A surface model for gentle algebras

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Abstract

The main purpose of this thesis is the study of bounded derived categories of gentle algebras and related problems by means of surface models.
In the first chapter we construct surface models for such categories and describe a variety of connections between the triangulated structure of the derived category and the geometry of its associated model.

In the subsequent chapter, we study equivalences among generalizations of the aforementioned categories which can be modelled by a surface in a sufficient way. One of the main results of the chapter asserts that we can attach a homeomorphism between the associated surfaces to every such equivalence. Furthermore, we prove that we can study auto-equivalences of a large class of gentle algebras in this way. As another application we obtain a complete derived invariant for gentle algebras.

In the final chapter, we apply the theory of surface models to a particular family of gentle algebras arising from algebraic geometry. In particular, we provide an answer to a question raised by Polishchuk [62]: we prove that the group of auto-equivalences of the bounded derived category of coherent sheaves on an arbitrary cycle of projective lines acts transitively on the set of spherical objects.

Zusammenfassung

Der Hauptgegenstand dieser Dissertation ist das Studium beschränkter, abgeleiteter Kategorien sanfter Algebren und verwandter Fragestellungen mit Hilfe von Flächenmodellen.

Im ersten Kapitel konstruieren wir eine Fläche zu jeder solchen Kategorie. Anschließend beschreiben wir eine Vielzahl von Zusammenhängen zwischen der triangulierten Struktur der abgeleiteten Kategorie und der Geometrie der zu ihr assoziierten Fläche.

Im folgenden Kapitel untersuchen wir Äquivalenzen zwischen Verallgemeinerungen der oben genannten Kategorien, welche durch eine Fläche in hinreichender Weise modelliert werden können. Eines der Hauptresultate des Kapitels besagt, dass wir zu jeder solchen Äquivalenz einen Homöomorphismus zwischen den entsprechenden Flächen konstruieren können. Darüber hinaus beweisen wir, dass sich auf diese Weise die Autoäquivalenzen beschränkter, abgeleiteter Kategorien einer großen Klasse sanfter Algebren studieren lassen. Als eine weitere Anwendung erhalten wir eine abgeleitete Invariante für die Klasse der sanften Algebren, die solche Algebren vollständig bis auf abgeleitete Äquivalenz klassifiziert.

Im dritten und letzten Kapitel wenden wir die gezeigten Resultate auf eine bestimmte Familie sanfter Algebren an. Insbesondere geben wir eine partielle Antwort auf eine Frage von Polishchuk [62]: Wir zeigen, dass die Gruppe der Autoäquivalenzen der beschränkten, abgeleiteten Kategorie kohärenter Garben auf einem beliebigen Zykel projektiver Geraden transitiv auf der Menge der sphärischen Objekte wirkt.

Contents

Introduction

Gentle algebras were introduced in [8] as generalizations of iterated tilted algebras of type A [6, 7] and type $\tilde{A}$ [8]. Since then they have appeared in a variety of contexts including dimer models [16, 17], categorical resolutions of curves [25], enveloping algebras of Lie algebras [51] and cluster theory. They appear in the form of m-cluster and m-Calabi–Yau tilted algebras as well as instances of Jacobian algebras associated to surfaces with marked points on the boundary [5, 40, 55].

Of particular interest is the appearance of the derived category of graded gentle algebras in the context of partially wrapped Fukaya categories [48] which are studied in the context of homological mirror symmetry.

Over the last few decades, gentle algebras have been subject of extensive research and many aspects of their bounded derived categories are now well understood, including

- a classification of their indecomposable objects ([12, 23, 21, 24, 22], see also [26] for a generalization to unbounded homotopy categories);
- an explicit basis for morphisms between their indecomposable objects ([4]);
- a description of the Auslander-Reiten triangles in their subcategories of perfect complexes ([13] and later in [4] via different methods);
- a description of isomorphism classes of mapping cones associated to basis elements, by means of manipulations of diagrams ([31]).

Being defined in terms of quivers and relations, gentle algebras are combinatoric in nature and, from this point of view, it is even more remarkable that the class of gentle algebras is closed under derived equivalences [66]. The introduction of a combinatorial derived invariant by Avella-Alaminos and Geiß (hereafter referred to as the AAG invariant) in [11] has sparked great interest in understanding when two gentle algebras are derived equivalent, see for instance [15, 10, 37, 36, 1, 2, 53, 14, 58]. However, there are gentle algebras with the same AAG-invariants that are not derived equivalent.

Another approach to the representation theory of gentle algebras arose from the theory of partially wrapped Fukaya categories of graded surfaces as considered in [48]. The authors showed that partially wrapped Fukaya categories are equivalent to the bounded derived category of certain (graded) gentle algebras. It was shown, independently in work of Lekili and Polishchuk [58] and in work of Plamondon, Schroll and the author of this thesis [61], that every (graded) gentle algebra admits a surface model and that the AAG-invariant of a gentle algebra is encoded in the boundary of its surface.

Furthermore, it follows from [58] that, if the associated surfaces of gentle algebras A, B of finite global dimension are homeomorphic in a compatible way, then A and B are derived equivalent. However, the authors did not show that derived equivalent algebras have equivalent surface models.

In what follows we describe the new results which are proved in this thesis.

Content of chapter I

In this chapter, for any gentle algebra A, we construct a geometric model of its bounded derived category $\mathcal{D}^b(A)$, which consists of

- an oriented surface with boundary,
- marked points on the boundary and a lamination (see Definition I.1.14) of the surface, and
- a function ω_A, which associates an integer (''winding number") to each loop on the surface.

In [67], see also [68], a ribbon graph was associated to every gentle algebra. Our model is based on the embedding of said ribbon graph into its ribbon surface. The marked points correspond to the vertices of the ribbon graph embedded into the boundary of the surface. The lamination corresponds to a form of dual ribbon graph within the surface. Furthermore, we show that the fundamental group of the surface is isomorphic to the fundamental group of the underlying graph of the quiver of A (Proposition I.1.20).

We give an explicit description of the correspondence between homotopy classes of (infinite) curves on the surface and the indecomposable objects (up to shift) in the derived category of a gentle algebra using methods based on the homotopy strings and bands of [12] (Theorem I.2.5).

We show that basis elements of morphisms between two indecomposable objects in $\mathcal{D}^b(A)$, as described in [4], correspond to crossings of curves (Theorem I.3.3). Based on the graphical mapping cone calculus given in [31], we show that the mapping cone of a map corresponding to a crossing of curves is given by the resolution of the crossing (Theorem I.4.3).

Furthermore, the Auslander-Reiten triangle corresponding to a bounded complex of projective modules (a "perfect object") is described in geometric terms. In particular, we show that the Auslander-Reiten translate of a perfect object corresponds to the rotation of the endpoints on the boundary of the corresponding curve in the surface (Corollary I.5.4).

We show that the AAG-invariant is encoded in the surface and can be expressed in terms of the number of boundary components, the number of marked points on each boundary component and the winding numbers of certain boundary curves. (see Theorem I.6.1 and Corollary II.1.11).

Proofs of these results already appeared in in joint work with Plamondon and Schroll [61].

In additio, we prove in Theorem I.7.1 that the composition of basis elements associated to intersections can be computed (up to a set of scalars) in terms of immersed triangles and other geometric figures ("forks"). Finally, we show that winding numbers of loops are closely related to certain sequences of morphisms, which we call *cycles of morphisms* (Theorem I.8.1). Furthermore, in Chapter II, we show that this connection leads to an entirely categorical interpretation of winding numbers based on degrees of distinguished self-extensions of objects.

Connections to other geometric models. The geometric model we describe in this thesis has various connections to other geometric models associated to gentle algebras. A classification of thick subcategories of discrete derived categories was obtained in [18] using a geometric model. Discrete derived categories were classified in [72], where it was shown that they are equivalent to bounded derived categories of a particular class of gentle algebras.

The geometric model constructed in [18] coincides with our model for the class of discrete derived algebras.

Jacobian algebras of (ideal) triangulations of marked surfaces, whose marked points are on the boundary, are gentle algebras [55, 5]. We note that the ribbon graph of such a gentle algebra corresponds exactly to the triangulation of the surface. In this context, the indecomposable objects of the associated cluster category were classified in [19] in terms of arcs and loops on the surface, and the Auslander-Reiten translation was described in [20]. Bases for the extension spaces were described in terms of crossings of arcs in [33]. The results in [19], [20] and [33] above were subsequently extended to surfaces with punctures (that is, marked points in the interior) and objects associated

to arcs [63]. A complete description of indecomposable objects by means of arcs and loops was given in [3]. Moreover, for gentle algebras associated to triangulations of surfaces with marked points on the boundary, the geometric description of the Auslander-Reiten translation turns out to coincide in both, the associated module category [19] and the cluster category [19]. We show in this thesis, that this holds true in the bounded derived category.
We also note that there is a correspondence between the triangulated surface S corresponding to a gentle Jacobian algebra A and the surface underlying our geometric model of $\mathcal{D}^b(A)$. However, the sets of marked points do not necessarily coincide.

Content of chapter II

In Chapter II we introduce the notion of *Fukaya-like* categories. These are triangulated categories, the structure of which can be modelled by a pair $(\mathcal{S}, \omega)$ consisting of an oriented S with marked points and a winding number function ω. As the name suggests, their definition is inspired by properties of partially wrapped Fukaya categories as in [9] and [48], the latter of which has been proved to generalize the class of bounded derived categories of gentle algebras. Indeed, as a consequence of the results of Chapter I, we prove in Proposition II.1.2 that the bounded derived category of every gentle algebra is an example of a Fukaya-like category. In this sense, Chapter II can be seen as a continutation of our studies of gentle algebras.
However, our focus shifts away from the combinatorial nature of gentle algebras to the geometric nature of their bounded derived categories, which, on one hand, allows us to study their auto-equivalences and find new derived invariants for gentle algebras and, on the other hand, makes the used techniques applicable to a (presumably) larger class of categories. In particular, we hope to apply the results developed in Chapter II to arbitrary partially wrapped Fukaya categories of surfaces as considered in [48] and [9].

The following new result are proved in Chapter II.

Theorem A. *Let $\mathcal{F}$ and $\mathcal{F}'$ be Fukaya-like categories and let $(\mathcal{S}_{\mathcal{F}}, \omega_{\mathcal{F}})$ and $(\mathcal{S}_{\mathcal{F}'}, \omega_{\mathcal{F}'})$ be surface models of $\mathcal{F}$ and $\mathcal{F}'$. Assume that neither $\mathcal{S}_{\mathcal{F}}$ nor $\mathcal{S}_{\mathcal{F}'}$ is a disc with at most* 3 *marked points. Then the following statements are true.*

1) *Every triangle equivalence $T : \mathcal{F} \rightarrow \mathcal{F}'$ induces a homeomorphism $\Psi(T) : S_{\mathcal{F}} \rightarrow S_{\mathcal{F}'}$ with the following properties:*

a) $\Psi(T)$ *preserves the orientation and restricts to a bijection between the sets of marked points,*

b) $\Psi(T)$ *preserves winding numbers, i.e. for all loops* γ *in* $S_{\mathcal{F}}$*,*

$$\omega_{\mathcal{F}'}\left(\Psi(T)\circ\gamma\right)=\omega_{\mathcal{F}}(\gamma),$$

c) if $\gamma_X \subset S_{\mathcal{F}}$ *is a curve that represents a quasi-linear indecomposable object* $X \in \mathcal{F}$ *and* $\gamma_{T(X)} \subset S_{\mathcal{F}'}$ *is a curve which represents* $T(X)$*, then*

$$\gamma_{T(X)} \simeq_* \Psi(T)\circ\gamma_X,$$

where $\simeq_*$ *is an equivalence relation slightly coarser than homotopy.*

2) *The map* $\Psi(-)$ *gives rise to a group homomorphism*

$$\Psi : \mathrm{Aut}(\mathcal{F}) \to \mathcal{MCG}(\mathcal{S}_{\mathcal{F}})$$

from the group of auto-equivalences of $\mathcal{F}$ *(modulo natural isomorphism) to the mapping class group of* $\mathcal{S}_{\mathcal{F}}$ *(see Section II.2.1).*

The construction of the map $\Psi(-)$ is given in Section II.2. Theorem A contains the assertions of Theorem II.3.1, Proposition II.3.24, Proposition II.3.25 as well as the results from Section II.4.1. The notion of a *quasi-linear* indecomposable object (as appeared in Theorem A 1c)) is introduced in Section II.1.1. If the ground field is algebraically closed, this is an empty condition and every indecomposable object is quasi-linear.

The equivalence relation $\simeq_*$ agrees with homotopy for most surfaces. In the other cases, the equivalence relation $\simeq_*$ identifies certain boundary loops with closed arcs, see Definition II.2.16.

From Theorem A we deduce that the geometric model of a Fukaya-like triangulated category is an invariant of the category.

Theorem B. *Let* $\mathcal{F}, \mathcal{F}'$ *be Fukaya-like categories and let* $(S_{\mathcal{F}}, \omega_{\mathcal{F}}), (S_{\mathcal{F}'}, \omega_{\mathcal{F}'})$ *be surface models of* $\mathcal{F}$ *and* $\mathcal{F}'$*. If* $\mathcal{F}'$ *and* $\mathcal{F}'$ *are triangle equivalent, then there exists an orientation preserving homeomorphism* $F:\ \mathcal{S}_{\mathcal{F}} \to \mathcal{S}_{\mathcal{F}'}$*, which preserves winding numbers and restricts to a bijection between the sets of marked points.*

Combining Theorem B with the results of Theorem II.4.1, we then deduce that the surface model of a gentle algebra is a complete derived invariant.

Corollary C. *Let A and A_2 be gentle algebras, let $\mathcal{S}_{A_i}$ denote the surface of A_i and let ω_{A_i} denote the corresponding winding number function.*
Then, the algebras A_1 and A_2 are derived equivalent if and only if there exists a orientation preserving homeomorphism $H : S_{A_1} \to S_{A_2}$, which restricts to a bijection between the sets of marked points and preserves winding numbers.

Although our version of winding numbers $\omega_A(-)$ is defined purely in terms of the combinatorics of a gentle algebra, it seems likely that this winding number arises from a *line field*.

Roughly speaking, a line field assigns a line (i.e. an unoriented tangent vector) to each point of the surface in a continuous way. Given an immersed closed curve, we can integrate the difference between the tangent line of the curve and the line field over the curve. In this way one can assign an integer to every closed curve and the result only depends on the regular homotopy class of the curve. This leads to the definition of a winding number with respect to a line field.

A result in [58] shows that if ω_{A_1} and ω_{A_2} arise from a line field, then a homeomorphism as in Corollary C exists if and only if the AAG-invariants of A_1 and A_2, and further, two additional invariants with values in $\{0, 1\}$ coincide. In particular, showing that ω_A arises from a line field on S_A, would imply that whether or not two given gentle algebras are derived equivalent, can be deduced entirely based on these three invariants.

In our final theorem, we investigate the kernel of Ψ, as appeared in Theorem A, for a large class of gentle algebras. It contains the assertion of Theorem II.5.1.

Theorem D. *Let $A = kQ/I$ be a gentle algebra, such that the quiver Q has no oriented cycles and let S_A denote its associated surface.*
Then the kernel $\ker \Psi$ *of* $\Psi : \mathrm{Aut}(\mathcal{D}^b(A)) \to \mathcal{MCG}(S_A)$ *as in Theorem A satisfies*

$$\ker \Psi \cong \begin{cases} PGL_2(k) \times k^\times \times \mathbb{Z}, & \text{if } Q \text{ is the Kronecker quiver;} \\ \mathcal{R} \times \mathbb{Z}, & \text{otherwise,} \end{cases}$$

where $\mathcal{R}$ is a group which can be computed explicitly, see Section II.5.

We further provide an explicit description of $\ker \Psi$ as a certain subgroup of the Picard group of A, see Theorem II.5.1.

Content of Chapter III

The third chapter focuses on applications of the results of Chapter II to problems related to algebraic geometry.

In our first result we apply the surface model of certain gentle algebras to give a parametrization of the set of so-called *spherical objects* in the category of perfect complexes over an $n-$cycle of projective lines E_n ($n \geq 1$) which is a singular algebraic curve. The following Theorem contains the result of Theorem III.0.1.

Theorem E. *Denote $\mathcal{S}$ the set of isomorphism classes of spherical objects in $\mathcal{D}^b(\operatorname{Coh} E_n)$ modulo the action of the shift functor and denote T_n the torus with n punctures. There is a one-to-one correspondence between $\mathcal{S}$ and pairs $(\gamma, \mathcal{V})$, where γ is the homotopy class of an embedded loop on T_n, whose complement is connected, and $\mathcal{V}$ is a 1-dimensional local system on γ.*

Roughly speaking, a local system on a loop is an generalization of a vector bundle to arbitrary fields. A precise definition is given in the preliminaries. The proof of Theorem E exploits the connections between the curves E_n and a certain gentle algebras Λ_n. We give a more detailed account on this approach during the discussion of Theorem F below.

The curves E_n prominently appear in the study of solutions to various types of Yang-Baxter equations, such as the classical, the quantum and the associative Yang-Baxter equation, e.g. see [27, 29, 62].

In [62], spherical objects over cycles of projective lines were used to construct solutions of the associative Yang-Baxter equation. Furthermore, spherical objects are known to induce so-called *spherical twists* (see Section III.2) which are interesting auto-equivalences of the derived category of E_n.

The second result of Chapter III is concerned with the following question by Polishchuk [62]:

Let C be a singular projective curve C of arithmetic genus 1 with trivial dualizing sheaf. Does the group of k-linear triangle auto-equivalences of $\mathcal{D}^b(\operatorname{Coh} C)$ act transitively on the set of isomorphism classes of spherical objects in $\mathcal{D}^b(\operatorname{Coh} C)$?

In general, this questions remains unanswered. We tackle this question in the case, where $C = E_n$ is an n-cycle of projective lines.

It was proved in [28] that a subgroup of $\operatorname{Aut}(\mathcal{D}^b(\operatorname{Coh} E_1))$ acts transitively on the set of isomorphism classes of spherical objects. In fact, this subgroup is generated by a certain Fourier-Mukai transform, the Picard group and the shift functor. In addition, the authors showed that every spherical object in the bounded derived category of E_1 is isomorphic to a shift of a simple vector bundle or to a shift of a structure sheaf of a smooth point.

Another partial answer to the question was given later in [57] for general cycles of projective lines by incorporating ideas from homological mirror symmetry. The authors proved that the subgroup generated by all spherical twist functors acts transitively on the set of isomorphism classes of shifts of simple vector bundles. However, for $n > 1$, it is known that not all spherical objects in $\mathcal{D}^b(\operatorname{Coh} E_n)$ are isomorphic to a shift of a skyscraper sheaf of a smooth point or a shift of a simple vector bundle. As a consequence, full transitivity in these cases remained open problems. The second theorem of Chapter III generalizes the result in [57] and establishes full transitivity.

Theorem F. *Assume that the ground field is algebraically closed and let $n \in \mathbb{N}$. Denote G the subgroup of $\operatorname{Aut}(\mathcal{D}^b(\operatorname{Coh} E_n))$ generated by*

- *the spherical twist $T_{\mathcal{O}_{E_n}}$ associated to the structure sheaf of E_n;*
- *the shift functor and the derived tensor products $- \otimes^{\mathbb{L}} \mathcal{L}(x)$, where $x \in E_n$ is smooth and $\mathcal{L}$ denotes the vector bundle associated to x.*

Then G acts transitively on the set of isomorphism classes of spherical objects in $\mathcal{D}^b(\operatorname{Coh} E_n)$.

Theorem F coincides with Theorem III.0.1.

As a corollary to Theorem F we prove:

Corollary G. *The subgroup of $\operatorname{Aut}(\mathcal{D}^b(\operatorname{Coh} E_n))$ generated by all spherical twists and the shift functor coincides with the group G from Theorem F.*

The statement of the corollary is the statement of Corollary III.6.4.
One of the challenges in the proof of Theorem F is to understand the action of the twist functor $T_{\mathcal{O}_{E_n}}$ on the level of objects. This requires a rather good understanding of morphisms and mapping cones in the derived category which seems to be unavailable at the moment.

To avoid calculations in the derived category of E_n, our approach to Theorem E and Theorem F exploits the connection between the curves E_n and a certain family of gentle algebras.

In [25] the authors defined for every positive natural number n a non-commutative curve $\mathbb{X}_n$ whose bounded derived category is a categorical resolution of the bounded derived category of coherent sheaves of an n-cycle of projective lines. They showed that $\mathcal{D}^b(\operatorname{Coh} \mathbb{X}_n)$ admits a tilting object whose endomorphism ring is isomorphic to the opposite of a gentle algebra Λ_n defined in Section III.1.

This yields an embedding of the category of perfect complexes over the n-cycle into the bounded derived category of Λ_n, which allows us to study spherical objects by means of their images under the embedding, where they become spherical objects in $\mathcal{D}^b(\Lambda_n)$. Via the correspondence between objects and curves for the surface of this gentle algebra (which is a torus with n boundary components) we establish the proof of Theorem E. Classical results about the mapping class group of the punctured torus are then used to prove Theorem F.

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Preliminaries

Conventions & general notation Unless stated otherwise, any algebra will be assumed to be finite-dimensional over a base field k and all modules over such an algebra will be assumed to be finite-dimensional. For any algebra A, we write $A - \mathrm{mod}$ (resp. $A - \mathrm{proj}$) for its category of finitely generated (projective) left A-modules. The bounded derived category of left A-modules is denoted by $\mathcal{D}^b(A)$ and the bounded homotopy category over $A - \mathrm{proj}$ is denoted by $\mathrm{Perf}(A)$. Objects in $\mathrm{Perf}(A)$ will also be referred to as perfect objects and $\mathrm{Perf}(A)$ is referred to as the category of perfect complexes. If $(X, \mathcal{O})$ is a ringed topological space, e.g. an algebraic (non-commutative) curve, we denote by $\mathrm{Coh}\, X$ its category of coherent sheaves of $\mathcal{O}$-modules. By $\mathrm{Perf}(X)$ we denote its category of perfect complexes.

Arrows in a quiver are composed from left to right, whereas morphisms and maps are composed from right to left. If α is an arrow in a quiver, we denote by $s(\alpha)$ (resp. $t(\alpha)$) the source (resp. the target) vertex of α.

The set of real numbers (resp. integers) is denoted by $\mathbb{R}$ (resp. $\mathbb{Z}$). The set of natural number is denoted by $\mathbb{N}$ and contains 0 by definition. If $a, b \in \mathbb{Z}$ and n is an integer valued variable, we write $n \in [a, b]$ instead of $a \leq n \leq b$ and similar for all other types of intervals.

Marked surfaces

For the main part of the present thesis we often consider surfaces with a set of distinguished points.

Definition. A **marked surface** is a pair $(S, \overline{\mathcal{M}})$, where S is an oriented, compact 2-dimensional real manifold with a non-empty boundary ∂S and $\overline{\mathcal{M}} \subseteq S$ is a subset of marked points, such that at least one boundary component of S contains a marked point.

If there is no ambiguity, we often refer to S as the marked surface instead of the pair $(S, \overline{\mathcal{M}})$.
For the rest of this section let $\mathcal{S} = (S, \overline{\mathcal{M}})$ denote a marked surface.

The set $\overline{\mathcal{M}}$ is a disjoint union of its elements on the boundary and its elements in the interior, the latter of which we refer to as **punctures**. A boundary component of S is called **unmarked** if it contains no points of $\overline{\mathcal{M}}$. The induced orientation on ∂S canonically induces a cyclic order on $\overline{\mathcal{M}} \cap B$ for each boundary component $B \subseteq \partial S$. Regarding each puncture as a cyclic ordered set with one element, this allows us to speak of successors and predecessors of a marked point (in $\overline{\mathcal{M}}$). In particular, each puncture is its own successor and predecessor.

We distinguish between various types of *curves* on marked surfaces.

- A **curve** on $\mathcal{S}$ is a continuous map of the form $\gamma : I \to S$, where I is an interval of the form $(0,1), (0,1], [0,1)$ or $(0,1)$, or $I = S^1$ is the unit circle, such that $\partial I = \gamma^{-1}(\overline{\mathcal{M}})$.
- If a curve is defined on an interval it is called an **arc** if either, it is defined on $[0,1]$, or, it is defined on an open or half-open interval and each of it unbounded ends wraps infinitely many times around an unmarked boundary component in counter-clockwise direction as shown in Figure 1.
- An arc is said to be **finite** if it is defined on $[0,1]$ and both of its endpoints are on the boundary; otherwise it is called **infinite**.
- A curve is a **loop** if it is defined on the circle.

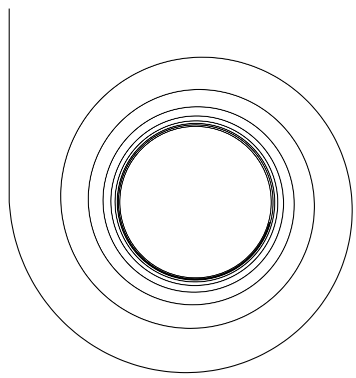


Figure 1: An infinite arc

Homotopies between curves. We say that two infinite arcs $\gamma : (0,1] \to S$ and $\gamma' : (0,1] \to S$ (and similar for infinite arcs defined on $[0,1)$) wrapping infinitely many times around the same unmarked boundary component B are homotopic if $\gamma(1) = \gamma'(1) \in \overline{\mathcal{M}}$ and if for every closed neighborhood N of B the induced maps $\gamma, \gamma' : [0,1] \to S/N$ are homotopic relative to their endpoints.

In a similar way, we say that two infinite arcs $\gamma : (0,1) \to S$ and $\gamma' : (0,1) \to S$ are equivalent if they wrap infinitely many times around the same unmarked boundary components B and B' on each end and if for every closed neighborhood N of B and N' of B' the induced maps $\gamma, \gamma' : [0,1] \to S/(N \cup N')$ are homotopic relative to their endpoints.

In all the remaining cases a homotopy of two curves is a homotopy between the corresponding maps $I \to S$ relative to the boundary of I and such a homotopy is required to restrict to a homotopy $I \setminus \partial I \to S \setminus \overline{\mathcal{M}}$. In other words, homotopy classes of loops are free homotopy classes and for all curves, homotopies are constant on end points. It means that, apart from end points, we consider all marked points as being removed from the surface.

A **boundary curve** is a curve, which is homotopic (in the previous sense) to a path in the boundary of S. Note that by definition, our curves come with an orientation of their domain. However, we often consider them as unoriented objects by identifying a curve γ and its inverse curve, which we denote by $\overline{\gamma}$.

Likewise, we often consider homotopy classes of unoriented curves, which are the union of the homotopy class of a curve and its inverse.

Remark. It will sometimes be useful to think of unmarked boundary components as punctures in the surface. Infinite arcs wrapping around such a boundary component can then be viewed as arcs going to the puncture.

Convention. If not said otherwise we always assume that a loop $\gamma : S^1 \to S$ is **primitive**, i.e. there exists no $m \in \mathbb{N} \cup \{\infty\}$, $m > 1$, such that γ is homotopic to a loop that factors through an m-fold covering map $S^1 \to S^1$.

Intersections and curves in minimal position Given distinct curves $\gamma_1 : I_1 \to S$ and $\gamma_2 : I_2 \to S$ on S, we write $\gamma_1 \cap \gamma_2$ for the set $\{(s_1, s_2) \in I_1 \times I_2 \,|\, \gamma_1(s_1) = \gamma_2(s_2)\}$ of intersections. If it causes no ambiguity, we identify $p = (s_1, s_2) \in \gamma_1 \cap \gamma_2$ and its image $\gamma_1(s_1) = \gamma_2(s_2) \in S$. A **self-intersection** of γ_i is a pair $(s,t) \in I_i^2$, such that $s \neq t$ and $\gamma_i(s) = \gamma_i(t)$.

Definition. A set of curves $\{\gamma_1, \ldots, \gamma_m\}$ is said to be in **minimal position** if for all (not necessarily distinct) $i, j \in [1, m]$, the number of (self-)intersections is minimal within their respective homotopy classes.

We often make use of the following results about curves in minimal position:

1. As pointed out in [71], it follows from [44] and [60], that every finite set of curves can be homotoped to a set of curves in minimal position and given a set $\{\gamma_1, \ldots, \gamma_m\}$ in minimal position and set of curves $\{\gamma_{m+1}, \ldots, \gamma_n\}$, there exists a set of curves $\{\gamma'_{m+1}, \ldots, \gamma'_n\}$, such that $\gamma'_i \simeq \gamma_i$ for all $i \in (m, n]$ and $\{\gamma_1, \ldots, \gamma_m, \gamma'_{m+1}, \ldots, \gamma'_n\}$ is in minimal position.

2. It is shown in Lemma I.3.4 that if $\{\gamma_1, \gamma_2\}$ are in minimal position, then any of their lifts to the universal cover of S intersect at most once in the interior and hence at most twice.

Oriented intersections. An important concept in all parts of this thesis is the relationship between intersections on surfaces and morphisms in certain categories. While the latter is a directed object (a morphism *starts* in its source and *ends* in its target), it is a-priori not clear in what sense the former allows for a notion of direction. However, it turns out that the following definition is the right concept for us.

Definition. Let γ_1 and γ_2 be curves on a marked surface $\mathcal{S}$. Then, we denote by $\gamma_1 \overrightarrow{\cap} \gamma_2$ the set of **oriented intersections** from γ_1 to γ_2. It is a subset of $\gamma_1 \cap \gamma_2$ and consists of the following intersections:

1) all interior intersections (including punctures);

2) boundary intersections of γ_1 and γ_2, such that locally around the intersection, γ_1 "lies before" γ_2 in the counter-clockwise orientation as shown Figure 2.

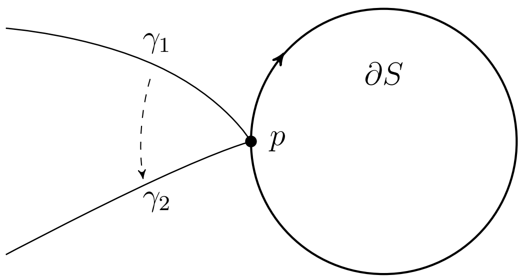


Figure 2

Local systems

In this section, let k be an arbitrary field, let $\mathcal{S} = (S, \overline{\mathcal{M}})$ be a marked surface and let $\gamma : S^1 \to S$ be a loop on S.

Definition. The **fundamental groupoid** $\pi_1(S)$ of S is the category whose set of objects is S and whose morphisms from x to y are the homotopy classes of all paths starting in x and ending in y. Composition of morphisms is defined via composition of paths.

For all $x \in \pi_1(S)$, the identity morphism $x \to x$ corresponds to the constant path with image x. Every morphism of $\pi_1(S)$ is invertible.

Definition. A k-linear **local system** on γ is a covariant functor $\mathcal{V}$ from $\pi_1(S^1)$ to the category of finite dimensional k-vector spaces.

In other words, a local system is given by a family of vector spaces $\mathcal{V} = (V_z)_{z \in S^1}$ and a family of isomorphisms ϕ_p for every path p in S^1 subject to certain compatibility conditions.

The category of local system By definition, the class of local systems becomes a category with morphisms given by natural transformations of functors. The construction of direct sums of vector spaces naturally extends to the definition of direct sums of local systems. In particular, we may talk about isomorphism classes and indecomposable local systems. We give a classification of indecomposable local systems below.

It is not difficult to see that a local system $\mathcal{V}'$ on γ is isomorphic to a local system $\mathcal{V}$, such that $\mathcal{V}_z = \mathcal{V}_1$ for all $z \in S^1$ and $\phi_p = \mathrm{Id}$ for all paths p of the form $t \mapsto \exp(2\pi i \cdot s)$ with $s \in [0, 1)$. Such a local system $\mathcal{V}$ however is uniquely determined by the isomorphism $\phi_1 : V_1 \to V_1$ associated with the path $[0, 1] \to S^1, t \mapsto \exp(2\pi i)$. In fact, isomorphism classes of local systems are in bijection with isomorphisms $V_1 \to V_1$ up to conjugation by such isomorphisms. Indeed, the assignment $\mathcal{V} \mapsto (\phi_1 : V_1 \to V_1)$ defines an equivalence from the category of local systems to the category of finite dimensional $k[X]$-modules. As a consequence, one obtains the following classification of indecomposable local systems.

Lemma. *The isomorphism classes of indecomposable local systems on γ over k are in bijection with powers of irreducible polynomials over k.*

Note that the bijection given above depends on our chosen basepoint of S^1 (1 in our case) and is therefore not canoncal. The subclasses of linear and quasi-linear local systems will be of special importance to us:

Definition. We call an indecomposable local system $\mathcal{V}$ on γ **irreducible** (resp. **linear**, resp. **quasi-linear**) if the associated characteristic polynomial of ϕ_1 is irreducible (resp. linear, resp. a power of a linear polynomial).

Note that the property of an indecomposable local system to be (quasi-)linear or irreducible does not depend on the choice of the basepoint of S^1 (which is 1 in our case).

Chapter I

The surface of a gentle algebra

Layout of the chapter In Section I.1 we present a construction of a marked surface S_A of a gentle algebra A, its ribbon graph as well as a lamination of S_A. The correspondence between homotopy classes of curves with objects in the bounded derived category $\mathcal{D}^b(A)$ is given in Section I.2.
In Section I.3 we establish a correspondence between the basis of morphisms in $\mathcal{D}^b(A)$ given in [4] and crossings of the corresponding curves in S_A.
The mapping cones of the basis of morphism in terms of resolutions of crossings is given in Section I.4, and it is shown in Section I.5 that the Auslander-Reiten translate corresponds to a rotation of both endpoints of the homotopy class of curves corresponding to an indecomposable object in $\mathcal{D}^b(A)$.
Section I.6 contains a description of the derived invariant of Avella-Alaminos and Geiss in terms of the surface.
We further establish a connection between the winding numbers of loops and an numbers attached to certain cyclic sequences of morphisms in Section I.8. Finally, we show in Section I.7 how compositions of morphisms can be re-intepreted geometrically.

I.1 From gentle algebras to surfaces with boundary

In this section, we recall the construction of a surface with boundary associated to a gentle algebra. Our main references in this section are [68] and [56].

I.1.1 Ribbon graphs and ribbon surfaces

A ribbon graph is an unoriented graph with a cyclic ordering of the edges around each vertex. In order to give a precise definition, it is useful to define a graph as a collection of vertices and **half-edges**, each of which is attached to a vertex and another half-edge. More precisely:

Definition I.1.1. A **graph** is a quadruple $\Gamma = (V, E, s, \iota)$, where

- V is a finite set, whose elements are called **vertices**;
- E is a finite set, whose elements are called **half-edges**;
- $s : E \to V$ is a function;
- $\iota : E \to E$ is an involution without fixed points.

We think of s as a function sending each half-edge to the vertex it is attached to, and of ι as sending each half-edge to the other half-edge it is glued to. This definition is equivalent to the usual definition of a graph, and in practice we will draw graphs in the usual way.

Definition I.1.2. A **ribbon graph** is a graph Γ endowed with a permutation $\sigma : E \to E$ whose orbits correspond to the sets $s^{-1}(v)$, for all $v \in V$. A **ribbon graph** is a graph Γ endowed with a permutation $\sigma : E \to E$ such that the cycles of σ correspond to the sets $s^{-1}(v)$, for all $v \in V$.

In other words, a ribbon graph is a graph endowed with a cyclic ordering of the half-edges attached to each vertex.

Any ribbon graph can be embedded in the interior of a canonical oriented surface with boundary, called the ribbon surface, in such a way that the orientation of the surface is induced by the cyclic orderings of the ribbon graph. Whenever we deal with oriented surfaces in this paper, we will call **clockwise orientation** the orientation of the surface, and **anti-clockwise orientation** the opposite orientation. When drawing surfaces or graphs in the plane, we will do so that locally, the orientation of the surface or graph becomes the clockwise orientation of the plane.

Definition I.1.3. Let Γ be a connected ribbon graph. The **ribbon surface** S_Γ is constructed by gluing polygons as follows.

- For any vertex $v \in V$ with valency $d(v) \geq 1$, let P_v be an oriented $2d(v)$-gon.
- Following the cyclic orientation, label every other side of P_v with the half-edges $e \in E$ such that $s(e) = v$.

- For any half-edge e of Γ, identify the side of P_v labelled e with the side of the polygon $P_{s(\iota(e))}$ labelled $\iota(e)$, respecting the orientations of the polygons.

In this definition, we exclude the degenerate case where Γ has only one vertex and no half-edges.

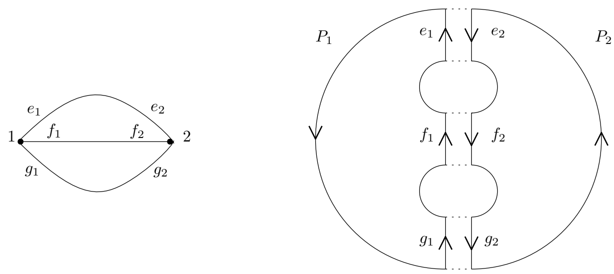


Figure I.1: Example of a ribbon graph Γ with orientation given by the planar embedding and with half edge labelling on the left and on the right the associated ribbon surface S_Γ obtained by gluing the two polygons P_1 and P_2 corresponding to vertices 1 and 2 of the ribbon graph.

Note that S_Γ is oriented, and that we can embed Γ in S_Γ as follows: the vertices of Γ are the centers of the polygons P_v, and the half edges of Γ are arcs joining the center of each P_v to the middle of the side with the same label. By [56, Corollary 2.2.11], S_Γ is, up to homeomorphism, the only oriented surface S in which we can embed Γ, preserving the cyclic ordering around each vertex, and such that the complement of the embedding of Γ in S is a disjoint union of discs (we say that Γ is **filling** for S). Moreover, by [56, Proposition 2.2.7], the number of boundary components of S_Γ is equal to the number of **faces** of Γ, according to the following definition.

Definition I.1.4. Let Γ be a ribbon graph. A **face** of Γ is an equivalence class, up to cyclic orientation, of tuples of half-edges $(e_1, \dots, e_n)$ such that

- $e_{p+1} = \begin{cases} \iota(e_p) & \text{if } s(e_p) = s(e_{p-1}), \\ \sigma(e_p) & \text{otherwise,} \end{cases}$ where the indices are taken modulo n;

- the tuple is non-repeating, in the sense that if $p \neq q$ and $e_p = e_q$, then $e_{p+1} \neq e_{q+1}$.

I.1.2 Marked ribbon graphs

When we study gentle algebras in Section I.1.3, we will obtain ribbon graphs endowed with one additional piece of information. We will call these **marked ribbon graph** and we define them as follows.

Definition I.1.5. A **marked ribbon graph** is a ribbon graph Γ together with a map $m : V \to E$, such that for every vertex $v \in V$, $m(v) \in s^{-1}(v)$.

In other words, a marked ribbon graph is a ribbon graph in which we have chosen one half-edge $m(v)$ around each vertex v.

If Γ is a marked ribbon graph, we can construct its ribbon surface S_Γ like in Definition I.1.3. Moreover, with the additional information given by the map m, we can do the following:

Proposition I.1.6. *There is an orientation-preserving embedding of Γ in S_Γ which sends all vertices of Γ to boundary components of S_Γ, such that for each vertex $v \in V$, the boundary component lies between $m(v)$ and $\sigma(m(v))$ in the clockwise orientation. This embedding is unique up to homotopy relative to ∂S_Γ.*

Proof. We use the notations of Definition I.1.3. In order to prove the existence of the embedding, it suffices to move v to the unlabelled side of P_v that lies between the sides labelled with $m(v)$ and $\sigma(m(v))$. Uniqueness follows from the fact that there is precisely one boundary component inside every face of Γ. $\square$

We call an embedding as in Proposition I.1.6 a **marked embedding of Γ in S_Γ**. We usually denote by M the set of marked points on S_Γ corresponding to the vertices of Γ.

I.1.3 The marked ribbon graph of a gentle algebra

Here, we follow [67], see also [68, Section 3]. Gentle algebras are finite-dimensional algebras having a particularly nice description in terms of generators and relations. Their representation theory is well understood and their study goes back to [46, 39, 73, 30]. Let us recall their definition:

Definition I.1.7. An algebra A is **gentle** if it is isomorphic to an algebra of the form kQ/I, where

1. Q is a finite quiver;

2. I is an admissible ideal of Q (that is, if R is the ideal generated by the arrows of Q, then there exists an integer $m \geq 2$, such that $R^m \subset I \subset R^2$);

3. I is generated by paths of length 2;

4. for every arrow α of Q, there is at most one arrow β, such that $\alpha\beta \in I \setminus \{0\}$; at most one arrow γ, such that $\gamma\alpha \in I \setminus \{0\}$; at most one arrow β', such that $\alpha\beta' \notin I$; and at most one arrow γ', such that $\gamma'\alpha \notin I$.

Definition I.1.8. For a gentle algebra $A = kQ/I$, let

- $\mathcal{M}$ be the set of maximal paths in (Q, I), that is, paths $w \notin I$, such that for any arrow α, $\alpha w \in I$ and $w\alpha \in I$;

- $\mathcal{M}_0$ be the set of trivial paths e_v, such that either v is the source or target of only one arrow, or v is the target of exactly one arrow α and the source of exactly one arrow β, and $\alpha\beta \notin I$;

- $\overline{\mathcal{M}} = \mathcal{M} \cup \mathcal{M}_0$.

We call $\overline{\mathcal{M}}$ the *augmented set of maximal paths of* A.
Then the **marked ribbon graph** Γ_A **of** A is defined as follows.

1. The set of vertices of Γ_A is $\overline{\mathcal{M}}$.

2. For every vertex of Γ_A corresponding to a path ω, there is a half-edge attached to ω and labelled by i for every vertex i of Q through which ω passes. Note that this includes the vertices at which ω starts and ends. Furthermore, if ω passes through i multiple times (at most 2), then there is one half-edge labelled by i for every such passage.

3. For every vertex i of Q, there are exactly two half-edges labelled with i. The involution ι sends each one to the other.

4. For each vertex ω of Γ_A, the vertices through which the path ω passes are ordered from starting point to ending point. The permutation σ sends each vertex in this ordering to the next, with the additional property that it sends the ending point of ω to its starting point.

5. The map m takes every ω to the half-edge labelled by its ending point.

Remark I.1.9. Instead of using $\overline{\mathcal{M}}$, the marked ribbon graph of a gentle algebra can also be defined via the augmented set of all paths in Q, such that any subpaths of length 2 is in I. This is the set of forbidden threads as defined in [11].

Using Section I.1.2, we can now define a surface with boundary and marked points for every gentle algebra.

Definition I.1.10. Let $A = kQ/I$ be a gentle algebra. Then, the **ribbon surface** of A is the marked surface $\mathcal{S}_A = (S_A, \overline{\mathcal{M}})$, where S_A is the ribbon surface of Γ_A and $\overline{\mathcal{M}}$ is given by the embedding of Γ_A as in Proposition I.1.6.

Thus, $\mathcal{S}_A$ contains no punctures and the marked points of $\mathcal{S}_A$ are in bijection with the vertices of Γ_A. Morover, the edges of Γ_A are in bijection with the vertices of Q.

Example I.1.11.

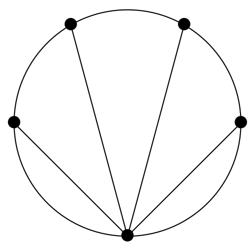
1. Let A be the algebra defined by the quiver

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} 4$$

with no relations. The ribbon graph Γ_A of this algebra is

$$\begin{tikzcd} & e_2 \arrow[dr, no head] & & e_3 \arrow[dl, no head] & \\ e_1 \arrow[rr, no head] & & \alpha_1\alpha_2\alpha_3 \arrow[rr, no head] & & e_4 \end{tikzcd}$$

and its ribbon surface $\mathcal{S}_A$ is a disc.



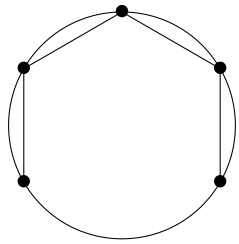
2. Let A be the algebra defined by the quiver

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} 4$$

with relations $\alpha_1\alpha_2$ and $\alpha_2\alpha_3$. The ribbon graph Γ_A of this algebra is

$$e_1 - \alpha_1 - \alpha_2 - \alpha_3 - e_4$$

and its ribbon surface $\mathcal{S}_A$ is, again, a disc.



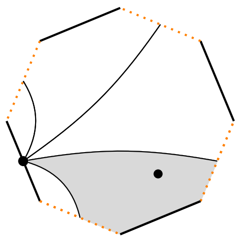
For any gentle algebra A, the edges of Γ_A cut S_A into polygons as follows.

Proposition I.1.12. *Let $A = kQ/I$ be a gentle algebra, and let Γ_A and $\mathcal{S}_A = (S_A, \overline{\mathcal{M}})$ be as in Definitions I.1.8 and I.1.10 with Γ_A being embedded into S_A by virtue of Proposition I.1.6. Then S_A is divided into two types of pieces glued together by their edges:*

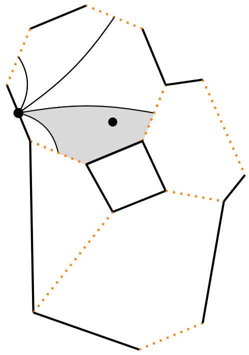
1) *polygons whose edges are edges of Γ_A, except for exactly one boundary edge, and whose interior contains no boundary component of S_A;*

2) *polygons whose edges are edges of Γ_A and whose interior contains exactly one boundary component of S_A with no marked points.*

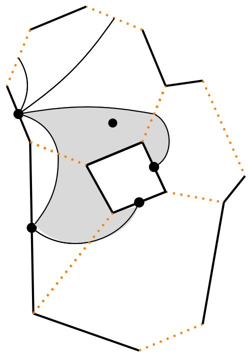
Proof. Take any point X in the interior of S_A which does not belong to any edge of Γ_A. Then this point belongs to a polygon P_v as in Definition I.1.3. This polygon has $2d$ sides (for a certain integer d) and contains exactly one marked point on one of its boundary segments, from which emanate d edges of Γ_A. Below is the local picture if P_v is an octogon:



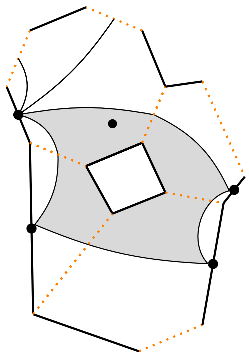
We see that X belongs to a region of P_v (grayed on the picture) that is partly bounded by a segment of a boundary component B of S_A. Around this boundary component are other polygons $P_{v_1}, P_{v_2}, \dots, P_{v_r}$, each containing exactly one marked point on one of its boundary segments. The picture around this boundary component B is as follows (in the example, B is a square).



Two cases arise.
Case 1: There is at least one marked point on B. In this case, the point X belongs to a polygon cut out by edges of Γ_A and by exactly one boundary edge on B, as illustrated in the following picture.



Case 2: There are no marked points on B. In this case, the point X belongs to a polygon on S_A cut out by edges of Γ_A and which contains the boundary component B, as illustrated below.



This finishes the proof. $\square$

Remark I.1.13. Let A be a gentle algebra with ribbon graph Γ_A and associated ribbon surface $\mathcal{S}_A = (S_A, \overline{\mathcal{M}})$. Suppose that Γ_A admits v vertices, $2e$

half-edges and f faces.
1) The complement of Γ_A in S_A is a disjoint union of open discs.
2) The Euler characteristic $\chi(\Gamma_A) = v - e + f$ of the ribbon graph Γ is equal to the Euler characteristic of $\widehat{S_A}$, where $\widehat{S_A}$ is the surface without boundary obtained from S_A by gluing an open disc to each of the boundary components of S_A.
3) The genus of S_A (as well as the genus of $\widehat{S_A}$) is equal to $1 - \chi(\Gamma_A)/2$ that is the genus of S_A is $(e - v - f + 2)/2$.

I.1.4 A lamination on the surface of a gentle algebra

On any surface with boundary and marked points on the boundary, the notion of lamination is defined in [43, Definiton 12.1]. We will need to modify the definition slightly for what follows.

Definition I.1.14. Let $\mathcal{S} = (S, M)$ be a marked surface . A **lamination** on $\mathcal{S}$ is a finite collection of non-selfintersecting and pairwise non-intersecting paths on S, considered up to isotopy relative to M. Each of these paths is one of the following:

- a closed path not homotopic to a point; or
- a path from one non-marked point to another non-marked point, both on the boundary of S. We exclude such paths that are isotopic to a part of the boundary of S containing no marked points.

A path that is part of a lamination is called a **laminate**.

Remark I.1.15. In [43, Definition 12.1], the case of a path from a non-marked point to another on the boundary that is isotopic to a part of the boundary containing exactly one marked point is also excluded. For our purposes, we need to allow such paths in our laminations.

We will now define a canonical lamination of the ribbon surface of a gentle algebra.

Proposition I.1.16. *Let $A = kQ/I$ be a gentle algebra, and let $\mathcal{S}_A = (S_A, \overline{\mathcal{M}})$ be its ribbon surface as in Definition I.1.10. There exists a unique lamination L of $\mathcal{S}_A$, such that*

1. *L contains no closed paths;*
2. *for every vertex i of Q (that is, every edge of Γ_A), there is a unique laminate $\gamma_i \in L$, such that γ_i crosses the edge labelled by i of the embedding of Γ_A once, and crosses no other edges;*

3. L contains no other laminates than those described in (2).

Proof. Every edge E of Γ_A is part of two (not necessarily distinct) faces, in the sense of Definition I.1.4, and each of these faces encloses a boundary component in S_A. Therefore, if a path γ in a lamination crosses E, then either it starts and ends on these two boundary components, or it has to cross at least another edge. Moreover, there is a unique path starting on one of these two boundary components and ending on the other that crosses E once and no other edges of Γ_A. □

Example I.1.17. We give the laminations for the two gentle algebras in Example I.1.11.

(1)

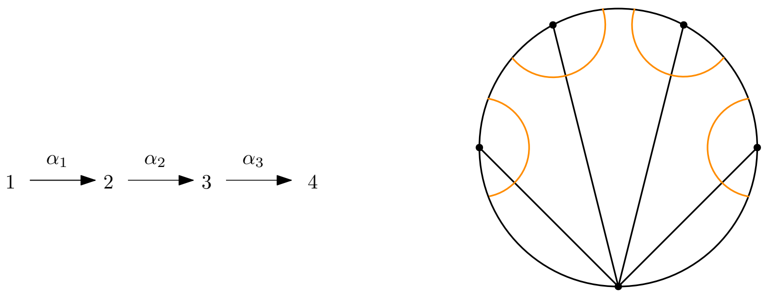


Figure I.2: On the right side is the ribbon graph embedded in the ribbon surface as well as the lamination of the hereditary gentle algebra on the left.

(2)

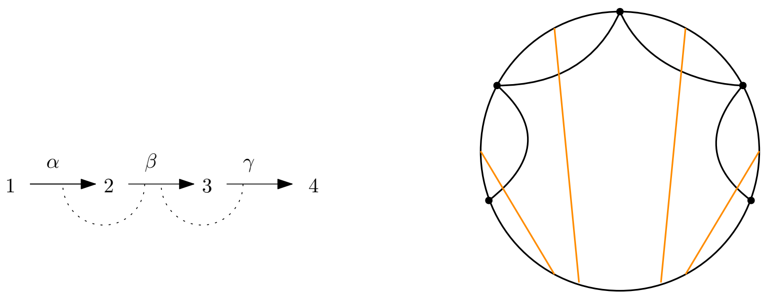


Figure I.3: On the right side is the ribbon graph embedded in the ribbon surface as well as the lamination of the gentle algebra on the left with relations $\alpha\beta$ and $\beta\gamma$.

Definition I.1.18. Let $A = kQ/I$ be a gentle algebra. Then we denote by L_A the lamination described in Proposition I.1.16, and we call it the **lamination of** A.

I.1.5 Recovering the gentle algebra from its lamination

The surface S_A and lamination L_A of a gentle algebra A contain, by construction, enough information to recover the algebra A. We record the procedure in the following proposition.

Proposition I.1.19. *Let $A = kQ/I$ be a gentle algebra, and let L_A be the associated lamination (see Definition I.1.18). Define a quiver Q_L as follows:*

- *its vertices correspond to paths in L_A;*
- *whenever two paths i and j in L_A both have an endpoint on the same boundary segment of S_A, so that no other curve has an endpoint in between, then there is an arrow from i to j if the endpoint of j follows that of i on the boundary in the clockwise order.*

Let I_L be the ideal of kQ_L defined by the following relations: whenever there are paths i, j and r in L_A that have an endpoint on the same boundary segment of S_A, so that the endpoint of r follows that of j, which itself follows that of i, and if $\alpha : i \to j$ and $\beta : j \to r$ are the corresponding arrows, then $\beta\alpha$ is a relation. Then $A \cong kQ_L/I_L$.

I.1.6 The fundamental group of the surface of a gentle algebra

We show that the fundamental group of the surface S_A of a gentle algebra $A = kQ/I$ is isomorphic to $\pi_1(Q)$, the fundamental group of the graph underlying its quiver Q.

Proposition I.1.20. *Let $A = kQ/I$ be a gentle algebra. Let $\pi_1(Q)$ be the fundamental group of its underlying graph. There exists an isomorphism $\pi_1(S_A) \cong \pi_1(Q)$.*

Corollary I.1.21. *Let $A = kQ/I$ be a gentle algebra. Then following are equivalent*
(i) the graph underlying Q is a tree
(ii) S_A is a disc
(iii) A is derived equivalent to a path algebra of Dynkin type $\mathbb{A}$.

Proof. The equivalence of (i) and (ii) directly follows from Proposition I.1.20, the equivalence of (i) and (iii) is due to [6]. □

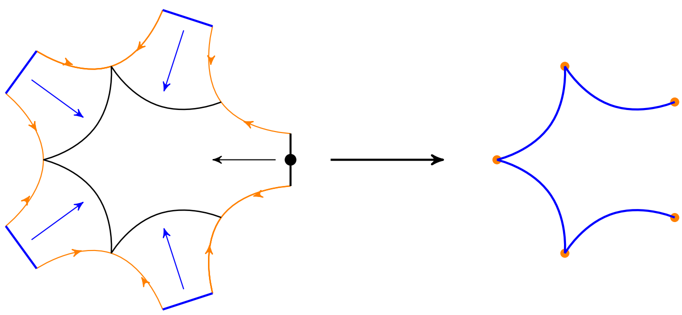
Corollary I.1.22. *Let $A = kQ/I$ be a gentle algebra. Then S_A is an annulus if and only if A has precisely one cycle.*

Proof. Follows directly from Proposition I.1.20 and the fact that the annulus is the only (compact oriented) surface with fundamental group $\mathbb{Z}$. □

By [6], the algebras appearing in Corollary I.1.21 are precisely the algebras which are derived equivalent to a path algebra of type A; by [11], those appearing in Corollary I.1.22 are determined, up to derived equivalence, by their AG-invariant (for more on the AG-invariant, see Section I.6).

Proof of Proposition I.1.20. It follows from Proposition I.1.19, that there exists an embedding of Q into S_A, such that each vertex is mapped to an interior point on the corresponding laminate and such that each arrow is mapped to a path with no intersection with the boundary and no intersection with any of the laminates apart from its endpoints. For our assertions it is sufficient to prove that this embedding is a strong deformation retract of the surface. We do this by gluing deformation retractions of the individual polygons cut out by the lamination.

For each polygon P_v, $v \in \overline{\mathcal{M}}$, denote by $Q(v)$ the subquiver of Q, which contains all arrows of the path v, if $v \in \mathcal{M}$, and, in case $v \in \mathcal{M}_0$, let $Q(v)$ be the subquiver with a single vertex corresponding to $v \in \mathcal{M}_0$. We define a strong deformation retraction of P_v onto the embedding of $Q(v)$, which contracts each laminate to a single point and projects each boundary segment onto an arrow.



This is done in two steps. Every arrow α of $Q(v)$ singles out a square in P_v bounded by the edge α, a boundary segment and segments of the laminates crossed by α. P_v is glued from these squares and another polygon P'_v, which contains the marked point.

For each $L \in L_A$ and for $a_L \in [0,1]$, such that $L(a_L) = p_L$, denote by H_L the homotopy from Id_L to the constant map p_L corresponding to $t \to a_L + (1-t)\cdot(t-a_L)$. Convex linear combinations enable us to extend any homotopy, which is constant on $\{0,1\}\times\{0\}$, from $\mathrm{Id}_{\{0,1\}\times[0,1]}$ to the map $(a,t)\mapsto(a,0)$ to a homotopy, which is constant on $\{0,1\}\times[0,1]$, from $\mathrm{Id}_{[0,1]^2}$ to the map $(a,t)\mapsto(a,0)$. In particular, we find a homotopy from the identity of each square to a map, which projects the square onto the corresponding arrow of $Q(v)$ and which extends the contractions of the segments of laminates L to the point p_L (as restrictions of H_L). We finally find a homotopy from $\mathrm{Id}_{P'_v}$, which is constant on all arrows of $Q(v)$, to a map, which projects each point of P'_v to a point of the embedding of $Q(v)$. By construction, we can glue all the homotopies showing that P_v strongly deformation retracts onto the embedding of $Q(v)$. All such homotopies can be glued at the laminates. This completes the proof. $\square$

I.2 Indecomposable objects in the derived category of a gentle algebra

Throughout this section let $A = kQ/I$ be a gentle algebra. We prove that the indecomposable objects of the bounded derived category $\mathcal{D}^b(A)$ are in bijection, up to shift, with certain curves on the surface $\mathcal{S}_A$.

I.2.1 Homotopy strings and bands

We briefly recall the classification of the indecomposable objects in the bounded derived category of a gentle algebra in terms of homotopy string and band complexes [12]. Another classification via a different approach was obtained in [23] (see also [24, 22, 26]).

Throughout this section let $A = kQ/I$ be a gentle algebra. Recall that there is a triangle equivalence $\mathcal{D}^b(A) \simeq \mathcal{K}^{-,b}(A - \mathrm{proj})$, where $A - \mathrm{proj}$ is the full subcategory of $A-\mathrm{mod}$ given by the finitely generated projective A-modules, $\mathcal{K}^{-,b}(A - \mathrm{proj})$ is the homotopy category of complexes of objects in $A - \mathrm{proj}$ which are bounded on the right and have bounded homology, and $\mathcal{D}^b(A)$ is the bounded derived category of $A - \mathrm{mod}$.

For every $a \in Q_1$, we define a formal inverse $\overline{a}$ where $s(\overline{\alpha}) = t(\alpha)$ and $t(\overline{\alpha}) = s(\alpha)$. We denote by $\overline{Q_1}$ the set of formal inverses of the elements in Q_1, and we extend the operation $\overline{(-)}$ to an involution of $Q_1 \cup \overline{Q_1}$ by setting $\overline{\overline{\alpha}} = \alpha$.

A **walk** is a sequence $w_1 \dots w_n$, where $w_i \in Q_1 \cup \overline{Q_1}$, such that $s(w_{i+1}) = t(w_i)$. We also allow **trivial walks** e_u for every vertex u of Q. A **string** is a walk w, such that $w_{i+1} \neq \overline{w_i}$ and such that for all substrings $w' = w_i w_{i+1} \cdots w_j$ of w with the $w_i, \dots, w_j$ all in Q_1 (or all in $\overline{Q_1}$), we have that $w' \notin I$ (or $\overline{w'} \notin I$, respectively). We say that $w = w_1 \dots w_n$ is a **direct** (resp. **inverse**) string if for all $1 \leq i \leq n$, we have $w_i \in Q_1$ (resp. $w_i \in \overline{Q_1}$).

A **generalized walk** is a sequence $\sigma_1 \dots \sigma_m$, such that each σ_i is a string, such that $s(\sigma_{i+1}) = t(\sigma_i)$.

Definition I.2.1. Let $A = kQ/I$ be a gentle algebra. A **finite homotopy string** $\sigma = w_1 \dots w_n$, where $w_i \in Q_1 \cup \overline{Q_1}$, is a (possibly trivial) walk in (Q, I) consisting of subwalks $\sigma_1, \dots, \sigma_r$ with $\sigma = \sigma_1 \dots \sigma_r$ and such that

1. σ_j is a direct or inverse string;
2. if σ_j, σ_{j-1} are both direct strings then $\sigma_{j-1}\sigma_j \in I$ (resp. if both $\overline{\sigma_j}, \overline{\sigma_{j-1}}$ are inverse strings, then $\overline{\sigma_{j-1}\sigma_j} \in I$).

If σ_k is a direct string it is called a **direct homotopy letter**, otherwise it is called an **inverse homotopy letter**.

A **homotopy band** is a finite homotopy string $\sigma = \sigma_1 \dots \sigma_r$ with an equal number of direct and inverse homotopy letters σ_i such that $t(\sigma_r) = s(\sigma_1)$ and $\sigma_1 \neq \overline{\sigma_r}$ and $\sigma \neq \tau^m$ for some homotopy string τ and $m > 1$.

A homotopy string or band $\sigma = \sigma_1 \dots \sigma_r$ is **reduced** if $\sigma_i \neq \overline{\sigma_{i+1}}$ for all $i \in \{1, \dots, r-1\}$.

A generalized walk is called a **direct** (resp. **inverse**) **antipath** if each homotopy letter is a direct (resp. inverse) arrow.

Definition I.2.2. A left (resp. right) infinite generalized walk $\sigma = \dots \sigma_{-2}\sigma_{-1}$ (resp. $\sigma = \sigma_1\sigma_2 \dots$) is called a **left (resp. right) infinite homotopy string** if there exists $k \geq 1$ such that $\dots \sigma_k\sigma_{k+1}$ (resp. $\sigma_{-k-1}\sigma_{-k} \dots$) is a direct (resp. inverse) antipath which is eventually periodic and eventually involves only homotopy letters of length 1.

A two-sided infinite generalized walk $\sigma = \dots \sigma_{-1}\sigma_0\sigma_1 \dots$ is called a **two-sided infinite homotopy string** if $\dots \sigma_{-1}\sigma_0$ is a left infinite homotopy string and $\sigma_0\sigma_1 \dots$ is a right infinite homotopy string.

To each homotopy string and homotopy band σ, as described above, there is an associated (possibly infinite) complex of projective modules $P^\bullet_\sigma$ [12]. We now recall this construction.

Definition I.2.3 ([12])**.** 1. Let $\sigma = \sigma_1 \cdots \sigma_r$ be a finite reduced homotopy string. Define $v_0 = s(\sigma_1)$ and $v_i = t(\sigma_i)$ for all $i \in \{1, \dots, r\}$. Define further $\mu_0 = 0$ and

$$\mu_{i+1} = \begin{cases} \mu_i + 1 & \text{if } \sigma_i \text{ is a direct homotopy letter;} \\ \mu_i - 1 & \text{if } \sigma_i \text{ is an inverse homotopy letter,} \end{cases}$$

and let $\mu(\sigma) := \min_{i\in\{0,1,\dots,r\}}(\mu_i)$. Then the complex

$$P_\sigma^\bullet = \ldots \to P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} P^1 \to \ldots$$

is given by

- for all $j \in \mathbb{Z}$,
$$P^j = \bigoplus_{\substack{0\leq i\leq r\\ \mu_i=j}} P_{v_i},$$
where each P_{v_i} is the indecomposable projective module associated to the vertex v_i;
- each direct (resp. inverse) homotopy letter σ_i defines a morphism $P_{v_{i-1}} \xrightarrow{\sigma_i} P_{v_i}$ (resp. $P_{v_i} \xrightarrow{\overline{\sigma_i}} P_{v_{i-1}}$). These form the components of the differentials d^j in the natural way. We call $P_\sigma^\bullet$ a **string object**.

2. The definition of $P_\sigma^\bullet$ when σ is an infinite reduced homotopy string is similar, and we again call $P_\sigma^\bullet$ a **string object**.

3. Let $\sigma = \sigma_1 \cdots \sigma_r$ be a reduced homotopy band. Let M be a finite-dimensional indecomposable $k[X]$-module, and let $m = \dim_K M$. Let F be the matrix of the multiplication by X for a given basis of M. Define $v_0, \dots, v_r, \mu_0, \dots, \mu_r$ as for homotopy strings.

   Then the complex $P_{\sigma,F}^\bullet$ is defined by

   - for all $j \in \mathbb{Z}$,
   $$P^j = \bigoplus_{\substack{0\leq i\leq r-1\\ \mu_i=j}} P_{v_i}^{\oplus m};$$
   - for all $i \in \{1, \dots, r-1\}$, the direct (resp. inverse) homotopy letter σ_i defines a morphism $P_{v_{i-1}}^{\oplus m} \xrightarrow{\sigma_i \mathrm{Id}_m} P_{v_i}^{\oplus m}$ (resp. $P_{v_i}^{\oplus m} \xrightarrow{\overline{\sigma_i}\mathrm{Id}_m} P_{v_{i-1}}^{\oplus m}$), where Id_m is the $m \times m$ identity matrix. These form the components of the differentials d^j in the natural way.
   - The homotopy letter σ_r defines a final component of the differential. If it is a direct letter, then the morphism used is $P_{v_{r-1}}^{\oplus m} \xrightarrow{\sigma_i F} P_{v_0}^{\oplus m}$; otherwise, the morphism is $P_{v_0}^{\oplus m} \xrightarrow{\overline{\sigma_i} F} P_{v_{r-1}}^{\oplus m}$.

In this case, $P_{\sigma,F}^\bullet$ is called a **band object**.

Furthermore, it is shown in [12] that the isomorphism classes of indecomposable objects in $\mathcal{D}^b(A)$ up to shift are in bijection with homotopy strings and bands up to inverse. More precisely, the equivalence is modulo the equivalence relation $\sigma \sim \overline{\sigma}$ for a homotopy string σ, infinite homotopy strings up to inverse, and pairs consisting of a homotopy band up to inverse and up to permutation and of an isomorphism class of indecomposable $k[X]$-modules. This bijection is the one described in Definition I.2.3.

Remark I.2.4. If the field k is algebraically closed, then the matrix F of Definition I.2.3 (3) can always be chosen to be a Jordan block $J_m(\lambda)$ of size m corresponding to a scalar $\lambda \in k$. Note that for $m = 1$, we have $P^\bullet_{\sigma,J_1(\lambda)} = P^\bullet_{\sigma,\lambda}$. In the text, if the result or proof does not depend on the scalar λ, we will sometimes omit it in our notation and we will write $P^\bullet_\sigma$ instead of $P^\bullet_{\sigma,\lambda}$.

I.2.2 Main result on indecomposable objects of the derived category

We are now ready to prove our classification of indecomposable objects (up to shift) in $\mathcal{D}^b(A)$ using curves on the marked surface $\mathcal{S}_A$. Before stating our main result, we recall, see for example [35, 54], the definition of the orbit category $\mathcal{D}^b(A)/[1]$ where $[1]$ is the shift functor. Namely the objects of $\mathcal{D}^b(A)/[1]$ are the same as the objects of $\mathcal{D}^b(A)$ and

$$\mathrm{Hom}_{D^b(A)/[1]}(X, Y) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{D^b(A)}(X, Y[n]).$$

Theorem I.2.5. *Let $A = kQ/I$ be a gentle algebra with marked ribbon graph Γ_A and a marked embedding in the associated ribbon surface $\mathcal{S}_A = (S_A, \overline{\mathcal{M}})$. Let $[1]$ be the shift functor in $\mathcal{D}^b(A)$. Then,*

1. *the isomorphism classes of the indecomposable string objects in $\mathcal{D}^b(A)/[1]$ are in bijection with homotopy classes of arcs on $\mathcal{S}_A$.*
2. *the isomorphism classes of the indecomposable band objects in $\mathcal{D}^b(A)/[1]$ are in bijection with pairs $([\gamma], \mathcal{V})$, where $[\gamma]$ is a homotopy class of loops on $\mathcal{S}_A$ satisfying condition (3) of Lemma I.2.11 below and $\mathcal{V}$ is an isomorphism class of indecomposable $K[X]$-modules.*

More precisely, finite arcs correspond to homotopy string complexes, infinite arcs correspond to infinite homotopy string complexes and loops correspond to homotopy bands.
We will sometimes refer to a loop, which represents indecomposable objects

as in Theorem I.2.5, as a **gradable loop**.

Before proving the theorem, we need some results on the geometry of the lamination.

Lemma I.2.6. *1. The lamination L_A subdivides S_A into polygons whose sides are laminates and boundary segments. The laminates of L_A can be chosen to be the "glued edges" of Definition I.1.3.*

2. Each polygon contains exactly one marked point.

3. Every boundary segment of S_A contains the endpoint of at least one laminate of L_A.

Proof. It suffices to observe that the "glued edges" of Definition I.1.3 cut the surface S_A into the polygons P_v of Definition I.1.3, which contain exactly one marked point each by definition. Moreover, every boundary segment of these polygons is adjacent to at least one laminate. □

Once we know that a surface is cut into polygons, then any arc is determined by the order in which it crosses the edges of the polygons. Note that the edges crossed correspond exactly to the laminates.

Assumption I.2.7. Whenever in this section we shall be dealing with collections of curves on a surface, we will assume that they are in minimal position.

Lemma I.2.8. *Let γ be a curve on S_A and assume that every laminate of L_A that γ crosses is crossed transversally (we can assume this, up to homotopy).*

1. *If γ is an arc, then its homotopy class is completely determined by the (possibly infinite) sequence of the laminates that it crosses.*
2. *If γ is a loop, then its homotopy class is completely determined by the sequence of the laminates that it crosses, up to cyclic ordering.*

The order in which a curve crosses the laminates gives rise to a homotopy string or band, as we will see in Lemma I.2.11.

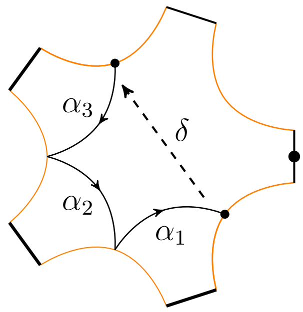
Definition I.2.9. Let P_v be a polygon on the surface S_A, as per Lemma I.2.6, and let M_v be the unique marked point in P_v. Let δ be a curve in P_v starting and ending on edges ℓ_1 and ℓ_2 of P_v which are laminates.

- If M_v lies between ℓ_2 and ℓ_1 in the clockwise order, then let $w_1, \ldots, w_r$ be the laminates between $\ell_1 = w_1$ and $\ell_2 = w_r$ in clockwise order. By Proposition I.1.19, these correspond to vertices of the quiver Q of A which are joined by arrows $\alpha_1, \ldots, \alpha_{r-1}$.

  Then define $\sigma(\delta) := \alpha_1 \cdots \alpha_{r-1}$.

- If M_v lies between ℓ_1 and ℓ_2 in the clockwise order, then let $w_1, \dots, w_r$ be the laminates between $\ell_2 = w_1$ and $\ell_1 = w_r$ in clockwise order. By Proposition I.1.19, these correspond to vertices of the quiver Q of A which are joined by arrows $\alpha_1, \dots, \alpha_{r-1}$.

  Then define $\sigma(\delta) := \overline{\alpha_1 \cdots \alpha_{r-1}}$.



Lemma I.2.10. *Let P_v and δ be as in Definition I.2.9. Then $\sigma(\delta)$ is a homotopy letter.*

Proof. By Proposition I.1.19, the compositions of the arrows of $\sigma(\delta)$ are not in the ideal of relations of A. $\square$

Lemma I.2.11.
1. *Let γ be a finite arc on S_A. Let $\ell_1, \ell_2, \dots, \ell_r$ be the laminates crossed (in that order) by γ, as per Lemma I.2.8. For every $i \in \{1, 2, \dots, r-1\}$, let γ_i be the part of γ between its crossing of ℓ_i and of ℓ_{i+1}. Let*

$$\sigma(\gamma) := \begin{cases} \prod_{i=1}^{r-1} \sigma(\gamma_i) & \text{if } r > 1; \\ e_{\ell_1} & \text{if } r=1. \end{cases}$$

   Then $\sigma(\gamma)$ is a homotopy string.

2. *Let γ be an infinite arc. Assume that on any infinite end of γ, the arc cycles infinitely many times around a boundary component in counter-clockwise orientation. Let (ℓ_i) be the sequence of laminates crossed by γ (this sequence can be infinite on either side). For every i, let γ_i be the part of γ between its crossing of ℓ_i and of ℓ_{i+1}. Let*

$$\sigma(\gamma) := \prod_i \sigma(\gamma_i).$$

   Then $\sigma(\gamma)$ is an infinite homotopy string.

3. *Let γ be a primitive loop on S_A. Let $\ell_1, \ell_2, \ldots, \ell_r$ be the laminates crossed (in that order) by γ. For every $i \in \{1, 2, \ldots, r-1\}$, let γ_i be the part of γ between its crossing of ℓ_i and of ℓ_{i+1}, and let γ_r be the part of γ between its crossing of ℓ_r and of ℓ_1. Let*

$$\sigma(\gamma) := \prod_{i=1}^{r} \sigma(\gamma_i).$$

If there is an equal number of inverse and direct homotopy letters among the $\sigma(\gamma_i)$, then $\sigma(\gamma)$ is a homotopy band.

Proof. In all three cases, for any index i, if $\sigma(\gamma_i)$ and $\sigma(\gamma_{i+1})$ are both direct homotopy letters, then by Proposition I.1.19, composition of the last arrow of $\sigma(\gamma_i)$ and of the first of $\sigma(\gamma_{i+1})$ form a relation. The argument for consecutive inverse homotopy letters is similar. This proves (1).

To prove (2), assume that γ is an infinite arc. Then γ eventually wraps around one of the boundary components without marked points. By Lemma I.2.6, there is at least one laminate with one endpoint on this boundary component. Thus, by Proposition I.1.19, every full turn of γ around the boundary component induces a subword of $\sigma(\gamma)$ of the form $\alpha_1 \cdots \alpha_r$, where the α_i form an oriented cycle of Q such that every composition is a relation. Thus $\sigma(\gamma)$ is eventually periodic, with homotopy letters of length one. Since the infinite ends of γ cycle around a boundary component in counter-clockwise direction, we get that the start (or the end) of $\sigma(\gamma)$, if infinite, is a direct (resp. inverse) antipath. This proves (2).

To prove (3), assume that γ is a primitive loop. Write $\sigma(\gamma) := \prod_{i=1}^{r} \sigma(\gamma_i)$ as in the statement of the Lemma. Clearly, $s(\sigma(\gamma_1)) = t(\sigma(\gamma_r))$ and $\sigma(\gamma_1) \neq \overline{\sigma(\gamma_r)}$. The condition on the number of inverse and direct homotopy letters among the $\sigma(\gamma_i)$ ensures that $\sigma(\gamma)$ is a homotopy band. $\square$

Remark I.2.12. In Lemma I.2.11 (3), it should be stressed that the condition on the number of inverse and direct homotopy letters among the $\sigma(\gamma_i)$ is not satisfied by all loops. In Section I.8 we give a more general interpretation of the difference between the numbers of direct and inverse homotopy letters.

Conversely, any homotopy string or band defines an arc or a loop on S_A.

Lemma I.2.13. 1. *For any finite homotopy string τ, there exists a unique finite arc γ on S_A (up to homotopy) such that $\tau = \sigma(\gamma)$.*

2. *For any infinite homotopy string τ, there exists a unique infinite arc γ on S_A (up to homotopy) such that $\tau = \sigma(\gamma)$.*

3. *For any homotopy band b, there exists a unique loop γ on S_A (up to homotopy) such that $b = \sigma(\gamma)$.*

Proof. We only prove (1); the proofs of (2) and (3) are similar. Write $\tau = \tau_1 \cdots \tau_r$, where each τ_i is a homotopy letter. Write $\tau_i = \alpha_i^1 \cdots \alpha_i^{s_i}$, where the α_i^j are either all arrows or all inverse arrows. By Proposition I.1.19, since there are no relations in the (possibly inverse) path $\alpha_i^1 \cdots \alpha_i^{s_i}$, then there are laminates $\ell_i^1, \dots, \ell_i^{s_i+1}$ inside a unique polygon P_v such that ℓ_i^j and ℓ_i^{j+1} have an endpoint on the same boundary segment of P_v and ℓ_i^{j+1} follows ℓ_i^j in the clockwise order if τ_i is a direct homotopy letter, and counter-clockwise order if τ_i is an inverse homotopy letter.

Define γ_i to be a segment in P_v going from ℓ_i^1 to $\ell_i^{s_i+1}$ if τ_i is a direct homotopy letter, or the other way around if τ_i is an inverse homotopy letter. We can assume that the endpoint of γ_i is the starting point of γ_{i+1}.

If we define $\gamma(\tau)$ to be the concatenation of $\gamma_1, \dots, \gamma_r$, then $\sigma(\gamma(\tau)) = \tau_1 \cdots \tau_r = \tau$. This proves the existence result.

To prove uniqueness, assume that γ and γ' are such that $\sigma(\gamma) = \sigma(\gamma')$. Let τ be the (unique) reduced expression of the homotopy string $\sigma(\gamma) = \sigma(\gamma')$. Then $\gamma(\tau)$ is homotopic to γ and γ'. Indeed, if $\sigma(\gamma)$ is reduced, then $\tau = \sigma(\gamma)$ and we are done. Otherwise, it means that in the expression $\sigma(\gamma)_1 \cdots \sigma(\gamma)_r$ of $\sigma(\gamma)$ as a product of homotopy letters, there are two adjacent letters $\sigma(\gamma)_i$ and $\sigma(\gamma)_{i+1}$ that are inverse to each other. Then the corresponding segments in the polygon P_v described above are the same path going in opposite directions; their concatenation is thus homotopic to a trivial path. Thus if we cancel the two inverse homotopy letters, we get that $\gamma\Big(\sigma(\gamma)_1 \cdots \sigma(\gamma)_{i-1}\sigma(\gamma)_{i+2} \cdots \sigma(\gamma)_r\Big)$ is homotopic to $\gamma\Big(\sigma(\gamma)\Big)$. By induction on the number of reduction steps to get from $\sigma(\gamma)$ to τ, we get that $\gamma\Big(\sigma(\gamma)\Big) = \gamma(\tau)$.
The same applies if we replace γ by γ'. This proves the uniqueness, and finishes the proof of the Lemma. □

With the help of the previous lemma, we can now prove Theorem I.2.5.

Proof of Theorem I.2.5. It follows from the results of [12] that indecomposable objects of $\mathcal{D}^b(A)$ are in bijection with homotopy strings (finite and infinite) and homotopy bands paired with an isomorphism class of indecomposable $k[X]$-modules. As explained in the preliminaries of this thesis, such an isomorphism class of $k[X]$-modules corresponds to an isomorphism class of an indecomposable local system. By Lemma I.2.11, we can associate a homotopy string or band to each of the curves listed in the statement of Theorem I.2.5. Then Lemma I.2.13 ensures that this defines the desired bijections. □

I.3 Homomorphisms in the derived category of a gentle algebra

The morphism spaces in the bounded derived category $\mathcal{D}^b(A)$ of a gentle algebra A were completely described in [4]. Our aim in this section is to describe a basis of the morphism spaces in the orbit category
$\mathcal{D}^b(A)/[1]$ in terms of curves on the surface $\mathcal{S}_A$ that was associated to A in Section I.1.

I.3.1 Bases for morphism spaces in the derived category

We now briefly recall the results of [4]. These results are proved in the case where the base field k is algebraically closed; for the rest of this section, we will assume that we are in this situation. Also, their results deal with morphisms in $\mathcal{D}^b(A)$, but we will immediately translate them to the setting of the orbit category $\mathcal{D}^b(A)/[1]$.
Let σ and τ be two homotopy strings or bands. Let $P^\bullet_\sigma$ and $P^\bullet_\tau$ be the associated indecomposable objects in $\mathcal{D}^b(A)/[1]$ (if σ is a homotopy band and $\lambda \in k^\times$, then we write $P^\bullet_\sigma$ instead of $P^\bullet_{\sigma,\lambda}$). In all that follows, we consider σ and τ only up to the action of the inverse operation $\overline{?}$; this means that whenever we are comparing σ and τ, we also need to compare $\overline{\sigma}$ and τ in order to get all morphisms.

Graph maps

Assume that σ and τ have a maximal subword in common, say $\sigma_i\sigma_{i+1}\cdots\sigma_j$ and $\tau_i\tau_{i+1}\cdots\tau_j$, with each σ_ℓ equal to τ_ℓ. We also allow this subword to be a trivial homotopy string.

Consider the following conditions.

LG1 Either the homotopy letters σ_{i-1} and τ_{i-1} are both direct and there exists a path p in Q, such that $p\tau_{i-1} = \sigma_{i-1}$, or they are both inverse letters and there exists a path p in Q, such that $\tau_{i-1} = \sigma_{i-1}p$.

LG2 The homotopy letter σ_{i-1} is either zero or inverse, and τ_{i-1} is either zero or direct.

RG1 Dual of (LG1).

RG2 Dual of (LG2).

If one of (LG1) and (LG2) holds, and one of (RG1) and (RG2) holds, then one can construct a morphism from $P_\sigma^\bullet$ to $P_\tau^\bullet$ called a **graph map**. Note that if σ and τ are infinite homotopy strings, then the definition above extends to the case where the strings have an infinite subword in common: for instance, if this subword is on the left, then one simply drops conditions (LG1) or (LG2).

We make following useful observation. Let $u = \alpha_1 \cdots \alpha_n$ be a cycle, i.e. u is an antipath, $\alpha_n \alpha_1 \in I$, and there exists no v with the same properties, such that u is a power of v. Let $\sigma = \ldots \sigma_j u_1 u_2 \ldots$ and $\tau = \ldots \tau_j u_1' u_2' \ldots$ (with $u_i = u = u_i'$) be infinite homotopy strings, which share a maximal infinite common subword $\sigma_i \ldots \sigma_j u_1 u_2 \ldots = \tau_i \ldots \tau_j u_1' u_2' \ldots$, such that u is not a suffix of $\sigma_i \ldots \sigma_j$. Then, this common subword satisfies one of the left end point conditions (and hence gives rise to a graph map) if and only if for all $l \in \mathbb{N}$ their maximal common subword

$$u_1 u_2 \ldots = u_{l+1}' u_{l+2}' \ldots$$

satisfies one of the left end point conditions. By applying a similar argument to the case when $\sigma = \tau$ are infinite homotopy strings, we see that graph maps between infinite string complexes, which correspond to infinite common subwords, occur in families. It is not difficult to see from the definition of the associated chain maps, that in the above situation, these families of graph maps correspond to morphisms $f_i : P_\sigma^\bullet \to P_\tau^\bullet[m + i \cdot n]$, where $i \geq 0$ and n is the length of the cycle as above.

Quasi-graph maps

Keep the above notations. If none of the conditions (LG1), (LG2), (RG1) and (RG2) hold, then one can construct a morphism in $\mathcal{D}^b(A)/[1]$ from $P_\sigma^\bullet$ to $P_\tau^\bullet$, called a **quasi-graph map**. Again, this definition extends to infinite homotopy strings in the natural way.
Note that a quasi-graph map gives rise to a homotopy class of single and double maps, defined in the next section. In fact, all single and double maps that are not singleton maps arise in this way, see [4].

Single maps

Assume that there are direct homotopy letters σ_i and τ_j and a non-trivial path p, such that $s(p) = t(\sigma_i)$ and $t(p) = t(\tau_j)$. (What follows also works if σ_i and τ_j are both inverse letters by working with $\overline{\sigma}$ and $\overline{\tau}$ instead).

Consider the following conditions:

L1 If σ_i is direct, then $\sigma_i p \in I$.

L2 If τ_i is inverse, then $p\overline{\tau}_i \in I$.

R1 If σ_{i+1} is inverse, then $\overline{\sigma}_{i+1}p \in I$.

R2 If τ_{i+1} is direct, then $p\tau_{i+1} \in I$.

If conditions (L1), (L2), (R1) and (R2) are satisfied, then p induces a morphism of complexes from $P^\bullet_\sigma$ to $P^\bullet_\tau$ called a **single map**.

Assume, moreover, that

- σ_{i+1} is zero or is a direct homotopy letter of the form $p\sigma'_{i+1}$, where σ'_{i+1} is a direct homotopy letter;
- τ_i is zero or is a direct homotopy letter of the form $\tau'_i p$, where τ'_i is a direct homotopy letter.
- p is neither a subword of σ_i nor a subword of τ_{i+1}.

If that is the case, then p induces a morphism from $P^\bullet_\sigma$ to $P^\bullet_\tau$ in $\mathcal{D}^b(A)/[1]$ called a **singleton single map**.

Double maps

Keeping the above notations, assume now that there are non-trivial paths p and q, such that $s(p) = s(\sigma_i)$, $t(p) = s(\tau_j)$, $s(q) = t(\sigma_i)$ and $t(q) = t(\tau_j)$, and such that $\sigma_i q = p\tau_i$.
If conditions (L1) and (L2) above are satisfied for p and conditions (R1) and (R2) are satisfied for q, then p and q induce a morphism of complexes from $P^\bullet_\sigma$ to (a shift of) $P^\bullet_\tau$ called a **double map**.
If, moreover, there exists a non-trivial path r, such that $\sigma_i = \sigma'_i r$ and $\tau_i = r\tau'_i$, with σ'_i and τ'_i direct homotopy letters, then p and q induce a morphism from $P^\bullet_\sigma$ to $P^\bullet_\tau$ in $\mathcal{D}^b(A)/[1]$ called a **singleton double map**.

The basis

We can now state the main result of [4].

Theorem I.3.1 (Theorem 3.15 of [4])**.** *A basis of the space of morphisms from $P^\bullet_\sigma$ to $P^\bullet_\tau$ in $\mathcal{D}^b(A)/[1]$ is given by all graph maps, quasi-graph maps, singleton single maps and singleton double maps in $\mathcal{D}^b(A)/[1]$.*

Definition I.3.2. The basis described in Theorem I.3.1 will be called the **standard basis**.

I.3.2 Morphisms as intersections

As before, let $A = kQ/I$ denote a fixed gentle algebra and let $\mathcal{S}_A = (S_A, \overline{\mathcal{M}})$ denote its marked surface (see Section I.1). As before, we assume all sets consisting of curves and laminates on S_A to be in minimal position.

The main result of this section (Theorem I.3.3) is that the set of intersection points of two curves δ_1 and δ_2 on S_A gives rise to the standard basis of the vector space of morphisms from $P^\bullet_{\sigma(\gamma_1)}$ to $P^\bullet_{\sigma(\gamma_2)}$ in $\mathcal{D}^b(A)/[1]$. Recall that S_A has no punctures and possibly boundary components without any marked points. Throughout this section, we will replace all boundary components of S_A without marked points by punctures, and consider infinite arcs wrapping around such a boundary component as infinite arcs going to the puncture as explained in the preliminaries. We can do this, since according to our conventions every infinite arc wraps around such a boundary component only in one direction, namely the counter-clockwise direction. In our new marked surface, every boundary component contains at least one marked point and the set of punctures is in bijection with the unmarked boundary components of our old surface.

Let us state the main result of this section.

Theorem I.3.3. *Let γ_1 and γ_2 be arcs or loops on $\mathcal{S}_A$, and let $\mathcal{B}$ be the standard basis of* $\operatorname{Hom} D^b(A)/[1](P^\bullet_{\sigma(\gamma_1)}, P^\bullet_{\sigma(\gamma_2)})$. *Then there exists an explicit injection*

$$\mathfrak{B} : \gamma_1 \overrightarrow{\cap} \gamma_2 \hookrightarrow \mathcal{B}.$$

Moreover, the following hold true.

i) The map $\mathfrak{B}$ is a bijection, unless γ_1 and γ_2 are the same loop and $P^\bullet_{\sigma(\gamma_1),\lambda_1}$ and $P^\bullet_{\sigma(\gamma_2),\lambda_2}$ are isomorphic, or γ_1 and γ_2 intersect at a puncture.

ii) If γ_1 and γ_2 are the same loop and $P^\bullet_{\sigma(\gamma_1),\lambda_1}$ and $P^\bullet_{\sigma(\gamma_2),\lambda_2}$ are isomorphic, then $\mathfrak{B}$ is not surjective, and the missing elements in its image are an invertible graph map and the quasi graph map ξ that appears in an Auslander-Reiten triangle

$$\tau P^\bullet_{\sigma(\gamma_1)} \to E \to P^\bullet_{\sigma(\gamma_1)} \xrightarrow{\xi} \tau P^\bullet_{\sigma(\gamma_1)}[1]$$

(keeping in mind that in this case, $\tau P^\bullet_{\sigma(\gamma_1)} = P^\bullet_{\sigma(\gamma_1)}[1] = P^\bullet_{\sigma(\gamma_1)}$ in $\mathcal{D}^b(A)/[1]$).

iii) For every puncture p at which γ_1 and γ_2 intersect, there exists an infinite family of elements $(\mathfrak{B}(p)(i))_{i\geq 0}$ in $\mathcal{B}$, where $\mathfrak{B}(p)(0) = \mathfrak{B}(p)$. Morover $\mathcal{B}$ is equal to the disjoint union of the image of $\mathfrak{B}$ and all elements of the form $\mathfrak{B}(q)(j)$, where $q \in \gamma_1 \cap \gamma_2$ is a puncture and $j \geq 1$.

The proof of Theorem I.3.3 occupies the rest of this section.

The Walk of an Intersection

It will be useful to lift intersection points of curves to a universal cover of S_A. Let $\pi : \tilde{S}_A \to S_A$ be a fixed universal covering map, and let $\tilde{L}_A$ be the set of all lifts of laminates $\ell \in L_A$. Note that $\tilde{S}_A$ is a union of polygons whose edges are either boundary segments or laminates in $\tilde{L}_A$. We lift arcs on S_A to arcs on $\tilde{S}_A$ and loops on S_A to infinite paths $(0,1) \to \tilde{S}_A$.

Let γ_1 and γ_2 be two arcs or loops on S_A. Let $q \in \gamma_1 \overrightarrow{\cap} \gamma_2$, and let $\tilde{q}$ be any lift of q on $\tilde{S}_A$. There are unique lifts $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ on $\tilde{S}_A$, such that $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ intersect at $\tilde{q}$.

Lemma I.3.4. *The curves $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ intersect only at $\tilde{q}$.*

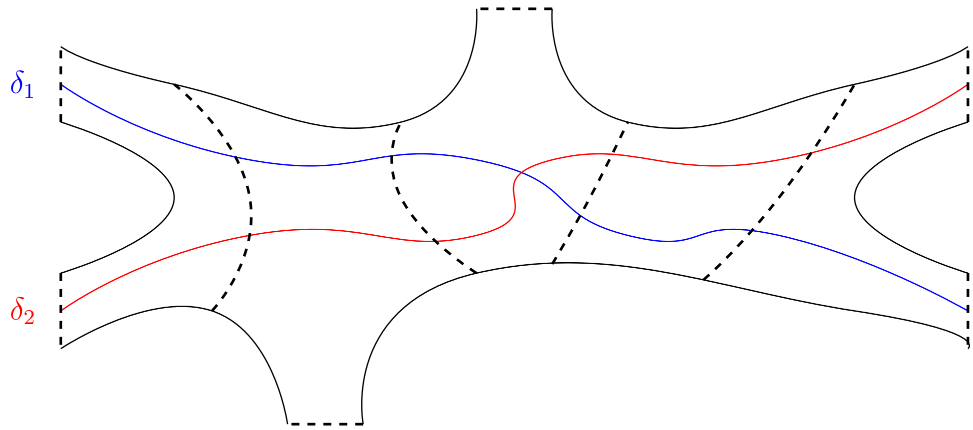
Proof. It follows from [69] that $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are simple. Assume that they intersect twice in succession, say at $\tilde{r}_1$ and $\tilde{r}_2$. Then $\tilde{r}_1$ and $\tilde{r}_2$ are not two lifts of the same intersection point of γ_1 and γ_2. Indeed, the sections of $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ between $\tilde{r}_1$ and $\tilde{r}_2$ form a disc. Around the boundary of this disc, we can assume without loss of generality that $\tilde{\gamma}_1$ comes before $\tilde{\gamma}_2$ at $\tilde{r}_1$ in the orientation of the surface. But then $\tilde{\gamma}_2$ comes before $\tilde{\gamma}_1$ at $\tilde{r}_2$, which is impossible if $\tilde{r}_1$ and $\tilde{r}_2$ are lifts of the same intersection point of γ_1 and γ_2.

Therefore, by the bigon criterion (see [42], Proposition 1.7), we can find a homotopy of $\tilde{\gamma}_1$ which descends to a homotopy of γ_1 that reduces the number of intersections with γ_2 - a contradiction with the assumption that the two are in minimal position.

□

Next, we define a region $S_{\tilde{q}}$ of $\tilde{S}_A$. Let P_0 be the polygon of $\tilde{S}_A$ containing $\tilde{q}$. Define a set of polygons $\mathcal{P}_n$ recursively by setting $\mathcal{P}_0 = \{P_0\}$ and by letting $\mathcal{P}_{n+1}$ contain all polygons of $\mathcal{P}_n$ and all polygons P_v adjacent to a polygon of $\mathcal{P}_n$, such that both $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ go through P_v. Then $S_{\tilde{q}}$ is defined to be the union of all polygons belonging to one of the $\mathcal{P}_n$. In other words, $S_{\tilde{q}}$ is the region of $\tilde{S}_A$ containing the laminates intersected by both $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ as well as $\tilde{q}$.

Lemma I.3.5. *The surface $S_{\tilde{q}}$ is a union of finitely many polygons.*



The surface $S_{\tilde{q}}$: Dashed curves belong to $\tilde{L}_A$, whereas the blue and the red curve show δ_1 and δ_2, respectively, and solid black lines belong to $\partial\tilde{S}_A$. In this example $\sigma(q)$ has 3 homotopy letters.

Proof. The result is trivial if γ_1 or γ_2 is an arc. Assume that γ_1 and γ_2 are loops and suppose that $S_{\tilde{q}}$ contains an infinite number of polygons. Since a fundamental domain of S_A in $\tilde{S}_A$ contains only finitely many polygons, one of the polygons in $S_{\tilde{q}}$ will contain another lift of q, say $\tilde{q}'$. At this lift, $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ must intersect. This contradicts Lemma I.3.4. □

Let δ_1 and δ_2 be the parts of $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ contained in $S_{\tilde{q}}$. In the interior of $S_{\tilde{q}}$, the curves δ_1 and δ_2 cross the same laminates of $\tilde{L}_A$ in the same order. These crossings define a (possibly empty and possibly infinite) homotopy string $\sigma(q)$. If $\sigma(q)$ is non-empty, it is a subwalk of $\sigma(\gamma_1)$ and $\sigma(\gamma_2)$, where we replaced γ_1 and γ_2 implicitely by the appropriate curves in the original unpunctured surface S_A in a canonical way. In case of a homotopy band $\sigma_1 \cdots \sigma_n$, we mean that $\sigma(q)$ is a subwalk of the cyclic two-sided infinite walk $\cdots \sigma_1 \cdots \sigma_n \sigma_1 \cdots$. If, on the other hand $\sigma(q)$ is empty, it means that $\pi \circ \delta_1$ and $\pi \circ \delta_2$ are contained in a single polygon P of S_A.

Remark I.3.6. Since we have replaced boundary components without marked points by punctures, one should note that if γ_1 and γ_2 meet at a puncture, then there exists lifts of them and an infinite number of lifted laminates that are crossed by both lifts. In this case, the walk $\sigma(q)$ is infinite on one side.

Lemma I.3.7. *If $\sigma(q)$ is empty, then the unique marked point in P as before is contained in $\mathcal{M}$ (see Definition I.1.8).*

Proof. Suppose the marked point in P is an element in $\mathcal{M}_0$. Then the only laminate on the boundary of $S_{\tilde{q}}$ is a lift of the laminate ℓ associated to a vertex $v \in Q_0$. But by definition of $S_{\tilde{q}}$, there exists $i \in \{1,2\}$, such that γ_i does not cross ℓ. Thus, γ_i is contained in P and homotopic to a constant path - a contradiction. $\square$

We now explain how an intersection gives rise to a morphism. For $j \in \{1,2\}$, denote $p_j^1, \ldots, p_j^m$ the ordered sequence of intersections of the curve δ_j with the boundary or the laminates of L_A. We may assume that if $\sigma(q)$ is non-empty, then for each $i \in (1,m)$, p_1^i and p_2^i lie on the same laminate.

Then δ_1 and δ_2 in $S_{\tilde{q}}$ are homeomorphic to two arcs crossing in a closed disc; their endpoints alternate on the boundary of $S_{\tilde{q}}$. We distinguish two cases:

a) if $\sigma(q)$ is non-empty, then either p_2^1 comes immediately before p_1^1 in the counter-clockwise orientation of $\partial S_{\tilde{q}}$, or vice versa;

b) if $\sigma(q)$ is empty (i.e. $m = 2$), let X be the unique marked point on the boundary of the polygon containing δ_1 and δ_2. Then either X is after an endpoint of δ_2 and before an endpoint of δ_1 in the counter-clockwise orientation of $\partial S_{\tilde{q}}$, or vice versa.

The subsequent lemmas prove that δ_1 and δ_2 encode a basis element of the corresponding homomorphism space in a natural way.

Lemma I.3.8. *Let $\mathcal{B}$ be the basis of* $\operatorname{Hom} \mathcal{D}^b(A)/[1](P^\bullet_{\sigma(\gamma_1)}, P^\bullet_{\sigma(\gamma_2)})$ *described in Theorem I.3.1, and let $q \in \gamma_1 \overrightarrow{\cap} \gamma_2$. Then q gives rise to an element $\mathfrak{B}(q) \in \mathcal{B}$. Furthermore,*

i) If $\sigma(q)$ is non-empty, then $\mathfrak{B}(q)$ is a graph map if p_2^1 comes immediately before p_1^1 in the counter-clockwise orientation and a quasi graph map otherwise.

ii) If $\sigma(q)$ is empty, then $\mathfrak{B}(q)$ is a singleton single or singleton double map.

Proof. Suppose first that $\sigma(q)$ is non-empty. Assume for the moment that q is not on the boundary of S_A and is not a puncture. Let P_v be the polygon in which $p_1^1, p_2^1, p_1^2, p_2^2$ are found, and let X be the marked point on its boundary. Note that a left (resp. right) endpoint condition is satisfied if and only if γ_1 and γ_2 are arranged as in Figure I.5.

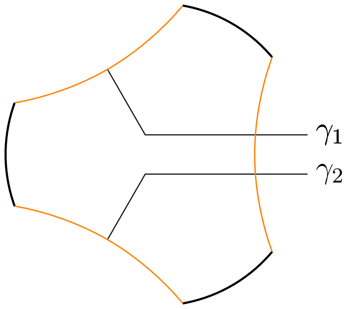


Figure I.5: The arcs γ_1 and γ_2 in P_v.

The position of X with respect to p_1^1 and p_2^1 decides which of the conditions (LG1) to (RG2) of Section I.3.1 is satisfied, as illustrated below. Note that this includes the case that q is a boundary intersection, that is p_1^1 and p_2^1 both coincide with the marked point.

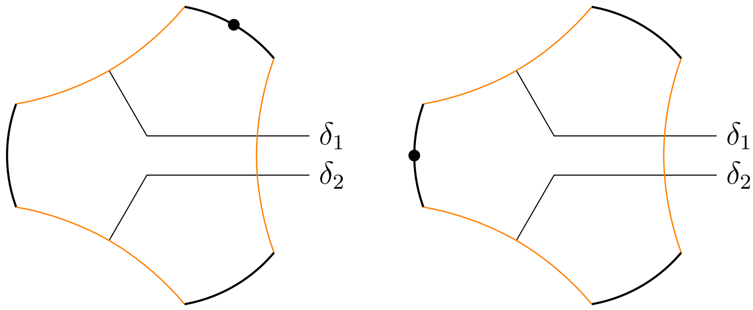


Figure I.6: The endpoint conditions LG1/RG1 (left) and LG2/RG2 (right)

As the analogous statements hold for $p_1^{m-1}, p_2^{m-1}, p_1^m$ and p_2^m, it follows that the intersection point q defines the data of a graph map if p_2^1 comes immediately before p_1^1 in the counter-clockwise orientation and the data of a quasi graph map otherwise.

Next, assume that q is on the boundary of S_A, and that $\delta_1 \neq \delta_2$. Then (LG2) is automatically satisfied for $\sigma(q)$, and we need only repeat the above argument for $p_1^{m-1}, p_2^{m-1}, p_1^m$ and p_2^m.

Now, assume that q is on the boundary of S_A, and that δ_1 and δ_2 are equal. This implies that γ_1 and γ_2 are the same arc, and that q is on the boundary of S_A. Then $\mathfrak{B}(q)$ will be an identity map from one of $P^\bullet_{\sigma(\gamma_1)}$ and $P^\bullet_{\sigma(\gamma_2)}$ to the other. Indeed, we can choose representatives of the arcs $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$, such that they only cross at their endpoints. In this case, one of the endpoints will be in $\gamma_1 \overrightarrow{\cap} \gamma_2$ and the other in $\gamma_2 \overrightarrow{\cap} \gamma_1$. The image of these points by $\mathfrak{B}$ will be the identity graph maps $P^\bullet_{\sigma(\gamma_1)} \to P^\bullet_{\sigma(\gamma_2)}$ and $P^\bullet_{\sigma(\gamma_2)} \to P^\bullet_{\sigma(\gamma_1)}$.

Finally, we note that in the case where δ_1 and δ_2 meet in one puncture, then $\sigma(q)$ is infinite on one side, and we need to look at the conditions (LG1) to (RG2) on one side only. If they meet at two punctures, then γ_1 and γ_2 are homotopic. Recall from Section I.3.1 that in any case, there exists a whole family of graph maps $(f_i)_{\geq 0}$ attached to $\sigma(q)$. These are the elements $\mathfrak{B}(q)(i)$ for $i \geq 0$. In particular, if γ_1 and γ_2 are homotopic, then $\mathfrak{B}(q)$ will be an identity morphism, as above.

This finishes the proof for case i).

To prove case ii), suppose that $\sigma(q)$ is empty and that $q \notin \partial S_A$. Consequently, δ_1 and δ_2 are contained in a single polygon P_v. Depending on the position of the marked point in P_v, q gives rise to different types of singleton maps. If the marked point lies between p_2^j and p_1^j in counter-clockwise order for some $j \in \{1, 2\}$, then q defines the data of a singleton single map. In

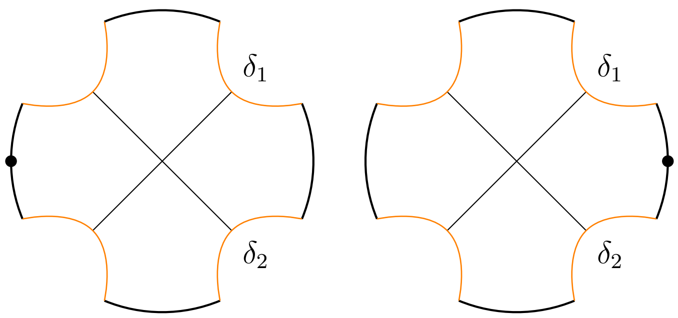


Figure I.7

the other situation, i.e. the marked point lies between p_1^j and p_2^j in counter-clockwise order, q gives rise to a singleton double map. In the previous picture this is the situation we obtain by interchanging the labels δ_1 and δ_2.

Finally, if δ_1 or δ_2 have a marked endpoint it can be seen that q gives rise to a singleton single map.

□

Remark I.3.9. The graph and quasi graph maps which occur in the previous Lemma as $\mathfrak{B}(q)$ cannot be invertible graph maps or maps of the form ξ occuring in Auslander-Reiten triangles as described in Theorem I.3.3 (2).

Remark I.3.10. The precise definition of $\mathfrak{B}$ depends on the homotopy representatives of the curves γ_1 and γ_2

Indeed, suppose that γ_1 and γ_2 are the same arc, and that q is on the boundary of S_A. Recall from the proof of Lemma I.3.8 that the identity morphisms between $P^\bullet_{\sigma(\gamma_1)}$ and $P^\bullet_{\sigma(\gamma_2)}$ are obtained as follows. First, choose

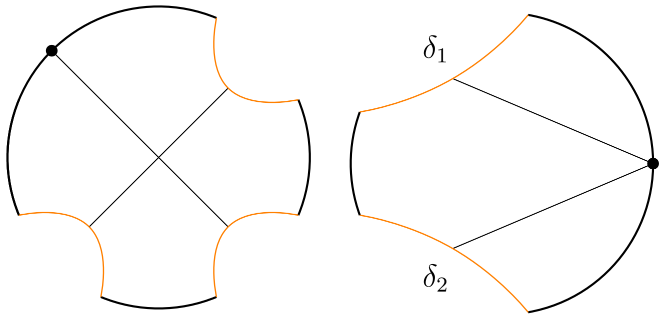


Figure I.8

representatives of the arcs $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ which only cross at their endpoints. Then one of the endpoints will be in $\gamma_1 \overrightarrow{\sqcap} \gamma_2$ and the other in $\gamma_2 \overrightarrow{\sqcap} \gamma_1$. The image of these points by $\mathfrak{B}$ will be the identity graph maps $P^\bullet_{\sigma(\gamma_1)} \to P^\bullet_{\sigma(\gamma_2)}$ and $P^\bullet_{\sigma(\gamma_2)} \to P^\bullet_{\sigma(\gamma_1)}$.
Choosing different representatives of $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ could lead to the first intersection point belonging to $\gamma_2 \overrightarrow{\sqcap} \gamma_1$ and the second one belonging to $\gamma_1 \overrightarrow{\sqcap} \gamma_2$. Then the images of these two points by $\mathfrak{B}$ would be permuted.

Lemma I.3.11. *Let $f \in \operatorname{Hom} \mathcal{D}^b(A)/[1](P^\bullet_{\sigma(\gamma_1)}, P^\bullet_{\sigma(\gamma_2)})$ be an element of $\mathcal{B}$ which is neither an invertible graph map nor a quasi graph map of the form ξ occuring in Auslander-Reiten triangles as described in Theorem I.3.3 (2). (see also Remark I.3.9). Then there exists a unique $q \in \gamma_1 \overrightarrow{\sqcap} \gamma_2$, and, in case q is a puncture, a unique $i \geq 0$, such that*

$$f = \begin{cases} \mathfrak{B}(q), & \text{if } q \text{ is not a puncture;} \\ \mathfrak{B}(q)(i), & \text{otherwise.} \end{cases}$$

Proof. Let $\tilde{\gamma}_1$ be a lift of γ_1. We distinguish two cases.
First, assume that f is a graph or quasi graph map and let σ be the maximal common subword associated to f as in Section I.3.1. Our assumptions imply that σ is finite or $\sigma(\gamma_1)$ and $\sigma(\gamma_2)$ are infinite homotopy strings, in which case σ is an (one- or two-)sided infinite homotopy string. The subword σ of $\sigma(\gamma_1)$ corresponds to a section δ_1 of $\tilde{\gamma}_1$. Let $\tilde{\gamma}_2$ be the unique lift of γ_2, such that the section δ_2 corresponding to the subword σ passes through the same polygons as δ_1.
As we have seen in the proof of Lemma I.3.8, the conditions (LG1), (LG2), (RG1) and (RG2) are equivalent to certain cofigurations of δ_1, δ_2 and of the marked point in the first and last polygons that δ_1 and δ_2 cross (see Figure I.6). These conditions force δ_1 and δ_2 to intersect in a (unique) point $\tilde{q}$. By contruction, $f = \mathfrak{B}(\pi(\tilde{q}))$.

Next, assume that f is a singleton single or singleton double map. If f is a single map, denote by p the non-trivial path which appears in the definition of single maps, see Section I.3.1. Otherwise, let p denote the non-trivial path which was denoted by r in the definition of singleton double maps, see Section I.3.1. There exists a polygon P of the surface $\tilde{S}_A$, which corresponds to p and is crossed by $\tilde{\gamma}_1$. We write $\tilde{\gamma}_2$ for the unique lift of γ_2 which crosses P, and denote δ_i the restriction of $\tilde{\gamma}_i$ to P. The combinatorial conditions in the definition of singleton single and singleton double maps are then equivalent to certain configurations of the marked point in P and the endpoints of δ_1 and δ_2 as shown in Figure I.7 and Figure I.8. As above, this proves that δ_1 and δ_2 intersect in a (unique) point $\tilde{q}$ such that $\mathfrak{B}(\pi(\tilde{q})) = f$. $\square$

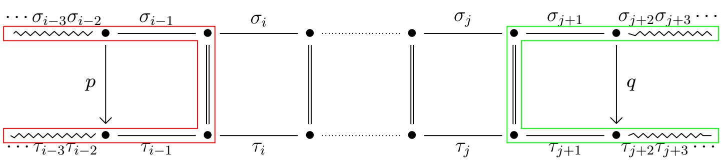
The previous lemma finishes the proof of Theorem I.3.3.

I.4 Mapping cones in the derived category of a gentle algebra

In this Section we will show that the mapping cone of a map in $\mathcal{D}^b(A)/[1]$ is given by the homotopy strings of the two curves resolving the corresponding crossing.

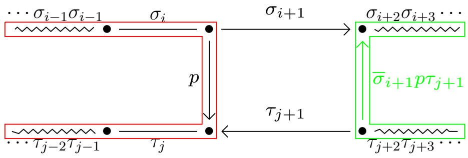
Theorem I.4.1. *[31] Let A be a gentle algebra and let $P^\bullet_\sigma$ and $P^\bullet_\tau$ be indecomposable objects in $\mathcal{D}^b(A)/[1]$ with homotopy strings or bands σ and τ. Let $f^\bullet \in \mathrm{Hom}_{D^b(A)}(P^\bullet_\sigma, P^\bullet_\tau)$ be an standard basis element which is neither and invertible graph map, nor a quasi map of the form ξ occuring in an Auslander-Reiten triangle as described in Theorem I.3.3. Then the indecomposable summands of the mapping cone $M^\bullet_{f^\bullet}$ are given by the homotopy strings and bands occurring in the green and red boxes resulting from the following graphical calculus.*

1. *Let $\sigma = \ldots\sigma_{i-2}\sigma_{i-1}\sigma_i\ldots\sigma_j\sigma_{j+1}\sigma_{j+2}\ldots$ and $\tau = \ldots\tau_{i-2}\tau_{i-1}\tau_i\ldots\tau_j\tau_{j+1}\tau_{j+2}\ldots$ and suppose $f^\bullet$ is a graph map with common homotopy substring $\sigma_i\ldots\sigma_j = \tau_i\ldots\tau_j$ Then $M^\bullet_{f^\bullet} = P^\bullet_{c_1} \oplus P^\bullet_{c_2}$ with homotopy strings $c_1 = \ldots\sigma_{i-2}\sigma_{i-1}\overline{\tau}_{i-1}\overline{\tau}_{i-2}\ldots$ and $c_2 = \ldots\overline{\tau}_{j+2}\overline{\tau}_{j+1}\sigma_{j+1}\sigma_{j+2}\ldots$ is given by:*

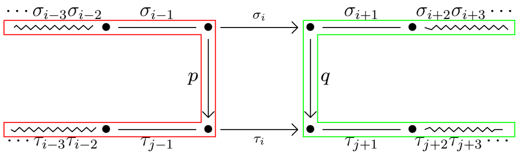


Similarly, if the one-sided infinite string $\sigma_i \ldots = \tau_i \ldots$ is the common substring of $f^\bullet$, then $M^\bullet_{f^\bullet}$ is isomorphic to $P^\bullet_c$, where $c = \ldots \sigma_{i-1}\overline{\tau}_{i-1}\overline{\tau}_{i-2}\ldots$

2. *Let $\sigma = \ldots \sigma_i\sigma_{i+1}\ldots$ and $\tau = \ldots \tau_j\tau_{j+1}\ldots$ and suppose $f^\bullet$ is a single map. Then $M^\bullet_{f^\bullet} = P^\bullet_{c_1} \oplus P^\bullet_{c_2}$ with homotopy strings $c_1 = \ldots \sigma_{i-1}\sigma_i p\overline{\tau}_j\overline{\tau}_{j-1}\ldots$ and $c_2 = \ldots \overline{\sigma}_{i+2}\overline{\sigma}_{i+1}p\tau_{j+1}\tau_{j+2}\ldots$ is given by:*



3. *Let $\sigma = \ldots \sigma_{i-2}\sigma_{i-1}\sigma_i\sigma_{i+1}\sigma_{i+2}\sigma_{i+3}\ldots$ and $\tau = \ldots \tau_{j-2}\tau_{j-1}\tau_j\tau_{j+1}\tau_{j+2}\ldots$ and suppose $f^\bullet$ is a double map. Then $M^\bullet_{f^\bullet} = P^\bullet_{c_1} \oplus P^\bullet_{c_2}$ with homotopy strings $c_1 = \ldots \sigma_{i-2}\sigma_{i-1}p\overline{\tau}_{j-1}\overline{\tau}_{j-2}\ldots$ and $c_2 = \ldots \overline{\sigma}_{i+2}\overline{\sigma}_{i+1}q\tau_{j+1}\tau_{j+2}\ldots$ is given by:*



Note that in the previous theorem, $P^\bullet_{c_i}$ is zero if the corresponding diagram obtained by graphical calculus is empty. In particular, if $f^\bullet : P^\bullet_\sigma \to P^\bullet_\tau$ is a standard basis element, then $M^\bullet_{f^\bullet}$ is indecomposable or zero if σ or τ is a homotopy band, or, if $f^\bullet$ is a graph map corresponding to an infinite common substring.

Remark I.4.2. Note that quasi-graph maps are implicitly treated in the previous theorem, since they give rise to homotopy classes of single and double maps.

Theorem I.4.3. *Let σ and τ be homotopy strings or bands and $f^\bullet : P^\bullet_\sigma \to P^\bullet_\tau$ be a map in $\mathcal{D}^b(A)/[1]$ associated to a crossing point $q \in \gamma(\sigma)\overrightarrow{\cap}\gamma(\tau)$ of the corresponding arcs $\gamma(\sigma)$ and $\gamma(\tau)$. Suppose further that $f^\bullet$ is different from the identity and if σ is a band, different from the map $P^\bullet_\sigma \to \tau P^\bullet_\sigma[1]$. Let $M^\bullet_{f^\bullet} = P^\bullet_{c_1} \oplus P^\bullet_{c_2}$ be its mapping cone (as described in Theorem I.4.1). Then the homotopy classes of arcs $\gamma(c_1)$ and $\gamma(c_2)$ corresponding to $P^\bullet_{c_1}$ and $P^\bullet_{c_2}$ are given by the following resolution of the crossing $\gamma(\sigma)\overrightarrow{\cap}\gamma(\tau)$.*

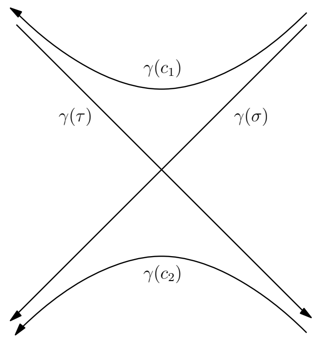


Figure I.9: Curves associated to the mapping cone $M^\bullet_{f^\bullet} = P^\bullet_{c_1} \oplus P^\bullet_{c_2}$ of a map $f^\bullet : P^\bullet_\sigma \to P^\bullet_\tau$.

Proof. We first consider the case when $f^\bullet$ is a graph map. If q is not a puncture but in the interior, then locally in the surface this corresponds to the following configuration.

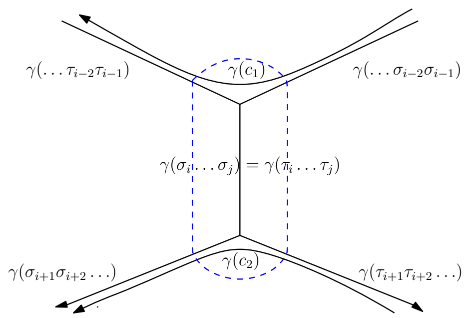


Figure I.10: Curves associated to the mapping cone $M^\bullet_{f^\bullet} = P^\bullet_{c_1} \oplus P^\bullet_{c_2}$ of a map $f^\bullet : P^\bullet_\sigma \to P^\bullet_\tau$ when $f^\bullet$ is a graph map. In this picture we have decomposed $\gamma(\sigma)$ into segments $\gamma(\ldots \sigma_{i-2}\sigma_{i-1}), \gamma(\sigma_i \ldots \sigma_j)$ and $\gamma(\sigma_{j+1}\sigma_{j+2}\ldots)$ and $\gamma(\tau)$ into segments $\gamma(\ldots \tau_{i-2}\tau_{i-2}), \gamma(\tau_i \ldots \tau_j)$ and $\gamma(\tau_{j+1}\tau_{j+2}\ldots)$.

The blue dotted region corresponds to the topological disc $S_{\tilde{q}}$ of Section I.3.2. Thus we see that the curve γ_{c_1} at the top is split into two subcurves, so that $\gamma(c_1) = \gamma(\ldots \sigma_{i-2}\sigma_{i-1})\gamma(\overline{\ldots \tau_{i-2}\tau_{i-1}})$. This proves that $P^\bullet_{c_1}$ is has the form in the statement of the theorem. A similar argument at the bottom of the picture proves the result for $P^\bullet_{c_2}$. The same arguments work if q is a puncture or a boundary intersection.

Next, we treat the case of single maps. In that case, $\gamma(\sigma)$ and $\gamma(\tau)$ meet in a polygon which forms the whole of $S_{\tilde{q}}$.

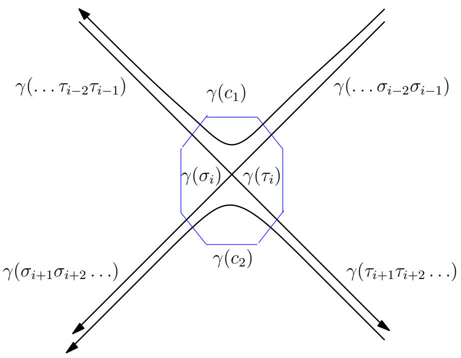


Figure I.11

We see that $\gamma(c_2)$ is obtained by $\gamma(\overline{\tau_{i+1}\tau_{i+2}\ldots})\gamma(\overline{p})\gamma(\sigma_{i+1}\sigma_{i+2}\ldots)$, as in the previous case. We also see that $\gamma(c_1)$ is obtained in a similar fashion, by noticing that $\sigma_i p \overline{\tau}_i$ contains one copy of p, since σ_i ends in (and $\overline{\tau}_i$ starts in) p^{-1}.
The remaining cases of a double map or of a single map arising from an intersection on the boundary of S_A are treated in a similar fashion. $\square$

I.5 Auslander-Reiten triangles

I.5.1 Reminder on Auslander-Reiten triangles

We recall the definition of Auslander-Reiten triangles.

Definition I.5.1. Let $\mathcal{F}$ be a triangulated category. A distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

is an **Auslander-Reiten triangle** if

1) X and Z are indecomposable, and

2) for all non-split morphisms $h : W \to Z$, $w \circ h = 0$.

The following is an immediate consequence of the definition

Lemma I.5.2. *Let*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

be an Auslander-Reiten triangle in a triangulated category $\mathcal{F}$ and let $T : \mathcal{F} \to \mathcal{F}'$ be a triangle equivalence. Then

$$T(X) \xrightarrow{T(u)} T(Y) \xrightarrow{T(v)} T(Z) \xrightarrow{T(w)} T(X)[1]$$

is an Auslander-Reiten triangle.

In case it exists, any of the objects X and Z determines the Auslander-Reiten triangle uniquely up to a non-unique isomorphism. In particular, the isomorphism class of X determines the isomorphism class of Z and vice versa. It gives rise to a partially defined bijection τ on the class of isomorphism classes in $\mathcal{F}$, such that τ applied to the isomorphism class of Z is by definition the isomorphism class of X.

Auslander-Reiten triangles of gentle algebras. It follows from a result in [49] that in the setting of $\mathcal{D}^b(A)$, where A is a finite dimensional Gorenstein algebra, τ is defined precisely for indecomposable objects in the category of perfect complexes $\mathrm{Perf}(A) = \mathcal{K}^b(A-\mathrm{proj})$. Furthermore it was shown that if$\mathbb{S}$ is a Serre functor on $\mathrm{Perf}(A)$ (e.g. the left derived functor of the Nakayama functor), then $\tau(X) \cong \mathbb{S}(X)[-1]$. Note that by a result in [45], gentle algebras are Gorenstein.
For the remainder of this section, let A be a gentle algebra. Its indecomposable perfect objects in $\mathcal{D}^b(A-\mathrm{mod})$ are given by the string objects with finite homotopy string and the band objects. We recall some general facts on Auslander-Reiten triangles in $\mathcal{K}^b(A-\mathrm{proj})$. The first explicit description of such triangles for gentle algebras was given in [13]. Let $P^\bullet_\sigma \in \mathcal{K}^b(A-\mathrm{proj})$ be an indecomposable object, where σ is a finite homotopy string or a homotopy band. Then, there exists an Auslander-Reiten triangle

$$P^\bullet_\sigma \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

in $\mathcal{K}^b(A-\mathrm{proj})$ and the following are true.

- The object Y is the direct sum of at most two indecomposable objects and up to isomorphism the entries in f, g, h are standard basis elements [4].
- If $P\bullet_\sigma$ is a band complex associated to a m-dimensional $K[X]$-module, then $Z \cong P^\bullet_\sigma$, that is $\tau^{-1}P^\bullet_\sigma = P^\bullet_\sigma$, and Y is (isomorphic to) a direct sum of band complexes associated to $m\pm 1$-dimensional $K[X]$-modules.

It can be shown that if $P^\bullet_\sigma$ is a band complex associated to a 1-dimensional $K[X]$-module, then h as above is not represented by an intersection. In particular, if the representing loop of P_σ is simple, then none of the maps in an Auslander-Reiten triangle are represented by an intersection.

I.5.2 Geometric description of Auslander-Reiten triangles

The purpose of this section is to show that an Auslander-Reiten triangle of a perfect string object $P^\bullet_\sigma$ is determined by the corresponding arc $\gamma(\sigma)$ on S_A.

For this, we follow the convention that given a finite homotopy string σ with corresponding arc $\gamma(\sigma)$, the start of $\gamma(\sigma)$ corresponds to $s(\sigma)$ and the end of $\gamma(\sigma)$ corresponds to $t(\sigma)$ and we define the following:

1. Let ${}_s\sigma$ be the homotopy string corresponding the arc ${}_s\gamma(\sigma)$ obtained from $\gamma(\sigma)$ by rotating its start clockwise to the next marked point on the boundary.

2. Let σ_e be the homotopy string corresponding the arc $\gamma(\sigma)_e$ obtained from $\gamma(\sigma)$ by rotating its end clockwise to the next marked point on the boundary.

3. Let ${}_s\sigma_e$ be the homotopy string corresponding the arc ${}_s\gamma(\sigma)_e$ obtained from $\gamma(\sigma)$ by rotating its end and it's start clockwise to the next marked point on the boundary.

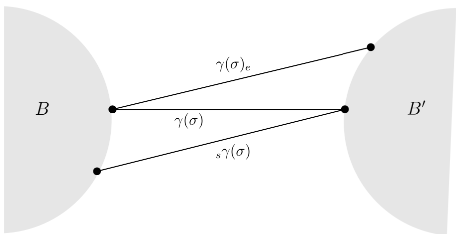


Figure I.12: The arcs ${}_s\gamma(\sigma)$, $\gamma(\sigma)_e$ associated to $\gamma(\sigma)$.

It follows from the above that ${}_s\gamma(\sigma) = \gamma({}_s\sigma)$, $\gamma(\sigma)_e = \gamma(\sigma_e)$ and ${}_s\gamma(\sigma)_e = \gamma({}_s\sigma_e)$. We can now state the main result of this Section.

Theorem I.5.3. *Let $P^\bullet_\sigma \in D^b(A-\mathrm{mod})/[1]$ be an indecomposable object with finite homotopy string σ. Then the Auslander-Reiten triangle starting in $P^\bullet_\sigma$ is given by*

$$P^\bullet_\sigma \xrightarrow{\binom{{}_sf}{f_e}} P^\bullet_{{}_s\sigma} \oplus P^\bullet_{\sigma_e} \xrightarrow{(\,{}_sg\;\; g_e\,)} P^\bullet_{{}_s\sigma_e} \xrightarrow{h} P^\bullet_\sigma[1]$$

Furthermore, every morphism in the above triangle can be given by a standard basis element corresponding to the distinguished intersection as described below.

It directly follows that the Auslander-Reiten translate $\tau P^\bullet_\sigma$ of $P^\bullet_\sigma$ corresponds to rotating both endpoints of the corresponding arc $\gamma(\sigma)$.

Corollary I.5.4. *Let $P^\bullet_\sigma \in D^b(A - \text{mod})/[1]$ be an indecomposable object with finite homotopy string. Let $\tau^{-1}\gamma(\sigma)$ be the arc corresponding to $\tau^{-1}P^\bullet_\sigma$. Then*

$$\tau^{-1}\gamma(\sigma) = \gamma({}_s\sigma_e).$$

In Figure I.13 we give an example of the geometric realisation of the Auslander-Reiten translate of $P^\bullet_\sigma$.

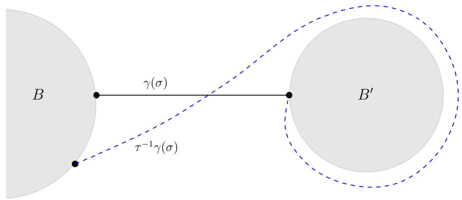


Figure I.13: The arcs associated to indecomposable perfect string objects $P^\bullet_\sigma$ and $\tau^{-1}P^\bullet_\sigma$in $\mathcal{D}^b(A)/[1]$.

Remark I.5.5. A version of Theorem I.5.3 holds for string complexes of homotopy strings which are infinite. Indeed, with a similar proof, one can show that these irreducible maps [4] are represented by intersections of arcs $\gamma(\sigma)$ and ${}_s\gamma(\sigma)$ (resp. $\gamma(\sigma)_e$), where ${}_s(-)$ (resp. $(-)_e$) is extended to arcs which end (resp. start) at a puncture. In this case, the corresponding intersection is at the puncture and the associated map is a graph map given by an infinite subword.

Distinguished intersections and fractional twists

By definition, γ shares its start point with γ_e and its end point with ${}_s\gamma$. They determine boundary intersections in $\gamma \overrightarrow{\cap} \gamma_e$ and $\gamma \overrightarrow{\cap} {}_s\gamma$, which we call **distinguished**.
Write $\tau^{-1}\gamma$ for ${}_s(\gamma_e)$. Then $\tau^{-1}\gamma$ and $({}_s\gamma)_e$ are homotopic and if $\delta \simeq \tau^{-1}\gamma$ and γ are in minimal position, then $\gamma \cap \delta$ contains a distinguished intersection as shown in Figure I.13. A more precise definition is given as follows.
Choose a lift $\widetilde{\gamma}$ of γ to the universal cover of S and denote $\widetilde{\delta}$ the unique lift of δ whose end points are successors of end points of $\widetilde{\gamma}$. Then, the distinguished intersection corresponds to the unique intersection of $\widetilde{\gamma}$ and $\widetilde{\delta}$.

In fact, the action of τ^{-1} on homotopy classes is induced by the inverse of a self-homeomorphism τ of S, which we describe next.

Definition I.5.6. Let $\|\|$ denote the norm in the complex plane. Let D be a tubular neighborhood of a boundary component $B \subseteq \partial S$ with $N := |B \cap \overline{\mathcal{M}}\,|$ marked points and $\phi : D \to \{z \in \mathbb{C} \,|\, 1 \leq \|z\| \leq 2\}$ an arbitrary orientation preserving homeomorphism such that $\phi(B \cap \overline{\mathcal{M}}) = \{x \in \mathbb{C} \,|\, x^N = 1\}$ is the set of roots of unity of order N. Let $\exp : \mathbb{C} \to \mathbb{C}\backslash\{0\}$ denote the exponential function. The **fractional twist** associated with B is a $\boldsymbol{\tau}_B : S \to S$ map, which restricts to the map

$$\begin{array}{ccc} D & \longrightarrow & D \\ \\ x & \longmapsto & \phi^{-1}\left(\phi(x) \cdot \exp\left(\frac{2\pi i}{|I|} \cdot (2 - \|\phi(x)\|)\right)\right). \end{array}$$

on D and which restricts to the identity of $\overline{S \setminus D}$.

The homeomorphism $\boldsymbol{\tau}_B$ rotates the surface in a neighborhood of B in counter-clockwise direction, as shown in Figure I.14.

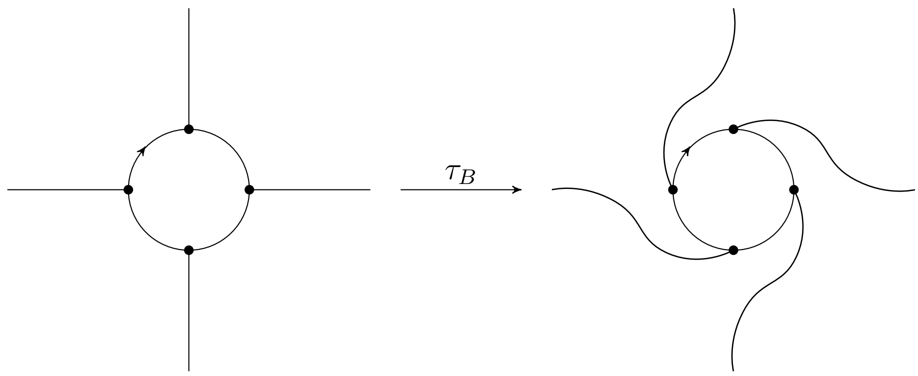


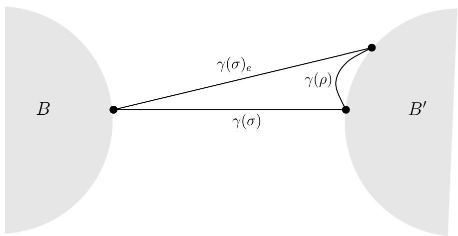
Figure I.14: The action of τ_B on arcs

The isotopy class of $\boldsymbol{\tau}_B$ relative to the boundary is independent of the choice of D and ϕ. Further, $\boldsymbol{\tau}_B$ and $\boldsymbol{\tau}_{B'}$ commute up to isotopy relative to the boundary. Assuming $B \neq B'$, this can be seen by choosing pairwise disjoint neighborhoods of B and B'.
The homeomorphism $\boldsymbol{\tau}$ is defined as $\prod_B \boldsymbol{\tau}_B$, where B is indexed by the set of boundary components of S. It is well-defined up to isotopy relative to the boundary. On the level of homotopy classes of arcs its inverse $\boldsymbol{\tau}^{-1}$ acts in the same way as the operation $\boldsymbol{\tau}^{-1}$ defined earlier.

Proof of Theorem I.5.3. We first note that since ${}_s\gamma(\sigma) = \gamma(\overline{\sigma})_e$, we have ${}_s\sigma = \overline{(\overline{\sigma}_e)}$. Therefore it is enough to prove the result for f_e, the proof for ${}_sf$

then follows. Furthermore, the proof for ${}_sg$ and g_e then also follows noting that ${}_s\sigma_e = {}_s(\sigma_e) = ({}_s\sigma)_e$.

To prove that f_e is an irreducible map of the required form we follow Algorithm 6.3 in [4] step by step. Algorithm 6.3 in [4] breaks down into five cases. In each case, it suffices to prove that σ_e is the homotopy string of the resolution of the boundary intersection of $\gamma(\sigma)$ with the arc γ_B containing in its homotopy class the boundary arc connecting the end of $\gamma(\sigma)$ and the end of $\gamma(\sigma)_e$. Let ρ be the homotopy string of γ_B, that is $\gamma_B = \gamma(\rho)$. Locally in the surface we have the following configuration



For what follows, write $\sigma = \sigma_1 \cdots \sigma_n$ with homotopy letters σ_i.

Case 1: Suppose that there exists a maximal path q in Q (which then corresponds to a single homotopy letter) and a maximal inverse antipath $\theta = \theta_1 \cdots \theta_m$, such that $\sigma q \theta$ is a homotopy string. Note that by Remark 6.5 (4) in [4], θ is finite. Furthermore, $\gamma(\sigma)$ ends on the marked point corresponding to the start of $\gamma(q)$. Since $t(q) = s(\theta)$ and $s(q) = t(\sigma)$, $\sigma q \theta$ is obtained from σ as the homotopy string of the mapping cone of the singleton single map $\phi^\bullet : P^\bullet_{\theta_1 \cdots \theta_n} \to P^\bullet_\sigma$ induced by q and thus $\sigma_e = \sigma q \theta$. By maximality of θ, it follows that $\gamma(\theta) \sim \gamma(\rho)$ and the single map $\phi^\bullet$ corresponds to the boundary intersection $\gamma(\rho) \overrightarrow{\cap} \gamma(\sigma)$. Then f_e is a graph map induced by the subword σ and hence is represented by the distinguished intersection $\gamma(\sigma) \overrightarrow{\cap} \gamma(\sigma)_e$ as claimed.

The other cases are treated in a similar way and we only give an outline for each.

Case 2: Suppose that $\sigma_{r+1} \ldots \sigma_n$ is a direct antipath and that σ_r is an inverse homotopy letter. Suppose further that there exists $\alpha \in Q_1$, such that $\alpha \overline{\sigma_r} \notin I$. Assume also that there exists a maximal inverse antipath θ, such that $\overline{\alpha}\theta$ is a homotopy string. Then we have $\rho = \overline{\sigma_n} \ldots \overline{\sigma_{r+1}}\ \overline{\alpha}\theta$ and $\sigma_e = \sigma_1 \ldots \sigma_r \overline{\alpha}\theta$, which is the homotopy string of the mapping cone of a graph map $P^\bullet_\rho \to P^\bullet_\sigma$ associated to the common subword $\sigma_{r+1} \ldots \sigma_n$ and f_e

is a graph map associated to the common subword $\sigma_1 \ldots \sigma_r$ of σ and σ_e.

Case 3: Suppose that $\sigma_{r+1} \ldots \sigma_n$ is a direct antipath and that σ_r is an inverse homotopy letter. Suppose further that there exists no $\alpha \in Q_1$ such that $\alpha\overline{\sigma_r} \notin I$. In this case, $\sigma_e = \sigma_1 \ldots \sigma_{r-1}$ and $\rho = \sigma_{r+1} \ldots \sigma_n$ where σ_e is the homotopy string of the mapping cone of the graph map $P_\rho \to P_\sigma$ associated to the common subword $\sigma_{r+1} \ldots \sigma_n$ and where f_e is a graph map associated to the common subword σ_e.

Case 4 Suppose that $\sigma_{r+1} \ldots \sigma_n$ is a direct antipath and that σ_r is a direct homotopy letter and write $\sigma_r = q\alpha$ where $\alpha \in Q_1$. Let θ be a maximal inverse antipath such that $\sigma_1 \ldots \sigma_{r-1} q\theta$ is a homotopy string. Then one verifies that $\sigma_e = \sigma_1 \ldots \sigma_{r+1} q\theta$ and $\rho = \overline{\theta}\alpha\sigma_{r+1} \ldots \sigma_n$, where σ_e is the homotopy string of a the mapping cone of the graph map $P^\bullet_\rho \to P^\bullet_\sigma$ associated to the common subword $\sigma_{r+1} \ldots \sigma_n$ and f_e is given by the graph map determined by the common subword $\sigma_1 \ldots \sigma_{r-1}$.

Case 5: Suppose that σ is a direct antipath and suppose that there exists $\alpha \in Q_1$ such that $\alpha\sigma_1 \in I$. Let θ be a maximal inverse antipath starting at $s(\alpha)$. Then $\sigma_e = \theta$, which is the mapping cone of the graph map $P^\bullet_\rho \to P^\bullet_\sigma$ associated to the subword σ, where $\rho = \overline{\theta}\alpha\sigma$. In that case, f_e is a singleton single map.

If none of the above cases hold, then σ_e is empty and $P^\bullet_{\sigma_e} = 0$, so there is nothing to show. $\square$

Remark I.5.7. We provide an alternative proof of a (slightly weaker) version of Theorem I.5.3 in Chapter 2 based on Section I.7 and Section I.8 below.

I.6 Avella-Alaminos–Geiss invariant on the surface

In [11] Avella-Alaminos and Geiss define invariants for derived equivalence classes of gentle algebras. We will refer to these invariants as AG-invariants. In this Section we show that these derived invariants are encoded in the ribbon surface of a gentle algebra. In their paper Avella-Alaminos and Geiss show that two gentle algebras that are derived equivalent have the same AG-invariant but they also give an example of two gentle algebras that are not derived equivalent yet have the same AG-invariant. Since then, many other examples of non-derived equivalent gentle algebras with the same AG-invariants have appeared in the literature, see for example [53, 1].

I.6.1 The Avella-Alaminos–Geiss invariant

We begin by briefly recalling the definition of the AG-invariants. Let $A = kQ/I$ be a gentle algebra with augmented set of maximal paths $\overline{\mathcal{M}} = \mathcal{M} \cup \mathcal{M}_0$ (see Definition I.1.8). Let $\mathcal{F}$ be the set of paths w in (Q, I) such that if $w = \alpha_1 \dots \alpha_n$ then $\alpha_i \alpha_{i+1} \in I$ for all $i \in \{1, \dots, n-1\}$, and such that w is maximal for this property, that is for all $\beta \in Q_1$, if $t(\beta) = s(\alpha_1)$ then $\beta\alpha_1 \notin I$ and if $t(\alpha_n) = s(\beta)$ then $\alpha_n \beta \notin I$. Let $\mathcal{F}_0 = \{e_v \mid v \in W_0\}$ where W_0 is the subset of Q_0 containing all vertices that are either the source or target of only one arrow and those vertices that are the target of exactly one arrow α and the source of exactly one arrow β and $\alpha\beta \in I$.

Let $H_0 = m_0$ with $m_0 \in \overline{\mathcal{M}}$. Set $F_0 = f_0$ where f_0 is the unique element in $\mathcal{F}$, if it exists, such that $t(f_0) = t(m_0)$ and such that if $m_0 = p\alpha$ is non-trivial with $\alpha \in Q_1$ then $f_0 = q\beta$ with $\beta \neq \alpha$ and $\beta \in Q_1$. If no such $f_0 \in \mathcal{F}$ exists then we set $f_0 = e_{t(m_0)}$. Note that in this case $e_{t(m_0)} \in \mathcal{F}_0$.

Now define $H_1 = m_1$ where m_1 is the unique element in $\mathcal{M}$, if it exists, such that $s(m_1) = s(f_0)$ and such that if $f_0 = \gamma q$ is non-trivial with $\gamma \in Q_1$ then $m_1 = \delta r$ with $\delta \neq \gamma$ and $\delta \in Q_1$. If no such m_1 exists then we set $m_1 = e_{s(f_0)}$ and we note that $e_{s(f_0)} \in \mathcal{M}_0$.

Define F_{i-1}, and H_i for $i \geq 2$ in an analogous way to the above. The algorithm stops as soon as $H_i = H_0$ and we set $k = i$. Set l equal to the number of arrows in $F_0, \dots, F_{k-1}$.

We repeat this process until every element of $\overline{\mathcal{M}}$ has appeared once as one of the H_i. This gives rise to a set of tuples (k, l). We add to this a pair $(0, n)$ for each full cycle of relations of length n.

The **AG-invariant** of A is the function $\phi_A : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by sending (i, j) to the number of pairs corresponding to these entries in the above algorithm. the boundary consists of as many boundary segments as there are marked points.

Theorem I.6.1. *Let A be a gentle algebra with associated ribbon surface S_A and lamination L_A. Let $B_1, \dots, B_n$ be the boundary components in S_A. Then the AG-invariant of A is given by the set of pairs (b_i, c_i) for $1 \leq i \leq n$ where*

- *b_i is given by the number of marked points on B_i,*
- *$c_i = l_i - b_i$ where l_i is equal to the number of laminates starting or ending on B_i.*

Furthermore, if $b_i \neq 0$, we also have $c_i = \sum_j k_j - 2$ where j runs over all k_j-gons which have at least one side isotopic with a boundary segment of B_i.

Note that in Theorem I.6.1, if a laminate ends and starts on the same boundary component, then it is counted twice.

Proof of Theorem I.6.1. First suppose that B is a boundary component with no marked points. Then, by Proposition I.1.12, B lies in the interior of an n-gon P which corresponds to an n-cycle with full relations in (Q, I). Therefore it corresponds to a pair $(0, n)$ in the algorithm of the AG-invariant. Furthermore, by construction each side of P corresponds to exactly one laminate incident with B.

Now let B be a boundary component with marked points $m_1, \ldots, m_r$ ordered in counter-clockwise occurrence on B. Then set H_0 to be the maximal path associated to the fan at m_1 or if i_1 is the only edge of Γ_A incident with m_1 set $H_0 = e_{i_1}$. Let F_0 be the inverse path corresponding to the arrows inscribed in the polygon P_1 with boundary segment between m_1 and m_2. Clearly if P_1 has k_1 edges (exactly one of which is a boundary segment by Proposition I.1.12) then there are $k_1 - 2$ arrows inscribed in that polygon giving an element in $\mathcal{F}$ except when $k_1 = 2$ in which case we set $F_0 = e_{j_1}$ where j_1 is the only internal edge of P_1. Now let H_1 be the path corresponding to the maximal fan at m_2 or if this fan consists of a single edge i_2 then set $H_1 = e_{i_2}$. We set F_1 to be equal to the inverse path consisting of $k_2 - 2$ inverse arrows inscribed in the k_2-gon P_1 with (unique) boundary segment between m_2 and m_3 where $F_1 = e_{j_2}$ with j_2 the only internal edge of P_1 if $k_2 = 2$. We continue in a similar fashion along the boundary component B in a counter-clockwise direction until we return to the fan at m_1. At this point the algorithm repeats and therefore stops and we move on to the next boundary component. The number of steps in each part of the algorithm is given by the number of fans on the boundary component which is equal to the number of marked points on B. The total number of arrows in the inverse paths at B corresponds to the sum of the arrows in the k_j-gons P_j incident with B, that is it is equal to $\sum_{i=j}^{r} k_j - 2$ as claimed. We repeat this for every boundary component, thus covering every element in $\overline{\mathcal{M}}$ exactly once.
Given a k_j-gon P_j with one side isotopic to a boundary component B_i, it follows from the construction of the lamination L_A that there are exactly $k_j - 1$ laminates incident with the only boundary edge of P_j and since there are as many marked points on a boundary component as there are boundary segments, we have $c_i = l_i - b_i$ as claimed. □

Remark I.6.2. We give an alternative proof of Theorem I.6.1 in Chapter II based on geometric arguments in the context of Fukaya-like triangulated categories, see Corollary II.1.11.

I.7 Composition of basis elements

In this section, we show that compositions of basis elements between string and band complexes can be modelled in terms of the topology of the surface S_A. We prove the following Theorem.

Theorem I.7.1. *Let ρ, σ and τ be homotopy strings or bands. Let $p \in \gamma(\rho)\overrightarrow{\cap}\gamma(\sigma)$, $q \in \gamma(\sigma)\overrightarrow{\cap}\gamma(\tau)$ be intersections and let $\mathfrak{B}(p) : P^\bullet_\rho \to P^\bullet_\sigma$ and $\mathfrak{B}(q) : P^\bullet_\sigma \to P^\bullet_\tau$ be associated standard basis elements. Assume that if ρ, σ and τ are identical, then they are homotopy strings.*
Then, the following are equivalent:

a) If $z \in \gamma(\rho)\overrightarrow{\cap}\gamma(\tau)$, then the coefficient of $\mathfrak{B}(z)$ in the decomposition of $\mathfrak{B}(q) \circ \mathfrak{B}(p)$ with respect to any standard basis containing $\mathfrak{B}(z)$ is non-zero.

b) there exist lifts $\widetilde{\gamma}(\rho)$, $\widetilde{\gamma}(\sigma)$ and $\widetilde{\gamma}(\tau)$ of $\gamma(\rho)$, $\gamma(\sigma)$ and $\gamma(\tau)$ to the universal cover of S_A which intersect in a triangle, a fork or a bigon as shown in Figure I.15 and Figure I.16 and Figure I.17.

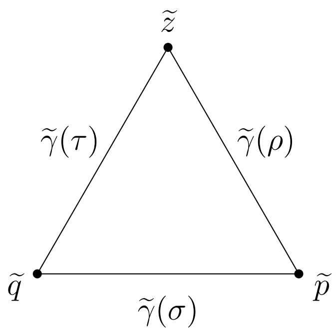


Figure I.15: $\widetilde{*}$ is a lift $* \in \{p, q, z\}$.

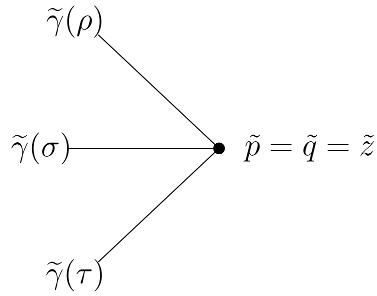


Figure I.16: A “fork”.

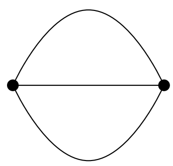


Figure I.17: A bigon

We like to stress that the orientation of the figures in Figure I.15 and Figure I.16 are important. Before we begin with the proof of Theorem I.7.1, we recollect some important facts about standard basis elements which will we use throughout this section. All of these observations were already made (at least implicitely) in [4]. We assume that our gentle algebra A is given by a quotient kQ/I as in Definition I.1.7.

1. The set of single, double and graph maps form a basis of the space of chain maps, see Proposition 4.1. in [4]. In what follows, we refer to any of such chain maps as a standard chain map.

2. Every standard chain map can be reconstructed from any of its components. By this we mean the following. Suppose σ, τ are homotopy strings or bands and $f : P^\bullet_\sigma \to P^\bullet_\tau$ is a standard basis element. Let $u : P \to P'$ be a map between indecomposable projectives P and P' of $P^\bullet_\sigma$ and $P^\bullet_\tau$,, such that u occurs as a component of f. In particular, u is a multiple of a map induced from an admissible path in Q. Reconstructing f from u now means that if $g : P^\bullet_\sigma \to P^\bullet_\tau$ is another basis element, such that u is a component of f, then $f = g$.

3. Given a chain map $f : P_\sigma \to P^\bullet_\tau$ and a component $u : P \to P'$ of f as before, then there is a unique chain map $g : P_\sigma \to P^\bullet_\tau$ with component

u and g is a summand of f when decomposed into standard chain maps. Thus, we can decompose f inductively into a sum of standard chain maps by looking at a single component of f at each time. Moreover, the decomposition obtained in this way is unique.

4. The homotoy relation in the space of chain maps with fixed source and target are generated by relations of the form $f \simeq 0$ and $f \simeq g$, where f and g are standard chain maps.

Summarizing the observations above we see that all information about a basis element in the homotopy category is stored in a single component of any of its representing standard chain maps.

Hoping that the following will clarify the proof of Theorem I.7.1 even further, we point out the following:

1. Let $f : P_\sigma^\bullet \to P_\tau^\bullet$ be a standard basis element. Let $p \in \gamma(\sigma) \overrightarrow{\cap} \gamma(\tau)$ be an intersection corresponding to f and let $\tilde{\gamma}(\sigma)$ and $\tilde{\gamma}(\tau)$ be lifts in the universal cover of S_A which intersect in a lift $\tilde{p}$ of p. Let be U be a polygon in the subsurface $S_{\tilde{p}}$ (see Figure I.4) and let P (resp. P') be an indecomposable projective module of $P_\sigma^\bullet$ (resp. $P_\tau^\bullet$) which corresponds to an intersection of $\tilde{\gamma}(\sigma)$ (resp. $\tilde{\gamma}(\tau)$) with a laminate on the boundary of U. If P and P' sit in the same degree of their complexes, then there exists a unique standard chain map $P_\sigma^\bullet \to P_\tau^\bullet$ in the homotopy class f which has a component $P \to P'$ as indicated by the dotted arrows in Figure I.18.

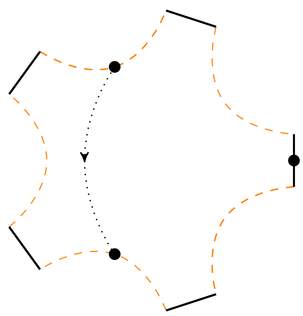


Figure I.18

This can be seen as a uniform – yet rather inexplicit – way to describe the homotopy class of chain maps associated to an intersection.

2. As a converse to the previous entry in this list, every component of a homotopically non-trivial standard chain map $P_\sigma^\bullet \to P_\tau^\bullet$ determines a lift $\tilde{\gamma}(\tau)$ of $\gamma(\tau)$ to the universal cover of S_A for every fixed lift $\tilde{\gamma}(\sigma)$ of $\gamma(\sigma)$. However, note that $\tilde{\gamma}(\tau)$ is only unique up to "full periods of $\gamma(\sigma)$". In more precise terms this means that $\tilde{\gamma}(\tau)$ is unique up to the action of the group of Deck transformations corresponding to the powers of $\gamma(\sigma)$ (considered as an element in a fundamental group). In particular, $\tilde{\gamma}(\sigma)$ is unique if σ is a homotopy string.

Proof. We fix standard chain maps $f : P_\rho^\bullet \to P_\sigma^\bullet$ and $g : P_\sigma^\bullet \to P_\tau^\bullet$ in the homotopy classes of $\mathfrak{B}(p)$ and $\mathfrak{B}(q)$. Let $u : P \to P'$ be a component of $g \circ f$. It determines a unique standard chain map h which occurs in the decomposition of $g \circ f$ into a sum of standard chain maps and u arises as a composition $u = u_2 \circ u_1$ of components u_1 of f and u_2 of g. On the universal cover this corresponds to a polygon U in the intersection of subsurfaces $S_{\tilde{p}}$ and $S_{\tilde{q}}$ associated to p and q. The components u_1, u_2 and u arise from intersections of lifts $\tilde{\gamma}(\rho)$, $\tilde{\gamma}(\sigma)$ and $\tilde{\gamma}(\tau)$, of $\gamma(\rho)$, $\gamma(\sigma)$ and $\gamma(\tau)$, see Figure I.19

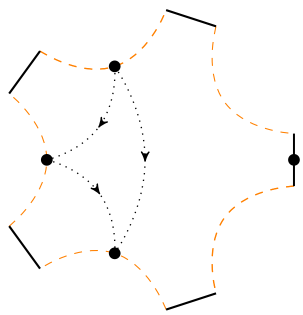


Figure I.19: The components u, u_1 and u_2.

Now, the curves $\tilde{\gamma}(\rho)$ and $\tilde{\gamma}(\sigma)$ intersect if and only if h is not homotopic to zero. This follows from Theorem I.3.3. Note that if they intersect, it means that $\tilde{\gamma}(\rho)$, $\tilde{\gamma}(\sigma)$ and $\tilde{\gamma}(\tau)$ intersect pairwise and only once. As the universal cover of S_A can be embedded into the plane, we see that the curves must form

- a triangle, if at least one of the intersections is neither on the boundary nor a puncture, and
- a fork otherwise or bigon otherwise.

It follows that condition a) implies condition b).

In order to show that condition b) implies condition a), it is sufficient to show that no cancellation can occur which is caused by two homotopic but distinct standard chain maps arising from different components of of $g \circ f$. First of all, note that such homotopic chain maps would give rise to the same lifts of $\gamma(\tau)$ (up to "full periods of $\gamma(\sigma)$" as discussed above). But this would require f and g to be graph maps and $g \circ f$ to be a quasi-map. Since every component of a a representative of a quasi-map is non-invertible, it is not difficult to see that this prevents the lifts from forming a triangle or a fork. $\square$

Corollary I.7.2. *Let ρ and τ be homotopy strings or bands and let σ be a homotopy string. Furthermore, let $f^\bullet : P^\bullet_\rho \to P^\bullet_\sigma$ and $g : P^\bullet_\sigma \to P^\bullet_\tau$ be standard basis elements and assume that g arises from a boundary intersection between the corresponding curves. Then, $g \circ f$ is zero or a multiple of a standard basis element.*

Proof. There exists at most one triangle (up to the action of Deck transformations) which contains a lift of p as a corner. $\square$

I.8 Winding numbers & cycles of morphisms

Let A be a gentle algebra. Suppose γ is a primitive loop on the surface S_A of A. As we have seen in Theorem I.2.5, γ corresponds to an indecomposable object in $\mathcal{D}^b(A)$ if and only if the number of direct homotopy letters in $\sigma = \sigma(\gamma)$ (see Lemma I.2.11), which was defined in terms of the intersections of γ with the laminates of A (see Proposition I.1.16), coincides with the number of its inverse homotopy letters. In that case, σ is the homotopy band of the indecomposable complexes associated to γ.

For aribtrary γ, define $\omega_A(\gamma)$ as the difference $i - d$, where d denote thes number of inverse homotopy letters and i denotes the number of inverse homotopy letters in σ. In other words, γ (equipped with an indecomposable local system) represents an object in $\mathcal{D}^b(A)$ if and only if $\omega_A(\gamma) = 0$. We extend ω_A to all loops by imposing the relation $\omega_A(\gamma^l) = l \cdot \omega_A(\gamma)$ for all $l \geq 0$. Note that by definition, $\omega_A(\gamma) = -\omega_A(\overline{\gamma})$, where $\overline{\gamma}$ denote the inverse loop of γ. We refer to the number $\omega_A(\gamma)$ as the **winding number of** γ.

In the main theorem of this section, we show that the values of ω_A on a all loops γ on S_A (not just the ones which represent objects) can be interpreted by means of $\mathcal{D}^b(A)$. Before we state the theorem we need to introduce some notation.

Cycles of morphisms. Let $\{\gamma_1, \dots, \gamma_m\}$ be a set consisting of finite arcs and gradable loops, i.e $\omega_A(\gamma) = 0$, on S in minimal position. Assume γ is an oriented loop without **teardrops**, i.e. γ contains no contractible subcurve. Assume further that γ is glued from arcs $\delta_1, \dots, \delta_m$ (in cyclic order), where each δ_i is a (possibly trivial) subarc of γ_i connecting crossings $p_i \in \gamma_{i-1} \cap \gamma_i$ and $p_{i+1} \in \gamma_i \cap \gamma_{i+1}$.

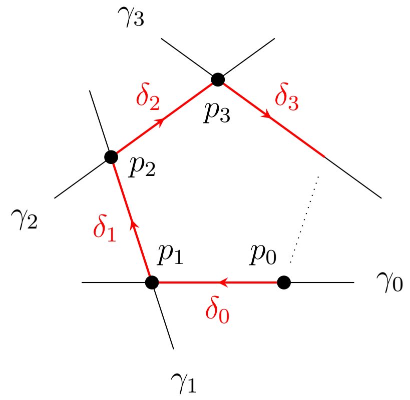


Figure I.20: A cycle of curves.

Let $Y_1, \dots, Y_{m+1} \in \mathcal{F}$ be string complexes or linear band complexes, such that for each $i \in [1, m+1]$,

a) $f_i = \mathfrak{B}(p_i) \in \mathrm{Hom}_{\mathcal{F}}(Y_{s_i}, Y_{t_i})$, where $\{s_i, t_i\} = \{i-1, i\}$, and,

b) $\gamma_i \in \gamma(Y_i)$, where $\gamma_{m+1} \coloneqq \gamma_1$.

Let $N \in \mathbb{Z}$ be the unique integer, such that $Y_1 \cong Y_{m+1}[N]$ and for each $i \in [2, m+1]$, set

$$\sigma_i := \begin{cases} 1, & \text{if } \gamma_{i-1} \text{ crosses } \gamma_i \text{ at } p_i \text{ from the right hand side;} \\ -1, & \text{otherwise.} \end{cases}$$

By convention, we set $\sigma_i := (t_i - s_i)$ in case $p_i = p_{i+1}$.

A sequence of morphisms as above is called a **cycle of morphisms** and the number N is called its **degree**.
In the case, where all p_i are boundary intersections, the numbers $t_i - s_i$ are uniquely determined by γ and we refer to the number

$$\omega_A(\gamma) + \frac{1}{2}\left(\sum_{i=1}^{m} \sigma_i + \sum_{i=1}^{m} (t_i - s_i)\right)$$

as the **weighted winding number** of the cut curve γ. Before we finally state the theorem, recall that loop γ has a **teardrop**, if there exists a closed contractible subpath of γ.

We prove the following relation between degrees of cycles of morphisms and the function ω_A.

Theorem I.8.1. *Let γ, γ_j, σ_j, p_j,Y_i,s_j, t_j and N be defined as above, where $i \in [1, m+1]$ and $j \in [1, m]$ and assume that γ has no teardrops. Then,*

$$\omega_A(\gamma) = N - \frac{1}{2}\left(\sum_{i=1}^{m} \sigma_i + \sum_{i=1}^{m} (t_i - s_i)\right).$$

Proof. We may assume that $\gamma_1, \dots, \gamma_m$ and the laminates of S_A are in minimal position. First of all, we observe that we may assume w.l.o.g. that $t_i - s_i = 1$ for all i, such that p_i is interior. This follows from the fact that if we replace $f_i : Y_{s_i} \to Y_{t_i}$ by the morphism $f_i' : Y_{t_i} \to Y_{s_i}[1]$ associated to p_i and each morphism f_j by $f_j' := f_j[1]$ for all $i < j \leq m+1$, then the assertion is true if and only if it is true for the new sequence of morphisms $f_1, \dots, f_{i-1}, f_i', \dots, f_{m+1}'$, $N' = N+1$ and the sequences $s_1, \dots, s_{i-1}, t_i, s_{i+1}, \dots, s_{m+1}$ and $t_1, \dots, t_{i-1}, s_i, t_{i+1} \dots, t_{m+1}$.

Denote $\widetilde{\gamma}_1$ a lift of γ_1 to the universal cover of S. Inductively one constructs lifts $\widetilde{\gamma}_i$ of γ_i for each $i \in [1, m]$, such that for all $i \in [1, m+1]$, $\widetilde{\gamma}_{i-1}$ and $\widetilde{\gamma}_i$ intersect in a lift $\widetilde{p}_i$ of p_i, where $\gamma_{m+1} := \gamma_1$. Note that in general, $\widetilde{\gamma}_{m+1} \neq \widetilde{\gamma}_1$. We label every intersection of γ_i with a laminate with the degree of the corresponding indecomposable projective in Y_i. In particular, if x, x' are consecutive intersections on γ_i, then their labels differ by 1. Let Δ_i denote the polygon, which is bounded by lifts of laminates and which contains $\widetilde{p}_i$. Then there exists $n \in \mathbb{N}$ and a component $u : P \to Q$ of f_i in degree n, such that $\widetilde{\gamma}_{i-1}$ and $\widetilde{\gamma}_i$ each have intersection with the boundary of Δ_i labelled by n and such that the corresponding projectives are P and Q as shown in Figure I.21.

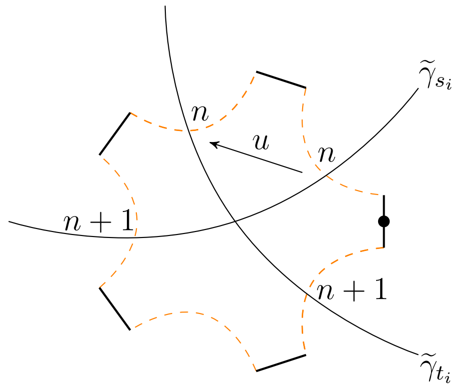


Figure I.21

Note that we obtain a lift of γ, if we resolve all the intersections $\widetilde{p}_i$ simultaneously as determined by the intersection indices σ_i. Thus in order to obtain $\omega_A(\gamma)$ from N we need to subtract a correction term for every p_i arising from the resolution of the crossing. This is where the conditions that γ has no teardrops is important. Suppose, i and j are, such that $\Delta_i = \Delta_{i+1} = \cdots = \Delta_j$. Identifying $\Delta := \Delta_i$ with a disc with one marked point, then $\tilde{\gamma}_i, \ldots, \tilde{\gamma}_j$, for example, intersect as shown in Figure I.22

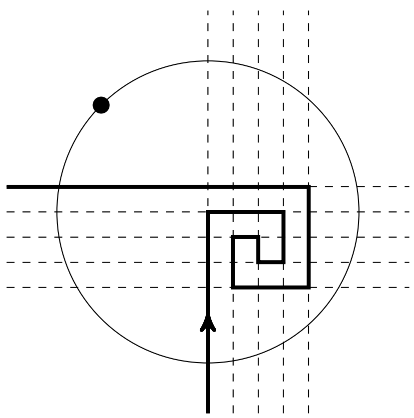


Figure I.22: A lift of γ (solid line) in the polygon Δ. Dashed lines indicate the curves $\widetilde{\gamma}_i, \ldots, \widetilde{\gamma}_j$

Indeed, since γ has no teardrops, its lift does not intersect itself in Δ. We may assume that i and j were chosen in a way, such that $i = 1$ or $\Delta_{i-1} \neq \Delta$ and $j = m+1$ or $\Delta_{j+1} \neq \Delta$. Suppose there exist $a \in [i, j-2]$, such that $\sigma_a = -\sigma_{a+1}$. It follows that $\widetilde{\gamma}_{a-1}$ and $\widetilde{\gamma}_{a+2}$ have a (unique) intersection, say q in Δ. For a better presentation of the argument, we assume that p_a and

p_{a+1} are interior. However, the other case is treated in an analogous way. We claim that the degree of the cycle corresponding to all the $p_0, \dots, p_m$ is equal to the degree of a new cycle corresponding to $p_0, \dots, p_{a-1}, q, p_{a+3}, \dots$ decreased by 1. Because the other case is dual with respect to flipping the orientation of γ, we may assume that $\sigma_a = 1 = -\sigma_{a+1}$. Denote $d_1, \dots, d_8$ denote the degrees of the projective modules in the complexes corresponding to the of intersections of $\widetilde{\gamma}_i$ $(i \in \{a-1, \dots, a+2\})$ with the boundary of Δ as shown in Figure I.23. Then, we observe that $d_7 = d_8 - 1$ regardless of the position of the marked point.

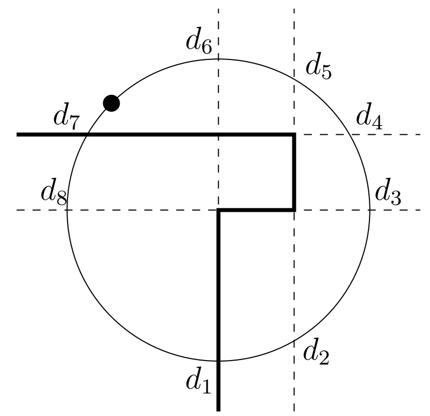


Figure I.23

If for example the marked point is placed in Figure I.22, then the situation is as in Figure I.24.

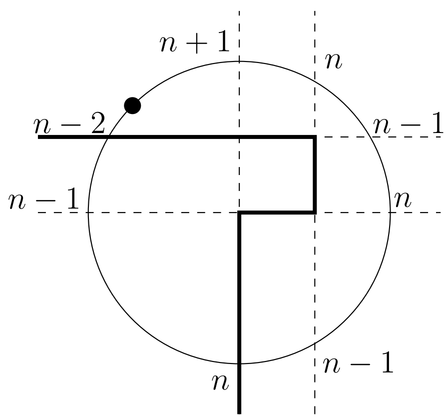


Figure I.24

The case $a = j-1$ follows in the same way. Note that by construction,

γ is homotopic to the curve corresponding to the new cycle and that the intersection index $\sigma(q)$ at q is equal to σ_{a+2}. Using that $t_j - s_j = 1$ for all $j \in \{a, a+1\}$ by assumption, the equation of the assertion is satisfied if and only if

$$\omega_A(\gamma) = (N+1) - \frac{1}{2}\left(\sum_{i \notin \{a,a+1\}} \sigma_i + \sum_{i \notin \{a,a+1\}} (t_i - s_i)\right).$$

From what we said before the former equation coincides with the assertion of the Theorem applied to our new cycle. By repeating this cancellation process several times for each polygon Δ_i, we are in the situation that if $\Delta_i = \Delta_{i+1} = \cdots = \Delta_j$, then $\sigma_i = \cdots = \sigma_j$. Again, since γ has no tear drops, this implies that $j - i \leq 1$ (otherwise we immediately find a self-intersection of γ). Therefore we have reduced the situation locally to two possible cases, which can be verified by hand similar as in Figure I.24.

□

Chapter II

Auto-equivalences and invariants of Fukaya-Like categories

Layout of the chapter

The present chapter is divided into three logical units.
In Section II.1, we introduce Fukaya-like categories and study the interplay between geometric and algebraic concepts in such categories. As examples of those correlations we prove the existence of Auslander-Reiten triangles for the class of so-called 'arc objects' (see Proposition II.1.5) and a morphism analogue of the dichotomy of boundary intersections and interior intersections (see Section II.1.3). The latter allows us to deduce a (preliminary) characterization of the winding number function ω_F in a Fukaya-like category, see Corollary II.1.22.
In the subsequent section, we present a way to assign a homeomorphism to every triangle equivalence between Fukaya-like categories. In Section II.3 we then prove that the isomorphism class of an indecomposable object is determined by a certain sequence of morphisms in the same way as the homotopy class of a curve is determined by its intersections with a triangulation. As a consequence, we deduce that bijections of indecomposable objects arising from triangle equivalences are realized by bijections of curves induced by homeomorphisms.
The final section of this chapter is devoted to the study of the question about how close the relationship between auto-equivalences of the derived category of a gentle algebra and their associated homeomorphisms of its surface $\mathcal{S}_A$ – as defined in Chapter I – is and we obtain a complete answer for the large class of triangular gentle algebras.

II.1 Fukaya-like categories

Let $\mathcal{F}$ be a k-linear triangulated category and let [1] denote its shift functor. We assume that $\mathcal{F}$ is Krull-Schmidt and its class of isomorphism classes of indecomposable objects is a set.
Recall from Section I.2.2 that the orbit category $\mathcal{F}/[1]$ of $\mathcal{F}$ has the same objects as $\mathcal{F}$ and morphisms given by the vector spaces $\bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}(X, Y[i])$. In particular, isomorphism classes of objects in $\mathcal{F}/[1]$ are naturally in bijection with [1]-orbits of isomorphism classes of objects in $\mathcal{F}$.

Let ω be a function, which associates an integer to every (not necessarily primitive) loop on a given marked surface $\mathcal{S}$, such that all of the following conditions are satisfied:

W1) The function ω is constant on homotopy classes;

W2) If $\pi : S^1 \to S^1$ is an m-fold covering map ($m \in \mathbb{N}$), then $\omega(\gamma \circ \pi) = m \cdot \omega(\gamma)$ for all loops on $\mathcal{S}$;

W3) If $\overline{\gamma}$ denotes the inverse of a loop γ on $\mathcal{S}$, then $\omega(\overline{\gamma}) = -\omega(\gamma)$.

W4) $\omega(\gamma) \neq 0$ for every loop γ that is contractible onto a puncture.

A loop γ on $\mathcal{S}$ is called ω-**gradable** if $\omega(\gamma) = 0$.
By analogy to winding numbers on the surface of a gentle algebra, we often refer to ω as a **winding number function** and we refer to $\omega(\gamma)$ as the *winding number* of the loop γ with respect to ω.

Next, we define Fukaya-like triples and Fukaya-like categories. All of their defining properties are inspired by the results of Chapter I on derived categories of gentle algebras.

Definition II.1.1. Let $\mathcal{S}$ be a marked surface and let ω be a winding number function on $\mathcal{S}$ satisfying conditions W1)–W4) above. The triple $(\mathcal{F}, \mathcal{S}, \omega)$ is called **Fukaya-like** (and $(\mathcal{S}, \omega$ is called a **surface model** for $\mathcal{F}$) if all of the following six relations between $\mathcal{F}$ and the pair $(\mathcal{S}, \omega)$ hold true.

1) Indecomposable Objects & Curves: There exists a bijection γ between the isomorphism classes of indecomposable objects in the orbit category $\mathcal{F}/[1]$ of $\mathcal{F}$ and unoriented homotopy classes of arcs and $\omega_{\mathcal{F}}$-gradable primitive loops, equipped with the isomorphism class of an indecomposable k-linear local system (as defined in the preliminaries)

In what follows let $X_1, X_2, X_3 \in T$ be indecomposable objects which are represented by an arc or a loop equipped with a linear local system.

2) Intersections & Morphisms: Let $\gamma_i \in \gamma(X_i)$ be a representative, such that γ_1 and γ_2 are in minimal position.
There exists an injection $\mathfrak{B}$ of the intersections in $\gamma_1 \overrightarrow{\cap} \gamma_2$ into a basis of $\mathrm{Hom}_{\mathcal{F}/[1]}(X_1, X_2)$ consisting of morphisms.
Furthermore, for every intersection $q \in \gamma_1 \overrightarrow{\cap} \gamma_2$ at a puncture there exists a family of morphisms $(\mathfrak{B}(q)(j))_{j \in \mathbb{N}}$ and $m \in \mathbb{Z}$, such that $\mathfrak{B}(q)(0) = \mathfrak{B}(q)$ and $\mathfrak{B}(q)(j) \in \mathrm{Hom}(X_1, X_2[m + j \cdot w_q])$, where w_q is the winding number of the simple loop, which winds around q once in clockwise direction.
Moreover the following hold true:

i) If γ_1 and γ_2 are neither homotopic loops nor intersect in a puncture, then $\mathfrak{B}$ is a bijection.

ii) If γ_1 and γ_2 are homotopic loops, then $\mathfrak{B}$ is not surjective and the quotient of $\mathrm{Hom}_{\mathcal{F}/[1]}(X_1, X_2)$ by the image of $\mathfrak{B}$ is spanned by the residue class of an isomorphism and the residue class of a connecting morphism h in an Auslander-Reiten triangle

$$X_2[n] \longrightarrow Y \longrightarrow X_1 \xrightarrow{h} X_2[n+1].$$

iii) If $\gamma_1 \overrightarrow{\cap} \gamma_2$ contains a puncture, then a basis of $\mathrm{Hom}^*(X_1, X_2)$ is given by the set consisting of

a) all morphisms $\mathfrak{B}(p)$ for all $p \in \gamma_1 \overrightarrow{\cap} \gamma_2$, which are not punctures, and

b) all morphisms $\mathfrak{B}(q)(j)$, where $q \in \gamma_1 \overrightarrow{\cap} \gamma_2$ is a puncture and $j \in \mathbb{N}$.

iv) If $Y \in T$ is represented by a loop with a non-linear local system, then

$$\dim \mathrm{Hom}^*(Y, Y) \;\geq\; 3.$$

3) Mapping Cones & Resolution of Crossings: Let γ_1 and γ_2 be as in 2) and let $p \in \gamma_1 \overrightarrow{\sqcap} \gamma_2$ be different from a puncture. Then the resolution of p (as shown in Figure II.1) is a representative of the mapping cone of $\mathfrak{B}(p)$.

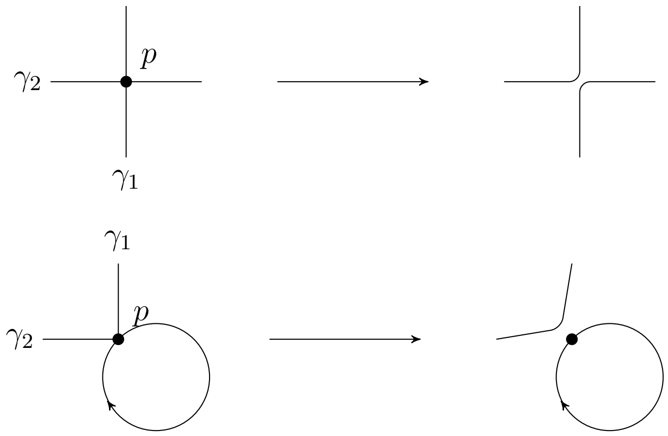


Figure II.1

4) Compositions & Immersed Triangles: Let $\gamma_i \in \gamma(X_i)$ be representatives in minimal position. For $i \in \{1, 2\}$, let $q_i \in \gamma_i \overrightarrow{\sqcap} \gamma_{i+1}$. Then the following statements are true.

i) Assume that if $\gamma(X_1) = \gamma(X_2) = \gamma(X_3)$ are identical, then they are arcs. Then, $\mathfrak{B}(q_2) \circ \mathfrak{B}(q_1)$ is a linear combination of precisely those morphisms $\mathfrak{B}(q_3)$ ($q_3 \in \gamma_1 \overrightarrow{\sqcap} \gamma_3$), such that there exist lifts $\widetilde{\gamma}_i$ of γ_i to the universal cover of $\mathcal{S}$ intersecting in a triangle, a fork or a bigon as shown in Figure II.2,

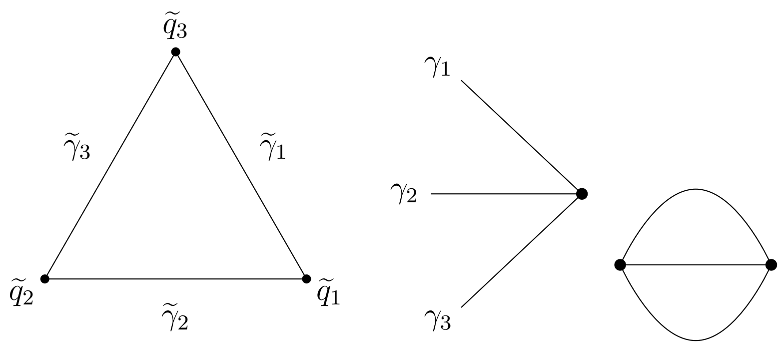


Figure II.2: $\widetilde{q}_i$ is a lift q_i

ii) Suppose $\gamma(X_1) = \gamma(X_3)$ are homotopy classes of loops and $q_1 = q_2 \in \mathcal{S}$. Then $\mathfrak{B}(q_2) \circ \mathfrak{B}(q_1)$ is not a linear combination of morphisms associated to intersections.

5) Winding numbers & Degrees of Cycles of Morphisms: Let $\gamma_1, \dots, \gamma_m$ be arcs or ω-gradable primitive loops on S equipped with linear local systems, such that $\{\gamma_1, \dots, \gamma_m\}$ is in minimal position. Assume γ is an oriented curve such that γ contains no contractible loop as a subcurve. Assume further that γ is glued from arcs $\delta_1, \dots, \delta_m$ (in cyclic order), where each δ_i is a (possibly contractible) subarc of γ_i connecting crossings $p_i \in \gamma_{i-1} \cap \gamma_i$ and $p_{i+1} \in \gamma_i \cap \gamma_{i+1}$ as shown in Figure II.3.

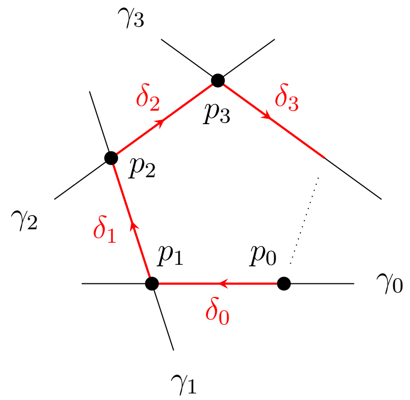


Figure II.3

Then,

$$w(\gamma) = N - \sum_{i=1}^{m} d_i,$$

where $d_i \in \{0, \pm 1\}$ and $N \in \mathbb{Z}$ are defined as follows. Let $Y_1, \dots, Y_{m+1} \in \mathcal{F}$ be indecomposable, such that for each $i \in [1, m+1]$,

a) $f_i = \mathfrak{B}(p_i) \in \operatorname{Hom}_{\mathcal{F}}(Y_{s_i}, Y_{t_i})$, where $\{s_i, t_i\} = \{i-1, i\}$, and

b) $\gamma_i \in \gamma(Y_i)$, where $\gamma_{m+1} := \gamma_1$.

Then let $N \in \mathbb{Z}$ be the unique integer, such that $Y_1 \cong Y_{m+1}[N]$ and

$$d_i := \frac{1}{2} \cdot (\sigma_i + t_i - s_i),$$

where is the index of the intersection p_i, i.e.

$$\sigma_i = \begin{cases} 1, & \text{if } \gamma_{i-1} \text{ crosses } \gamma_i \text{ at } p_i \text{ from the right hand side;} \\ -1, & \text{otherwise.} \end{cases}$$

6) Auslander-Reiten Triangles:

- Let γ be a ω-gradable loop on $\mathcal{S}$. Every object X represented by γ is τ-invariant, i.e. there exists an Auslander-Reiten triangle of the form
$$X \longrightarrow Y \longrightarrow X \longrightarrow X[1].$$
Moreover, it is contained in a homogeneous tube in the Auslander-Reiten quiver of the form
$$\begin{tikzcd} \cdots \arrow[r, bend right] & X_3 \arrow[l, bend right] \arrow[r, bend right] & X_2 \arrow[l, bend right] \arrow[r, bend right] & X_1. \arrow[l, bend right] \end{tikzcd}$$
There exists an irreducible polynomial $P \in k[X]$, such that for all $d \in \mathbb{N}$, $\gamma(X_d) = (\gamma, \mathcal{V}_{P^d})$, where $\mathcal{V}_{P^d}$ denotes the isomorphism class of indecomposable local systems associated to P^d.

- If $X \in \mathcal{F}$ is indecomposable, such that $\gamma(X)$ contains arcs whose end points are punctures, then there exists no irreducible morphism starting in X.
- Auslander-Reiten triangles do neither start nor end in indecomposable objects which are represented by infinite arcs.

A triangulated category $\mathcal{F}$ is called **Fukaya-like** if there exists a Fukaya-like triple of the form $(\mathcal{F}, \mathcal{S}, \omega)$.

Convention. In most parts of this chapter we only consider indecomposable objects in a Fukaya-like triangulated category, which are represented by either arcs or loops with a linear local system. This is due to the observation that, if k is algebraically closed, the behavior of the homogeneous tube associated to a loop is essentially determined by the behavior of its object at the base. That being said, we stick to the convention that 'indecomposable' in a Fukaya-like triangulated category shall refer to an indecomposable object of the above type unless mentioned otherwise.

As pointed out at the beginning of this chapter, bounded derived categories of gentle algebras are prototypical examples of Fukaya-like categories. We prove the following.

Proposition II.1.2. *Let A be a gentle algebra. Let $\mathcal{S}_A$ denote its ribbon surface (see Definition I.1.10), where we replace each boundary component without marked points by a puncture, and let ω_A denote the function defined in Section I.8. Then $(\mathcal{D}^b(A), \mathcal{S}_A, \omega_A)$ is a Fukaya-like triple. In particular, $\mathcal{D}^b(A)$ is Fukaya-like.*

Proof. We have to verify 1)–6) of Definition II.1.1. Property 1) follows from Theorem I.2.5, Property 2) is the statement of Theorem I.3.3 and Property 3) is the assertion of Theorem I.4.3. Finally, Property 4) and 5) follow from Theorem I.8.1 and Theorem I.7.1, whereas Property 6) is a consequence of the description of Auslander-Reiten triangles for band complexes, as recalled in Section I.5.1, and Corollary 6.8 in [4]. $\square$

II.1.1 Properties of Fukaya-like categories

In this section we collect and prove a variety of useful observations about Fukaya-like categories.

Let $(\mathcal{F}, \mathcal{S}, \omega)$ be a Fukaya-like triple, where $\mathcal{S} = (S, \overline{\mathcal{M}})$. By its very definition, there are two types of indecomposable objects in $\mathcal{F}$ and we say that an indecomposable object $X \in \mathcal{F}$ is a (finite or infinite) **arc object** if $\gamma(X)$ is a homotopy class of (finite or infinite) arcs. Otherwise we call it a **loop object**.
This dichotomy roughly coincides with the dichotomy between τ-invariant and non-τ-invariant objects and the approximative picture is that loop objects consitute 'almost all' τ-invariant indecomposables, i.e. up to the action of the shift functor there exist only a finite number of homogeneous tubes consisting of arc objects. We will make this more precise in Lemma II.1.10, where we classify fractionally Calabi-Yau objects geometrically.
For convenience we refer to an arc object or a loop object associated to a linear local system as a **linear indecomposable**. An indecomposable object is called **quasi-linear** if it is an arc object or a loop object equipped associated with a quasi-linear local system.

Our first consequence of Definition II.1.1 is the following.

Lemma II.1.3. *Let $X_1, X_2 \in \mathcal{F}$ be indecomposable and let $\{\gamma_1, \gamma_2\}$ be curves in minimal position, such that $\gamma_i \in \gamma(X_i)$. If $p \in \gamma_1 \overrightarrow{\cap} \gamma_2$ is interior but not a puncture and $\mathfrak{B}(p) \in \operatorname{Hom}(X_1, X_2[n])$, then $\mathfrak{B}(p) \in \operatorname{Hom}(X_2, X_1[1-n])$ regarding p as an element in $\gamma_2 \overrightarrow{\cap} \gamma_1$*

Proof. This is a special case of Property 5) in Definition II.1.1 with $m = 2$ and δ_1, δ_2 contractible paths. In particular, the winding number of the curve glued from δ_1 and δ_2 vanishes. $\square$

The following lemma is a generalization of the fact that the identity morphism of a string complex in a bounded derived category of a gentle algebra corresponds to the tautological boundary self-intersection of its associated arc, see Remark I.3.10.

Lemma II.1.4. *Let γ_1, γ_2 be homotopic arcs on $\mathcal{S}$ in minimal position and let $p \in \gamma_1 \overrightarrow{\sqcap} \gamma_2$ be a marked point. If $p \in \partial S$, then $\mathfrak{B}(p)$ is invertible. If p is a puncture, then $\mathfrak{B}(p)(0)$ is invertible.*

Proof. We prove the assertion when p is on the boundary. The other case is similar. Set $f \coloneqq \mathfrak{B}(p)$. It suffices to show that $\operatorname{Hom}(Y, f)$ and $\operatorname{Hom}(f, Y)$ are surjective maps as this implies that there exist $g, g' \in \operatorname{Hom}(Y, X)$, such that $g \circ f = \operatorname{Id}_X$ and $f \circ g' = \operatorname{Id}_Y$ and $g = g \circ f \circ g' = g'$. As the arguments are the same in both cases we only show that $\operatorname{Hom}(Y, f)$ is surjective.
Let $\gamma_3 \simeq \gamma_1$, such that $\{\gamma_1, \gamma_2, \gamma_3\}$ is in minimal position. The elements $\gamma_3 \overrightarrow{\sqcap} \gamma_2$ correspond to a basis of $\operatorname{Hom}^*(X, Y)$ consisting of morphisms. Let $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ be lifts of γ_1 and γ_2 to the universal cover of S, which intersect at their end points. By replacing γ_3 by another representative we may assume that the lift $\widetilde{p} \in \widetilde{\gamma}_1 \overrightarrow{\sqcap} \widetilde{\gamma}_2$ of p is an element in $\delta \overrightarrow{\sqcap} \widetilde{\gamma}_1$ and $\delta \overrightarrow{\sqcap} \widetilde{\gamma}_2$ for some lift δ of γ_3. Let $q \in \gamma_3 \overrightarrow{\sqcap} \gamma_2$ and denote $\widetilde{\gamma}_3$ the lift of γ_3, which intersects $\widetilde{\gamma}_2$ in a lift $\widetilde{q} \in \widetilde{\gamma}_3 \overrightarrow{\sqcap} \widetilde{\gamma}_2$ of q. As all arcs are in minimal position, $\widetilde{\gamma}_3 \overrightarrow{\sqcap} \widetilde{\gamma}_1$ contains a single element $\widetilde{q}'$ and $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ bound a disc. By construction, $\widetilde{p}, \widetilde{q}$ and $\widetilde{q}'$ form a fork or the corners of a triangle in clockwise order. Thus, $\mathfrak{B}(q) \circ \mathfrak{B}(p)$ is a non-zero multiple of $\mathfrak{B}(q')$, where $q' \in \gamma_3 \overrightarrow{\sqcap} \gamma_1$ is the projection of $\widetilde{q}'$. Since q was arbitrary and $\operatorname{Hom}(Y, f)$ is linear, this shows that $\operatorname{Hom}(Y, f)$ is surjective. $\square$

As generalization of Theorem I.5.3, we show the existence of Auslander-Reiten triangles for arc objects in a Fukaya-like category and that the homeomorphism $\boldsymbol{\tau}$ on $\mathcal{S}$ is a geometric incarnation of the Auslander-Reiten translation. Recall from Section I.5.2 that γ and $\boldsymbol{\tau}\gamma$ have a distinguished intersection if in minimal position.

Proposition II.1.5. *Let γ be a finite arc on $\mathcal{S}$. If $p \in \gamma \overrightarrow{\sqcap} \boldsymbol{\tau}\gamma$ is the distinguished intersection, then $\mathfrak{B}(p)$ is the connecting morphism of an Auslander-Reiten triangle in $\mathcal{F}$.*

Proof. Choose a representative X of γ and a representative Y of $\boldsymbol{\tau}\gamma$ and set $h \coloneqq \mathfrak{B}(p)$, i.e. $h \in \operatorname{Hom}(X, Y[m])$ for some $m \in \mathbb{Z}$. To begin with, observe that by Property 3) in Definition II.1.1, the mapping cone Z of $\mathfrak{B}(p)$ is represented by a boundary segment connecting consecutive marked points. In particular, Z is indecomposable. Since $\mathcal{F}$ is Krull-Schmidt, it suffices to show that $h \circ f$ vanishes for all non-invertible morphisms $f : X' \to X$, where X' is any type of indecomposable object in $\mathcal{F}$, as in this case, being non-split and being non-invertible are equivalent. By Definition II.1.1 2ii), it therefore suffices to prove that $h \circ \mathfrak{B}(q) = 0$ for all intersections $q \in \delta \overrightarrow{\sqcap} \gamma$ for any curve δ, such that $\{\gamma, \delta\}$ is in minimal position. Suppose, p is an interior point. By Definition II.1.1 4), such a composition is then non-zero only if there exists a triangle in the universal cover of $\mathcal{S}$ bounded by subarcs of lifts $\widetilde{\delta}$, $\widetilde{\gamma}$ and $\widetilde{\tau\gamma}$ of δ, γ and $\tau\gamma$ in clockwise order. Since $\{\gamma, \tau\gamma, \delta\}$ are in minimal position, $\widetilde{\delta}$ intersects each of $\widetilde{\gamma}$ and $\widetilde{\tau\gamma}$ at most once in the interior. Since $\widetilde{\tau\gamma}$ and $\widetilde{\gamma}$ have neighboring end points we therefore conclude that at least one of the corners of the triangle lies on the boundary. Suppose that two of the corners of such a triangle were boundary intersections, then lifts of $\gamma, \tau\gamma$ and δ would be arranged in one of the ways shown in Figure II.4.

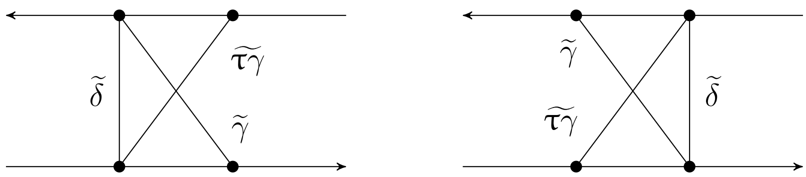


Figure II.4

Note that we do not assume that the shown boundary components are distinct. However, in order for p to be interior, all endpoints of the lifts must be pairwise distinct. In any case, we observe that the unique intersection of $\widetilde{\tau\gamma}$ and $\widetilde{\delta}$ only defines an element in $\tau\gamma \overrightarrow{\sqcap} \delta$ but not in $\delta \overrightarrow{\sqcap} \tau\gamma$, implying that $h \circ \mathfrak{B}(q) = 0$.
In case only one of the corners lies on the boundary, then again, because every pair of lifts intersects at most once in the interior, it follows that $\delta \simeq \gamma$ or $\delta \simeq \tau\gamma$ showing that $\mathfrak{B}(q)$ is invertible as shown in Lemma II.1.4.

Similarly, if $p \in \partial \mathcal{S}$, then γ and $\tau\gamma$ are arcs connecting consecutive marked points on the boundary and at all lifts of p to a universal cover of $\mathcal{S}$, there exist lifts of γ and $\tau\gamma$, which intersect as in Figure II.5.

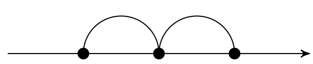


Figure II.5

In particular, if $h \circ \mathfrak{B}(q) \neq 0$, for some q and δ as above, q is required to be a boundary intersection and certain lifts of $\gamma, \boldsymbol{\tau}\gamma$ and δ are required to form a fork. However, because $\mathfrak{B}(q)$ is a morphism from an object representing δ to an object representing γ, q cannot coincide with the unique intersection of the lifts of γ and $\boldsymbol{\tau}\gamma$, unless $\delta \simeq \gamma$, in which case $\mathfrak{B}(q)$ is invertible according to Lemma II.1.4. In any case, it follows that $h \circ \mathfrak{B}(q) = 0$. $\square$

Corollary II.1.6. *Let $X \in \mathcal{F}$ be an arc object and let $\gamma \in \gamma(X)$. Then the Auslander-Reiten translate of X exists and $\boldsymbol{\tau}\gamma \in \gamma(\tau X)$.*

In analogy to Corollary 6.3 in [13], the previous Corollary enables us to identify the arcs which give rise to Auslander-Reiten triangle with indecomposable middle term:

Corollary II.1.7. *Let $X \in T$ be an arc object and let*

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

be an Auslander-Reiten triangle. Then Y is indecomposable if and only if $\gamma(X)$ (or, equivalently, $\gamma(Z)$) contains a simple boundary arc which connects consecutive marked points.

Proof. It follows from Lemma II.1.5 and Definition II.1.1 3) that the middle term of such a triangle is indecomposable if and only if the distinguished intersection in $\gamma \overrightarrow{\cap} \tau(\gamma)$ lies on the boundary which happens if and only if $\gamma(X)$ contains a simple arc connecting consecutive marked points. $\square$

Remark II.1.8. By similar arguments as used in the proof of Proposition II.1.5 one can show that the distinguished intersection of an arc γ and γ_e (resp. ${}_s\gamma$), as defined in Section I.5.2, corresponds to irreducible morphisms. This is analogous to Theorem I.5.3.

Corollary II.1.6 motivates the following definition of a perfect subcategory in every Fukaya-like category.

Definition II.1.9. Let $(\mathcal{F}, \mathcal{S}, \omega)$ be a Fukaya-like triple. Define the **category of perfect objects** $\mathrm{Perf}(\mathcal{F})$ as the full subcategory of $\mathcal{F}$ containing all objects, which are isomorphic to finite direct sums of direct summands of objects occuring in an Auslander-Reiten triangle in $\mathcal{F}$.

It is then a consequence of Definition II.1.1 6), Lemma II.1.5 and property 6) in Definition II.1.1 that $\mathrm{Perf}(\mathcal{F})$ is the full subcategory of objects, which are isomorphic to finite direct sums of finite arc objects and loop objects.
In particular, if $\mathcal{F} = \mathcal{D}^b(A)$ for some gentle algebra A, then $\mathrm{Perf}(\mathcal{F})$ coincides

with the usual definition of the category of perfect complexes.
The important feature of $\mathrm{Perf}(\mathcal{F})$ for general $\mathcal{F}$ is that every triangle equivalence between Fukaya-like categories restricts to an equivalence between their perfect subcategories. However, we do not claim that $\mathrm{Perf}(\mathcal{F})$ is closed under arbitrary mapping cones in general and hence a triangulated subcategory.

II.1.2 Boundary objects & segment objects

In this section we explore how the natural distinction between boundary and interior of a surface can be re-interpreted algebraically as a dichotomy of two classes of morphisms, so-called boundary morphisms and interior morphisms. In order to do so, we begin with the study of 'fractionally Calabi-Yau' objects in Fukaya-like categories and their connection with this idea.

Lemma II.1.10. *Let $X \in \mathcal{F}$ be an arc object. There exist $m, n \in \mathbb{Z}$, such that $\tau^m X \cong X[n]$ if and only if $\gamma(X)$ contains a boundary arc γ with image in a component $B \subseteq \partial S$ and, in this case, $m = |B \cap \overline{\mathcal{M}}\,|$ and $n = n_B := w(\gamma_B)$, where γ_B is the simple boundary loop which winds around B in counter-clockwise direction. In particular, $\tau X \cong X$ if and only if $\gamma(X)$ contains a boundary arc with image on a boundary component B with a single marked point and such that $n_B = 0$.*

Proof. Lifting $\gamma \in \gamma(X)$ to the universal cover, we see that $\boldsymbol{\tau}^m \gamma \simeq \gamma$ implies that γ is homotopic to a boundary arc with image in a component $B \subseteq \partial S$ and $\boldsymbol{\tau}^m \gamma \simeq \gamma$ if and only if m equals the number of marked points on B. Suppose $\tau^m X \cong X[n]$ and let $\gamma \in \gamma(X)$ be a boundary arc contained in a component $B \subset \partial S$. For each $i \in [0, m)$, denote p_i the distinguished intersection in $\tau^i \gamma \overrightarrow{\cap} \tau^{i+1} \gamma$ corresponding to a connecting morphism $\tau^i\ X \ \to\ \tau^{i+1}\ X[1]$ of an Auslander-Reiten triangle ending in $\tau^i X$. We obtain a whole sequence of morphisms

$$X \to \tau X[1] \to \tau^2 X[2] \to \cdots \to \tau^m X[m] \cong X[n+m].$$

The intersections $p_0, \dots, p_m$ determine a loop which is homotopic to a simple loop δ which winds around B in counter-clockwise direction. According to property 5) in Definition II.1.1, we have $n_B = (n+m) - m \cdot 1 = n$.

□

The previous lemma shows that shift orbits of fractionally Calabi-Yau objects are encoded in the number of marked points on the boundary components and winding numbers of boundary curves. However, as shown in [11], this is what the AAG-invariant of a gentle algebra counts. We obtain the following variant of Theorem I.6.1.

Corollary II.1.11. *Let A be a gentle algebra and let $B_1, \dots, B_n$ be the boundary components of S_A. Let n_{B_i} be defined as in Lemma II.1.10. Then the AG-invariant of A is given by the set of pairs (b_i, c_i) ($i \in [1, n]$), where*

- *b_i is given by the number of marked points on B_i, and*
- *$c_i - b_i = n_{B_i}$.*

If $\mathcal{F}$ is the bounded derived category of a gentle algebra, then the property that $\tau^m X \cong X[n]$ for an object $X \in \operatorname{Perf}(\mathcal{F})$ and integers $m, n \in \mathbb{Z}$ is equivalent to X being fractionally Calabi-Yau, i.e. if $\mathbb{S}$ is a Serre functor for $\mathcal{F}$, then there exist $m', n' \in \mathbb{Z}$, such that $\mathbb{S}^{m'} X \cong X[n']$, see Section I.5.1. The assertion of Lemma II.1.10 suggest the following definition.

Definition II.1.12. Let $X \in \operatorname{Perf}(\mathcal{F})$ be an arbitrary indecomposable. We say that X is a **boundary object** if the following hold true:

1) There exist integers $m, n \in \mathbb{N}$ such that $\tau^m X \cong X[n]$.
2) If X is τ-invariant then for all indecomposable objects $Z \in \mathcal{F}$,

$$\max\{\dim \operatorname{Hom}^*(Y, Z), \dim \operatorname{Hom}^*(Z, Y)\} \leq 2.$$

A boundary object X is further called a **segment object** if the middle term of any Auslander-Reiten triangle starting in X is indecomposable.

Note that if a loop object is a boundary object, then by Property 2 ii) of Definition II.1.1, it sits at the mouth of its homogeneous tube.

Proposition II.1.13. *Let $X \in \operatorname{Perf}(\mathcal{F})$ be an arbitrary indecomposable. Then X is a boundary object if and only if $\gamma(X)$ contains a simple boundary curve with a linear local system. If X is a boundary object, then it is a segment object if and only if it is represented by a loop with linear local system or $\gamma(X)$ contains a boundary arc connecting consecutive marked points (a 'boundary segment').*

Proof. Due to Lemma II.1.10, simplicity of the representing arc of an arc object is equivalent to condition 2) of II.1.12. By Corollary II.1.7 the condition that the middle term of an Auslander-Reiten triangle starting (or equivalently, ending) in X is indecomposable, translates to the property that the end points of an arc γ representing X have to be consecutive elements. In other words γ is a boundary segment.
We therefore assume that X is represented by a loop γ and hence is τ-invariant. If γ is homotopic to a boundary loop, it does not intersect any

other loop, which is in minimal position with γ. Furthermore, it intersects arcs only in a neighborhood of their end points and hence at most twice. If the local system associated to X is linear, then each intersection determines a single basis element $\mathrm{Hom}^*(X, Z)$ (resp. $\mathrm{Hom}^*(Z, X)$). Note that the simplicity of γ implies that $\dim \mathrm{Hom}^*(X, X) \leq 2$.
Conversely, let γ be a gradable loop and $U \in \mathrm{Perf}(\mathcal{F})$ be a representative of γ. In particular, U is τ-invariant. Assume that U satisfies condition 2) and therefore that $\dim \mathrm{Hom}^*(U, U) \leq 2$. It follows from Definition II.1.1 2 ii) that the local system associated to U is linear and that γ can be chosen to be simple. Suppose γ is not homotopic to a boundary curve. We cut S along γ and denote by S' the resulting surface, i.e. S' contains distinct boundary components B_1, B_2, such that $S' \setminus (B_1 \cup B_2) = S \setminus \gamma$. Let $m \in \mathbb{N}$. We construct a simple arc δ on S, such that $\{\gamma, \delta\}$ is in minimal position and $|\gamma \cap \delta| \geq m$ – yielding a contradiction.
W.l.o.g. we may assume that the connected component V of S', which contains B_1 on its boundary, contains a marked point of S which is not a puncture. By our assumptions and since D is not nullhomotopic, each connected component of S' is neither homeomorphic to a disc nor to a cylinder. As a result, there exist m disjoint simple arcs $E_1^1, \dots, E_1^m$ in S' connecting distinct points in B_1, such that neither of them is homotopic to an arc with image in B_1. For example, choose any such simple arc E_1^1 with this property and let $E_1^2, \dots, E_1^m$ be small perturbations of E_1^1. In the same way, we may choose disjoint simple arcs $E_2^1, \dots, E_2^m$, such that $s(E_2^i) = t(E_1^i)$ and $t(E_2^i) = s(E_1^{i+1})$ for all $i \in (1, m)$. Furthermore, we choose simple arcs W_1, W_2 in V, such that

- for all $i \in [1, m]$, $W_1 \cap E_1^i = \emptyset$;
- the arcs W_1 and W_2 have no interior intersections;
- the end point (resp. start point) of W_1 (resp. W_2) agrees with the start point (resp. end point) of E_2^1 (resp. E_2^m);
- $s(W_1)$ and $t(W_2)$ are marked boundary points.

The concatenation $\delta \coloneqq W_2 * E_2^m * E_1^m * \cdots * E_2^1 * W_1$, as indicated by the symbol $*$, is a simple arc in S and $|\gamma \cap \delta| = m$. Suppose $\{\gamma, \delta\}$ is not in minimal position, then by [42], Proposition 1.7, γ and δ form a bigon. By the assumptions we made on the E_j^i, such a bigon must be formed by a subarc γ' of γ and an arc E_b^a for some $a \in [1, m]$ and some $b \in [1, 2]$. But this implies that E_b^a is homotopic to δ' which contradicts our assumptions on E_b^a. $\square$

Remark II.1.14. Proposition II.1.13 implies that if $\tau U[1] \ncong U$, then $\dim \mathrm{Hom}(U, \tau U[1]) = 1$ for every segment object $U \in \mathcal{F}$ showing that every connecting morphism in an Auslander-Reiten triangle starting (or ending) in U is represented by the distinguished intersection. In case $U \cong \tau U[1]$, note that, since $\operatorname{End}(U)$ is local, it follows $\dim \operatorname{rad} \operatorname{End}(U) = 1$.

By Proposition II.1.13 we can characterize boundary and segment objects purely in terms of geometric properties of their representing curves. The notion of such objects is the key to distinguish morphisms "arising from boundary intersection" from morphisms "arising from interior intersections". The precise definitions for this (almost correct) dichotomy are introduced in the subsequent section.

II.1.3 The spaces of interior morphisms

Based on the previous section we define the distinguished subspaces of "interior" morphisms between objects in Fukaya-like categories.

Definition II.1.15. Let $X, Y \in \mathcal{F}$ be indecomposable. A morphism $f : X \to Y[i]$ is called **interior**, if X or Y is a τ-invariant boundary object, or, both of the following two conditions are satisfied.

i) The connecting morphism of any Auslander-Reiten triangle ending in a segment object $U \in \operatorname{Perf}(\mathcal{F})$ does not factor through f.

ii) The morphism f factors through an object in $\operatorname{Perf}(\mathcal{F})$.

If f is not an interior morphism, we call it a **boundary morphism**.

Notation II.1.16. For indecomposable objects $X_1, X_2 \in \mathcal{F}$, let $\operatorname{Hom}_{\mathrm{Int}}(X_1, X_2)$ denote the subset of interior morphisms.

By definition, being an interior (or a boundary morphism) is invariant under triangle equivalences.

Lemma II.1.17. *Let $(\mathcal{F}, \mathcal{S}_{\mathcal{F}}, \omega_{\mathcal{F}}), (\mathcal{F}', \mathcal{S}_{\mathcal{F}'}, \omega_{\mathcal{F}'})$ be Fukaya-like triples and let $T : \mathcal{F} \to \mathcal{F}'$ be a triangle equivalence. Then, the isomorphisms*

$$\operatorname{Hom}_{\mathcal{F}}(-,-) \longrightarrow \operatorname{Hom}_{\mathcal{F}'}(T(-), T(-))$$

restrict to bijections between the subsets of interior morphisms.

Our next goal is to show that interior morphisms form a subvector space and are spanned by the basis elements associated to interior intersections. In order to do so, we need the following lemma which characterizes morphisms which factor through the category of perfect objects.

Lemma II.1.18. *Let $X, Y \in \mathcal{F}$ be indecomposable and let $f : X \to Y$ be a morphism, such that $f = \sum_{i=1}^{m} \lambda_i \cdot \mathfrak{B}(p_i)$ for intersections $p_0, \dots, p_m$. Then f factors through* $\mathrm{Perf}(\mathcal{F})$ *if and only if none of the p_i is a puncture.*

Proof. Suppose p_0 is a puncture and suppose there exists $Z \in \mathrm{Perf}(\mathcal{F})$ and morphisms $g : X \to Z$ and $h : Z \to Y$, such that $h \circ g = f$. In particular, $X, Y \notin \mathrm{Perf}(\mathcal{F})$. To arrive at a contradiction, it is sufficient to show that for any two morphisms $g' : X \to Z'$ and $h' : Z' \to Y$ with $Z' \in \mathrm{Perf}(\mathcal{F})$ indecomposable, $\mathfrak{B}(p_0)$ cannot occur as a summand of $h' \circ g'$ if written as a linear combination of basis vectors arising from intersections. However, since Z' is represented by a finite arc or loop, g' and h' are sums of morphisms associated to intersections which are not punctures. By Definition II.1.1 4), a composition of such basis elements is a sum of morphisms associated to intersections, which are not punctures.
Next, assume that none of the p_i is a puncture. Because f factors through the identity morphisms of X and Y, we may assume that $X, Y \notin \mathrm{Perf}(\mathcal{F})$. Since $\mathrm{Perf}(\mathcal{F})$ is closed under finite direct sums, it is sufficient to show that each $\mathfrak{B}(p_i)$ factors through $\mathrm{Perf}(\mathcal{F})$. Let $\widetilde{\gamma}_X, \widetilde{\gamma}_Y$ be arcs on the universal cover $\widetilde{S}$ of $\mathcal{S}_{\mathcal{F}}$ representing X and Y, such that $\widetilde{\gamma}_X$ and $\widetilde{\gamma}_Y$ intersect in a single lift of p_i. Let p_X (resp. p_Y) be an interior end point of $\widetilde{\gamma}_X$ (resp. $\widetilde{\gamma}_Y$), i.e. p_X and p_Y are punctures. In case p_i lies in the interior, denote by γ a simple finite arc in $\widetilde{S}$, such that$p_X, p_Y \in \widetilde{S} \setminus \gamma$ has a connected component homeomorphic to a disc. We may further assume that the image of $\widetilde{\gamma}_X, \widetilde{\gamma}_Y$ and $\widetilde{\gamma}$ are in minimal position. By construction, p_i and the unique interior intersections q_X and q_Y of $\widetilde{\gamma}$ with $\widetilde{\gamma}_X$ and $\widetilde{\gamma}_Y$ form a triangle. Hence $\widetilde{\gamma}$, p_X and p_Y give rise to an object Z and morphisms $\mathfrak{B}(q_X) : X \to Z$ and $\mathfrak{B}(q_Y) : Z \to Y$, such that $\mathfrak{B}(q_Y) \circ \mathfrak{B}(q_X)$ is a multiple of $\mathfrak{B}(p_i)$.
If p_i lies on the boundary, we choose lifts $\widetilde{\gamma}_X$ and $\widetilde{\gamma}_Y$ as above, which intersect exactly once in a lift q of p and choose $\widetilde{\gamma}$ to be a finite arc which intersects $\widetilde{\gamma}_X$ and $\widetilde{\gamma}_Y$ only in its end point q and which lies between $\widetilde{\gamma}_X$ and $\widetilde{\gamma}_Y$ locally around q. Then f factors through the morphisms determined by q. $\square$

The following lemma justifies the term 'interior morphism'.

Lemma II.1.19. *Let $X_1, X_2 \in \mathcal{F}$ and $\gamma_i \in \gamma(X_i)$, such that $\{\gamma_1, \gamma_2\}$ is in minimal position. Let $f = \sum_{i=1}^{m} \lambda_i \cdot \mathfrak{B}(p_i) \in \mathrm{Hom}(X_1, X_2)$, where $\lambda_i \in k^\times$ and $p_1, \dots, p_m \in \gamma_1 \overrightarrow{\cap} \gamma_2$ are pairwise distinct intersections.*

1) *If X_1 or X_2 is τ-invariant, then f is interior.*

2) *If both X_1 and X_2 are not τ-invariant, then f is interior if and only if all p_i are interior intersections different from a puncture.*

Proof. If X_1 or X_2 is a τ-invariant boundary object, then by definition f is interior. We therefore assume that neither X_1, nor X_2 is a τ-invariant boundary object.

It follows from Lemma II.1.18 that f factors through $\mathrm{Perf}(\mathcal{F})$ if and only if none of the p_i is a puncture. Set $f_i \coloneqq \lambda_i \cdot \mathfrak{B}(p_i)$. Let U be a segment object, γ_U a representing boundary segment or a boundary loop. Let $h : U \to \tau U[1]$ be non-zero. As $\mathrm{Hom}(U, \tau U[1])$ is 1-dimensional, h is unique up to a scalar and a connecting morphism in an Auslander-Reiten triangle. Suppose h factors through f. Let $i \in [1, m]$ such that f_i is represented by an interior intersection.

We distinguish two cases. First, assume that U is an arc object. Then it follows from property 4) in Definition II.1.1, that each composition $f_i \circ \alpha$ and $\beta \circ f_i$, with $\alpha : U \to X$, $\beta : Y \to \tau U[1]$ being morphisms associated to intersections, is either zero, or a sum of morphisms associated to interior intersections. The same applies to $\beta \circ f_i \circ \alpha$. But a multiple of a morphism in $\mathrm{Hom}(U, \tau U[1])$ is represented by the distinguished (boundary) intersection in $\gamma_U \overrightarrow{\cap} \tau\gamma_U$, proving $\beta \circ f_i \circ \alpha = 0$. Hence, if h factors through f as a map $\beta \circ f \circ \alpha$, then h equals the sum of all $\beta \circ f_j \circ \alpha$, such that $p_j \notin \partial\mathcal{S}_{\mathcal{F}}$.

Next, suppose that U is a loop object and denote by B the boundary component which contains γ_U. None of the morphisms in $\mathrm{Hom}(U, U[1])$ is represented by an intersection. Thus, for β, α as before, $\beta \circ f_i \circ \alpha$ is zero, unless $\beta \circ f_i$ and α correspond to the same interior intersection (see Definition II.1.1 4 ii)) and in this case, p_i is the unique marked point on B. This shows that f is interior if all p_j are interior.

Conversely, suppose that for some $i \in [1, m]$, p_i lies on a component $B \subseteq \partial S$. By assumption, X_1 and X_2 are not τ-invariant arc objects as such objects are boundary objects according to Lemma II.1.10. We show that f is a boundary morphism. Let δ be a simple boundary arc with image in B, connecting p_i and its successor, and let $U \in \mathcal{F}$ be a representative of δ. By Definition II.1.1 4 i), p_i encodes morphisms $\alpha : U[m] \to X$ and $\beta : Y \to \tau U[m+1]$ for some $m \in \mathbb{Z}$ and $\beta \circ f_i \circ \alpha$ is a multiple of $\mathfrak{B}(q)$, where $q \in \delta \overrightarrow{\cap} \tau\delta$ is the distinguished intersection (see Section I.5.2). Finally, we observe that for all p_j with $j \neq i$, $\beta \circ f_j \circ \alpha = 0$ as none of the corresponding intersections form a fork but $\mathrm{Hom}^*(U, U) = \mathrm{Hom}(U, U[1])$ is represented by a boundary intersection. Hence, $\beta \circ f \circ \alpha$ is a multiple of $\mathfrak{B}(q)$ and the proof is complete. □

Corollary II.1.20. *Let $X_1, X_2 \in \mathcal{F}$ be indecomposable and not represented by the same homotopy class of loops. Then $\mathrm{Hom}_{\mathrm{Int}}(X_1, X_2)$ is a subvector space of $\mathrm{Hom}(X_1, X_2)$.*

Proof. According to Lemma II.1.19 a morphism $f : X_1 \to X_2$ is interior if and only if $f = \sum_{i=1}^{m} \lambda_i \cdot \mathfrak{B}(p_i)$, where $\{p_0, \dots, p_m\}$ is the set of interior intersections (different from punctures) between representing curves of X_1 and X_2. $\square$

Notation II.1.21. In the situation of the previous corollary, set

$$\mathrm{Hom}^*_{\mathrm{Int}}(X_1, X_2) := \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{Int}}(X_1, X_2[n]).$$

A categorical characterization of winding numbers As an application of the dichotomy of interior and boundary morphisms, we present a categorical characterization of the winding number function ω.
Let γ be a loop on S and let γ' be any finite closed arc, such that γ' is homotopic to γ if regarded as a loop in the natural way. The boundary self-intersection p of γ gives rise to a boundary intersection of γ and a homotopic arc γ'. In this way, it gives rise to a self-extension $\mathfrak{B}(p) : X \to X[d]$ for every representative $X \in \mathcal{F}$ of γ'. The degree d of $\mathfrak{B}(p)$ is well-defined and, according to Definition II.1.1, is equal to $\sigma \cdot \omega(\gamma) + 1$, where σ is equal to 1 (resp. -1) if γ' meets itself at p from the right (resp. left) hand side.
Let γ be a finite closed arc. It follows from Lemma II.1.19, that the degree d of a morphism $\mathfrak{B}(p) \in \mathrm{Hom}(X, X[d])$ associated to the unique boundary self-intersection p of γ is characterized as the degree of *every* non-invertible boundary morphism from X to any shift of X.

Corollary II.1.22. *Let $(\mathcal{F}, \mathcal{S}_{\mathcal{F}}, \omega_{\mathcal{F}}), (\mathcal{F}', \mathcal{S}_{\mathcal{F}'}, \omega_{\mathcal{F}'})$ be Fukaya-like triples and let $T : \mathcal{F} \to \mathcal{F}'$ be a triangle equivalence. Assume that*

- *$\gamma \subset \mathcal{S}_{\mathcal{F}}$ is a closed finite arc in minimal position;*
- *$X \in \mathrm{Perf}(\mathcal{F})$, such that $\gamma \in \gamma(X)$;*
- *$\gamma' \in \gamma(T(X))$ is in minimal position, such that both γ and γ' intersect themselves at the boundary from the same side.*

Then, $\omega_{\mathcal{F}}(\gamma) = \omega_{\mathcal{F}'}(\gamma')$, where we consider γ and γ' as loops in the natural way.

We will refine the statement of Corollary II.1.22 in the subsequent sections.

II.2 Homeomorphisms induced by auto-equivalences

This section is mainly concerned with the construction of homeomorpisms from triangle equivalences between Fukaya-like categories. It is based on the study of automorphisms of the so-called arc complex of a surface and their connection to homeomorphisms of the surface as studied in [38]. After discussing the observation that being a loop object is not preserved under triangle equivalences, we propose a definition of families of τ-invariant objects, which remedy this failure (as shown in Theorem II.3.2). Throughout this section, let $\mathcal{S} = (S, \overline{\mathcal{M}})$ and $\mathcal{S}' = (S', \overline{\mathcal{M}}')$ denote marked surfaces.

A **homeomorphism of marked surfaces** $F : \mathcal{S} \to \mathcal{S}'$, is a homeomorphism $F : S \to S'$, which restricts to a bijection $\overline{\mathcal{M}} \to \overline{\mathcal{M}}'$.

II.2.1 Arc complexes and essential objects

Before we give the definition of an arc complex, we recall the necessary definitions.

Definition II.2.1. An (abstract) **simplicial complex** is a set V and a collection K of finite subsets of V, such that for all $v \in V$, $\{v\} \in K$ and if M is an element of K, then so is every subset of M. The sets $\{v\}$ are called the **vertices** of K. An m-simplex of K is a set $M \in K$ with $m+1$ elements. The 1-simplices are also called **edges**.

Remark II.2.2. To every simplicial complex, one can associate a topological space. For example, in this construction every vertex becomes a point and every 1-simplex becomes a line. Likewise every 2-simplex becomes a triangle and every 3-simplex a tetrahedron. These elementary topological spaces are then glued according to the subset relationship between the simplices. For example, two edges E_1, E_2 are glued at one of their endpoints q if $\{v\} = E_1 \cap E_2$ and in this case q is the point associated to the vertex $\{v\}$.

Next, we present the definition of an arc complex and its building blocks, essential arcs.

Definition II.2.3 (see [38]). A finite arc on $\mathcal{S}$ is called **essential** if it is simple, i.e. has no interior self-intersections, and it is not homotopic to a boundary segment connecting consecutive marked points. The **arc complex** $A_* = A_*(\mathcal{S})$ of $\mathcal{S}$ is the cell complex defined as follows. The vertices of A_* are in one-to-one correspondence with homotopy classes of essential arcs on $\mathcal{S}$. A set of distinct vertices $\{v_0, \ldots, v_n\}$ of A_* is contained in an n-simplex if and only if there exists a set of simple representatives γ_i of v_i, such that γ_i and γ_j have no interior intersections for all $i \neq j$.

Note that $A_*(\mathcal{S})$ is empty if and only if S is a disc and $\overline{\mathcal{M}}$ contains at most 3 marked points but no punctures.

Remark II.2.4. It is a theorem that two essential arcs are homotopic if and only if they are isotopic, i.e. there exists a continuous family of homeomorphisms, which restricts to a homotopy of the given arcs. For a prove of the related case of loops without self-intersections see [42].

Remark II.2.5. It follows from [41], Theorem 3.1 that $A_*(\mathcal{S})$ is a *flag complex*, i.e. $n+1$ vertices of $A_*(\mathcal{S})$ constitute an n-simplex if and only if they are pairwise connected by an edge.

Isomorphisms of arc complexes and homeomorphisms

If $A_*(\mathcal{S})$ is non-empty, then homeomorphisms of $\mathcal{S}$ act naturally on $A_*(\mathcal{S})$ by simplicial automorphisms. A **simplicial automorphism** of a simplicial complex K over a set V is a bijection $F: V \to V$, such that $F(M) \in K$ if and only if $M \in K$. Isomorphisms between simplicial complexes are defined in the analogous way.
Given any homeomorphism $F: \mathcal{S} \to \mathcal{S}'$, it induces a bijection between homotopy classes of essential arcs on $\mathcal{S}$ and essential arcs on $\mathcal{S}'$, which, by the previous remark, can be extended to a simplicial isomorphism $A_*(F): A_*(\mathcal{S}) \to A_*(\mathcal{S}')$. It is clear that $A_*(F)$ does not change if we replace F by an isotopic homeomorphism.
Let $\mathrm{Homeo}(\mathcal{S}, \mathcal{S}')$ denote the equivalence classes of homeomorphisms $\mathcal{S} \to \mathcal{S}'$ modulo isotopies, which fix marked points. For two simplicial complexes C, C' denote by $\mathrm{Simp}(C, C')$ the set of simplicial isomorphisms. The assignment $F \mapsto A_*(F)$ determines a map

$$\Phi: \mathrm{Homeo}(\mathcal{S}, \mathcal{S}') \longrightarrow \mathrm{Simp}(A_*(\mathcal{S}), A_*(\mathcal{S}')),$$

which is a homomorphism of groups if $\mathcal{S} = \mathcal{S}'$ with multiplication being given by composition. If $\mathcal{S} = \mathcal{S}'$, the subgroup $\mathcal{MCG}(\mathcal{S}) \subset \mathrm{Homeo}(\mathcal{S}, \mathcal{S})$ consisting of classes of all orientation preserving homeomorphisms is called the **mapping class group** of $\mathcal{S}$.

A natural question to ask is whether Φ is surjective or injective and an answer to this question is given in [38]. In order to state it, it is convenient to introduce the following definition.

Definition II.2.6. We call a marked surface $\mathcal{S}$ **special** if its arc complex is empty or is of dimension at most 1.

Theorem II.2.7 (Theorem 1.1 & Theorem 1.2, [38])**.** *Let $\mathcal{S}, \mathcal{S}'$ be marked surfaces, such that $A_*(\mathcal{S}), A_*(\mathcal{S}') \neq \emptyset$. Then Φ is surjective. If $\mathcal{S}$ is non-special then Φ is a bijection.*

We write $\overline{\Phi}$ for the induced bijective map

$$\overline{\Phi} : \mathrm{Homeo}(\mathcal{S}, \mathcal{S}')/_{\sim} \longrightarrow \mathrm{Simp}(A_*(\mathcal{S}), A_*(\mathcal{S}')),$$

where $H \sim H'$ if and only if $\Phi(H) = \Phi(H')$.
We discuss special marked surfaces and provide a complete list of such in Section II.4.1.

Essential objects

The main ingredient of our approach to study equivalences is the construction of a map which associates with every exact equivalence $T : \mathcal{F} \to \mathcal{F}'$ an element

$$A_*(T) \in \mathrm{Simp}(A_*(\mathcal{S}), A_*(\mathcal{S}')).$$

The idea is to find a homeomorphism $\mathcal{S} \to \mathcal{S}'$, which realizes the action of T on isomorphism classes in $\mathcal{F}/[1]$ in terms of its action on homotopy classes on curves. This is the approach we pursue below.

For $X, Y \in \mathcal{F}$, denote $\mathrm{Hom}(X, Y)_{\mathrm{Perf}(\mathcal{F})}$ the set of all $f : X \to Y$, such that f factors through $\mathrm{Perf}(\mathcal{F})$. Since $\mathrm{Perf}(\mathcal{F})$ is closed under finite direct sums, it is a subvector space of $\mathrm{Hom}(X, Y)$. As usual, we write $\mathrm{Hom}^*(X, Y)_{\mathrm{Perf}(\mathcal{F})}$ as the direct sum of all $\mathrm{Hom}(X, Y[i])_{\mathrm{Perf}(\mathcal{F})}$ ($i \in \mathbb{Z}$). If X or Y is an arc object, then by Lemma II.1.18, $\mathrm{Hom}^*(X, Y)_{\mathrm{Perf}(\mathcal{F})}$ is spanned by all basis elements corresponding to intersections different from punctures.
In order to define $A_*(T)$, we need to consider an algebraic counterpart of essential arcs. It is defined as follows.

Definition II.2.8. Let $X \in \mathcal{F}$ be an arbitrary indecomposable. Then X is called **essential** if it satisfies the following conditions.

1) The dimension of $\mathrm{Hom}^*(X, X)$ is at most 2, if $X \in \mathrm{Perf}(\mathcal{F})$, and if $X \notin \mathrm{Perf}(\mathcal{F})$, then the dimension of $\mathrm{Hom}^*(X, X)_{\mathrm{Perf}(\mathcal{F})}$ is at most one.

2) If $X \in \mathrm{Perf}(\mathcal{F})$, then the middle term Z of any Auslander-Reiten triangle

$$X \longrightarrow Z \longrightarrow \tau^{-1}(X) \longrightarrow X[1]$$

is decomposable.

It follows immediately that if $X \in \mathcal{F}$ is essential and $T : \mathcal{F} \to \mathcal{F}'$ a triangle equivalence, then $T(X)$ is essential. Our first observation about essential objects is the following.

Lemma II.2.9. *Let $X \in \operatorname{Perf}(\mathcal{F})$ be essential. Then $\tau X \ncong X$ in $\mathcal{F}/[1]$. In particular, essential objects are arc objects.*

Proof. Suppose $\tau X \cong X[n]$ for some $n \in \mathbb{Z}$. By Lemma II.1.10, X must be represented by a loop or a boundary arc γ contained in a boundary component with a single marked point. Suppose X is represented by a loop. By condition 2) in Definition II.2.8 and condition 3) in Definition II.1.1, it has to sit at the base of its homogeneous tube, as otherwise $\dim \operatorname{Hom}^*(X, X) > 2$. However, this violates condition 2). If γ is an arc, it must be homotopic to a simple arc because of condition 1). Again, this implies that X violates condition 2). □

Corollary II.2.10. *Let $X \in \mathcal{F}$ be an arbitrary indecomposable. Then X is essential if and only if it is represented by an essential arc.*

Proof. If X is represented by a simple arc, then $\dim \operatorname{Hom}^*(X, X) \leq 2$, if X is perfect, due to the at most two boundary intersections. If X is not perfect, then none of its (graded) endomorphisms factors through $\operatorname{Perf}(\mathcal{F})$ by Lemma II.1.18 and hence $\operatorname{Hom}^*(X, X)_{\operatorname{Perf}(\mathcal{F})} = 0$. The condition that an arc object X' is represented by a boundary segment connecting consecutive marked points is equivalent to condition 2) in Definition II.2.8. By Lemma II.2.9, this proves that every essential object is represented by an essential arc.
For the converse, note that every interior self-intersection of a curve γ gives rise to two distinct intersections of γ and γ', if $\gamma \simeq \gamma'$ and $\{\gamma, \gamma'\}$ is in minimal position. In particular, the presence of an interior self-intersection forces $\dim \operatorname{Hom}^*(X, X)_{\operatorname{Perf}(\mathcal{F})} \geq 2$. □

Next, we define the map

$$A_*(-) : \operatorname{Equ}(\mathcal{F}, \mathcal{F}') \longrightarrow \operatorname{Simp}(A_*(\mathcal{S}), A_*(\mathcal{S}')),$$

where $\operatorname{Equ}(\mathcal{F}, \mathcal{F}')$ denotes the class of equivalence classes of k-linear triangle equivalences modulo natural isomorphisms. The set $\operatorname{Aut}(\mathcal{F}) := \operatorname{Equ}(\mathcal{F}, \mathcal{F})$ is a group with multiplication given by composition.

Definition II.2.11. Let $(\mathcal{F}, \mathcal{S}, w), (\mathcal{F}', \mathcal{S}', w')$ be Fukaya-like triples and $T : \mathcal{F} \to T'$ a triangle equivalence. Define $A_*(T) \in \operatorname{Simp}(A_*(\mathcal{S}), A_*(\mathcal{S}'))$ as the unique simplicial isomorphism, which sends a vertex $\gamma(X) \in A_*(\mathcal{S})$, $X \in \mathcal{F}$ essential, to $\gamma(T(X)) \in A_*(\mathcal{S}')$.

Lemma II.2.12. *Let $(\mathcal{F}, \mathcal{S}, w), (\mathcal{F}', \mathcal{S}', w')$ be Fukaya-like triples, such that $A_*(\mathcal{S}) \neq \emptyset$. Then, $A_*(-)$ is well-defined and is a group homomorphism if $\mathcal{S} = \mathcal{S}'$.*

Proof. Let γ be an essential arc. By Corollary II.2.10, γ is represented by the isomorphism class of essential objects $X_\gamma \in \mathcal{F}/[1]$. Let $T : \mathcal{F} \to T'$ be a triangle equivalence. Then, every $T(X_\gamma)$ is essential and hence represented by a homotopy class of essential arcs. The assignment $\gamma \mapsto \gamma(T(X_\gamma))$ gives rise to a well defined (invertible) map between the vertices of $A_*(\mathcal{S})$ and $A_*(\mathcal{S}')$. We want to show that this map extends to a simplicial isomorphism $A_*(T) : A_*(\mathcal{S}) \to A_*(\mathcal{S}')$. Since $A_*(-)$ is a flag complex (see Remark II.2.5)it is sufficient to prove that the condition that γ and γ' are connected by an edge in $A_*(\mathcal{S})$ is equivalent to the condition that $\gamma(T(X_\gamma))$ and $\gamma(T(X_{\gamma'}))$ are connected by an edge of $A_*(\mathcal{S}')$.
By Lemma II.1.19, γ and γ' are connected by an edge if and only if the space of interior morphisms $\mathrm{Hom}^*_{\mathrm{Int}}(X_\gamma, X_{\gamma'})$ trivial. Thus, the equivalence of the two conditions follows from Lemma II.1.17, which implies that if $X, Y \in \mathcal{F}$ are essential, then $\mathrm{Hom}^*_{\mathrm{Int}}(X, Y) = 0$ if and only if $\mathrm{Hom}^*_{\mathrm{Int}}(T(X), T(Y)) = 0$ is.
Note that $A_*(T)$ is homotopic to the identity if T is isomorphic to the identity functor, proving that $A_*(-)$ is a well-defined map $\mathrm{Equ}(\mathcal{F}, \mathcal{F}') \to \mathrm{Simp}\left(A_*(\mathcal{S}), A_*(\mathcal{S}')\right)$. The second assertion follows immediately from the definition of $A_*(-)$. $\square$

Given an equivalence $T \in \mathrm{Equ}(\mathcal{F}, \mathcal{F}')$ and assuming that $\mathcal{S}$ is non-special, we set $\Psi(T) \coloneqq \Phi^{-1} \circ A_*(T)$. This defines the desired map Ψ in case $A_*(\mathcal{S}) \neq \emptyset$. We discuss how to extend Ψ to the case of special surfaces in Section II.4.1.

As this is a composition of group homomorphisms in case $\mathcal{S} = \mathcal{S}'$, the second assertion of Theorem A follows immediately. By construction, Ψ satisfies the following.

Corollary II.2.13. *Let $(\mathcal{F}, \mathcal{S}, w), (\mathcal{F}', \mathcal{S}', w')$ be Fukaya-like triples and let $T : \mathcal{F} \to \mathcal{F}'$ be a triangle equivalence. Then for all $H \in \Psi(T)$, we have that $H(\gamma(X)) = \gamma(T(X))$ for all essential objects $X \in \mathcal{F}$.*

We will discuss in Section II.2.2 to what extent the assertion of Corollary II.2.13 holds true for other objects in a Fukaya-like category.

As a second an important consequence of the existence of Ψ, we obtain that the surface of a gentle algebra is a derived invariant.

Corollary II.2.14. *Let $A = kQ/I$ and $A' = kQ'/I'$ be gentle algebras and $\mathcal{S}_A$, $\mathcal{S}_{A'}$ their associated marked surfaces (see Chapter I). If A and A' are derived equivalent, then $\mathcal{S}_A \cong \mathcal{S}_{A'}$.*

II.2.2 Families of τ-invariant objects

It seems natural to expect that the assertion of Corollary II.2.13 holds true for all objects of a Fukaya-like triangulated category. However, in the presence of τ-invariant arc objects this might not be the case, as illustrated by the Kronecker quiver.

Example II.2.15. Let

$$Q = \qquad \begin{tikzcd} x \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & y, \end{tikzcd}$$

denote the Kronecker quiver and let A denote its associated (gentle) path algebra. Its surface S_A is a cylinder with one marked point on each boundary component and $\omega_A = 0$. There exists a natural embedding $\mathrm{PGL}_2(k) \hookrightarrow \mathrm{Aut}(\mathcal{D}^b(A))$, the elements of the image of which we call **coordinate transformations**. They are defined as follows. Let

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(k)$$

be an invertible matrix. Denote $\sigma = \sigma_M$ the k-linear automorphism of A, such that

$$\begin{array}{rcl} \sigma(x) & = & x \\ \sigma(y) & = & y \\ \sigma(\alpha) & = & a \cdot \alpha + c \cdot \beta \\ \sigma(\beta) & = & b \cdot \alpha + d \cdot \beta. \end{array}$$

Denote ${}_{\sigma}A$ the A-bimodule which as a right module is the regular A-module and as a left module is the regular left A-module twisted by σ, i.e. $a \in A$ acts on $x \in {}_{\sigma}A = A$ by $a.x := \sigma(a) \cdot x$. The derived functor ${}_{\sigma_M}A \otimes^{\mathbb{L}} -$ defines an element in $\mathrm{Aut}(\mathcal{D}^b(A))$ and two of such functors associated to σ_M and $\sigma_{M'}$ are isomorphic if and only if $M = \lambda \cdot M'$ for some $\lambda \in k^{\times}$. Consider the family of A-modules

$$B_{\lambda,\mu} = \qquad \begin{tikzcd} k \arrow[r, bend left, "\cdot\lambda"] \arrow[r, bend right, "\cdot\mu"'] & k, \end{tikzcd}$$

where $\lambda, \mu \in k$ are not both zero. Then $B_{\lambda,\mu}$ is a band object if and only if $\lambda, \mu \neq 0$ and, in this case, is represented by the primitive loop equipped with a 1-dimensional indecomposable local system. The string complexes $B_{0,\mu}$ and $B_{\lambda,0}$ are represented by any of the two closed arcs, which are homotopic (as loops) to the primitve loop. It is not difficult to see that $\mathrm{GL}_2(k)$ acts transitively on the family $B_{\lambda,\mu}$ in this way.

The previous example shows that loop objects may be sent to 'degenerations' of themselves implying that whether a τ-invariant object is represented by an arc or a loop can not be distinguished within the triangulated category.

However, we will show that this is the worst case and that every triangle equivalence preserves families of isomorphism classes of τ-invariant objects consisting of band objects and their degenerations.
The above considerations motivate the following definitions.

Definition II.2.16. Let $(\mathcal{F}, \mathcal{S}, \overline{\mathcal{M}})$ be a Fukaya-like triple and $\mathcal{S} = (S, \overline{\mathcal{M}})$. Then the equivalence relation $\sim_*$ is defined as the relation which identifies a gradable boundary curve γ with image in a boundary component $B \subseteq S$ with $|B \cap \overline{\mathcal{M}}\,| = 1$ with the simple closed boundary arc with image in B which is homotopic to γ as a loop. The relation $\simeq_*$ is defined as the equivalence relation on the set of curves on S generated by homotopy and $\sim_*$.

Given an object $X \in \mathcal{F}$, we write $\gamma_*(X)$ for the equivalence class of curves in $\gamma(X)$ with respect to $\simeq_*$.

Definition II.2.17. A **family of τ-invariant objects** in $\mathcal{F}$ is a collection $\mathcal{X} = \{X_i\}_{i \in I} \subset \mathcal{F}$ of indecomposable τ-invariant objects, such that there exists an $\simeq_*$-equivalence class of curves $\gamma_*(\mathcal{X})$ which contains $\gamma_*(X_i)$ for all $i \in I$.
Such a family is further called **complete** if for every τ-invariant indecomposable object $X \in \mathcal{F}$, which is represented by a curve in $\gamma_*(\mathcal{X})$, there exists $i \in I$, such that $X \cong X_i$.
An arc object in $\{X_i\}_{i \in I}$ is called a **degenerated object** of the family.

Every $\simeq_*$-equivalence class of a fixed loop γ contains at most two distinct homotopy classes of closed boundary arcs.
This suggests the following trichotomy for families of τ-invariant objects.

Definition II.2.18. Let $\mathcal{X}$ be a complete family of τ-invariant objects.
It is called

- $k^{\times}$**-family** if $\gamma_*(\mathcal{X})$ contains no homotopy classes of arcs;
- $\mathbb{A}^1_k$**-family** if $\gamma_*(\mathcal{X})$ contains precisely one homotopy class of arcs;
- $\mathbb{P}^1_k$**-family** if $\gamma_*(\mathcal{X})$ contains two distinct homotopy classes of arcs.

The following lemma shows that the presence of $\mathbb{P}^1_k$-families is a very rare phenomenon.

Lemma II.2.19. *A $\mathbb{P}^1_k$-family of τ-invariant objects exists if and only if $(S, \overline{\mathcal{M}})$ is a cylinder with a single marked point on each boundary component and $\omega = 0$. There exists at most one such family (up to isomorphisms of objects).*

Proof. Note that the cylinder is the only compact surface with two distinct but homotopic boundary components meaning that the corresponding simple boundary curves (or their inverses) are homotopic. Thus, the assertion follows from Lemma II.1.10. $\square$

In case $\mathcal{F}$ arises from a gentle algebra we can prove even more.

Lemma II.2.20. *Let $A = kQ/I$ be a gentle algebra, such that $\omega_A = 0$ and that S_A is homeomorphic to a cylinder with a single marked point on each boundary component. Then, Q is the Kronecker quiver.*

Proof. Let $\phi_A : \mathbb{N}^2 \to \mathbb{Z}$ denote the Avella-Alaminos-Geiss invariant of A, see Section I.6.1. By Theorem I.6.1 (see also Theorem 3.2.2 in [58]), it then follows that $\sum_{t \in \mathbb{N}^2} \phi_A(t)$ coincides with the number of boundary components of S_A and, by Remark 8 in [11], coincides with the number of arrows in Q. Thus, $|Q_1| = 2$. Since A is connected and the fundamental group of Q is $\pi_1(S_A) \cong \mathbb{Z}$ (Proposition I.1.20), Q has precisely two vertices. Furthermore, A has finite global dimension because every boundary component contains at least one marked point. This implies that Q contains no loops. Altogether we conclude that the only gentle algebras in question for A are the Kronecker algebra and gentle algebras given by quotients of kQ, where Q is the quiver

$$\bullet \leftrightarrows \bullet.$$

Quotients of Q are derived discrete (no band complexes), whereas the Kronecker algebra is not. In addition, one can see easily that their surfaces do not coincide with S_A. $\square$

II.3 Triangulations and characteristic sequences of objects

The main result of this section is a generalization of Corollary II.2.13. It asserts the statement of Theorem A 1 c) of the introduction.

Theorem II.3.1. *Let $u = (\mathcal{F}, \mathcal{S}, w)$ and $u' = (\mathcal{F}', \mathcal{S}', w')$ be Fukaya-like triples and $T : \mathcal{F} \to \mathcal{F}'$ be a triangle equivalence. If $\mathcal{S}$ is non-special, then for every linear indecomposable $X \in \mathcal{F}$,*

$$\Psi(T)(\gamma_*(X)) = \gamma_*(T(X)).$$

The proof of Theorem II.3.1 is given in Section II.3.2. We extend its result to special surfaces in Section II.4.1. As a corollary of Theorem II.3.1 we obtain:

Corollary II.3.2. *Let u, u' and T be as in Theorem II.3.1. Suppose $\mathcal{X}$ is an I-family of τ-invariant objects as defined in Section II.2.2, where $I \in \{k^\times, \mathbb{A}^1_k, \mathbb{P}^1_k\}$. Then $T(\mathcal{X})$ is an I-family of τ-invariant objects.*

Strategy of the proof of Theorem II.3.1

Our strategy for the proof of is to reduce the general case to the case of essential objects, which is already covered by Corollary II.2.13. We achieve this in the following way.
It is well-known that the homotopy class of a curve is uniquely determined by its sequence of (signed) intersections with a triangulation assuming that the edges of the triangulation and the curves are in minimal position. Pursuing the approach that homeomorphism invariant geometric properties correspond to properties which are invariant under triangle equivalences, we will define the notion of a *triangulation* of $\mathcal{F}$ and translate intersections into well-behaved *characteristic sequences of morphisms* between X and objects in a triangulation. We show that these sequences are sufficient to recover the isomorphism class of an object X up to shift, if X is not τ-invariant, and sufficient to recover the corresponding family of τ-invariant objects otherwise. Although the idea is straightforward, there is no reason for $\gamma(-)$ and the correspondence between morphisms and intersections to be canonical. This failure requires a rather careful choice of (technical) assumptions on such sequences of morphisms. However, we hope to convey the idea behind all of this along the way.

II.3.1 Triangulations of Fukaya-like categories

We introduce the notion of a triangulation of a Fukaya-like category which mimicks the concept of a triangulation of a surface.
By a **triangulation** of a marked surface $\mathcal{S} = (S, \overline{\mathcal{M}})$, we mean a collection of homotopy classes of essential arcs on $\mathcal{S}$, which form the vertices of a simplex in $A_*(\mathcal{S})$ of maximal dimension. The arcs of a triangulation cut S into triangles. Note that besides embedded triangles this possibly includes self-folded triangles. For technical reasons we will restrict ourselves to particular types triangulations of the following kind. Let Δ be a triangulation and denote $\Delta' \subseteq \Delta$ its subset of finite arcs. The triangulation Δ is said to

separate punctures if Δ' is a triangulation of $(S, \overline{\mathcal{M}} \cap \partial S)$ and each triangle of Δ' contains at most one puncture.

Lemma II.3.3. *Let Δ be a triangulation of a marked surface $(S, \overline{\mathcal{M}})$, such that the set of finite arcs in Δ is a triangulation of $(S, \overline{\mathcal{M}} \cap \partial S)$. Then Δ separates punctures if and only if Δ contains no infinite arc, both end points of which are punctures.*

Proof. Let γ be an infinite arc of Δ. Since γ does not cross any other arc of Δ anywhere except at its end points, γ is contained in a single triangle of Δ', where $\Delta' \subset \Delta$ denotes the subset of finite arcs. In particular, if Δ contains an arc both end points of which are punctures, it follows that Δ does not separate punctures. Conversely, suppose that Δ does not separate punctures. Thus, there exists a polygon P of Δ', which contains at least two distinct punctures. By definition they lie in the interior of P. Let p be a puncture in P and let $\Delta'' \subseteq \Delta$ denote the set of arcs every end point of which is different from p. As $\Delta' \subseteq \Delta''$, Δ'' cuts S into (possibly folded) polygons. Denote Q the polygon of Δ'' which contains p. By definition, p is the only puncture in the interior of Q. We further observe that Q can not be a polygon of Δ', as otherwise $P = Q$ proving that P contains only a single puncture – a contradiction. In particular, the boundary of Q must contain a puncture. Since Δ is a triangulation, the set $\Delta \setminus \Delta''$ must contain all arcs connecting p with any marked point on the boundary of Q proving that Δ contains an arc both end points of which are punctures. $\square$

A well-known property of triangulations is that they allow us to translate homotopy classes of curves into sequences of their intersections with the triangulation. We state it in the following Lemma without further proof.

Lemma II.3.4. *Let Δ be a triangulation of a marked surface $(S, \overline{\mathcal{M}})$ and let γ be a curve, such that $\Delta \cup \{\gamma\}$ is in minimal position and such that γ is not homotopic to an arc in Δ. Then the following is true.*

1. *The homotopy class of γ is uniquely determined by the linear or cyclic sequence of interior intersections of γ with arcs in Δ.*

2. *If $E_{i_1}, \dots, E_{i_m}$ is a linear or cyclic sequence of elements in Δ, such that for each j, E_{i_j} and $E_{i_{j+1}}$ are contained in the boundary of a single triangle, then there exists a unique homotopy class of curves γ, such that $E_{i_1}, \dots, E_{i_m}$ is the sequence of interior intersections of γ with Δ.*

Next, we introduce an algebraic counterpart of triangulations in the context of Fukaya-like categories.

Definition II.3.5. Let $(\mathcal{F}, S, w)$ be a Fukaya-like triple. A **triangulation** of $\mathcal{F}$ is a set $\mathcal{X} = \{X_i \,|\, i \in I\} \subset \mathcal{F}$ of essential objects with the following properties.

1) For all $i, j \in I$, $\mathrm{Hom}^*_{\mathrm{Int}}(X_i, X_j) = 0$ and $X_i \ncong X_j$ in $\mathcal{F}/[1]$ for all $i \neq j$.

2) $\mathcal{X}$ is maximal among all sets of essential objects satisfying 1).

A triangulation $\mathcal{X} \subset \mathcal{F}$ **separates punctures** if further

a) $\mathcal{X} \cap \mathrm{Perf}(\mathcal{F})$ is a maximal subset of essential objects in $\mathrm{Perf}(\mathcal{F})$ satisfying condition 1) above and,

b) for all $X \in \mathcal{X}$, X is not the source of a irreducible morphism.

The first observation is that if $\mathcal{X} \subset \mathcal{F}$ is a triangulation and $T : \mathcal{F} \to \mathcal{F}'$ is a triangle equivalence between Fukaya-like categories, then $T(\mathcal{X})$ is a triangulation and $\mathcal{X}$ separates punctures if and only if $T(\mathcal{X})$ separates punctures. Further, Lemma II.1.19 and the fact that no irreducible morphism starts in an object represented by an arc which starts and ends at punctures, imply the following.

Lemma II.3.6. *Let $(\mathcal{F}, S, w)$ be a Fukaya-like triple. A collection of indecomposable objects $\mathcal{X} \subseteq \mathcal{F}$ is a triangulation if and only if the associated collection Δ of homotopy classes is a triangulation of S. Moreover, $\mathcal{X}$ separates punctures if and only if Δ separates punctures.*

Every marked surface $(S, \overline{\mathcal{M}})$ with at least one marked point on the boundary admits a triangulation which separates punctures. First, note that the marked surface $(S, \overline{\mathcal{M}} \cap \partial S)$ has a triangulation Δ. As Δ cuts S into topological discs, the proof of the existence of a triangulation of $(S, \overline{\mathcal{M}})$ which separates punctures reduces to the assertion that any marked disc admits such a triangulation. The proof of the latter is left as an exercise. We obtain:

Corollary II.3.7. *Every Fukaya-like triangulated category admits a triangulation which separates punctures.*

II.3.2 Characteristic sequences

The next concept we introduce are so-called *characteristic sequences* of an object. These are sequences of morphisms which allow us to recover the isomorphism class (or an associated family of τ-invariant objects) of an object in $\mathcal{F}$ but before we do so, we want to explain the idea that underlies their

definition.
As before and unless stated otherwise, we assume that every indecomposable object, which appears in any of the subsequent sections of the present chapter, is either a string object or a linear loop object, i.e. a loop object associated to a linear local system.

The intuition behind characteristic sequences Suppose γ is a gradable loop or arc on S, Δ is a triangulation of S, such that $\Delta \cup \{\gamma\}$ is in minimal position, and p_1 and p_2 are consecutive interior intersections of γ with edges $E_1, E_2 \in \Delta$. There exists a unique intersection q of E_1 and E_2, such that p_1, p_2 and q lift to an embedded triangle in the universal cover of S. Assuming that $q \in E_1 \overrightarrow{\cap} E_2$ and regarding p_i as an element in $\gamma \overrightarrow{\cap} E_i$, we have $\mathfrak{B}(q) \circ \mathfrak{B}(p_1) = \lambda \cdot \mathfrak{B}(p_2)$ for some $\lambda \neq 0$ as implied by Definition II.1.1. Similarly, regarding p_i as an element in $E_i \overrightarrow{\cap} \gamma$, we have $\mu \cdot \mathfrak{B}(p_1) = \mathfrak{B}(p_2) \circ \mathfrak{B}(q)$ for some $\mu \neq 0$. It suggest that our desired sequence of morphisms should be such that every two consecutive morphisms in this sequence are related via composition with a morphism between the corresponding objects of a triangulation.

Arrow morphisms As there might be several boundary intersections in $E_1 \overrightarrow{\cap} E_2$ we have to distinguish morphisms between essential objects which are multiples of basis elements and proper linear combinations of these.
This motivates the following definition.

Definition II.3.8. Let $f : X \to Y[m]$ be a non-zero morphism between distinct objects X, Y of a triangulation. The morphism f is called **pure** if it is a multiple of $\mathfrak{B}(p)$ for some intersection p of representing curves of X and Y. A pure morphism $f : X \to Y[m]$ is called an **arrow morphism** of $\mathcal{X}$ if f does not factor through a pure morphism $g : X \to X'[n]$ where $X' \in \mathcal{X} \setminus \{X, Y\}$.

So far it is not clear if the notion of pureness is independent of the chosen representing curves. The following is a direct consequence of Definition II.3.8

Lemma II.3.9. *Let $\mathcal{X}$ be a triangulation. Arrow morphisms between objects $X, Y \in \mathcal{X}$ are precisely the multiples of morphisms $\mathfrak{B}(p)$, where p is the corner of a triangle in the corresponding triangulation of S.*

Our goal is to prove that being pure is invariant under triangle equivalences. It follows from the subsequent lemmas. We distinguish between perfect and non-perfect objects.

Lemma II.3.10. *Let $f : X \to Y[m]$ be a non-zero morphism between distinct objects X, Y of a triangulation. If $X, Y \in \mathrm{Perf}(\mathcal{F})$, then f is pure if and only if there exist segment objects X', Y' and non-zero morphisms $g : X' \to X$ and $h : Y[m] \to Y'$, such that $f \circ g = 0 = h \circ f$.*

Proof. There is nothing to show if $|\gamma_X \overrightarrow{\cap} \gamma_Y| = 1$. Assume therefore that $q, q' \in \gamma_X \overrightarrow{\cap} \gamma_Y$ are distinct intersections. In means that in a universal cover of S, there exists a lift $\widetilde{\gamma}_X$ of γ_X and lifts of γ_Y arranged as in Figure II.6

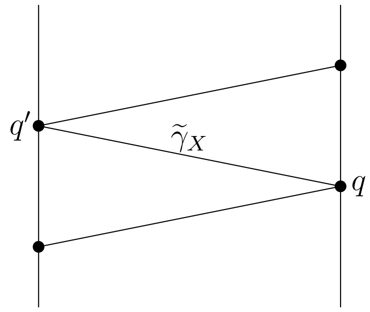


Figure II.6

We like to stress that the boundaries containing q and q' might coincide. However in that case, q and q' are not consecutive marked points. As the situation is analogous for q', it is sufficient to prove the assertion for $\mathfrak{B}(q)$. Let δ denote the boundary arc connecting q' with its predecessor in $\overline{\mathcal{M}}$. Then δ and $\widetilde{\gamma}_X$ have a unique intersection $p \in \widetilde{\gamma}_Y \overrightarrow{\cap} \delta$. As δ and $\widetilde{\gamma}_Y$ lie on different sides of $\widetilde{\gamma}_X$ it follows from Definition II.1.1 4) that $\mathfrak{B}(p) \circ \mathfrak{B}(q) = 0$. Similarly, if p' denotes the unique intersection in $\delta' \overrightarrow{\cap} \widetilde{\gamma}_X$, where δ' is the boundary arc connecting q' and its successor in $\overline{\mathcal{M}}$, then $\mathfrak{B}(q) \circ \mathfrak{B}(p') = 0$. This proves that every multiple of $\mathfrak{B}(q)$ and $\mathfrak{B}(q')$ satisfies the condition of the Lemma. To prove the other implication, note that p and q' give rise to a fork (see property 4) in Definition II.1.1) and hence $\mathfrak{B}(p) \circ \mathfrak{B}(q')$ is a multiple of the morphisms associated to the unique intersection $\widetilde{\gamma}_X \overrightarrow{\cap} \delta$. Similarly, if ϵ denotes the boundary arc which connects q and its predecessor and $u \in \widetilde{\gamma}_Y \overrightarrow{\cap} \epsilon$ is the unique intersection, then $\mathfrak{B}(u) \circ \mathfrak{B}(q)$ is a multiple of $\mathfrak{B}(u')$, where $u' \in \widetilde{\gamma}_X \overrightarrow{\cap} \epsilon$ denotes the unique intersection.
Now suppose $g : Y \to Y'$ is a morphism to a segment object Y' and $g \circ f = 0$. Then Y' is represented by δ or ϵ and by what we said before, $g \circ f = 0$ implies that f is a multiple of $\mathfrak{B}(q)$ or a multiple of $\mathfrak{B}(q')$. □

Lemma II.3.11. *Let $f : X \to Y[m]$ be a non-zero morphism between distinct objects X, Y of a triangulation, which separates punctures, such that X or Y*

is not an object of $\mathrm{Perf}(\mathcal{F})$*. Then* f *is pure if and only if one of the following conditions is satisfied:*

i) f *factors through* $\mathrm{Perf}(\mathcal{F})$*;*

ii) f *does not factor through* $\mathrm{Perf}(\mathcal{F})$ *and there exists a segment object* U *and a morphism* $g : U \to X$*, such that* $\mathrm{Hom}(g, Y) \neq 0$ *and* $f \circ g = 0$*.*

Proof. Let γ_X and γ_Y be representing arcs of X and Y in minimal position. By symmetry of the following arguments we may assume that γ_X is an infinite arc. In particular, γ_X and γ_Y share at most one boundary point p and, since the triangulation separates punctures, at most one puncture q. It follows from Lemma II.1.18 that condition i) is equivalent to the condition that f is a multiple of $\mathfrak{B}(p)$.
We show that the second condition is equivalent to f being a multiple of $\mathfrak{B}(q)$. As in the proof of Lemma II.3.10, we see that there exists a segment object U represented by a boundary segment, which connects p and its successor, and a morphism $g : U \to X$ (unique up to a scalar), such that $h \circ g \neq 0$ for a morphism $h : X \to \tau U[1]$. Write $f = \lambda \cdot \mathfrak{B}(q) + \mu \cdot \mathfrak{B}(p)$. Consequently, $g \circ f = 0$ if and only if $\mu = 0$ if and only if f is a multiple of $\mathfrak{B}(q)$. This finishes the proof. □

We showed that being a pure morphism attached to a triangulation can be phrased in terms of the triangulated category. As usually, one shows:

Corollary II.3.12. *Let* X, Y *be distinct objects in a triangulation of* $\mathcal{F}$ *(resp.* $\mathrm{Perf}(\mathcal{F})$*) which separates punctures. Let* $f : X \to Y$ *be a morphism and* $T : \mathcal{F} \to \mathcal{F}'$ *a triangle equivalence. Then* $T(f)$ *is pure (resp. an arrow morphism) if and only if* f *is pure (resp. an arrow morphism).*

Definition and properties of characteristic sequences

We are finally prepared to state the definition of the sequence of morphisms which will allow us to recover the isomorphism class (resp. its family of τ-invariant objects).

In order to unify notation, we extend the definition of τ to non-perfect objects X by $\tau X := X$.

Definition II.3.13. Let $\mathcal{X}$ be a triangulation of $\mathcal{F}$, which separates punctures. Let $X \in \mathcal{F}$ be indecomposable and τ-invarant (resp. not τ-invariant), such that X is not isomorphic to the shift of an object in $\mathcal{X}$.

A cyclic (resp. linear) sequence $(\phi_0, \overline{\phi}_0), \ldots, (\phi_m, \overline{\phi}_m)$ of pairs of non-zero morphisms

$$(\phi_j, \overline{\phi}_j) \in \mathrm{Hom}_{\mathrm{Int}}(X, Y_j[n_j]) \times \mathrm{Hom}_{\mathrm{Int}}(\tau^{-1} Y_j[n_j - 1], X),$$

where $Y_0, \ldots, Y_m \in \mathcal{X}$ is called a **characteristic sequence of** X (with respect to $\mathcal{X}$) if it satisfies all of the following conditions.

1) The set $\{\phi_j \,|\, j \in [0, m]\}$ is a basis of

$$\bigoplus_{i \in I} \mathrm{Hom}^*_{\mathrm{Int}}(X, Y_i)$$

and the set $\{\overline{\phi}_j[-n_j + 1] \,|\, j \in [0, m]\}$ is a basis of

$$\bigoplus_{i \in I} \mathrm{Hom}^*_{\mathrm{Int}}(\tau^{-1} Y_i, X);$$

2) For every $j \in [0, m]$ (resp. $j \in [0, m)$), there exists a pair $\{\sigma_j^1, \sigma_j^2\} = \{j, j+1\}$ with the following properties:

   a) There exists an arrow morphism

   $$\alpha_j : Y_{\sigma_j^1}[n_{\sigma_j^1}] \to Y_{\sigma_j^2}[n_{\sigma_j^2}],$$

   such that $\alpha_j \circ \phi_{\sigma_j^1}$ is a non-zero multiple of $\phi_{\sigma_j^2}$.

   b) There exists an arrow morphism

   $$\overline{\alpha}_j : \tau^{-1} Y_{\sigma_j^1}[n_{\sigma_j^1} - 1] \to \tau^{-1} Y_{\sigma_j^2}[n_{\sigma_j^2} - 1],$$

   such that $\overline{\phi}_j \circ \overline{\alpha}_j$ is a non-zero multiple of $\overline{\phi}_{\sigma_j^2}$.

3) For all $j \in [0, m]$, such that $Y_j \in \mathrm{Perf}(\mathcal{F})$, $\phi_j \circ \overline{\phi}_j$ is a connecting morphism in an Auslander-Reiten triangle.

We refer to the cyclic (resp. linear) sequence $Y_0, \ldots, Y_m$ as a **characteristic sequence of objects** of X.

Part 1) and 2) of Definition II.3.13 resemble our previous considerations. Although this is not quite the case, 3) should be thought of as the algebraic equivalent of the conditions that ϕ_i and $\overline{\phi}_i$ are morphisms associated to the same interior intersection up to the action of $\boldsymbol{\tau}^{-1}$.

The following is straighforward.

Lemma II.3.14. *Let $(\mathcal{F}, S, w), (\mathcal{F}', S', w')$ be Fukaya-like triples, let $\mathcal{X}$ a triangulation of $\mathcal{F}$, which separates punctures. Let $X \in \mathcal{F}$ be indecomposable. If $(\phi_0, \overline{\phi}_0), \dots, (\phi_m, \overline{\phi}_m)$ is a characteristic sequence of X with respect to $\mathcal{X}$ and $T : \mathcal{F} \to \mathcal{F}'$ is a triangle equivalence, then*

$$(T(\phi_0), T(\overline{\phi}_0)), \dots, (T(\phi_m), T(\overline{\phi}_m)),$$

is a characteristic sequence of $T(X)$ with respect to $T(\mathcal{X})$.

Existence of characteristic sequences The following lemma shows that characteristic sequences exist under suitable assumptions.

Lemma II.3.15. *Let $(\mathcal{F}, S, w)$ be a Fukaya-like triple, $\mathcal{X} = \{X_i\}_{i \in I}$ be a triangulation of $\mathcal{F}$, which separates punctures, and let $X \in T$ be indecomposable, such that X is not isomorphic to any shift of any object in $\mathcal{X}$. Then X has a characteristic sequence with respect to $\mathcal{X}$.*

Proof. Let Δ be a complete collection of representatives of the homotopy classes in $\gamma(\mathcal{X})$. Since X is not isomorphic to a shift of an object in $\mathcal{X}$, γ is not homotopic to any arc of Δ and therefore intersects at least one arc of Δ in the interior.
If X is τ-invariant, let γ be a loop, which represents the family of τ-invariant objects associated to X. Otherwise, let $\gamma \in \gamma(X)$ be an arc. In any case, we may assume that $\{\gamma\} \cup \Delta$ is in minimal position. Let $p_0, \dots, p_m$ be the cyclic (resp. linear) ordered sequence of interior intersection points of γ with arcs in Δ. For each $j \in [0, m]$, let $\delta_j \in \Delta$, such that $p_i \in \gamma \cap \delta_i$ and denote $Y_j \in \mathcal{X}$, such that $\delta_j \in \gamma(Y_j)$.
If X is a loop object or not τ-invariant, set $\phi_j := \mathfrak{B}(p_j) \in \operatorname{Hom}^*(X, Y_j)$. If X is a τ-invariant arc object, then any homotopy H from γ to a representative γ' of X (which is a closed arc), such that $\{\gamma'\} \cup \Delta$ is in minimal position, induces a bijection $\hat{H} : \gamma \overrightarrow{\cap} \delta_j \to \gamma' \overrightarrow{\cap} \delta_j$ and ϕ_j is defined as the morphism associated to $\hat{H}(p_i)$.
Every isotopy ψ (not necessarily relative to the boundary) from $\boldsymbol{\tau}^{-1}$ to the identity induces a bijection $\hat{\psi}$ from $\gamma \overrightarrow{\cap} \delta_i$ to the subset of all interior intersections in $\gamma \overrightarrow{\cap} \boldsymbol{\tau}^{-1} \delta_i$. If X is a loop object or not τ-invariant, define $\overline{\phi}_j := \mathfrak{B}(\hat{\psi}(p_j)) \in \operatorname{Hom}^*(\tau^{-1} Y_j[n_j - 1], X)$. If X is a τ-invariant arc object, one uses a homotopy as before to construct $\overline{\phi}_j$ from $\hat{\psi}(p_j)$. By construction, $p_j, \hat{\psi}(p_j)$ and the distinguished intersection of δ_j and $\boldsymbol{\tau}^{-1} \delta_j$ bound a triangle if X is a loop object or not τ-invariant and they form a fork otherwise. Thus, Definition II.1.1 4) and Proposition II.1.5 imply that $\phi_j \circ \overline{\phi}_j$ is a connecting morphism in an Auslander-Reiten triangle, whenever δ_j is a finite arc. In

particular, $\overline{\phi}_j \in \mathrm{Hom}(\tau^{-1}Y_j[n_j - 1], X)$ in those cases. Property 1) of Definition II.3.13 follows from Lemma II.1.19 and Definition II.1.1 2) i) and iii). Since p_{j+1} and p_j lie on the boundary of the same triangle of Δ, there exist σ_j^1, σ_j^2 with $\{\sigma_j^1, \sigma_j^2\} = \{j, j+1\}$ and a (unique) intersection $q_j \in \delta_{\sigma_j^1} \overrightarrow{\cap} \delta_{\sigma_j^2}$, such that p_j, p_{j+1} and q_j form a triangle. Set $\alpha_j := \mathfrak{B}(q_j)$. In particular, $\alpha_j \in \mathrm{Hom}^*(Y_{\sigma_j(1)}[n_{\sigma_j(1)}], Y_{\sigma_j(2)})$. By assumption, the composition $\alpha_j \circ \phi_{\sigma_j(1)}$ is a multiple of the morphism associated to $p_{\sigma_j(2)}$ and hence $\alpha_j \in \mathrm{Hom}(Y_{\sigma_j(1)}[n_{\sigma_j(1)}], Y_{\sigma_j(2)}[n_{\sigma_j(2)}])$. The morphism $\overline{\alpha}_j$ is defined as the morphism associated to $\boldsymbol{\tau}^{-1}(q_j)$ and Property 2b) in Definition II.3.13 follows in a similar way. □

II.3.3 Reconstructing an object from its characteristic sequence

After we established their existence, we shall prove the uniqueness of characteristic sequences up to inversion and rotation. As an application, we show that the isomorphism class of an object, or, at least its τ-invariant family, can be reconstructed from its characteristic sequence.

In what follows, let $\mathcal{X}$ be a triangulation of $\mathcal{F}$, which separates punctures and denote by Δ a corresponding triangulation of S. Let further $X \in \mathcal{F}$ be an indecomposable object and let $\gamma \in \gamma_*(X)$ be a curve, such that $\{\gamma\} \cup \Delta$ is in minimal position. We write $(p_i)_i$ for the linear or cyclic sequence of intersections of γ with arcs in Δ.

Notation We introduce some notation for later reference. For the remainder of this section, let $(\phi_j, \overline{\phi}_j)_j$ be a characteristic sequence of X, let (Y_j) denote the corresponding sequence of objects in $\mathcal{X}$ and write $\delta_j \in \Delta$ for the representative of Y_j.

Depending on whether X is τ-invariant or not, we identify the parametrizing set of indices j with either an interval of integers or elements in a cyclic group in the natural way. For each j, let σ_j^1, σ_j^2, α_j and $\overline{\alpha}_j$ be as in Definition II.3.13.

Finally for each j, we write

$$\phi_j = \sum_{l \in I_j} \lambda_l \cdot b_l,$$

where I_j is some finite index set, $\lambda_l \in k^\times$ and b_l is the morphism associated to an interior intersection of γ and Δ.

In other words, $\{b_l\}_{l \in I_j}$ is the set of basis elements that appear, when decomposing ϕ_j as a linear combination. The notation for the coefficients will not be used anywhere in the rest of this section.

In a similar fashion, let $\overline{I}_j$ be a finite index set and let $\{\overline{b}_l\}_{l\in\overline{I}_j}$ be the set of basis elements associated to interior intersections of γ and arcs in the set $\tau^{-1}\Delta$ that appear in a decomposition of $\overline{\phi}_j$ as a linear combination of such basis elements.

An important observation is that if α is an arrow morphism, such that $\alpha \circ b_l$ is defined, then either $\alpha \circ b_l = 0$ or it is a multiple of an element $b_{l'}$. This serves as motivation for the following lemma.

Lemma II.3.16. *Let $j \in J$.*

1) *If $(\sigma_j^1, \sigma_j^2) = (j+1, j)$, there exists a canonical injection $u^j : I_j \hookrightarrow I_{j+1}$, such that $\alpha_j \circ b_{u^j(l)}$ is a multiple of b_l for all $l \in I_j$;*

2) *If $(\sigma_j^1, \sigma_j^2) = (j, j+1)$, there exists a canonical injection $u^j : I_{j+1} \hookrightarrow I_j$, such that $\alpha_j \circ b_{u^j(l)}$ is a multiple of b_l for all $l \in I_{j+1}$.*

Proof. If $(\sigma_j^1, \sigma_j^2) = (j+1, j)$, then ϕ_j is a multiple of $\alpha_j \circ \phi_{j+1}$. Otherwise, ϕ_{j+1} is a multiple of $\alpha_j \circ \phi_j$. In the former case, note that for $l \in I_{j+1}$, $\alpha_j \circ b_l$ is zero or a multiple of some $b_{f(l)}$ for some unique $f(l) \in I_j$ and it is not difficult to verify that if $f(l)$ and $f(l')$ are both defined and agree, then $l = l'$. By the above mentioned relation between ϕ_{i+1} and ϕ_i, every $l' \in I_j$ is of the form $l' = f(l)$ for some $l \in I_j$ and u^j is then defined via $u^j(f(l)) := l$ for all $l \in I_j$. The proof of the other case is analogous. $\square$

The previous lemma has a dual analogue which we state without further proof.

Lemma II.3.17. *Let $j \in J$. Then,*

1) *If $(\sigma_j^1, \sigma_j^2) = (j+1, j)$, there exists a canonical injection $\overline{u}^j : \overline{I}_{j+1} \hookrightarrow \overline{I}_j$, such that $\overline{b}_{\overline{u}^j(l)} \circ \overline{\alpha}_j$ is a multiple of b_l for all $l \in \overline{I}_{j+1}$;*

2) *If $(\sigma_j^1, \sigma_j^2) = (j, j+1)$, there exists a canonical injection $\overline{u}^j : \overline{I}_j \hookrightarrow I_{j+1}$, such that $\overline{b}_{\overline{u}^j(l)} \circ \overline{\alpha}_j$ is a multiple of b_l for all $l \in \overline{I}_j$.*

The functions u^j and $\overline{u}^j$ allow us to pass from an intersection of γ with the triangulation Δ to its successor or predecessor. Ultimately, this would allow us to reconstruct the homotopy class of γ.

However, the problem is that only one of u^j and $\overline{u}^j$ might be defined for some j. This requires us to be able to switch between the sets I_j and $\overline{I}_j$ by attaching a sort of dual in $\overline{I}_j$ to every in I_j.

To be more explicit, note that I_j (resp. $\overline{I}_j$) does not appear as the domain of some u^i (resp. $\overline{u}^i$) if and only if $(\sigma^1_{j-1}, \sigma^2_{j-1}) = (j, j-1)$ and $(\sigma^1_j, \sigma^2_j) = (j, j+1)$ (resp. $(\sigma^1_{j-1}, \sigma^2_{j-1}) = (j-1, j)$ and $(\sigma^1_j, \sigma^2_j) = (j+1, j)$).

On the other hand, I_j (resp. $\overline{I}_j$) does appear as the domain of definition of more than one u^i (resp. $\overline{u}^i$) if and only if $(\sigma^1_{j-1}, \sigma^2_{j-1}) = (j-1, j)$ and $(\sigma^1_j, \sigma^2_j) = (j+1, j)$ (resp. $(\sigma^1_{j-1}, \sigma^2_{j-1}) = (j, j-1)$ and $(\sigma^1_j, \sigma^2_j) = (j, j+1)$). The notion of duality is made precise in terms of Auslander-Reiten triangles:

Definition II.3.18. We say that elements $l \in I_j$ and $l' \in \overline{I}_j$ are **dual** if $b_l \circ \overline{b}_{l'}$ is the connecting morphism of an Auslander-Reiten triangle.

Condition 3) of Definition II.3.13 implies that there exist dual elements $l \in I_j$ and $l' \in \overline{I}_j$ for every j, such that X_j is finite.

Remark II.3.19. The elements $l \in I_j$ and $l' \in \overline{I}_j$ are dual if and only if their corresponding intersections are related via the bijection between $\gamma \overrightarrow{\cap} \delta_j$ and the set of interior intersections in $\gamma \overrightarrow{\cap} \tau^{-1}\delta_i$, which is induced by any isotopy ψ from τ^{-1} to the identity. In particular, the dual of an element is unique if it exists.

Lemma II.3.20. *The following hold true.*

1) *Suppose $l \in I_j$ has a dual. If $(\sigma^1_j, \sigma^2_j) = (j+1, j)$, then $u^j(l)$ has a dual and if $(\sigma^1_{j-1}, \sigma^2_{j-1}) = (j-1, j)$, then $u^{j-1}(l)$ has a dual.*

2) *Suppose $\bar{l} \in \overline{I}_j$ has a dual. If $(\sigma^1_{j-1}, \sigma^2_{j-1}) = (j, j-1)$, then $\overline{u}^{j-1}(\bar{l})$ has a dual and if $(\sigma^1_j, \sigma^2_j) = (j, j+1)$, then $\overline{u}^j(\bar{l})$ has a dual-*

Proof. We only prove 1). The proof of 2) is analogous. The conditions on (σ^1_j, σ^2_j) and $(\sigma^1_{j-1}, \sigma^2_{j-1})$ are the necessary conditions so that u^j (resp. u^{j-1}) is defined on I_j. The dual of $u^{j'}(l)$ for $j' \in \{j, j-1\}$ is the unique element $\bar{l}' \in I_{j-1} \sqcup I_j$, such that $\overline{b}_{\bar{l}'}$ is a multiple of $\overline{b}_{\bar{l}} \circ \overline{\alpha}_j$, where $\bar{l}$ is the dual of l. $\square$

Note that for every pair $(l, \bar{l})$ of dual elements, there exists j', such that $u^{j'}(l)$ is defined or there exists j'', such that $\overline{u}^{j''}(\bar{l})$ is defined. In other words, the family of maps $\{u^j\}$ and $\{\overline{u}^j\}$ allow us to produce a whole sequence of dual pairs.

In light of Remark II.3.19 we observe the following.

Lemma II.3.21. *For every j the restriction of the maps u^j and $\overline{u}^j$ to elements which admit a dual are inverse up to taking duals. That is, if $\bar{l}$ is the dual of some l in the domain of definition of u^j and $\bar{l}'$ denotes the dual of $u^j(l)$, then $\bar{l} = \overline{u}^j(\bar{l}')$.*

We are now prepared to prove the main result of this section. It is an algebraic analogue of Lemma II.3.4.

Proposition II.3.22. *Let $(\mathcal{F}, S, w)$ be a Fukaya-like triple. Further, let $\mathcal{X}$ be a triangulation of $\mathcal{F}$, which separates punctures, and denote by Δ an associated triangulation of S. Let $X, X' \in T$ be indecomposable and not isomorphic to any shift of any object in $\mathcal{X}$. For $Y_0, \dots, Y_n \in \mathcal{X}$ a characteristic sequence of objects of X and $\delta_j \in \gamma(Y_j) \in \Delta$ the corresponding arcs, $\gamma_*(X)$ contains the class of a curve γ, which is in minimal position with Δ, such that the sequence of interior intersections of γ with Δ is given by $\delta_0, \dots, \delta_n$.*

Proof. First, assume there exists j, such that δ_j is a finite. It is sufficient to produce a cyclic (resp. linear) sequence $(l_0, \bar{l}_0), \dots, (l_m, \bar{l}_m)$ of pairwise distinct dual pairs $(l_j, \bar{l}_j) \in I_j \times \overline{I}_j$, such that $l_i = u^{i'}(l_{i+1})$ or $l_{i+1} = u^{i'}(l_i)$ for some i'. Clearly, such a sequence is sufficient to recover the characteristic sequence of objects $Y_0, \dots, Y_m$ up to rotation and inversion, respectively. However, it is also sufficient to recover the sequence of intersections of γ (resp. the inverse of γ) with the triangulation Δ. As a matter of fact, such a sequence tells us, that the intersection points q_i and q_{i+1} of γ with arcs in Δ, which correspond to l_i and l_{i+1}, are consecutive elements in the whole sequence of intersections. Because all l_i are pairwise distinct it follows that $q_0, \dots, q_m$ coincides with $p_0, \dots, p_m$ up to rotating and inverting the sequence.
The sequence is constructed by virtue of the following algorithm.
Choose a dual pair $(l_j, \bar{l}_j) \in I_j \times \overline{I}_j$. Its existence follows from condition 3) in Definition II.3.13. Suppose we have already constructed a sequence $(l_j, \bar{l}_j), \dots, (l_s, \bar{l}_s)$ of dual pairs $(l_i, \bar{l}_i) \in I_i \times \overline{I}_i$. If $(\sigma_s^1, \sigma_s^2) = (s+1, s)$, define $l_{s+1} \coloneqq u(l_s)$ and $\bar{l}_{s+1}$ as the dual of l_{s+1} (which exists by Lemma II.3.20). Otherwise, set $\bar{l}_{s+1} \coloneqq \overline{u}(\bar{l}_s)$ and define l_{s+1} as the dual of $\bar{l}_{s+1}$. In the latter case, we know that $l_s = u(l_{s+1})$ (Lemma II.3.21). If X is not τ-invariant, we proceed in the analogous way to produce pairs $(l_{j-1}, \bar{l}_{j-1}), (l_{j-2}, \bar{l}_{j-2}) \dots, (l_0, \bar{l}_0)$.
In case X is τ-invariant, the finiteness of I guarantees, that $(l_j, \bar{l}_j) = (l_n, \bar{l}_n)$ for some $n = j + r$. W.l.o.g. we assume that r is minimal with this property. By condition 1) of Definition II.3.13, $r \leq m$. We claim that all pairs are pairwise distinct. Denote q_i the intersection corresponding to l_i and observe that, by construction, either for all $j \in [0, n)$, q_{i+1} is the successor of q_i or, for all $j \in [0, n)$, q_i is the successor of q_{i+1}. Thus, if X is not τ-invariant, all pairs are distinct and the first property in Definition II.3.13 implies that the homotopy class of γ is the unique class associated to the series of intersections $q_0, \dots, q_n$.
Consequently, the existence of identical pairs requires γ to be in the $\simeq_*$-class of a curve and contradicts the minimality assumption on r. In this case X

is necessarily τ-invariant and the sequence $q_0, \dots, q_r$ is a cyclic subsequence of $p_0, \dots, p_m$ up to inversion. The two sequences therefore coincide up to inversion and rotation. This finishes the proof in case γ has at least one intersection with a finite arc of Δ.
Finally, suppose all δ_j are infinite, i.e. $Y_j \notin \operatorname{Perf}(\mathcal{F})$. Since $\mathcal{X}$ (and hence Δ) separates punctures, there exists a unique puncture $p \in S$, such that p is an end point of all δ_j. In other words, γ is contained in a polygon of the triangulation $\Delta' = \{\delta \in \Delta \mid \delta \text{ finite}\}$ and winds around p. In case X is τ-invariant this already determines the $\simeq_*$-class of γ as the class of the simple loop winding around p. Suppose therefore that X is not τ-invariant, we then see that (depending on whether γ winds around p clockwise or counter-clockwise) one of the following conditions holds true.

1) $(\sigma_j^1, \sigma_j^2) = (j, j+1)$ for all $j \in [0, n)$ and there exists $l_0 \in I_0$, such that $b_l = \mathfrak{B}(p_0)$.

2) $(\sigma_j^1, \sigma_j^2) = (j+1, j)$ for all $j \in [0, n)$ and there exists $l_n \in I_n$, such that $b_l = \mathfrak{B}(p_n)$.

Suppose for example that $(\sigma_j^1, \sigma_j^2) = (j, j+1)$ for all $j \in [0, n)$. Then

$$0 \neq \phi_n = \alpha_{n-1} \circ \cdots \circ \alpha_0 \circ \phi_0,$$

but

$$\alpha_{n-1} \circ \cdots \circ \alpha_1 \circ \mathfrak{B}(p_i) = 0$$

for all $i \neq 0$, such that $Y_i = Y_0$. Hence $\mathfrak{B}(p_0) = b_l$ for some $l \in I_0$. As a consequence, we obtain a sequence $l_0, \dots, l_n$ ($l_j \in I_j$), such that for all $j \in (0, n]$ (resp. $j \in [0, n)$) $u^{j-1}(l_j) = l_{j-1}$ (resp. $u^j(l_j) = l_{j+1}$).
We have proved that the sequence $p_0, \dots, p_n$ can be reconstructed from the morphisms $\phi_0, \dots, \phi_n$. This finishes the proof. $\square$

The following is a consequence of Lemma II.3.4 and the previous proposition:

Corollary II.3.23. *Let $(\mathcal{F}, S, w)$ be a Fukaya-like triple and let $\mathcal{X}$ be a triangulation of $\mathcal{F}$ which separates punctures. Let further $X, X' \in \mathcal{F}$ be indecomposable and not isomorphic to any shift of any object in $\mathcal{X}$. If X and X' are not τ-invariant, then $X \cong X'$ in $\mathcal{F}/[1]$ if and only if their characteristic sequences coincide up to orientation. Furthermore, X and X' belong to the same family of τ-invariant objects if and only if their characteristic sequences of objects coincide up to rotation and inversion.*

We are prepared to finalize the proof of Theorem II.3.1.

Proof of Theorem II.3.1. Let $X \in \mathcal{F}$ be indecomposable, let $\mathcal{X}$ be any triangulation of $\mathcal{F}$ and denote by $\mathcal{A} \coloneqq \gamma(\mathcal{X})$ the associated triangulation of $\mathcal{S}_{\mathcal{F}}$. By Corollary II.2.13, we may assume that X is not isomorphic to any shift of any of the objects in $\mathcal{X}$. The corollary also implies that the triangulation $\Psi(T)(\mathcal{A})$ of $\mathcal{S}_{\mathcal{F}'}$ coincides with $\gamma(\mathcal{X}')$, where $\mathcal{X}' \coloneqq \{T(Y) \mid Y \in \mathcal{X}\}$. By Lemma II.3.14, every characteristic sequence of X with respect to $\mathcal{X}$ is mapped to a characteristic sequence of $T(X)$ with respect to $\mathcal{X}'$. In particular, the characteristic sequence of objects $Y_0, \dots, Y_m$ of X is mapped to a characteristic sequence $T(Y_0), \dots, T(Y_m)$ of $T(X)$. Proposition II.3.22 therefore implies that $\Psi(T)\left(\gamma_*(X)\right) = \gamma_*(T(X))$. $\square$

II.3.4 Induced homeomorphisms preserve orientations

It turns out that the homeomorphism associated to a triangle equivalence preserves the orientation as a consequence of the fact that the functor is covariant. Together with Theorem II.3.1 the following proposition implies Theorem A 1 a).

Proposition II.3.24. *Let $(\mathcal{F}, \mathcal{S}_{\mathcal{F}}, \omega_{\mathcal{F}})$ and $(\mathcal{F}', \mathcal{S}_{\mathcal{F}'}, \omega_{\mathcal{F}'})$ be Fukaya-like triples and assume that $\mathcal{S}$ non-special. Then for all triangle equivalences $T : \mathcal{F} \to \mathcal{F}'$, the homeomorphism $\Psi(T)$ is orientation preserving.*

Proof. Write $\mathcal{S}_{\mathcal{F}} = (S, \overline{\mathcal{M}})$. Let $\Delta = \{\gamma_1, \dots, \gamma_m\}$ be a triangulation of $(S, \overline{\mathcal{M}} \cap \partial S)$ and let $\{X_1, \dots, X_m\} \subset \mathcal{F}$ be a representing set of objects for Δ. By assumption, Δ contains no self-folded triangles. We claim the following. If $\gamma_{\sigma_0}, \gamma_{\sigma_1}, \gamma_{\sigma_2}$ are edges of a triangle in Δ, then the given order coincides with the clockwise order in the orientation if and only if there exists an arrow morphism in at least one of the sets $\mathrm{Hom}^*(X_{\sigma_i}, X_{\sigma_{i+1}})$, where $i \in \{0, 1, 2\}$ (indices modulo 3). Recall, that by Lemma II.3.9, arrow morphisms (up to a scalar) between objects corresponding to a triangle are in bijection with the corners of said triangle. Let U be a triangle of Δ and let $\widetilde{U}$ be a lift of U to the universal cover of S. Then $\widetilde{U}$ is an embedded triangle and at least two sides of it are arcs. Such a pair of arcs has a unique boundary intersection. Moreover, this intersection defines a morphism $X_i \to X_j$ (which is an arrow morphism by Lemma II.3.9) if and only if γ_i comes immediately before γ_j in clockwise order.
To prove the claim it is sufficient to show that $\widetilde{U}$ is not bounded by more than one lift of the same arc γ_i. Suppose this was not the case, then the two lifts, which we denote by δ and δ', intersect at the boundary in a point p. By uniqueness of lifts, p is the start point of δ and the end point of δ' or vice versa. But this implies that U contains an embedded Möbius strip in contradiction to the orientability of S.

The assertion follows from the fact that any homeomorphism $H : S \to S$ is orienation preserving if it preserves the order of the edges of each triangle in Δ. However, by what we mentioned before, this is a consequence of Corollary II.3.12 $\square$

As a second application of Theorem II.3.1 we are able to prove Theorem A 1b):

Proposition II.3.25. *Let $u = (\mathcal{F}, \mathcal{S}_{\mathcal{F}}, \omega_{\mathcal{F}})$ and $u' = (\mathcal{F}', \mathcal{S}_{\mathcal{F}'}, \omega_{\mathcal{F}'})$ be Fukaya-like triples and assume that $\mathcal{S}$ non-special. Let further $T : \mathcal{F} \to \mathcal{F}'$ be a triangle equivalence. Then, for all loops γ on $\mathcal{S}_{\mathcal{F}}$,*

$$\omega_{\mathcal{F}'}(\Psi(T) \circ \gamma) = \omega_{\mathcal{F}}(\gamma).$$

Proof. Set $\omega := \omega_{\mathcal{F}}(\gamma)$. Let δ be any closed arc, such that $\gamma \simeq \delta$ regarded as loops. Let $X \in \mathcal{F}$ be a representative of δ. According to Corollary II.1.22, we know that there exists a non-invertible boundary morphism $f \in \mathrm{Hom}(X, X[d])$ of degree $d = \sigma \cdot \omega + 1$, where $\sigma \in \{\pm 1\}$ and $\sigma = 1$ if and only if δ intersects itself at the boundary from the right hand side. Since T sends boundary morphisms to boundary morphisms, it follows that $T(f) \in \mathrm{Hom}(T(X), T(X)[d])$ is a non-invertible boundary morphism. By Theorem II.3.1, if X is not τ-invariant, then $T(X)$ is represented by $\delta' := \Psi(T)(\delta)$ and if X is τ-invariant, then an object in the family of τ-invariant objects associated to $T(X)$ is represented by δ'. Consequently, $d = \sigma' \cdot \omega_{\mathcal{F}'}(\Psi(T) \circ \gamma)) + 1$, where $\sigma' \in \{\pm 1\}$ and $\sigma' = 1$ if and only if $\Psi(T) \circ \gamma$ intersects itself at the boundary from the right hand side. Finally, $\Psi(T)$ is orientation preserving (Proposition II.3.24) and hence preserves intersection indices. Thus, $\sigma = \sigma'$ and the proof is complete. $\square$

II.4 Homeomorphisms induce derived equivalences

In this section we prove that gentle algebras with equivalent surface models are derived equivalent. It is a generalization of Corollary 3.2.4, [58], to the case of (ungraded) gentle algebras of arbitrary global dimension. More precisely, the main result of this section reads as follows.

Theorem II.4.1. *Let A, B be gentle algebras and let $(\mathcal{S}_A, \omega_A)$ and $(\mathcal{S}_B, \omega_B)$ be their associated surface models as defined in Chapter I. Assume that there exists an orientation preserving homeomorphism $F : \mathcal{S}_A \to \mathcal{S}_B$, such that $\omega_A = \omega_B \circ F$. Then, A and B are derived equivalent.*

The idea of the proof is simple. We choose a collection of arcs which represent the indecomposable projective modules of A. In other words, our collection might be thought of as a representative of the regular A-module. Since A is a tilting object in $\mathcal{D}^b(A)$, we expect that the image of the collection of arcs under F represents a tilting object in $\mathcal{D}^b(B)$ with endomorphism ring A^{op}. In what follows, we prove that this intuition indeed correct.

Tilting complexes in derived categories of gentle algebras We recall the definition of a tilting object in a form which is convenient for us. The original definition is due to Rickard [64].

Definition II.4.2. Let A be a finite dimensional algebra. An object $X \in \mathrm{Perf}(A)$ is a **tilting object** if all of the following conditions are satisfied:

1) $\mathrm{Hom}^*(X, X)$ is concentrated in degree zero.

2) Let $\mathcal{T}$ denote the smallest triangulated subcategory of $\mathcal{D}^b(A)$ which contains X and is closed under taking direct summands. Then $A \in \mathcal{T}$.

Remark II.4.3. The standard definition of a tilting object assumes that X is perfect, satisfies the first condition of Definition II.4.2 and has the property that $\mathcal{D}(A - \mathrm{Mod})$ is the smallest localizing subcategory of the full derived category $\mathcal{D}(A - \mathrm{Mod})$ ($A - \mathrm{Mod}$ denoting the category of all A-modules) which contains X. By Proposition 1.6.8 in [59], every localizing subcategory of $\mathcal{D}(A - \mathrm{Mod})$ is closed under direct summands. Since A is always a tilting object, Definition II.4.2 is equivalent to the original definition.

The following lemma provides a sufficient condition for a collection of arcs to represent the direct summands of a tilting object.

Lemma II.4.4. *Let $T = \{\gamma_1, \dots, \gamma_n\}$ be a set consisting of pairwise non-homotopic simple finite arcs on S_A which intersect each other only in the boundary. Assume further that the following two conditions are satisfied:*

1) The weighted winding number of every cycle of arcs in T (see Section I.8) is zero.

2) For every finite arc δ on S_A, there exists a sequence of finite arcs $\delta_1, \dots, \delta_m \simeq \delta$, such that $\delta_1 \in T$ and such that for all $i \in (1, m)$, δ_{i+1} is the concatenation of δ_i and an arc of T at one of their end points.

Then, there exists a tilting object $X = \bigoplus_{i=1}^n X_i \in \mathrm{Perf}(A)$ with indecomposable direct summands X_i, such that γ_i represents X_i.

Proof. Let $U_1, \dots, U_n \in \mathcal{D}^b(A)$ be representatives of $\gamma_1, \dots, \gamma_n$. We construct integers $a_0, \dots, a_n$, such that $X \coloneqq \bigoplus_{i=1}^n U_i[a_i]$ has only self-extensions in degree 0. Let $i \in [1, n]$ and denote $Z_0(i) := \{U_i\}$. Via induction, define

$$Z_{j+1}(i) \coloneqq \{U_l \mid l \in [1,n], \exists Z \in Z_j : \operatorname{Hom}^*(U_l, Z) \neq 0 \text{ or } \operatorname{Hom}^*(Z, U_l) \neq 0\}.$$

This is an increasing sequence of subsets of $\{U_1, \dots, U_n\}$ and hence stabilizes to a subset

$$Z(i) := \bigcup_{j \in \mathbb{N}} Z_j(i).$$

Note that either $Z(i) = Z(j)$ or $Z(i) \cap Z(j) = \emptyset$. The set $Z(i)$ has a simple description on the surface. Namely, regarding T as a graph G embedded in S_A with edges γ_i labelled by the elements in T, the objects in $Z(i)$ are those objects in $\{U_1, \dots, U_n\}$ which correpond to the edges in the connected component of γ_i in G.
Depending on a choice for $a_i \in \mathbb{Z}$ we specify a_l for every $l \in [1, n]$ such that $U_l \in Z(i)$. This is achieved as follows. By definition, for each $U_l \in Z(i)$, there exist a sequence $U_i[a_i] = U_{i_0}[b_{i_0}], \dots, U_{i_s}[b_{i_s}] = U_l[b_{i_s}]$ and morphisms $f_0, \dots, f_s$, such that for each $j \in [0, s)$,

- f_j is a morphism from $U_{i_j}[b_{i_j}]$ to $U_{i_{j+1}}[b_{i_{j+1}}]$ or vice versa, and
- f_j corresponds to an intersection of γ_{i_j} and $\gamma_{i_{j+1}}$.

Set $a_l \coloneqq b_{i_s}$. We claim that if $U_l \in Z(i)$, then the definition of a_l only depends on the choice of a_i and not on the choice of the sequences above. Let $U_{j_0}[c_{j_0}], \dots, U_{j_q}[c_{j_t}]$ and $g_0, \dots, g_t$ be another chain of objects and morphisms as before. Then, the sequences

$$U_l[b_{i_s}] = U_{i_s}[b_{i_s}], \dots, U_{i_0}[b_{i_0}], U_{j_0}[c_{j_1}], \dots, U_{j_t}[c_{j_t}] = U_l[c_{j_t}]$$

and

$$f_s, \dots, f_0, g_0, \dots, g_t$$

correspond to a cycle of arcs in T. By assumption, its weighted winding number vanishes and Theorem I.8.1 implies $b_{i_s} = c_{j_t}$. The same argument also shows that if $f : U_a \to U_b[i]$ and $g : U_a \to U_b[j]$ are two morphisms associated with intersections of γ_a and γ_b, then $i = j$ and, by definition of a_i and a_j, it follows $i = j = 0$. Thus, the algebra of self-extensions of $X \coloneqq \bigoplus_{i=1}^n U_i[a_i]$ is concentrated in degree 0.
Theorem I.4.3 and the second condition imply that the smallest triangulated subcategory of $\mathcal{D}^b(A)$, which is closed under direct summands and which

contains X, also contains every perfect indecomposable object which is not τ-invariant (and hence is represented by a finite arc). In particular, it contains all indecomposable projective modules of A and hence A. Altogether, we showed that X is tilting. $\square$

For the regular A-module the converse of Lemma II.4.4 holds true.

Lemma II.4.5. *Let $P_1, \ldots, P_n$ denote a complete set of indecomposable projective modules of A and let $T = \{\gamma_1, \ldots, \gamma_n\}$ be a set representatives of $P_1, \ldots, P_n$ in minimal position. Then T consists of simple finite arcs with disjoint interiors. Moreover, T satisfies conditions 1) and 2) of Lemma II.4.4.*

Proof. The first assertion follows from the construction of the surface of A and its lamination.
The proof of Lemma II.4.4 actually shows that condition 1) in Lemma II.4.4 is equivalent to the fact that A has no self-extensions in non-zero degrees.
To prove that every arc is obtained by a series of concatenations from arcs in T, we use the combinatorial description of string objects. We obtain each string complex X over A by a series of mapping cones. We may assume that A is given by a quotient kQ/I of a quiver Q with respect to gentle relations. Suppose $\sigma = \sigma_1 \cdots \sigma_n$ is a homotopy string with homotopy letters σ_j. Then for each $i \in [1, n]$, we set $u_i \coloneqq \sigma_1 \cdots \sigma_i$ and define u_0 to be the trivial homotopy string attached to $s(\sigma_1)$. For each $i \in [0, n)$ and depending on whether σ_{i+1} is inverse or direct, $P^\bullet_{u_{i+1}}$ is the mapping cone of a map $P^\bullet_{u_i} \to P_{t(u_{i+1})}$ or a map $P_{t(u_{i+1})} \to P^\bullet_{u_i}$ which corresponds to a boundary intersection of the arcs representing $P_{t(u_{i+1})}$ and $P^\bullet_{u_i}$.
This finishes the proof since each such mapping cone corresponds to the resolution of a crossing at a boundary. $\square$

As a consequence of the previous lemmas we show the following.

Proposition II.4.6. *Let A and B be gentle algebras and $H : \mathcal{S}_A \to \mathcal{S}_B$ be an orientation preserving homeomorphism of marked surfaces, such that $\omega_A = \omega_B \circ H$. Let $T = \{\gamma_1, \ldots, \gamma_n\}$ be a set of arcs representing all isomorphism classes of indecomposable projective A-modules. Then, there exists a tilting object $X = \bigoplus_{i=1}^n X_i$ in $\operatorname{Perf}(B)$, such that each X_i is indecomposable and is represented by $\gamma_i \circ H$.*

Proof. Note that $T' \coloneqq \{\gamma_1 \circ H, \ldots, \gamma_n \circ H\}$ is a set of simple arcs with pairwise disjoint interior. Since H is a homeomorphism and T satisfies the conditions of Lemma II.4.4 it follows from Lemma II.4.5 that T' satisfies the conditions of Lemma II.4.4 as well. Thus, there exists a tilting object with the desired properties. $\square$

Next, we show that the endomorphism ring of a tilting object can be recovered from a representing set of arcs. Together with Proposition II.4.6, the following lemma concludes the proof of Theorem II.4.1.

Lemma II.4.7. *Let $X \in \mathrm{Perf}(A)$ be a tilting object and assume that $X = \bigoplus_{i=1}^{n} X_i$ (X_i indecomposable) is represented by a set of arcs $T = \{\gamma_1, \dots, \gamma_n\}$. Then,* $\mathrm{Hom}(X, X)$ *is isomorphic to the algebra $k\Gamma/R$, where Γ is a quiver and R is an ideal generated by quadratic zero relations, given as follows:*

- *Γ has vertices $\{x_1, \dots, x_n\}$ and the arrows from x_i to x_j are in one-to-one correspondence with the directed intersections $p \in \gamma_i \overrightarrow{\cap} \gamma_j$, such that there is no other arc of T ending between γ_i and γ_j.*
- *R is generated by all expressions pq, where p and q are composable arrows of Γ and $p \neq q$ as points in S_A.*

Proof. There exists an algebra homomorphism $\varphi : k\Gamma/_R \to B$, which sends each vertex x_i of Γ to the identity morphism of X_i which we regard as an element in B in the natural way. We require also that φ sends an arrow $p \in \gamma_i \overrightarrow{\cap} \gamma_j$ to a morphism $X_i \to X_j$ associated to p. It follows from the correspondences between intersections and morphisms on one hand and the correspondence between compositions and the existence of forks (Theorem I.7.1) on the other hand, that φ is surjective. By comparing dimensions (which we can express in terms of intersections) we see that φ is an isomorphism. $\square$

Remark II.4.8. Note that Lemma II.4.7 provides a geometric proof of the result in [66] that the class of gentle algebras is closed under derived equivalences.

II.4.1 Special surfaces

In the final section of this chapter we study special surfaces and prove variants of our results on non-special surfaces. Recall from Definition II.2.6 that a marked surface is special if its arc complex is empty or has dimension at most 1.

The following is a complete list of special marked surfaces and is taken from [38], Figure 1. A marked surface $\mathcal{S} = (S, \overline{\mathcal{M}})$ is special if and only if

- S is a disc with no punctures and $|\,\overline{\mathcal{M}}\,| \leq 5$, or
- S is a disc with one puncture and at most two marked boundary points, or

- S is a cylinder with no puncture and a single marked point on each boundary component.

Special surfaces occur as surface models of prominent examples of gentle algebras. The surface of a quiver of type A_n is a disc with $n+1$ marked points on the boundary and no punctures and hence is special for $n \leq 4$. The second case in the list above is obtained as the surface of the algebra of dual numbers $k[X]/(X^2)$. As shown in Lemma II.2.20, the Kronecker is the only quiver which realizes the third entry of the list.

In what follows we describe the group of homeomorphisms of $\mathcal{S}$ up to isotopy and the kernel of the group homomorphism

$$\Phi : \operatorname{Homeo}(\mathcal{S}, \mathcal{S}) \longrightarrow \operatorname{Simp}(A_*(\mathcal{S}), A_*(\mathcal{S})),$$

for every special surface $\mathcal{S} = (S, \overline{\mathcal{M}})$, such that $A_*(\mathcal{S}) \neq \emptyset$. Note that the mapping class group is a subgroup of index 2 in $\operatorname{Homeo}(\mathcal{S}, \mathcal{S})$. Its non-trivial coset contains the isotopy classes of orientation-reversing homeomorphisms. In particular, $\operatorname{Homeo}(\mathcal{S}, \mathcal{S})$ is generated by the mapping class group of $\mathcal{S}$ and an orientation-reversing homeomorphism. The mapping class group can be described using standard techniques as for example can be found in [42].

1) If $\mathcal{S}$ is a disc with 4 marked boundary points, then $A_*(\mathcal{S})$ consists of two disconnected points, $\mathcal{MCG}(\mathcal{S})$ is generated by τ (see Section I.5.2) and $\ker \Phi \cap \mathcal{MCG}(\mathcal{S})$ is generated by τ^2.

2) If $\mathcal{S}$ is a disc with 5 marked boundary points, then $\mathcal{MCG}(\mathcal{S})$ is generated by τ and $A_*(\mathcal{S})$ is a 5-gon. Assume that all marked points are evenly distributed on the boundary and denote by ρ the reflection at the line through an arbitrary fixed marked point and the center of the disc. The image of Φ is non-trivial which proves that the restriction of Ψ to the mapping class group is injective. On the other hand $\mathcal{MCG}(\mathcal{S})$ and $\operatorname{Simp}(A_*(\mathcal{S}), A_*(\mathcal{S}))$ have the same cardinality proving that the restriction is an isomorphism.
   It follows immediately that Φ has a kernel of order 2 which is generated by ρ.

3) If $\mathcal{S}$ is a disc with two marked points, one of which is a puncture, then $A_*(\mathcal{S})$ is a point and $\operatorname{Simp}(A_*(\mathcal{S}), A_*(\mathcal{S}))$ and $\mathcal{MCG}(\mathcal{S})$ are both trivial. Assuming that the puncture coincides with the center of the disc, $\operatorname{Homeo}(\mathcal{S}, \mathcal{S})$ is generated by the reflection through the line connecting the marked points.

4) If $\mathcal{S}$ is a disc with 2 marked points on its boundary and one puncture, then $A_*(\mathcal{S})$ is a graph of type A_4 and hence its simplicial automorphism group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ the generator given by the reflection. Assuming that the puncture is the center of the disc and that all marked points lie on a single line, $\mathrm{Homeo}(\mathcal{S}, \mathcal{S})$ is generated by the reflection ρ at that said line and τ. The kernel of Φ is generated by ρ.
The mapping class group of $\mathcal{S}$ is generated by τ and has order 2. Φ is non-zero, thus its restriction to $\mathcal{MCG}(\mathcal{S})$ is an isomorphism.

5) If $\mathcal{S}$ is a cylinder with a single marked point on each boundary component, then, $A_*(\mathcal{S})$ is an infinite line of vertices and has automorphism group $\mathbb{Z}$. Identifying $\mathcal{S}$ with $(S^1 \times [0,1], \{(0,0),(0,1)\})$, $\mathcal{MCG}(\mathcal{S})$ is generated by τ_B for any of its boundary components B and therefore isomorphic to $\mathbb{Z}$. Since $\Phi(\tau_B)$ is a generator it follows that Φ is an isomorphism.

As a consequence of the previous considerations we see that, with the exception of the disc with 4 marked points, the restriction $\Phi|_{\mathcal{MCG}(\mathcal{S})}$ is invertible and we may define Ψ as $\Phi|^{-1}_{\mathcal{MCG}(\mathcal{S})} \circ A_*$. The same proof as in the general case show that $\Psi(T)(\gamma(X)) \simeq \gamma(T(X))$ for all indecomposables $X \in \mathcal{F}$ and all equivalences $T : \mathcal{F} \to \mathcal{F}$.

In case of a disc with 4 marked points, every mapping class is uniquely determined by its action on the set of marked points which itself is uniquely determined by the action on the set of boundary segments. Thus, for every auto-equivalence T of $\mathcal{F}$, there is a unique mapping class $\Psi(T)$, such that $\Psi(T)(\gamma(X)) \simeq \Psi(T(X))$. Similar, if $\mathcal{S}$ is a disc with at most 3 marked points, then there for each auto-equivalence T of $\mathcal{F}$ (which has at most 3 indecomposable objects), there exists a unique mapping class of $\Psi(T)$ of $\mathcal{S}$, such that $\Psi(T)(\gamma(X)) \simeq \gamma(T(X))$ for all $X \in \mathcal{F}$.

Altogether we have seen that Ψ can be defined for special surfaces and has the desired properties.

II.5 The kernel of Ψ

We study the kernel of Ψ, where $u = (\mathcal{D}^b(A), \mathcal{S}_A, \omega_A)$ is the Fukaya-like triple associated to a gentle algebra A. The main result of this section is the following proposition (Theorem E in the introduction).

Theorem II.5.1. *Let $A = kQ/I$ be a gentle algebra, such that Q is connected and has no oriented cycles. Then* $\ker \Psi$ *is generated by rescaling equivalences* *if Q is not isomorphic to the Kronecker quiver. If Q is isomorphic to the Kronecker quiver, then* $\ker \Psi$ *is generated by* $[1]$*, all rescaling equivalences and all coordinate transformations.*

The definition of rescaling equivalences is given below. Note that in order for Ψ to be defined, we henceforth assume that S_A is not a disc with at most 3 marked points. This is a very mild condition as it rules out precisely all path algebras of type A_1 or A_2. Indeed, this follows from Proposition I.1.20 and the classification of iterated-tilted algebras of type A, see [6]. Our proof of Theorem II.5.1 builds on the ideas of [47].
Set $\mathcal{D} := \mathcal{D}^b(A)$. By the very construction of Ψ, $T \in \mathrm{Aut}(\mathcal{D})$ is contained in the kernel of Ψ if and only if T fixes the isomorphism class of every essential object up to a shift. It shows the following:

Lemma II.5.2. *The shift functor is an element of* $\ker \Psi$*.*

From Theorem II.3.1 and the discussion on special surfaces in Section II.4.1, we deduce the even stronger statement that $T \in \ker \Psi$ if and only if T fixes the isomorphism class (in $\mathcal{F}/[1]$) of every indecomposable object in $\mathcal{F}$, which is not τ-invariant, and T fixes all families of τ-invariant objects.

Examples of elements in $\ker \Psi$ We present further typical examples of auto-equivalences in the kernel of Ψ. Recall that the map which attaches to any algebra automorphism $\sigma : A \to A$ its corresponding equivalence ${}_{\sigma^{-1}}A \otimes_{\mathbb{L}} - : \mathcal{D} \to \mathcal{D}$, defines a group homomorphism

$$\mathcal{O} : \mathrm{Aut}_k(A) \longrightarrow \mathrm{Aut}\,(\mathcal{D})\,.$$

A short calculations shows that two such equivalences associated to automorphisms σ, σ' are naturally isomorphic as functors if and only if ${}_{\sigma}A \cong {}_{\sigma'}A$ as A-A-bimodules. It is well-known (see Chapter VII in [65]) that $\ker \mathcal{O}$ consists of the set of inner automorphisms. Thus, $\mathcal{O}$ descends to an embedding of the group $\mathrm{Out}(A)$ of outer automorphisms of A into $\mathrm{Aut}(\mathcal{D})$.

As an important special case, suppose $f : A \to A$ is an algebra isomorphism that fixes every vertex of Q, such that $f(\alpha) = \lambda_\alpha \cdot \alpha$ for some $\lambda_\alpha \in k^\times$ for all $\alpha \in Q_1$. In other words, f multiplies every arrow by a non-zero element in k. The set of such automorphisms forms a subgroup of $\mathrm{Aut}_k(A)$ which is isomorphic to $(k^\times)\,|Q_1|$. Its elements we call **rescaling automorphisms** and their associated equivalences $\mathcal{O}(f)$ are called **rescaling equivalences**. We denote by $\mathcal{R}$ the subgroup of $\mathrm{Aut}(\mathcal{D})$ consisting of all rescaling equivalences (up to natural isomorphism).
Rescaling automorphisms are special cases of the broader class of so-called

linear changes of variables, as defined in [47], and it follows from there that there exists a short exact sequence of groups

$$0 \longrightarrow (k^\times)^{|Q_0|}/k^\times \xrightarrow{\phi} (k^\times)^{|Q_1|} \xrightarrow{\mathcal{O}} \mathcal{R} \longrightarrow 0,$$

where the quotient on the left is taken with respect to the diagonal embedding and the map ϕ is defined by

$$\phi\left(\overline{(\lambda_x)_{x\in Q_0}}\right) := (\lambda_{s(\alpha)}^{-1} \cdot \lambda_{t(\alpha)})_{\alpha \in Q_1}.$$

The map on the right hand side is the canonical projection sending a rescaling automorphism to its associated element in $\mathcal{R}$. The image of ϕ are the so-called *acyclic characters* of Q, see [47].

Lemma II.5.3. *$\mathcal{R}$ is a subset of $\ker \Psi$.*

Proof. Let $f \in \mathrm{Aut}(A)$ be a rescaling automorphism and $F = \mathcal{O}(f) \in \mathrm{Aut}(\mathcal{D})$ the corresponding equivalence. Then F maps projective modules to projective modules and since f sends arrows to multiples of themselves, F sends string complexes to string complexes with rescaled components of its differential. This does not change the isomorphism class of a string complex. For the same reasons, $\mathcal{F}$ may change the isomorphism classes of band complexes, but only to a band complex in the same family of band complexes (a family of τ-invariant objects). In particular, the representing homotopy class of every indecomposable object in $\mathcal{D}$ remains unchanged under the action of F showing that $\mathcal{F} \in \ker \Psi$. □

In a similar way one proves:

Lemma II.5.4. *Let Q be the Kronecker quiver, $B = kQ$ and let $T \in \mathrm{Aut}(\mathcal{D}^b(B))$ be a coordinate transformation as defined in Example II.2.15). Then $T \in \ker \Psi$.*

Note that if Q is the Kronecker quiver, then rescaling automorphisms are special cases of coordinate transformations.

The proof of Theorem II.5.1 The following lemma asserts that under certain conditions every element of $\ker \Psi$ is a shift of an outer automorphism.

Lemma II.5.5. *Let $A = kQ/I$ be a gentle algebra, such that Q is connected and has no oriented cycles. For every $T \in \ker \Psi$, there exists an integer $n \in \mathbb{Z}$, such that $T \circ [n] \in \mathrm{Out}(A)$.*

Proof. Let $T \in \ker \Psi$. Every indecomposable projective module is an essential object and, consequently, T preserves their isomorphism classes up to a shift. For $P \in \mathcal{F}$ a shift of an indecomposable projective A-module, let $m_P \in \mathbb{Z}$, such that $T(P) \cong P[m_P]$. If P and P' are shifts of indecomposable projective A-modules, such that $\mathrm{Hom}^*(P, P') \neq 0$, then $\mathrm{Hom}^*(P, P')$ is concentrated in a single degree and it follows that $m_P = m_{P'}$. Since Q is connected, it follows inductively that $m_P = m_{P'}$ for all shifts of indecomposable projective A-modules P and P'. Therefore, by composing T with an appropriate shift, we may assume that $T(P) \cong P$ for all indecomposable projective A-modules P. Since Q has no oriented cycles, it follows from Proposition 2.4 in [34] that T is naturally isomorphic to a derived tensor product ${}_{\sigma}A \otimes_L -$ for some $\sigma \in \mathrm{Aut}_k(A)$ □

Together with Lemma II.5.5 the following lemma provides a proof of Theorem II.5.1.

Lemma II.5.6. *Let $A = kQ/I$ be a gentle algebra. Then,* $\mathrm{Out}(A) \cap \ker \Psi = \mathcal{R}$.

Proof. The inclusion "$\subseteq$" was shown in Lemma II.5.3. To prove the opposite inclusion, it is convenient to replace T with the derived functor T' of the functor which sends an A-module M and its associated algebra homomorphism $\phi_M : A \to \mathrm{End}_k(M)$ to the A-module M equipped with the structure map $\phi_M \circ \sigma : A \to \mathrm{End}_k(M)$.
The map σ induces automorphisms on all powers of the radical of A. In particular, it induces an automorphism $\overline{\sigma}$ on the semisimple quotient $B := A/\operatorname{rad}(A)$ ($\operatorname{rad}(A)$ denoting the radical). As an algebra, B naturally decomposes into the vector spaces spanned by the projections of idempotents associated to vertices of Q. Since every T (and hence T') preserves the isomorphism class of every indecomposable projective A-module it follows from [47] that we may assume $\sigma(x) = x$ for all $x \in Q_0$ after composition with an inner automorphism. Note that since A is gentle, it is naturally graded by its radical, i.e. A is isomorphic to the graded algebra associated with the filtration of A by the powers of $\operatorname{rad}(A)$.
Suppose that for some $\alpha \in Q_1$, $\{\sigma(\alpha), \alpha\}$ is linearly independent. Since A is gentle, it follows that there exists at most one path $\beta \notin I$ in Q different from α which is parallel to α. As $\sigma(x) = x$ for all $x \in Q_0$, it follows $\sigma(\alpha) = a \cdot \alpha + b \cdot \beta$ for some $a, b \in k$ and $b \neq 0$. Note that, if $a = 0$, then $T(P_{\alpha}^{\bullet}) \cong P_{\beta}^{\bullet} \ncong P_{\alpha}^{\bullet}$. Therefore, if $a = 0$, it follows from $T \in \ker \Psi$ that $P_{\alpha}^{\bullet}$ and $P_{\beta}^{\bullet}$ are τ-invariant, which implies that Q is the Kronecker quiver (Lemma II.2.20).

If $a \neq 0$, then $T'(P^\bullet_\alpha)$ is isomorphic to a τ-invariant band complex

$$\cdots \longrightarrow 0 \longrightarrow P_{t(\alpha)} \xrightarrow{a\cdot\alpha+b\cdot\beta} P_{s(\alpha)} \longrightarrow 0 \longrightarrow \cdots, \tag{II.1}$$

implying that $P^\bullet_\alpha$ must be τ-invariant as well. In particular, if $P^\bullet_\alpha$ is not τ-invariant, then $b = 0$ and $\sigma(\alpha) = a \cdot \alpha$. Applying a rescaling equivalence we may assume that $\sigma(\alpha) = \alpha$ for all $\alpha \in Q_1$, such that $P^\bullet_\alpha$ is not τ-invariant.

Next, suppose there exists a path $\beta \notin I$ in Q, such that $P^\bullet_\beta$ is τ-invariant. According to Lemma II.1.10, $P^\bullet_\beta$ is represented by a boundary arc on a boundary component B which contains a single marked point. By construction of γ (see [61]), $P^\bullet_\beta$ must sit at the base of the homogeneous tube, or equivalently, it is represented by a simple boundary arc. In particular, the middle term Y in an Auslander-Reiten triangle

$$P^\bullet_\beta \longrightarrow Y \longrightarrow P^\bullet_\beta \longrightarrow P^\bullet_\beta[1]$$

is indecomposable. By Corollary 6.3. in [13] (see also Corollary II.1.7), β is a maximal antipath, i.e. if $\alpha \in Q_1$ and $\alpha\beta$ (resp. $\beta\alpha$) is a path in Q, then $\alpha\beta \in I$ (resp. $\beta\alpha \in I$). The simple boundary loop around B represents a family of band complexes as in (II.1) above for some path (of arbitrary length) α parallel to β. Since T' preserves the family of τ-invariant objects associated to X_β, there exist $a, b, c, d \in k$, such that $\sigma(\alpha) = a \cdot \alpha + c \cdot \beta$ and $\sigma(\beta) = b \cdot \alpha + d \cdot \beta$. Since σ is invertible, so is the matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If Q is the Kronecker quiver, it means that T is isomorphic to a coordinate transformation.

For the rest of this proof we assume that Q is not isomorphic to the Kronecker quiver. Then, $P^\bullet_\beta$ is contained in an $\mathbb{A}^1_k$-family of τ-invariant objects (see Lemma II.2.19). In particular, X_α is not τ-invariant, as otherwise it would belong to the same family of τ-invariant objects as $P^\bullet_\beta$ yielding a $\mathbb{P}^1_k$-family. Consequently, $\sigma(\alpha) = \alpha$. Suppose M is not diagonal, then M^{-1} is not diagonal and there exists a (τ-invariant!) band complex which is mapped under T' to $P^\bullet_\alpha$ – a contradiction. Thus, M is diagonal. This shows that for all β, such that $P^\bullet_\beta$ is τ-invariant, $\sigma(\beta) = \lambda_\beta \cdot \beta$ for some $\lambda_\beta \in k^\times$. Thus, σ is a rescaling automorphism and the proof is complete. $\square$

Chapter III

Spherical objects on cycles of projective lines

The following Theorems are proved in the present chapter.

Theorem III.0.1. *Let $n \in \mathbb{N}$, $n > 1$. There exists a one-to-one correspondence between isomorphism classes of spherical objects on an n-cycle of projective lines, up to shift, and pairs $(\gamma, \mathcal{V})$, where γ is the homotopy class of a simple loop on the torus T_n with n punctures and $\mathcal{V}$ is a 1-dimensional local system on γ, such that the complement of γ in T_n is connected.*

Theorem III.0.2. *Assume that k is algebraically closed and let $n \in \mathbb{N}$. The group of auto-equivalences of the category of perfect complexes acts transitively on the set of isomorphism classes of spherical objects over an n-cycle of projective lines.*

Theorem III.0.1 and Theorem III.0.2 coincide with Theorem E and Theorem F of the introduction.

III.1 Categorical resolutions of cycles of projective lines

We recall the definition of a cycle of projective lines and some of the results in [25].

Definition III.1.1. Let $n \in \mathbb{N}$, $n \geq 1$. By an n**-cycle of projective lines** we mean a reduced rational projective curve E_n of arithmetic genus 1, which is a union of n copies of $\mathbb{P}^1$ glued transversally in a configuration of type $\tilde{A}_{n-1}$.

By definition, E_n is the union of n irreducible components $\mathbb{P}^1_i$ ($i \in \mathbb{Z}/n\mathbb{Z}$), each of which is isomorphic to $\mathbb{P}^1$, such that for all $i \in \mathbb{Z}/n\mathbb{Z}$, $\mathbb{P}^1_i$ and $\mathbb{P}^1_{i+1}$ intersect in a nodal singularity.
In particular, E_1 is the Weierstraß nodal cubic. The curve E_3 is shown in Figure III.1.

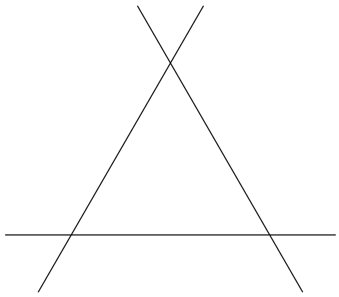


Figure III.1: A 3-cycle of projective lines

Burban and Drozd constructed a fully faithful and exact functor

$$\mathrm{Perf}(E_n) \longrightarrow \mathcal{D}^b(\mathrm{Coh}\, \mathbb{X}_n),$$

where $\mathbb{X}_n$ is a certain non-commutative curve. They proved that $\mathcal{D}^b(\mathrm{Coh}\, \mathbb{X}_n)$ contains a tilting complex, the endomorphism algebra Λ_n of which is isomorphic to the quotient of the path algebra of the quiver

$$\begin{tikzcd}
\bullet & \bullet & \cdots & \bullet \\
\bullet \arrow[u, "d_0"] \arrow[urrr, bend left, "b_0"] & \bullet \arrow[u, "d_1"] \arrow[ul, "b_1"] & \cdots \arrow[ul, "b_2"] & \bullet \arrow[u, "d_{n-1}"'] \arrow[ul, "b_{n-1}"] \\
\bullet \arrow[u, bend left, "a_0"] \arrow[u, bend right, "c_0"'] & \bullet \arrow[u, bend left, "a_1"] \arrow[u, bend right, "c_1"'] & \cdots & \bullet \arrow[u, bend left, "a_{n-1}"] \arrow[u, bend right, "c_{n-1}"']
\end{tikzcd}$$

by the ideal generated by the set $R := \{a_i b_i, c_i d_i \mid i \in [0, n)\}$. For example, Λ_1 is the quotient of the path algebra of

$$\begin{tikzcd}
3 \arrow[r, shift left, "a"] \arrow[r, shift right, "c"'] & 2 \arrow[r, shift left, "b"] \arrow[r, shift right, "d"'] & 1
\end{tikzcd}$$

by the ideal (ba, dc). The algebras Λ_n are gentle of global dimension 2 and, as a consequence, the Auslander-Reiten translation is defined for all its indecomposable objects.

From the above, it follows that there exists an embedding of triangulated categories

$$\mathbb{F} : \mathrm{Perf}(E_n) \hookrightarrow \mathcal{D}^b(\Lambda_n).$$

Burban and Drozd proved that the image of $\mathbb{F}$ is the full subcategory of τ-invariant objects, i.e. all objects $X \in \mathcal{D}^b(\Lambda_n)$ satisfying $X \cong \tau X$. They further computed the image of the Jacobian $\mathrm{Pic}^0(E_1) \cong k^\times$ as well as the images of the skyscraper sheaves $k(x)$ of smooth points $x \in E_1$.
The functor $\mathbb{F}$ identifies isomorphism classes of line bundle of degree 1 with the isomorphism of the following family $(\mathcal{O}(\lambda))_{\lambda \in k^\times}$ of complexes.

$$\mathcal{O}(\lambda) = \;\cdots \longrightarrow 0 \longrightarrow P_1 \xrightarrow{\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right)} \underline{P_2^2} \xrightarrow{(a \;\; \lambda c)} P_3 \longrightarrow 0 \longrightarrow \cdots .$$

The set $\{k(x) \,|\, x \in E_1 \text{ smooth}\}$ is identified with the set of complexes $\{k(\lambda) \,|\, \lambda \in k^\times\}$, where

$$k(\lambda) = \;\cdots \longrightarrow 0 \longrightarrow P_2 \xrightarrow{a+\lambda c} \underline{P_3} \longrightarrow 0 \longrightarrow \cdots .$$

The underlined projectives indicate the degree zero part of $\mathcal{O}(\lambda)$ and $k(\lambda)$. In fact, $k(\lambda)$ is quasi-isomorphic to the Λ_1-module

$$k \underset{1}{\overset{\lambda}{\rightrightarrows}} k \rightrightarrows 0$$

It was communicated to us by Igor Burban, that for general $n \in \mathbb{N}$, one obtains the following descriptions of the images of $\mathrm{Pic}^{\mathbb{1}}(E_n)$ and $k(x)$, where $\mathrm{Pic}^{\mathbb{1}}(E_n)$ denotes the set of all line bundles on E_n of multi-degree $\mathbb{1} = (1, \ldots, 1) \in \mathbb{Z}^n$.

Theorem III.1.2 (Burban)**.** *Let $n \in \mathbb{N}$ and let $Y \in \mathcal{D}^b(\Lambda_n)$. Then there exists $\mathbb{L} \in \mathrm{Pic}^{\mathbb{1}}(E_n)$, such that $Y \cong \mathbb{F}(\mathbb{L})$ if and only if Y is isomorphic to a band complex $P^\bullet_{\sigma_{\mathrm{Pic}^{\mathbb{1}}},\lambda}$, which is concentrated in degrees 0 and 1, where $\lambda \in k^\times$ and $\sigma_{\mathrm{Pic}^{\mathbb{1}}} = d_0\overline{b_1}d_1 \cdots d_{n-1}\overline{b_{n-1}}$.*

$$
\begin{tikzcd}
P_{2[0]} \arrow[r, "\lambda d_0"] \arrow[rdddddddd, bend left=60, "b_0"] & P_{1[0]} \\
\oplus & \oplus \\
P_{2[1]} \arrow[uur, "b_1"] \arrow[r, "d_1"] & P_{1[1]} \\
\oplus & \oplus \\
{} \arrow[uur, "b_2"] & {} \\
\vdots & \vdots \\
{} & {} \\
\oplus & \oplus \\
P_{2[n-1]} \arrow[uur, "b_{n-1}"] \arrow[r, "d_{n-1}"] & P_{1[n-1]}
\end{tikzcd}
$$

Figure III.2: The images of $\mathrm{Pic}^{\mathbb{1}}$ under $\mathbb{F}$.

Theorem III.1.3 (Burban)**.** *Let $n \in \mathbb{N}$ and let $Y \in \mathcal{D}^b(\Lambda_n)$. Then there exists a smooth point $x \in \mathbb{P}^1_i$, such that $Y \cong \mathbb{F}(k(x))$ if and only if Y is isomorphic to a band complex $P^\bullet_{\sigma_{k(x),i},\lambda}$, which is concentrated in degree -1 and 0, where $\lambda \in k^\times$ and $\sigma_{k(x),i} = a_i\overline{c_i}$.*

$$\cdots \longrightarrow 0 \longrightarrow P_{2[i]} \xrightarrow{\lambda a_i + c_i} P_{3[i]} \longrightarrow 0 \longrightarrow \cdots .$$

Figure III.3: The images of $k(x), x \in \mathbb{P}^1_i$, under $\mathbb{F}$.

The proofs of Theorem III.1.2 and Theorem III.1.3 were communicated to the author by Igor Burban and can be found in the Appendix.

III.2 Spherical objects and spherical twists

In this section we recall the definition of a spherical object in a triangulated category and their associated auto-equivalences, the so-called *spherical* *twists*. Spherical objects and spherical twists were first introduced in [70]. Throughout this section we fix a k-linear, Hom-finite triangulated category $\mathcal{T}$, i.e. $\mathrm{Hom}^*(X,Y)$ is finite dimensional for all $X, Y \in \mathcal{T}$.

Definition III.2.1. A **Serre dual** of an object $X \in \mathcal{T}$ is an object $\mathcal{S}(X) \in \mathcal{T}$, such that there exists a k-linear isomorphism of functors $\mathrm{Hom}(X,-) \cong \mathrm{Hom}(-,\mathcal{S}(X))^*$, where $(-)^*$ denotes the duality over the ground field k.

By Yoneda's lemma, the Serre dual of an object is unique up to unique isomorphism if it exists. It further follows that the mapping $X \mapsto \mathcal{S}(X)$ is functorial on the subcategory of objects it is defined for. If $\mathcal{S}$ is defined for all objects of $\mathcal{T}$, then any such functor is called a **Serre functor**. In addition to being unique up to unique isomorphism, Serre functors commute with every k-linear triangle equivalence up to natural isomorphism.

Remark III.2.2. The categories $\mathrm{Perf}(E_n)$ and $\mathcal{D}^b(\Lambda_n)$ have Serre functors. In the former case, it is the left derived tensor product $- \otimes^{\mathbb{L}} \omega[1]$, where ω denotes the canonical sheaf. In fact, ω is a trivial line bundle, showing that the Serre functor is isomorphic to the shift functor [1].
Since Λ_n has finite global dimension, the left derived functor of the Nakayama functor $\nu = (\mathrm{Hom}_{\Lambda_n}(-, {}_{\Lambda_n}\Lambda_n))^* : \Lambda_n - \mathrm{mod} \to \Lambda_n - \mathrm{mod}$ is a Serre functor of $\mathcal{D}^b(\Lambda_n)$.

Definition III.2.3. An object $X \in \mathcal{T}$ is called d**-Calabi-Yau** ($d \in \mathbb{Z}$) if $X[d]$ is a Serre dual of X. A d-Calabi-Yau object $X \in \mathcal{T}$ is d**-spherical** if $\dim \mathrm{Hom}^*(X,X) = 2$ and is concentrated in degrees 0 and d, and if moreover $\mathrm{char}\, k = 2$ and $d = 0$, then $\mathrm{Hom}(X,X) \cong k[U]/(U^2)$.

It follows from Remark III.2.2 that every object $X \in \mathrm{Perf}(E_n)$ is 1-Calabi-Yau. In particular, all of its spherical objects are 1-spherical.

Example III.2.4. The following are examples of spherical objects.

1) Every simple vector bundle $\mathcal{L} \in \mathrm{Perf}(E_n)$ is 1-spherical since $\mathrm{Hom}(\mathcal{L},\mathcal{L}) \cong k$ and hence by Serre duality,

$$\mathrm{Hom}(\mathcal{L},\mathcal{L}[i]) \cong \mathrm{Hom}(\mathcal{L},\mathcal{L}[1-i])^* \cong \begin{cases} k, & \text{if } i = 1; \\ 0 & \text{if } i \geq 2. \end{cases}$$

   In particular, all line bundles on E_n are spherical.

2) Let $x \in E_n$ be smooth. By a similar argument as before, the associated skyscraper sheaf $k(x)$ is an object in $\mathrm{Perf}(E_n)$ and is 1-spherical.

Lemma III.2.5. *Let $X \in \mathcal{T}$ be spherical. Then, X is indecomposable.*

Proof. If $X \in \mathcal{T}$ is d-Calabi-Yau, then by virtue of the isomorphism

$$\operatorname{Hom}(X, X[i]) \cong \operatorname{Hom}(X[i], X[d])^* \cong \operatorname{Hom}(X, X[d-i])^*,$$

$\operatorname{End}^*(X)$ is concentrated in degrees 0 and d. Note that if $\operatorname{char} k \neq 2$ and $d = 0$, then $\operatorname{Hom}^*(X, X) \cong k[U]/(U^2)$ by definition. In particular, $\operatorname{End}(X)$ is local for any characteristic and hence X is indecomposable. $\square$

It was shown in [70] (see also [50]) that if $\mathcal{T}$ is algebraic (or more general, has a DG-enhancement), every spherical object $X \in \mathcal{T}$ gives rise to a k-linear exact auto-equivalence T_X of $\mathcal{T}$, called a **spherical twist**. By its definition, for every $Y \in \mathcal{T}$, the object $T_X(Y)$ sits in a distinguished triangle of the form

$$\operatorname{Hom}^*(X, Y) \otimes_k X \xrightarrow{\text{ev}} Y \longrightarrow T_X(Y) \longrightarrow \operatorname{Hom}^*(X, Y) \otimes_k X[1]$$

where ev denotes the canonical evaluation map.
In particular, given a basis $f_1, \ldots, f_m$ of $\bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}(X[i], Y)$ consisting of morphisms $f_i : X[n_i] \to Y$, then the above triangle is isomorphic to the triangle

$$\bigoplus_{i=1}^m X[n_i] \xrightarrow{\bigoplus f_i} Y \longrightarrow T_X(Y) \longrightarrow \bigoplus_{i=1}^m X[n_i][1]$$

Besides being useful for computations in Section III.5, it shows that twist functors commute with embeddings on the level of objects.

Corollary III.2.6. *Let $\mathcal{T}'$ be a k-linear,* hom-*finite and algebraic triangulated category and let $\mathbb{F} : \mathcal{T} \to \mathcal{T}'$ be a k-linear, exact and fully faithful functor. Let $X \in \mathcal{T}$ be spherical and assume that $\mathbb{F}(X)$ is spherical. Then for all $Y \in \mathcal{T}$,*

$$\mathbb{F} \circ T_X(Y) \cong T_{\mathbb{F}(X)} \circ \mathbb{F}(Y).$$

In particular, if $\mathbb{F}$ is an equivalence and $\mathbb{F}^{-1}$ a quasi-inverse of $\mathbb{F}$, then $\mathbb{F} \circ T_X \circ \mathbb{F}^{-1}(Y) \cong T_{\mathbb{F}(X)}(\mathbb{F}(Y))$.

The previous corollary allows us to analyze the twist functor of any spherical object $X \in \operatorname{Perf}(E_n)$ implicitely by analyzing the twist functor of $\mathbb{F}(X) \in \mathcal{D}^b(\Lambda_n)$.

III.3 The surface model of Λ_n

We describe the surface S_{Λ_n}, its laminations and the loops associated to $P^\bullet_{\sigma_{\mathrm{Pic}\mathbb{1}}}$ and $P^\bullet_{\sigma_{k(x),i}}$.

The surface associated to Λ_n is a torus T_n, which we define in the following way. Let $n \in \mathbb{N}$ and for all integer valued points $(i,j) \in \mathbb{R}^2$ denote $B_i^j \subseteq \mathbb{R}^2$ the open disc with radius $\frac{1}{4}$ and center $(i+\frac{1}{2}, \frac{1}{2}+j)$. Denote T_n the torus with n removed open discs, i.e. T_n is the quotient of

$$\mathbb{R}^2 \setminus \left(\bigsqcup_{(i,j)\in\mathbb{Z}^2} B_i^j \right),$$

with respect to the equivalence relation generated by $(t,l) \sim (t, l+1)$ and $(m,s) \sim (m+n, s)$, where $l,m,s,t \in \mathbb{R}$. Denote by ρ the corresponding quotient map. Its fundamental domain is $[0,n) \times [0,1)$. For convenience we often refer to a point $x \in T_n$ via a representative in $\mathbb{R}^2$.

Set $\overline{\mathcal{M}} := \{(i+\frac{1}{2}, \frac{3}{4}), (i+\frac{1}{2}, \frac{1}{4}) \,|\, 0 \leq i < n\} \subseteq \partial T_n$. This is the set of marked points. They are in bijection with the set of maximal admissible paths $\{a_i d_i, c_i b_{i+1} \,|\, i \in [0,n)\}$, i.e. the path $a_i d_i$ corresponds to the point $(i+\frac{1}{2}, \frac{1}{4})$ and the path $c_i b_{i+1}$ to the point $(i+\frac{1}{2}, \frac{3}{4})$. The lamination of Λ_n is shown in Figure III.4

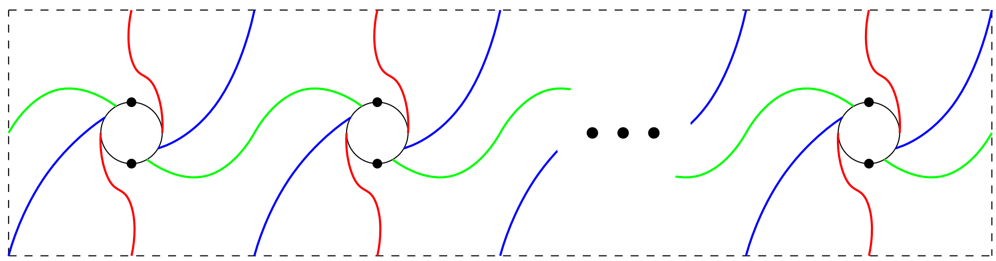


Figure III.4: Laminates of Λ_n

The laminates shown on the right hand side of Figure III.5 correspond to the vertices $s(a_i) = s(c_i)$ and laminates as shown on the left hand side correspond to vertices $t(d_i) = t(b_{i+1})$. The remaining laminates in Figure III.4 represent the vertices $t(a_i) = t(c_i) = s(d_i) = s(b_{i+1})$.

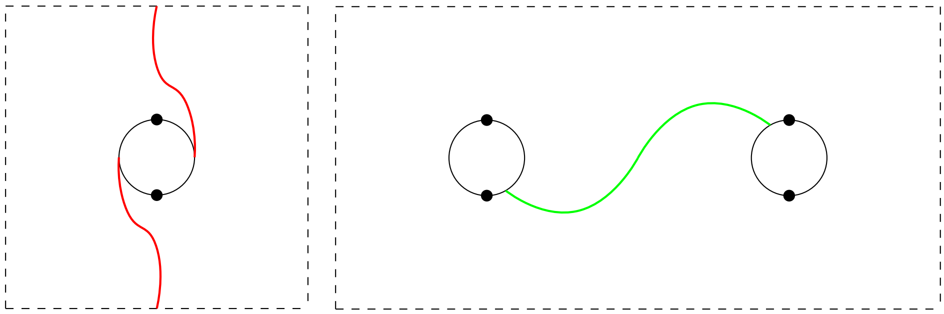


Figure III.5

The lamination cuts T_n into $2n$ connected components, each of which is homeomorphic to a 6-gon.

A representative of the homotopy class of loops associated to $\sigma_{\mathrm{Pic}^{\mathbb{1}}}$ is the loop $\gamma_{\mathrm{Pic}^{\mathbb{1}}} : S^1 \to T_n$ defined by $\gamma_{\mathrm{Pic}^{\mathbb{1}}}(\exp(t)) := (n \cdot t, \frac{1}{4})$, where exp denotes the usual exponential map. Similarly, the loop $\gamma^i_{k(x)} : S^1 \to T_n$ defined as $\gamma^i_{k(x)}(\exp(t)) := (i+1, t)$ is a representative of $\gamma(\sigma_{k(x),i})$.

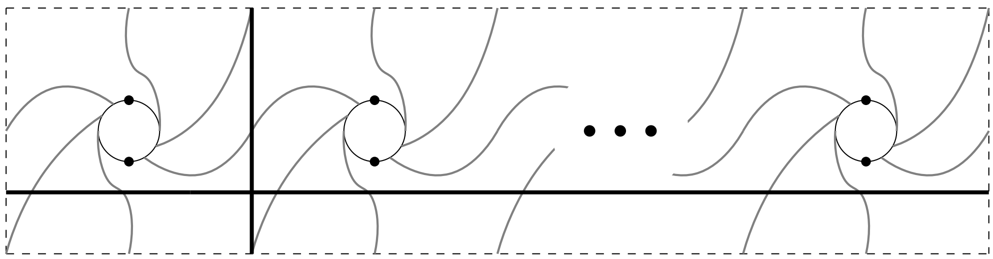


Figure III.6: The loops $\gamma_{\mathrm{Pic}^{\mathbb{1}}}$ and $\gamma^1_{k(x)}$.

III.3.1 Spherical objects in $\mathcal{D}^b(\Lambda_n)$ and simple loops

We show that, up to shifts, isomorphism classes of spherical objects in $\mathcal{D}^b(\Lambda_n)$ are in bijection with a certain set of simple loops.

Let $X \in \mathrm{Perf}(E_n)$ be spherical. By Corollary 6 in [25], $Y := \mathbb{F}(X)$ is τ-invariant and since $\mathbb{F}$ is fully faithful, Y is spherical.

Lemma III.3.1. *Let $Y \in \mathcal{D}^b(\Lambda_n)$ be indecomposable. Then Y is Calabi-Yau if and only if it is isomorphic to a band complex.*

Proof. All band complexes are τ-invariant (see Section I.5.1) and hence 1-Calabi-Yau. It follows from Lemma II.1.10 and the relation $\mathbb{S} = \tau[1]$ that the only Calabi-Yau arc complexes are represented by boundary arcs of boundary components with a single marked point showing that there are no such string complexes in $\mathcal{D}^b(\Lambda_n)$. $\square$

The previous proposition has the following consequence.

Corollary III.3.2. *If $Y \in \mathcal{D}^b(\Lambda_n)$ is spherical, then $Y \cong \mathbb{F}(X)$ for some spherical object $X \in \mathrm{Perf}(E_n)$.*

Proof. By Lemma III.3.1, Y is τ-invariant and hence in the essential image of $\mathbb{F}$. Since $\mathbb{F}$ is fully faithful, every (essential) preimage must be spherical. $\square$

Recall that a loop γ is called **simple** if it has no self-intersections.

Lemma III.3.3. *Let $Y \in \mathcal{D}^b(\Lambda_n)$ be indecomposable. Then Y is spherical if and only if it is represented by a simple loop.*

Proof. Suppose Y be spherical. By Lemma III.3.1, Y is represented by a homotopy class of loops. Let γ, γ' be representatives of this homotopy class in minimal position. Then by Theorem I.3.3, γ and γ' are disjoint. In particular, γ can not have self-intersections, i.e. γ is simple.
Conversely, suppose γ is a simple loop which represents Y. In particular, we find a simple loop γ' homotopic to γ, such that γ and γ' are disjoint. Again by Theorem I.3.3, we conclude that $\bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}(Y, Y[i])$ has dimension 2 and is concentrated in degree 0 and 1 the generators being the identity and a connecting morphism of an Auslander-Reiten triangle. Since Y is τ-invariant, Y is spherical. $\square$

As a weaker version of Theorem III.0.1 we obtain:

Corollary III.3.4. *The functor $\mathbb{F} : \mathrm{Perf}(E_n) \to \mathcal{D}^b(\Lambda_n)$ gives rise to a bijection between the set of isomorphism classes of spherical objects in $\mathrm{Perf}(E_n)$ and those homotopy classes of simple loops on T_n, which represent objects of $\mathcal{D}^b(\Lambda_n)$.*

The surprisingly difficult part is, however, to determine, which loops on T_n actually represent an indecomposable object of $\mathcal{D}^b(\Lambda_n)$. We will do this in Section III.6.

III.4 The mapping class group of T_n

Recall from Section II.2.1 that the mapping class group $\mathcal{MCG}(T_n)$ of T_n is the group consisting of all orientation-preserving homeomorphisms T_n which restrict to a bijection of marked points modulo isotopy. Its subgroup of all homeomorphisms, which restrict to the identity on ∂T_n, is called the **pure mapping class group** and is denoted by $\mathcal{PMCG}(T_n)$. Similarly, it is useful to consider the subgroup $\mathcal{MCG}(T_n)_\partial$ of the mapping class group consisting of all f, which preserve each boundary component of T_n as a set. We have the following relationships between these groups.
There exist short exact sequences (see [42])

$$0 \longrightarrow \mathcal{PMCG}(T_n) \longrightarrow \mathcal{MCG}(T_n) \longrightarrow (\mathbb{Z}/2\mathbb{Z})^n \times \Sigma_n \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{PMCG}(T_n) \longrightarrow \mathcal{MCG}(T_n)_\partial \longrightarrow (\mathbb{Z}/2\mathbb{Z})^n \longrightarrow 0.$$

Here Σ_n denotes the symmetric group on n elements. In the above sequences, a mapping class f is projected to the induced bijections of the sets of marked points and the set of boundary components, respectively.

Dehn twists A typical example of a pure mapping class is the class of a so-called **Dehn twist** D_γ about a given (oriented) simple loop γ on S. If W is a tubular neighborhood of γ, i.e. a neighborhood W with a homeomorphism $\phi : S^1 \times [-1, 1] \to W$, such that $\phi|_{S^1\times\{0\}} = \gamma$, then $D_\gamma : S \to S$ is defined by $D_\gamma|_{S\setminus W} := \mathrm{Id}_{S\setminus W}$ on $S \setminus W$ and on W by

$$D_\gamma(\phi(z, t)) := \phi(z \cdot e^{\pi i(t+1)}, t).$$

While D_γ depends on W, its mapping class is well-defined. On the level of homotopy classes, the twist D_γ sends the homotopy class of a curve δ, which is in minimal position with γ, to the homotopy class of the curve obtained by resolving all intersections of γ and δ at once, following the direction of γ. That is, whenever δ crosses γ from the right (resp. left) hand side of γ, we turn right (resp. left) at the intersection. When compared to the definition of spherical twists while keeping in mind that resolution of crossings correspond to mapping cones, it reveals the strong formal similarity between Dehn twists and spherical twist which seems to be one of the intentions behind their introduction in [70].
In analogy to spherical twists (see Corollary III.2.6) the set of mapping classes of Dehn twists is closed under conjugation.

Lemma III.4.1 ([42])**.** *Let γ be a simple loop on an oriented surface S and let $F : S \to S$ be a homeomorphism. Then $D_{F\circ\gamma}$ and $F \circ D_\gamma \circ F^{-1}$ define the same mapping classes.*

The following is a special case of a Theorem due to Humphries which can be found in [42].

Theorem III.4.2 (Humphries)**.** *$\mathcal{PMCG}(T_n)$ is generated by the mapping classes of $n+1$ Dehn twists about the loops $\gamma_{\mathrm{Pic}^{\mathbb{1}}}$ and $\gamma^0_{k(x)}, \ldots, \gamma^{n-1}_{k(x)}$ as defined in Section III.3.*

The Dehn twist of the previous theorem are referred to as the **Humphries generators** of $\mathcal{PMCG}(T_n)$.

The action of $\mathcal{PMCG}(T_n)$ on simple curves With Theorem III.0.2 and the expected correspondence between spherical twists and Dehn twists in mind, we want to analyze the action of $\mathcal{PMCG}(T_n)$ on simple curves. We characterize its orbits in terms of cut surfaces. The **cut surface** of a simple loop γ on a compact surface S is the surface M with boundary components $B \neq B'$, such that $M \setminus (B \cup B') = S \setminus \gamma$. In particular, a subset of the boundary components of the cut surface is canonically identified with the boundary components of S.
The natural action of homeomorphisms on curves on a compact surface S with boundary gives rise to an action of $\mathcal{MCG}(S)$ on the set of isotopy classes, or equivalently homotopy classes, of simple loops.

Definition III.4.3. Let γ, γ' be simple loops with image in the interior of a compact oriented surface S with boundary components $B_0, \ldots, B_m \subseteq \partial S$. Denote M (resp. M') the cut surface of γ (resp. γ'). We say that γ and γ' have the same **strong topological type** if there exists a homeomorphism $F : M \to M'$, which restricts to homeomorphisms $B_i \to B_i$ for every $i \in [0, m]$.

Remark III.4.4. In the literature the term topological type is used to refer to simple loops γ, γ', such that $S \setminus \gamma$ and $S \setminus \gamma'$ are homeomorphic without further assumptions on the homeomorphism.

The group $\mathcal{MCG}(S)$ acts naturally on the set of homotopy classes of all simple loops of any fixed topological type and it is well-known that this action is transitive. Similarly, $\mathcal{PMCG}(S)$ acts naturally on the set of homotopy classes of all simple loops of the same strong topological type. We adapt the proof of the classical result (which can be found in Section 1.3 of [42]) and show that this action is transitive as well.

Proposition III.4.5. *Let S be a compact, oriented surface with boundary and γ, γ' be a simple loops of the same strong topological type. Then there exists a pure mapping class $F \in \mathcal{PMCG}(S)$ such that $F \circ \gamma \simeq \gamma'$ up to change of orientation of γ'.*

Proof. Denote $\{B_0, \ldots, B_m\}$ the set of boundary components of S. Set $M := S \setminus \gamma$ and $M' := S \setminus \gamma'$ and equip them with the orientation induced from S. By assumption, there exists a homeomorphism $H : M \to M'$, which restricts to homeomorphisms $B_i \to B_i$ for all $i \in [0, m]$. By composing H with an orientation reversing homeomorphism of M to itself, possibly followed by other orientation preserving homeomorphisms that permute boundary components (so-called *half-twists*, see [42]), we may assume that H is orientation preserving. By Lemma III.4.6 below we can extend every orientation preserving homeomorphism $B_i \to B_i$ to an orientation preserving homeomorphism of M, which restricts to identities on all its other boundary components. We may therefore assume that for all $i \in [1, n]$, H restricts to the identity of B_i for all $i \in [0, m]$.
H induces a homeomorphism $S \to S$ with the desired property by gluing those boundary components of M and M' back together which correspond to γ and γ', respectively. $\square$

Next, we prove Lemma III.4.6, which was already used in the previous proof.

Lemma III.4.6. *Let S be a compact oriented surface, B a boundary component of S and $f : B \to B$ an orientation preserving homeomorphism, where B is equipped with the induced orientation. Then for every closed neighborhood W of B, f extends to a homeomorphism $F : S \to S$, such that F restricts to the identity outside the complement of W.*

Proof. The mapping class group of the circle is trivial. In other words, every orientation preserving homeomorphism $S^1 \to S^1$ is isotopic to the identity. In particular, we find an isotopy $\psi : [0,1] \times B \to B$ of f to Id_B. For $U \subseteq W$ a tubular neighborhood of B with homeomorphism $\phi : U \to [0,1] \times B$ define F as the extension by the identity of the map $\psi \circ \phi : U \to U$. $\square$

We are mainly interested in a particular strong topological type formed by the so-called non-separating loops. A simple loop γ is called **non-separating** if its complement $S \setminus \gamma$, or equivalently its cut surface, is connected. It is clear from the definition, the set of non-separating loops splits into orbits of the action of $\mathcal{PMCG}(T_n)$. However, as the following lemma asserts, they all belong to a single orbit.

Lemma III.4.7. *All non-separating loops have the same strong topological type.*

Proof. Assume γ is a non-separating simple loop on a surface S with genus g and boundary components $B_1, \dots, B_m$. Then the cut surface M of γ is a connected surface of genus $g-1$ and $m+2$ boundary components, which follows from the additivity of Euler characteristics. In particular, if γ' is another non-separating loop on S, its cut surface is homeomorphic to M by an orientation preserving homeomorphism. By a series of half-twists (compare Lemma III.4.5), we can permute any two boundary components on M by an orientation preserving homeomorphism which restricts to the identity on all other boundary components. In this way we obtain a homeomorphism $M \to M'$, which restricts to homeomorphisms $B_i \to B_i$ for every $i \in [1, m]$. We then proceed as in the proof of Proposition III.4.5 to obtain the desired homeomorphism $M \to M'$. $\square$

Corollary III.4.8. $\mathcal{PMCG}(S)$ *acts transitively on the set of non-separating simple loops.*

Corollary III.4.8 tells us that all we need to prove in order to finish the proof of Theorem III.0.2 is the following:

1. A simple loop on T_n represents an object if and only if it is non-separating.

2. For every Humphries generator H of $\mathcal{PMCG}(T_n)$, find $\mathcal{F} \in \operatorname{Aut}(\mathcal{D}^b(\Lambda_n)$ such that $\Psi(\mathcal{F})$ acts on simple loops in the same way as H.

Remark III.4.9. If $n \neq 1, 2$, then the action of a homeomorphism $T_n \to T_n$ on simple loops completely determines its mapping class and hence the second entry in the previous list implies that in those cases $\mathcal{PMCG}(T_n)$ is contained in the image of the homomorphism $\Psi : \operatorname{Aut}(\mathcal{D}^b(\Lambda_n)) \to \mathcal{MCG}(T_n)$ (see Section II.2). This follows from a result of Ivanov ([52]) which shows that the group of simplicial automorphisms of the curve complex (which is a variant of the arc complex introduced in Definition II.2.3) is equal to the mapping class group for almost all types of (unmarked!) surfaces with negative euler characteristic.

Example III.4.10. We provide a list of examples of separating loops on T_n of different strong topological types. In fact, this list is exhaustive as we will show below.
Let I be any (possibly empty) subset of components of ∂T_n. If I contains all components of ∂T_n, let δ_I be any separating simple loop with image in the interior of the fundamental domain $(0, n) \times (0, 1)$, which cuts the fundamental polygon into a region with no boundary components and a region, which contains all boundary components. Otherwise, let $i_0, \dots, i_m$ be the elements

in $\mathbb{Z}/n\mathbb{Z}$ in cyclic order, such that for all $j \in [0, m]$, $i_j \in I$ and $i_j - 1 \notin I$. Denote δ_I the homotopy class of loops as shown in Figure III.7 .

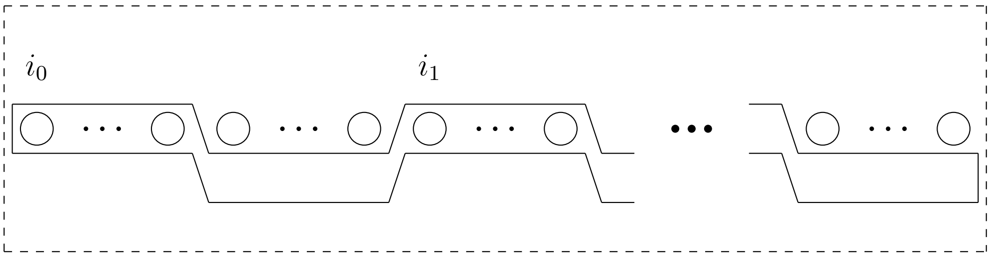


Figure III.7: Prototype of a separating simple loop

Lemma III.4.11. *Every strong topological type of simple, separating loops in T_n is represented by a loop from Example III.4.10.*

Proof. Suppose γ, γ' are separating, simple loops on T_n with cut surfaces M and M'. By assumption M (resp. M') consists of two connected components. Since the Euler characteristic of M (resp. M') and T_n agree and is additive on disjoint unions, it follows that $M = M_1 \sqcup M_2$ (resp. $M' = M_1' \sqcup M_2'$), where M_1 (resp. M_1') has genus 1 and M_2 (resp. M_2') has genus 0. In particular, M_2 and M_2' determine canonical subsets I and I' of the set of boundary components of T_n and γ and γ' have the same strong topological type if and only if $I = I'$. $\square$

Lemma III.4.12. *Let I be a subset of boundary components of T_n and let δ_I denote the loop from Example III.4.10. Then δ_I does not represent an object in $\mathcal{D}^b(\Lambda_n)$.*

Proof. It is not difficult to verify that the number of clockwise and counterclockwise arrows in $\sigma(\delta_I)$ differs by 2. Thus, by Theorem I.2.5, it does not correspond to an object in $\mathcal{D}^b(\Lambda_n)$. $\square$

In the subsequent section we relate Dehn twists of the Humphries generators of $\mathcal{PMCG}(T_n)$ and the spherical twists of the band complexes $P^{\bullet}_{\sigma}$, where $\sigma \in \{\sigma_{\mathrm{Pic}^1}, \sigma_{k(x),0}, \dots, \sigma_{k(x),n-1}\}$. With this in mind, Corollary III.4.8 should be thought of as the geometric version of Theorem III.0.2.

III.5 Spherical twists in $\mathcal{D}^b(\Lambda_n)$

The main portion of this section is occupied with the proof of the following result.

Proposition III.5.1. *Let $\lambda \in k^\times$ and $P^\bullet = P^\bullet_{\sigma,\lambda}$ for any*

$$\sigma \in \left\{\sigma_{\mathrm{Pic}^{\mathbb{1}},\lambda},\ \sigma_{k(x),0},\ \dots,\ \sigma_{k(x),n-1}\right\}.$$

Let Y be a linear band complex and let further

$$\delta \in \left\{\gamma_{\mathrm{Pic}^{\mathbb{1}}},\ \gamma^{0}_{k(x)},\ \dots,\ \gamma^{n-1}_{k(x)}\right\}$$

be the simple arc corresponding to σ. Then $\gamma(T_{P^\bullet}(Y))$ coincides with the image of the homotopy class $\gamma(Y)$ under D_δ.

Below we will verify Proposition III.5.1 by direct computations. However, before we do so, we use it to prove the following generalization of Corollary 5.8. in [28].

Theorem III.5.2. *Let $n \in \mathbb{N} \setminus \{1,2\}$. Then there exists a short exact sequence*

$$0 \longrightarrow (k^\times)^{2n-1} \times \mathbb{Z} \longrightarrow \mathrm{Aut}\left(\mathcal{D}^b(\Lambda_n)\right) \xrightarrow{\Psi} \mathrm{Im}\,\Psi \longrightarrow 0,$$

and $\mathcal{MCG}(T_n)_\partial \subseteq \mathrm{Im}\,\Psi \subseteq \mathcal{MCG}(T_n)$. In particular, $\mathrm{Im}\,\Psi$ is a finite index subgroup $\mathcal{MCG}(T_n)$.

Proof. It follows from Proposition III.5.1 and results of Ivanov [52] on the simplicial automorphism group of arc complexes that $\mathrm{Im}\,\Psi$ contains the pure mapping class group. Note that the condition $n \neq 1, 2$ is necessary in order to apply Ivanov's result. The images of fractional twists T_B (see Definition I.5.6) under Ψ, where $B \subset \partial T_n$ is a connected component, are elements of $\mathcal{MCG}(T_n)_\partial$. This follows from the description of mapping cones by resolution of crossings. Due to the second short exact sequence on page 131, it follows that $\mathrm{Im}\,\Psi$ contains $\mathcal{MCG}(T_n)_\partial$ hence is of finite index in $\mathcal{MCG}(T_n)$ according to the first sequence on page 131. The kernel is obtained by Theorem II.5.1 and direct computations of the group rescaling equivalences via the short exact sequence presented on page 119). □

Remark III.5.3. The natural $\mathbb{Z}/n\mathbb{Z}$-action on the underlying quiver of Λ_n induces an action on the mapping class group of T_n. The generator is mapped to the mapping class $g : T_n \to T_n$, $g(x,i) := (x, i+1)$. However, it is not clear to us which other permutations of boundary components can be realized by restricting images of Ψ to the boundary.

As a first step in our proof of Proposition III.5.1, we give a description of the ALP basis of the morphisms from any of the above spherical objects to a linear band complex.

Lemma III.5.4. *Let $Y = P^\bullet_\rho$ be a linear band complex associated to a homotopy band $\rho \neq \sigma_{k(x),i}$. Then, for $P^\bullet = P^\bullet_{\sigma_{k(x),i}}$ the standard basis of* $\mathrm{Hom}^*(P^\bullet, Y)$ *consists of graph maps and quasi maps. Moreover,*

1) *the graph maps are in bijection with substrings of ρ of the form $\overline{d_i}\,(\overline{c_i}a_i)^m\, b_i$ or its inverse, where $m \geq 0$, and*

2) *the quasi maps are in bijection with substrings of ρ of the form $\overline{(a_i d_i)}\,(c_i\overline{a_i})^m\,(c_i b_i)$ or its inverse, where $m \geq 0$.*

Proof. As all homotopy letters of $\sigma = \sigma_{k(x),i} = a_i\overline{c_i}$ are (inverse) paths of length one, there exists no singleton single map and no singleton double map from $P^\bullet$ to any shift of Y. Suppose ρ and $\sigma_{k(x),i}$ have a maximal (possibly trivial) subword in common, say $\rho_a \cdots \rho_b$ and $\sigma_a \cdots \sigma_b$. Then this common subword satisfies condition LG2 if and only if $\sigma_a \in \{\overline{c_i}, \overline{a_i}\}$ and it satisfies no left endpoint condition if and only if $\sigma_a \in \{a_i, c_i\}$. As the right endpoint conditions are dual, we conclude that graph maps correspond to maximal subwords $\overline{c_i}a_i \cdots \overline{c_i}a_i$ of ρ. By maximality (and since ρ is a homotopy band) it follows that $\overline{d_i}\,(\overline{c_i}a_i)^m\, b_i$ is a subword of ρ. Conversely, every such subword determines a graph map. The proof is analogous for quasi maps. $\square$

Due to the structural similarity of $\sigma_{k(x),i}$ and $\sigma_{\mathrm{Pic}^{\mathbb{1}}}$ we obtain

Lemma III.5.5. *Let $Y = P^\bullet_\rho$ be a linear band complex associated to a homotopy band $\rho \neq \sigma_{\mathrm{Pic}^{\mathbb{1}}}$. Then, for $P^\bullet = P^\bullet_{\sigma_{\mathrm{Pic}^{\mathbb{1}}}}$ the standard basis of* $\mathrm{Hom}^*(P^\bullet, Y)$ *consists of graph maps and quasi maps. Moreover,*

1) *the graph maps are in bijection with substrings of ρ of the form $(a_i d_i)\overline{b_{i+1}}d_{i+1} \cdots \overline{b_{j-1}}d_{j-1}\overline{(c_j b_j)}$ or its inverse, and*

2) *the quasi maps are in bijection with substrings of ρ of the form $c_i d_i \overline{b_{i+1}} \cdots d_{j-1}\overline{b_j}\,\overline{a_j}$, or its inverse.*

Proof. The proof is analogous to the proof of Lemma III.5.4. As before, by the fact that all homotopy letters of $\sigma_{\mathrm{Pic}^{\mathbb{1}}}$ are (inverse) paths of length one, there exists no singleton single map and no singleton double map. We observe that a maximal common subword $\rho_a \cdots \rho_b$ of ρ and σ satisfies LG1 if and only if $\rho_a \in \{\overline{d_i}, \overline{b_i}\}$ for some $i \in [0, n]$ and no left endpoint condition if $\rho_a \in \{d_i, b_i\}$. The assertion follows in the same way as in the proof of Lemma III.5.4. $\square$

By Theorem I.4.3, it follows that the mapping cone of a graph map or quasi map $P^\bullet \to Y$ as in Lemma III.5.4 and Lemma III.5.5 is represented by the curve obtained by resolving the crossing of loops. As mentioned above (see p.131), the image of a homotopy class under a Dehn twist about a loop γ can be computed by resolving all intersections of γ with a representative of said homotopy class. As a consequence of this description, we deduce the following algorithm for computing the homotopy band of a twisted loop.

Lemma III.5.6. *Let σ be a homotopy band and $\gamma = \gamma(\sigma)$ be a loop. For each $i \in [0, n)$, the homotopy band $\sigma(D_{\gamma^i_{k(x)}} \circ \gamma)$ is obtained from σ by the following rules.*

a) Replace every substring of σ of the form (up to inverting the substring)

$$\overline{d_i}\,(\overline{c_i}a_i)^m\, b_i$$

by the substring

$$\begin{cases} \overline{d_i}\,(\overline{c_i}a_i)^{m-1}\, b_i, & \text{if } m \geq 1; \\ \overline{(a_i d_i)}(c_i b_i), & \text{if } m = 0. \end{cases}$$

b) Replace every substring of σ of the form (up to inverting the substring)

$$\overline{(a_i d_i)}\,(c_i\overline{a_i})^m\,(c_i b_i),$$

with $m \geq 0$, by the substring

$$\overline{(a_i d_i)}\,(c_i\overline{a_i})^{m+1}\,(c_i b_i).$$

The Dehn twist about $\gamma_{\mathrm{Pic}^{\mathbb{1}}}$ is described in a similar fashion.

Lemma III.5.7. *Let σ be a homotopy band and $\gamma = \gamma(\sigma)$ be a loop, which is in minimal position with the laminates on T_n. Then, $\sigma(D_{\gamma_{\mathrm{Pic}^{\mathbb{1}}}} \circ \gamma)$ is a homotopy band and is obtained from σ by the following rules.*

a) Replace every substring of σ of the form (up to inverting the substring)

$$(a_i d_i)u^m\overline{b_{i+1}}d_{i+1}\cdots\overline{b_{j-1}}d_{j-1}\overline{(c_j b_j)},$$

where $u = \overline{b_{i+1}}d_{i+1}\cdots\overline{b_i}d_i$, by the substring

$$\begin{cases} (a_i d_i)u^{m-1}\overline{b_{i+1}}d_{i+1}\cdots\overline{b_{j-1}}d_{j-1}\overline{(c_j b_j)}, & \text{if } m \geq 1; \\ a_i b_i\overline{d_{i-1}}\cdots b_{j+1}\overline{d_j}\overline{c_j}, & \text{if } m = 0. \end{cases}$$

b) Replace every substring of σ of the form (up to inverting the substring)

$$a_i b_i v^m \overline{d_{i-1}} \cdots b_{j+1} \overline{d_j} \overline{c_j},$$

where $v = b_i \overline{d_{i-1}} \cdots b_{i+1} \overline{d_i}$ and $m \geq 0$, by the substring

$$a_i b_i v^{m+1} \overline{d_{i-1}} \cdots b_{j+1} \overline{d_j} \overline{c_j}.$$

Our strategy to describe the isomorphism class of $T_{P^\bullet}(Y)$ with $P^\bullet$ and Y as in Proposition III.5.1 is to prove that the mapping cone, which determines $T_{P^\bullet}(Y)$, can be computed by computing mapping cones of basis elements locally. The necessary (however tedious) computations are subject of the subsequent lemmas.

Lemma III.5.8. *Let $m \in \mathbb{N}$ and let $\sigma = u\overline{d_i}\,(\overline{c_i} a_i)^m\, b_i v$ be a homotopy string with u and v denoting homotopy letters of σ. Let $f : P^\bullet_{\sigma_{k(x),i}} \to P^\bullet_\sigma$ be a graph map associated to the common subword $(\overline{c_i} a_i)^m$. Then its mapping cone* $\mathrm{Cone}(f)$ *is isomorphic as a chain complex to $P^\bullet_\tau \oplus Q$, where*

$$\tau = \begin{cases} u\overline{d_i}\,(\overline{c_i} a_i)^{m-1}\, b_i v, & \text{if } m \geq 1; \\ u\overline{(a_i d_i)}(c_i b_i) v, & \text{if } m = 0. \end{cases}$$

and $Q \cong 0$ in the homotopy category. The isomorphism can be chosen in such a way that it restricts to a multiple of the identity on the indecomposable projective summands corresponding to $t(u)$ and $s(v)$ occuring as prefix or suffix in both τ and σ, respectively.

Proof. This is a special case of the standard procedure used to transform a complex of projective modules into a minimal complex, see Lemma 2.4 in [32]. □

Note that the previous lemma (with the exception of the assumption on the isomorphism) is essentially the assertion of Theorem I.4.1. However, in order to reduce all computations to computations of 'local' nature the choice of the isomorphism is important. By the same argument as before we obtain:

Lemma III.5.9. *Let $m \in \mathbb{N}$ and let*

$$\sigma = p(a_i d_i) u^m \overline{b_{i+1}} d_{i+1} \cdots \overline{b_{j-1}} d_{j-1} \overline{(c_j b_j)} q,$$

where $u = \overline{b_{i+1}} d_{i+1} \cdots \overline{b_i} d_i$, be a homotopy string with p and q denoting homotopy letters of σ. Let $f : P^\bullet_{\sigma_{\overline{\mathrm{Pic}\mathbb{1}}}} \to P^\bullet_\sigma$ be a graph map associated to the common subword $u^m \overline{b_{i+1}} d_{i+1} \cdots \overline{b_{j-1}} d_{j-1}$. Then its mapping cone $\mathrm{Cone}(f)$

$$\tau = \begin{cases} p(a_i d_i) u^{m-1} \overline{b_{i+1}} d_{i+1} \cdots \overline{b_{j-1}} d_{j-1} \overline{(c_j b_j)} q, & \text{if } m \geq 1; \\ p a_i b_i \overline{d_{i-1}} \cdots b_{j+1} \overline{d_j} \overline{c_j} q, & \text{if } m = 0. \end{cases}$$

and $Q \cong 0$ in the homotopy category. The isomorphism can be chosen in such a way that it restricts to a multiple of the identity on the indecomposable projective summands corresponding to $t(p)$ and $s(q)$ occuring as prefix or suffix in both τ and σ, respectively.

Finally, we prove analogues of the previous lemmas for quasi maps.

Lemma III.5.10. *Let $m \in \mathbb{N}$ and let $\sigma = u(a_i d_i)\,(c_i\overline{a_i})^m\,(c_i b_i)v$ be a homotopy string with u and v denoting homotopy letters of σ. Let $f : P^\bullet_{\sigma_{k(x),i}} \to P^\bullet_\sigma$ be a quasi map associated to the common subword $(c_i\overline{a_i})^m$. Then its mapping cone $\operatorname{Cone}(f)$ is isomorphic as a chain complex to $P^\bullet_\tau$, where*

$$\tau = \overline{(a_i d_i)}\,(c_i\overline{a_i})^{m+1}\,(c_i b_i)$$

The isomorphism can be chosen in such a way that it restricts to a multiple of the identity on the indecomposable projective summands corresponding to $t(u)$ and $s(v)$ occuring as prefix or suffix in both τ and σ, respectively.

Proof. We present a series of basis transformations, which will gradually turn the differential of $\operatorname{Cone}(f)$ into the desired differential. First, suppose $m = 0$. Then, the matrix of the differential of $\operatorname{Cone}(f)$ (which is the matrix on the left) is transformed as follows. In order to improve readability we omitted the indices of the arrows.

$$\begin{pmatrix} \overline{u} & 0 & 0 \\ ad & cb & a \\ 0 & 0 & -(a+\lambda c) \\ 0 & 0 & v \end{pmatrix} \rightsquigarrow \begin{pmatrix} \overline{u} & 0 & 0 \\ 0 & cb & a \\ ad & 0 & -(a+\lambda c) \\ 0 & 0 & v \end{pmatrix} \rightsquigarrow \begin{pmatrix} \overline{u} & 0 & 0 \\ 0 & cb & a \\ ad & -cb & -(a+\lambda c) \\ 0 & 0 & v \end{pmatrix} \rightsquigarrow \begin{pmatrix} \overline{u} & 0 & 0 \\ 0 & cb & a \\ ad & 0 & -\lambda c \\ 0 & 0 & v \end{pmatrix}$$

In the first step we added the $(-d)$-multiple of the third column to the first. In the second step we added the $(\lambda^{-1}b)$-multiple of the third column to the second and in the last step we added the second row to the third row. The case $m > 0$ requires some more work.

Consider the following sequence of transformations applied to the differential of Cone(f).

$$
\begin{pmatrix}
\overline{u} \\
& -(a+\lambda c) \\
ad & a & c \\
& & a & c \\
& & & a & \ddots \\
& & & & \ddots & c \\
& & & & & a & cb \\
& & & & & & v
\end{pmatrix}
\rightsquigarrow
\begin{pmatrix}
\overline{u} \\
ad & -\lambda c & c \\
& a & c \\
& & a & c \\
& & & a & \ddots \\
& & & & \ddots & c \\
& & & & & a & cb \\
& & & & & & v
\end{pmatrix}
$$

$$
\rightsquigarrow
\begin{pmatrix}
\overline{u} \\
ad & -\lambda c \\
& a & \lambda^{-1}(a+\lambda c) \\
& & a & c \\
& & & a & \ddots \\
& & & & \ddots & c \\
& & & & & a & cb \\
& & & & & & v
\end{pmatrix}
\rightsquigarrow
\begin{pmatrix}
\overline{u} \\
ad & -\lambda c \\
& -\lambda a & -(a+\lambda c) \\
& & a & c \\
& & & a & \ddots \\
& & & & \ddots & c \\
& & & & & a & cb \\
& & & & & & v
\end{pmatrix}
$$

$$
\rightsquigarrow
\begin{pmatrix}
\overline{u} \\
ad & c \\
& a & -(a+\lambda c) \\
& & a & c \\
& & & a & \ddots \\
& & & & \ddots & c \\
& & & & & a & cb \\
& & & & & & v
\end{pmatrix}
$$

In the first step, we added the second row to the first and afterwards substracted the d-multiple of the second column from the first column. In the second step, we added the λ^{-1}-multiple of the second column to the third column. The fourth matrix is then obtained from the third by multiplying the second row by $-\lambda$. Finally, we multiply the second column by $-\lambda^{-1}$. Observe that the last matrix contains a submatrix (obtained by deleting the first row and the first column) which has almost the same form as the matrix we started with. By repeating the above transformations several times, we end up with the matrix

$$
\begin{pmatrix}
\overline{u} \\
ad & c \\
& a & c \\
& & a & \ddots \\
& & & \ddots & c \\
& & & & a & -(a+\lambda c) \\
& & & & & a & cb \\
& & & & & & v
\end{pmatrix}
\rightsquigarrow
\begin{pmatrix}
\overline{u} \\
ad & c \\
& a & c \\
& & a & \ddots \\
& & & \ddots & c \\
& & & & a & -\lambda c & cb \\
& & & & & a & cb \\
& & & & & & v
\end{pmatrix}
$$

$$
\rightsquigarrow
\begin{pmatrix}
\overline{u} \\
ad & c \\
& a & c \\
& & a & \ddots \\
& & & \ddots & c \\
& & & & a & -\lambda c \\
& & & & & a & cb \\
& & & & & & v
\end{pmatrix}
\rightsquigarrow
\begin{pmatrix}
\overline{u} \\
ad & c \\
& a & c \\
& & a & \ddots \\
& & & \ddots & c \\
& & & & a & c \\
& & & & & a & -\lambda cb \\
& & & & & & v
\end{pmatrix}
$$

All transformations were chosen carefully such that, beside a multiplication by $-\lambda^{-1}$, no further transformations have been applied to the projective modules corresponding to $t(u)$ and $s(v)$. □

Via the same approach one proves an analogue of the previous lemma for quasi maps $P^{\bullet}_{\sigma_{\mathrm{Pic}^{\mathbb{1}}}} \to Y$. We finally give the proof of Proposition III.5.1.

Proof of Proposition III.5.1. By Lemma III.5.5 and Lemma III.5.4, we see that the common subwords u, v of two distinct basis elements $f, f' : P^{\bullet} \to Y$ can only intersect at their end vertices. Therefore, and by Lemma III.5.8, Lemma III.5.9, Lemma III.5.10 and its analogue, it follows that the mapping cone which defines $T_{P^{\bullet}}(Y)$ is computed locally and it follows that the homotopy band defining $T_{P^{\bullet}}(Y)$ is obtained by applying the replacement rules for substrings in the lemmas above one after another (or all at the same time). The description of the action of the corresponding Dehn twist in Lemma III.5.6 and Lemma III.5.7 then shows that $T_{P^{\bullet}}$ and D_{δ} act on homotopy bands in the same way. This finishes the proof. □

III.6 Proofs of Theorem III.0.1 and Theorem III.0.2

Corollary III.3.4 and the following Lemma imply Theorem III.0.1.

Lemma III.6.1. *Let γ be a simple loop in T_n. Then, γ represents an object in $\mathcal{D}^b(\Lambda_n)$ if and only if γ is non-separating.*

Proof. It follows from Proposition III.5.1, that the Dehn twists associated to the loops $\gamma_{\mathrm{Pic}^{\mathbb{1}}}$ and $\gamma^0_{k(x)}, \dots, \gamma^{n-1}_{k(x)}$ preserve the set of homotopy classes of loops in T_n which represent objects in $\mathcal{D}^b(\Lambda_n)$ and hence Theorem III.4.2 implies that every pure mapping class preserves this set as well. We have seen that the pure mapping class group acts transitively on all homotopy classes of a fixed strong topological type (Proposition III.4.5) and that all homotopy classes of non-separating simple loops belong to the same strong topological type (Lemma III.4.7). It therefore follows from the fact that $\gamma_{\mathrm{Pic}^{\mathbb{1}}}$ represents an object in $\mathcal{D}^b(\Lambda_n)$ (namely images of lines bundles of multi-degree $\mathbb{1}$), that γ represents an object as well. The same arguments together with Lemma III.4.11 and Lemma III.4.12 imply that no separating simple loop represents an object of $\mathcal{D}^b(\Lambda_n)$. □

In what follows, denote $\mathrm{Tw}(\Lambda_n)$ the group of auto-equivalences generated by all twists $T_{P^{\bullet}}$, where $P^{\bullet} = P^{\bullet}_{\sigma,\lambda}$ for $\lambda \in k^{\times}$ and $\sigma \in \{\sigma_{\mathrm{Pic}^{\mathbb{1}}}, \sigma_{k(x),i} \mid i \in [0, n)\}$.

We show that all spherical twists by objects in $\mathcal{D}^b(\Lambda_n)$ correspond to the Dehn twist of their associated simple curves.

Proposition III.6.2. *Let $X, Y \in \mathcal{D}^b(\Lambda_n)$, such that X is spherical and Y is a linear band complex. Let further $\gamma \in \gamma(X)$ be simple. Then $\gamma(T_X(Y))$ is equal to the image of the homotopy class of $\gamma(Y)$ under D_γ and there exists an equivalence $\mathcal{G} \in \mathrm{Tw}(\Lambda_n)$, such that $\mathcal{G}(X) \cong P^\bullet_{\sigma_{\mathrm{Pic}^{\mathbb{1}}},\lambda}[m]$ for some pair $(\lambda, m) \in k^\times \times \mathbb{Z}$.*

Proof. Set $P^\bullet(\lambda) := P^\bullet_{\sigma_{\mathrm{Pic}^{\mathbb{1}}},\lambda}$. By Lemma III.6.1 and Corollary III.4.8, there exists $F \in \mathcal{PMCG}(T_n)$, such that $F \circ \gamma \simeq \gamma_{\mathrm{Pic}^{\mathbb{1}}}$ and, by Theorem III.4.2, F is isotopic to a composition $D^{\sigma_1}_{\gamma_1} \cdots D^{\sigma_m}_{\gamma_m}$ for loops

$$\gamma_1, \dots, \gamma_m \in \{\gamma_{\mathrm{Pic}^{\mathbb{1}}}, \gamma^0_{k(x)}, \dots, \gamma^{n-1}_{k(x)}\},$$

where $\sigma_1, \dots, \sigma_m \in \{\pm 1\}$. It follows from Lemma III.4.1 that $D_\gamma = D_{F^{-1}(\gamma_{\mathrm{Pic}^{\mathbb{1}}})}$ is isotopic to

$$F^{-1} \circ D_{\gamma_{\mathrm{Pic}^{\mathbb{1}}}} \circ F = D^{-\sigma_m}_{\gamma_m} \circ \cdots \circ D^{-\sigma_1}_{\gamma_1} \circ D_{\gamma_{\mathrm{Pic}^{\mathbb{1}}}} \circ D^{\sigma_1}_{\gamma_1} \circ \cdots \circ D^{\sigma_m}_{\gamma_m}.$$

For all $i \in [1, m]$, denote X_i a spherical object corresponding to γ_i. Applying Proposition III.5.1 $2m+1$ times then shows that for all $\mu \in k^\times$

$$D_\gamma(\gamma(Y)) = \gamma\left(\mathcal{G}^{-1} \circ T_{P^\bullet(\mu)} \circ T(Y)\right) = \gamma(T_{\mathcal{G}^{-1}(P^\bullet(\mu))}(Y)), \tag{III.1}$$

where

$$\mathcal{G} = T^{\sigma_1}_{X_1} \circ \cdots \circ T^{\sigma_m}_{X_m} \in \mathrm{Tw}(\Lambda_n).$$

By Proposition III.5.1, it follows $F \circ \gamma \simeq \gamma_{\mathrm{Pic}^{\mathbb{1}}}$ and hence $\mathcal{G}(X) \cong P^\bullet(\lambda)[m]$ for a unique pair $(\lambda, m) \in k^\times \times \mathbb{Z}$. This proves the second assertion. Choosing $\mu := \lambda$ in (III.1) proves the first assertion. □

Lemma III.6.3. *Let $\lambda, \mu \in k^\times$. Denote by $P^\bullet(\alpha)$ the complex $P^\bullet_{\sigma_{\mathrm{Pic}^{\mathbb{1}}},\alpha}$. Then, there exists $\mathcal{G} \in \mathrm{Tw}(\Lambda_n)$, such that $\mathcal{G}(P^\bullet(\lambda)) \cong P^\bullet(\mu)$.*

Proof. For $\alpha \in k^\times$ set $X^\bullet(\alpha) := P^\bullet_{\sigma_{k(x),0},\alpha}$ and

$$\mathcal{G}_\alpha := T^{-1}_{X^\bullet(1)} \circ T_{X^\bullet(\alpha)}.$$

Then, for all $\alpha \in k^\times$, $\dim \mathrm{Hom}^*(X^\bullet(\alpha), P^\bullet(\lambda)) = 1$ as it is equal to the number of intersections of the corresponding loops. In particular the twist $T_{X^\bullet(\alpha)}(P^\bullet(\lambda))$ is given by the mapping cone of a single graph map. By Proposition 4.5, [31], it follows that $T_{X^\bullet(\alpha)}(P^\bullet(\lambda)) \cong T_{X^\bullet(1)}(P^\bullet(\alpha^{-1}\lambda))$ and therefore $\mathcal{G}_{\lambda^{-1}\mu}(P^\bullet(\lambda)) \cong P^\bullet(\mu)$.

□

Corollary III.6.4. *The group* $\mathrm{Tw}(\Lambda_n)$ *coincides with the group of auto-equivalences generated by all Dehn twists.*

Proof. Proposition III.6.2 and Lemma III.6.3 show that $\mathrm{Tw}(\Lambda_n)$ acts transitively on isomorphism classes of spherical objects up to shift. Thus, by Corollary III.2.6, every spherical twist is conjugate to an element in $\mathrm{Tw}(\Lambda_n)$ by an element in $\mathrm{Tw}(\Lambda_n)$. $\square$

Finally, we obtain the following result (see Corollary G in introduction):

Corollary III.6.5. *The group of auto-equivalences of* $\mathcal{D}^b(\mathrm{Coh}\, E_n)$ *generated by all spherical twists is generated by* $T_{\mathcal{O}_{E_n}}$ *and the functors* $- \otimes \mathbb{L}\mathcal{L}(x)$, $x \in E_n$ *smooth, where* $\mathcal{L}(x)$*, denotes the line bundle associated to* x*.*

Proof. By Corollary III.6.4, it follows that the twist functors associated to smooth points together with the twist functor of an arbitrary line bundle $\mathcal{L}$ of multi-degree $\mathbb{1} = (1, \ldots, 1)$ generate the group of auto-equivalences generated by all twist functors. As pointed out in Example 2.12 in [28], it follows from [70], that $T_{k(x)} \cong - \otimes_L \mathcal{L}(x)$. Finally, note that that there exist $x_0 \in \mathbb{P}^1_0, \ldots, x_{n-1} \in \mathbb{P}^1_{n-1}$, such that $\mathcal{L} \cong \bigotimes_{i=0}^{n-1} \mathcal{L}(x_i)$. $\square$

Appendix

Images of skyscraper sheaves and line bundles

This section contains the proofs of Theorem III.1.2 and Theorem III.1.3 which were explained to us by Igor Burban. We frequently use results from [25]. For convenience of the reader we repeat the assertions of the theorems. Recall that the gentle algebra Λ_n is the quotient of the path algebra of the quiver Γ_n, given by

$$\begin{tikzcd}
\bullet & \bullet & \cdots & \bullet \\
\bullet \arrow[u, "d_0"] \arrow[urrr, bend left, "b_0"] & \bullet \arrow[u, "d_1"] \arrow[ul, "b_1"] & \cdots \arrow[ul, "b_2"] & \bullet \arrow[u, "d_{n-1}"'] \arrow[ul, "b_{n-1}"] \\
\bullet \arrow[u, bend left, "a_0"] \arrow[u, bend right, "c_0"'] & \bullet \arrow[u, bend left, "a_1"] \arrow[u, bend right, "c_1"'] & \cdots & \bullet \arrow[u, bend left, "a_{n-1}"] \arrow[u, bend right, "c_{n-1}"']
\end{tikzcd}$$

by the ideal generated by the relations $\{a_i b_i, c_i d_i \mid i \in [0, n)\}$. The curve E_n denotes an n-cycle of projective lines as defined in Section III.1.

Theorem (Burban)**.** *Let $n \in \mathbb{N}$ and let $Y \in \mathcal{D}^b(\Lambda_n)$. Then there exists $\mathbb{L} \in \mathrm{Pic}^{\mathbb{1}}(E_n)$, such that $Y \cong \mathbb{F}(\mathbb{L})$ if and only if Y is isomorphic to a band complex $P^\bullet_{\sigma_{\mathrm{Pic}^{\mathbb{1}}},\lambda}$, which is concentrated in degrees 0 and 1, where $\lambda \in k^\times$ and $\sigma_{\mathrm{Pic}^{\mathbb{1}}} = d_0\overline{b_1}d_1 \cdots d_{n-1}\overline{b_{n-1}}$.*

Theorem (Burban)**.** *Let $n \in \mathbb{N}$ and let $Y \in \mathcal{D}^b(\Lambda_n)$. Then there exists a smooth point $x \in \mathbb{P}^1_i$, such that $Y \cong \mathbb{F}(k(x))$ if and only if Y is isomorphic to a band complex $P^\bullet_{\sigma_{k(x),i},\lambda}$, which is concentrated in degree -1 and 0, where $\lambda \in k^\times$ and $\sigma_{k(x),i} = a_i\overline{c_i}$.*

Notation & Setup For the rest of this section fix $n \in \mathbb{N}$ and set $X := E_n$. X is a union of irreducible component $X_0, \dots, X_{n-1}$. Let $\pi : \widetilde{X} \to X$ denote a normalization map. Since X is rational, $\widetilde{X} = \sqcup_{i=0}^{n-1} \widetilde{X}_i$, where $\widetilde{X}_i \cong \mathbb{P}^1$

is the normalization of X_i. By changing coordinates, we may assume that $0, \infty \in \mathbb{P}^1$ are the preimages of the intersection of the singular locus with X_i. Set $\widetilde{\mathcal{O}} := \pi_*(\mathcal{O})$, where $\mathcal{O} = \mathcal{O}_X$ denotes the sheaf of regular functions on X and denote $\mathcal{I}$ the ideal sheaf of the singular locus on X. Note that $\mathcal{O}$ is a subsheaf of $\widetilde{\mathcal{O}}$. In [25] the authors defined the so-called *Auslander-sheaf* $\mathscr{A}$ as the sheaf of orders on X given by

$$\mathscr{A} = \begin{pmatrix} \mathcal{O} & \widetilde{\mathcal{O}} \\ \mathcal{I} & \widetilde{\mathcal{O}} \end{pmatrix}.$$

$\mathcal{O}$ is embedded into $\mathscr{A}$ diagonally. In [25] the authors proved that the bounded derived category $\mathcal{D}^b(\operatorname{Coh}\mathbb{X})$ of coherent sheaves over the non-commutative curve $\mathbb{X} = \mathbb{X}_n = (E_n, \mathscr{A})$ admits a tilting object $\mathscr{H}$ defined a follows. Let $\mathcal{S}$ denote the torsion sheaf $\mathscr{A}$-modules defined as the cokernel of the canonical inclusion of $\begin{pmatrix} \mathcal{I} \\ \mathcal{I} \end{pmatrix}$ in $\begin{pmatrix} \mathcal{O} \\ \mathcal{I} \end{pmatrix}$. Furthermore, set

$$\mathcal{F} := \mathscr{A}e_1 = \begin{pmatrix} \mathcal{O} \\ \mathcal{I} \end{pmatrix} \quad \text{and} \quad \mathcal{P} := \mathscr{A}e_2 = \begin{pmatrix} \widetilde{\mathcal{O}} \\ \widetilde{\mathcal{O}} \end{pmatrix}.$$

The tensor product $\mathcal{F} \otimes_{\mathcal{O}} -$ defines a functor $\mathbb{J} : \operatorname{Coh} X \to \operatorname{Coh}\mathbb{X}$. Moreover, $\mathbb{J}$ is fully faithful and has a right adjoint. By definition, we have $\mathcal{P} = \bigoplus_{i=0}^{n-1} \mathcal{P}_i$, where $\mathcal{P}_i := \mathbb{J}(\pi_*(\mathcal{O}_{\widetilde{X_i}})) =$. Then

$$\mathcal{H} := \mathcal{S}[-1] \oplus \mathcal{P}(-1) \oplus \mathcal{P}.$$

Here, for every integer $m \in \mathbb{Z}$ we define

$$\mathcal{P}(m) := \bigoplus_{i=0}^{n-1} \mathcal{P}_i \otimes_{\mathcal{O}} \mathcal{L},$$

where there tensor product is defined componentwise and $\mathcal{L}$ is any line bundle of multi-degree $(m, \ldots, m)$ on X. As shown in Lemma 4, [25], the isomorphism class of $\mathcal{P}(m)$ (and hence the isomorphism class of $\mathcal{H}$) is independent of the choice of $\mathcal{L}$. Burban and Drozd showed that Λ_n^{op} is isomorphic to the endomorphism algebra of $\mathcal{H}$ by identifying the indecomposable summands of $\mathcal{H}$ with the vertices of the quiver Γ_n.

Images of skyscraper sheaves of smooth points: Let $x \in X_i$ be a smooth point corresponding to a point $(\lambda : \mu) \in \mathbb{P}^1$. Note that by our assumption on the chosen coordinates, $\lambda, \mu \neq 0$. Let $\widetilde{x} \in \widetilde{X_i}$ denote the unique preimage of x under π. Then, $\pi_*(k(\widetilde{x})) \cong k(x)$ and if $z_0^i, z_\infty^i : \mathcal{O} \to$

$\mathcal{O}(1)$ correspond to the chosen coordinates on $\widetilde{X}_i$ vanishing at 0 and ∞ respectively, then the cokernel of the map $\mu z_0(-1) - \lambda z_\infty(-1)$ ($-(m)$ denoting the Serre twist) is $k\,(\lambda : \mu)$. Keeping in mind that $\mathbb{J} \circ \pi_*(\mathcal{O}_{\widetilde{X}_i}(m)) = \mathcal{P}_i(m)$, we deduce that there is a short exact sequence

$$0 \longrightarrow \mathcal{P}_i(-1) \xrightarrow{\mu z_0^i - \lambda z_\infty^i} \mathcal{P}_i \longrightarrow \mathbb{J}(k(x)) \longrightarrow 0.$$

The tilting functor associated with $\mathcal{H}$ sends $\mathcal{P}_i(-1)$ to the indecomposable projective of $s(a_i)$ and $\mathcal{P}_i$ to the indecomposable projective of $t(a_i)$ proving that $\mathbb{F}(k(x))$ is isomorphic to the complex described in the first theorem of this section.

Images of line bundles of multi-degree $(1, \dots, 1)$: Let $\mathcal{L}$ be a line bundle on X of multi-degree $(1, \dots, 1)$ and set $\mathcal{G} := \mathbb{J}(\mathcal{L}) = \mathcal{F} \otimes_{\mathcal{O}} \mathcal{L}$. We want to compute the dimension of the vector spaces $\mathrm{Ext}^i(\mathcal{S}, \mathcal{G})$, $\mathrm{Ext}^i(\mathcal{P}, \mathcal{G})$ and $\mathrm{Ext}^i(\mathcal{P}(-1), \mathcal{G})$ for $i \geq 0$. Note that by Theorem 2, [25], the global dimension of $\mathrm{Coh}\,\mathbb{X}$ is 2.

Since $\mathcal{S}$ is a torsion sheaf and $\mathcal{G}$ is torsion-free it follows $\mathcal{H}om_{\mathscr{A}}(\mathcal{S}, \mathcal{G}) \cong 0$. Therefore it global sections vanish, i.e. $\mathrm{Hom}_{\mathscr{A}}(\mathcal{S}, \mathcal{G}) = 0$. In a similar spirit, we can deduce from local computation of $\mathcal{E}xt(\mathcal{S}, \mathcal{G})$ and the local-to-global spectral sequence that $\mathrm{Ext}^1(\mathcal{S}, \mathcal{G}) \cong k$. Since $\mathcal{S}$ has a locally projective resolution of length one (by definition it is the cokernel of an injective map between locally projective sheaves) $\mathcal{E}xt^i_{\mathscr{A}}(\mathcal{S}, \mathcal{G}) = 0$ for all $i \geq 2$. An application of the local-to-global spectral sequence implies $\mathcal{E}xt^i_{\mathscr{A}}(\mathcal{S}, \mathcal{G}) = 0$ for all $i \geq 2$.

By Corollary 4, [25], there exists an isomorphism of $\mathcal{O}$-modules

$$\mathcal{I} \cong \mathcal{H}om_{\mathscr{A}}(\mathcal{P}, \mathcal{F}).$$

We have a sequence of isomorphisms

$$\mathcal{H}om_{\mathscr{A}}(\mathcal{P}(-1), \mathcal{G}) \cong \mathcal{H}om_{\mathscr{A}}(\mathcal{P}(-1) \otimes_{\mathcal{O}} \mathcal{L}^\vee, \mathcal{F}) \cong \mathcal{H}om_{\mathscr{A}}(\mathcal{P}(-2), \mathcal{F}),$$

where $\mathcal{L}^\vee$ denotes the dual of $\mathcal{L}$. Using that $\mathcal{P}(-2) \cong \begin{pmatrix} \mathcal{I} \\ \mathcal{I} \end{pmatrix}$ and that both $\mathcal{F}$ and $\mathcal{P}(-2)$ are torsion free, it follows from Proposition 6, [25], that $\mathcal{H}om_{\mathscr{A}}(\mathcal{P}(-1), \mathcal{G}) \cong \widetilde{\mathcal{O}}$ as $\mathcal{O}$-modules. Since $k \cong \Gamma(X, \mathcal{O}) \cong \Gamma(X, \widetilde{\mathcal{O}})$, we deduce $\mathrm{Hom}_{\mathscr{A}}(\mathcal{P}(-1), \mathcal{G}) \cong k$. Moreover, it follows from Corollary 3, [25], that $\mathrm{Ext}^i(\mathcal{P}(-1), \mathcal{G}) = 0$ for all $i \geq 1$.

Since $\mathbb{F}(\mathcal{L})$ is indecomposable, we conclude by the classification of all τ-invariant indecomposable objects in $\mathcal{D}^b(\Lambda_n)$ (these are precisely the band complexes) that $\mathbb{F}(\mathcal{L})$ is a band complex of the shape described in the second theorem above.

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Köln, 28.05.2019, Sebastian Opper

Preprints:

- S. Opper, P-G. Plamondon and S. Schroll, *A geometric model for the derived category of gentle algebras*, arXiv preprint (2018): `https://arxiv.org/abs/1801.09659`
- S. Opper, *On auto-equivalences and complete derived invariants of gentle algebras*, arXiv preprint (2019): `https://arxiv.org/abs/1904.04859`