The Hamiltonian Vlasov Equation

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Kurzzusammenfassung

Ein Vlasov System beschreibt auf statistische Weise die Zeitentwicklung einer großen Zahl von ununterscheidbaren Teilchen, die gegenseitig in Wechselwirkung stehen. Beispiele sind etwa Elektronengase oder Sternenwolken. In dieser Arbeit wird eine Hamiltonsche Struktur für das Vlasov System untersucht.

Es wird ein allgemeines Schema entwickelt, mit dessen Hilfe auf dem Raum der komplexwertigen quadratintegrablen Funktionen $\alpha \in \mathcal{L}^2$ auf dem Einteilchenphasenraum $\mathbb{R}^{2d}_{\mathbf{z}}$ ein neues Hamiltonsches System, genannt das Hamiltonsche Vlasov System, definiert wird. Für jede Lösung $t \mapsto \alpha(t)$ dieses Systems ist dann $t \mapsto f(t) \equiv |\alpha(t)|^2$ eine Lösung des zugehörigen Vlasov Systems. Die einzige Voraussetzung ist die hinreichende Regularität des Energiefunktionals auf einem Unterraum der Dichtefunktionen $f \in \mathcal{L}^1$. Dieses Vorgehen verallgemeinert bekannte Ergebnisse von Fröhlich, Knowles und Schwarz [7].

Für dieses Hamiltonsche Vlasov System wird die globale Wohlgestelltheit des Problems anhand zweier Wechselwirkungspotentiale bewiesen, einer Klasse von regulären Potentialen einerseits, sowie des Coulomb-Potentials, welches zum namhaften Vlasov–Poisson System führt, andererseits.

Man nutzt eine strukturelle Parallele zwischen dem Hamiltonschen Vlasov System und dem quantenmechanischen Hartree System, welche durch Anwendung einer Fouriertransformation in den Geschwindigkeitsvariablen offenkundig wird. Anhand dieses neuen Hamiltonschen Hartree Systems wird ein Vielteilchenlimes für den Fall nichtrelativistischer kinetischer Energie und regulärer Wechselwirkung begründet. Die verwendete Methode hat ihren Ursprung in der Quantenmechanik, kann hier aber für den klassischen Fall erweitert werden.

Schlussendlich wird ein allgemeiner Ansatz diskutiert, um periodische Lösungen des Vlasov Systems unter Ausnutzung des Hamiltonschen Formalismus zu finden. Dieser erlaubt etwa, solche Lösungen erstmals als kritische Punkte eines Wirkungsfunktionals zu charakterisieren. Des Weiteren kann die Erhaltung von Masse und Impuls erstmals als Konjugierte zu Noethersymmetrien identifiziert werden. Die Methode der Marsden–Weinstein Symmetriereduktion [12] hilft uns, die Phaseninvarianz zu elimi-

nieren, indem das System auf die Quotientenmannigfaltigkeit $\mathbb{S}^{\mathcal{L}^2}/\mathbb{S}^1$ projiziert wird. Es zeigt sich, dass dieser Prozess nötig ist, um die Trajektorien zu schließen. Um die Vorteile der Reduktion zu demonstrieren, wird sie am vereinfachten Beispiel des Harmonischen Vlasov Systems, einem nichtrelativistischen Vlasov System mit anziehendem harmonischen Wechselwirkungspotential, explizit durchgeführt.

Abstract

The Vlasov system describes the dynamics of large collections of indistinguishable particles, which interact with each other, in a statistical manner. Important examples are electron gas or stellar clusters. In this thesis, we study a general Hamiltonian structure of the Vlasov system.

We give a formal guideline to derive a new system, the Hamiltonian Vlasov system, on the space of complex-valued square-integrable functions $\alpha \in \mathcal{L}^2$ on the one-particle phase space $\mathbb{R}^{2d}_{\mathbf{z}}$. This system's key property is that for any solution $t \mapsto \alpha(t)$ of this system, the trajectory $t \mapsto f(t) = |\alpha(t)|^2$ is a solution of the Vlasov system. The only requirement is a sufficiently regular energy functional on a subspace of density functions $f \in \mathcal{L}^1$, greatly generalizing the previous findings of Fröhlich, Knowles, and Schwarz [7].

For this newly obtained Hamiltonian Vlasov system, we prove global well-posedness for the two cases of regular and Coulomb interaction potentials, where the latter yields the famous Vlasov–Poisson system.

We exploit a peculiar parallel between the Hamiltonian Vlasov system and the quantum mechanical Hartree system, achieved by a Fourier transform in the velocity coordinates. In the context of this newly obtained Hamilton Hartree system, we establish a mean field limit for the case of non-relativistic kinetic energy and regular two-body interaction. The used method originates from the study of quantum mechanical systems, but allows extension to a classical problem here.

Finally, we discuss a general approach for finding periodic solutions bifurcating from equilibrium points of classical Vlasov systems, utilizing the Hamiltonian Vlasov framework. In particular, the Hamiltonian structure allows to characterize periodic solutions as critical points of an action functional, previously not possible. Also, the formalism allows to identify some conserved quantities, such as mass and linear momentum, as Noether conjugate to symmetries of the Hamiltonian. Through the Marsden–Weinstein symmetry reduction [12], we specifically reduce the phase equivariance and map the problem to a Hamiltonian system on the quotient manifold $\mathbb{S}^{\mathcal{L}^2}/\mathbb{S}^1$. This process proves to be necessary to close many trajectories of the dynamics. We use the Harmonic Vlasov

system, a non-relativistic Vlasov equation with attractive harmonic two-body interaction potential, as a toy model to apply the method to. The simple structure of this model allows to compute all of its solutions directly and therefore, test the benefits of the Hamiltonian formalism and symmetry reduction in Vlasov systems.

Contents

Chapter 1

Introduction

The following chapter contains results from the author's work published or submitted in [14, 15, 16].

The Vlasov system has been introduced by the Russian physicist Anatoly Vlasov [23] in order to describe the collective behavior of electron gas. It is a classical model in the sense that it does not account for quantum mechanical effects. The guiding idea behind the equation is to reduce the information about a large number of particles to a statistical space density, which is certainly justified, if the observation scale is too coarse to distinguish between individual particles. The result is a first order partial differential equation (PDE), rather than a huge system of first order ordinary differential equations (ODEs).

In the decades since its introduction, the model has not only been used to describe the collective electron dynamics, but also to model galactic dynamics. Given the Newtonian interaction potential in both cases and the system's effective potential, given by the solution of the Poisson equation in three dimensions, this specific system has been known as the *Vlasov–Poisson system*. Other famous Vlasov systems feature electromagnetic fields (*Vlasov–Maxwell*) or relativistic motion (*Vlasov–Einstein*).

Generally, Vlasov systems are embedded into the bigger class of *Boltzmann equations*, which model time evolution of non-equilibrium thermodynamical systems. Vlasov equations are a special class of *collisionless* Boltzmann equations, as they do not model dissipative collision or diffusion. This is a key property, motivating the whole perspective of this work, which is to put those conservative systems into a symplectic framework. The equation has the structure of a non-linear transport equation, with non-linearities arising from the interaction between the particles.

Given a non-dissipative system, a natural question to ask is whether the system is equipped with a Hamiltonian structure. This structure provides access to techniques and results from finite dimensional Hamiltonian dynamics, e.g., analysis of closed orbits or stability. It also allows to derive the system's evolution equation as the extremal condition of an action functional, in mathematical physics known as *Hamilton's principle*.

An early paper by Ye, Morrison, and Crawford [25] contributes to the subject of Hamiltonian structure of Vlasov systems by formal derivation of a Lie-Poisson bracket for the linear space of density functions. They argue to remove the degeneracy of this bracket stemming from certain conserved quantities called Casimir functions by foliating the linear space into symplectic leaves which are the orbits of the symplectomorphism group, acting by composition.

Conjecturing that all these symplectomorphisms can be obtained as time-1 maps of autonomous Hamiltonian systems, they compute a Poisson bracket restricted on the leaf, which is claimed to be no longer degenerate. They expect that this non-degenerate bracket is the cosymplectic form of the Kirillov–Kostant–Souriau form from finite-dimensional theory, turning the leaf into a symplectic manifold.

As intriguing as these ideas are, a rigorous verification of the formal computation is still missing. In addition, the derived bracket on the symplectic leaf is by its implicit definition so difficult to handle, that no further results have grown from it.

A different ansatz occurs in the work of Fröhlich, Knowles, and Schwarz [7] as starting point to compute mean-field limits of bosonic quantum systems. Therein, denoting the density by $f = |\alpha|^2$, where $\alpha \in \mathcal{L}^2$ is complex valued, the symplectic form $\omega(\alpha,\beta) \equiv \Im \langle \alpha,\beta\rangle_{\mathcal{L}^2}$ on $\mathcal{L}^2$ allows to find a new Hamiltonian functional $\mathcal{H}_{\mathrm{Vl}}(\alpha)$ which can no longer be seen as the system's energy, but leads to a Hamiltonian evolution equation for α, that reproduces the Vlasov equation for $|\alpha|^2$. In [7] the explicit case of non-relativistic dynamics with velocity-independent two-body interaction potential is treated. In addition to this, we find an underlying structure that generalizes to any Vlasov system and we contribute to the subject with the first rigorous discussion of the new *Hamiltonian Vlasov system*.

This thesis is structured as follows. In the introduction, we present the standard derivation of the Vlasov system. We continue by giving a formal concept for the symplectization of *any* such system, starting with the energy functional $\mathcal{H}(f)$, where f is a non-negative density function on the one-particle phase space, extending findings in [7]. In addition, we introduce a new system, which we call the *Hamilton Hartree system* due to its structural analogy to the quantum mechanical Hartree system. These structural results are adapted from the three publications [14, 15, 16]. The introduction closes with a discussion of the main results.

We then use the derived formalism to discuss various questions rigorously. In Chapter 3, we prove global well-posedness for two different Hamiltonian Vlasov systems based on the publications [14, 16]. Chapter 4 focuses on the existence of a mean field limit for the Hamiltonian Vlasov system. The main results stem from [16]. Chapter 5 discusses the method of symplectic symmetry reduction by Marsden–Weinstein, applied to the Hamiltonian Vlasov setup. The reduction method is discussed specifically in the context of periodic solutions. The findings are adapted from [15].

The final chapter gives a summary of the achieved results and an outlook on possible further studies in the field of the Hamiltonian Vlasov system.

In the following sections, we formally derive a Hamiltonian evolution equation for a general Vlasov equation. The positive integer $d \in \mathbb{N}$ denotes the dimension of the underlying physical system, usually $d = 3$. The generic coordinates $\mathbf{z} = (\mathbf{x}, \mathbf{v})$ parameterize the phase space consisting of location ($\mathbf{x}$) and velocity ($\mathbf{v}$) component.

1.1 The Classical Vlasov System

We begin by deriving the Vlasov equation from first principles. For the most general setup we consider a density function $f : \mathbb{R}^{2d}_{\mathbf{z}} \to \mathbb{R}$ on the one-particle phase space $\mathbb{R}^{2d}_{\mathbf{z}} = \mathbb{R}^{d}_{\mathbf{x}} \times \mathbb{R}^{d}_{\mathbf{v}}$. We assume that the full system's energy functional depending on the density f is given by $\mathcal{H}(f) \in \mathbb{R}$. A typical example is given by

$$\mathcal{H}(f) \equiv \int_{\mathbb{R}^{2d}_{\mathbf{z}}} \epsilon(\mathbf{v})\, f(\mathbf{z})\, \mathrm{d}\mathbf{z} + \frac{1}{2} \int_{\mathbb{R}^{2d}_{\mathbf{z}}} \int_{\mathbb{R}^{2d}_{\mathbf{z}}} \Gamma(\mathbf{x}_1 - \mathbf{x}_2)\, f(\mathbf{z}_1)\, f(\mathbf{z}_2)\, \mathrm{d}\mathbf{z}_1\, \mathrm{d}\mathbf{z}_2. \tag{1.1}$$

The first term contributes the kinetic energy with the velocity dependent energy density $\epsilon : \mathbb{R}^{d}_{\mathbf{v}} \to \mathbb{R}$. The second term, which is quadratic in f, represents the self-consistent potential energy stemming from a symmetric two-body interaction potential $\Gamma : \mathbb{R}^{d}_{\mathbf{x}} \to \mathbb{R}$. In this example, the potential only depends on the relative position of the particles, an assumption satisfied in most physical systems. The most interesting classical systems are all in this form.

We restrict our general choice of $\mathcal{H}(f)$ by the technical requirement that the first functional derivative

$$\mathrm{D}^1\mathcal{H}(f)(\delta f) \equiv \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(\mathcal{H}(f + \varepsilon\, \delta f) - \mathcal{H}(f)\right) = \int_{\mathbb{R}^{2d}_{\mathbf{z}}} H^{(1)}_f(\mathbf{z})\, \delta f(\mathbf{z})\, \mathrm{d}\mathbf{z} \tag{1.2}$$

exists for a kernel function $H^{(1)}_f : \mathbb{R}^{2d}_{\mathbf{z}} \to \mathbb{R}$. For (1.1), this means

$$H^{(1)}_f(\mathbf{z}) = \epsilon(\mathbf{v}) + \int_{\mathbb{R}^{2d}_{\mathbf{z}}} \Gamma(\mathbf{x} - \bar{\mathbf{x}})\, f(\bar{\mathbf{z}})\, \mathrm{d}\bar{\mathbf{z}}.$$

For a static background density f, the kernel function $H_f^{(1)}$ can physically be interpreted as the unit point mass one-particle autonomous Hamiltonian. If one replaces f by some time-dependent $f(t)$, the Hamiltonian becomes non-autonomous.

The derivation of the classical Vlasov dynamics now requires solving the self-consistent system of characteristic equations

$$\begin{aligned}\partial_t X(t,s,\mathbf{z}) &= \left(\nabla_{\mathbf{v}} H^{(1)}_{f(t)}\right)(X(t,s,\mathbf{z}), V(t,s,\mathbf{z})),\\ \partial_t V(t,s,\mathbf{z}) &= \left(-\nabla_{\mathbf{x}} H^{(1)}_{f(t)}\right)(X(t,s,\mathbf{z}), V(t,s,\mathbf{z}))\end{aligned} \tag{1.3}$$

subject to the conditions

$$(X(t=s,s,\mathbf{z}), V(t=s,s,\mathbf{z})) = \mathbf{z} \quad \text{and} \quad f(t,\mathbf{z}) = \mathring{f}(X(0,t,\mathbf{z}), V(0,t,\mathbf{z})). \tag{1.4}$$

These conditions combined force the density $f(t)$ to follow the same trajectories as a test particle under the influence of the background density $f(t)$. This fact justifies to call the model self-consistent. The given set of equations (1.3) and (1.4) can be reduced to the single transport equation

$$\begin{aligned}0 &= \partial_t f(t,\mathbf{z}) + \left(\nabla_{\mathbf{v}} H^{(1)}_{f(t)}\right)(\mathbf{z}) \cdot \nabla_{\mathbf{x}} f(t,\mathbf{z}) + \left(-\nabla_{\mathbf{x}} H^{(1)}_{f(t)}\right)(\mathbf{z}) \cdot \nabla_{\mathbf{v}} f(t,\mathbf{z})\\ &= \partial_t f(t,\mathbf{z}) + \left[f(t), H^{(1)}_{f(t)}\right](\mathbf{z})\end{aligned} \tag{Vl}$$

with initial datum $f(0) = \mathring{f}$. The expression $[g,h](\mathbf{z}) \equiv \nabla_{\mathbf{x}} g(\mathbf{z}) \cdot \nabla_{\mathbf{v}} h(\mathbf{z}) - \nabla_{\mathbf{v}} g(\mathbf{z}) \cdot \nabla_{\mathbf{x}} h(\mathbf{z})$ is the standard Poisson bracket for differentiable functions $g,h : \mathbb{R}^{2d}_{\mathbf{z}} = \mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{v}} \to \mathbb{C}$. Equation (Vl) is known as the **Vlasov equation** which has been studied for many different potentials and phase spaces. The following example is the most famous one.

Example 1.1 (The Vlasov–Poisson System). The Vlasov–Poisson system is the Vlasov system with non-relativistic kinetic energy and Coulomb two-body interaction. In $d \geq 3$ dimensions it fits exactly into the given framework with kinetic term $\epsilon(\mathbf{v}) \equiv \frac{|\mathbf{v}|^2}{2}$, Coulomb interaction $\Gamma(\mathbf{x}) \equiv \frac{|\mathbf{x}|^{2-d}}{d(2-d)\omega_d}$, energy functional

$$\mathcal{H}(f) = \int_{\mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{v}}} \frac{|\mathbf{v}|^2}{2} f(\mathbf{x},\mathbf{v}) \,\mathrm{d}(\mathbf{x},\mathbf{v}) + \frac{1}{2} \int_{\mathbb{R}^{2d}_{\mathbf{z}} \times \mathbb{R}^{2d}_{\mathbf{z}}} \frac{f(\mathbf{x}_1,\mathbf{v}_1)\, f(\mathbf{x}_2,\mathbf{v}_2)}{d(2-d)\omega_d\, |\mathbf{x}_1 - \mathbf{x}_2|^{d-2}} \,\mathrm{d}(\mathbf{z}_1,\mathbf{z}_2),$$

and Vlasov equation

$$\partial_t f(t,\mathbf{x},\mathbf{v}) = -\mathbf{v} \cdot \nabla_{\mathbf{x}} f(t,\mathbf{x},\mathbf{v}) + (\nabla\Gamma * f(t))(\mathbf{x}) \cdot \nabla_{\mathbf{v}} f(t,\mathbf{x},\mathbf{v}),$$

where ∇ is the gradient and $*$ is the convolution on $\mathbb{R}^{2d}_{\mathbf{z}}$.

Remark 1.2. The derivation of the Vlasov equation can be generalized to multiple particle species, if one identifies the densities $(f_1, \ldots, f_n)$ on $\mathbb{R}^{2d}_{\mathbf{z}}$ as a single density f on the disjoint union $\sqcup_{i=1}^n \mathbb{R}^{2d}_{\mathbf{z}}$.

1.2 The Hamiltonian Vlasov System

Now our focus shifts towards constructing a symplectic evolution equation from some Vlasov equation. We assume that the Hamiltonian $\mathcal{H}(f)$ is real-valued and twice differentiable, i.e., not only $\mathrm{D}^1\mathcal{H}(f)(\delta f)$, but also

$$\begin{aligned}\mathrm{D}^2\mathcal{H}(f)\,(\delta f_1,\delta f_2) \equiv&\ \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big(\mathrm{D}^1\mathcal{H}(f+\varepsilon\,\delta f_1)(\delta f_2)-\mathrm{D}^1\mathcal{H}(f)(\delta f_2)\Big)\\ =&\ \int_{\mathbb{R}^{2d}_{\mathbf{z}}}\int_{\mathbb{R}^{2d}_{\mathbf{z}}} H^{(2)}_f(\mathbf{z}_1,\mathbf{z}_2)\,\delta f_1(\mathbf{z}_1)\,\delta f_2(\mathbf{z}_2)\,\mathrm{d}\mathbf{z}_1\,\mathrm{d}\mathbf{z}_2\end{aligned}$$

exists. We describe a canonical way to find a complex valued Vlasov type equation, which allows to apply methods from symplectic geometry.

In most applications, it is physically reasonable to assume that the overall particle mass is bounded, i.e., $f\in\mathcal{L}^1_{\mathbf{z}}$, and non-negative (otherwise consider $f=f^+-f^-$ with the same characteristic equations). Both these restrictions are naturally fulfilled for $f=|\alpha|^2$ with some complex valued function $\alpha\in\mathcal{L}^2_{\mathbf{z}}$. In addition, $\mathcal{L}^2_{\mathbf{z}}$ admits the translation invariant symplectic form, for complex valued $\alpha,\beta\in\mathcal{L}^2_{\mathbf{z}}$ given by

$$\omega\,(\alpha,\beta)\equiv\Im\,\langle\alpha,\beta\rangle=\Im\int_{\mathbb{R}^{2d}_{\mathbf{z}}}\alpha(\mathbf{z})\,\bar{\beta}(\mathbf{z})\,\mathrm{d}\mathbf{z}\in\mathbb{R},\tag{SyF}$$

where $\Im$ denotes the imaginary part and $\bar{\cdot}$ the complex conjugate, respectively. It is now possible to define (on a set of suitable functions $\alpha\in\mathcal{L}^2_{\mathbf{z}}$) the new **Hamiltonian Vlasov functional**

$$\mathcal{H}_{\mathrm{Vl}}(\alpha)\equiv\ \frac{1}{2}\Im\ \mathrm{D}^1\mathcal{H}\left(|\alpha|^2\right)\!\left([\bar{\alpha},\alpha]\right)=\frac{1}{2}\Im\int_{\mathbb{R}^{2d}_{\mathbf{z}}}H^{(1)}_{|\alpha|^2}(\mathbf{z})\ \,[\bar{\alpha},\alpha]\,(\mathbf{z})\,\mathrm{d}\mathbf{z}\tag{HVlF}$$

from the system's given energy functional $\mathcal{H}$. Now, formal computation of the derivative shows that

$$\begin{aligned}\mathrm{D}^1\mathcal{H}_{\mathrm{Vl}}(\alpha)(\delta\alpha)=&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\mathcal{H}_{\mathrm{Vl}}(\alpha+\varepsilon\,\delta\alpha)-\mathcal{H}_{\mathrm{Vl}}(\alpha)\right)\\ =&\ \frac{1}{2}\Im\ \mathrm{D}^1\mathcal{H}\left(|\alpha|^2\right)\Big[2\mathrm{i}\ \Im\,[\delta\bar{\alpha},\alpha]\Big]+\frac{1}{2}\Im\ \mathrm{D}^2\mathcal{H}\left(|\alpha|^2\right)\Big[2\Re\ (\alpha\,\delta\bar{\alpha})\,,[\bar{\alpha},\alpha]\Big]\\ =&\ \Im\int_{\mathbb{R}^{2d}_{\mathbf{z}}}\left(\left[\alpha,H^{(1)}_{|\alpha|^2}\right](\bar{\mathbf{z}})+\alpha(\bar{\mathbf{z}})\ \underbrace{\int_{\mathbb{R}^{2d}_{\mathbf{z}}}[\bar{\alpha},\alpha]\,(\mathbf{z})\,H^{(2)}_{|\alpha|^2}(\bar{\mathbf{z}},\mathbf{z})\,\mathrm{d}\mathbf{z}}_{\equiv K_\alpha(\bar{\mathbf{z}})}\right)\delta\bar{\alpha}(\bar{\mathbf{z}})\,\mathrm{d}\bar{\mathbf{z}}\\ \equiv&-\omega\left(X_{\mathcal{H}_{\mathrm{Vl}}}(\alpha),\delta\alpha\right),\end{aligned}$$

defining the **Hamiltonian vector field** $X_{\mathcal{H}_{\mathrm{Vl}}}(\alpha)$ in the usual manner. Finally, this gives the **Hamiltonian Vlasov equation** for α, namely,

$$\begin{aligned}\partial_t\alpha(t,\mathbf{z}) = X_{\mathcal{H}_{\mathrm{Vl}}}(\alpha(t))(\mathbf{z}) &= \left[H^{(1)}_{|\alpha(t)|^2}, \alpha(t)\right](\mathbf{z}) - K_{\alpha(t)}(\mathbf{z})\,\alpha(t,\mathbf{z}) \qquad \text{(HVl)}\\ &= -\nabla_{\mathbf{x}}\alpha(t,\mathbf{z})\cdot\left(\nabla_{\mathbf{v}}H^{(1)}_{|\alpha(t)|^2}\right)(\mathbf{z}) - \nabla_{\mathbf{v}}\alpha(t,\mathbf{z})\cdot\left(-\nabla_{\mathbf{x}}H^{(1)}_{|\alpha(t)|^2}\right)(\mathbf{z}) - K_{\alpha(t)}(\mathbf{z})\,\alpha(t,\mathbf{z}),\end{aligned}$$

where the function K is imaginary valued.

Example 1.3. Given that the energy is of the form (1.1), the Hamiltonian Vlasov functional is

$$\mathcal{H}_{\mathrm{Vl}}(\alpha) = \frac{1}{2\mathrm{i}}\int_{\mathbb{R}^{2d}_{\mathbf{z}}}\left(\epsilon(\mathbf{v}) + \left(\Gamma * |\alpha|^2\right)(\mathbf{x})\right)\,[\bar{\alpha},\alpha]\,(\mathbf{z})\,\mathrm{d}\mathbf{z},$$

yielding the Hamiltonian Vlasov equation

$$\partial_t\alpha(t,\mathbf{z}) = \left[\epsilon + \left(\Gamma * |\alpha(t)|^2\right), \alpha(t)\right](\mathbf{z}) - (\Gamma * [\bar{\alpha}(t),\alpha(t)])\,(\mathbf{x})\,\alpha(t,\mathbf{z}).$$

Examining (HVl) one can see that the Hamiltonian Vlasov equation in its general form is actually just a complex valued Vlasov equation with an additional non-linear phase oscillation term. This provides the following two formal results, which should fit into any rigorous topological framework, as they rely more on the structure of the equation than the specific rigorous setup.

Proposition 1.4. *Let $I \subseteq \mathbb{R}$ be an interval and $\alpha : I \times \mathbb{R}^{2d}_{\mathbf{z}} \to \mathbb{C}$ be a local solution of* (HVl). *Then $f(t,\mathbf{z}) \equiv |\alpha(t,\mathbf{z})|^2$ is a local solution of the classical Vlasov equation* (Vl).

Proof. Multiply (HVl) with $2\bar{\alpha}$ and take the real part $\mathfrak{R}$. Noticing that $2\mathfrak{R}\,\bar{\alpha}\,\partial\alpha = \partial\,|\alpha|^2$ for any first order derivative ∂ and $\mathfrak{R}\,K_\alpha = 0$, we recover (Vl) for $f = |\alpha|^2$:

$$\begin{aligned}\partial_t f(t) = \partial_t\,|\alpha(t)|^2 = 2\mathfrak{R}\,\bar{\alpha}(t)\,\partial_t\alpha(t) &= 2\mathfrak{R}\,\bar{\alpha}(t)\left(\left[H^{(1)}_{|\alpha(t)|^2},\alpha(t)\right] - K_{\alpha(t)}\alpha(t)\right)\\ &= \mathfrak{R}\left[H^{(1)}_{|\alpha(t)|^2}, |\alpha(t)|^2\right] - 2\mathfrak{R}\,K_{\alpha(t)}\,|\alpha(t)|^2 = \left[H^{(1)}_{f(t)}, f(t)\right]. \qquad \square\end{aligned}$$

Proposition 1.5. *Let $0 \in I \subseteq \mathbb{R}$ be an interval and $\alpha : I \times \mathbb{R}^{2d}_{\mathbf{z}} \to \mathbb{C}$ be some function. It is a local solution of* (HVl) *if and only if*

$$\alpha(t,\mathbf{z}) = \alpha(0, Z(0,t,\mathbf{z}))\,\exp\left(-\int_0^t K_{\alpha(\tau)}(Z(\tau,t,\mathbf{z}))\,\mathrm{d}\tau\right), \tag{1.5}$$

where $Z = (X,V)$ is the solution map of the general characteristic system (1.3).

Proof. **⇒.** Let α be a local solution and Z be the solution map of the characteristic system. For any $t, s \in I$, we find

$$\begin{aligned} \mathrm{d}_t(\alpha(t, Z(t,s,\mathbf{z}))) &= (\partial_t \alpha)(t, Z(t,s,\mathbf{z})) + \left[\alpha, H^{(1)}_{|\alpha(t)|^2}\right](Z(t,s,\mathbf{z})) \\ &\overset{\text{(HVl)}}{=} -K_{\alpha(t)}(Z(t,s,\mathbf{z}))\,\alpha(t, Z(t,s,\mathbf{z})), \end{aligned}$$

which is a linear ODE with a unique solution for fixed $s, \mathbf{z}$, given by

$$\alpha(t, Z(t,s,\mathbf{z})) = \alpha(0, Z(0,s,\mathbf{z}))\ \exp\left(-\int_0^t K_{\alpha(\tau)}(Z(\tau,s,\mathbf{z}))\,\mathrm{d}\tau\right).$$

⇐. Let α satisfy (1.5) and let Z be the solution map of (1.3). We then compute

$$\alpha(t, Z(t,0,\mathbf{z})) = \alpha(0,\mathbf{z})\ \exp\left(-\int_0^t K_{\alpha(\tau)}(Z(\tau,0,\mathbf{z}))\,\mathrm{d}\tau\right).$$

Differentiating both sides w.r.t. t yields exactly (HVl) again. □

Equation (1.5) can provide the structure for an iterative scheme in order to solve the Hamiltonian Vlasov system. This is elaborated rigorously in Chapter 3 for the Vlasov–Poisson system.

It is worth noting that the choice of a Hamiltonian functional is not unique[1]. In fact, if we assume that $\tilde{\mathcal{G}} : \mathcal{L}^1_{\mathbf{z}} \to \mathbb{R}$ is any observable of the classical Vlasov system, not necessarily a constant of motion, then

$$\widetilde{\mathcal{H}_{\mathrm{Vl}}}(\alpha) \equiv \mathcal{H}_{\mathrm{Vl}}(\alpha) + \tilde{\mathcal{G}}\left(|\alpha|^2\right)$$

yields a different Hamiltonian $\widetilde{\mathcal{H}_{\mathrm{Vl}}}$ whose trajectories are different in $\mathcal{L}^2_{\mathbf{z}}$, but coincide with those trajectories from $\mathcal{H}_{\mathrm{Vl}}$ in $\mathcal{L}^1_{\mathbf{z}}$ under the map $\alpha \mapsto |\alpha|^2$.

While this degeneracy of the problem might offer exciting opportunities for stability analysis or related issues, for all the problems treated in this work, the canonical choice $\tilde{\mathcal{G}} \equiv 0$ proved to be the most natural.

We close this section by formally classifying this gauging degeneracy.

Proposition 1.6 (Gauging Classification)**.** *Let $\mathcal{H}_{Vl} : \mathcal{L}^2_{\mathbf{z}} \to \mathbb{R}$ be given by* (HVlF) *and $\tilde{\mathcal{G}} : \mathcal{L}^1_{\mathbf{z}} \to \mathbb{R}$ be a smooth functional. Then*

$$\widetilde{\mathcal{H}_{Vl}}(\alpha) \equiv \mathcal{H}_{Vl}(\alpha) + \tilde{\mathcal{G}}\left(|\alpha|^2\right)$$

[1] The author thanks one of the referees of [14] for pointing this out.

also generates trajectories satisfying Proposition 1.4. Conversely, if

$$\widetilde{\mathcal{H}_{Vl}}(\alpha) \equiv \mathcal{H}_{Vl}(\alpha) + \mathcal{G}(\alpha)$$

generates trajectories complying with Proposition 1.4 and $X_{\mathcal{G}}$ is pointwise phase equivariant, i.e.,

$$\forall \alpha \in \mathcal{L}^2_{\mathbf{z}},\ \zeta : \mathbb{R}^{2d}_{\mathbf{z}} \to \mathbb{S}^1 \subset \mathbb{C}, \quad \mathbf{z} \in \mathbb{R}^{2d}_{\mathbf{z}} : \quad X_{\mathcal{G}}(\zeta\alpha)(\mathbf{z}) = \zeta(\mathbf{z})\, X_{\mathcal{G}}(\alpha)(\mathbf{z}),$$

then $\mathcal{G}(\alpha) = \tilde{\mathcal{G}}\left(|\alpha|^2\right)$ for some $\tilde{\mathcal{G}} : \mathcal{L}^1_{\mathbf{z}} \to \mathbb{R}$.

Proof. **⇒.** In this case, by the linearity of the Hamiltonian vector field, we have

$$X_{\widetilde{\mathcal{H}_{\mathrm{Vl}}}}(\alpha) = X_{\mathcal{H}_{\mathrm{Vl}}}(\alpha) + \frac{1}{\mathrm{i}} \tilde{G}^{(1)}_{|\alpha|^2}\, \alpha$$

and the latter term only contributes by a phase oscillation, which is invisible under $\alpha \mapsto |\alpha|^2$.

⇐. Let $X_{\widetilde{\mathcal{H}_{\mathrm{Vl}}}}$ generate trajectories satisfying Proposition 1.4, i.e., the different gauging only contributes by a phase oscillation. Then for every $\alpha \in \mathcal{L}^2_{\mathbf{z}}$ at any $\mathbf{z} \in \mathbb{R}^{2d}_{\mathbf{z}}$, we find that

$$X_{\mathcal{G}}(\alpha)(\mathbf{z}) \in \mathbb{R}\ \mathrm{i}\alpha(\mathbf{z}).$$

Therefore, at any α from the dense subset $\subset \mathcal{L}^2_{\mathbf{z}}$ where $\{\alpha = 0\}$ is a null set,

$$\forall \mathbf{z} \notin \{\alpha = 0\} : \quad \tilde{G}^{(1)}_{|\alpha|^2}(\mathbf{z}) \equiv -\mathrm{i}\, \frac{X_{\mathcal{G}}(\alpha)(\mathbf{z})}{2\alpha(\mathbf{z})} \in \mathbb{R}$$

is a well-defined function, because the pointwise phase equivariance shows that the right-hand side only depends on $|\alpha|$. Thus

$$\mathrm{D}^1\mathcal{G}(\alpha)(\delta\alpha) = -\omega\left(X_{\mathcal{G}}(\alpha), \delta\alpha\right) = \int_{\mathbb{R}^{2d}_{\mathbf{z}}} \tilde{G}^{(1)}_{|\alpha|^2}(\mathbf{z})\, 2\, \Re\bar{\alpha}(\mathbf{z})\, \delta\alpha(\mathbf{z})\, \mathrm{d}\mathbf{z},$$

showing that actually $\mathcal{G}(\alpha) = \tilde{\mathcal{G}}\left(|\alpha|^2\right)$. □

1.3 The Hamilton Hartree System

Another remarkable feature of the Hamiltonian Vlasov equation is its structural proximity to the mean field equations of quantum theory, specifically the Hartree equation as a self-consistent description of a free electron wave function [9]. This formal equivalence inspires the studies of Chapter 4 and Section 5.2, in particular.

To see this relation, we denote the Fourier transform in the velocity coordinate by

$$\mathcal{F}: \quad \mathcal{L}^2_{\mathbf{z}} \to \mathcal{L}^2_{\hat{\mathbf{z}}}, \quad \alpha \mapsto \hat{\alpha}, \quad \text{where} \quad \hat{\alpha}(\mathbf{x}, \boldsymbol{\xi}) \equiv \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d_{\mathbf{v}}} \alpha(\mathbf{x}, \mathbf{v})\, e^{-\mathrm{i}\mathbf{v}\cdot\boldsymbol{\xi}}\, \mathrm{d}\mathbf{v}, \tag{FT}$$

in particular the conjugate variable of $\mathbf{v} \in \mathbb{R}^d_{\mathbf{v}}$ is consistently denoted by $\boldsymbol{\xi} \in \mathbb{R}^d_{\boldsymbol{\xi}}$, further $\hat{\mathbf{z}} \equiv (\mathbf{x}, \boldsymbol{\xi})$. As the (partial) Fourier transform is an $\mathcal{L}^2$ isometry and therefore also preserves the symplectic form (SyF), it is always possible to apply this transform and obtain a system equivalent to the Hamiltonian Vlasov one.

A particularly interesting case occurs if the Vlasov energy functional $\mathcal{H}$ is in the form of (1.1) with non-relativistic kinetic energy $\epsilon(\mathbf{v}) = \frac{|\mathbf{v}|^2}{2}$. The corresponding Hamiltonian Vlasov functional $\mathcal{H}_{\mathrm{Vl}}$ expressed in $\hat{\alpha} \equiv \mathcal{F}\alpha$ instead of α reveals a remarkably simple structure, motivating further studies. Introducing the notation

$$\hat{V}(\mathbf{x}, \boldsymbol{\xi}) \equiv -\nabla\Gamma(\mathbf{x}) \cdot \boldsymbol{\xi},$$

integrating by parts, and recalling the Plancherel Theorem for the Fourier transform a.e. in $\mathbf{x}$, one computes

$$\begin{aligned}
\mathcal{H}_{\mathrm{Vl}}(\alpha) &= \frac{1}{2\mathrm{i}} \int_{\mathbb{R}^{2d}_{\mathbf{z}}} \frac{|\mathbf{v}|^2}{2}\; [\bar{\alpha}, \alpha]\,(\mathbf{x}, \mathbf{v})\, \mathrm{d}\mathbf{z} \\
&\quad + \frac{1}{2\mathrm{i}} \int_{\mathbb{R}^{2d}_{\mathbf{z}}} \int_{\mathbb{R}^{2d}_{\mathbf{z}}} |\alpha(\mathbf{z}_1)|^2\; \Gamma(\mathbf{x}_1 - \mathbf{x}_2)\; [\bar{\alpha}, \alpha]\,(\mathbf{z}_2)\, \mathrm{d}\mathbf{z}_1\, \mathrm{d}\mathbf{z}_2 \\
&= \frac{1}{2\mathrm{i}} \int_{\mathbb{R}^{2d}_{\mathbf{z}}} \nabla_{\mathbf{x}}\alpha(\mathbf{x}, \mathbf{v}) \cdot \mathbf{v}\; \bar{\alpha}(\mathbf{x}, \mathbf{v})\, \mathrm{d}\mathbf{z} \\
&\quad + \frac{1}{2\mathrm{i}} \int_{\mathbb{R}^{2d}_{\mathbf{z}}} \int_{\mathbb{R}^{2d}_{\mathbf{z}}} |\alpha(\mathbf{z}_1)|^2\; \bar{\alpha}(\mathbf{z}_2)\, \nabla\Gamma(\mathbf{x}_1 - \mathbf{x}_2)\, \nabla_{\mathbf{v}}\alpha(\mathbf{z}_2)\, \mathrm{d}\mathbf{z}_1\, \mathrm{d}\mathbf{z}_2 \\
&\overset{\text{F.T., Planch.}}{=} -\frac{1}{2} \int_{\mathbb{R}^{2d}_{\hat{\mathbf{z}}}} \nabla_{\mathbf{x}}\hat{\alpha}(\mathbf{x}, \boldsymbol{\xi}) \cdot \nabla_{\boldsymbol{\xi}}\bar{\hat{\alpha}}(\mathbf{x}, \boldsymbol{\xi})\, \mathrm{d}\hat{\mathbf{z}} \\
&\quad + \frac{1}{4} \int_{\mathbb{R}^{2d}_{\hat{\mathbf{z}}}} \int_{\mathbb{R}^{2d}_{\hat{\mathbf{z}}}} |\hat{\alpha}(\hat{\mathbf{z}}_1)|^2\; |\hat{\alpha}(\hat{\mathbf{z}}_2)|^2\; \nabla\Gamma(\mathbf{x}_1 - \mathbf{x}_2) \cdot (\boldsymbol{\xi}_2 - \boldsymbol{\xi}_1)\; \mathrm{d}\hat{\mathbf{z}}_1\, \mathrm{d}\hat{\mathbf{z}}_2 \\
&= -\frac{1}{2} \int_{\mathbb{R}^{2d}_{\hat{\mathbf{z}}}} \nabla_{\mathbf{x}}\hat{\alpha}(\mathbf{x}, \boldsymbol{\xi}) \cdot \nabla_{\boldsymbol{\xi}}\bar{\hat{\alpha}}(\mathbf{x}, \boldsymbol{\xi})\, \mathrm{d}\hat{\mathbf{z}} + \frac{1}{4} \int_{\mathbb{R}^{2d}_{\hat{\mathbf{z}}}} \hat{\alpha}(\hat{\mathbf{z}})\, \left(\hat{V} * |\hat{\alpha}|^2\right)(\hat{\mathbf{z}})\; \bar{\hat{\alpha}}(\hat{\mathbf{z}})\, \mathrm{d}\hat{\mathbf{z}} \\
&= \frac{1}{2} \left\langle \hat{\alpha}, \left(\nabla_{\mathbf{x}} \cdot \nabla_{\boldsymbol{\xi}} + \frac{1}{2}\left(\hat{V} * |\hat{\alpha}|^2\right)\right) \hat{\alpha} \right\rangle_{\mathcal{L}^2_{\hat{\mathbf{z}}}} \equiv \mathcal{H}_{\mathrm{Ht}}(\hat{\alpha}),
\end{aligned}$$

which is structurally intriguingly close to the Hartree energy functional of an electron without ion background [9], given by

$$\mathcal{L}^2_{\mathbf{x}} \to \mathbb{R}, \quad \psi \mapsto \frac{1}{2}\left\langle \psi, \left(-\Delta + \frac{1}{2}\left(\Gamma * |\psi|^2\right)\right) \psi \right\rangle.$$

Hence, we call $\mathcal{H}_{\mathrm{Ht}}(\hat{\alpha})$ the **Hamilton Hartree functional**. Nevertheless, there are important differences between these two. The kinetic term is now ultrahyperbolic instead of elliptic and the underlying space is $\mathbb{R}^{2d}_{\hat{\mathbf{z}}}$ instead of $\mathbb{R}^d_{\mathbf{x}}$. We remark that, for different forms of the kinetic energy ϵ, the operator ∇_{ξ} is replaced by the Fourier conjugate operator of $\nabla_{\mathbf{v}}\epsilon(\mathbf{v})$.

From the first derivative of this functional, the Hamiltonian vector field and the corresponding **Hamilton Hartree equation** is computed to be

$$\mathrm{i}\,\partial_t \hat{\alpha}(t, \hat{\mathbf{z}}) = \left(\nabla_{\mathbf{x}} \cdot \nabla_{\xi} + \left(\hat{V} * |\hat{\alpha}(t)|^2\right)(\hat{\mathbf{z}})\right) \hat{\alpha}(t, \hat{\mathbf{z}}), \tag{HHt}$$

which can also be directly recovered from the velocity Fourier transform (FT) of (HVl).

Chapter 2

Outline of Main Results

While the theory developed in Chapter 1 relies on formal computations and gives results of structural type, we also provide some new rigorous results. These are organized in three chapters each of which is summarized in one of the following sections.

Global Well-Posedness Theory

In Chapter 3 we establish well-posedness for three different systems. Section 3.2 treats the Hamiltonian Vlasov–Poisson system from Example 1.1. The entire section is based on the published paper [14]. For a real parameter $\kappa \geq 2d$, the Banach space $\mathcal{B}_{\mathbf{z}}^{1,\kappa,2}$ consists of some continuously differentiable functions with integrable local supremum. This new space is discussed in depth in Appendix A. It allows to prove local well-posedness of the system.

Theorem I (3.9, 3.20, Local Well-Posedness for Hamiltonian Vlasov–Poisson)**.** *Let $d \geq 3$ be the dimension of the underlying physical system and let $\kappa \geq 2d$ be a real parameter. For every $M > 0$, there is a positive time of existence $T(M) > 0$, s.t. any initial datum $\mathring{\alpha} \in \mathcal{B}_{\mathbf{z}}^{1,\kappa,2}$ with $\|\mathring{\alpha}\|_{\mathcal{B}_{\mathbf{z}}^{1,\kappa,2}} \leq M$ gives rise to a unique local solution*

$$[0, T(M)) \to \mathcal{B}_{\mathbf{z}}^{1,\kappa,2}, \quad t \mapsto \alpha(t)$$

of

$$\partial_t \alpha(t, \mathbf{z}) = \left[\frac{|\mathbf{v}|^2}{2} + \left(\Gamma * |\alpha(t)|^2\right), \alpha(t) \right](\mathbf{z}) - (\Gamma * [\bar{\alpha}(t), \alpha(t)])\,(\mathbf{z})\,\alpha(t, \mathbf{z}),$$

$$\Gamma(\mathbf{x}) \equiv \frac{|\mathbf{x}|^{2-d}}{d(2-d)\omega_d}$$

with $\alpha(0) = \mathring{\alpha}$. The solution depends continuously on the initial datum.

We remark that for a subclass of these initial values, we obtain global existence based on the conditions by Pfaffelmoser–Schaeffer [17, 21] and Lions–Perthame [11].

Based on the submitted paper [16], Section 3.1 is dedicated to the Regular Vlasov system, where the two-body interaction $\Gamma : \mathbb{R}^d_{\mathbf{x}} \to \mathbb{R}$ is $C^3_{\mathbf{x}}$ and its first three derivatives are bounded. We remind the reader that $\mathcal{W}^{1,2}_{\mathbf{z}}$ is the notation for the first Sobolev space on $\mathbb{R}^{2d}_{\mathbf{z}}$.

Theorem II (3.1, Global Well-Posedness for Regular Hamiltonian Vlasov)**.** *Any initial datum $\mathring{\alpha} \in \mathcal{W}^{1,2}_{\mathbf{z}}$ gives rise to a unique global solution*

$$\mathbb{R}_{\geq 0} \to \mathcal{W}^{1,2}_{\mathbf{z}}, \quad t \mapsto \alpha(t)$$

of

$$\partial_t \alpha(t,\mathbf{z}) = \left[\frac{|\mathbf{v}|^2}{2} + \left(\Gamma * |\alpha(t)|^2\right), \alpha(t)\right](\mathbf{z}) - \left(\Gamma * [\bar{\alpha}(t), \alpha(t)]\right)(\mathbf{z})\, \alpha(t,\mathbf{z}) \tag{2.1}$$

with $\alpha(0) = \mathring{\alpha}$. The solution depends continuously on the initial datum.

As outlined in Section 1.3, the velocity Fourier transform gives a parallel result for the Hamilton Hartree system. By defining $\mathcal{M}^1_{\hat{\mathbf{z}}} \equiv \mathcal{F}\left(\mathcal{W}^{1,2}_{\mathbf{z}}\right)$ to be the isometric image of the first Sobolev space under the velocity Fourier transform, the following result holds:

Theorem III (3.6, Global Well-Posedness for Regular Hamilton Hartree)**.** *Any initial datum $\mathring{\hat{\alpha}} \in \mathcal{M}^1_{\hat{\mathbf{z}}}$ gives rise to a unique global solution*

$$\mathbb{R}_{\geq 0} \to \mathcal{M}^1_{\hat{\mathbf{z}}}, \quad t \mapsto \hat{\alpha}(t)$$

of

$$\mathrm{i}\, \partial_t \hat{\alpha}(t,\hat{\mathbf{z}}) = \left(\nabla_{\mathbf{x}} \cdot \nabla_{\boldsymbol{\xi}} + \left(\hat{V} * |\hat{\alpha}(t)|^2\right)(\hat{\mathbf{z}})\right) \hat{\alpha}(t,\hat{\mathbf{z}}), \qquad \hat{V}(\mathbf{x},\boldsymbol{\xi}) \equiv -\nabla\Gamma(\mathbf{x}) \cdot \boldsymbol{\xi}$$

with $\hat{\alpha}(0) = \mathring{\hat{\alpha}}$. The solution depends continuously on the initial datum.

Mean Field Limit

In Chapter 4, based on the submitted paper [16], we discuss the role of the Hamiltonian Vlasov system as an effective limiting equation for a many-particle system. This is reasonable, since the statistical interpretation known for the Vlasov system has no direct $\mathcal{L}^2$ counterpart. Then again, given the $\mathcal{L}^2$ structure of the equation, we transfer a method from the quantum mechanical mean field literature [18], which helps to count the average share of particles in the mean field state $\alpha \in \mathcal{L}^2_{\mathbf{z}}$ with a bounded self-adjoint operator

$$\mathfrak{m}^\alpha_\lambda : \quad \mathcal{L}^2_{\vec{\mathbf{z}}} \to \mathcal{L}^2_{\vec{\mathbf{z}}}, \quad 0 < \lambda < 1$$

on the N particle state space $\mathcal{L}^2_{\vec{\mathbf{z}}} \equiv \mathcal{L}^2_{\mathbf{z}_1,\ldots,\mathbf{z}_N}$. This operator $\mathfrak{m}^\alpha_\lambda$ is a sum of weighted orthogonal projections with weights in the interval $[0, 1]$. They are chosen, s.t. among other restrictions, the condition $\left\langle \alpha_N, \mathfrak{m}^\alpha_\lambda \alpha_N \right\rangle = 0$ on an N particle state $\alpha_N \in \mathcal{L}^2_{\vec{\mathbf{z}}}$ implies that $\alpha_N \propto \alpha^{\otimes N}$ is a pure product state.

Theorem IV (4.12, Regular Hamiltonian Vlasov Mean Field Limit)**.** *Assume that the potential Γ is $C^3_{\mathbf{x}}$ and its first three derivatives are bounded. Let $\mathring{\alpha} \in \mathcal{W}^{2,2}_{\mathbf{z}}$ be an $\mathcal{L}^2_{\mathbf{z}}$ normalized initial state and $\mathring{\alpha}_N \in \mathcal{W}^{2,2}_{\vec{\mathbf{z}}}$ a sequence of $\mathcal{L}^2_{\vec{\mathbf{z}}}$ normalized initial states which are also invariant under particle index permutation.*

Then there exist $\alpha : \mathbb{R}_{\geq 0} \to \mathcal{W}^{2,2}_{\mathbf{z}}$, solving the Regular Hamiltonian Vlasov equation (2.1) *with $\alpha(0) = \mathring{\alpha}$, and a sequence of $\alpha_N : \mathbb{R}_{\geq 0} \to \mathcal{W}^{2,2}_{\vec{\mathbf{z}}}$, solving the complex valued Liouville equation*

$$\partial_t \alpha_N(t, \vec{\mathbf{z}}) = \sum_{m=1}^{N} \left(-\mathbf{v}_m \cdot \nabla_{\mathbf{x}_m} \alpha_N(t, \vec{\mathbf{z}}) + \left(\frac{1}{(N-1)} \sum_{n=1, n \neq m}^{N} \nabla\Gamma(\mathbf{x}_m - \mathbf{x}_n) \right) \cdot \nabla_{\mathbf{v}_m} \alpha_N(t, \vec{\mathbf{z}}) \right),$$

with $\vec{\mathbf{z}} = (\mathbf{z}_1, \ldots, \mathbf{z}_N)$ and $\alpha_N(0) = \mathring{\alpha}_N$. If there is some $M \geq 1$, s.t.

$$\sup_N \left\| \mathrm{D}^1_{\mathbf{z}_1} \mathring{\alpha}_N \right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}} \leq M, \quad \sup_N \left\| \mathrm{D}^2_{(\mathbf{z}_1, \mathbf{z}_2)} \mathring{\alpha}_N \right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}} \leq M, \quad \textit{and} \quad \|\mathring{\alpha}\|_{\mathcal{W}^{2,2}_{\mathbf{z}}} \leq M,$$

then for every $\lambda \in (0, 1)$, there exist continuous, non-decreasing functions $B_{1,M}, B_{2,M}$: $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, both independent of N, s.t.

$$\begin{aligned} &\left| \left\langle \alpha_N(t), \mathfrak{m}^{\alpha(t)}_\lambda \alpha_N(t) \right\rangle - \left\langle \mathring{\alpha}_N, \mathfrak{m}^{\mathring{\alpha}}_\lambda \mathring{\alpha}_N \right\rangle \right| \\ &\quad \leq \left(N^{-\lambda} + N^{-\frac{1-\lambda}{4}} \right) \left(\int_0^t B_{2,M}(\tau)\, \mathrm{d}\tau \right) \exp\left(\int_0^t B_{1,M}(\tilde{\tau})\, \mathrm{d}\tilde{\tau} \right). \end{aligned}$$

In other words, a small share of particles outside the mean field state remains small on compact time intervals for the limit $N \to \infty$. We also remark that this convergence result implies convergence of density matrices in full analogy to quantum mechanical theory, which is an intriguing example where quantum methods are used to conclude results for classical systems.

Symmetry Reduction and Periodicity

The final Chapter 5 is based on the submitted work [15]. Therein we present the general concept of symplectic symmetry reduction by Marsden–Weinstein [12] in the context of Hamiltonian PDEs in general and the Hamiltonian Vlasov system in particular. In this method one first restricts the motion to a manifold of conserved quantities and then contracts orbits of the group action. In order to implement this procedure, we identify mass and linear momentum as Noether conjugate conserved quantities of certain phase oscillation symmetries.

We apply the formalism to the idealistic example of the Harmonic Vlasov system, a non-relativistic Vlasov system with attractive harmonic two-body interaction. Its associated Hamilton Hartree equation is

$$\mathrm{i}\,\partial_t\hat{\alpha}(t,\hat{\mathbf{z}}) = \left(\nabla_{\mathbf{x}}\cdot\nabla_{\boldsymbol{\xi}} + \left(\hat{V} * |\hat{\alpha}(t)|^2\right)(\hat{\mathbf{z}})\right)\,\hat{\alpha}(t,\hat{\mathbf{z}}), \qquad \hat{V}(\mathbf{x},\boldsymbol{\xi}) \equiv -\mathbf{x}\cdot\boldsymbol{\xi}.$$

The reduction by the global $\mathbb{S}^1$ phase equivariance, which is a Noether conjugate symmetry to mass conservation, gives the unit sphere projection map

$$\Pi : \quad \mathbb{S}^{\mathcal{L}^2_{\hat{\mathbf{z}}}} \to \mathbb{S}^{\mathcal{L}^2_{\hat{\mathbf{z}}}}/\mathbb{S}^1, \quad \hat{\alpha} \mapsto \mathbb{S}^1\hat{\alpha},$$

which contracts orbits of the global phase multiplication. On this quotient manifold, we classify equilibria and families of periodic solutions, which bifurcate around them. Emphasizing the quotient operation, they are referred to as *relative* equilibria and *relatively periodic*, respectively. We note that the operator

$$\mathfrak{N} \equiv \nabla_{\mathbf{x}}\cdot\nabla_{\boldsymbol{\xi}} - \mathbf{x}\cdot\boldsymbol{\xi}$$

is unitarily equivalent to the sum of $2d$ independent quantum mechanical harmonic oscillators in pairs of opposite signs. Therefore, the operator $\mathfrak{N}$ has a complete basis of eigenfunctions and discrete spectrum $\mathbb{Z}$, with each eigenvalue infinitely degenerate. The main findings of the chapter are summarized in the following theorem.

Theorem V (5.10, Spectral Parameterization of Periodic Families in Harmonic Vlasov)**.** *Let $\hat{\alpha} \in \mathbb{S}^{\mathcal{L}^2_{\hat{\mathbf{z}}}}$ be a unit eigenvector of $\mathfrak{N}$ with eigenvalue $N \in \mathbb{Z}$. If $X_{\bar{\mathcal{H}}_{Ht}} \equiv \Pi_* X_{\mathcal{H}_{Ht}}$ is the push-forward of the Hamiltonian vector field, then we find that:*

(i) $X_{\bar{\mathcal{H}}_{Ht}}(\Pi\hat{\alpha})$ *is zero and* $\Pi\hat{\alpha}$ *is a relative equilibrium.*

(ii) There exists a subspace $\mathcal{W}_{\Pi\hat{\alpha}} \subseteq \mathrm{T}_{\Pi\hat{\alpha}}\left(\mathbb{S}^{\mathcal{L}^2_{\hat{\mathbf{z}}}}/\mathbb{S}^1\right)$ *with $4d$-dimensional complement, s.t.*

$$\mathrm{D}X_{\bar{\mathcal{H}}_{Ht}}(\Pi\hat{\alpha})\Big|_{\mathcal{W}_{\Pi\hat{\alpha}}} = \frac{1}{\mathrm{i}}\,(\mathfrak{N} - N)$$

and in direction of every eigenvector of $\mathrm{D}X_{\bar{\mathcal{H}}_{Ht}}(\Pi\hat{\alpha})\big|_{\mathcal{W}_{\Pi\hat{\alpha}}}$ *bifurcates a family of relatively periodic solutions, whose constant angular frequency is given by modulus of the eigenvalue.*

Chapter 3

Global Well-Posedness Theory

One of the first steps in the analysis of an equation occurring in mathematical physics is to discuss its well-posedness, which provides a qualitative justification of the equation's relevance for the particular physical system it describes. For example, an equation without unique or even without any solution is unsuitable to predict the system's behavior for obvious reasons. Furthermore, the continuous dependence on initial conditions is quite important, since a mathematical theory should take into account that in any application scenario, controlled initial conditions are subject to small deviations due to inaccuracies, however small, in any experimental setup.

In this chapter we therefore discuss the global well-posedness for the Hamiltonian Vlasov equation

$$\partial_t \alpha(t, \mathbf{z}) = \left[\epsilon + \left(\Gamma * |\alpha(t)|^2\right), \alpha(t)\right](\mathbf{z}) - \left(\Gamma * [\bar{\alpha}(t), \alpha(t)]\right)(\mathbf{x})\, \alpha(t, \mathbf{z}) \tag{3.1}$$

for the following two cases:

(i) **Regular Hamiltonian Vlasov system** in $d \in \mathbb{N}$ dimensions with kinetic energy and regular interaction given by

$$\epsilon(\mathbf{v}) \equiv \frac{|\mathbf{v}|^2}{2} \quad \text{and} \quad \Gamma \in C^3_{\mathbf{x}},$$

s.t. the first three derivatives of Γ, but not necessarily Γ itself, are bounded. In this case we prove global existence and continuous dependence on initial data in the Sobolev space $\mathcal{W}^{1,2}_{\mathbf{z}}$. In addition, we transfer the results to the associated Hamilton Hartree system and prove some higher derivative bounds that are needed for the results of Chapter 4. Exploiting the duality of the Hamiltonian Vlasov and the Hamilton Hartree systems through the invertible Fourier transform (FT), we also establish global well-posedness for the **Regular Hamilton Hartree system** (3.10) explicitly.

(ii) **Hamiltonian Vlasov–Poisson system** in $d \geq 3$ dimensions with the kinetic energy and the Coulomb interaction given by

$$\epsilon(\mathbf{v}) \equiv \frac{|\mathbf{v}|^2}{2} \quad \text{and} \quad \Gamma(\mathbf{x}) \equiv \frac{|\mathbf{x}|^{2-d}}{d(2-d)\omega_d},$$

where ω_d is the unit sphere surface in $\mathbb{R}^d_{\mathbf{x}}$. For this system we establish local existence and continuous dependence on initial data for a newly defined Banach space $\mathcal{B}^{1,\kappa,2}_{\mathbf{z}} \subseteq \mathcal{W}^{1,2}_{\mathbf{z}} \cap C^1_{\mathbf{z}}$ of continuously differentiable functions with integrable local supremum. We also adapt the global existence criteria for $d = 3$ from the celebrated work of Pfaffelmoser–Schaeffer [17, 21] and Lions–Perthame [11].

3.1 On the Regular Hamiltonian Vlasov System

This section is in large parts adapted from the author's work [16, Sec.3].

Throughout this section the positive integer $d \in \mathbb{N}$ denotes a fixed dimension. The Regular Hamiltonian Vlasov system is a non-relativistic system with the restriction that the physical force $\nabla\Gamma \in C^2_{\mathbf{x}}$ is bounded, i.e., the potential $\Gamma \in C^3_{\mathbf{x}}$, which is assumed to be even, satisfies

$$C_\Gamma \equiv \max\left\{\left\|\mathrm{D}^1\Gamma\right\|_{\mathcal{L}^\infty_{\mathbf{x}}}, \left\|\mathrm{D}^2\Gamma\right\|_{\mathcal{L}^\infty_{\mathbf{x}}}, \left\|\mathrm{D}^3\Gamma\right\|_{\mathcal{L}^\infty_{\mathbf{x}}}\right\} < \infty. \tag{Pot}$$

Under these assumptions, global well-posedness can be established in the first Sobolev space $\mathcal{W}^{1,2}_{\mathbf{z}} = \left\{\alpha \in \mathcal{L}^2_{\mathbf{z}} : \nabla\alpha \in \mathcal{L}^2_{\mathbf{z}}\right\}$.

Theorem 3.1 (Global Well-Posedness for Regular Hamiltonian Vlasov)**.** *Let $\mathring{\alpha} \in \mathcal{W}^{1,2}_{\mathbf{z}}$ be given. Then there is a unique global solution $\mathbb{R}_{\geq 0} \times \mathbb{R}^{2d}_{\mathbf{z}} \ni (t, \mathbf{z}) \mapsto \alpha(t, \mathbf{z})$ of* (3.1) *with initial datum $\alpha(0) = \mathring{\alpha}$. For any $T < \infty$, the solution map*

$$\begin{array}{ccc} \mathcal{W}^{1,2}_{\mathbf{z}} & \to & \mathcal{L}^\infty_t \mathcal{L}^2_{\mathbf{z}} \\ \mathring{\alpha} & \mapsto & \alpha : \begin{array}{ccc} [0,T] & \to & \mathcal{L}^2_{\mathbf{z}} \\ t & \mapsto & \alpha(t) \end{array} \end{array}$$

is locally Lipschitz continuous. Additionally, the image of this map is in $\mathcal{L}^\infty_t \mathcal{L}^2_{\mathbf{z}} \cap \mathcal{L}^\infty_{t,loc} \mathcal{W}^{1,2}_{\mathbf{z}}$ for $T = \infty$.

Proof. **(i) Iteration scheme.** Consider $\mathring{\alpha} \in \mathcal{W}^{1,2}_{\mathbf{z}} \cap C^1_{\mathbf{z},c}$, $\|\mathring{\alpha}\|_{\mathcal{W}^{1,2}_{\mathbf{z}}} \leq M$, compactly supported. Choose any $T > 0$. We define $\alpha_0(t, \mathbf{z}) \equiv \mathring{\alpha}(\mathbf{z})$ and given α_n for some $n \in \mathbb{N}_0$, we define the objects

$$\begin{aligned}
\rho_n(t, \mathbf{x}) &\equiv \int_{\mathbb{R}^d_{\mathbf{v}}} |\alpha_n(t, \mathbf{z})|^2 \, \mathrm{d}\mathbf{v}, \\
F_n(t, \mathbf{x}) &\equiv (-\nabla\Gamma * \rho_n(t))(\mathbf{x}), \\
K_n(t, \mathbf{x}) &\equiv (\Gamma * [\bar{\alpha}_n(t), \alpha_n(t)])(\mathbf{x}) = \left(\nabla\Gamma * \int_{\mathbb{R}^d_{\mathbf{v}}} \bar{\alpha}_n(t, \cdot, \mathbf{v}) \, \nabla_{\mathbf{v}}\alpha_n(t, \cdot, \mathbf{v}) \, \mathrm{d}\mathbf{v}\right)(\mathbf{x}),
\end{aligned}$$

which are all well-defined for compactly supported $\alpha(t) \in C^1_{\mathbf{z},c}$. In fact, $F_n(t), K_n(t) \in C^2_{\mathbf{x}}$ and all their derivatives are continuous in $(t, \mathbf{x})$. Furthermore, let $Z_n(t, t_0, \mathbf{z}) = (X_n, V_n)(t, t_0, \mathbf{z})$ be the solution map of the non-autonomous Hamiltonian system

$$\partial_t X_n(t, t_0, \mathbf{z}) = V_n(t, t_0, \mathbf{z}), \quad \partial_t V_n(t, t_0, \mathbf{z}) = F_n(t, X_n(t, t_0, \mathbf{z})). \tag{3.2}$$

Now α_{n+1} is defined iteratively from α_n as the unique solution of the linear equation

$$\partial_t\alpha_{n+1}(t, \mathbf{z}) = \left[\frac{|\mathbf{v}|^2}{2} + \left(\Gamma * |\alpha_n(t)|^2\right)(\mathbf{x}), \alpha_{n+1}(t)\right](\mathbf{z}) - (\Gamma * [\bar{\alpha}_n(t), \alpha_n(t)])(\mathbf{x}) \, \alpha_{n+1}(t, \mathbf{z})$$

with initial datum $\alpha_{n+1}(0) = \mathring{\alpha}$. In fact, this equation integrates to

$$\alpha_{n+1}(t, \mathbf{z}) = \mathring{\alpha}(Z_n(0, t, \mathbf{z})) \, \exp\left(-\int_0^t K_n(\tau, X_n(\tau, t, \mathbf{z})) \, \mathrm{d}\tau\right), \tag{3.3}$$

which proves that also $\alpha_{n+1}(t)$ is compactly supported and $C^1_{\mathbf{z},c}$ for any t. Therefore, the iteration scheme is well-defined.

(ii) Uniform bounds. As a next step, one determines bounds of the transport coefficients which are uniform in n and depend on M only. With C_Γ from (Pot) one readily finds

$$\begin{aligned}
\|\rho_n(t)\|_{\mathcal{L}^1_{\mathbf{x}}} &= \|\alpha_n(t)\|^2_{\mathcal{L}^2_{\mathbf{z}}} \leq M^2, \\
\|F_n(t)\|_{\mathcal{L}^\infty_{\mathbf{x}}} &= \|(\nabla\Gamma * \rho_n(t))\|_{\mathcal{L}^\infty_{\mathbf{x}}} \leq \|\nabla\Gamma\|_{\mathcal{L}^\infty_{\mathbf{x}}} \|\rho_n(t)\|_{\mathcal{L}^1_{\mathbf{x}}} \leq C_\Gamma M^2, \quad \text{and} \\
\|\nabla F_n(t)\|_{\mathcal{L}^\infty_{\mathbf{x}}} &= \left\|\left(\mathrm{D}^2\Gamma * \rho_n(t)\right)\right\|_{\mathcal{L}^\infty_{\mathbf{x}}} \leq \left\|\mathrm{D}^2\Gamma\right\|_{\mathcal{L}^\infty_{\mathbf{x}}} \|\rho_n(t)\|_{\mathcal{L}^1_{\mathbf{x}}} \leq C_\Gamma M^2.
\end{aligned}$$

This yields a bound on the solution map's differential in matrix operator norm:

$$\begin{aligned}
& Z_n(t, t_0, \mathbf{z}) = \mathbf{z} + \int_{t_0}^t \begin{pmatrix} V_n(\tau, t_0, \mathbf{z}) \\ F_n(\tau, X_n(\tau, t_0, \mathbf{z})) \end{pmatrix} \mathrm{d}\tau \\
\Rightarrow \quad & \mathrm{D}Z_n(t, t_0, \mathbf{z}) = \mathbb{1} + \int_{t_0}^t \begin{pmatrix} 0 & \mathbb{1} \\ \nabla F_n(\tau, X_n(\tau, t_0, \mathbf{z})) & 0 \end{pmatrix} \cdot \mathrm{D}Z_n(\tau, t_0, \mathbf{z}) \, \mathrm{d}\tau \\
\Rightarrow \quad & \|\mathrm{D}Z_n(t, t_0)\|_{\mathcal{L}^\infty_{\mathbf{z}}} \leq 1 + \int_{t_0}^t (1 + C_\Gamma M^2) \, \|\mathrm{D}Z_n(\tau, t_0)\|_{\mathcal{L}^\infty_{\mathbf{z}}} \, \mathrm{d}\tau \leq e^{(1+C_\Gamma M^2)|t-t_0|},
\end{aligned}$$

where $\mathbb{1}$ is the identity matrix. Further, one gets the following inequalities for K_n, namely,

$$\|K_n(t)\|_{\mathcal{L}^\infty_{\mathbf{x}}} \leq \left\|\mathrm{D}^1\Gamma\right\|_{\mathcal{L}^\infty_{\mathbf{x}}} \|\alpha_n(t)\|_{\mathcal{L}^2_{\mathbf{z}}} \|\nabla\alpha_n(t)\|_{\mathcal{L}^2_{\mathbf{z}}} \leq C_\Gamma M \|\nabla\alpha_n(t)\|_{\mathcal{L}^2_{\mathbf{z}}} \quad \text{and}$$
$$\|\nabla K_n(t)\|_{\mathcal{L}^\infty_{\mathbf{x}}} \leq \left\|\mathrm{D}^2\Gamma\right\|_{\mathcal{L}^\infty_{\mathbf{x}}} \|\alpha_n(t)\|_{\mathcal{L}^2_{\mathbf{z}}} \|\nabla\alpha_n(t)\|_{\mathcal{L}^2_{\mathbf{z}}} \leq C_\Gamma M \|\nabla\alpha_n(t)\|_{\mathcal{L}^2_{\mathbf{z}}}.$$

Combining all these results, one computes an iterative Grönwall scheme with the aid of Equation (3.3), that is,

$$\begin{aligned}\|\alpha_{n+1}(t)\|_{\mathcal{W}^{1,2}_{\mathbf{z}}} &= \left(\|\alpha_{n+1}(t)\|^2_{\mathcal{L}^2_{\mathbf{z}}} + \|\nabla\alpha_{n+1}(t)\|^2_{\mathcal{L}^2_{\mathbf{z}}}\right)^{\frac{1}{2}} \leq \|\alpha_{n+1}(t)\|_{\mathcal{L}^2_{\mathbf{z}}} + \|\nabla\alpha_{n+1}(t)\|_{\mathcal{L}^2_{\mathbf{z}}} \\ &\leq \|\mathring{\alpha}\|_{\mathcal{L}^2_{\mathbf{z}}} + \|\nabla\mathring{\alpha}\|_{\mathcal{L}^2_{\mathbf{z}}} \|\mathrm{D}Z_n(0,t)\|_{\mathcal{L}^\infty_{\mathbf{z}}} + \|\mathring{\alpha}\|_{\mathcal{L}^2_{\mathbf{z}}} \int_0^t \|\nabla K_n(\tau)\|_{\mathcal{L}^\infty_{\mathbf{x}}} \|\mathrm{D}Z_n(\tau,t)\|_{\mathcal{L}^\infty_{\mathbf{z}}} \,\mathrm{d}\tau \\ &\leq M\left(1 + e^{(1+C_\Gamma M^2)t}\right) + C_\Gamma M^2 \int_0^t \|\alpha_n(\tau)\|_{\mathcal{W}^{1,2}_{\mathbf{z}}} e^{(1+C_\Gamma M^2)(t-\tau)} \,\mathrm{d}\tau.\end{aligned}$$

Hence, if $Q_M : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is the fixed point of the integral equation

$$Q_M(t) = M\left(1 + e^{(1+C_\Gamma M^2)t}\right) + C_\Gamma M^2 \int_0^t Q_M(\tau)\, e^{(1+C_\Gamma M^2)(t-\tau)} \,\mathrm{d}\tau$$

uniquely given by

$$Q_M(t) = M\frac{1 + C_\Gamma M^2 + C_\Gamma M^2\, e^{(1+2C_\Gamma M^2)t}}{1 + 2C_\Gamma M^2}, \tag{3.4}$$

one obviously finds that $\|\alpha_0(t)\|_{\mathcal{W}^{1,2}_{\mathbf{z}}} \leq M \leq Q_M(t)$ and inductively $\|\alpha_n(t)\|_{\mathcal{W}^{1,2}_{\mathbf{z}}} \leq Q_M(t), \forall n$.

(iii) Cauchy sequence. The previous steps allow to compute an iterative Grönwall scheme in $\mathcal{L}^\infty_t\mathcal{L}^2_{\mathbf{z}}$, $t \in [0,T]$:

$$\begin{aligned}\|\alpha_{n+2}(t) - \alpha_{n+1}(t)\|^2_{\mathcal{L}^2_{\mathbf{z}}} &= \int_0^t \partial_\tau \|\alpha_{n+2}(\tau) - \alpha_{n+1}(\tau)\|^2_{\mathcal{L}^2_{\mathbf{z}}} \,\mathrm{d}\tau \\ = 2\Re \int_0^t &\langle \alpha_{n+2}(\tau) - \alpha_{n+1}(\tau), \partial_\tau\alpha_{n+2}(\tau) - \partial_\tau\alpha_{n+1}(\tau)\rangle \,\mathrm{d}\tau \\ = 2\Re \int_0^t &\left\langle \alpha_{n+2}(\tau) - \alpha_{n+1}(\tau), \left[\frac{|\mathbf{v}|^2}{2} + \left(\Gamma * |\alpha_{n+1}(\tau)|^2\right), \alpha_{n+2}(\tau)\right]\right\rangle \mathrm{d}\tau \\ - 2\Re \int_0^t &\left\langle \alpha_{n+2}(\tau) - \alpha_{n+1}(\tau), \left[\frac{|\mathbf{v}|^2}{2} + \left(\Gamma * |\alpha_{n}(\tau)|^2\right), \alpha_{n+1}(\tau)\right]\right\rangle \mathrm{d}\tau \\ - 2\Re \int_0^t &\langle \alpha_{n+2}(\tau) - \alpha_{n+1}(\tau), K_{n+1}(\tau)\, \alpha_{n+2}(\tau) - K_n(\tau)\, \alpha_{n+1}(\tau)\rangle \,\mathrm{d}\tau\end{aligned}$$

$$
\begin{aligned}
&= 2\Re \int_0^t \left\langle \alpha_{n+2}(\tau) - \alpha_{n+1}(\tau), \left[\frac{|\mathbf{v}|^2}{2} + \left(\Gamma * |\alpha_{n+1}(\tau)|^2\right), \alpha_{n+2}(\tau) - \alpha_{n+1}(\tau)\right]\right\rangle \mathrm{d}\tau \\
&\quad + 2\Re \int_0^t \left\langle \alpha_{n+2}(\tau) - \alpha_{n+1}(\tau), \left[\left(\Gamma * \left(|\alpha_{n+1}(\tau)|^2 - |\alpha_n(\tau)|^2\right)\right), \alpha_{n+1}(\tau)\right]\right\rangle \mathrm{d}\tau \\
&\quad - 2\Re \int_0^t \langle \alpha_{n+2}(\tau) - \alpha_{n+1}(\tau), K_{n+1}(\tau)\ (\alpha_{n+2}(\tau) - \alpha_{n+1}(\tau))\rangle\ \mathrm{d}\tau \\
&\quad - 2\Re \int_0^t \langle \alpha_{n+2}(\tau) - \alpha_{n+1}(\tau), (K_{n+1}(\tau) - K_n(\tau))\ \alpha_{n+1}(\tau)\rangle\ \mathrm{d}\tau \\
&\leq \int_0^t \underbrace{\int_{\mathbb{R}^{2d}_{\mathbf{z}}} \left[\frac{|\mathbf{v}|^2}{2} + \left(\Gamma * |\alpha_{n+1}(\tau)|^2\right), |\alpha_{n+2}(\tau) - \alpha_{n+1}(\tau)|^2\right] \mathrm{d}\mathbf{z}}_{=0}\ \mathrm{d}\tau \\
&\quad + 2\int_0^t \|\alpha_{n+2}(\tau) - \alpha_{n+1}(\tau)\|_{\mathcal{L}^2_{\mathbf{z}}} \left\|\left[\left(\Gamma * \left(|\alpha_{n+1}(\tau)|^2 - |\alpha_n(\tau)|^2\right)\right), \alpha_{n+1}(\tau)\right]\right\|_{\mathcal{L}^2_{\mathbf{z}}} \mathrm{d}\tau \\
&\quad \underbrace{- 2\,\Re \int_0^t \int_{\mathbb{R}^{2d}_{\mathbf{z}}} \underbrace{K_{n+1}(\tau, \mathbf{x})}_{\in \mathrm{i}\mathbb{R}}\ |\alpha_{n+2}(\tau, \mathbf{z}) - \alpha_{n+1}(\tau, \mathbf{z})|^2\ \mathrm{d}\mathbf{z}\,\mathrm{d}\tau}_{=0} \\
&\quad + 2\int_0^t \|\alpha_{n+2}(\tau) - \alpha_{n+1}(\tau)\|_{\mathcal{L}^2_{\mathbf{z}}} \|(K_{n+1}(\tau) - K_n(\tau))\ \alpha_{n+1}(\tau)\|_{\mathcal{L}^2_{\mathbf{z}}}\ \mathrm{d}\tau \\
&\leq 2\int_0^t \|\alpha_{n+2}(\tau) - \alpha_{n+1}(\tau)\|^2_{\mathcal{L}^2_{\mathbf{z}}}\ \mathrm{d}\tau + \int_0^t \left\|\left[\left(\Gamma * \left(|\alpha_{n+1}(\tau)|^2 - |\alpha_n(\tau)|^2\right)\right), \alpha_{n+1}(\tau)\right]\right\|^2_{\mathcal{L}^2_{\mathbf{z}}} \mathrm{d}\tau \\
&\quad + \int_0^t \|(K_{n+1}(\tau) - K_n(\tau))\ \alpha_{n+1}(\tau)\|^2_{\mathcal{L}^2_{\mathbf{z}}}\ \mathrm{d}\tau,
\end{aligned}
$$

where for the last two terms, we use that

$$
\begin{aligned}
&\left\|\left[\left(\Gamma * \left(|\alpha_{n+1}(\tau)|^2 - |\alpha_n(\tau)|^2\right)\right), \alpha_{n+1}(\tau)\right]\right\|_{\mathcal{L}^2_{\mathbf{z}}} \\
&\leq \|\nabla\Gamma\|_{\mathcal{L}^\infty_{\mathbf{x}}} \left(\|\alpha_{n+1}(\tau)\|_{\mathcal{L}^2_{\mathbf{z}}} + \|\alpha_n(\tau)\|_{\mathcal{L}^2_{\mathbf{z}}}\right) \|\alpha_{n+1}(\tau) - \alpha_n(\tau)\|_{\mathcal{L}^2_{\mathbf{z}}} \|\nabla\alpha_{n+1}(\tau)\|_{\mathcal{L}^2_{\mathbf{z}}} \\
&\leq 2C_\Gamma M Q_M(\tau)\, \|\alpha_{n+1}(\tau) - \alpha_n(\tau)\|_{\mathcal{L}^2_{\mathbf{z}}}
\end{aligned}
$$

and

$$
\begin{aligned}
&\|(K_{n+1}(\tau) - K_n(\tau))\ \alpha_{n+1}(\tau)\|_{\mathcal{L}^2_{\mathbf{z}}} \leq \|K_{n+1}(\tau) - K_n(\tau)\|_{\mathcal{L}^\infty_{\mathbf{x}}} \|\alpha_{n+1}(\tau)\|_{\mathcal{L}^2_{\mathbf{z}}} \\
&\leq \left\|\left(\nabla\Gamma * \int_{\mathbb{R}^d_{\mathbf{v}}} (\bar{\alpha}_{n+1}(\tau)\nabla_{\mathbf{v}}\alpha_{n+1}(\tau) - \bar{\alpha}_n(\tau)\nabla_{\mathbf{v}}\alpha_n(\tau))\ \mathrm{d}\mathbf{v}\right)\right\|_{\mathcal{L}^\infty_{\mathbf{x}}} \|\alpha_{n+1}(\tau)\|_{\mathcal{L}^2_{\mathbf{z}}} \\
&\leq \|\nabla\Gamma\|_{\mathcal{L}^\infty_{\mathbf{x}}} \|\alpha_{n+1}(\tau)\|_{\mathcal{L}^2_{\mathbf{z}}} \left(\|\nabla\alpha_{n+1}(\tau)\|_{\mathcal{L}^2_{\mathbf{z}}} + \|\nabla\alpha_n(\tau)\|_{\mathcal{L}^2_{\mathbf{z}}}\right) \|\alpha_{n+1}(\tau) - \alpha_n(\tau)\|_{\mathcal{L}^2_{\mathbf{z}}}
\end{aligned}
$$

$$\leq 2C_\Gamma M Q_M(\tau) \left\|\alpha_{n+1}(\tau) - \alpha_n(\tau)\right\|_{\mathcal{L}^2_{\mathbf{z}}}.$$

Combining the previous estimates, along with an inductive scheme of Grönwall type and $\|\alpha_1(\tau) - \alpha_0(\tau)\|^2_{\mathcal{L}^2_{\mathbf{z}}} \leq 4M^2$, one finds that for any $0 \leq t \leq T$,

$$\begin{aligned}
\|\alpha_{n+2}(t) - \alpha_{n+1}(t)\|^2_{\mathcal{L}^2_{\mathbf{z}}} &\leq 8C_\Gamma^2 M^2 \int_0^t Q_M^2(\tau) \left\|\alpha_{n+1}(\tau) - \alpha_n(\tau)\right\|^2_{\mathcal{L}^2_{\mathbf{z}}} \,\mathrm{d}\tau \\
&\quad + 2\int_0^t \|\alpha_{n+2}(\tau) - \alpha_{n+1}(\tau)\|^2_{\mathcal{L}^2_{\mathbf{z}}} \,\mathrm{d}\tau \\
&\leq 8C_\Gamma^2 M^2 Q_M^2(t) e^{2t} \int_0^t \|\alpha_{n+1}(\tau) - \alpha_n(\tau)\|^2_{\mathcal{L}^2_{\mathbf{z}}} \,\mathrm{d}\tau \\
&\leq \left(8C_\Gamma^2 M^2 Q_M^2(T) e^{2T}\right)^{n+1} \frac{T^{n+1}}{(n+1)!} 4M^2.
\end{aligned}$$

As the square root of the right-hand side is summable in n, (α_n) is a Cauchy sequence in the Banach space $\mathcal{L}^\infty\left([0,T]; \mathcal{L}^2_{\mathbf{z}}\right)$ with a unique limit $\alpha \equiv \lim_n \alpha_n$. As T is arbitrary, α is in fact in $\mathcal{L}^\infty_t \mathcal{L}^2_{\mathbf{z}}$, where t ranges in $\mathbb{R}_{\geq 0}$.

(iv) Regularity. Fix some $t \in \mathbb{R}_{\geq 0}$. (iii) proves that $\alpha_n(t) \to \alpha(t)$ strongly in $\mathcal{L}^2_{\mathbf{z}}$ and $\|\alpha_n(t)\|_{\mathcal{W}^{1,2}_{\mathbf{z}}} \leq Q_M(t)$. By the reflexivity of the Hilbert space $\mathcal{W}^{1,2}_{\mathbf{z}}$, some subsequence converges weakly to the same limit $\alpha(t)$, hence, $\alpha(t) \in \mathcal{W}^{1,2}_{\mathbf{z}}$ and $\|\alpha(t)\|_{\mathcal{W}^{1,2}_{\mathbf{z}}} \leq Q_M(t)$.

In addition, we remark that by their $\mathcal{L}^\infty_{t,\mathrm{loc}} \mathcal{L}^2_{\mathbf{z}}$ norm convergence, F_n and K_n, as well as their first two spatial derivatives, converge in $\mathcal{L}^\infty_{(t,\mathbf{x})}$, i.e., their limits are still continuous. Taking the $n \to \infty$ limit of (3.2) and (3.3) proves that $\alpha \in C^1_{(t,\mathbf{z})}$ is a classical solution.

(v) Dense extension and uniqueness. Now assume that $t \mapsto \alpha(t)$ and $t \mapsto \beta(t)$ are two classical solutions with compactly supported $C^1_{\mathbf{z},c}$ initial values $\|\mathring{\alpha}\|_{\mathcal{W}^{1,2}_{\mathbf{z}}}, \left\|\mathring{\beta}\right\|_{\mathcal{W}^{1,2}_{\mathbf{z}}} \leq M$. Using the bounds from (ii), which remain valid because solutions are fixed points of the iteration, and estimating a Grönwall type inequality similar to (iii), we find that

$$\begin{aligned}
&\|\alpha(t) - \beta(t)\|^2_{\mathcal{L}^2_{\mathbf{z}}} - \left\|\mathring{\alpha} - \mathring{\beta}\right\|^2_{\mathcal{L}^2_{\mathbf{z}}} - 2\int_0^t \|\alpha(\tau) - \beta(\tau)\|^2_{\mathcal{L}^2_{\mathbf{z}}} \,\mathrm{d}\tau \\
&\leq \int_0^t \left\|\left[\left(\Gamma * \left(|\alpha(\tau)|^2 - |\beta(\tau)|^2\right)\right), \alpha(\tau)\right]\right\|^2_{\mathcal{L}^2_{\mathbf{z}}} \mathrm{d}\tau \\
&\quad + \int_0^t \left\|\left(\nabla\Gamma * \int_{\mathbb{R}^d_{\mathbf{v}}} \left(\bar{\alpha}(\tau)\nabla_{\mathbf{v}}\alpha(\tau) - \bar{\beta}(\tau)\nabla_{\mathbf{v}}\beta(\tau)\right) \,\mathrm{d}\mathbf{v}\right) \alpha(\tau)\right\|^2_{\mathcal{L}^2_{\mathbf{z}}} \mathrm{d}\tau \\
&\leq \int_0^t \|\nabla\Gamma\|^2_{\mathcal{L}^\infty_{\mathbf{x}}} \|\nabla\alpha(\tau)\|^2_{\mathcal{L}^2_{\mathbf{z}}} \left(\|\alpha(\tau)\|_{\mathcal{L}^2_{\mathbf{z}}} + \|\beta(\tau)\|_{\mathcal{L}^2_{\mathbf{z}}}\right)^2 \|\alpha(\tau) - \beta(\tau)\|^2_{\mathcal{L}^2_{\mathbf{z}}} \,\mathrm{d}\tau
\end{aligned}$$

$$+ \int_0^t \|\nabla\Gamma\|^2_{\mathcal{L}^\infty_{\mathbf{x}}} \|\alpha(\tau)\|^2_{\mathcal{L}^2_{\mathbf{z}}} \left(\|\nabla\alpha(\tau)\|_{\mathcal{L}^2_{\mathbf{z}}} + \|\nabla\beta(\tau)\|_{\mathcal{L}^2_{\mathbf{z}}}\right)^2 \|\alpha(\tau) - \beta(\tau)\|^2_{\mathcal{L}^2_{\mathbf{z}}} \, \mathrm{d}\tau$$
$$\leq 8C_\Gamma^2 M^2 \int_0^t Q_M^2(\tau) \|\alpha(\tau) - \beta(\tau)\|^2_{\mathcal{L}^2_{\mathbf{z}}} \, \mathrm{d}\tau.$$

Grönwall's Lemma therefore yields

$$\|\alpha(t) - \beta(t)\|_{\mathcal{L}^2_{\mathbf{z}}} \leq \left\|\mathring{\alpha} - \mathring{\beta}\right\|_{\mathcal{L}^2_{\mathbf{z}}} \exp\left(\int_0^t \left(1 + 4C_\Gamma^2 M^2 Q_M^2(\tau)\right) \mathrm{d}\tau\right). \tag{3.5}$$

Indeed, we have constructed a solution map

$$\mathcal{W}^{1,2}_{\mathbf{z}} \cap C^1_{\mathbf{z},c} \to \mathcal{L}^\infty_{t,\mathrm{loc}}\mathcal{L}^2_{\mathbf{z}}, \quad \mathring{\alpha} \mapsto \alpha,$$

which is locally Lipschitz and uniquely extensible to the entire space $\mathcal{W}^{1,2}_{\mathbf{z}}$. This gives a notion of solution for every $\mathcal{W}^{1,2}_{\mathbf{z}}$ initial value and (3.5) also proves the uniqueness of classical solutions with compact support. By the same regularity argument as in (iv), $\|\alpha(t)\|_{\mathcal{W}^{1,2}_{\mathbf{z}}} \leq Q_{\|\mathring{\alpha}\|_{\mathcal{W}^{1,2}_{\mathbf{z}}}}(t)$ remains valid for any initial datum. Hence, solutions are actually in $\mathcal{L}^\infty_{t,\mathrm{loc}}\mathcal{W}^{1,2}_{\mathbf{z}} \cap \mathcal{L}^\infty_t\mathcal{L}^2_{\mathbf{z}}$. □

Theorem 3.1 yields a couple of Corollaries. It states that solutions are bounded in $\mathcal{W}^{1,2}_{\mathbf{z}}$ and explicitly gives this bound as well as the dependence of the Lipschitz constant of the solution map on the initial data.

Corollary 3.2. *Let $\alpha : \mathbb{R}_{\geq 0} \to \mathcal{W}^{1,2}_{\mathbf{z}}$ be a solution of* (3.1). *If $\|\alpha(0)\|_{\mathcal{W}^{1,2}_{\mathbf{z}}} \leq M$, then for any $t \geq 0$, the solution α satisfies*

$$\|\alpha(t)\|_{\mathcal{W}^{1,2}_{\mathbf{z}}} \leq \mathfrak{b}_M^{\mathcal{W}^{1,2}_{\mathbf{z}}}(t) = M \frac{1 + C_\Gamma M^2 + C_\Gamma M^2 \, e^{(1+2C_\Gamma M^2)t}}{1 + 2C_\Gamma M^2}. \tag{3.6}$$

Corollary 3.3. *Let $M > 0$ be arbitrary and $\mathring{\alpha}, \mathring{\beta} \in \mathcal{W}^{1,2}_{\mathbf{z}}$ with $\|\mathring{\alpha}\|_{\mathcal{W}^{1,2}_{\mathbf{z}}}, \left\|\mathring{\beta}\right\|_{\mathcal{W}^{1,2}_{\mathbf{z}}} \leq M$. For the corresponding solutions $t \mapsto \alpha(t)$ and $t \mapsto \beta(t)$ of* (3.1), *we have*

$$\|\alpha(t) - \beta(t)\|_{\mathcal{L}^2_{\mathbf{z}}} \leq \left\|\mathring{\alpha} - \mathring{\beta}\right\|_{\mathcal{L}^2_{\mathbf{z}}} \exp\left(\int_0^t \left(1 + 4C_\Gamma^2 M^2 \left(\mathfrak{b}_M^{\mathcal{W}^{1,2}_{\mathbf{z}}}(\tau)\right)^2\right) \mathrm{d}\tau\right).$$

In preparation of Chapter 4, we state bounds on the second derivatives of the solutions of (3.1).

Corollary 3.4. *There is a family of continuous, non-decreasing functions*

$$\left\{\mathfrak{b}_{M,\mathrm{D}^2}^{\mathcal{L}^2_{\mathbf{z}}} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}\right\}_{M \geq 0},$$

s.t. for every solution $\alpha : \mathbb{R}_{\geq 0} \to \mathcal{W}^{1,2}_{\mathbf{z}}$ *of* (3.1) *with initial value* $\alpha(0) = \mathring{\alpha} \in \mathcal{W}^{2,2}_{\mathbf{z}}$ *and* $\|\mathring{\alpha}\|_{\mathcal{W}^{2,2}_{\mathbf{z}}} \leq M$*, we have*

$$\left\|\mathrm{D}^2\alpha(t)\right\|_{\mathcal{L}^2_{\mathbf{z}}} \leq \mathfrak{b}^{\mathcal{L}^2_{\mathbf{z}}}_{M,\mathrm{D}^2}(t)$$

and the solution is in fact in $\mathcal{L}^{\infty}_{t,loc}\mathcal{W}^{2,2}_{\mathbf{z}}$.

Proof. A similar claim for the $\mathcal{W}^{1,2}_{\mathbf{z}}$ norm is already explicitly given in the proof of Theorem 3.1. Hence, it suffices to consider $\left\|\mathrm{D}^2\alpha(t)\right\|_{\mathcal{L}^2_{\mathbf{z}}}$. At first, we assume $\mathring{\alpha} \in C^{\infty}_{\mathbf{z},c}$ compactly supported. Using $F(t,\mathbf{x}) \equiv \left(-\nabla\Gamma * |\alpha(t)|^2\right)(\mathbf{x})$ again, it is necessary to bound the second derivatives of the solution map $Z(t,t_0)$ of the characteristic system, namely,

$$Z(t,t_0,\mathbf{z}) = \mathbf{z} + \int_{t_0}^{t} \begin{pmatrix} V(\tau,t_0,\mathbf{z}) \\ F(\tau,X(\tau,t_0,\mathbf{z})) \end{pmatrix} \mathrm{d}\tau.$$

Because of the regularity assumptions on Γ, $F(t) \in C^2_{\mathbf{x}}$ and clearly $Z(t,t_0)$ is in $C^2_{\mathbf{z}}$. Differentiation and applying the $\mathcal{L}^{\infty}_{\mathbf{z}}$ norm of the tensor operator norms yields for the first two derivatives that

$$\begin{aligned} \|\mathrm{D}Z(t,t_0)\|_{\mathcal{L}^{\infty}_{\mathbf{z}}} &\leq 1 + \int_{t_0}^{t} \left(1 + \left\|\left(\mathrm{D}^2\Gamma * |\alpha(\tau)|^2\right)\right\|_{\mathcal{L}^{\infty}_{\mathbf{x}}}\right) \|\mathrm{D}Z(\tau,t_0)\|_{\mathcal{L}^{\infty}_{\mathbf{z}}} \ \mathrm{d}\tau \\ &\leq \exp\left(\left(1 + C_{\Gamma} \|\mathring{\alpha}\|^2_{\mathcal{L}^2_{\mathbf{z}}}\right)|t-t_0|\right) \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \left\|\mathrm{D}^2Z(t,t_0)\right\|_{\mathcal{L}^{\infty}_{\mathbf{z}}} &\leq \int_{t_0}^{t} \left\|\left(\mathrm{D}^3\Gamma * |\alpha(\tau)|^2\right)\right\|_{\mathcal{L}^{\infty}_{\mathbf{x}}} \|\mathrm{D}Z(\tau,t_0)\|^2_{\mathcal{L}^{\infty}_{\mathbf{z}}} \ \mathrm{d}\tau \\ &\quad + \int_{t_0}^{t} \left(1 + \left\|\left(\mathrm{D}^2\Gamma * |\alpha(\tau)|^2\right)\right\|_{\mathcal{L}^{\infty}_{\mathbf{x}}}\right) \left\|\mathrm{D}^2Z(\tau,t_0)\right\|_{\mathcal{L}^{\infty}_{\mathbf{z}}} \ \mathrm{d}\tau \\ &\leq C_{\Gamma} \|\mathring{\alpha}\|^2_{\mathcal{L}^2_{\mathbf{z}}} \frac{\exp\left(2\left(1 + C_{\Gamma}\|\mathring{\alpha}\|^2_{\mathcal{L}^2_{\mathbf{z}}}\right)|t-t_0|\right) - 1}{2\left(1 + C_{\Gamma}\|\mathring{\alpha}\|^2_{\mathcal{L}^2_{\mathbf{z}}}\right)} \exp\left(\left(C_{\Gamma}\|\mathring{\alpha}\|^2_{\mathcal{L}^2_{\mathbf{z}}}\right)|t-t_0|\right) \\ &\leq C_{\Gamma} \|\mathring{\alpha}\|^2_{\mathcal{L}^2_{\mathbf{z}}} |t-t_0| \ \exp\left(3\left(1 + C_{\Gamma}\|\mathring{\alpha}\|^2_{\mathcal{L}^2_{\mathbf{z}}}\right)|t-t_0|\right). \end{aligned} \tag{3.8}$$

For the second derivative of $\alpha(t)$, we set $K(t,\mathbf{x}) \equiv (\Gamma * [\bar{\alpha}(t),\alpha(t)])(\mathbf{x})$ and recall the transport formula

$$\alpha(t,\mathbf{z}) = \mathring{\alpha}(Z(0,t,\mathbf{z})) \ \exp\left(-\int_0^t K(\tau,X(\tau,t,\mathbf{z}))\,\mathrm{d}\tau\right)$$

for the general Hamiltonian Vlasov systems from Proposition 1.5. With this at hand, we determine for any pair k, l and summation indices i, j in the range $\{1, \ldots, 2d\}$ that

$$
\begin{aligned}
&(\partial_l \partial_k \alpha(t, \mathbf{z})) \ \exp\left(\int_0^t K(\tau, X(\tau, t, \mathbf{z}))\, \mathrm{d}\tau \right) \\
= &\sum_{i,j} \partial_i \partial_j \mathring{\alpha}(Z(0, t, \mathbf{z}))\, \partial_l Z_i(0, t, \mathbf{z})\, \partial_k Z_j(0, t, \mathbf{z}) \\
&+ \sum_j \partial_j \mathring{\alpha}(Z(0, t, \mathbf{z}))\, \partial_k \partial_l Z_j(0, t, \mathbf{z}) \\
&- \sum_i \partial_i \mathring{\alpha}(Z(0, t, \mathbf{z}))\, \partial_l Z_i(0, t, \mathbf{z}) \ \int_0^t \sum_j \partial_j K(\tau, X(\tau, t, \mathbf{z}))\, \partial_k X_j(\tau, t, \mathbf{z})\, \mathrm{d}\tau \\
&- \sum_i \partial_i \mathring{\alpha}(Z(0, t, \mathbf{z}))\, \partial_k Z_i(0, t, \mathbf{z}) \ \int_0^t \sum_j \partial_j K(\tau, X(\tau, t, \mathbf{z}))\, \partial_l X_j(\tau, t, \mathbf{z})\, \mathrm{d}\tau \\
&- \mathring{\alpha}(Z(0, t, \mathbf{z})) \ \int_0^t \sum_{i,j} \partial_i \partial_j K(\tau, X(\tau, t, \mathbf{z}))\, \partial_k X_j(\tau, t, \mathbf{z})\, \partial_l X_i(\tau, t, \mathbf{z})\, \mathrm{d}\tau \\
&- \mathring{\alpha}(Z(0, t, \mathbf{z})) \ \int_0^t \sum_j \partial_j K(\tau, X(\tau, t, \mathbf{z}))\, \partial_l \partial_k X_j(\tau, t, \mathbf{z})\, \mathrm{d}\tau \\
&+ \mathring{\alpha}(Z(0, t, \mathbf{z})) \ \int_0^t \sum_i \partial_i K(\tau, X(\tau, t, \mathbf{z}))\, \partial_k X_i(\tau, t, \mathbf{z})\, \mathrm{d}\tau \\
&\quad \cdot \int_0^t \sum_j \partial_j K(\tilde{\tau}, X(\tilde{\tau}, t, \mathbf{z}))\, \partial_l X_j(\tilde{\tau}, t, \mathbf{z})\, \mathrm{d}\tilde{\tau}.
\end{aligned}
$$

Hence, the $\mathcal{L}^2_{\mathbf{z}}$ norm is computed to be

$$
\begin{aligned}
\left\| \mathrm{D}^2 \alpha(t) \right\|_{\mathcal{L}^2_{\mathbf{z}}} \leq{}& \left\| \mathrm{D}^2 \mathring{\alpha} \right\|_{\mathcal{L}^2_{\mathbf{z}}} \|\mathrm{D}Z(0, t)\|^2_{\mathcal{L}^\infty_{\mathbf{z}}} + \|\nabla \mathring{\alpha}\|_{\mathcal{L}^2_{\mathbf{z}}} \left\| \mathrm{D}^2 Z(0, t) \right\|_{\mathcal{L}^\infty_{\mathbf{z}}} \\
&+ 2\, \|\nabla \mathring{\alpha}\|_{\mathcal{L}^2_{\mathbf{z}}} \|\mathrm{D}Z(0, t)\|_{\mathcal{L}^\infty_{\mathbf{z}}} \int_0^t \|(\nabla \Gamma * [\bar{\alpha}(\tau), \alpha(\tau)])\|_{\mathcal{L}^\infty_{\mathbf{x}}} \|\mathrm{D}Z(\tau, t)\|_{\mathcal{L}^\infty_{\mathbf{z}}} \ \mathrm{d}\tau \\
&+ \|\mathring{\alpha}\|_{\mathcal{L}^2_{\mathbf{z}}} \int_0^t \left\| \left(\mathrm{D}^2 \Gamma * [\bar{\alpha}(\tau), \alpha(\tau)] \right) \right\|_{\mathcal{L}^\infty_{\mathbf{x}}} \|\mathrm{D}Z(\tau, t)\|^2_{\mathcal{L}^\infty_{\mathbf{z}}} \ \mathrm{d}\tau \\
&+ \|\mathring{\alpha}\|_{\mathcal{L}^2_{\mathbf{z}}} \int_0^t \|(\nabla \Gamma * [\bar{\alpha}(\tau), \alpha(\tau)])\|_{\mathcal{L}^\infty_{\mathbf{x}}} \left\| \mathrm{D}^2 Z(\tau, t) \right\|_{\mathcal{L}^\infty_{\mathbf{z}}} \ \mathrm{d}\tau \\
&+ \|\mathring{\alpha}\|_{\mathcal{L}^2_{\mathbf{z}}} \left(\int_0^t \|(\nabla \Gamma * [\bar{\alpha}(\tau), \alpha(\tau)])\|_{\mathcal{L}^\infty_{\mathbf{x}}} \|\mathrm{D}Z(\tau, t)\|_{\mathcal{L}^\infty_{\mathbf{z}}} \ \mathrm{d}\tau \right)^2 \leq \mathfrak{b}^{\mathcal{L}^2_{\mathbf{z}}}_{M, \mathrm{D}^2}(t) \quad (3.9)
\end{aligned}
$$

for a continuous non-decreasing function $\mathfrak{b}^{\mathcal{L}^2_{\mathbf{z}}}_{M, \mathrm{D}^2} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, explicitly computable from (3.7), (3.8), and $\mathfrak{b}_M^{\mathcal{W}_{\mathbf{z}}^{1,2}}$. In particular, it only depends on a bound M of the $\mathcal{W}_{\mathbf{z}}^{2,2}$ norm of $\mathring{\alpha}$. Now if we approximate $\mathring{\alpha} \in \mathcal{W}_{\mathbf{z}}^{2,2}$ by a sequence $\mathring{\alpha}^{(k)}$ of test functions with norm $\leq M$, we have the norm convergence $\left\|\alpha(t) - \alpha^{(k)}(t)\right\|_{\mathcal{L}_{\mathbf{z}}^2} \to 0$ for any $t \geq 0$ by Theorem 3.1. By the inequalities (3.6) and (3.9) given above, $\left\|\alpha^{(k)}(t)\right\|_{\mathcal{W}_{\mathbf{z}}^{2,2}}$ is bounded and the reflexivity of $\mathcal{W}_{\mathbf{z}}^{2,2}$ gives a weakly converging subsequence. Its limit coincides with the $\mathcal{L}_{\mathbf{z}}^2$ limit $\alpha(t)$, which is already in $\mathcal{W}_{\mathbf{z}}^{2,2}$ and its norm is dominated by the same bound (3.9), i.e., $\left\|\mathrm{D}^2\alpha(t)\right\|_{\mathcal{L}_{\mathbf{z}}^2} \leq \mathfrak{b}_{M,\mathrm{D}^2}^{\mathcal{L}_{\mathbf{z}}^2}(t)$. □

In this particular setup, it is very natural to restate the global well-posedness specifically for the Hamilton Hartree system (3.10), since the Sobolev spaces nicely transform under the velocity Fourier transform (FT) given by

$$\mathcal{F}: \quad \mathcal{L}_{\mathbf{z}}^2 \to \mathcal{L}_{\hat{\mathbf{z}}}^2, \quad \alpha \mapsto \hat{\alpha}, \quad \text{where} \quad \hat{\alpha}(\mathbf{x}, \boldsymbol{\xi}) \equiv \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}_{\mathbf{v}}^d} \alpha(\mathbf{x}, \mathbf{v})\, e^{-\mathrm{i}\mathbf{v}\cdot\boldsymbol{\xi}}\, \mathrm{d}\mathbf{v},$$

as known from standard Fourier theory [6, Sec.5.8.4]. We recall that the Hamiltonian Vlasov equation (3.1) yields for $\hat{\alpha} = \mathcal{F}\alpha$ the Hamilton Hartree equation

$$\mathrm{i}\,\partial_t \hat{\alpha}(t, \hat{\mathbf{z}}) = \left(\nabla_{\mathbf{x}} \cdot \nabla_{\boldsymbol{\xi}} + \left(\hat{V} * |\hat{\alpha}(t)|^2\right)(\hat{\mathbf{z}})\right) \hat{\alpha}(t, \hat{\mathbf{z}}) \tag{3.10}$$

for $\hat{V}(\mathbf{x}, \boldsymbol{\xi}) \equiv -\nabla\Gamma(\mathbf{x}) \cdot \boldsymbol{\xi}$.

Definition 3.5. For any $k \in \mathbb{N}_0$, define $\mathcal{M}_{\hat{\mathbf{z}}}^k \equiv \mathcal{F}\left(\mathcal{W}_{\mathbf{z}}^{k,2}\right)$ to be the set of functions $\hat{\alpha} : \mathbb{R}_{\hat{\mathbf{z}}}^{2d} \to \mathbb{C}$ which arise from the velocity Fourier transform (FT) of the respective Sobolev functions with isometric norm $\|\hat{\alpha}\|_{\mathcal{M}_{\hat{\alpha}}^k} \equiv \left\|\mathcal{F}^{-1}\hat{\alpha}\right\|_{\mathcal{W}_{\mathbf{z}}^{k,2}}$. In the case of $k = 1$, the norm is

$$\|\hat{\alpha}\|_{\mathcal{M}_{\hat{\mathbf{z}}}^1} \equiv \left(\|\hat{\alpha}\|_{\mathcal{L}_{\hat{\mathbf{z}}}^2}^2 + \|\boldsymbol{\xi}\,\hat{\alpha}\|_{\mathcal{L}_{\hat{\mathbf{z}}}^2}^2 + \|\nabla_{\mathbf{x}}\hat{\alpha}\|_{\mathcal{L}_{\hat{\mathbf{z}}}^2}^2\right)^{\frac{1}{2}}.$$

Utilizing Definition 3.5, we are now ready to state the following result.

Theorem 3.6 (Global Well-Posedness for Regular Hamilton Hartree)**.** *Let $\mathring{\hat{\alpha}} \in \mathcal{M}_{\hat{\mathbf{z}}}^1$ be arbitrary. Then there is a unique global solution $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\hat{\mathbf{z}}}^{2d} \ni (t, \hat{\mathbf{z}}) \mapsto \hat{\alpha}(t, \hat{\mathbf{z}})$ of (3.10) with initial datum $\hat{\alpha}(0) = \mathring{\hat{\alpha}}$. For any $T < \infty$, the solution map*

$$\begin{array}{ccc} \mathcal{M}_{\hat{\mathbf{z}}}^1 & \to & \mathcal{L}_t^\infty \mathcal{L}_{\hat{\mathbf{z}}}^2 \\ \mathring{\hat{\alpha}} & \mapsto & \hat{\alpha} : \begin{array}{ccc} [0,T] & \to & \mathcal{L}_{\hat{\mathbf{z}}}^2 \\ t & \mapsto & \hat{\alpha}(t) \end{array} \end{array}$$

is locally Lipschitz continuous. In fact, the image of this map is in $\mathcal{L}_t^\infty \mathcal{L}_{\hat{\mathbf{z}}}^2 \cap \mathcal{L}_{t,loc}^\infty \mathcal{M}_{\hat{\mathbf{z}}}^1$ for $T = \infty$.

Proof. This is immediate from the norm conserving properties of the Fourier transform (FT), $\|\alpha\|_{\mathcal{W}^{1,2}_{\mathbf{z}}} = \|\hat{\alpha}\|_{\mathcal{M}^1_{\hat{\mathbf{z}}}}$ and $\|\alpha\|_{\mathcal{L}^2_{\mathbf{z}}} = \|\hat{\alpha}\|_{\mathcal{L}^2_{\hat{\mathbf{z}}}}$, and Theorem 3.1. We also remark that classical solutions of (3.1) in the class of Schwartz functions transform into classical solutions of (3.10), since the Fourier transform is a bijection of Schwartz functions. Therefore, the notion of a solution given here is a meaningful extension, even without proper regularity restrictions for the derivatives. □

3.2 On the Hamiltonian Vlasov–Poisson System

This section is in large parts adapted from the author's work published in [14, Sec.3].

Since the original purpose of the Vlasov equation is to describe collective behavior of matter under the influence of gravitational or electric forces, the most fundamental model is genuinely the Vlasov–Poisson system from Example 1.1.

As opposed to the regular potential assumption of Section 3.1, the singularity of the Newtonian interaction potential

$$\Gamma(\mathbf{x}) = \frac{|\mathbf{x}|^{2-d}}{d(2-d)\omega_d}, \quad d \geq 3,$$

requires a more sophisticated choice of initial data in order to solve Equation (3.1).

Let us recall the Hamiltonian Vlasov functional from Example 1.3. Integrating by parts yields

$$\begin{aligned} \mathcal{H}_{\mathrm{Vl}}(\alpha) \equiv \frac{1}{2}\Re \int \bar{\alpha}(\mathbf{x}, \mathbf{v})\, \mathbf{v} \cdot \frac{1}{\mathrm{i}} \nabla_{\mathbf{x}} \alpha(\mathbf{x}, \mathbf{v})\, \mathrm{d}(\mathbf{x}, \mathbf{v}) \\ - \frac{1}{2}\Re \int \bar{\alpha}(\mathbf{x}, \mathbf{v}) \left(\int \nabla\Gamma(\mathbf{x}-\mathbf{y})\ |\alpha(\mathbf{y}, \mathbf{w})|^2\ \mathrm{d}(\mathbf{y}, \mathbf{w}) \right) \cdot \frac{1}{\mathrm{i}} \nabla_{\mathbf{v}} \alpha(\mathbf{x}, \mathbf{v})\, \mathrm{d}(\mathbf{x}, \mathbf{v}), \end{aligned} \tag{3.11}$$

while the corresponding **Hamiltonian Vlasov–Poisson equation** is easily determined to be

$$\begin{aligned} \partial_t \alpha(t, \mathbf{x}, \mathbf{v}) = - \mathbf{v} \cdot \nabla_{\mathbf{x}} \alpha(t, \mathbf{x}, \mathbf{v}) + \left(\int \nabla\Gamma(\mathbf{x}-\mathbf{y})\, |\alpha(t, \mathbf{y}, \mathbf{w})|^2\ \mathrm{d}(\mathbf{y}, \mathbf{w}) \right) \cdot \nabla_{\mathbf{v}} \alpha(t, \mathbf{x}, \mathbf{v}) \\ - \alpha(t, \mathbf{x}, \mathbf{v}) \int \bar{\alpha}(t, \mathbf{y}, \mathbf{w})\, \nabla\Gamma(\mathbf{x}-\mathbf{y}) \cdot \nabla_{\mathbf{w}} \alpha(t, \mathbf{y}, \mathbf{w})\, \mathrm{d}(\mathbf{y}, \mathbf{w}). \end{aligned} \tag{3.12}$$

The initial condition is given by $\alpha(0, \cdot) = \mathring{\alpha}$ for some function $\mathring{\alpha} : \mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{v}} \to \mathbb{C}$. Formulae (3.11) and (3.12) exactly coincide with the results in [7, Sec.2], motivating to write these equations in this form. We use the following definition in order to simplify the notation.

Definition 3.7 (Characteristic Tuple)**.** Let $\alpha : \mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{v}} \to \mathbb{C}$ be some function. If all of the functions

$$\rho(\mathbf{x}) \equiv \int_{\mathbb{R}^d_{\mathbf{v}}} |\alpha(\mathbf{x}, \mathbf{v})|^2 \, \mathrm{d}\mathbf{v}, \qquad F(\mathbf{x}) \equiv -(\nabla\Gamma * \rho)(\mathbf{x}),$$

$$\varphi(\mathbf{x}) \equiv \int_{\mathbb{R}^d_{\mathbf{v}}} \bar{\alpha}(\mathbf{x}, \mathbf{v}) \, \nabla_{\mathbf{v}}\alpha(\mathbf{x}, \mathbf{v}) \, \mathrm{d}\mathbf{v}, \qquad K(\mathbf{x}) \equiv -(\nabla\Gamma * \varphi)(\mathbf{x})$$

are well-defined, then α has a **characteristic tuple** (ρ, F, φ, K). In parallel notation to ρ and F, φ is called the **phase density** and K the **phase force**.

Equation (3.12) now reduces to

$$\partial_t \alpha(t, \mathbf{x}, \mathbf{v}) = -\mathbf{v} \cdot \nabla_{\mathbf{x}}\alpha(t, \mathbf{x}, \mathbf{v}) - F(t, \mathbf{x}) \cdot \nabla_{\mathbf{v}}\alpha(t, \mathbf{x}, \mathbf{v}) + K(t, \mathbf{x}) \, \alpha(t, \mathbf{x}, \mathbf{v}).$$

3.2.1 Local Existence and Well-Posedness

The well-posedness theory is developed in a Banach space of $C^k_{\mathbf{z}}$ functions with $\mathcal{L}^p_{\mathbf{z}}$ integrable local supremum of all derivatives up to order k, as given in Definition 3.8. For a more detailed discussion of the properties of this Banach space we refer the reader to Appendix A.

Definition 3.8. Let $f : \mathbb{R}^d \to \mathbb{R}$ be some measurable function. For any real parameters $p \geq 1$ and $\kappa \geq d$, consider the norm

$$\|f\|_{\mathcal{A}^{\kappa,p}} \equiv \sup_{R \geq 0} (1+R)^{-\frac{\kappa}{p}} \left(\int_{\mathbb{R}^d} \sup_{|\bar{\mathbf{z}}| \leq R} |f(\mathbf{z} + \bar{\mathbf{z}})|^p \, \mathrm{d}\mathbf{z} \right)^{\frac{1}{p}}. \tag{3.13}$$

Denote by $\mathcal{A}^{\kappa,p}$ the set of measurable functions with finite norm. In addition, we define the linear space

$$\mathcal{B}^{k,\kappa,p} \equiv \left\{ f \in C^k \left(\mathbb{R}^d \to \mathbb{C} \right) : \forall l \leq k, |\alpha| = l : \mathrm{D}^\alpha f \in \mathcal{A}^{\kappa,p} \right\} \tag{3.14}$$

with the norm

$$\|f\|_{\mathcal{B}^{k,\kappa,p}} \equiv \left(\sum_{|\alpha| \leq k} \|\mathrm{D}^\alpha f\|^p_{\mathcal{A}^{\kappa,p}} \right)^{\frac{1}{p}}.$$

Throughout the section, the real parameter $\kappa \geq \dim \mathbb{R}^{2d}_{\mathbf{z}} = 2d$ is fixed. The central result of this section is the following statement:

Theorem 3.9 (Local Existence for Hamiltonian Vlasov–Poisson)**.** *Let $\mathring{\alpha} \in \mathcal{B}^{1,\kappa,2}_{\mathbf{z}}$ be an initial datum. Then there is a positive time of existence $T = T\left(\|\mathring{\alpha}\|_{\mathcal{B}^{1,\kappa,2}_{\mathbf{z}}}\right) > 0$, such that the Hamiltonian Vlasov–Poisson equation gives rise to a unique solution on $[0, T)$.*

Proof. This is an implication of Proposition 3.20. □

Remark 3.10. **(i).** If for some $\gamma > d$ we have $\sup_{\mathbf{z}}(1 + |\mathbf{z}|)^\gamma(|\mathring{\alpha}(\mathbf{z})| + |\nabla_{\mathbf{z}}\mathring{\alpha}(\mathbf{z})|) < \infty$, then $\mathring{\alpha} \in \mathcal{B}_{\mathbf{z}}^{1,\kappa,2}$. Therefore, Theorem 3.9 applies to a wide class of appropriately decaying data.

(ii). For $|\mathbf{z}| \to \infty$, we always have $|\mathring{\alpha}(\mathbf{z})| \to 0$. This is indeed the case because an unbounded sequence of $(\mathbf{z}_n)$ with $|\alpha(\mathbf{z}_n)| \geq \epsilon > 0$ would imply an infinite norm. Nevertheless, the explicit decay rate is not prescribed. In particular, there need not exist any $\gamma > 0$, s.t. $\sup_{\mathbf{z}} |\alpha(\mathbf{z})| (1 + |\mathbf{z}|)^\gamma < \infty$.

Similarly, $|\nabla_{\mathbf{z}}\mathring{\alpha}(\mathbf{z})| \to 0$ for $|\mathbf{z}| \to \infty$. This forbids oscillations at infinity, even if they have decreasing amplitude. Toward the boundary of the phase space, matter has to be locally *almost* equidistributed. Again, no specific decay rate is implied.

The remainder of the section is devoted to the proof of Theorem 3.9. At first, we specify the class of solutions we are interested in.

Definition 3.11 (Solution)**.** Let $T > 0$ and $\alpha \in C^1\left([0,T) \times \mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{v}}; \mathbb{C}\right)$ be some function. α is called **admissible** if and only if

(i) for every $t \in [0,T)$, the characteristic tuple $(\rho(t), F(t), \varphi(t), K(t))$ of $\alpha(t)$ is well-defined,

(ii) $F, \nabla_{\mathbf{x}}F, K, \nabla_{\mathbf{x}}K \in C^0_{(t,\mathbf{x})}$ exist and are continuous as functions of time and space, and

(iii) $\sup_{\tau\leq t} \|F(\tau)\|_{\mathcal{L}^\infty_{\mathbf{x}}} < \infty$ for every $t \in [0,T)$.

An admissible function is called a **local solution** if it satisfies the Hamiltonian Vlasov–Poisson equation. The solution is called **global** in the case of $T = \infty$.

In preparation of our proof, we need three technical lemmata. We introduce the notation

$$A(a,b) \equiv \inf_{R>0} \frac{(1+R)^a}{\tau_b R^b} = \frac{a^a(a-b)^{b-a}}{\tau_b b^b}, \tag{3.15}$$

where τ_b represents the Lebesgue volume of the b dimensional unit ball.

Lemma 3.12. *Let $\alpha \in \mathcal{B}_{\mathbf{z}}^{1,\kappa,2}$ be some function. Then the following holds for the characteristic tuple of α:*

$$(\rho, F, \varphi, K) \in \left(\mathcal{W}^{1,1}_{\mathbf{x}} \cap C^1_{\mathbf{x}}\right) \times C^{1,1}_{\mathbf{x}} \times \left(\mathcal{W}^{1,1}_{\mathbf{x}} \cap C^1_{\mathbf{x}}\right) \times C^{1,1}_{\mathbf{x}}.$$

Proof. **(i) Density ρ.** Since $\alpha \in \mathcal{B}^{1,\kappa,2}_{\mathbf{z}} \subseteq \mathcal{A}^{\kappa,2}_{\mathbf{z}} \subseteq \mathcal{L}^2_{\mathbf{z}}$, we get that $\rho \in \mathcal{L}^1_{\mathbf{x}}$. On the other hand, because $\alpha \in \mathcal{A}^{\kappa,2}_{\mathbf{z}}$, for any $R > 0$ and $\mathbf{x}$, we have that

$$\begin{aligned}
|\rho(\mathbf{x})| \leq & \frac{1}{\tau_d R^d} \int_{B_R(\mathbf{x}) \subset \mathbb{R}^d_{\mathbf{x}}} \sup_{|\bar{\mathbf{x}}| \leq R} |\rho(\tilde{\mathbf{x}} + \bar{\mathbf{x}})| \, \mathrm{d}\tilde{\mathbf{x}} \\
\leq & \frac{1}{\tau_d R^d} \int_{\mathbb{R}^d_{\mathbf{x}}} \int_{\mathbb{R}^d_{\mathbf{v}}} \sup_{|(\bar{\mathbf{x}}, \bar{\mathbf{v}})| \leq R} |\alpha(\tilde{\mathbf{x}} + \bar{\mathbf{x}}, \tilde{\mathbf{v}} + \bar{\mathbf{v}})|^2 \, \mathrm{d}(\tilde{\mathbf{x}}, \tilde{\mathbf{v}}) \\
\leq & \frac{(1+R)^\kappa}{\tau_d R^d} \, \|\alpha\|^2_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} \overset{R \text{ opt.}}{=} A(\kappa, d) \, \|\alpha\|^2_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} \,,
\end{aligned}$$

implying that $\rho \in \mathcal{L}^\infty_{\mathbf{x}}$. Since $\nabla_{\mathbf{x}} \alpha \in \mathcal{A}^{\kappa,2}_{\mathbf{z}} \subseteq \mathcal{L}^2_{\mathbf{z}}$, the natural candidate for the derivative is

$$\nabla_{\mathbf{x}} \rho(\mathbf{x}) = 2\Re \int_{\mathbb{R}^d_{\mathbf{v}}} \bar{\alpha}(\mathbf{x}, \mathbf{v}) \, \nabla_{\mathbf{x}} \alpha(\mathbf{x}, \mathbf{v}) \, \mathrm{d}\mathbf{v}.$$

This is indeed the case, because for any $R > 0$ and $\mathbf{x}$,

$$\begin{aligned}
& \int_{\mathbb{R}^d_{\mathbf{v}}} \sup_{|\bar{\mathbf{x}}| \leq R} |\bar{\alpha}(\mathbf{x} + \bar{\mathbf{x}}, \mathbf{v})| \; |\nabla_{\mathbf{x}} \alpha(\mathbf{x} + \bar{\mathbf{x}}, \mathbf{v})| \, \mathrm{d}\mathbf{v} \\
\leq & \frac{1}{\tau_d R^d} \int_{\mathbb{R}^{2d}_{\mathbf{z}}} \sup_{|\bar{\mathbf{z}}| \leq 2R} |\bar{\alpha}(\tilde{\mathbf{z}} + \bar{\mathbf{z}})| \; |\nabla_{\mathbf{x}} \alpha(\tilde{\mathbf{z}} + \bar{\mathbf{z}})| \; \mathrm{d}\tilde{\mathbf{z}} \\
\leq & \frac{(1+2R)^\kappa}{\tau_d R^d} \, \| |\bar{\alpha}| \, |\nabla_{\mathbf{x}} \alpha| \|_{\mathcal{A}^{\kappa,1}_{\mathbf{z}}} \overset{\text{Hölder}}{\leq} \frac{(1+2R)^\kappa}{\tau_d R^d} \, \|\alpha\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} \, \|\nabla_{\mathbf{x}} \alpha\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} \\
\overset{R \text{ opt.}}{=} & 2^d A(\kappa, d) \, \|\alpha\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} \, \|\nabla_{\mathbf{x}} \alpha\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} \,.
\end{aligned}$$

Hence, $\rho \in C^1_{\mathbf{x}}$. Finally $\nabla_{\mathbf{x}} \rho \in \mathcal{L}^1_{\mathbf{x}}$ as $\alpha, \nabla_{\mathbf{x}} \alpha \in \mathcal{L}^2_{\mathbf{z}}$, yielding that $\rho \in \mathcal{W}^{1,1}_{\mathbf{x}}$.

(ii) Force F. By (i), the density ρ is in $\mathcal{L}^1_{\mathbf{x}} \cap C^1_{\mathbf{x}}$. By Lemma B.1, the force $F = -(\nabla\Gamma * \rho)$ then is in $C^{1,1}_{\mathbf{x}}$.

(iii) Phase density φ. Since $\alpha, \nabla_{\mathbf{v}} \alpha \in \mathcal{L}^2_{\mathbf{z}}$, we have that $\varphi \in \mathcal{L}^1_{\mathbf{x}}$. In addition, $\alpha, \nabla_{\mathbf{v}} \alpha \in \mathcal{A}^{\kappa,2}_{\mathbf{z}}$ implies that $\varphi \in \mathcal{L}^\infty_{\mathbf{x}}$. If $\alpha \in C^\infty_{\mathbf{z},c}$, the derivative is then

$$\nabla_{\mathbf{x}} \varphi(\mathbf{x}) = 2\mathrm{i}\Im \int_{\mathbb{R}^d_{\mathbf{v}}} \nabla_{\mathbf{x}} \bar{\alpha}(\mathbf{x}, \mathbf{v}) \, \nabla_{\mathbf{v}} \alpha(\mathbf{x}, \mathbf{v}) \, \mathrm{d}\mathbf{v},$$

with similar computations as in (i). This relation can be extended by a density argument to $\alpha \in \mathcal{B}^{1,\kappa,2}_{\mathbf{z}}$. By identical arguments as in (i), we find that $\varphi \in \mathcal{W}^{1,1}_{\mathbf{x}} \cap C^1_{\mathbf{x}}$.

(iv) Phase force K. As φ is in the same space as ρ, we use the same argumentation as in (ii). $\square$

Lemma 3.13. *Let $\alpha_1, \alpha_2 \in \mathcal{B}_{\mathbf{z}}^{1,\kappa,2}$ be two functions. If $(\rho_i, F_i, \varphi_i, K_i)$ denote the characteristic tuples, then the following inequalities hold:*

(i) $\|\rho_1 - \rho_2\|_{\mathcal{L}^\infty_{\mathbf{x}}} \leq A(\kappa, d)\left(\|\alpha_1\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} + \|\alpha_2\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}}\right)\|\alpha_1 - \alpha_2\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}}$,

(ii) $\|\rho_1 - \rho_2\|_{\mathcal{L}^1_{\mathbf{x}}} \leq \left(\|\alpha_1\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} + \|\alpha_2\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}}\right)\|\alpha_1 - \alpha_2\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}}$,

(iii) $\|F_1 - F_2\|_{\mathcal{L}^\infty_{\mathbf{x}}} \leq c_{1,d} A(\kappa, d)^{1-\frac{1}{d}}\left(\|\alpha_1\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} + \|\alpha_2\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}}\right)\|\alpha_1 - \alpha_2\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}}$,

(iv) $\|\varphi_1 - \varphi_2\|_{\mathcal{L}^\infty_{\mathbf{x}}} \leq A(\kappa, d)\left(\|\nabla_{\mathbf{v}}\alpha_1\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} + \|\nabla_{\mathbf{v}}\alpha_2\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}}\right)\|\alpha_1 - \alpha_2\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}}$,

(v) $\|\varphi_1 - \varphi_2\|_{\mathcal{L}^1_{\mathbf{x}}} \leq \left(\|\nabla_{\mathbf{v}}\alpha_1\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} + \|\nabla_{\mathbf{v}}\alpha_2\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}}\right)\|\alpha_1 - \alpha_2\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}}$, *and*

(vi) $\|K_1 - K_2\|_{\mathcal{L}^\infty_{\mathbf{x}}} \leq c_{1,d} A(\kappa, d)^{1-\frac{1}{d}}\left(\|\nabla_{\mathbf{v}}\alpha_1\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} + \|\nabla_{\mathbf{v}}\alpha_2\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}}\right)\|\alpha_1 - \alpha_2\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}}$.

Proof. **(i), (ii), & (iii).** For any $\mathbf{x}$, $R > 0$,

$$\begin{aligned}|\rho_1(\mathbf{x}) - \rho_2(\mathbf{x})| &\leq \frac{1}{\tau_d R^d}\int_{\mathbb{R}^{2d}_{\mathbf{z}}} \sup_{|\bar{\mathbf{z}}|\leq R}\left(|\alpha_1(\tilde{\mathbf{z}} + \bar{\mathbf{z}})| + |\alpha_1(\tilde{\mathbf{z}} + \bar{\mathbf{z}})|\right)|\alpha_1(\tilde{\mathbf{z}} + \bar{\mathbf{z}}) - \alpha_2(\tilde{\mathbf{z}} + \bar{\mathbf{z}})|\,\mathrm{d}\tilde{\mathbf{z}}\\ &\overset{R\text{ opt.}}{\leq} A(\kappa, d)\left(\|\alpha_1\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} + \|\alpha_2\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}}\right)\|\alpha_1 - \alpha_2\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}},\end{aligned}$$

and upon integration,

$$\|\rho_1 - \rho_2\|_{\mathcal{L}^1_{\mathbf{x}}} \leq \left(\|\alpha_1\|_{\mathcal{L}^2_{\mathbf{z}}} + \|\alpha_2\|_{\mathcal{L}^2_{\mathbf{z}}}\right)\|\alpha_1 - \alpha_2\|_{\mathcal{L}^2_{\mathbf{z}}} \leq \left(\|\alpha_1\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} + \|\alpha_2\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}}\right)\|\alpha_1 - \alpha_2\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}}.$$

In combination with inequality (B.5), one gets that

$$\begin{aligned}\|F_1 - F_2\|_{\mathcal{L}^\infty_{\mathbf{x}}} &\leq c_{1,d}\,\|\rho_1 - \rho_2\|^{\frac{1}{d}}_{\mathcal{L}^1_{\mathbf{x}}}\,\|\rho_1 - \rho_2\|^{1-\frac{1}{d}}_{\mathcal{L}^\infty_{\mathbf{x}}}\\ &\leq c_{1,d} A(\kappa, d)^{1-\frac{1}{d}}\left(\|\alpha_1\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} + \|\alpha_2\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}}\right)\|\alpha_1 - \alpha_2\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}}.\end{aligned}$$

(iv), (v), & (vi). For any $\mathbf{x}$,

$$\begin{aligned}
|\varphi_1(\mathbf{x}) - \varphi_2(\mathbf{x})| &\le \left|\int_{\mathbb{R}^d_{\mathbf{v}}} (\bar{\alpha}_1(\mathbf{x},\mathbf{v}) - \bar{\alpha}_2(\mathbf{x},\mathbf{v}))\, \nabla_{\mathbf{v}}\alpha_1(\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{v}\right| \\
&\quad + \left|\int_{\mathbb{R}^d_{\mathbf{v}}} \bar{\alpha}_2\, (\nabla_{\mathbf{v}}\alpha_1(\mathbf{x},\mathbf{v}) - \nabla_{\mathbf{v}}\alpha_2(\mathbf{x},\mathbf{v}))\, \mathrm{d}\mathbf{v}\right| \\
&\le \int_{\mathbb{R}^d_{\mathbf{v}}} (|\nabla_{\mathbf{v}}\alpha_1(\mathbf{x},\mathbf{v})| + |\nabla_{\mathbf{v}}\alpha_2(\mathbf{x},\mathbf{v})|)\, |\alpha_1(\mathbf{x},\mathbf{v}) - \alpha_2(\mathbf{x},\mathbf{v})|\, \mathrm{d}\mathbf{v} \\
&\le \frac{1}{\tau_d R^d} \int_{\mathbb{R}^{2d}_{\mathbf{z}}} \sup_{|\bar{\mathbf{z}}|\le R} (|\nabla_{\mathbf{v}}\alpha_1(\mathbf{z}+\bar{\mathbf{z}})| + |\nabla_{\mathbf{v}}\alpha_2(\mathbf{z}+\bar{\mathbf{z}})|)\, |\alpha_1(\mathbf{z}+\bar{\mathbf{z}}) - \alpha_2(\mathbf{z}+\bar{\mathbf{z}})|\, \mathrm{d}\mathbf{z} \\
&\overset{R\text{ opt.}}{\le} A\,(\kappa, d) \left(\|\nabla_{\mathbf{v}}\alpha_1\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} + \|\nabla_{\mathbf{v}}\alpha_2\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}}\right) \|\alpha_1 - \alpha_2\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}}
\end{aligned}$$

and upon integration,

$$\begin{aligned}
\|\varphi_1 - \varphi_2\|_{\mathcal{L}^1_{\mathbf{x}}} &\le \left(\|\nabla_{\mathbf{v}}\alpha_1\|_{\mathcal{L}^2_{\mathbf{z}}} + \|\nabla_{\mathbf{v}}\alpha_2\|_{\mathcal{L}^2_{\mathbf{z}}}\right) \|\alpha_1 - \alpha_2\|_{\mathcal{L}^2_{\mathbf{z}}} \\
&\le \left(\|\nabla_{\mathbf{v}}\alpha_1\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} + \|\nabla_{\mathbf{v}}\alpha_2\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}}\right) \|\alpha_1 - \alpha_2\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}}\,.
\end{aligned}$$

Finally, by inequality (B.5),

$$\|K_1 - K_2\|_{\mathcal{L}^\infty_{\mathbf{x}}} \le c_{1,d} A\,(\kappa, d)^{1-\frac{1}{d}} \left(\|\nabla_{\mathbf{v}}\alpha_1\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} + \|\nabla_{\mathbf{v}}\alpha_2\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}}\right) \|\alpha_1 - \alpha_2\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}}\,. \qquad \square$$

Lemma 3.14. *Let $Z = (X, V), \bar{Z} = (\bar{X}, \bar{V}) : \mathbb{R}\times\mathbb{R}\times\mathbb{R}^d_{\mathbf{x}}\times\mathbb{R}^d_{\mathbf{v}} \to \mathbb{R}^d_{\mathbf{x}}\times\mathbb{R}^d_{\mathbf{v}}$ be the solution maps of the C^1-ODE systems*

$$\begin{aligned}
&\partial_s X(s,t,\mathbf{x},\mathbf{v}) = V(s,t,\mathbf{x},\mathbf{v}), \quad \partial_s V(s,t,\mathbf{x},\mathbf{v}) = F(s, X(s,t,\mathbf{x},\mathbf{v})), \quad \textit{and} \\
&\partial_s \bar{X}(s,t,\mathbf{x},\mathbf{v}) = \bar{V}(s,t,\mathbf{x},\mathbf{v}), \quad \partial_s \bar{V}(s,t,\mathbf{x},\mathbf{v}) = \bar{F}(s, \bar{X}(s,t,\mathbf{x},\mathbf{v})),
\end{aligned}$$

where $F, \bar{F} \in C^0_t C^{1,1}_{\mathbf{x}}$. Then we have that for any $s \le t$,

$$\left\|Z(s,t) - \bar{Z}(s,t)\right\|_{\mathcal{L}^\infty_{\mathbf{z}}} \le \int_s^t \left\|F(\tau) - \bar{F}(\tau)\right\|_{\mathcal{L}^\infty_{\mathbf{x}}} \exp\left(\int_s^\tau \left(1 + \|\nabla_{\mathbf{x}} F(\tilde{\tau})\|_{\mathcal{L}^\infty_{\mathbf{x}}}\right) \mathrm{d}\tilde{\tau}\right) \mathrm{d}\tau. \quad (3.16)$$

Proof. Let $s \le t$ and $\mathbf{z}$ be arbitrary, then we get

$$\begin{aligned}
&\left|X(s,t,\mathbf{z}) - \bar{X}(s,t,\mathbf{z})\right| + \left|V(s,t,\mathbf{z}) - \bar{V}(s,t,\mathbf{z})\right| \\
&\le \int_s^t \left(\left|V(\tau,t,\mathbf{z}) - \bar{V}(\tau,t,\mathbf{z})\right| + \left|F(\tau, X(\tau,t,\mathbf{z})) - \bar{F}(\tau, \bar{X}(\tau,t,\mathbf{z}))\right|\right) \mathrm{d}\tau \\
&\le \int_s^t \left(\left|V(\tau,t,\mathbf{z}) - \bar{V}(\tau,t,\mathbf{z})\right| + \left|F(\tau, X(\tau,t,\mathbf{z})) - F(\tau, \bar{X}(\tau,t,\mathbf{z}))\right|\right) \mathrm{d}\tau
\end{aligned}$$

$$
\begin{aligned}
&+ \int_s^t \left\| F(\tau) - \bar{F}(\tau) \right\|_{\mathcal{L}^\infty_{\mathbf{x}}} \,\mathrm{d}\tau \\
&\leq \int_s^t \left(1 + \|\nabla_{\mathbf{x}} F(\tau)\|_{\mathcal{L}^\infty_{\mathbf{x}}}\right) \left(\left| X(\tau,t,\mathbf{z}) - \bar{X}(\tau,t,\mathbf{z}) \right| + \left| V(\tau,t,\mathbf{z}) - \bar{V}(\tau,t,\mathbf{z}) \right| \right) \,\mathrm{d}\tau \\
&+ \int_s^t \left\| F(\tau) - \bar{F}(\tau) \right\|_{\mathcal{L}^\infty_{\mathbf{x}}} \,\mathrm{d}\tau,
\end{aligned}
$$

yielding a bound by the Grönwall Lemma

$$
\begin{aligned}
\left| Z(s,t,\mathbf{z}) - \bar{Z}(s,t,\mathbf{z}) \right| &\leq \left| X(s,t,\mathbf{z}) - \bar{X}(s,t,\mathbf{z}) \right| + \left| V(s,t,\mathbf{z}) - \bar{V}(s,t,\mathbf{z}) \right| \\
&\leq \int_s^t \left\| F(\tau) - \bar{F}(\tau) \right\|_{\mathcal{L}^\infty_{\mathbf{x}}} \exp\left(\int_s^\tau \left(1 + \|\nabla_{\mathbf{x}} F(\tilde{\tau})\|_{\mathcal{L}^\infty_{\mathbf{x}}}\right) \,\mathrm{d}\tilde{\tau} \right) \mathrm{d}\tau. \qquad \square
\end{aligned}
$$

Lemma 3.15 (Transport Formula). *Let $\alpha \in C^1\left([0,T) \times \mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{v}}; \mathbb{C}\right)$ be an admissible function in the sense of Definition 3.11. Then the following statements are equivalent:*

(i) α is a local solution of the Hamiltonian Vlasov–Poisson equation.

(ii) For any $(t,\mathbf{z})$, we have

$$
\alpha(t,\mathbf{z}) = \alpha(0, Z(0,t,\mathbf{z})) \, \exp\left(\int_0^t K(\tau, X(\tau,t,\mathbf{z})) \,\mathrm{d}\tau \right), \tag{3.17}
$$

where $Z(s,t,\mathbf{z}) = (X,V)(s,t,\mathbf{z})$ is the solution map of the characteristic system

$$
\frac{\partial}{\partial s} X(s,t,\mathbf{x},\mathbf{v}) = V(s,t,\mathbf{x},\mathbf{v}), \qquad \frac{\partial}{\partial s} V(s,t,\mathbf{x},\mathbf{v}) = F(s, X(s,t,\mathbf{x},\mathbf{v}))
$$

with initial condition $Z(t,t,\mathbf{x},\mathbf{v}) = (\mathbf{x},\mathbf{v})$.

Proof. As α is admissible, we get that $F \in C^0_t C^1_{\mathbf{x}}$ and the solution map Z is C^1. The explicit computation is identical to the formal proof of Proposition 1.5 $\square$

Lemma 3.16 (Existence). *Let $M > 0$ be a positive real number. Then there is a positive time of existence $T(M) > 0$, s.t. every compactly supported initial datum $\mathring{\alpha} \in C^1_{\mathbf{z},c}$ with $\|\mathring{\alpha}\|_{\mathcal{B}^{1,\kappa,2}_{\mathbf{z}}} \leq M$ gives rise to a C^1 solution*

$$
\alpha : \quad [0, T(M)) \times \mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{v}} \to \mathbb{C}
$$

of the Hamiltonian Vlasov–Poisson equation.

Proof. **(i) Iterative scheme.** The solution is obtained from an iterative scheme which produces a convergent sequence. At first, we define $\alpha_0(t,\mathbf{x},\mathbf{v}) \equiv \mathring{\alpha}(\mathbf{x},\mathbf{v})$ on $\mathbb{R}_{\geq 0} \times \mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{v}}$.

If α_n is defined, it has a characteristic tuple $(\rho_n, F_n, \varphi_n, K_n)$ due to its compact support on any compact time interval. Then let $Z_n(s,t,\mathbf{x},\mathbf{v}) = (X_n, V_n)(s,t,\mathbf{x},\mathbf{v})$ be the solution map of the characteristic system

$$\frac{\partial}{\partial s} X_n(s,t,\mathbf{x},\mathbf{v}) = V_n(s,t,\mathbf{x},\mathbf{v}), \qquad \frac{\partial}{\partial s} V_n(s,t,\mathbf{x},\mathbf{v}) = F_n(s, X_n(s,t,\mathbf{x},\mathbf{v})),$$

with initial condition $Z_n(t,t,\mathbf{x},\mathbf{v}) = (\mathbf{x},\mathbf{v})$. Finally, the iteration step is completed by defining

$$\alpha_{n+1}(t,\mathbf{x},\mathbf{v}) \equiv \mathring{\alpha}(Z_n(0,t,\mathbf{x},\mathbf{v})) \ \exp\left(\int_0^t K_n(\tau, X_n(\tau,t,\mathbf{x},\mathbf{v}))\,\mathrm{d}\tau\right).$$

(ii) Well-definedness. Following instructive ideas similar to [20, pp.396], we want to verify that the iteration scheme is well-defined, that $\alpha_n \in C^1_{t,\mathbf{z}}$, and that for any t, we have $\alpha_n(t) \in C^1_{\mathbf{z},c}$. For the latter claim we control the size of the support. Therefore, we define

$$R_n(t) \equiv \sup\{|\mathbf{x}| : (\mathbf{x},\mathbf{v}) \in \operatorname{supp} \alpha_n(t)\} \quad \text{and} \quad P_n(t) \equiv \sup\{|\mathbf{v}| : (\mathbf{x},\mathbf{v}) \in \operatorname{supp} \alpha_n(t)\}$$

and claim that they both remain bounded on finite time intervals.

The regularity condition is fulfilled for α_0 by construction and by the choice of the initial value. The size of the support is found to be $R_0(t) = \mathring{R}, P_0(t) = \mathring{P} > 0$ for the minimal numbers, s.t. $\operatorname{supp} \mathring{\alpha} \subseteq \overline{B_{\mathring{R}}(\mathbf{0})} \times \overline{B_{\mathring{P}}(\mathbf{0})} \subseteq \mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{v}}$.

Now assume that the claims hold for some $n \geq 0$. From the iterative scheme it is obvious that $\rho_n(t)$ has compact support and is continuously differentiable. We as well see that $F_n(t)$ is continuously differentiable and bounded on $\mathbb{R}^d_{\mathbf{x}}$. The phase density φ_n is also continuously differentiable. To see this, consider the following. For C^2 functions $\alpha_n(t,\mathbf{x},\mathbf{v})$ we can differentiate φ_n once w.r.t. $(t,\mathbf{x})$ and then integrate by parts to obtain

$$\partial\varphi_n(t,\mathbf{x}) = \int_{\mathbb{R}^d_{\mathbf{v}}} (\partial\bar{\alpha}_n(t,\mathbf{x},\mathbf{v})\, \nabla_{\mathbf{v}}\alpha_n(t,\mathbf{x},\mathbf{v}) - \nabla_{\mathbf{v}}\bar{\alpha}_n(t,\mathbf{x},\mathbf{v})\, \partial\alpha_n(t,\mathbf{x},\mathbf{v}))\,\mathrm{d}\mathbf{v}.$$

This identity is easily extended to the C^1 case by standard approximation arguments. Finally, also $K_n \in C^0_t C^1_{\mathbf{x}}$.

By standard theory of ordinary differential equations, the solution map Z_n is then C^1. Therefore, $\alpha_{n+1} \in C^1\left(\mathbb{R}_{\geq 0} \times \mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{v}}\right)$ and because $K_n(t,\mathbf{x})$ is imaginary,

$$|\alpha(t,\mathbf{x},\mathbf{v})| = |\mathring{\alpha}(Z_n(0,t,\mathbf{x},\mathbf{v}))|\,. \tag{3.18}$$

Hence, as $Z_n(s,t,\cdot)$ is a diffeomorphism of $\mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{v}}$,

$$\begin{aligned}
P_{n+1}(t) &= \sup\{|\mathbf{v}| : (\mathbf{x},\mathbf{v}) \in \operatorname{supp} \alpha_n(t)\} = \sup\{|V_n(t,0,\mathbf{x},\mathbf{v})| : (\mathbf{x},\mathbf{v}) \in \operatorname{supp} \mathring{\alpha}\} \\
&\leq \sup\left\{|V_n(0,0,\mathbf{x},\mathbf{v})| + \int_0^t |F_n(s, X_n(s,0,\mathbf{x},\mathbf{v}))| \, \mathrm{d}s : (\mathbf{x},\mathbf{v}) \in \operatorname{supp} \mathring{\alpha}\right\} \\
&\leq \sup\left\{|\mathbf{v}| + \int_0^t \|F_n(s)\|_{\mathcal{L}^\infty_{\mathbf{x}}} \, \mathrm{d}s : (\mathbf{x},\mathbf{v}) \in \operatorname{supp} \mathring{\alpha}\right\} \\
&\overset{\text{Lem.B.1-(B.5)}}{\leq} \mathring{P} + c_{1,d} \|\mathring{\alpha}\|^{\frac{2}{d}}_{\mathcal{L}^2_{\mathbf{z}}} \left(\tau_d \|\mathring{\alpha}\|^2_{\mathcal{L}^\infty_{\mathbf{z}}}\right)^{1-\frac{1}{d}} \int_0^t P_n(s)^{d-1} \, \mathrm{d}s
\end{aligned}$$

and

$$\begin{aligned}
R_{n+1}(t) &= \sup\{|\mathbf{x}| : (\mathbf{x},\mathbf{v}) \in \operatorname{supp} \alpha_{n+1}(t)\} = \sup\{|X_n(t,0,\mathbf{x},\mathbf{v})| : (\mathbf{x},\mathbf{v}) \in \operatorname{supp} \mathring{\alpha}\} \\
&\leq \sup\left\{|X_n(0,0,\mathbf{x},\mathbf{v})| + \int_0^t |V_n(s,0,\mathbf{x},\mathbf{v})| \, \mathrm{d}s : (\mathbf{x},\mathbf{v}) \in \operatorname{supp} \mathring{\alpha}\right\} \\
&\leq \mathring{R} + \int_0^t P_n(s) \, \mathrm{d}s,
\end{aligned}$$

yielding the second claim. The iterative scheme is well-defined and all objects exist.

(iii) Preserved $\mathcal{L}^p$ norms. As $Z_n(s,t,\cdot)$ is a symplectomorphism of $\mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{v}}$, it preserves the Lebesgue measure. By relation (3.18), one finds

$$\|\alpha_n(t)\|_{\mathcal{L}^p_{\mathbf{z}}} \overset{(3.18)}{=} \left(\int_{\mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{v}}} |\mathring{\alpha}(Z_{n-1}(0,t,\mathbf{z}))|^p \, \mathrm{d}\mathbf{z}\right)^{\frac{1}{p}} = \|\mathring{\alpha}\|_{\mathcal{L}^p_{\mathbf{z}}} \tag{3.19}$$

and

$$\|\rho_n(t)\|_{\mathcal{L}^1_{\mathbf{x}}} = \int_{\mathbb{R}^d_{\mathbf{x}}} \left(\int_{\mathbb{R}^d_{\mathbf{v}}} |\alpha_n(t,\mathbf{x},\mathbf{v})|^2 \, \mathrm{d}\mathbf{v}\right) \mathrm{d}\mathbf{x} \overset{(3.19)}{=} \|\mathring{\alpha}\|^2_{\mathcal{L}^2_{\mathbf{z}}}.$$

(iv) Time of existence. We want to prove convergence in the space $\mathcal{L}^\infty_t \mathcal{H}^{\kappa,2}_{\mathbf{z}}$. Therefore, it is necessary to show that all elements $\alpha_n(t)$ are uniformly bounded in this norm.

At first, we need to compute the difference of the flow Z_n generated under the external potential of $\alpha_n(t)$ and the free flow $\bar{Z} = (\bar{X}, \bar{V})$, solving the ODE system

$$\frac{\partial}{\partial s}\bar{X}(s,t,\mathbf{x},\mathbf{v}) = \bar{V}(s,t,\mathbf{x},\mathbf{v}), \quad \frac{\partial}{\partial s}\bar{V}(s,t,\mathbf{x},\mathbf{v}) = 0.$$

By Lemma 3.14,

$$\left\|Z_n(s,t) - \bar{Z}(s,t)\right\|_{\mathcal{L}^\infty_{\mathbf{z}}} \leq \int_s^t \|F_n(\tau)\|_{\mathcal{L}^\infty_{\mathbf{x}}} e^{\tau - s} \mathrm{d}\tau. \tag{3.20}$$

This implies that for any $t \geq 0$,

$$
\begin{aligned}
\|\alpha_{n+1}(t)\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} &= \sup_{R\geq 0}(1+R)^{-\frac{\kappa}{2}} \left(\int_{\mathbb{R}^{2d}_{\mathbf{z}}} \sup_{|\bar{\mathbf{z}}|\leq R} |\alpha_{n+1}(t, \mathbf{z}+\bar{\mathbf{z}})|^2 \,\mathrm{d}\mathbf{z} \right)^{\frac{1}{2}} \\
&= \sup_{R\geq 0}(1+R)^{-\frac{\kappa}{2}} \left(\int_{\mathbb{R}^{2d}_{\mathbf{z}}} \sup_{|\bar{\mathbf{z}}|\leq R} |\mathring{\alpha}(Z_n(0,t,\mathbf{z}+\bar{\mathbf{z}})|^2 \,\mathrm{d}\mathbf{z} \right)^{\frac{1}{2}} \\
&\leq \sup_{R\geq 0}(1+R)^{-\frac{\kappa}{2}} \left(\int_{\mathbb{R}^{2d}_{\mathbf{z}}} \sup_{|\bar{\mathbf{z}}|\leq R} \sup_{|\bar{\bar{\mathbf{z}}}|\leq \|Z_n(0,t)-\bar{Z}(0,t)\|_{\mathcal{L}^\infty_{\mathbf{z}}}} \left|\mathring{\alpha}(\bar{Z}(0,t,\mathbf{z}+\bar{\mathbf{z}})+\bar{\bar{\mathbf{z}}})\right|^2 \,\mathrm{d}\mathbf{z} \right)^{\frac{1}{2}} \\
&\leq \sup_{R\geq 0}(1+R)^{-\frac{\kappa}{2}} \left(\int_{\mathbb{R}^{2d}_{\mathbf{z}}} \sup_{|\bar{\mathbf{z}}|\leq R} \sup_{|\bar{\bar{\mathbf{z}}}|\leq \int_0^t \|F_n(\tau)\|_{\mathcal{L}^\infty_{\mathbf{x}}} e^\tau \mathrm{d}\tau} \left|\mathring{\alpha}(\mathbf{x}-t\mathbf{v}+\bar{\mathbf{x}}-t\bar{\mathbf{v}}+\bar{\bar{\mathbf{x}}}, \mathbf{v}+\bar{\mathbf{v}}+\bar{\bar{\mathbf{v}}})\right|^2 \,\mathrm{d}\mathbf{z} \right)^{\frac{1}{2}} \\
&\overset{(*)}{\leq} \sup_{R\geq 0}(1+R)^{-\frac{\kappa}{2}} \left(\int_{\mathbb{R}^{2d}_{\mathbf{z}}} \sup_{|\bar{\mathbf{z}}|\leq (1+t)R+\int_0^t \|F_n(\tau)\|_{\mathcal{L}^\infty_{\mathbf{x}}} e^\tau \mathrm{d}\tau} |\mathring{\alpha}(\mathbf{z}+\bar{\mathbf{z}})|^2 \,\mathrm{d}\mathbf{z} \right)^{\frac{1}{2}} \\
&\leq \sup_{R\geq 0} \left(\frac{(1+t)R + \int_0^t \|F_n(\tau)\|_{\mathcal{L}^\infty_{\mathbf{x}}} e^\tau \mathrm{d}\tau}{1+R} \right)^{\frac{\kappa}{2}} \|\mathring{\alpha}\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} \\
&\leq \left(1+t+\int_0^t \|F_n(\tau)\|_{\mathcal{L}^\infty_{\mathbf{x}}} e^\tau \mathrm{d}\tau \right)^{\frac{\kappa}{2}} \|\mathring{\alpha}\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} \qquad (3.21) \\
&\leq \left(1+t+c_{1,d} A(\kappa,d)^{1-\frac{1}{d}} \int_0^t \|\alpha_n(\tau)\|^2_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} e^\tau \mathrm{d}\tau \right)^{\frac{\kappa}{2}} \|\mathring{\alpha}\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} ,
\end{aligned}
$$

where we apply the volume preserving substitution $(\mathbf{x},\mathbf{v}) \mapsto (\mathbf{x}+t\mathbf{v},\mathbf{v})$ at $(*)$. Inspired by this integral inequality, we infer that $\mathfrak{b}^{\kappa,2}_{\alpha}(t) \equiv \sup_n \|\alpha_n(t)\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}}$ is uniformly bounded by the solution $\mathfrak{B}^{\kappa,2}_{M,\kappa,\alpha}(t)$ of the integral equation

$$
\mathfrak{B}^{\kappa,2}_{M,\kappa,\alpha}(t) = M\left(1+t+c_{1,d} A(\kappa,d)^{1-\frac{1}{d}} \int_0^t \mathfrak{B}^{\kappa,2}_{M,\kappa,\alpha}(\tau)^2 e^\tau \mathrm{d}\tau \right)^{\frac{\kappa}{2}} .
$$

The maximal solution of this integral equation exists on some finite time interval defining $T = T(M)$. We call this interval $I = I(M) = [0, T(M))$. We remark that all bounds further derived in this proof can be ultimately bounded by some expression in $\mathfrak{B}^{\kappa,2}_{M,\kappa,\alpha}$. Therefore, they only depend on the bound M of the initial datum $\mathring{\alpha}$ and the fundamental parameter κ.

(v) Uniform bounds and Lipschitz bounds on density and force. For technical reasons, we need to prove various bounds. We start by computing uniform global bounds for ρ and F, namely,

$$\mathfrak{b}_\rho^\infty(t) \equiv \sup_n \|\rho_n(t)\|_{\mathcal{L}_{\mathbf{x}}^\infty} \le A(\kappa,d) \sup_n \|\alpha_n(t)\|^2_{\mathcal{A}_{\mathbf{z}}^{\kappa,2}} \le A(\kappa,d)\mathfrak{B}^{\kappa,2}_{M,\kappa,\alpha}(t)^2 \equiv \mathfrak{B}^\infty_{M,\kappa,\rho}(t),$$
$$\mathfrak{b}_\rho^1(t) \equiv \sup_n \|\rho_n(t)\|_{\mathcal{L}_{\mathbf{x}}^1} \overset{Z_{n-1}\ \text{sympl.}}{=} \|\mathring{\alpha}\|^2_{\mathcal{L}_{\mathbf{z}}^2} \le \|\mathring{\alpha}\|^2_{\mathcal{A}_{\mathbf{z}}^{\kappa,2}} \le M^2 \equiv \mathfrak{B}^1_{M,\kappa,\rho}(t),$$

and

$$\mathfrak{b}_F^\infty(t) \equiv \sup_n \|F_n(t)\|_{\mathcal{L}_{\mathbf{x}}^\infty} \le c_{1,d} A(\kappa,d)^{1-\frac{1}{d}} \mathfrak{b}^{\kappa,2}_\alpha(t)^2 \le c_{1,d} A(\kappa,d)^{1-\frac{1}{d}} \mathfrak{B}^{\kappa,2}_{M,\kappa,\alpha}(t)^2 \equiv \mathfrak{B}^\infty_{M,\kappa,F}(t).$$

Upon applying (3.20), we find that for any $s \le t$,

$$\mathfrak{b}^\infty_{Z-\bar{Z}}(s,t) \equiv \sup_n \left\|Z_n(s,t) - \bar{Z}(s,t)\right\|_{\mathcal{L}_{\mathbf{z}}^\infty} \overset{(3.20)}{\le} \int_s^t \mathfrak{b}_F^\infty(\tau)\, e^{\tau-s}\, \mathrm{d}\tau$$
$$\le \int_s^t \mathfrak{B}^\infty_{M,\kappa,F}(\tau)\, e^{\tau-s}\, \mathrm{d}\tau \equiv \mathfrak{B}^\infty_{M,\kappa,Z-\bar{Z}}(s,t).$$

Similarly, we want to prove bounds for

$$\mathfrak{b}^\infty_{\nabla\rho}(t) \equiv \sup_n \|\nabla_{\mathbf{x}}\rho_n(t)\|_{\mathcal{L}_{\mathbf{x}}^\infty} \quad \text{and} \quad \mathfrak{b}^\infty_{\nabla F}(t) \equiv \sup_n \|\nabla_{\mathbf{x}} F_n(t)\|_{\mathcal{L}_{\mathbf{x}}^\infty}.$$

For $R > 0$, we have that

$$\begin{aligned}
|\nabla_{\mathbf{x}}\rho_{n+1}(t,\mathbf{x})| &\le \left|\nabla_{\mathbf{x}} \|\alpha_{n+1}(t,\mathbf{x},\mathbf{v})\|^2_{\mathcal{L}_{\mathbf{v}}^2}\right| = \left|\nabla_{\mathbf{x}} \|\mathring{\alpha}(Z_n(0,t,\mathbf{x},\mathbf{v}))\|^2_{\mathcal{L}_{\mathbf{v}}^2}\right| \\
&\le \int_{\mathbb{R}^d_{\mathbf{v}}} \left|(\nabla_{\mathbf{z}} |\mathring{\alpha}|^2)(Z_n(0,t,\mathbf{x},\mathbf{v}))\right| \, |\nabla_{\mathbf{x}} Z_n(0,t,\mathbf{x},\mathbf{v})| \, \mathrm{d}\mathbf{v} \\
&\le \frac{1}{\tau_d R^d} \int_{\mathbb{R}^{2d}_{\mathbf{z}}} \sup_{|\bar{\mathbf{z}}|\le R} \left|(\nabla_{\mathbf{z}} |\mathring{\alpha}|^2)(Z_n(0,t,\mathbf{z}+\bar{\mathbf{z}}))\right| \mathrm{d}\mathbf{z} \;\; \|\nabla_{\mathbf{x}} Z_n(0,t)\|_{\mathcal{L}_{\mathbf{z}}^\infty} \\
&\le \frac{1}{\tau_d R^d} \int_{\mathbb{R}^{2d}_{\mathbf{z}}} \sup_{|\bar{\mathbf{z}}|\le(1+t)R} \; \sup_{|\bar{\bar{\mathbf{z}}}|\le \mathfrak{b}^\infty_{Z-\bar{Z}}(0,t)} \left|(\nabla_{\mathbf{z}} |\mathring{\alpha}|^2)(\bar{Z}(0,t,\mathbf{z}) + \bar{\mathbf{z}} + \bar{\bar{\mathbf{z}}})\right| \mathrm{d}\mathbf{z} \;\; \|\nabla_{\mathbf{x}} Z_n(0,t)\|_{\mathcal{L}_{\mathbf{z}}^\infty} \\
&= \frac{1}{\tau_d R^d} \int_{\mathbb{R}^{2d}_{\mathbf{z}}} \sup_{|\bar{\mathbf{z}}|\le(1+t)R + \mathfrak{b}^\infty_{Z-\bar{Z}}(0,t)} \left|(\nabla_{\mathbf{z}} |\mathring{\alpha}|^2)(\bar{Z}(0,t,\mathbf{z}) + \bar{\mathbf{z}})\right| \mathrm{d}\mathbf{z} \;\; \|\nabla_{\mathbf{x}} Z_n(0,t)\|_{\mathcal{L}_{\mathbf{z}}^\infty} \\
&\le 2\frac{1}{\tau_d R^d} \left((1+t)R + \mathfrak{b}^\infty_{Z-\bar{Z}}(0,t)\right)^\kappa \|\nabla_{\mathbf{z}}\mathring{\alpha}\|_{\mathcal{A}_{\mathbf{z}}^{\kappa,2}} \|\mathring{\alpha}\|_{\mathcal{A}_{\mathbf{z}}^{\kappa,2}} \|\nabla_{\mathbf{x}} Z_n(0,t)\|_{\mathcal{L}_{\mathbf{z}}^\infty} \\
&\overset{R\ \text{opt.}}{\le} 2A(\kappa,d) \left(1 + t + \mathfrak{b}^\infty_{Z-\bar{Z}}(0,t)\right)^\kappa M^2 \|\nabla_{\mathbf{x}} Z_n(0,t)\|_{\mathcal{L}_{\mathbf{z}}^\infty}.
\end{aligned}$$

This can be used to estimate the equations of variation of $Z_n = (X_n, V_n)$,

$$\frac{\partial}{\partial s}\nabla_{\mathbf{z}} X_n(s,t,\mathbf{z}) = \nabla_{\mathbf{z}} V_n(s,t,\mathbf{z}), \quad \frac{\partial}{\partial s}\nabla_{\mathbf{z}} V_n(s,t,\mathbf{z}) = \nabla_{\mathbf{x}} F_n(s, X_n(s,t,\mathbf{z})) \cdot \nabla_{\mathbf{z}} X_n(s,t,\mathbf{z}),$$

which can be rewritten in integral form for any $\mathbf{z} = (\mathbf{x}, \mathbf{v}) \in \mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{v}}$, $0 \le s \le t$,

$$\begin{pmatrix} \nabla_{\mathbf{z}} X_n(s,t,\mathbf{z}) \\ \nabla_{\mathbf{z}} V_n(s,t,\mathbf{z}) \end{pmatrix} = \mathrm{id}_{\mathbb{R}^{2d}_{\mathbf{z}}} - \int_s^t \begin{pmatrix} 0 & \mathbb{1}_n \\ \nabla_{\mathbf{x}} F_n(\tau, X_n(\tau,t,\mathbf{z})) & 0 \end{pmatrix} \cdot \begin{pmatrix} \nabla_{\mathbf{z}} X_n(\tau,t,\mathbf{z}) \\ \nabla_{\mathbf{z}} V_n(\tau,t,\mathbf{z}) \end{pmatrix} \mathrm{d}\tau$$

to obtain the following for the matrix operator norm:

$$\begin{aligned} |\nabla_{\mathbf{z}} Z_n(s,t,\mathbf{z})| &= \left| \begin{pmatrix} \nabla_{\mathbf{z}} X_n(s,t,\mathbf{z}) \\ \nabla_{\mathbf{z}} V_n(s,t,\mathbf{z}) \end{pmatrix} \right| \\ &\le \left| \mathrm{id}_{\mathbb{R}^{2d}_{\mathbf{z}}} \right| + \int_s^t (1 + |\nabla_{\mathbf{x}} F_n(\tau, X_n(\tau,t,\mathbf{z}))|) \left| \begin{pmatrix} \nabla_{\mathbf{z}} X_n(\tau,t,\mathbf{z}) \\ \nabla_{\mathbf{z}} V_n(\tau,t,\mathbf{z}) \end{pmatrix} \right| \mathrm{d}\tau \\ &\le 1 + \int_s^t \left(1 + \|\nabla_{\mathbf{x}} F_n(\tau)\|_{\mathcal{L}^\infty_{\mathbf{x}}}\right) |\nabla_{\mathbf{z}} Z_n(\tau,t,\mathbf{z})| \ \mathrm{d}\tau. \end{aligned}$$

A Grönwall argument therefore gives the bounds

$$\|\nabla_{\mathbf{z}} Z_n(s,t)\|_{\mathcal{L}^\infty_{\mathbf{z}}} \le \exp\left(\int_s^t \left(1 + \|\nabla_{\mathbf{x}} F_n(\tau)\|_{\mathcal{L}^\infty_{\mathbf{x}}}\right) \mathrm{d}\tau \right).$$

Combining the estimates, one has

$$\|\nabla_{\mathbf{x}} \rho_{n+1}(t)\|_{\mathcal{L}^\infty_{\mathbf{x}}} \le \underbrace{2A(\kappa,d) \left(1 + t + \mathfrak{b}^\infty_{Z-\bar{Z}}(0,t)\right)^\kappa M^2}_{\le 2A(\kappa,d)\left(1+t+\mathfrak{B}^\infty_{M,\kappa,Z-\bar{Z}}(0,t)\right)^\kappa M^2 \equiv H_{M,\kappa}(t)} \exp\left(\int_0^t \left(1 + \|\nabla_{\mathbf{x}} F_n(\tau)\|_{\mathcal{L}^\infty_{\mathbf{x}}}\right) \mathrm{d}\tau \right). \tag{3.22}$$

Applying Lemma B.1 yields

$$\begin{aligned} \|\nabla_{\mathbf{x}} F_{n+1}(t)\|_{\mathcal{L}^\infty_{\mathbf{x}}} &\le c_d \left[\left(1 + \|\rho_{n+1}(t)\|_{\mathcal{L}^\infty_{\mathbf{x}}}\right)\left(1 + \ln_+ \|\nabla_{\mathbf{x}} \rho_{n+1}(t)\|_{\mathcal{L}^\infty_{\mathbf{x}}}\right) + \|\rho_{n+1}(t)\|_{\mathcal{L}^1_{\mathbf{x}}} \right] \\ &\le c_d \left[\left(1 + \mathfrak{B}^\infty_{M,\kappa,\rho}(t)\right) \left(1 + \ln_+ H_{M,\kappa}(t) + t + \int_0^t \|\nabla_{\mathbf{x}} F_n(\tau)\|_{\mathcal{L}^\infty_{\mathbf{x}}} \ \mathrm{d}\tau \right) + \mathfrak{B}^1_{M,\kappa,\rho}(t) \right]. \end{aligned}$$

By an inductive Grönwall argument, we can find a bound

$$\begin{aligned} \mathfrak{b}^\infty_{\nabla F}(t) \equiv \sup_n \|\nabla_{\mathbf{x}} F_n(t)\|_{\mathcal{L}^\infty_{\mathbf{x}}} &\le c_d \left[\left(1 + \mathfrak{B}^\infty_{M,\kappa,\rho}(t)\right)(1 + \ln_+ H_{M,\kappa}(t) + t) + \mathfrak{B}^1_{M,\kappa,\rho} \right] \\ &\quad \cdot \exp\left(c_d \left(1 + \mathfrak{B}^\infty_{M,\kappa,\rho}(t)\right) t\right) \equiv \mathfrak{B}^\infty_{M,\kappa,\nabla F}(t). \end{aligned}$$

At last, $\mathfrak{b}^\infty_{\nabla F}$ and estimate (3.22) prove the finiteness of $\mathfrak{b}^\infty_{\nabla\rho}$, i.e.,

$$\mathfrak{b}^\infty_{\nabla\rho}(t) \le 2A(\kappa,d)\left(1 + t + \mathfrak{B}^\infty_{M,\kappa,Z-\bar{Z}}(0,t)\right)^\kappa M^2 \equiv \mathfrak{B}^\infty_{M,\kappa,\nabla\rho}(t).$$

Finally, we have shown that for any $0 \le s \le t < T(M)$,

$$\mathfrak{b}^{\infty}_{\nabla Z}(s,t) \equiv \sup_n \|\nabla_{\mathbf{z}} Z_n(s,t)\|_{\mathcal{L}^\infty_{\mathbf{z}}} \le \exp\left(\int_s^t \left(1 + \mathfrak{B}^{\infty}_{M,\kappa,\nabla F}(\tau)\right) \mathrm{d}\tau\right) \equiv \mathfrak{B}^{\infty}_{M,\kappa,\nabla Z}(s,t) \quad (3.23)$$

is finite.

(vi) Uniform bounds on phase density φ and phase force K. We want to prove uniform global bounds $\mathfrak{b}^{\infty}_{\varphi}, \mathfrak{b}^{\infty}_{K}$, i.e., that

$$\mathfrak{b}^{\infty}_{\varphi}(t) \equiv \sup_n \|\varphi_n(t)\|_{\mathcal{L}^\infty_{\mathbf{x}}} \quad \text{and} \quad \mathfrak{b}^{\infty}_{K}(t) \equiv \sup_n \|K_n(t)\|_{\mathcal{L}^\infty_{\mathbf{x}}} \quad (3.24)$$

remain finite on I. We find

$$\begin{aligned}
\varphi_{n+1}(t,\mathbf{x}) &= \int_{\mathbb{R}^d_{\mathbf{v}}} \bar{\alpha}_{n+1}(t,\mathbf{x},\mathbf{v})\, \nabla_{\mathbf{v}} \alpha_{n+1}(t,\mathbf{x},\mathbf{v})\, \mathrm{d}\mathbf{v} \\
&= \int_{\mathbb{R}^d_{\mathbf{v}}} \overline{\mathring{\alpha}(Z_n(0,t,\mathbf{x},\mathbf{v}))}\, \nabla_{\mathbf{v}} \left(\mathring{\alpha}(Z_n(0,t,\mathbf{x},\mathbf{v}))\right)\, \mathrm{d}\mathbf{v} \\
&\quad + \int_{\mathbb{R}^d_{\mathbf{v}}} |\mathring{\alpha}(Z_n(0,t,\mathbf{x},\mathbf{v}))|^2\, \nabla_{\mathbf{v}} \left(\int_0^t K_n(s, X_n(s,t,\mathbf{x},\mathbf{v}))\, \mathrm{d}s\right) \mathrm{d}\mathbf{v} \\
&= \int_{\mathbb{R}^d_{\mathbf{v}}} \overline{\mathring{\alpha}(Z_n(0,t,\mathbf{x},\mathbf{v}))}\, \nabla_{\mathbf{v}} \left(\mathring{\alpha}(Z_n(0,t,\mathbf{x},\mathbf{v}))\right)\, \mathrm{d}\mathbf{v} \\
&\quad - \int_{\mathbb{R}^d_{\mathbf{v}}} \nabla_{\mathbf{v}} \left(|\mathring{\alpha}(Z_n(0,t,\mathbf{x},\mathbf{v}))|^2\right) \left(\int_0^t K_n(s, X_n(s,t,\mathbf{x},\mathbf{v}))\, \mathrm{d}s\right) \mathrm{d}\mathbf{v}.
\end{aligned}$$

Upon estimating the absolute value and multiply using the Hölder inequality, one derives

$$\begin{aligned}
&|\varphi_{n+1}(t,\mathbf{x})| \le |\rho_{n+1}(t,\mathbf{x})|^{\frac{1}{2}}\, \|\nabla_{\mathbf{v}} \left(\mathring{\alpha}(Z_n(0,t,\mathbf{x},\mathbf{v}))\right)\|_{\mathcal{L}^2_{\mathbf{v}}} \left(1 + 2\int_0^t \|K_n(\tau)\|_{\mathcal{L}^\infty_{\mathbf{x}}}\, \mathrm{d}\tau\right) \\
&\le |\rho_{n+1}(t,\mathbf{x})|^{\frac{1}{2}}\, \|(\nabla_{\mathbf{z}} \mathring{\alpha})\,(Z_n(0,t,\mathbf{x},\mathbf{v}))\|_{\mathcal{L}^2_{\mathbf{v}}}\, \|\nabla_{\mathbf{v}} Z_n(0,t)\|_{\mathcal{L}^\infty_{\mathbf{z}}} \left(1 + 2\int_0^t \|K_n(\tau)\|_{\mathcal{L}^\infty_{\mathbf{x}}}\, \mathrm{d}\tau\right) \\
&\le \mathfrak{b}^{\infty}_{\rho}(t)^{\frac{1}{2}} \left(1 + t + \mathfrak{b}^{\infty}_{Z-\bar{Z}}(0,t)\right)^{\frac{\kappa}{2}} A(\kappa,d)^{\frac{1}{2}}\, \|\nabla_{\mathbf{z}} \mathring{\alpha}\|_{\mathcal{A}^{\kappa,2}}\, \|\nabla_{\mathbf{v}} Z_n(0,t)\|_{\mathcal{L}^\infty_{\mathbf{z}}} \left(1 + 2\int_0^t \|K_n(\tau)\|_{\mathcal{L}^\infty_{\mathbf{x}}}\, \mathrm{d}\tau\right) \\
&\le \mathfrak{B}^{\infty}_{M,\kappa,\rho}(t)^{\frac{1}{2}} \left(1 + t + \mathfrak{B}^{\infty}_{M,\kappa,Z-\bar{Z}}(0,t)\right)^{\frac{\kappa}{2}} A(\kappa,d)^{\frac{1}{2}}\, M\, \mathfrak{B}^{\infty}_{M,\kappa,\nabla Z}(0,t) \left(1 + 2\int_0^t \|K_n(\tau)\|_{\mathcal{L}^\infty_{\mathbf{x}}}\, \mathrm{d}\tau\right).
\end{aligned}$$

For the integral norm, one finds

$$\begin{aligned}
&\|\varphi_{n+1}(t)\|_{\mathcal{L}^1_{\mathbf{x}}} \\
&\le \|\mathring{\alpha} \circ Z_n(0,t)\|_{\mathcal{L}^2_{\mathbf{z}}} \, \|(\nabla_{\mathbf{z}}\mathring{\alpha}) \circ Z_n(0,t)\|_{\mathcal{L}^2_{\mathbf{z}}} \, \|\nabla_{\mathbf{v}} Z_n(0,t)\|_{\mathcal{L}^\infty_{\mathbf{z}}} \left(1 + 2\int_0^t \|K_n(\tau)\|_{\mathcal{L}^\infty_{\mathbf{x}}} \, \mathrm{d}\tau\right) \\
&= \|\mathring{\alpha}\|_{\mathcal{L}^2_{\mathbf{z}}} \, \|\nabla_{\mathbf{z}}\mathring{\alpha}\|_{\mathcal{L}^2_{\mathbf{z}}} \, \|\nabla_{\mathbf{v}} Z_n(0,t)\|_{\mathcal{L}^\infty_{\mathbf{z}}} \left(1 + 2\int_0^t \|K_n(\tau)\|_{\mathcal{L}^\infty_{\mathbf{x}}} \, \mathrm{d}\tau\right) \\
&\le M^2 \, \mathfrak{B}^\infty_{M,\kappa,\nabla Z}(0,t) \left(1 + 2\int_0^t \|K_n(\tau)\|_{\mathcal{L}^\infty_{\mathbf{x}}} \, \mathrm{d}\tau\right).
\end{aligned}$$

As K_n is only an (imaginary valued) convolution, by Lemma B.1,

$$\begin{aligned}
\|K_{n+1}(t)\|_{\mathcal{L}^\infty_{\mathbf{x}}} \le \; & c_{1,d} \, \|\varphi_{n+1}(t)\|^{\frac{1}{d}}_{\mathcal{L}^1_{\mathbf{x}}} \, \|\varphi_{n+1}(t)\|^{1-\frac{1}{d}}_{\mathcal{L}^\infty_{\mathbf{x}}} \le c_{1,d} M^{1+\frac{1}{d}} \mathfrak{B}^\infty_{M,\kappa,\nabla Z}(0,t) \\
& \cdot \left(\mathfrak{B}^\infty_{M,\kappa,\rho}(t)^{\frac{1}{2}} \left(1 + t + \mathfrak{B}^\infty_{M,\kappa,Z-\bar{Z}}(0,t)\right)^{\frac{\kappa}{2}} A(\kappa,d)^{\frac{1}{2}}\right)^{1-\frac{1}{d}} \left(1 + 2\int_0^t \|K_n(\tau)\|_{\mathcal{L}^\infty_{\mathbf{x}}} \, \mathrm{d}\tau\right),
\end{aligned}$$

which proves the uniform finiteness of $\mathfrak{b}^\infty_K$ on the minimal interval of existence. Now we find by an inductive Grönwall argument that

$$\begin{aligned}
&\sup_n \|K_n(t)\|_{\mathcal{L}^\infty_{\mathbf{x}}} \\
&\le c_{1,d} M^{1+\frac{1}{d}} \mathfrak{B}^\infty_{M,\kappa,\nabla Z}(0,t) \left(\mathfrak{B}^\infty_{M,\kappa,\rho}(t)^{\frac{1}{2}} \left(1 + t + \mathfrak{B}^\infty_{M,\kappa,Z-\bar{Z}}(0,t)\right)^{\frac{\kappa}{2}} A(\kappa,d)^{\frac{1}{2}}\right)^{1-\frac{1}{d}} \\
&\quad \cdot \exp\left(2c_{1,d} M^{1+\frac{1}{d}} \mathfrak{B}^\infty_{M,\kappa,\nabla Z}(0,t) \left(\mathfrak{B}^\infty_{M,\kappa,\rho}(t)^{\frac{1}{2}} \left(1 + t + \mathfrak{B}^\infty_{M,\kappa,Z-\bar{Z}}(0,t)\right)^{\frac{\kappa}{2}} A(\kappa,d)^{\frac{1}{2}}\right)^{1-\frac{1}{d}} t\right) \\
&\equiv \mathfrak{B}^\infty_{M,\kappa,K}(t).
\end{aligned}$$

The estimates on φ_{n+1} in turn imply the finiteness of

$$\begin{aligned}
\mathfrak{b}^\infty_\varphi(t) \equiv \sup_n \|\varphi_n(t)\|_{\mathcal{L}^\infty_{\mathbf{x}}} \le \; & \mathfrak{B}^\infty_{M,\kappa,\rho}(t)^{\frac{1}{2}} \left(1 + t + \mathfrak{B}^\infty_{M,\kappa,Z-\bar{Z}}(0,t)\right)^{\frac{\kappa}{2}} \\
& \cdot A(\kappa,d)^{\frac{1}{2}} M \, \mathfrak{B}^\infty_{M,\kappa,\nabla Z}(0,t) \left(1 + 2\int_0^t \mathfrak{B}^\infty_{M,\kappa,K}(\tau) \, \mathrm{d}\tau\right) \equiv \mathfrak{B}^\infty_{M,\kappa,\varphi}(t)
\end{aligned}$$

and

$$\mathfrak{b}^1_\varphi(t) \equiv \sup_n \|\varphi_n(t)\|_{\mathcal{L}^1_{\mathbf{x}}} \le M^2 \, \mathfrak{B}^\infty_{M,\kappa,\nabla Z}(0,t) \left(1 + 2\int_0^t \mathfrak{B}^\infty_{M,\kappa,K}(\tau) \, \mathrm{d}\tau\right) \equiv \mathfrak{B}^1_{M,\kappa,\varphi}(t).$$

(vii) Uniform Lipschitz bounds on phase density φ and phase force K. Finally, we need to show that $\mathfrak{b}^{\infty}_{\nabla\varphi}(t) \equiv \sup_n \|\nabla_{\mathbf{x}}\varphi_n(t)\|_{\mathcal{L}^{\infty}_{\mathbf{x}}}$ and $\mathfrak{b}^{\infty}_{\nabla K}(t) \equiv \sup_n \|\nabla_{\mathbf{x}} K_n(t)\|_{\mathcal{L}^{\infty}_{\mathbf{x}}}$ remain finite. Indeed, we compute the matrix valued derivative of φ_{n+1} as follows:

$$
\begin{aligned}
\nabla_{\mathbf{x}}\varphi_{n+1}(t,\mathbf{x}) &= 2\mathrm{i}\ \Im \int_{\mathbb{R}^d_{\mathbf{v}}} \nabla_{\mathbf{x}}\bar{\alpha}_{n+1}(t,\mathbf{x},\mathbf{v}) \cdot \nabla_{\mathbf{v}}\alpha_{n+1}(t,\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{v} \\
&= 2\mathrm{i}\ \Im \int_{\mathbb{R}^d_{\mathbf{v}}} \Bigg(\nabla_{\mathbf{x}}(\bar{\mathring{\alpha}}(Z_n(0,t,\mathbf{x},\mathbf{v}))) + \bar{\mathring{\alpha}}(Z_n(0,t,\mathbf{x},\mathbf{v})) \\
&\qquad \cdot \int_0^t (\nabla_{\mathbf{x}}\bar{K}_n)(\tau, X_n(\tau,t,\mathbf{x},\mathbf{v})) \cdot (\nabla_{\mathbf{x}}X_n)(\tau,t,\mathbf{x},\mathbf{v})\,\mathrm{d}\tau\Bigg) \\
&\qquad \cdot \Bigg(\nabla_{\mathbf{v}}(\mathring{\alpha}(Z_n(0,t,\mathbf{x},\mathbf{v}))) + \mathring{\alpha}(Z_n(0,t,\mathbf{x},\mathbf{v})) \\
&\qquad \cdot \int_0^t (\nabla_{\mathbf{x}}K_n)(\tau, X_n(\tau,t,\mathbf{x},\mathbf{v})) \cdot (\nabla_{\mathbf{v}}X_n)(\tau,t,\mathbf{x},\mathbf{v})\,\mathrm{d}\tau\Bigg)\,\mathrm{d}\mathbf{v}.
\end{aligned}
$$

Estimating the absolute value using the Hölder and Minkowski inequalities yields

$$
\begin{aligned}
|\nabla_{\mathbf{x}}\varphi_{n+1}(t,\mathbf{x})| &\leq 2\int_{\mathbb{R}^d_{\mathbf{v}}} \Bigg(|(\nabla_{\mathbf{z}}\mathring{\alpha}) \circ Z_n(0,t,\mathbf{x},\mathbf{v})|\,\mathfrak{b}^{\infty}_{\nabla Z}(0,t) \\
&\qquad + |\mathring{\alpha} \circ Z_n(0,t,\mathbf{x},\mathbf{v})| \int_0^t \|\nabla_{\mathbf{x}}K_n(\tau)\|_{\mathcal{L}^{\infty}_{\mathbf{x}}}\ \mathfrak{b}^{\infty}_{\nabla Z}(\tau,t)\,\mathrm{d}\tau\Bigg)^2\,\mathrm{d}\mathbf{v} \\
&\leq 2\frac{(1+R)^{\kappa}}{\tau_d R^d}\left(\frac{(1+t)R + \mathfrak{b}^{\infty}_{Z-\bar{Z}}(0,t)}{1+R}\right)^{\kappa}\Bigg(\mathfrak{b}^{\infty}_{\nabla Z}(0,t)\,\|\nabla_{\mathbf{z}}\mathring{\alpha}\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} \\
&\qquad + \int_0^t \|\nabla_{\mathbf{x}}K_n(\tau)\|_{\mathcal{L}^{\infty}_{\mathbf{x}}}\ \mathfrak{b}^{\infty}_{\nabla Z}(\tau,t)\,\mathrm{d}\tau\ \|\mathring{\alpha}\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}}\Bigg)^2 \\
&\overset{R\text{ opt.}}{\leq}\ 2A(\kappa,d)\,M^2\,\left(1+t+\mathfrak{B}^{\infty}_{M,\kappa,Z-\bar{Z}}(0,t)\right)^{\kappa}\ \mathfrak{B}^{\infty}_{M,\kappa,\nabla Z}(0,t)^2 \\
&\qquad \cdot \left(1 + \int_0^t \|\nabla_{\mathbf{x}}K_n(\tau)\|_{\mathcal{L}^{\infty}_{\mathbf{x}}}\ \frac{\mathfrak{B}^{\infty}_{M,\kappa,\nabla Z}(\tau,t)}{\mathfrak{B}^{\infty}_{M,\kappa,\nabla Z}(0,t)}\mathrm{d}\tau\right)^2.
\end{aligned}
$$

Taking the positive logarithm $\ln_+$ of this equation and applying Lemma B.1, we find

$$
\begin{aligned}
\ln_+\|\nabla_{\mathbf{x}}\varphi_{n+1}(t)\|_{\mathcal{L}^{\infty}_{\mathbf{x}}} \leq\ &\ln_+\left(2A(\kappa,d)\,M^2\,\left(1+t+\mathfrak{B}^{\infty}_{M,\kappa,Z-\bar{Z}}(0,t)\right)^{\kappa}\mathfrak{B}^{\infty}_{M,\kappa,\nabla Z}(0,t)^2\right) \\
&+ 2\ln_+\left(1 + \int_0^t \|\nabla_{\mathbf{x}}K_n(\tau)\|_{\mathcal{L}^{\infty}_{\mathbf{x}}}\ \underbrace{\frac{\mathfrak{B}^{\infty}_{M,\kappa,\nabla Z}(\tau,t)}{\mathfrak{B}^{\infty}_{M,\kappa,\nabla Z}(0,t)}}_{\leq 1}\mathrm{d}\tau\right)
\end{aligned}
$$

$$\leq \ln_+ \Big(2A(\kappa,d)\, M^2\, \left(1+t+\mathfrak{B}^\infty_{M,\kappa,Z-\bar{Z}}(0,t)\right)^\kappa \mathfrak{B}^\infty_{M,\kappa,\nabla Z}(0,t)^2\Big)$$
$$+ 2\int_0^t c_d \left[\left(1+\mathfrak{B}^\infty_{M,\kappa,\varphi}(\tau)\right)\left(1+\ln_+ \|\nabla_{\mathbf{x}}\varphi_n(\tau)\|_{\mathcal{L}^\infty_{\mathbf{x}}}\right) + \mathfrak{B}^1_{M,\kappa,\varphi}(\tau)\right] \mathrm{d}\tau.$$

An inductive Grönwall argument proves the finiteness of $\sup_n \ln_+ \|\nabla_{\mathbf{x}}\varphi_n(t)\|_{\mathcal{L}^\infty_{\mathbf{x}}}$, namely,

$$\mathfrak{b}^\infty_{\nabla\varphi}(t) \leq \exp\left(\sup_n \ln_+ \|\nabla_{\mathbf{x}}\varphi_n(t)\|_{\mathcal{L}^\infty_{\mathbf{x}}}\right)$$
$$\leq \exp\Big(\Big(1+\ln_+ \Big(2A(\kappa,d)\, M^2\, \left(1+t+\mathfrak{B}^\infty_{M,\kappa,Z-\bar{Z}}(0,t)\right)^\kappa \mathfrak{B}^\infty_{M,\kappa,\nabla Z}(0,t)^2\Big) + 2c_d t \mathfrak{B}^1_{M,\kappa,\varphi}(t)\Big)$$
$$\cdot \exp\left(\left(\int_0^t \left(1+\mathfrak{B}^\infty_{M,\kappa,\varphi}(\tau)\right)\mathrm{d}\tau\right) - 1\right) \equiv \mathfrak{B}^\infty_{M,\kappa,\nabla\varphi}(t).$$

Hence, Equation (B.7) ensures the finiteness of

$$\mathfrak{b}^\infty_{\nabla K}(t) \leq c_d \left[\left(1+\mathfrak{B}^\infty_{M,\kappa,\varphi}(t)\right)\left(1+\ln_+ \mathfrak{B}^\infty_{M,\kappa,\nabla\varphi}\right) + \mathfrak{B}^1_{M,\kappa,\varphi}(t)\right] \equiv \mathfrak{B}^\infty_{M,\kappa,\nabla K}(t).$$

(viii) Uniform supremum and integral bounds on $\nabla_{\mathbf{z}}\alpha_n$. For any $(t,\mathbf{x},\mathbf{v})$ and $n \in \mathbb{N}_0$ we find that

$$|\nabla_{\mathbf{z}}\alpha_{n+1}(t,\mathbf{x},\mathbf{v})|$$
$$\leq |(\nabla_{\mathbf{z}}\mathring{\alpha})(Z_n(0,t,\mathbf{x},\mathbf{v}))|\,|\nabla_{\mathbf{z}}Z_n(0,t,\mathbf{x},\mathbf{v})|$$
$$+ |\mathring{\alpha}(Z_n(0,t,\mathbf{x},\mathbf{v}))| \int_0^t |\nabla_{\mathbf{x}}K_n(\tau, X(\tau,t,\mathbf{x},\mathbf{v}))|\,|\nabla_{\mathbf{z}}X_n(\tau,t,\mathbf{x},\mathbf{v})|\,\mathrm{d}\tau$$
$$\leq |(\nabla_{\mathbf{z}}\mathring{\alpha})\,(Z_n(0,t,\mathbf{x},\mathbf{v})|\,\mathfrak{b}^\infty_{\nabla Z}(0,t) + |\mathring{\alpha}(Z_n(0,t,\mathbf{x},\mathbf{v}))| \int_0^t \mathfrak{b}^\infty_{\nabla K}(\tau)\mathfrak{b}^\infty_{\nabla Z}(\tau,t)\,\mathrm{d}\tau,$$

which yields the following for the $\mathcal{A}^{\kappa,2}_{\mathbf{z}}$-norm,

$$\mathfrak{b}^{\kappa,2}_{\nabla\alpha}(t) \equiv \sup_n \|\nabla_{\mathbf{z}}\alpha_n(t)\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}}$$
$$\leq \left(1+t+\mathfrak{b}^\infty_{Z-\bar{Z}}(0,t)\right)^{\frac{\kappa}{2}} \left(\|\nabla_{\mathbf{z}}\mathring{\alpha}\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}}\,\mathfrak{b}^\infty_{\nabla Z}(0,t) + \|\mathring{\alpha}\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} \int_0^t \mathfrak{b}^\infty_{\nabla K}(\tau)\mathfrak{b}^\infty_{\nabla Z}(\tau,t)\,\mathrm{d}\tau\right)$$
$$\leq \left(1+t+\mathfrak{B}^\infty_{M,\kappa,Z-\bar{Z}}(0,t)\right)^{\frac{\kappa}{2}} M\left(\mathfrak{B}^\infty_{M,\kappa,\nabla Z}(0,t) + \int_0^t \mathfrak{B}^\infty_{M,\kappa,\nabla K}(\tau)\mathfrak{B}^\infty_{M,\kappa,\nabla Z}(\tau,t)\mathrm{d}\tau\right)$$
$$\equiv \mathfrak{B}^{\kappa,2}_{M,\kappa,\nabla\alpha}(t),$$

proving the finiteness on the interval I. This proves that $\mathfrak{b}^{1,\kappa,2}_\alpha(t) \equiv \sup_n \|\mathring{\alpha}\|_{\mathcal{B}^{1,\kappa,2}_{\mathbf{z}}} < \infty$ in I and the curve of each convergent remains in $\mathcal{B}^{1,\kappa,2}_{\mathbf{z}}$.

(ix) Difference estimates. We now want to show that α_n converges to some limit α in the $\mathcal{L}^\infty_t \mathcal{A}^{\kappa,2}_{\mathbf{z}}$ norm. At first, we estimate the difference $\alpha_{n+1}(t) - \alpha_n(t)$ of the elements at $\mathbf{z} \in \mathbb{R}^{2d}_{\mathbf{z}}$, namely,

$$\begin{aligned}
&|\alpha_{n+2}(t,\mathbf{z}) - \alpha_{n+1}(t,\mathbf{z})| \\
&\leq |\mathring{\alpha}(Z_{n+1}(0,t,\mathbf{z})) - \mathring{\alpha}(Z_n(0,t,\mathbf{z}))| \left|\exp\left(\int_0^t K_{n+1}(\tau, X_{n+1}(\tau,t,\mathbf{z}))\,\mathrm{d}\tau\right)\right| \\
&\quad + |\mathring{\alpha}(Z_n(0,t,\mathbf{z}))| \left|\exp\left(\int_0^t K_{n+1}(\tau, X_{n+1}(\tau,t,\mathbf{z}))\,\mathrm{d}\tau\right) - \exp\left(\int_0^t K_n(\tau, X_n(\tau,t,\mathbf{z}))\,\mathrm{d}\tau\right)\right| \\
&\leq \int_0^1 |(\nabla_{\mathbf{z}}\mathring{\alpha})\,((1-s)Z_n(0,t,\mathbf{z}) + sZ_{n+1}(0,t,\mathbf{z}))|\ \mathrm{d}s\,|Z_{n+1}(0,t,\mathbf{z}) - Z_n(0,t,\mathbf{z})| \\
&\quad + |\mathring{\alpha}(Z_n(0,t,\mathbf{z}))| \left|\int_0^t (K_{n+1}(\tau, X_{n+1}(\tau,t,\mathbf{z})) - K_n(\tau, X_{n+1}(\tau,t,\mathbf{z})))\ \mathrm{d}\tau\right| \\
&\quad + |\mathring{\alpha}(Z_n(0,t,\mathbf{z}))| \left|\int_0^t (K_n(\tau, X_{n+1}(\tau,t,\mathbf{z})) - K_n(\tau, X_n(\tau,t,\mathbf{z})))\ \mathrm{d}\tau\right| \\
&\leq \int_0^1 |(\nabla_{\mathbf{z}}\mathring{\alpha})\,((1-s)Z_n(0,t,\mathbf{z}) + sZ_{n+1}(0,t,\mathbf{z}))|\ \mathrm{d}s\,\|Z_{n+1}(0,t) - Z_n(0,t)\|_{\mathcal{L}^\infty_{\mathbf{z}}} \\
&\quad + |\mathring{\alpha}(Z_n(0,t,\mathbf{z}))| \int_0^t \|K_{n+1}(\tau) - K_n(\tau)\|_{\mathcal{L}^\infty_{\mathbf{x}}}\,\mathrm{d}\tau \\
&\quad + |\mathring{\alpha}(Z_n(0,t,\mathbf{z}))| \int_0^t \|\nabla_{\mathbf{x}} K_n(\tau)\|_{\mathcal{L}^\infty_{\mathbf{x}}}\,\|X_{n+1}(\tau,t) - X_n(\tau,t)\|_{\mathcal{L}^\infty_{\mathbf{z}}}\,\mathrm{d}\tau.
\end{aligned}$$

Now we treat all the terms differently. Primarily, we want to remark that by Lemma 3.14, we have

$$\begin{aligned}
&\sup_n \left\|(1-s)(Z_n(0,t) - \bar{Z}(0,t)) + s(Z_{n+1}(0,t) - \bar{Z}(0,t))\right\|_{\mathcal{L}^\infty_{\mathbf{z}}} \\
&\leq (1-s)\sup_n \left\|Z_n(0,t) - \bar{Z}(0,t)\right\|_{\mathcal{L}^\infty_{\mathbf{z}}} + s\sup_n \left\|Z_{n+1}(0,t) - \bar{Z}(0,t)\right\|_{\mathcal{L}^\infty_{\mathbf{z}}} \\
&\leq \sup_n \left\|Z_n(0,t) - \bar{Z}(0,t)\right\|_{\mathcal{L}^\infty_{\mathbf{z}}} = \mathfrak{b}^\infty_{Z-\bar{Z}}(0,t) \leq \mathfrak{B}^\infty_{M,\kappa,Z-\bar{Z}}(0,t),
\end{aligned}$$

yielding for any $R > 0$,

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}_{\mathbf{z}}} \sup_{|\bar{\mathbf{z}}|\leq R} \left(\int_0^1 |(\nabla_{\mathbf{z}}\mathring{\alpha})\,(sZ_n(0,t,\mathbf{z}+\bar{\mathbf{z}}) + (1-s)Z_{n+1}(0,t,\mathbf{z}+\bar{\mathbf{z}}))|\ \mathrm{d}s\right)^2 \mathrm{d}\mathbf{z} \\
&\leq \int_{\mathbb{R}^{2d}_{\mathbf{z}}} \sup_{|\bar{\mathbf{z}}|\leq R} \left(\sup_{|\bar{\bar{\mathbf{z}}}|\leq \mathfrak{B}^\infty_{M,\kappa,Z-\bar{Z}}(0,t)} \left|(\nabla_{\mathbf{z}}\mathring{\alpha})\,(\bar{Z}(0,t,\mathbf{z}+\bar{\mathbf{z}}) + \bar{\bar{\mathbf{z}}})\right|\right)^2 \mathrm{d}\mathbf{z}
\end{aligned}$$

$$\le \int_{\mathbb{R}^{2d}_{\mathbf{z}}} \sup_{|\bar{\mathbf{z}}|\le(1+t)R+\mathfrak{B}^{\infty}_{M,\kappa,Z-\bar{Z}}(0,t)} \left|(\nabla_{\mathbf{z}}\mathring{\alpha})\,(\bar{Z}(0,t,\mathbf{z})+\bar{\mathbf{z}})\right|^2 \mathrm{d}\mathbf{z}$$
$$= \int_{\mathbb{R}^{2d}_{\mathbf{z}}} \sup_{|\bar{\mathbf{z}}|\le(1+t)R+\mathfrak{B}^{\infty}_{M,\kappa,Z-\bar{Z}}(0,t)} |(\nabla_{\mathbf{z}}\mathring{\alpha})\,(\mathbf{z}+\bar{\mathbf{z}})|^2 \,\mathrm{d}\mathbf{z}.$$

Dividing by $(1+R)^{\kappa}$ and applying the supremum and the square root, we find that

$$\sup_{R\ge 0}(1+R)^{-\frac{\kappa}{2}}$$
$$\cdot\left(\int_{\mathbb{R}^{2d}_{\mathbf{z}}} \sup_{|\bar{\mathbf{z}}|\le R}\left(\int_0^1 |(\nabla_{\mathbf{z}}\mathring{\alpha})\,(sZ_n(0,t,\mathbf{z}+\bar{\mathbf{z}})+(1-s)Z_{n+1}(0,t,\mathbf{z}+\bar{\mathbf{z}}))|\;\mathrm{d}s\right)^2 \mathrm{d}\mathbf{z}\right)^{\frac{1}{2}}$$
$$\le \left(1+t+\mathfrak{B}^{\infty}_{M,\kappa,Z-\bar{Z}}(0,t)\right)^{\frac{\kappa}{2}} \|\nabla_{\mathbf{z}}\mathring{\alpha}\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}}\,.$$

Similarly, for the second and third terms we get

$$\sup_{R\ge 0}(1+R)^{-\frac{\kappa}{2}}\left(\int_{\mathbb{R}^{2d}_{\mathbf{z}}} \sup_{|\bar{\mathbf{z}}|\le R} |\mathring{\alpha}(Z_n(0,t,\mathbf{z}+\bar{\mathbf{z}}))|^2\,\mathrm{d}\mathbf{z}\right)^{\frac{1}{2}} \le \left(1+t+\mathfrak{B}^{\infty}_{M,\kappa,Z-\bar{Z}}(0,t)\right)^{\frac{\kappa}{2}} \|\mathring{\alpha}\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}}\,.$$

Upon combining these estimates with the Lipschitz estimates of Lemma 3.13, one derives

$$\left(1+t+\mathfrak{B}^{\infty}_{M,\kappa,Z-\bar{Z}}(0,t)\right)^{-\frac{\kappa}{2}} \|\alpha_{n+2}(t)-\alpha_{n+1}(t)\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}}$$
$$\le \|\nabla_{\mathbf{z}}\mathring{\alpha}\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} \|Z_{n+1}(0,t)-Z_n(0,t)\|_{\mathcal{L}^{\infty}_{\mathbf{z}}} + \|\mathring{\alpha}\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} \int_0^t \|K_{n+1}(\tau)-K_n(\tau)\|_{\mathcal{L}^{\infty}_{\mathbf{x}}}\,\mathrm{d}\tau$$
$$+ \|\mathring{\alpha}\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} \int_0^t \|\nabla_{\mathbf{x}}K_n(\tau)\|_{\mathcal{L}^{\infty}_{\mathbf{x}}} \|X_{n+1}(\tau,t)-X_n(\tau,t)\|_{\mathcal{L}^{\infty}_{\mathbf{z}}}\,\mathrm{d}\tau$$
$$\le \|\nabla_{\mathbf{z}}\mathring{\alpha}\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} \int_0^t \|F_{n+1}(\tau)-F_n(\tau)\|_{\mathcal{L}^{\infty}_{\mathbf{x}}} \exp\left(\int_0^{\tau}\left(1+\|\nabla_{\mathbf{x}}F_n(\tilde{\tau})\|_{\mathcal{L}^{\infty}_{\mathbf{x}}}\right)\mathrm{d}\tilde{\tau}\right)\mathrm{d}\tau$$
$$+ \|\mathring{\alpha}\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} \int_0^t \|K_{n+1}(\tau)-K_n(\tau)\|_{\mathcal{L}^{\infty}_{\mathbf{x}}}\,\mathrm{d}\tau$$
$$+ \|\mathring{\alpha}\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} \int_0^t \|\nabla_{\mathbf{x}}K_n(\tau)\|_{\mathcal{L}^{\infty}_{\mathbf{x}}} \int_{\tau}^t \|F_{n+1}(\tilde{\tau})-F_n(\tilde{\tau})\|_{\mathcal{L}^{\infty}_{\mathbf{x}}}$$
$$\cdot\exp\left(\int_{\tau}^{\tilde{\tau}}\left(1+\left\|\nabla_{\mathbf{x}}F_n(\tilde{\tilde{\tau}})\right\|_{\mathcal{L}^{\infty}_{\mathbf{x}}}\right)\mathrm{d}\tilde{\tilde{\tau}}\right)\mathrm{d}\tilde{\tau}\,\mathrm{d}\tau$$
$$\le \|\nabla_{\mathbf{z}}\mathring{\alpha}\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} \int_0^t \|F_{n+1}(\tau)-F_n(\tau)\|_{\mathcal{L}^{\infty}_{\mathbf{x}}} \exp\left(\int_0^{\tau}\left(1+\mathfrak{B}^{\infty}_{M,\kappa,\nabla F}(\tilde{\tau})\right)\mathrm{d}\tilde{\tau}\right)\mathrm{d}\tau$$
$$+ \|\mathring{\alpha}\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} \int_0^t \|K_{n+1}(\tau)-K_n(\tau)\|_{\mathcal{L}^{\infty}_{\mathbf{x}}}\,\mathrm{d}\tau$$

$$
\begin{aligned}
&+ \|\mathring{\alpha}\|_{\mathcal{A}_{\mathbf{z}}^{\kappa,2}} \int_0^t \|F_{n+1}(\tau) - F_n(\tau)\|_{\mathcal{L}_{\mathbf{x}}^\infty} \\
&\quad \cdot \left(\int_0^\tau \mathfrak{B}_{M,\kappa,\nabla K}^\infty(\tilde{\tau}) \exp\left(\int_{\tilde{\tau}}^\tau \left(1 + \mathfrak{B}_{M,\kappa,\nabla F}^\infty(\tilde{\tilde{\tau}})\right) \mathrm{d}\tilde{\tilde{\tau}} \right) \mathrm{d}\tilde{\tau} \right) \mathrm{d}\tau \\
\leq{}& M \int_0^t \|K_{n+1}(\tau) - K_n(\tau)\|_{\mathcal{L}_{\mathbf{x}}^\infty} \,\mathrm{d}\tau \\
&+ M \int_0^t \|F_{n+1}(\tau) - F_n(\tau)\|_{\mathcal{L}_{\mathbf{x}}^\infty} \left(\exp\left(\int_0^\tau \left(1 + \mathfrak{B}_{M,\kappa,\nabla F}^\infty(\tilde{\tau})\right) \mathrm{d}\tilde{\tau} \right) \right. \\
&\left. + \int_0^\tau \mathfrak{B}_{M,\kappa,\nabla K}^\infty(\tilde{\tau}) \exp\left(\int_{\tilde{\tau}}^\tau \left(1 + \mathfrak{B}_{M,\kappa,\nabla F}^\infty(\tilde{\tilde{\tau}})\right) \mathrm{d}\tilde{\tilde{\tau}} \right) \mathrm{d}\tilde{\tau} \right) \mathrm{d}\tau \\
\leq{}& c_{1,d} A(\kappa,d)^{1-\frac{1}{d}}\, M \int_0^t \left(\|\nabla_{\mathbf{v}} \alpha_n(\tau)\|_{\mathcal{A}_{\mathbf{z}}^{\kappa,2}} + \|\nabla_{\mathbf{v}} \alpha_{n+1}(\tau)\|_{\mathcal{A}_{\mathbf{z}}^{\kappa,2}} \right) \|\alpha_{n+1}(\tau) - \alpha_n(\tau)\|_{\mathcal{A}_{\mathbf{z}}^{\kappa,2}} \,\mathrm{d}\tau \\
&+ c_{1,d} A(\kappa,d)^{1-\frac{1}{d}} M \int_0^t \left(\|\alpha_n(\tau)\|_{\mathcal{A}_{\mathbf{z}}^{\kappa,2}} + \|\alpha_{n+1}(\tau)\|_{\mathcal{A}_{\mathbf{z}}^{\kappa,2}} \right) \|\alpha_{n+1}(\tau) - \alpha_n(\tau)\|_{\mathcal{A}_{\mathbf{z}}^{\kappa,2}} \\
&\quad \cdot \left(\exp\left(\int_0^\tau \left(1 + \mathfrak{B}_{M,\kappa,\nabla F}^\infty(\tilde{\tau})\right) \mathrm{d}\tilde{\tau} \right) \right. \\
&\qquad \left. + \int_0^\tau \mathfrak{B}_{M,\kappa,\nabla K}^\infty(\tilde{\tau}) \exp\left(\int_{\tilde{\tau}}^\tau \left(1 + \mathfrak{B}_{M,\kappa,\nabla F}^\infty(\tilde{\tilde{\tau}})\right) \mathrm{d}\tilde{\tilde{\tau}} \right) \mathrm{d}\tilde{\tau} \right) \mathrm{d}\tau \\
\leq{}& \mathfrak{B}_{M,\kappa,1}(t) \int_0^t \mathfrak{B}_{M,\kappa,2}(\tau) \, \|\alpha_{n+1}(\tau) - \alpha_n(\tau)\|_{\mathcal{A}_{\mathbf{z}}^{\kappa,2}} \,\mathrm{d}\tau,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathfrak{B}_{M,\kappa,1}(t) &\equiv \left(1 + t + \mathfrak{B}_{M,\kappa,Z-\bar{Z}}^\infty(0,t)\right)^{\frac{\kappa}{2}}, \\
\mathfrak{B}_{M,\kappa,2}(\tau) &\equiv 2c_{1,d} A(\kappa,d)^{1-\frac{1}{d}} M \left[\mathfrak{B}_{M,\kappa,\nabla\alpha}^{\kappa,2}(\tau) + \mathfrak{B}_{M,\kappa,\alpha}^{\kappa,2}(\tau) \left(\exp\left(\int_0^\tau \left(1 + \mathfrak{B}_{M,\kappa,\nabla F}^\infty(\tilde{\tau})\right) \mathrm{d}\tilde{\tau} \right) \right. \right. \\
&\qquad \left. \left. + \int_0^\tau \mathfrak{B}_{M,\kappa,\nabla K}^\infty(\tilde{\tau}) \exp\left(\int_{\tilde{\tau}}^\tau \left(1 + \mathfrak{B}_{M,\kappa,\nabla F}^\infty(\tilde{\tilde{\tau}})\right) \mathrm{d}\tilde{\tilde{\tau}} \right) \mathrm{d}\tilde{\tau} \right) \right].
\end{aligned}
$$

Upon estimating $\mathfrak{B}_{M,\kappa,2}(\tau) \leq \mathfrak{B}_{M,\kappa,2}(t)$ for $\tau \leq t$, we even deduce that for $\mathfrak{B}_{M,\kappa,1+2}(t) \equiv \mathfrak{B}_{M,\kappa,1}(t)\, \mathfrak{B}_{M,\kappa,2}(t)$, we have

$$
\|\alpha_{n+2}(t) - \alpha_{n+1}(t)\|_{\mathcal{A}_{\mathbf{z}}^{\kappa,2}} \leq \mathfrak{B}_{M,\kappa,1+2}(t) \int_0^t \|\alpha_{n+1}(\tau) - \alpha_n(\tau)\|_{\mathcal{A}_{\mathbf{z}}^{\kappa,2}} \,\mathrm{d}\tau.
$$

By induction, one immediately verifies that

$$
\sup_{\tau \in [0,t]} \|\alpha_{n+1}(\tau) - \alpha_n(\tau)\|_{\mathcal{A}_{\mathbf{z}}^{\kappa,2}} \leq 2 \sup_{\tau \in [0,t]} \mathfrak{b}_\alpha^{\kappa,2}(\tau) \frac{\mathfrak{B}_{M,\kappa,1+2}(t)^n t^n}{n!} \leq 2\mathfrak{B}_{M,\kappa,\alpha}^{\kappa,2}(t) \frac{\mathfrak{B}_{M,\kappa,1+2}(t)^n t^n}{n!}.
$$

As the right-hand side is summable in n for any $t < T(M)$, the sequence α_n converges uniformly on $[0,t] \times \mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{v}}$ to some limit $\alpha \in C^0_{t,\mathbf{z}}$ and $\alpha = \lim_{n\to\infty} \alpha_n \in \mathcal{L}^\infty_t \mathcal{A}^{\kappa,2}_{\mathbf{z}}$.

(x) Regularity. Using Lemmata 3.13 and 3.14, we can estimate the Cauchy property of all the sequences $Z_n, \rho_n, F_n, \varphi_n, K_n$ uniformly on the set $[0,t] \times \mathbb{R}^{2d}_{\mathbf{z}}$ for any $t \in I$. In addition, ρ_n and φ_n are Cauchy sequences in the $\mathcal{L}^1_{\mathbf{x}}$ norm. As the space of continuous functions with the $\mathcal{L}^\infty$ norm is complete, we find the continuous limits Z, ρ, F, φ, K. We need to show that these limits are continuously differentiable. This is achieved by proving the uniform Cauchy property of the first derivatives.

We start by examining F_n. Let $0 \le \tau \le t < T$ and $\epsilon > 0$ be arbitrary. For $R = \epsilon / \left(4 c_d \mathfrak{B}^\infty_{M,\kappa,\nabla\rho}(t)\right)$ in the estimate (B.6) applied on $F_n(\tau) - F_m(\tau)$, one finds, regarding the monotonicity of $\mathfrak{B}^\infty_{M,\kappa,\nabla\rho}(\cdot)$ that

$$\begin{aligned}
\|\nabla_{\mathbf{x}} F_n(\tau) - \nabla_{\mathbf{x}} F_m(\tau)\|_{\mathcal{L}^\infty_{\mathbf{x}}} \le\ & c_d \left[\left(1 + \ln \frac{r}{R}\right) \|\rho_n(\tau) - \rho_m(\tau)\|_{\mathcal{L}^\infty_{\mathbf{x}}} + \frac{1}{r^d} \|\rho_n(\tau) - \rho_m(\tau)\|_{\mathcal{L}^1_{\mathbf{x}}} \right. \\
& \left. + R \|\nabla_{\mathbf{x}} \rho_n(\tau) - \nabla_{\mathbf{x}} \rho_m(\tau)\|_{\mathcal{L}^\infty_{\mathbf{x}}} \right] \\
\overset{\text{Lem.3.13}}{\le}\ & \left[c_d \left(1 + \ln \frac{r}{R}\right) A(\kappa, d) + \frac{c_d}{r^d}\right] 2\mathfrak{B}^{\kappa,2}_{M,\kappa,\alpha}(\tau) \|\alpha_n(\tau) - \alpha_m(\tau)\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} + \frac{\epsilon}{2}.
\end{aligned}$$

where the first term can be made $\le \frac{\epsilon}{2}$, choosing n, m large enough by the Cauchy property of $\alpha_n \in \mathcal{A}^{\kappa,2}_{\mathbf{z}}$ and the estimates from (ix). Hence, $\nabla_{\mathbf{x}} F_n$ is uniformly Cauchy on $[0,t] \times \mathbb{R}^d_{\mathbf{x}}$ and converges to a continuous limit, which is known to be $\nabla_{\mathbf{x}} F$, and F is continuously differentiable. An identical argument proves the existence and continuity of $\nabla_{\mathbf{x}} K$.

Now taking the limit $n \to \infty$ of the integral equation

$$Z_n(s,t,\mathbf{z}) = \mathbf{z} + \int_t^s (V_n(\tau,t,\mathbf{z}), F_n(\tau, X_n(\tau,t,\mathbf{z})))\ \mathrm{d}\tau$$

proves that Z solves an ODE system with right-hand side $(t, \mathbf{x}, \mathbf{v}) \mapsto (\mathbf{v}, F(t,\mathbf{x}))$. The continuity of $\nabla_{\mathbf{x}} F$ proves that $Z \in C^1\left(I \times I \times \mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{v}}\right)$ by standard theory [2, Sec.32].

Combining these results, one finds

$$\alpha(t,\mathbf{z}) = \mathring{\alpha}(Z(0,t,\mathbf{z}))\ \exp\left(\int_0^t K(\tau, X(\tau,t,\mathbf{z}))\ \mathrm{d}\tau\right) \tag{3.25}$$

to be continuously differentiable as a composition of such functions.

(xi) Solution. This is a consequence of Lemma 3.15 and Equation (3.25). □

Corollary 3.17. *Let $\alpha \in C^1\left(I \times \mathbb{R}^{2d}_{\mathbf{z}}; \mathbb{C}\right)$ be a local solution of the Hamiltonian Vlasov–Poisson equation with initial datum $\alpha(0) = \mathring{\alpha} \in B_M(\mathbf{0}) \subseteq \mathcal{B}^{1,\kappa,2}_{\mathbf{z}}$. Then all the bounds $\mathfrak{B}^{\cdot}_{M,\kappa,\cdot}$ derived in the proof of Lemma 3.16 also hold for the characteristic tuple of α.*

Proof. As α is a solution, the transport formula (3.17) holds. In turn this implies that α with its characteristic tuple is a fixed point of the iterative scheme in Lemma 3.16. Therefore, all the inductive Grönwall arguments used to prove Lemma 3.16 can be replaced by non-inductive Grönwall estimates that deliver the same bounds. □

Corollary 3.18. *Let $\alpha_1, \alpha_2 \in C^1\left(I \times \mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{v}}\right)$ be two local solutions of the Hamiltonian Vlasov–Poisson equation with initial data $\mathring{\alpha}_1, \mathring{\alpha}_2 \in B_M(\mathbf{0}) \subseteq \mathcal{B}^{1,\kappa,2}_{\mathbf{z}}$. Then*

$$\|\alpha_1(t) - \alpha_2(t)\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} \leq \|\mathring{\alpha}_1 - \mathring{\alpha}_2\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} \left(1 + t + \mathfrak{B}^{\infty}_{M,\kappa,Z-\bar{Z}}(0,t)\right)^{\frac{\kappa}{2}} \exp\left(\int_0^t \mathfrak{B}_{M,\kappa,1+2}(\tau)\mathrm{d}\tau\right). \tag{3.26}$$

Proof. By Corollary 3.17, all the bounds for the solutions α_1, α_2 are uniformly valid. We remark that by the transport formula (3.17), the solutions are fixed points of the iterative scheme. Therefore, we can estimate

$$\begin{aligned} |\alpha_1(t,\mathbf{z}) - \alpha_2(t,\mathbf{z})| \leq{}& |\mathring{\alpha}_1(Z_1(0,t,\mathbf{z})) - \mathring{\alpha}_2(Z_1(0,t,\mathbf{z}))| + |\mathring{\alpha}_2(Z_1(0,t,\mathbf{z})) - \mathring{\alpha}_2(Z_2(0,t,\mathbf{z}))| \\ &+ |\mathring{\alpha}_2(Z_2(0,t,\mathbf{z}))| \int_0^t |K_1(\tau, X_1(\tau,t,\mathbf{z})) - K_2(\tau, X_1(\tau,t,\mathbf{z}))|\,\mathrm{d}\tau \\ &+ |\mathring{\alpha}_2(Z_2(0,t,\mathbf{z}))| \int_0^t |K_2(\tau, X_1(\tau,t,\mathbf{z})) - K_2(\tau, X_2(\tau,t,\mathbf{z}))|\,\mathrm{d}\tau. \end{aligned}$$

All terms except for the first one can be treated exactly as in the proof of Lemma 3.16. For the first term, we find that in the $\mathcal{A}^{\kappa,2}_{\mathbf{z}}$-norm, we have

$$\begin{aligned} &\sup_{R\geq 0}(1+R)^{-\frac{\kappa}{2}} \left(\int_{\mathbb{R}^{2d}_{\mathbf{z}}} \sup_{|\bar{\mathbf{z}}|\leq R} |(\mathring{\alpha}_1 - \mathring{\alpha}_2)(Z_1(0,t,\mathbf{z}+\bar{\mathbf{z}}))|^2\,\mathrm{d}\mathbf{z}\right)^{\frac{1}{2}} \\ &\leq \sup_{R\geq 0}(1+R)^{-\frac{\kappa}{2}} \left(\int_{\mathbb{R}^{2d}_{\mathbf{z}}} \sup_{|\bar{\mathbf{z}}|\leq(1+t)R+\mathfrak{B}^{\infty}_{M,\kappa,Z-\bar{Z}}(0,t)} \left|(\mathring{\alpha}_1 - \mathring{\alpha}_2)(\bar{Z}(0,t,\mathbf{z})+\bar{\mathbf{z}})\right|^2\,\mathrm{d}\mathbf{z}\right)^{\frac{1}{2}} \\ &\leq \left(1 + t + \mathfrak{B}^{\infty}_{M,\kappa,Z-\bar{Z}}(0,t)\right)^{\frac{\kappa}{2}} \|\mathring{\alpha}_1 - \mathring{\alpha}_2\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} = \mathfrak{B}_{M,\kappa,1}(t)\,\|\mathring{\alpha}_1 - \mathring{\alpha}_2\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}}. \end{aligned}$$

Combining this estimate with the estimates of the other terms yields

$$\begin{aligned} \|\alpha_1(t) - \alpha_2(t)\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} \leq{}& \mathfrak{B}_{M,\kappa,1}(t)\,\|\mathring{\alpha}_1 - \mathring{\alpha}_2\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} \\ &+ \mathfrak{B}_{M,\kappa,1}(t) \int_0^t \mathfrak{B}_{M,\kappa,2}(\tau)\,\|\alpha_1(\tau) - \alpha_2(\tau)\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}}\,\mathrm{d}\tau. \end{aligned}$$

A short Grönwall application delivers the desired inequality. □

Proposition 3.19. *Let $\{\mathring{\alpha}_n\} \subseteq B_M(\mathbf{0}) \subseteq \mathcal{B}^{1,\kappa,2}_{\mathbf{z}}$, $\|\mathring{\alpha}_n - \mathring{\alpha}\|_{\mathcal{B}^{1,\kappa,2}_{\mathbf{z}}} \to 0$ be a convergent sequence of compactly supported initial data with solutions α_n on $I(M) = [0, T(M))$. Then there is a solution α of the Hamiltonian Vlasov–Poisson equation for the initial datum $\mathring{\alpha}$ and for every $t < T(M)$, the following holds:*

$$\sup_{\tau\in[0,t]} \|\alpha(\tau) - \alpha_n(\tau)\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} \overset{n\to\infty}{\to} 0.$$

Proof. **(i) Limit.** At first, we remark that $\{\mathring{\alpha}_n\}$ is Cauchy in $\mathcal{B}^{1,\kappa,2}_{\mathbf{z}}$ and so is $\{\alpha_n\}$ in $\mathcal{L}^\infty_t \mathcal{A}^{\kappa,2}_{\mathbf{z}}$ by Corollary 3.18, if one restricts t to any compact subinterval $J \subseteq I(M)$. As all the α_n are continuous, there is a limit $\alpha \in C^0\left(I(M) \times \mathbb{R}^{2d}_{\mathbf{z}}; \mathbb{C}\right)$, s.t.

$$\sup_{t\in J} \|\alpha(t) - \alpha_n(t)\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} \overset{n\to\infty}{\longrightarrow} 0.$$

(ii) Tuple convergence. We now want to prove that $\{\rho_n\}$ and $\{F_n\}$ converge to the density ρ and the force F induced by the limit α, respectively. In fact, this is an immediate consequence of Lemma 3.13-(i)-(iii). We remark that $\nabla_{\mathbf{x}} F$ exists as well and is continuous, because $\{\nabla_{\mathbf{x}} F_n\}$ is a Cauchy sequence in $\mathcal{L}^\infty_{(t,\mathbf{x})}$. We proceed similarly to (x) in the proof of Lemma 3.16. By Corollary 3.17, for all n we have $\sup_{\tau\le t} \|\nabla_{\mathbf{x}}\rho_n(\tau)\|_{\mathcal{L}^\infty_{\mathbf{x}}} \le \mathfrak{B}^\infty_{M,\kappa,\nabla\rho}(t)$. By (B.6) in Lemma B.1, one finds

$$\begin{aligned}\|\nabla_{\mathbf{x}} F_n(t) - \nabla_{\mathbf{x}} F_m(t)\|_{\mathcal{L}^\infty_{\mathbf{x}}} \le c_d \Bigg[&\left(1 + \ln \frac{r}{R}\right) \|\rho_n(t) - \rho_m(t)\|_{\mathcal{L}^\infty_{\mathbf{x}}} \\ &+ \frac{1}{r^d} \|\rho_n - \rho_m\|_{\mathcal{L}^1_{\mathbf{x}}} + R\, \|\nabla_{\mathbf{x}}\rho_n(t) - \nabla_{\mathbf{x}}\rho_m(t)\|_{\mathcal{L}^\infty_{\mathbf{x}}}\Bigg].\end{aligned}$$

For any given $\epsilon > 0$, we choose $R = \epsilon/(4c_d \mathfrak{B}^\infty_{M,\kappa,\nabla\rho}(t))$ and some $r \ge R$. The first two terms are controlled by Lemma 3.13-(i)-(ii) and the Cauchy property of α_n.

For φ, we only need to remark that for all n, $\sup_{\tau\le t} \|\nabla_{\mathbf{z}}\alpha_n(\tau)\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} \le \mathfrak{B}^{\kappa,2}_{M,\kappa,\nabla\alpha}(t)$. By Lemma 3.13-(iv)-(vi), φ_n and K_n are Cauchy in their respective spaces. Denote the limits by φ_∞ and K_∞. After noticing $\sup_n \sup_{\tau\in[0,t]} \|\nabla_{\mathbf{x}}\varphi_n(\tau)\|_{\mathcal{L}^\infty_{\mathbf{x}}} \le \mathfrak{B}^\infty_{M,\kappa,\nabla\varphi}(t)$, the argument of $\nabla_{\mathbf{x}} F$ can be copied to prove the convergence $\|\nabla_{\mathbf{x}} K_n(t) - \nabla_{\mathbf{x}} K_\infty(t)\|_{\mathcal{L}^\infty_{\mathbf{x}}} \to 0$.

Now let Z_n denote the solution map of the characteristic system. By Lemma 3.14, we find that

$$\|Z_n(s,t) - Z_m(s,t)\|_{\mathcal{L}^\infty_{\mathbf{z}}} \le \int_s^t \|F_n(\tau) - F_m(\tau)\|_{\mathcal{L}^\infty_{\mathbf{x}}} \exp\left(\int_s^\tau \left(1 + \mathfrak{B}^\infty_{M,\kappa,\nabla F}(\tilde{\tau})\right) \mathrm{d}\tilde{\tau}\right) \mathrm{d}\tau,$$

proving the uniform Cauchy property of Z_n and thus, its convergence. This allows to compute

$$\begin{aligned} Z(s,t,\mathbf{z}) &\equiv \lim_n Z_n(s,t,\mathbf{z}) = \lim_n (X_n, V_n)(s,t,\mathbf{z}) \\ &= \mathbf{z} + \lim_n \int_t^s (V_n(\tau,t,\mathbf{z}), F_n(\tau, X_n(\tau,t,\mathbf{z})))\,\mathrm{d}\tau \\ &= \mathbf{z} + \int_t^s (V(\tau,t,\mathbf{z}), F(\tau, X(\tau,t,\mathbf{z})))\,\mathrm{d}\tau. \end{aligned}$$

Hence, Z is the solution map of a characteristic system with a $C_t^0 C_{\mathbf{z}}^1$ right-hand side. By standard theory [2, Sec.32], $Z \in C^1$.

(iii) Solution. As all the α_n are solutions, they satisfy the transport formula

$$\alpha_n(t,\mathbf{z}) = \mathring{\alpha}_n(Z_n(0,t,\mathbf{z}))\exp\left(\int_0^t K_n(\tau, X_n(\tau,t,\mathbf{z}))\,\mathrm{d}\tau\right).$$

In the previous part we have shown that all the sequences converge for the limit $n \to \infty$ sufficiently nice to derive

$$\alpha(t,\mathbf{z}) = \mathring{\alpha}(Z(0,t,\mathbf{z}))\exp\left(\int_0^t K_\infty(\tau, X(\tau,t,\mathbf{z}))\,\mathrm{d}\tau\right)$$

and indeed, the right-hand side is $C^1_{(t,\mathbf{z})}$. The function α is a local solution for the initial datum $\mathring{\alpha}$ by Lemma 3.15 if $(\rho, F, \varphi_\infty, K_\infty)$ is indeed its characteristic tuple. At first, we verify that φ is well-defined. In fact,

$$\begin{aligned} \|\nabla_{\mathbf{z}}\alpha(t)\|_{\mathcal{A}_{\mathbf{z}}^{\kappa,2}} \le{}& \left(1 + t + \left\|Z(0,t) - \bar{Z}(0,t)\right\|_{\mathcal{L}_{\mathbf{z}}^\infty}\right)^{\frac{\kappa}{2}} \\ &\cdot \left(\|\nabla_{\mathbf{z}}\mathring{\alpha}\|_{\mathcal{A}_{\mathbf{z}}^{\kappa,2}} \|\nabla_{\mathbf{z}} Z(0,t)\|_{\mathcal{L}_{\mathbf{z}}^\infty} + \|\mathring{\alpha}\|_{\mathcal{A}_{\mathbf{z}}^{\kappa,2}} \int_0^t \|\nabla_{\mathbf{x}} K_\infty(\tau)\|_{\mathcal{L}_{\mathbf{x}}^\infty} \|\nabla_{\mathbf{z}} Z(\tau,t)\|_{\mathcal{L}_{\mathbf{z}}^\infty}\,\mathrm{d}\tau\right) \\ \le{}& 2\left(1 + t + \left\|Z(0,t) - \bar{Z}(0,t)\right\|_{\mathcal{L}_{\mathbf{z}}^\infty}\right)^{\frac{\kappa}{2}} \\ &\cdot \left(\|\nabla_{\mathbf{z}}\mathring{\alpha}\|_{\mathcal{A}_{\mathbf{z}}^{\kappa,2}} \exp\left(\int_0^t \left(1 + \|\nabla_{\mathbf{x}} F(\tau)\|_{\mathcal{L}_{\mathbf{x}}^\infty}\right)\mathrm{d}\tau\right)\right. \\ &\left. + \|\mathring{\alpha}\|_{\mathcal{A}_{\mathbf{z}}^{\kappa,2}} \int_0^t \|\nabla_{\mathbf{x}} K_\infty(\tau)\|_{\mathcal{L}_{\mathbf{x}}^\infty} \exp\left(\int_\tau^t \left(1 + \|\nabla_{\mathbf{x}} F(\tilde{\tau})\|_{\mathcal{L}_{\mathbf{x}}^\infty}\right)\mathrm{d}\tilde{\tau}\right)\mathrm{d}\tau\right) \\ \le{}& 2\left(1 + t + \mathfrak{B}^\infty_{M,\kappa,Z-\bar{Z}}(0,t)\right)^{\frac{\kappa}{2}} \left(\|\nabla_{\mathbf{z}}\mathring{\alpha}\|_{\mathcal{A}_{\mathbf{z}}^{\kappa,2}} \exp\left(\int_0^t \left(1 + \mathfrak{B}^\infty_{M,\kappa,\nabla F}(\tau)\right)\mathrm{d}\tau\right)\right. \\ &\left. + \|\mathring{\alpha}\|_{\mathcal{A}_{\mathbf{z}}^{\kappa,2}} \int_0^t \mathfrak{B}^\infty_{M,\kappa,\nabla K}(\tau) \exp\left(\int_\tau^t \left(1 + \mathfrak{B}^\infty_{M,\kappa,\nabla F}(\tilde{\tau})\right)\mathrm{d}\tilde{\tau}\right)\mathrm{d}\tau\right), \end{aligned}$$

proving that $\alpha(t)$ has a well-defined characteristic tuple. Now for the phase force K, we indeed find that

$$\begin{aligned}\|K(t)-K_\infty(t)\|_{\mathcal{L}^\infty_{\mathbf{x}}} &\le \|K(t)-K_n(t)\|_{\mathcal{L}^\infty_{\mathbf{x}}} + \|K_\infty(t)-K_n(t)\|_{\mathcal{L}^\infty_{\mathbf{x}}}\\ &\le {c_{1,d}}^{1-\frac{1}{d}} A(\kappa,d)^{1-\frac{1}{d}} \left(\|\nabla_{\mathbf{v}}\alpha(t)\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} + \|\nabla_{\mathbf{v}}\alpha_n(t)\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}}\right) \|\alpha(t)-\alpha_n(t)\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}}\\ &\quad + \|K_\infty(t)-K_n(t)\|_{\mathcal{L}^\infty_{\mathbf{x}}} \overset{n\to\infty}{\to} 0.\end{aligned}$$

Therefore, α satisfies the transport formula and is a solution with initial datum $\alpha(0) = \mathring{\alpha}$ by Lemma 3.15. □

Proposition 3.20 (Well-posedness)**.** *For every $M > 0$ and $t < T(M)$, there is a Lipschitz continuous time evolution operator*

$$\mathfrak{U}_{M,t} : \begin{array}{ccc} B_M(\mathbf{0}) \subseteq \mathcal{B}^{1,\kappa,2}_{\mathbf{z}} & \to & \mathcal{A}^{\kappa,2}_{\mathbf{z}} \\ \mathring{\alpha} & \mapsto & \alpha(t), \end{array} \tag{3.27}$$

which maps the initial datum to the state at time t. The function $\alpha(t,\mathbf{z}) \equiv (\mathfrak{U}_{M,t}\mathring{\alpha})(\mathbf{z})$ then is the unique local solution of the Hamiltonian Vlasov–Poisson equation with initial datum $\mathring{\alpha}$. Indeed, we have $\mathfrak{U}_{M,t}(B_M(\mathbf{0})) \subseteq \mathcal{B}^{1,\kappa,2}_{\mathbf{z}} \subseteq \mathcal{A}^{\kappa,2}_{\mathbf{z}}$.

Proof. Let $M > 0$ be arbitrary. At first, restrict the domain of definition to the set of compactly supported initial data. By Lemma 3.16, there is the uniform time of existence $T(M)$, s.t. there is a solution for any of these initial data. By Corollary 3.18, the time evolution operator is globally Lipschitz on the compactly supported data in $B_M(\mathbf{0}) \subseteq \mathcal{B}^{1,\kappa,2}_{\mathbf{z}}$. By Lemma A.3, these initial data are dense. Because $\mathcal{B}^{1,\kappa,2}_{\mathbf{z}}$ is complete by Theorem A.4, there is a unique Lipschitz continuous extension of $\mathfrak{U}_{M,t}$ on $B_M(\mathbf{0}) \subseteq \mathcal{B}^{1,\kappa,2}_{\mathbf{z}}$. Proposition 3.19 finally proves that the functions obtained by this technical extension truly are solutions of the Hamiltonian Vlasov–Poisson equation.

The uniqueness of these solutions is immediately provided by Corollary 3.18, as any two solutions with the same initial datum coincide.

The final statement follows from the fact that the finiteness of $\|\alpha(t)\|_{\mathcal{B}^{1,\kappa,2}_{\mathbf{z}}}$ is given up to $T(M)$. □

3.2.2 Global Existence

Due to the close relation with the classical Vlasov–Poisson equation, the Hamiltonian Vlasov–Poisson equation admits almost the same continuation and global existence criteria as those due to Pfaffelmoser–Schaeffer [17, 21] and Lions–Perthame [11] for the classical case. For completeness, we include the leading steps before adapting their central results.

Proposition 3.21 (Maximality Criterion). *Let $\alpha \in C^1\left([0,T)\times\mathbb{R}^{2d}_{\mathbf{z}};\mathbb{C}\right)$ be a local solution of the Hamiltonian Vlasov–Poisson equation with initial datum $\mathring{\alpha} \in \mathcal{B}^{1,\kappa,2}_{\mathbf{z}}$. It is maximal if either $T=\infty$, or $\|\alpha(t)\|_{\mathcal{B}^{1,\kappa,2}_{\mathbf{z}}} \to \infty$ for $t \uparrow T$. In any other case, there exists an extension. In particular, we find a maximal solution for every initial datum $\mathring{\alpha}$.*

Proof. Maximality is obvious in both cases. Hence, assume that none of the cases holds. Then, by definition, we know that

$$\exists M \geq 0 : \forall \bar{T} \in [0,T) : \exists t \in (\bar{T},T) : \quad \|\alpha(t)\|_{\mathcal{B}^{1,\kappa,2}_{\mathbf{z}}} \leq M.$$

In conclusion, one can pick some $t_0 \in \left[T-\frac{T(M)}{2},T\right)$, where $T(M)$ is the lower bound for the existence interval of the solution, and continue the solution to at least $T+\frac{T(M)}{2}$, contradicting maximality. □

Definition 3.22 (Velocity Moments). Let $k \geq 0$ be arbitrary. For any measurable function $\alpha : \mathbb{R}^{2d}_{\mathbf{z}} \to \mathbb{C}$, we define the k**-th local velocity moment** by

$$\mathfrak{m}_k(\alpha)(\mathbf{x}) \equiv \int_{\mathbb{R}^d_{\mathbf{v}}} |\mathbf{v}|^k\,|\alpha(\mathbf{x},\mathbf{v})|^2\,\mathrm{d}\mathbf{v} \tag{3.28}$$

and the k**-th velocity moment** as

$$\mathfrak{M}_k(\alpha) \equiv \int_{\mathbb{R}^d_{\mathbf{x}}} \mathfrak{m}_k(\alpha)(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^d_{\mathbf{x}}}\int_{\mathbb{R}^d_{\mathbf{v}}} |\mathbf{v}|^k\,|\alpha(\mathbf{x},\mathbf{v})|^2\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{x}. \tag{3.29}$$

Lemma 3.23. *[20, Lem.1.8] Let $k \geq 0$ be arbitrary and $\alpha : \mathbb{R}^{2d}_{\mathbf{z}} \to \mathbb{C}$ be measurable. Now let $p \in [1,\infty]$ be some exponent and $0 \leq l \leq k < \infty$. Define the exponent*

$$r \equiv \frac{k + d\frac{p-1}{p}}{l + \frac{k-l}{p} + d\frac{p-1}{p}}.$$

Then the following estimate holds

$$\|\mathfrak{m}_l(\alpha)\|_{\mathcal{L}^r_{\mathbf{x}}} \leq\ c_{k,l,p}\,\|\alpha\|_{\mathcal{L}^{2p}_{\mathbf{z}}}^{\frac{2p(k-l)}{pk+(p-1)d}}\,\mathfrak{M}_k(\alpha)^{\frac{lp+d(p-1)}{pk+(p-1)d}}. \tag{3.30}$$

Proof. Let $\mathbf{x}$ be arbitrary and call $q \equiv \frac{p}{p-1}$ the Hölder conjugate of p. Then we find that for any $R > 0$,

$$
\begin{aligned}
\mathfrak{m}_l(\alpha)(\mathbf{x}) &\leq \int_{B_R(\mathbf{0})} |\mathbf{v}|^l\, |\alpha(\mathbf{x},\mathbf{v})|^2 \,\mathrm{d}\mathbf{v} + \int_{\mathbb{R}^d_{\mathbf{v}} - B_R(\mathbf{0})} |\mathbf{v}|^l\, |\alpha(\mathbf{x},\mathbf{v})|^2 \,\mathrm{d}\mathbf{v} \\
&\leq \|\alpha(\mathbf{x},\cdot)\|^2_{\mathcal{L}^{2p}_{\mathbf{v}}} \left(\int_{B_R(\mathbf{0})} |\mathbf{v}|^{lq} \,\mathrm{d}\mathbf{v} \right)^{\frac{1}{q}} + R^{l-k} \int_{\mathbb{R}^d_{\mathbf{v}}} |\mathbf{v}|^k\, |\alpha(\mathbf{x},\mathbf{v})|^2 \,\mathrm{d}\mathbf{v} \\
&\leq \|\alpha(\mathbf{x},\cdot)\|^2_{\mathcal{L}^{2p}_{\mathbf{v}}} \left(\frac{\omega_d}{lq+d} \right)^{\frac{1}{q}} R^{l+\frac{d}{q}} + \mathfrak{m}_k(\alpha)(\mathbf{x})\, R^{l-k} \\
&\overset{R \text{ opt.}}{=} \underbrace{\left(\left(\frac{k-l}{l+\frac{d}{q}} \right)^{\frac{l+\frac{d}{q}}{k+\frac{d}{q}}} + \left(\frac{l+\frac{d}{q}}{k-l} \right)^{\frac{k-l}{k+\frac{d}{q}}} \right) \left(\frac{\omega_d}{lq+d} \right)^{\frac{k-l}{kq+d}}}_{\equiv c_{k,l,p}} \|\alpha(\mathbf{x},\cdot)\|^{2\frac{k-l}{k+\frac{d}{q}}}_{\mathcal{L}^{2p}_{\mathbf{v}}}\, \mathfrak{m}_k(\alpha)(\mathbf{x})^{\frac{l+\frac{d}{q}}{k+\frac{d}{q}}} .
\end{aligned}
$$

Now taking the r-th power and integrating this expression, where the right-hand side is estimated by the Hölder inequality once more, we obtain

$$
\begin{aligned}
\|\mathfrak{m}_l(\alpha)^r\|_{\mathcal{L}^1_{\mathbf{x}}} &\leq c^r_{k,l,p} \left\| \left(\|\alpha\|^{2p}_{\mathcal{L}^{2p}_{\mathbf{v}}} \right)^{\frac{r}{p}\frac{k-l}{k+\frac{d}{q}}} \mathfrak{m}_k(\alpha)^{r\frac{l+\frac{d}{q}}{k+\frac{d}{q}}} \right\|_{\mathcal{L}^1_{\mathbf{x}}} \\
&= c^r_{k,l,p} \left\| \left(\|\alpha\|^{2p}_{\mathcal{L}^{2p}_{\mathbf{v}}} \right)^{\frac{k-l}{k-l+lp+(p-1)d}} \mathfrak{m}_k(\alpha)^{\frac{lp+d(p-1)}{lp+(p-1)d+k-l}} \right\|_{\mathcal{L}^1_{\mathbf{x}}} \\
&\leq c^r_{k,l,p}\, \|\alpha\|^{\frac{2p(k-l)}{k-l+lp+(p-1)d}}_{\mathcal{L}^{2p}_{\mathbf{z}}}\, \mathfrak{M}_k(\alpha)^{\frac{lp+d(p-1)}{lp+(p-1)d+k-l}} .
\end{aligned}
$$

Upon taking the r-th root of this expression, we find the asserted inequality

$$
\|\mathfrak{m}_l(\alpha)\|_{\mathcal{L}^r_{\mathbf{x}}} \leq\; c_{k,l,p}\, \|\alpha\|^{\frac{2p(k-l)}{pk+(p-1)d}}_{\mathcal{L}^{2p}_{\mathbf{z}}}\, \mathfrak{M}_k(\alpha)^{\frac{lp+d(p-1)}{pk+(p-1)d}} . \qquad \square
$$

Proposition 3.24 (Global Existence Criteria)**.** *Let $\alpha \in C^1\left([0,T) \times \mathbb{R}^{2d}_{\mathbf{z}}; \mathbb{C}\right)$ be a solution of the Hamiltonian Vlasov–Poisson equation on a maximal interval of existence with characteristic tuple (ρ, F, φ, K) and initial datum $\|\mathring{\alpha}\|_{\mathcal{B}^{1,\kappa,2}_{\mathbf{z}}} < \infty$. If there is a continuous $h : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, s.t. either*

(i) $\|F(t)\|_{\mathcal{L}^\infty_{\mathbf{x}}} \leq h(t)$ *on* $t \in [0,T)$, *or*

(ii) $\|\rho(t)\|_{\mathcal{L}^p_{\mathbf{x}}} \leq h(t)$ *on* $t \in [0,T)$ *for some* $p \in d\left[1 - \frac{1}{\kappa}, 1\right)$, *or*

(iii) $\mathfrak{M}_k(\alpha(t)) \leq h(t)$ *on* $t \in [0,T)$ *for some* $k \geq \left[\left(1 - \frac{1}{\kappa}\right)d - 1\right]d \geq \left(d - \frac{3}{2}\right)d$,

then $T = \infty$ and the solution is global.

Proof. The idea is to replace the key bound $\mathfrak{B}^{\kappa,2}_{M,\kappa,\alpha}$ in Lemma 3.16 by a continuous function that remains finite for finite times. Therefore, we need to manipulate the estimate (3.21):

$$\begin{aligned}
\|\alpha(t)\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} &\overset{(3.21)}{\leq} \left(1 + t + \int_0^t \|F(\tau)\|_{\mathcal{L}^\infty_{\mathbf{x}}} e^\tau \mathrm{d}\tau\right)^{\frac{\kappa}{2}} \|\mathring{\alpha}\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} \\
&\leq \left(1 + t + \int_0^t c_{p,d} \|\rho(\tau)\|^{\frac{p}{d}}_{\mathcal{L}^p_{\mathbf{x}}} \|\rho(\tau)\|^{1-\frac{p}{d}}_{\mathcal{L}^\infty_{\mathbf{x}}} e^\tau \mathrm{d}\tau\right)^{\frac{\kappa}{2}} \|\mathring{\alpha}\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} \\
&\leq \left(1 + t + \int_0^t c_{p,d} A(\kappa, d) \|\rho(\tau)\|^{\frac{p}{d}}_{\mathcal{L}^p_{\mathbf{x}}} e^\tau \|\alpha(\tau)\|^{2\left(1-\frac{p}{d}\right)}_{\mathcal{A}^{\kappa,2}_{\mathbf{x}}} \mathrm{d}\tau\right)^{\frac{\kappa}{2}} \|\mathring{\alpha}\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} .
\end{aligned}$$

Statement (i) follows from the first inequality. The last inequality shows that whenever $p \in d\left[1 - \frac{1}{\kappa}, 1\right)$ and (ii) holds, a Grönwall argument proves the existence of $g \in C^0(\mathbb{R}_{\geq 0}; \mathbb{R}_{\geq 0})$, s.t. $\|\alpha(t)\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}} \leq g(t)$. By replacing the general bound $\mathfrak{B}^{\kappa,2}_{M,\kappa,\alpha}$ with the specific upper bound g in the proof of Lemma 3.16, one infers that even $\|\nabla_{\mathbf{z}}\alpha(t)\|_{\mathcal{A}^{\kappa,2}_{\mathbf{z}}}$ and therefore $\|\alpha(t)\|_{\mathcal{B}^{1,\kappa,2}_{\mathbf{z}}}$ remains bounded on any bounded interval. By the Maximality Criterion 3.24, the solution is global.

If we now assume (iii), we can reduce this to the condition (i) by Lemma 3.23, i.e.,

$$\begin{aligned}
\|\rho(t)\|_{\mathcal{L}^p_{\mathbf{x}}} = \|\mathfrak{m}_0(\alpha(t))\|_{\mathcal{L}^p_{\mathbf{x}}} &\overset{\text{Lem.3.23}}{\leq} c_{k,l,q} \|\alpha(t)\|^{\frac{2qk}{qk+(q-1)d}}_{\mathcal{L}^{2q}_{\mathbf{z}}} \mathfrak{M}_k(\alpha(t))^{\frac{(q-1)d}{qk+(q-1)d}} \\
&\leq c_{k,l,q} \|\mathring{\alpha}\|^{\frac{2qk}{qk+(q-1)d}}_{\mathcal{L}^{2q}_{\mathbf{z}}} \mathfrak{M}_k(\alpha(t))^{\frac{(q-1)d}{qk+(q-1)d}} ,
\end{aligned}$$

given that

$$\left(1 - \frac{1}{\kappa}\right) d \leq p = \frac{qk + (q-1)d}{k + (q-1)d} < d.$$

By adjusting $1 \leq q \leq \infty$, we find that if $\mathfrak{M}_k(\alpha(t))$ is bounded for some fixed $k \geq 0$, then so is $\|\rho(t)\|_{\mathcal{L}^p_{\mathbf{x}}}$ for any $1 \leq p \leq 1 + \frac{k}{d}$. In order to satisfy the condition (ii), we find the natural restriction for k, given by

$$\left(1 - \frac{1}{\kappa}\right) d \leq 1 + \frac{k}{d}. \qquad \square$$

Theorem 3.25 (Pfaffelmoser–Schaeffer Adaptation)**.** *Let $d = 3$, $\kappa \geq 2d = 6$ be fixed and $\mathring{\alpha} \in \mathcal{B}^{1,\kappa,2}_{\mathbf{z}}$ be an initial datum, which in addition satisfies*

(i) $\left\| \|\mathring{\alpha}\|_{\mathcal{L}^\infty_{\mathbf{x}}} \right\|_{\mathcal{A}^{d,2}_{\mathbf{v}}} < \infty,$

(ii) $\exists \lambda \geq d : \left\| \|\nabla_{\mathbf{z}} \mathring{\alpha}\|_{\mathcal{L}^\infty_{\mathbf{x}}} \right\|_{\mathcal{A}^{\lambda,2}_{\mathbf{v}}} < \infty$, *and*

(iii) $\mathfrak{M}_2(\mathring{\alpha}) < \infty$.

Then the maximal solution for this initial datum is global.

Proof. Let α be the maximal solution for the given initial datum and $f \equiv |\alpha|^2$ its corresponding solution of the classical Vlasov–Poisson equation. It is easily seen that $\mathring{f} \equiv |\mathring{\alpha}|^2$ then satisfies the conditions given in [17, Gen.Ass.1]. Further, [17, Thm.19] in combination with [17, Lem.10] proves that there exists $h \in C^0(\mathbb{R}_{\geq 0}; \mathbb{R}_{\geq 0})$, s.t. $\|F(t)\|_{\mathcal{L}^\infty_{\mathbf{x}}} \leq h(t)$, yielding global existence of α by Proposition 3.24-(i). □

Theorem 3.26 (Lions–Perthame Adaptation)**.** *Let $d = 3$ be fixed, $\kappa \geq 2d = 6$, and $\mathring{\alpha} \in \mathcal{B}^{1,\kappa,2}_{\mathbf{z}}$ be an initial datum, s.t. in addition $\mathfrak{M}_k(\mathring{\alpha}) < \infty$ for some $k > 9\left(1 - \frac{1}{\kappa}\right) - 3 \geq \frac{9}{2}$. Then the maximal solution for this initial datum is global.*

Proof. Let α be the maximal solution for the given initial datum and $f \equiv |\alpha|^2$ be the corresponding classical solution of the Vlasov–Poisson system. The statement of [11, Thm.1] proves that there is $h \in C^0(\mathbb{R}_{\geq 0}; \mathbb{R}_{\geq 0})$, s.t. $\mathfrak{M}_k(\alpha(t)) \leq h(t)$, yielding global existence for α by Proposition 3.24-(iii). □

Remark 3.27. **(i) Pfaffelmoser–Schaeffer.** While the relation between local-sup integrability conditions given in [17, Gen.Ass.1] and the conditions (i), (ii) in Theorem 3.25 are very close, it is worth noting the subtle difference. While $\alpha \in \mathcal{B}^{1,\kappa,2}_{\mathbf{z}}$ only requires the integrability of the local supremum on balls in $\mathbb{R}^{2d}_{\mathbf{z}}$, the conditions in [17, Gen.Ass.1] require integrability of the local supremum on balls in $\mathbb{R}^d_{\mathbf{v}}$ and the full space $\mathbb{R}^d_{\mathbf{x}}$. Hence, (i) and (ii) are true restrictions, which are not genuinely fulfilled for any $\alpha \in \mathcal{B}^{1,\kappa,2}_{\mathbf{z}}$.

(ii) Lions–Perthame. In the Vlasov picture where $\mathring{f} = |\mathring{\alpha}|^2$, Theorem 1 in [11] provides the global extension of solutions under the sole restriction of $\mathfrak{M}_k(\mathring{\alpha}) < \infty$ for some $k > 3$. On the other hand, uniqueness of f is only established for the case of $k > 6$, but on a much larger class of solutions, essentially satisfying $\rho \in \mathcal{L}^\infty_{t,\mathrm{loc}} \mathcal{L}^\infty_{\mathbf{x}}$. We should remark that any $\mathring{f}$ derived from $\mathring{\alpha} \in \mathcal{B}^{1,\kappa,2}_{\mathbf{z}}$ is in $\mathcal{B}^{1,\kappa,1}_{\mathbf{z}}$ and indeed satisfies all the regularity and integrability conditions of [11, Thm.6]. Therefore, on the Vlasov level, the solution f is unique in that much larger class, as long as $k > 6$. Nevertheless, as the map $\mathring{\alpha} \mapsto |\mathring{\alpha}|^2$ is not one to one, the results are not instantly transferable to the Hamiltonian Vlasov setup.

Chapter 4

Mean Field Limit

This chapter mainly consists of results by the author submitted in [16].

In mathematical physics it is often the case that one has to model a physical system consisting of many indistinguishable particles whose state can only be detected in a statistical manner. Mathematical models can then be developed on the *microscopic* scale of N particles and on the *macroscopic* scale of their statistical ensemble. In order to make the transition from the microscopic to the macroscopic description, the mean field approach replaces direct particle interactions with an effective background field, typically in the form of a convolution as in Equation (4.3). A big justification for the macroscopic theory is achieved when its time evolution is consistent with the microscopic evolution at least under some limit procedure $N \to \infty$.

This is the idea behind computing a mean field limit. In order to define a mean field scheme, we require a sequence of physical systems Σ_N with respective evolution equations, describing the microscopic dynamics of N particles, as well as a physical system $\bar{\Sigma}$ with an evolution equation, parameterizing the macroscopic state of the same system on a *coarse* scale, which no longer identifies different particles, but rather carries the statistical information of their ensemble. Given a projection map

$$\Pi : \quad \Sigma_1 \times \Sigma_2 \times \cdots \to \bar{\Sigma}^{\mathbb{N}}, \quad (\sigma_N)_{N\in\mathbb{N}} \mapsto (\Pi\sigma_N)_{N\in\mathbb{N}},$$

which assigns a sequence of states in $\bar{\Sigma}$ to a sequence of microscopic states in the Σ_N, a **mean field limit** is a statement which, given the convergence $\Pi\sigma_N \to \bar{\sigma}$, $N \to \infty$ in some sense, implies the convergence $\Pi\sigma_N(t) \to \bar{\sigma}(t)$, $N \to \infty$ as time evolves in the respective systems. This can be expressed in a commutative diagram, see Figure 4.1.

$$\begin{tikzcd} \Sigma_1 \times \Sigma_2 \times \cdots \arrow[r, "\Pi"] \arrow[d, "\text{time evolution}"] & \bar{\Sigma}^{\mathbb{N}} \arrow[r, "N\to\infty"] & \bar{\Sigma} \arrow[d, "\text{time evolution}"] \\ \Sigma_1 \times \Sigma_2 \times \cdots \arrow[r, "\Pi"] & \bar{\Sigma}^{\mathbb{N}} \arrow[r, "N\to\infty"] & \bar{\Sigma} \end{tikzcd}$$

Figure 4.1: Scheme of a mean field limit. A sequence of many-particle states $\Sigma_1\times\Sigma_2\times\cdots$ is mapped to a limit in the mean field system $\bar{\Sigma}$. This map must commute with time evolution.

The Vlasov formalism fits precisely in such a framework, as it has been proven to be the mean field limit of numerous classical many-particle systems. The most seminal work in this context is [5]. In order to intrinsically motivate the study of the Hamiltonian Vlasov system, in this chapter we show that the Hamiltonian Vlasov equation indeed arises as a mean field equation of a many-particle problem.

Throughout the chapter, we restrict our attention to the case of the Regular Vlasov non-relativistic model with regular interaction potential $\Gamma : \mathbb{R}^d_{\mathbf{x}} \to \mathbb{R}$, i.e., the energy functional is

$$\mathcal{H}(f) \equiv \int_{\mathbb{R}^{2d}_{\mathbf{z}}} \left(\frac{|\mathbf{v}|^2}{2} + \frac{1}{2} (\Gamma * f)(\mathbf{x}) \right) f(\mathbf{x}, \mathbf{v})\, \mathrm{d}\mathbf{z}$$

and $\Gamma \in C^3_{\mathbf{x}}$ satisfies

$$C_\Gamma \equiv \max\left\{ \left\|\mathrm{D}^1\Gamma\right\|_{\mathcal{L}^\infty_{\mathbf{x}}}, \left\|\mathrm{D}^2\Gamma\right\|_{\mathcal{L}^\infty_{\mathbf{x}}}, \left\|\mathrm{D}^3\Gamma\right\|_{\mathcal{L}^\infty_{\mathbf{x}}} \right\} < \infty. \tag{Pot}$$

We recall that the global well-posedness for the associated Hamiltonian Vlasov and Hamilton Hartree equations is established in Section 3.1.

4.1 The Many-Particle Model

In the classical Vlasov formalism, one way to derive the evolution equation is via the requirement that the density is constant, i.e.,

$$\mathrm{d}_t f(t, X(t,0,\mathbf{z}), V(t,0,\mathbf{z})) = 0 \tag{Vl}$$

along the trajectories (X, V) of the non-autonomous Hamiltonian system

$$\begin{aligned} \partial_t X(t,0,\mathbf{z}) &= \left(\nabla_{\mathbf{v}} H^{(1)}_{f(t)}\right)(X(t,0,\mathbf{z}), V(t,0,\mathbf{z})), & X(0,0,\mathbf{z}) &= \mathbf{x}, \\ \partial_t V(t,0,\mathbf{z}) &= -\left(\nabla_{\mathbf{x}} H^{(1)}_{f(t)}\right)(X(t,0,\mathbf{z}), V(t,0,\mathbf{z})), & V(0,0,\mathbf{z}) &= \mathbf{v}, \end{aligned}$$

where $H^{(1)}_{f(t)}$ is the kernel of the first functional derivative of the energy functional $\mathcal{H} : \mathcal{L}^1_{\mathbf{z}} \to \mathbb{R}$.

A helpful interpretation is that any infinitesimal mass $f(t,\mathbf{z})\,\mathrm{d}\mathbf{z}$ follows the trajectory of a unit mass test particle under the non-autonomous Hamiltonian $H^{(1)}_{f(t)}$ generated from the background $f(t)$. In this sense, the evolution equation is self-consistent.

On the macroscopic scale, the Vlasov equation describes a continuous cloud of matter, which on the microscopic scale consists of many point particles. It is therefore natural to search for a link to the classical many-body problem

$$\begin{aligned}\partial_t X_m(t,0,\mathbf{z}_1,\dots,\mathbf{z}_N) &= (\nabla_{\mathbf{v}_m} H_N)\left((X_1,V_1,\dots,X_N,V_N)(t,0,\mathbf{z}_1,\dots,\mathbf{z}_N)\right), \\ \partial_t V_m(t,0,\mathbf{z}_1,\dots,\mathbf{z}_N) &= -(\nabla_{\mathbf{x}_m} H_N)\left((X_1,V_1,\dots,X_N,V_N)(t,0,\mathbf{z}_1,\dots,\mathbf{z}_N)\right), \quad 1\le m\le N\end{aligned} \tag{4.1}$$

of N particles with autonomous Hamiltonian

$$H_N: \quad \mathbb{R}^{2dN}_{\vec{\mathbf{z}}} \to \mathbb{R}, \quad (\mathbf{z}_1,\dots,\mathbf{z}_N) \mapsto \sum_{m=1}^{N} \frac{|\mathbf{v}_m|^2}{2} + \frac{1}{2(N-1)} \sum_{\substack{m,n=1\\ m\neq n}}^{N} \Gamma(\mathbf{x}_m - \mathbf{x}_n). \tag{4.2}$$

The state of such a system of many indistinguishable particles is most efficiently described in a probabilistic manner. This can be achieved by assuming that the initial data have the joint distribution $\mathring{f}_N : \mathbb{R}^{2dN}_{\vec{\mathbf{z}}} \to \mathbb{R}_{\geq 0}$ and this distribution follows the trajectories $\vec{Z} \equiv (X_1,V_1,\dots,X_N,V_N)$ of the ODE system (4.1). From both these assumptions, one easily deduces that the joint distribution evolves according to the following **Liouville equation**

$$\begin{aligned} 0 &= \partial_t f_N(t,\vec{\mathbf{z}}) + \sum_{m=1}^{N} \left((\nabla_{\mathbf{x}_m} f_N)(t,\vec{\mathbf{z}}) \cdot (\nabla_{\mathbf{v}_m} H_N)(\vec{\mathbf{z}}) - (\nabla_{\mathbf{v}_m} f_N)(t,\vec{\mathbf{z}}) \cdot (\nabla_{\mathbf{x}_m} H_N)(\vec{\mathbf{z}}) \right) \\ &= \partial_t f_N(t,\vec{\mathbf{z}}) + [f_N(t), H_N]_N(\vec{\mathbf{z}}), \end{aligned} \tag{Lvl$^{\mathrm{N}}$}$$

where $[\cdot,\cdot]_N$ indicates the Poisson bracket on $\mathbb{R}^{2dN}_{\vec{\mathbf{z}}}$.

The link between the two distributions $f_N(t)$ and $f(t)$ can be heuristically well explained. While $f(t)$ is a deterministic model, $f_N(t)$ arises from a probabilistic interpretation. For $m = 1$ in (4.1) one finds that

$$\begin{pmatrix} \partial_t X_1 \\ \partial_t V_1 \end{pmatrix} = \begin{pmatrix} (\nabla_{\mathbf{v}_1} H_N)(\vec{Z}(t,0,\vec{\mathbf{z}})) \\ -(\nabla_{\mathbf{x}_1} H_N)(\vec{Z}(t,0,\vec{\mathbf{z}})) \end{pmatrix} = \begin{pmatrix} V_1 \\ -\frac{1}{N-1}\sum_{n=2}^{N} \nabla\Gamma(X_1 - X_n) \end{pmatrix}.$$

The second component is expected to behave like

$$-\frac{1}{N-1}\sum_{n=2}^{N} \nabla\Gamma(X_1 - X_n) \overset{N \text{ large}}{\approx} -(\nabla\Gamma * f(t))(X_1) \tag{4.3}$$

on regions of the phase space where $f_N(t) \gg 0$, since the marginal distributions of $\{X_2, \ldots, X_N\}$ should remain close to $f(t)$ if the initial condition is close to the independent identical distribution (i.i.d.) $\mathring{f}_N \approx \mathring{f}^{\otimes N}$.

Existing results are then often in this form: If the joint initial distribution $\mathring{f}_N$ converges to the distribution $\mathring{f}^{\otimes N}$, where each particle is i.i.d. $\propto \mathring{f}$, then for every $k \in \mathbb{N}$ and $t > 0$, the k marginals

$$f_N^{(k)}(t, \mathbf{z}_1, \ldots, \mathbf{z}_k) \equiv \int_{\mathbb{R}^{2d}_{\mathbf{z}}} \cdots \int_{\mathbb{R}^{2d}_{\mathbf{z}}} f_N(t, \vec{\mathbf{z}})\, \mathrm{d}\mathbf{z}_N \cdots \mathrm{d}\mathbf{z}_{k+1}$$

of $f_N(t)$ converge to the i.i.d. product $f(t)^{\otimes k}$ for $N \to \infty$ in some space of probability distributions. This was done for regular forces in the seminal paper [5] and has encouraged a rich literature, the main focus today lying on optimal regularization procedures for singular potentials, mostly Coulomb [10].

Guided by the general symplectization principle of any Hamiltonian Vlasov, one can now derive a Hamiltonian equation for both the mean field system (Vl) as well as the many-particle system ($\mathrm{Lvl^N}$). In the case of the mean field system, the result is the Hamiltonian Vlasov equation

$$\partial_t \alpha(t, \mathbf{z}) = \left[\frac{|\mathbf{v}|^2}{2} + \left(\Gamma * |\alpha(t)|^2\right), \alpha(t) \right](\mathbf{z}) - \left(\Gamma * [\bar{\alpha}(t), \alpha(t)]\right)(\mathbf{x})\, \alpha(t, \mathbf{z}). \tag{HVl}$$

Its equivalent Hamilton Hartree equation is obtained by the velocity Fourier transform (FT) of Equation (HVl)

$$\mathrm{i}\, \partial_t \hat{\alpha}(t, \hat{\mathbf{z}}) = \left(\nabla_{\mathbf{x}} \cdot \nabla_{\boldsymbol{\xi}} + \left(\hat{V} * |\hat{\alpha}(t)|^2\right)(\hat{\mathbf{z}})\right) \hat{\alpha}(t, \hat{\mathbf{z}}), \quad \hat{V}(\mathbf{x}, \boldsymbol{\xi}) \equiv -\nabla\Gamma(\mathbf{x}) \cdot \boldsymbol{\xi}, \tag{HHt}$$

as already outlined in the discussion of their global well-posedness in Section 3.1.

On the other hand, the energy functional for the N particle problem is

$$\mathcal{H}_N(f_N) = \int_{\mathbb{R}^{2dN}_{\vec{\mathbf{z}}}} H_N(\vec{\mathbf{z}})\, f_N(\vec{\mathbf{z}})\, \mathrm{d}\vec{\mathbf{z}}.$$

Applying the Vlasov symplectization scheme for $\mathcal{L}^1_{\vec{\mathbf{z}}}$ and $\mathbb{R}^{2dN}_{\vec{\mathbf{z}}}$, rather than $\mathcal{L}^1_{\mathbf{z}}$ and $\mathbb{R}^{2d}_{\mathbf{z}}$, yields the corresponding N particle Hamiltonian Vlasov functional

$$\mathcal{H}_{N,\mathrm{Vl}}(\alpha_N) \equiv \frac{1}{2\mathrm{i}} \mathrm{D}^1 \mathcal{H}_N\left(|\alpha_N|^2\right)([\bar{\alpha}_N, \alpha_N]_N) = \frac{1}{2\mathrm{i}} \int_{\mathbb{R}^{2dN}_{\vec{\mathbf{z}}}} H_N(\vec{\mathbf{z}})\ [\bar{\alpha}_N, \alpha_N]_N(\vec{\mathbf{z}})\, \mathrm{d}\vec{\mathbf{z}}.$$

This functional yields the **Hamiltonian Liouville equation**

$$\partial_t \alpha_N(t, \mathbf{z}_1, \ldots, \mathbf{z}_N) = [H_N, \alpha_N(t)]_N(\mathbf{z}_1, \ldots, \mathbf{z}_N), \tag{$\mathrm{HLv^N}$}$$

for α_N. Note that by the linearity of the energy functional $\mathcal{H}_N$, Equation HLv$^{\mathrm{N}}$ is the complex-valued $\mathcal{L}^2$ equivalent of (Lvl$^{\mathrm{N}}$).

Under the velocity Fourier transform $\mathcal{F}^{\otimes N}$ in every coordinate, Equation (HLv$^{\mathrm{N}}$) becomes

$$\mathrm{i}\,\partial_t \hat{\alpha}_N(t, \hat{\mathbf{z}}_1, \dots, \hat{\mathbf{z}}_N) = \left(\sum_{m=1}^{N} \nabla_{\mathbf{x}_m} \cdot \nabla_{\xi_m} + \frac{1}{2(N-1)} \sum_{n \neq m} \hat{V}(\hat{\mathbf{z}}_n - \hat{\mathbf{z}}_m) \right) \hat{\alpha}_N(t, \hat{\mathbf{z}}_1, \dots, \hat{\mathbf{z}}_N), \tag{PsQM$^{\mathrm{N}}$}$$

which is called the **Pseudo-Schrödinger system**, since its structure is similar to the bosonic many-particle Schrödinger equation, the only difference being in the kinetic term and the underlying space, which is $\mathbb{R}^{2dN}_{\hat{\vec{\mathbf{z}}}}$ as opposed to $\mathbb{R}^{dN}_{\vec{\mathbf{x}}}$. In agreement with bosonic systems we are only interested in the **symmetric** states $\hat{\alpha}_N$, that is, states which are invariant under coordinate permutations.

The connection between the various systems introduced above is nicely represented in a schematic, *commutative* diagram, as seen in Fig. 4.2. The diagram indicates two key features.

At first, the structural analogy between the Hamilton Hartree / Pseudo-Schrödinger ensemble and the quantum mechanical Hartree / Schrödinger system offers exciting opportunities to develop a mean field limit similar to the one for bosonic quantum systems. Up to some restrictions and adaptations, it is possible to follow the guidelines of [18], where a quantum mean field limit of bosonic many-particle Schrödinger to the Hartree equation is proven. Following the strategy presented therein we show that $\hat{\alpha}(t)$ arises as a mean field limit from a sequence of many-particle pseudo wave functions $\hat{\alpha}_N(t)$, solving (PsQM$^{\mathrm{N}}$). The sense of convergence is given by the average number of particles in the mean field state. This convergence also implies convergence of the respective pure density matrices in the operator norm. The computations are rigorously carried out in Section 4.3.

Secondly, the diagram shows, that any mean field limit in the Hamiltonian Vlasov or Pseudo-Hartree picture yields a mean field limit for the classical Vlasov case by reversing the velocity Fourier transform and mapping $\alpha \mapsto |\alpha|^2$. It is a peculiar example, where quantum $\mathcal{L}^2$ methods can be used to infer properties of a classical system. Due to the reversibility of the velocity Fourier transform, the $\mathcal{L}^2$ mean field limits in both pictures are equivalent.

Mean field		Many-particle
Classical Vlasov $f(\mathbf{x}, \mathbf{v})$ $\mathcal{H}(f)$ $\partial_t f = \left[H_f, f\right]$	stochastic many particle model $\rightharpoonup$ $\leftharpoondown$ convergence of marginal distributions	**Liouville** $f_N(\vec{\mathbf{x}}, \vec{\mathbf{v}})$ $\mathcal{H}_N(f_N)$ linear $\partial_t f_N = [H_N, f_N]_N$
$\uparrow$ $f=\lvert\alpha\rvert^2$		$\uparrow$ $f_N=\lvert\alpha_N\rvert^2$
Hamiltonian Vlasov $\alpha(\mathbf{x}, \mathbf{v})$ $\mathcal{H}_{\mathrm{Vl}}(\alpha)$ $\partial_t \alpha = \left[H_{\lvert\alpha\rvert^2}, \alpha\right] - (\Gamma * [\bar{\alpha}, \alpha]) \cdot \alpha$		***N* Hamiltonian Liouville** $\alpha_N(\vec{\mathbf{x}}, \vec{\mathbf{v}})$ $\mathcal{H}_{N,\mathrm{Vl}}(\alpha_N)$ quadratic $\partial_t \alpha_N = [H_N, \alpha_N]_N$
$\updownarrow$ F.T.		$\updownarrow$ F.T.
Hamilton Hartree $\hat{\alpha}(\mathbf{x}, \boldsymbol{\xi})$ $\mathcal{H}_{\mathrm{Ht}}(\hat{\alpha})$ $\mathrm{i}\partial_t \hat{\alpha} = \left(\nabla_{\mathbf{x}} \cdot \nabla_{\boldsymbol{\xi}} + \left(\hat{V} * \lvert\hat{\alpha}\rvert^2\right)\right) \hat{\alpha}$	many-particle bosonic Schrödinger $\rightharpoonup$ $\leftharpoondown$ Thm. 4.12	***N* Pseudo-Schrödinger** $\hat{\alpha}_N(\vec{\mathbf{x}}, \vec{\boldsymbol{\xi}})$ $\mathcal{H}_{N,\mathrm{Ht}}(\hat{\alpha}_N)$ $\mathrm{i}\partial_t \hat{\alpha}_N = \left(\nabla_{\vec{\mathbf{x}}} \cdot \nabla_{\vec{\boldsymbol{\xi}}} + \frac{1}{N} \sum_{n,m} \hat{V}_{nm}\right) \hat{\alpha}_N$

Figure 4.2: The hierarchy of the discussed equations with their name, phase space variable, energy functional, and equation of motion. Expressions in $[\cdot, \cdot]$ and $[\cdot, \cdot]_N$ denote the Poisson bracket on $\mathbb{R}^{2d}_{\mathbf{z}}$ and $\mathbb{R}^{2dN}_{\vec{\mathbf{z}}}$ respectively. All mean field theories are in the left column, all many-particle systems in the right one. The velocity Fourier transform (F.T.) is reversible and therefore yields equivalent equations. The Vlasov mean field theory in the first row is widely discussed in the literature, e.g. [5, 10]. The dashed lines indicate newly established results.

4.2 Solutions and Estimates

In preparation of the mean field result in Section 4.3, we need to state some properties and estimates of the solutions of the many-particle system (HLv$^{\mathrm{N}}$). Although not explicitly mentioned here, the results clearly translate to solutions of (PsQM$^{\mathrm{N}}$) through properties of the velocity Fourier transform (FT).

Theorem 4.1 (Global Well-Posedness for the Many-Particle System)**.** *Under the assumptions in* (Pot)*, the Liouville equation* (HLv$^{\mathrm{N}}$) *has a global solution for any $N \in \mathbb{N}$ and any initial value. The solution preserves regularity of the initial state up to $C^2_{\vec{\mathbf{z}}}$, as* *well as its invariance under particle permutations. If the solution is classical in $C^1_{(t,\vec{\mathbf{z}})}$, it is unique therein. Any two solutions keep their $\mathcal{L}^2_{\vec{\mathbf{z}}}$ distance constant for all times.*

Proof. In fact, (HLv$^{\mathrm{N}}$) is a linear transport equation and its characteristics are independent of the initial value $\mathring{\alpha}_N$. They are given by the solution map of

$$\partial_t X_m(t,s,\mathbf{z}) = V_m(t,s,\mathbf{z}), \quad \partial_t V_m(t,s,\mathbf{z}) = \frac{1}{N-1}\sum_{n\neq m} \nabla\Gamma\left(X_n(t,s,\mathbf{z}) - X_m(t,s,\mathbf{z})\right).$$

Assumption (Pot) assures that the solution map $\vec{Z}_N = (X_1, V_1, \ldots, X_N, V_N)$ is well-defined on $\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{2dN}_{\vec{\mathbf{z}}}$ and, because Γ is $C^3_{\mathbf{x}}$, any transformation $\vec{Z}_N(t,s)$ is $C^2_{\vec{\mathbf{z}}}$. Hence, for any initial datum $\mathring{\alpha}_N \in \mathcal{L}^2_{\vec{\mathbf{z}}}$, there is a solution given by $\alpha_N(t,\vec{\mathbf{z}}) \equiv \mathring{\alpha}_N(\vec{Z}_N(0,t,\vec{\mathbf{z}}))$. The solution is classical if $\mathring{\alpha}_N \in C^1_{\vec{\mathbf{z}}}$. If the initial datum is in $C^2_{\vec{\mathbf{z}}}$ or $\mathcal{W}^{2,2}_{\vec{\mathbf{z}}}$, the solution remains in that respective space.

In addition, the Hamiltonian structure of the characteristic equations implies that all $\mathcal{L}^p_{\vec{\mathbf{z}}}$ norms are conserved because $\vec{Z}_N$ is a family of volume preserving symplectomorphisms. Moreover, since the characteristic system is symmetric, i.e., it is invariant under permutation of particles, such symmetric initial states conserve this property under time evolution.

The uniqueness of classical solutions is immediate, because the transport equation and the characteristic equations are equivalent on that class.

For the last claim, we remark that all solutions arise from composition with the very same solution map. It is volume preserving due to its Hamiltonian structure. Therefore one obtains $\|\alpha_N(t) - \beta_N(t)\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}} = \left\|\mathring{\alpha}_N - \mathring{\beta}_N\right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}}$ for any two initial data $\mathring{\alpha}_N, \mathring{\beta}_N$ and their respective solutions. □

Proving a mean field limit requires uniform bounds in N on the first and second phase space derivatives of the solution. They can be naturally obtained by a respective condition on the sequence of N particle Hamiltonians.

Definition 4.2 (Mean Field Consistency)**.** Let $\{H_N : \mathbb{R}^{2dN}_{\vec{\mathbf{z}}} \to \mathbb{R}\}_{N\in\mathbb{N}}$ be a sequence of Hamiltonian functions on the N particle phase spaces. Then the functions H_N are **mean field consistent**, if and only if

(i) each H_N is invariant under particle permutation and

(ii) there are constants $C_1, C_2, C_3 > 0$, s.t. for any $N \geq 2$ and any permutation invariant

$\alpha_N \in \mathcal{W}^{2,2}_{\vec{\mathbf{z}}}$, the following inequalities hold:

$$\begin{aligned}\left\|\left[\mathrm{D}^1_{\mathbf{z}_1} H_N, \alpha_N\right]_N\right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}} &\leq C_1 \left\|\mathrm{D}^1_{\mathbf{z}_1}\alpha_N\right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}},\\ \left\|\left[\mathrm{D}^2_{(\mathbf{z}_1,\mathbf{z}_2)} H_N, \alpha_N\right]_N\right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}} &\leq C_2 \left\|\mathrm{D}^1_{\mathbf{z}_1}\alpha_N\right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}},\\ \left\|\left[\mathrm{D}^1_{(\mathbf{z}_1,\mathbf{z}_2)} H_N, \mathrm{D}^1_{(\mathbf{z}_1,\mathbf{z}_2)}\alpha_N\right]_N\right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}} &\leq C_3 \left\|\mathrm{D}^2_{(\mathbf{z}_1,\mathbf{z}_2)}\alpha_N\right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}}.\end{aligned} \tag{MFC}$$

This definition is tailored for our purposes.

Lemma 4.3 (Non-Relativistic Two-Body Interaction)**.** *The sequence of Hamiltonians* (4.2) *along with Condition* (Pot) *is mean field consistent.*

Proof. For any $N \geq 2$ and α_N permutation invariant, w.l.o.g. $C^\infty_{\vec{\mathbf{z}}}$ and compactly supported, we have that

$$\begin{aligned}\left\|\left[\mathrm{D}^1_{\mathbf{z}_1} H_N, \alpha_N\right]_N\right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}} &= \left(\int_{\mathbb{R}^{2dN}_{\vec{\mathbf{z}}}} \left|\left[\begin{pmatrix} \frac{1}{N-1}\sum_{m=2}^N \nabla\Gamma(\mathbf{x}_1-\mathbf{x}_m) \\ \mathbf{v}_1\end{pmatrix}, \alpha_N\right]_N\right|^2 \mathrm{d}\vec{\mathbf{z}}\right)^{\frac{1}{2}}\\ &\overset{(*)}{=} \left(\int_{\mathbb{R}^{2dN}_{\vec{\mathbf{z}}}} \sum_{i=1}^d \left(\left|\partial_{x_{1,i}}\alpha_N\right|^2 + \left|\sum_{j=1}^d \frac{1}{N-1}\sum_{m=2}^N \partial_i\partial_j\Gamma(\mathbf{x}_1-\mathbf{x}_m)\left(\partial_{v_{1,j}}\alpha_N - \partial_{v_{m,j}}\alpha_N\right)\right|^2\right)\mathrm{d}\vec{\mathbf{z}}\right)^{\frac{1}{2}}\\ &\overset{(**)}{\leq} \left\|\mathrm{D}^1_{\mathbf{x}_1}\alpha_N\right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}} + 2\left\|\mathrm{D}^2\Gamma\right\|_{\mathcal{L}^\infty_{\mathbf{x}}}\left\|\mathrm{D}^1_{\mathbf{v}_1}\alpha_N\right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}} \leq \left(1+2\left\|\mathrm{D}^2\Gamma\right\|_{\mathcal{L}^\infty_{\mathbf{x}}}\right)\left\|\mathrm{D}^1_{\mathbf{z}_1}\alpha_N\right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}},\end{aligned}$$

where at $(*)$ we evaluate the Poisson bracket and norm square and at $(**)$ apply the matrix operator norm of $\mathrm{D}^2\Gamma$, as well as exploit the permutation invariance of α_N in order to use the derivative on the coordinates of the first particle.

The computations for the other two inequalities of (MFC) can be carried out very similarly. The bounds on the first three derivatives of Γ are crucial here. By standard approximation, these inequalities are extended to permutation invariant functions of $\mathcal{W}^{2,2}_{\vec{\mathbf{z}}}$. □

Lemma 4.4. *Let* $\alpha_N : \mathbb{R}\times\mathbb{R}^{2dN}_{\vec{\mathbf{z}}} \to \mathbb{C}$ *be an invariant w.r.t. particle permutation solution of* (HLv$^{\mathrm{N}}$)*,* $\mathring{\alpha}_N = \alpha_N(0) \in \mathcal{W}^{2,2}_{\vec{\mathbf{z}}}$*,* $\left\|\mathrm{D}^1_{\mathbf{z}_1}\mathring{\alpha}_N\right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}}, \left\|\mathrm{D}^2_{(\mathbf{z}_1,\mathbf{z}_2)}\mathring{\alpha}_N\right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}} \leq M$*. Let* H_N *be mean field consistent. Then for any* $N \geq 2$ *with* $C_1, C_2, C_3 > 0$ *from* (MFC)*, we have that*

$$\left\|\mathrm{D}^1_{\mathbf{z}_1}\alpha_N(t)\right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}} \leq \left\|\mathrm{D}^1_{\mathbf{z}_1}\mathring{\alpha}_N\right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}} e^{\frac{1}{2}\left(1+C_1^2\right)t} \leq M e^{\frac{1}{2}\left(1+C_1^2\right)t} \equiv \mathfrak{b}^{\mathcal{L}^2_{\vec{\mathbf{z}}}}_{M,\mathrm{D}^1_{\mathbf{z}_1}}(t),$$

$$\left\|\mathrm{D}^2_{(\mathbf{z}_1,\mathbf{z}_2)}\alpha_N(t)\right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}} \leq M\left(1+\frac{(4d)^2C_2^2}{1+C_1^2}\left(e^{\left(1+C_1^2\right)t}-1\right)\right)^{\frac{1}{2}} e^{(4d)^2\left(\frac{3}{2}+C_3^2\right)t} \equiv \mathfrak{b}^{\mathcal{L}^2_{\vec{\mathbf{z}}}}_{M,\mathrm{D}^2_{(\mathbf{z}_1,\mathbf{z}_2)}}(t).$$

Proof. Let α_N be the solution for some permutation invariant initial datum $\mathring{\alpha}_N$, which for now is supposed to be $C^\infty_{\vec{\mathbf{z}}}$ with compact support. Then for any $t \geq 0$, we find that

$$
\begin{aligned}
&\left\|\mathrm{D}^1_{\mathbf{z}_1}\alpha_N(t)\right\|^2_{\mathcal{L}^2_{\vec{\mathbf{z}}}} - \left\|\mathrm{D}^1_{\mathbf{z}_1}\mathring{\alpha}_N\right\|^2_{\mathcal{L}^2_{\vec{\mathbf{z}}}} \\
&= \int_0^t \partial_\tau \left\|\mathrm{D}^1_{\mathbf{z}_1}\alpha_N(\tau)\right\|^2_{\mathcal{L}^2_{\vec{\mathbf{z}}}} \mathrm{d}\tau = 2\Re \int_0^t \left\langle \mathrm{D}^1_{\mathbf{z}_1}\alpha_N(\tau), \partial_\tau \mathrm{D}^1_{\mathbf{z}_1}\alpha_N(\tau)\right\rangle \mathrm{d}\tau \\
&= 2\Re \int_0^t \int_{\mathbb{R}^{2dN}_{\vec{\mathbf{z}}}} \left(\mathrm{D}^1_{\mathbf{z}_1}\bar{\alpha}_N(\tau,\vec{\mathbf{z}}) \cdot \mathrm{D}^1_{\mathbf{z}_1}\left[H_N, \alpha_N(\tau)\right]_N(\vec{\mathbf{z}})\right) \mathrm{d}\vec{\mathbf{z}}\,\mathrm{d}\tau \\
&= \int_0^t \underbrace{\int_{\mathbb{R}^{2dN}_{\vec{\mathbf{z}}}} \left[H_N, \left|\mathrm{D}^1_{\mathbf{z}_1}\alpha_N(\tau)\right|^2\right]_N(\vec{\mathbf{z}})\,\mathrm{d}\vec{\mathbf{z}}}_{=0}\,\mathrm{d}\tau \\
&\quad + 2\Re \int_0^t \int_{\mathbb{R}^{2dN}_{\vec{\mathbf{z}}}} \left(\mathrm{D}^1_{\mathbf{z}_1}\bar{\alpha}_N(\tau,\vec{\mathbf{z}}) \cdot \left[\mathrm{D}^1_{\mathbf{z}_1}H_N, \alpha_N(\tau)\right]_N(\vec{\mathbf{z}})\right) \mathrm{d}\vec{\mathbf{z}}\,\mathrm{d}\tau \\
&\leq 2\int_0^t \left\|\mathrm{D}^1_{\mathbf{z}_1}\alpha_N(\tau)\right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}} \left\|\left[\mathrm{D}^1_{\mathbf{z}_1}H_N, \alpha_N(\tau)\right]_N\right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}} \mathrm{d}\tau \\
&\leq \int_0^t \left(\left\|\mathrm{D}^1_{\mathbf{z}_1}\alpha_N(\tau)\right\|^2_{\mathcal{L}^2_{\vec{\mathbf{z}}}} + \left\|\left[\mathrm{D}^1_{\mathbf{z}_1}H_N, \alpha_N(\tau)\right]_N\right\|^2_{\mathcal{L}^2_{\vec{\mathbf{z}}}}\right) \mathrm{d}\tau \\
&\leq (1 + C_1^2)\int_0^t \left\|\mathrm{D}^1_{\mathbf{z}_1}\alpha_N(\tau)\right\|^2_{\mathcal{L}^2_{\vec{\mathbf{z}}}} \mathrm{d}\tau,
\end{aligned}
$$

yielding the first claim by Grönwall's Lemma. For the second claim due to differentiability constraints, we need to replace the matrix operator norm $|\cdot|$ with the trace norm first:

$$
\forall A \in \mathbb{C}^{4d\times 4d}: \quad |A|_{\mathrm{tr}} \equiv \left(\mathrm{tr}\,(A^*A)\right)^{\frac{1}{2}}, \quad \text{implying} \quad |A| \leq |A|_{\mathrm{tr}} \leq 4d\,|A|\,.
$$

With $\|\cdot\|_* \equiv \||\cdot|_{\mathrm{tr}}\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}}$, we have in a similar way

$$
\begin{aligned}
&\left\|\mathrm{D}^2_{(\mathbf{z}_1,\mathbf{z}_2)}\alpha_N(t)\right\|^2_* - \left\|\mathrm{D}^2_{(\mathbf{z}_1,\mathbf{z}_2)}\mathring{\alpha}_N\right\|^2_* = \int_0^t \partial_\tau \left\|\mathrm{D}^2_{(\mathbf{z}_1,\mathbf{z}_2)}\alpha_N(\tau)\right\|^2_* \mathrm{d}\tau \\
&= 2\Re\int_0^t \left\langle \mathrm{D}^2_{(\mathbf{z}_1,\mathbf{z}_2)}\alpha_N(\tau), \mathrm{D}^2_{(\mathbf{z}_1,\mathbf{z}_2)}\partial_\tau\alpha_N(\tau)\right\rangle \mathrm{d}\tau = \int_0^t \underbrace{\int_{\mathbb{R}^{2dN}_{\vec{\mathbf{z}}}} \left[H_N, \left|\mathrm{D}^2_{(\mathbf{z}_1,\mathbf{z}_2)}\alpha_N(\tau)\right|^2_{\mathrm{tr}}\right](\vec{\mathbf{z}})\,\mathrm{d}\vec{\mathbf{z}}}_{=0}\,\mathrm{d}\tau \\
&\quad + 2\Re\int_0^t \left\langle \mathrm{D}^2_{(\mathbf{z}_1,\mathbf{z}_2)}\alpha_N(\tau), 2\left[\mathrm{D}^1_{(\mathbf{z}_1,\mathbf{z}_2)}H_N, \mathrm{D}^1_{(\mathbf{z}_1,\mathbf{z}_2)}\alpha_N(\tau)\right] + \left[\mathrm{D}^2_{(\mathbf{z}_1,\mathbf{z}_2)}H_N, \alpha_N(\tau)\right]\right\rangle \mathrm{d}\tau \\
&\leq 2\int_0^t \left\|\mathrm{D}^2_{(\mathbf{z}_1,\mathbf{z}_2)}\alpha_N(\tau)\right\|_* \left(2\left\|\left[\mathrm{D}^1_{(\mathbf{z}_1,\mathbf{z}_2)}H_N, \mathrm{D}^1_{(\mathbf{z}_1,\mathbf{z}_2)}\alpha_N(\tau)\right]\right\|_* + \left\|\left[\mathrm{D}^2_{(\mathbf{z}_1,\mathbf{z}_2)}H_N, \alpha_N(\tau)\right]\right\|_*\right) \mathrm{d}\tau
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_0^t \Big(3 \left\| \mathrm{D}^2_{(\mathbf{z}_1,\mathbf{z}_2)} \alpha_N(\tau) \right\|_*^2 + 2 \left\| \left[\mathrm{D}^1_{(\mathbf{z}_1,\mathbf{z}_2)} H_N, \mathrm{D}^1_{(\mathbf{z}_1,\mathbf{z}_2)} \alpha_N(\tau) \right] \right\|_*^2 + \left\| \left[\mathrm{D}^2_{(\mathbf{z}_1,\mathbf{z}_2)} H_N, \alpha_N(\tau) \right] \right\|_*^2 \Big) \, \mathrm{d}\tau \\
&\leq (4d)^2 \int_0^t \Big(3 \left\| \mathrm{D}^2_{(\mathbf{z}_1,\mathbf{z}_2)} \alpha_N(\tau) \right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}}^2 + 2 \left\| \left[\mathrm{D}^1_{(\mathbf{z}_1,\mathbf{z}_2)} H_N, \mathrm{D}^1_{(\mathbf{z}_1,\mathbf{z}_2)} \alpha_N(\tau) \right] \right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}}^2 \\
&\qquad + \left\| \left[\mathrm{D}^2_{(\mathbf{z}_1,\mathbf{z}_2)} H_N, \alpha_N(\tau) \right] \right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}}^2 \Big) \, \mathrm{d}\tau \\
&\leq (4d)^2 C_2^2 \int_0^t \left\| \mathrm{D}^1_{\mathbf{z}_1} \alpha_N(\tau) \right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}}^2 \, \mathrm{d}\tau + (4d)^2 (3 + 2C_3^2) \int_0^t \left\| \mathrm{D}^2_{(\mathbf{z}_1,\mathbf{z}_2)} \alpha_N(\tau) \right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}}^2 \, \mathrm{d}\tau \\
&\leq (4d)^2 C_2^2 \int_0^t \left\| \mathrm{D}^1_{\mathbf{z}_1} \alpha_N(\tau) \right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}}^2 \, \mathrm{d}\tau + (4d)^2 (3 + 2C_3^2) \int_0^t \left\| \mathrm{D}^2_{(\mathbf{z}_1,\mathbf{z}_2)} \alpha_N(\tau) \right\|_*^2 \, \mathrm{d}\tau.
\end{aligned}
$$

As intermediate steps in the computation require $C^3_{\vec{\mathbf{z}}}$ regularity of α_N, it is necessary to extend this inequality in two steps. At first, consider a smoothened potential $\Gamma_\varepsilon \to \Gamma$ in the Hamiltonian H_N, s.t. the solution of the characteristic system is $C^3_{\vec{\mathbf{z}}}$ at least. As the support of α_N is compact in $I \times \mathbb{R}^{2dN}_{\vec{\mathbf{z}}}$ for any compact time interval $I \subset \mathbb{R}$, both sides of this inequality converge. In a second step, the inequality can be extended to $\mathcal{W}^{2,2}_{\vec{\mathbf{z}}}$, because the computation actually proves boundedness of the linear map

$$
\mathcal{W}^{2,2}_{\vec{\mathbf{z}}} \to \mathcal{L}^\infty_{t,\mathrm{loc}} \mathcal{L}^2_{\vec{\mathbf{z}}}, \quad \mathring{\alpha}_N \mapsto \mathrm{D}^2_{(\mathbf{z}_1,\mathbf{z}_2)} \alpha_N, \qquad \square
$$

once recalling that $|\cdot| \leq |\cdot|_{\mathrm{tr}}$ implies $\|\cdot\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}} \leq \|\cdot\|_*$.

4.3 The Counting Method

This section is devoted to stating and proving the mean field convergence, proving that the Hamilton Hartree system (HHt) arises as effective limit for the pseudo Schrödinger system (PsQM$^{\mathrm{N}}$) and, through the duality of the velocity Fourier transform, the Hamiltonian Vlasov equation (HVl) is the limit for the Liouville equation (HLv$^{\mathrm{N}}$), respectively.

The mean field derivation heavily follows ideas and uses techniques of [18] with slight adjustments to fit the new requirements of an unbounded, non-integrable pair interaction potential. For the sake of completeness, the key definitions are included here. The main result is Theorem 4.12 at the end of the section.

Given the quantum type nature of the result, the effective one-particle state $\hat{\alpha}$ lives in $\mathcal{L}^2_{\hat{\mathbf{z}}}$, while the many-particle state $\hat{\alpha}_N$ is an element of the functions $\mathcal{L}^2_{\vec{\hat{\mathbf{z}}}} \equiv \bigotimes_{m=1}^N \mathcal{L}^2_{\hat{\mathbf{z}}_m}$. As mean field analysis is only reasonable for large collections of indistinguishable particles, a natural restriction is to demand that the many-particle state is invariant under particle permutation, also referred to as being symmetric. In addition, all states are consistently assumed to be normalized in $\|\cdot\|_{\mathcal{L}^2}$.

Definition 4.5 (N Particle Projections and Counting Operators)**.** [18, Def.2.1] Fix some normalized function $\hat{\alpha} \in \mathcal{L}^2_{\hat{\mathbf{z}}}$ with $\|\hat{\alpha}\|_{\mathcal{L}^2_{\hat{\mathbf{z}}}} = 1$ and some positive integer $N \in \mathbb{N}$.

(i) For $j \in \{1, \ldots, N\}$ and $k \in \mathbb{Z}$, define the projections in $\mathcal{L}^2_{\vec{\hat{\mathbf{z}}}}$ by

$$\begin{aligned} \left(p^{\hat{\alpha}}_j \hat{\alpha}_N\right)(\vec{\hat{\mathbf{z}}}) &\equiv \hat{\alpha}(\hat{\mathbf{z}}_j) \int_{\mathbb{R}^{2d}_{\hat{\mathbf{z}}}} \bar{\hat{\alpha}}(\tilde{\hat{\mathbf{z}}}_j)\, \hat{\alpha}_N(\hat{\mathbf{z}}_1, \ldots, \tilde{\hat{\mathbf{z}}}_j, \ldots, \hat{\mathbf{z}}_N)\, \mathrm{d}\tilde{\hat{\mathbf{z}}}_j, \\ q^{\hat{\alpha}}_j &\equiv \mathrm{id} - p^{\hat{\alpha}}_j, \\ P^{\hat{\alpha}}_{k,j} &\equiv \sum_{\substack{\mathbf{a} \in \{0,1\}^j \\ \sum_i a_i = k}} \prod_{i=1}^{j} \left(p^{\hat{\alpha}}_{N-j+i}\right)^{1-a_i} \left(q^{\hat{\alpha}}_{N-j+i}\right)^{a_i}, \end{aligned}$$

all commuting pairwisely.

(ii) Now let $f : \{0, \ldots, N\} \to \mathbb{R}$ be a map. We define the associated operator on $\mathcal{L}^2_{\vec{\hat{\mathbf{z}}}}$ by

$$f^{\hat{\alpha}} \equiv \sum_{k=0}^{N} f(k)\, P^{\hat{\alpha}}_{k,N},$$

i.e., f defines the spectral decomposition of $f^{\hat{\alpha}}$ and the operator commutes with all projections from (i). In addition, for $l \in \mathbb{Z}$, define the shifted operator

$$\left(f^{\hat{\alpha}}\right)_l \equiv \sum_{k=l}^{N+l} f(k-l)\, P^{\hat{\alpha}}_{k,N} = \sum_{k=0}^{N} f(k) P^{\hat{\alpha}}_{k+l,N}.$$

(iii) In particular, we define for any $\lambda \in [0, 1]$ the **counting functions** on $\{0, \ldots, N\}$, namely,

$$\mathfrak{n}(k) \equiv \sqrt{\frac{k}{N}} \quad \text{and} \quad \mathfrak{m}_\lambda(k) \equiv \min\left\{\frac{k}{N^\lambda}, 1\right\}.$$

Remark 4.6. **(i).** The operator $p^{\hat{\alpha}}_j$ projects onto the subspace of functions with the j-th particle in state $\hat{\alpha}$. Their symmetric product $P^{\hat{\alpha}}_{k,j}$ projects onto the subspace, where exactly $j - k$ out of the last j particles are in state $\hat{\alpha}$ and the other k particles are in an orthogonal state.

(ii). Note, that $P^{\hat{\alpha}}_{k,j} \neq 0$ only for $0 \leq k \leq j \leq N$. In addition, $\sum_{k=0}^{j} P^{\hat{\alpha}}_{k,j} = \mathrm{id}$.

(iii). The expectation value $\left\|\mathfrak{n}^{\hat{\alpha}} \hat{\alpha}_N\right\|^2 = \left\langle \hat{\alpha}_N, (\mathfrak{n}^{\hat{\alpha}})^2 \hat{\alpha}_N \right\rangle$ counts the share of particles from $\hat{\alpha}_N$ not in state $\hat{\alpha}$. Since $\mathfrak{n}(k)^2 \leq \mathfrak{m}_\lambda(k)$, the quantity $\beta_N \equiv \left\langle \hat{\alpha}_N, \mathfrak{m}^{\hat{\alpha}}_\lambda \hat{\alpha}_N \right\rangle$ is an upper bound for that share.

The following lemma summarizes important computation rules for these operators.

Lemma 4.7. *[18, Lem.2.3] Let $N \in \mathbb{N}$ be fixed and $\hat{\alpha} \in \mathcal{L}^2_{\hat{\mathbf{z}}}$ be normalized.*

(i) For any two $f, g : \{0, \dots, N\} \to \mathbb{R}_{\geq 0}$ we have

$$f^{\hat{\alpha}}\, g^{\hat{\alpha}} = g^{\hat{\alpha}}\, f^{\hat{\alpha}}, \quad f^{\hat{\alpha}} p_j^{\hat{\alpha}} = p_j^{\hat{\alpha}} f^{\hat{\alpha}}, \quad f^{\hat{\alpha}} P_{k,j}^{\hat{\alpha}} = P_{k,j}^{\hat{\alpha}} f^{\hat{\alpha}} \quad \forall j, k.$$

(ii) The following identities hold

$$\frac{1}{N}\sum_{j=1}^{N} q_j^{\hat{\alpha}} = \frac{1}{N}\sum_{j=1}^{N} q_j^{\hat{\alpha}} \sum_{k=0}^{N} P_{k,N}^{\hat{\alpha}} = \sum_{k=0}^{N} \frac{k}{N}\, P_{k,N}^{\hat{\alpha}} = \left(\mathfrak{n}^{\hat{\alpha}}\right)^2.$$

(iii) For any $f : \{0, \dots, N\} \to \mathbb{R}_{\geq 0}$ and $\alpha_N \in \mathcal{L}^2_{\vec{\hat{\mathbf{z}}}}$ symmetric, we have

$$\left\|f^{\hat{\alpha}} q_1^{\hat{\alpha}} \hat{\alpha}_N\right\| = \left\|f^{\hat{\alpha}} \mathfrak{n}^{\hat{\alpha}} \hat{\alpha}_N\right\| \quad \text{and} \quad \left\|f^{\hat{\alpha}} q_1^{\hat{\alpha}} q_2^{\hat{\alpha}} \hat{\alpha}_N\right\| \leq \sqrt{\frac{N}{N-1}} \left\|f^{\hat{\alpha}} \left(\mathfrak{n}^{\hat{\alpha}}\right)^2 \hat{\alpha}_N\right\|.$$

(iv) For any $f : \{0, \dots, N\} \to \mathbb{R}_{\geq 0}$, any multiplication operator $\hat{v} : \left(\mathbb{R}^{2d}_{\hat{\mathbf{z}}}\right)^2 \to \mathbb{R}$, and any $j, k \in \{0, 1, 2\}$, we have

$$f^{\hat{\alpha}} Q_j^{\hat{\alpha}}\, \hat{v}(\hat{\mathbf{z}}_1, \hat{\mathbf{z}}_2)\, Q_k^{\hat{\alpha}} = Q_j^{\hat{\alpha}}\, \hat{v}(\hat{\mathbf{z}}_1, \hat{\mathbf{z}}_2)\, Q_k^{\hat{\alpha}} \left(f^{\hat{\alpha}}\right)_{k-j},$$

where $Q_0^{\hat{\alpha}} = p_1^{\hat{\alpha}} p_2^{\hat{\alpha}}$, $Q_1^{\hat{\alpha}} = p_1^{\hat{\alpha}} q_2^{\hat{\alpha}}$, and $Q_2^{\hat{\alpha}} = q_1^{\hat{\alpha}} q_2^{\hat{\alpha}}$.

Proof. See reference. □

Lemma 4.8. *Let $\hat{\alpha}_N : \mathbb{R}_{\geq 0} \to \mathcal{M}^1_{\vec{\hat{\mathbf{z}}}}$ be some $\mathcal{L}^2_{\vec{\hat{\mathbf{z}}}}$ normalized symmetric solution of* (PsQM$^\text{N}$). *Let $\hat{\alpha} : \mathbb{R}_{\geq 0} \to \mathcal{M}^1_{\hat{\mathbf{z}}}$ be some $\mathcal{L}^2_{\hat{\mathbf{z}}}$ normalized solution of* (HHt). *Then we have*

$$\begin{aligned}&\left\langle \hat{\alpha}_N(t), \mathfrak{m}_\lambda^{\hat{\alpha}(t)} \hat{\alpha}_N(t)\right\rangle - \left\langle \hat{\alpha}_N(0), \mathfrak{m}_\lambda^{\hat{\alpha}(0)} \hat{\alpha}_N(0)\right\rangle \\ &= N \int_0^t \Im \left\langle \hat{\alpha}_N(\tau), \mathfrak{m}_\lambda^{\hat{\alpha}(\tau)} \left(\hat{V}(\hat{\mathbf{z}}_1 - \hat{\mathbf{z}}_2) - \left(\hat{V} * |\hat{\alpha}(\tau)|^2\right)(\hat{\mathbf{z}}_1) - \left(\hat{V} * |\hat{\alpha}(\tau)|^2\right)(\hat{\mathbf{z}}_2)\right) \hat{\alpha}_N(\tau)\right\rangle \,\mathrm{d}\tau.\end{aligned}$$

Proof. At first we assume that $\hat{\alpha}(0)$ and $\hat{\alpha}_N(0)$ are Schwartz functions. Their smoothness and decay properties are then uniform on compact time intervals. The time derivative then can be pulled into the inner product and be explicitly evaluated. Equations (HHt) and (PsQM$^\text{N}$) immediately apply, that is,

$$\begin{aligned}\left\langle \hat{\alpha}_N(t), \mathfrak{m}_\lambda^{\hat{\alpha}(t)} \hat{\alpha}_N(t)\right\rangle - \left\langle \hat{\alpha}_N(0), \mathfrak{m}_\lambda^{\hat{\alpha}(0)} \hat{\alpha}_N(0)\right\rangle &= \int_0^t \partial_\tau \left\langle \hat{\alpha}_N(\tau), \mathfrak{m}_\lambda^{\hat{\alpha}(\tau)} \hat{\alpha}_N(\tau)\right\rangle \,\mathrm{d}\tau \\ = \int_0^t \left(2\Re \left\langle \hat{\alpha}_N(\tau), \mathfrak{m}_\lambda^{\hat{\alpha}(\tau)} \partial_\tau \hat{\alpha}_N(\tau)\right\rangle + \left\langle \hat{\alpha}_N(\tau), \left(\partial_\tau \mathfrak{m}_\lambda^{\hat{\alpha}(\tau)}\right) \hat{\alpha}_N(\tau)\right\rangle\right) \mathrm{d}\tau &= (*).\end{aligned}$$

Utilizing the short hand notation $\hat{H}_m^{\hat{\alpha}} \equiv \nabla_{\mathbf{x}_m} \cdot \nabla_{\xi_m} + \left(\hat{V} * |\hat{\alpha}|^2\right)(\hat{\mathbf{z}}_m)$ and substituting $\partial_\tau \hat{\alpha}(\tau) = \frac{1}{\mathrm{i}} \hat{H}^{\hat{\alpha}(\tau)} \hat{\alpha}(\tau)$, we get that

$$\partial_\tau p_m^{\hat{\alpha}(\tau)} = \frac{1}{\mathrm{i}} \left(\hat{H}_m^{\hat{\alpha}(\tau)} p_m^{\hat{\alpha}(\tau)} - p_m^{\hat{\alpha}(\tau)} \hat{H}_m^{\hat{\alpha}(\tau)}\right) \quad \text{and} \quad \partial_\tau P_{k,N}^{\hat{\alpha}(\tau)} = \frac{1}{\mathrm{i}} \sum_{m=1}^N \left(\hat{H}_m^{\hat{\alpha}(\tau)} P_{k,N}^{\hat{\alpha}(\tau)} - P_{k,N}^{\hat{\alpha}(\tau)} \hat{H}_m^{\hat{\alpha}(\tau)}\right).$$

This yields along with $\partial_\tau \hat{\alpha}_N(\tau) = \frac{1}{\mathrm{i}} \hat{H}_N \hat{\alpha}_N(\tau)$ and the symmetry of $\hat{H}_N$ and $\hat{\alpha}_N$ that

$$\begin{aligned}
(*) &= 2 \int_0^t \Im \left\langle \hat{\alpha}_N(\tau), \mathfrak{m}_\lambda^{\hat{\alpha}(\tau)} \left(\sum_m \hat{H}_m^{\hat{\alpha}(\tau)} - \hat{H}_N\right) \hat{\alpha}_N(\tau)\right\rangle \mathrm{d}\tau \\
&= 2 \int_0^t \Im \left\langle \hat{\alpha}_N(\tau), \mathfrak{m}_\lambda^{\hat{\alpha}(\tau)} \left(\sum_m \left(\hat{V} * |\hat{\alpha}(\tau)|^2\right)(\hat{\mathbf{z}}_m) - \frac{1}{2(N-1)} \sum_{m \neq n} \hat{V}(\hat{\mathbf{z}}_m - \hat{\mathbf{z}}_n)\right) \hat{\alpha}_N(\tau)\right\rangle \mathrm{d}\tau \\
&= N \int_0^t \Im \left\langle \hat{\alpha}_N(\tau), \mathfrak{m}_\lambda^{\hat{\alpha}(\tau)} \left(\left(\hat{V} * |\hat{\alpha}(\tau)|^2\right)(\hat{\mathbf{z}}_1) + \left(\hat{V} * |\hat{\alpha}(\tau)|^2\right)(\hat{\mathbf{z}}_2) - \hat{V}(\hat{\mathbf{z}}_1 - \hat{\mathbf{z}}_2)\right) \hat{\alpha}_N(\tau)\right\rangle \mathrm{d}\tau,
\end{aligned}$$

where we made use of the symmetry of α_N at the end. The claim is completed by standard approximation on $\mathcal{M}_{\hat{\mathbf{z}}}^1$ for the initial data and implied $\mathcal{L}_{\hat{\mathbf{z}}}^2$ convergence of their respective solutions on compact time intervals. □

Lemma 4.9. *[18, Sec.3.2] Define the counting functions on* $\{0, \ldots, N\}$ *by*

$$\Delta_l \mathfrak{m}_\lambda(k) \equiv \begin{cases} \mathfrak{m}_\lambda(k) - \mathfrak{m}_\lambda(k-l) & : \quad -|l| \le k \le N^\lambda + |l|, \\ 0 & : \quad else. \end{cases}$$

For any normalized $\hat{\alpha} \in \mathcal{L}_{\hat{\mathbf{z}}}^2$ *and fixed* $N \ge 2$, *we have*

$$\mathfrak{m}_\lambda^{\hat{\alpha}} = \left(\Delta_{-2} \mathfrak{m}_\lambda^{\hat{\alpha}}\right) p_1^{\hat{\alpha}} p_2^{\hat{\alpha}} + \left(\Delta_{-1} \mathfrak{m}_\lambda^{\hat{\alpha}}\right) \left(p_1^{\hat{\alpha}} q_2^{\hat{\alpha}} + q_1^{\hat{\alpha}} p_2^{\hat{\alpha}}\right) + \sum_{k=0}^N \mathfrak{m}_\lambda(k) P_{k-2,N-2}^{\hat{\alpha}}.$$

Proof. Writing $1 = p_1^{\hat{\alpha}} p_2^{\hat{\alpha}} + q_1^{\hat{\alpha}} p_2^{\hat{\alpha}} + p_1^{\hat{\alpha}} q_2^{\hat{\alpha}} + q_1^{\hat{\alpha}} q_2^{\hat{\alpha}}$ and reordering gives

$$\begin{aligned}
\mathfrak{m}_\lambda^{\hat{\alpha}} &= \sum_{k=0}^N \mathfrak{m}_\lambda(k) \left(p_1^{\hat{\alpha}} p_2^{\hat{\alpha}} P_{k,N-2}^{\hat{\alpha}} + \left(p_1^{\hat{\alpha}} q_2^{\hat{\alpha}} + q_1^{\hat{\alpha}} p_2^{\hat{\alpha}}\right) P_{k-1,N-2}^{\hat{\alpha}} + q_1^{\hat{\alpha}} q_2^{\hat{\alpha}} P_{k-2,N-2}^{\hat{\alpha}}\right) \\
&= \sum_{k=0}^N \mathfrak{m}_\lambda(k) \left(P_{k,N}^{\hat{\alpha}} - P_{k-2,N}^{\hat{\alpha}}\right) p_1^{\hat{\alpha}} p_2^{\hat{\alpha}} + \sum_{k=0}^N \mathfrak{m}_\lambda(k) \left(P_{k,N}^{\hat{\alpha}} - P_{k-1,N}^{\hat{\alpha}}\right) \left(p_1^{\hat{\alpha}} q_2^{\hat{\alpha}} + q_1^{\hat{\alpha}} p_2^{\hat{\alpha}}\right) \\
&\quad + \sum_{k=0}^N \mathfrak{m}_\lambda(k) P_{k-2,N-2}^{\hat{\alpha}} \\
&= \left(\Delta_{-2} \mathfrak{m}_\lambda^{\hat{\alpha}}\right) p_1^{\hat{\alpha}} p_2^{\hat{\alpha}} + \left(\Delta_{-1} \mathfrak{m}_\lambda^{\hat{\alpha}}\right) \left(p_1^{\hat{\alpha}} q_2^{\hat{\alpha}} + q_1^{\hat{\alpha}} p_2^{\hat{\alpha}}\right) + \sum_{k=0}^N \mathfrak{m}_\lambda(k) P_{k-2,N-2}^{\hat{\alpha}}.
\end{aligned}$$

□

The main idea of dealing with the unbounded interaction potential, which is linear in the $\boldsymbol{\xi}$ variables, is to introduce a cutoff. This cutoff in the $\boldsymbol{\xi}$ variables corresponds in turn to the regularity of the Fourier transform. Therefore, the following Lemma is essential to treat the large $\boldsymbol{\xi}$ term.

Lemma 4.10. *Let $\alpha_N \in \mathcal{W}^{2,2}_{\vec{\mathbf{z}}}$ be $\mathcal{L}^2_{\vec{\mathbf{z}}}$ normalized and symmetric, $\hat{\alpha}_N \equiv \mathcal{F}^{\otimes N}\alpha_N \in \mathcal{M}^2_{\vec{\mathbf{z}}}$. Likewise, consider $\alpha \in \mathcal{W}^{2,2}_{\mathbf{z}}$ and $\hat{\alpha} \equiv \mathcal{F}\alpha$. Assume that $\hat{\chi} : \mathbb{R}^d_{\boldsymbol{\xi}} \to [0,1]$ is smooth, invariant under rotations, and $\hat{\chi}\big|_{\{|\cdot|\le 1\}} = 1$, $\hat{\chi}\big|_{\{|\cdot|\ge 2\}} = 0$. Define $\chi \equiv \mathcal{F}^{-1}\hat{\chi}$ and for any $R > 0$, set $\hat{\chi}_R(\boldsymbol{\xi}) \equiv \hat{\chi}(\frac{\boldsymbol{\xi}}{R})$. This yields*

$$\|(1-\hat{\chi}_R(\boldsymbol{\xi}_1))\,\boldsymbol{\xi}_1\hat{\alpha}_N\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}} \le \frac{1}{R}\left\|\mathrm{D}^2_{\mathbf{z}_1}\alpha_N\right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}}, \tag{4.4}$$

$$\left\|(1-\hat{\chi}_R(\boldsymbol{\xi}_1))\,\boldsymbol{\xi}_1 p_1^{\hat{\alpha}}\hat{\alpha}_N\right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}} \le \frac{1}{R}\left\|\mathrm{D}^2_{\mathbf{z}}\alpha\right\|_{\mathcal{L}^2_{\mathbf{z}}}. \tag{4.5}$$

Proof. Let $\alpha_N \in C^2_{\vec{\mathbf{z}}} \cap \mathcal{W}^{2,2}_{\vec{\mathbf{z}}}$ be some function, then for $\hat{\alpha}_N \equiv \mathcal{F}^{\otimes N}\alpha_N$ we have that

$$\begin{aligned}\|(1-\hat{\chi}_R(\boldsymbol{\xi}_1))\,\boldsymbol{\xi}_1\hat{\alpha}_N\|^2_{\mathcal{L}^2_{\vec{\mathbf{z}}}} &= \int_{\mathbb{R}^{2dN}_{\vec{\mathbf{z}}}} (1-\hat{\chi}_R(\boldsymbol{\xi}_1))^2\,|\boldsymbol{\xi}_1|^2\left|\hat{\alpha}_N(\vec{\mathbf{z}})\right|^2\,\mathrm{d}\vec{\mathbf{z}}\\ &\le \frac{1}{R^2}\int_{\mathbb{R}^{2dN}_{\vec{\mathbf{z}}}} |\boldsymbol{\xi}_1|^4\left|\hat{\alpha}_N(\vec{\mathbf{z}})\right|^2\,\mathrm{d}\vec{\mathbf{z}} = \frac{1}{R^2}\left\||\boldsymbol{\xi}_1|^2\,\hat{\alpha}_N\right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}} \le \frac{1}{R^2}\left\|\mathrm{D}^2_{\mathbf{z}_1}\alpha_N\right\|^2_{\mathcal{L}^2_{\vec{\mathbf{z}}}}.\end{aligned}$$

For the second claim, we analogously compute

$$\left\|(1-\hat{\chi}_R(\boldsymbol{\xi}_1))\,\boldsymbol{\xi}_1 p_1^{\hat{\alpha}}\hat{\alpha}_N\right\|^2_{\mathcal{L}^2_{\vec{\mathbf{z}}}} = \|(1-\hat{\chi}_R(\boldsymbol{\xi}))\,\boldsymbol{\xi}\,\hat{\alpha}\|^2_{\mathcal{L}^2_{\hat{\mathbf{z}}}}\left\|p_1^{\hat{\alpha}}\hat{\alpha}_N\right\|^2_{\mathcal{L}^2_{\vec{\mathbf{z}}}} \overset{(4.4)\text{ for } N=1}{\le} \frac{1}{R^2}\left\|\mathrm{D}^2_{\mathbf{z}}\alpha\right\|^2_{\mathcal{L}^2_{\mathbf{z}}}. \qquad \square$$

Lemma 4.11. *Let $\hat{\alpha} \in \mathcal{M}^1_{\hat{\mathbf{z}}}$ be some $\mathcal{L}^2_{\hat{\mathbf{z}}}$ normalized function, $\alpha = \mathcal{F}^{-1}\hat{\alpha}$. Then we have the following three inequalities for the operator norms, namely,*

$$\left\|\hat{V}(\hat{\mathbf{z}}_1-\hat{\mathbf{z}}_2)p_1^{\hat{\alpha}}p_2^{\hat{\alpha}}\right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}\to\mathcal{L}^2_{\vec{\mathbf{z}}}} \le C_\Gamma\,\|\hat{\alpha}\|^2_{\mathcal{M}^1_{\hat{\mathbf{z}}}} = C_\Gamma\,\|\alpha\|^2_{\mathcal{W}^{1,2}_{\mathbf{z}}}, \tag{4.6}$$

$$\left\|\left(\hat{V} * |\hat{\alpha}|^2\right)(\hat{\mathbf{z}}_1)p_1^{\hat{\alpha}}\right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}\to\mathcal{L}^2_{\vec{\mathbf{z}}}} \le C_\Gamma\,\|\hat{\alpha}\|^2_{\mathcal{M}^1_{\hat{\mathbf{z}}}} = C_\Gamma\,\|\alpha\|^2_{\mathcal{W}^{1,2}_{\mathbf{z}}}, \quad \text{and} \tag{4.7}$$

$$\left\|\boldsymbol{\xi}_1 p_1^{\hat{\alpha}}\right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}\to\mathcal{L}^2_{\vec{\mathbf{z}}}} = \||\boldsymbol{\xi}|\,\hat{\alpha}\|_{\mathcal{L}^2_{\hat{\mathbf{z}}}} = \|\nabla\alpha\|_{\mathcal{L}^2_{\mathbf{z}}}. \tag{4.8}$$

Proof. Let $\hat{\alpha}_N \in \mathcal{L}^2_{\vec{\mathbf{z}}}$ be arbitrary. Using the estimate $|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2| \le \left(1+|\boldsymbol{\xi}_1|^2\right)^{\frac{1}{2}}\left(1+|\boldsymbol{\xi}_2|^2\right)^{\frac{1}{2}}$, one finds

$$\begin{aligned}\left\|\hat{V}(\hat{\mathbf{z}}_1-\hat{\mathbf{z}}_2)p_1^{\hat{\alpha}}p_2^{\hat{\alpha}}\hat{\alpha}_N\right\|^2_{\mathcal{L}^2_{\vec{\mathbf{z}}}} &= \left\langle\hat{\alpha}_N, p_1^{\hat{\alpha}}p_2^{\hat{\alpha}}\left|\hat{V}(\hat{\mathbf{z}}_1-\hat{\mathbf{z}}_2)\right|^2 p_1^{\hat{\alpha}}p_2^{\hat{\alpha}}\hat{\alpha}_N\right\rangle\\ &= \left\|\left(\left|\hat{V}\right|^2 * |\hat{\alpha}|^2\right)|\hat{\alpha}|^2\right\|_{\mathcal{L}^1_{\hat{\mathbf{z}}}}\left\|p_1^{\hat{\alpha}}p_2^{\hat{\alpha}}\hat{\alpha}_N\right\|^2_{\mathcal{L}^2_{\vec{\mathbf{z}}}} \le \|\nabla\Gamma\|^2_{\mathcal{L}^\infty_{\mathbf{x}}}\,\|\hat{\alpha}\|^4_{\mathcal{M}^1_{\hat{\mathbf{z}}}}\,\|\hat{\alpha}_N\|^2_{\mathcal{L}^2_{\vec{\mathbf{z}}}}.\end{aligned}$$

The other two claims hold similarly. □

Now we have all necessary ingredients to prove our main result.

Theorem 4.12 (Regular Hamilton Hartree / Hamiltonian Vlasov Mean Field Limit)**.** *Assume that the regularity condition* (Pot) *of the potential* Γ *holds. Let* $\mathring{\alpha} \in \mathcal{W}^{2,2}_{\mathbf{z}}$ *be an* $\mathcal{L}^2_{\mathbf{z}}$ *normalized initial state and* $\mathring{\alpha}_N \in \mathcal{W}^{2,2}_{\vec{\mathbf{z}}}$ *be a sequence of symmetric,* $\mathcal{L}^2_{\vec{\mathbf{z}}}$ *normalized initial states.*

According to Theorems 3.1 and 4.1 there exist

$$\begin{array}{rclcll} \alpha & : & \mathbb{R}_{\geq 0} & \to & \mathcal{W}^{2,2}_{\mathbf{z}} & \textit{solving } (\text{HVl}) \textit{ with } \alpha(0) = \mathring{\alpha} \quad \textit{and} \\ \alpha_N & : & \mathbb{R}_{\geq 0} & \to & \mathcal{W}^{2,2}_{\vec{\mathbf{z}}} & \textit{solving } (\text{HLv}^{\text{N}}) \textit{ with } \alpha_N(0) = \mathring{\alpha}_N. \end{array}$$

Applying the velocity Fourier transform leads to solutions

$$\begin{array}{rll} \hat{\alpha}: & \mathbb{R}_{\geq 0} \to \mathcal{M}^2_{\hat{\mathbf{z}}}, \quad t \mapsto \mathcal{F}\alpha(t), & \textit{of } (\text{HHt}) \textit{ with } \quad \hat{\alpha}(0) = \mathring{\hat{\alpha}} \equiv \mathcal{F}\mathring{\alpha} \quad \textit{and} \\ \hat{\alpha}_N: & \mathbb{R}_{\geq 0} \to \mathcal{M}^2_{\vec{\hat{\mathbf{z}}}}, \quad t \mapsto \mathcal{F}^{\otimes N}\alpha_N(t), & \textit{of } (\text{PsQM}^{\text{N}}) \textit{ with } \quad \hat{\alpha}_N(0) = \mathring{\hat{\alpha}}_N \equiv \mathcal{F}^{\otimes N}\mathring{\alpha}_N. \end{array}$$

If there is some $M \geq 1$*, s.t.*

$$\sup_N \left\| \mathrm{D}^1_{\mathbf{z}_1} \mathring{\alpha}_N \right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}} \leq M, \quad \sup_N \left\| \mathrm{D}^2_{(\mathbf{z}_1,\mathbf{z}_2)} \mathring{\alpha}_N \right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}} \leq M, \quad \textit{and} \quad \|\mathring{\alpha}\|_{\mathcal{W}^{2,2}_{\mathbf{z}}} \leq M,$$

then for every $\lambda \in (0,1)$ *there are continuous, non-decreasing functions* $B_{1,M}, B_{2,M}$: $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$*, both independent of N, s.t.*

$$\begin{aligned} \left| \left\langle \hat{\alpha}_N(t), \mathfrak{m}^{\hat{\alpha}(t)}_{\lambda} \hat{\alpha}_N(t) \right\rangle - \left\langle \mathring{\hat{\alpha}}_N, \mathfrak{m}^{\mathring{\hat{\alpha}}}_{\lambda} \mathring{\hat{\alpha}}_N \right\rangle \right| &= \left| \left\langle \alpha_N(t), \mathfrak{m}^{\alpha(t)}_{\lambda} \alpha_N(t) \right\rangle - \left\langle \mathring{\alpha}_N, \mathfrak{m}^{\mathring{\alpha}}_{\lambda} \mathring{\alpha}_N \right\rangle \right| \\ &\leq \left(N^{-\lambda} + N^{-\frac{1-\lambda}{4}} \right) \left(\int_0^t B_{2,M}(\tau)\, \mathrm{d}\tau \right) \exp\left(\int_0^t B_{1,M}(\tilde{\tau})\, \mathrm{d}\tilde{\tau} \right). \end{aligned}$$

Remark 4.13. Assuming that $\left\| \mathrm{D}^2_{\mathbf{z}_1} \alpha_N(t) \right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}}$ and $\|\nabla\alpha(t)\|_{\mathcal{L}^\infty_{\mathbf{z}}}$ are bounded uniformly in N and t, it is sufficient to assume that $\nabla\Gamma \in \mathcal{L}^2_{\mathbf{x}}$ to prove the validity of Theorem 4.12. Note, that while proving the assumption on $\alpha_N(t)$ might be quite difficult for singular potentials, the assumption on $\alpha(t)$ is proven even for the more singular Coulomb interaction locally in time for initial data in $\mathcal{B}^{1,\kappa,2}_{\mathbf{z}}$ and globally in time under some additional restrictions, see Section 3.2.

Although these additional assumptions are not completely satisfactory, it is still remarkable that they allow for a derivation of the effective description under the presence of interaction potentials with mild singularities.

Proof. Let $\mathring{\alpha}$ and $\mathring{\alpha}_N$ fulfill the conditions of the theorem. Let α, $\hat{\alpha} \equiv \mathcal{F}\alpha$, α_N, and $\hat{\alpha}_N \equiv \mathcal{F}^{\otimes N}\alpha_N$ denote the solutions to the respective initial value problems.

At first, we remark that for every $t \geq 0$, we have

$$\left\langle \hat{\alpha}_N(t), \mathfrak{m}_\lambda^{\hat{\alpha}(t)} \hat{\alpha}_N(t) \right\rangle = \left\langle \alpha_N(t), \mathfrak{m}_\lambda^{\alpha(t)} \alpha_N(t) \right\rangle,$$

using the Plancherel theorem for the velocity Fourier transform and $\mathcal{F}_m p_m^{\alpha(t)} \mathcal{F}_m^{-1} = p_m^{\hat{\alpha}(t)}$. Therefore, it suffices to prove the theorem only for the counting functional

$$\beta_N : \quad \mathbb{R}_{\geq 0} \to [0,1], \quad t \mapsto \left\langle \hat{\alpha}_N(t), \mathfrak{m}_\lambda^{\hat{\alpha}(t)} \hat{\alpha}_N(t) \right\rangle.$$

For the sake of readability, we use the shortened notation

$$\mathfrak{m}_\lambda \equiv \mathfrak{m}_\lambda^{\hat{\alpha}(t)}, \quad \Delta_l \mathfrak{m}_\lambda \equiv \Delta_l \mathfrak{m}_\lambda^{\hat{\alpha}(t)}, \quad \hat{V}_{m,n} \equiv \hat{V}(\hat{\mathbf{z}}_m - \hat{\mathbf{z}}_n),$$
$$\bar{V}_m \equiv \left(\hat{V} * |\hat{\alpha}(t)|^2\right)(\hat{\mathbf{z}}_m), \quad p_m \equiv p_m^{\hat{\alpha}(t)}.$$

By Lemma 4.8, the map $t \mapsto \beta_N(t)$ is differentiable and therefore

$$\begin{aligned}
\partial_t \beta_N(t) = \partial_t \langle \hat{\alpha}_N(t), \mathfrak{m}_\lambda \hat{\alpha}_N(t) \rangle &\overset{\text{Lem.4.8}}{=} N\, \Im \left\langle \hat{\alpha}_N(t), \mathfrak{m}_\lambda \left(\bar{V}_1 + \bar{V}_2 - \hat{V}_{1,2}\right) \hat{\alpha}_N(t) \right\rangle \\
\overset{\text{Lem.4.9}}{=} N\, \Im &\left\langle \hat{\alpha}_N(t), (\Delta_{-2}\mathfrak{m}_\lambda)\, p_1 p_2 \left(\bar{V}_1 + \bar{V}_2 - \hat{V}_{1,2}\right) \hat{\alpha}_N(t) \right\rangle \\
&+ 2N\, \Im \left\langle \hat{\alpha}_N(t), (\Delta_{-1}\mathfrak{m}_\lambda)\, p_1 q_2 \left(\bar{V}_1 + \bar{V}_2 - \hat{V}_{1,2}\right) \hat{\alpha}_N(t) \right\rangle \\
&+ N\, \Im \left\langle \hat{\alpha}_N(t), \left(\sum_{k=0}^{N} \mathfrak{m}_\lambda(k) P_{k-2,N-2}^{\hat{\alpha}} \right) \left(\bar{V}_1 + \bar{V}_2 - \hat{V}_{1,2}\right) \hat{\alpha}_N(t) \right\rangle.
\end{aligned}$$

We remark that $\Im \langle \hat{\alpha}_N(t), A \hat{\alpha}_N(t) \rangle = 0$ for any symmetric operator A. As the two operators in the third term commute and are symmetric, their product is symmetric and the term is canceled out. Now, interchanging particle indices 1 and 2 in the second term and inserting $1 = p_1 p_2 + p_1 q_2 + q_1 p_2 + q_1 q_2$ in both remaining terms, one obtains

$$\begin{aligned}
\partial_t \beta_N(t) = N\, \Im &\left\langle \hat{\alpha}_N(t), (\Delta_{-2}\mathfrak{m}_\lambda)\, p_1 p_2 \left(\bar{V}_1 + \bar{V}_2 - \hat{V}_{1,2}\right) p_1 p_2 \hat{\alpha}_N(t) \right\rangle && (1) \\
+ N\, \Im &\left\langle \hat{\alpha}_N(t), (\Delta_{-2}\mathfrak{m}_\lambda)\, p_1 p_2 \left(\bar{V}_1 + \bar{V}_2 - \hat{V}_{1,2}\right) p_1 q_2 \hat{\alpha}_N(t) \right\rangle && (2) \\
+ N\, \Im &\left\langle \hat{\alpha}_N(t), (\Delta_{-2}\mathfrak{m}_\lambda)\, p_1 p_2 \left(\bar{V}_1 + \bar{V}_2 - \hat{V}_{1,2}\right) q_1 p_2 \hat{\alpha}_N(t) \right\rangle && (3) \\
+ N\, \Im &\left\langle \hat{\alpha}_N(t), (\Delta_{-2}\mathfrak{m}_\lambda)\, p_1 p_2 \left(\bar{V}_1 + \bar{V}_2 - \hat{V}_{1,2}\right) q_1 q_2 \hat{\alpha}_N(t) \right\rangle && (4) \\
+ 2N\, \Im &\left\langle \hat{\alpha}_N(t), (\Delta_{-1}\mathfrak{m}_\lambda)\, q_1 p_2 \left(\bar{V}_1 + \bar{V}_2 - \hat{V}_{1,2}\right) p_1 p_2 \hat{\alpha}_N(t) \right\rangle && (5) \\
+ 2N\, \Im &\left\langle \hat{\alpha}_N(t), (\Delta_{-1}\mathfrak{m}_\lambda)\, q_1 p_2 \left(\bar{V}_1 + \bar{V}_2 - \hat{V}_{1,2}\right) p_1 q_2 \hat{\alpha}_N(t) \right\rangle && (6) \\
+ 2N\, \Im &\left\langle \hat{\alpha}_N(t), (\Delta_{-1}\mathfrak{m}_\lambda)\, q_1 p_2 \left(\bar{V}_1 + \bar{V}_2 - \hat{V}_{1,2}\right) q_1 p_2 \hat{\alpha}_N(t) \right\rangle && (7) \\
+ 2N\, \Im &\left\langle \hat{\alpha}_N(t), (\Delta_{-1}\mathfrak{m}_\lambda)\, q_1 p_2 \left(\bar{V}_1 + \bar{V}_2 - \hat{V}_{1,2}\right) q_1 q_2 \hat{\alpha}_N(t) \right\rangle. && (8)
\end{aligned}$$

Along with Lemma 4.7-(iv), we see that

$$\begin{aligned}
(1) &= N\,\Im\left\langle \hat{\alpha}_N(t), (\Delta_{-2}\mathfrak{m}_\lambda)\, p_1 p_2 \left(\bar{V}_1 + \bar{V}_2 - \hat{V}_{1,2}\right) p_1 p_2 \hat{\alpha}_N(t)\right\rangle \\
&= N\,\Im\left\langle \hat{\alpha}_N(t), p_1 p_2 \left(\bar{V}_1 + \bar{V}_2 - \hat{V}_{1,2}\right) p_1 p_2\, (\Delta_{-2}\mathfrak{m}_\lambda) \hat{\alpha}_N(t)\right\rangle^* \\
&= -\,N\,\Im\left\langle \hat{\alpha}_N(t), (\Delta_{-2}\mathfrak{m}_\lambda)\, p_1 p_2 \left(\bar{V}_1 + \bar{V}_2 - \hat{V}_{1,2}\right) p_1 p_2 \hat{\alpha}_N(t)\right\rangle = 0,
\end{aligned}$$

and by the very same argument terms (6) and (7) also vanish. Acknowledging the invariance w.r.t. particle permutation, we can also recombine terms (2) and (3), yielding

$$\begin{aligned}
\partial_t \beta_N(t) = 2N\,\Im&\left\langle \hat{\alpha}_N(t), (\Delta_{-2}\mathfrak{m}_\lambda)\, p_1 p_2 \left(\bar{V}_1 + \bar{V}_2 - \hat{V}_{1,2}\right) q_1 p_2 \hat{\alpha}_N(t)\right\rangle && \text{(1')} \\
+ N\,\Im&\left\langle \hat{\alpha}_N(t), (\Delta_{-2}\mathfrak{m}_\lambda)\, p_1 p_2 \left(\bar{V}_1 + \bar{V}_2 - \hat{V}_{1,2}\right) q_1 q_2 \hat{\alpha}_N(t)\right\rangle && \text{(2')} \\
+ 2N\,\Im&\left\langle \hat{\alpha}_N(t), (\Delta_{-1}\mathfrak{m}_\lambda)\, q_1 p_2 \left(\bar{V}_1 + \bar{V}_2 - \hat{V}_{1,2}\right) p_1 p_2 \hat{\alpha}_N(t)\right\rangle && \text{(3')} \\
+ 2N\,\Im&\left\langle \hat{\alpha}_N(t), (\Delta_{-1}\mathfrak{m}_\lambda)\, q_1 p_2 \left(\bar{V}_1 + \bar{V}_2 - \hat{V}_{1,2}\right) q_1 q_2 \hat{\alpha}_N(t)\right\rangle. && \text{(4')}
\end{aligned}$$

Finally, we compute for the third term (3') that

$$\begin{aligned}
\text{(3')} &= -\,2N\,\Im\left\langle \hat{\alpha}_N(t), p_1 p_2 \left(\bar{V}_1 + \bar{V}_2 - \hat{V}_{1,2}\right) q_1 p_2\, (\Delta_{-1}\mathfrak{m}_\lambda) \hat{\alpha}_N(t)\right\rangle \\
&= -\,2N\,\Im\left\langle \hat{\alpha}_N(t), (\Delta_{-1}\mathfrak{m}_\lambda)_{-1}\, p_1 p_2 \left(\bar{V}_1 + \bar{V}_2 - \hat{V}_{1,2}\right) q_1 p_2 \hat{\alpha}_N(t)\right\rangle,
\end{aligned}$$

which recombines with term (1') and with $(\Delta_{-2}\mathfrak{m}_\lambda) - (\Delta_{-1}\mathfrak{m}_\lambda)_{-1} = (\Delta_{-1}\mathfrak{m}_\lambda)$ gives that

$$\begin{aligned}
\partial_t \beta_N(t) = 2N\,\Im&\left\langle \hat{\alpha}_N(t), (\Delta_{-1}\mathfrak{m}_\lambda)\, p_1 p_2 \left(\bar{V}_1 + \bar{V}_2 - \hat{V}_{1,2}\right) q_1 p_2 \hat{\alpha}_N(t)\right\rangle \\
+ N\,\Im&\left\langle \hat{\alpha}_N(t), (\Delta_{-2}\mathfrak{m}_\lambda)\, p_1 p_2 \left(\bar{V}_1 + \bar{V}_2 - \hat{V}_{1,2}\right) q_1 q_2 \hat{\alpha}_N(t)\right\rangle \\
+ 2N\,\Im&\left\langle \hat{\alpha}_N(t), (\Delta_{-1}\mathfrak{m}_\lambda)\, q_1 p_2 \left(\bar{V}_1 + \bar{V}_2 - \hat{V}_{1,2}\right) q_1 q_2 \hat{\alpha}_N(t)\right\rangle.
\end{aligned}$$

The first term vanishes because $p_1 \bar{V}_2 q_1 = p_1 q_1 \bar{V}_2 = 0$ and $p_2 \hat{V}_{1,2} p_2 = p_2 \bar{V}_1 p_2$, where the first equality can also be used to simplify the other two expressions by omitting $\bar{V}_1$ and $\bar{V}_2$, and $\bar{V}_1$ respectively, resulting in

$$\begin{aligned}
\partial_t \beta_N(t) = -\,N\,\Im&\left\langle \hat{\alpha}_N(t), (\Delta_{-2}\mathfrak{m}_\lambda)\, p_1 p_2 \hat{V}_{1,2} q_1 q_2 \hat{\alpha}_N(t)\right\rangle && \text{(1'')} \\
+ 2N\,\Im&\left\langle \hat{\alpha}_N(t), (\Delta_{-1}\mathfrak{m}_\lambda)\, q_1 p_2 \left(\bar{V}_2 - \hat{V}_{1,2}\right) q_1 q_2 \hat{\alpha}_N(t)\right\rangle. && \text{(2'')}
\end{aligned}$$

Estimating the absolute value of (1''), we use the notation $\bar{\mathfrak{n}}(k) \equiv \sqrt{\frac{N}{k}}$ for $1 \le k \le N$ and $\bar{\mathfrak{n}}(k) = 0$ else and observe that $\bar{\mathfrak{n}}\mathfrak{n} = \mathrm{id} - P_{0,N}$ is *almost* the inverse of $\mathfrak{n}$. Since that implies $\bar{\mathfrak{n}}\mathfrak{n}\, q_1 q_2 = q_1 q_2$, we compute

$$
\begin{aligned}
|(1")| &\leq N \left| \left\langle \hat{\alpha}_N(t), (\Delta_{-2}\mathfrak{m}_\lambda)\, p_1 p_2 \hat{V}_{1,2} \bar{\mathfrak{n}}\mathfrak{n} q_1 q_2 \hat{\alpha}_N(t) \right\rangle \right| \\
&\overset{\text{Lem.4.7-(iv)}}{=} N \left| \left\langle \hat{\alpha}_N(t), (-\Delta_{-2}\mathfrak{m}_\lambda)^{\frac{1}{2}}\, (\mathfrak{n})_{-2}\, p_1 p_2 \hat{V}_{1,2} \left((-\Delta_{-2}\mathfrak{m}_\lambda)^{\frac{1}{2}} \right)_2 \bar{\mathfrak{n}} q_1 q_2 \hat{\alpha}_N(t) \right\rangle \right| \\
&\leq N \left\| (-\Delta_{-2}\mathfrak{m}_\lambda)^{\frac{1}{2}}\, (\mathfrak{n})_{-2}\ \hat{\alpha}_N(t) \right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}} \left\| \hat{V}_{1,2} p_1 p_2 \right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}} \to \mathcal{L}^2_{\vec{\mathbf{z}}}} \left\| \left((-\Delta_{-2}\mathfrak{m}_\lambda)^{\frac{1}{2}} \right)_2 \bar{\mathfrak{n}} q_1 q_2 \hat{\alpha}_N(t) \right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}} \\
&\overset{\text{Lem.4.7-(iii)}}{\leq} N \left\| (-\Delta_{-2}\mathfrak{m}_\lambda)^{\frac{1}{2}}\, (\mathfrak{n})_{-2}\ \hat{\alpha}_N(t) \right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}} \left\| \hat{V}_{1,2} p_1 p_2 \right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}} \to \mathcal{L}^2_{\vec{\mathbf{z}}}} \\
&\qquad \cdot \left(\frac{N}{N-1} \right)^{\frac{1}{2}} \left\| \left((-\Delta_{-2}\mathfrak{m}_\lambda)^{\frac{1}{2}} \right)_2 \bar{\mathfrak{n}}\mathfrak{n}^2 \hat{\alpha}_N(t) \right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}} \\
&\overset{\bar{\mathfrak{n}}\mathfrak{n}\leq 1}{\leq} N \left\langle \hat{\alpha}_N(t), (-\Delta_{-2}\mathfrak{m}_\lambda)\, (\mathfrak{n})^2_{-2} \hat{\alpha}_N(t) \right\rangle^{\frac{1}{2}} \left\| \hat{V}_{1,2} p_1 p_2 \right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}} \to \mathcal{L}^2_{\vec{\mathbf{z}}}} \\
&\qquad \cdot \left(\frac{N}{N-1} \right)^{\frac{1}{2}} \left\langle \hat{\alpha}_N(t), (-\Delta_{-2}\mathfrak{m}_\lambda)_2\, \mathfrak{n}^2 \hat{\alpha}_N(t) \right\rangle^{\frac{1}{2}},
\end{aligned}
$$

via $(\mathfrak{m}_\lambda(k+2) - \mathfrak{m}_\lambda(k))\, \frac{k+2}{N} \leq \frac{2}{N}\left(\mathfrak{m}_\lambda(k) + \frac{2}{N^\lambda} \right)$ and $(\mathfrak{m}_\lambda(k) - \mathfrak{m}_\lambda(k-2))\, \frac{k}{N} \leq \frac{2}{N}\left(\mathfrak{m}_\lambda(k) + \frac{2}{N^\lambda} \right)$ in the first and second inner products, respectively. We conclude for the estimate that

$$
\overset{(4.6)}{\leq} 2 \left(\frac{N}{N-1} \right)^{\frac{1}{2}} C_\Gamma \, \|\alpha(t)\|^2_{\mathcal{W}^{1,2}_{\mathbf{z}}} \left(\beta_N(t) + \frac{2}{N^\lambda} \right).
$$

In order to estimate (2"), one picks a smooth cutoff χ with the properties as in Lemma 4.10. Regrouping $\hat{V}_{1,2} = -\nabla\Gamma_{1,2} \cdot (\hat{\chi}_R(\boldsymbol{\xi}_1)\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) - \nabla\Gamma_{1,2} \cdot (1 - \hat{\chi}_R(\boldsymbol{\xi}_1))\boldsymbol{\xi}_1$ yields

$$
\begin{aligned}
|(2")| &\leq 2N \left| \left\langle \hat{\alpha}_N(t), (\Delta_{-1}\mathfrak{m}_\lambda)\, q_1 p_2 \left(\bar{V}_2 + \nabla\Gamma_{1,2} \cdot (\hat{\chi}_R(\boldsymbol{\xi}_1)\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \right) q_1 q_2 \hat{\alpha}_N(t) \right\rangle \right| && (1''') \\
&\quad + 2N \left| \left\langle \hat{\alpha}_N(t), (\Delta_{-1}\mathfrak{m}_\lambda)\, q_1 p_2 \left(\nabla\Gamma_{1,2} \cdot (1 - \hat{\chi}_R(\boldsymbol{\xi}_1))\boldsymbol{\xi}_1 \right) q_1 q_2 \hat{\alpha}_N(t) \right\rangle \right|. && (2''')
\end{aligned}
$$

As the multiplication operator in (1"') times p_2 is bounded due to Lemma 4.11, we find that

$$
\begin{aligned}
&\left\| \left(\bar{V}_2 + \nabla\Gamma_{1,2} \cdot (\hat{\chi}_R(\boldsymbol{\xi}_1)\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \right) p_2 \right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}} \to \mathcal{L}^2_{\vec{\mathbf{z}}}} \\
&\leq \left\| \bar{V}_2 p_2 \right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}} \to \mathcal{L}^2_{\vec{\mathbf{z}}}} + \|\nabla\Gamma\|_{\mathcal{L}^\infty_{\mathbf{x}}} \left(\|\hat{\chi}_R(\boldsymbol{\xi}_1)\boldsymbol{\xi}_1\|_{\mathcal{L}^\infty_{\vec{\mathbf{z}}}} + \left\| \boldsymbol{\xi}_2 p_2 \right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}} \to \mathcal{L}^2_{\vec{\mathbf{z}}}} \right) \\
&\overset{\text{Lem.4.11}}{\leq} C_\Gamma \left(\|\alpha(t)\|^2_{\mathcal{W}^{1,2}_{\mathbf{z}}} + 2R + \|\nabla\alpha(t)\|_{\mathcal{L}^2_{\mathbf{z}}} \right).
\end{aligned}
$$

This can be used to estimate

$$
\begin{aligned}
(1''') &= 2N \left|\left\langle \hat{\alpha}_N(t), (-\Delta_{-1}\mathfrak{m}_\lambda)\, q_1 p_2 \left(\bar{V}_2 + \nabla\Gamma_{1,2}\cdot(\hat{\chi}_R(\boldsymbol{\xi}_1)\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)\right) q_1 q_2 \hat{\alpha}_N(t)\right\rangle\right| \\
&\le 2N \left\|(-\Delta_{-1}\mathfrak{m}_\lambda)^{\frac{1}{2}}\, q_1 \hat{\alpha}_N(t)\right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}} \left\|\left(\bar{V}_2 + \nabla\Gamma_{1,2}\cdot(\hat{\chi}_R(\boldsymbol{\xi}_1)\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)\right) p_2\right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}\to\mathcal{L}^2_{\vec{\mathbf{z}}}} \\
&\quad \cdot \left\|\left((-\Delta_{-1}\mathfrak{m}_\lambda)^{\frac{1}{2}}\right)_1 q_1 q_2 \hat{\alpha}_N(t)\right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}} \\
&\overset{\text{Lem.4.7-(iii)}}{\le} 2N \left\|(-\Delta_{-1}\mathfrak{m}_\lambda)^{\frac{1}{2}}\, \mathfrak{n}\hat{\alpha}_N(t)\right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}} \left\|\left(\bar{V}_2 + \nabla\Gamma_{1,2}\cdot(\hat{\chi}_R(\boldsymbol{\xi}_1)\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)\right) p_2\right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}\to\mathcal{L}^2_{\vec{\mathbf{z}}}} \\
&\quad \cdot \left(\frac{N}{N-1}\right)^{\frac{1}{2}} \left\|\left((-\Delta_{-1}\mathfrak{m}_\lambda)^{\frac{1}{2}}\right)_1 \mathfrak{n}^2 \hat{\alpha}_N(t)\right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}} \\
&= 2N \left\langle \hat{\alpha}_N(t), (-\Delta_{-1}\mathfrak{m}_\lambda)\, \mathfrak{n}^2 \hat{\alpha}_N(t)\right\rangle^{\frac{1}{2}} \left\|\left(\bar{V}_2 + \nabla\Gamma_{1,2}\cdot(\hat{\chi}_R(\boldsymbol{\xi}_1)\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)\right) p_2\right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}\to\mathcal{L}^2_{\vec{\mathbf{z}}}} \\
&\quad \cdot \left(\frac{N}{N-1}\right)^{\frac{1}{2}} \left\langle \hat{\alpha}_N(t), (-\Delta_{-1}\mathfrak{m}_\lambda)_1\, \mathfrak{n}^4 \hat{\alpha}_N(t)\right\rangle^{\frac{1}{2}},
\end{aligned}
$$

via $(\mathfrak{m}_\lambda(k+1) - \mathfrak{m}_\lambda(k))\,\frac{k}{N} \le \frac{1}{N}\mathfrak{m}_\lambda(k)$ and $(\mathfrak{m}_\lambda(k) - \mathfrak{m}_\lambda(k-1))\,\frac{k^2}{N^2} \le \frac{N^\lambda+1}{N^2}\left(\mathfrak{m}_\lambda(k) + \frac{1}{N^\lambda}\right)$ in the first and second inner products, respectively. We conclude for the estimate that

$$
\begin{aligned}
&\le 2 \left(\frac{N^\lambda+1}{N-1}\right)^{\frac{1}{2}} C_\Gamma \left(\|\alpha(t)\|^2_{\mathcal{W}^{1,2}_{\mathbf{z}}} + 2R + \|\nabla\alpha(t)\|_{\mathcal{L}^2_{\mathbf{z}}}\right)\left(\beta_N(t) + \frac{1}{N^\lambda}\right)^{\frac{1}{2}} \beta_N(t)^{\frac{1}{2}} \\
&\le 2\, N^{\frac{\lambda-1}{2}} \left(\frac{1+N^{-\lambda}}{1-N^{-1}}\right)^{\frac{1}{2}} C_\Gamma \left(\|\alpha(t)\|^2_{\mathcal{W}^{1,2}_{\mathbf{z}}} + 2R + \|\nabla\alpha(t)\|_{\mathcal{L}^2_{\mathbf{z}}}\right)\left(\beta_N(t) + \frac{1}{N^\lambda}\right).
\end{aligned}
$$

Ultimately, term (2''') can be controlled with the regularity estimates on the many-particle solution. With the aid of Lemmata 4.7-(iv) and 4.10, we get

$$
\begin{aligned}
(2''') &\le 2N\, \|\nabla\Gamma\|_{\mathcal{L}^\infty_{\mathbf{x}}} \left\|(1-\hat{\chi}_R(\boldsymbol{\xi}_1))\boldsymbol{\xi}_1(1-p_1)\hat{\alpha}_N(t)\right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}} \left\|(\Delta_{-1}\mathfrak{m}_\lambda)_1\, q_1 q_2 \hat{\alpha}_N(t)\right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}} \\
&\le 2\, C_\Gamma \frac{1}{R}\left(\left\|\mathrm{D}^2_{\mathbf{z}_1}\alpha_N(t)\right\|_{\mathcal{L}^2_{\vec{\mathbf{z}}}} + \left\|\mathrm{D}^2_{\mathbf{z}}\alpha(t)\right\|_{\mathcal{L}^2_{\mathbf{z}}}\right) N \left(\frac{N}{N-1}\right)^{\frac{1}{2}} \\
&\qquad \cdot \left\langle \hat{\alpha}_N(t), (\Delta_{-1}\mathfrak{m}_\lambda)^2_1\, \mathfrak{n}^4 \hat{\alpha}_N(t)\right\rangle^{\frac{1}{2}},
\end{aligned}
$$

yielding with $(\mathfrak{m}_\lambda(k) - \mathfrak{m}_\lambda(k-1))\,\frac{k^2}{N^2} \le \left(1+N^{-\lambda}\right)^2 \frac{1}{N^2}$ that

$$
\overset{\text{Lem.4.4,Cor.3.4}}{\le} 2\, C_\Gamma \frac{1+N^{-\lambda}}{(1-N^{-1})^{\frac{1}{2}}} \frac{1}{R}\left(\mathfrak{b}^{\mathcal{L}^2_{\vec{\mathbf{z}}}}_{M,\mathrm{D}^2_{(\mathbf{z}_1,\mathbf{z}_2)}}(t) + \mathfrak{b}^{\mathcal{L}^2_{\mathbf{z}}}_{M,\mathrm{D}^2}(t)\right).
$$

Combining all the estimates for (1”), (1”’), and (2”’), we obtain the Grönwall type estimate

$$
\begin{aligned}
|\partial_t \beta_N(t)| \leq\ & 2\, \frac{C_\Gamma}{(1-N^{-1})^{\frac{1}{2}}} \left(\left(\mathfrak{b}_M^{\mathcal{W}_{\mathbf{z}}^{1,2}}(t) \right)^2 + N^{\frac{\lambda-1}{2}} \left(1+N^{-\lambda}\right)^{\frac{1}{2}} \left(2R + \left(1 + \mathfrak{b}_M^{\mathcal{W}_{\mathbf{z}}^{1,2}}(t)\right)^2 \right) \right) \beta_N(t) \\
& + 4\, \frac{C_\Gamma}{(1-N^{-1})^{\frac{1}{2}}} \left(\mathfrak{b}_M^{\mathcal{W}_{\mathbf{z}}^{1,2}}(t) \right)^2 N^{-\lambda} + 2\, C_\Gamma \frac{1+N^{-\lambda}}{(1-N^{-1})^{\frac{1}{2}}} \left(\mathfrak{b}^{\mathcal{L}_{\mathbf{z}}^2}_{M,\mathrm{D}^2_{(\mathbf{z}_1,\mathbf{z}_2)}}(t) + \mathfrak{b}^{\mathcal{L}_{\mathbf{z}}^2}_{M,\mathrm{D}^2}(t) \right) \frac{1}{R} \\
& + 2\, C_\Gamma \left(\frac{1+N^{-\lambda}}{1-N^{-1}} \right)^{\frac{1}{2}} \left(2R + \left(1 + \mathfrak{b}_M^{\mathcal{W}_{\mathbf{z}}^{1,2}}(t) \right)^2 \right) N^{-\frac{1+\lambda}{2}}.
\end{aligned}
$$

If we choose $R(N) \equiv N^\delta$, then for the limit $N \to \infty$ we have

$$
|\partial_t \beta_N(t)| \leq \left(\mathcal{O}(1) + \mathcal{O}\left(N^{\frac{\lambda-1}{2}}\right) + \mathcal{O}\left(N^{\delta+\frac{\lambda-1}{2}}\right) \right) \beta_N(t) + \mathcal{O}\left(N^{-\lambda}\right) + \mathcal{O}\left(N^{-\delta}\right) + \mathcal{O}\left(N^{-\frac{1+\lambda}{2}}\right).
$$

This clearly requires the conditions $0 < \lambda < 1$ and $0 < \delta < \frac{1-\lambda}{2}$. For the optimal choice of $\delta = \frac{1-\lambda}{4}$, the convergence rate of all error terms is $\mathcal{O}\left(N^{-\frac{1-\lambda}{4}}\right) + \mathcal{O}\left(N^{-\lambda}\right)$. The explicit bounds $B_{1,M}, B_{2,M} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ can now be inferred from the estimate on $|\partial_t \beta_N(t)|$ above. □

As mentioned in Remark 4.6, the result controls the share of particles outside the mean field state $\hat{\alpha}(t)$ or $\alpha(t)$ in the Hamilton Hartree or Hamiltonian Vlasov setup, respectively. Of course this is not a direct convergence in a metric space. Nevertheless, controlling this share of inaccurately described particles actually implies convergence of the reduced density matrix of the many-particle system to the pure density matrix constructed from the mean field state [19, Lem.2.3-(a)]. In order to give a precise formulation of the result, we adapt the following definition from standard quantum theory [22, Sec.20.2].

Definition 4.14 (Density Matrix)**.** Let $\mathcal{H}$ be some separable Hilbert space.

(i) For a positive semi-definite Hermitian operator $Q : \mathcal{H} \to \mathcal{H}$, its **trace** is given by

$$
\operatorname{tr} Q \equiv \sum_{k=1}^{\infty} \langle Q e_k, e_k \rangle \in [0, \infty],
$$

where $\{e_k\}_{k\in\mathbb{N}}$ is any orthonormal basis of $\mathcal{H}$.

(ii) A positive semi-definite Hermitian operator $Q : \mathcal{H} \to \mathcal{H}$ with trace $\operatorname{tr} Q = 1$ is called a **density matrix**. It is called **pure**, if and only if it is an orthogonal projection onto a one dimensional subspace.

(iii) Let $\mathcal{K}$ be another separable Hilbert space with some orthonormal basis $\{f_k\}$ and let $\mathcal{H} \otimes \mathcal{K}$ be their tensor product. If $Q : \mathcal{H} \otimes \mathcal{K} \to \mathcal{H} \otimes \mathcal{K}$ is a density matrix, then the **partial trace**

$$(\mathrm{tr}_{\mathcal{K}}\, Q) : \quad \mathcal{H} \to \mathcal{H}, \quad h \mapsto \sum_{k \in \mathbb{N}} \langle Q(h \otimes f_k), f_k \rangle_{\mathcal{K}}\,,$$

defines a density matrix on $\mathcal{H}$. It is called the **reduced density matrix**.

Remark 4.15. If $Q : \mathcal{H} \to \mathcal{H}$ is a density matrix, then Q is a compact Hermitian operator and has a spectral representation [24, Sec.VI.3] given by

$$Q(h) = \sum_{k=1}^{\infty} \mu_k\, q_k \,\langle h, q_k \rangle \quad \text{with} \quad \mu_k \geq 0 \quad \text{and} \quad \{q_k\} \text{ orthonormal.}$$

In quantum systems the μ_k are interpreted as the probability of the system to be in state q_k, since they are non-negative quantities which add up to 1.

In light of this quantum mechanical language, we present the following alternative interpretation of Theorem 4.12.

Corollary 4.16. *Let* $\hat{\alpha} : \mathbb{R}_{\geq 0} \to \mathcal{M}^1_{\hat{\mathbf{z}}}$ *be a solution of* (HHt) *and* $\hat{\alpha}_N : \mathbb{R}_{\geq 0} \to \mathcal{M}^1_{\hat{\vec{\mathbf{z}}}}$ *a sequence of solutions, all subject to the constraints of Theorem 4.12. Define the pure density matrices*

$$Q_N(t) : \quad \mathcal{L}^2_{\hat{\mathbf{z}}_1,\ldots,\hat{\mathbf{z}}_N} \to \mathcal{L}^2_{\hat{\mathbf{z}}_1,\ldots,\hat{\mathbf{z}}_N}, \quad h \mapsto \langle h, \hat{\alpha}_N(t) \rangle\, \hat{\alpha}_N(t)$$

and

$$Q(t) : \quad \mathcal{L}^2_{\hat{\mathbf{z}}} \to \mathcal{L}^2_{\hat{\mathbf{z}}}, \quad h \mapsto \langle h, \hat{\alpha}(t) \rangle\, \hat{\alpha}(t).$$

Then, via the representation $\mathcal{L}^2_{\hat{\mathbf{z}}_1,\ldots,\hat{\mathbf{z}}_N} = \mathcal{L}^2_{\hat{\mathbf{z}}} \otimes \mathcal{L}^2_{\hat{\mathbf{z}}_2,\ldots,\hat{\mathbf{z}}_N}$*, for any* $t \geq 0$*, we have that*

$$\left\langle \hat{\alpha}_N(0), \mathfrak{m}_\lambda^{\hat{\alpha}(0)} \hat{\alpha}_N(0) \right\rangle \overset{N\to\infty}{\to} 0 \quad \Rightarrow \quad \left\| \mathrm{tr}_{\mathcal{L}^2_{\hat{\mathbf{z}}_2,\ldots,\hat{\mathbf{z}}_N}} Q_N(t) - Q(t) \right\|_{\mathcal{L}^2_{\hat{\mathbf{z}}} \to \mathcal{L}^2_{\hat{\mathbf{z}}}} \overset{N\to\infty}{\to} 0.$$

Proof. Let $t \geq 0$ be given. By Theorem 4.12, we get that

$$\left\| q_1^{\hat{\alpha}(t)} \hat{\alpha}_N(t) \right\|^2_{\mathcal{L}^2_{\hat{\vec{\mathbf{z}}}}} \overset{\text{Lem.4.7-(ii)}}{=} \left\langle \hat{\alpha}_N(t), \left(\mathfrak{n}^{\hat{\alpha}(t)}\right)^2 \hat{\alpha}_N(t) \right\rangle \overset{\text{Rmk.4.6-(ii)}}{\leq} \left\langle \hat{\alpha}_N(t), \mathfrak{m}_\lambda^{\hat{\alpha}(t)} \hat{\alpha}_N(t) \right\rangle$$
$$\leq \left| \left\langle \hat{\alpha}_N(t), \mathfrak{m}_\lambda^{\hat{\alpha}(t)} \hat{\alpha}_N(t) \right\rangle - \left\langle \hat{\alpha}_N(0), \mathfrak{m}_\lambda^{\hat{\alpha}(0)} \hat{\alpha}_N(0) \right\rangle \right| + \left\langle \hat{\alpha}_N(0), \mathfrak{m}_\lambda^{\hat{\alpha}(0)} \hat{\alpha}_N(0) \right\rangle \overset{N\to\infty}{\to} 0.$$

Therefore, we omit the time argument t for the rest of the proof. Using the $\mathcal{L}^2$ kernel function representation of the density matrix following [19, Lem.2.3-(a)], we compute

$$\begin{aligned}
\operatorname{tr}_{\mathcal{L}^2_{\hat{\mathbf{z}}_2,\dots,\hat{\mathbf{z}}_N}} Q_N &= \int_{\mathbb{R}^{2d}_{\hat{\mathbf{z}}}} \cdots \int_{\mathbb{R}^{2d}_{\hat{\mathbf{z}}}} \bar{\hat{\alpha}}_N(\cdot, \hat{\mathbf{z}}_2, \dots, \hat{\mathbf{z}}_N)\, \hat{\alpha}_N(\cdot, \hat{\mathbf{z}}_2, \dots, \hat{\mathbf{z}}_N)\, \mathrm{d}\hat{\mathbf{z}}_2 \cdots \mathrm{d}\hat{\mathbf{z}}_N \\
&= \int_{\mathbb{R}^{2d}_{\hat{\mathbf{z}}}} \cdots \int_{\mathbb{R}^{2d}_{\hat{\mathbf{z}}}} p_1^{\hat{\alpha}} \bar{\hat{\alpha}}_N(\cdot, \hat{\mathbf{z}}_2, \dots, \hat{\mathbf{z}}_N)\, p_1^{\hat{\alpha}} \hat{\alpha}_N(\cdot, \hat{\mathbf{z}}_2, \dots, \hat{\mathbf{z}}_N)\, \mathrm{d}\hat{\mathbf{z}}_2 \cdots \mathrm{d}\hat{\mathbf{z}}_N \\
&\quad + \int_{\mathbb{R}^{2d}_{\hat{\mathbf{z}}}} \cdots \int_{\mathbb{R}^{2d}_{\hat{\mathbf{z}}}} p_1^{\hat{\alpha}} \bar{\hat{\alpha}}_N(\cdot, \hat{\mathbf{z}}_2, \dots, \hat{\mathbf{z}}_N)\, q_1^{\hat{\alpha}} \hat{\alpha}_N(\cdot, \hat{\mathbf{z}}_2, \dots, \hat{\mathbf{z}}_N)\, \mathrm{d}\hat{\mathbf{z}}_2 \cdots \mathrm{d}\hat{\mathbf{z}}_N \\
&\quad + \int_{\mathbb{R}^{2d}_{\hat{\mathbf{z}}}} \cdots \int_{\mathbb{R}^{2d}_{\hat{\mathbf{z}}}} q_1^{\hat{\alpha}} \bar{\hat{\alpha}}_N(\cdot, \hat{\mathbf{z}}_2, \dots, \hat{\mathbf{z}}_N)\, p_1^{\hat{\alpha}} \hat{\alpha}_N(\cdot, \hat{\mathbf{z}}_2, \dots, \hat{\mathbf{z}}_N)\, \mathrm{d}\hat{\mathbf{z}}_2 \cdots \mathrm{d}\hat{\mathbf{z}}_N \\
&\quad + \int_{\mathbb{R}^{2d}_{\hat{\mathbf{z}}}} \cdots \int_{\mathbb{R}^{2d}_{\hat{\mathbf{z}}}} q_1^{\hat{\alpha}} \bar{\hat{\alpha}}_N(\cdot, \hat{\mathbf{z}}_2, \dots, \hat{\mathbf{z}}_N)\, q_1^{\hat{\alpha}} \hat{\alpha}_N(\cdot, \hat{\mathbf{z}}_2, \dots, \hat{\mathbf{z}}_N)\, \mathrm{d}\hat{\mathbf{z}}_2 \cdots \mathrm{d}\hat{\mathbf{z}}_N.
\end{aligned}$$

The first term is exactly $\left\| p_1^{\hat{\alpha}} \hat{\alpha}_N \right\|^2_{\mathcal{L}^2_{\vec{\hat{\mathbf{z}}}}} Q$, the others are dominated in operator norm by the respective $\mathcal{L}^2$ norms, yielding

$$\begin{aligned}
\left\| \operatorname{tr}_{\mathcal{L}^2_{\hat{\mathbf{z}}_2,\dots,\hat{\mathbf{z}}_N}} Q_N - Q \right\|_{\mathcal{L}^2_{\hat{\mathbf{z}}} \to \mathcal{L}^2_{\hat{\mathbf{z}}}} &\le \left| \left\| p_1^{\hat{\alpha}} \hat{\alpha}_N \right\|^2_{\mathcal{L}^2_{\vec{\hat{\mathbf{z}}}}} - 1 \right| \|Q\|_{\mathcal{L}^2_{\hat{\mathbf{z}}} \to \mathcal{L}^2_{\hat{\mathbf{z}}}} + 2 \left\| p_1^{\hat{\alpha}} \hat{\alpha}_N \right\|_{\mathcal{L}^2_{\vec{\hat{\mathbf{z}}}}} \left\| q_1^{\hat{\alpha}} \hat{\alpha}_N \right\|_{\mathcal{L}^2_{\vec{\hat{\mathbf{z}}}}} \\
&\quad + \left\| q_1^{\hat{\alpha}} \hat{\alpha}_N \right\|^2_{\mathcal{L}^2_{\vec{\hat{\mathbf{z}}}}} \\
&\le 4 \left\| q_1^{\hat{\alpha}} \hat{\alpha}_N \right\|_{\mathcal{L}^2_{\vec{\hat{\mathbf{z}}}}} \to 0,
\end{aligned}$$

proving the convergence in the operator norm of the reduced density matrices to the pure state Q. □

In a way, the partial trace operation is the $\mathcal{L}^2$ counterpart of the integration of the unobserved variables yielding the marginal distributions for the classical Vlasov system, discussed at the beginning of this chapter.

Chapter 5

Symmetry Reduction and Periodicity

This chapter is mainly adapted from the author's work [15].

The problem of finding periodic solutions of Vlasov type equations has been around for a couple of decades. While there exist some results for solutions on periodic domains [3], or periodic solutions under boundary conditions [4], the existence of periodic solutions on the full domain $\mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{v}}$, in particular $d = 3$, for any type of interaction potential has not been treated yet.

The construction of periodic solutions can be achieved mainly via two methods, both of which require a Hamiltonian formulation of the dynamical system. One option is to find extremal points of the action functional

$$\Phi(\lambda, \alpha) = \int_0^1 \left(\frac{\lambda}{2}\, \omega\left(\alpha(t), \partial_t \alpha(t)\right) + \mathcal{H}_{\mathrm{Vl}}(\alpha(t)) \right) \mathrm{d}t, \quad \alpha : \mathbb{R}/\mathbb{Z} \to \mathcal{L}^2_{\mathbf{z}}, \quad \lambda > 0,$$

evaluated on a Banach space of $\mathcal{L}^2_{\mathbf{z}}$ valued closed curves. Given a semi-bounded Hamiltonian, one could potentially apply some mountain pass techniques. However, in this particular case, the mountain pass method turns out to be inappropriate as in general $\mathcal{H}_{\mathrm{Vl}}(\bar{\alpha}) = -\mathcal{H}_{\mathrm{Vl}}(\alpha)$ and the functional is not semi-bounded.

Alternatively, one can first try to identify stationary points of the dynamics and find non-resonant eigenvalues of the second derivative of the Hamiltonian. This allows to construct families of closed curves oscillating around the stationary point. The bifurcation equation of interest

$$\lambda\, \partial_t \alpha(t) - X_{\mathcal{H}_{\mathrm{Vl}}}(\alpha(t)) = 0, \quad \alpha : \mathbb{R}/\mathbb{Z} \to \mathcal{L}^2_{\mathbf{z}}, \quad \lambda > 0 \qquad \text{(Bfc)}$$

is exactly the extremal condition of the functional Φ. This method is well developed for many finite and infinite dimensional systems [1, Chap.5-7]. The existence of such periodic bifurcations strongly relies on a non-resonance condition of the respective eigenvalues of $\mathrm{D}X_{\mathcal{H}_{\mathrm{Vl}}}(\mathring{\alpha})$ at some equilibrium point $\mathring{\alpha}$. Finding and classifying equilibrium states based on their stability is also an active topic of research [13, 20].

The Hamiltonian Vlasov system reveals a couple of remarkable features that underline its potential relevance for further studies. For example, it is possible to identify conserved quantities of the Vlasov system such as mass, linear momentum, and angular momentum as Noether conjugates of continuous symmetries for the first time.

Identifying and removing continuous symmetries of the system is a major prerequisite to solve the bifurcation equation (Bfc). Naturally, symmetries embed every solution into a continuous family of solutions. This disrupts the use of the implicit function theorem in all methods designed to solve the bifurcation problem. A very strategic approach to deal with this sort of degeneracy gives the method of symplectic symmetry reduction usually attributed to Marsden–Weinstein [12]. It is a two-step procedure, where at first the motion is restricted to level sets of the conserved quantities and then the orbits of the group action are contracted in a quotient manifold, thereby removing this degeneracy. As it has already proven its value in many other examples, we want to explore its advantages in this new PDE setup.

In adapting the method, we find two types of symmetries. The first type ($\mathbb{S}^1$ phase invariance) acts smoothly on the model Banach space ($\subseteq \mathcal{L}^2$) and can be treated by symplectic symmetry reduction following Marsden and Weinstein [12]. As it turns out, reduction of the phase symmetry is not only convenient, but actually necessary in order to identify periodic solutions.

The second type (translation invariance) is much more difficult to overcome. The key issue is that the group action is not smooth anymore, only continuous. That defies any chance to apply global symmetry reduction and projection onto a symplectic quotient manifold based on existing results as a key condition is violated. Nevertheless there is hope to overcome the presented difficulties with some local reduction principle around equilibria points, if one is able to choose an appropriate topological setup. Still, this remains as an open problem as technical difficulties with regularities arise.

The phase invariance of the Hamiltonian yields mass conservation and motivates restriction to the sphere $\mathbb{S}^{\mathcal{L}^2}$. Projection onto the quotient space $\mathbb{S}^{\mathcal{L}^2}/\mathbb{S}^1$ not only increases the number of stationary points, but also predicts the correct periodic families bifurcating around them in the studied example of the Harmonic Vlasov system. This is a system with non-relativistic kinetic energy and harmonic two-body interaction. Indeed, these bifurcating families are invisible for other methods, since in the unreduced system they are also governed by a global phase oscillation usually in irrational relation to the profile oscillation, hence, only relatively periodic in the notion of [12]. For our simplified example of the Harmonic Vlasov system, we find a one-to-one correspondence between the spectrum of the Hamiltonian vector field and families of bifurcating periodic solutions in the direction of the eigenvectors (Theorem 5.10). This indicates the method's potential impact on more complicated problems.

The chapter is structured as follows. Section 5.1 gives a short introduction to the principle of symplectic symmetry reduction alongside its application to Hamiltonian PDEs in general. It also discusses the setup of the Hamiltonian Vlasov formulation. In Section 5.2 the method is applied to the Harmonic Vlasov system. At first, the spectra of the linearized Hamiltonian vector field are evaluated around stationary points, then the associated families of periodic solutions are explicitly computed.

5.1 Symplectic Symmetry Reduction

The idea of symplectic symmetry reduction in the sense of Marsden and Weinstein [12] is the reduction of the phase space of a Hamiltonian system according to its continuous symmetries and the associated conserved quantities. This procedure is particularly useful as it removes degeneracies around bifurcation points. For completeness, we give a short overview of the method, before applying it in the specific setup of the Hamiltonian Vlasov system. We remark that the method applies to all Hamiltonian PDEs if they fit into an appropriate topological framework. More precisely, a smooth group action is a necessary requirement. Among others, this is the case for $\mathbb{S}^1$ phase multiplication in the NLS equation, the Hartree equation, and the massive Thirring model.

5.1.1 On Linear Symplectic Spaces

Consider a symplectic phase space (M, ω), for the sake of simplicity assumed to be a linear space M with a constant symplectic form ω. Let $H : M \to \mathbb{R}$ be a smooth Hamiltonian and the associated Hamiltonian vector field $X_H : M \to \mathrm{T}M$ be implicitly given by

$$\mathrm{D}H(\alpha)(\delta\alpha) = -\omega\,(X_H(\alpha), \delta\alpha) \quad \forall(\alpha, \delta\alpha) \in \mathrm{T}M, \tag{5.1}$$

where $\mathrm{T}M$ denotes the tangent bundle. The Hamiltonian equation of motion is now given by

$$\partial_t \alpha(t) = X_H(\alpha(t)). \tag{5.2}$$

Suppose there is a Lie group G, for the sake of simplicity assumed to be Abelian and finite-dimensional, acting smoothly on the phase space (M, ω) by

$$\mu : \quad G \times M \to M, \quad (g, \alpha) \mapsto \mu(g, \alpha) \equiv g.\alpha,$$

leaving the symplectic form invariant, i.e.,

$$\forall g \in G : \quad \mu(g, \cdot)_* \omega = \omega$$

under the push-forward with respect to any map $\mu(g, \cdot)$. This is sometimes denoted as $G \subseteq \mathrm{Symp}(M)$. Also assume that G is a continuous symmetry of the Hamiltonian, i.e.,

$$\forall g \in G,\ \alpha \in M : \quad H(g.\alpha) = H(\alpha). \tag{5.3}$$

Example 5.1 (Nonlinear Schrödinger Equation). The NLS equation fits into the given framework. The manifold is $M = \mathcal{L}^2_x$, equipped with the symplectic form

$$\omega\,(u, v) \equiv \Im \int_{\mathbb{R}} u\,\bar{v}\,\mathrm{d}x,$$

the group action

$$\mathbb{S}^1 \times \mathcal{L}^2_x \to \mathcal{L}^2_x, \quad (\zeta, u) \mapsto \zeta\, u,$$

and the Hamiltonian functional

$$\mathcal{H}(u) \equiv \frac{1}{2} \int_{\mathbb{R}} \left(|\partial_x u|^2 - \frac{1}{2}\,|u|^4 \right) \mathrm{d}x.$$

The Hamiltonian formalism yields the well-known NLS equation

$$\mathrm{i}\,\partial_t u = -\partial_x^2 u - |u|^2\, u.$$

The symplectic form and the Hamiltonian are also invariant under translation in x, but this group action is not smooth, only continuous.

The existence of such a symmetry group G raises two major concerns for our analysis. Firstly, it embeds any equilibrium into a continuous family, as shifting by group elements yields more equilibria of the same type. This causes technical problems especially for bifurcation theory, as these families contribute to the kernel of $\mathrm{D}X_H$, disallowing to solve

$$\lambda\,\partial_t \alpha(t) - X_H(\alpha(t)) = 0, \quad \alpha : \mathbb{R}/\mathbb{Z} \to M, \quad \lambda > 0$$

locally and thereby to prove existence of periodic families for certain frequencies λ.

On the other hand, the symmetry operations are often of no physical interest, but can easily destroy periodicity. For example, a global $\mathbb{S}^1$ phase oscillation is usually irrelevant as it does not change the profile of the wave package. The same holds for profiles traveling at constant speed, such as soliton solutions of the NLS equation.

The symplectic symmetry reduction solves both of these problems, removing the degeneracies of equilibria arising from the action of G and deleting the non-relevant motion along orbits of the group action. The newly generated equilibria in the quotient system are then referred to as *relative equilibria*. They still carry all the qualitatively interesting information.

The process of symmetry reduction is summarized in the following steps.

(i) From Equations (5.1), (5.2), and (5.3) one computes the full set of conserved quantities implied by the Lie group G (Noether's Theorem). The number is equal to the (real) dimension q of G. The resulting map

$$C : M \to \mathbb{R}^q.$$

is usually referred to as the *moment map*.

(ii) From the conservation laws we know that the trajectories of (5.2) remain in level sets of C. Therefore it suffices to restrict ourselves to the level set $C^{-1}(\mathbf{c})$ of a regular value $\mathbf{c} \in \mathbb{R}^q$, enforcing the set to carry the structure of a manifold.

(iii) The Marsden–Weinstein reduction now states that there is a smooth quotient map

$$\Pi : C^{-1}(\mathbf{c}) \to C^{-1}(\mathbf{c})/G,$$

contracting orbits of the group action, such that $C^{-1}(\mathbf{c})/G$ has a symplectic form Ω, which satisfies

$$\iota^*\omega = \Pi^*\Omega$$

for the embedding $\iota : C^{-1}(\mathbf{c}) \to M$, where ι^* denotes the pullback.

(iv) Finally, if $\bar{H} : C^{-1}(\mathbf{c})/G \to \mathbb{R}$ is chosen to satisfy $H \circ \iota = \bar{H} \circ \Pi$, which is possible by (5.3), we obtain that the following diagram commutes:

$$\begin{array}{ccc} C^{-1}(\mathbf{c}) & \xrightarrow{\Pi} & \left(C^{-1}(\mathbf{c})/G, \Omega\right) \\ \big\downarrow \iota & & \big\downarrow \bar{H} \\ (M, \omega) & \xrightarrow{H} & \mathbb{R}. \end{array}$$

If $X_{\bar{H}}$ is the Hamiltonian vector field of $\bar{H}$ w.r.t. Ω, then

$$\forall \alpha \in C^{-1}(\mathbf{c}) : \quad X_{\bar{H}}(\Pi\alpha) = \mathrm{d}\Pi_\alpha(X_H(\alpha)).$$

This key relation indicates how to compute the new Hamiltonian vector field locally in charts of the quotient manifold, which is otherwise quite challenging in applications. It also shows that motion purely governed by symmetry transformations of G is canceled, yielding new *relative equilibria*, not even detectable in the unreduced system. In charts around these points it is now possible to use bifurcation theory in search of *relatively periodic families*.

5.1.2 On the Hamiltonian Vlasov System

While the structural approach of Section 5.1.1 is very general, we want to study the special case of the Vlasov system in its Hamiltonian form. Even though the Hamiltonian formalism is widely used to understand equations such as the NLS or Hartree equation, this is not the case for the Vlasov system yet. However, the concepts of this section are applicable to all the aforementioned systems.

Let us consider a classical Vlasov system with an energy functional $\mathcal{H}(f) \in \mathbb{R}$, kinetic energy $\epsilon : \mathbb{R}^d_{\mathbf{v}} \to \mathbb{R}$, and two-body interaction potential $\Gamma : \mathbb{R}^d_{\mathbf{x}} \to \mathbb{R}$, that is,

$$\begin{aligned}\mathcal{H}(f) &\equiv \int_{\mathbb{R}^{2d}_{\mathbf{z}}} \left(\epsilon\,(\mathbf{v}) + \frac{1}{2}\,(\Gamma * f)\,(\mathbf{x})\right)\, f(\mathbf{x},\mathbf{v})\;\mathrm{d}(\mathbf{x},\mathbf{v}),\\ \mathcal{H}_{\mathrm{Vl}}(\alpha) &\equiv \frac{1}{2\mathrm{i}}\mathrm{D}^1\mathcal{H}\left(|\alpha|^2\right)([\bar{\alpha},\alpha]) = \frac{1}{2\mathrm{i}}\int_{\mathbb{R}^{2d}_{\mathbf{z}}} \left(\epsilon(\mathbf{v}) + \left(\Gamma * |\alpha|^2\right)(\mathbf{x})\right)\;[\bar{\alpha},\alpha]\;\mathrm{d}\mathbf{z}.\end{aligned} \tag{5.4}$$

The corresponding equation of motion then is

$$\begin{aligned}\partial_t\alpha(t) = X_{\mathcal{H}_{\mathrm{Vl}}}(\alpha(t)) &= \left[\epsilon(\mathbf{v}) + \left(\Gamma * |\alpha(t)|^2\right)(\mathbf{x}), \alpha(t)\right] - \left(\Gamma * [\bar{\alpha}(t),\alpha(t)]\right)\;\alpha(t) \qquad (5.5)\\ &= -\,(\nabla_{\mathbf{v}}\epsilon)(\mathbf{v})\cdot\nabla_{\mathbf{x}}\alpha(t) + \left(\nabla\Gamma * |\alpha(t)|^2\right)(\mathbf{x})\cdot\nabla_{\mathbf{v}}\alpha(t) - \left(\Gamma * [\bar{\alpha}(t),\alpha(t)]\right)(\mathbf{x})\;\alpha(t).\end{aligned}$$

If the system's energy is structured as in (5.4), then there is a number of general continuous symmetries. These symmetries naturally imply conserved quantities as predicted by Noether's Theorem.

Proposition 5.2 (Continuous Symmetries and Noether's Theorem)**.** *Any Vlasov Hamiltonian $\mathcal{H}_{Vl}$ structured as in* (5.4) *is invariant under the continuous action of the* $2d+1$ *dimensional Lie group $G = \mathbb{S}^1 \times \mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\boldsymbol{\xi}}$ given by*

$$G \times \mathcal{L}^2_{\mathbf{z}} \to \mathcal{L}^2_{\mathbf{z}}, \quad (g,\alpha) = \left((\zeta,\mathbf{x}_0,\boldsymbol{\xi}_0),\alpha\right) \mapsto (g.\alpha)\,(\mathbf{x},\mathbf{v}) \equiv \zeta\;\alpha(\mathbf{x}-\mathbf{x}_0,\mathbf{v})\;\exp\left(\mathrm{i}\mathbf{v}\cdot\boldsymbol{\xi}_0\right).$$

This implies that any solution $t \mapsto \alpha(t)$ of (5.5) *conserves the* $2d+1$ *quantities*

$$t \mapsto \|\alpha(t)\|^2_{\mathcal{L}^2_{\mathbf{z}}}, \quad t \mapsto \langle\alpha(t), \mathrm{i}\,\nabla_{\mathbf{x}}\alpha(t)\rangle, \quad t \mapsto \langle\alpha(t), \mathbf{v}\,\alpha(t)\rangle,$$

if they are well defined.

Proof. **(i) Constant phase invariance and mass conservation.** The $\mathbb{S}^1$ phase invariance of $\mathcal{H}_{\text{Vl}}$ is immediate from (5.4). Let $t \mapsto \alpha(t)$ solve (5.5), then

$$\frac{1}{2}\,\partial_t\,\|\alpha(t)\|^2_{\mathcal{L}^2_{\mathbf{z}}} = \Re\,\langle\partial_t\alpha(t),\alpha(t)\rangle = \Re\,\langle X_{\mathcal{H}_{\text{Vl}}}(\alpha(t)),\alpha(t)\rangle = \omega\left(X_{\mathcal{H}_{\text{Vl}}}(\alpha(t)),\mathrm{i}\alpha(t)\right)$$
$$\overset{\text{Def.}}{=} -\,\mathrm{D}\mathcal{H}_{\text{Vl}}(\alpha(t))(\mathrm{i}\alpha(t)) = -\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0}\mathcal{H}_{\text{Vl}}\left(\left(e^{\mathrm{i}\tau},\mathbf{0},\mathbf{0}\right).\alpha(t)\right) = 0.$$

(ii) Translation invariance and pseudo momentum conservation. The translation invariance in $\mathbf{x}$ is also immediate from (5.4). For any $1 \le i \le d$, one computes

$$\frac{1}{2}\,\partial_t\,\langle\alpha(t),\mathrm{i}\,\partial_{x_i}\alpha(t)\rangle = \Re\,\langle\partial_t\alpha(t),\mathrm{i}\,\partial_{x_i}\alpha(t)\rangle = \omega\left(X_{\mathcal{H}_{\text{Vl}}}(\alpha(t)),\partial_{x_i}\alpha(t)\right)$$
$$\overset{\text{Def.}}{=} -\,\mathrm{D}\mathcal{H}_{\text{Vl}}(\alpha(t))(\partial_{x_i}\alpha(t)) = -\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0}\mathcal{H}_{\text{Vl}}((1,\tau\mathbf{e}_i,\mathbf{0}).\alpha(t)) = 0.$$

(iii) Linear phase invariance and linear momentum conservation. This symmetry is not directly obvious, though its implication, conservation of linear momentum, is well known. For $\boldsymbol{\xi}_0 \in \mathbb{R}^d_{\boldsymbol{\xi}}$, one gets

$$\left[e^{-\mathrm{i}\mathbf{v}\cdot\boldsymbol{\xi}_0}\bar{\alpha},e^{\mathrm{i}\mathbf{v}\cdot\boldsymbol{\xi}_0}\alpha\right] - [\bar{\alpha},\alpha] = [\bar{\alpha},\mathrm{i}\mathbf{v}\cdot\boldsymbol{\xi}_0]\,\alpha + [-\mathrm{i}\mathbf{v}\cdot\boldsymbol{\xi}_0,\alpha]\,\bar{\alpha} = \mathrm{i}\left[|\alpha|^2,\mathbf{v}\cdot\boldsymbol{\xi}_0\right] = \mathrm{i}\,\nabla_{\mathbf{x}}\,|\alpha|^2\cdot\boldsymbol{\xi}_0,$$

which contributes neither to the kinetic, nor to the potential energy term of $\mathcal{H}_{\text{Vl}}$. The associated conserved quantity is derived as in (i) and (ii). □

Remark 5.3. **(i).** For an SO(d) invariant kinetic energy ϵ and interaction potential Γ, the quantities

$$t \mapsto \left\langle\alpha,\left(x_i\,\mathrm{i}\,\partial_{x_j} - x_j\,\mathrm{i}\,\partial_{x_i}\right)\alpha\right\rangle,\quad 1 \le i < j \le d$$

are also (Noether conjugate) conserved.

(ii). In contrast with classical mechanics, momentum conservation here is not a consequence of translation invariance, but rather implied by linear phase invariance. This linear phase invariance is actually translation invariance in the Fourier conjugate of the velocity coordinates.

Remark 5.4 (Moment Map). In the geometric literature, the key quantity for the symmetry reduction principle is the moment map. It is usually defined as a map $M \to \mathfrak{g}^*$ with values in the dual space of the Lie algebra $\mathfrak{g} = \mathrm{T}_{1_G}G$.

We want to show that in our application the moment map is equivalent to the conserved quantities from Proposition 5.2. Given $G = \mathbb{S}^1 \times \mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\boldsymbol{\xi}}$, for any $(s, \mathbf{y}, \boldsymbol{\eta}) \in \mathbb{R} \times \mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\boldsymbol{\xi}} = \mathfrak{g}$ define the vector field

$$X_{(s,\mathbf{y},\boldsymbol{\eta})}(\alpha) \equiv \left.\frac{\partial}{\partial \tau}\right|_{\tau=0} \exp\left(\tau(s, \mathbf{y}, \boldsymbol{\eta})\right).\alpha = \left(\mathrm{i}s + \mathbf{y} \cdot \nabla_{\mathbf{x}} + \mathrm{i}\boldsymbol{\eta} \cdot \mathbf{v}\right)\alpha,$$

given through the derivative of the exponential map $\exp : \mathfrak{g} \to G$. In this setup, each of these vector fields is Hamiltonian in the sense that it is generated by a real valued function on (a dense subspace of) $\mathcal{L}^2_{\mathbf{z}}$, explicitly given by

$$-\omega\left(X_{(s,\mathbf{y},\boldsymbol{\eta})}(\alpha), \delta\alpha\right) = \Re \left\langle s\,\alpha + \mathbf{y} \cdot \mathrm{i}\nabla_{\mathbf{x}}\alpha + \boldsymbol{\eta} \cdot \mathbf{v}\,\alpha, \delta\alpha \right\rangle = \mathrm{D}C_{(s,\mathbf{y},\boldsymbol{\eta})}(\alpha)(\delta\alpha)$$

for

$$C_{(s,\mathbf{y},\boldsymbol{\eta})}(\alpha) = \frac{s}{2}\,\|\alpha\|^2_{\mathcal{L}^2_{\mathbf{z}}} + \frac{\mathbf{y}}{2} \cdot \langle \alpha, \mathrm{i}\nabla_{\mathbf{x}}\alpha \rangle + \frac{\boldsymbol{\eta}}{2} \cdot \langle \alpha, \mathbf{v}\alpha \rangle = \frac{1}{2} \begin{pmatrix} \|\alpha\|^2_{\mathcal{L}^2_{\mathbf{z}}} \\ \langle \alpha, \mathrm{i}\,\nabla_{\mathbf{x}}\alpha \rangle \\ \langle \alpha, \mathbf{v}\,\alpha \rangle \end{pmatrix} \cdot \begin{pmatrix} s \\ \mathbf{y} \\ \boldsymbol{\eta} \end{pmatrix},$$

which is linear in $(s, \mathbf{y}, \boldsymbol{\eta})$ and naturally defines an element in the dual space $\mathfrak{g}^*$ for every α. Hence, under a linear isomorphism $\mathfrak{g}^* \simeq \mathbb{R}^{2d+1}$, a vector of the conserved quantities from Proposition 5.2 equals the moment map, justifying our choice of notation.

At first sight, the phase invariance is an unwanted degeneracy, because it does not seem to reflect physical properties of the classical Vlasov system. All phase information is lost upon the mapping $\alpha \mapsto |\alpha|^2$ anyway. Nevertheless, the phase information is not artificial at all, since in this Hamiltonian formulation, it provides two continuous symmetries that are the Noether conjugates of mass and momentum conservation.

Following the guidelines of the Marsden–Weinstein reduction, it is desirable to cancel out the group action, which corresponds to the phase oscillation $(d+1)$ and translation (d) in the dynamics of the system. The huge technical problem here is that the group action is not smooth. Smoothness is required to ensure that the quotient space is a smooth manifold and to define its differentiable structure. It seems that this obstruction is not easily removed. From the structure of the derivative it is also obvious that this is not possible for a space of non-smooth functions.

Here, we restrict ourselves to applying the reduction only with respect to a global phase multiplication which is smooth. Clearly, the global phase multiplication is invisible in the classical Vlasov picture after mapping solutions under $\alpha \mapsto |\alpha|^2$.

Proposition 5.5 (Phase Equivariant Symplectic Reduction)**.** *Consider the symplectic vector space $\left(\mathcal{L}^2_{\mathbf{z}}, \omega\right)$ with the group action*

$$\mathbb{S}^1 \times \mathcal{L}^2_{\mathbf{z}} \to \mathcal{L}^2_{\mathbf{z}}, \quad (\zeta, \alpha) \mapsto \zeta\,\alpha.$$

Given its associated moment map

$$C : \mathcal{L}^2_{\mathbf{z}} \mapsto \mathbb{R}, \quad \alpha \mapsto \frac{1}{2} \|\alpha\|^2_{\mathcal{L}^2_{\mathbf{z}}},$$

$1/2 \in \mathbb{R}$ is a regular value. By the Marsden–Weinstein reduction, there is a unique symplectic form Ω on the smooth Hilbert manifold $\mathbb{S}^{\mathcal{L}^2_{\mathbf{z}}}/\mathbb{S}^1$, s.t. under the embedding and projection maps

$$\iota : \mathbb{S}^{\mathcal{L}^2_{\mathbf{z}}} \to \mathcal{L}^2_{\mathbf{z}} \quad \textit{and} \quad \Pi : \mathbb{S}^{\mathcal{L}^2_{\mathbf{z}}} \to \mathbb{S}^{\mathcal{L}^2_{\mathbf{z}}}/\mathbb{S}^1,$$

the relation

$$\iota^* \omega = \Pi^* \Omega$$

holds. Also, given a $\mathbb{S}^1$-invariant Hamiltonian $\mathcal{H}$ on $\mathcal{L}^2_{\mathbf{z}}$ and its push-forward $\bar{\mathcal{H}}$ on $\mathbb{S}^{\mathcal{L}^2_{\mathbf{z}}}/\mathbb{S}^1$, the respective Hamiltonian vector fields satisfy

$$\forall \alpha \in \mathbb{S}^{\mathcal{L}^2_{\mathbf{z}}} : \quad X_{\bar{\mathcal{H}}}(\Pi\alpha) = (\Pi_* X_{\mathcal{H}})(\Pi\alpha) = \mathrm{d}\Pi_\alpha(X_{\mathcal{H}}(\alpha)).$$

Proof. The group action is smooth, free, and proper (since $\mathbb{S}^1$ is compact). It is also symplectic as it leaves the symplectic form ω invariant. The moment map is constructed as in Remark 5.4. The rest is a straightforward application of the reduction theorem [12, Thm.1]. The claim on the Hamiltonian vector fields is just an application of [12, Cor.3]. □

With explicit computations in mind, it is useful to give coordinate representations for the manifolds and important maps between them.

Lemma 5.6 (Coordinate Representations)**.** *We have the following explicit coordinate representations:*

(i) The tangent bundle $\mathrm{T}\mathbb{S}^{\mathcal{L}^2_{\mathbf{z}}}$ can be parameterized as a subset of $\mathcal{L}^2_{\mathbf{z}}$ by

$$\mathrm{T}\mathbb{S}^{\mathcal{L}^2_{\mathbf{z}}} \simeq \left\{ (\alpha, \delta\alpha) \in \left(\mathcal{L}^2_{\mathbf{z}}\right)^2 : \|\alpha\|_{\mathcal{L}^2_{\mathbf{z}}} = 1, \Re \langle \alpha, \delta\alpha \rangle = 0 \right\},$$

(ii) its quotient space $\mathrm{T}\left(\mathbb{S}^{\mathcal{L}^2_{\mathbf{z}}}/\mathbb{S}^1\right)$ allows the natural chart

$$\mathrm{T}\left(\mathbb{S}^{\mathcal{L}^2_{\mathbf{z}}}/\mathbb{S}^1\right) \simeq \left\{ \begin{array}{c} \mathbb{S}^1(\alpha, \delta\alpha) \in \left(\mathbb{S}^{\mathcal{L}^2_{\mathbf{z}}} \times \mathcal{L}^2_{\mathbf{z}}\right)/\mathbb{S}^1 : \\ \langle \alpha, \delta\alpha \rangle = 0 \end{array} \right\},$$

(iii) yielding for the projection map

$$\begin{array}{cccc} & \mathrm{T}\mathbb{S}^{\mathcal{L}^2_{\mathbf{z}}} & \to & \mathrm{T}\left(\mathbb{S}^{\mathcal{L}^2_{\mathbf{z}}}/\mathbb{S}^1\right) \\ \Pi : & \left\{ \begin{array}{c} (\alpha, \delta\alpha) \in \left(\mathcal{L}^2_{\mathbf{z}}\right)^2 : \\ \|\alpha\|_{\mathcal{L}^2_{\mathbf{z}}} = 1, \Re \langle \alpha, \delta\alpha \rangle = 0 \end{array} \right\} & \to & \left\{ \begin{array}{c} \mathbb{S}^1(\alpha, \delta\alpha) \in \left(\mathbb{S}^{\mathcal{L}^2_{\mathbf{z}}} \times \mathcal{L}^2_{\mathbf{z}}\right)/\mathbb{S}^1 : \\ \langle \alpha, \delta\alpha \rangle = 0 \end{array} \right\} \\ & (\alpha, \delta\alpha) & \mapsto & (\Pi\alpha, \mathrm{d}\Pi_\alpha \delta\alpha) = \mathbb{S}^1 (\alpha, \delta\alpha - \langle \delta\alpha, \alpha \rangle \alpha). \end{array}$$

(iv) Finally, the symplectic form Ω on $\mathbb{S}^{\mathcal{L}^2_{\mathbf{z}}}/\mathbb{S}^1$ is given by

$$\Omega : \begin{array}{ccc} \left\{ \begin{array}{c} \mathbb{S}^1(\alpha, \delta\alpha_1, \delta\alpha_2) \in \left(\mathbb{S}^{\mathcal{L}^2_{\mathbf{z}}} \times \left(\mathcal{L}^2_{\mathbf{z}}\right)^2\right)/\mathbb{S}^1 : \\ \langle \alpha, \delta\alpha_1 \rangle = \langle \alpha, \delta\alpha_2 \rangle = 0 \end{array} \right\} & \to & \mathbb{R} \\ \mathbb{S}^1(\alpha, \delta\alpha_1, \delta\alpha_2) & \mapsto & \Im \langle \delta\alpha_1, \delta\alpha_2 \rangle . \end{array}$$

Proof. All claims are self-evident. □

5.1.3 Remarks on Bifurcation

The main motivation behind the transformations made in Section 5.1.2 is to simplify the search for stationary points and bifurcating periodic solutions, as they are known to yield periodic solutions for the underlying classical Vlasov system.

The explored method has two notable advantages. Firstly, switching from the classical $\mathcal{L}^1$ formulation to a Hamiltonian $\mathcal{L}^2$ language grants access to methods relying on spectral theory. This links the Vlasov systems to a rich field of periodic bifurcations, previously applied to many Hamiltonian PDEs [1].

Secondly, canceling the symmetry of the phase equivariance by projecting onto the quotient manifold $\mathbb{S}^{\mathcal{L}^2}/\mathbb{S}^1$ actually increases the potential number of stationary points and periodic solutions of the Hamiltonian Vlasov system, since a pure uniform phase oscillation is now invisible to the dynamics. This has the great advantage that for any relative equilibrium on the quotient manifold, one can now try to solve the bifurcation equation

$$\lambda\, \partial_t \alpha(t) - X_{\tilde{\mathcal{H}}}(\alpha(t)) = 0, \quad \alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{S}^{\mathcal{L}^2}/\mathbb{S}^1, \quad \lambda > 0$$

in a small neighborhood on the tangent bundle $\mathrm{T}\left(\mathbb{S}^{\mathcal{L}^2}/\mathbb{S}^1\right)$, locally linearized to a $\mathbb{C}$-codimension 1 subspace of $\mathcal{L}^2$. Taking care of the phase oscillation degeneracy on the full $\mathcal{L}^2$ space is topologically complicated.

5.2 Example: the Harmonic Vlasov System

As a toy model, where the developed toolbox works almost perfectly, we want to classify the periodic solutions of the *Harmonic Vlasov system*. This is the system of non-relativistic motion with an attractive harmonic two-body interaction potential, i.e.,

$$\epsilon(\mathbf{v}) \equiv \frac{|\mathbf{v}|^2}{2} \quad \text{and} \quad \Gamma(\mathbf{x}) \equiv \frac{|\mathbf{x}|^2}{2}.$$

The system's energy functional is then

$$\mathcal{H}(f) = \int_{\mathbb{R}^{2d}_{\mathbf{z}}} \frac{|\mathbf{v}|^2}{2} f(\mathbf{z})\,\mathrm{d}\mathbf{z} + \frac{1}{2}\int_{\mathbb{R}^{2d}_{\mathbf{z}}\times\mathbb{R}^{2d}_{\mathbf{z}}} f(\mathbf{z}_1)\,\frac{(\mathbf{x}_1-\mathbf{x}_2)^2}{2}\,f(\mathbf{z}_2)\,\mathrm{d}(\mathbf{z}_1,\mathbf{z}_2), \tag{5.6}$$

yielding the Vlasov equation

$$\partial_t f(t) = \left[\frac{|\mathbf{v}|^2}{2} + (\Gamma * f(t))\,(\mathbf{x}), f(t)\right] = -\mathbf{v}\cdot\nabla_{\mathbf{x}} f(t) + (\nabla\Gamma * f(t))\cdot\nabla_{\mathbf{v}} f(t). \tag{Vl}$$

Since the condition of a fixed center of mass at $\mathbf{x} = \mathbf{0}$ along with unit mass $\|f\|_{\mathcal{L}^1_{\mathbf{z}}} = 1$ automatically implies that $(\nabla\Gamma * f(t))\,(\mathbf{x}) = \mathbf{x}$, the existence of a large variety of periodic solutions is not at all surprising for this model.

Proposition 5.7 (Solutions of the Harmonic Vlasov System)**.** *Let* $\mathring{f} : \mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\mathbf{v}} \to \mathbb{R}_{\geq 0}$ *be a differentiable function, s.t.*

$$\int_{\mathbb{R}^{2d}_{\mathbf{z}}} \mathring{f}(\mathbf{x},\mathbf{v})\,\mathrm{d}(\mathbf{x},\mathbf{v}) = 1 \quad \textit{and} \quad \int_{\mathbb{R}^{2d}_{\mathbf{z}}} (|\mathbf{x}| + |\mathbf{v}|)\,\mathring{f}(\mathbf{x},\mathbf{v})\,\mathrm{d}(\mathbf{x},\mathbf{v}) < \infty.$$

Then up to translation, $\mathring{f}$ *gives rise to a solution with period* 2π*. This period is not necessarily minimal.*

Proof. W.l.o.g. assume (by appropriate translation) that

$$\int_{\mathbb{R}^{2d}_{\mathbf{z}}} \begin{pmatrix} \mathbf{x} \\ \mathbf{v} \end{pmatrix} \mathring{f}(\mathbf{x},\mathbf{v})\,\mathrm{d}(\mathbf{x},\mathbf{v}) = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}.$$

One directly verifies that

$$f(t,\mathbf{x},\mathbf{v}) \equiv \mathring{f}\,(\mathbf{x}\cos t - \mathbf{v}\sin t, \mathbf{x}\sin t + \mathbf{v}\cos t)$$

is the corresponding solution of (Vl). Adding the center of mass motion for general initial data completes the proof. □

The improvement achieved by the symmetry reduction method in this work is to algebraically characterize the *minimal* period and stationary solutions. Nevertheless, the focus really lies on the method itself and the helpful insights it gives in order to solve more complicated systems.

5.2.1 Phase Equivariant Reduction and Transformations

The associated Vlasov Hamiltonian of the energy functional (5.6) constructed from (5.4) is

$$\mathcal{H}_{\mathrm{Vl}}(\alpha) = \frac{1}{2\mathrm{i}} \int_{\mathbb{R}^{2d}_{\mathbf{z}}} \left(\frac{|\mathbf{v}|^2}{2} + \left(\Gamma * |\alpha|^2\right)(\mathbf{x}) \right) [\bar{\alpha}, \alpha](\mathbf{z})\, \mathrm{d}\mathbf{z}.$$

As already introduced in Section 1.3, it is reasonable to switch to the equivalent setup of the *Hamilton Hartree system* via the partial Fourier transform in the velocity coordinates $\mathbf{v}$, i.e.,

$$\mathcal{F}: \quad \mathcal{L}^2_{\mathbf{z}} \to \mathcal{L}^2_{\hat{\mathbf{z}}}, \quad \alpha \mapsto \hat{\alpha}, \quad \text{where} \quad \hat{\alpha}(\hat{\mathbf{z}}) \equiv \hat{\alpha}(\mathbf{x}, \boldsymbol{\xi}) \equiv \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d_{\mathbf{v}}} \alpha(\mathbf{x}, \mathbf{v})\, e^{-\mathrm{i}\mathbf{v}\cdot\boldsymbol{\xi}}\, \mathrm{d}\mathbf{v}.$$

For this system the transformation significantly simplifies to compute spectra around certain equilibria. By the Plancherel Theorem it is also an isometry of $\mathcal{L}^2$ and thereby a symplectic diffeomorphism. The Hamiltonian formalism therefore is preserved under this transformation. As computed in Section 1.3, it yields the new Hamilton Hartree functional

$$\mathcal{H}_{\mathrm{Ht}}(\hat{\alpha}) = \frac{1}{2} \left\langle \hat{\alpha}, \left(\nabla_{\mathbf{x}} \cdot \nabla_{\boldsymbol{\xi}} + \frac{1}{2} \left(\hat{V} * |\hat{\alpha}|^2 \right) \right) \hat{\alpha} \right\rangle \quad \text{with} \quad \hat{V}(\mathbf{x}, \boldsymbol{\xi}) \equiv -\nabla\Gamma(\mathbf{x}) \cdot \boldsymbol{\xi} = -\mathbf{x} \cdot \boldsymbol{\xi}.$$

In this particular case, a second transformation simplifies the structure even more. In fact, the linear, self-inverse (hence volume-preserving) phase space transformation

$$\tau: \quad \mathbb{R}^{2d}_{\hat{\mathbf{z}}} \to \mathbb{R}^{2d}_{\mathbf{w}}, \quad (\mathbf{x}, \boldsymbol{\xi}) \mapsto \mathbf{w} \equiv (\mathbf{q}, \mathbf{p}) \equiv \left(\frac{1}{\sqrt{2}}(\mathbf{x} + \boldsymbol{\xi}), \frac{1}{\sqrt{2}}(\mathbf{x} - \boldsymbol{\xi}) \right)$$

and the notation $\beta = \tau^* \hat{\alpha} \equiv \hat{\alpha} \circ \tau^{-1}$, $W(\mathbf{q}, \mathbf{p}) \equiv \hat{V} \circ \tau^{-1}(\mathbf{q}, \mathbf{p}) = -\frac{|\mathbf{q}|^2}{2} + \frac{|\mathbf{p}|^2}{2}$, result in the new Hamiltonian

$$\begin{aligned} \mathcal{H}_{\mathrm{Ht}}(\beta) &= \frac{1}{2} \left\langle \beta, \left(\frac{1}{2}\Delta_{\mathbf{q}} - \frac{1}{2}\Delta_{\mathbf{p}} + \frac{1}{2} \left(W * |\beta|^2 \right) \right) \beta \right\rangle \qquad (5.7) \\ &= \frac{1}{2} \left\langle \beta, \left(\frac{1}{2}\Delta_{\mathbf{q}} - \frac{|\mathbf{q}|^2}{2} \|\beta\|^2_{\mathcal{L}^2_{\mathbf{w}}} - \frac{1}{2}\Delta_{\mathbf{p}} + \frac{|\mathbf{p}|^2}{2} \|\beta\|^2_{\mathcal{L}^2_{\mathbf{w}}} \right) \beta \right\rangle + \frac{1}{4} |\langle \beta, \mathbf{q}\, \beta \rangle|^2 - \frac{1}{4} |\langle \beta, \mathbf{p}\, \beta \rangle|^2 . \end{aligned}$$

Also $\tau^* : \mathcal{L}^2_{\hat{\mathbf{z}}} \to \mathcal{L}^2_{\mathbf{w}}$ is a symplectic diffeomorphism, because τ preserves volume. Ultimately, this yields the Hamiltonian equation of motion for β with $X_{\mathcal{H}_{\mathrm{Ht}}}$, denoting the Hamiltonian vector field by

$$\begin{aligned} X_{\mathcal{H}_{\mathrm{Ht}}}(\beta) = &\frac{1}{\mathrm{i}} \left(\frac{1}{2}\Delta_{\mathbf{q}} - \frac{|\mathbf{q}|^2}{2} - \frac{1}{2}\Delta_{\mathbf{p}} + \frac{|\mathbf{p}|^2}{2} \right) \beta + \frac{1}{\mathrm{i}} \left(\langle \beta, \mathbf{q}\, \beta \rangle \cdot \mathbf{q} - \langle \beta, \mathbf{p}\, \beta \rangle \cdot \mathbf{p} \right) \beta \\ &+ \frac{1}{\mathrm{i}} \left(-\left\langle \beta, \frac{|\mathbf{q}|^2}{2} \beta \right\rangle + \left\langle \beta, \frac{|\mathbf{p}|^2}{2} \beta \right\rangle \right) \beta + \frac{1}{\mathrm{i}} \left(\|\beta\|^2_{\mathcal{L}^2_{\mathbf{w}}} - 1 \right) \frac{-|\mathbf{q}|^2 + |\mathbf{p}|^2}{2} \beta. \qquad (5.8) \end{aligned}$$

From this representation one sees that the restriction to $\mathbb{S}^{\mathcal{L}^2_{\mathbf{w}}}$ and projection Π from Proposition 5.5 greatly simplify the Hamiltonian vector field, since the restriction cancels the last summand and all terms in $\mathrm{i}\mathbb{R}\beta$ lie in the kernel of the projection.

The form of Equation (5.8) allows us to reformulate the evolution equation for the Hamilton Hartree system as

$$\partial_t \beta = X_{\mathcal{H}_{\mathrm{Ht}}}(\beta), \quad \beta : \mathbb{R} \to \mathcal{L}^2_{\mathbf{w}}, \tag{HrHt}$$

called the **Harmonic Hartree equation**.

It turns out extremely useful to exploit the structural equivalence to the quantum mechanical harmonic oscillator and rewrite the equation with the algebraic ladder operators, a well-known formalism discussed in many standard textbooks from quantum mechanics [22, Sec.3.1], i.e.,

$$\mathfrak{a}_i = \frac{1}{\sqrt{2}}\left(q_i + \partial_{q_i}\right), \quad \mathfrak{a}_i^* = \frac{1}{\sqrt{2}}\left(q_i - \partial_{q_i}\right), \quad \mathfrak{b}_i = \frac{1}{\sqrt{2}}\left(p_i + \partial_{p_i}\right), \quad \mathfrak{b}_i^* = \frac{1}{\sqrt{2}}\left(p_i - \partial_{p_i}\right),$$

along with the complete basis of eigenfunctions $\left\{|\mathbf{a}, \mathbf{b}\rangle : \mathbf{a}, \mathbf{b} \in \mathbb{N}_0^d\right\}$, which simultaneously diagonalize the commuting counting operators $\mathfrak{a}_i^*\mathfrak{a}_i$ and $\mathfrak{b}_i^*\mathfrak{b}_i$. We recall that this implies that

$$\mathfrak{a}_i \ |\mathbf{a}, \mathbf{b}\rangle = \sqrt{a_i}\ |\mathbf{a} - \mathbf{e}_i, \mathbf{b}\rangle, \quad \mathfrak{a}_i^*\ |\mathbf{a}, \mathbf{b}\rangle = \sqrt{a_i + 1}\ |\mathbf{a} + \mathbf{e}_i, \mathbf{b}\rangle, \quad \mathfrak{a}_i^*\mathfrak{a}_i\ |\mathbf{a}, \mathbf{b}\rangle = a_i\ |\mathbf{a}, \mathbf{b}\rangle,$$

and for $\mathfrak{b}$ likewise. The vectors $\mathbf{e}_i$ denote the standard basis of $\mathbb{R}^d$. Due to its multiple occurrence, it is also useful to define the **linear excitation operator**

$$\mathfrak{N} \equiv \sum_{i=1}^{d} \left(\mathfrak{b}_i^*\mathfrak{b}_i - \mathfrak{a}_i^*\mathfrak{a}_i\right).$$

For further convenience we define the simultaneous eigenspaces of $\sum_i \mathfrak{a}_i^*\mathfrak{a}_i$ and $\sum_i \mathfrak{b}_i^*\mathfrak{b}_i$ by

$$\mathcal{F}_{A,B} \equiv \operatorname{span}_{\mathbb{C}}\left\{|\mathbf{a}, \mathbf{b}\rangle \;\middle|\; \sum_{i=1}^{d} a_i = A, \sum_{i=1}^{d} b_i = B\right\}, \quad \mathcal{F}_N \equiv \bigoplus_{B-A=N} \mathcal{F}_{A,B}, \quad \mathcal{L}^2_{\mathbf{w}} = \bigoplus_{N=-\infty}^{\infty} \mathcal{F}_N.$$

In particular, $\mathfrak{N}$ is diagonalizable with

$$\forall N \in \mathbb{Z} : \ker(\mathfrak{N} - N) = \mathcal{F}_N \quad \text{and} \quad \sigma(\mathfrak{N}) = \mathbb{Z}.$$

In this algebraic notation, upon replacing $q_i = \frac{1}{\sqrt{2}}(\mathfrak{a}_i + \mathfrak{a}_i^*)$ and $p_i = \frac{1}{\sqrt{2}}(\mathfrak{b}_i + \mathfrak{b}_i^*)$ in (5.8), the Hamiltonian vector field is represented by

$$\begin{aligned} X_{\mathcal{H}_{\mathrm{Ht}}}(\beta) &= \frac{1}{\mathrm{i}}\left(\mathfrak{N} + \frac{1}{2}\langle\beta, \mathfrak{N}\beta\rangle\right)\beta + \frac{1}{2\mathrm{i}}\mathfrak{R}\left\langle\beta, \sum_{i=1}^{d}(\mathfrak{b}_i^*\mathfrak{b}_i^* - \mathfrak{a}_i\mathfrak{a}_i)\beta\right\rangle\beta \\ &\quad - \frac{1}{\mathrm{i}}\sum_{i=1}^{d}\left(\mathfrak{R}\langle\beta, \mathfrak{b}_i\beta\rangle(\mathfrak{b}_i + \mathfrak{b}_i^*) - \mathfrak{R}\langle\beta, \mathfrak{a}_i\beta\rangle(\mathfrak{a}_i + \mathfrak{a}_i^*)\right)\beta \\ &\quad + \frac{1}{4\mathrm{i}}\left(\|\beta\|^2_{\mathcal{L}^2} - 1\right)\left(2\mathfrak{N} + \sum_{i=1}^{d}(\mathfrak{b}_i\mathfrak{b}_i + \mathfrak{b}_i^*\mathfrak{b}_i^* - \mathfrak{a}_i\mathfrak{a}_i - \mathfrak{a}_i^*\mathfrak{a}_i^*)\right)\beta. \end{aligned} \tag{5.9}$$

This is a well-defined $\mathcal{L}^2_{\mathbf{w}}$ valued vector-field on the dense domain

$$\mathcal{V} \equiv \left\{\beta = \sum_{A,B=0}^{\infty}\beta_{A,B} \in \mathcal{L}^2_{\mathbf{w}} : \beta_{A,B} \in \mathcal{F}_{A,B}, \sum_{A,B=0}^{\infty}(A^2 + B^2)\left\|\beta_{A,B}\right\|^2_{\mathcal{L}^2_{\mathrm{w}}} < \infty\right\} \subset \mathcal{L}^2_{\mathbf{w}}. \tag{5.10}$$

With these bosonic creation and annihilation operators on $\mathcal{L}^2_{\mathbf{w}}$, an application of Proposition 5.5 provides the Hamiltonian functional $\bar{\mathcal{H}}_{\mathrm{Ht}}$ and the Hamiltonian vector field $X_{\bar{\mathcal{H}}_{\mathrm{Ht}}}$ at $\beta \in \mathbb{S}^{\mathcal{L}^2_{\mathrm{w}}} \cap \mathcal{V}$, given by

$$\bar{\mathcal{H}}_{\mathrm{Ht}}(\Pi\beta) = \mathcal{H}_{\mathrm{Ht}}(\beta) \overset{\|\beta\|_{\mathcal{L}^2_{\mathrm{w}}}=1}{=} \frac{1}{2}\langle\beta, \mathfrak{N}\beta\rangle + \frac{1}{2}\sum_{i=1}^{d}\left|\mathfrak{R}\langle\beta, \mathfrak{a}_i\beta\rangle\right|^2 - \frac{1}{2}\sum_{i=1}^{d}\left|\mathfrak{R}\langle\beta, \mathfrak{b}_i\beta\rangle\right|^2 \tag{5.11}$$

and

$$\begin{aligned} X_{\bar{\mathcal{H}}_{\mathrm{Ht}}}(\Pi\beta) &= \mathrm{d}\Pi_\beta\left(X_{\mathcal{H}_{\mathrm{Ht}}}(\beta)\right) \\ &= \mathbb{S}^1\Bigg(\beta, \frac{1}{\mathrm{i}}(\mathfrak{N} - \langle\beta, \mathfrak{N}\beta\rangle)\beta - \frac{1}{\mathrm{i}}\sum_{i=1}^{d}\left(\mathfrak{R}\langle\beta, \mathfrak{b}_i\beta\rangle(\mathfrak{b}_i^* + \mathfrak{b}_i) - 2\left|\mathfrak{R}\langle\beta, \mathfrak{b}_i\beta\rangle\right|^2\right)\beta \\ &\qquad + \frac{1}{\mathrm{i}}\sum_{i=1}^{d}\left(\mathfrak{R}\langle\beta, \mathfrak{a}_i\beta\rangle(\mathfrak{a}_i^* + \mathfrak{a}_i) - 2\left|\mathfrak{R}\langle\beta, \mathfrak{a}_i\beta\rangle\right|^2\right)\beta\Bigg). \end{aligned} \tag{5.12}$$

The **phase-reduced Hamilton Hartree equation** is denoted by

$$\partial_t\Pi\beta(t) = X_{\bar{\mathcal{H}}_{\mathrm{Ht}}}(\Pi\beta(t)), \quad \Pi\beta : \mathbb{R} \to \mathbb{S}^{\mathcal{L}^2_{\mathrm{w}}}/\mathbb{S}^1. \tag{HrHt$^{\mathrm{red}}$}$$

5.2.2 Stationary Points and Spectral Classification

A convenient way to find periodic solutions is by bifurcating around stationary points of the Hamiltonian dynamics. Many results utilizing this principle exist. We want to compute relative equilibria $\Pi\beta \in \mathbb{S}^{\mathcal{L}^2_{\mathrm{w}}}/\mathbb{S}^1$ and identify possible periods of bifurcating families from the spectrum of

$$\mathrm{D}X_{\bar{\mathcal{H}}_{\mathrm{Ht}}}(\Pi\beta) : \mathrm{T}_{\Pi\beta}\Big(\mathbb{S}^{\mathcal{L}^2_{\mathrm{w}}}/\mathbb{S}^1\Big) \to \mathrm{T}_{\Pi\beta}\Big(\mathbb{S}^{\mathcal{L}^2_{\mathrm{w}}}/\mathbb{S}^1\Big).$$

Indeed, in this highly symmetric model, every eigenvector of the differential at a stationary point can be linked to a bifurcating periodic family or to a continuous symmetry, as is outlined at the end of this subsection.

Lemma 5.8 (Relative Equilibria)**.** *The phase equivariant projection* $\Pi\beta$ *of any unit eigenvector* $\beta \in \mathcal{V}$ *from* (5.10) *of* $\mathfrak{N}$ *is a stationary point of* (HrHt$^{\mathrm{red}}$).

Proof. This is immediate from the representation (5.12), because $\mathfrak{R}\,\langle\beta, \mathfrak{a}_i\beta\rangle = \mathfrak{R}\,\langle\beta, \mathfrak{b}_i\beta\rangle = 0$, as the operators $\mathfrak{a}_i$ ($\mathfrak{b}_i$) map β to the eigenspace of the next higher (lower) eigenvalue of $\mathfrak{N}$, as seen from the properties stated in Section 5.2.1, i.e.,

$$\forall N \in \mathbb{Z} : \quad \mathfrak{a}_i(\mathcal{F}_N) \subseteq \mathcal{F}_{N+1}, \quad \mathfrak{b}_i(\mathcal{F}_N) \subseteq \mathcal{F}_{N-1},$$

and those eigenspaces are orthogonal to one another. □

Proposition 5.9 (Spectral Properties)**.** *For* $N \in \mathbb{Z}$ *at any unit eigenvector* $\mathring{\beta} \in \ker(\mathfrak{N} - N) \cap \mathbb{S}^{\mathcal{L}^2_{\mathrm{w}}}$ *of* $\mathfrak{N}$*, on the subspace*

$$\mathcal{W}_{\Pi\mathring{\beta}} \equiv \mathbb{S}^1 \left\{ \begin{array}{c} (\mathring{\beta}, \delta\beta) : \delta\beta \in \mathcal{V}, \left\langle \delta\beta, \mathring{\beta} \right\rangle = 0, \forall 1 \leq i \leq d : \\ \mathfrak{R}\left\langle \delta\beta, \mathfrak{a}_i\mathring{\beta} \right\rangle = \mathfrak{R}\left\langle \delta\beta, \mathfrak{a}_i^*\mathring{\beta} \right\rangle = 0, \\ \mathfrak{R}\left\langle \delta\beta, \mathfrak{b}_i\mathring{\beta} \right\rangle = \mathfrak{R}\left\langle \delta\beta, \mathfrak{b}_i^*\mathring{\beta} \right\rangle = 0 \end{array} \right\} \subseteq \mathrm{T}_{\Pi\mathring{\beta}}\Big(\mathbb{S}^{\mathcal{L}^2_{\mathrm{w}}}/\mathbb{S}^1\Big),$$

which has real codimension $4d$ *in* $\mathrm{T}_{\Pi\beta}\Big(\mathbb{S}^{\mathcal{L}^2_{\mathrm{w}}}/\mathbb{S}^1\Big)$*, one finds that*

$$\mathrm{D}X_{\bar{\mathcal{H}}_{Ht}}(\Pi\mathring{\beta})\Big|_{\mathcal{W}_{\Pi\mathring{\beta}}} \left(\mathbb{S}^1(\mathring{\beta}, \delta\beta)\right) = \mathbb{S}^1\left(\mathring{\beta}, \frac{1}{\mathrm{i}}(\mathfrak{N} - N)\delta\beta\right) \quad \textit{and} \quad \sigma\left(\mathrm{D}X_{\bar{\mathcal{H}}_{Ht}}(\Pi\mathring{\beta})\Big|_{\mathcal{W}_{\Pi\mathring{\beta}}}\right) = \mathrm{i}\mathbb{Z}.$$

Proof. Carrying out spectral analysis on the quotient manifold $\mathbb{S}^{\mathcal{L}^2_{\mathrm{w}}}/\mathbb{S}^1$ is possible in the chart of Lemma 5.6-(ii). In this chart $X_{\bar{\mathcal{H}}_{\mathrm{Ht}}}$ is represented by the vector field at $\beta \in \mathcal{V} \cap \mathbb{S}^{\mathcal{L}^2_{\mathrm{w}}}$, given by

$$\begin{aligned} \mathfrak{X}(\beta) = \frac{1}{\mathrm{i}}\left(\mathfrak{N} - \langle\beta, \mathfrak{N}\beta\rangle\right)\beta - \frac{1}{\mathrm{i}}\sum_{i=1}^{d}\left(\mathfrak{R}\,\langle\beta, \mathfrak{b}_i\beta\rangle\,(\mathfrak{b}_i^* + \mathfrak{b}_i) - 2\left|\mathfrak{R}\,\langle\beta, \mathfrak{b}_i\beta\rangle\right|^2\right)\beta \\ + \frac{1}{\mathrm{i}}\sum_{i=1}^{d}\left(\mathfrak{R}\,\langle\beta, \mathfrak{a}_i\beta\rangle\,(\mathfrak{a}_i^* + \mathfrak{a}_i) - 2\left|\mathfrak{R}\,\langle\beta, \mathfrak{a}_i\beta\rangle\right|^2\right)\beta, \end{aligned}$$

s.t. $\langle \mathfrak{X}(\beta), \beta \rangle = 0$. It now suffices to compute the spectrum of $\mathrm{D}\mathfrak{X}(\mathring{\beta}) : \{\mathring{\beta}\}^\perp \to \{\mathring{\beta}\}^\perp$, because its spectrum equals the one of $\mathrm{D}X_{\bar{\mathcal{H}}_{\mathrm{Ht}}}$ and their eigenspaces map to one another.

For the linearization of $\mathfrak{X}$ at its zero $\mathring{\beta}$ with $\mathfrak{N}\mathring{\beta} = N\mathring{\beta}$, $\left\langle \mathring{\beta}, \delta\beta \right\rangle = 0$, one computes

$$\mathrm{D}\mathfrak{X}(\mathring{\beta})(\delta\beta) = \frac{1}{\mathrm{i}} (\mathfrak{N} - N)\, \delta\beta - \frac{1}{\mathrm{i}} \sum_{i=1}^{d} \mathfrak{R} \left\langle \delta\beta, (\mathfrak{b}_i^* + \mathfrak{b}_i)\mathring{\beta} \right\rangle (\mathfrak{b}_i^* + \mathfrak{b}_i)\mathring{\beta}$$
$$+ \frac{1}{\mathrm{i}} \sum_{i=1}^{d} \mathfrak{R} \left\langle \delta\beta, (\mathfrak{a}_i^* + \mathfrak{a}_i)\mathring{\beta} \right\rangle (\mathfrak{a}_i^* + \mathfrak{a}_i)\mathring{\beta}.$$

The two sums combined are a nuclear perturbation of the operator $\frac{1}{\mathrm{i}}(\mathfrak{N} - N)$ with finite-dimensional range. Indeed,

$$\mathcal{W}_{\mathring{\beta}} \equiv \left\{ \delta\beta \in \mathcal{V} : \left\langle \delta\beta, \mathring{\beta} \right\rangle = 0, \forall 1 \le i \le d : \begin{array}{c} \mathfrak{R} \left\langle \delta\beta, \mathfrak{a}_i\mathring{\beta} \right\rangle = \mathfrak{R} \left\langle \delta\beta, \mathfrak{a}_i^*\mathring{\beta} \right\rangle = 0, \\ \mathfrak{R} \left\langle \delta\beta, \mathfrak{b}_i\mathring{\beta} \right\rangle = \mathfrak{R} \left\langle \delta\beta, \mathfrak{b}_i^*\mathring{\beta} \right\rangle = 0 \end{array} \right\}$$

lies in the kernel of the perturbation, proving that $\mathrm{D}\mathfrak{X}(\mathring{\beta})\big|_{\mathcal{W}_{\mathring{\beta}}} = \frac{1}{\mathrm{i}}(\mathfrak{N} - N)$. Hence, every imaginary integer is an infinitely degenerate eigenvalue of $\mathrm{D}\mathfrak{X}(\mathring{\beta})$. As this is a computation in a chart, this result transfers to $\mathrm{D}X_{\bar{\mathcal{H}}_{\mathrm{Ht}}}(\Pi\mathring{\beta})\big|_{\mathcal{W}_{\Pi\mathring{\beta}}}$, given that $\mathcal{W}_{\Pi\mathring{\beta}} = \Pi(\mathring{\beta}, \mathcal{W}_{\mathring{\beta}})$. □

The spectral decomposition hints at candidates for periodic solutions around the stationary point $\Pi\mathring{\beta}$. Nevertheless, in this highly degenerate problem, periodic families also bifurcate in directions outside the eigenspaces[1], as can be seen in Theorem 5.16.

However, one derives a satisfying spectral characterization of periodic families as the $4d$-dimensional complement of $\mathcal{W}_{\Pi\mathring{\beta}}$ in Theorem 5.10 reflects the kernel generated by the unreduced symmetry group $\mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^d_{\boldsymbol{\xi}}$ from Proposition 5.2, as for example,

$$0 = \frac{1}{\sqrt{2}} \mathfrak{R} \left\langle \delta\beta, (\mathfrak{a}_i - \mathfrak{a}_i^*)\mathring{\beta} \right\rangle = \mathfrak{R} \left\langle \delta\beta, \partial_{q_i}\mathring{\beta} \right\rangle$$

shows orthogonality of $\delta\beta \in \mathcal{W}_{\mathring{\beta}}$ and the local generator of translation in q_i.

The following result proves that all spectrally admitted bifurcating periodic solutions around the identified relative equilibria actually exist. Although this problem's highly degenerated spectrum obstructs us from obtaining these solutions implicitly from bifurcation theory, we at least get a full spectral mapping, justified from the explicit computations in Section 5.2.3.

[1] Which corresponds to an oscillation index > 2, see Section 5.2.3.

Theorem 5.10 (Spectral Parameterization of Periodic Families). *Let $\mathring{\beta} \in \mathcal{V} \cap \mathbb{S}^{\mathcal{L}^2_{\mathrm{w}}}$ be a unit eigenvector of $\mathfrak{N}$ and choose the subspace $\mathcal{W}_{\Pi\mathring{\beta}} \subseteq \mathrm{T}_{\Pi\mathring{\beta}}\left(\mathbb{S}^{\mathcal{L}^2_{\mathrm{w}}}/\mathbb{S}^1\right)$ as in Proposition 5.9. Then there is a canonical injection*

$$\left\{\begin{array}{c} (\mathbb{S}^1\tilde{\beta}, L) : \tilde{\beta} \in \mathcal{W}_{\Pi\mathring{\beta}}, \\ \left\|\tilde{\beta}\right\|_{\mathcal{L}^2_{\mathrm{w}}} = 1, L \in \mathbb{C}, \\ \tilde{\beta} \in \ker\left(\mathrm{D}X_{\bar{\mathcal{H}}_{Ht}}(\Pi\mathring{\beta}) - \mathrm{i}L\right) \end{array}\right\} \to \left\{\begin{array}{c} \textit{Continuous families of periodic solutions} \\ \textit{and their periods bifurcating from } \Pi\mathring{\beta} \end{array}\right\}$$

$$\left(\mathbb{S}^1\tilde{\beta}, L\right) \mapsto \begin{array}{ccc} [0,\pi] & \to & \left\{\textit{Closed curves in } \mathbb{S}^{\mathcal{L}^2}/\mathbb{S}^1\right\} \times \mathbb{R}_{\geq 0} \\ \gamma & \mapsto & \left(\mathbb{S}^1\left(\cos\frac{\gamma}{2}\,\mathring{\beta} + \sin\frac{\gamma}{2}\,\mathbb{S}^1\tilde{\beta}\right), \frac{2\pi}{L}\right). \end{array}$$

Proof. This is a consequence of the explicit computations in Section 5.2.3. It is immediately implied by Corollary 5.17 and illustrated in Figure 5.1. □

5.2.3 Algebraic Computation of Periodic Solutions

The highly resonant spectrum $\mathrm{i}\mathbb{Z} \subseteq \sigma\left(\mathrm{D}X_{\bar{\mathcal{H}}_{\mathrm{Ht}}}(\Pi\beta)\right)$ at the listed relative equilibria of Section 5.2.2 is a difficult obstacle to overcome by applying classical bifurcation theory to prove the existence of periodic families around them. However, the system fortunately admits an explicit computation of periodic solutions, since it leaves a lot of *finite-dimensional* subspheres in $\mathbb{S}^{\mathcal{L}^2_{\mathrm{w}}}$ invariant by the dynamics. They are the key to computing a rich variety of periodic solutions. The spheres are constructed as intersections of $\mathbb{S}^{\mathcal{L}^2_{\mathrm{w}}}$ with finite-dimensional complex subspaces of $\mathcal{L}^2_{\mathbf{w}}$.

Definition 5.11 (Centered Subspaces). Let $W \subset \mathcal{L}^2_{\mathbf{w}}$ be a finite-dimensional complex subspace. Then W is called **centered** if and only if

(i) $W = \bigoplus_{N\in\mathbb{Z}} W_N$, where $W_N \subseteq \mathcal{F}_N = \ker(\mathfrak{N} - N)$, (or equivalently $\mathfrak{N}(W) \subseteq W$) and

(ii) $\forall\beta \in W, \forall 1 \leq i \leq d$: $\mathfrak{R}\langle\beta, \mathfrak{a}_i\beta\rangle = \mathfrak{R}\langle\beta, \mathfrak{b}_i\beta\rangle = 0$.

The number of non-zero subspaces W_N in the decomposition is called the **decomposition index** of W, i.e.,

$$\mathrm{ind}_{\mathrm{dec.}}(W) \equiv \#\{N \in \mathbb{Z} : \dim_{\mathbb{C}} W_N > 0\} \in \mathbb{N}_0 \cup \{\infty\}.$$

Example 5.12. **(i).** Any $W = \bigoplus_{N\in\mathbb{Z}} W_N$, s.t. $W_N, W_M \neq \{\mathbf{0}\} \Rightarrow |N - M| \neq 1$, is centered. For $\beta = \sum_N \beta_N \in W$, using $\mathfrak{a}_i(\mathcal{F}_N) \subseteq \mathcal{F}_{N+1}$ and the orthogonality of $\mathcal{F}_N$, one has

$$\mathfrak{R}\langle\beta, \mathfrak{a}_i\beta\rangle = \sum_{M,N\in\mathbb{Z}} \mathfrak{R}\langle\beta_N, \mathfrak{a}_i\beta_M\rangle = \sum_{N\in\mathbb{Z}} \mathfrak{R}\langle\beta_N, \mathfrak{a}_i\beta_{N-1}\rangle = 0.$$

(ii). If $\beta = \sum_{N\in\mathbb{Z}} \beta_N$ is contained in some centered subspace $W = \bigoplus_{N\in\mathbb{Z}} W_N$, then necessarily $\mathbb{C}\beta_N \subseteq W_N$ by the vector space property of the W_N, hence $\bigoplus_{N\in\mathbb{Z}} \mathbb{C}\beta_N \subseteq W$. Also $\bigoplus_{N\in\mathbb{Z}} \mathbb{C}\beta_N$ obviously satisfies the decomposition condition. The second condition is clearly true for any subset of W. Therefore, $\bigoplus_{N\in\mathbb{Z}} \mathbb{C}\beta_N$ is the minimal centered subspace containing β.

Although it seems a little technical at first, Condition 5.11-(ii) is actually very natural. In fact,

$$\Re\langle\beta, \mathfrak{a}_i\beta\rangle = \frac{1}{\sqrt{2}}\langle\beta, q_i\,\beta\rangle \quad \text{and} \quad \Re\langle\beta, \mathfrak{b}_i\beta\rangle = \frac{1}{\sqrt{2}}\langle\beta, p_i\,\beta\rangle,$$

therefore this condition simply allows to cancel the translation invariance in both $(\mathbf{q}, \mathbf{p})$ and to center them at $(\mathbf{0}, \mathbf{0})$.

One quickly checks that the family of centered subspaces is $\cap$-stable. This admits the following definition.

Definition 5.13 (Oscillation Index)**.** Let $\beta \in \mathbb{S}^{\mathcal{L}^2_{\mathbf{w}}}$ be a function. The **oscillation index** of β is

$$\mathrm{ind}_{\mathrm{osc.}}(\beta) \equiv \inf\left\{\mathrm{ind}_{\mathrm{dec.}}(W) : W \subset \mathcal{L}^2_{\mathbf{w}} \text{ centered, } \beta \in W\right\} \in \mathbb{N}_0 \cup \{\infty\}.$$

The following Lemma shows that the centered subspaces actually encode some *hidden* conservation laws, because their unit spheres are stable under the dynamics.

Lemma 5.14 ($\mathbb{S}^{\mathcal{L}^2}$ Vector Field)**.** *Let $\mathfrak{Z} : \mathcal{V} \to \mathcal{L}^2_{\mathbf{w}}$ be the vector field defined by*

$$\mathfrak{Z}(\beta) = \frac{1}{\mathrm{i}}\sum_{i=1}^{d}\left(\mathfrak{b}_i^*\mathfrak{b}_i + \frac{1}{2}\Re\langle\beta, (\mathfrak{b}_i\mathfrak{b}_i + \mathfrak{b}_i^*\mathfrak{b}_i)\beta\rangle - \Re\langle\beta, \mathfrak{b}_i\beta\rangle(\mathfrak{b}_i + \mathfrak{b}_i^*)\right)\beta$$
$$-\frac{1}{\mathrm{i}}\sum_{i=1}^{d}\left(\mathfrak{a}_i^*\mathfrak{a}_i + \frac{1}{2}\Re\langle\beta, (\mathfrak{a}_i\mathfrak{a}_i + \mathfrak{a}_i^*\mathfrak{a}_i)\beta\rangle - \Re\langle\beta, \mathfrak{a}_i\beta\rangle(\mathfrak{a}_i + \mathfrak{a}_i^*)\right)\beta.$$

Then the following claims hold:

(i) $\mathfrak{Z} = X_{\mathcal{H}_{\mathrm{Ht}}}$ on $\mathcal{V} \cap \mathbb{S}^{\mathcal{L}^2_{\mathbf{w}}}$, hence $\mathfrak{Z} : \mathcal{V} \cap \mathbb{S}^{\mathcal{L}^2_{\mathbf{w}}} \to \mathrm{T}\mathbb{S}^{\mathcal{L}^2_{\mathbf{w}}}$ is well-defined.

(ii) For all centered subspaces $W \subset \mathcal{L}^2_{\mathbf{w}}$, we have $\mathfrak{Z}(W) \subseteq W$. In particular, $\mathfrak{Z} : \mathbb{S}^W \to \mathrm{T}\mathbb{S}^W$ is a smooth vector field.

Proof. **(i).** Note that $\mathfrak{Z}$ equals $X_{\mathcal{H}_{\mathrm{Ht}}}$ from (5.9) up to a term containing $\left(\|\beta\|^2_{\mathcal{L}^2_{\mathbf{w}}} - 1\right)$ in the product. Thus, they are identical on $\mathbb{S}^{\mathcal{L}^2_{\mathbf{w}}}$.

(ii). Let $W \subset \mathcal{L}^2_{\mathbf{w}}$ be a finite-dimensional centered subspace. Pick any $\beta = \sum_N \beta_N \in W$ from the decomposition of W, in particular just finitely many $\beta_N \neq 0$. Then using Condition (ii) from Definition 5.11, we get

$$\begin{aligned}
\mathfrak{Z}(\beta) &= \frac{1}{\mathrm{i}} \sum_{N\in\mathbb{Z}} \left(\mathfrak{N} + \frac{1}{2}\langle\beta, \mathfrak{N}\beta\rangle\right)\beta_N + \frac{1}{\mathrm{i}}\left(\frac{1}{2}\mathfrak{R}\left\langle\beta, \sum_{i=1}^{d}(\mathfrak{b}_i^*\mathfrak{b}_i^* - \mathfrak{a}_i\mathfrak{a}_i)\beta\right\rangle\right)\beta \\
&= \frac{1}{\mathrm{i}} \sum_{N\in\mathbb{Z}} \left(N + \frac{1}{2}\langle\beta, \mathfrak{N}\beta\rangle + \frac{1}{2}\mathfrak{R}\sum_{M\in\mathbb{Z}}\left\langle\beta_M, \left(\sum_{i=1}^{d}(\mathfrak{b}_i^*\mathfrak{b}_i^* - \mathfrak{a}_i\mathfrak{a}_i)\right)\beta_{M-2}\right\rangle\right)\beta_N \\
&\in \sum_{N\in\mathbb{Z}} \mathrm{i}\mathbb{R}\beta_N \subseteq W.
\end{aligned}$$

As the vector field is in fact only a polynomial in the (finitely many) coefficients β_N, its smoothness restricted on W is immediate. □

In order to learn more about the value of the symmetry reduction, we compute solutions not only in the quotient manifold $\mathbb{S}^{\mathcal{L}^2}/\mathbb{S}^1$, but also in the sphere $\mathbb{S}^{\mathcal{L}^2}$.

The given example of the Harmonic Vlasov system already proves that many trajectories are only closed by passing on to the quotient, highlighting the value of the symmetry reduction. An inspection of the proof of Theorem 5.15 even shows that the global phase oscillation is not generally separable by an $e^{\mathrm{i}\lambda t}$ ansatz, emphasizing the necessity of the reduction method over this ansatz.

We remind the reader that *relatively periodic* means periodic in the quotient manifold.

Theorem 5.15 (Relatively Periodic Families of the Harmonic Hartree System)**.** *Let $\mathring{\beta} \in \mathbb{S}^{\mathcal{L}^2_{\mathrm{w}}}$ have a finite oscillation index* $\mathrm{ind}_{osc.}\left(\mathring{\beta}\right) = J \in \mathbb{N}$ *and be contained in the minimal centered subspace $W = \oplus_N W_N$. Then $\mathring{\beta}$ gives rise to a global solution of* (HrHt)*, which always remains in $\mathbb{S}^W$, is relatively periodic for $J > 1$, and is relatively constant for* $J = 1$.

In the case of $J \geq 2$, the relative period T_{rel} is given by

$$T_{rel}\left(\mathring{\beta}\right) \equiv \frac{2\pi}{\gcd\left(\{|N - M| : \dim_{\mathbb{C}} W_M, \dim_{\mathbb{C}} W_N > 0\}\right)}.$$

In particular, the solution is classically periodic if and only if

$$\sum_{i=1}^{d} \langle\beta, (\mathfrak{b}_i^*\mathfrak{b}_i - \mathfrak{a}_i^*\mathfrak{a}_i)\beta\rangle = \langle\beta, \mathfrak{N}\beta\rangle \in \mathbb{Q}.$$

The classical period is a multiple of the relative period.

Proof. Let $\mathring{\beta} = \sum_N \mathring{\beta}_N$ be a decomposition according to the minimal centered space W containing $\mathring{\beta}$. Let $\mathcal{N} \equiv \{N \in \mathbb{Z} : \dim_{\mathbb{C}} W_N > 0\}$ denote the finite set of indices with non-trivial W_N. By Lemma 5.14, the initial value problem

$$\partial_t \beta(t) = \mathfrak{Z}(\beta(t)), \quad \beta(0) = \mathring{\beta} \in \mathbb{S}^W$$

is well-posed on the finite-dimensional compact manifold $\mathbb{S}^W \simeq \mathbb{S}^{2J-1} \subset \mathbb{C}^J$ and has a global smooth solution by the smoothness and boundedness of $\mathfrak{Z}$ on $\mathbb{S}^W$. By Lemma 5.14-(i), this is actually a solution of (HrHt).

As seen in Example 5.12-(ii), the minimality of W already implies the strict form $W_N = \mathbb{C}\mathring{\beta}_N$. Hence, one rewrites the equations of motion in the orthogonal decomposition

$$\begin{aligned} \forall N \in \mathcal{N}: \quad \mathrm{i}\,\dot{\beta}_N(t) = &\left(N + \frac{1}{2}\langle \beta(t), \mathfrak{N}\beta(t)\rangle\right)\beta_N(t) \\ &+ \left(\frac{1}{2}\mathfrak{R} \sum_{M, M-2 \in \mathcal{N}} \left\langle \beta_M(t), \left(\sum_{i=1}^{d} (\mathfrak{b}_i^* \mathfrak{b}_i^* - \mathfrak{a}_i \mathfrak{a}_i)\right)\beta_{M-2}(t)\right\rangle\right)\beta_N(t). \end{aligned} \tag{5.13}$$

Now by (5.13) we have $\dot{\beta}_N \in \mathrm{i}\mathbb{R}\beta_N$, thus the $\mathcal{L}^2$ norms of all β_N are conserved and equal to $\|\mathring{\beta}_N\|_{\mathcal{L}^2_{\mathrm{w}}}$ for $t = 0$. In particular, there is only phase oscillation inside W_N. Consequently, $t \mapsto \langle \beta(t), \mathfrak{N}\beta(t)\rangle \equiv \langle \mathfrak{N}\rangle$ is constant. We therefore choose the ansatz

$$\forall N \in \mathcal{N}: \quad \beta_N(t) = \mathring{\beta}_N \exp\left(-\mathrm{i}\left(N + \frac{1}{2}\langle \mathfrak{N}\rangle\right)t - \mathrm{i}\varphi(t)\right), \quad \varphi : \mathbb{R} \to \mathbb{R}, \quad \varphi(0) = 0,$$

which reduces the full system of ODEs (5.13) to the single ODE

$$\dot{\varphi}(t) = \frac{1}{2}\mathfrak{R} e^{2\mathrm{i}t} \sum_{M \in \mathbb{Z}} \left\langle \mathring{\beta}_M, \left(\sum_{i=1}^{d} (\mathfrak{b}_i^* \mathfrak{b}_i^* - \mathfrak{a}_i \mathfrak{a}_i)\right)\mathring{\beta}_{M-2}\right\rangle, \quad \varphi(0) = 0,$$

with a unique global solution, given by a linear combination of $\sin(2t)$ and $\cos(2t)$.

We remark that the common phase $\frac{1}{2}\langle \mathfrak{N}\rangle t + \varphi(t)$ is canceled by the $\mathbb{S}^1$ modulo operation. The minimal common relative period of the $t \mapsto \beta_N(t)$ is now found by the formula given in the statement of the theorem. In the special case of $J = 1$, the dynamics is only a phase oscillation and therefore relatively constant.

Given that φ has period π, $\langle \mathfrak{N}\rangle \in \mathbb{Q}$ is equivalent to the solution being classically periodic. □

Theorem 5.16 (Topological Characterization of Periodic Orbits)**.** *Let $W \subset \mathcal{L}^2_{\mathrm{w}}$ be a centered subspace of finite dimension $I = \dim_{\mathbb{C}} W \in \mathbb{N}$. The vector field*

$$\mathfrak{Z} : \mathbb{S}^W \to \mathrm{T}\mathbb{S}^W$$

from Lemma 5.14 is equivariant under the action of $\mathbb{S}^1$ *and therefore can be pushed forward under the projection*

$$\Pi : \mathbb{S}^W \to \mathbb{S}^W/\mathbb{S}^1 \simeq \mathbb{C}P^{I-1}.$$

The projection onto $\mathbb{S}^W/\mathbb{S}^1$ *of each trajectory of* (HrHt) *through* $\beta \in \mathbb{S}^W$ *is closed and has constant velocity*

$$v(\beta) \equiv \sqrt{\langle \beta, \mathfrak{N}^2\beta\rangle - \langle \beta, \mathfrak{N}\beta\rangle^2}.$$

The projections of two trajectories either coincide, or are disjoint.

Proof. Let $W \subset \mathcal{L}^2_{\mathrm{w}}$ be a centered subspace. The $\mathbb{S}^1$ equivariance of $\mathfrak{Z}$ follows from

$$\forall \zeta \in \mathbb{S}^1,\ \forall \beta \in \mathbb{S}^W : \quad \zeta^{-1}\,\mathfrak{Z}(\zeta\,\beta) = \mathfrak{Z}(\beta).$$

Let $\Pi_*\mathfrak{Z}$ denote the push-forward. In particular, for solutions of (HrHt), we have

$$\partial_t\beta = \mathfrak{Z}(\beta) \quad \Rightarrow \quad \partial_t\Pi\beta = \mathrm{d}\Pi_\beta(\partial_t\beta) = \mathrm{d}\Pi_\beta(\mathfrak{Z}(\beta)) = (\Pi_*\mathfrak{Z})\,(\Pi\beta)\,.$$

We conclude that trajectories in $\mathbb{S}^W/\mathbb{S}^1$ cannot intersect because they solve a first order autonomous ODE with a smooth vector field. Their closedness is shown in Theorem 5.15.

It remains to compute the velocity in the metric of the surrounding space. Let $\beta \in \mathbb{S}^W$ be given. We can model the differential of Π at β by the charts of Lemma 5.6-(i) as

$$\mathrm{d}\Pi_\beta : \begin{array}{ccc} \mathrm{T}_\beta\mathbb{S}^W = \{\gamma \in W : \mathfrak{R}\,\langle\beta,\gamma\rangle = 0\} & \to & \mathrm{T}_{\Pi\beta}\Big(\mathbb{S}^W/\mathbb{S}^1\Big) = \{\gamma \in W : \langle\beta,\gamma\rangle = 0\} \\ \gamma & \mapsto & \gamma - \langle\gamma,\beta\rangle\,\beta. \end{array}$$

This yields the following expression for the velocity of a trajectory $t \mapsto \beta(t) \in \mathbb{S}^W$ in $\mathbb{S}^W/\mathbb{S}^1$, namely,

$$\begin{aligned} |\partial_t\Pi\beta|^2_{\mathrm{T}_{\Pi\beta}(\mathbb{S}^W/\mathbb{S}^1)} &= \left|\mathrm{d}\Pi_\beta(\partial_t\beta)\right|^2_{\mathrm{T}_{\Pi\beta}(\mathbb{S}^W/\mathbb{S}^1)} = \left\|\dot\beta - \left\langle\dot\beta,\beta\right\rangle\beta\right\|^2_{\mathcal{L}^2_{\mathrm{w}}} = \left\|\dot\beta\right\|^2_{\mathcal{L}^2_{\mathrm{w}}} - \left|\left\langle\dot\beta,\beta\right\rangle\right|^2 \\ &= \|\mathfrak{Z}(\beta)\|^2_{\mathcal{L}^2_{\mathrm{w}}} - |\langle\mathfrak{Z}(\beta),\beta\rangle|^2 = (*). \end{aligned}$$

Using the representation of solutions $\beta = \sum_N \beta_N$ from Theorem 5.15 and the representation of $\mathfrak{Z}$ from the proof of Lemma 5.14, one computes

$$\begin{aligned} (*) &= \sum_{N\in\mathbb{Z}} \|\beta_N\|^2_{\mathcal{L}^2_{\mathrm{w}}} \left(N + \frac{1}{2}\langle\beta,\mathfrak{N}\beta\rangle + \frac{1}{2}\mathfrak{R}\sum_{M\in\mathbb{Z}}\left\langle\beta_M,\left(\sum_{i=1}^d\left(\mathfrak{b}_i^*\mathfrak{b}_i^* - \mathfrak{a}_i\mathfrak{a}_i\right)\right)\beta_{M-2}\right\rangle\right)^2 \\ &\quad - \left(\sum_{N\in\mathbb{Z}} \|\beta_N\|^2_{\mathcal{L}^2_{\mathrm{w}}} \left(N + \frac{1}{2}\langle\beta,\mathfrak{N}\beta\rangle + \frac{1}{2}\mathfrak{R}\sum_{M\in\mathbb{Z}}\left\langle\beta_M,\left(\sum_{i=1}^d\left(\mathfrak{b}_i^*\mathfrak{b}_i^* - \mathfrak{a}_i\mathfrak{a}_i\right)\right)\beta_{M-2}\right\rangle\right)\right)^2 \\ &= \left\langle\beta,\mathfrak{N}^2\beta\right\rangle - \langle\beta,\mathfrak{N}\beta\rangle^2\,. \end{aligned}$$

□

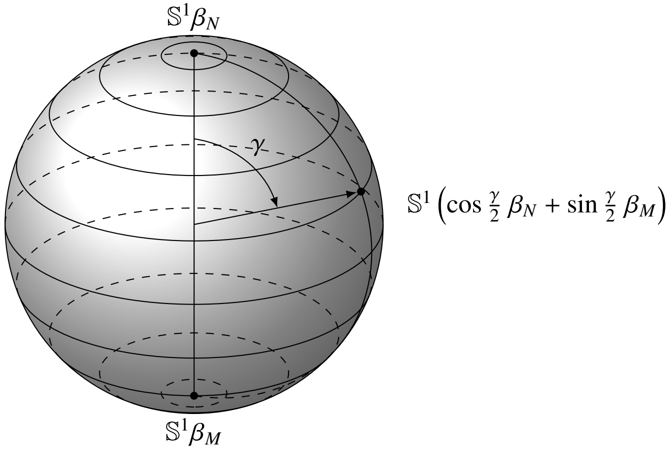

Figure 5.1: The phase portrait of an interpolating family given in Corollary 5.17. For representation issues a diffeomorphic map between $\mathbb{S}^W/\mathbb{S}^1 \simeq \mathbb{C}P^1 \simeq \mathbb{S}^2 \subset \mathbb{R}^3$ is chosen. The north and south pole are relatively constant solutions. The relatively periodic solutions flow along the latitudes with $\Pi_*\mathfrak{Z}$, whose trajectories can be parameterized by the angle $\gamma \in [0, \pi]$.

Corollary 5.17 (Interpolating Families)**.** *Let $\beta_N, \beta_M \in \mathbb{S}^{\mathcal{L}^2_{\mathbf{w}}}$ be two eigenvectors of the linear excitation operator $\mathfrak{N}$ with different eigenvalues $M \neq N$, s.t. in addition $W \equiv \mathbb{C}\beta_M \oplus \mathbb{C}\beta_N$ is centered. Then there exists a continuous family of relatively periodic solutions of the phase equivariant Harmonic Hartree* (HrHt$^{\text{red}}$)*, s.t. all trajectories have the same period.*

Proof. The space $W \equiv \mathbb{C}\beta_N \oplus \mathbb{C}\beta_M$ has complex dimension and decomposition index 2. Therefore, all elements of W which are not a multiple of the eigenvectors lead to relatively periodic solutions with period $\frac{2\pi}{|M-N|}$ by Theorem 5.15. The continuous interpolation parameter can be an angle γ, as illustrated in Figure 5.1. □

By the hierarchy of equations, we can transform any solution of (HrHt) back to the $\mathcal{L}^1_{\mathbf{z}}$ picture of the classical Vlasov system. This leads to a rich variety of periodic solutions, previously difficult to classify.

Theorem 5.18 (Periodic Solutions of the Harmonic Vlasov System)**.** *Under the chain of transformations*

$$\mathcal{L}^2_{\mathbf{q},\mathbf{p}} \quad \overset{\tau^*}{\to} \quad \mathcal{L}^2_{\mathbf{x},\xi} \quad \overset{F.T.}{\to} \quad \mathcal{L}^2_{\mathbf{x},\mathbf{v}} \quad \overset{|\cdot|^2}{\to} \quad \mathcal{L}^1_{\mathbf{x},\mathbf{v}},$$

the set of initial conditions

$$\mathbb{S}^{\mathcal{L}^2_{\mathbf{q},\mathbf{p}}} \cap \bigcup_{W \text{ centered}} W$$

leads to periodic solutions of the classical Harmonic Vlasov system (Vl).

Proof. Following the given chain of transformations, the periodic curves obtained from Theorem 5.15 yield the result immediately. In particular, the relative periodicity turns into classical periodicity under the last transformation. □

Actually, many periodic solutions from Theorem 5.18 can be computed very conveniently, if one chooses an initial state of the form

$$\mathring{\beta} \propto (\text{polynomial in } \mathbf{q}, \mathbf{p}) \cdot \exp\left(-\frac{|\mathbf{q}|^2 + |\mathbf{p}|^2}{2}\right),$$

as is highlighted by the following example.

Example 5.19. Choose the space dimension $d = 1$ and the two explicit eigenfunctions

$$\mathring{\beta}_{0,0}(q,p) \equiv \pi^{-\frac{1}{2}} e^{-\frac{q^2+p^2}{2}} \quad \text{and} \quad \mathring{\beta}_{2,0} \equiv \pi^{-\frac{1}{2}} e^{-\frac{q^2+p^2}{2}} \frac{1}{\sqrt{2}} \left(q^2 - 1\right)$$

of the linear excitation operator $\mathfrak{N}$. Using the angular parameterization $\gamma \in [0, \pi]$ consistent with Fig. 5.1, we compute the time-dependent solution

$$\mathbb{S}^1 \beta_\gamma(t, q, p) = \mathbb{S}^1 \left(\beta_{0,0}(q,p) \, \cos\frac{\gamma}{2} + e^{2\mathrm{i}t} \beta_{2,0}(q,p) \, \sin\frac{\gamma}{2}\right)$$

of (HrHt$^{\text{red}}$), the corresponding solution

$$\mathbb{S}^1 \hat{\alpha}_\gamma(t, x, \xi) = \mathbb{S}^1 \pi^{-\frac{1}{2}} e^{-\frac{x^2+\xi^2}{2}} \left(\cos\frac{\gamma}{2} + e^{2\mathrm{i}t} \frac{1}{\sqrt{2}} \left(\frac{(x+\xi)^2}{2} - 1\right) \sin\frac{\gamma}{2}\right)$$

of the reduced Hamilton Hartree equation, the solution

$$\mathbb{S}^1 \alpha_\gamma(t, x, v) = \mathbb{S}^1 \pi^{-\frac{1}{2}} e^{-\frac{x^2+v^2}{2}} \left(\cos\frac{\gamma}{2} + \frac{1}{2\sqrt{2}} (x + \mathrm{i}v)^2 \, \sin\frac{\gamma}{2}\right)$$

of the reduced Hamiltonian Vlasov system, and, finally, the Vlasov density

$$f_\gamma(t, x, v) = \pi^{-1} e^{-(x^2+v^2)} \left(\cos^2\frac{\gamma}{2} + \frac{1}{2}\left(x^2 + v^2\right)^2 \, \sin^2\frac{\gamma}{2} + \frac{1}{\sqrt{2}} \, \mathfrak{R} e^{2\mathrm{i}t} \, (x + \mathrm{i}v)^2 \, \sin\gamma\right)$$

with its oscillating space density

$$\rho_\gamma(t, x) = \pi^{-\frac{1}{2}} e^{-x^2} \left(\cos^2\frac{\gamma}{2} + \frac{1}{2}\left(x^4 + x^2 + \frac{3}{4}\right) \sin^2\frac{\gamma}{2} + \sqrt{2}\cos(2t)\left(x^2 - \frac{1}{2}\right) \sin\gamma\right).$$

We conclude this chapter by outlining the steps for solving the Harmonic Vlasov system. We begin to do so by following the general construction principle in order to define the corresponding Hamiltonian Vlasov system. With the aid of the velocity Fourier transform and the new choice of coordinates, the system is structurally brought into a form which is analogous to the quantum mechanical harmonic oscillator. This given structure allows us to compute the relative equilibria and then linearize the vector field at these points. Hence, we determine the spectral decomposition of the respective linearizations. Finally, we identify the finite dimensional subspheres left invariant by the dynamics. On these subspheres, the obtained global phase oscillation is more complicated than a separation ansatz $e^{i\lambda t}$, highlighting the additional value of the employed machinery.

Chapter 6

Conclusions

The main result of this work is a mechanism, which endows any Vlasov system with a Hamiltonian structure. This allows to embed Vlasov systems into a collection of well-studied Hamiltonian PDEs, such as the NLS or Hartree equation.

As expected, the Hamiltonian formalism proves to make valuable contributions to the field of Vlasov systems. The first notable findings are of structural type. Firstly, the general symplectization procedure outlined in Section 1.2 unveils some peculiar features. For example, the derivation of the Vlasov Hamiltonian

$$\mathcal{H}_{\mathrm{Vl}}(\alpha) = \frac{1}{2\mathrm{i}}\mathrm{D}^1\mathcal{H}(|\alpha|^2)([\bar{\alpha}, \alpha])$$

from the energy functional as well as its gauge freedom lead to a variety of fundamental questions about the physical meaning of these quantities. With the aid of this Hamiltonian formalism, the Vlasov equation is derived from an action principle for the very first time. Namely, the Hamiltonian Vlasov equation is exactly the extremal condition of the action functional

$$\Phi(\alpha) \equiv \int_0^T \left(\frac{1}{2}\omega\left(\alpha(t), \partial_t\alpha(t)\right) + \mathcal{H}_{\mathrm{Vl}}(\alpha(t))\right) \mathrm{d}t, \quad \alpha : [0, T] \to \mathcal{L}^2_{\mathbf{z}},$$

under variations, vanishing at the interval boundaries. This proves that the Vlasov system stems from a system, which obeys Hamilton's principle. This is historically an important validation point of any equation in mathematical physics.

Secondly, the velocity Fourier transform from Section 1.3 exposes the Vlasov system in a whole new light. Although a relation between the Hartree equation on the quantum scale and the Vlasov equation on the classical scale is heuristically not surprising, the technical realization is stunning. In particular, the Hamilton Hartree equation

$$\mathrm{i}\,\partial_t\hat{\alpha}(t,\hat{\mathbf{z}}) = \left(\nabla_{\mathbf{x}}\cdot\nabla_{\xi} + \left(\hat{V} * |\hat{\alpha}(t)|^2\right)(\hat{\mathbf{z}})\right)\,\hat{\alpha}(t,\hat{\mathbf{z}})$$

with its ultrahyperbolic kinetic term suggests that further studies of this type of equations seem profoundly promising. For example, finding equilibria of this equation is incomparably harder due to the lack of a minimization principle, as opposed to the well-known Hartree energy functional.

Going beyond the formal viewpoint, a number of rigorous results are achieved, which all attribute to new structural analogies to systems already studied in more depth.

The fundamental question, regarding any equation in mathematical physics, is well-posedness. This is answered in a satisfying manner in Chapter 3. Therein, a global well-posedness theory is discussed for two Hamiltonian Vlasov systems with non-relativistic kinetic energy, the Regular Vlasov system with a bounded Lipschitz interaction force in any dimension, as well as the classical Vlasov–Poisson system with Coulomb interaction in at least three dimensions. For both systems, global unique existence is established under reasonable conditions on the initial data. For the discussion of the latter one, a new Banach space of continuous functions with integrable local supremum is introduced, see Appendix A.

The second aspect of this work studies the role of the Hamiltonian Vlasov equation as a possible limiting equation in Chapter 4. Inspired by the structural analogy to the Schrödinger / Hartree ensemble, a mean field limit for the Regular Vlasov system is established. Nevertheless, the method might not be unfolded to its full capability, since it is able to deal with singular potentials in quantum theory, an open question for classical mechanics so far. As our results show, this obstacle reduces to a (missing) regularity estimate on the many-particle state. Ultimately, we prove that, in coherence with quantum mechanical notation, the partial trace of the many-particle density matrix converges in the operator norm to the pure mean field state. This is consistent with existing results of marginal convergence for the classical Vlasov system.

Finally, we discuss the problem of symmetries and periodic solutions of the Hamiltonian Vlasov system in Chapter 5. We give a rare example of symplectic symmetry reduction on a Hamiltonian PDE, namely, the Harmonic Vlasov system. Although the explicit results obtained for the simplistic and idealistic Harmonic Vlasov system are not very surprising and do not easily generalize to other systems, there are some key conclusions to draw.

For the very first time the existence of periodic solutions for any classical Vlasov system without boundary constraints is established. Utilizing the Hamiltonian formalism for the Hamiltonian Vlasov system has proven invaluable in this area.

Secondly, the method of symplectic symmetry reduction for phase equivariant Hamiltonian PDEs turns out to be profoundly effective while looking for periodic solutions.

The explicit solutions of the Harmonic Vlasov system prove that without this symmetry reduction, the trajectories are rarely closed and cannot be found by simply trying to solve the unreduced equation

$$\lambda\, \partial_t \alpha(t) - X_{\mathcal{H}_{\mathrm{Vl}}}(\alpha(t)) = 0, \quad \alpha : \mathbb{R}/\mathbb{Z} \to \mathcal{L}^2, \quad \lambda > 0$$

around some equilibrium. Nevertheless, this is possible on the quotient manifold, where all periods predicted by the spectrum of the reduced vector field at the relative equilibrium are naturally identified in periodic families, bifurcating at that equilibrium.

On the other hand, the method still leaves some open problems. For example, it is not yet able to deal with the translation invariance of the Hamiltonian Vlasov system. Since this degeneracy contributes to the kernel of the spectrum at critical points, it opposes the application of general bifurcation theorems. While the systematic approach of Marsden–Weinstein seems to be formally valid for this type of symmetry as well, as shown by the $4d$ dimensional complement in Theorem 5.10, the heavily used technical assumption of a smooth group action is not easily removed.

Moreover, the method encourages the discussion of other systems around known equilibria as many Hamiltonian PDEs are endowed with the structure of global $\mathbb{S}^1$ phase invariance. Probably the largest obstacle is to find a representation of the equilibrium, explicit enough to compute the spectrum of the first derivative of the reduced Hamiltonian vector field. As the degeneracies arising from symmetries mentioned above are common to many Hamiltonian PDEs, the formalism puts the Hamiltonian Vlasov equation into this much larger framework, thereby encouraging to develop the symplectic symmetry reduction on a weaker topological setup in the future.

Finally, in order to give a more speculative outlook on future applications, the $\mathcal{L}^2$ setup of the Hamiltonian Vlasov equation might allow to transfer more methods to the field of Vlasov systems from fields where they have been developed. Examples could include application of numerical methods, designed to solve Hilbert space valued equations, as well as, in an even more speculative way, adaptation of some inverse scattering method, in parallel to the theory of the NLS equation.

We conclude that the Hamiltonian Vlasov equation leads to an exciting variety of applications and infuses the field of Vlasov systems with a huge set of new methods. While this thesis contributes to the subject with the two examples of a mean field limit and the symplectic symmetry reduction, there are many more inspiring open problems to be tackled.

Appendix A

Functions with Integrable Local Supremum

This chapter is adapted from the author's published paper [14, App.A].

As it is used in many estimates throughout this chapter, for any $a \geq b$, we define the constant

$$A(a,b) \equiv \inf_{R>0} \frac{(1+R)^a}{\tau_b R^b} = \frac{1}{\tau_b} \frac{a^a}{b^b}(a-b)^{b-a}. \tag{A.1}$$

Definition A.1. Let $f : \mathbb{R}^d \to \mathbb{R}$ be some measurable function. For any $p \geq 1$ and $\kappa \geq d$, consider the norm

$$\|f\|_{\mathcal{A}^{\kappa,p}} \equiv \sup_{R\geq 0}(1+R)^{-\frac{\kappa}{p}} \left(\int_{\mathbb{R}^d} \sup_{|\bar{\mathbf{z}}|\leq R} |f(\mathbf{z}+\bar{\mathbf{z}})|^p \, \mathrm{d}\mathbf{z} \right)^{\frac{1}{p}}. \tag{A.2}$$

Denote by $\mathcal{A}^{\kappa,p}$ the set of measurable functions where this norm is finite. In addition, we define

$$\mathcal{B}^{k,\kappa,p} \equiv \left\{ f \in C^k \left(\mathbb{R}^d \to \mathbb{C} \right) : \forall l \leq k, |\alpha| = l : \mathrm{D}^\alpha f \in \mathcal{A}^{\kappa,p} \right\} \tag{A.3}$$

with the norm

$$\|f\|_{\mathcal{B}^{k,\kappa,p}} \equiv \left(\sum_{|\alpha|\leq k} \|\mathrm{D}^\alpha f\|^p_{\mathcal{A}^{\kappa,p}} \right)^{\frac{1}{p}}.$$

Lemma A.2. *For $d \in \mathbb{N}$, $p \geq 1$, and $\kappa \geq d$, we have:*

(i) The following inclusions hold: $C^0_c \subseteq \mathcal{A}^{\kappa,p} \subseteq \mathcal{L}^p \cap \mathcal{L}^\infty$.

(ii) *If* $f \in \mathcal{L}^p$ *and* $\nabla f \in \mathcal{A}^{\kappa,p}$*, then* $f \in \mathcal{A}^{\kappa+p,p}$*.*

(iii) *For* $\kappa \leq \lambda$*,* $\|f\|_{\mathcal{A}^{\lambda,p}} \leq \|f\|_{\mathcal{A}^{\kappa,p}}$*. Hence,* $\mathcal{A}^{\kappa,p} \subseteq \mathcal{A}^{\lambda,p}$*.*

(iv) *For* $\frac{1}{p} + \frac{1}{q} = 1$*,* $\|f \cdot g\|_{\mathcal{A}^{\frac{\kappa}{p}+\frac{\lambda}{q},1}} \leq \|f\|_{\mathcal{A}^{\kappa,p}} \|g\|_{\mathcal{A}^{\lambda,q}}$*, i.e., a generalized Hölder inequality holds.*

Proof. **(i).** Let $f \in C_c^0$ have compact support. Then there is $R_0 > 0$, s.t. $\operatorname{supp} f \subseteq B_{R_0}(\mathbf{0})$. Thus,

$$\begin{aligned}
\|f\|_{\mathcal{A}^{\kappa,p}} &= \sup_{R\geq 0}(1+R)^{-\frac{\kappa}{p}} \left(\int_{\mathbb{R}^d} \sup_{|\bar{\mathbf{z}}|\leq R} |f(\mathbf{z}+\bar{\mathbf{z}})|^p \; \mathrm{d}\mathbf{z} \right)^{\frac{1}{p}} \\
&\leq \|f\|_{\mathcal{L}^\infty} \sup_{R\geq 0}(1+R)^{-\frac{\kappa}{p}} \left(\int_{\mathbb{R}^d} \sup_{|\bar{\mathbf{z}}|\leq R} \mathbb{1}_{B_{R_0}(\mathbf{0})}(\mathbf{z}+\bar{\mathbf{z}}) \mathrm{d}\mathbf{z} \right)^{\frac{1}{p}} \\
&= \|f\|_{\mathcal{L}^\infty} \sup_{R\geq 0}(1+R)^{-\frac{\kappa}{p}} (R_0+R)^{\frac{d}{p}} \tau_d^{\frac{1}{p}} < \infty.
\end{aligned}$$

The inequality $\|f\|_{\mathcal{L}^p} \leq \|f\|_{\mathcal{A}^{\kappa,p}}$ is obvious by taking $R = 0$. In addition, for any $R > 0$ we have

$$\begin{aligned}
\|f\|_{\mathcal{L}^\infty} &\leq \left(\frac{1}{\tau_d R^d} \int_{\mathbb{R}^d} \sup_{|\bar{\mathbf{z}}|\leq R} |f(\mathbf{z}+\bar{\mathbf{z}})|^p \; \mathrm{d}\mathbf{z} \right)^{\frac{1}{p}} \leq \left(\frac{(1+R)^\kappa}{\tau_d R^d} \right)^{\frac{1}{p}} \|f\|_{\mathcal{A}^{\kappa,p}} \\
&\overset{R \text{ opt.}}{=} A(\kappa,d)^{\frac{1}{p}} \|f\|_{\mathcal{A}^{\kappa,p}} .
\end{aligned}$$

(ii). Let $f \in \mathcal{L}^p$ and $\nabla f \in \mathcal{A}^{\kappa,p}$, then for any $R > 0$, we get

$$\begin{aligned}
(1+R)^{-\frac{\kappa+p}{p}} \left(\int_{\mathbb{R}^d} \sup_{|\bar{\mathbf{z}}|\leq R} |f(\mathbf{z}+\bar{\mathbf{z}})|^p \mathrm{d}\mathbf{z} \right)^{\frac{1}{p}} &\leq (1+R)^{-\frac{\kappa+p}{p}} \left(\int_{\mathbb{R}^d} \left(|f(\mathbf{z})| + R \sup_{|\bar{\mathbf{z}}|\leq R} |\nabla f(\mathbf{z}+\bar{\mathbf{z}})| \right)^p \mathrm{d}\mathbf{z} \right)^{\frac{1}{p}} \\
&\leq \|f\|_{\mathcal{L}^p} + \|\nabla f\|_{\mathcal{A}^{\kappa,p}} .
\end{aligned}$$

(iii). The statement is clear, since $(1+R)^{-\lambda} \leq (1+R)^{-\kappa}$.

(iv). For $R > 0$, we have

$$\begin{aligned}
&(1+R)^{-\frac{\kappa}{p}-\frac{\lambda}{q}} \int_{\mathbb{R}^d} \sup_{|\bar{\mathbf{z}}|\leq R} |f(\mathbf{z}+\bar{\mathbf{z}}) \, g(\mathbf{z}+\bar{\mathbf{z}})| \, \mathrm{d}\mathbf{z} \\
&\leq (1+R)^{-\frac{\kappa}{p}-\frac{\lambda}{q}} \int_{\mathbb{R}^d} \left(\sup_{|\bar{\mathbf{z}}|\leq R} |f(\mathbf{z}+\bar{\mathbf{z}})| \right) \left(\sup_{|\bar{\mathbf{z}}|\leq R} |g(\mathbf{z}+\bar{\mathbf{z}})| \right) \mathrm{d}\mathbf{z} \overset{\text{Hölder}}{\leq} \|f\|_{\mathcal{A}^{\kappa,p}} \|g\|_{\mathcal{A}^{\lambda,q}} .
\end{aligned}$$

□

Lemma A.3. *Let $k \in \mathbb{N}$, $\kappa \geq d$, and $p \geq 1$ be arbitrary. Then the functions of compact support C_c^k are dense in $\mathcal{B}^{k,\kappa,d}$.*

Proof. **(i)** $C_c^k \subseteq \mathcal{B}^{k,\kappa,p}$**.** As all derivatives have compact support, the claim follows from Lemma A.2-(i).

(ii) $\mathcal{B}^{k,\kappa,p} \subseteq \overline{C_c^k}$**.** Let $f \in \mathcal{B}^{k,\kappa,p}$ be some function. Let $\eta : \mathbb{R}^d \to [0,1]$ be some C_c^k function, such that $\eta \equiv 1$ on $B_1(\mathbf{0})$ and $\eta \equiv 0$ on $B_2(\mathbf{0})^c$. For any $\epsilon > 0$ we then define $\eta_\epsilon(\mathbf{x}) \equiv \eta(\epsilon\mathbf{x})$. Obviously, $f\eta_\epsilon \in C_c^k$. We want to show that for any $\alpha \in \mathbb{N}_0^d, |\alpha| \leq k$, $\|\mathrm{D}^\alpha(f\eta_\epsilon) - \mathrm{D}^\alpha f\|_{\mathcal{A}^{\kappa,p}} \to 0$ for $\epsilon \downarrow 0$. In fact, by the Leibniz rule, we have

$$\mathrm{D}^\alpha(f\eta_\epsilon) = \sum_{\beta\in\mathbb{N}_0^d,\beta\leq\alpha} \binom{\alpha}{\beta}\left(\mathrm{D}^{\alpha-\beta}f\right)\left(\mathrm{D}^\beta\eta_\epsilon\right) = (\mathrm{D}^\alpha f)\,\eta_\epsilon + \sum_{\beta\in\mathbb{N}_0^d,\beta\leq\alpha,\beta\neq\mathbf{0}} \binom{\alpha}{\beta}\left(\mathrm{D}^{\alpha-\beta}f\right)\left(\mathrm{D}^\beta\eta_\epsilon\right)$$

with the usual notation $\binom{\alpha}{\beta} \equiv \prod_i \binom{\alpha_i}{\beta_i}$. The first term converges to $\mathrm{D}^\alpha f$ and as $\left(\mathrm{D}^\beta\eta_\epsilon\right)(\mathbf{x}) = \epsilon^{|\beta|}\left(\mathrm{D}^\beta\eta\right)(\epsilon\mathbf{x})$, all the other terms are integrably dominated for bounded ϵ and their $\mathcal{A}^{\kappa,p}$ norms vanish in the limit $\epsilon \downarrow 0$. This proves that $\lim_{\epsilon\downarrow 0}\|\mathrm{D}^\alpha(f\eta_\epsilon) - \mathrm{D}^\alpha f\|_{\mathcal{A}^{\kappa,p}} = 0$. □

Theorem A.4. *For any $k \in \mathbb{N}_0$, $\kappa \geq d$, and $p \geq 1$, $\mathcal{B}^{k,\kappa,p}$ is a Banach space.*

Proof. It remains to prove completeness. Let $\{f_n\}$ be a Cauchy sequence in $\mathcal{B}^{k,\kappa,p}$. Since $\mathcal{A}^{\kappa,p}$ embeds continuously into C^0, $\mathcal{B}^{k,\kappa,p}$ embeds into C^k. Therefore, $\{f_n\}$ is Cauchy in C^k and has a uniform limit in this Banach space, say f.

We now want to show that $\|f_n - f\|_{\mathcal{B}^{k,\kappa,p}} \to 0$ and therefore, $f \in \mathcal{B}^{k,\kappa,p}$ is the right limit. It suffices to check the case $k = 0$, as all the arguments can be repeated for the higher order derivatives.

Let $\epsilon > 0$ be arbitrary and choose $N \in \mathbb{N}$, s.t. by the Cauchy property for $n, m \geq N$, $\|f_n - f_m\|_{\mathcal{A}^{\kappa,p}} \leq \epsilon$. Now let $\mathbf{x} \in \mathbb{R}^d$ and by the reverse triangle inequality, we get the following for the supremum norm on $B_R(\mathbf{x}) \subseteq \mathbb{R}^d$, that is,

$$\begin{aligned}\left|\sup_{|\bar{\mathbf{x}}|\leq R}|f_n(\mathbf{x}+\bar{\mathbf{x}}) - f_m(\mathbf{x}+\bar{\mathbf{x}})| - \sup_{|\bar{\mathbf{x}}|\leq R}|f(\mathbf{x}+\bar{\mathbf{x}}) - f_m(\mathbf{x}+\bar{\mathbf{x}})|\right| &\leq \sup_{|\bar{\mathbf{x}}|\leq R}|f_n(\mathbf{x}+\bar{\mathbf{x}}) - f(\mathbf{x}+\bar{\mathbf{x}})| \\ &\leq \|f_n - f\|_{\mathcal{L}^\infty_{\mathbf{x}}} \overset{n\to\infty}{\to} 0,\end{aligned}$$

in other words,

$$\lim_{n\to\infty}\sup_{|\bar{\mathbf{x}}|\leq R}|f_n(\mathbf{x}+\bar{\mathbf{x}}) - f_m(\mathbf{x}+\bar{\mathbf{x}})| = \sup_{|\bar{\mathbf{x}}|\leq R}|f(\mathbf{x}+\bar{\mathbf{x}}) - f_m(\mathbf{x}+\bar{\mathbf{x}})|\,.$$

Now, for any $R > 0$, using this equation at every $\mathbf{x}$ and in combination with Fatou's Lemma, for any $m \geq N$, we derive

$$
\begin{aligned}
&\frac{1}{(1+R)^{\frac{\kappa}{p}}} \left(\int_{\mathbb{R}^d} \sup_{|\bar{\mathbf{x}}| \leq R} |f(\mathbf{x}+\bar{\mathbf{x}}) - f_m(\mathbf{x}+\bar{\mathbf{x}})|^p \, \mathrm{d}\mathbf{x} \right)^{\frac{1}{p}} \\
&= \frac{1}{(1+R)^{\frac{\kappa}{p}}} \left(\int_{\mathbb{R}^d} \liminf_{n\to\infty} \sup_{|\bar{\mathbf{x}}| \leq R} |f_n(\mathbf{x}+\bar{\mathbf{x}}) - f_m(\mathbf{x}+\bar{\mathbf{x}})|^p \, \mathrm{d}\mathbf{x} \right)^{\frac{1}{p}} \\
&\leq \liminf_{n\to\infty} \frac{1}{(1+R)^{\frac{\kappa}{p}}} \left(\int_{\mathbb{R}^d} \sup_{|\bar{\mathbf{x}}| \leq R} |f_n(\mathbf{x}+\bar{\mathbf{x}}) - f_m(\mathbf{x}+\bar{\mathbf{x}})|^p \, \mathrm{d}\mathbf{x} \right)^{\frac{1}{p}} \\
&\leq \liminf_{n\to\infty} \|f_n - f_m\|_{\mathcal{A}^{\kappa,p}} \leq \epsilon.
\end{aligned}
$$

Therefore, $\|f - f_m\|_{\mathcal{A}^{\kappa,p}} \leq \epsilon$ and $f \in \mathcal{A}^{\kappa,p}$ is indeed the limit. □

Appendix B

Inequalities

We collect some important inequalities well-known from potential theory.

Lemma B.1 (Newtonian Potential)**.** *Define the Newtonian interaction potential in space dimension $d \geq 3$ by*

$$\Gamma(\mathbf{x}) \equiv \frac{|\mathbf{x}|^{2-d}}{d\,(2-d)\,\omega_d}. \tag{B.1}$$

Let $\rho \in C^{0,\alpha}_{\mathbf{x}} \cap \mathcal{L}^1_{\mathbf{x}}$ for some $\alpha \in (0,1]$ be Hölder continuous and $U_\rho : \mathbb{R}^d_{\mathbf{x}} \to \mathbb{R}$ be given by

$$U_\rho(\mathbf{x}) = (\Gamma * \rho)(\mathbf{x}) = \int_{\mathbb{R}^d_{\mathbf{x}}} \Gamma(\mathbf{x}-\mathbf{y})\,\rho(\mathbf{y})\,\mathrm{d}\mathbf{y}.$$

Then $U_\rho \in C^{2,\alpha}_{\mathbf{x}}$ and the following representation formulae hold for any $\mathbf{x}_0 \in \mathbb{R}^d_{\mathbf{x}}$ and $|\mathbf{x}-\mathbf{x}_0| < \frac{R}{2}$:

$$\partial_j U_\rho(\mathbf{x}) = \int_{\mathbb{R}^d_{\mathbf{x}}} \partial_j \Gamma(\mathbf{x}-\mathbf{y})\,\rho(\mathbf{y})\,\mathrm{d}\mathbf{y}, \tag{B.2}$$

$$\begin{aligned}\partial_i\partial_j U_\rho(\mathbf{x}) = &\int_{\mathbb{R}^d_{\mathbf{x}}-B_R(\mathbf{x}_0)} \partial_i\partial_j\Gamma(\mathbf{x}-\mathbf{y})\,\rho(\mathbf{y})\,\mathrm{d}\mathbf{y} + \int_{B_R(\mathbf{x}_0)} \partial_i\partial_j\Gamma(\mathbf{x}-\mathbf{y})\,(\rho(\mathbf{y})-\rho(\mathbf{x}))\,\mathrm{d}\mathbf{y}\\ &-\rho(\mathbf{x})\int_{\partial B_R(\mathbf{x}_0)} \partial_j\Gamma(\mathbf{x}-\mathbf{y})\,\nu_i(\mathbf{y})\,\mathrm{d}s(\mathbf{y}).\end{aligned} \tag{B.3}$$

In addition, for any $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^d_{\mathbf{x}}$ we get that

$$\left|\nabla^2_{\mathbf{x}} U_\rho(\mathbf{x}_1) - \nabla^2_{\mathbf{x}} U_\rho(\mathbf{x}_2)\right| \leq L(d,\alpha) \sup_{\boldsymbol{\xi}_1 \neq \boldsymbol{\xi}_2} \frac{|\rho(\boldsymbol{\xi}_1)-\rho(\boldsymbol{\xi}_2)|}{|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2|^\alpha}\,|\mathbf{x}_1-\mathbf{x}_2|^\alpha\,. \tag{B.4}$$

In particular, $U_\rho \in C^{2,\alpha}_{\mathbf{x}}$ and we find the estimates:

$$\forall p \in [1,d)\ \exists c_{p,d} > 0 : \left\|\nabla_{\mathbf{x}} U_\rho\right\|_{\mathcal{L}^\infty_{\mathbf{x}}} \leq c_{p,d} \left\|\rho\right\|^{1-\frac{p}{d}}_{\mathcal{L}^\infty_{\mathbf{x}}} \ \left\|\rho\right\|^{\frac{p}{d}}_{\mathcal{L}^p_{\mathbf{x}}} . \tag{B.5}$$

Finally, in the case $\alpha = 1$, there exists $c_d > 0$, s.t. for every $0 < R \leq r$,

$$\left\|\nabla^2_{\mathbf{x}} U_\rho\right\|_{\mathcal{L}^\infty_{\mathbf{x}}} \leq c_d \left[\left(1 + \ln \frac{r}{R}\right) \left\|\rho\right\|_{\mathcal{L}^\infty_{\mathbf{x}}} + \frac{1}{r^d} \left\|\rho\right\|_{\mathcal{L}^1_{\mathbf{x}}} + R \left\|\nabla_{\mathbf{x}}\rho\right\|_{\mathcal{L}^\infty_{\mathbf{x}}}\right], \tag{B.6}$$

$$\left\|\nabla^2_{\mathbf{x}} U_\rho\right\|_{\mathcal{L}^\infty_{\mathbf{x}}} \leq c_d \left[\left(1 + \left\|\rho\right\|_{\mathcal{L}^\infty_{\mathbf{x}}}\right)\left(1 + \ln_+ \left\|\nabla_{\mathbf{x}}\rho\right\|_{\mathcal{L}^\infty_{\mathbf{x}}}\right) + \left\|\rho\right\|_{\mathcal{L}^1_{\mathbf{x}}}\right], \tag{B.7}$$

where $\ln_+ x = \ln \max\{1, x\}$.

Proof. Formulae (B.2) and (B.3) are well-known from potential theory and are proven for example in [8, Lem.4.1&4.2].

For the second derivative, the proof of [8, Lem.4.4] contains the estimate for any $R > 0$ and $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in B_R(\mathbf{x}_0)$, that is,

$$\begin{aligned}\left|\partial_i\partial_j U_\rho(\boldsymbol{\xi}_1) - \partial_i\partial_j U_\rho(\boldsymbol{\xi}_2)\right| \leq \tilde{L}(d,\alpha)\left(R^{-\alpha}\left\|\rho\right\|_{\mathcal{L}^\infty_{\mathbf{x}}} + \sup_{\mathbf{x}\neq\bar{\mathbf{x}}} \frac{|\rho(\mathbf{x}) - \rho(\bar{\mathbf{x}})|}{|\mathbf{x} - \bar{\mathbf{x}}|^\alpha}\right)|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|^\alpha \\ + C(d) R^{-d} \left\|\rho\right\|_{\mathcal{L}^1_{\mathbf{x}}},\end{aligned}$$

where $\tilde{L}$ depends only on the dimension and the Hölder index, and C only on the dimension. Taking the limit $R \to \infty$ proves the estimate (B.4).

In order to prove (B.5), we divide the area of integration into $B_R(\mathbf{x})$ and its complement, yielding for $\frac{1}{p} + \frac{1}{q} = 1$

$$\begin{aligned}\left|\nabla_{\mathbf{x}} U_\rho(\mathbf{x})\right| &\leq \int_{\mathbb{R}^d_{\mathbf{x}} - B_R(\mathbf{x})} |\nabla_{\mathbf{x}}\Gamma(\mathbf{x}-\mathbf{y})|\,|\rho(\mathbf{y})|\ \mathrm{d}\mathbf{y} + \int_{B_R(\mathbf{x})} |\nabla_{\mathbf{x}}\Gamma(\mathbf{x}-\mathbf{y})|\,|\rho(\mathbf{y})|\ \mathrm{d}\mathbf{y} \\ &\leq \left(\int_{\mathbb{R}^d_{\mathbf{x}} - B_R(\mathbf{0})} \frac{\mathrm{d}\mathbf{y}}{\left(\omega_d d\,|\mathbf{y}|^{d-1}\right)^q}\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^d_{\mathbf{x}}} |\rho(\mathbf{y})|^p\ \mathrm{d}\mathbf{y}\right)^{\frac{1}{p}} + \int_{B_R(\mathbf{0})} \frac{\mathrm{d}\mathbf{y}}{\omega_d d\,|\mathbf{y}|^{d-1}} \left\|\rho\right\|_{\mathcal{L}^\infty_{\mathbf{x}}} \\ &= \frac{1}{d\omega_d^{\frac{1}{p}}}\left(\frac{d-p}{p-1}\right)^{\frac{p-1}{p}} R^{-\frac{d-p}{p}} \left\|\rho\right\|_{\mathcal{L}^p_{\mathbf{x}}} + \frac{R}{d}\left\|\rho\right\|_{\mathcal{L}^\infty_{\mathbf{x}}} \overset{R\text{ optimal}}{=} c_{p,d} \left\|\rho\right\|^{\frac{p}{d}}_{\mathcal{L}^p_{\mathbf{x}}} \left\|\rho\right\|^{1-\frac{p}{d}}_{\mathcal{L}^\infty_{\mathbf{x}}},\end{aligned}$$

requiring $p \in [1, d)$.

Inequality (B.6) is derived as follows. Use (B.3) with $\mathbf{x} = \mathbf{x}_0$ and divide the first integral into small ($\leq r$) and large ($> r$) radial distance. On the compact sets, ρ is bounded by the $\mathcal{L}^\infty_{\mathbf{x}}$ norm, $\rho(\mathbf{x}) - \rho(\mathbf{y})$ by the derivative. On the unbounded complement, the interaction potential is bounded and ρ is integrable. □

Appendix C

Notation Index

Symbol	Meaning	Ref.
d	dimension of the physical system	
$\mathbf{x}, \mathbf{v}, \boldsymbol{\xi}$	coordinates in one-particle space $\mathbb{R}^d$	
$\vec{\mathbf{x}}, \vec{\mathbf{v}}, \vec{\boldsymbol{\xi}}$	coordinates in the N particle space $\mathbb{R}^{dN}$	
$\mathbf{z}$	$= (\mathbf{x}, \mathbf{v})$, coordinates in the one-particle phase space $\mathbb{R}^{2d}_{\mathbf{z}}$	
$\hat{\mathbf{z}}$	$= (\mathbf{x}, \boldsymbol{\xi})$, coordinates in the Fourier conjugate space $\mathbb{R}^{2d}_{\hat{\mathbf{z}}}$ of the one-particle phase space $\mathbb{R}^{2d}_{\mathbf{z}}$	
$\vec{\mathbf{z}}, \vec{\hat{\mathbf{z}}}$	coordinates in the respective N particle spaces $\mathbb{R}^{2dN}$	
$\nabla_{\mathbf{z}}, \nabla_{\mathbf{x}}, \nabla_{\mathbf{v}}$	vector valued gradient of a scalar function or matrix valued gradient of a vector field	
$\partial_t, \partial_{x_j}$	scalar partial derivative	
$\Re, \Im$	real and imaginary part	
$\mathbb{S}^1$	unit sphere in $\mathbb{C}$	
$\mathbf{v} \cdot \boldsymbol{\xi}$	Euclidean inner product of $\mathbb{R}^d$	
$\lvert\cdot\rvert$	Euclidean norm on $\mathbb{R}$, $\mathbb{R}^d$, $\mathbb{C}$, or operator norm on matrices	
$\mathbb{1}$	identity matrix	
ω_d	Lebesgue surface of the unit sphere in $\mathbb{R}^d$	
τ_d	Lebesgue volume of the unit ball in $\mathbb{R}^d$	
$A(a, b)$	specific positive constant depending on $a, b > 0$	(3.15)

Symbol	**Meaning**	**Ref.**
$\mathcal{L}^p_{\mathbf{z}}$ ($\mathcal{L}^p_{\mathbf{z},\mathrm{loc}}$)	space of (locally) p-integrable functions over the coordinate space of the given variable	
$\mathcal{W}^{k,p}_{\mathbf{z}}$	Sobolev space of p-integrable functions with weak derivatives up to order k	
$\mathcal{A}^{\kappa,p}_{\mathbf{z}}$	a Banach space of p-integrable continuous functions	Def. A.1
$\mathcal{B}^{k,\kappa,p}_{\mathbf{z}}$	a Banach space of p-integrable k times continuously differentiable functions	Def. A.1
$C^k_{\mathbf{z}}$ ($C^k_{\mathbf{z},c}$)	space of k times continuously differentiable functions (with compact support)	
$C^{k,\alpha}_{\mathbf{z}}$	space of k times continuously differentiable functions with α-Hölder continuous k-th derivatives	
$\mathrm{D}^1\mathcal{H}(f)(\delta f)$	first derivative of $\mathcal{H}$ at f in direction of δf	(1.2)
$[f, g]$	$\equiv \nabla_{\mathbf{x}} f \cdot \nabla_{\mathbf{v}} g - \nabla_{\mathbf{v}} f \cdot \nabla_{\mathbf{x}} g$, Poisson bracket of functions on $\mathbb{R}^{2d}_{\mathbf{z}}$	
$[f, g]_N$	$\equiv \nabla_{\vec{\mathbf{x}}} f \cdot \nabla_{\vec{\mathbf{v}}} g - \nabla_{\vec{\mathbf{v}}} f \cdot \nabla_{\vec{\mathbf{x}}} g$, Poisson bracket of functions on $\mathbb{R}^{2dN}_{\vec{\mathbf{z}}}$	
$(f * g)$	standard convolution on $\mathbb{R}^d_{\mathbf{x}}$ or $\mathbb{R}^{2d}_{\mathbf{z}}$, whatever the smallest common domain of definition of f, g is	
$\langle\cdot,\cdot\rangle$	complex valued $\mathcal{L}^2$ inner product, antilinear in the second argument	
$\omega(\cdot,\cdot)$	$\equiv \Im\langle\cdot,\cdot\rangle$, symplectic form on $\mathcal{L}^2_{\mathbf{z}}$	(SyF)
$\mathbb{S}^{\mathcal{L}^2}$	unit sphere in $\mathcal{L}^2$	

Bibliography

[1] A. Ambrosetti and G. Prodi. *A Primer of Nonlinear Analysis*, volume 34 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1993.

[2] V. Arnold. *Ordinary Differential Equations*. The MIT Press, Cambridge, Massachussetts, 1973.

[3] J. Batt and G. Rein. Global classical solutions of the periodic Vlasov–Poisson system in three dimensions. *C. R. Acad. Sci. Paris Sér. I Math.*, 313(6):411–416, 1991.

[4] M. Bostan. Boundary value problem for the N-dimensional time periodic Vlasov–Poisson system. *Math. Methods Appl. Sci.*, 29(15):1801–1848, 2006.

[5] W. Braun and K. Hepp. The Vlasov dynamics and its fluctuations in the $1/N$ limit of interacting classical particles. *Comm. Math. Phys.*, 56(2):101–113, 1977.

[6] L. Evans. *Partial Differential Equations*. American Mathematical Society, Providence, 1998.

[7] J. Fröhlich, A. Knowles, and S. Schwarz. On the mean-field limit of bosons with Coulomb two-body interaction. *Comm. Math. Phys.*, 288(3):1023–1059, 2009.

[8] D. Gilbarg and N. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Grundlehren der mathematischen Wissenschaften. Springer, Zürich, 2nd edition, 1983.

[9] D. Hartree. *The calculation of atomic structures*. John Wiley & Sons, Inc., New York, 1957.

[10] D. Lazarovici and P. Pickl. A mean field limit for the Vlasov–Poisson system. *Arch. Ration. Mech. Anal.*, 225(3):1201–1231, 2017.

[11] P.-L. Lions and B. Perthame. Propagation of moments and regularity for the 3-dimensional Vlasov–Poisson system. *Invent. Math.*, 105(2):415–430, 1991.

[12] J. Marsden and A. Weinstein. Reduction of symplectic manifolds with symmetry. *Reports on Mathematical Physics*, 5(1):121 – 130, 1974.

[13] C. Mouhot. Stabilité orbitale pour le système de Vlasov–Poisson gravitationnel (d'après Lemou–Méhats–Raphaël, Guo, Lin, Rein et al.). *Astérisque*, (352):Exp. No. 1044, vii, 35–82, 2013. Séminaire Bourbaki. Vol. 2011/2012. Exposés 1043–1058.

[14] R. Neiss. Generalized symplectization of Vlasov dynamics and application to the Vlasov–Poisson system. *Archive for Rational Mechanics and Analysis*, 231(1):115–151, Jan 2019.

[15] R. Neiss. Symmetry reduction and periodic solutions in Hamiltonian Vlasov systems. *arXiv:1901.09571*, 2019.

[16] R. Neiss and P. Pickl. A mean field limit for the Hamiltonian Vlasov system. *arXiv:1811.12011*, 2018.

[17] K. Pfaffelmoser. Global classical solutions of the Vlasov–Poisson system in three dimensions for general initial data. *J. Differential Equations*, 95(2):281–303, 1992.

[18] P. Pickl. On the time dependent Gross Pitaevskii- and Hartree equation. *arXiv:0808.1178v1*, 2008.

[19] P. Pickl. A simple derivation of mean field limits for quantum systems. *Letters in Mathematical Physics*, 97(2):151–164, Aug 2011.

[20] G. Rein. *Collisionless kinetic equations from astrophysics – the Vlasov–Poisson system*, volume 3 pp. 383–476 of *Handbook of differential equations: evolutionary equations*. Elsevier / North-Holland, Amsterdam, 2007.

[21] J. Schaeffer. Global existence of smooth solutions to the Vlasov–Poisson system in three dimensions. *Comm. Partial Differential Equations*, 16(8-9):1313–1335, 1991.

[22] F. Schwabl. *Quantenmechanik (QM I): Eine Einführung*. Springer Berlin Heidelberg, Berlin, Heidelberg, 2007.

[23] A. Vlasov. The vibrational properties of an electron gas. *Phys. Usp.*, 10(6):721–733, 1968.

[24] D. Werner. *Funktionalanalysis*. Springer, 7th edition, 2011.

[25] H. Ye, P. Morrison, and J. Crawford. Poisson bracket for the Vlasov equation on a symplectic leaf. *Phys. Lett. A*, 156:96–100, 1991.

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Erklärung

Ich versichere, dass ich die von mir vorgelegte Dissertation selbständig angefertigt, die benutzten Quellen und Hilfsmittel vollständig angegeben und die Stellen der Arbeit – einschließlich Tabellen, Karten und Abbildungen –, die anderen Werken im Wortlaut oder dem Sinn nach entnommen sind, in jedem Einzelfall als Entlehnung kenntlich gemacht habe; dass diese Dissertation noch keiner anderen Fakultät oder Universität zu Prüfung vorgelegen hat; dass sie – abgesehen von unten angegebenen Teilpublikationen – noch nicht veröffentlicht worden ist sowie, dass ich eine solche Veröffentlichung vor Abschluss des Promotionsverfahrens nicht vornehmen werde. Die Bestimmungen der Promotionsordnung sind mir bekannt. Die von mir vorgelegte Dissertation ist von Univ.-Prof. Dr. Markus Kunze betreut worden.

Robert Axel Neiss   Köln, 5. April 2019

Liste der Teilpublikationen

[1] R. Neiss. Generalized symplectization of Vlasov dynamics and application to the Vlasov–Poisson system. *Archive for Rational Mechanics and Analysis*, 231(1):115–151, Jan 2019.

[2] R. Neiss. Symmetry reduction and periodic solutions in Hamiltonian Vlasov systems. *arXiv:1901.09571*, 2019.

[3] R. Neiss and P. Pickl. A mean field limit for the Hamiltonian Vlasov system. *arXiv:1811.12011*, 2018.