{aligned} \mathcal{P}^+ - \mathcal{P}^- &= Id_{L^p(\mathbb{R})}, \\ \mathcal{P}^+ + \mathcal{P}^- &= -i\mathcal{H}, \end{aligned} \tag{3.2.3}$$

where $\mathcal{H} : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ is the Hilbert transform given by

$$\mathcal{H}[f](z) := \lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\left(\int_{-\infty}^{z-\varepsilon} + \int_{z+\varepsilon}^{\infty}\right)\frac{f(s)}{s-z}ds, \quad z \in \mathbb{R}\,.$$

(v) *Let f_+ and f_- functions defined in the upper (lower) $\mathbb{C}$-plane. If $f_\pm$ is analytic in $\mathbb{C}^\pm$ and $f_\pm(z) \to 0$ as $|z| \to \infty$ for $\pm\,\mathrm{Im}(z) > 0$, then*

$$\mathcal{P}^\pm[f_\mp](z) = 0, \qquad \mathcal{P}^\pm[f_\pm](z) = \pm f_\pm(z), \quad z \in \mathbb{R}\,. \tag{3.2.4}$$

The proof of this proposition is omitted here due to space limitations. The jump condition of Riemann–Hilbert problem 2.8.1, $M_+ = M_-(1+R)$ on $\mathbb{R}$, can be rewritten in the following form:

$$[M_+(x;w)-1] - [M_-(x;w)-1] = M_-(x;w)\begin{bmatrix} w|r(w)|^2 & \overline{r(w)}e^{-\frac{i}{2}(w-w^{-1})x} \\ wr(w)e^{\frac{i}{2}(w-w^{-1})x} & 0 \end{bmatrix} \tag{3.2.5}$$

Assuming $(u,v) \in \mathcal{G}_0$, the function $M(x;\cdot)$ is analytic on $\mathbb{C}\setminus\mathbb{R}$. Thus, by (3.2.4) applying of $\mathcal{P}^\pm$ to both sides of (3.2.5) leads to

$$\begin{aligned} M_+(x;w) - 1 &= \mathcal{P}^+[M_-(x;\diamond)R(x;\diamond)](w), \\ M_-(x;w) - 1 &= \mathcal{P}^-[M_-(x;\diamond)R(x;\diamond)](w). \end{aligned}$$

In particular, the function $(M_-(x;\cdot)-1)$ is a solution to the following integral equation

$$(1 - C_R)[M_-(x;\diamond) - 1](w) = \mathcal{P}^-[R(x;\diamond)](w), \tag{3.2.6}$$

with the x-dependent operator C_R acting on matrix valued functions M by

$$C_R[M](w) := \mathcal{P}^-[M(\diamond)R(x;\diamond)](w). \tag{3.2.7}$$

If $R(x;\cdot) \in L^\infty(\mathbb{R})$, the operator $C_R : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is bounded and the operator norm coincides with the L^∞-norm of $R(x;\cdot)$ by Proposition 3.2.1 (iii). For the same reason, if $R(x;\cdot) \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, then $\mathcal{P}^-[R(x;\diamond)](\cdot) \in L^2(\mathbb{R})$. Thus, under the assumption $R(x;\cdot) \in L^2(\mathbb{R})\cap L^\infty(\mathbb{R})$, the integral equation (3.2.6) makes sense as an equation in $L^2(\mathbb{R})$, i.e. an equation for the unknown function $(M_-(x;\cdot)-1) \in L^2(\mathbb{R})$. On the other hand, for any solution M_- of (3.2.6), we directly find a solution to Riemann–Hilbert problem 2.8.1 (with $N = 0$) by means of the formula

$$M(x;w) = 1 + \frac{1}{2\pi i}\int_{\mathbb{R}}\frac{M_-(x;s)R(x;s)}{s-w}ds, \quad z \in \mathbb{C}\setminus\mathbb{R}\,.$$

Indeed, denoting the right hand side by $H(x;w)$ and using the notation $H_\pm(x;w) = \lim_{\varepsilon\downarrow 0} H(x;w\pm i\,\varepsilon)$. we have by assumption

$$\begin{aligned} H_-(x;w) &= 1 + \mathcal{P}^-[M_-(x;\diamond)R(x;\diamond)](w) \\ &= M_-(x;w), \end{aligned}$$

and by the Sokhotski-Plemelj theorem

$$\begin{aligned} H_+(x;w) &= 1 + \mathcal{P}^+[M_-(x;\diamond)R(x;\diamond)](w) \\ &= 1 + \mathcal{P}^-[M_-(x;\diamond)R(x;\diamond)](w) + M_-(x;w)R(xw) \\ &= M_-(x;w) + M_-(x;w)R(x;w) \\ &= H_-(x;w)(1+R(x;w)). \end{aligned}$$

We conclude that the integral equation (3.2.6) is equivalent to Riemann–Hilbert problem 2.8.1 with no eigenvalues. Thus, we need to study the operator C_R defined in (3.2.7). We make the following observation: the second line $(M_-(x;\diamond)-1)_2$ of $M_-(x;w)-1$ satisfies

$$(1-C_R)[(M_-(x;\diamond)-1)_2](w) = \left(\mathcal{P}^-[\diamond r(\diamond)e^{\frac{i}{2}(\diamond-\diamond^{-1})x}](w),\ 0\right). \tag{3.2.8}$$

Rewriting the jump relation once again as

$$[M_-(x;w)-1] - [M_+(x;w)-1] = M_+(x;w)\begin{bmatrix} 0 & -\overline{r(w)}e^{-\frac{i}{2}(w-w^{-1})x} \\ -wr(w)e^{\frac{i}{2}(w-w^{-1})x} & w|r(w)|^2 \end{bmatrix},$$

we find

$$\begin{aligned} M_+(x;w) - 1 &= \mathcal{P}^+[M_+(x;\diamond)\widetilde{R}(x;\diamond)](w), \\ M_-(x;w) - 1 &= \mathcal{P}^-[M_+(x;\diamond)\widetilde{R}(x;\diamond)](w), \end{aligned}$$

for

$$\widetilde{R} = \begin{bmatrix} 0 & \overline{r(w)}e^{-\frac{i}{2}(w-w^{-1})x} \\ wr(w)e^{\frac{i}{2}(w-w^{-1})x} & -w|r(w)|^2 \end{bmatrix}.$$

This implies that the first row of $M_+ - 1$ satisfies

$$(1-C_{\widetilde{R}})[(M_+(x;\diamond)-1)_1](w) = \left(0\,, \mathcal{P}^+\left[\overline{r(\diamond)}e^{-\frac{i}{2}(\diamond-\diamond^{-1})x}\right](w)\right), \tag{3.2.9}$$

where the operator $C_{\widetilde{R}}$ is defined by

$$C_{\widetilde{R}}[M](w) := \mathcal{P}^+[M(\diamond)\widetilde{R}(x;\diamond)](w). \tag{3.2.10}$$

By the jump relation $M_+ = M_-(1+R)$ it suffices to have knowledge of $(M_+(x;\diamond)-1)_1$ and $(M_-(x;\diamond)-1)_2$ in order to solve the Riemann–Hilbert problem completely. Since in both equations, (3.2.8) and (3.2.9), the right hand side has one zero entry, the operators C_R and $C_{\widetilde{R}}$ do not need to be fully inverted but only on the corresponding subspaces. From [PS18b, Lemma 9] we take the following.

Lemma 3.2.2. *Let $r \in X^{2,1}_{-2,1}$ be such that inequality (3.1.1) is satisfied. Then, for each $h \in L^2(\mathbb{R})$, the integral equations*

$$(1-C_R)[m(\diamond)](w) = (h(w),0), \qquad (1-C_{\widetilde{R}})[n(\diamond)](w) = (0,h(w)),$$

admit unique column-vector-valued solutions $m = (m_1,m_2) \in L^2(\mathbb{R})$ and $n = (n_1,n_2) \in L^2(\mathbb{R})$. Moreover, there exists a positive constant C that only depends on $\|r\|_{L^\infty(\mathbb{R})}$ such that the unique solutions satisfy

$$\|m\|_{L^2(\mathbb{R})} + \|n\|_{L^2(\mathbb{R})} \le C\|h\|_{L^2}.$$

The proof is not presented here because it can be copied word by word from [PS18b]. However, we need the following conclusion:

Corollary 3.2.3. *Let $r \in X^{2,1}_{-2,1}$ be such that inequality (3.1.1) is satisfied. Then, for each $x \in \mathbb{R}$, Riemann–Hilbert problem 2.8.1 admits a unique solution $M(x;w)$ satisfying*

$$\|M_\pm(x;\cdot) - 1\|_{L^2(\mathbb{R})} \le C\|r\|_{L^{2,1}(\mathbb{R})}.$$

Actually, applying Lemma 3.2.2 to (3.2.8) and (3.2.9) yields the following more precise estimates:

$$\begin{aligned}\|(M_-(x;\cdot)-1)_2\|_{L^2(\mathbb{R})} &\le C\left\|\mathcal{P}^-[\diamond r(\diamond)e^{\frac{i}{2}(\diamond-\diamond^{-1})x}](w)\right\|_{L^2_w(\mathbb{R})},\\ \|(M_+(x;\cdot)-1)_1\|_{L^2(\mathbb{R})} &\le C\left\|\mathcal{P}^+[\overline{r(\diamond)}e^{-\frac{i}{2}(\diamond-\diamond^{-1})x}](w)\right\|_{L^2_w(\mathbb{R})}.\end{aligned} \tag{3.2.11}$$

Making use of these estimates is the main subject of the following section.

3.3 Some technical results

The first technical result we present is the analogue of Proposition 7 in [PS14] for another oscillatory phase factor, namely e^{ixs} instead of $e^{ix(s-s^{-1})}$. Even though there is only the small phase factor difference, the proofs of the two propositions are very different. Proposition 7 in [PS14] is proven via standard Fourier analysis, whereas Proposition 3.3.1 in the present work requires a bit more than that. It is also remarkable that we can not assume that $r \in H^1(\mathbb{R})$ which is usually the case in inverse scattering theory for the NLS or DNLS equation. We need to mention that, more formally and for Schwartz class functions, a similar proposition can be found in [Zho89, Lemma 3.7] and also in [CVZ99, page 1207] in the context of the sine-Gordon equation.

Proposition 3.3.1. *There exists a positive constant c such that for all $f \in X^{0,0}_{-1,1}$, we have*

$$\sup_{x\in[0,\infty)} \|\langle x\rangle \mathcal{P}^\pm[f(\diamond)e^{\mp ix\Theta(\diamond)}]\|_{L^2(\mathbb{R})} \le c\|f\|_{X^{0,0}_{-1,1}}, \tag{3.3.1}$$

where $\langle x\rangle := (1+x^2)^{1/2}$. In addition, if $f \in X^{0,0}_{-1,1}$, then

$$\sup_{x\in\mathbb{R}} \|\mathcal{P}^\pm[f(\diamond)e^{ix\Theta(\diamond)}]\|_{L^\infty(\mathbb{R})} \le c\|f\|_{X^{0,0}_{-1,1}}. \tag{3.3.2}$$

Furthermore, if $f \in L^{2,-1}(\mathbb{R}) \cap L^{2,1}(\mathbb{R})$, then

$$\sup_{x\in\mathbb{R}} \|\mathcal{P}^\pm[(\diamond-\diamond^{-1})f(\diamond)e^{ix\Theta(\diamond)}]\|_{L^2(\mathbb{R})} \le c\|f\|_{L^{2,-1}(\mathbb{R})\cap L^{2,1}(\mathbb{R})}. \tag{3.3.3}$$

Proof. Due to the linearity of $\mathcal{P}^\pm$ and since $f(s)e^{\pm ix\Theta(s)} = f(s)e^{\pm ix\Theta(s)}\mathbb{1}_{\mathbb{R}_-}(s) + f(s)e^{\pm ix\Theta(s)}\mathbb{1}_{\mathbb{R}_+}(s)$ we can prove the proposition separately for functions f that either vanish entirely on $\mathbb{R}_+$ or on $\mathbb{R}_-$. In the following we give an estimate of $\|\mathcal{P}^+[f(\diamond)e^{-ix\Theta(\diamond)}\mathbb{1}_{\mathbb{R}_+}(\diamond)]\|_{L^2(\mathbb{R})}$. The other cases are handled analogously. The proof relies on a further decomposition $e^{ix\Theta(s)}\mathbb{1}_{\mathbb{R}_+}(s) = h_I(x,s) + h_{II}(x,s)$, see (3.3.7) below. Here, h_I is a function whose $L^2(\mathbb{R}_+)$-norm is decaying like x^{-1} as $x \to \infty$, whereas h_{II} admits an analytic extension into the lower half plane. In order to construct this decomposition, we consider the following operator.

$$\mathfrak{a}[f](k) := \int_0^\infty e^{-ik(s-s^{-1})}\frac{1+s^2}{s^2}f(s)ds. \tag{3.3.4}$$

Using the following change of coordinates

$$y(s) = s - s^{-1}, \qquad s(y) = \frac{y}{2} + \sqrt{1+\frac{y^2}{4}}, \qquad s'(y) = \frac{1}{2} + \frac{y}{4}\left(\sqrt{1+\frac{y^2}{4}}\right)^{-1} = \frac{s(y)^2}{1+s(y)^2}, \tag{3.3.5}$$

it is seen that $\mathfrak{a}[f](k) = \mathfrak{F}[\tilde{f}](k)$, where the function $\tilde{f}$ is given by

$$\tilde{f}(y) = f(s(y)), \quad y \in \mathbb{R},$$

and $\mathfrak{F}$ denotes the Fourier transform

$$\mathfrak{F}[\tilde{r}](k) = \int_{-\infty}^{\infty} e^{-iky}\tilde{r}(y)dy.$$

We get that

$$\|\tilde{f}\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} |f(s(y))|^2 dy = \int_0^{\infty} \frac{1+s^2}{s^2}|f(s)|^2 ds \leq \|f\|^2_{L^{2,-1}(\mathbb{R}_+)} \leq \|f\|^2_{X^{0,0}_{-1,1}}.$$

In addition,

$$\|\tilde{f}'\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \left(\frac{s(y)^2}{1+s(y)^2}\right)^2 |f'(s(y))|^2 dy = \int_0^{\infty} \frac{s^2}{1+s^2}|f'(s)|^2 ds \leq \|f\|^2_{X^{0,0}_{-1,1}}.$$

It follows that $\tilde{f} \in H^1(\mathbb{R})$ and thus $\mathfrak{a}[f](k) \in L^{2,1}_k(\mathbb{R})$ with $\|\mathfrak{a}[f]\|_{L^{2,1}(\mathbb{R})} \leq \|f\|_{X^{0,0}_{-1,1}}$. Using the inverse Fourier transform

$$\mathfrak{F}^{-1}[g](y) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{iyk} g(k)dk,$$

we find for $s > 0$:

$$f(s) = \tilde{f}(y(s)) = \mathfrak{F}^{-1}[\mathfrak{a}[f]](y(s)) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ik(s-s^{-1})}\mathfrak{a}[f](k)dk. \tag{3.3.6}$$

Now, let $x > 0$ and consider the following decomposition for $s > 0$,

$$f(s)e^{-ix\Theta(s)} = h_I(x,s) + h_{II}(x,s), \tag{3.3.7}$$

with

$$h_I(x,s) = e^{-ix\Theta(s)}\frac{1}{2\pi}\int_{x/4}^{\infty} e^{ik(s-s^{-1})}\mathfrak{a}[f](k)dk$$

and

$$h_{II}(x,s) = e^{-i\frac{x}{4}(s-s^{-1})}\frac{1}{2\pi}\int_{-\infty}^{x/4} e^{i(k-\frac{x}{4})(s-s^{-1})}\mathfrak{a}[f](k)dk.$$

Using that $s'(y) < 1$, for the functions h_I we get

$$\|h_I(x,\cdot)\|^2_{L^2(\mathbb{R}_+)} \leq \left\| \frac{1}{2\pi}\int_{x/4}^{\infty} e^{iky}\mathfrak{a}[f](k)dk \right\|_{L^2_y(\mathbb{R})} = \int_{x/4}^{\infty} |\mathfrak{a}[f](k)|^2 dk \leq c\frac{\|\mathfrak{a}[f]\|^2_{L^{2,1}(\mathbb{R})}}{1+x^2}. \tag{3.3.8}$$

The function $h_{II}(x,\cdot)$ is analytic in the domain $\{\mathrm{Im}(s) < 0\}$ and additionally for $s = -i\xi$ with $\xi \in \mathbb{R}_+$ we have

$$|h_{II}(x,s)| \leq c\|\mathfrak{a}[f]\|_{L^{2,1}(\mathbb{R})}e^{-\frac{x}{4}(\xi+\xi^{-1})}$$

and we can conclude by an elementary computation that $\|h_{II}(x,\cdot)\|_{L^2(i\,\mathbb{R}_-)}$ is decaying exponentially as $x \to \infty$. Now we have

$$\|\mathcal{P}^+[f(\diamond)e^{-ix\Theta(\diamond)}\mathbb{1}_{\mathbb{R}_+}(\diamond)]\|_{L^2(\mathbb{R})} \leq \|\mathcal{P}^+[h_I(x,\diamond)\mathbb{1}_{\mathbb{R}_+}(\diamond)]\|_{L^2(\mathbb{R})} + \|\mathcal{P}^+[h_{II}(x,\diamond)\mathbb{1}_{\mathbb{R}_+}(\diamond)]\|_{L^2(\mathbb{R})}$$

Since $\mathcal{P}^+$ is a bounded operator $L^2_w(\mathbb{R}_+) \to L^2_w(\mathbb{R})$, it follows by (3.3.8) that

$$\|\mathcal{P}^+[h_I(x,\diamond)\mathbb{1}_{\mathbb{R}_+}(\diamond)]\|^2_{L^2(\mathbb{R})} \leq \|h_I(x,\cdot)\|^2_{L^2(\mathbb{R}_+)} \leq c\|f\|^2_{X^{0,0}_{-1,1}}\langle x\rangle^{-2}.$$

Using a suitable path of integration and the analyticity of h_{II} we find that

$$\mathcal{P}^+[h_{II}(x,\diamond)](z) = -\mathcal{P}_{i\,\mathbb{R}_-}[h_{II}(x,\diamond)](z),$$

where

$$\mathcal{P}_{i\,\mathbb{R}_-}[h](z) := \frac{1}{2\pi i}\int_{-\infty}^{0}\frac{h(is)}{is-z}ds, \quad z\in\mathbb{R},$$

for a function $h : i\,\mathbb{R}_- \to \mathbb{C}$. Since $\mathcal{P}_{i\,\mathbb{R}_-}$ is a bounded operator $L^2(i\,\mathbb{R}_-) \to L^2(\mathbb{R})$ (see for instance estimate (23.11) in [BDT88]) and because $\|h_{II}(x,\cdot)\|_{L^2(i\,\mathbb{R}_-)}$ is decaying exponentially as $x\to\infty$, the proof of (3.3.1) is completed. In order to prove estimate (3.3.2) we firstly note that for $z \le 0$,

$$\begin{aligned}
|\mathcal{P}^{\pm}[e^{ix\Theta(\diamond)}f(\diamond)\mathbb{1}_{\mathbb{R}_+}(\diamond)](z)| &\le \int_0^\infty \frac{|f(s)|}{s}ds \\
&\le \left(\int_0^1 \frac{|f(s)|^2}{s^2}ds\right)^{1/2} + \left(\int_1^\infty \frac{1}{s^2}ds\right)^{1/2}\left(\int_1^\infty |f(s)|^2 ds\right)^{1/2} \\
&\le c\|f\|_{L^{2,-1}}.
\end{aligned}$$

Thus it remains to estimate $|\mathcal{P}^{\pm}[e^{-ix\Theta(\diamond)}f(\diamond)\mathbb{1}_{\mathbb{R}_+}(\diamond)](z)|$ for $z>0$. Using (3.3.6) we decompose

$$f(s) = h_+(s) + h_-(s), \qquad h_\pm(s) := \pm\frac{1}{2\pi}\int_0^{\pm\infty} e^{ik(s-s^{-1})}\mathfrak{a}[f](k)dk,$$

where $h_\pm$ has an analytic extension within the domain $\{s\in\mathbb{C} : \mathrm{Re}(s)>0, \pm\,\mathrm{Im}(s)>0\}$ and for $\xi>0$, we have

$$|h_\pm(\pm i\xi)| \le c\|e^{-k(\xi+\xi^{-1})}\|_{L^2_k(\mathbb{R}_+)}\|\mathfrak{a}[f]\|_{L^2_k(\mathbb{R}_\pm)} = \frac{c}{\sqrt{2}}\sqrt{\frac{\xi}{1+\xi^2}}\|\mathfrak{a}[f]\|_{L^2_k(\mathbb{R}_\pm)}. \tag{3.3.9}$$

Using a residue calculation, for $z>0$, we obtain that

$$\begin{aligned}
\mathcal{P}^{\pm}[f(\diamond)\mathbb{1}_{\mathbb{R}_+}(\diamond)](z) &= \lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_0^\infty \frac{h_+(s)+h_-(s)}{s-(z\pm i\,\varepsilon)}ds \\
&= \mathcal{P}_{i\,\mathbb{R}_+}[h_+](z) - \mathcal{P}_{i\,\mathbb{R}_-}[h_-](z) + h_\pm(z).
\end{aligned}$$

Thanks to the bound (3.3.9), we find

$$\begin{aligned}
\sup_{z\in\mathbb{R}}|\mathcal{P}_{i\,\mathbb{R}_\pm}[h_\pm](z)| &\le \frac{1}{2\pi}\left|\int_0^\infty \frac{|h_\pm(\pm i\xi)|}{\xi}d\xi\right| \\
&\le \frac{1}{2\pi}\int_0^\infty \frac{1}{\sqrt{\xi}\sqrt{1+\xi^2}}d\xi\|\mathfrak{a}[f]\|_{L^2_k(\mathbb{R}_\pm)} \\
&\le c\|\mathfrak{a}[f]\|_{L^2_k(\mathbb{R}_\pm)}.
\end{aligned}$$

In addition, for $z>0$ we have $|h_\pm(z)| \le \|\mathfrak{a}[f]\|_{L^1_k(\mathbb{R}_\pm)}$. We conclude:

$$\sup_{z\in\mathbb{R}_+}|\mathcal{P}^{\pm}[f(\diamond)\mathbb{1}_{\mathbb{R}_+}(\diamond)](z)| \le c\|\mathfrak{a}[f]\|_{L^1(\mathbb{R})\cap L^2(\mathbb{R})} < \infty. \tag{3.3.10}$$

From the definition of $\mathfrak{a}$ it follows that

$$\mathfrak{a}[e^{ix\Theta(\diamond)}f(\diamond)](k) = \mathfrak{a}[f(\diamond)](k-\frac{x}{2})$$

and thus, the $L^1(\mathbb{R})\cap L^2(\mathbb{R})$-norm with respect to k of $\mathfrak{a}[e^{ix\Theta(\diamond)}f(\diamond)](k)$ does not depend on x. Therefore, (3.3.10) yields

$$\sup_{z\in\mathbb{R}_+}|\mathcal{P}^{\pm}[e^{ix\Theta(\diamond)}f(\diamond)\mathbb{1}_{\mathbb{R}_+}(\diamond)](z)| \le \|\mathfrak{a}[e^{ix\Theta(\diamond)}f(\diamond)]\|_{L^1(\mathbb{R})\cap L^2(\mathbb{R})} = \|\mathfrak{a}[f]\|_{L^1(\mathbb{R})\cap L^2(\mathbb{R})}, \tag{3.3.11}$$

which completes the proof of (3.3.2). The bound (3.3.3) follows from $\|\mathcal{P}^{\pm}\|_{L^2\to L^2}=1$ and the fact that $(s-s^{-1})f(s)\in L^2_s(\mathbb{R})$ if $f\in L^{2,-1}(\mathbb{R})\cap L^{2,1}(\mathbb{R})$. □

Combining (3.2.11) and (3.3.1) we can prove the following lemma.

Lemma 3.3.2. *For every $r \in X^{1,1}_{-1,1}$, the unique solution M of Riemann–Hilbert problem 2.8.1 satisfies the estimates*

$$\sup_{x\in[0,\infty)} \|\langle x\rangle \, [M_-(x;\cdot)]_{21}\|_{L^2(\mathbb{R})} \leq C\|r\|_{X^{1,1}_{0,2}} \tag{3.3.12}$$

and

$$\sup_{x\in[0,\infty)} \|\langle x\rangle \, [M_+(x;\cdot)]_{12}\|_{L^2(\mathbb{R})} \leq C\|r\|_{X^{0,0}_{-1,1}}, \tag{3.3.13}$$

where C is a positive constant that depends on $\|r\|_{L^\infty}$. Moreover, if $r \in X^{2,1}_{-2,1}$, then

$$\sup_{x\in\mathbb{R}} \|\partial_x \, [M_-(x;\cdot)]_{21}\|_{L^2(\mathbb{R})} \leq C\|r\|_{X^{2,1}_{-2,1}} \tag{3.3.14}$$

and

$$\sup_{x\in\mathbb{R}} \|\partial_x \, [M_+(x;\cdot)]_{12}\|_{L^2(\mathbb{R})} \leq C\|r\|_{X^{2,1}_{-2,1}}, \tag{3.3.15}$$

where C is another positive constant depending on $\|r\|_{L^\infty}$.

Proposition 3.3.3. *There exists a constant $c > 0$ such that for every function $f \in X^{1,0}_{0,1}$ we have*

$$\left\| \int_{\mathbb{R}} e^{ix(s-s^{-1})/2} f(s)ds \right\|_{H^1_x(\mathbb{R})\cap L^{2,1}_x(\mathbb{R})} \leq c\|f\|_{X^{1,0}_{0,1}}. \tag{3.3.16}$$

Proof. Using the same change of variables $y = s - s^{-1} =: y(s)$ as in (3.3.5) we obtain

$$\int_0^\infty e^{ix(s-s^{-1})/2} f(s)ds = \int_{\mathbb{R}} e^{ixy/2}\widetilde{f}(y)ds, \quad \text{where } \widetilde{f}(y) = s'(y)f(s(y)).$$

Thus, if $\widetilde{f}(y) \in H^1_y(\mathbb{R}) \cap L^{2,1}_y(\mathbb{R})$, then

$$\left\| \int_0^\infty e^{ix(s-s^{-1})/2} f(s)ds \right\|_{H^1_x(\mathbb{R})\cap L^{2,1}_x(\mathbb{R})} \leq c\|\widetilde{f}\|_{H^1_y(\mathbb{R})\cap L^{2,1}_y(\mathbb{R})}. \tag{3.3.17}$$

We compute

$$\begin{aligned}
\|\widetilde{f}\|^2_{L^{2,1}_y(\mathbb{R})} &= \int_{\mathbb{R}} (1+y^2)\left|\frac{s(y)^2}{1+s(y)^2} f(s(y))\right|^2 dy \\
&= \int_0^\infty \left(1+\left(s-s^{-1}\right)^2\right)\frac{s^2}{1+s^2}|f(s)|^2 ds \leq c\|f\|^2_{L^{2,1}(\mathbb{R}_+)}
\end{aligned}$$

and

$$\begin{aligned}
\|\widetilde{f}'\|^2_{L^2_y(\mathbb{R})} &\leq c\int_{\mathbb{R}}\left|\left(\frac{s(y)^2}{1+s(y)^2}\right)^2 f'(s(y))\right|^2 dy + c\int_{\mathbb{R}}\left|\left(\frac{s(y)^2}{1+s(y)^2}\right)^3 f(s(y))\right|^2 dy \\
&= c\int_{\mathbb{R}}\left(\frac{s^2}{1+s^2}\right)^3 \left|f'(s)\right|^2 ds + c\int_{\mathbb{R}}\left(\frac{s^2}{1+s^2}\right)^5 |f(s)|^2 ds \leq c\|f\|^2_{X^{0,0}_{5,3}} \leq c\|f\|^2_{X^{0,0}_{0,1}}.
\end{aligned}$$

It follows that we can replace $\|\widetilde{f}\|_{H^1_y(\mathbb{R})\cap L^{2,1}_y(\mathbb{R})}$ in (3.3.17) by $\|f\|^2_{L^{2,1}(\mathbb{R}_+)\cap X^{0,0}_{0,1}} \sim \|f\|^2_{X^{1,0}_{0,1}}$, which yields (3.3.16) for the integral over the positive real axis. Repeating similar computations we can derive the same estimate for the integral over the negative real axis. □

3.4 Estimates on the positive half line

The main result of the present section is the following lemma:

Lemma 3.4.1. *Under the same assumptions as in Corollary 3.2.3, we have*

$$\lim_{|w|\to\infty} w \cdot [M(x;w)]_{12} \in H^1_x(\mathbb{R}_+) \cap L^{2,1}_x(\mathbb{R}_+), \qquad \lim_{|w|\to\infty} w \cdot [M(x;w)]_{21} \in H^1_x(\mathbb{R}_+) \cap L^{2,1}_x(\mathbb{R}_+). \tag{3.4.1}$$

In addition, for any $w_1 \in \mathbb{C}\setminus\mathbb{R}$, we have

$$(M(x;w_1) - 1) \in H^1_x(\mathbb{R}_+) \cap L^{2,1}_x(\mathbb{R}_+). \tag{3.4.2}$$

Proof. The proof uses (3.4.7) below. In order to prove this equation, we consider the factorization $1 + R = (1 + R_+)(1 + R_-)$ with

$$R_+(x;w) = \begin{pmatrix} 0 & \overline{r(w)}e^{-ix\Theta(w)} \\ 0 & 0 \end{pmatrix}, \qquad R_-(x;w) = \begin{pmatrix} 0 & 0 \\ wr(w)e^{ix\Theta(w)} & 0 \end{pmatrix}, \tag{3.4.3}$$

where

$$\Theta(w) = \frac{1}{2}\left(w - \frac{1}{w}\right). \tag{3.4.4}$$

Then, the jump relation $M_+ = M_-(1 + R)$ of Riemann–Hilbert problem 2.8.1 can be rewritten as $M_+ - M_- = M_-R_+ + M_+R_-$. Applying $\mathcal{P}^+$ and $\mathcal{P}^-$ to this equation yields the following expression for the solution M of the RHP 2.8.1:

$$M(x;w) = 1 + \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{M_-(x;s)R_+(x;s) + M_+(x;s)R_-(x;s)}{s-w}ds. \tag{3.4.5}$$

In component form, for the non-tangential limits $w \to \mathbb{R}$, we find

$$M_\pm(x;w) = 1 + \begin{bmatrix} \mathcal{P}^\pm\left[[M_+(x;\diamond)]_{12}\diamond r(\diamond)e^{ix\Theta(\diamond)}\right](w) & \mathcal{P}^\pm\left[[M_-(x;\diamond)]_{11}\overline{r(\diamond)}e^{-ix\Theta(\diamond)}\right](w) \\ \mathcal{P}^\pm\left[[M_+(x;\diamond)]_{22}\diamond r(\diamond)e^{ix\Theta(\diamond)}\right](w) & \mathcal{P}^\pm\left[[M_-(x;\diamond)]_{21}\overline{r(\diamond)}e^{-ix\Theta(\diamond)}\right](w) \end{bmatrix}. \tag{3.4.6}$$

Now, using the representation formula (3.4.5) and the limit (3.2.2), we get

$$\lim_{|w|\to\infty} w \cdot [M(x;w)]_{12} = -\frac{1}{2\pi i}\int_{\mathbb{R}} [M_-(x;s)]_{11}\overline{r(s)}e^{-ix\Theta(s)}ds$$

From the component-wise representation (3.4.6) we learn that

$$[M_-(x;s)]_{11} = 1 + \mathcal{P}^-\left[[M_+(x;\diamond)]_{12}\diamond r(\diamond)e^{ix\Theta(\diamond)}\right](s)$$

so that

$$\begin{aligned} \lim_{|w|\to\infty} w \cdot [M(x;w)]_{12} = &-\frac{1}{2\pi i}\int_{\mathbb{R}} \overline{r(s)}e^{-ix\Theta(s)}ds \\ &+ \frac{1}{2\pi i}\int_{\mathbb{R}} [M_+(x;s)]_{12}\, sr(s)e^{ix\Theta(s)}\mathcal{P}^+\left[\overline{r(\diamond)}e^{-ix\Theta(\diamond)}\right](s)ds, \end{aligned} \tag{3.4.7}$$

where we also have to use integration by parts for any two functions $g, h \in L^2(\mathbb{R})$, that is,

$$\int_{\mathbb{R}} \mathcal{P}^-[h](s)g(s)ds = -\int_{\mathbb{R}} h(s)\mathcal{P}^+[g](s)ds.$$

In a similar way we find

$$\begin{aligned} \lim_{|w|\to\infty} w \cdot [M(x;w)]_{21} = &-\frac{1}{2\pi i}\int_{\mathbb{R}} sr(s)e^{+ix\Theta(s)}ds \\ &+ \frac{1}{2\pi i}\int_{\mathbb{R}} [M_-(x;s)]_{21}\,\overline{r(s)}e^{-ix\Theta(s)}\mathcal{P}^-\left[\diamond r(\diamond)e^{ix\Theta(\diamond)}\right](s)ds. \end{aligned} \tag{3.4.8}$$

The first terms in (3.4.7) and (3.4.8) are controlled in $H^1_x(\mathbb{R}_+)\cap L^{2,1}_x(\mathbb{R}_+)$ by Proposition 3.3.3. To analyse the second terms we denote

$$\begin{aligned} I_1(x) &:= \int_{\mathbb{R}} [M_+(x;s)]_{12}\, sr(s)e^{ix\Theta(s)}\mathcal{P}^+\left[\overline{r(\diamond)}e^{-ix\Theta(\diamond)}\right](s)ds, \\ I_2(x) &:= \int_{\mathbb{R}} [M_-(x;s)]_{21}\, \overline{r(s)}e^{-ix\Theta(s)}\mathcal{P}^-\left[\diamond r(\diamond)e^{ix\Theta(\diamond)}\right](s)ds. \end{aligned} \tag{3.4.9}$$

By bound (3.3.1) of Proposition 3.3.1, bound (3.3.13) of Lemma 3.4.1, and the Hölder inequality, we have

$$\begin{aligned} \sup_{x\in[0,\infty)} |\langle x\rangle^2 I_1(x)| &\leq \|(\cdot)r(\cdot)\|_{L^\infty(\mathbb{R})} \sup_{x\in[0,\infty)} \|\langle x\rangle\,[M_+(x;\cdot)]_{12}\|_{L^2(\mathbb{R})} \sup_{x\in[0,\infty)} \left\|\langle x\rangle\mathcal{P}^+\left[\overline{r(\diamond)}e^{-ix\Theta(\diamond)}\right]\right\|_{L^2(\mathbb{R})} \\ &\leq c\|(\cdot)r(\cdot)\|_{L^\infty(\mathbb{R})}\|r\|^2_{X^{2,1}_{-2,1}}. \end{aligned} \tag{3.4.10}$$

This bound is sufficient to conclude that $I_1(\cdot)\in L^{2,1}(\mathbb{R}_+)$. In order to show that $I_1(\cdot)\in H^1(\mathbb{R}_+)$, we differentiate I_1 in x and obtain.

$$\begin{aligned} I_1'(x) &= \int_{\mathbb{R}} \partial_x[M_+(x;s)]_{12}\, sr(s)e^{ix\Theta(s)}\mathcal{P}^+\left[\overline{r(\diamond)}e^{-ix\Theta(\diamond)}\right](s)ds \\ &\quad +i\int_{\mathbb{R}} [M_+(x;s)]_{12}\, sr(s)\Theta(s)e^{ix\Theta(s)}\mathcal{P}^+\left[\overline{r(\diamond)}e^{-ix\Theta(\diamond)}\right](s)ds \\ &\quad -i\int_{\mathbb{R}} [M_+(x;s)]_{12}\, sr(s)e^{ix\Theta(s)}\mathcal{P}^+\left[\Theta(\diamond)\overline{r(\diamond)}e^{-ix\Theta(\diamond)}\right](s)ds. \end{aligned}$$

By bounds (3.3.1)–(3.3.3) of Proposition 3.3.1, bounds (3.3.13) and (3.3.15) of Lemma 3.4.1, and the Hölder inequality, we find

$$\begin{aligned} \sup_{x\in[0,\infty)} |\langle x\rangle I_1'(x)| &\leq \|(\cdot)r(\cdot)\|_{L^\infty(\mathbb{R})} \sup_{x\in[0,\infty)} \|\partial_x\,[M_+(x;\cdot)]_{12}\|_{L^2(\mathbb{R})} \sup_{x\in[0,\infty)} \|\langle x\rangle\mathcal{P}^+[\overline{r(\diamond)}e^{-ix\Theta(\diamond)}]\|_{L^2(\mathbb{R})} \\ &\quad +\|(\cdot)r(\cdot)\Theta(\cdot)\|_{L^2(\mathbb{R})} \sup_{x\in[0,\infty)} \|\langle x\rangle\,[M_+(x;\cdot)]_{12}\|_{L^2(\mathbb{R})} \sup_{x\in[0,\infty)} \|\mathcal{P}^+[\overline{r(\diamond)}e^{-ix\Theta(\diamond)}]\|_{L^\infty(\mathbb{R})} \\ &\quad +\|(\cdot)r(\cdot)\|_{L^\infty(\mathbb{R})} \sup_{x\in[0,\infty)} \|\langle x\rangle\,[M_+(x;\cdot)]_{12}\|_{L^2(\mathbb{R})} \sup_{x\in[0,\infty)} \|\mathcal{P}^+[\Theta(\diamond)\overline{r(\diamond)}e^{-ix\Theta(\diamond)}]\|_{L^2(\mathbb{R})} \\ &\leq c\|r\|^3_{X^{2,1}_{-2,1}} \end{aligned}$$

This is sufficient for $I_1'(\cdot)\in L^2(\mathbb{R}_+)$ and hence, $I_1\in H^1_x(\mathbb{R}_+)\cap L^{2,1}_x(\mathbb{R}_+)$. Repeating the same analysis for I_2 completes the proof of (3.4.1).

For the proof of (3.4.2) we firstly remark that

$$\begin{aligned} [M(x;w_1)]_{12} &= -\frac{1}{2\pi i}\int_{\mathbb{R}} \overline{r(s)}e^{-ix\Theta(s)}ds \\ &\quad +\frac{1}{2\pi i}\int_{\mathbb{R}} [M_+(x;s)]_{12}\, sr(s)e^{ix\Theta(s)}\mathcal{P}^+\left[\frac{\overline{r(\diamond)}e^{-ix\Theta(\diamond)}}{\diamond - w_1}\right](s)ds \end{aligned}$$

and

$$\begin{aligned} [M(x;w_1)]_{21} &= -\frac{1}{2\pi i}\int_{\mathbb{R}} sr(s)e^{+ix\Theta(s)}ds \\ &\quad +\frac{1}{2\pi i}\int_{\mathbb{R}} [M_-(x;s)]_{21}\, \overline{r(s)}e^{-ix\Theta(s)}\mathcal{P}^-\left[\frac{\diamond r(\diamond)e^{ix\Theta(\diamond)}}{\diamond - w_1}\right](s)ds. \end{aligned}$$

Analogously to the derivation of (3.4.7) and (3.4.8), these two formulas are derived by making use of (3.4.5). In each formula the first line of the right hand side is controlled in $H^1\cap L^{2,1}$ by Proposition

3.3.3. Each second line is an expression similar to I_1 and I_2 defined in (3.4.9) and thus, also controlled in $H^1 \cap L^{2,1}$. We conclude that

$$[M(\cdot;w_1)]_{12} \in H^1(\mathbb{R}_+) \cap L^{2,1}(\mathbb{R}_+), \qquad [M(\cdot;w_1)]_{21} \in H^1(\mathbb{R}_+) \cap L^{2,1}(\mathbb{R}_+).$$

For the diagonal entries we compute:

$$\begin{aligned}
[M(\cdot;w_1)]_{11} - 1 &= \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{[M_+(x;s)]_{12} s r(s) e^{ix\Theta(s)}}{s-w_1} ds \\
&= \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\mathcal{P}^+\left[[M_-(x;\diamond)]_{11}\overline{r(\diamond)}e^{-ix\Theta(\diamond)}\right](s)\, s r(s) e^{ix\Theta(s)}}{s-w_1} ds \\
&= -\frac{1}{2\pi i}\int_{\mathbb{R}} \mathcal{P}^+\left[[M_-(x;\diamond)]_{11}\overline{r(\diamond)}e^{-ix\Theta(\diamond)}\right](s)\, \mathcal{P}^-\left[\frac{\diamond r(\diamond)e^{ix\Theta(\diamond)}}{\diamond - w_1}\right](s) ds \\
&= -\frac{1}{2\pi i}\int_{\mathbb{R}} [M_+(x;s)]_{12}\, \mathcal{P}^-\left[\frac{\diamond r(\diamond)e^{ix\Theta(\diamond)}}{\diamond - w_1}\right](s) ds.
\end{aligned}$$

In addition, by similar manipulations:

$$[M(\cdot;w_1)]_{22} - 1 = -\frac{1}{2\pi i}\int_{\mathbb{R}} [M_-(x;s)]_{21}\, \mathcal{P}^+\left[\frac{\overline{r(\diamond)}e^{-ix\Theta(\diamond)}}{\diamond - w_1}\right](s) ds.$$

These formulas show that $[M(\cdot;w_1)]_{11} - 1$ and $[M(\cdot;w_1)]_{22} - 1$ can be controlled in $H^1(\mathbb{R}_+) \cap L^{2,1}(\mathbb{R}_+)$ by the the same arguments we used for I_1 and I_2. Thus, the proof of the Lemma is completed. □

Remark 3.4.2. Recalling

$$\begin{aligned}
[M(x;w_1)]_{12} = &-\frac{1}{2\pi i}\int_{\mathbb{R}} \overline{r(s)} e^{-ix\Theta(s)} ds \\
&+ \frac{1}{2\pi i}\int_{\mathbb{R}} [M_+(x;s)]_{12}\, s r(s) e^{ix\Theta(s)} \mathcal{P}^+\left[\frac{\overline{r(\diamond)}e^{-ix\Theta(\diamond)}}{\diamond - w_1}\right](s) ds
\end{aligned}$$

and using $r \in L^1(\mathbb{R})$, $\|[M_+(x;\cdot)]_{12}\|_{L^2(\mathbb{R})} \leq c\|r\|_{L^2}$ (see Lemma 3.2.2) and $\|\mathcal{P}^+\left[\frac{\overline{r(\diamond)}e^{-ix\Theta(\diamond)}}{\diamond - w_1}\right](\cdot)\|_{L^2(\mathbb{R})} \leq c\|r\|_{L^2}$, we find

$$|\,[M(x;w_1)]_{12}\,| \leq c$$

for a constant not depending on x. Repeating analogous computations for the remaining entries of M we obtain

$$\|M(x;\cdot)\|_{L^\infty(K)} \leq c$$

for any compact subset $K \subset \mathbb{C} \setminus \mathbb{R}$. The constant c may depend on K.

3.5 Estimates on the negative half line

Estimates on the positive half-line are given in Lemma 3.4.1. These estimates rely for instance on formulas (3.4.7) and (3.4.8), which are not controllable in $H^1(\mathbb{R}_-) \cap L^{2,1}(\mathbb{R}_-)$ by the same methods as in the proof of Lemma 3.4.1. However, if we rewrite the Riemann–Hilbert problem 2.8.1 in an equivalent form, we can find formulas similar to (3.4.7) and (3.4.8), which are useful on the negative half-line of x. The reformulation of the Riemann–Hilbert problem is based on a factorization of the jump matrix $R(x;w)$. Therefore, let us consider the following scalar Riemann–Hilbert problem :

Riemann-Hilbert problem 3.5.1. For a given function $r \in X^{2,1}_{-2,1}$ find a scalar function $\mathbb{C} \setminus \mathbb{R} \ni w \mapsto d(w)$ which satisfies

1. $d(w)$ is analytic in $\mathbb{C} \setminus \mathbb{R}$.
2. $d(w) = 1 + \mathcal{O}\left(w^{-1}\right)$ as $|w| \to \infty$.
3. The non-tangential boundary values $d_\pm(w) := \lim_{\varepsilon \downarrow 0} d(w \pm i\,\varepsilon)$ exist for $w \in \mathbb{R}$ and satisfy the jump relation
$$d_+(w) = d_-(w)\left(1 + w|r(w)|^2\right). \tag{3.5.1}$$

Assuming that such a function $d(w)$ exists we may look at

$$M_d(x;w) := M(x;w)\begin{bmatrix} 1/d(w) & 0 \\ 0 & d(w) \end{bmatrix}, \tag{3.5.2}$$

where $M(x;w)$ is supposed to be the unique solution of Riemann–Hilbert problem 2.8.1. We have

$$M_{\pm,d}(x;w) = M_\pm(x;w)\begin{bmatrix} 1/d_\pm(w) & 0 \\ 0 & d_\pm(w) \end{bmatrix},$$

such that by (2.8.2)

$$\begin{aligned} M_{+,d}(x;w) &= M_-(x;w)(1+R(x;w))\begin{bmatrix} 1/d_+(w) & 0 \\ 0 & d_+(w) \end{bmatrix} \\ &= M_{-,d}(x;w)\begin{bmatrix} d_-(w) & 0 \\ 0 & 1/d_-(w) \end{bmatrix}(1+R(x;w))\begin{bmatrix} 1/d_+(w) & 0 \\ 0 & d_+(w) \end{bmatrix}. \end{aligned}$$

Using (3.5.1), this can be written equivalently as $M_{+,d}(x;w) = M_{-,d}(x;w)(1+\widetilde{R}_d(x;w))$, where

$$\widetilde{R}_d(x;w) = \begin{bmatrix} 0 & d_+(w)d_-(w)\overline{r(w)}e^{-ix\Theta(w)} \\ \frac{wr(w)}{d_+(w)d_-(w)}e^{ix\Theta(w)} & w|r(w)|^2 \end{bmatrix}. \tag{3.5.3}$$

Assuming that M solves RHP 2.8.1 and d solves RHP 3.5.1, the new function $M_d(x;\cdot)$ is analytic in $\mathbb{C} \setminus \mathbb{R}$ and has the same limit 1 as $|w| \to \infty$. Therefore, M_d is a solution of Riemann–Hilbert problem 2.8.1 with jump matrix $\widetilde{R}_d(x;w)$ instead of $R(x;w)$. For further analysis of the RHP, let us note some properties of $d_\pm$.

Proposition 3.5.2. *Let $r \in X^{2,1}_{-2,1}$ satisfy (3.1.1). Then the unique solution of Riemann–Hilbert problem 3.5.1 is given by*

$$d(w) = \exp\left\{\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\log(1+s|r(s)|^2)}{s-w}ds\right\}, \tag{3.5.4}$$

such that in particular

$$d_\pm(w) = \exp\left\{\mathcal{P}^\pm\left[\log(1+\diamond|r(\diamond)|^2)\right](w)\right\}.$$

Proof. Since $1 + s|r(s)|^2 > c_1$ by assumption, we have $\log(1+s|r(s)|^2)$ being a real-valued function. Furthermore, there exists a constant C, depending on c_1, such that

$$|\log(1+s|r(s)|^2)| \leq C|s|\,|r(s)|^2$$

for all $s \in \mathbb{R}$. From $r \in X^{2,1}_{-2,1}$ we conclude that $s|r(s)|^2 \in L^2(\mathbb{R})$ and hence also $\log(1+s|r(s)|^2) \in L^2(\mathbb{R})$. Thus, by Proposition 3.2.1 (i)–(ii), the expression on the right-hand side of (3.5.4) defines an analytic function on $\mathbb{C} \setminus \mathbb{R}$ with limit $d(w) \to 1$ as $|w| \to \infty$. The condition (3.5.1) is satisfied due to the Sokhotski-Plemelj theorem, see (3.2.3). □

From (3.5.4) we learn that

$$d_+(w)d_-(w) = \exp\left\{-i\mathcal{H}\left[\log(1+\diamond|r(\diamond)|^2)\right](w)\right\}, \tag{3.5.5}$$

where the Sokhotski-Plemelj theorem is used, see (3.2.3). The Hilbert transform of a real-valued function is again a real-valued function and for this reason we conclude that

$$|d_+(w)d_-(w)| = 1 \tag{3.5.6}$$

for all $w \in \mathbb{R}$. In particular, $\overline{d_+(w)d_-(w)} = (d_+(w)d_-(w))^{-1}$. Thus, using the notation

$$r^{(d)}(w) := \frac{r(w)}{d_+(w)d_-(w)}, \tag{3.5.7}$$

we have the following equivalent equation for $\widetilde{R}_d$ defined in (3.5.3):

$$\widetilde{R}_d(x;w) = \begin{bmatrix} 0 & \overline{r^{(d)}(w)}e^{-ix\Theta(w)} \\ wr^{(d)}(w)e^{ix\Theta(w)} & w|r^{(d)}(w)|^2 \end{bmatrix}. \tag{3.5.8}$$

Proposition 3.5.3. *Let $r \in X^{2,1}_{-2,1}$ satisfy (3.1.1). Then $r^{(d)} \in X^{2,1}_{-2,1}$ and $1 + w|r^{(d)}(w)|^2 \geq c_1 > 0$ with the same constant c_1 as in (3.1.1).*

Proof. The assertion $1+w|r^{(d)}(w)|^2 \geq c_1$ follows directly from (3.1.1) and (3.5.6). In addition, from (3.5.6) it follows that $|r(w)| = |r^{(d)}(w)|$. Thus, $r \in L^{2,2}(\mathbb{R}) \cap L^{2,-2}(\mathbb{R})$ implies that $r^{(d)} \in L^{2,2}(\mathbb{R}) \cap L^{2,-2}(\mathbb{R})$. Since the Hilbert transform commutes with the drivative, see [Duo01], we have

$$\left|w\frac{\partial}{\partial w}r^{(d)}(w)\right| \leq |wr'(w)| + \left|\mathcal{H}\left[\frac{\partial_w(\diamond|r(\diamond)|^2)}{1+\diamond|r(\diamond)|^2}\right](w)\right| |wr(w)|.$$

Using

$$\left\|\frac{\partial_w(w|r(w)|^2)}{1+w|r(w)|^2}\right\|_{L^2_w(\mathbb{R})} \leq C(\|r\|_{L^2}\|r\|_{L^\infty} + 2\|wr'(w)\|_{L^2_w}\|r\|_{L^\infty}) \leq C'\|r\|^2_{X^{2,1}_{-2,1}}$$

and $\|\mathcal{H}\|_{L^2\to L^2} = 1$, we finally find

$$\left\|w\frac{\partial}{\partial w}r^{(d)}(w)\right\|_{L^2_w(\mathbb{R})} \leq c\left(\|wr'(w)\|_{L^2} + \|r\|^2_{X^{2,1}_{-2,1}}\|wr(w)\|_{L^\infty}\right).$$

The proof of the proposition is now completed. □

As an analogue to Lemma 3.3.2 we have:

Lemma 3.5.4. *For every $r^{(d)} \in X^{2,1}_{-2,1}$, the unique solution M_d of Riemann–Hilbert problem 2.8.1 with R replaced by $\widetilde{R}_d$ satisfies the estimates*

$$\sup_{x\in(-\infty,0]}\left\|\langle x\rangle\left[M_{-,d}(x;\cdot)\right]_{12}\right\|_{L^2(\mathbb{R})} \leq C\|r^{(d)}\|_{X^{0,0}_{-1,1}} \tag{3.5.9}$$

and

$$\sup_{x\in(-\infty,0]}\left\|\langle x\rangle\left[M_{+,d}(x;\cdot)\right]_{21}\right\|_{L^2(\mathbb{R})} \leq C\|r^{(d)}\|_{X^{1,1}_{0,2}}, \tag{3.5.10}$$

where C is a positive constant that depends on $\|r\|_{L^\infty(\mathbb{R})}$. Moreover, if $r \in X^{2,1}_{-2,1}$, then

$$\sup_{x\in\mathbb{R}}\left\|\partial_x\left[M_{-,d}(x;\cdot)\right]_{12}\right\|_{L^2(\mathbb{R})} \leq C\|r^{(d)}\|_{X^{2,1}_{-2,1}} \tag{3.5.11}$$

and

$$\sup_{x\in\mathbb{R}}\left\|\partial_x\left[M_{+,d}(x;\cdot)\right]_{12}\right\|_{L^2(\mathbb{R})} \leq C\|r^{(d)}\|_{X^{2,1}_{-2,1}}, \tag{3.5.12}$$

where C is another positive constant depending on $\|r\|_{L^\infty(\mathbb{R})}$.

Lemma 3.5.5. *Under the same assumptions as in Lemma 3.4.1 we have*

$$\lim_{|w|\to\infty} w\cdot [M_d(x;w)]_{12} \in H^1_x(\mathbb{R}_-)\cap L^{2,1}_x(\mathbb{R}_-), \qquad \lim_{|w|\to\infty} w\cdot [M_d(x;w)]_{21} \in H^1_x(\mathbb{R}_-)\cap L^{2,1}_x(\mathbb{R}_-). \quad (3.5.13)$$

In addition, for any $w_1 \in \mathbb{C}\setminus\mathbb{R}$, we have

$$(M_d(x;w_1)-1) \in H^1_x(\mathbb{R}_-)\cap L^{2,1}_x(\mathbb{R}_-). \quad (3.5.14)$$

From the construction of M_d it follows that

$$\lim_{|w|\to\infty} w\cdot [M_d(x;w)]_{12} = \lim_{|w|\to\infty} w\cdot [M(x;w)]_{12}$$

and

$$\lim_{|w|\to\infty} w\cdot [M_d(x;w)]_{21} = \lim_{|w|\to\infty} w\cdot [M(x;w)]_{21}\,.$$

Therefore, Theorem 3.1.1, for the case $N=0$, is proven by Lemmas 3.4.1 and 3.5.5.

3.6 Bäcklund transformation: adding eigenvalues

In general, the terminology *Bäcklund transformation* is used to describe transformations mapping one solution of a nonlinear PDE to another solution of another (or the same) nonlinear PDE. Relying on computations that can be found in [RS02, DP11], in what follows we construct a solution of Riemann–Hilbert problem 2.8.1 with $N=1$ out of another solution of Riemann–Hilbert problem 2.8.1 with $N=0$. The same transformation was also used in [Saa17b] in the context of the derivative NLS equation.

We start by defining the matrix

$$A(x) = \begin{bmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{bmatrix}$$

with

$$\begin{pmatrix} a_{11}(x) \\ a_{21}(x) \end{pmatrix} := M^{(0)}(x;w_1)\begin{pmatrix} 1 \\ \frac{w_1 c_1 e^{ix\Theta(w_1)}}{\overline{w}_1 - w_1} \end{pmatrix}, \quad \begin{pmatrix} a_{12}(x) \\ a_{22}(x) \end{pmatrix} := M^{(0)}(x;\overline{w}_1)\begin{pmatrix} \frac{-\overline{c}_1 e^{-ix\Theta(\overline{w}_1)}}{w_1-\overline{w}_1} \\ 1 \end{pmatrix}. \quad (3.6.1)$$

In order to define the Bäcklund transformation it is necessary to know that there is no x such that the determinant of $A(x)$ vanishes.

Proposition 3.6.1. *The matrix A is invertible for all $x\in\mathbb{R}$. Moreover,*

$$|\det(A(x))|^{-1} \le C_M, \quad \textit{for all } x>0, \quad (3.6.2)$$

where the constant C_M does not depend on x and r.

Proof. Using the symmetry (2.8.12) we find

$$\begin{aligned}
\begin{pmatrix} a_{12}(x) \\ a_{22}(x) \end{pmatrix} &= \frac{1}{\overline{w}_1}\begin{bmatrix} -\Omega(x) & 1 \\ -\overline{w}_1 - |\Omega(x)|^2 & \overline{\Omega(x)} \end{bmatrix} \overline{M^{(0)}(x;w_1)} \begin{bmatrix} 0 & -1 \\ \overline{w}_1 & 0 \end{bmatrix} \begin{pmatrix} \frac{-\overline{c}_1 e^{-ix\Theta(\overline{w}_1)}}{w_1-\overline{w}_1} \\ 1 \end{pmatrix} \\
&= \frac{-1}{\overline{w}_1}\begin{bmatrix} -\Omega(x) & 1 \\ -\overline{w}_1 - |\Omega(x)|^2 & \overline{\Omega(x)} \end{bmatrix} \overline{M^{(0)}(x;w_1)} \begin{pmatrix} 1 \\ \frac{\overline{w}_1\overline{c}_1 e^{-ix\Theta(\overline{w}_1)}}{w_1-\overline{w}_1} \end{pmatrix} \\
&= \frac{-1}{\overline{w}_1}\begin{bmatrix} -\Omega(x) & 1 \\ -\overline{w}_1 - |\Omega(x)|^2 & \overline{\Omega(x)} \end{bmatrix} \begin{pmatrix} \overline{a_{11}(x)} \\ \overline{a_{21}(x)} \end{pmatrix}.
\end{aligned}$$

It follows directly that

$$\det(A(x)) = |a_{11}(x)|^2 + \frac{1}{\overline{w}_1}\left|\overline{\Omega(x)}a_{11}(x) - a_{21}(x)\right|^2. \quad (3.6.3)$$

The case $\det(A(x)) = 0$ is impossible, since $\mathrm{Im}(w_1) \neq 0$ which would imply that $a_{11}(x) = a_{21}(x) = 0$ and hence, $\left(1, \frac{w_1 c_1 e^{ix\Theta(w_1)}}{\overline{w}_1 - w_1}\right)^T \in \ker[M^{(0)}]$. This contradicts $\det(M^{(0)}(x;w)) \equiv 1$ (see Remark 3.1.3). Now we turn to the proof of (3.6.2). We first introduce the constant

$$\kappa(w_1) := \sup_{x\in[0,\infty)} \left\{ \left|[M^{(0)}(x;w_1)]_{12}\right| + \left|[M^{(0)}(x;w_1)]_{22}\right| \right\}.$$

Since by Lemma 3.4.1 we have $M^{(0)}(x;w_1) - 1 \in H^1(\mathbb{R}_+) \cap L^{2,1}(\mathbb{R}_+)$, the constant $\kappa(w_1)$ is finite. Next, we use the definition of $a_{11}(x)$ and $a_{21}(x)$ to compute

$$\begin{aligned} a_{11}(x)[M^{(0)}(x;w_1)]_{22} - [M^{(0)}(x;w_1)]_{12}a_{21}(x) = \\ [M^{(0)}(x;w_1)]_{11}[M^{(0)}(x;w_1)]_{22} - [M^{(0)}(x;w_1)]_{12}[M^{(0)}(x;w_1)]_{21} = 1, \end{aligned}$$

where we refer again to Remark 3.1.3 for the last equality. It follows that

$$\begin{aligned} \frac{1}{|a_{11}(x)| + |a_{21}(x)|} &= \frac{\left|a_{11}(x)[M^{(0)}(x;w_1)]_{22} - [M^{(0)}(x;w_1)]_{12}a_{21}(x)\right|}{|a_{11}(x)| + |a_{21}(x)|} \\ &\leq \frac{|a_{11}(x)|\left|[M^{(0)}(x;w_1)]_{22}\right| + \left|[M^{(0)}(x;w_1)]_{12}\right||a_{21}(x)|}{|a_{11}(x)| + |a_{21}(x)|} \\ &\leq \kappa(w_1). \end{aligned}$$

Since formula (3.6.3) implies that $|\det A(x)|$ is bounded from below by a constant times $(|a_{11}(x)| + |a_{21}(x)|)^2$ we get

$$|\det A(x)|^{-1} \leq C\kappa(w_1)^2.$$

Here, C is some constant depending on w_1 and $\|\Omega(\cdot)\|_{L^\infty(\mathbb{R}_+)}$. □

Lemma 3.6.2. *For any scattering data $\mathcal{S}_w(u^{(1)}, v^{(1)}) = (r^{(1)}; \{w_1, c_1\})$ such that $r^{(1)} \in X^{2,1}_{-2,1}$ satisfies (3.1.1) and $\mathrm{Im}(w_1) > 0$, Riemann–Hilbert problem 2.8.1 admits a unique solution $M^{(1)}(x;w)$. This solution is explicitly given by*

$$M^{(1)}(x;w) = A(x)\mu(w)A^{-1}(x)M^{(0)}(x;w)\mu^{-1}(w), \tag{3.6.4}$$

where

$$\mu(w) = \begin{bmatrix} w - w_1 & 0 \\ 0 & w - \overline{w}_1 \end{bmatrix},$$

and $M^{(0)}(x;w)$ being the unique solution of Riemann–Hilbert problem 2.8.1 subject to pure radiation data $\mathcal{S}_w(u^{(0)}, v^{(0)}) = (r^{(0)})$, where

$$r^{(0)}(w) := r^{(1)}(w)\frac{w - w_1}{w - \overline{w}_1}. \tag{3.6.5}$$

Proof. Firstly we note that the definition (3.6.5) implies that $r^{(0)} \in X^{2,1}_{-2,1}$ and $r^{(0)}$ satisfies (3.1.1). Thus, by Lemma 3.2.2 the solution $M^{(0)}(x;w)$ indeed exists. Now, let us denote by $\widetilde{M}(x;w)$ the right hand side of (3.6.4) and set

$$\begin{bmatrix} \tau_{11}(w) & \tau_{12}(w) \\ \tau_{21}(w) & \tau_{22}(w) \end{bmatrix} := A^{-1}(x)M^{(0)}(x;w).$$

We find

$$\begin{aligned} \operatorname*{Res}_{w=w_1} \widetilde{M}(x;w) &= A(x)\begin{bmatrix} 0 & 0 \\ (w_1 - \overline{w}_1)\tau_{21}(w_1) & 0 \end{bmatrix}, \\ \operatorname*{Res}_{w=\overline{w}_1} \widetilde{M}(x;w) &= A(x)\begin{bmatrix} 0 & (\overline{w}_1 - w_1)\tau_{12}(\overline{w}_1) \\ 0 & 0 \end{bmatrix}, \end{aligned} \tag{3.6.6}$$

and on the other hand

$$\begin{aligned}
&\lim_{w\to w_1} \widetilde{M}(x;w)\begin{pmatrix} 0 & 0 \\ w_1 c_1 e^{ix\Theta(w_1)} & 0 \end{pmatrix} = A(x)\begin{bmatrix} 0 & 0 \\ w_1 c_1 e^{ix\Theta(w_1)}\tau_{22}(w_1) & 0 \end{bmatrix}, \\
&\lim_{w\to \overline{w}_1} \widetilde{M}(x;w)\begin{pmatrix} 0 & -\overline{c}_1 e^{-ix\Theta(\overline{w}_1)} \\ 0 & 0 \end{pmatrix} = A(x)\begin{bmatrix} 0 & -\overline{c}_1 e^{-ix\Theta(\overline{w}_1)}\tau_{11}(\overline{w}_1) \\ 0 & 0 \end{bmatrix}.
\end{aligned} \tag{3.6.7}$$

Using $\det M^{(0)} \equiv 1$ we get

$$\tau_{21}(w_1) = \frac{1}{\det A(x)} \frac{w_1 c_1 e^{ix\Theta(w_1)}}{w_1 - \overline{w}_1}, \quad \tau_{22}(w_1) = \frac{1}{\det A(x)},$$

and

$$\tau_{11}(\overline{w}_1) = \frac{1}{\det A(x)}, \quad \tau_{12}(\overline{w}_1) = \frac{-1}{\det A(x)} \frac{\overline{c}_1 e^{-ix\Theta(\overline{w}_1)}}{\overline{w}_1 - w_1},$$

and thus by comparing (3.6.6) and (3.6.7) we justify that $\widetilde{M}$ satisfies the residue conditions (2.8.3) of Riemann–Hilbert problem 2.8.1.

Now let us check if $\widetilde{M}$ admits the correct jump behavior on $\mathbb{R}$. By assumption, $M^{(0)}$ satisfies

$$M_+^{(0)}(x;w) = M_-^{(0)}(x;w)\begin{bmatrix} 1 + w|r^{(0)}(w)|^2 & \overline{r^{(0)}(w)}e^{-ix\Theta(w)} \\ wr^{(0)}(w)e^{ix\Theta(w)} & 1 \end{bmatrix}.$$

Using the definition (3.6.5) of $r_\pm^{(0)}$, for $w \in \mathbb{R}$, we get that

$$\begin{aligned}
\widetilde{M}_+(w;x) &= \widetilde{M}_-(w;x)\mu(w)\begin{bmatrix} 1 + w|r^{(0)}(w)|^2 & \overline{r^{(0)}(w)}e^{-ix\Theta(w)} \\ wr^{(0)}(w)e^{ix\Theta(w)} & 1 \end{bmatrix}\mu^{-1}(w) \\
&= \widetilde{M}_-(w;x)\begin{bmatrix} 1 + w|r^{(1)}(w)|^2 & \overline{r^{(1)}(w)}e^{-ix\Theta(w)} \\ wr^{(1)}(w)e^{ix\Theta(w)} & 1 \end{bmatrix}.
\end{aligned}$$

Next we observe

$$\widetilde{M}(w;x) = \left[1 + \frac{A(x)\,\mu(0)\,A^{-1}(x)}{w}\right] M^{(0)}(x;w)\begin{bmatrix} \frac{w}{w-w_1} & 0 \\ 0 & \frac{w}{w-\overline{w}_1} \end{bmatrix}. \tag{3.6.8}$$

It follows that $\widetilde{M} \to 1$ as $|w| \to \infty$. This concludes the proof of our claim that $\widetilde{M}$ solves Riemann–Hilbert problem 2.8.1 with data $(r^{(1)}, \{w_1, c_1\})$. ☐

The Bäcklund transformation formula (3.6.4) is a nice construction of solutions for Riemann–Hilbert problem 2.8.1 with singularities at w_1 and $\overline{w}_1$. It does not only prove existence, but also allows a precise analysis of the solution.

Lemma 3.6.3. *Under the same assumptions of Lemma 3.6.2, we have*

$$\lim_{|w|\to\infty} w\cdot\left[M^{(1)}(x;w)\right]_{12} \in H_x^1(\mathbb{R}_+)\cap L_x^{2,1}(\mathbb{R}_+), \qquad \lim_{|w|\to\infty} w\cdot\left[M^{(1)}(x;w)\right]_{21} \in H_x^1(\mathbb{R}_+)\cap L_x^{2,1}(\mathbb{R}_+). \tag{3.6.9}$$

In addition, for any $w_2 \in \mathbb{C}\setminus\mathbb{R}$, we have

$$(M^{(1)}(x;w_2) - 1) \in H_x^1(\mathbb{R}_+) \cap L_x^{2,1}(\mathbb{R}_+). \tag{3.6.10}$$

Proof. We use (3.6.8) and the expansion

$$\begin{bmatrix} \frac{w}{w-w_1} & 0 \\ 0 & \frac{w}{w-\overline{w}_1} \end{bmatrix} = 1 - \frac{\mu(0)}{w} + \mathcal{O}(w^{-2}), \quad \text{as } |w| \to \infty$$

in order to find

$$\begin{aligned}\lim_{|w|\to\infty} w\cdot\left[M^{(1)}(x;w)\right]_{12} &= \lim_{|w|\to\infty} w\cdot\left[M^{(0)}(x;w)\right]_{12} + \left[A(x)\,\mu(0)\,A^{-1}(x)\right]_{12},\\ \lim_{|w|\to\infty} w\cdot\left[M^{(1)}(x;w)\right]_{21} &= \lim_{|w|\to\infty} w\cdot\left[M^{(0)}(x;w)\right]_{21} + \left[A(x)\,\mu(0)\,A^{-1}(x)\right]_{21}.\end{aligned} \tag{3.6.11}$$

Note that due to $\operatorname{Im}(w_1)>0$, the factors $e^{ix\Theta(w_1)}$ and $e^{-ix\Theta(\overline{w}_1)}$ arising in (3.6.1) are decaying exponentially as $x\to+\infty$. By the exponential decay and (3.4.2) it follows that

$$A(x)-1\in H^1_x(\mathbb{R}_+)\cap L^{2,1}_x(\mathbb{R}_+). \tag{3.6.12}$$

Taking also into account (3.6.2), we conclude that

$$\left[A(x)\,\mu(0)\,A^{-1}(x)\right]_{12} = \frac{2i\operatorname{Im}(w_1)a_{11}(x)a_{12}(x)}{\det(A(x))}\in H^1_x(\mathbb{R}_+)\cap L^{2,1}_x(\mathbb{R}_+)$$

and

$$\left[A(x)\,\mu(0)\,A^{-1}(x)\right]_{21} = \frac{-2i\operatorname{Im}(w_1)a_{21}(x)a_{22}(x)}{\det(A(x))}\in H^1_x(\mathbb{R}_+)\cap L^{2,1}_x(\mathbb{R}_+).$$

As for the proof of (3.6.10), we rewrite (3.6.4) as

$$\begin{aligned}M^{(1)}(x;w_2) &= M^{(0)}(x;w_2)\\ &- 2i\operatorname{Im}(w_1)A(x)\begin{bmatrix} 0 & \frac{a_{21}(x)[M^{(0)}(x;w_2)]_{11}-a_{11}(x)[M^{(0)}(x;w_2)]_{21}}{(w_2-w_1)\det(A(x))}\\ \frac{a_{22}(x)[M^{(0)}(x;w_2)]_{12}-a_{12}(x)[M^{(0)}(x;w_2)]_{22}}{(w_2-\overline{w}_1)\det(A(x))} & 0\end{bmatrix}.\end{aligned}$$

Now, (3.6.10) is a direct consequence of $M^{(0)}(x;w_2)-1\in H^1(\mathbb{R}_+)\cap L^{2,1}(\mathbb{R}_+)$ (see Lemma 3.4.1), (3.6.2) and (3.6.12). □

In order to analyze Riemann–Hilbert problem 2.8.1 for negative x, we proceed as in Section 3.5. Let $M^{(0)}(x;w)$ be the solution of Riemann–Hilbert problem 2.8.1 associated to data $\mathcal{S}_w(u^{(0)},v^{(0)})=(r^{(0)})$ as in Lemma 3.6.2. Furthermore, let $M^{(1)}(x;w)$ be the solution of Riemann–Hilbert problem 2.8.1 associated to data $\mathcal{S}_w(u^{(0)},v^{(0)})=(r^{(1)},\{w_1,c_1\})$ provided by Lemma 3.6.2. Analogously to (3.5.2) we set

$$M^{(0)}_d(x;w) := M^{(0)}(x;w)\begin{bmatrix} 1/d(w) & 0\\ 0 & d(w)\end{bmatrix}. \tag{3.6.13}$$

It follows that $M^{(0)}_d(x;w)$ satisfies estimates (3.5.13) and (3.5.14) of Lemma 3.5.5. We define the matrix

$$A^{(d)}(x)=\begin{bmatrix} a^{(d)}_{11}(x) & a^{(d)}_{12}(x)\\ a^{(d)}_{21}(x) & a^{(d)}_{22}(x)\end{bmatrix}$$

by

$$\begin{aligned}\begin{pmatrix} a^{(d)}_{11}(x)\\ a^{(d)}_{21}(x)\end{pmatrix} &:= M^{(0)}_d(x;\overline{w}_1)\begin{pmatrix} 1\\ -\frac{(w_1-\overline{w}_1)e^{ix\Theta(\overline{w}_1)}}{\overline{c}_1 d(\overline{w}_1)^2}\end{pmatrix},\\ \begin{pmatrix} a^{(d)}_{12}(x)\\ a^{(d)}_{22}(x)\end{pmatrix} &:= M^{(0)}_d(x;w_1)\begin{pmatrix} \frac{d(w_1)^2(\overline{w}_1-w_1)}{w_1c_1}e^{-ix\Theta(w_1)}\\ 1\end{pmatrix}.\end{aligned} \tag{3.6.14}$$

We observe

$$\begin{aligned}\begin{pmatrix} a^{(d)}_{12}(x)\\ a^{(d)}_{22}(x)\end{pmatrix} &= M^{(0)}_d(x;w_1)\begin{pmatrix} \frac{d(w_1)^2(\overline{w}_1-w_1)}{w_1c_1}e^{-ix\Theta(w_1)}\\ 1\end{pmatrix}\\ &= M^{(0)}(x;w_1)\begin{pmatrix} \frac{d(w_1)(\overline{w}_1-w_1)}{w_1c_1}e^{-ix\Theta(w_1)}\\ d(w_1)\end{pmatrix}\\ &= \begin{pmatrix} a_{11}(x)\\ a_{21}(x)\end{pmatrix}\frac{d(w_1)(\overline{w}_1-w_1)}{w_1c_1}e^{-ix\Theta(w_1)}\end{aligned}$$

and in the same way

$$\begin{pmatrix} a_{11}^{(d)}(x) \\ a_{21}^{(d)}(x) \end{pmatrix} = -\begin{pmatrix} a_{12}(x) \\ a_{22}(x) \end{pmatrix} \frac{(w_1 - \overline{w}_1)e^{ix\Theta(\overline{w}_1)}}{\overline{c}_1 d(\overline{w}_1)}.$$

We conclude that

$$A^{(d)}(x) = A(x) \begin{bmatrix} 0 & \frac{d(w_1)(\overline{w}_1 - w_1)}{w_1 c_1} e^{-ix\Theta(w_1)} \\ -\frac{(w_1-\overline{w}_1)e^{ix\Theta(\overline{w}_1)}}{\overline{c}_1 d(\overline{w}_1)} & 0 \end{bmatrix}$$

so that by the diagonality of $\mu(w)$, we have that

$$A(x)\mu(w)\,[A(x)]^{-1} = A^{(d)}(x)\widetilde{\mu}(w)\left[A^{(d)}(x)\right]^{-1}, \qquad \widetilde{\mu}(w) := \begin{bmatrix} w - \overline{w}_1 & 0 \\ 0 & w - w_1 \end{bmatrix}$$

Multiplying (3.6.4) from the right by

$$\begin{bmatrix} \frac{1}{d(w)}\frac{w-w_1}{w-\overline{w}_1} & 0 \\ 0 & d(w)\frac{w-\overline{w}_1}{w-w_1} \end{bmatrix}$$

and substituting the definitions (3.6.13), yields

$$M_d^{(1)}(x;w) = A^{(d)}(x)\widetilde{\mu}(w)\left[A^{(d)}(x)\right]^{-1} M_d^{(0)}(x;w)\widetilde{\mu}^{-1}(w), \tag{3.6.15}$$

for

$$M_d^{(1)}(x;w) := M^{(1)}(x;w) \begin{bmatrix} \frac{1}{d(w)}\frac{w-w_1}{w-\overline{w}_1} & 0 \\ 0 & d(w)\frac{w-\overline{w}_1}{w-w_1} \end{bmatrix}. \tag{3.6.16}$$

Hence, $M_d^{(1)}$ is obtained from $M_d^{(0)}$ in similar way as $M^{(1)}$ is obtained from $M^{(0)}$ by (3.6.4). But since $A^{(d)}(x)$ is defined through $M_d(x;w)$ and the exponential factors arising in (3.6.14) decay exponentially as $x \to -\infty$, we have

$$|\det A^{(d)}(x)|^{-1} \leq C$$

for all $x < 0$ and

$$(A^{(d)}(x) - 1) \in H_x^1(\mathbb{R}_-) \cap L_x^{2,1}(\mathbb{R}_-).$$

As a result we can copy the arguments and computations of the foregoing proof to extend Lemma 3.6.3 to the negative half-line:

Lemma 3.6.4. *Under the same assumptions as in Lemma 3.6.2, the matrix-valued function defined in (3.6.16)satisfies*

$$\lim_{|w|\to\infty} w \cdot \left[M_d^{(1)}(x;w)\right]_{12} \in H_x^1(\mathbb{R}_-) \cap L_x^{2,1}(\mathbb{R}_-), \qquad \lim_{|w|\to\infty} w \cdot \left[M_d^{(1)}(x;w)\right]_{21} \in H_x^1(\mathbb{R}_-) \cap L_x^{2,1}(\mathbb{R}_-). \tag{3.6.17}$$

In addition, for any $w_2 \in \mathbb{C} \setminus \mathbb{R}$, we have

$$(M_d^{(1)}(x;w_2) - 1) \in H_x^1(\mathbb{R}_-) \cap L_x^{2,1}(\mathbb{R}_-). \tag{3.6.18}$$

It is clear that one can also use the Bäcklund transformation (3.6.4) to construct the solution of Riemann–Hilbert problem 2.8.1 in the presence of $N \geq 2$ eigenvalues, i.e. according to scattering data $\mathcal{S}_w(u,v) = (r; \{w_j, c_j\}_{j=1}^N)$. Therefore, one needs to apply (3.6.4) N-times which adds successively more and more eigenvalues. We do not provide further details, except to note that in the first step $M^{(0)} \mapsto M^{(1)}$, the function $M^{(0)}$ should be associated to the reflection coefficient

$$r^{(0)}(w) = r(w) \prod_{j=1}^{N} \frac{w - w_j}{w - \overline{w}_j}$$

and the norming constant used in the first step should read

$$\widetilde{c}_1 := c_1 \prod_{j=2}^{N} \frac{w_1 - w_j}{w_1 - \overline{w}_j}.$$

Thus, we have proven Theorem 3.1.1.

Remark 3.6.5. Let $M(x; w)$ be the solution of Riemann–Hilbert problem 2.8.1 subject to scattering data $\mathcal{S}_w(u, v) = (r; \{w_j, c_j\}_{j=1}^N)$. Fix some $w' \in \mathbb{C} \setminus (\mathbb{R} \cup \{w_1, ..., w_N, \overline{w}_1, ..., \overline{w}_N\})$ and consider a smooth matrix-valued function g satisfying

$$g(x) = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & x \geq 1, \\ \begin{bmatrix} d(w') \displaystyle\prod_{j=1}^{N} \frac{w' - \overline{w}_j}{w' - w_j} & 0 \\ 0 & \dfrac{1}{d(w')} \displaystyle\prod_{j=1}^{N} \frac{w' - w_j}{w' - \overline{w}_j} \end{bmatrix}, & x \leq -1. \end{cases}$$

Then, it follows from (3.6.10) and (3.6.18) that

$$(M(x; w') - g(x)) \in H_x^1(\mathbb{R}) \cap L_x^{2,1}(\mathbb{R}).$$

However, this fact has no relevance for the inverse scattering transform. The only purpose for presenting statements (3.6.10) and (3.6.18) was their usefulness in the analysis of the Bäcklund transformation formula (3.6.4).

Remark 3.6.6. Combining Remark 3.4.2 and the explicit formula in the end of the proof of Lemma 3.6.2 we are able to prove that

$$\|M(x; \cdot)\|_{L^\infty(K)} \leq c$$

for any compact subset $K \subset \mathbb{C} \setminus (\mathbb{R} \cup \{w_1, ..., w_N, \overline{w}_1, ..., \overline{w}_N\})$. The constant c may depend on K and $\|r\|_{L^2}$. In particular, since due to Corollary 2.9.2 the L^2-norm of r is constant in time, we may even bound

$$\|M(t, x; \cdot)\|_{L^\infty(K)} \leq c,$$

uniformly in t and x.

3.7 New coordinates

Let

$$\Theta(s) = \frac{1}{2}\left(s - \frac{1}{s}\right), \qquad Z(\zeta) = \frac{1}{2}\left(\zeta + \frac{1}{\zeta}\right). \tag{3.7.1}$$

We note that the function Θ was introduced earlier in this chapter. For instance, we can rewrite the jump matrix (2.8.2) of Riemann–Hilbert problem 2.8.1 as

$$R(x; w) = \begin{bmatrix} w|r(w)|^2 & \overline{r(w)} e^{-ix\Theta(w)} \\ w r(w) e^{ix\Theta(w)} & 0 \end{bmatrix}.$$

The function Z, which is also called *Joukowsky* transform, appears in the time evolution of the scattering data. For instance, according to Corollary 2.9.2, the time evolution of the reflection coefficent r is given by

$$r(t; w) = r(w) e^{-itZ(w)}.$$

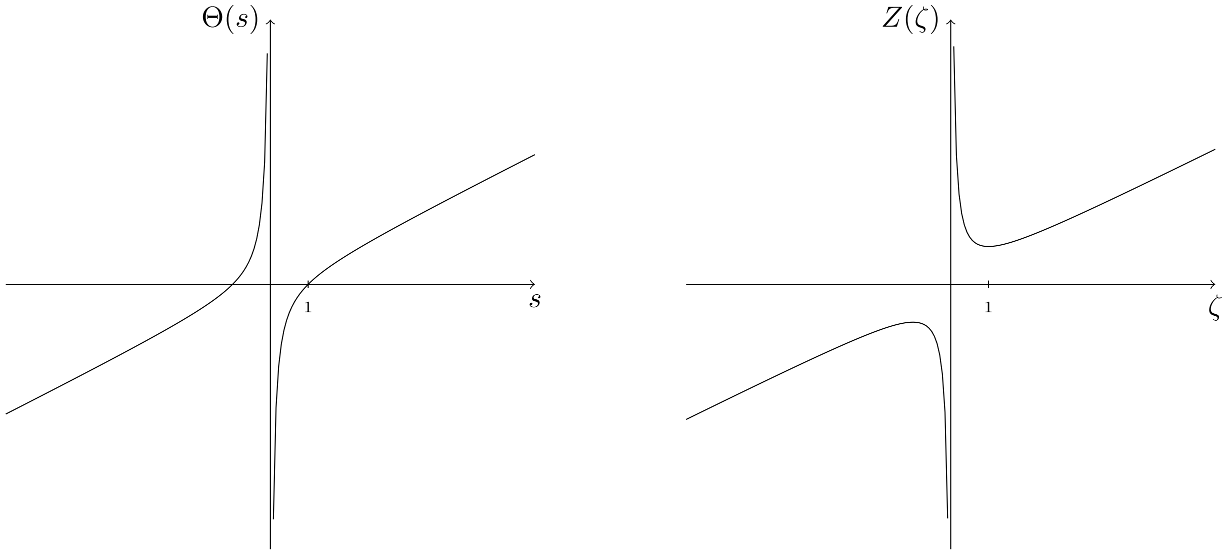


Figure 3.1: *Graphs of* $\Theta(s) = \frac{1}{2}(s - s^{-1})$ *and* $Z(\zeta) = \frac{1}{2}(\zeta + \zeta^{-1})$.

Thus, if we want to find the solution $(u(t,x), v(t,x))$ of the MTM system at time $t \neq 0$, we have to substitute the time evolution of the reflection coefficient into the matrix $R(x;w)$. It follows that Riemann–Hilbert problem 2.8.1 becomes time dependent with jump matrix

$$R(t,x;w) = \begin{bmatrix} w|r(w)|^2 & \overline{r(w)}e^{-ix\Theta(w)+itZ(w)} \\ wr(w)e^{ix\Theta(w)-itZ(w)} & 0 \end{bmatrix}.$$

We recall from the reconstruction formula (2.8.5) and the formulation given in (3.4.7), that, to first order in r,

$$|u(t,x)| \sim \frac{1}{2\pi}\left|\int_{\mathbb{R}} \overline{r(w)}e^{-ix\Theta(w)+itZ(w)}dw\right|.$$

By the method of stationary phase (or, equivalently, by the method of linear steepest descent), we know that

$$\left|\int_{\mathbb{R}} \overline{r(w)}e^{-ix\Theta(w)+itZ(w)}dw\right| \sim \begin{cases} |x|^{-k}, & \text{as } |x| \to \infty \text{ and } t \text{ fixed}, \\ |t|^{-1/2}, & \text{as } |t| \to \infty \text{ and } x \text{ fixed}. \end{cases} \tag{3.7.2}$$

Here, the number $k > 0$ depends on the regularity of r. For instance, if $r(w)$ and $r(1/w)$ are of the Schwartz class, then k can be chosen arbitrarily large. On the other hand, the decay $\sim |t|^{-1/2}$ as $|t| \to \infty$ cannot be improved by increasing the regularity of r. The two different rates of decay can be traced back to the differences between the functions Θ and Z. As it can be seen in Figure 3.1, the function Θ has no critical point on the real axis. On the other hand for the Joukowsky transform we have $Z'(\pm 1) = 0$ and $Z''(\pm 1) \neq 0$. For the method of stationary phase we refer to [LP14, Corollary 1.1]. In order to understand the behavior along paths where $|x| \to \infty$ and $|t| \to \infty$ simultaneously, we define the following parameters, which are illustrated in Figure 3.2:

$$\tau := \sqrt{|t^2 - x^2|} \in \mathbb{R}^+, \qquad w_0 := \sqrt{\left|\frac{t+x}{t-x}\right|} \in \mathbb{R}^+. \tag{3.7.3}$$

Assuming $|x| \neq |t|$, it is now easy to see that

$$-ix\Theta(w) + itZ(w) = \begin{cases} \pm i\tau Z\left(\dfrac{w}{w_0}\right), & \pm t > |x|, \\ \mp i\tau\Theta\left(\dfrac{w}{w_0}\right), & \pm x > |t|, \end{cases} \tag{3.7.4}$$

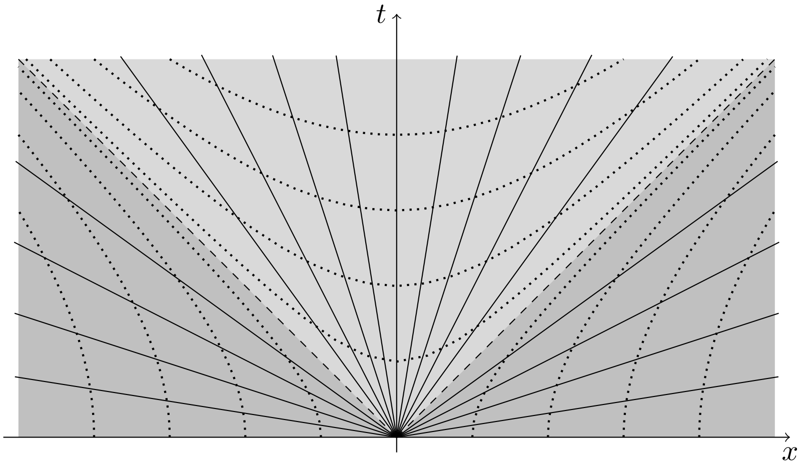


Figure 3.2: *Illustration of the parameters (τ, w_0) introduced in (3.7.3). Along the dotted curves, parameter τ is constant. The rays running out of the origin show constant levels for w_0. On the diagonals $x = t$ and $x = -t$ the new parameters are not defined and relation (3.7.4) does not hold.*

which shows that the x-t-plane ($= \mathbb{R} \times \mathbb{R}$) is basically divided into two disjoint regions. We call them *interior* and *exterior* region. The precise definitions are given by

$$\text{"interior region"} := \left\{(x,t) \in \mathbb{R}^2 : |t| > |x|\right\}, \qquad \text{"exterior region"} := \left\{(x,t) \in \mathbb{R}^2 : |x| > |t|\right\}. \tag{3.7.5}$$

From (3.7.4) we learn that in the interior region the long-time behavior is determined by $e^{i\tau Z}$ as oscillatory factor, whereas it is determined by $e^{i\tau\Theta}$ in the exterior region. Analogously to (3.7.2) we find

$$\left| \int_{\mathbb{R}} \overline{r(w)} e^{-ix\Theta(w)+itZ(w)} dw \right| \sim \begin{cases} |\tau|^{-k}, & \text{as } |\tau| \to \infty,\ (x,t) \in \{\text{exterior region}\} \text{ and } w_0 \text{ fixed}, \\ |\tau|^{-1/2}, & \text{as } |\tau| \to \infty,\ (x,t) \in \{\text{interior region}\} \text{ and } w_0 \text{ fixed}. \end{cases} \tag{3.7.6}$$

Note that (3.7.6) contains (3.7.2) as a special case. The fast decay in the exterior region has a simple physical interpretation: since the MTM system arises in the context of general relativity nothing can travel faster than the speed of light. In this model the speed of light is given by 1 and thus, eventually everything will enter the interior region. From the analytical point of view in the exterior region as well as in the interior region the question is twofold:

- How to extend the method of linear steepest descend to a nonlinear equation? More precisely, how to analyze the second term of (3.4.7) which is nonlinear in r?
- In which way is it possible to have control over the dependence on w_0 in (3.7.6)?

The detailed analysis of $(u(t,x), v(t,x))$ as $\tau \to \infty$ is the main subject of Chapters 5 and 6. We end the section with some remarks:

Remark 3.7.1. The above defined parameters w_0 and τ are useful for understanding the time-dependence of Riemann–Hilbert problem 2.8.1. But for the reconstruction of $v(t,x)$ we use Riemann–Hilbert problem 2.8.2 instead. Here, the time-dependent jump matrix can be written as

$$\widehat{R}(t,x;z) = \begin{bmatrix} 0 & -\overline{\widehat{r}(z)} e^{ix\Theta(z)+itZ(z)} \\ -z\widehat{r}(z) e^{-ix\Theta(z)-itZ(z)} & z|\widehat{r}(z)|^2 \end{bmatrix},$$

which follows from the time evolution of $\widehat{r}$ given in Corollary 2.9.2. Defining

$$z_0 := w_0^{-1} = \sqrt{\left|\frac{t-x}{t+x}\right|}, \tag{3.7.7}$$

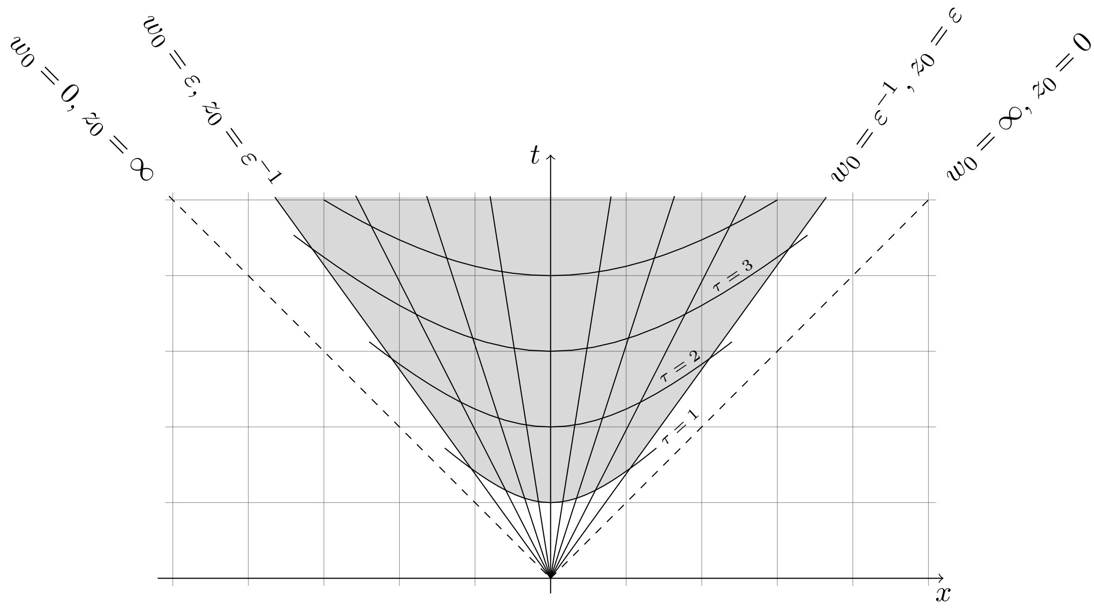


Figure 3.3: *Illustration of* (τ, w_0) *and* (τ, z_0) *in the interior region and with the additional restriction* $\varepsilon < w_0 < \varepsilon^{-1}$.

we find analogously to (3.7.4) that

$$ix\Theta(z) + itZ(z) = \begin{cases} \pm i\tau Z\left(\dfrac{z}{z_0}\right), & \pm t > |x|, \\ \pm i\tau\Theta\left(\dfrac{z}{z_0}\right), & \pm x > |t|, \end{cases}$$

so that the terminology of exterior and interior region can be used for both functions u and v.

Remark 3.7.2. As it can be seen from Figure 3.2, it is not possible to use (τ, w_0) as alternative coordinates because they would be singular on the diagonals $\{x = t\}$ and $\{x = -t\}$. But if we consider only points (x, t) in the interior region with $t > 0$, $\tau > 1$ and $\varepsilon < w_0 < \varepsilon^{-1}$ with an arbitrarily small $\varepsilon > 0$, say, then w_0 and τ make sense as proper coordinates, see Figure 3.3. Another way of avoiding the singularity of (τ, w_0) on the diagonals $\{x = t\}$ and $\{x = -t\}$ is to introduce the characteristic coordinates, see Figure 3.4,

$$\xi = (x + t)/2, \qquad \eta = (x - t)/2.$$

We will use these coordinates later in Theorem 5.4.1.

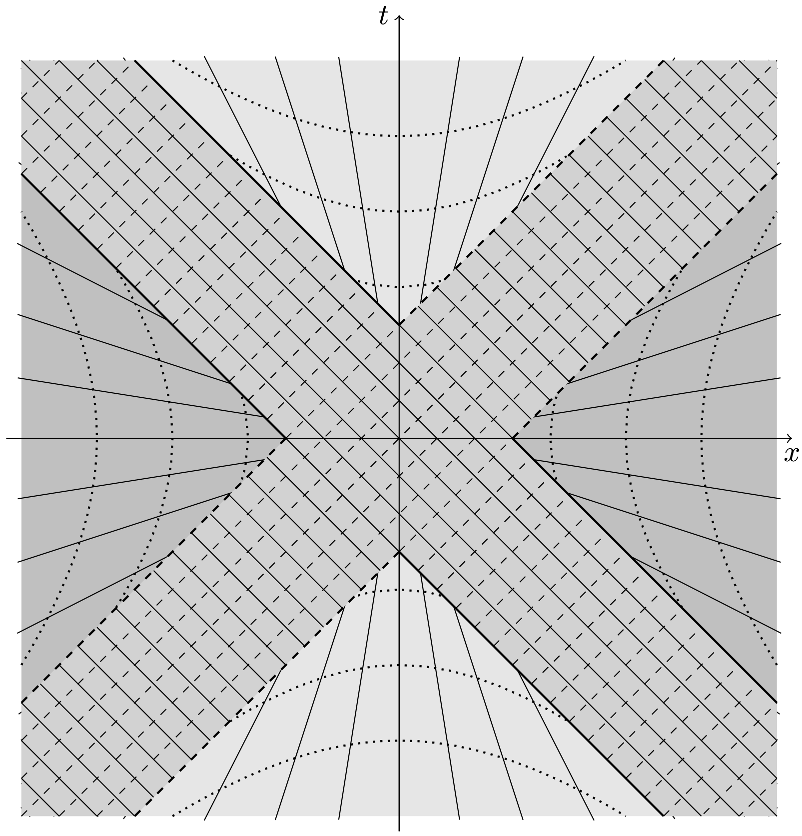


Figure 3.4: *Illustration of the parameters (τ, w_0) as in Figure 3.2 in combination with the characteristic coordinates $\xi = (x+t)/2$ and $\eta = (x-t)/2$ which can be defined everywhere but are most useful near the boundary of the light cone.*

Chapter 4

Solitons

4.1 Characterization of solitons

In order to provide an explicit characterization of solitons, we begin by recalling an example of a dispersive partial differential equation, namely, the Klein-Gordon equation,

$$u_{tt} - u_{xx} + u = 0.$$

Looking for plane wave solutions of the form $u(t,x) = Ae^{i(kx-\omega t)}$ we obtain the dispersion relation

$$\omega^2 = k^2 + 1.$$

Writing ω as a function of k we obtain $\omega''(k) \neq 0$ and hence, the quantity $\omega'(k)$, also called the *group velocity*, is not constant in k. In the physical context, this means that the speed of the waves varies according to frequency. In particular, different waves disperse in the medium. It is not possible that a single hump maintains its shape when time evolves. Once nonlinear effects are included, this dispersive property can be lost. For example, for the MTM system (1.1.1) this phenomenon can be directly observed, since

$$\begin{cases} u(t,x) = \operatorname{sech}(x + i\frac{\pi}{4}), \\ v(t,x) = -\operatorname{sech}(x - i\frac{\pi}{4}), \end{cases}$$

provides an explicit solution of (1.1.1), that is obviously of permanent form. Thus, it seems that the dispersive effects are cancelled by the presence of the nonlinearity. And it turns out that there exists a large class of solutions that do not disperse but maintain their envelope. Such solutions are called *solitons*. In their introductory textbook [DJ89] the authors P. G. Drazin and R. S. Johnson associate the term *soliton* with any solution of a nonlinear equation, admitting the following three properties:

(a) The solution represents a wave of permanent form.

(b) The solution is localized, so that it decays or approaches a constant at infinity.

(c) The solution can strongly interact with other solitons and retain its identity, except for a phase shift.

A single, more rigorous definition of a soliton for a general nonlinear equation is difficult to find. But thanks to the inverse scattering machinery as a mathematical framework there exists the following formal definition of multi-solitons for the MTM system. We recall Definition 2.7.3: to any initial data $(u_0, v_0) \in \mathcal{G}_N$ we can associate the "original" scattering data $\mathcal{S}(u_0, v_0) = (p, \{\lambda_j, C_j\}_{j=1}^N)$.

Definition 4.1.1. In the case where the initial data generate pairwise distinct eigenvalues $\lambda_1, ..., \lambda_N$, but $p(\lambda) = 0$ for all $\lambda \in (\mathbb{R} \cup i\,\mathbb{R}) \setminus \{0\}$ we call the solution of (1.1.1) an *N-soliton solution* or *multi-soliton solution*. For $\mathcal{D} = \{\lambda_j, C_j\}_{j=1}^N \subset (\mathbb{C}_{II})^N \times (\mathbb{C}^*)^N$ we use the notation

$$(u_{sol}(t,x;\mathcal{D}), v_{sol}(t,x;\mathcal{D}))$$

for the associated N-soliton.

According to this definition, the zero solution $(u, v) \equiv (0, 0)$ is a zero-soliton. We will see later in this chapter that in the generic case an N-soliton $(u_{sol}(t, x; \mathcal{D}), v_{sol}(t, x; \mathcal{D}))$ $(N \leq 2)$ represents N solitons travelling at different speeds. Each soliton collides with each of the other solitons and emerges from the collision unchanged, except for a phase shift. Using Riemann–Hilbert techniques we will analyse this interaction very precisely. Thus, we justify the nomenclature of Definition 4.1.1 in the sense of Drazin and Johnson, [DJ89].

4.2 The one-soliton

Let us start with examining the one-soliton or, shortly, soliton. That is, we fix some λ_1 with $\mathrm{Im}(\lambda_1) > 0$ and $\mathrm{Re}(\lambda_1) < 0$ and a non-zero complex number C_1. Then we want to find the unique solution (u, v) of (1.1.1) which generates scattering data $(0, \{\lambda_1, C_1\})$ at time $t = 0$. Therefore, we shall find a solution $M(t, x; w)$ to Riemann–Hilbert problem 2.8.1 with

$$r(w) = 0, \quad w_1 = \lambda_1^{-2}, \quad \text{and} \quad c_1 = \frac{-2C_1}{\lambda_1^2},$$

see (2.7.1) and (2.7.5) for the transformation $\{\lambda_1, C_1\} \to \{w_1, c_1\}$. Note that we can use Remark 2.8.3 to conclude that

$$u_{sol}(t, x; \{\lambda_1, C_1\}) = [M(t, x; 0)]_{21}, \qquad v_{sol}(t, x; \{\lambda_1, C_1\}) = \overline{[M(t, x; 0)]_{12}}. \tag{4.2.1}$$

With the help of the Bäcklund transformation given in Lemma 3.6.2, equation (3.6.4), we can derive explicit expressions for $M(t, x; w)$ and thus for (u, v). In principle, in order to apply formula (3.6.4) we firstly have to solve Riemann–Hilbert problem 2.8.1 without eigenvalues. But since we assume that $r = 0$ this yields the trivial solution $M^{(0)} \equiv 1$. Plugging the latter in into (3.6.4) we find

$$M(t, x; w) = A(t, x)\mu(w)A^{-1}(t, x)\mu^{-1}(w),$$

where

$$A(t, x) = \begin{bmatrix} a_{11}(t, x) & a_{12}(t, x) \\ a_{21}(t, x) & a_{22}(t, x) \end{bmatrix} := \begin{bmatrix} 1 & \frac{-\overline{c}_1 e^{-ix\Theta(\overline{w}_1)+itZ(\overline{w}_1)}}{w_1-\overline{w}_1} \\ \frac{w_1 c_1 e^{ix\Theta(w_1)-itZ(w_1)}}{\overline{w}_1-w_1} & 1 \end{bmatrix}.$$

Note that we also used the time evolution of the norming constants given by Corollary 2.9.2. A direct calculation yields

$$[M(t, x; 0)]_{21} = \frac{2i\,\mathrm{Im}(w_1)a_{21}(t, x)}{w_1 \det A(t, x)}, \qquad [M(t, x; 0)]_{12} = \frac{-2i\,\mathrm{Im}(w_1)a_{12}(t, x)}{\overline{w}_1 \det A(t, x)}. \tag{4.2.2}$$

For the determinant of A we find

$$\begin{aligned} \det A(t, x) &= 1 + \frac{w_1}{4(\mathrm{Im}(w_1))^2}|c_1|^2 \left|e^{ix\Theta(w_1)-itZ(w_1)}\right|^2 \\ &= \frac{|c_1|\left|e^{ix\Theta(w_1)-itZ(w_1)}\right|}{\lambda_1 \mathrm{Im}(w_1)} \cosh\left\{\mathrm{Re}(ix\Theta(w_1) - itZ(w_1)) + \log\left(\frac{|c_1||\lambda_1|^{-1}}{2\,\mathrm{Im}(w_1)}\right) - i\arg\lambda_1\right\}. \end{aligned}$$

It is seen that

$$\log\left(\frac{|c_1||\lambda_1|^{-1}}{2\,\mathrm{Im}(w_1)}\right) = \log\left(\frac{|C_1|}{|\,\mathrm{Im}(\lambda_1^2)||\lambda_1|}\right).$$

Furthermore,

$$\begin{aligned} \mathrm{Re}(ix\Theta(w_1) - itZ(w_1)) &= \frac{-\,\mathrm{Im}(w_1)}{|w_1|}\frac{1 + |w_1|^2}{2|w_1|}\left(x + \frac{1 - |w_1|^2}{1 + |w_1|^2}t\right) \\ &= \sin(2\arg\lambda_1)\,\frac{|\lambda_1|^2 + |\lambda_1|^{-2}}{2}\left(x + \frac{|\lambda_1|^2 - |\lambda_1|^{-2}}{|\lambda_1|^2 + |\lambda_1|^{-2}}t\right) \end{aligned}$$

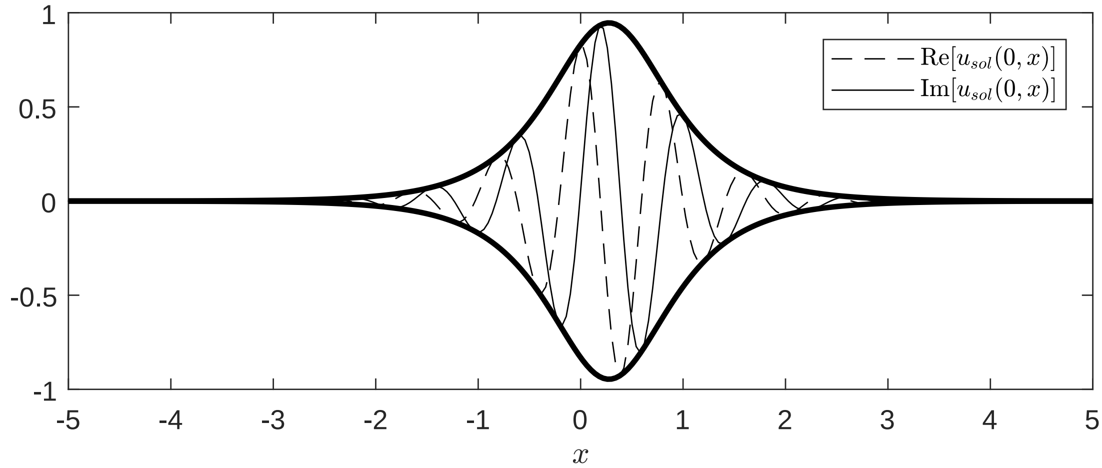


Figure 4.1: *This graphic shows real and imaginary part of* $u_{sol}(t,x;\{\lambda_1,C_1\})$ *and the envelope at time* $t=0$. *The parameters are set to* $\lambda_1 = -0.25+0.03i$ *and* $C_1 = 0.1$.

and

$$\begin{aligned}\operatorname{Im}(ix\Theta(w_1) - itZ(w_1)) &= \frac{-\operatorname{Re}(w_1)}{|w_1|}\frac{1+|w_1|^2}{2|w_1|}\left(t + \frac{1-|w_1|^2}{1+|w_1|^2}x\right)\\ &= -\cos(2\arg\lambda_1)\frac{|\lambda_1|^2+|\lambda_1|^{-2}}{2}\left(t+\frac{|\lambda_1|^2-|\lambda_1|^{-2}}{|\lambda_1|^2+|\lambda_1|^{-2}}x\right).\end{aligned}$$

Using the following physical paramters

$$\begin{gathered}\nu = \tfrac{|\lambda_1|^{-2}-|\lambda_1|^2}{|\lambda_1|^{-2}+|\lambda_1|^2}, \quad E = -\tfrac{|\lambda_1|^2+|\lambda_1|^{-2}}{2}\sin(2\arg\lambda_1), \quad \beta = \tfrac{|\lambda_1|^2+|\lambda_1|^{-2}}{2}\cos(2\arg\lambda_1),\\ x_0 = \tfrac{1}{E}\log\left(\tfrac{|C_1|}{\operatorname{Im}(\lambda_1^2)|\lambda_1|}\right), \quad \phi_1 = \arg\left(\tfrac{-C_1}{\lambda_1}\right),\end{gathered} \tag{4.2.3}$$

via (4.2.1) and (4.2.2), we get

$$\begin{cases} u_{sol}(t,x;\{\lambda_1,C_1\}) = |\lambda_1|^{-1}\sin(2\arg\lambda_1)\operatorname{sech}(E(x-\nu t-x_0)+i\arg\lambda_1)\, e^{-i\beta(t-\nu x)+i\phi_1}, \\ v_{sol}(t,x;\{\lambda_1,C_1\}) = -|\lambda_1|\sin(2\arg\lambda_1)\operatorname{sech}(E(x-\nu t-x_0)-i\arg\lambda_1)\, e^{-i\beta(t-\nu x)+i\phi_1}, \end{cases} \tag{4.2.4}$$

which coincide with the formulas to be found in the literature. We refer to Figures 4.1 and 4.2 for graphical representations of single solitons with specific parameters. In these plots and as well in the explicit formulas (4.2.4), it is seen that the general 1-soliton describes a single wave package propagating with velocity ν determined by $|\lambda_1|$. More precisely, if $|\lambda_1| > 1$, then $\nu \in (-1,0)$, which means that the soliton propagates to the left. On the other hand, if $|\lambda_1| < 1$, then $\nu \in (0,1)$ and the soliton travels to the right. Finally, for $|\lambda_1| = 1$ the soliton does not move ($\nu = 0$). In contrast to the velocity, the shape of the soliton is not only affected by $|\lambda_1|$, but also by $\arg(\lambda_1)$. Let us define

$$A(\lambda_1) := \max_{(t,x)\in\mathbb{R}^2}\sqrt{|u_{sol}(t,x;\{\lambda_1,C_1\})|^2 + |v_{sol}(t,x;\{\lambda_1,C_1\})^2|}$$

such that A gives the amplitude of the soliton. Making use of

$$\max_{a\in\mathbb{R}}|\operatorname{sech}(a\pm ib)| = |\operatorname{sech}(ib)| = \frac{1}{|\cos(b)|},$$

we see from (4.2.4) that the amplitude is given by

$$A(\lambda_1) = \sqrt{|\lambda_1|^{-2}+|\lambda_1|^2}\left|\frac{\sin(2\arg\lambda_1)}{\cos(\arg\lambda_1)}\right|. \tag{4.2.5}$$

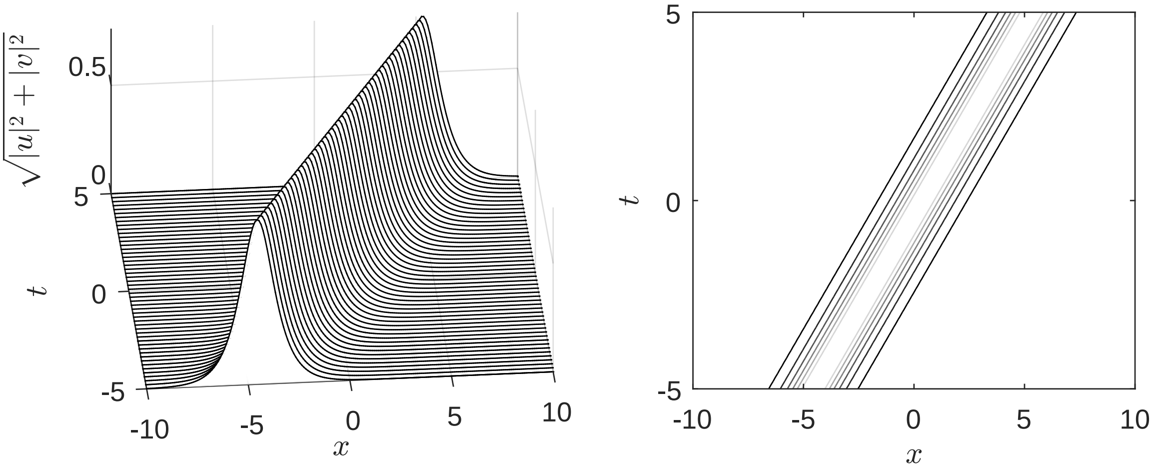


Figure 4.2: *The left frame is a plot of the amplitude* $\sqrt{|u|^2+|v|^2}$ *of the soliton associated to scattering data* $\{\lambda_1, C_1\} = \{-0.28+0.03i, 0.1\}$. *The amplitude as computed in (4.2.5) is given by* $A(\lambda_1) = 0.7590$. *The right frame is a contour plot of the same amplitude.*

Hence, we conclude that

$$A(\lambda_1) \to \begin{cases} 2\sqrt{|\lambda_1|^{-2}+|\lambda_1|^2}, & \text{as } \arg\lambda_1 \to \frac{\pi}{2}, \\ 0, & \text{as } \arg\lambda_1 \to \pi. \end{cases}$$

Another parameter of the soliton is E. It determines the width in the sense that the bigger E, the narrower the profile of the soliton. We end our discussion on the one-soliton with several remarks.

Remark 4.2.1. The norming constant C_1 has influence only on the spatial position and the phase of the soliton.

Remark 4.2.2. Let us recall that $C_1 = \gamma_1/\alpha'(\lambda_1)$. It can be shown that for a one-soliton the scattering coefficient α is given by

$$\alpha(\lambda) = \frac{\overline{\lambda}_1}{\lambda_1}\frac{\lambda^2-\lambda_1^2}{\lambda^2-\overline{\lambda}_1^2}$$

and thus $\alpha'(\lambda_1) = -i\overline{\lambda}_1/\operatorname{Im}(\lambda_1^2)$. It follows that

$$x_0 = \frac{1}{E}\log(|\gamma_1|), \qquad \phi_0 = \arg(\gamma_1) + \frac{\pi}{2},$$

Note that in [KM77, page 197] one can find the formula $x_0 = \frac{1}{E}\log(|c_1|)$. This discrepancy is due to different notations: γ_1 in our notation is equivalent to c_1 in the notation of [KM77].

4.3 Multi-solitons

This section is devoted to the study of multi-solitons with more than one eigenvalue. In principle, in order to compute an N-soliton in the sense of Definition 4.1.1, we can use the Bäcklund transformation (3.6.4) N times to solve the corresponding RHP. But in practice, if $N \geq 2$, this will not lead to a nice formula such as (4.2.4). However, very nice numerical plots of N-solitons can be obtained with the Bäcklund transformation. As a matter of fact each plot presented in the present chapter is generated with Matlab using nothing but the Bäcklund transformation (3.6.4).

If one wants to understand multi-solitons with analytical tools, it is necessary to compute their behavior as $t \to -\infty$ and $t \to +\infty$. Surprisingly, Theorem 4.3.2 below also holds in the presence of a non-vanishing reflection coefficient $p(\lambda) \neq 0$. In the ensuing Corollary 4.3.3 we extract the statement that any multi-soliton breaks up into single solitons.

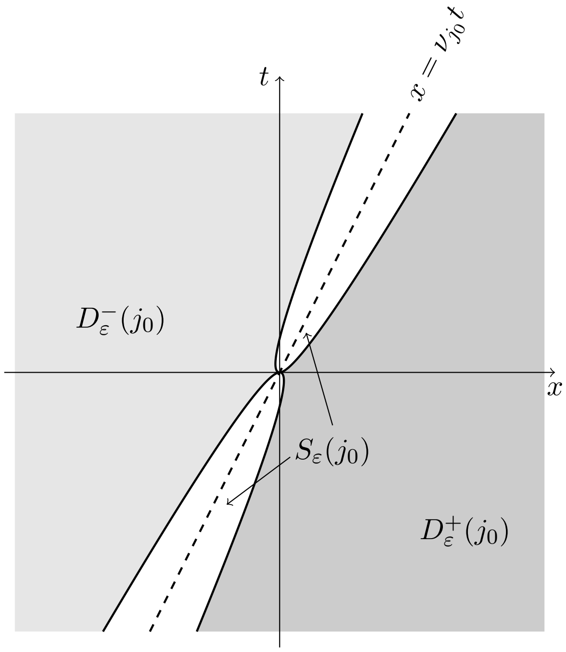


Figure 4.3: *An illustration of the partition* $\mathbb{R}^2 = D_\varepsilon^-(j_0) \cup S_\varepsilon(j_0) \cup D_\varepsilon^+(j_0)$.

Let us consider scattering data $\mathcal{S}(u,v) = (p, \{\lambda_j, C_j\}_{j=1}^N)$. We pick one eigenvalue, say λ_{j_0}, for an arbitrary $j_0 \in \{1, ..., N\}$. From the explicit formula for one-solitons, see (4.2.4), we know that, if $p \equiv 0$ and $N = 1$, then this eigenvalue would correspond to a soliton propagating at speed ν_{j_0}, where

$$\nu_{j_0} := \frac{|\lambda_{j_0}|^{-2} - |\lambda_{j_0}|^2}{|\lambda_{j_0}|^{-2} + |\lambda_{j_0}|^2}. \tag{4.3.1}$$

Furthermore, this soliton will eventually be localized in the region

$$S_\varepsilon(j_0) := \left\{(t,x) \in \mathbb{R}^2 : |x - \nu_{j_0} t| \le \sqrt{|t|}\,\varepsilon\right\}, \tag{4.3.2}$$

where $\varepsilon > 0$ can be chosen arbitrarily. The statement of Theorem 4.3.2 below is, that (up to an exponentially decaying correction term) outside of $S_\varepsilon(j_0)$, the particular eigenvalue λ_{j_0} is actually not 'visible'. Only the eigenvalues λ_j corresponding to velocities $\nu_j \neq \nu_{j_0}$ are affecting $(u(t,x), v(t,x))$ if $(t,x) \notin S_\varepsilon(j_0)$. Therefore, we define the the following index sets

$$\begin{aligned} \triangle(j_0) &:= \{j \in \{1, ..., N\} : |\lambda_j| > |\lambda_{j_0}|\}, \\ \square(j_0) &:= \{j \in \{1, ..., N\} : |\lambda_j| = |\lambda_{j_0}|\}, \\ \triangledown(j_0) &:= \{j \in \{1, ..., N\} : |\lambda_j| < |\lambda_{j_0}|\}, \\ \Lambda(j_0) &:= \triangle(j_0) \cup \triangledown(j_0). \end{aligned} \tag{4.3.3}$$

Note that $\#\Lambda(j_0) \le N - 1$. Since the case where different eigenvalues have identical absolute values is not excluded, it is possible that $\#\Lambda(j_0) < N - 1$. As it is seen in Figure 4.3, $S_\varepsilon(j_0)$ forms a narrow set. The line $\left\{(t,x) \in \mathbb{R}^2; x = \nu t\right\}$ is contained in $S_\varepsilon(j_0)$ if and only if $\nu = \nu_{j_0}$. Furthermore, $\mathbb{R}^2$ is split by $S_\varepsilon(j_0)$ into three disjoint domains. We have $\mathbb{R}^2 = D_\varepsilon^-(j_0) \cup S_\varepsilon(j_0) \cup D_\varepsilon^+(j_0)$, where

$$D_\varepsilon^-(j_0) := \left\{(t,x) \in \mathbb{R}^2 : x < \nu_{j_0} t - \sqrt{|t|}\,\varepsilon\right\}, \qquad D_\varepsilon^+(j_0) := \left\{(t,x) \in \mathbb{R}^2 : x > \nu_{j_0} t + \sqrt{|t|}\,\varepsilon\right\}. \tag{4.3.4}$$

We shall now give a technical statement to be used in the proof of Theorem 4.3.2.

Proposition 4.3.1. *Let $\lambda_{j_0} \in \mathbb{C}$ such that* $\mathrm{Im}(\lambda_{j_0}) > 0$ *and* $\mathrm{Re}(\lambda_{j_0}) < 0$. *Define the quantities* ν_{j_0} *and* $D_\varepsilon^{\pm}(j_0)$ *as in (4.3.1) and (4.3.4) and set* $w_{j_0} = \lambda_{j_0}^{-2}$. *Then there exists a* $t_0 > 0$ *such that for all* $(t,x) \in D_\varepsilon^+(j_0)$ *with* $|t| > t_0$, *we have*

$$\left| e^{ix\Theta(w_{j_0}) - itZ(w_{j_0})} \right| \le e^{-c\,\varepsilon\sqrt{|t|}}.$$

On the other hand, for all $(t,x) \in D_\varepsilon^-(j_0)$ *with* $|t| > t_0$, *we have*

$$\left| e^{-ix\Theta(w_{j_0}) + itZ(w_{j_0})} \right| \le e^{-c\,\varepsilon\sqrt{|t|}}.$$

In both inequalities the constant c *does not depend on* x *and* t.

Proof. The proof of the proposition is very technical but still elementary. Let us start with $(t,x) \in D_\varepsilon^+(j_0) \cap \{x > |t|\}$. We recall the parameters τ and w_0 from Section 3.7. By (3.7.4) we have

$$e^{ix\Theta(w_{j_0}) - itZ(w_{j_0})} = e^{i\tau\Theta\left(\frac{w_{j_0}}{w_0}\right)}.$$

We recall the definition $\Theta(s) = (s - s^{-1})/2$ and find

$$\mathrm{Re}\left(i\tau\Theta\left(\frac{w_{j_0}}{w_0}\right)\right) = -\tau\frac{\mathrm{Im}(w_{j_0})}{2}\frac{|w_{j_0}|^2 + w_0^2}{w_0|w_{j_0}|^2}.$$

Using $\tau = |t - x|w_0$ we get

$$\mathrm{Re}\left(i\tau\Theta\left(\frac{w_{j_0}}{w_0}\right)\right) = -|t-x|\frac{\mathrm{Im}(w_{j_0})}{2}\frac{|w_{j_0}|^2 + w_0^2}{|w_{j_0}|^2} < -|t-x|\frac{\mathrm{Im}(w_{j_0})}{2}.$$

If $t < 0$ and $x > |t|$, then $|t| < |t-x|/2$. Therefore, we can conclude that

$$|e^{ix\Theta(w_{j_0}) - itZ(w_{j_0})}| \le e^{-|t|\,\mathrm{Im}(w_{j_0})}, \qquad x > -t > 0.$$

Analogously, by $\tau = |t + x|/w_0$ we have

$$\mathrm{Re}\left(i\tau\Theta\left(\frac{w_{j_0}}{w_0}\right)\right) = -|t+x|\frac{\mathrm{Im}(w_{j_0})}{2}\frac{|w_{j_0}|^2 + w_0^2}{w_0^2|w_{j_0}|^2} < -|t+x|\frac{\mathrm{Im}(w_{j_0})}{2|w_{j_0}|^2}.$$

If $t > 0$ and $x > |t|$, then $|t| < |t+x|/2$. Therefore we can conclude that

$$|e^{ix\Theta(w_{j_0}) - itZ(w_{j_0})}| \le e^{-|t|\frac{\mathrm{Im}(w_{j_0})}{|w_{j_0}|^2}}, \qquad x > t > 0.$$

Now we turn to the interior region. That means that we consider $(t,x) \in D_\varepsilon^+(j_0) \cap \{x < |t|\}$. If $t > 0$, the condition $x > \nu_{j_0} t + \sqrt{|t|}\,\varepsilon$ is equivalent to

$$1 + \frac{x}{t} > 1 + \nu_{j_0} + \frac{\varepsilon}{\sqrt{t}} = \frac{2|w_{j_0}|}{|w_{j_0}| + |w_{j_0}|^{-1}} + \frac{\varepsilon}{\sqrt{t}}$$

and

$$1 - \frac{x}{t} < 1 - \nu_{j_0} - \frac{\varepsilon}{\sqrt{t}} = \frac{2|w_{j_0}|^{-1}}{|w_{j_0}| + |w_{j_0}|^{-1}} - \frac{\varepsilon}{\sqrt{t}},$$

such that

$$w_0^2 = \frac{t+x}{t-x} = \frac{1 + \frac{t}{x}}{1 - \frac{t}{x}} > \frac{\frac{2|w_{j_0}|}{|w_{j_0}|+|w_{j_0}|^{-1}} + \frac{\varepsilon}{\sqrt{t}}}{\frac{2|w_{j_0}|^{-1}}{|w_{j_0}|+|w_{j_0}|^{-1}} - \frac{\varepsilon}{\sqrt{t}}} = |w_{j_0}|^2 + \frac{\varepsilon}{\sqrt{t}}\left(\frac{1 + |w_{j_0}|^2}{\frac{2|w_{j_0}|^{-1}}{|w_{j_0}|+|w_{j_0}|^{-1}} - \frac{\varepsilon}{\sqrt{t}}}\right).$$

Thus, we can find constants $c, t_0 > 0$ depending only on $|w_{j_0}|$ such that if $t > t_0$, then

$$w_0^2 - |w_{j_0}|^2 > c\frac{\varepsilon}{\sqrt{t}}. \tag{4.3.5}$$

From (3.7.4) we know that in the domain $D_\varepsilon^+(j_0) \cap \{x < |t|\} \cap \{t > 0\}$ we have

$$ix\Theta(w_{j_0}) - itZ(w_{j_0}) = -i\tau Z\left(\frac{w_{j_0}}{w_0}\right)$$

and it is easy to compute that

$$\mathrm{Re}\left(-i\tau Z\left(\frac{w_{j_0}}{w_0}\right)\right) = -\tau\frac{\mathrm{Im}(w_{j_0})}{2}\frac{w_0^2 - |w_{j_0}|^2}{w_0|w_{j_0}|^2} = -\tau w_0\frac{\mathrm{Im}(w_{j_0})}{2|w_{j_0}|^2}\left(\frac{w_0^2 - |w_{j_0}|^2}{w_0^2}\right).$$

By (4.3.5), the factor $(w_0^2 - |w_{j_0}|^2)/w_0^2$ satisfies

$$\frac{w_0^2 - |w_{j_0}|^2}{w_0^2} > \begin{cases} \frac{1}{2}, & \text{if } w_0^2 > 2|w_{j_0}|^2 \\ \dfrac{\varepsilon}{\sqrt{t}}\dfrac{c}{2|w_{j_0}|^2}, & \text{if } w_0^2 \le 2|w_{j_0}|^2. \end{cases}$$

Using $\tau w_0 = (t + x) = t(1 + \frac{x}{t}) > t\nu_{j_0}$ we can finally estimate as follows:

$$\left|e^{ix\Theta(w_{j_0}) - itZ(w_{j_0})}\right| \le \begin{cases} e^{-t\nu_{j_0}\frac{\mathrm{Im}(w_{j_0})}{4|w_{j_0}|^2}}, & \text{if } (t,x) \in D_\varepsilon^+(j_0) \cap \{x < |t|\} \cap \{t > 0\} \text{ and } w_0^2 > 2|w_{j_0}|^2 \\ e^{-\sqrt{t}\nu_{j_0}c\frac{\mathrm{Im}(w_{j_0})}{4|w_{j_0}|^4}}, & \text{if } (t,x) \in D_\varepsilon^+(j_0) \cap \{x < |t|\} \cap \{t > 0\} \text{ and } w_0^2 \le 2|w_{j_0}|^2. \end{cases}$$

If $(t, x) \in D_\varepsilon^+(j_0) \cap \{x < |t|\} \cap \{t < 0\}$, then we have

$$ix\Theta(w_{j_0}) - itZ(w_{j_0}) = +i\tau Z\left(\frac{w_{j_0}}{w_0}\right)$$

and

$$|w_{j_0}|^2 - w_0^2 > c\frac{\varepsilon}{\sqrt{t}}.$$

Using these observations, it is possible to complete the proof of the first part of the proposition. The proof of the $D_\varepsilon^-(j_0)$ case is analogous. □

Now we state the main result of the present chapter. The meaning of this theorem in the context of multi-solitons will become clear by remarks and the corollary thereafter.

Theorem 4.3.2. *Suppose that $(u_0, v_0) \in \mathcal{G}_N$ with scattering data $\mathcal{S}(u_0, v_0) = (p; \{\lambda_k, C_k\}_{k=1}^N)$ and consider the solution (u, v) of (1.1.1) with initial data (u_0, v_0). Then for any $\varepsilon > 0$ and $j_0 \in \{1, ..., N\}$ there exist positive constants c and t_0 such that the following two statements are true.*

(i) Denote by $(u_{D_\varepsilon^+(j_0)}, v_{D_\varepsilon^+(j_0)}) \in \mathcal{G}_{\#\Lambda(j_0)}$ the solution of (1.1.1) with modified scattering data

$$(p; \{\lambda_k, C_k\}_{k \in \Lambda(j_0)}).$$

Then for all $(t, x) \in D_\varepsilon^+(j_0)$ with $|t| > t_0$ we have

$$|u(t,x) - u_{D_\varepsilon^+(j_0)}(t,x)| + |v(t,x) - v_{D_\varepsilon^+(j_0)}(t,x)| \le ce^{-c\varepsilon\sqrt{|t|}}. \tag{4.3.6}$$

(ii) Denote by $(u_{D_\varepsilon^-(j_0)}, v_{D_\varepsilon^-(j_0)})$ the solution of (1.1.1) with modified scattering data

$$(\widetilde{p}; \{\lambda_k, \widetilde{C}_k\}_{k \in \Lambda(j_0)}),$$

where

$$\widetilde{p}(\lambda) = p(\lambda) \prod_{j\in\square(j_0)} \frac{\overline{\lambda}_j^2}{\lambda_j^2}\left(\frac{\lambda^2-\lambda_j^2}{\lambda^2-\overline{\lambda}_j^2}\right)^2, \qquad \widetilde{C}_k = C_k \prod_{j\in\square(j_0)} \frac{\overline{\lambda}_j^2}{\lambda_j^2}\left(\frac{\lambda_k^2-\lambda_j^2}{\lambda_k^2-\overline{\lambda}_j^2}\right)^2, \tag{4.3.7}$$

Then for all $(t,x) \in D_\varepsilon^-(j_0)$ *with* $|t| > t_0$ *we have*

$$|u(t,x) - u_{D_\varepsilon^-(j_0)}(t,x)| + |v(t,x) - v_{D_\varepsilon^-(j_0)}(t,x)| \le ce^{-c\varepsilon\sqrt{|t|}}.$$

Proof. Instead of $(p;\{\lambda_k, C_k\}_{k=1}^N)$ let us consider the transformed scattering data $(r;\{w_k, c_k\}_{k=1}^N)$ and the corresponding solution $M(t,x;w)$ of Riemann–Hilbert problem 2.8.1. Now we define for $j_0 \in \{1,...,N\}$ a new function $\widetilde{M}$ by

$$\widetilde{M}(t,x;w) := \begin{cases} M(t,x;w)\begin{pmatrix} 1 & 0 \\ \frac{-w_j c_j e^{ix\Theta(w_j)-itZ(w_j)}}{w-w_j} & 1 \end{pmatrix}, & \text{if } |w_j| = |w_{j_0}| \text{ and } |w-w_j| < \varrho, \\ M(t,x;w)\begin{pmatrix} 1 & \frac{\overline{c}_j e^{-ix\Theta(\overline{w}_j)+itZ(\overline{w}_j)}}{w-\overline{w}_j} \\ 0 & 1 \end{pmatrix}, & \text{if } |w_j| = |w_{j_0}| \text{ and } |w-\overline{w}_j| < \varrho, \\ M(t,x;w), & \text{else.} \end{cases} \tag{4.3.8}$$

The constant ϱ is chosen sufficiently small such that the balls $B_\varrho(w_j)$ and $B_\varrho(\overline{w}_j)$ do not intersect. The new unknown $\widetilde{M}$ differs from M only in small neighborhoods of such eigenvalues w_j and $\overline{w}_j$ whose absolute values coincide with the absolute value of $|w_{j_0}|$. It is a standard computation (see for instance [Saa17a, page 465]) to show that $\widetilde{M}$ does not admit any singularities at w_j and $\overline{w}_j$ if $|w_j| = |w_{j_0}|$. In other words, these singularities are removed by the definition (4.3.8). On the other hand, the singularities at w_j and $\overline{w}_j$ with $j \in \Lambda(j_0)$ are still present. The jump on $\mathbb{R}$ is also unchanged. That is, $\widetilde{M}_+ = \widetilde{M}_-(1+R)$ for the same matrix R as in Riemann–Hilbert problem 2.8.1. But the definition (4.3.8) also produces a new jump on the contour

$$\Sigma := \bigcup_{j\in\square(j_0)} \partial B_\varrho(w_j) \cup \partial B_\varrho(\overline{w}_j).$$

For $w \in \Sigma$ let us denote by $\widetilde{M}_\pm(t,x;w)$ the limit of $\widetilde{M}(t,x;w')$ when w' approaches w from the interior/exterior of the ball $B_\varrho(w_j)$ or $B_\varrho(\overline{w}_j)$, respectively. Using this notation, we have $\widetilde{M}_+ = \widetilde{M}_-(1+\widetilde{R})$ on Σ, where

$$\widetilde{R}(t,x;w) = \begin{cases} \begin{pmatrix} 0 & 0 \\ \dfrac{-w_j c_j e^{ix\Theta(w_j)-itZ(w_j)}}{w-w_j} & 0 \end{pmatrix}, & \text{if } |w_j| = |w_{j_0}| \text{ and } |w-w_j| = \varrho, \\ \begin{pmatrix} 0 & \dfrac{\overline{c}_j e^{-ix\Theta(\overline{w}_j)+itZ(\overline{w}_j)}}{w-\overline{w}_j} \\ 0 & 0 \end{pmatrix}, & \text{if } |w_j| = |w_{j_0}| \text{ and } |w-\overline{w}_j| = \varrho. \end{cases}$$

Now let us denote by $M^\#(t,x;w)$ the solution of Riemann–Hilbert problem 2.8.1 associated with the scattering data $(r;\{w_k,c_k\}_{k\in\Lambda(j_0)})$. We want to find $E(t,x;w)$ such that

$$M^\#(t,x;w) = E(t,x;w)\widetilde{M}(t,x;w). \tag{4.3.9}$$

By the above explanations it is clear that E has to be analytic everywhere in $\mathbb{C}\setminus\Sigma$. On Σ, E has to satisfy a jump condition $E_+ = E_-(1+R^{(err)})$. Since $M^\#$ is continuous across Σ, we find the condition

$$E_+(t,x;w)\widetilde{M}_+(t,x;w) = E_-(t,x;w)\widetilde{M}_-(t,x;w),$$

which is satisfied if

$$R^{(err)}(t,x;w) = -\widetilde{M}_+(t,x;w)\widetilde{R}(t,x;w)\left[\widetilde{M}_+(t,x;w)\right]^{-1}.$$

By Remark 3.6.6 we know that $\|\widetilde{M}_+(t,x;\cdot)\|_{L^\infty(\Sigma)}$ can be bounded uniformly in x and t by a constant C that only depends on the scattering data and the constant ϱ. Thus, by Proposition 4.3.1 we can conclude that for all $(t,x) \in D_\varepsilon^+(j_0)$ with $|t| > t_0$ we have

$$\|R^{(err)}(t,x;\cdot)\|_{L^\infty(\Sigma)\cap L^1(\Sigma)} \leq Ce^{-c\varepsilon\sqrt{|t|}}.$$

From the small norm theory for RHP's (see Theorem A.1.3 in Appendix A.1) it follows that for all $(t,x) \in D_\varepsilon^+(j_0)$ with $|t| > t_0$ we have

$$|E(t,x;0) - 1| \leq Ce^{-c\varepsilon\sqrt{|t|}}. \tag{4.3.10}$$

Recalling Remark 2.8.3, we have

$$u(t,x) = [M(t,x;0)]_{21} = [\widetilde{M}(t,x;0)]_{21}.$$

Furthermore, using the notation $(u_{D_\varepsilon^+(j_0)}, v_{D_\varepsilon^+(j_0)})$ as introduced in the theorem we have

$$u_{D_\varepsilon^+(j_0)}(t,x) = [M^\#(t,x;0)]_{21}.$$

Thus, thanks to (4.3.9) and (4.3.10) we finally find for $(t,x) \in D_\varepsilon^+(j_0)$ with $|t| > t_0$,

$$\begin{aligned} |u(t,x) - u_{D_\varepsilon^+(j_0)}(t,x)| &= \left|[\widetilde{M}(t,x;0)]_{21} - [M^\#(t,x;0)]_{21}\right| \\ &= \left|[\widetilde{M}(t,x;0)]_{21} - \left[E(t,x;0)\widetilde{M}(t,x;0)\right]_{21}\right| \\ &\leq |E(t,x;0) - 1| \cdot |\widetilde{M}(t,x;0)| \\ &\leq ce^{-c\varepsilon\sqrt{|t|}}. \end{aligned}$$

Analogously, we find $|v(t,x) - v_{D_\varepsilon^+(j_0)}(t,x)| \leq ce^{-c\varepsilon\sqrt{|t|}}$, such that the first assertion of the theorem is proven.
For the proof of the second assertion we can proceed in a similar way. But first we have to define

$$D(w) = \begin{pmatrix} \prod\limits_{j\in\square(j_0)} \dfrac{w - w_j}{w - \overline{w}_j} & 0 \\ 0 & \prod\limits_{j\in\square(j_0)} \dfrac{w - \overline{w}_j}{w - w_j} \end{pmatrix}.$$

As above, let $M(t,x;w)$ be the solution of Riemann–Hilbert problem 2.8.1 subject to the transformed scattering data $(r;\{w_k,c_k\}_{k=1}^N)$. Set

$$\check{M}(t,x;w) := \begin{cases} M(t,x;w)\begin{pmatrix} 1 & \frac{w-w_j}{-w_jc_je^{ix\Theta(w_j)-itZ(w_j)}} \\ 0 & 1 \end{pmatrix} D(w), & \text{if } |w_j| = |w_{j_0}| \text{ and } |w - w_j| < \varrho, \\ M(t,x;w)\begin{pmatrix} 1 & 0 \\ \frac{w-w_j}{\overline{c}_je^{-ix\Theta(\overline{w}_j)+itZ(\overline{w}_j)}} & 1 \end{pmatrix} D(w), & \text{if } |w_j| = |w_{j_0}| \text{ and } |w - \overline{w}_j| < \varrho, \\ M(t,x;w)D(w), & \text{else.} \end{cases} \tag{4.3.11}$$

It can be checked by a straight forward computation (see once more [Saa17a, page 465]) that singularities at w_j and $\overline{w}_j$ are removed for all j such that $|w_j| = |w_{j_0}|$. The singularities at w_j and $\overline{w}_j$ for $j \in \Lambda(j_0)$ are still present and we have the following residue conditions:

$$\begin{aligned} \operatorname*{Res}_{w=w_j} \check{M}(t,x;w) &= \lim_{w\to w_j} \check{M}(t,x;w)\,[D(w)]^{-1} \begin{bmatrix} 0 & 0 \\ w_jc_je^{ix\Theta(w_j)-itZ(w_j)} & 0 \end{bmatrix} D(w), \\ \operatorname*{Res}_{w=\overline{w}_j} \check{M}(t,x;w) &= \lim_{w\to\overline{w}_j} \check{M}(t,x;w)\,[D(w)]^{-1} \begin{bmatrix} 0 & -\overline{c}_je^{-ix\Theta(\overline{w}_j)+itZ(\overline{w}_j)} \\ 0 & 0 \end{bmatrix} D(w). \end{aligned} \tag{4.3.12}$$

On $\mathbb{R}$, $\check{M}$ satisfies the jump condition

$$\check{M}_+(t,x;w) = \check{M}_-(t,x;w)\left(1 + [D(w)]^{-1}\begin{bmatrix} w|r(w)|^2 & \overline{r(w)}e^{-ix\Theta(w)+itZ(w)} \\ wr(w)e^{ix\Theta(w)-itZ(w)} & 0 \end{bmatrix}D(w)\right). \quad (4.3.13)$$

Defining new scattering data

$$(\check{r}; \{w_k, \check{c}_k\}_{k\in\Lambda(j_0)}), \quad (4.3.14)$$

where

$$\check{r}(w) = r(w)\prod_{j\in\square(j_0)}\left(\frac{w-w_j}{w-\overline{w}_j}\right)^2, \qquad \check{c}_k = c_k\prod_{j\in\square(j_0)}\left(\frac{w_k-w_j}{w_k-\overline{w}_j}\right)^2, \quad (4.3.15)$$

the residue condition (4.3.12) can be rewritten as

$$\operatorname*{Res}_{w=w_j}\check{M}(t,x;w) = \lim_{w\to w_j}\check{M}(t,x;w)\begin{bmatrix} 0 & 0 \\ w_j\check{c}_j e^{ix\Theta(w_j)-itZ(w_j)} & 0 \end{bmatrix},$$
$$\operatorname*{Res}_{w=\overline{w}_j}\check{M}(t,x;w) = \lim_{w\to \overline{w}_j}\check{M}(t,x;w)\begin{bmatrix} 0 & -\overline{\check{c}}_j e^{-ix\Theta(\overline{w}_j)+itZ(\overline{w}_j)} \\ 0 & 0 \end{bmatrix},$$

and the jump condition (4.3.13) is equivalent to

$$\check{M}_+(t,x;w) = \check{M}_-(t,x;w)\left(1 + \begin{bmatrix} w|\check{r}(w)|^2 & \overline{\check{r}(w)}e^{-ix\Theta(w)+itZ(w)} \\ w\check{r}(w)e^{ix\Theta(w)-itZ(w)} & 0 \end{bmatrix}\right). \quad (4.3.16)$$

Furthermore, from the definition of $\check{M}$ it follows that $\check{M}$ admits a discontinuity on Σ, that is $\check{M}_+ = \check{M}_-(1+\check{R})$ on Σ, where

$$\check{R}(t,x;w) = \begin{cases} [D(w)]^{-1}\begin{pmatrix} 0 & \frac{w-w_j}{-w_jc_je^{ix\Theta(w_j)-itZ(w_j)}} \\ 0 & 0 \end{pmatrix}D(w), & \text{if } |w_j| = |w_{j_0}| \text{ and } |w-w_j| = \varrho, \\ [D(w)]^{-1}\begin{pmatrix} 0 & 0 \\ \frac{w-w_j}{\overline{c}_je^{-ix\Theta(\overline{w}_j)+itZ(\overline{w}_j)}} & 0 \end{pmatrix}D(w), & \text{if } |w_j| = |w_{j_0}| \text{ and } |w-\overline{w}_j| = \varrho. \end{cases}$$

Now we proceed as in the first part of the proof: we define $M^\flat(t,x;w)$ to be the solution of Riemann–Hilbert problem 2.8.1 with scattering data (4.3.15). Hence, $\check{M}$ and $M^\flat$ satisfy identical residue conditions at w_j and $\overline{w}_j$ for $j\in\Lambda(j_0)$. Furthermore, they satisfy identical jump conditions on $\mathbb{R}$. The difference is only given by the additional discontinuity of $\check{M}$ on Σ. Thus, if we want to write

$$M^\flat(t,x;w) = E'(t,x;w)\check{M}(t,x;w), \quad (4.3.17)$$

we find that on Σ, E' has to satisfy a jump condition $E'_+ = E'_-(1+R^{(err)'})$ with

$$R^{(err)'}(t,x;w) = -\check{M}_+(t,x;w)\check{R}(t,x;w)\left[\check{M}_+(t,x;w)\right]^{-1}.$$

Since the exponential factors in the definition of $\check{R}$ are reversed, we can apply the second part of Proposition 4.3.1 which tells us that for all $(t,x)\in D^-_\varepsilon(j_0)$ with $|t|>t_0$ we have

$$\|R^{(err)'}(t,x;\cdot)\|_{L^\infty(\Sigma)\cap L^1(\Sigma)} \le ce^{-c\varepsilon\sqrt{|t|}}.$$

Hence, from the small norm theory for RHP's (see the appendix) we conclude

$$|E'(t,x;0)-1| \le ce^{-c\varepsilon\sqrt{|t|}}. \quad (4.3.18)$$

Recalling Remark 2.8.3, we have that

$$u(t,x) = [M(t,x;0)]_{21} = \prod_{j\in\square(j_0)}\frac{\overline{w}_j}{w_j}[\check{M}(t,x;0)]_{21}.$$

For

$$u^\flat(t,x) := [M^\flat(t,x;0)]_{21},$$

and thanks to (4.3.17) and (4.3.18) we finally find for $(t,x) \in D_\varepsilon^+(j_0)$ with $|t| > t_0$,

$$\begin{aligned}
\left| u(t,x) - \left(\prod_{j \in \Box(j_0)} \frac{\overline{w}_j}{w_j} \right) u^\flat(t,x) \right| &= \left| [\check{M}(t,x;0)]_{21} - [M^\flat(t,x;0)]_{21} \right| \\
&= \left| [\check{M}(t,x;0)]_{21} - \left[E'(t,x;0) \widetilde{M}(t,x;0) \right]_{21} \right| \\
&\leq |E'(t,x;0) - 1| \cdot |\check{M}(t,x;0)| \\
&\leq c e^{-c\,\varepsilon \sqrt{|t|}}.
\end{aligned}$$

Recalling the notation $u_{D_\varepsilon^-(j_0)}(t,x)$ from the theorem, it remains to show that

$$u_{D_\varepsilon^-(j_0)}(t,x) = \left(\prod_{j \in \Box(j_0)} \frac{\overline{w}_j}{w_j} \right) u^\flat(t,x). \tag{4.3.19}$$

By definition, $u^\flat(t,x)$ belongs to the scattering data $(\check{r}; \{w_k, \check{c}_k\}_{k \in \Lambda(j_0)})$. From Remark 2.9.4 and since $|\prod \frac{\overline{w}_j}{w_j}| = 1$, we know that

$$\left(\prod_{j \in \Box(j_0)} \frac{\overline{w}_j}{w_j} \right) u^\flat(t,x)$$

belongs to the scattering data $(\widetilde{r}; \{w_k, \widetilde{c}_k\}_{k \in \Lambda(j_0)})$, where

$$\widetilde{r}(w) = \check{r}(w) \prod_{j \in \Box(j_0)} \frac{\overline{w}_j}{w_j} = r(w) \prod_{j \in \Box(j_0)} \frac{\overline{w}_j}{w_j} \left(\frac{w - w_j}{w - \overline{w}_j} \right)^2,$$

and

$$\widetilde{c}_k = \check{c}_k \prod_{j \in \Box(j_0)} \frac{\overline{w}_j}{w_j} = c_k \prod_{j \in \Box(j_0)} \frac{\overline{w}_j}{w_j} \left(\frac{w_k - w_j}{w_k - \overline{w}_j} \right)^2.$$

Finally we remark that by

$$\frac{\overline{w}_j}{w_j} \left(\frac{w - w_j}{w - \overline{w}_j} \right)^2 = \frac{\overline{\lambda}_j^2}{\lambda_j^2} \left(\frac{\lambda^2 - \lambda_j^2}{\lambda^2 - \overline{\lambda}_j^2} \right)^2,$$

which holds for all complex $\lambda^2 = w^{-1}$ and $\lambda_j^2 = w_j^{-1}$, it follows that the transformed modified scattering data $(\widetilde{r}; \{w_k, \widetilde{c}_k\}_{k \in \Lambda(j_0)})$ are equivalent to the modified data $(\widetilde{p}; \{\lambda_k, \widetilde{C}_k\}_{k \in \Lambda(j_0)})$ as defined in (4.3.7). Hence, (4.3.19) is verified. Deriving $|v(t,x) - v_{D_\varepsilon^-(j_0)}(t,x)| \leq c e^{-c\,\varepsilon \sqrt{|t|}}$ in an analogous way we complete the proof of the theorem. $\Box$

In order to understand the meaning of Theorem 4.3.2, we consider the following particular case. Let $(u_0, v_0) \in \mathcal{G}_2$ with scattering data $\mathcal{S}(u_0, v_0) = (p(\lambda); \{\lambda_1, \lambda_2; C_1, C_2\})$ Furthermore we assume $|\lambda_1| < |\lambda_2|$. For both eigenvalues we can compute the quantities ν_j as in (4.3.1) and we find $\nu_1 > \nu_2$. The crucial observation is the following: for large positive t, the set $S_\varepsilon(1)$ is contained in $D_\varepsilon^+(2)$, while for large negative t it is contained in $D_\varepsilon^-(2)$. More precisely, there exists a constant $t_0 > 0$ such that

$$S_\varepsilon(1) \cap \{\pm t > t_0\} \subset D_\varepsilon^\pm(2).$$

We refer to Figure 4.4 for an illustration of the arrangements of the sets $S_\varepsilon(1)$, $D_\varepsilon^\pm(1)$, $S_\varepsilon(2)$ and $D_\varepsilon^\pm(2)$. Theorem4.3.2 applied to $j = 1$ yields:

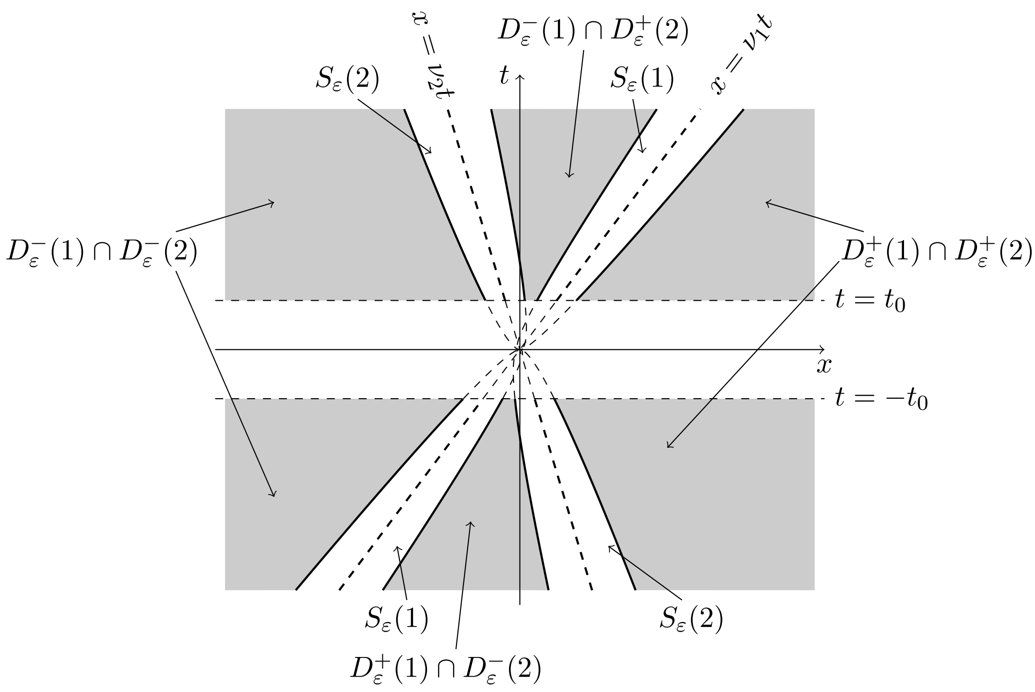


Figure 4.4: *The picture shows the situation of two eigenvalues with* $|\lambda_1| < |\lambda_2|$.

$$\begin{aligned}(x,t) \in S_\varepsilon(1), t > t_0 : &\quad (u(t,x), v(t,x)) \sim (u_{D_\varepsilon^+(2)}(t,x), v_{D_\varepsilon^+(2)}(t,x)),\\ (x,t) \in S_\varepsilon(1), t < -t_0 : &\quad (u(t,x), v(t,x)) \sim (u_{D_\varepsilon^-(2)}(t,x), v_{D_\varepsilon^-(2)}(t,x)),\end{aligned}$$

and we learn that in $S_\varepsilon(1)$, for the two cases $t \to \infty$ and $t \to -\infty$, the solution (u, v) converges to different solutions. The same holds for the set $S_\varepsilon(2)$: we have

$$S_\varepsilon(2) \cap \{\pm t > t_0\} \subset D_\varepsilon^\mp(1),$$

and by an application of Theorem 4.3.2 to the case $j = 2$ we obtain

$$\begin{aligned}(x,t) \in S_\varepsilon(2), t > t_0 : &\quad (u(t,x), v(t,x)) \sim (u_{D_\varepsilon^-(1)}(t,x), v_{D_\varepsilon^-(1)}(t,x)),\\ (x,t) \in S_\varepsilon(2), t < -t_0 : &\quad (u(t,x), v(t,x)) \sim (u_{D_\varepsilon^+(1)}(t,x), v_{D_\varepsilon^+(1)}(t,x))\end{aligned}$$

In order to describe (u, v) in the complement of $S_\varepsilon(1) \cup S_\varepsilon(2)$, we have to apply Theorem 4.3.2 twice. Let us consider for example the region $D_\varepsilon^-(1) \cap D_\varepsilon^-(2)$. Applying the theorem to the index $j = 2$, we know that

$$(x,t) \in D_\varepsilon^-(2) : (u(x,t), v(x,t)) \sim (u_{D_\varepsilon^-(2)}(t,x), v_{D_\varepsilon^-(2)}(t,x)),$$

where $(u_{D_\varepsilon^-(2)}(t,x), v_{D_\varepsilon^-(2)}(t,x)) \in \mathcal{G}_1$ belongs to scattering data

$$\left(p(\lambda) \frac{\overline{\lambda}_2^2}{\lambda_2^2} \left(\frac{\lambda^2 - \lambda_2^2}{\lambda^2 - \overline{\lambda}_2^2} \right)^2 ; \left\{ \lambda_1, C_1 \frac{\overline{\lambda}_2^2}{\lambda_2^2} \left(\frac{\lambda_1^2 - \lambda_2^2}{\lambda_1^2 - \overline{\lambda}_2^2} \right)^2 \right\} \right).$$

Now we can apply Theorem 4.3.2 again to the solution $(u_{D_\varepsilon^-(2)}(t,x), v_{D_\varepsilon^-(2)}(t,x))$ and we find that in $D_\varepsilon^-(1)$ for large $|t|$, $(u_{D_\varepsilon^-(2)}(t,x), v_{D_\varepsilon^-(2)}(t,x))$ is approximated by a pure radiation solution, that is a function in $\mathcal{G}_0$ which has only a reflection coefficient and no eigenvalues. This reflection coefficient is given by

$$p^{(--)}(\lambda) = p(\lambda) \prod_{j=1}^{2} \frac{\overline{\lambda}_j^2}{\lambda_j^2} \left(\frac{\lambda^2 - \lambda_j^2}{\lambda^2 - \overline{\lambda}_j^2} \right)^2 .$$

From this example it is clear that the procedure can be repeated for the remaining connected components of $\mathbb{R}^2 \setminus (S_\varepsilon(1) \cup S_\varepsilon(2))$. For each component one will eventually end up with a pure radiation solution whose reflection coefficient depends on the component as it is seen in the following table:

domain	reflection coefficient
$D_\varepsilon^-(1) \cap D_\varepsilon^-(2)$	$p^{(--)}(\lambda) = p(\lambda) \prod_{j=1}^{2} \frac{\overline{\lambda}_j^2}{\lambda_j^2} \left(\frac{\lambda^2 - \lambda_j^2}{\lambda^2 - \overline{\lambda}_j^2} \right)^2$
$D_\varepsilon^-(1) \cap D_\varepsilon^+(2)$	$p^{(-+)}(\lambda) = p(\lambda) \frac{\overline{\lambda}_1^2}{\lambda_1^2} \left(\frac{\lambda^2 - \lambda_1^2}{\lambda^2 - \overline{\lambda}_1^2} \right)^2$
$D_\varepsilon^+(1) \cap D_\varepsilon^-(2)$	$p^{(+-)}(\lambda) = p(\lambda) \frac{\overline{\lambda}_2^2}{\lambda_2^2} \left(\frac{\lambda^2 - \lambda_2^2}{\lambda^2 - \overline{\lambda}_2^2} \right)^2$
$D_\varepsilon^+(1) \cap D_\varepsilon^+(2)$	$p^{(++)}(\lambda) = p(\lambda)$

In the case of more than two eigenvalues, one can proceed in a similar way. We summarize our observations in the following corollary.

Corollary 4.3.3. *Suppose that $(u_0, v_0) \in \mathcal{G}_N$ with scattering data $\mathcal{S}(u_0, v_0) = (p; \{\lambda_k, C_k\}_{k=1}^N)$ and consider the solution (u, v) of (1.1.1) with initial data (u_0, v_0).*

1. *Then for any $\varepsilon > 0$ and $j_0 \in \{1, ..., N\}$, there exist positive constants c and t_0 such that the following two statements are true.*

   (i) Denote by $(u_{j_0,+}, v_{j_0,+})$ the solution of (1.1.1) with modified scattering data

$$(\widetilde{p}^+; \{\lambda_k, \widetilde{C}_k^+\}_{k \in \square(j_0)}),$$

   where

$$\widetilde{p}^+(\lambda) = p(\lambda) \prod_{j \in \nabla(j_0)} \frac{\overline{\lambda}_j^2}{\lambda_j^2} \left(\frac{\lambda^2 - \lambda_j^2}{\lambda^2 - \overline{\lambda}_j^2} \right)^2, \qquad \widetilde{C}_k^+ = C_k \prod_{j \in \nabla(j_0)} \frac{\overline{\lambda}_j^2}{\lambda_j^2} \left(\frac{\lambda_k^2 - \lambda_j^2}{\lambda_k^2 - \overline{\lambda}_j^2} \right)^2. \tag{4.3.20}$$

   Then for all $(t, x) \in S_\varepsilon(j_0)$ with $t > t_0$, we have

$$|u(t, x) - u_{j_0,+}(t, x)| + |v(t, x) - v_{j_0,+}(t, x)| \leq c e^{-c\varepsilon\sqrt{|t|}}.$$

   (ii) Denote by $(u_{j_0,-}, v_{j_0,-})$ the solution of (1.1.1) with modified scattering data

$$(\widetilde{p}^-; \{\lambda_k, \widetilde{C}_k^-\}_{k \in \square(j_0)}),$$

   where

$$\widetilde{p}^-(\lambda) = p(\lambda) \prod_{j \in \triangle(j_0)} \frac{\overline{\lambda}_j^2}{\lambda_j^2} \left(\frac{\lambda^2 - \lambda_j^2}{\lambda^2 - \overline{\lambda}_j^2} \right)^2, \qquad \widetilde{C}_k^- = C_k \prod_{j \in \triangle(j_0)} \frac{\overline{\lambda}_j^2}{\lambda_j^2} \left(\frac{\lambda_k^2 - \lambda_j^2}{\lambda_k^2 - \overline{\lambda}_j^2} \right)^2. \tag{4.3.21}$$

   Then for all $(t, x) \in S_\varepsilon(j_0)$ with $t < -t_0$, we have

$$|u(t, x) - u_{j_0,+}(t, x)| + |v(t, x) - v_{j_0,+}(t, x)| \leq c e^{-c\varepsilon\sqrt{|t|}}.$$

2. *Moreover, for each connected component D of*

$$\mathbb{R}^2 \setminus \bigcup_{j=1}^{N} S_\varepsilon(j),$$

there exists an index set $\mathcal{N}(D) \subset \{1, ..., N\}$ such that the pure radiation solution $(u_D, v_D) \in \mathcal{G}_0$ associated to the reflection coefficient

$$\widetilde{p}_D(\lambda) = p(\lambda) \prod_{j \in \mathcal{N}(D)} \frac{\overline{\lambda}_j^2}{\lambda_j^2} \left(\frac{\lambda^2 - \lambda_j^2}{\lambda^2 - \overline{\lambda}_j^2} \right)^2$$

approximates (u, v) in the sense that

$$|u(t,x) - u_D(t,x)| + |v(t,x) - v_D(t,x)| \le ce^{-c\varepsilon\sqrt{|t|}},$$

for $(t,x) \in D$ and $|t|$ sufficiently large.

Remark 4.3.4. Since for any index j_0 we have $|\nu_{j_0}| < 1$, the set $\{(x,t) \in \mathbb{R}^2 : |x| > \max\{|t|, t_0\}\}$ (which coincides with the exterior region for large t and x) is contained entirely in the complement of each $S_\varepsilon(j_0)$. As a consequence, by Corollary 4.3.3 it follows that we can limit ourself to pure radiation solutions if we want to study the long-time behavior in the exterior region.

Let us now return to solitons. We consider $\mathcal{D} = \{\lambda_j, C_j\}_{j=1}^N$ and recall the notation

$$(u_{sol}(t,x;\mathcal{D}), v_{sol}(t,x;\mathcal{D}))$$

from Definition 4.1.1. Since $p(\lambda) = 0$, the second part of Corollary 4.3.3 tells us that for all

$$(t,x) \in \mathbb{R}^2 \setminus \bigcup_{j=1}^N S_\varepsilon(j)$$

and $|t| > t_0$, we have

$$|u_{sol}(t,x;\mathcal{D})| + |v_{sol}(t,x;\mathcal{D})| \le Ce^{-c\varepsilon\sqrt{|t|}}.$$

Thus, the multi-soliton is localized in the sets $S_\varepsilon(j)$. For fixed $j_0 \in \{1, ..., N\}$ we define

$$\mathcal{D}_{j_0}^{\pm} = \{\lambda_j, C_j^{\pm}\}_{j \in \square(j_0)},$$

$$C_k^+ = C_k \prod_{j \in \nabla(j_0)} \frac{\overline{\lambda}_j^2}{\lambda_j^2} \left(\frac{\lambda_k^2 - \lambda_j^2}{\lambda_k^2 - \overline{\lambda}_j^2} \right)^2, \qquad C_k^- = C_k \prod_{j \in \triangle(j_0)} \frac{\overline{\lambda}_j^2}{\lambda_j^2} \left(\frac{\lambda_k^2 - \lambda_j^2}{\lambda_k^2 - \overline{\lambda}_j^2} \right)^2,$$

and find for $(t,x) \in S_\varepsilon(j_0)$ and $\pm t > t_0$ that

$$|u_{sol}(t,x;\mathcal{D}) - u_{sol}(t,x;\mathcal{D}_{j_0}^{\pm})| + |v_{sol}(t,x;\mathcal{D}) - v_{sol}(t,x;\mathcal{D}_{j_0}^{\pm})| \le ce^{-c\varepsilon\sqrt{|t|}}. \tag{4.3.22}$$

One might wonder if this formula yields any advantage because it expresses a multi-soliton in terms of other multi-solitons. But what is nice in (4.3.22), is the case where any two eigenvalues λ_j and λ_k have different absolute values. That is

$$|\lambda j| \neq |\lambda_k|, \quad \text{if } j \neq k. \tag{4.3.23}$$

In this case we have $\#\square(j_0) = 1$ for each $j_0 \in \{1, ..., N\}$ and thus

$$\mathcal{D}_{j_0}^{\pm} = \{\lambda_{j_0}, C_{j_0}^{\pm}\}.$$

Hence, each multi-soliton $(u_{sol}(t,x;\mathcal{D}_{j_0}^{\pm}), v_{sol}(t,x;\mathcal{D}_{j_0}^{\pm}))$ in the formula (4.3.22) is actually a single soliton for which we can use the explicit formula (4.2.4). It is then possible to rewrite (4.3.22) in the following form:

$$\begin{cases} u_{sol}(t,x;\mathcal{D}) \sim \displaystyle\sum_{j=1}^N u_{sol}(t,x;\{\lambda_j, C_j^{\pm}\}), \\ v_{sol}(t,x;\mathcal{D}) \sim \displaystyle\sum_{j=1}^N v_{sol}(t,x;\{\lambda_j, C_j^{\pm}\}), \end{cases} \quad \text{as } t \to \pm\infty. \tag{4.3.24}$$

Thus, under the assumption (4.3.23) an N-soliton breaks up into N individual solitons. This implies that an N-soliton describes the interaction of single solitons. To make this more precise, we denote by $x_{0,j}^{\pm}$ the center of the 1-soliton $(u_{sol}(t,x;\{\lambda_j,C_j^{\pm}\}),v_{sol}(t,x;\{\lambda_j,C_j^{\pm}\}))$ at time $t=0$. Thanks to (4.2.3) we have the very explicit formula

$$x_{0,j}^{\pm}=\frac{1}{E_j}\log\left(\frac{|C_j^{\pm}|}{\operatorname{Im}(\lambda_j^2)|\lambda_j|}\right).$$

The interaction of the solitons can be determined by the quantity $\Delta x_{0,j}:=x_{0,j}^{+}-x_{0,j}^{-}$. We directly find

$$\Delta x_{0,j}=\frac{1}{E_j}\log\left(\frac{|C_j^{+}|}{|C_j^{-}|}\right)=\frac{2}{E_j}\left(\sum_{k\in\nabla(j)}\log\left|\frac{\lambda_j^2-\lambda_k^2}{\lambda_j^2-\overline{\lambda}_k^2}\right|-\sum_{k\in\triangle(j)}\log\left|\frac{\lambda_j^2-\lambda_k^2}{\lambda_j^2-\overline{\lambda}_k^2}\right|\right). \tag{4.3.25}$$

In particular, if we consider the collision of two solitons corresponding to eigenvalues λ_1 and λ_2 with $|\lambda_1|<|\lambda_2|$, the shift of the centers of the two solitons is given by

$$\Delta x_{0,1}=-\frac{2}{E_1}\log\left|\frac{\lambda_1^2-\lambda_2^2}{\lambda_1^2-\overline{\lambda}_2^2}\right|,\qquad \Delta x_{0,2}=\frac{2}{E_2}\log\left|\frac{\lambda_2^2-\lambda_1^2}{\lambda_2^2-\overline{\lambda}_1^2}\right|.$$

Since for any $j\neq k$ we have

$$\left|\frac{\lambda_j^2-\lambda_k^2}{\lambda_j^2-\overline{\lambda}_k^2}\right|<1,$$

we get that $\Delta x_{0,1}>0$ and $\Delta x_{0,2}<0$ which means that the first soliton is pushed into positive x-direction whereas the second is shifted into negative x-direction. Note that the case $|\lambda_1|<|\lambda_2|$ which is equivalent to $\nu_1>\nu_2$ can be identified with to one of the following three cases:

(i) $\nu_1>\nu_2>0$: both solitons travel from the left to the right. As $t\to-\infty$ the first soliton is on the left of the second. As $t\to+\infty$ they have changed places which means that the first soliton overtook the second. In this setting, the faster soliton is shifted forward by the interaction.

(ii) $\nu_1>0>\nu_2$: the first soliton travels from the left to the right, the second from the right to the left. In the meantime they collide. Caused by the collision, the first one is shifted into positive x-direction, the second is shifted into the negative x-direction. But according to direction of travel, for both solitons the shift can be regarded as a forward shift. We refer to Figure 4.5 for an illustration of this kind of soliton interaction.

(iii) $0>\nu_1>\nu_2$: both solitons travel from the right to the left. As $t\to-\infty$ the first soliton is on the left of the second. As $t\to+\infty$ they have changed places which means that the second soliton overtook the first. Because of $|\nu_1|<|\nu_2|$ the second soliton can be regarded as the faster one. It is shifted into negative x-direction which can be interpreted as a forward shift. Thus, there is an analogy between the cases (i) and (iii).

Equation (4.3.25) can be analyzed as follows: If N solitons interact, they actually interact pairwisely. In other words, every soliton collides with all others (here, by collision we also mean overtaking manoeuvres). (4.3.25) tells us that the total soliton shift is equal to the sum of its paired collisions. Hence, there is no effect of multiparticle collisions at all. We refer to Figure 4.6 for the situation, when $N=3$. Each soliton has to interact with the two other solitons. In total, three collisions occur before the solitons continue travelling separately. The norming constants are chosen in such a way that these three collisions take place at different points in the t-x-plane. Furthermore, one can see that between two collisions each soliton behaves like a soliton after one collision.

Additionally to (4.3.25) we can also compute the phase shift $\Delta\phi_j:=\phi_j^{+}-\phi_j^{-}$, where $\phi_j^{\pm}$ denotes the phase of the soliton $(u_{sol}(t,x;\{\lambda_j,C_j^{\pm}\}),v_{sol}(t,x;\{\lambda_j,C_j^{\pm}\}))$ as given in (4.2.3). We compute

$$\Delta\phi_j=-2\sum_{k\in\nabla(j)}\arg\left(\frac{\overline{\lambda}_k}{\lambda_k}\frac{\lambda_j^2-\lambda_k^2}{\lambda_j^2-\overline{\lambda}_k^2}\right)+2\sum_{k\in\triangle(j)}\arg\left(\frac{\overline{\lambda}_k}{\lambda_k}\frac{\lambda_j^2-\lambda_k^2}{\lambda_j^2-\overline{\lambda}_k^2}\right).$$

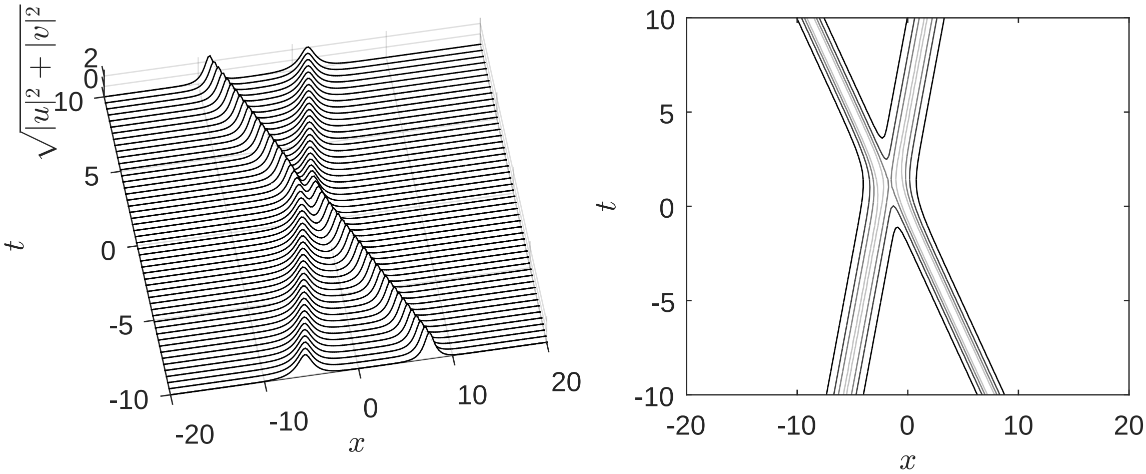


Figure 4.5: *The left frame is a plot of the amplitude* $\sqrt{|u|^2+|v|^2}$ *of the* 2*-soliton associated to scattering data* $\{\lambda_1, \lambda_2; C_1, C_2\} = \{-0.6 + 0.6i, -1.2 + 1.2i; 0.1, 1\}$. *The right frame is a contour plot of the same amplitude.*

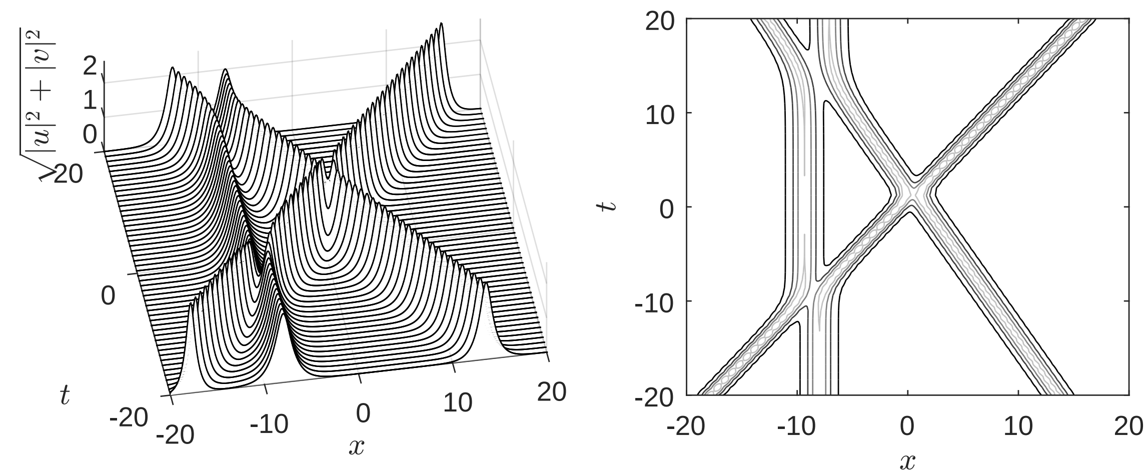


Figure 4.6: *The left frame is a plot of the amplitude* $\sqrt{|u|^2+|v|^2}$ *of the* 3*-soliton associated to scattering data* $\{\lambda_1, \lambda_2, \lambda_3; C_1, C_2, C_3\} = \{-0.4+0.4i, -\sqrt{2}+\sqrt{2}i, -1+i; 0.1, .001, 10\}$. *The right frame is a contour plot of the same amplitude.*

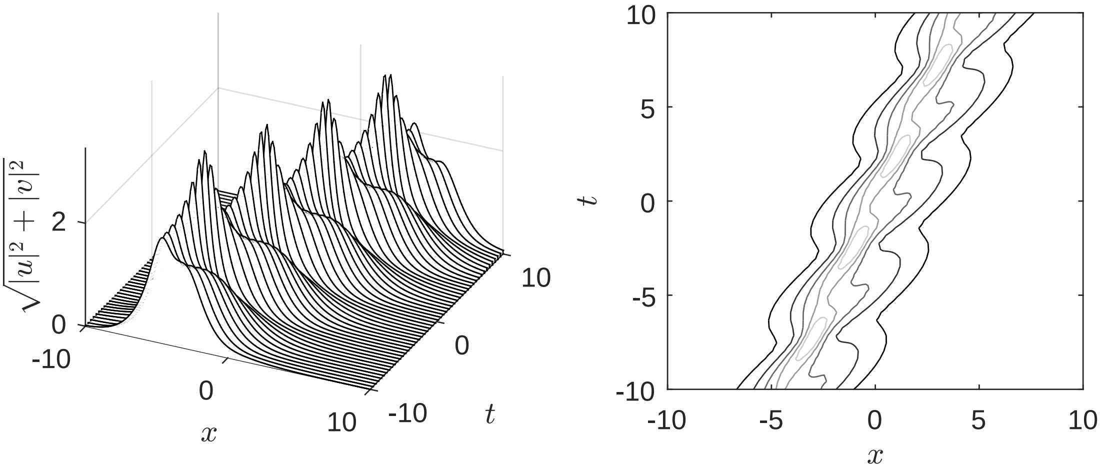


Figure 4.7: *The left frame is a plot of the amplitude* $\sqrt{|u|^2+|v|^2}$ *of the breather associated to scattering data* $\{\lambda_1, \lambda_2; C_1, C_2\} = \{-0.7538 + 0.2680i, -0.3329 + 0.7274i; 1, 1\}$. *The right frame is a contour plot of the same amplitude. Thanks to (4.4.2) we can compute the time period:* $\omega = 4.8414$

Again, we realize that the total shift is the sum of shifts that come from two-body interactions.

The conclusion of this section is the following: assuming (4.3.23), for large negative t, an N-soliton is approximated by the sum of N single solitons with different velocities. When time t varies from $-\infty$ to $+\infty$, each soliton collides with all of the others. But after $N-1$ collisions the soliton is the same as before, except for a different norming constant. In terms of physical parameters, the change of the norming constant entails a spatial shift $\Delta x_{0,j}$ and a phase shift $\Delta\phi_j$. But any other parameter appearing in the list (4.2.3) remains unchanged. In this sense we can justify that the term *soliton* as introduced in Definition 4.1.1 by requiring $p = 0$ is on par with the general physical definition of [DJ89], as reported in Section 4.1.

If we assume that $|\lambda_j| = |\lambda_k|$ for some $j \neq k$, an N-soliton does not break up into N single solitons. (4.3.22) shows that in the region $S_\varepsilon(j) = S_\varepsilon(k)$ the N-soliton is approximated by a 2-soliton. In the following section we give an answer to the question what a 2-soliton with $|\lambda_j| = |\lambda_k|$ is looking like.

4.4 Breather solutions

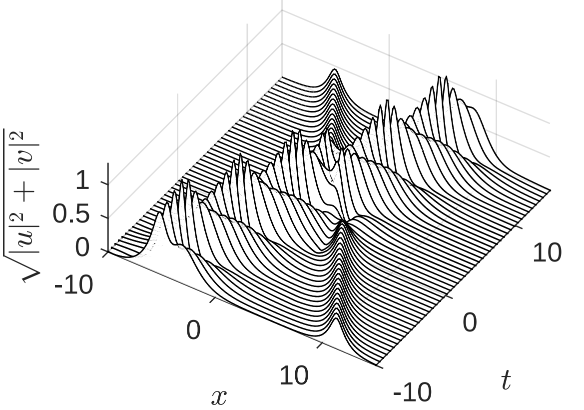
Using the Bäcklund transformation of Section 3.6 one could compute an expression for a 2-soliton with eigenvalues satisfying $|\lambda_1| = |\lambda_2|$. However, this is a very lengthy algebraic computation that we prefer to perform using a computer. See for instance Figure 4.7, where the amplitude of a 2-soliton in the case of $|\lambda_1| = |\lambda_2|$ is computed numerically. From the picture one may presume that the resulting object is periodic in time while it travels at constant speed. And indeed, one can prove that this is the general behavior of such particular 2-solitons. We do not give more details except for the following observation: one can be convinced by the Bäcklund transformation formula (3.6.4) that $|u_{sol}(t, x; \{\lambda_j, C_j\}_{j=1}^2)|$ is a rational function of the following expressions:

$$k_1\overline{k}_1, \qquad k_1\overline{k}_2, \qquad k_2\overline{k}_1, \qquad k_2\overline{k}_2, \tag{4.4.1}$$

where $k_j(t, x) = c_j e^{ix\Theta(w_j)-itZ(w_j)}$ for $j = 1$ and $j = 2$. As it known from Section 4.2, expression $k_j\overline{k}_j$ is constant along a path $x = x_0 + t\nu_j$. So, if $|w_1| = |w_2|$, then $\nu_1 = \nu_2$ and they are constant along the same pathes. In order to understand the objects $k_1\overline{k}_2$ and $k_2\overline{k}_1$, we use the following proposition.

Proposition 4.4.1. *Let* $\lambda_1, \lambda_2 \in \mathbb{C}$ *such that* $\mathrm{Re}(\lambda_j) < 0$ *and* $\mathrm{Im}(\lambda_j) > 0$ *and* $|\lambda_1| = |\lambda_2|$, *but* $\lambda_1 \neq \lambda_2$. *Let* $w_j = \lambda_j^{-2}$, *let* $A, B \in \mathbb{C}$ *and define*

$$f(t, x) = Ae^{ix\Theta(w_1)-itZ(w_1)}\overline{Be^{ix\Theta(w_2)-itZ(w_2)}}.$$



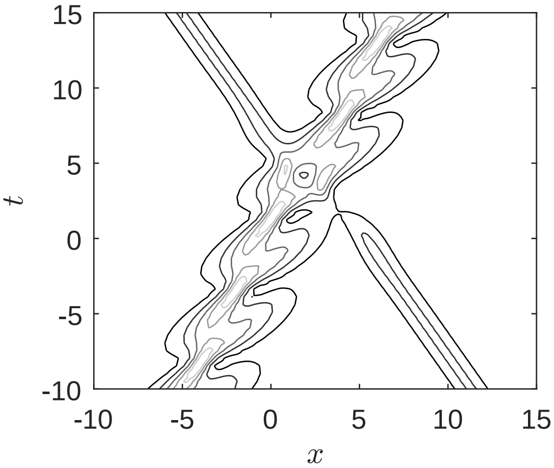


Figure 4.8: *The left frame is a plot of the amplitude* $\sqrt{|u|^2+|v|^2}$ *of the* 3*-soliton associated to scattering data* $\{\lambda_1, \lambda_2, \lambda_3; C_1, C_2, C_3\} = \{-0.7538 + 0.2680i, -0.3329 + 0.7274i, -1 + i; 1, 1, 1000\}$. *The right frame is a contour plot of the same amplitude.*

Then, for all $(t, x) \in \mathbb{R}^2$,

$$f(t, x) = f(t + \omega, x + \nu\omega),$$

where $\nu = (|\lambda_1|^{-2} - |\lambda_1|^2)/(|\lambda_1|^{-2} + |\lambda_1|^2) = (|\lambda_2|^{-2} - |\lambda_2|^2)/(|\lambda_2|^{-2} + |\lambda_2|^2)$ *and*

$$\omega = \frac{\pi(|\lambda_1|^{-4} + |\lambda_1|^4)}{|\cos(2\arg\lambda_1) - \cos(2\arg\lambda_2)|}. \tag{4.4.2}$$

Hence, along paths where $x = x_0 + \nu t$, *the function* f *is periodic in* t *with period* ω.

The proof is elementary and omitted here.
Returning to the study of the 2-soliton we conclude, that each function in (4.4.1) is periodic in t with period ω. Hence, $|u_{sol}(t, x; \{\lambda_j, C_j\}_{j=1}^2)|$ is also periodic in t with the same period.

The periodicity obtained above is the reason why multi-solitons that do not diverge, but form a bound state, are commonly called *breathers*. We finish the chapter with some remarks.

Remark 4.4.2. Assume $N = 3$ and $|\lambda_1| = |\lambda_2| = |\lambda_3|$. Then, analogously to (4.4.1) one needs to consider

$$k_1\overline{k}_1, \quad k_1\overline{k}_2, \quad k_1\overline{k}_3, \quad k_2\overline{k}_1, \quad k_2\overline{k}_2, \quad k_2\overline{k}_3, \quad k_3\overline{k}_1, \quad k_3\overline{k}_2, \quad k_3\overline{k}_3.$$

All these expressions are periodic in t, but they have different periods. Thus, the resulting breather solution is periodic if all periods are rational multiples of each other and quasi-periodic otherwise. The same can be obtained for $N \geq 4$.

Remark 4.4.3. Breathers are structurally unstable in the sense that almost every perturbation of a breather leads to scattering data that satisfy (4.3.23). Thus, the perturbed breather splits into N single solitons.

Remark 4.4.4. Clearly, we can also have the combination of single solitons and breathers. Then, Corollary 4.3.3 tells us that the single solitons interact with the breathers in the same way as they interact with solitons. See Figure 4.8, where we show how a soliton collides with a second order breather.

Chapter 5

Long-time asymptotics in the exterior region

5.1 Main result for the exterior region

In this chapter, our goal is to find the asymptotic behavior of solutions of the massive Thirring model in the exterior region $|x| > |t|$. First of all we can make use of Corollary 4.3.3 (see also Remark 4.3.4), which tells us that in the exterior region any solution converges exponentially to a pure radiation solution. For this reason it is sufficient to consider the RHP's 2.8.1 and 2.8.2 without poles ($N = 0$). Using the time evolution of the reflection coefficient (see Corollary 2.9.2) we find that the time-dependent jump matrix of Riemann–Hilbert problem 2.8.1 is given by

$$R(t,x;w) = \begin{bmatrix} w|r(w)|^2 & \overline{r(w)}e^{-ix\Theta(w)+itZ(w)} \\ wr(w)e^{ix\Theta(w)-itZ(w)} & 0 \end{bmatrix}.$$

Using the new parameters w_0 and τ as defined in (3.7.3) and the relation $-ix\Theta(w)+itZ(w) = -i\tau\Theta(w/w_0)$ for $x > |t|$, we get that

$$R(t,x;w) = \mathbf{R}_\tau(w/w_0), \tag{5.1.1}$$

where $\mathbf{R}_\tau$ is given by

$$\mathbf{R}_\tau(s) = \begin{bmatrix} \rho(s)\breve{\rho}(s) & \breve{\rho}(s)e^{-i\tau\Theta(s)} \\ \rho(s)e^{i\tau\Theta(s)} & 0 \end{bmatrix} \tag{5.1.2}$$

with

$$\rho(s) := w_0 \cdot s \cdot r(w_0 \cdot s), \qquad \breve{\rho}(s) := \overline{r(w_0 \cdot s)}. \tag{5.1.3}$$

Now, denoting by $M(t,x;w)$ the solution of Riemann–Hilbert problem 2.8.1 we can construct by

$$\mathbf{M}(\tau;s) := M(t,x;w_0 \cdot s) \tag{5.1.4}$$

a solution of the following problem for fixed w_0:

Riemann-Hilbert problem 5.1.1. For given functions $\rho, \breve{\rho}$ and $\tau \in \mathbb{R}$, find a 2×2-matrix valued function $\mathbb{C} \setminus \mathbb{R} \ni s \mapsto \mathbf{M}(\tau;s)$ which satisfies

1. $\mathbf{M}(\tau;\cdot)$ is analytic in $\mathbb{C} \setminus \mathbb{R}$.
2. $\mathbf{M}(\tau;s) = 1 + \mathcal{O}\left(\frac{1}{s}\right)$ as $|s| \to \infty$.
3. The non-tangential boundary values $\mathbf{M}_\pm(\tau;s)$ exist for $s \in \mathbb{R}$ and satisfy the jump relation

$$\mathbf{M}_+ = \mathbf{M}_-(1 + \mathbf{R}_\tau),$$

where $\mathbf{R}_\tau$ is as in (5.1.2).

It is important to note that the scaling $s \to w_0 \cdot s$ affects the reconstruction formula (2.8.5) in the following way:

$$|u(t,x)| = \left|\lim_{w\to\infty} w[M(t,x;w)]_{12}\right| = w_0 \left|\lim_{s\to\infty} s[\mathbf{M}(\tau;s)]_{12}\right|. \tag{5.1.5}$$

In order to describe the behavior of $\mathbf{M}(\tau;s)$ as $\tau \to \infty$, we need the following definitions. Firstly, for a function $\breve{\rho} \in X^{0,1}_{-2,1}$ we set:

$$\begin{aligned} \Gamma_1(\breve{\rho}) &:= \int_{-1}^{1} s^2 |\breve{\rho}'(s)| ds, \quad \Gamma_2(\breve{\rho}) := \int_{\mathbb{R}\setminus[-1,1]} |\breve{\rho}'(s)| ds, \\ \Gamma_3(\breve{\rho}) &:= \int_{-1}^{1} s|\breve{\rho}(s)| ds, \quad \Gamma_4(\breve{\rho}) := \int_{\mathbb{R}\setminus[-1,1]} s^{-1} |\breve{\rho}(s)| ds. \end{aligned} \tag{5.1.6}$$

Additionally, for $\rho, \breve{\rho} \in X^{1,1}_{-2,0}$ we define

$$\begin{aligned} \mathcal{C}(\rho,\breve{\rho}) := \Bigg\{ &\int_{-1}^{1} \frac{1}{|s|}(|\rho(s)|^2 + |\breve{\rho}(s)|^2) + |s|(|\rho'(s)|^2 + |\breve{\rho}'(s)|^2) ds \\ &+ \int_{\mathbb{R}\setminus[-1,1]} \frac{1}{|s|^2}(|\rho(s)|^2 + |\breve{\rho}(s)|^2) + (|\rho'(s)|^2 + |\breve{\rho}'(s)|^2) ds \Bigg\}^{1/2}. \end{aligned} \tag{5.1.7}$$

Our main result of the present chapter is the following:

Lemma 5.1.2. *Let $\rho \in X^{1,1}_{-3,-1}$ and $\breve{\rho} \in X^{2,2}_{-2,0}$ and denote by $\boldsymbol{M}(\tau;s)$ the solution of Riemann–Hilbert problem 5.1.1. Then, there exist positive constants ε_0 and C such that for all $\tau > 0$ satisfying*

$$\mathcal{C}(\rho,\breve{\rho})(\tau^{-1/4} + \tau^{-1/2}) < \varepsilon_0, \tag{5.1.8}$$

the following holds:

$$\left|\lim_{|s|\to\infty} s\,[\boldsymbol{M}(\tau;s)]_{12}\right| \le C|\tau|^{-1} \sum_{k=1}^{4} \Gamma_k(\breve{\rho}). \tag{5.1.9}$$

The two constants are independent of ρ, $\breve{\rho}$ and τ and could be computed explicitly.

The detailed proof of this essential lemma is presented in Section 5.3. It is based on a $\overline{\partial}$ argument and it turns out that (5.1.8) is the sufficient condition which allows us to use the $\overline{\partial}$ method. Note that the assumption $\rho \in X^{1,1}_{-3,-1}$ and $\breve{\rho} \in X^{2,2}_{-2,0}$ follows from (5.1.3) only if $r \in X^{2,2}_{-2,0}$. However, according to Corollary 2.6.5, the latter does not follow from our minimal assumption $u_0, v_0 \in H^2 \cap H^{1,1}$. As discussed in Remark 2.6.7, we need to require $u_0, v_0 \in H^{2,1}$.

In the following we present two technical propositions. Thanks to Proposition 5.1.3 we can determine for which t and x in the exterior region the technical condition (5.1.8) is fulfilled. For a better understanding of the right hand side of (5.1.9), we will use Proposition 5.1.4.

Proposition 5.1.3. *Let $r \in X^{2,2}_{-2,0}$, $w_0 \in \mathbb{R}_+$ and set $\rho(s) := w_0 \cdot s \cdot r(w_0 \cdot s)$ and $\breve{\rho}(s) := \overline{r(w_0 \cdot s)}$. Then,*

$$\mathcal{C}(\rho,\breve{\rho}) \le c \min\{\sqrt{w_0}, 1\} \|r\|_{X^{2,2}_{-2,0}}. \tag{5.1.10}$$

The proof can be found in the appendix. For a given function r we conclude from Proposition 5.1.3 that

$$\mathcal{C}(\rho,\breve{\rho}) \le C,$$

where the constant C is determined by r but is independent of w_0. It follows that there exists a positive constant τ_0 such that (5.1.8) is satisfied for all $\tau > \tau_0$. On the other hand, if $w_0 < 1$, we learn from (5.1.10) that

$$\mathcal{C}(\rho,\breve{\rho})(\tau^{-1/4} + \tau^{-1/2}) \le c\|r\|_{X^{2,2}_{-2,0}} \left(\frac{w_0^{1/4}}{\tau^{1/4}} + \frac{w_0^{1/2}}{\tau^{1/2}}\right) = c\|r\|_{X^{2,2}_{-2,0}} \left(\frac{1}{|t-x|^{1/4}} + \frac{1}{|t-x|^{1/2}}\right).$$

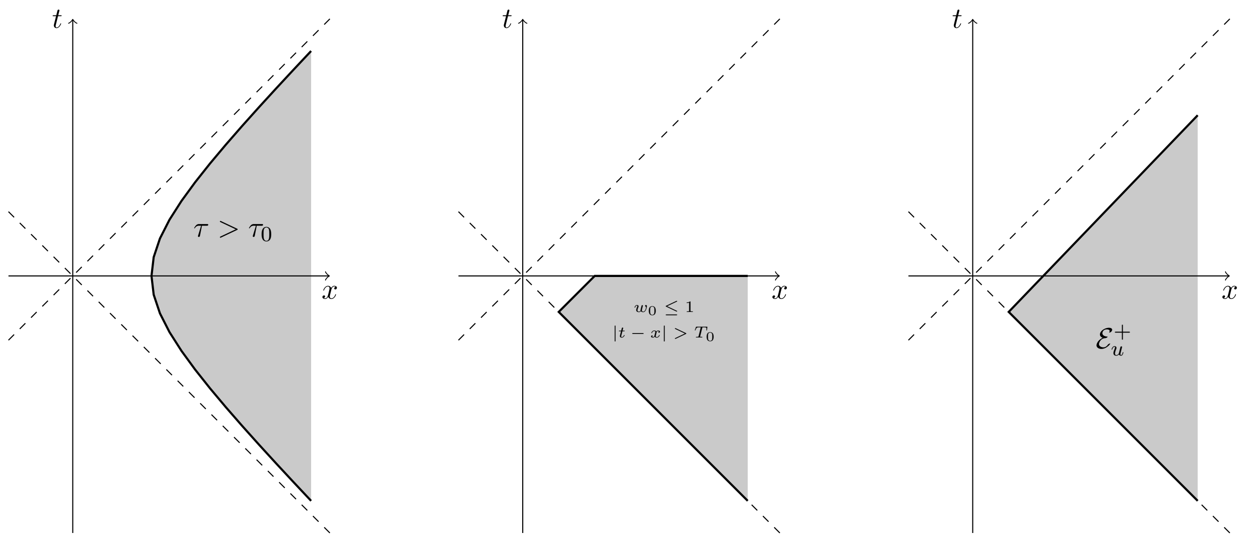


Figure 5.1: *The left and the middle graphic show the two possible regions in the exterior region, where the technical condition (5.1.8) is satisfied. The right illustration shows the set $\mathcal{E}_u^+$ which is defined in (5.1.11).*

It follows that there exists another constant T_0, such that (5.1.8) is satisfied for all $|t-x| > T_0$. In Figure 5.1, we have sketched the two possible regions in the t-x-plane where the technical condition (5.1.8) is satisfied. From this pictures it is seen directly that the union

$$\{x > |t|, \tau > \tau_0\} \cup \{x > |t|, w_0 < 1, |t-x| > T_0\}$$

contains the set

$$\mathcal{E}_u^+ := \left\{(t,x) \in \mathbb{R}^2 : x > |t|, |t-x| > T_1\right\} \tag{5.1.11}$$

with a suitable $T_1 > 0$. We continue with another technical statement:

Proposition 5.1.4. *Let $r \in X^{2,2}_{-2,0}$, $w_0 \in \mathbb{R}_+$ and set $\breve{\rho}(s) := \overline{r(w_0 \cdot s)}$. Then,*

$$\sum_{k=1}^{4} \Gamma_k(\breve{\rho}) \leq c \min\left\{1, \frac{1}{w_0^{3/2}}\right\} \|r\|_{X^{2,2}_{-2,0}}. \tag{5.1.12}$$

Again, the proof can be found in the appendix. Let us now put together the results of Lemma 5.1.2 and the above propositions. Taking into account (5.1.5), it follows from (5.1.9) and (5.1.12) that for $(t,x) \in \mathcal{E}_u^+$ we have

$$\begin{aligned} |u(t,x)| &= w_0 \left| \lim_{|s|\to\infty} s\,[\mathbf{M}(\tau;s)]_{12} \right| \\ &\leq c w_0 \tau^{-1} \min\left\{1, \frac{1}{w_0^{3/2}}\right\} \|r\|_{X^{2,2}_{-2,0}} \\ &= c \min\left\{\frac{w_0}{\tau}, \frac{1}{\tau\sqrt{w_0}}\right\} \|r\|_{X^{2,2}_{-2,0}}. \end{aligned}$$

Now we can firstly use $w_0/\tau = |t-x|^{-1}$. Secondly, by $|t-x| > T_1$ (see the definition of $\mathcal{E}_u^+$, (5.1.11)) we find that

$$\frac{1}{\tau\sqrt{w_0}} = \frac{1}{\sqrt{\tau}\sqrt{\tau w_0}} = \frac{1}{|t+x|^{1/4}|t-x|^{1/4}\sqrt{|t+x|}} \leq \frac{1}{T_1^{1/4}|t+x|^{3/4}}, \qquad (t,x) \in \mathcal{E}_u^+,$$

and conclude that

$$|u(t,x)| \le c \min\left\{|t-x|^{-1}, |t+x|^{-3/4}\right\} \|r\|_{X^{2,2}_{-2,0}}, \qquad (t,x) \in \mathcal{E}_u^+.$$

This is our final result for the part $\{x > |t|\}$ of the exterior region. For the region $\{-x > |t|\}$ we observe that by (3.7.4),

$$R(t,x;w) = \begin{bmatrix} \rho(s)\breve{\rho}(s) & \breve{\rho}(s)e^{i\tau\Theta(s)} \\ \rho(s)e^{-i\tau\Theta(s)} & 0 \end{bmatrix}, \qquad -x > |t|.$$

Note that compared to (5.1.2), the exponential factors $e^{\pm i\tau\Theta(s)}$ are replaced by $e^{\mp i\tau\Theta(s)}$. Thus, we cannot apply Lemma 5.1.2 and we need to rewrite the Riemann–Hilbert problem 2.8.1 in an equivalent form. Therefore, we recall the transformation (see (3.5.2) in Section 3.5)

$$M_d(t,x;w) := M(t,x;w) \begin{bmatrix} 1/d(w) & 0 \\ 0 & d(w) \end{bmatrix},$$

which entails the jump matrix (see (3.5.8))

$$\widetilde{R}_d(t,x;w) = \begin{bmatrix} 0 & \overline{r^{(d)}(w)}e^{-ix\Theta(w)+itZ(w)} \\ wr^{(d)}(w)e^{ix\Theta(w)-itZ(w)} & w|r^{(d)}(w)|^2 \end{bmatrix}.$$

Based on this jump matrix we can reproduce the above procedure with the final result that

$$|u(t,x)| \le c \min\left\{|t-x|^{-1}, |t+x|^{-3/4}\right\} \|r^{(d)}\|_{X^{2,2}_{-2,0}}, \qquad (t,x) \in \mathcal{E}_u^-,$$

where

$$\mathcal{E}_u^- := \left\{(t,x) \in \mathbb{R}^2 : -x > |t|, |t-x| > T_1\right\}. \tag{5.1.13}$$

If we want to study the long-time behavior of $v(t,x)$ in the exterior region we have to recall the relation

$$ix\Theta(z) + itZ(z) = \pm i\tau\Theta\left(\frac{z}{z_0}\right), \qquad \pm x > |t|.$$

Here, $\tau = \sqrt{|x^2 - t^2|}$ as above, and $z_0 = w_0^{-1}$. Hence, $w_0 \to \infty$ is equivalent to $z_0 \to 0$ and vice versa. As a consequence we shall consider $v(t,x)$ in the regions

$$\mathcal{E}_v^\pm := \left\{(t,x) \in \mathbb{R}^2 : \pm x > |t|, |t+x| > T_1\right\}. \tag{5.1.14}$$

All results of this section are summarized in the following theorem.

Theorem 5.1.5. *Suppose that $(u_0, v_0) \in \mathcal{G}_N$ with scattering data $\mathcal{S}(u_0,v_0) = (p; \{\lambda_k, C_k\}_{k=1}^N)$ and consider the solution (u,v) of (1.1.1) with initial data (u_0,v_0). Additionally, assume that the transformed reflection coefficients satisfy $r, \widehat{r} \in X^{2,2}_{-2,0}$. Then, there exists a positive number T_1 depending on (u_0,v_0) and a positive number C not depending on (u_0,v_0) such that*

$$|u(t,x)| \le C \min\left\{|t-x|^{-1}, |t+x|^{-3/4}\right\} \|r\|_{X^{2,2}_{-2,0}}, \qquad (t,x) \in \mathcal{E}_u^+ \cup \mathcal{E}_u^-, \tag{5.1.15}$$

and

$$|v(t,x)| \le C \min\left\{|t+x|^{-1}, |t-x|^{-3/4}\right\} \|\widehat{r}\|_{X^{2,2}_{-2,0}}, \qquad (t,x) \in \mathcal{E}_v^+ \cup \mathcal{E}_v^-. \tag{5.1.16}$$

Note that the number T_1 determines the sets $\mathcal{E}_u^\pm$ and $\mathcal{E}_v^\pm$.

5.2 Some remarks

Comparing Riemann–Hilbert problem 5.1.1 at $t = 0$ with Riemann–Hilbert problem 2.8.1 we observe that they are identical except for the two changes

$$x \to \tau \quad \text{and} \quad \left(wr(w), \overline{r(w)}\right) \to (\rho(s), \breve{\rho}(s)).$$

But this actually means that all estimates made in Section 3.4 can be used for the study of Riemann–Hilbert problem 5.1.1. For example, from the computations in the proof of Lemma 3.4.1 we know that

$$\begin{aligned}\lim_{|s|\to\infty} s \cdot [\mathbf{M}(\tau; s)]_{12} = &-\frac{1}{2\pi i}\int_{\mathbb{R}} \breve{\rho}(s) e^{-i\tau\Theta(s)} ds \\ &- \frac{1}{2\pi i}\int_{\mathbb{R}} \mathcal{P}^- \left[[\mathbf{M}_+(\tau;\diamond)]_{12}\rho(\diamond)e^{i\tau\Theta(\diamond)}\right](s)\, \breve{\rho}(s)e^{-i\tau\Theta(s)} ds.\end{aligned} \tag{5.2.1}$$

It follows from (3.4.10) that the second line can be estimated by

$$\tau^{-2}\|\rho\|_{L^\infty(\mathbb{R})}\|\breve{\rho}\|^2_{X^{2,1}_{-2,1}}. \tag{5.2.2}$$

However, this is not completely satisfactory for the following reason: on one hand, assuming $r \in X^{0,0}_{-2,1}$, then (5.1.3) implies that $\breve{\rho} \in X^{0,0}_{-2,1}$ for any $w_0 \in \mathbb{R}_+$. But on the other hand, the respective norm is not controllable uniformly in w_0. This can be seen by the following elementary computation using the substitution $s = w/w_0$:

$$\begin{aligned}\int_0^\infty \frac{1+s^2}{s^2}|\breve{\rho}(s)|^2 ds &\le \int_0^1 \frac{1}{s^2}|\breve{\rho}(s)|^2 ds + \int_1^\infty |\breve{\rho}(s)|^2 ds \\ &= w_0\int_0^{w_0} \frac{1}{w^2}|r(w)|^2 dw + \frac{1}{w_0}\int_{w_0}^\infty |r(w)|^2 dw.\end{aligned}$$

Thus, we have

$$\lim_{w_0\to 0}\|\breve{\rho}\|^2_{X^{0,0}_{-2,1}} = \lim_{w_0\to\infty}\|\breve{\rho}\|^2_{X^{0,0}_{-2,1}} = \infty.$$

Moreover, it is not possible to enforce the existence of these limits by simply increasing the regularity of r.

Now, let us consider the first line of (5.2.1). We have the following result.

Proposition 5.2.1. *Let $\breve{\rho} \in X^{2,1}_{-2,1}$ and $\Theta(s) = (s - s^{-1})/2$. Then*

$$\left|\int_{\mathbb{R}} \breve{\rho}(s)e^{-i\tau\Theta(s)} ds\right| \le 2|\tau|^{-1}\sum_{k=1}^4 \Gamma_k(\breve{\rho}). \tag{5.2.3}$$

Proof. Note that

$$e^{i\tau\Theta(s)} = \frac{\partial}{\partial s}\left(\frac{1}{i\tau\Theta'(s)}e^{i\tau\Theta(s)}\right) - \frac{\Theta''(s)}{i\tau(\Theta'(s))^2}e^{i\tau\Theta(s)}.$$

Integration by parts yields

$$\begin{aligned}\left|\int_0^\infty e^{i\tau\Theta(s)} f(s) ds\right| &\le \frac{1}{|\tau|}\left(\int_0^\infty \frac{1}{\Theta'(s)}|f'(s)|ds + \int_0^\infty \frac{|\Theta''(s)|}{(\Theta'(s))^2}|f(s)|ds\right) \\ &= \frac{4}{|\tau|}\left(\int_0^\infty \frac{s^2}{1+s^2}|f'(s)|ds + \int_0^\infty \frac{s}{(1+s^2)^2}|f(s)|ds\right),\end{aligned}$$

which proves the proposition. □

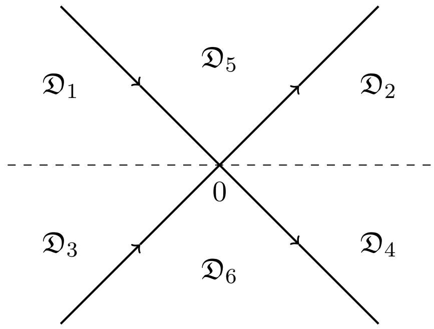


Figure 5.2: *The definition of the domains* $\mathfrak{D}_1, ..., \mathfrak{D}_6$.

It is interesting to observe that the bound (5.2.3) for the linear summand in (5.2.1) coincides with the bound of Lemma 5.1.2.

Combining the bound (5.2.2) and Proposition 5.2.1 we find that for $x > |t|$ there exists a constant C depending on r and w_0 such that

$$|u(t,x)| \le C\tau^{-1}. \tag{5.2.4}$$

But, on a path (x,t), where $t = \sqrt{x^2 - c^2}$, the parameter τ is constant by its definition and as $x \to \pm\infty$ the parameter either tends to zero or to infinity. Hence, despite the fact, that $|x|, |t| \to \infty$ on this path (as $x \to \pm\infty$), our formula (5.2.4) does not even yield an upper bound for $|u|$ in the exterior region.

This demonstrates that the $\overline{\partial}$ argument presented in the subsequent section is indeed necessary and yields significantly better results. But we should also mention that on paths where $w_0 = const.$, the bound (5.2.4) yields a decay $\mathcal{O}(\tau^{-1})$. This coincides with Theorem 5.1.5. But in contrast to the assumption $r \in X^{2,2}_{-2,0}$ of Theorem 5.1.5, the decay (5.2.4) already follows for $r \in X^{2,1}_{-2,1}$.

5.3 Proof of Lemma 5.1.2

Let $\mathbf{M}(\tau; s)$ be the solution of Riemann–Hilbert problem 5.1.1 and denote by $\mathfrak{D}_1, ..., \mathfrak{D}_6$ the domains as depicted in Figure 5.2. We define a new unknown

$$\mathbf{M}^{(1)}(\tau; s) := \mathbf{M}(\tau; s)\mathbf{W}(\tau; s), \tag{5.3.1}$$

where

$$\mathbf{W}(\tau; s) := \begin{cases} \begin{bmatrix} 1 & 0 \\ -\left(1 - \left|\frac{\mathrm{Im}(s)}{\mathrm{Re}(s)}\right|\right) \rho(\mathrm{Re}(s)) e^{i\tau\Theta(s)} & 1 \end{bmatrix}, & \text{if } s \in \mathfrak{D}_1 \cup \mathfrak{D}_2, \\ \begin{bmatrix} 1 & \left(1 - \left|\frac{\mathrm{Im}(s)}{\mathrm{Re}(s)}\right|\right) \breve{\rho}(\mathrm{Re}(s)) e^{-i\tau\Theta(s)} \\ 0 & 1 \end{bmatrix}, & \text{if } s \in \mathfrak{D}_3 \cup \mathfrak{D}_4, \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & \text{if } s \in \mathfrak{D}_5 \cup \mathfrak{D}_6. \end{cases} \tag{5.3.2}$$

For $s \in \mathbb{R}$ we can define non-tangential boundary values in the usual way

$$\mathbf{W}_\pm(\tau; s) := \lim_{\varepsilon \downarrow 0} \mathbf{W}(\tau; s \pm i\,\varepsilon).$$

It is clear from the definition that these limits are explicitly given by

$$\mathbf{W}_+(\tau; s) = \begin{bmatrix} 1 & 0 \\ -\rho(s)e^{i\tau\Theta(s)} & 1 \end{bmatrix}, \qquad \mathbf{W}_-(\tau; s) = \begin{bmatrix} 1 & \breve{\rho}(s)e^{-i\tau\Theta(s)} \\ 0 & 1 \end{bmatrix},$$

which yield

$$\mathbf{W}_-(\tau; s)\,[\mathbf{W}_+(\tau; s)]^{-1} = 1 + \mathbf{R}_\tau(s).$$

Recall that $\mathbf{R}_\tau$ is the jump matrix of the Riemann–Hilbert problem 5.1.1. By the following algebraic computation it follows that $\mathbf{M}^{(1)}$ defined in (5.3.1) is continuous on $\mathbb{R}$:

$$\mathbf{M}^{(1)}_+ = \mathbf{M}_+\mathbf{W}_+ = \mathbf{M}_-(1+\mathbf{R}_\tau)\mathbf{W}_+ = \mathbf{M}_-\mathbf{W}_- = \mathbf{M}^{(1)}_-.$$

On the boundaries where $|\operatorname{Im}(s)| = |\operatorname{Re}(s)|$, we obviously find that $\mathbf{W}$ is continuous with $\mathbf{W}(\tau; s) = 1$. Thus, $\mathbf{M}^{(1)}$ is continuous everywhere in $\mathbb{C}$. On the other hand, one cannot expect, that $\mathbf{W}$ is an analytic function. The lack of analyticity can be measured by means of the $\overline{\partial}$-operator which acts on differentiable functions $f : \mathbb{C} \to \mathbb{C}$ by

$$\overline{\partial} f(s) := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f(x+iy), \qquad s = x+iy \in \mathbb{C}.$$

Defining

$$\mathbf{Y}(\tau; s) := \overline{\partial}\, \mathbf{W}(\tau; s) \tag{5.3.3}$$

it follows by triangularity that $\mathbf{WY} = \mathbf{Y}$. Furthermore, using the analyticity of $\mathbf{M}$ which is equivalent to $\overline{\partial}\, \mathbf{M}(\tau; s) = 0$ for $s \in \mathbb{C}\setminus\mathbb{R}$, we find

$$\overline{\partial}\, \mathbf{M}^{(1)}(\tau; s) = \mathbf{M}^{(1)}(\tau; s)\mathbf{Y}(\tau; s). \tag{5.3.4}$$

Altogether we have found that $\mathbf{M}^{(1)}$ is a solution of the following problem:

$\overline{\partial}$-Problem 5.3.1. For each $\tau \in \mathbb{R}^+$, find a 2×2-matrix valued function $\mathbb{C} \ni s \mapsto \mathbf{M}^{(1)}(\tau; s)$ which satisfies

1. $\mathbf{M}^{(1)}(\tau; s)$ is continuous in $\mathbb{C}$ (with respect to the parameter s).
2. $\mathbf{M}^{(1)}(\tau; s) \to 1$ as $s \to \infty$.
3. The relation (5.3.4) is satisfied.

As it is explained for example in [AF03, Lemma 7.6.1], the above $\overline{\partial}$-problem is equivalent to

$$\mathbf{M}^{(1)}(\tau; s) = 1 + \frac{1}{\pi}\int_{\mathbb{C}} \frac{\mathbf{M}^{(1)}(\tau; k)\mathbf{Y}(\tau; k)}{k-s} dA(k), \tag{5.3.5}$$

which can be written as $\mathbf{M}^{(1)} = 1 + \mathbf{J}[\mathbf{M}^{(1)}]$, where the operator $\mathbf{J} : L^\infty(\mathbb{C}) \to L^\infty(\mathbb{C})$ is given by

$$\mathbf{J}[H](s) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{H(k)\mathbf{Y}(\tau; k)}{k-s} dA(k). \tag{5.3.6}$$

In order to solve the integral equation, we need to show that $\mathbf{J}$ is small in norm. The following proposition provides the needed estimate.

Proposition 5.3.2. *Under the same assumptions as in Lemma 5.1.2, if $\tau > 1$, then*

$$\|\boldsymbol{J}\|_{L^\infty(\mathbb{C})\to L^\infty(\mathbb{C})} \le c(\tau^{-1/4} + \tau^{-1/2})\mathcal{C}(\rho, \breve{\rho}), \tag{5.3.7}$$

where $\mathcal{C}(\rho, \breve{\rho})$ is given in (5.1.7).

Proof. In order to prove the bound (5.3.7), we have to estimate

$$\int_{\mathfrak{D}_j} \frac{|\mathbf{Y}(\tau; k)|}{|k-s|} dA(k)$$

for $j = 1, .., 4$. We give the details for $\mathfrak{D}_3$ only since the other sectors can be handled with appropriate modifications. It is easy to see that for $k = x + iy$ we have

$$\left|\bar{\partial}\left[\left(1 - \left|\frac{\mathrm{Im}(k)}{\mathrm{Re}(k)}\right|\right) \breve{\rho}(\mathrm{Re}(k))\right]\right| \leq \frac{|\breve{\rho}(x)|}{|x|} + |\breve{\rho}'(x)|. \tag{5.3.8}$$

Furthermore,

$$|e^{-i\tau\Theta(k)}| = e^{\tau y \frac{1+x^2+y^2}{x^2+y^2}},$$

such that for $y < 0$ and $\tau > 0$, the following two estimates hold

$$|e^{i\tau\Theta(k)}| \leq e^{\tau y}, \qquad |e^{i\tau\Theta(k)}| \leq e^{\frac{\tau y}{x^2+y^2}}. \tag{5.3.9}$$

Writing $s = \alpha + i\beta$ these observations yield

$$\left|\int_{\mathfrak{D}_3} \frac{|\mathbf{Y}(\tau; k)|}{|k - s|} dA(k)\right| \leq \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4,$$

where

$$\begin{aligned}
\mathcal{I}_1 &= \int_{-1}^{0}\int_{-1}^{y} \frac{|\breve{\rho}(x)| e^{\frac{\tau y}{x^2+y^2}}}{|x|\sqrt{(x-\alpha)^2 + (y-\beta)^2}} dxdy,\\
\mathcal{I}_2 &= \int_{-\infty}^{0}\int_{-\infty}^{\min\{y,-1\}} \frac{|\breve{\rho}(x)| e^{\tau y}}{|x|\sqrt{(x-\alpha)^2 + (y-\beta)^2}} dxdy,\\
\mathcal{I}_3 &= \int_{-1}^{0}\int_{-1}^{y} \frac{|\breve{\rho}'(x)| e^{\frac{\tau y}{x^2+y^2}}}{\sqrt{(x-\alpha)^2 + (y-\beta)^2}} dxdy,\\
\mathcal{I}_4 &= \int_{-\infty}^{0}\int_{-\infty}^{\min\{y,-1\}} \frac{|\breve{\rho}'(x)| e^{\tau y}}{\sqrt{(x-\alpha)^2 + (y-\beta)^2}} dxdy.
\end{aligned}$$

Before estimating $\mathcal{I}_1, ..., \mathcal{I}_4$, we present the following useful facts:

$$\begin{aligned}
\|((x-\alpha)^2 + (y-\beta)^2)^{-1/2}\|_{L^2_x(\mathbb{R})} &\leq c|y-\beta|^{-1/2}.\\
\int_{-\infty}^{0} \frac{e^{\tau y}}{\sqrt{|y-\beta|}} &\leq c\tau^{-1/2}, \quad (\tau > 0).
\end{aligned} \tag{5.3.10}$$

In both inequalities the constant c is independent of y, β and τ. The proofs of these estimates can be found for example in [CP14, eq. (3.30) and (3.33)]. For the analysis of $\mathcal{I}_1$ and $\mathcal{I}_3$ we will also need the following estimate which holds for $y < 0$:

$$\max_{x<y} \frac{1}{\sqrt{|x|}} e^{\frac{\tau y}{x^2+y^2}} \leq c\tau^{-1/4}|y|^{-1/4}. \tag{5.3.11}$$

We shall give a quick proof of this statement. For fixed $y < 0$ and $\tau > 0$ it follows that

$$\max_{x<y} \frac{1}{\sqrt{|x|}} e^{\frac{\tau y}{x^2+y^2}} \leq \max_{x\in\mathbb{R}} \frac{1}{\sqrt{|x|}} e^{\frac{\tau y}{2x^2}}.$$

Hence, we consider the function $g(x) = e^{\frac{\tau y}{2x^2}}/\sqrt{|x|}$ for which we find that $g'(x_0) = 0$ if and only if $x_0 = \pm\sqrt{-2\tau y}$. Thus, $|g(x)| \leq |g(\pm\sqrt{-2\tau y})| = c\tau^{-1/4}|y|^{-1/4}$ which proves (5.3.11).

Now we estimate $\mathcal{I}_1$. Using the Hölder inequality, the first line of (5.3.10) and (5.3.11) we find

$$\begin{aligned}
\mathcal{I}_1 &\leq \||x|^{-1/2}\breve{\rho}(x)\|_{L^2_x(-1,0)} \int_{-1}^{0} \max_{x\in[-1,y]} \left(\frac{1}{\sqrt{|x|}} e^{\frac{\tau y}{x^2+y^2}}\right) \|((x-\alpha)^2 + (y-\beta)^2)^{-1/2}\|_{L^2_x(\mathbb{R})} dy\\
&\leq c\tau^{-1/4}\||x|^{-1/2}\breve{\rho}(x)\|_{L^2_x(-1,0)} \int_{-1}^{0} |y|^{-1/4}|y-\beta|^{-1/2} dy\\
&\leq c\tau^{-1/4}\||x|^{-1/2}\breve{\rho}(x)\|_{L^2_x(-1,0)}.
\end{aligned}$$

Using the Hölder inequality once again and using the second line of (5.3.10) we obtain

$$\begin{aligned}
\mathcal{I}_2 &\leq \||x|^{-1}\breve{\rho}(x)\|_{L^2_x(-\infty,-1)} \int_{-\infty}^{0} e^{\tau y}\|((x-\alpha)^2+(y-\beta)^2)^{-1/2}\|_{L^2_x(\mathbb{R})} dy \\
&\leq c\||x|^{-1}\breve{\rho}(x)\|_{L^2_x(-\infty,-1)} \int_{-\infty}^{0} e^{\tau y}|y-\beta|^{-1/2} dy \\
&\leq c\tau^{-1/2}\||x|^{-1}\breve{\rho}(x)\|_{L^2_x(-\infty,-1)}.
\end{aligned}$$

By replacing $|x|^{-1}\breve{\rho}(x)$ with $\breve{\rho}'(x)$ we can copy the above arguments for $\mathcal{I}_1$ and $\mathcal{I}_2$ to derive

$$\mathcal{I}_3 \leq c\tau^{-1/4}\||x|^{1/2}\breve{\rho}'(x)\|_{L^2_x(-1,0)}, \qquad \mathcal{I}_4 \leq c\tau^{-1/2}\|\breve{\rho}'(x)\|_{L^2_x(-\infty,-1)}.$$

Repeating similar computations for the other domains $\mathfrak{D}_j$, we finally prove the bound (5.3.7) with the constant $\mathcal{C}(\rho,\breve{\rho})$ as defined in (5.1.7). □

The proposition just proved guarantees that the integral equation $\mathbf{M}^{(1)} = 1+\mathbf{J}[\mathbf{M}^{(1)}]$ may be inverted by Neumann series if τ is sufficiently large, say $\tau > \tau_0$. Furthermore, in this case we have

$$\|\mathbf{M}^{(1)}(\tau;\cdot)\|_{L^\infty(\mathbb{C})} \leq C$$

for all $\tau > \tau_0$. Thus, from (5.3.5) it follows that

$$\begin{aligned}
\left|\lim_{|\operatorname{Im}(s)|\to\infty} s\left[\mathbf{M}^{(1)}(\tau;s)\right]_{12}\right| &= \left|\left[\frac{1}{\pi}\int_{\mathbb{C}} \mathbf{M}^{(1)}(\tau;k)\mathbf{Y}(\tau;k)dA(k)\right]_{12}\right| \\
&\leq c\frac{1}{\pi}\int_{\mathbb{C}} |[\mathbf{Y}(\tau;k)]_2|\, dA(k) \\
&\leq c\int_{\mathfrak{D}_3\cup\mathfrak{D}_4} \left|\bar{\partial}\left[\left(1-\left|\frac{\operatorname{Im}(k)}{\operatorname{Re}(k)}\right|\right)\breve{\rho}(\operatorname{Re}(k))\right]\right| |e^{-i\tau\Theta(k)}|dA(k).
\end{aligned}$$

Here we denote by $[\mathbf{Y}]_2$ the second column of $\mathbf{Y}$ which vanishes outside the domains $\mathfrak{D}_3\cup\mathfrak{D}_4$. Since we have $\mathbf{M}(\tau;s) = \mathbf{M}^{(1)}(\tau;s)$ for $s\in\mathfrak{D}_5\cup\mathfrak{D}_6$ by definition, we conclude

$$\lim_{|\operatorname{Im}(s)|\to\infty} s\left[\mathbf{M}^{(1)}(\tau;s)\right]_{12} = \lim_{|\operatorname{Im}(s)|\to\infty} s\left[\mathbf{M}(\tau;s)\right]_{12}$$

and thus by the above computation and by (5.3.8)

$$\left|\lim_{|\operatorname{Im}(s)|\to\infty} s\left[\mathbf{M}(\tau;s)\right]_{12}\right| \leq c\int_{\mathfrak{D}_3\cup\mathfrak{D}_4}\left(\frac{|\breve{\rho}(\operatorname{Re}(k))|}{|k|} + |\breve{\rho}'(\operatorname{Re}(k))|\right)e^{-i\tau\Theta(k)}dA(k). \tag{5.3.12}$$

We finally reached the formula that will be used for the proof of Lemma 5.1.2.

Proof of Lemma 5.1.2. The proof relies on the formula (5.3.12) and we will prove estimates only for the integral over the domain $\mathfrak{D}_3$. The remaining integral over $\mathfrak{D}_4$ is handled analogously. Similar to the proof of Proposition 5.3.2 we use

$$\int_{\mathfrak{D}_3}\left(\frac{|\breve{\rho}(\operatorname{Re}(k))|}{|k|} + |\breve{\rho}'(\operatorname{Re}(k))|\right)|e^{-i\tau\Theta(k)}|dA(k) \leq \mathcal{I}_1' + \mathcal{I}_2'$$

with

$$\mathcal{I}_1' = \int_{-1}^{0}\int_{x}^{0}\left(\frac{|\breve{\rho}(x)|}{|x|} + |\breve{\rho}'(x)|\right)e^{\frac{\tau y}{x^2}}dydx,$$

and

$$\mathcal{I}_2' = \int_{-\infty}^{-1}\int_{x}^{0}\left(\frac{|\breve{\rho}(x)|}{|x|} + |\breve{\rho}'(x)|\right)e^{\tau y}dydx.$$

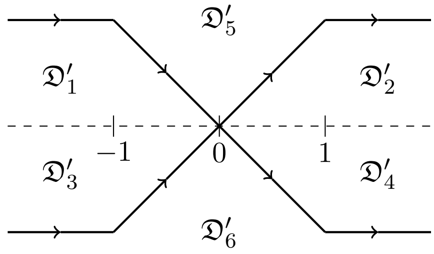


Figure 5.3: *The definition of the domains* $\mathfrak{D}_1', \ldots, \mathfrak{D}_6'$.

Using

$$\int_{-x}^{0} e^{\frac{\tau y}{x^2}} dy = \frac{x^2}{\tau} \int_{-\tau/x}^{0} e^z dz \leq c\frac{x^2}{\tau}, \tag{5.3.13}$$

we find

$$\mathcal{I}_1' \leq c\tau^{-1} \int_{-1}^{0} \left(|x||\breve{\rho}(x)| + |x|^2|\breve{\rho}'(x)|\right) dx.$$

On the other hand, by

$$\int_{x}^{0} e^{\tau y} dy \leq \frac{1}{\tau}$$

we may conclude that

$$\mathcal{I}_2' \leq c\tau^{-1} \int_{-\infty}^{-1} \left(\frac{|\breve{\rho}(x)|}{|x|} + |\breve{\rho}'(x)|\right) dx.$$

Repeating these arguments for the integral over $\mathfrak{D}_4$ we finally end up with

$$\left|\lim_{|\operatorname{Im}(s)|\to\infty} s\left[\mathbf{M}(\tau;s)\right]_{12}\right| \leq c\tau^{-1}\left\{\int_{-1}^{1}\left(|x||\breve{\rho}(x)| + |x|^2|\breve{\rho}'(x)|\right) dx + \int_{\mathbb{R}\setminus[-1,1]}\left(\frac{|\breve{\rho}(x)|}{|x|} + |\breve{\rho}'(x)|\right) dx\right\},$$

which is in turn equivalent to (5.1.9). Thus, the proof of the lemma is completed. □

5.4 Near the boundary of the light cone

Theorem 5.1.5 leaves open the question how $u(t,x)$ behaves if $|t-x| < T_1$ and $|t+x| \to \infty$. Analogously, the theorem does not provide enough information about the behaviour of $v(t,x)$ if $|t+x| < T_1$ and $|t-x| \to \infty$. Using the characteristic coordinates

$$\xi = \frac{x+t}{2}, \qquad \eta = \frac{x-t}{2},$$

we can provide an answer to this question. It is easy to see that the jump matrix of Riemann–Hilbert problem 2.8.1 can be rewritten as

$$R(t,x;w) = \begin{bmatrix} w|r(w)|^2 & \overline{r(w)}e^{-ix\Theta(w)+itZ(w)} \\ wr(w)e^{ix\Theta(w)-itZ(w)} & 0 \end{bmatrix} = \mathfrak{R}_\xi(w),$$

where

$$\mathfrak{R}_\xi(w) := \begin{bmatrix} \mathfrak{r}(w)\breve{\mathfrak{r}}(w) & \breve{\mathfrak{r}}(w)e^{i\xi/w} \\ \mathfrak{r}(w)e^{-i\xi/w} & 0 \end{bmatrix} \tag{5.4.1}$$

and

$$\mathfrak{r}(w) = wr(w)e^{i\eta w}, \qquad \breve{\mathfrak{r}}(w) = \overline{r(w)}e^{-i\eta w}. \tag{5.4.2}$$

Proceeding as in Section 5.3 we can transform the Riemann–Hilbert problem with jump matrix $\mathfrak{R}_\xi(w)$

to a $\overline{\partial}$-problem with respect to the partition of $\mathbb{C}$ as shown in Figure 5.3. The $\overline{\partial}$-problem in turn will be equivalent to the following integral equation

$$\mathfrak{M}^{(1)}(\xi;w) = 1 + \frac{1}{\pi}\int_{\mathbb{C}} \frac{\mathfrak{M}^{(1)}(\xi;k)Y(\xi;k)}{k-w}dA(k), \tag{5.4.3}$$

where

$$Y(\xi;w) := \begin{cases} \begin{bmatrix} 0 & 0 \\ -\overline{\partial}\left[\chi(w)\mathfrak{r}(\mathrm{Re}(w))\right]e^{-i\xi/w} & 0 \end{bmatrix}, & \text{if } w \in \mathfrak{D}_1' \cup \mathfrak{D}_2', \\ \begin{bmatrix} 0 & \overline{\partial}\left[\chi(w)\breve{\mathfrak{r}}(\mathrm{Re}(w))\right]e^{i\xi/w} \\ 0 & 0 \end{bmatrix}, & \text{if } w \in \mathfrak{D}_3' \cup \mathfrak{D}_4', \\ 0, & \text{if } w \in \mathfrak{D}_5' \cup \mathfrak{D}_6', \end{cases}$$

with

$$\chi(w) := \begin{cases} 1 - \frac{|\mathrm{Im}(w)|}{|\mathrm{Re}(w)|}, & \text{if } |\mathrm{Re}(w)| \le 1, \\ 1 - |\mathrm{Im}(w)|, & \text{if } |\mathrm{Re}(w)| > 1. \end{cases}$$

The solution of (5.4.3) exists if we can bound the operator $\mathfrak{J} : L^\infty(\mathbb{C}) \to L^\infty(\mathbb{C})$ defined by

$$\mathfrak{J}[H](w) := \frac{1}{\pi}\int_{\mathbb{C}} \frac{H(k)Y(\xi;k)}{k-w}dA(k).$$

Analogously to the proof of Proposition 5.3.2 we find for H supported in $\mathfrak{D}_3'$:

$$|\mathfrak{J}[H](\alpha+i\beta)| \le \mathfrak{I}_1 + \mathfrak{I}_2,$$

where for $\xi > 1$,

$$\begin{aligned} \mathfrak{I}_1 &= \int_{-\xi^{1/2}}^{0}\int_{-1}^{y} \frac{(|x|^{-1}|\breve{\mathfrak{r}}(x)| + |\breve{\mathfrak{r}}'(x)|)e^{\frac{\xi y}{x^2+y^2}}}{\sqrt{(x-\alpha)^2+(y-\beta)^2}}dxdy \\ &\le c\xi^{-1/4}\left(\||x|^{-1/2}\breve{\mathfrak{r}}(x)\|_{L^2_x(-\xi^{-1/2},0)} + \||x|^{1/2}\breve{\mathfrak{r}}'(x)\|_{L^2_x(-\xi^{-1/2},0)}\right) \\ &\le c\xi^{-1/4}\|\breve{\mathfrak{r}}\|_{X^{1,1}_{-2,0}}, \end{aligned}$$

$$\begin{aligned} \mathfrak{I}_2 &= \int_{-1}^{0}\int_{-\infty}^{-\xi^{1/2}} \frac{|\breve{\mathfrak{r}}(x)| + |\breve{\mathfrak{r}}'(x)|}{\sqrt{(x-\alpha)^2+(y-\beta)^2}}dxdy \\ &\le c\left(\|\breve{\mathfrak{r}}(x)\|_{L^2_x(-\infty,-\xi^{-1/2})} + \|\breve{\mathfrak{r}}'(x)\|_{L^2_x(-\infty,-\xi^{-1/2})}\right) \\ &\le c\xi^{-1/2}\left(\||x|\breve{\mathfrak{r}}(x)\|_{L^2_x(-\infty,-\xi^{-1/2})} + \||x|\breve{\mathfrak{r}}'(x)\|_{L^2_x(-\infty,-\xi^{-1/2})}\right) \\ &\le c\xi^{-1/2}\|\breve{\mathfrak{r}}\|_{X^{1,1}_{-2,0}}. \end{aligned}$$

Note that we have

$$|e^{-i\xi/w}| = e^{\frac{\xi y}{x^2+y^2}}, \qquad w \in \mathfrak{D}_3',$$

which is the same as in (5.3.9) for small $|x|$. But for large $|x|$ we estimate the exponential factor simply by 1. For this reason it does not appear in the definition of $\mathfrak{I}_2$. Moreover it is important to note that

$$|\overline{\partial}\chi(w)| \le \begin{cases} |w|^{-1}, & \text{if } |\mathrm{Re}(w)| \le 1, \\ 1, & \text{if } |\mathrm{Re}(w)| > 1, \end{cases}$$

with the consequence that there is no multiplication of $|\breve{\mathfrak{r}}(x)|$ by $|x|^{-1}$ in the definition of $\mathfrak{I}_2$.

This tells us that for ξ sufficiently large, the operator $(1-\mathfrak{J})$ is invertible. Thus, analogously to (5.3.12), we can conclude that

$$\left|\lim_{|\mathrm{Im}(w)|\to\infty} w\left[\mathfrak{M}^{(1)}(\xi;w)\right]_{12}\right| \le c\int_{\mathfrak{D}_3'\cup\mathfrak{D}_4'}\left(\frac{|\breve{\mathfrak{r}}(\mathrm{Re}(k))|}{|k|} + |\breve{\mathfrak{r}}'(\mathrm{Re}(k))|\right)e^{i\xi/k}dA(k).$$

The integral over $\mathfrak{D}_3'$ can be estimated by a constant times $\mathfrak{I}_1' + \mathfrak{I}_2'$, where

$$\mathfrak{I}_1' = \int_{-\xi^{1/2}}^{0} \int_{x}^{0} \left(\frac{|\breve{\mathfrak{r}}(x)|}{|x|} + |\breve{\mathfrak{r}}'(x)| \right) e^{\frac{\xi y}{x^2}} \, dydx,$$

and

$$\mathfrak{I}_2' = \int_{-\infty}^{-\xi^{1/2}} \int_{-1}^{0} |\breve{\mathfrak{r}}(x)| + |\breve{\mathfrak{r}}'(x)| dydx.$$

Using (5.3.13), we find

$$\mathfrak{I}_1' \le \xi^{-1} \int_{-\xi^{1/2}}^{0} \left(|x||\breve{\mathfrak{r}}(x)| + |x|^2 |\breve{\mathfrak{r}}'(x)| \right) dx \le \xi^{-3/4} \|\breve{\mathfrak{r}}\|_{X^{2,2}_{-2,0}}.$$

Additionally,

$$\mathfrak{I}_2' = \int_{-\infty}^{-\xi^{1/2}} |\breve{\mathfrak{r}}(x)| + |\breve{\mathfrak{r}}'(x)| dx \le c \left(\int_{-\infty}^{-\xi^{1/2}} \frac{1}{w^4} \right)^{1/2} \|\breve{\mathfrak{r}}\|_{X^{2,2}_{-2,0}} \le c\xi^{-3/4} \|\breve{\mathfrak{r}}\|_{X^{2,2}_{-2,0}}.$$

This is sufficient to conclude

$$\left| \lim_{|\operatorname{Im}(w)|\to\infty} w \left[\mathfrak{M}^{(1)}(\xi; w) \right]_{12} \right| \le c\xi^{-3/4} \|\breve{\mathfrak{r}}\|_{X^{2,2}_{-2,0}}. \tag{5.4.4}$$

Since there is no scaling as in (5.1.5), we can use this bound directly for $|u(t,x)|$. But we shall also recall that the bound also depends on η through $\breve{\mathfrak{r}}$, see (5.4.2). Moreover, as long as $|\eta| \le T_1$ for some constant T_1, we have

$$\|\breve{\mathfrak{r}}\|_{X^{2,2}_{-2,0}} \le C \|r\|_{X^{2,2}_{-2,0}}$$

for some constant C depending on T_1 only. In this sense, the bound (5.4.4) is uniform in η. Using the equivalent jump matrix (3.5.8) as in Section 3.5, we can also extend (5.4.4) to negative ξ. Additionally, the whole procedure can be reproduced for $v(t,x)$. In this case, ξ and η swap places. The following summarizes our results.

Theorem 5.4.1. *Suppose that $(u_0, v_0) \in \mathcal{G}_N$ with scattering data $\mathcal{S}(u_0, v_0) = (p; \{\lambda_k, C_k\}_{k=1}^N)$ and consider the solution (u,v) of (1.1.1) with initial data (u_0, v_0). Additionally, assume that the transformed reflection coefficients satisfy $r, \widehat{r} \in X^{2,2}_{-2,0}$. For any positive number T_1 there exists a constant C depending on (u_0, v_0) such that*

$$|u(t,x)| \le C|t+x|^{-3/4} \|r\|_{X^{2,2}_{-2,0}}, \qquad |t-x| < T_1, \tag{5.4.5}$$

and

$$|v(t,x)| \le C|t+x|^{-3/4} \|\widehat{r}\|_{X^{2,2}_{-2,0}}, \qquad |t+x| < T_1. \tag{5.4.6}$$

This theorem extends Theorem 5.1.5 in a perfect way. Combining these two theorems we obtain:

Corollary 5.4.2. *Suppose that $(u_0, v_0) \in \mathcal{G}_N$ with scattering data $\mathcal{S}(u_0, v_0) = (p; \{\lambda_k, C_k\}_{k=1}^N)$ and consider the solution (u,v) of (1.1.1) with initial data (u_0, v_0). Additionally, assume that the transformed reflection coefficients satisfy $r, \widehat{r} \in X^{2,2}_{-2,0}$. Then, there exists a positive number T_0 depending on (u_0, v_0) and a positive number C independent of (u_0, v_0) such that*

$$|u(t,x)| \le C \min\left\{ |t-x|^{-1}, |t+x|^{-3/4} \right\} \|r\|_{X^{2,2}_{-2,0}}, \qquad |x| > \max\{|t|, T_0\}$$

and

$$|v(t,x)| \le C \min\left\{ |t+x|^{-1}, |t-x|^{-3/4} \right\} \|\widehat{r}\|_{X^{2,2}_{-2,0}}, \qquad |x| > \max\{|t|, T_0\}.$$

In particular we have shown that

$$|u(t,x)| + |v(t,x)| \le C|x|^{-3/4}, \qquad |x| > \max\{|t|, T_0\}.$$

We would like to mention that this improves the result in [CL18], since therein it is shown that $|u(t,x)| + |v(t,x)| \le C|x|^{-N/2}$ under the following assumptions on the initial data:

$$\langle x \rangle^{3+N/2} u_0(x) \in H^{N+4}(\mathbb{R}), \qquad \langle x \rangle^{3+N/2} v_0(x) \in H^{N+4}(\mathbb{R}).$$

Chapter 6

Long-time asymptotics in the interior region without solitons: nonlinear steepest descent

6.1 The main results for the interior region

We recall that by (3.7.4), for $t > |x|$, the jump matrix of Riemann–Hilbert problem 2.8.1 is given by

$$R(t,x;w) = \begin{bmatrix} w|r(w)|^2 & \overline{r(w)}e^{i\tau Z(w/w_0)} \\ wr(w)e^{-i\tau Z(w/w_0)} & 0 \end{bmatrix}.$$

Using the function $d(w)$ of Proposition 3.5.2, we can define $M_d = M[d]^{-\sigma_3}$ as in (3.5.2) which solves a Riemann–Hilbert problem with jump matrix

$$\widetilde{R}_d(t,x;w) = \begin{bmatrix} 0 & \overline{r^{(d)}(w)}e^{i\tau Z(w/w_0)} \\ wr^{(d)}(w)e^{-i\tau Z(w/w_0)} & w|r^{(d)}(w)|^2 \end{bmatrix}.$$

Thus, setting

$$M^{(0)}(\tau;\zeta) := M_d(t,x;w_0\cdot\zeta), \tag{6.1.1}$$

and

$$\rho(\zeta) := \frac{w_0\cdot\zeta\cdot r(w_0\cdot\zeta)}{d_-(w_0\cdot\zeta)\cdot d_+(w_0\cdot\zeta)}, \qquad \breve{\rho}(\zeta) := \overline{r(w_0\cdot\zeta)}\cdot d_-(w_0\cdot\zeta)\cdot d_+(w_0\cdot\zeta), \tag{6.1.2}$$

we find that $M^{(0)}(\tau;\zeta)$ is a solution of the following Riemann–Hilbert problem .

Riemann-Hilbert problem 6.1.1. For given functions $\rho, \breve{\rho}$ and $\tau \in \mathbb{R}$, find a 2×2-matrix valued function $\mathbb{C}\setminus\mathbb{R} \ni \zeta \mapsto M^{(0)}(\tau;\zeta)$ which satisfies

1. $M^{(0)}(\tau;\cdot)$ is analytic in $\mathbb{C}\setminus\mathbb{R}$.
2. $M^{(0)}(\tau;\zeta) = 1 + \mathcal{O}\left(\frac{1}{\zeta}\right)$ as $|\zeta| \to \infty$.
3. The non-tangential boundary values $M^{(0)}_\pm(\tau;\zeta)$ exist for $\zeta \in \mathbb{R}$ and satisfy the jump relation

$$M^{(0)}_+ = M^{(0)}_-(1 + R^{(0)}_\tau),$$

where

$$R^{(0)}_\tau(\zeta) = \begin{bmatrix} 0 & \breve{\rho}(\zeta)e^{i\tau Z(\zeta)} \\ \rho(\zeta)e^{-i\tau Z(\zeta)} & \rho(\zeta)\breve{\rho}(\zeta) \end{bmatrix}. \tag{6.1.3}$$

Note that we can write (6.1.2) in the following equivalent way,

$$\rho(\zeta) := w_0 \cdot \zeta \cdot r^{(d)}(w_0 \cdot \zeta), \qquad \breve{\rho}(\zeta) := \overline{r^{(d)}(w_0 \cdot \zeta)},$$

where we use the notation as introduced in (3.5.7). By a slight modification of Proposition 3.5.3 we find that

$$\|r^{(d)}\|_{X^{2,2}_{-2,0}} \leq c\|r\|_{X^{2,2}_{-2,0}}. \tag{6.1.4}$$

Now, let us recall the jump matrix of Riemann–Hilbert problem 2.8.2 for $t > |x|$:

$$\widehat{R}(t, x; z) = \begin{bmatrix} 0 & -\overline{\widehat{r}(z)}e^{i\tau Z(z/z_0)} \\ -z\widehat{r}(z)e^{-i\tau Z(z/z_0)} & z|\widehat{r}(z)|^2 \end{bmatrix}.$$

It follows that

$$M^{(0)}(\tau; \zeta) := \widehat{M}(t, x, z_0 \cdot \zeta), \tag{6.1.5}$$

solves the same Riemann–Hilbert problem 6.1.1, but for

$$\rho(\zeta) := -z_0 \cdot \zeta \cdot \widehat{r}(z_0 \cdot \zeta), \qquad \breve{\rho}(\zeta) := -\overline{\widehat{r}(z_0 \cdot \zeta)}. \tag{6.1.6}$$

In what follows we assume that ρ and $\breve{\rho}$ are given either by (6.1.2), or by (6.1.6). If $r, \widehat{r} \in X^{2,2}_{-2,0}$ and $\inf_{w\in\mathbb{R}}(1+w|r(w)|^2) = \inf_{z\in\mathbb{R}}(1+z|\widehat{r}(z)|^2) \geq c_1 > 0$, then in each case we obtain the following properties for ρ and $\breve{\rho}$:

$$\rho \in X^{1,1}_{-3,-1}, \qquad \breve{\rho} \in X^{2,2}_{-2,0}, \tag{6.1.7}$$

$$\rho(\zeta)\breve{\rho}(\zeta) \in \mathbb{R} \text{ for } \zeta \in \mathbb{R}, \qquad \rho(0)\breve{\rho}(0) = 0, \tag{6.1.8}$$

$$\inf_{\zeta\in\mathbb{R}} (1 + \rho(\zeta)\breve{\rho}(\zeta)) \geq c_1 > 0. \tag{6.1.9}$$

In order to formulate the essential Lemma 6.1.2 below, it is useful to define the following functions:

$$\begin{aligned} p_1(\zeta) &:= p_4(\zeta) := \breve{\rho}(\zeta), \\ p_2(\zeta) &:= p_3(\zeta) := \frac{\rho(\zeta)}{1 + \rho(\zeta)\breve{\rho}(\zeta)}, \\ p_5(\zeta) &:= p_8(\zeta) := \rho(\zeta), \\ p_6(\zeta) &:= p_7(\zeta) := \frac{\breve{\rho}(\zeta)}{1 + \rho(\zeta)\breve{\rho}(\zeta)}. \end{aligned} \tag{6.1.10}$$

We also define the following quantity which is equivalent to $\mathcal{C}(\rho, \breve{\rho})$ as defined in (5.1.7):

$$\widetilde{\mathcal{C}}(\rho, \breve{\rho}) := \sum_{k=1}^{8} \left\{ \int_{-1/2}^{1/2} \frac{1}{|\zeta|}|p_k(\zeta)|^2 + |\zeta||p_k'(\zeta)|^2 d\zeta + \int_{\mathbb{R}\setminus[-\frac{3}{2},\frac{3}{2}]} \frac{1}{|\zeta|^2}|p_k(\zeta)|^2 + |\breve{\rho}'(\zeta)|^2 d\zeta \right\}^{1/2}. \tag{6.1.11}$$

Moreover, we recall the quantities $\Gamma_1(\rho), ..., \Gamma_4(\rho)$ from the defining equation (5.1.6) and define two further quantities:

$$\begin{aligned} \Gamma_5(p_k) &:= \left(\int_{-3/2}^{-1/2} |p_k'(\zeta)|^2 d\zeta + \int_{1/2}^{3/2} |p_k'(\zeta)|^2 d\zeta \right)^{1/2}, \\ \Gamma_6(p_k) &:= \|p_k\|_{L^\infty(-\frac{3}{2},-\frac{1}{2})} + \|p_k\|_{L^\infty(\frac{1}{2},\frac{3}{2})}. \end{aligned} \tag{6.1.12}$$

Also, it is convenient to work with the following set of definitions:

$$\nu(\zeta) := \frac{1}{2\pi} \log\left(1 + \rho(\zeta)\breve{\rho}(\zeta)\right), \qquad \nu_0^{\pm} := \nu(\pm 1), \tag{6.1.13}$$

$$\delta(\zeta) := \exp\left\{ \frac{1}{i} \int_{-1}^{1} \frac{\nu(s)}{s - \zeta} ds \right\}, \quad \zeta \in \mathbb{C} \setminus [-1, 1], \tag{6.1.14}$$

$$\delta_0^{\pm} = \exp\left\{\pm\frac{1}{i}\int_0^{\pm 1}\frac{\nu(s)\mp s\cdot\nu_0^{\pm}}{s\mp 1}ds \mp \frac{1}{i}\int_0^{\mp 1}\frac{\nu(s)}{s\mp 1}ds \mp i\nu_0^{\pm}\right\}. \tag{6.1.15}$$

Over the course of the present chapter we repeat each of the definitions (6.1.13)–(6.1.15) and explain their meanings. The integral appearing in the expression for $\delta(\zeta)$ is well-defined for $\zeta = 0$ due to (6.1.8). Hence, we may write $\delta(0)$ if needed. Our list of definitions ends with the following:

$$\mathfrak{C}(\nu) := e^{\|\nu\|_{L^\infty}} + \left\{\left(\int_{-1}^{-1/10} + \int_{1/10}^{1}\right)|\nu'(\zeta)|^2 d\zeta\right\}^{1/2} + \int_{-1}^{1}|\nu(\zeta)|d\zeta. \tag{6.1.16}$$

For the moment we need all the definitions (6.1.10)–(6.1.16) only in order to express the following result which is the main lemma for the interior region:

Lemma 6.1.2. *Let ρ and $\breve{\rho}$ satisfy the assumptions (6.1.7)–(6.1.9) and denote by $M^{(0)}(\tau;\zeta)$ the solution of Riemann–Hilbert problem 6.1.1. Then, there exist positive constants ε_0 and C such that for all $\tau > 0$ satisfying*

$$\mathfrak{C}(\nu)\left(\widetilde{C}(\rho,\breve{\rho})(\tau^{-1/4}+\tau^{-1/2}) + \sum_{k=1}^{8}(\Gamma_5(p_k)+\Gamma_6(p_k))\tau^{-1/4}\right) < \varepsilon_0, \tag{6.1.17}$$

the following holds:

$$\begin{aligned}\left|M^{(0)}(\tau;0) - [\delta(0)]^{-\sigma_3}\right| &\le c\sum_{k=1}^{8}\Gamma_6(p_k)\tau^{-1/2}\\ &+ \mathfrak{C}(\nu)\left(\sum_{k=1}^{8}\left[\Gamma_5(p_k)+\Gamma_6(p_k)\right]\tau^{-3/4} + \sum_{\substack{j\in\{2,4\}\\ 1\le k\le 8}}\Gamma_j(p_k)\tau^{-1} + \sum_{\substack{j\in\{1,3\}\\ k\in\{2,3,6,7\}}}\Gamma_j(\widetilde{p}_k)\tau^{-1}\right),\end{aligned} \tag{6.1.18}$$

where $\widetilde{p}_k(\zeta) := p_k(\zeta)/|\zeta|$. Moreover, the function $q^{(0)}(\tau)$ defined by

$$q^{(0)}(\tau) := \lim_{\zeta\to\infty}\zeta\left[M^{(0)}(\tau;\zeta)\right]_{12} \tag{6.1.19}$$

satisfies,

$$\begin{aligned}|q^{(0)}(\tau) - q^{(as)}(\tau)| \le c&\sum_{k=1}^{8}\Gamma_6(p_k)\tau^{-1}\\ \times\mathfrak{C}(\nu)&\left\{\left(\sum_{k\in\{1,4,6,7\}}\left[\Gamma_5(p_k)+\Gamma_6(p_k)\right]\tau^{-3/4} + \sum_{\substack{1\le j\le 4\\ k\in\{1,4,6,7\}}}\Gamma_j(p_k)\tau^{-1}\right)\right.\\ &\left.+\sum_{k=1}^{8}\Gamma_6(p_k)\left(\sum_{k\in\{2,3,5,8\}}\left[\Gamma_5(p_k)+\Gamma_6(p_k)\right]\tau^{-3/4} + \sum_{\substack{1\le j\le 4\\ k\in\{2,3,5,8\}}}\Gamma_j(p_k)\tau^{-1}\right)\right\},\end{aligned} \tag{6.1.20}$$

where the limit function $q^{(as)}(\tau)$ is given by

$$q^{(as)}(\tau) = \frac{e^{-i\tau}e^{i\nu_0^-\ln(\tau)}}{\tau^{1/2}}\frac{\sqrt{2\pi}e^{\pi\nu_0^-/2}e^{-i\pi/4}}{\rho(-1)(\delta_0^-)^2\Gamma(i\nu_0^-)} + \frac{e^{i\tau}e^{-i\nu_0^+\ln(\tau)}}{\tau^{1/2}}\frac{\sqrt{2\pi}e^{\pi\nu_0^+/2}e^{i\pi/4}}{\rho(1)(\delta_0^+)^2\Gamma(-i\nu_0^+)} \tag{6.1.21}$$

in the case of $\rho(\pm 1)\neq 0$. If either $\rho(-1) = 0$ or $\rho(1) = 0$, the corresponding summand in (6.1.21) has to be set to zero.

The proof of this lemma is the main part of the present chapter. For the interpretation of the lemma we firstly need to understand, for which t and x in the interior region the technical condition (6.1.17) is fulfilled. Next we shall simplify (6.1.18) and (6.1.20). For these purposes, we have the following proposition.

Proposition 6.1.3. *Let $r \in X^{2,2}_{-2,0}$ satisfy $\inf_{w\in\mathbb{R}}(1 + w|r(w)|^2) \geq c_1 > 0$ and define for $w_0 \in \mathbb{R}_+$ the functions ρ and $\breve{\rho}$ as in (6.1.2). Then,*

$$\widetilde{\mathcal{C}}(\rho, \breve{\rho}) \leq c \min\{\sqrt{w_0}, 1\} \|r\|_{X^{2,2}_{-2,0}}, \tag{6.1.22}$$

with a constant that depends on c_1 only. Furthermore, for $k \in \{1, 4, 6, 7\}$,

$$\Gamma_5(p_k) \leq c \min\left\{\sqrt{w_0}, \frac{1}{w_0^{3/2}}\right\} \|r\|_{X^{2,2}_{-2,0}}, \qquad \Gamma_6(p_k) \leq c \min\left\{w_0, \frac{1}{w_0^2}\right\} \|r\|_{X^{2,2}_{-2,0}}, \tag{6.1.23}$$

while for $k \in \{2, 3, 5, 8\}$,

$$\Gamma_5(p_k) \leq c \min\left\{\sqrt{w_0}, \frac{1}{\sqrt{w_0}}\right\} \|r\|_{X^{2,2}_{-2,0}}, \qquad \Gamma_6(p_k) \leq c \min\left\{w_0^2, \frac{1}{w_0}\right\} \|r\|_{X^{2,2}_{-2,0}}. \tag{6.1.24}$$

Again, the constant c depends on c_1 only. Finally, there exists a constant C depending on r and c_1 only, such that

$$\mathfrak{C}(\nu) \leq C, \tag{6.1.25}$$

for all $w_0 \in \mathbb{R}_+$. Each estimate of this proposition also holds if we replace r with $\widehat{r}$ and w_0 with z_0 and define ρ and $\breve{\rho}$ as in (6.1.6).

The proof of the proposition can be found in the appendix. Let us now assume that ρ and $\breve{\rho}$ are defined through r via (6.1.2). By (6.1.22), one possibility that (6.1.17) is satisfied, is

$$\tau > \tau_0 \tag{6.1.26}$$

for some sufficiently large $\tau_0 > 1$. In this case, thanks to Proposition 6.1.3, (6.1.20) can be simplified as

$$|q^{(0)}(\tau) - q^{(as)}(\tau)| \leq C \min\left\{1, \frac{1}{w_0^{3/2}}\right\} \tau^{-3/4}.$$

On the other hand, (6.1.18) becomes

$$\left|M^{(0)}(\tau; 0) - [\delta(0)]^{-\sigma_3}\right| \leq c \min\left\{\sqrt{w_0}, \frac{1}{\sqrt{w_0}}\right\} \tau^{-1/2}.$$

Setting

$$u^{(as)}(t, x) := w_0 \delta^{-1}(0) d(0) \overline{q^{(as)}(\tau)}, \tag{6.1.27}$$

using the reconstruction formula (2.8.11) and using (6.1.1), we find

$$\begin{aligned}
\left|u(t,x) - u^{(as)}(t,x)\right| &= \left|w_0[M^{(0)}(\tau;0)]_{11} d(0)\overline{q^{(0)}(\tau)} - w_0\delta^{-1}(0)d(0)\overline{q^{(as)}(\tau)}\right| \\
&\leq Cw_0|\delta^{-1}(0)| \left|\overline{q^{(0)}(\tau)} - \overline{q^{(as)}(\tau)}\right| + \left|\overline{q^{(as)}(\tau)}\right| |[M^{(0)}(\tau;0)]_{11} - \delta^{-1}(0)| \\
&\leq C \min\left\{w_0, \frac{1}{\sqrt{w_0}}\right\} \tau^{-3/4}.
\end{aligned}$$

Here, for the last inequality we have used that by (6.1.25) and Proposition 6.4.1 (iii), $|\delta^{-1}(0)|$ is bounded uniformly in w_0 and $|q^{(as)}(\tau)| \leq c w_0^{1/2}(|\nu(-1)| + |\nu(1)|)\tau^{-1/2} \leq c \min\left\{w_0^{7/2}, w_0^{-5/2}\right\} \tau^{-1/2}$, see (6.1.31) below.

If we assume that ρ and $\breve{\rho}$ are defined through $\widehat{r}$ via (6.1.6), then we can set

$$v^{(as)}(t,x) := z_0 \delta^{-1}(0) \overline{q^{(as)}(\tau)} \tag{6.1.28}$$

and find

$$\left| v(t,x) - v^{(as)}(t,x) \right| \le C \min \left\{ z_0, \frac{1}{\sqrt{z_0}} \right\} \tau^{-3/4}.$$

Using

$$\tau = |t-x| w_0 = \frac{|t+x|}{w_0} = |t+x| z_0 = \frac{|t-x|}{z_0},$$

as in Chapter 5, we can summarize our computations as follows:

Theorem 6.1.4. *Let $(u_0, v_0) \in \mathcal{G}_0$ and assume in addition that the transformed scattering coefficients satisfy $r, \widehat{r} \in X^{2,2}_{-2,0}$. Then there exist positive constants $C = C(u_0, v_0)$ and $\tau_0 = \tau_0(u_0, v_0) > 1$ such that the solution (u, v) of (1.1.1) satisfies*

$$\begin{aligned} \left| u(t,x) - u^{(as)}(t,x) \right| &\le C \min \left\{ |t-x|^{-3/4}, |t+x|^{-1/2} \right\}, \\ \left| v(t,x) - v^{(as)}(t,x) \right| &\le C \min \left\{ |t+x|^{-3/4}, |t-x|^{-1/2} \right\}, \end{aligned} \tag{6.1.29}$$

for all $\sqrt{t^2 - x^2} > \tau_0$ and $t > 0$.

Substituting (6.1.2), (6.1.6) and (6.1.21) into (6.1.27) and (6.1.28) and after some lengthy and extensive calculations (see Appendix B.4), we get that $u^{(as)}$ and $v^{(as)}$ can be rewritten as

$$\begin{aligned} u^{(as)}(t,x) &= \frac{1}{\sqrt{t-x}} \left(e^{i\tau + i|f_-(\frac{x}{t})|^2 \ln(\tau)} f_-\left(\frac{x}{t}\right) + e^{-i\tau + i|f_+(\frac{x}{t})|^2 \ln(\tau)} f_+\left(\frac{x}{t}\right) \right), \\ v^{(as)}(t,x) &= \frac{1}{\sqrt{t+x}} \left(e^{i\tau + i|f_-(\frac{x}{t})|^2 \ln(\tau)} f_-\left(\frac{x}{t}\right) - e^{-i\tau + i|f_+(\frac{x}{t})|^2 \ln(\tau)} f_+\left(\frac{x}{t}\right) \right). \end{aligned} \tag{6.1.30}$$

The functions $f_\pm$ are given by

$$\left| f_\pm \left(\frac{x}{t} \right) \right|^2 = \pm \widehat{\kappa}(\pm z_0) \tag{6.1.31}$$

and

$$\begin{aligned} \arg\left(f_\pm\left(\frac{x}{t}\right)\right) &= \mp \frac{\pi}{4} + \arg(\widehat{r}(\pm z_0)) + \arg(\Gamma(\mp i \widehat{\kappa}(\pm z_0))) \\ &\mp 2 \int_0^{\pm z_0} \frac{\widehat{\kappa}(s) \mp \frac{s}{z_0} \widehat{\kappa}(\pm z_0)}{s \mp z_0} ds \pm 2 \int_0^{\mp z_0} \frac{\widehat{\kappa}(s)}{s \mp z_0} ds \mp \widehat{\kappa}(\pm z_0) + \int_{-z_0}^{z_0} \frac{\widehat{\kappa}(s)}{s} ds, \end{aligned} \tag{6.1.32}$$

where

$$\widehat{\kappa}(z) = \frac{1}{2\pi} \log(1 + z |\widehat{r}(z)|^2). \tag{6.1.33}$$

Moreover, in (6.1.32),

$$\Gamma(k) = \int_0^\infty t^{k-1} e^{-t} dt$$

is the standard gamma function.

Let us briefly explain how one can compute the limit functions $u_-^{(as)}(t,x)$ and $v_-^{(as)}(t,x)$ if $t \to -\infty$. For this we have to use the observations of Remark 2.9.5. They tell us that

$$u_-^{(as)}(t,x) = \overline{v_+^{(as)}(-t,x)}, \qquad v_-^{(as)}(t,x) = \overline{u_+^{(as)}(-t,x)},$$

where $\overline{u_+^{(as)}}$ and $\overline{v_+^{(as)}}$ are the limit functions as $t \to +\infty$ for initial data $\tilde{u}_0(x) = \overline{v_0(x)}$ and $\tilde{v}_0(x) := \overline{u_0(x)}$. Thus, as explained in Remark 2.9.5, the formulas for $\overline{u_+^{(as)}}$ and $\overline{v_+^{(as)}}$ can be obtained by replacing $\widehat{r}$ with r in (6.1.31) in (6.1.32). After complex conjugation and replacing t with $-t$ we arrive at

$$u_-^{(as)}(t,x) = \frac{1}{\sqrt{|t-x|}}\left(e^{-i\tau - i|g_-(\frac{x}{t})|^2\ln(\tau)} g_-\left(\frac{x}{t}\right) + e^{i\tau - i|g_+(\frac{x}{t})|^2\ln(\tau)} g_+\left(\frac{x}{t}\right)\right),$$
$$v_-^{(as)}(t,x) = \frac{1}{\sqrt{|t+x|}}\left(e^{-i\tau - i|g_-(\frac{x}{t})|^2\ln(\tau)} g_-\left(\frac{x}{t}\right) - e^{i\tau - i|g_+(\frac{x}{t})|^2\ln(\tau)} g_+\left(\frac{x}{t}\right)\right),$$

where
$$|g_\pm\left(\frac{x}{t}\right)| = |f_\pm\left(\frac{x}{t}\right)|.$$

In principle, another lengthy explicit expression for $\arg(g_\pm)$ is available. However, we only mention that $\arg(g_\pm) \neq \arg(f_\pm)$.

From Proposition 6.1.3 it follows that the technical condition (6.1.17) can also be satisfied if

$$\frac{w_0}{\tau} < \varepsilon,$$

even if τ is very small. If $\tau < 1$ and $w_0 < 1$, then one can find by the above considerations that

$$|u(t,x) - u^{(as)}(t,x)| \leq c\frac{w_0}{\tau} = c\frac{1}{|t-x|}.$$

Analogously, if $\tau < 1$ and $z_0 < 1$, then

$$|v(t,x) - v^{(as)}(t,x)| \leq c\frac{z_0}{\tau} = c\frac{1}{|t+x|}.$$

Combining these observations with Corollary 5.4.2 of Chapter 5, we prove the following theorem.

Theorem 6.1.5. *Let $(u_0, v_0) \in \mathcal{G}_0$ and assume in addition that the transformed scattering coefficients satisfy $r, \widehat{r} \in X_{-2,0}^{2,2}$. Then there exist positive constants $C = C(u_0, v_0)$ and $t_0 = t_0(u_0, v_0) > 1$ such that the solution (u, v) of (1.1.1) satisfies*

$$|u(t,x)| + |v(t,x)| \leq C|t|^{-1/2}, \tag{6.1.34}$$

for all $|t| > \max\{t_0, |x|\}$.

6.2 Summary of the proof of Lemma 6.1.2

The long-time behavior results (6.1.18) and (6.1.20) are obtained through a sequence of transformations of RHP's. The initial RHP is RHP 6.1.1 above and it has contour $\mathbb{R}$ and jump matrix $R_\tau^{(0)}$. We use the notation RHP($\Sigma^{(j)}$,$R^{(j)}$) to denote the Riemann–Hilbert problem 6.1.1, where $\mathbb{R}$ is replaced by the contour $\Sigma^{(j)}$ and the jump matrix $R_\tau^{(0)}$ is replaced by $R_\tau^{(j)}$. Then, by $M^{(j)}(\tau;\zeta)$ we denote the solution of RHP($\Sigma^{(j)}$,$R_\tau^{(j)}$) and set

$$q^{(j)}(\tau) := \lim_{|\zeta|\to\infty} \zeta \cdot [M^{(j)}(\tau;\zeta)]_{12}. \tag{6.2.1}$$

The sequence of the assigned functions $q^{(j)}$ is thus determined by the sequence of pairs of contours and jump matrices which reads as follows:

$$(\mathbb{R}, R^{(0)}) \to (\mathbb{R}, R^{(1)}) \to (\Sigma^{(3)}, R^{(3)}) \begin{array}{l} \nearrow (\Sigma^{(4-)}, R^{(4-)}) \to (\Sigma^{(5-)}, R^{(5-)}) \\ \searrow (\Sigma^{(4+)}, R^{(4+)}) \to (\Sigma^{(5+)}, R^{(5+)}) \end{array}$$

The pair $(\Sigma^{(2)}, R^{(2)})$ does not appear in this schematic graph, since in the second step it is necessary to consider a mixed $\overline{\partial}$-RHP instead of a pure RHP. In what follows we give a summary of the computations without many details. We refer to the subsequent sections for full calculations.

Step 1: The first step is standard in proofs of long-time behavior of oscillatory Riemann-Hilbert problems. In order to prepare the initial Riemann-Hilbert problem RHP($\mathbb{R}$,$R_\tau^{(0)}$) for the method of steepest descent by (6.2.2) below, we first have to solve the following scalar Riemann-Hilbert problem.

Riemann-Hilbert problem 6.2.1. For given functions $\rho, \breve{\rho} \in L^2(\mathbb{R})$ with $1+\rho\breve{\rho} > 0$, find a scalar function $\mathbb{C}\setminus\mathbb{R} \ni \zeta \mapsto \delta(\zeta)$ which satisfies

1. $\delta(\zeta)$ is analytic in $\mathbb{C}\setminus\mathbb{R}$.
2. $\delta(\zeta) = 1 + \mathcal{O}\left(\zeta^{-1}\right)$ as $|\zeta| \to \infty$.
3. The non-tangential boundary values $\delta_\pm(\zeta)$ exist for $\zeta \in \mathbb{R}$ and satisfy the jump relation

$$\delta_+(\zeta) = \begin{cases} \delta_-(\zeta), & \zeta \in \mathbb{R}\setminus[-1,1], \\ \delta_-(\zeta)\left(1+\rho(\zeta)\breve{\rho}(\zeta)\right), & \zeta \in [-1,1]. \end{cases}$$

In Section 6.4.1 we list properties of δ and give details about the solvability of the scalar Riemann-Hilbert problem 6.2.1. In particular, we find an explicit solution formula for $\delta(\zeta)$, see (6.4.2). The function δ is finally used to define the following transfomation:

$$M^{(1)}(\tau;\zeta) := M^{(0)}(\tau;\zeta)[\delta(\zeta)]^{\sigma_3}. \tag{6.2.2}$$

It is easy to verify that we obtain a solution of a new Riemann-Hilbert problem RHP($\mathbb{R}$,$R_\tau^{(1)}$), where

$$R_\tau^{(1)}(\zeta) = \begin{cases} \begin{bmatrix} 0 & \breve{\rho}\delta^{-2}e^{i\tau Z} \\ \rho\delta^{+2}e^{-i\tau Z} & \rho\breve{\rho} \end{bmatrix}, & \text{if } |\zeta| \geq 1, \\ \begin{bmatrix} \rho\breve{\rho} & \frac{\breve{\rho}\delta_-^{-2}}{1+\rho\breve{\rho}}e^{i\tau Z} \\ \frac{\rho\delta_+^2}{1+\rho\breve{\rho}}e^{-i\tau Z} & 0 \end{bmatrix}, & \text{if } |\zeta| < 1. \end{cases} \tag{6.2.3}$$

Since the factor $[\delta]^{\sigma_3}$ is diagonal, the manipulation (6.2.2) does not affect the reconstruction formula (6.2.1) and we have $q^{(1)}(\tau) = q^{(0)}(\tau)$ and moreover, $M^{(1)}(\tau;0) = M^{(0)}(\tau;0)[\delta(0)]^{\sigma_3}$.

Step 2: The next transformation deforms the contour $\mathbb{R}$ to a new contour $\Sigma^{(2)}$, a picture of which is given in Figure 6.1. The transformation is based on the fact that $R_\tau^{(1)}$ defined in (6.2.3) admits a factorisation of the form

$$1 + R_\tau^{(1)} = (1 + R_L^{(1)})(1 + R_R^{(1)}), \tag{6.2.4}$$

where

$$(R_L^{(1)}, R_R^{(1)}) = \begin{cases} \left(\begin{bmatrix} 0 & 0 \\ \rho\delta^2 e^{-i\tau Z} & 0 \end{bmatrix}, \begin{bmatrix} 0 & \breve{\rho}\delta^{-2}e^{i\tau Z} \\ 0 & 0 \end{bmatrix} \right), & \text{if } |\zeta| \geq 1, \\ \left(\begin{bmatrix} 0 & \frac{\breve{\rho}}{1+\rho\breve{\rho}}\delta_-^{-2}e^{i\tau Z} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ \frac{\rho}{1+\rho\breve{\rho}}\delta_+^2 e^{-i\tau Z} & 0 \end{bmatrix} \right), & \text{if } |\zeta| < 1. \end{cases} \tag{6.2.5}$$

As we specify in Section 6.4.2, there exists a matrix-valued function $\zeta \to \mathcal{W}(\tau;\zeta)$ of the form (6.4.12), which is continuous on $\mathbb{C}\setminus(\mathbb{R}\cup\Sigma^{(2)})$ and satisfies for $\zeta \in \mathbb{R}$ the following:

$$\begin{aligned} \mathcal{W}_+ &= 1 - R_R^{(1)}, \\ \mathcal{W}_- &= 1 + R_L^{(1)}. \end{aligned} \tag{6.2.6}$$

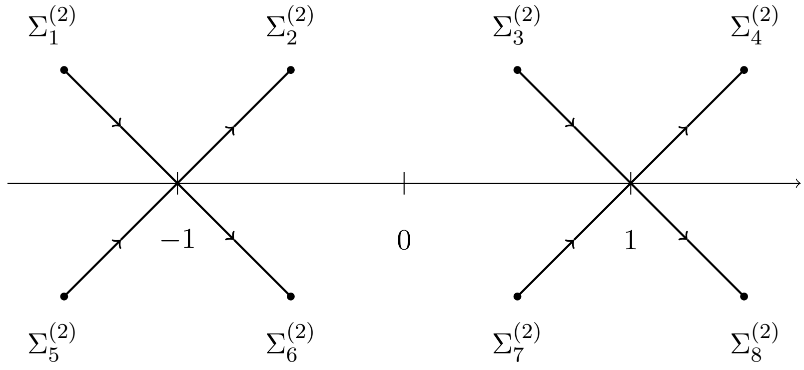


Figure 6.1: *The augmented contour* $\Sigma^{(2)} = \Sigma_1^{(2)} \cup ... \cup \Sigma_8^{(2)}$.

Here $\mathcal{W}_\pm$ are the boundary values of $\mathcal{W}$ as $\pm\,\mathrm{Im}(\zeta) \downarrow 0$. It can be verified easily by the triangularity of $R_R^{(1)}$ and $R_L^{(1)}$ that the new unknown

$$M^{(2)}(\tau;\zeta) := M^{(1)}(\tau;\zeta)\mathcal{W}(\tau;\zeta) \tag{6.2.7}$$

has no jump on the real axis. The discontinuity of $\mathcal{W}$ on $\Sigma^{(2)}$ can be arranged in such a way that

$$M_+^{(2)}(\tau;\zeta) = M_-^{(2)}(\tau;\zeta)(1 + R_\tau^{(2)}(\zeta)(1-\chi(\zeta)))$$

on $\Sigma^{(2)}$ with

$$R_\tau^{(2)}(\zeta) := \begin{cases} \begin{bmatrix} 0 & \breve{\rho}(-1)(\delta_0^-)^{-2}(-(\zeta+1))^{-2i\nu_0^-}e^{i\tau Z(\zeta)} \\ 0 & 0 \end{bmatrix}, & \zeta \in \Sigma_1^{(2)}, \\ \begin{bmatrix} 0 & 0 \\ \frac{\rho(-1)}{1+\rho(-1)\breve{\rho}(-1)}(\delta_0^-)^2(-(\zeta+1))^{2i\nu_0^-}e^{-i\tau Z(\zeta)} & 0 \end{bmatrix}, & \zeta \in \Sigma_2^{(2)}, \\ \begin{bmatrix} 0 & 0 \\ \rho(-1)(\delta_0^-)^2(-(\zeta+1))^{2i\nu_0^-}e^{-i\tau Z(\zeta)} & 0 \end{bmatrix}, & \zeta \in \Sigma_5^{(2)}, \\ \begin{bmatrix} 0 & \frac{\breve{\rho}(-1)}{1+\rho(-1)\breve{\rho}(-1)}(\delta_0^-)^{-2}(-(\zeta+1))^{-2i\nu_0^-}e^{i\tau Z(\zeta)} \\ 0 & 0 \end{bmatrix}, & \zeta \in \Sigma_6^{(2)}, \end{cases} \tag{6.2.8}$$

and

$$R_\tau^{(2)}(\zeta) := \begin{cases} \begin{bmatrix} 0 & 0 \\ \frac{\rho(1)}{1+\rho(1)\breve{\rho}(1)}(\delta_0^+)^2(\zeta-1)^{-2i\nu_0^+}e^{-i\tau Z(\zeta)} & 0 \end{bmatrix}, & \zeta \in \Sigma_3^{(2)}, \\ \begin{bmatrix} 0 & \breve{\rho}(1)(\delta_0^+)^{-2}(\zeta-1)^{2i\nu_0^+}e^{i\tau Z(\zeta)} \\ 0 & 0 \end{bmatrix}, & \zeta \in \Sigma_4^{(2)}, \\ \begin{bmatrix} 0 & \frac{\breve{\rho}(1)}{1+\rho(1)\breve{\rho}(1)}(\delta_0^+)^{-2}(\zeta-1)^{2i\nu_0^+}e^{i\tau Z(\zeta)} \\ 0 & 0 \end{bmatrix}, & \zeta \in \Sigma_7^{(2)}, \\ \begin{bmatrix} 0 & 0 \\ \rho(1)(\delta_0^+)^2(\zeta-1)^{-2i\nu_0^+}e^{-i\tau Z(\zeta)} & 0 \end{bmatrix}, & \zeta \in \Sigma_8^{(2)}, \end{cases} \tag{6.2.9}$$

and a cutoff function $\chi \in C_0^\infty(\mathbb{C},[0,1])$ supported on a neighborhood of each end point of $\Sigma^{(2)}$. We notice that $R_\tau^{(2)}$ is determined as follows. Scattering data are replaced by their values at -1, see (6.2.8), or by their values at $+1$, see (6.2.9). Powers of δ are replaced by their asymptotic forms near -1, see (6.2.8), or by their asymptotic forms near $+1$, see (6.2.9). We refer to Proposition 6.4.2 which provides these asymptotics of δ near ± 1. The crucial point in the definition of $R_\tau^{(2)}$, (6.2.8) and (6.2.9), is the fact

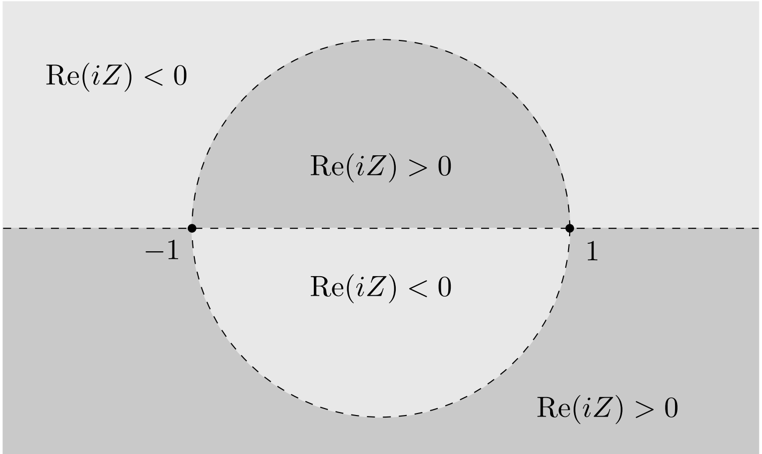


Figure 6.2: *Signature table for* $\mathrm{Re}(iZ)$.

that on $\Sigma^{(2)}$, each of the factors $e^{\pm i\tau Z(\zeta)}$ is decaying exponentially as $\tau \to \infty$. Indeed, let us make the following observation, see Figure 6.2,

$$\mathrm{Re}(iZ(\zeta)) \begin{cases} > 0, & \text{if} \begin{cases} & \mathrm{Im}(\zeta) > 0 \text{ and } |\zeta| < 1, \\ \text{or} & \mathrm{Im}(\zeta) < 0 \text{ and } |\zeta| > 1, \end{cases} \\ < 0, & \text{if} \begin{cases} & \mathrm{Im}(\zeta) > 0 \text{ and } |\zeta| > 1, \\ \text{or} & \mathrm{Im}(\zeta) < 0 \text{ and } |\zeta| < 1. \end{cases} \end{cases} \tag{6.2.10}$$

We want to mention that the purpose of the first step (6.2.2) is exactly to allow this construction and the contour $\Sigma^{(2)}$ fits in an optimal way to the signature table for $\mathrm{Re}(iZ)$. Note that in (6.2.4) the sign of $\pm i\tau Z(\zeta)$ in $R_L^{(1)}$ and $R_R^{(1)}$ depends on whether $\zeta \in [-1, 1]$ or not.
In general $\mathcal{W}$ cannot be chosen as a holomorphic function. Due to its specific form (6.4.12) we find

$$\overline{\partial}\, M^{(2)} = M^{(1)}\, \overline{\partial}\, \mathcal{W} = M^{(2)} \mathcal{W}^{-1}\, \overline{\partial}\, \mathcal{W} = M^{(2)}\, \overline{\partial}\, \mathcal{W}. \tag{6.2.11}$$

The jump condition given by (6.2.8) and (6.2.9) and the lack of analyticity are summarized in the following mixed $\overline{\partial}$-RHP:

$\overline{\partial}$-Riemann-Hilbert problem 6.2.2. For given functions $\rho, \breve{\rho}$ and $\tau \in \mathbb{R}$, find a 2×2-matrix valued function $\mathbb{C} \setminus \Sigma^{(2)} \ni \zeta \mapsto M^{(2)}(\tau; \zeta)$ which satisfies

1. $M^{(2)}(\tau; \cdot)$ has continuous first partial derivatives in $\mathbb{C} \setminus \Sigma^{(2)}$ (with respect to ζ).
2. $M^{(2)}(\tau; \zeta) = 1 + \mathcal{O}\left(\frac{1}{\zeta}\right)$ as $|\zeta| \to \infty$.
3. The non-tangential boundary values $M_\pm^{(2)}(\tau; \zeta)$ exist for $\zeta \in \Sigma^{(2)}$ and satisfy the jump relation

$$M_+^{(2)} = M_-^{(2)}(1 + R_\tau^{(2)}(1 - \chi)), \tag{6.2.12}$$

   where $R_\tau^{(2)} = R_\tau^{(2)}(\zeta)$ is given in (6.2.8) and (6.2.9) and χ is defined later in (6.4.16).
4. The relation (6.2.11) holds in $\mathbb{C} \setminus \Sigma^{(2)}$.

Step 3: The idea of the third step is to split the mixed $\overline{\partial}$-RHP 6.2.2 into a pure RHP and a pure $\overline{\partial}$-problem. For consistency of notation we set

$$\Sigma^{(3)} := \Sigma^{(2)}, \qquad R_\tau^{(3)} := R_\tau^{(2)}, \tag{6.2.13}$$

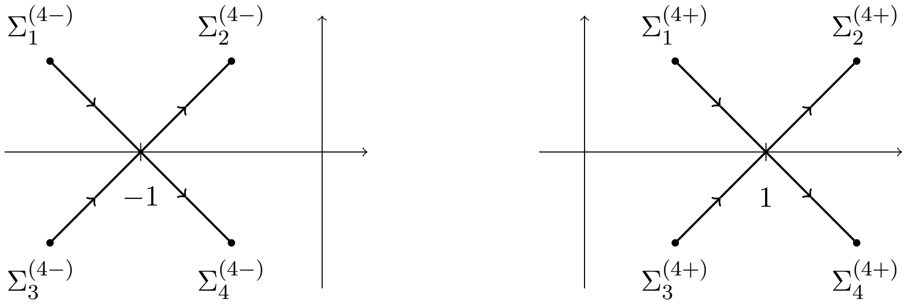


Figure 6.3: *The two crosses* $\Sigma^{(4-)}$ *and* $\Sigma^{(4+)}$.

and define $M^{(3)}$ to be the solution of the normalized pure Riemann-Hilbert problem RHP($\Sigma^{(3)}$,$R^{(3)}$). Notice that RHP($\Sigma^{(3)}$,$R^{(3)}$) and $\overline{\partial}$-RHP 6.2.2 would coincide if $\mathcal{W}$ was analytic. In general we have $M^{(2)} \neq M^{(3)}$ and seek a function D such that

$$M^{(2)}(\tau;\zeta) = D(\tau;\zeta)M^{(3)}(\tau;\zeta). \tag{6.2.14}$$

It follows that D has to be continuous in the entire $\mathbb{C}$-plane. Furthermore, the following equation must hold:

$$\overline{\partial}\, D(\tau;\zeta) = D(\tau;\zeta)\Upsilon(\tau;\zeta), \quad \text{where } \Upsilon(\tau;\zeta) := M^{(3)}(\tau;\zeta)\,\overline{\partial}\,\mathcal{W}(\tau;\zeta)\left[M^{(3)}(\tau;\zeta)\right]^{-1}. \tag{6.2.15}$$

It is easy to check (6.2.15) by direct calculations and we arrive at the following pure $\overline{\partial}$-problem for D.

$\overline{\partial}$-Problem 6.2.3. For each $\tau \in \mathbb{R}^+$, find a 2×2-matrix valued function $\mathbb{C} \ni \zeta \mapsto D(\tau;\zeta)$ which satisfies

1. $D(\tau;\zeta)$ is continuous in $\mathbb{C}$ (with respect to the parameter ζ).
2. $D(\tau;\zeta) \to 1$ as $|\zeta| \to \infty$.
3. The relation (6.2.15) is satisfied.

We solve $\overline{\partial}$-Problem 6.2.3 in Section 6.4.3. It turns out that the contribution of D in (6.2.14) is of order $\tau^{-3/4}$, see Lemma 6.4.6. Thus, the asymptotic (6.1.21) is mainly determined by $M^{(3)}$ which represents the Riemann-Hilbert part of the mixed $\overline{\partial}$-RHP 6.2.2. Section 6.3 is devoted to RHP($\Sigma^{(3)}$,$R^{(3)}$). In contrast to, for instance, the NLS and DNLS equation, at this point RHP($\Sigma^{(3)}$,$R^{(3)}$) is not solvable directly and some further work is required. The following two remaining steps are necessary.

Step 4: The next step separates out the influence of the jumps on the two crosses $\Sigma^{(4-)}$ and $\Sigma^{(4+)}$, see Figure 6.3. Let us split the jump matrix $R_\tau^{(3)}$ ($= R_\tau^{(2)}$) in the following way.

$$R_\tau^{(3)}(\zeta) = R_\tau^{(4-)}(\zeta) + R_\tau^{(4+)}(\zeta),$$

where

$$R_\tau^{(4-)}(\zeta) = 0 \quad \text{for } \zeta \in \Sigma^{(4+)}, \qquad R_\tau^{(4+)}(\zeta) = 0 \quad \text{for } \zeta \in \Sigma^{(4-)}.$$

To each cross and corresponding jump matrix we associate a Riemann-Hilbert problem. That is, we look at solutions $M^{(4\pm)}$ of RHP($\Sigma^{(4\pm)}$,$R^{(4\pm)}$) and consider the assigned functions $q^{(4\pm)}(\tau)$. In Proposition 6.3.4 we show that $M^{(3)}$ is approximated by the product $M^{(4-)}M^{(4+)}$ and thus, $q^{(3)}(\tau)$ is approximated by the sum $q^{(4-)}(\tau) + q^{(4+)}(\tau)$. Since there are no explicit expressions available for $M^{(4-)}$ and $M^{(4+)}$,

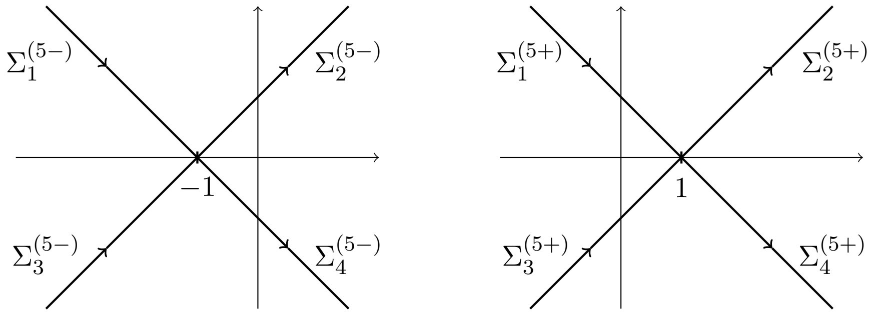


Figure 6.4: *The unbounded augmented crosses $\Sigma^{(5-)}$ and $\Sigma^{(5+)}$.*

we introduce the final step 5.

Step 5: In the last step we end up with two model RHP's for which explicit solutions are known. The first model RHP is obtained from RHP($\Sigma^{(4-)}$,$R_\tau^{(4-)}$), by replacing the phase $Z(\zeta)$ occurring in $R_\tau^{(4-)}$ with its second-order Taylor expansion around the negative stationary phase point -1:

$$Z(\zeta) = -1 - \frac{1}{2}(\zeta+1)^2 + \mathcal{O}\left(|\zeta+1|^3\right), \quad \text{as } \zeta \to -1.$$

We also enlarge the rays $\Sigma_2^{(4-)}$ and $\Sigma_4^{(4-)}$ and define, see Figure 6.4,

$$\Sigma^{(5-)} = \Sigma_1^{(5-)} \cup \ldots \cup \Sigma_4^{(5-)} := \Big[(e^{-i\pi/4}\,\mathbb{R}_-) \cup (e^{i\pi/4}\,\mathbb{R}_+) \cup (e^{i\pi/4}\,\mathbb{R}_-) \cup (e^{-i\pi/4}\,\mathbb{R}_+)\Big] - 1.$$

Now by extending $R_\tau^{(4-)}$ in a natural way, we set

$$R_\tau^{(5-)}(\zeta) := \begin{cases} \begin{bmatrix} 0 & \breve{\rho}(-1)(\delta_0^-)^{-2}(-(\zeta+1))^{-2i\nu_0^-}\, e^{-i\tau\left(1+\frac{1}{2}(\zeta+1)^2\right)} \\ 0 & 0 \end{bmatrix}, & \zeta \in \Sigma_1^{(5-)}, \\ \begin{bmatrix} 0 & 0 \\ \frac{\rho(-1)}{1+\rho(-1)\breve{\rho}(-1)}(\delta_0^-)^{2}(-(\zeta+1))^{2i\nu_0^-}\, e^{i\tau\left(1+\frac{1}{2}(\zeta+1)^2\right)} & 0 \end{bmatrix}, & \zeta \in \Sigma_2^{(5-)}, \\ \begin{bmatrix} 0 & 0 \\ \rho(-1)(\delta_0^-)^{2}(-(\zeta+1))^{2i\nu_0^-}\, e^{i\tau\left(1+\frac{1}{2}(\zeta+1)^2\right)} & 0 \end{bmatrix}, & \zeta \in \Sigma_3^{(5-)}, \\ \begin{bmatrix} 0 & \frac{\breve{\rho}(-1)}{1+\rho(-1)\breve{\rho}(-1)}(\delta_0^-)^{-2}(-(\zeta+1))^{-2i\nu_0^-}\, e^{-i\tau\left(1+\frac{1}{2}(\zeta+1)^2\right)} \\ 0 & 0 \end{bmatrix}, & \zeta \in \Sigma_4^{(5-)}. \end{cases} \tag{6.2.16}$$

Analogously we derive a model RHP from RHP($\Sigma^{(4+)}$,$R^{(4+)}$) by setting

$$\Sigma^{(5+)} = \Sigma_1^{(5+)} \cup \ldots \cup \Sigma_4^{(5+)} := \Big[(e^{-i\pi/4}\,\mathbb{R}_-) \cup (e^{i\pi/4}\,\mathbb{R}_+) \cup (e^{i\pi/4}\,\mathbb{R}_-) \cup (e^{-i\pi/4}\,\mathbb{R}_+)\Big] + 1$$

and defining

$$R_\tau^{(5+)}(\zeta) := \begin{cases} \begin{bmatrix} 0 & 0 \\ \frac{\rho(1)}{1+\rho(1)\breve{\rho}(1)}(\delta_0^+)^{2}(\zeta-1)^{-2i\nu_0^+}\, e^{-i\tau\left(1+\frac{1}{2}(\zeta-1)^2\right)} & 0 \end{bmatrix}, & \zeta \in \Sigma_1^{(5+)}, \\ \begin{bmatrix} 0 & \breve{\rho}(1)(\delta_0^+)^{-2}(\zeta-1)^{2i\nu_0^+}\, e^{i\tau\left(1+\frac{1}{2}(\zeta-1)^2\right)} \\ 0 & 0 \end{bmatrix}, & \zeta \in \Sigma_2^{(5+)}, \\ \begin{bmatrix} 0 & \frac{\breve{\rho}(1)}{1+\rho(1)\breve{\rho}(1)}(\delta_0^+)^{-2}(\zeta-1)^{2i\nu_0^+}\, e^{i\tau\left(1+\frac{1}{2}(\zeta-1)^2\right)} \\ 0 & 0 \end{bmatrix}, & \zeta \in \Sigma_3^{(5+)}, \\ \begin{bmatrix} 0 & 0 \\ \rho(1)(\delta_0^+)^{2}(\zeta-1)^{-2i\nu_0^+}\, e^{-i\tau\left(1+\frac{1}{2}(\zeta-1)^2\right)} & 0 \end{bmatrix}, & \zeta \in \Sigma_4^{(5+)}. \end{cases} \tag{6.2.17}$$

Computing the solution of RHP($\Sigma^{(5-)}$,$R^{(5-)}$) is presented in Subsection 6.3.1. In subsection 6.3.2 we show that the two solutions $M^{(5\pm)}$ approximate the two solutions $M^{(4\pm)}$ in the sense that $M^{(4\pm)} = F^{(\pm)}M^{(5\pm)}$ with matrix functions $F^{(\pm)}$ close to identity, see (6.3.21) and Proposition 6.3.3.

Regrouping of the transformations: We close this summary with regrouping the above explained transformations. Recalling successively (6.2.2), (6.2.7) and (6.2.14) we obtain

$$M^{(0)}(\tau;\zeta) = D(\tau;\zeta)\, M^{(3)}(\tau;\zeta)\ [\mathcal{W}(\tau;\zeta)]^{-1}\, [\delta(\zeta)]^{-\sigma_3}.$$

Using $\mathcal{W}(\tau;\zeta) = 1$ for $\zeta = 0$ and $\zeta \in \Omega_9 \cup \Omega_{10}$, see Figure 6.8, and making use of the fact that $[\delta(\zeta)]^{-\sigma_3}$ is diagonal, we derive the following two solution formulas for the expressions we want to evaluate in Lemma 6.1.2:

$$\begin{aligned} M^{(0)}(\tau;0) &= D(\tau;0)\, M^{(3)}(\tau;0)\, [\delta(0)]^{-\sigma_3}, \\ q^{(0)}(\tau) &= \lim_{\zeta\to\infty} \zeta\, [D(\tau;\zeta)]_{12} + \lim_{\zeta\to\infty} \zeta \left[M^{(3)}(\tau;\zeta)\right]_{12}. \end{aligned} \tag{6.2.18}$$

Based on these formulas, Lemma 6.1.2 is a direct consequence of Propositions 6.3.2 – 6.3.4 and Lemma 6.4.6 below. Therein the following is shown:

Propositions 6.3.2 – 6.3.4	$M^{(3)}(\tau;0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \mathcal{O}(\tau^{-1/2})$ $\lim_{\zeta\to\infty} \zeta \left[M^{(3)}(\tau;\zeta)\right]_{12} = q^{(as)}(\tau) + \mathcal{O}(\tau^{-1})$
Lemma 6.4.6	$D(\tau;0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \mathcal{O}(\tau^{-3/4})$ $\lim_{\zeta\to\infty} \zeta\, [D(\tau;\zeta)]_{12} = \mathcal{O}(\tau^{-3/4})$

The estimates (6.1.18) and (6.1.20) of Lemma 6.1.2 follow easily by substituting these results into (6.2.18).

6.3 Analysis of the pure RHP

The RHP contribution (6.2.12) is mainly responsible for the long-time asymptotics of $q(\tau)$ stated in Lemma 6.1.2. We can provide this explicit result since the function $q^{(3)}(\tau)$ associated to RHP($\Sigma^{(3)}$,$R_\tau^{(3)}$) converges to the sum $q^{(5-)}(\tau) + q^{(5+)}(\tau)$, where both $q^{(5\pm)}$ are associated to model RHP's which can be solved explicitly.

6.3.1 Two model RHPs

The Riemann–Hilbert problems RHP($\Sigma^{(5\pm)}$,$R_\tau^{(5\pm)}$) with $\Sigma^{(5\pm)}$ depicted in Figure 6.4 and $R_\tau^{(5\pm)}$ given in (6.2.16) and (6.2.17) are explicitly solvable. Usually the solution procedure is presented in the following way. We begin with defining a change of variables

$$\eta(\zeta) = -\sqrt{\tau}(\zeta + 1), \quad \zeta(\eta) = \frac{-1}{\sqrt{\tau}}\eta - 1, \tag{6.3.1}$$

and set

$$M^{(PC-)}(\tau;\eta) = M^{(5-)}(\tau;\zeta(\eta)), \quad \eta \in \mathbb{C} \setminus \Sigma^{(PC)}, \tag{6.3.2}$$

with the new contour

$$\Sigma^{(PC)} := (e^{i\pi/4}\,\mathbb{R}) \cup (e^{-i\pi/4}\,\mathbb{R}). \tag{6.3.3}$$

As depicted in Figure 6.5, we let $\Sigma^{(PC)}$ inherit the orientation of $\mathbb{R}$. Thus, it is important to notice that $\Sigma^{(PC)}$ is not simply the image of $\Sigma^{(5-)}$ under the transformation $\zeta \to \eta$ defined in (6.3.1). This is because the change of variables in (6.3.1) would also rotate the contour by an angle of π and thus, reverse the

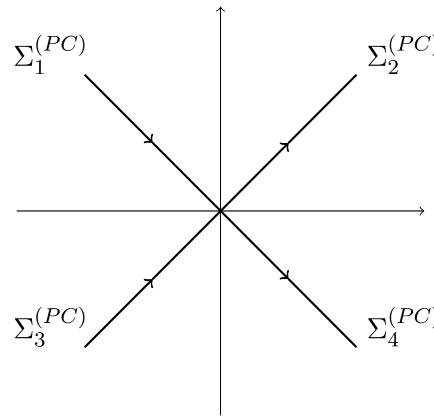


Figure 6.5: *The contour $\Sigma^{(PC)}$ of the model Riemann-Hilbert problems.*

orientation. As an important consequence we have $M_\pm^{(PC-)}(\tau;\eta) = M_\mp^{(5-)}(\tau;\zeta(\eta))$ for $\eta \in \Sigma^{(PC)}$. It follows that the condition

$$M_+^{(5-)}(\tau;\zeta) = M_-^{(5-)}(\tau;\zeta)\left(1 + R_\tau^{(5-)}(\zeta)\right), \qquad \zeta \in \Sigma^{(5-)},$$

is transformed into the jump condition

$$\begin{aligned} M_+^{(PC-)}(\tau;\eta) &= M_-^{(PC-)}(\tau;\eta)\left(1 + R_\tau^{(5-)}(\zeta(\eta))\right)^{-1} \\ &= M_-^{(PC-)}(\tau;\eta)\left(1 - R_\tau^{(5-)}(\zeta(\eta))\right), \qquad \eta \in \Sigma^{(PC)}. \end{aligned}$$

This gives rise to the definition

$$R_\tau^{(PC-)}(\eta) := -R_\tau^{(5-)}(\zeta(\eta)) \qquad \eta \in \Sigma^{(PC)}. \tag{6.3.4}$$

With a view towards (6.2.16), the following identities are useful:

$$e^{i\tau\left(1+\frac{1}{2}(\zeta+1)^2\right)} = e^{i\eta^2/2}e^{i\tau}, \qquad (-(\zeta+1))^{2i\nu_0^-} = \eta^{2i\nu_0^-}e^{-i\nu_0^-\ln(\tau)}.$$

Note that we define $-\pi < \arg(\eta) < \pi$ so that the branch cut of $\eta^{2i\nu_0^-}$ is given by $\mathbb{R}^-$ whereas $(-(\zeta+1))^{2i\nu_0^-}$ is cut along $(-1,\infty)$. Define new parameters

$$\begin{aligned} \rho_0^- &:= \rho(-1)(\delta_0^-)^2 e^{i\tau}e^{-i\nu_0^-\ln(\tau)}, \\ \breve{\rho}_0^- &:= \breve{\rho}(-1)(\delta_0^-)^{-2}e^{-i\tau}e^{i\nu_0^-\ln(\tau)}, \end{aligned} \tag{6.3.5}$$

such that $\rho_0^-\breve{\rho}_0^- = \rho(-1)\breve{\rho}(-1)$ and $\nu_0^- = \frac{1}{2\pi}\log(1+\rho_0^-\breve{\rho}_0^-)$. Using the notation just introduced, we find for $\eta \in \Sigma^{(PC)}$

$$R_\tau^{(PC-)}(\eta) = \begin{cases} \begin{bmatrix} 0 & \frac{-\breve{\rho}_0^-}{1+\rho_0^-\breve{\rho}_0^-}\eta^{-2i\nu_0^-}e^{-i\eta^2/2} \\ 0 & 0 \end{bmatrix}, & \eta \in \Sigma_1^{(PC)} := e^{-i\pi/4}\,\mathbb{R}_-, \\ \begin{bmatrix} 0 & 0 \\ -\rho_0^-\eta^{2i\nu_0^-}e^{i\eta^2/2} & 0 \end{bmatrix}, & \eta \in \Sigma_2^{(PC)} := e^{i\pi/4}\,\mathbb{R}_+, \\ \begin{bmatrix} 0 & 0 \\ \frac{-\rho_0^-}{1+\rho_0^-\breve{\rho}_0^-}\eta^{2i\nu_0^-}e^{i\eta^2/2} & 0 \end{bmatrix}, & \eta \in \Sigma_3^{(PC)} := e^{i\pi/4}\,\mathbb{R}_-, \\ \begin{bmatrix} 0 & -\breve{\rho}_0^-\eta^{-2i\nu_0^-}e^{-i\eta^2/2} \\ 0 & 0 \end{bmatrix}, & \eta \in \Sigma_4^{(PC)} := e^{-i\pi/4}\,\mathbb{R}_+\,. \end{cases} \tag{6.3.6}$$

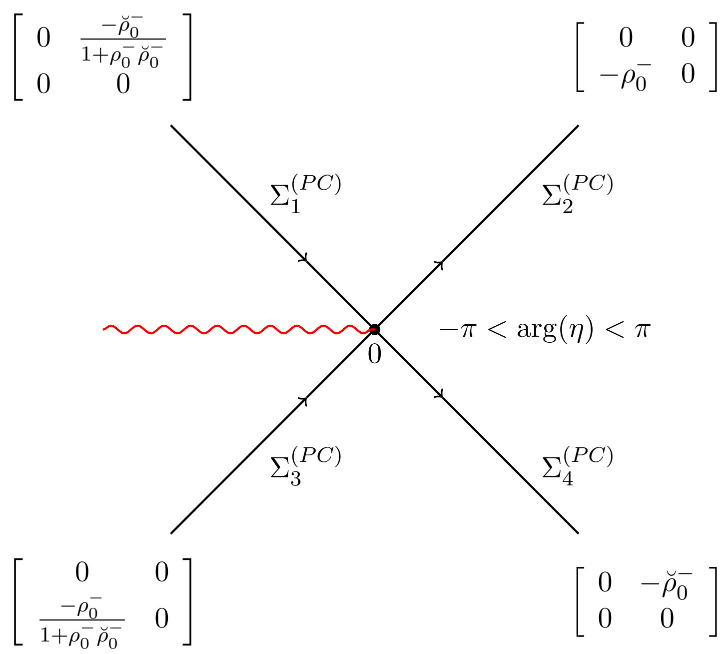


Figure 6.6: *The jump matrix R_0^-. The curly line illustrates the branch cut of $\eta^{2i\nu_0^-}$.*

We may write $R_\tau^{(PC-)}(\eta)$ in the form

$$R_\tau^{(PC-)}(\eta) = S(\eta) R_0^- [S(\eta)]^{-1}, \qquad S(\eta) := \eta^{-i\nu_0^- \sigma_3} e^{-\frac{i}{4}\eta^2 \sigma_3}, \tag{6.3.7}$$

where R_0^- is shown in Figure 6.6 . Now let us repeat the above procedure for the positive stationary phase point $+1$. That means to transform RHP($\Sigma^{(5+)}$,$R_\tau^{(5+)}$) into a Riemann-Hilbert problem on $\Sigma^{(PC)}$. We start with introducing another change of variables

$$\eta(\zeta) = \sqrt{\tau}(\zeta - 1), \quad \zeta(\eta) = \frac{1}{\sqrt{\tau}}\eta + 1, \tag{6.3.8}$$

and set

$$M^{(PC+)}(\tau; \eta) := M^{(5+)}(\tau; \zeta(\eta)). \tag{6.3.9}$$

In contrast to the scaling (6.3.1) used for the model RHP at the negative stationary phase point we do not rotate the contour if we use the new variable η defined in (6.3.8). For that reason, $M^{(PC+)}(\tau, \eta)$ defined in (6.3.9) is a solution of RHP($\Sigma^{(PC)}$,$R_\tau^{(PC+)}$), where we recall the definition of $\Sigma^{(PC)}$, see (6.3.3), and the jump is simply given by $R_\tau^{(PC+)}(\zeta) := R_\tau^{(5+)}(w(\zeta))$. The following identities hold:

$$e^{i\tau\left(1+\frac{1}{2}(\zeta-1)^2\right)} = e^{i\eta^2/2} e^{i\tau}, \qquad (\zeta - 1)^{2i\nu_0^+} = \eta^{2i\nu_0^+} e^{-i\nu_0^+ \ln(\tau)}.$$

We set

$$\begin{aligned} \rho_0^+ &:= \rho(1)(\delta_0^+)^2 e^{-i\tau} e^{i\nu_0^+ \ln(\tau)}, \\ \breve{\rho}_0^+ &:= \breve{\rho}(1)(\delta_0^+)^{-2} e^{i\tau} e^{-i\nu_0^- \ln(\tau)}, \end{aligned} \tag{6.3.10}$$

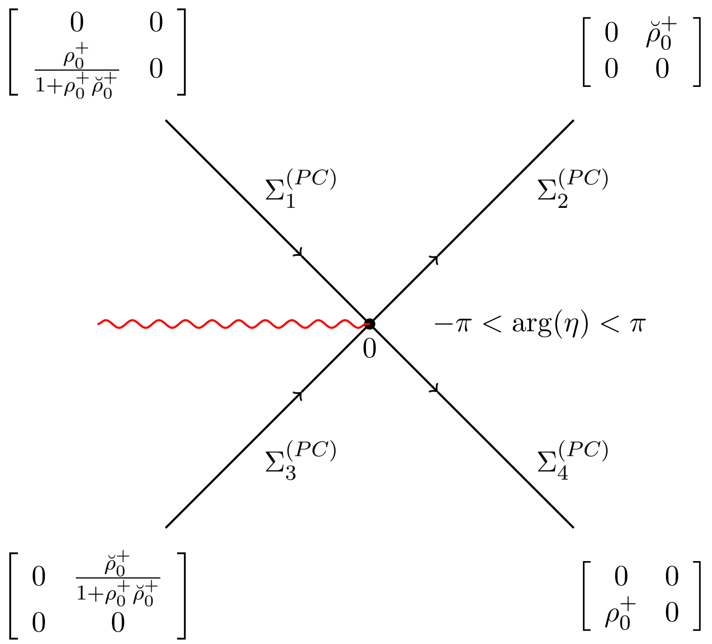


Figure 6.7: *The jump matrix R_0^+. The curly line illustrates the branch cut of $\eta^{2i\nu_0^+}$.*

so that $\rho_0^+\breve{\rho}_0^+ = \rho(1)\breve{\rho}(1)$ and $\nu_0^+ = \frac{1}{2\pi}\log(1+\rho_0^+\breve{\rho}_0^+)$. We find

$$R_\tau^{(PC+)}(\zeta) = \begin{cases} \begin{bmatrix} 0 & 0 \\ \frac{\rho_0^+}{1+\rho_0^+\breve{\rho}_0^+}\zeta^{-2i\nu_0^+}e^{-i\zeta^2/2} & 0 \end{bmatrix}, & w \in \Sigma_1^{(PC)} \\ \begin{bmatrix} 0 & \breve{\rho}_0^+\zeta^{2i\nu_0^+}e^{i\zeta^2/2} \\ 0 & 0 \end{bmatrix}, & w \in \Sigma_2^{(PC)}, \\ \begin{bmatrix} 0 & \frac{\breve{\rho}_0^+}{1+\rho_0^+\breve{\rho}_0^+}\zeta^{2i\nu_0^+}e^{i\zeta^2/2} \\ 0 & 0 \end{bmatrix}, & w \in \Sigma_3^{(PC)}, \\ \begin{bmatrix} 0 & 0 \\ \rho_0^+\zeta^{-2i\nu_0^+}e^{-i\zeta^2/2} & 0 \end{bmatrix}, & w \in \Sigma_4^{(PC)}. \end{cases} \tag{6.3.11}$$

In Figure 6.7 we illustrate the matrix R_0^+ defined on $\Sigma^{(PC)}$, where R_0^+ has the meaning that

$$1 + R_\tau^{(PC+)}(\eta) = S(\eta)R_0^+[S(\eta)]^{-1}, \quad S(\eta) = \eta^{i\nu_0^+\sigma_3}e^{\frac{i}{4}\eta^2\sigma_3}.$$

The solution of the two Riemann-Hilbert problems RHP($\Sigma^{(PC)}$,$R_\tau^{(PC\pm)}$) is not worked out in this paper. However, the reader might wonder why the superscript (PC) is used for the model Riemann-Hilbert problems. Here, PC stands for Parabolic Cylinder functions. This well-known class of functions plays a certain role in the derivation of the following which we take from [LPS18]. See also Lemma 3.5 in [CP14].

Proposition 6.3.1. *Given complex non-zero constants $\rho_0^\pm$ and $\breve{\rho}_0^\pm$ such that $1+\rho_0^\pm\breve{\rho}_0^\pm \in \mathbb{R}^+$, the Riemann-Hilbert problems RHP($\Sigma^{(PC)}$,$R_\tau^{(PC\pm)}$) with $R_\tau^{(PC\pm)}$ defined in (6.3.6) and (6.3.11) and with $\Sigma^{(PC)}$ depicted in Figure 6.5 are solvable. The solutions take the form*

$$M^{(PC\pm)}(\tau;\eta) = 1 + \frac{1}{\eta}\begin{bmatrix} 0 & \mp i\beta_{12}^\pm \\ \pm i\beta_{21}^\pm & 0 \end{bmatrix} + \mathcal{O}\left(\frac{1}{\eta^2}\right), \tag{6.3.12}$$

as $|\eta| \to \infty$, where the constants $\beta_{12}^{\pm}$ and $\beta_{21}^{\pm}$ can be computed from $\rho_0^{\pm}$ and $\breve{\rho}_0^{\pm}$ through

$$\begin{aligned} \beta_{12}^{-} &= \frac{\sqrt{2\pi} e^{\pi\nu_0^-/2} e^{i\pi/4}}{\rho_0^- \Gamma(i\nu_0^-)}, & \beta_{21}^{-} &= \frac{\sqrt{2\pi} e^{\pi\nu_0^-/2} e^{-i\pi/4}}{\breve{\rho}_0^- \Gamma(-i\nu_0^-)}, \\ \beta_{12}^{+} &= \frac{\sqrt{2\pi} e^{\pi\nu_0^+/2} e^{3\pi i/4}}{\rho_0^+ \Gamma(-i\nu_0^+)}, & \beta_{21}^{+} &= \frac{\sqrt{2\pi} e^{\pi\nu_0^+/2} e^{-3\pi i/4}}{\breve{\rho}_0^+ \Gamma(i\nu_0^+)}. \end{aligned} \tag{6.3.13}$$

In addition,

$$\|M^{(PC\pm)}(\tau;\cdot) - 1\| \le C \left[\left| \frac{\rho_0^{\pm}}{1+\rho_0^{\pm}\breve{\rho}_0^{\pm}} \right| + |\breve{\rho}_0^{\pm}| + \left| \frac{\breve{\rho}_0^{\pm}}{1+\rho_0^{\pm}\breve{\rho}_0^{\pm}} \right| + |\rho_0^{\pm}| \right], \tag{6.3.14}$$

for some constant $C > 0$.

Using the relations (6.3.2) and (6.3.9), and the two different scalings (6.3.1) and (6.3.8) we obtain $M^{(5\pm)}(\tau;\zeta) = M^{(PC\pm)}(\tau; \pm\sqrt{\tau}(\zeta \mp 1))$ and then we find that

$$\lim_{\zeta\to\infty} \zeta \cdot \left(M^{(5\pm)}(\tau;\zeta) - 1 \right) = \frac{\pm 1}{\sqrt{\tau}} \lim_{\eta\to\infty} \eta \cdot \left(M^{(PC\pm)}(\tau;\eta) - 1 \right)$$

and

$$M^{(5\pm)}(\tau;0) = M^{(PC\pm)}(\tau; -\sqrt{\tau}).$$

Thus, if we substitute the definitions of $\rho_0^{\pm}$ and $\breve{\rho}_0^{\pm}$, see (6.3.5) and (6.3.10), into the formulas for β_{12}^{+} and β_{12}^{-}, see (6.3.13), we find the following proposition as a corollary to Proposition 6.3.1:

Proposition 6.3.2. *Under the assumptions that $|\log(1+\rho(\pm 1)\breve{\rho}(\pm 1)| \le c$ for some constant $c > 0$, and that $\rho(\pm 1) \neq 0$ and $\breve{\rho}(\pm 1) \neq 0$, the Riemann-Hilbert problems RHP($\Sigma^{(5\pm)}$,$R^{(5\pm)}$) are uniquely solvable and the functions*

$$q^{(5\pm)}(\tau) = \lim_{\zeta\to\infty} \zeta \cdot \left[M^{(5\pm)}(\tau;\zeta) \right]_{12}$$

are explicitly given by

$$\begin{aligned} q^{(5-)}(\tau) &= \frac{e^{-i\tau} e^{i\nu_0^- \ln(\tau)}}{\tau^{1/2}} \frac{\sqrt{2\pi} e^{\pi\nu_0^-/2} e^{-i\pi/4}}{\rho(-1)(\delta_0^-)^2 \Gamma(i\nu_0^-)}, \\ q^{(5+)}(\tau) &= \frac{e^{i\tau} e^{-i\nu_0^+ \ln(\tau)}}{\tau^{1/2}} \frac{\sqrt{2\pi} e^{\pi\nu_0^+/2} e^{i\pi/4}}{\rho(1)(\delta_0^+)^2 \Gamma(-i\nu_0^+)}. \end{aligned} \tag{6.3.15}$$

Moreover, we have

$$|M^{(5\pm)}(\tau;0) - 1| \le \tau^{-1/2} \sum_{k=1}^{8} \Gamma_6(p_k) \tag{6.3.16}$$

and

$$\|M^{(5\pm)}(\tau;\cdot) - 1\|_{L^\infty(\mathbb{C})} \le \sum_{k=1}^{8} \Gamma_6(p_k). \tag{6.3.17}$$

6.3.2 Truncated crosses

This subsection is devoted to the Riemann–Hilbert problems RHP($\Sigma^{(4\pm)}$,$R_\tau^{(4\pm)}$), where the contours $\Sigma^{(4+)}$ and $\Sigma^{(4-)}$ are depicted in Figure 6.3. We recall that $R_\tau^{(4\pm)}$ are given by

$$R_\tau^{(4\pm)} = R_\tau^{(2)}\Big|_{\Sigma^{(4\pm)}}.$$

See (6.2.8) and (6.2.9) for the definition of $R_\tau^{(2)}$. Our next basic result is the following proposition.

Proposition 6.3.3. *Under the same assumptions as in Proposition 6.3.2, there exist a constant $\varepsilon_0 > 0$ such that for all $\tau > 0$ satisfying*

$$\sum_{k=1}^{8} \Gamma_6(p_k)\tau^{-1/2} < \varepsilon_0,$$

Riemann–Hilbert problems RHP($\Sigma^{(4\pm)}$,$R^{(4\pm)}$) are uniquely solvable. Moreover, there exist positive constants C_1 and C_2 such that the functions

$$q^{(4\pm)}(\tau) = \lim_{\zeta\to\infty} \zeta \cdot \left[M^{(4\pm)}(\tau;\zeta)\right]_{12}$$

satisfy

$$\left|q^{(4\pm)}(\tau) - q^{(5\pm)}(\tau)\right| \le C_1 \sum_{k=1}^{8} \Gamma_6(p_k)\tau^{-1}, \tag{6.3.18}$$

where $q^{(5\pm)}(\tau)$ are given in (6.3.15). We also have

$$\|M^{(4\pm)}(\tau;\cdot) - 1\|_{L^\infty(\mathbb{C})} \le C_1 \sum_{k=1}^{8} \Gamma_6(p_k), \tag{6.3.19}$$

and for a fixed ζ_0 with $\operatorname{dist}(\Sigma^{(4\mp)},\zeta_0) > 0$,

$$\left|M^{(4\pm)}(\tau;\zeta_0) - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right| \le \frac{C_2}{\operatorname{dist}(\Sigma^{(4\mp)},\zeta_0)} \sum_{k=1}^{8} \Gamma_6(p_k)\tau^{-1/2}. \tag{6.3.20}$$

In particular, (6.3.20) holds for $\zeta_0 = 0$.

Proof. We only give a proof for the "$-$" case of (6.3.18)–(6.3.20). The idea is to construct the solution of RHP($\Sigma^{(4-)}$,$R_\tau^{(4-)}$) from the solution of RHP($\Sigma^{(5-)}$,$R_\tau^{(5-)}$) which is provided by Proposition 6.3.2. Therefore we seek a matrix-valued function F such that

$$M^{(4-)}(\tau;\zeta) = F(\tau;\zeta)M^{(5-)}(\tau;\zeta). \tag{6.3.21}$$

A direct computation shows that F needs to be the solution of a normalized Riemann-Hilbert problem RHP($\Sigma^{(5-)}$,$R_\tau^{(F)}$), where the jump is given by

$$1 + R_\tau^{(F)} = M_-^{(5-)}(1 + R_\tau^{(4-)})(1 - R_\tau^{(5-)})\left[M_-^{(5-)}\right]^{-1}, \quad \zeta \in \Sigma^{(5-)}. \tag{6.3.22}$$

Here we set $R_\tau^{(4-)}(\zeta) = 0$ for $\zeta \in \Sigma^{(5-)} \setminus \Sigma^{(4-)}$. Otherwise (6.3.22) would not make sense because $R_\tau^{(4-)}$ is not defined everywhere on $\Sigma^{(5-)}$. Our goal is to apply the small norm RHP theory presented in Appendix A.1, see Theorem A.1.3. This requires bounds for the L^∞ and L^1 norms of $R_\tau^{(F)}$. For this purpose we use the triangularity of $1 + R_\tau^{(4-)}$ and $1 + R_\tau^{(5-)}$ and arrange (6.3.22) in the following way:

$$R_\tau^{(F)}(\zeta) = M_-^{(5-)}(\tau;\zeta)(R_\tau^{(4-)}(\zeta) - R_\tau^{(5-)}(\zeta))\left[M_-^{(5-)}(\tau;\zeta)\right]^{-1}$$

We learn that for all $w \in \Sigma^{(5-)}$,

$$|R_\tau^{(F)}(w)| \le c|R_\tau^{(4-)}(w) - R_\tau^{(5-)}(w)|. \tag{6.3.23}$$

The constant c is determined by $\|M_-^{(5-)}\|_{L^\infty(\Sigma^{(5-)})}$ and thus independent of τ, see (6.3.17). Using the notation

$$\widetilde{Z}(\zeta) = -1 - \frac{1}{2}(\zeta+1)^2,$$

we can find a constant c such that

$$|Z(\zeta) - \widetilde{Z}(\zeta)| \leq c|\zeta+1|^3, \quad \text{for all } |\zeta+1| \leq \frac{1}{\sqrt{2}}.$$

It follows that for all $\zeta \in \Sigma^{(4-)} \subset \Sigma^{(5-)}$ we have

$$\left|e^{\pm i\tau Z(\zeta)} - e^{\pm i\tau \widetilde{Z}(\zeta)}\right| \leq c\tau \left|e^{\pm i\tau \widetilde{Z}(\zeta)}\right| |\zeta+1|^3.$$

Taking $\zeta \in \Sigma_2^{(4-)} \subset \Sigma_2^{(5-)}$ and parameterizing $\zeta = se^{i\pi/4} - 1$ with $0 \leq s \leq 1/\sqrt{2}$, we obtain

$$|e^{i\tau Z(\zeta)} - e^{i\tau \widetilde{Z}(\zeta)}| \leq c\gamma_\tau(s), \qquad \text{with } \gamma_\tau(s) = \tau e^{-\tau s^2/2} s^3.$$

As it is shown easily, γ_τ has the following properties

$$\|\gamma_\tau\|_{L^\infty(\mathbb{R}_+)} \leq c_1 \tau^{-1/2}, \qquad \|\gamma_\tau\|_{L^1(0,1/2)} \leq c_2 \tau^{-1}.$$

From these observations and similar estimates on the other rays $\Sigma_1^{(4-)}$, $\Sigma_3^{(4-)}$ and $\Sigma_4^{(4-)}$, we can deduce that

$$\|R_\tau^{(4-)} - R_\tau^{(5-)}\|_{L^\infty(\Sigma^{(4-)})} \leq c \sum_{k=1}^{8} \Gamma_6(p_k)\tau^{-1/2}$$

and

$$\|R_\tau^{(4-)} - R_\tau^{(5-)}\|_{L^1(\Sigma^{(4-)})} \leq c \sum_{k=1}^{8} \Gamma_6(p_k)\tau^{-1}.$$

In order to show that equivalent estimates are also available on the contour $\Sigma^{(5-)} \setminus \Sigma^{(4-)}$ we use that we have set $R_\tau^{(4-)} \equiv 0$ on $\Sigma^{(5-)} \setminus \Sigma^{(4-)}$. Elementary computations show that the L^∞-norm and also the L^1-norm of $R_\tau^{(5-)}$ over $\Sigma^{(5-)} \setminus \Sigma^{(4-)}$ decays exponentially as $\tau \to \infty$. Thus, combining all estimates, we finally find

$$\|R_\tau^{(F)}\|_{L^\infty(\Sigma^{(5-)})} \leq \sum_{k=1}^{8} \Gamma_6(p_k)\tau^{-1/2}$$

and

$$\|R_\tau^{(F)}\|_{L^1(\Sigma^{(5-)})} \leq c_2 \sum_{k=1}^{8} \Gamma_6(p_k)\tau^{-1}$$

with constants independent of τ. By (6.3.23), Theorem A.1.3 and (6.3.21), these estimates are sufficient for (6.3.18)–(6.3.20) to be valid. The Proposition is proven. □

6.3.3 Combining the two crosses

In this subsection we discuss the Riemann-Hilbert part of the mixed $\overline{\partial}$-RHP 6.2.2 Therefore we will construct the solution $M^{(3)}$ of RHP($\Sigma^{(3)}$,$R_\tau^{(3)}$) from the two solutions of RHP($\Sigma^{(4\pm)}$,$R_\tau^{(4\pm)}$), as constructed in the proof of Proposition 6.3.3 from the two model Riemann-Hilbert problems. We recall that the contour $\Sigma^{(3)} = \Sigma^{(2)}$ is depicted in Figure 6.1 and we also recall that

$$R_\tau^{(3)}(\zeta) = R_\tau^{(2)}(\zeta), \qquad \zeta \in \Sigma^{(3)}.$$

See (6.2.8) and (6.2.9) for the definition of $R_\tau^{(2)}$. We have the following proposition.

Proposition 6.3.4. *Under the same assumptions as in Proposition 6.3.2, there exists a constant $\varepsilon_0 > 0$ such that for all $\tau > 0$ satisfying*

$$\sum_{k=1}^{8} \Gamma_6(p_k)\tau^{-1/2} < \varepsilon_0,$$

the Riemann-Hilbert problem RHP($\Sigma^{(3)}$,$R^{(3)}$) is uniquely solvable. Moreover, there exists a positive constant C, such that the function

$$q^{(3)}(\tau) = \lim_{\zeta\to\infty} \zeta \cdot \left[M^{(3)}(\tau;\zeta)\right]_{12}$$

satisfies

$$\left|q^{(3)}(\tau) - \left(q^{(4-)}(\tau) + q^{(4+)}(\tau)\right)\right| \le \sum_{k=1}^{8} \Gamma_6(p_k)\tau^{-1}, \tag{6.3.24}$$

where $q^{(4\pm)}(\tau)$ satisfy (6.3.18). We also have

$$\|M^{(3)}(\tau;\cdot) - 1\|_{L^\infty(\mathbb{C})} \le C \sum_{k=1}^{8} \Gamma_6(p_k), \tag{6.3.25}$$

and for a fixed ζ_0 with $\mathrm{dist}(\Sigma^{(2)},\zeta_0) > 0$,

$$\left|M^{(3)}(\tau;0) - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right| \le \sum_{k=1}^{8} \frac{C}{\mathrm{dist}(\Sigma^{(2)},\zeta_0)} \Gamma_6(p_k)\tau^{-1/2}, \tag{6.3.26}$$

In particular, (6.3.26) holds for $\zeta_0 = 0$.

Proof. We consider RHP($\Sigma^{(4-)}$,$R_\tau^{(E)}$), where the jump matrix $R_\tau^{(E)}$ is given by

$$1 + R_\tau^{(E)} = M_-^{(4-)} M^{(4+)} (1 + R_\tau^{(3)})[M^{(4+)}]^{-1}(1 - R_\tau^{(3)})[M_-^{(4-)}]^{-1}, \quad \zeta \in \Sigma^{(4-)}.$$

Denoting the solution of RHP($\Sigma^{(4-)}$,$R_\tau^{(E)}$) by $E(\tau,\zeta)$, we then have

$$M^{(3)}(\tau;w) = E(\tau;w)M^{(4-)}(\tau;w)M^{(4+)}(\tau;w),$$

which is verified by computing explicitly the jumps on $\Sigma^{(3)} = \Sigma^{(4-)} \cup \Sigma^{(4+)}$. Furthermore, it follows that

$$q^{(3)}(\tau) = [E_1(\tau)]_{12} + q^{(4-)}(\tau) + q^{(4+)}(\tau),$$

where

$$E(\tau;\zeta) = 1 + \frac{E_1(\tau)}{\zeta} + \mathcal{O}\left(\zeta^{-2}\right), \qquad \text{as } \zeta \to \infty.$$

Thus, it suffices to show that RHP($\Sigma^{(4-)}$,$R^{(E)}$) is indeed solvable and we have to prove estimates on E_1. Similar to the above proof of Proposition 6.3.3, we intend to apply theory for RHPs with jump matrix $R^{(E)}$ near zero, see Appendix A.1. It follows from Theorem A.1.3 that (6.3.24)–(6.3.26) are proven if the following estimates can be verified.

$$\|R_\tau^{(E)}(\cdot)\|_{L^\infty(\Sigma^{(4-)})} \le c \sum_{k=1}^{8} \Gamma_6(p_k)\tau^{-1/2} \tag{6.3.27}$$

and

$$\|R_\tau^{(E)}(\cdot)\|_{L^1(\Sigma^{(4-)})} \le \sum_{k=1}^{8} \Gamma_6(p_k)\tau^{-1}. \tag{6.3.28}$$

We will provide the proof of (6.3.27) and (6.3.28) in what follows. Writing

$$\begin{aligned} R_\tau^{(E)} \;=\;& M_-^{(4-)}\left(M^{(4+)} - 1\right) R_\tau^{(3)}[M^{(4+)}]^{-1}\left(1 - R_\tau^{(3)}\right)[M_-^{(4-)}]^{-1} \\ &+ M_-^{(4-)} R_\tau^{(3)}\left(1 - R_\tau^{(3)}\right)\left([M^{(4+)}]^{-1} - 1\right)[M_-^{(4-)}]^{-1}, \qquad \zeta \in \Sigma^{(4-)}, \end{aligned}$$

where we also used that $[R_\tau^{(3)}]^2 = 0$, we find out that

$$\left|R_\tau^{(E)}(\zeta)\right| \leq c\left|M^{(4+)}(\zeta) - 1\right| \cdot \left|R_\tau^{(3)}(\zeta)\right|. \tag{6.3.29}$$

Here the constant c is determined by the $L^\infty(\Sigma^{(4-)})$-norm of $M_-^{(4-)}$, $M^{(4+)}$ and $R^{(3)}$ and is, thus, independent of τ, see (6.3.19). For any $\zeta \in \Sigma^{(4-)}$, we have $\operatorname{dist}(\Sigma^{(4+)}, \zeta) > 1/3$. Making use of this observation and considering estimate (6.3.20) we realize that (6.3.29) implies

$$\left|R_\tau^{(E)}(\zeta)\right| \leq c\tau^{-1/2}\left|R_\tau^{(3)}(\zeta)\right|. \tag{6.3.30}$$

Thus, we need to calculate $\|R_\tau^{(3)}\|_{L^1(\Sigma^{(4-)})}$. For this purpose we calculate exemplarily the L^1-norm on $\Sigma_1^{(4-)} = \Sigma_1^{(3)} = \Sigma_1^{(2)}$. For that we use the parametrization $\zeta = x + iy$ with $x = -1 - y$ and $0 \leq y \leq \frac{1}{2}$. Similar to the third line of (B.5.1) it follows that

$$\zeta = x + iy \in \Sigma_1^{(4-)} \quad \Rightarrow \quad |e^{i\tau Z(\zeta)}| \leq e^{\frac{-\tau y^2}{6}}.$$

Furthermore, since $(\zeta+1)^{-2i\nu_0^+}$ is bounded on $\Sigma_1^{(4-)}$, we find that

$$\|R_\tau^{(3)}\|_{L^1(\Sigma_1^{(4-)})} \leq c\,\Gamma_6(p_1)\int_{\Sigma_1^{(4-)}}\left|e^{i\tau Z(\zeta)}\right| d\zeta = c\,\Gamma_6(p_1)\sqrt{2}\int_0^{\frac{1}{2}} e^{\frac{-\tau y^2}{6}}dy \leq c\,\Gamma_6(p_1)\tau^{-1/2}.$$

The same argument can be used to estimate $\|R_\tau^{(3-)}\|_{L^1(\Sigma_j^{(4-)})}$ for $j \in \{2, 3, 4\}$, so that

$$\|R_\tau^{(3-)}\|_{L^1(\Sigma_1^{(4-)})} \leq c\sum_{k=1}^{8}\Gamma_6(p_k)\tau^{-1/2}. \tag{6.3.31}$$

Combining (6.3.30) and (6.3.31), we obtain (6.3.28). □

6.4 Analysis of the $\overline{\partial}$-problem

6.4.1 A scalar Riemann–Hilbert problem

Scalar Riemann-Hilbert problems such as RHP 6.2.1 are well understood and explicit representations for their solution are available in terms of the Cauchy operator, see (6.4.2) below. We will state important global properties of the solution δ of RHP 6.2.1 below. Afterwards we compute the asymptotic behavior at the stationary phase points ± 1.

As defined in the beginning of this chapter we set

$$\nu(\zeta) := \frac{1}{2\pi}\log\left(1 + \rho(\zeta)\breve{\rho}(\zeta)\right), \tag{6.4.1}$$

and consider the following function

$$\delta(\zeta) := \exp\left\{\frac{1}{i}\int_{-1}^{1}\frac{\nu(s)}{s-\zeta}ds\right\}, \quad \zeta \in \mathbb{C}\setminus[-1,1]. \tag{6.4.2}$$

The following can be found in many works, see for instance [DZ94].

Proposition 6.4.1. *The function δ defined in (6.4.2) satisfies the following:*

(i) δ is a solution of Riemann-Hilbert problem 6.2.1.

(ii) For $\mp\operatorname{Im}(\zeta) > 0$, we have $|\delta^{\pm 1}(\zeta)| \leq 1$.

(iii) For $\zeta \notin [-1,1]$, we have $e^{-\|\nu\|_{L^\infty}/2} \leq |\delta(\zeta)| \leq e^{\|\nu\|_{L^\infty}/2}$.

Recalling from (6.1.16) that

$$\mathfrak{C}(\nu) = e^{\|\nu\|_{L^\infty}} + \left\{ \left(\int_{-1}^{-1/10} + \int_{1/10}^{1} \right) |\nu'(\zeta)|^2 d\zeta \right\}^{1/2} + \int_{-1/10}^{1/10} |\nu(\zeta)| d\zeta,$$

we conclude from Proposition 6.4.1 that

$$\|\delta^{-2}\|_{L^\infty(\mathbb{C})} \leq \mathfrak{C}(\nu), \qquad \|\delta^2\|_{L^\infty(\mathbb{C})} \leq \mathfrak{C}(\nu). \tag{6.4.3}$$

This is relevant for the proof of Lemma 6.4.3.

Next, we address the asymptotic behavior of $\delta(\zeta)$ near the stationary phase points -1 and $+1$. Using the notation introduced in (6.4.1), we set

$$\nu_0^{\pm} := \nu(\pm 1). \tag{6.4.4}$$

Now, let $\zeta \in \mathbb{C} \setminus [-1,\infty)$ and use $-\pi < \arg(-(\zeta+1)) < \pi$ to define $(-(\zeta+1))^{i\nu_0^-}$. Then, for $\zeta \in [-1,\infty)$ we compute,

$$\lim_{\varepsilon \downarrow 0} (-(\zeta + i\,\varepsilon + 1))^{i\nu_0^-} = \left(\lim_{\varepsilon \downarrow 0} (-(\zeta - i\,\varepsilon + 1))^{i\nu_0^-} \right) (1 + \rho(-1)\breve{\rho}(-1)),$$

and we learn that locally for $\zeta \to -1$ the two functions $(-(\,\cdot\, + 1))^{i\nu_0^-}$ and $\delta(\,\cdot\,)$ as given in (6.4.2) satisfy the same jump condition. Analogously, let $\zeta \in \mathbb{C} \setminus (-\infty, 1]$ and use $-\pi < \arg(\zeta - 1) < \pi$ to define $(\zeta - 1)^{i\nu_0^+}$. It then turns out that the function $(\,\cdot\, - 1)^{i\nu_0^+}$ fulfills for $\zeta \to +1$ the same jump condition as the function δ defined in (6.4.2). In fact, we have the following asymptotics.

Proposition 6.4.2. *Let $\rho, \breve{\rho} \in H^1(\mathbb{R})$ and δ given by (6.4.2). There exist constants $\delta_0^{\pm} \in \mathbb{C}$ with $|\delta_0^{\pm}| = 1$ and $c > 0$ such that*

$$\left| \delta(\zeta) - \delta_0^- \cdot (-(\zeta+1))^{i\nu_0^-} \right| \leq c\mathfrak{C}(\nu)|\zeta+1|^{1/2}, \quad \text{for all } \zeta \in \mathbb{C} \setminus [-1,\infty) \text{ with } |\zeta+1| < \frac{1}{\sqrt{2}}, \tag{6.4.5}$$

and

$$\left| \delta(\zeta) - \delta_0^+ \cdot (\zeta-1)^{-i\nu_0^+} \right| \leq c\mathfrak{C}(\nu)|\zeta-1|^{1/2}, \quad \text{for all } \zeta \in \mathbb{C} \setminus (-\infty,1] \text{ with } |\zeta-1| < \frac{1}{\sqrt{2}}. \tag{6.4.6}$$

Here the complex powers of $\mp(\zeta \pm 1)$ are defined with the branch of the logarithm as described above: $-\pi < \arg(-(\zeta+1)) < \pi$ in (6.4.5) and $-\pi < \arg(\zeta-1) < \pi$ in (6.4.6), respectively. The constant $\mathfrak{C}(\nu)$ is defined in (6.1.16).

Proof. The computations are somewhat standard. Let us define the following two functions,

$$\varphi(s) := -s\nu_0^- \mathbb{1}_{[-1,0]}(s) + s\nu_0^+ \mathbb{1}_{[0,1]}(s), \qquad \beta(\zeta) := \frac{1}{i} \int_{-1}^{1} \frac{\nu(s) - \varphi(s)}{s - \zeta} ds$$

and denote the exponent in (6.4.2) by γ such that $\delta(\zeta) = \exp(\gamma(\zeta))$ and

$$\gamma(\zeta) = \beta(\zeta) - \frac{\nu_0^-}{i} \int_{-1}^{0} \frac{s}{s-\zeta} ds + \frac{\nu_0^+}{i} \int_{0}^{1} \frac{s}{s-\zeta} ds, \qquad \zeta \notin [-1,1].$$

Using the notation

$$\mathfrak{b}^{(-)}(\zeta) := -\frac{\nu_0^-}{i} \int_{-1}^{0} \frac{s}{s-\zeta} ds, \qquad \mathfrak{b}^{(+)}(\zeta) := \frac{\nu_0^+}{i} \int_{0}^{1} \frac{s}{s-\zeta} ds,$$

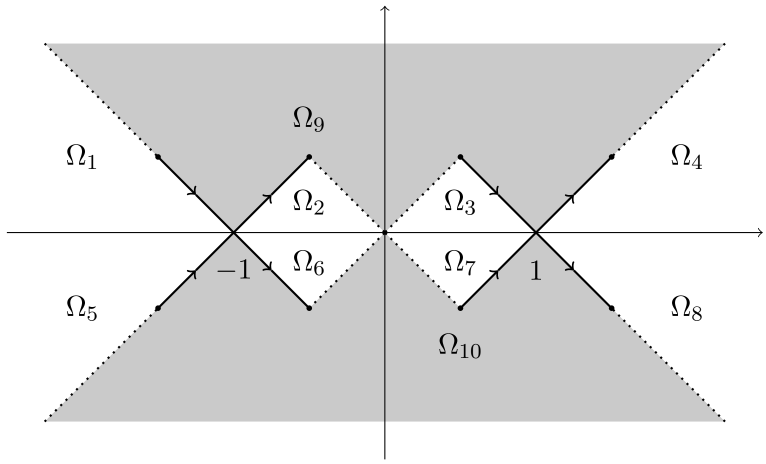


Figure 6.8: *Decomposition of* $\mathbb{C}\setminus(\mathbb{R}\cup\Sigma^{(2)})$ *into ten connected components.* $\mathcal{W}$ *is continuous across the dotted lines.*

we obtain $\gamma(\zeta) = \beta(\zeta) + \mathfrak{b}^{(-)}(\zeta) + \mathfrak{b}^{(+)}(\zeta)$. The functions $\mathfrak{b}^{(\pm)}$ are continuous around ∓ 1. Indeed, we have

$$|\mathfrak{b}^{(\pm)}(\zeta) - \mathfrak{b}^{(\pm)}(\mp 1)| \leq c|\nu_0^{\pm}| \cdot |\zeta \pm 1|, \tag{6.4.7}$$

if ζ is close to ∓ 1. Since the function $s \mapsto \nu(s) - \varphi(s)$ is in $H^1(\mathbb{R})$ and has zeroes at ± 1 of at least order $1/2$, the values $\beta(\pm 1)$ exist and, moreover, as shown in Section A.2 in the Appendix, we have for $|\zeta \pm 1| \leq \frac{1}{\sqrt{2}}$, that

$$|\beta(\zeta) - \beta(\mp 1)| \leq c\|\nu'\|_{L^2((-1,-\frac{1}{10})\cup(\frac{1}{10},1))}|\zeta \pm 1|^{1/2} + \|\nu\|_{L^1(-\frac{1}{10},\frac{1}{10})}|\zeta \pm 1|. \tag{6.4.8}$$

Now, let $\zeta \in \mathbb{C}\setminus[-1,\infty)$ and choose $-\pi < \arg(-(\zeta+1)) < \pi$. An explicit calculation of $\mathfrak{b}^{(-)}$ yields

$$\mathfrak{b}^{(-)}(\zeta) = i\nu_0^- + i\nu_0^-\zeta\log(-\zeta) - i\nu_0^-(\zeta+1)\log(-(\zeta+1)) + i\nu_0\log(-(\zeta+1)),$$

and we learn that $|\mathfrak{b}^{(-)}(\zeta) - i\nu_0^- - i\nu_0\log(-(\zeta+1))| \leq c|\nu_0^-| \cdot |\zeta+1|^{1/2}$ if ζ is close to -1. Making use of (6.4.7) and (6.4.8), we obtain

$$|\gamma(\zeta) - \beta(-1) - \mathfrak{b}^{(+)}(-1) - i\nu_0^- - i\nu_0\log(-(\zeta+1))| \leq c\mathfrak{C}(\nu)|\zeta+1|^{1/2}.$$

This in turn implies that (6.4.5) holds for

$$\delta_0^- = \exp\{\beta(-1) + \mathfrak{b}^{(+)}(-1) + i\nu_0^-\}. \tag{6.4.9}$$

Using very similar computations around $+1$ one can show that (6.4.6) holds for

$$\delta_0^+ = \exp\{\beta(1) + \mathfrak{b}^{(-)}(1) - i\nu_0^+\}. \tag{6.4.10}$$

The property $|\delta_0^{\pm}| = 1$ is obvious and this concludes the proof. □

6.4.2 $\overline{\partial}$-extensions of jump factorization

In this section we specify the transformation $M^{(1)} \to M^{(2)}$, given explicitly by (6.2.7). The purpose of this deformation is to remove the jump along the real axis and introduce jumps on $\Sigma^{(2)}$, where $\Sigma^{(2)} = \Sigma_1^{(2)} \cup ... \cup \Sigma_8^{(2)}$ is depicted in Figure 6.1. Therefore we need to construct the function $\mathcal{W}$ piecewisely on $\Omega_1, ..., \Omega_{10}$, where we denote by Ω_j the components of $\mathbb{C}\setminus(\mathbb{R}\cup\Sigma^{(2)})$ as sketched in Figure 6.8. Assume that the functions $R_k : \overline{\Omega}_k \to \mathbb{C}$, $k = 1, ..., 8$ satisfy the following boundary conditions:

$$R_1(\zeta) = \begin{cases} \breve{\rho}(\zeta)\delta^{-2}(\zeta), & \zeta \in (-\infty,-1), \\ \breve{\rho}(-1)(\delta_0^-)^{-2}(-(\zeta+1))^{-2i\nu_0^-}, & \zeta \in \Sigma_1^{(2)}, \end{cases} \tag{6.4.11a}$$

$$R_2(\zeta) = \begin{cases} \dfrac{\rho(\zeta)}{1+\rho(\zeta)\breve{\rho}(\zeta)}\delta_+^2(\zeta), & \zeta \in (-1,0), \\ \dfrac{\rho(-1)}{1+\rho(-1)\breve{\rho}(-1)}(\delta_0^-)^2(-(\zeta+1))^{2i\nu_0^-}(1-\chi(\zeta)), & \zeta \in \Sigma_2^{(2)}, \end{cases} \tag{6.4.11b}$$

$$R_3(\zeta) = \begin{cases} \dfrac{\rho(\zeta)}{1+\rho(\zeta)\breve{\rho}(\zeta)}\delta_+^2(\zeta), & \zeta \in (0,1), \\ \dfrac{\rho(1)}{1+\rho(1)\breve{\rho}(1)}(\delta_0^+)^2(\zeta-1)^{-2i\nu_0^+}(1-\chi(\zeta)), & \zeta \in \Sigma_3^{(2)}, \end{cases} \tag{6.4.11c}$$

$$R_4(\zeta) = \begin{cases} \breve{\rho}(\zeta)\delta^{-2}(\zeta), & \zeta \in (1,\infty), \\ \breve{\rho}(1)(\delta_0^+)^{-2}(\zeta-1)^{2i\nu_0^+}, & \zeta \in \Sigma_4^{(2)}, \end{cases} \tag{6.4.11d}$$

$$R_5(\zeta) = \begin{cases} \rho(\zeta)\delta^{2}(\zeta), & \zeta \in (-\infty,-1), \\ \rho(-1)(\delta_0^-)^{2}(-(\zeta+1))^{2i\nu_0^-}, & \zeta \in \Sigma_5^{(2)}, \end{cases} \tag{6.4.11e}$$

$$R_6(\zeta) = \begin{cases} \dfrac{\breve{\rho}(\zeta)}{1+\rho(\zeta)\breve{\rho}(\zeta)}\delta_-^{-2}(\zeta), & \zeta \in (-1,0), \\ \dfrac{\breve{\rho}(-1)}{1+\rho(-1)\breve{\rho}(-1)}(\delta_0^-)^{-2}(-(\zeta+1))^{-2i\nu_0^-}(1-\chi(\zeta)), & \zeta \in \Sigma_6^{(2)}, \end{cases} \tag{6.4.11f}$$

$$R_7(\zeta) = \begin{cases} \dfrac{\breve{\rho}(\zeta)}{1+\rho(\zeta)\breve{\rho}(\zeta)}\delta_-^{-2}(\zeta), & \zeta \in (0,1), \\ \dfrac{\breve{\rho}(1)}{1+\rho(1)\breve{\rho}(1)}(\delta_0^+)^{-2}(\zeta-1)^{2i\nu_0^+}(1-\chi(\zeta)), & \zeta \in \Sigma_7^{(2)}, \end{cases} \tag{6.4.11g}$$

$$R_8(\zeta) = \begin{cases} \rho(\zeta)\delta^{2}(\zeta), & \zeta \in (1,\infty), \\ \rho(1)(\delta_0^+)^{2}(\zeta-1)^{-2i\nu_0^+}, & \zeta \in \Sigma_8^{(2)}. \end{cases} \tag{6.4.11h}$$

We set

$$\mathcal{W}(\tau;\zeta) := \begin{cases} \begin{bmatrix} 1 & -R_k(\zeta)e^{i\tau Z(\zeta)} \\ 0 & 1 \end{bmatrix}, & \zeta \in \Omega_k, \quad k \in \{1,4\}, \\ \begin{bmatrix} 1 & 0 \\ -R_k(\zeta)e^{-i\tau Z(\zeta)} & 1 \end{bmatrix}, & \zeta \in \Omega_k, \quad k \in \{2,3\}, \\ \begin{bmatrix} 1 & 0 \\ R_k(\zeta)e^{i\tau Z(\zeta)} & 1 \end{bmatrix}, & \zeta \in \Omega_k, \quad k \in \{5,8\}, \\ \begin{bmatrix} 1 & R_k(\zeta)e^{i\tau Z(\zeta)} \\ 0 & 1 \end{bmatrix}, & \zeta \in \Omega_k, \quad k \in \{6,7\}, \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & \zeta \in \Omega_9 \cup \Omega_{10}, \end{cases} \tag{6.4.12}$$

and by (6.4.11a)–(6.4.11h) (first line in each case), we find that $\mathcal{W}$ satisfies (6.2.6). Moreover it follows from (6.4.11a)–(6.4.11h) (second line in each case) that $M^{(2)}$ admits jumps on $\Sigma^{(2)}$ of the form $M_+^{(2)} = M_-^{(2)}(1+R_\tau^{(2)})$ with $R_\tau^{(2)}$ precisely given by (6.2.8) and (6.2.9).
The lack of analyticity is measured by means of the following differential operator:

$$\overline{\partial} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \quad (\zeta = x+iy).$$

We would have $\overline{\partial}\mathcal{W} = 0$ if $\mathcal{W}$ was analytic. The general case is

$$\overline{\partial}\mathcal{W}(\tau;\zeta) := \begin{cases} \begin{bmatrix} 0 & -\overline{\partial} R_k(\zeta)e^{i\tau Z(\zeta)} \\ 0 & 0 \end{bmatrix}, & \zeta \in \Omega_k, \quad k \in \{1,4\}, \\ \begin{bmatrix} 0 & 0 \\ -\overline{\partial} R_k(\zeta)e^{-i\tau Z(\zeta)} & 0 \end{bmatrix}, & \zeta \in \Omega_k, \quad k \in \{2,3\}, \\ \begin{bmatrix} 0 & 0 \\ \overline{\partial} R_k(\zeta)e^{i\tau Z(\zeta)} & 0 \end{bmatrix}, & \zeta \in \Omega_k, \quad k \in \{5,8\}, \\ \begin{bmatrix} 0 & \overline{\partial} R_k(\zeta)e^{i\tau Z(\zeta)} \\ 0 & 0 \end{bmatrix}, & \zeta \in \Omega_k, \quad k \in \{6,7\}, \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, & \zeta \in \Omega_9 \cup \Omega_{10}. \end{cases} \tag{6.4.13}$$

Similar to [LPS18, CJ14, CP14, DM08] the following Lemma can be obtained by explicit construction.

Lemma 6.4.3. *Let $\rho, \breve{\rho}$ satisfy assumptions (6.1.7)–(6.1.9). Then for $k = 1, ..., 8$, there exist functions $R_k : \overline{\Omega}_k \to \mathbb{C}$ satisfying (6.4.11a)–(6.4.11h) and vanishing on $\overline{\Omega}_k \cap \partial(\Omega_9 \cup \Omega_{10})$, so that for $k \in \{1,2,5,6\}$ and $\zeta \in \Omega_k$,*

$$|\overline{\partial} R_k(\zeta)| \leq c\mathfrak{C}(\nu) \begin{cases} \dfrac{\Gamma_5(p_k) + \Gamma_6(p_k)}{|\zeta+1|^{1/2}} + |p_k'(\mathrm{Re}(\zeta))| + \Gamma_6(p_k)|\overline{\partial}\chi(\zeta)|, & \text{if } |\zeta+1| \leq \dfrac{1}{2}, \\ \dfrac{|p_k(\mathrm{Re}(\zeta))|}{|\zeta|} + |p_k'(\mathrm{Re}(\zeta))| + \Gamma_6(p_k)|\overline{\partial}\chi(\zeta)|, & \text{if } |\zeta+1| \geq \dfrac{1}{2}, \end{cases} \tag{6.4.14}$$

whereas for $k \in \{3,4,7,8\}$ and $\zeta \in \Omega_k$,

$$|\overline{\partial} R_k(\zeta)| \leq c\mathfrak{C}(\nu) \begin{cases} \dfrac{\Gamma_5(p_k) + \Gamma_6(p_k)}{|\zeta-1|^{1/2}} + |p_k'(\mathrm{Re}(\zeta))| + \Gamma_6(p_k)|\overline{\partial}\chi(\zeta)|, & \text{if } |\zeta-1| \leq \dfrac{1}{2}, \\ \dfrac{|p_k(\mathrm{Re}(\zeta))|}{|\zeta|} + |p_k'(\mathrm{Re}(\zeta))| + \Gamma_6(p_k)|\overline{\partial}\chi(\zeta)|, & \text{if } |\zeta-1| \geq \dfrac{1}{2}. \end{cases} \tag{6.4.15}$$

Proof. The functions R_k can be defined explicitly. Let us exemplarily consider the cases $k = 3$ and $k = 4$. We start with defining the cutoff function $\chi \in C_0^\infty(\mathbb{C}, [0,1])$. Therefore, let us denote by

$$E = \left\{ \frac{1}{2} \pm \frac{i}{2}, \frac{3}{2} \pm \frac{i}{2}, \frac{-1}{2} \pm \frac{i}{2}, \frac{-3}{2} \pm \frac{i}{2} \right\}$$

the eight end points of Σ_2. Now, set

$$\chi(\zeta) := \begin{cases} 1, & \text{if } \operatorname{dist}(E,\zeta) < \frac{1}{4}, \\ 0, & \text{if } \operatorname{dist}(E,\zeta) > \frac{3}{8}. \end{cases} \tag{6.4.16}$$

The reason for introducing this function will become clear later, see also Remark 6.4.4. Next, let $g : [0,\pi/4] \to [0,1]$ be a smooth function satisfying $g(\varphi) = 1$ for all $\varphi \in [0,\pi/6]$ and $g(\pi/4) = 0$, see Figure 6.9. Using $0 < \arg(\zeta) < \pi/4$ and $3\pi/4 < \arg(\zeta-1) < \pi$ for $\zeta \in \Omega_3$ and $0 < \arg(\zeta-1) < \pi/4$ for $\zeta \in \Omega_3$, we define:

$$\begin{aligned} G_{3,1}(\zeta) &= g(\pi - \arg(\zeta-1)), \\ G_{3,2}(\zeta) &= g(\arg(\zeta)), \\ G_4(\zeta) &= g(\arg(\zeta-1)). \end{aligned}$$

We refer to Figure 6.10 which shows the decomposition of Ω_3 and Ω_4 in further subregions. In these domains, the functions $G_{3,1}$, $G_{3,2}$ and G_4 satisfy

$$\begin{aligned} \zeta \in \Omega_{3,1}: &\quad |\overline{\partial} G_{3,1}(\zeta)| \leq c|\zeta-1|^{-1}, \\ \zeta \in \Omega_{3,2}: &\quad |\overline{\partial} G_{3,2}(\zeta)| \leq c|\zeta|^{-1}, \\ \zeta \in \Omega_4: &\quad |\overline{\partial} G_4(\zeta)| \leq c|\zeta-1|^{-1}, \end{aligned} \tag{6.4.17}$$

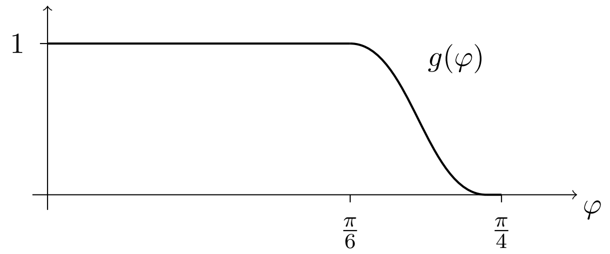


Figure 6.9: *The function g used in the construction of the functions R_k in the proof of Lemma 6.4.3*

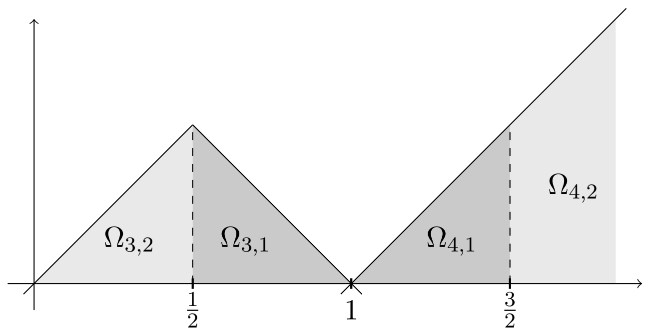


Figure 6.10: *Decompositon of Ω_3 and Ω_4 for the construction of R_3 and R_4 in Lemma 6.4.3.*

with a constant c that depends on the particular choice of g only. Now we are prepared to define the functions R_3 and R_4 by

$$R_3(\zeta) := \begin{cases} G_{3,1}(\zeta)\dfrac{\rho(\mathrm{Re}(\zeta))}{1+\rho(\mathrm{Re}(\zeta))\breve{\rho}(\mathrm{Re}(\zeta))}\delta^2(\zeta)(1-\chi(\zeta)) & \\ \quad + (1-G_{3,1}(\zeta))\,\dfrac{\rho(1)}{1+\rho(1)\breve{\rho}(1)}(\delta_0^+)^2(\zeta-1)^{-2i\nu_0^+}(1-\chi(\zeta)), & \zeta\in\Omega_{3,1}, \\ G_{3,2}(\zeta)\dfrac{\rho(\mathrm{Re}(\zeta))}{1+\rho(\mathrm{Re}(\zeta))\breve{\rho}(\mathrm{Re}(\zeta))}\delta^2(\zeta)(1-\chi(\zeta)), & \zeta\in\Omega_{3,2}, \end{cases} \tag{6.4.18}$$

and

$$R_4(\zeta) := \begin{cases} G_4(\zeta)\breve{\rho}(\mathrm{Re}(\zeta))\delta^{-2}(\zeta)(1-\chi(\zeta)) & \\ \quad + (1-G_4(\zeta))\,\breve{\rho}(1)(\delta_0^+)^{-2}(\zeta-1)^{2i\nu_0^+}(1-\chi(\zeta)), & \zeta\in\Omega_{4,1}, \\ G_4(\zeta)\breve{\rho}(\mathrm{Re}(\zeta))\delta^{-2}(\zeta)(1-\chi(\zeta)), & \zeta\in\Omega_{4,2}. \end{cases} \tag{6.4.19}$$

Immediately from this definition and from the definitions of $G_{3,1}$, $G_{3,2}$, G_4 and χ we can verify (6.4.11c) and (6.4.11d). Also, it is easy to verify that R_3 vanishes on $\{re^{i\pi/4} : r\in(0,1/\sqrt{2})\}$ and R_4 vanishes on $\{1+re^{i\pi/4} : r>1/\sqrt{2}\}$. Note that these lines are the dotted lines in Figure 6.8. Thus, it remains to prove the desired bounds (6.4.14). We start with R_3. Firstly, let $\zeta\in\Omega_{3,1}$. A direct computation yields $|\,\overline{\partial}\,R_3(\zeta)|\le W_1+W_2+W_3$ with

$$\begin{aligned} W_1 &= |\,\overline{\partial}\,G_{3,1}(\zeta)|\left|\frac{\rho(\mathrm{Re}(\zeta))}{1+\rho(\mathrm{Re}(\zeta))\breve{\rho}(\mathrm{Re}(\zeta))}\delta^2(\zeta)-\frac{\rho(1)}{1+\rho(1)\breve{\rho}(1)}(\delta_0^+)^2(\zeta-1)^{-2i\nu_0^+}\right|, \\ W_2 &= \|\delta\|^2_{L^\infty(\mathbb{C})}\left|\overline{\partial}\left(\frac{\rho(\mathrm{Re}(\zeta))}{1+\rho(\mathrm{Re}(\zeta))\breve{\rho}(\mathrm{Re}(\zeta))}\right)\right|, \\ W_3 &= \|\delta\|^2_{L^\infty(\mathbb{C})}\left\|\frac{\rho}{1+\rho\breve{\rho}}\right\|_{L^\infty(\frac{1}{2},1)}|\,\overline{\partial}\,\chi(\zeta)|, \end{aligned}$$

Using $|f(a)-f(b)|\le c\|f'\|_{L^2(a,b)}|a-b|^{1/2}$ for all functions $f\in H^1(\mathbb{R})$, Proposition 6.4.2 and (6.4.17) we find

$$\begin{aligned}
W_1 &\le |\,\overline{\partial}\,G_{3,1}(\zeta)|\left\{\left|\frac{\rho(1)}{1+\rho(1)\breve{\rho}(1)}\right|\left|\delta^2(\zeta)-(\delta_0^+)^2(\zeta-1)^{-2i\nu_0^+}\right|\right.\\
&\qquad\left.+\,|\delta^2(\zeta)|\left|\frac{\rho(\mathrm{Re}(\zeta))}{1+\rho(\mathrm{Re}(\zeta))\breve{\rho}(\mathrm{Re}(\zeta))}-\frac{\rho(1)}{1+\rho(1)\breve{\rho}(1)}\right|\right\}\\
&\le c\left(\mathfrak{C}(\nu)\left|\frac{\rho(1)}{1+\rho(1)\breve{\rho}(1)}\right|+\left\|\partial_\zeta\left(\frac{\rho}{1+\rho\breve{\rho}}\right)\right\|_{L^2(\frac12,1)}\|\delta\|^2_{L^\infty(\mathbb{C})}\right)|\zeta-1|^{-1/2}\\
&\le \mathfrak{C}(\nu)\left(\Gamma_5(p_3)+\Gamma_5(p_3)\right)|\zeta-1|^{-1/2}.
\end{aligned}$$

Here, we have also made use of (6.4.3). Now, let us assume that $\zeta\in\Omega_{3,2}$. Then, $|\,\overline{\partial}\,R_3(\zeta)|\le V_1+V_2+V_3$ with

$$\begin{aligned}
V_1 &= |\,\overline{\partial}\,G_{3,2}(\zeta)|\left|\frac{\rho(\mathrm{Re}(\zeta))}{1+\rho(\mathrm{Re}(\zeta))\breve{\rho}(\mathrm{Re}(\zeta))}\delta^2(\zeta)\right|,\\
V_2 &= \|\delta\|^2_{L^\infty(\mathbb{C})}\left|\overline{\partial}\left(\frac{\rho(\mathrm{Re}(\zeta))}{1+\rho(\mathrm{Re}(\zeta))\breve{\rho}(\mathrm{Re}(\zeta))}\right)\right|,\\
V_3 &= \|\delta\|^2_{L^\infty(\mathbb{C})}\left\|\frac{\rho}{1+\rho\breve{\rho}}\right\|_{L^\infty(\frac14,\frac12)}|\,\overline{\partial}\,\chi(\zeta)|
\end{aligned}$$

As for V_3, note that the support of $\overline{\partial}\,\chi$ stays away from the origin. Therefore, it is justified to use $\|\rho/(1+\rho\breve{\rho})\|_{L^\infty(\frac14,\frac12)}$ instead of $\|\rho/(1+\rho\breve{\rho})\|_{L^\infty(0,\frac12)}$. This completes the proof of (6.4.15) for $k=3$. R_4 is estimated in a similar way and the other R_k-s can be defined similarly to (6.4.18) and (6.4.19). The idea in each case is to use $\arg(\zeta\pm 1)$ and the function g to interpolate between the first and second line of (6.4.11a)–(6.4.11h). For technical reasons it is necessary to include the function $\chi(\zeta)$, see Remark 6.4.4 below. □

Remark 6.4.4. Since, for example, R_3 is defined piecewisely on $\Omega_{3,1}$ and $\Omega_{3,2}$, one would expect a discontinuity on the separating line $\{1/2+i\,t: 0\le t\le 1/2\}$. But the reader might check that the presence of the cutoff function χ, introduced in (6.4.16), and the fact that the function g is constant on $[0,\pi/6]$, as depicted in Figure 6.9, helps us to avoid this kind of discontinuity for R_3. The same holds for the other functions R_k.

6.4.3 Solvability of the $\overline{\partial}$ problem

The remaining part of the proof of Lemma 6.1.2 is the analysis of the $\overline{\partial}$-part of $M^{(2)}$ in (6.2.7), see (6.2.11). We show that the contribution of $\overline{\partial}\,\mathcal{W}$ in (6.2.11) goes to 0 to higher order and the asymtotics of $q^{(2)}$ through equation (6.2.1) are determined by the RHP part of $M^{(2)}$, see (6.2.12). $\overline{\partial}$-problem 6.2.3 is solved by finding a solution of $D=1+J[D]$, where

$$J[D](k) := \frac{1}{\pi}\int_{\mathbb{C}}\frac{D(\zeta)\Upsilon(\zeta)}{\zeta-k}dA(\zeta). \tag{6.4.20}$$

The definition of Υ was given in (6.2.15). For J we can prove the following.

Proposition 6.4.5. *For the operator in (6.4.20) we have $J: L^\infty(\mathbb{C})\to L^\infty(\mathbb{C})$ and there exists a constant $c>0$ such that*

$$\|J\|_{L^\infty(\mathbb{C})\to L^\infty(\mathbb{C})}\le c\mathfrak{C}(\nu)\left(\widetilde{\mathcal{C}}(\rho,\breve{\rho})(\tau^{-1/4}+\tau^{-1/2})+\sum_{k=1}^{8}(\Gamma_5(p_k)+\Gamma_6(p_k))\tau^{-1/4}\right) \tag{6.4.21}$$

for all $\tau\in\mathbb{R}^+$. The constant c is independent of τ, ρ and $\breve{\rho}$.

Proof. We decompose the operator into $J = J_1 + ... + J_8$, where

$$J_j[D](k) := \frac{1}{\pi}\int_{\Omega_j}\frac{D(\zeta)\Upsilon(\zeta)}{\zeta - k}dA(\zeta), \qquad j \in \{1, ..., 8\},$$

and prove (6.4.21) first for J_3 and afterwards for J_4. Other values for j are similar to one of these. We recall the decomposition of Ω_3 and Ω_4 defined by Figure 6.10 and claim that the following holds:

$$\begin{aligned}
\zeta = x+iy \in \Omega_{3,1}: &\qquad |e^{-i\tau Z(\zeta)}| \le e^{\frac{-\tau y(1-x)}{2}},\\
\zeta = x+iy \in \Omega_{3,2}: &\qquad |e^{-i\tau Z(\zeta)}| \le e^{\frac{-\tau y}{8x^2}},\\
\zeta = x+iy \in \Omega_{4,1}: &\qquad |e^{i\tau Z(\zeta)}| \le e^{\frac{-\tau y(x-1)}{6}},\\
\zeta = x+iy \in \Omega_{4,2}: &\qquad |e^{i\tau Z(\zeta)}| \le e^{\frac{-\tau y}{4}}.
\end{aligned} \tag{6.4.22}$$

These estimates are derived by elementary computations. For the sake of completeness one can find a proof of (6.4.22) in Section B.5 of the Appendix.

From Lemma 6.4.3 we know that for an arbitrary $k \in \mathbb{C}$,

$$\left|\int_{\Omega_{3,1}}\frac{D(\zeta)\Upsilon(\zeta)}{\zeta - k}dA(\zeta)\right| \le \|D\|_{L^\infty(\mathbb{C})}\mathfrak{C}(\nu)\left(\Big[\Gamma_5(p_3) + \Gamma_6(p_3)\Big]\,\mathbf{I}_1 + \mathbf{I}_2 + \Gamma_6(p_3)\,\mathbf{I}_3\right),$$

where $\mathbf{I}_1$, $\mathbf{I}_2$ and $\mathbf{I}_3$ are defined by

$$\mathbf{I}_1 := \int_{\Omega_{3,1}}\frac{|\zeta-1|^{-1/2}\,|e^{-i\tau Z(\zeta)}|}{|\zeta - k|}dA(\zeta), \quad \mathbf{I}_2 := \int_{\Omega_{3,1}}\frac{|p_3'(\mathrm{Re}(\zeta)||e^{-i\tau Z(\zeta)}|}{|\zeta - k|}dA(\zeta),$$

$$\mathbf{I}_3 := \int_{\Omega_{3,1}}\frac{\left|\overline{\partial}\,\chi(\zeta)\right|\,|e^{-i\tau Z(\zeta)}|}{|\zeta - k|}dA(\zeta).$$

Let us begin with estimating $\mathbf{I}_1$. Writing $\zeta = x + iy$ and $k = \alpha + i\beta$ and using (6.4.22), we obtain

$$\begin{aligned}
\mathbf{I}_1 \le& \int_0^{1/2}\int_{1/2}^{1-y}\frac{((x-1)^2+y^2)^{-1/4}e^{\frac{-\tau y(1-x)}{2}}}{\sqrt{(x-\alpha)^2+(y-\beta)^2}}dxdy\\
\le& \int_0^{1/2} e^{\frac{-\tau y^2}{2}}\left\|\frac{1}{(x^2+y^2)^{1/4}}\right\|_{L^p_x(-\frac{1}{2},-y)}\left\|\frac{1}{\sqrt{(x-\alpha)^2+(y-\beta)^2}}\right\|_{L^q_x(\frac{1}{2},1-y)}dy,
\end{aligned}$$

where $1 < q < 2 < p < \infty$ are supposed to satisfy $\frac{1}{p} + \frac{1}{q} = 1$. A direct computation shows that

$$\left\|(x^2+y^2)^{-1/4}\right\|_{L^p_x(-1/2,-y)} \le |y|^{-1/2+1/p}. \tag{6.4.23}$$

In addition, analogously to the first line of (5.3.10), we find that

$$\left\|((x-\alpha)^2+(y-\beta)^2)^{-1/2}\right\|_{L^q_x(\frac{1}{2},1-y)} \le c|y-\beta|^{\frac{1}{q}-1}$$

and hence,

$$\mathbf{I}_1 \le c\int_0^{1/2} e^{\frac{-\tau y^2}{2}}|y-\beta|^{\frac{1}{q}-1}|y|^{-\frac{1}{2}+\frac{1}{p}}dy.$$

Using $|y-\beta|^a|y|^b \le c(|y-\beta|^{a+b} + |y|^{a+b})$, it follows that

$$\mathbf{I}_1 \le c\int_0^{1/2} e^{\frac{-\tau y^2}{2}}\left(|y-\beta|^{-\frac{1}{2}} + |y|^{-\frac{1}{2}}\right)dy.$$

Furthermore, for any $\beta \in \mathbb{R}$,

$$\int_{\mathbb{R}} \frac{e^{\frac{-\tau y^2}{2}}}{|y-\beta|^{1/2}} dy \leq \int_{|y|\leq|y-\beta|} \frac{e^{\frac{-\tau y^2}{2}}}{|y|^{1/2}} dy + \int_{|y|\geq|y-\beta|} \frac{e^{\frac{-\tau(y-\beta)^2}{2}}}{|y-\beta|^{1/2}} dy \leq 2\int_{\mathbb{R}} \frac{e^{\frac{-\tau y^2}{2}}}{|y|^{1/2}} dy \leq c\tau^{-1/4}, \tag{6.4.24}$$

such that, finally, $\mathbf{I}_1 \leq c\tau^{-1/4}$. Next, we obtain

$$\begin{aligned}
\mathbf{I}_2 &\leq \int_0^{1/2}\int_{1/2}^{1-y} \frac{|p_3'(x)| e^{\frac{-\tau y(1-x)}{2}}}{\sqrt{(x-\alpha)^2+(y-\beta)^2}} dx dy \\
&\leq \int_0^{1/2} e^{\frac{-\tau y^2}{2}} \|p_3'\|_{L^2_x(\frac{1}{2},1-y)} \left\| \frac{1}{\sqrt{(x-\alpha)^2+(y-\beta)^2}} \right\|_{L^2_x(\frac{1}{2},1-y)} dy \\
&\leq c\Gamma_5(p_3) \int_0^{1/2} e^{\frac{-\tau y^2}{2}} |y-\beta|^{-1/2} dy \\
&\leq c\Gamma_5(p_3)\tau^{-1/4},
\end{aligned}$$

where we could use (6.4.24) again. In order to estimate $\mathbf{I}_3$, we observe that, if $\zeta \in F_3$, where

$$F_3 := \Omega_3 \cap \left\{\zeta' \in \mathbb{C} : \overline{\partial}\chi(\zeta') \neq 0\right\},$$

then $|e^{-i\tau Z(\zeta)}| \leq e^{-\tau c_0}$ with some positive constant c_0. Since $\overline{\partial}\chi$ is bounded, it follows that

$$\mathbf{I}_3 \leq ce^{-\tau c_0} \int_{F_3} \frac{1}{|\zeta-k|} dA(\zeta) \leq ce^{-\tau c_0}.$$

In order to complete the estimates for J_3, we use Lemma 6.4.3 again, which tells us that for an arbitrary $k \in \mathbb{C}$,

$$\left| \int_{\Omega_{3,2}} \frac{D(\zeta)\Upsilon(\zeta)}{\zeta-k} dA(\zeta) \right| \leq \|D\|_{L^\infty(\mathbb{C})} \mathfrak{C}(\nu) \Big(\mathbf{J}_1 + \mathbf{J}_2 + \Gamma_6(p_3)\, \mathbf{J}_3 \Big),$$

where $\mathbf{J}_1$, $\mathbf{J}_2$ and $\mathbf{J}_3$ are defined by

$$\mathbf{J}_1 := \int_{\Omega_{3,2}} \frac{|p_3(\mathrm{Re}(\zeta)|\, |e^{-i\tau Z(\zeta)}|}{|\zeta||\zeta-k|} dA(\zeta), \quad \mathbf{J}_2 := \int_{\Omega_{3,2}} \frac{|p_3'(\mathrm{Re}(\zeta)||e^{-i\tau Z(\zeta)}|}{|\zeta-k|} dA(\zeta),$$

$$\mathbf{J}_3 := \int_{\Omega_{3,2}} \frac{|\overline{\partial}\chi(\zeta)|\, |e^{-i\tau Z(\zeta)}|}{|\zeta-k|} dA(\zeta).$$

Using the second line of 6.4.22 we observe that $\mathbf{J}_1$ and $\mathbf{J}_2$ are similar to the integrals $\mathcal{I}_1$ and $\mathcal{I}_3$ in the proof of Proposition 5.3.2. Therefore, by the same arguments as in the proof of Proposition 5.3.2, we obtain

$$\mathbf{J}_1 \leq c\tau^{-1/4} \||x|^{-1/2} p_3(x)\|_{L^2_x(0,\frac{1}{2})}, \qquad \mathbf{J}_2 \leq c\tau^{-1/4} \||x|^{1/2} p_3'(x)\|_{L^2_x(0,\frac{1}{2})}.$$

In addition, $\mathbf{J}_3$ is estimated in the same way as $\mathbf{I}_3$ above. That means, $\mathbf{J}_3 \leq ce^{-\tau c_0}$. Summarizing our previous estimates, we find

$$\|J_3\|_{L^\infty(\mathbb{C})\to L^\infty(\mathbb{C})} \leq c\mathfrak{C}(\nu) \left(\widetilde{\mathcal{C}}(\rho,\breve{\rho}) + \Gamma_5(p_3) + \Gamma_6(p_3) \right) \tau^{-1/4}.$$

Let us now turn to J_3. From Lemma 6.4.3 we know that for an arbitrary $k \in \mathbb{C}$,

$$\left| \int_{\Omega_{4,1}} \frac{D(\zeta)\Upsilon(\zeta)}{\zeta-k} dA(\zeta) \right| \leq \|D\|_{L^\infty(\mathbb{C})} \mathfrak{C}(\nu) \Big(\Big[\Gamma_5(p_4) + \Gamma_6(p_4)\Big] \mathbf{K}_1 + \mathbf{K}_2 + \Gamma_6(p_4)\, \mathbf{K}_3 \Big),$$

where $\mathbf{K}_1$, $\mathbf{K}_2$ and $\mathbf{K}_3$ are defined by

$$\mathbf{K}_1 := \int_{\Omega_{4,1}} \frac{|\zeta-1|^{-1/2}\, |e^{i\tau Z(\zeta)}|}{|\zeta-k|} dA(\zeta), \quad \mathbf{K}_2 := \int_{\Omega_{4,1}} \frac{|p_4'(\mathrm{Re}(\zeta)||e^{i\tau Z(\zeta)}|}{|\zeta-k|} dA(\zeta),$$

$$\mathbf{K}_3 := \int_{\Omega_{4,1}} \frac{\left|\overline{\partial}\,\chi(\zeta)\right| \left|e^{i\tau Z(\zeta)}\right|}{|\zeta - k|} dA(\zeta).$$

Analogously to the above estimate of $\mathbf{I}_1$ we find by (6.4.22), that

$$\begin{aligned}
\mathbf{K}_1 \leq & \int_0^{1/2} \int_{y+1}^{3/2} \frac{((x-1)^2+y^2)^{-1/4} e^{\frac{-\tau y(x-1)}{6}}}{\sqrt{(x-\alpha)^2+(y-\beta)^2}} dxdy \\
\leq & \int_0^{1/2} e^{\frac{-\tau y^2}{6}} \left\| \frac{1}{(x^2+y^2)^{1/4}} \right\|_{L_x^p(y,\frac{1}{2})} \left\| \frac{1}{\sqrt{(x-\alpha)^2+(y-\beta)^2}} \right\|_{L_x^q(y+1,\frac{3}{2})} dy.
\end{aligned}$$

Proceeding as for $\mathbf{I}_1$, we finally end up with $\mathbf{K}_1 \leq c\tau^{-1/4}$. For $\mathbf{K}_2$ find

$$\begin{aligned}
\mathbf{K}_2 \leq & \int_0^{1/2} \int_{y+1}^{3/2} \frac{|p_4'(x)| e^{\frac{\tau y(x-1)}{6}}}{\sqrt{(x-\alpha)^2+(y-\beta)^2}} dxdy \\
\leq & \int_0^{1/2} e^{\frac{-\tau y^2}{6}} \|p_4'\|_{L_x^2(\frac{1}{2},1-y)} \left\| \frac{1}{\sqrt{(x-\alpha)^2+(y-\beta)^2}} \right\|_{L_x^2(\frac{1}{2},1-y)} dy \\
\leq & c\Gamma_5(p_4) \int_0^{1/2} e^{\frac{-\tau y^2}{6}} |y-\beta|^{-1/2} dy \\
\leq & c\Gamma_5(p_4)\tau^{-1/4}.
\end{aligned}$$

Finally, we have $\mathbf{K}_3 \leq ce^{-c_0\tau}$ for the same reasons as for $\mathbf{I}_3$ and $\mathbf{J}_3$. In order to complete the estimates for J_4, it remains to consider the integral over $\Omega_{4,2}$. We have

$$\left| \int_{\Omega_{3,2}} \frac{D(\zeta)\Upsilon(\zeta)}{\zeta - k} dA(\zeta) \right| \leq \|D\|_{L^\infty(\mathbb{C})} \mathfrak{C}(\nu) \left(\mathbf{L}_1 + \mathbf{L}_2 + \Gamma_6(p_3)\,\mathbf{L}_3 \right),$$

where $\mathbf{L}_1$, $\mathbf{L}_2$ and $\mathbf{L}_3$ are defined by

$$\mathbf{L}_1 := \int_{\Omega_{4,2}} \frac{|p_4(\mathrm{Re}(\zeta)|\left|e^{i\tau Z(\zeta)}\right|}{|\zeta||\zeta - k|} dA(\zeta), \quad \mathbf{L}_2 := \int_{\Omega_{4,2}} \frac{|p_4'(\mathrm{Re}(\zeta)||e^{i\tau Z(\zeta)}|}{|\zeta - k|} dA(\zeta),$$

$$\mathbf{L}_3 := \int_{\Omega_{4,2}} \frac{\left|\overline{\partial}\,\chi(\zeta)\right| \left|e^{i\tau Z(\zeta)}\right|}{|\zeta - k|} dA(\zeta).$$

Using the last line of 6.4.22 we observe that $\mathbf{L}_1$ and $\mathbf{L}_2$ are similar to the integrals $\mathcal{I}_2$ and $\mathcal{I}_4$ in the proof of Proposition 5.3.2. Therefore, by the same arguments as in the proof of Proposition 5.3.2, we obtain

$$\mathbf{L}_1 \leq c\tau^{-1/2} \||x|^{-1} p_4(x)\|_{L_x^2(\frac{3}{2},\infty)}, \qquad \mathbf{L}_2 \leq c\tau^{-1/2} \|p_4'(x)\|_{L_x^2(0,\frac{1}{2})}.$$

In addition, $\mathbf{L}_3$ is estimated in the same way as $\mathbf{I}_3$, $\mathbf{J}_3$ and $\mathbf{K}_3$ above. This means that, $\mathbf{L}_3 \leq ce^{-\tau c_0}$. Summarizing our previous estimates, we find

$$\|J_4\|_{L^\infty(\mathbb{C})\to L^\infty(\mathbb{C})} \leq c\mathfrak{C}(\nu) \left(\widetilde{\mathcal{C}}(\rho,\breve{\rho})\tau^{-1/2} + (\Gamma_5(p_4) + \Gamma_6(p_4))\tau^{-1/4} \right).$$

As mentioned at the beginning of the proof, the remaining operators J_j are handled in a similar way, such that (6.4.21) is now proven. □

6.4.4 Estimates on the solution of the $\overline{\partial}$ problem

Lemma 6.4.6. *There exists an ε_0 such that for alle $\tau > 0$ satisfying*

$$\mathfrak{C}(\nu) \left(\widetilde{\mathcal{C}}(\rho,\breve{\rho})(\tau^{-1/4} + \tau^{-1/2}) + \sum_{k=1}^{8} (\Gamma_5(p_k) + \Gamma_6(p_k))\tau^{-1/4} \right) < \varepsilon_0, \tag{6.4.25}$$

there exists a unique solution D for $\overline{\partial}$-problem 6.2.3. For $|\mathrm{Im}(\zeta)| \to \infty$, D has the property that

$$D(\tau;\zeta) = 1 + \frac{D_1(\tau)}{\zeta} + \mathcal{O}\left(\zeta^{-2}\right), \tag{6.4.26}$$

where

$$|D_1(\tau)| \leq C\|M^{(3)}\|_{L^\infty(\mathbb{C})}\mathfrak{C}(\nu)\left(\sum_{k=1}^{8}\left[\Gamma_5(p_k)+\Gamma_6(p_k)\right]\tau^{-3/4} + \sum_{\substack{1\leq j\leq 4\\ 1\leq k\leq 8}}\Gamma_j(p_k)\tau^{-1}\right) \tag{6.4.27}$$

and

$$\begin{aligned} |[D_1(\tau)]_{12}| \leq C\|M^{(3)}\|_{L^\infty(\mathbb{C})}\mathfrak{C}(\nu)&\left\{\left(\sum_{k\in\{1,4,6,7\}}\left[\Gamma_5(p_k)+\Gamma_6(p_k)\right]\tau^{-3/4} + \sum_{\substack{1\leq j\leq 4\\ k\in\{1,4,6,7\}}}\Gamma_j(p_k)\tau^{-1}\right)\right.\\ &\left.+\|[M^{(3)}]_{12}\|_{L^\infty(\mathbb{C})}\left(\sum_{k\in\{2,3,5,8\}}\left[\Gamma_5(p_k)+\Gamma_6(p_k)\right]\tau^{-3/4} + \sum_{\substack{1\leq j\leq 4\\ k\in\{2,3,5,8\}}}\Gamma_j(p_k)\tau^{-1}\right)\right\}. \end{aligned} \tag{6.4.28}$$

Moreover,

$$\begin{aligned} |D(\tau;0)-1| &\leq C\|M^{(3)}\|_{L^\infty(\mathbb{C})}\mathfrak{C}(\nu)\\ &\times\left(\sum_{k=1}^{8}\left[\Gamma_5(p_k)+\Gamma_6(p_k)\right]\tau^{-3/4} + \sum_{\substack{j\in\{2,4\}\\ 1\leq k\leq 8}}\Gamma_j(p_k)\tau^{-1} + \sum_{\substack{j\in\{1,3\}\\ k\in\{2,3,6,7\}}}\Gamma_j(\widetilde{p}_k)\tau^{-1}\right), \end{aligned} \tag{6.4.29}$$

where $\widetilde{p}_k(\zeta) := p_k(\zeta)/|\zeta|$.

Proof. We find a solution of $\overline{\partial}$-problem 6.2.3 by solving the integral equation $D = 1 + J[D]$. Thanks to Proposition 6.4.5, this equation is uniquely solvable in the space $L^\infty(\mathbb{C})$ for all $\tau > 0$ satisfying (6.4.25). Moreover, we have $\|D\|_{L^\infty(\mathbb{C})} \leq c$ uniformly for these τ. Therefore, the coefficient D_1 in the expansion (6.4.26) can be expressed by $D_1 = \frac{1}{\pi}\int_{\mathbb{C}} D\Upsilon dA$. It follows that in order to bound $|D_1|$, it suffices to bound $\int_{\Omega_j}\Upsilon dA$ for $j = 1,...,8$. Let us consider the case of $j = 4$. Analogously to the proof of Proposition 6.4.5 we have

$$\left|\int_{\Omega_{4,1}} D(\zeta)\Upsilon(\zeta)dA(\zeta)\right| \leq \|D\|_{L^\infty(\mathbb{C})}\|M^{(3)}\|^2_{L^\infty(\mathbb{C})}\mathfrak{C}(\nu)\left(\left[\Gamma_5(p_4)+\Gamma_6(p_4)\right]\mathbf{K}_1' + \mathbf{K}_2' + \Gamma_6(p_4)\,\mathbf{K}_3'\right)$$

where $\mathbf{K}_1'$, $\mathbf{K}_2'$ and $\mathbf{K}_3'$ are defined by

$$\mathbf{K}_1' := \int_{\Omega_{4,1}}|\zeta-1|^{-1/2}\left|e^{i\tau Z(\zeta)}\right|dA(\zeta), \quad \mathbf{K}_2' := \int_{\Omega_{4,1}}|p_4'(\mathrm{Re}(\zeta)||e^{i\tau Z(\zeta)}|dA(\zeta),$$

$$\mathbf{K}_3' := \int_{\Omega_{4,1}}\left|\overline{\partial}\,\chi(\zeta)\right|\left|e^{i\tau Z(\zeta)}\right|dA(\zeta).$$

Using $|e^{i\tau Z(\zeta)}| \leq e^{\frac{-\tau y(x-1)}{6}}$ for $\zeta = x + iy \in \Omega_{4,1}$ (see (6.4.22)), we find

$$\begin{aligned} \mathbf{K}_1' &\leq \int_0^{1/2}\int_{y+1}^{3/2}((x-1)^2+y^2)^{-1/4}e^{\frac{-\tau y(x-1)}{6}}dxdy\\ &\leq \int_0^{1/2}\left\|\frac{1}{(x^2+y^2)^{1/4}}\right\|_{L^p_x(y,\frac{1}{2})}\left\|e^{\frac{-\tau y(x-1)}{6}}\right\|_{L^q_x(y+1,\frac{3}{2})}dy, \end{aligned}$$

where $1 < q < 2 < p < \infty$ are supposed to satisfy $\frac{1}{p} + \frac{1}{q} = 1$. By (6.4.23) and

$$\left\| e^{\frac{-\tau y(x-1)}{6}} \right\|_{L^q_x(y+1,\frac{3}{2})} \leq \left\| e^{\frac{-\tau yx}{6}} \right\|_{L^q_x(y,\infty)} = \left(\frac{6}{\tau y q} \right)^{1/q} e^{\frac{-\tau y^2}{6}},$$

it follows that

$$\mathbf{K}_1' \leq c\tau^{-1/q} \int_0^{1/2} |y|^{-1/2+1/p-1/q} e^{\frac{-\tau y^2}{6}} dy.$$

Let us choose explicitly $p = 3$ and $q = \frac{3}{2}$. We find

$$\mathbf{K}_1' \leq c\tau^{-2/3} \int_0^{1/2} |y|^{-5/6} e^{\frac{-\tau y^2}{6}} dy = c\tau^{-2/3-1/2+5/12} \int_0^{\sqrt{\tau}/2} |z|^{-5/6} e^{\frac{-z^2}{6}} dz \leq c\tau^{-3/4}.$$

Continuing,

$$\begin{aligned}
\mathbf{K}_2' \leq & \int_0^{1/2} \int_{y+1}^{3/2} |p_4'(x)| e^{\frac{-\tau y(x-1)}{6}} dx dy \\
\leq & \int_0^{1/2} \|p_4'\|_{L^2_x(y+1,\frac{3}{2})} \left\| e^{\frac{-\tau y(x-1)}{6}} \right\|_{L^2_x(y+1,\frac{3}{2})} dy \\
\leq & \; c\tau^{-1/2} \Gamma_5(p_4) \int_0^{1/2} y^{-1/2} e^{\frac{-\tau y^2}{6}} dy \\
\leq & \; c\tau^{-3/4} \Gamma_5(p_4).
\end{aligned}$$

For $\mathbf{K}_3'$ we argue as in the proof of Proposition 6.4.5 for $\mathbf{I}_3$, $\mathbf{J}_3$, $\mathbf{K}_3$ and $\mathbf{L}_3$. It is clear from the above estimates of $\mathbf{K}_j'$ that the remaining integrals such as

$$\left| \int_{\Omega_{4,2}} D(\zeta)\Upsilon(\zeta) dA(\zeta) \right|,$$

can be handled similarly to the proof of Proposition 6.4.5. Hence,

$$\left| \int_{\Omega_{4,2}} \frac{D(\zeta)\Upsilon(\zeta)}{\zeta - k} dA(\zeta) \right| \leq \|D\|_{L^\infty(\mathbb{C})} \|M^{(3)}\|_{L^\infty(\mathbb{C})} \mathfrak{C}(\nu) \left(\mathbf{L}_1' + \mathbf{L}_2' + \Gamma_6(p_3)\, \mathbf{L}_3' \right),$$

where

$$\mathbf{L}_1' \leq c\tau^{-1}\Gamma_4(p_4), \qquad \mathbf{L}_2' \leq c\tau^{-1}\Gamma_2(p_2) \qquad \mathbf{L}_3' \leq ce^{-c\tau}.$$

This yields the proof of (6.4.27). In order to show (6.4.27) we recall from (6.4.13) that for $k \in \{2, 3, 5, 8\}$ and $\zeta \in \Omega_k$, the matrix $\overline{\partial}\, \mathcal{W}(\zeta)$ is of the form

$$\begin{bmatrix} 0 & 0 \\ \overline{\partial}\, R_k(\zeta) e^{\pm i\tau Z(\zeta)} & 0 \end{bmatrix}.$$

Hence, the function $\overline{\partial}\, R_k$ can contribute to the 1-2-entry of

$$DM^{(3)}\, \overline{\partial}\, \mathcal{W} [M^{(3)}]^{-1}$$

only in combination with $[M^{(3)}]_{12}$. This observation yields (6.4.28).

For the estimate of

$$D(\tau; 0) = \int_{\Omega_{3,1}} \frac{D(\zeta)\Upsilon(\zeta)}{\zeta} dA(\zeta),$$

we make the following observation. Integrals such as

$$\left| \int_{\Omega_{3,1}} \frac{D(\zeta)\Upsilon(\zeta)}{\zeta} dA(\zeta) \right|, \quad \left| \int_{\Omega_{4,1}} \frac{D(\zeta)\Upsilon(\zeta)}{\zeta} dA(\zeta) \right|, \quad \left| \int_{\Omega_{4,2}} \frac{D(\zeta)\Upsilon(\zeta)}{\zeta} dA(\zeta) \right|$$

are analyzed in the same way as the integrals above. This is because $|\zeta|^{-1}$ is bounded in the domains $\Omega_{3,1}$, $\Omega_{4,1}$ and $\Omega_{4,2}$. On the other hand, $|\zeta|^{-1}$ is clearly unbounded on $\Omega_{3,2}$. As a consequence we have to replace the function p_3 by

$$\widetilde{p}_3(\zeta) := \frac{p_3(\zeta)}{|\zeta|}.$$

In the end this change only affects the quantities Γ_1 and Γ_3. $\square$

Chapter 7

Soliton resolution problem

7.1 Steepest descent with solitons

We begin this section by recalling several concepts, introduced in Section 4.3, namely: assume $(u(t,x), v(t,x))$ solves the massive Thirring model and admits scattering data $\mathcal{S}(u,v) = (p, \{\lambda_j, C_j\}_{j=1}^N)$. Then, for given $\nu_0 \in (-1,1)$, in regions of the form

$$\left\{(x,t) \in \mathbb{R}^2 : |x - \nu_0 t| \leq \sqrt{|t|}\right\},$$

the long-time behavior of $(u(t,x), v(t,x))$ depends on whether there exists at least one index $j \in \{1,..,N\}$ such that

$$\frac{|\lambda_j|^{-2} - |\lambda_j|^2}{|\lambda_j|^{-2} + |\lambda_j|^2} = \nu_0, \tag{7.1.1}$$

or not. In the latter case, (u,v) behaves like a pure radiation solution and thus scatters to a linear solution modulo phase correction as shown in Theorem 6.1.4. In the former case there exist some eigenvalues λ_j satisfying (7.1.1) which is equivalent to $|\lambda_j|^2 = L_0$, where

$$L_0 := \sqrt{\frac{1-\nu_0}{1+\nu_0}}.$$

Other eigenvalues λ_k satisfying $|\lambda_k|^2 \neq L_0$ may also be contained in the scattering data, but they are not 'visible' in the narrow set $\left\{(x,t) \in \mathbb{R}^2 : |x - \nu_0 t| \leq \sqrt{|t|}\right\}$. As it was shown in the proof of Theorem 4.3.2, this set contains the set

$$\widetilde{S}(L_0) := \left\{(x,t) \in \mathbb{R}^2 : |w_0^{-1} - L_0| \leq \varepsilon / \sqrt{\tau}\right\} \tag{7.1.2}$$

for some sufficiently small $\varepsilon > 0$. Extending the steepest descent method of Chapter 6 to the presence of eigenvalues, the following theorem shows that in $\widetilde{S}(L_0)$ the solution converges to a pure soliton solution.

Theorem 7.1.1. *Let $L_0 \in \mathbb{R}^+$, $n \geq 1$ and suppose that $(u_0, v_0) \in \mathcal{G}_n$ with scattering data*

$$\mathcal{S}(u_0, v_0) = (p, \{\lambda_j, C_j\}_{j=1}^n)$$

such that

$$L_0 = |\lambda_1|^2 = ... = |\lambda_n|^2. \tag{7.1.3}$$

In addition, assume that $r, \widehat{r} \in X^{2,2}_{-2,0}$. Define $\mathcal{D}^\pm = \{\lambda_j, \widetilde{C}_j\}_{j=1}^n$ by

$$\begin{aligned} \widetilde{C}_j^+ &= C_j \exp\left\{\frac{-1}{\pi i}\left(\int_{-\infty}^{-L_0^{-1}} + \int_{L_0^{-1}}^{\infty}\right) \log(1 + w|r(w)|^2)\left(\frac{1}{w - w_j} - \frac{1}{2w}\right) dw\right\} \\ &= C_j \exp\left\{\frac{1}{\pi i}\int_{-L_0}^{L_0} \log(1 + z|\widehat{r}(z)|^2)\left(\frac{1}{z - z_j} - \frac{1}{2z}\right) dz\right\}, \end{aligned} \tag{7.1.4}$$

$$\begin{aligned}\widetilde{C}_j^- &= C_j \exp\left\{\frac{1}{\pi i}\left(\int_{-\infty}^{-L_0} + \int_{L_0}^{\infty}\right)\log(1+z|\widehat{r}(z)|^2)\left(\frac{1}{z-z_j}-\frac{1}{2z}\right)dz\right\} \\ &= C_j \exp\left\{\frac{-1}{\pi i}\int_{-L_0^{-1}}^{L_0^{-1}}\log(1+w|r(w)|^2)\left(\frac{1}{w-w_j}-\frac{1}{2w}\right)dw\right\}\end{aligned} \tag{7.1.5}$$

Then, using the notation as in Definition 4.1.1 and assuming $\pm t > t_0$ and $|w_0^{-1} - L_0| \le 1/\sqrt{\tau}$ (equivalently: $|z_0 - L_0| \le 1/\sqrt{\tau}$) we have that

$$|u(t,x) - u_{sol}(t,x;\mathcal{D}^{\pm})| + |v(x,t) - v_{sol}(t,x;\mathcal{D}^{\pm})| \le c|t|^{-1/2} \tag{7.1.6}$$

The constants t_0 and c depend on the initial data $\mathcal{S}(u_0, v_0)$ and on ϵ.

Proof. The proof has an identical structure to the one of Lemma 6.1.2. What makes the difference is the presence of singularities in the RHP's. If RHP 2.8.1 admits poles at $w_1, ..., w_n, \overline{w}_1, ..., \overline{w}_n$ (recall $w_j = \lambda_j^{-2}$), then the function $M^{(0)}(\zeta;\tau)$ defined in terms of M by (6.1.1) admits poles at $\zeta_1, ..., \zeta_n, \overline{\zeta}_1, ..., \overline{\zeta}_n$, where $\zeta_j = w_j/w_0$. The precise residuum relations that we have to add to RHP 6.1.1 are the following:

$$\begin{aligned}\operatorname*{Res}_{\zeta=\zeta_j} M^{(0)}(\tau;\zeta) &= \lim_{\zeta\to\zeta_j} M^{(0)}(\tau;\zeta)\begin{bmatrix} 0 & 0 \\ \frac{\zeta_j c_j}{d(w_j)^2}e^{-i\tau Z(\zeta_j)} & 0\end{bmatrix}, \\ \operatorname*{Res}_{\zeta=\overline{\zeta}_j} M^{(0)}(\tau;\zeta) &= \lim_{\zeta\to\overline{\zeta}_j} M^{(0)}(\tau;\zeta)\begin{bmatrix} 0 & \frac{-\overline{c}_j d(\overline{w}_j)^2}{w_0}e^{i\tau Z(\overline{\zeta}_j)} \\ 0 & 0\end{bmatrix}.\end{aligned} \tag{7.1.7}$$

Our assumption is that

$$|L_0^{-1} - w_0| \le \tau^{-1/2} \tag{7.1.8}$$

and thus, using (7.1.3), we know that the poles $\zeta_1, ..., \zeta_n$ lie in the region Ω_9, see Figure 6.8. This latter fact is useful because it guarantees that the modifications $M^{(0)} \mapsto M^{(1)} \mapsto M^{(2)}$ given by the explicit formulas (6.2.2) and (6.2.7) lead to a matrix valued function $M^{(2)}$, which is still meromorphic around $\zeta_1, ..., \zeta_n, \overline{\zeta}_1, ..., \overline{\zeta}_n$. This is because $\mathcal{W} = 1$ on $\Omega_9 \cup \Omega_{10}$. In fact, $M^{(2)}$ is a solution for the mixed $\overline{\partial}$-RHP 6.2.2, amended by

$$\begin{aligned}\operatorname*{Res}_{\zeta=\zeta_j} M^{(2)}(\tau;\zeta) &= \lim_{\zeta\to\zeta_j} M^{(2)}(\tau;\zeta)\begin{bmatrix} 0 & 0 \\ \frac{\zeta_j c_j \delta(\zeta_j)^2}{d(w_j)^2}e^{-i\tau Z(\zeta_j)} & 0\end{bmatrix}, \\ \operatorname*{Res}_{\zeta=\overline{\zeta}_j} M^{(2)}(\tau;\zeta) &= \lim_{\zeta\to\overline{\zeta}_j} M^{(2)}(\tau;\zeta)\begin{bmatrix} 0 & \frac{-\overline{c}_j d(\overline{w}_j)^2}{w_0\delta(\overline{\zeta}_j)^2}e^{i\tau Z(\overline{\zeta}_j)} \\ 0 & 0\end{bmatrix}.\end{aligned} \tag{7.1.9}$$

In contrast to the soliton-free case where we use decomposition (6.2.14) to seperate the $\overline{\partial}$-part and the RHP-part of $M^{(2)}$, here we need the following decomposition:

$$M^{(2)}(\tau;\zeta) = D(\tau;\zeta)M^{(mer)}(\tau;\zeta)M^{(3)}(\tau;\zeta). \tag{7.1.10}$$

The functions D, $M^{(mer)}$ and $M^{(3)}$ are chosen in the following way:

(i) $M^{(3)}$ is the solution for RHP($\Sigma^{(3)}$,$R_\tau^{(3)}$) with $R_\tau^{(3)}$ as in (6.2.13). In particular, it is possible to apply Proposition 6.3.4.

(ii) $M^{(mer)}(\tau;\zeta)$ is a meromorphic function on $\mathbb{C}$ with singularities at $\zeta_1, ..., \zeta_n, \overline{\zeta}_1, ..., \overline{\zeta}_n$ satisfying

$$\begin{aligned}\operatorname*{Res}_{\zeta=\zeta_j} M^{(mer)}(\tau;\zeta) &= \lim_{\zeta\to\zeta_j} M^{(mer)}(\tau;\zeta)[M^{(3)}(\tau;\zeta)]^{-1}\begin{bmatrix} 0 & 0 \\ \frac{\zeta_j c_j \delta(\zeta_j)^2}{d(w_j)^2}e^{-i\tau Z(\zeta_j)} & 0\end{bmatrix}M^{(3)}(\tau;\zeta), \\ \operatorname*{Res}_{\zeta=\overline{\zeta}_j} M^{(mer)}(\tau;\zeta) &= \lim_{\zeta\to\overline{\zeta}_j} M^{(mer)}(\tau;\zeta)[M^{(3)}(\tau;\zeta)]^{-1}\begin{bmatrix} 0 & \frac{-\overline{c}_j d(\overline{w}_j)^2}{w_0\delta(\overline{\zeta}_j)^2}e^{i\tau Z(\overline{\zeta}_j)} \\ 0 & 0\end{bmatrix}M^{(3)}(\tau;\zeta).\end{aligned} \tag{7.1.11}$$

(ii) $D(\tau;\zeta)$ solves the $\overline{\partial}$-problem 6.2.3 with

$$\Upsilon(\tau;\zeta) = M^{(3)}(\tau;\zeta)M^{(mer)}(\tau;\zeta)\,\overline{\partial}\,\mathcal{W}(\tau;\zeta)\left[M^{(3)}(\tau;\zeta)M^{(mer)}(\tau;\zeta)\right]^{-1} \tag{7.1.12}$$

replacing the expression given for Υ in (6.2.15). Since $\|M^{(mer)}(\tau;\cdot)\|_{L^\infty(\mathbb{C}\setminus(\Omega_9\cup\Omega_{10}))} \sim \mathcal{O}(1)$, all estimates for Υ that are presented in the analysis of (6.4.20) also hold for (7.1.12). As a consequence, we can use Lemma 6.4.6.

By the explicit transformation formulas (6.2.2) and (6.2.7), by Proposition 6.3.4 and by Lemma 6.4.6 it follows that

$$\begin{aligned}\left|\lim_{\zeta\to\infty}\zeta\left[M^{(0)}(\tau;\zeta)\right]_{12} - \lim_{\zeta\to\infty}\zeta\left[M^{(mer)}(\tau;\zeta)\right]_{12}\right| &\leq c\tau^{-1/2},\\ \left|M^{(0)}(\tau,0) - M^{(mer)}(\tau,0)[\delta(0)]^{-\sigma_3}\right| &\leq c\tau^{-1/2}.\end{aligned} \tag{7.1.13}$$

The remaining part of the proof is to analyze $M^{(mer)}$ but we also notice that (7.1.11) does not describe the residuum condition of soliton solutions of the MTM system because, for instance, in the first line of (7.1.11) the term

$$[M^{(3)}(\tau;\zeta)]^{-1}\begin{bmatrix} 0 & 0 \\ \frac{\zeta_j c_j \delta(\zeta_j)^2}{d(w_j)^2}e^{-i\tau Z(\zeta_j)} & 0\end{bmatrix}M^{(3)}(\tau;\zeta)$$

is not of the same form as the original one in (7.1.7). However, we have $M^{(3)}(\tau;\zeta)\to 1$ as $\tau\to\infty$, see (6.3.26), and also $\delta(\zeta_j) = \Delta_j + \mathcal{O}(\tau^{-1/2})$ for

$$\Delta_j := \exp\left\{\frac{1}{2\pi i}\int_{-L_0^{-1}}^{L_0^{-1}}\frac{\log(1+w|r(w)|^2)}{w-w_j}dw\right\}, \tag{7.1.14}$$

which is a consequence of our assumption (7.1.8) as shown in Proposition 7.2.1 below. Then, by Proposition 7.2.2 below, we obtain the following result: the meromorphic function $M^{(sol-0)}(\tau;\zeta)$ with singularities at $\zeta_1,...,\zeta_n,\overline{\zeta}_1,...,\overline{\zeta}_n$ satisfying

$$\begin{aligned}\operatorname*{Res}_{\zeta=\zeta_j} M^{(sol-0)}(\tau;\zeta) &= \lim_{\zeta\to\zeta_j} M^{(sol-0)}(\tau;\zeta)\begin{bmatrix} 0 & 0\\ \frac{\zeta_j c_j \Delta_j^2}{d(w_j)^2}e^{-i\tau Z(\zeta_j)} & 0\end{bmatrix},\\ \operatorname*{Res}_{\zeta=\overline{\zeta}_j} M^{(sol-0)}(\tau;\zeta) &= \lim_{\zeta\to\overline{\zeta}_j} M^{(sol-0)}(\tau;\zeta)\begin{bmatrix} 0 & \frac{-\overline{c}_j d(\overline{w}_j)^2\overline{\Delta}_j^2}{w_0}e^{i\tau Z(\overline{\zeta}_j)}\\ 0 & 0\end{bmatrix},\end{aligned} \tag{7.1.15}$$

has, under the assumption (7.1.8), the following properties:

$$\begin{aligned}\left|\lim_{\zeta\to\infty}\zeta\left[M^{(mer)}(\tau;\zeta)\right]_{12} - \lim_{\zeta\to\infty}\zeta\left[M^{(sol-0)}(\tau;\zeta)\right]_{12}\right| &\leq c\tau^{-1/2},\\ \left|M^{(mer)}(\tau,0) - M^{(sol-0)}(\tau,0)\right| &\leq c\tau^{-1/2}.\end{aligned} \tag{7.1.16}$$

Now, let us go back to the original coordinates t and x. We recall formula (6.1.1) and set analogously

$$M^{(sol)}(t,x;w) := M^{(sol-0)}(\tau;w/w_0).$$

From (7.1.15) it follows that $M^{(sol)}(t,x;w)$ satisfies exactly the RHP for the multi-soliton $u_{sol}(t,x;\mathcal{D}')$ with data $\mathcal{D}' = \{\lambda_j, C_j'\}_{j=1}^n$, where

$$C_j' = C_j\frac{\Delta_j^2}{d(w_j)^2}.$$

Moreover, by combining (7.1.13) and (7.1.16) and using the reconstruction formula (2.8.11), we find

$$\begin{aligned} u(t,x) &= [M(t,x;0)]_{11}\overline{\lim_{|w|\to\infty} w\cdot[M(t,x;w)]_{12}}\\ &= \frac{d(0)}{\delta(0)}[M^{(sol)}(t,x;0)]_{11}\overline{\lim_{w\to\infty} w\cdot[M^{(sol)}(t,x;0)]_{12}} + \mathcal{O}(\tau^{-1/2})\\ &= \frac{d(0)}{\delta(0)}u_{sol}(t,x;\mathcal{D}') + \mathcal{O}(t^{-1/2}).\end{aligned}$$

Similarly to (7.1.14) we find $\delta(0) = \Delta_0 + \mathcal{O}(\tau^{-1/2})$ for

$$\Delta_0 := \exp\left\{ \frac{1}{2\pi i} \int_{-L_0^{-1}}^{L_0^{-1}} \frac{\log(1 + w|r(w)|^2)}{w} dw \right\}. \tag{7.1.17}$$

We mention that (7.1.8) is again a necessary condition. So far we have $u(t,x) = \frac{d(0)}{\Delta_0} u_{sol}(t,x;\mathcal{D}') + \mathcal{O}(t^{-1/2})$. Additionally, making use of Remark 2.9.4, we end up with $u(t,x) = u_{sol}(t,x;\mathcal{D}^+) + \mathcal{O}(t^{-1/2})$ as $t \to \infty$. Here the scattering data are given by $\mathcal{D}^+ = \{\lambda_j, \widetilde{C}_j^+\}_{j=1}^n$ with modified norming constants

$$\widetilde{C}_j^+ = C_j' \frac{d(0)}{\Delta_0} = C_j \frac{d(0)}{\Delta_0} \frac{\Delta_j^2}{d(w_j)^2}.$$

It is left to the reader to verify that the latter expression is equivalent to both lines in (7.1.4).

Repeating the above line of argument for RHP 2.8.2, it follows that

$$v(t,x) = v_{sol}(t,x;\widetilde{D}) + \mathcal{O}(t^{-1/2})$$

with the same scattering data as for u.

Let us finish the proof with considering the case $t \to -\infty$. According to Remark 2.9.5, the norming constants $\widetilde{C}^-$ can be obtained from as $\widetilde{C}^+$ follows. Replace r with $\overline{\breve{r}}$ and vice versa, replace L_0 with L_0^{-1}, replace w_j with $\overline{z}_j$ and vice versa and, finally, conjugate the exponential factor in (7.1.4). We obviously obtain (7.1.5). □

7.2 Detailed calculation

Since the proof of Theorem 7.1.1 is in parts fragmentary, we present two propositions in order to close the gaps.

Proposition 7.2.1. *Assume* $|L_0^{-1} - w_0| \leq \tau^{-1/2}$, *then there exists a constant* $C > 0$ *depending on* L_0, $\|r\|_{L^\infty}$, $\inf_{w\in\mathbb{R}}(1 + w|r(w)|^2)$ *and* $\mathrm{Im}(w_j)$ *such that*

$$|\delta(\zeta_j) - \Delta_j| \leq C\tau^{-1/2},$$

where $\zeta_j = w_j/w_0$. *The function* δ *is given in (6.4.2) and* Δ_j *is defined in (7.1.14). Additionally,*

$$|\delta(0) - \Delta_0| \leq C\tau^{-1/2},$$

where Δ_0 *is defined in (7.1.17).*

Proof. The first assertion follows from the following calculation.

$$\begin{aligned}
|\delta(\zeta_j) - \Delta_j| &= |\Delta_j| \left| \frac{\delta(\zeta_j)}{\Delta_j} - 1 \right| \\
&= |\Delta_j| \left| \exp\left\{ \frac{1}{2\pi i} \left(\int_{-1}^{1} \frac{\log\left(1 + \rho(\zeta)\breve{\rho}(\zeta)\right)}{\zeta - \zeta_j} d\zeta - \int_{-L_0^{-1}}^{L_0^{-1}} \frac{\log(1 + w|r(w)|^2)}{w - w_j} dw \right) \right\} - 1 \right| \\
&= |\Delta_j| \left| \exp\left\{ \frac{1}{2\pi i} \left(\int_{-L_0^{-1}}^{-w_0} + \int_{L_0^{-1}}^{w_0} \right) \frac{\log(1 + w|r(w)|^2)}{w - w_j} dw \right\} - 1 \right| \\
&\leq |\Delta_j| \frac{|L_0^{-1} - w_0|}{\pi\, \mathrm{Im}(w_j)} \| \log(1 + w|r(w)|^2)\|_{L_w^\infty(\mathbb{R})} \\
&\leq C\tau^{-1/2}.
\end{aligned}$$

The second claim of the proposition follows by replacing $\| \log(1 + w|r(w)|^2)\|_{L_w^\infty(\mathbb{R})}$ with

$$\left\| \frac{\log(1 + w|r(w)|^2)}{w} \right\|_{L_w^\infty(\mathbb{R}\setminus[-L_0^{-1}+\tau^{-1/2}, L_0^{-1}-\tau^{-1/2}])},$$

which is finite for sufficiently large τ. □

In order to complete the proof of Theorem 7.1.1, we need to prove (7.1.16). The following proposition uses arguments of Lemmas 6.1 and 6.3 in [Saa17a].

Proposition 7.2.2. *Let $M^{(mer)}(\tau;\zeta)$ and $M^{(sol-0)}(\tau;\zeta)$ be meromorphic functions as defined in the proof of Theorem 7.1.1. Then, under the same assumptions as in Proposition 7.2.1,*

$$\left|\lim_{\zeta\to\infty}\zeta\left[M^{(mer)}(\tau;\zeta)\right]_{12}-\lim_{\zeta\to\infty}\zeta\left[M^{(sol-0)}(\tau;\zeta)\right]_{12}\right|\leq c\tau^{-1/2},$$
$$\left|M^{(mer)}(\tau,0)-M^{(sol-0)}(\tau,0)\right|\leq c\tau^{-1/2}.$$

Proof. Define $\zeta_j' := w_0L_0\zeta_j$ such that $|\zeta_j'|=1$ and $|\zeta_j-\zeta_j'|\leq c\tau^{-1/2}$. Fix some $\epsilon>0$ small enough so that the balls $B_\epsilon(\zeta_1'),...,B_\epsilon(\zeta_n')$ are pairwise disjoint and do not intersect the real line. For $\tau>0$ large enough that $\zeta_j\in B_\epsilon(\zeta_j')$ for $j=1,...,n$, define a new unknown by

$$\widetilde{M}^{(mer)}(\tau;\zeta):=\begin{cases} M^{(mer)}(\tau;\zeta)\left[M^{(3)}(\tau;\zeta_j)\right]^{-1}\begin{bmatrix}\frac{\Delta_j}{\delta(\zeta_j)} & 0\\ 0 & \frac{\delta(\zeta_j)}{\Delta_j}\end{bmatrix}, & \text{if } \zeta\in B_\epsilon(\zeta_1'),\\ M^{(mer)}(\tau;\zeta)\left[M^{(3)}(\tau;\overline{\zeta}_j)\right]^{-1}\begin{bmatrix}\frac{1}{\overline{\Delta}_j\delta(\overline{\zeta}_j)} & 0\\ 0 & \overline{\Delta}_j\delta(\overline{\zeta}_j)\end{bmatrix}, & \text{if } \zeta\in B_\epsilon(\overline{\zeta}_1'),\\ M^{(mer)}(\tau;\zeta), & \text{else.}\end{cases}$$

Note that we can write $\delta(\overline{\zeta}_j)=1/\overline{\delta(\zeta_j)}$. Using (7.1.11), a direct computations shows that

$$\operatorname*{Res}_{\zeta=\zeta_j}\widetilde{M}^{(mer)}(\tau;\zeta)=\lim_{\zeta\to\zeta_j}\widetilde{M}^{(mer)}(\tau;\zeta)\begin{bmatrix}0 & 0\\ \frac{\zeta_jc_j\Delta_j^2}{d(w_j)^2}e^{-i\tau Z(\zeta_j)} & 0\end{bmatrix},$$
$$\operatorname*{Res}_{\zeta=\overline{\zeta}_j}\widetilde{M}^{(mer)}(\tau;\zeta)=\lim_{\zeta\to\overline{\zeta}_j}\widetilde{M}^{(mer)}(\tau;\zeta)\begin{bmatrix}0 & \frac{-\overline{c}_jd(\overline{w}_j)^2\overline{\Delta}_j^2}{w_0}e^{i\tau Z(\overline{\zeta}_j)}\\ 0 & 0\end{bmatrix}.$$

Comparing these residue conditions with (7.1.15), we conclude that $\widetilde{M}^{(mer)}(\tau;\zeta)$ and $M^{(sol)}(\tau;\zeta)$ have identical properties, the only difference being that $\widetilde{M}^{(mer)}$ admits a discontinuity on the contour

$$\Sigma^{(mer)}:=\partial B_\epsilon(\zeta_1')\cup...\cup\partial B_\epsilon(\zeta_n')\cup\partial B_\epsilon(\overline{\zeta}_1')\cup...\cup\partial B_\epsilon(\overline{\zeta}_n').$$

Thus, writing

$$\widetilde{M}^{(mer)}(\tau;\zeta)=G(\tau;\zeta)M^{(sol)}(\tau;\zeta),\tag{7.2.1}$$

we have to require that G is a solution to the normalized Riemann–Hilbert problem $G_+=G_-(1+R^{(mer)})$ on $\Sigma^{(mer)}$, where

$$1+R^{(mer)}(\tau;\zeta):=\begin{cases} M^{(sol)}(\tau;\zeta)\left[M^{(3)}(\tau;\zeta_j)\right]^{-1}\begin{bmatrix}\frac{\Delta_j}{\delta(\zeta_j)} & 0\\ 0 & \frac{\delta(\zeta_j)}{\Delta_j}\end{bmatrix}[M^{(sol)}(\tau;\zeta)]^{-1}, & \text{if } \zeta\in\partial B_\epsilon(\zeta_1'),\\ M^{(sol)}(\tau;\zeta)\left[M^{(3)}(\tau;\overline{\zeta}_j)\right]^{-1}\begin{bmatrix}\frac{\overline{\delta(\zeta_j)}}{\overline{\Delta}_j} & 0\\ 0 & \frac{\overline{\Delta}_j}{\overline{\delta(\zeta_j)}}\end{bmatrix}[M^{(sol)}(\tau;\zeta)]^{-1}, & \text{if } \zeta\in\partial B_\epsilon(\overline{\zeta}_1').\end{cases}$$

As in the proof of Theorem 4.3.2, for $\zeta\in\Sigma^{(mer)}$, we use the notation $G_\pm(\tau;\zeta)$ to denote the limit of $G(\tau;\zeta')$ when ζ' approaches ζ from the interior/exterior of the ball $B_\epsilon(\zeta_j')$ or $B_\epsilon(\overline{\zeta}_j')$, respectively. We find

$$\begin{aligned}|R^{(mer)}(\tau;\zeta)| &\leq c\|M^{(sol)}(\tau;\cdot)\|_{L^\infty(\Sigma^{(mer)})}\sum_{j=1}^n\left|\left[M^{(3)}(\tau;\zeta_j)\right]^{-1}\begin{bmatrix}\frac{\Delta_j}{\delta(\zeta_j)} & 0\\ 0 & \frac{\delta(\zeta_j)}{\Delta_j}\end{bmatrix}-\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right|\\ &\leq c\tau^{-1/2}.\end{aligned}$$

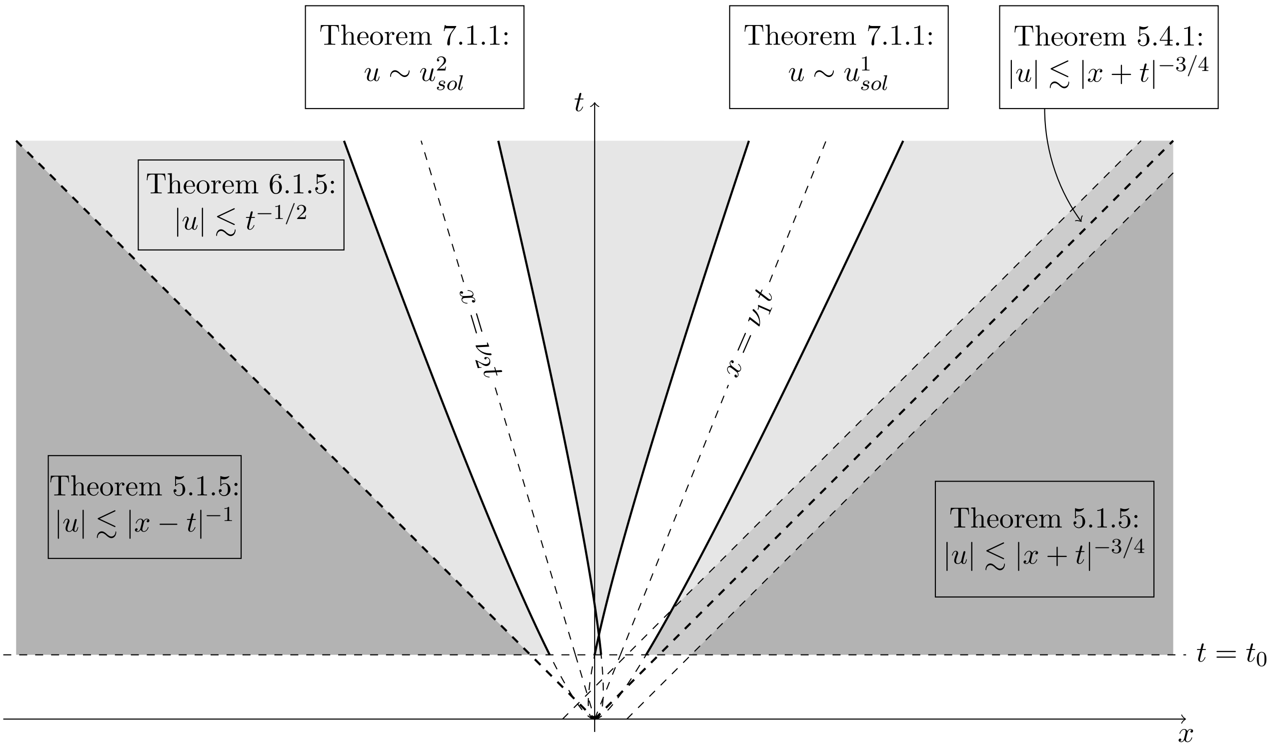


Figure 7.1: *This figure illustrates how Theorem 7.3.1 with two eigenvalues $|\lambda_1| < |\lambda_2|$ is obtained from Theorems 5.1.5, 5.4.1, 6.1.4 and 7.1.1. The abbreviation u^j_{sol} stands for $u_{sol}(t, x; \{\lambda_j, \widetilde{C}^+_j\})$. The second component $v(t, x)$ obeys a similar principle.*

Here, the last step follows from (6.3.26) and Proposition 7.2.1. Since $\Sigma^{(mer)}$ is a compact contour, we can conclude that

$$\|R^{(mer)}(\tau;\cdot)\|_{L^1(\Sigma^{(mer)})\cap L^\infty(\Sigma^{(mer)})} \leq c\tau^{-1/2},$$

which implies

$$\left|\lim_{\zeta\to\infty}\zeta\,[G(\tau;\zeta)]_{12}\right| \leq c\tau^{-1/2}, \qquad |G(\tau,0)-1| \leq c\tau^{-1/2},$$

according to the small norm theory for RHP's reviewed in Appendix A.1. Combining these estimates on G with (7.2.1), we find

$$\left|\lim_{\zeta\to\infty}\zeta\left[\widetilde{M}^{(mer)}(\tau;\zeta)\right]_{12} - \lim_{\zeta\to\infty}\zeta\left[M^{(sol-0)}(\tau;\zeta)\right]_{12}\right| \leq c\tau^{-1/2},$$
$$\left|\widetilde{M}^{(mer)}(\tau,0) - M^{(sol-0)}(\tau,0)\right| \leq c\tau^{-1/2}.$$

By the construction of $\widetilde{M}^{(mer)}$, it is clear that these inequalities are equivalent to the assertion of the proposition. $\square$

7.3 Soliton resolution

Combining Theorems 4.3.2, 5.1.5, 5.4.1, 6.1.4 and 7.1.1 we can prove the soliton resolution conjecture for the massive Thirring model. For the convenience of the reader, we refer to Figure 7.1 and provide a more detailed explanation as follows. We assume that our initial data admits scattering data $(p; \{\lambda_k, C_k\}_{k=1}^N)$. For some $j_0 \in \{1, ..., N\}$ we recall the set

$$S_\varepsilon(j_0) = \left\{(x,t)\in\mathbb{R}^2 : |x-\nu_{j_0}t| \leq \sqrt{|t|}\,\varepsilon\right\},$$

as was defined in (4.3.2). Corollary 4.3.3 states that inside of $S_\varepsilon(j_0)$, as $t \to \pm\infty$, the solution (u, v) is asymptotically determined by scattering data $(\widetilde{p}^{\pm}; \{\lambda_k, \widetilde{C}_k^{\pm}\}_{k\in\Box(j_0)})$. For any $k \in \Box(j_0)$ we have $|\lambda_k| = |\lambda_{j_0}|$. Hence, Theorem 7.1.1 is applicable and we conclude that (u, v) is approximated by a breather or single-soliton solution associated to scattering data

$$(0; \{\lambda_k, \widetilde{\widetilde{C}}{}_k^{\pm}\}_{k\in\Box(j_0)}).$$

If there exists no $j_0 \in \{1, ..., N\}$ such that $(t, x) \in S_\varepsilon(j_0)$, then we know from Corollary 4.3.3 that (u, v) is approximated by a pure-radiation solution and, thus, either $|u| + |v| \lesssim |t|^{-1/2}$ by Theorem 6.1.4, or $|u| \lesssim \min\{|x - t|^{-1}, |x + t|^{-3,4}\}$ and $|v| \lesssim \min\{|x + t|^{-1}, |x - t|^{-3,4}\}$ by Theorems 5.1.5 and 5.4.1. The following Theorem summarizes these statements and can be interpreted as the main result of the present dissertation:

Theorem 7.3.1. *Suppose that $(u_0, v_0) \in \mathcal{G}_N$ with scattering data $\mathcal{S}(u_0, v_0) = (p; \{\lambda_k, C_k\}_{k=1}^N)$, assume that $r, \widehat{r} \in X^{2,2}_{-2,0}$ and consider the solution (u, v) of (1.1.1) with initial data (u_0, v_0).*

1. *Then, for any $\varepsilon > 0$ and $j_0 \in \{1, ..., N\}$, there exist positive constants c and t_0 such that the following statement is true. Denote by $(u_{sol}(t, x; \mathcal{D}^{\pm}_{j_0}), v_{sol}(t, x; \mathcal{D}^{\pm}_{j_0}))$ the soliton solution of (1.1.1) with parameters $\mathcal{D}^{\pm}_{j_0} = \{\lambda_k, \widetilde{C}_k^{\pm}\}_{k\in\Box(j_0)}$, where*

$$\begin{aligned} \widetilde{C}_k^{+} &= C_k \prod_{j\in\bigtriangledown(j_0)} \frac{\overline{\lambda}_j^2}{\lambda_j^2} \left(\frac{\lambda_k^2 - \lambda_j^2}{\lambda_k^2 - \overline{\lambda}_j^2}\right)^2 \exp\left\{\frac{1}{\pi i}\int_{-L_0}^{L_0} \log(1 + z|\hat{r}(z)|^2)\left(\frac{1}{z - z_j} - \frac{1}{2z}\right) dz\right\}, \\ \widetilde{C}_k^{-} &= C_k \prod_{j\in\triangle(j_0)} \frac{\overline{\lambda}_j^2}{\lambda_j^2} \left(\frac{\lambda_k^2 - \lambda_j^2}{\lambda_k^2 - \overline{\lambda}_j^2}\right)^2 \exp\left\{\frac{-1}{\pi i}\int_{-L_0^{-1}}^{L_0^{-1}} \log(1 + w|r(w)|^2)\left(\frac{1}{w - w_j} - \frac{1}{2w}\right) dw\right\} \end{aligned} \tag{7.3.1}$$

Then for all $(t, x) \in S_\varepsilon(j_0)$ with $|t| > t_0$, we have

$$|u(t, x) - u_{sol}(t, x; \mathcal{D}^{\pm}_{j_0})| + |v(t, x) - v_{sol}(t, x; \mathcal{D}^{\pm}_{j_0})| \le c|t|^{-1/2}. \tag{7.3.2}$$

2. *Moreover, if*

$$(t, x) \in \mathbb{R}^2 \setminus \bigcup_{j=1}^{N} S_\varepsilon(j),$$

then

$$|u(t, x)| + |v(t, x)| \le c|t|^{-1/2}, \tag{7.3.3}$$

for $|t|$ sufficiently large.

As we have seen in Chapter 4, any multi-soliton itself behaves like (u, v) as stated in (7.3.2) and (7.3.3). As a consequence, we can rewrite Theorem 7.3.1 in the following way:

Theorem 7.3.2. *Fix $M > 0$. Then for any $(u_0, v_0) \in \mathcal{G}_N$ with scattering data $\mathcal{S}(u_0, v_0) = (p; \{\lambda_k, C_k\}_{k=1}^N)$, such that $r, \widehat{r} \in X^{2,2}_{-2,0}$ with $\|r\|_{X^{2,2}_{-2,0}} + \|\widehat{r}\|_{X^{2,2}_{-2,0}} < M$, the solution (u, v) of (1.1.1) with initial data (u_0, v_0) satisfies*

$$|u(t, x) - u_{sol}(t, x; \mathcal{D}^{\pm})| + |v(t, x) - v_{sol}(t, x; \mathcal{D}^{\pm})| \le c\|r\|_{X^{2,2}_{-2,0}}|t|^{-1/2}, \quad \pm t > t_0. \tag{7.3.4}$$

Here we denote by $(u_{sol}(t, x; \mathcal{D}^{\pm}), v_{sol}(t, x; \mathcal{D}^{\pm}))$ the multi-soliton solution of (1.1.1) with parameters $\mathcal{D}^{\pm} = \{\lambda_k, \widetilde{C}_k^{\pm}\}_{k=1}^N$, where

$$\begin{aligned} \widetilde{C}_k^{+} &= C_k \exp\left\{\frac{1}{\pi i}\int_{-L_0}^{L_0} \log(1 + z|\hat{r}(z)|^2)\left(\frac{1}{z - z_j} - \frac{1}{2z}\right) dz\right\}, \\ \widetilde{C}_k^{-} &= C_k \exp\left\{\frac{-1}{\pi i}\int_{-L_0^{-1}}^{L_0^{-1}} \log(1 + w|r(w)|^2)\left(\frac{1}{w - w_j} - \frac{1}{2w}\right) dw\right\}. \end{aligned} \tag{7.3.5}$$

The constants t_0 and c depend on M only.

For the special case, where all eigenvalues have distinct absolute values, Theorem 7.3.2 can be restated as follows.

Theorem 7.3.3. *Suppose that $(u_0, v_0) \in \mathcal{G}_N$ with scattering data $\mathcal{S}(u_0, v_0) = (p; \{\lambda_k, C_k\}_{k=1}^N)$, where $|\lambda_j| \neq |\lambda_k|$ for $j \neq k$. In addition, assume that $r, \widehat{r} \in X^{2,2}_{-2,0}$ and consider the solution (u, v) of (1.1.1) with initial data (u_0, v_0). Then for the norming constants as given in (7.3.1), we have*

$$\begin{aligned} \left| u(t,x) - \sum_{k=1}^N u_{sol}(t,x;\{\lambda_k, \widetilde{C}_k^{\pm}\}) \right| &\le c|t|^{-1/2}, \\ \left| v(t,x) - \sum_{k=1}^N v_{sol}(t,x;\{\lambda_k, \widetilde{C}_k^{\pm}\}) \right| &\le c|t|^{-1/2}. \end{aligned} \tag{7.3.6}$$

Thus, assuming $|\lambda_j| \neq |\lambda_k|$, the solution eventually resolves into N single solitons, moving at different speeds. This can be considered as a generic case, since assuming that (u_0, v_0) does not satisfy the assumptions of Theorem 7.3.1, then almost any perturbation of the initial data will make the assumptions satisfied. By (4.2.4), we even have an explicit expression for the solitons in (7.3.6). Thus, knowing the scattering data, we are able to compute the full solution of the MTM system up to an error of $|t|^{-1/2}$.

By the translation invariance of (1.1.1) we can also interpret Theorem 7.3.3 as follows. If the initial data is of the form

$$u_0(x) = \sum_{k=1}^N u_{sol}(t_0, x; \{\lambda_k, \widetilde{C}_k^-\}), \qquad v_0(x) = \sum_{k=1}^N v_{sol}(t_0, x; \{\lambda_k, \widetilde{C}_k^-\}),$$

for some $t_0 \ll 1$, then for some $t_1 \gg 1$, we find

$$u(t_1, x) \sim \sum_{k=1}^N u_{sol}(t_1, x; \{\lambda_k, \widetilde{C}_k^+\}), \qquad v(t_1, x) \sim \sum_{k=1}^N v_{sol}(t_1, x; \{\lambda_k, \widetilde{C}_k^-\}).$$

This implies that the initial solitons interact with each other but emerge from the collisions unchanged, except that each soliton (and its phase) has been shifted. As in Section 4.3, we denote by $\Delta x_{0,j}^{\pm}$ the total shift of the j-th soliton and by computing $|\widetilde{C}_j^+|/|\widetilde{C}_j^-|$, we obtain

$$\begin{aligned} \Delta x_{0,j} = \frac{2}{E_j} \Bigg(& \sum_{k \neq j} \operatorname{sgn}(|\lambda_k| - |\lambda_j|) \log \left| \frac{\lambda_j^2 - \lambda_k^2}{\lambda_j^2 - \overline{\lambda}_k^2} \right| \\ & + \frac{\operatorname{Im}(w_j)}{\pi} \int_{\mathbb{R}} \operatorname{sgn}(|w_j| - |w|) \frac{\log(1 + w|r(w)|^2)}{(w - \operatorname{Re}(w_j))^2 + \operatorname{Im}(w_j)^2} dw \Bigg). \end{aligned}$$

This formula shows that in contrast to pure multi-soliton solutions, in the case where the initial data is the sum of single solitons and thus only very close to a multi-soliton, each soliton interacts with both, the other solitons and the radiation represented by the non-vanishing reflection coefficient.

Let us close the chapter with a further (and equivalent) description of the soliton resolution. Therefore we preassign velocities $-1 < \nu_1 \le \nu_2 < 1$ and initial points $-\infty < x_1 \le x_2 < \infty$. Utilizing the results of the present work, we can compute the asymptotic behavior of solutions (u, v) for (1.1.1) in the space-time cone

$$\mathcal{K}(\nu_1, \nu_2, x_1, x_2) = \{(t,x) : t > 0 \text{ and } x = \xi + \eta t \text{ for } \xi \in [x_1, x_2], \eta \in [\nu_1, \nu_2]\}. \tag{7.3.7}$$

For arbitrary $\epsilon > 0$ and sufficiently large t, the set $\mathcal{K}$ corresponds to those (t,x) for which $z_0 = w_0^{-1} \in (L_1 - \epsilon, L_2 + \epsilon)$ with

$$L_j = \sqrt{\frac{1 - \nu_j}{1 + \nu_j}}, \qquad j \in \{1, 2\}. \tag{7.3.8}$$

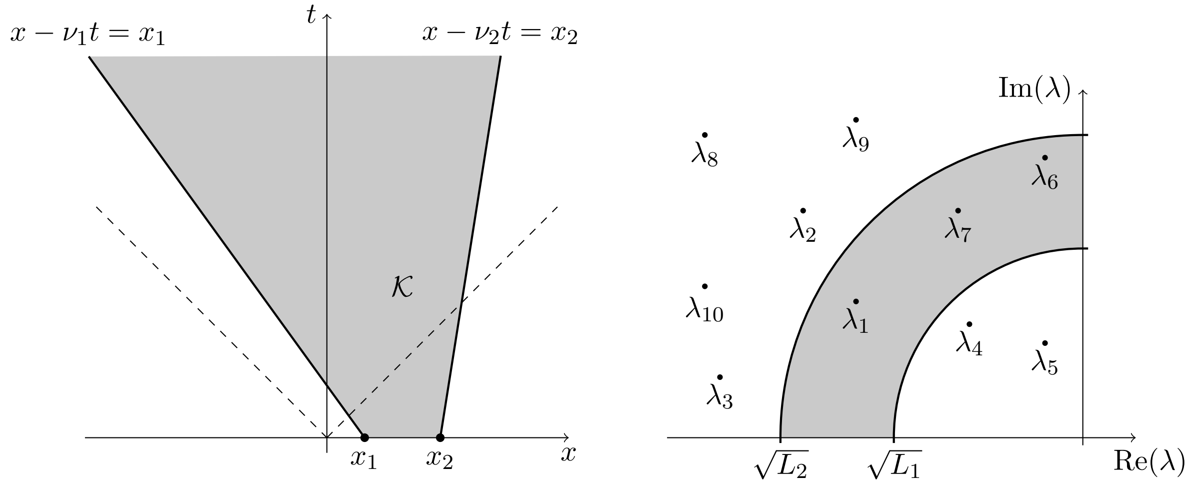


Figure 7.2: *Inside a cone $\mathcal{K}(\nu_1, \nu_2, x_1, x_2)$ the asymptotic behavior of $u(t,x)$ and $v(t,x)$ is preassigned by the reduced spectrum shaded on the right which only contains eigenvalues λ_j such that $L_1 \le |\lambda_j|^2 \le L_2$, where L_1 and L_2 are numbers determined from ν_1 and ν_2, (7.3.8).*

We set

$$\Lambda(\mathcal{K}) = \{k : L_1 \le |\lambda_k|^2 \le L_2\} \tag{7.3.9}$$

and it is clear from our observations of Section 4.3 that only solitons corresponding to eigenvalues λ_k with $k \in \Lambda(\mathcal{K})$ will be visible in $\mathcal{K}$. The remaining solitons corresponding to eigenvalues λ_k with k belonging to one of the sets

$$\Lambda^{\leftarrow}(\mathcal{K}) = \{k : |\lambda_k|^2 > L_1\}, \qquad \Lambda^{\rightarrow}(\mathcal{K}) = \{k : |\lambda_k|^2 < L_2\}, \tag{7.3.10}$$

will eventually leave $\mathcal{K}$ to the left (if $k \in \Lambda^{\leftarrow}(\mathcal{K})$) or to the right (if $k \in \Lambda^{\rightarrow}(\mathcal{K})$). We have:

Theorem 7.3.4. *Suppose that $(u_0, v_0) \in \mathcal{G}_N$ with scattering data $\mathcal{S}(u_0, v_0) = (p; \{\lambda_k, C_k\}_{k=1}^N)$, where $|\lambda_j| \ne |\lambda_k|$ for $j \ne k$. In addition, assume that $r, \widehat{r} \in X_{-2,0}^{2,2}$ and consider the solution (u,v) of (1.1.1) with initial data (u_0, v_0). For a given space-time cone $\mathcal{K}(\nu_1, \nu_2, x_1, x_2)$ of the form (7.3.7), define*

$$\mathcal{D}_{\mathcal{K}} = \left\{\lambda_k, \widetilde{C}_k\right\}_{k \in \Lambda(\mathcal{K})}$$

by

$$\widetilde{C}_k = C_k \prod_{j \in \Lambda^{\leftarrow}(\mathcal{K})} \frac{\overline{\lambda}_j^2}{\lambda_j^2} \left(\frac{\lambda_k^2 - \lambda_j^2}{\lambda_k^2 - \overline{\lambda}_j^2}\right)^2 \times \exp\left\{\frac{1}{\pi i} \int_{-L_0}^{L_0} \log(1 + z|\hat{r}(z)|^2) \left(\frac{1}{z - z_j} - \frac{1}{2z}\right) dz\right\}.$$

Then, for all $(t,x) \in \mathcal{K}$ with $t > t_0$ we have

$$|u(t,x) - u_{sol}(t,x;\mathcal{D}_{\mathcal{K}})| + |v(t,x) - v_{sol}(t,x;\mathcal{D}_{\mathcal{K}})| \le ct^{-1/2}.$$

The formulation of Theorem 7.3.4 has an advantage, because the description in cones $\mathcal{K}(\nu_1, \nu_2, x_1, x_2)$ accommodates many situations at once. For instance, Theorem 7.3.1 can be obtained by applying Theorem 7.3.4 repeatedly to cones $\mathcal{K}$ which contain at most one soliton speed ν_{j_0}. On the other hand, choosing ν_1 very close to -1 and ν_2 very close to $+1$, then Theorem 7.3.4 immediately implies Theorem 7.3.2.

Chapter 8

Conclusion

In this thesis, we gave functional-analytical details on how the direct and inverse scattering transforms can be applied to solve the initial-value problem for the MTM system in laboratory coordinates. We showed that initial data $(u_0, v_0) \in X_{2,1}$ admitting no resonances and at most finitely many simple eigenvalues uniquely define two reflection coefficients $r, \widehat{r} \in X^{2,1}_{-2,1}$. With the time evolution added, the reflection coefficients r and $\widehat{r}$ remain in the space $X^{2,1}_{-2,1}$ and together with the eigenvalues and norming constants they uniquely determine the solution (u, v) to the MTM system (1.1.1) in the space $X_{2,1}$. We have seen how solitons interact and how the interaction can be analysed in a rigorous setting with Riemann–Hilbert techniques. Using the nonlinear steepest descent of Deift and Zhou with its $\overline{\partial}$ extension introduced in[DM08], we were able to compute the long-time asymptotics of the MTM system up to an order of $|t|^{-1/2}$. In particular, we proved the soliton resolution conjecture. The present thesis demonstrates the usefulness of the inverse scattering transform, since other results concerning the long-time behavior of the MTM system obtained from PDE methods, could be significantly improved.

However, since any result of the present thesis requires the initial data to be *generic*, a natural question is whether the results can be extended to the entire space $X^{2,1}_{-2,1}$. But it is not so easy to include resonances and other spectral singularities (multiple eigenvalues) in the inverse scattering transform. Spectral singularities are subject of the very recent work [JLPS18b] in the context of the derivative NLS equation and one has to check if the methods can be adapted to the MTM system.

One can also see a fast growing interest in developing a theory of integrable systems on graphs. Also the MTM system on a graph could be used to model more realistic situations.

Finally, another interesting question is to consider the inverse scattering transform for the initial data decaying to constant (nonzero) boundary conditions. The MTM system (1.1.1) admits solitary waves over the nonzero background [BGK93, BG93] and analysis of spectral and orbital stability of such solitary waves is currently at the infancy stage.

Appendices

Appendix A

Riemann–Hilbert problems and Cauchy operator

A.1 Existence theory for RHPs

The following is presented for the convenience of the reader and does not contain something new. Let $\Sigma \subset \mathbb{C}$ be a finite union of smooth curves equipped with an orientation. Commonly, such objects are called *oriented contours*. To each contour Σ we can associate the Cauchy operator

$$C_\Sigma[f](\zeta) := \frac{1}{2\pi i}\int_\Sigma \frac{f(s)}{s-\zeta}ds, \quad \zeta \in \mathbb{C}\setminus\Sigma.$$

Note that in Section 3.2 we have introduced the Cauchy operator on the contour $\mathbb{R}$. For $\zeta \in \Sigma$ it is a fact that $C_\Sigma[f](\zeta')$ can approach different values as $\zeta' \to \zeta$, depending on the side of Σ on which the limit is taken. If one moves along the contour in the direction of the orientation, it is a convention to say that the $\oplus$-side lies to the left. The $\ominus$-side lies to the right, respectively. See Figure A.1 for an example. This gives rise to the following definition.

$$C_\Sigma^+[f](\zeta) := \lim_{\substack{\zeta'\to\zeta \\ \zeta'\in\oplus\text{-side} \\ \text{of } \Sigma}} C_\Sigma[f](\zeta'), \qquad C_\Sigma^-[f](\zeta) := \lim_{\substack{\zeta'\to\zeta \\ \zeta'\in\ominus\text{-side} \\ \text{of } \Sigma}} C_\Sigma[f](\zeta'), \qquad \zeta \in \Sigma. \tag{A.1.1}$$

which coincides with the definitions in 3.2.1. As an analogue to Proposition 3.2.1 we have the following for general oriented contours.

Proposition A.1.1. *(i) For every $f \in L^p(\Sigma)$, $1 \le p < \infty$, $\zeta \mapsto C_\Sigma[f](\zeta)$ is analytic for $\zeta \in \mathbb{C}\setminus\Sigma$ and satisfies*

$$\lim_{\substack{|\zeta|\to\infty \\ \zeta\in\mathbb{C}\setminus\Sigma}} \zeta \cdot C_\Sigma[f](\zeta) = -\frac{1}{2\pi i}\int_\Sigma f(s)ds \tag{A.1.2}$$

(ii) For $f \in L^p(\Sigma)$, $1 \le p < \infty$, the values $C_\Sigma^\pm[f](\zeta)$ exist for almost every $\zeta \in \Sigma$.

(iii) If $1 < p < \infty$, then there exists a positive constant C_p such that

$$\|C_\Sigma^\pm[f]\|_{L^p(\Sigma)} \le C_p\|f\|_{L^p(\Sigma)}. \tag{A.1.3}$$

(iv) (Sokhotski-Plemelj theorem) The following relation holds:

$$C_\Sigma^\pm[f](\zeta) = \pm\frac{1}{2}f(\zeta) - \frac{i}{2}\mathcal{H}[f](\zeta), \quad \zeta \in \Sigma, \tag{A.1.4}$$

where the Hilbert transform $\mathcal{H}$ is given by

$$\mathcal{H}[f](\zeta) := \frac{1}{\pi}\lim_{\varepsilon\searrow 0}\int_{\Sigma\setminus B_\varepsilon(\zeta)} \frac{f(s)}{s-\zeta}ds, \quad \zeta \in \Sigma.$$

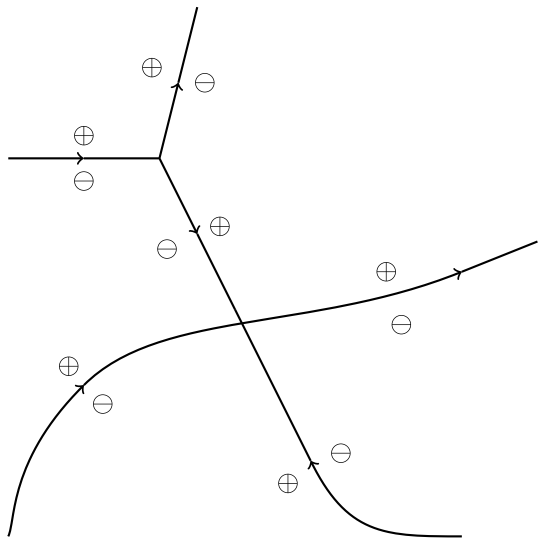


Figure A.1

Riemann–Hilbert problems can be considered for general contours. We will denote by RHP(Σ,R) the following Riemann–Hilbert problem.

Riemann-Hilbert problem A.1.2. For a given matrix-valued function $R : \Sigma \to \mathbb{C}^{n\times n}$, find a function $\mathbb{C}\setminus\Sigma \ni \zeta \mapsto M(\zeta) \in \mathbb{C}^{n\times n}$ which satisfies

1. $M(\zeta)$ is analytic in $\mathbb{C}\setminus\Sigma$.
2. $M(\zeta) = 1 + \mathcal{O}\left(\zeta^{-1}\right)$ as $|\zeta| \to \infty$.
3. The non-tangential boundary values

$$M_+(\zeta) = \lim_{\substack{\zeta'\to\zeta \\ \zeta'\in\oplus\text{-side} \\ \text{of } \Sigma}} M(\zeta'), \qquad M_-(\zeta) = \lim_{\substack{\zeta'\to\zeta \\ \zeta'\in\ominus\text{-side} \\ \text{of } \Sigma}} M(\zeta'), \qquad \zeta \in \Sigma,$$

exist and satisfy the jump relation

$$M_+(\zeta) = M_-(\zeta)(1 + R(\zeta)).$$

Theorem A.1.3. *For any given contour $\Sigma \subset \mathbb{C}$, there exists a constant Λ_Σ such that for all functions $R : \Sigma \to \mathbb{C}^{2\times 2}$ satisfying* $\det(1 + R) \equiv 1$, $R \in L^1(\Sigma) \cap L^\infty(\Sigma)$ *and*

$$\|R\|_{L^\infty(\Sigma)} \leq \Lambda_\Sigma, \tag{A.1.5}$$

the corresponding Riemann-Hilbert problem RHP(Σ,R) is uniquely solvable. Moreover, there exists another constant c_Σ such that for the solution M of RHP(Σ,R) we have

$$\left|\lim_{|\zeta|\to\infty} \zeta \cdot (M(\zeta) - 1)\right| \leq c_\Sigma \|R\|_{L^1(\Sigma)} \tag{A.1.6}$$

and

$$|M(\zeta_0) - 1| \leq \frac{c_\Sigma}{\operatorname{dist}(\Sigma, \zeta_0)} \|R\|_{L^1(\Sigma)}, \qquad \zeta_0 \in \mathbb{C}\setminus\Sigma. \tag{A.1.7}$$

If, in addition, R is locally analytic, then

$$\|M(\cdot)-1\|_{L^\infty(\mathbb{C})} \le c_\Sigma \|R\|_{L^1(\Sigma)}. \tag{A.1.8}$$

Proof. By (A.1.3) and (A.1.5),

$$C_{\Sigma,R}[f] := C_\Sigma^-[f\cdot R]$$

defines a bounded operator $C_{\Sigma,R}: L^2(\Sigma,\mathbb{C}^{2\times 2}) \to L^2(\Sigma,\mathbb{C}^{2\times 2})$. It satisfies $\|C_{\Sigma,R}\|_{L^2\to L^2} \le c_\Sigma' \|R\|_{L^\infty(\Sigma)}$ and thus, if $\|R\|_{L^\infty(\Sigma)}$ is sufficiently small we know, that $(1-C_{\Sigma,R})^{-1}$ exists as a bounded operator $L^2(\Sigma,\mathbb{C}^{2\times 2}) \to L^2(\Sigma,\mathbb{C}^{2\times 2})$ with $\|(1-C_{\Sigma,R})^{-1}\|_{L^2\to L^2} \le c\|R\|_{L^\infty(\Sigma)}$ with some constant $c>0$ not depending on $\|R\|_{L^\infty(\Sigma)}$ if (A.1.5) holds for some sufficiently small Λ_Σ. By assumption we have $R\in L^2(\Sigma)$ and thus $C_\Sigma^-[R]\in L^2(\Sigma)$ with $\|C_\Sigma^-[R]\|_{L^2(\Sigma)} \le c_\Sigma'\|R\|_{L^2(\Sigma)}$. Thus, for $\mu := (1-C_{\Sigma,R})^{-1}C_\Sigma^-[R]$ we find

$$\|\mu\|_{L^2(\Sigma)} \le c\|R\|_{L^\infty(\Sigma)}\|R\|_{L^2(\Sigma)}.$$

Now we claim that the solution M of Riemann-Hilbert problem RHP(Σ,R) is precisely given by

$$M(\zeta) = 1 + \frac{1}{2\pi i}\int_\Sigma \frac{(\mu(s)+1)R(s)}{s-\zeta}ds. \tag{A.1.9}$$

Assuming (A.1.9) for a moment, we can prove (A.1.6) as follows:

$$\begin{aligned}
\left|\lim_{|\zeta|\to\infty} \zeta\cdot(M(\zeta)-1)\right| &= \frac{1}{2\pi}\left|\int_\Sigma (\mu(s)+1)R(s)ds\right| \\
&\le \frac{1}{2\pi}\left(\|\mu\|_{L^2}\|R\|_{L^2} + \|R\|_{L^1}\right) \\
&\le c\left(\|R\|_{L^\infty}\|R\|_{L^2}^2 + \|R\|_{L^1}\right) \\
&\le c_\Sigma \|R\|_{L^1}.
\end{aligned}$$

The last inequality follows from the standard inclusion $L^2 \subset L^1\cap L^\infty$ and (A.1.5). In a very similar way we can prove (A.1.7). In order to understand why (A.1.9) is indeed a solution formula for RHP(Σ,R) we first note that $\mu = C_\Sigma^-[(\mu+1)R]$ by defintion. Next let us denote the right hand side of (A.1.9) by $\widetilde{M}$ such that $\widetilde{M} = 1 + C_\Sigma[(\mu+1)R]$. We obviously have $\widetilde{M}_- = 1+\mu$. Furthermore, using (A.1.4) we find

$$\begin{aligned}
\widetilde{M}_+ &= 1 + C_\Sigma^+[(\mu+1)R] \\
&= 1 + (\mu+1)R + C_\Sigma^-[(\mu+1)R] \\
&= 1 + (\mu+1)R + \mu \\
&= (\mu+1)(1+R) \\
&= \widetilde{M}_-(1+R).
\end{aligned}$$

By Proposition A.1.1 (i) we also have analyticity of $\widetilde{M}$ on $\mathbb{C}\setminus\Sigma$ and $\widetilde{M}(\zeta) = 1 + \mathcal{O}(\zeta^{-1})$ as $\zeta\to\infty$. Hence, $\widetilde{M}$ is a solution of RHP(Σ,R). For the proof of (A.1.8) we refer to [JLPS18b, page 1031]. □

A.2 The Cauchy operator on a half line

The following Proposition is used in the proof of Proposition 6.4.2.

Proposition A.2.1. *Let $\Sigma = (-\infty, 1)$ and assume, that $h\in H^1(\Sigma)$ with $h(1)=0$. Then, the following holds.*

(i) $C_\Sigma[h](1)$ *exists.*

(ii) $|C_\Sigma[h](\zeta) - C_\Sigma[h](1)| \le c(\|h\|_{L^2(\mathbb{R}_-)} + \|h'\|_{L^2(0,1)})|\zeta-1|^{1/2}$ *for all* $\zeta\notin(0,1)$ *satisfying* $|\zeta-1|<\frac{1}{2}$.

Proof. (i) follows from $|h(\zeta)| \leq \|h\|_{H^1(\Sigma)}|\zeta - 1|^{1/2}$. (ii) follows from

$$C_\Sigma[h](\zeta) = C_{\mathbb{R}_-}[h](\zeta) + C_{(0,1)}[h](\zeta).$$

Indeed, on one hand we have $|C_{\mathbb{R}_-}(\zeta) - C_{\mathbb{R}_-}(1)| \leq c\|h\|_{L^2(\mathbb{R}_-)}|\zeta - 1|$ if $|\zeta - 1| < \frac{1}{2}$. On the other hand, for $z = e^{i\theta}$, $-\pi < \theta < \pi$, by $h(1) = 0$ we can integrate by parts and get

$$\partial_r C_{(0,1)}[h](1 + zr) = C_{(0,1)}[h'](1 + zr).$$

Furthermore, by [BDT88, Lemma 23.3], $\|C_{(0,1)}[h'](1 + zr)\|_{L^2_r(\mathbb{R}_+)} \leq c\|h'\|_{L^2(0,1)}$ for a constant not depending on θ. We finally find

$$|C_{(0,1)}[h](1 + rz) - C_{(0,1)}[h](1)| = \left| \int_0^r C_{(0,1)}[h'](1 + zr')dr' \right| \leq c\|h'\|_{L^2(0,1)}|\zeta - 1|^{1/2}.$$

This completes the proof of the proposition. □

Appendix B

Several technical proofs

B.1 Proof of Proposition 5.1.3

Proposition B.1.1. *Let $r \in X^{2,2}_{-2,0}$, $w_0 \in \mathbb{R}_+$ and set $\rho(s) := w_0 \cdot s \cdot r(w_0 \cdot s)$ and $\breve{\rho}(s) := \overline{r(w_0 \cdot s)}$. Then,*

$$\mathcal{C}(\rho, \breve{\rho}) \leq c \min\{\sqrt{w_0}, 1\} \|r\|_{X^{2,2}_{-2,0}}. \tag{B.1.1}$$

Proof. The assertion follows from the following estimates:

$$\int_0^1 \frac{1}{s}|\breve{\rho}(s)|^2 ds = \int_0^{w_0} \frac{1}{w}|r(w)|^2 dw \leq \begin{cases} \|r\|^2_{X^{2,2}_{-2,0}}, \\ w_0 \displaystyle\int_0^{w_0} \frac{1}{w^2}|r(w)|^2 dw \leq w_0 \|r\|^2_{X^{2,2}_{-2,0}}, \end{cases}$$

$$\int_0^1 s|\breve{\rho}'(s)|^2 ds = \int_0^{w_0} w|r'(w)|^2 dw \leq \begin{cases} \|r\|^2_{X^{2,2}_{-2,0}}, \\ w_0 \displaystyle\int_0^{w_0} |r'(w)|^2 dw \leq w_0 \|r\|^2_{X^{2,2}_{-2,0}}, \end{cases}$$

$$\int_1^\infty \frac{1}{s^2}|\breve{\rho}(s)|^2 ds = w_0 \int_{w_0}^\infty \frac{1}{w^2}|r(w)|^2 dw \leq \begin{cases} w_0\|r\|^2_{X^{2,2}_{-2,0}}, \\ \dfrac{1}{w_0} \displaystyle\int_{w_0}^\infty |r(w)|^2 dw \leq w_0^{-1} \|r\|^2_{X^{2,2}_{-2,0}}, \end{cases}$$

$$\int_1^\infty |\breve{\rho}'(s)|^2 ds = w_0 \int_{w_0}^\infty |r'(w)|^2 dw \leq \begin{cases} w_0\|r\|^2_{X^{2,2}_{-2,0}}, \\ \dfrac{1}{w_0} \displaystyle\int_{w_0}^\infty w^2|r'(w)|^2 dw \leq w_0^{-1} \|r\|^2_{X^{2,2}_{-2,0}}. \end{cases}$$

In addition, one has to repeat these computations with $\breve{\rho}$ replaced by ρ. Due to the different definitions, it is clear that the corresponding integrals containing ρ are estimated by the $X^{2,2}_{-1,1}$-norm of $wr(w)$. Hence,

$$\mathcal{C}(\rho, \breve{\rho}) \leq c \min\{\sqrt{w_0}, 1\} (\|r(w)\|_{X^{1,1}_{-2,0}} + \|wr(w)\|_{X^{1,1}_{-2,0}}),$$

which is equivalent to the bound (B.1.1). □

B.2 Proof of Proposition 5.1.4

Proposition B.2.1. *Let $r \in X^{2,2}_{-2,0}$, $w_0 \in \mathbb{R}_+$ and set $\breve{\rho}(s) := r(w_0 \cdot s)$. Then,*

$$\Gamma_1(\breve{\rho}) + \Gamma_2(\breve{\rho}) + \Gamma_3(\breve{\rho}) + \Gamma_4(\breve{\rho}) \leq c \min\left\{1, \frac{1}{w_0^{3/2}}\right\} \|r\|_{X^{2,2}_{-2,0}}. \tag{B.2.1}$$

Proof. Throughout the proof we assume w.l.o.g that $\breve{\rho}$ vanishes on $\mathbb{R}_-$. By the chain rule it follows that

$$\Gamma_1(\breve{\rho}) = \frac{1}{w_0^2} \int_0^{w_0} w^2 |r'(w)| dw,$$

which can be used for $w_0 \geq 1$:

$$\Gamma_1(\breve{\rho}) \leq \frac{1}{w_0^{3/2}} \left(\int_0^{w_0} w^4 |r'(w)|^2 dw \right)^{1/2} \leq \frac{1}{w_0^{3/2}} \|r\|_{X^{2,2}_{-2,0}}.$$

If $w_0 \leq 1$, then

$$\Gamma_1(\breve{\rho}) = \int_0^1 s^2 |\breve{\rho}'(s)| ds \leq \int_0^1 |\breve{\rho}'(s)| ds = \int_0^{w_0} |r'(w)| dw \leq \sqrt{w_0} \int_0^{w_0} |r'(w)|^2 dw \leq \|r\|_{X^{2,2}_{-2,0}}.$$

Similarly, for $w_0 \geq 1$,

$$\Gamma_2(\breve{\rho}) = \int_{w_0}^\infty |r'(w)| dw \leq \left(\int_{w_0}^\infty \frac{1}{w^4} dw \right)^{1/2} \left(\int_{w_0}^\infty w^4 |r'(w)|^2 dw \right)^{1/2} \leq c \frac{1}{w_0^{3/2}} \|r\|_{X^{2,2}_{-2,0}}.$$

and on the other hand, for $w_0 \leq 1$,

$$\Gamma_2(\breve{\rho}) \leq \int_0^\infty |r'(w)| dw \leq c \left(\int_0^\infty (1+w^2) |r'(w)|^2 dw \right)^{1/2} \leq c \|r\|_{X^{2,2}_{-2,0}}.$$

We compute

$$\Gamma_3(\breve{\rho}) = \frac{1}{w_0^2} \int_0^{w_0} w |r(w)| dw.$$

Thus, assuming $w_0 \leq 1$, we obtain

$$\begin{aligned} \Gamma_3(\breve{\rho}) &\leq \frac{1}{w_0^2} \left(\int_0^{w_0} w^{10} dw \right)^{1/2} \left(\int_0^1 \frac{1}{w^4} |r(w)|^2 dw \right)^{1/2} \\ &\leq c w_0^{7/2} \|r\|_{X^{2,1}_{-2,1}} \leq c \|r\|_{X^{2,1}_{-2,1}}. \end{aligned}$$

On the other hand, if $w_0 \geq 1$, then

$$\begin{aligned} \Gamma_3(\breve{\rho}) &\leq \frac{1}{w_0^2} \left(\int_0^1 w |r(w)| dw + \int_1^{w_0} w |r(w)| dw \right) \\ &\leq \frac{1}{w_0^2} \left(\left(\int_0^1 w^2 |r(w)|^2 dw \right)^{1/2} + \left(\int_1^{w_0} \frac{1}{w^2} dw \right)^{1/2} \left(\int_1^{w_0} w^4 |r(w)|^2 dw \right)^{1/2} \right) \\ &\leq c \frac{1}{w_0^2} \|r\|_{X^{2,1}_{-2,1}}. \end{aligned}$$

Finally, we find

$$\Gamma_4(\breve{\rho}) = \int_{w_0}^\infty w^{-1} |r(w)| dw,$$

such that for $w_0 \leq 1$,

$$\begin{aligned} \Gamma_4(\breve{\rho}) &\leq \left(\int_{w_0}^1 w^{-2} |r(w)|^2 dw \right)^{1/2} + \left(\int_1^\infty w^{-2} dw \right)^{1/2} \left(\int_1^\infty |r(w)|^2 dw \right)^{1/2} \\ &\leq c \|r\|_{X^{2,1}_{-2,1}}, \end{aligned}$$

and for $w_0 \geq 1$,

$$\begin{aligned} \Gamma_4(\breve{\rho}) &\leq \left(\int_{w_0}^\infty w^{-6} dw \right)^{1/2} \left(\int_{w_0}^\infty w^4 |r(w)|^2 dw \right)^{1/2} \\ &\leq c \frac{1}{w_0^{5/2}} \|r\|_{X^{2,1}_{-2,1}}. \end{aligned}$$

□

B.3 Proof of Proposition 6.1.3

Proposition B.3.1. *Let $r \in X^{2,2}_{-2,0}$ satisfy $\inf_{w\in\mathbb{R}}(1 + w|r(w)|^2) \geq c_1 > 0$ and define for $w_0 \in \mathbb{R}_+$ the functions ρ and $\breve{\rho}$ as in (6.1.2). Then,*

$$\widetilde{\mathcal{C}}(\rho, \breve{\rho}) \leq c \min\{\sqrt{w_0}, 1\} \|r\|_{X^{2,2}_{-2,0}}, \tag{B.3.1}$$

with a constant that depends on c_1 only. Furthermore, for $k \in \{1, 4, 6, 7\}$,

$$\Gamma_5(p_k) \leq c \min\left\{\sqrt{w_0}, \frac{1}{w_0^{3/2}}\right\} \|r\|_{X^{2,2}_{-2,0}}, \qquad \Gamma_6(p_k) \leq c \min\left\{w_0, \frac{1}{w_0^2}\right\} \|r\|_{X^{2,2}_{-2,0}}, \tag{B.3.2}$$

while for $k \in \{2, 3, 5, 8\}$,

$$\Gamma_5(p_k) \leq c \min\left\{\sqrt{w_0}, \frac{1}{\sqrt{w_0}}\right\} \|r\|_{X^{2,2}_{-2,0}}, \qquad \Gamma_6(p_k) \leq c \min\left\{w_0^2, \frac{1}{w_0}\right\} \|r\|_{X^{2,2}_{-2,0}}. \tag{B.3.3}$$

Again, the constant c depends on c_1 only. Finally, there exists a constant C depending on r and c_1 only, such that

$$\mathfrak{C}(\nu) \leq C \tag{B.3.4}$$

for all $w_0 \in \mathbb{R}_+$. Each estimate of this proposition also holds if we replace r with $\widehat{r}$ and w_0 with z_0 and define ρ and $\breve{\rho}$ as in (6.1.6).

Proof. For $k \in \{1, 4, 5, 8\}$ we can estimate

$$\int_{-1/2}^{1/2} \frac{1}{|\zeta|}|p_k(\zeta)|^2 + |\zeta||p_k'(\zeta)|^2 d\zeta + \int_{\mathbb{R}\setminus[-\frac{3}{2},\frac{3}{2}]} \frac{1}{|\zeta|^2}|p_k(\zeta)|^2 + |\breve{\rho}'(\zeta)|^2 d\zeta$$

in the same way as in the proof of Proposition B.1.1. For $k \in \{2, 3, 6, 7\}$ we consider exemplarily $k = 2$. First, we observe that by (6.1.9)

$$|p_2(\zeta)| \leq \frac{|\rho(\zeta)|}{c_1} = \frac{|p_5(\zeta)|}{c_1}, \qquad |p_2'(\zeta)| \leq \frac{|\rho'(\zeta)| + |\rho(\zeta)|^2|\breve{\rho}'(\zeta)|}{c_1^2} \tag{B.3.5}$$

Since $\rho, \breve{\rho} \in L^\infty$ we conclude

$$|p_2'(\zeta)| \leq c(|p_1'(\zeta)| + |p_5'(\zeta)|).$$

Therefore, each summand in the the definition of $\widetilde{\mathcal{C}}$ is estimated in the same way as in the proof of Proposition B.1.1 and thus (B.3.1) is proven. For the estimates of Γ_5 consider the following:

$$\begin{aligned}\int_{1/2}^{3/2} |\rho'(\zeta)|^2 d\zeta &\leq 2w_0 \int_{w_0/2}^{3w_0/2} w^2|\partial_w r^{(d)}(w)|^2 + |r^{(d)}(w)|^2 dw \\ &\leq \frac{1}{2w_0}\int_{w_0/2}^{3w_0/2} w^4|\partial_w r^{(d)}(w)|^2 + w^2|r^{(d)}(w)|^2 dw \leq \frac{1}{2w_0}\|r^{(d)}\|^2_{X^{2,2}_{-2,0}} \leq \frac{c}{w_0}\|r\|^2_{X^{2,2}_{-2,0}},\end{aligned}$$

$$\begin{aligned}\int_{1/2}^{3/2} |\breve{\rho}'(\zeta)|^2 d\zeta &\leq w_0 \int_{w_0/2}^{3w_0/2} |\partial_w r^{(d)}(w)|^2 dw \\ &\leq \frac{1}{w_0^3}\int_{w_0/2}^{3w_0/2} w^4|\partial_w r^{(d)}(w)|^2 dw \leq \frac{1}{w_0^3}\|r^{(d)}\|^2_{X^{2,2}_{-2,0}} \leq \frac{c}{w_0^3}\|r\|^2_{X^{2,2}_{-2,0}}.\end{aligned}$$

These results are sufficient to prove (B.3.2) for $k = 1, 4$ and (B.3.3) for $k = 5, 8$. Next, using (B.3.5) we find that

$$\begin{aligned}
\int_{1/2}^{3/2} |p_2'(\zeta)|^2 d\zeta &\leq c \int_{1/2}^{3/2} |\rho'(\zeta)|^2 + |\rho(\zeta)|^4 |\breve{\rho}'(\zeta)|^2 d\zeta \\
&\leq c w_0 \int_{w_0/2}^{3w_0/2} w^2 |\partial_w r^{(d)}(w)|^2 + |r^{(d)}(w)|^2 + w^4 |r^{(d)}(w)|^4 |\partial_w r^{(d)}(w)|^2 dw \\
&\leq c \frac{1}{w_0} \int_{w_0/2}^{3w_0/2} w^4 |\partial_w r^{(d)}(w)|^2 + w^2 |r^{(d)}(w)|^2 + w^6 |r^{(d)}(w)|^4 |\partial_w r^{(d)}(w)|^2 dw \\
&\leq \frac{c}{w_0} \|r^{(d)}\|^2_{X^{2,2}_{-2,0}} \leq \frac{c}{w_0} \|r\|^2_{X^{2,2}_{-2,0}},
\end{aligned}$$

and

$$\begin{aligned}
\int_{1/2}^{3/2} |p_6'(\zeta)|^2 d\zeta &\leq c \int_{1/2}^{3/2} |\breve{\rho}'(\zeta)|^2 + |\breve{\rho}(\zeta)|^4 |\rho'(\zeta)|^2 d\zeta \\
&\leq c w_0 \int_{w_0/2}^{3w_0/2} |\partial_w r^{(d)}(w)|^2 + |r^{(d)}(w)|^4 (w^2 |\partial_w r^{(d)}(w)|^2 + |r^{(d)}(w)|^2) dw \\
&\leq c \frac{1}{w_0^3} \int_{w_0/2}^{3w_0/2} w^4 |\partial_w r^{(d)}(w)|^2 + w^2 |r^{(d)}(w)|^4 (w^4 |\partial_w r^{(d)}(w)|^2 + w^2 |r^{(d)}(w)|^2) dw \\
&\leq \frac{c}{w_0^3} \|r^{(d)}\|^2_{X^{2,2}_{-2,0}} \leq \frac{c}{w_0} \|r\|^2_{X^{2,2}_{-2,0}}.
\end{aligned}$$

In both of the above chains of inequalities we have used in the end that $\sup_{w \in \mathbb{R}} |w||r(w)|^2 \leq c$, see (B.3.6) below.

In order to prove (B.3.2)–(B.3.3) for Γ_6, we consider the following estimates. Assume $w_0 > 0$, then

$$|r(w_0)|^2 \leq 2 \int_0^{w_0} |r(w)||r'(w)| dw \leq 2w_0^2 \left(\int_0^{w_0} \frac{1}{w^4} |r(w)|^2 dw \right)^{1/2} \|r'\|_{L^2(\mathbb{R})} \leq 2w_0^2 \|r\|^2_{X^{2,2}_{-2,0}}.$$

On the other hand,

$$|r(w_0)|^2 \leq 2 \int_{w_0}^{\infty} |r(w)||r'(w)| dw \leq \frac{2}{w_0^4} \left(\int_{w_0}^{\infty} w^4 |r(w)|^2 dw \cdot \int_{w_0}^{\infty} w^4 |r'(w)|^2 dw \right)^{1/2} \leq \frac{2}{w_0^4} \|r\|^2_{X^{2,2}_{-2,0}}.$$

which leads to

$$|r(w_0)| \leq c \min \left\{ w_0, \frac{1}{w_0^2} \right\} \|r\|_{X^{2,2}_{-2,0}}. \tag{B.3.6}$$

This leads directly to the estimates of Γ_6.

It remains to show (B.3.4). Clearly, $\|\nu\|_{L^\infty}$ does not depend on w_0. Furthermore, we find

$$\begin{aligned}
\int_{1/10}^{1} |\nu'(\zeta)|^2 d\zeta &\leq c \frac{w_0}{\log(c_1)} \int_{w_0/10}^{w_0} |r(w)|^2 + 2w^2 |r(w)|^2 |r'(w)|^2 dw \\
&\leq \frac{c}{w_0 \log(c_1)} \int_{w_0/10}^{w_0} w^2 |r(w)|^4 + 2w^4 |r(w)|^2 |r'(w)|^2 dw,
\end{aligned}$$

which shows that $\int_{1/10}^{1} |\nu'(\zeta)|^2 d\zeta$ is bounded uniformly in w_0. Finally,

$$\begin{aligned}
\int_{-1/10}^{1/10} |\nu(\zeta)| d\zeta &\leq c \frac{1}{w_0} \int_{-w_0/10}^{w_0/10} |w||r(w)|^2 dw \\
&\leq c \frac{1}{10} \int_{-w_0/10}^{w_0/10} |r(w)|^2 dw,
\end{aligned}$$

which shows that $\int_{1/10}^{1} |\nu'(\zeta)|^2 d\zeta$ is bounded uniformly in w_0. □

B.4 Proof of (6.1.31) and (6.1.32)

We define

$$\widehat{\kappa}(z) = \frac{1}{2\pi}\log(1 + z|\widehat{r}(z)|^2), \qquad \kappa(w) = \frac{1}{2\pi}\log(1 + w|r(w)|^2), \tag{B.4.1}$$

such that $\widehat{\kappa}$ coincides with that one defined in (6.1.33). Substituting (6.1.2) into the definitions (6.1.13)–(6.1.15) we can use (6.1.21), (6.1.18), (6.1.20) and (6.1.1) to find

$$\begin{aligned} u(t,x) &= w_0\overline{q^{(as)}(\tau)}\,[M(t,x;0)]_{11}\,d(0) + \mathcal{O}(\tau^{-3/4}) \\ &= \sqrt{\frac{w_0}{\tau}}\left(e^{i\tau - i\kappa(-w_0)\log(\tau)}b_-(w_0) - e^{i\tau - i\kappa(w_0)\log(\tau)}b_+(w_0)\right) + \mathcal{O}(\tau^{-3/4}) \end{aligned}$$

with

$$\begin{aligned} \arg(b_\pm(w_0)) = {} & \mp\frac{\pi}{4} + \arg(r(\pm w_0)) - \arg(d_-(\pm w_0)d_+(\pm w_0)) + \arg(\Gamma(\mp i\kappa(\pm w_0)) \\ & \mp 2\int_0^{\pm w_0}\frac{\kappa(s) \mp \frac{s}{w_0}\kappa(\pm w_0)}{s \mp w_0}ds \pm 2\int_0^{\mp w_0}\frac{\kappa(s)}{s \mp w_0}ds \mp \kappa(\pm w_0) \\ & + \int_{-w_0}^{w_0}\frac{\kappa(s)}{s}ds - \int_{-\infty}^{\infty}\frac{\kappa(s)}{s}ds \end{aligned}$$

and

$$\begin{aligned} |b_\pm(w_0)|^2 &= \left|\frac{\sqrt{2\pi}d_-(\pm w_0)d_+(\pm w_0)e^{\pi\kappa(\pm w_0)/2}}{\sqrt{w_0}r(\pm w_0)\Gamma(\mp i\kappa(\pm w_0))}\right|^2 \\ &= \frac{\kappa(\pm w_0)}{w_0|r(\pm w_0)|^2}(e^{2\pi\kappa(\pm w_0)} - 1)|d_-(\pm w_0)d_+(\pm w_0)|^2 \\ &= \pm\kappa(\pm w_0)|d_-(\pm w_0)d_+(\pm w_0)|^2 \end{aligned}$$

where the identity $|\Gamma(\pm i\kappa)|^2 = \pi/(\kappa\sinh(\pi\kappa))$ is useful to obtain the second equality. Analogously, substituting (6.1.6) into the definitions (6.1.13)–(6.1.15) we can use (6.1.21), (6.1.18), (6.1.20) and (6.1.1) to find

$$\begin{aligned} v(t,x) &= z_0\overline{q^{(as)}(\tau)}\left[\widehat{M}(t,x;0)\right]_{11} + \mathcal{O}(\tau^{-3/4}) \\ &= \sqrt{\frac{z_0}{\tau}}\left(e^{i\tau - i\widehat{\kappa}(-z_0)\log(\tau)}\widehat{b}_-(z_0) - e^{i\tau - i\widehat{\kappa}(z_0)\log(\tau)}\widehat{b}_+(z_0)\right) + \mathcal{O}(\tau^{-3/4}) \end{aligned}$$

with

$$\begin{aligned} \arg\left(\widehat{b}_\pm(z_0)\right) = {} & \mp\frac{\pi}{4} + \arg(\widehat{r}(\pm z_0)) + \arg(\Gamma(\mp i\widehat{\kappa}(\pm z_0)) \\ & \mp 2\int_0^{\pm z_0}\frac{\widehat{\kappa}(s) \mp \frac{s}{z_0}\widehat{\kappa}(\pm z_0)}{s \mp z_0}ds \pm 2\int_0^{\mp z_0}\frac{\widehat{\kappa}(s)}{s \mp z_0}ds \mp \widehat{\kappa}(\pm z_0) + \int_{-z_0}^{z_0}\frac{\widehat{\kappa}(s)}{s}ds \end{aligned}$$

and

$$|\widehat{b}_\pm(z_0)|^2 = \pm\widehat{\kappa}(\pm z_0)$$

Lemma B.4.1. *We have $b_\pm(w_0) = \widehat{b}_\pm(z_0)$. In particular, since $w_0 = z_0^{-1}$ is constant along rays with $x/t = const.$, it is possible to define the function $f_\pm$ in (6.1.30) through*

$$f_\pm\left(\frac{x}{t}\right) = b_\pm(w_0) = \widehat{b}_\pm(z_0).$$

Proof. The first important observation is

$$\kappa(z^{-1}) = \widehat{\kappa}(z)$$

and follows directly from relation (2.6.8). In particular we have

$$\kappa(\pm w_0) = \widehat{\kappa}(\pm z_0). \tag{B.4.2}$$

Next, let us recall that by (3.5.4) we can write

$$d_-(w)d_+(w) = \exp\{C_{\mathbb{R}}^+[\kappa](w) + C_{\mathbb{R}}^-[\kappa](w)\}.$$

Thus, using (A.1.4) it follows that

$$|d_-(w)d_+(w)| = 1, \qquad \text{for all } w \in \mathbb{R}, \tag{B.4.3}$$

and

$$-\arg(d_-(\pm w_0)d_+(\pm w_0)) = 2\lim_{\varepsilon\searrow 0}\left(\int_{-\infty}^{\pm w_0-\varepsilon} + \int_{\pm w_0+\varepsilon}^{\infty}\right)\frac{\kappa(s)}{s\mp w_0}ds. \tag{B.4.4}$$

Equality (B.4.2) together with (B.4.3) yields $|b_\pm(w_0)| = |\widehat{b}_\pm(z_0)|$. Formula (B.4.4) can be used to calculate the following:

$$\begin{aligned}
-\arg&(d_-(\pm w_0)d_+(\pm w_0)) \mp 2\int_0^{\pm w_0}\frac{\kappa(s)\mp\frac{s}{w_0}\kappa(\pm w_0)}{s\mp w_0}ds \pm 2\int_0^{\mp w_0}\frac{\kappa(s)}{s\mp w_0}ds \\
&= 2\lim_{\varepsilon\searrow 0}\left[\left(\int_{-\infty}^{\pm w_0-\varepsilon} + \int_{\pm w_0+\varepsilon}^{\infty}\right)\frac{\kappa(s)}{s\mp w_0}ds \mp \int_{\mp w_0}^{\pm w_0\mp\varepsilon}\frac{\kappa(s)}{s\mp w_0}ds + \int_0^{\pm w_0\mp\varepsilon}\frac{\frac{s}{w_0}\kappa(\pm w_0)}{s\mp w_0}ds\right] \\
&= \pm 2\int_{\mp\infty}^{\mp w_0}\frac{\kappa(s)}{s\mp w_0}ds \pm 2\int_{\pm w_0}^{\pm\infty}\frac{\kappa(s)\mp\frac{w_0}{s}\kappa(\pm w_0)}{s\mp w_0}ds \\
&\quad + 2\lim_{\varepsilon\searrow 0}\left[\int_{\pm w_0\pm\varepsilon}^{\pm\infty}\frac{\frac{w_0}{s}\kappa(\pm w_0)}{s\mp w_0}ds + \int_0^{\pm w_0\mp\varepsilon}\frac{\frac{s}{w_0}\kappa(\pm w_0)}{s\mp w_0}ds\right] \\
&= \pm 2\int_{\mp\infty}^{\mp w_0}\frac{\kappa(s)}{s\mp w_0}ds \pm 2\int_{\pm w_0}^{\pm\infty}\frac{\kappa(s)\mp\frac{w_0}{s}\kappa(\pm w_0)}{s\mp w_0}ds + 2\kappa(\pm w_0) \\
&= \mp 2\int_0^{\mp z_0}\frac{\widehat{\kappa}(s)}{s} - \frac{\widehat{\kappa}(s)}{s\mp z_0}ds \mp 2\int_{\pm z_0}^{0}\frac{\widehat{\kappa}(s)}{s} - \frac{\widehat{\kappa}(s)\mp\frac{s}{z_0}\kappa(\pm z_0)}{s\mp z_0}ds \\
&= \mp 2\int_0^{\pm z_0}\frac{\widehat{\kappa}(s)\mp\frac{s}{z_0}\widehat{\kappa}(\pm z_0)}{s\mp z_0}ds \pm 2\int_0^{\mp z_0}\frac{\widehat{\kappa}(s)}{s\mp z_0}ds + 2\int_{-z_0}^{z_0}\frac{\widehat{\kappa}(s)}{s}ds.
\end{aligned}$$

Using, $\arg(r(\pm w_0)) = \arg(\widehat{r}(\pm z_0))$,

$$\int_{-w_0}^{w_0}\frac{\widehat{\kappa}(s)}{s}ds - \int_{-\infty}^{\infty}\frac{\widehat{\kappa}(s)}{s}ds = -\int_{-z_0}^{z_0}\frac{\widehat{\kappa}(s)}{s}ds$$

and (B.4.2) again we can finally conclude that $\arg(b_\pm(w_0)) = \arg(\widehat{b}_\pm(z_0))$. □

B.5 Proof of (6.4.22)

Proposition B.5.1. *For $Z(\zeta) = \frac{1}{2}(\zeta + \zeta^{-1})$ and $\zeta = x + iy$ we have*

$$\begin{aligned}
\zeta = x+iy \in \Omega_{3,1}: &\qquad |e^{-i\tau Z(\zeta)}| \le e^{\frac{-\tau y(1-x)}{2}}, \\
\zeta = x+iy \in \Omega_{3,2}: &\qquad |e^{-i\tau Z(\zeta)}| \le e^{\frac{-\tau y}{8x^2}}, \\
\zeta = x+iy \in \Omega_{4,1}: &\qquad |e^{i\tau Z(\zeta)}| \le e^{\frac{-\tau y(x-1)}{6}}, \\
\zeta = x+iy \in \Omega_{4,2}: &\qquad |e^{i\tau Z(\zeta)}| \le e^{\frac{-5\tau y}{18}},
\end{aligned} \tag{B.5.1}$$

where the domains $\Omega_{j,k}$ are depicted in Figure 6.10.

Proof. In general, we have

$$|e^{-i\tau Z(\zeta)}| = e^{-\tau y \frac{1-x^2-y^2}{2(x^2+y^2)}}, \qquad |e^{i\tau Z(\zeta)}| = e^{-\tau y \frac{x^2+y^2-1}{2(x^2+y^2)}}.$$

Let $\zeta = x + iy \in \Omega_{3,1}$. Then we have $0 \le y \le 1 - x$ and $1/2 \le x \le 1$ and it follows that

$$x^2 + y^2 \le x^2 + (1-x)^2 = 1 + 2(x^2 - x) \le x$$

and thus

$$y\frac{1-x^2-y^2}{2(x^2+y^2)} \ge \frac{y(1-x)}{2}.$$

From this follows that the first line of (B.5.1) holds.

Let $\zeta = x + iy \in \Omega_{3,2}$. Then we have $0 \le y \le x$ and $x^2 + y^2 \le \frac{1}{2}$ and it follows that

$$1 - x^2 - y^2 \ge \frac{1}{2}, \qquad 2(x^2+y^2) \le 4x^2$$

and thus

$$y\frac{1-x^2-y^2}{2(x^2+y^2)} \ge \frac{y}{8x^2}.$$

From this follows that the second line of (B.5.1) holds.

Let $\zeta = x + iy \in \Omega_{4,1}$. Then we have $1 \le x \le \frac{3}{2}$ and $x^2 + y^2 < 3$ and it follows that

$$x^2 + y^2 \ge x^2 \ge x$$

and thus

$$y\frac{x^2+y^2-1}{2(x^2+y^2)} \ge \frac{y(x-1)}{6}.$$

From this follows that the third line of (B.5.1) holds.

Let $\zeta = x + iy \in \Omega_{4,2}$. Then we have $x^2 + y^2 \ge \frac{9}{4}$ and it follows that

$$y\frac{x^2+y^2-1}{2(x^2+y^2)} = \frac{y}{2}\left(1 - \frac{1}{x^2+y^2}\right) \ge \frac{y}{2}\cdot\frac{5}{9}.$$

From this follows that the fourth line of (B.5.1) holds. □

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