Bergman Kernel Asymptotics for Partially Positive Line Bundles

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Zusammenfassung

Ende des letzten Jahrhunderts lieferten Catlin und Zelditch eine vollständige Beschreibung des folgenden Phänomens: Betrachtet man die Bergman-Kern-Funktion B_k des k-ten Tensorproduktes eines positiven holomorphen Geradenbündels L über einer kompakten, geschlossenen, komplexen Mannigfaltigkeit M, so hat diese Funktion eine asymptotische Entwicklung in k, d.h. B_k lässt sich als formale Summe

$$B_k \sim a_0 k^n + a_1 k^{n-1} + a_2 k^{n-2} + a_3 k^{n-3} \dots, \text{ für } k \to \infty$$

schreiben. Hierbei kodieren die Funktionen $a_0, a_1, \dots$ lokale geometrische Eigenschaften der zugrundeliegenden Objekte M und L. In diesem Sinne untersuchen wir sowohl das asymptotische Verhalten der Bergman-Kern-Funktion als auch das Verhalten des zugehörigen Bergman-Kerns für den Fall, dass die Mannigfaltigkeit nicht notwendigerweise kompakt, die Metrik nur teilweise positiv, und die zugrundeliegende Geometrie nicht beliebig regulär ist.

Im ersten Teil der Arbeit nehmen wir an, dass M ein beschränktes Gebiet des euklidischen Raums ist und konstruieren einen asymptotisch reproduzierenden Integralkern für quadratintegrable holomorphe Funktionen bezüglich eines beliebigen gewichteten inneren Produkts. Wir zeigen, dass zu jeder natürlichen Zahl N eine offene Teilmenge M_N existiert, auf welcher sowohl der Bergman-Kern als auch die Bergman-Kern-Funktion eine asymptotische Entwicklung bis hin zur Ordnung N besitzen, abhängig von der Regularität der zugrundeliegenden geometrischen Objekte. Es stellt sich heraus, dass die Krümmung der Metrik außerhalb von M_N beliebig seien kann. Unsere Methode liefert außerdem eine explizite Darstellung der Koeffizienten dieser asymptotischen Entwicklung für jede Wahl von Koordinaten.

Im zweiten Teil der Arbeit nehmen wir an, dass M ein Gebiet einer vollständigen Kähler-Mannigfaltigkeit X ist und dass sich L zu einem holomorphen Geradenbündel L_0 über X fortsetzen lässt. Wir beginnen mit einer oberhalbstetigen Metrik auf L und betrachten die Menge M_∞ bestehend aus Punkten, an denen diese Metrik durch positive Metriken auf L_0 niedergehalten wird. Unter Verwendung der Resultate des ersten Teils beweisen wir, dass der Bergman-Kern und die Bergman-Kern-Funktion eine asymptotische Entwicklung auf M_∞ haben, wobei die Ordnung dieser Entwicklung nur durch die Regularität der Geometrie begeschränkt wird.

Abstract

A famous result of Catlin and Zelditch developed in the end of the last century gives a complete description for the following phenomena: Given a positive holomorphic line bundle L over a closed compact complex manifold M the Bergman kernel function B_k for the k-th tensor power of L has a full asymptotic expansion. More precisely, B_k can be written as a formal sum

$$B_k \sim a_0 k^n + a_1 k^{n-1} + a_2 k^{n-2} + a_3 k^{n-3} \ldots, \quad \text{for } \ k \to \infty$$

where the coefficients $a_0, a_1, \ldots$ purely depend on the local geometric data of X and L. In that sense, we study the asymptotic behavior of the Bergman kernel function and the related Bergman kernel when M is not necessarily compact, L is only partially positive and the geometric data fail to be smooth.

In the first part of this thesis we consider M to be a bounded domain in the Euclidean space and establish a local asymptotically reproducing kernel for square integrable holomorphic functions with respect to a weighted inner product. From this method we deduce that for any non-negative integer N the Bergman kernel and the Bergman kernel function have an asymptotic expansion on some set M_N up to some order less than or equal to N depending on the regularity of the geometric data. It turns out that the curvature of the metric can be arbitrary in the complement of M_N. In addition, our method provides an explicit formula for the coefficients in this expansion which holds for any choice of coordinates.

In the second part we assume that M is a domain contained in a complete Kähler manifold X and that L can be holomorphically extended to a holomorphic line bundle L_0 over X. We start with an upper semi-continuous metric on L and consider the set M_∞ consisting of points where the metric of L can be suppressed by positive metrics defined on L_0. Using the results obtained in the first part, we prove that the Bergman kernel and the Bergman kernel function have an asymptotic expansion on M_∞ where the order of the expansion is just limited by the regularity of the geometric data.

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Chapter 1

Introduction and Statement of the Results

1.1 Introduction

Let E be a holomorphic line bundle over a complex manifold M and let $\mathcal{O}(M, E)$ be the space of global holomorphic sections. The choice of a Hermitian metric on E and a volume form dV_M on M defines the space $L^2(M, E)$, which is the space of L^2-integrable sections in E with respect to the norm $\|\cdot\|_h$ coming from the inner product

$$(u, v)_h := \int_M h(u, v) dV_M \text{ , for } u, v \in L^2(M, E).$$

For simplicity we will assume that h and dV_M are smooth objects. Let

$$H_2^0(M, E) = L^2(M, E) \cap \mathcal{O}(M, E)$$

denote the space of holomorphic sections with finite L^2-norm. One can check that this is a closed d-dimensional subspace of $L^2(M, E)$ where $d \in \mathbb{N}_0 \cup \{\infty\}$ strongly depends on the choice of the inner product when M is non-compact. Since $H_2^0(M, E)$ is closed there exists a unique orthogonal projection $L^2(M, E) \to H_2^0(M, E)$. It turns out that this projection can be represented by an integral kernel P_{h,dV_M} called Bergman kernel which is a smooth section $M \times M \to E \boxtimes E^*$ and by restricting P_{h,dV_M} along the diagonal in $M \times M$ we obtain the so called Bergman kernel function $B_{h,dV_M} \in C^\infty(M, \mathbb{R})$. Given an orthonormal basis $\{s_j\}_{j=1}^d$ of $H_2^0(M, E)$ we have the following representations

$$\begin{aligned} P_{h,dV_M}(x, y) &= \sum_{j=1}^{d} s_j(x) \otimes s_j(y)^* \in E_x \otimes E_y^*, \\ B_{h,dV_M}(x) &= \sum_{j=1}^{d} |s_j(x)|_h^2 \in \mathbb{R}, \quad x, y \in M. \end{aligned} \tag{1.1}$$

From the construction it is already clear that (1.1) does not depend on the choice of the orthonormal basis. Furthermore, we have $\int_M B_{h,dV_M} dV_M = d$, so B_{h,dV_M} should be seen as a dimension density of $H_2^0(M, E)$. The Bergman kernel was first introduced by Stefan Bergman in 1922 for domains in $\mathbb{C}^n$ [1]. Because of its strong connection to many subjects in complex geometry, complex analysis and quantum physics it has attracted a lot of attention during the last century till now (see [30], [11]). The Bergman kernel and the Bergman kernel function appear for example in the context of pseudoconvex boundaries [7], [19], [28] and extensions of holomorphic maps [18], holomorphic embeddings [21], distribution of zeros of random holomorphic sections [35], existence and approximation of Kähler metrics [36], vanishing theorems [14] as well as in quantization theory [34], [25], [18], [9], [32] and the computation of path integrals [15]. For a complete reference see the book of Ma–Marinescu [30]. A very important subject is to study the Bergman kernel for the line bundle $L_k = L^k \otimes E$, $k \in \mathbb{N}$, where L^k is the k-th tensor power of a holomorphic Hermitian line bundle L over M. The metrics of L and E induce a metric h_k on L_k and one tries to understand the asymptotic behavior of P_{h_k,dV_M} and B_{h_k,dV_M} for $k \to \infty$. When L is positive, i.e. its metric has a positive curvature, M is compact and h_k and dV_M are smooth it follows from a result of Catlin [10] and Zelditch [38] that B_{h_k,dV_M} has an asymptotic expansion that is

$$B_{h_k,dV_M} \sim a_0 k^n + a_1 k^{n-1} + a_2 k^{n-2} + \dots \tag{1.2}$$

where $a_0, a_1, \dots$ are smooth functions with $a_0 > 0$. More precisely, for any $N, r \in \mathbb{N}_0$ there exists a constant $C = C_{N,r} > 0$ such that

$$\left\| B_{h_k,dV_M} - \sum_{j=0}^{N} a_j k^{n-j} \right\|_{C^r(M)} \leq C_{N,r} k^{-N-1+n}$$

holds for all $k \in \mathbb{N}$. Furthermore, they calculated a_0. Note that the computation of the coefficients a_j in terms of local geometric data is an interesting but difficult subject (see Section 1.3 and the discussion therein).

In general, the asymptotic behavior of P_{h_k,dV_M} and B_{h_k,dV_M} in the case when L is globally positive or semi-positive and M possess a complete Kähler metric is well understood due to Dai-Lu-Ma [13] and Ma-Marinescu [30]. Their results even hold for symplectic manifolds and orbifolds and can be generalized to the case where h_k is singular in some specific cases (see [31], [12], [20]).

When L fails to be globally semi-positive there are - compared to the wide range of literature for the globally positive case - only a few results known. The most general results for that case are due to Berman [5] and Hsiao–Marinescu [20]. In [4] and [5] Berman studies big line bundles over compact projective manifolds. He established criteria on the existence of an asymptotic expansion in terms of equilibrium weights. Hsiao–Marinescu proved in [20] the existence of an asymptotic expansion under the assumption that the Kodaira-Laplacian has a small (local) spectral gap. Given a point $p \in M$ where the curvature of L fails to be positive it follows from a result of Berman [2] (see also [20]) that

$$\limsup_{k\to\infty} k^{-n} B_{h_k,dV_M}(p) = 0.$$

This shows that one should consider the set of points where L is positive in order to get an asymptotic expansion as in (1.2).

In this thesis we consider the case when the metric of L is arbitrary and study the asymptotic behavior of P_{h_k,dV_M} and B_{h_k,dV_M} at points where the curvature of L is positive. Starting with bounded domains D in $\mathbb{C}^n$ and globally trivial line bundles, assuming that the metric and the volume form is continuous up to the boundary, we give a self contained proof for Bergman kernel expansion based on elementary methods from complex analysis in combination with the asymptotic expansion of oscillatory integrals in a version proven by Hörmander [24] which allows us to weaken the regularity assumptions on the metric h_k. It turns out that for any $N \in \mathbb{N}_0$ we have an expansion up to order N on a set D_N (see Definition 1.1) where the curvature is positive, sufficiently regular and the metric satisfies some growth condition on D but is not necessarily semi-positive in the complement of D_N. Furthermore, thanks to Hörmander's method of stationary phase, our approach leads also to an explicit formula for all coefficients a_j in the asymptotic expansion (1.2) (see Definition 1.12). The main result in this part of the thesis is the construction of a reproducing kernel which asymptotically recovers the value of any holomorphic function at any given point in D_N up to some error which is an $O(k^{-N-1+n+\varepsilon})$ (see Theorem 1.3). The set D_N can be very small or even empty strongly depending on D and N. However, by shrinking D we can always ensure that D_N is non-empty. In this sense our results provide a local reproducing kernel which can be used to study the Bergman kernel in a more general case which leads to the second part of the thesis.

In the second part we will combine our results of local reproducing kernels with the L^2 estimates of Hörmander [22] given in a generalized version by Demailly [14]. We study the Bergman kernel for a domain M contained in a complete Kähler manifold X for line bundles which can be globally extended to X. Given a continuous volume form dV_M on M and an upper semi-continuous Hermitian metric h on L we obtain quite general results on Bergman kernel expansion for some set M_h which consists of points where the metric h is suppressed by a semi-positive metric h_0 of a holomorphic line bundle L_0 over X with $L_0|_M = L$ (see Definition 1.14, Theorem 1.15 and Theorem 1.16). An important feature of this method is that we can assume very weak regularity conditions on h and dV_M in the complement of M_h (see Example 1.19).

The thesis is organized as follows: this chapter provides an overview on our results (Section 1.2 and Section 1.3) as well as a discussion of their relation to previous results (Section 1.4). Furthermore, we give a sketch of the proofs in Section 1.5 pointing out the main ideas. Chapter 2 contains a detailed study of Bergman kernels and their asymptotics on domains in $\mathbb{C}^n$. In Chapter 3 we will prove the results announced in Section 1.2 and Section 1.3.

1.2 Local Expansion

Let $D \subset \mathbb{C}^n$ be a bounded domain $\varphi, \rho \in C^0(\overline{D})$ such that $\rho > 0$ on $\overline{D}$. We will consider the setting $(D \times \mathbb{C}, h) \to (D, dV_D)$ where the projection of the line bundle is given by $(z, \lambda) \mapsto z$, $dV_D = \rho dV_{\mathbb{C}^n}$ and h is defined by $|(z,\lambda)|_h^2 = |\lambda|^2 e^{-\varphi(z)}$. We identify holomorphic sections with holomorphic functions on D via the trivialization $z \mapsto (z, 1)$, that is we consider the space $H^0_{k\varphi,\rho}(D) = \{f \in \mathcal{O}(D) \mid \|f\|_{k\varphi,\rho} < \infty\}$ where the norm $\|\cdot\|_{k\varphi,\rho}$ is induced by the inner product

$$(f,g)_{k\varphi,\rho} = \int_D f\overline{g}e^{-k\varphi}dV_D, \quad f, g \in H^0_{k\varphi,\rho}(D).$$

Assume $\varphi \in C^{N+2}(D) \cap C^0(\overline{D})$ for some non-negative integer $N \in \mathbb{N}_0$. We define the functions $\gamma_N, \tilde{\varphi}_N : D \times \overline{D} \to \mathbb{C}$,

$$\begin{aligned} \gamma_N(z,w) &= \frac{\varphi(z)}{2} + \sum_{1 \le |\alpha| \le N+2} \frac{1}{\alpha!} \frac{\partial^{|\alpha|}\varphi(z)}{\partial^\alpha z}(w-z)^\alpha, \\ \tilde{\varphi}_N(z,w) &= \varphi(w) - \gamma_N(z,w) - \overline{\gamma_N(z,w)}. \end{aligned}$$

The complex Hessian of φ in $z \in D$ is the Hermitian $n \times n$ matrix given by

$$H_\varphi(z) = \left(\frac{\partial^2\varphi(z)}{\partial z_j \partial \overline{z}_l}\right)_{1 \le l,j \le n}.$$

Let $D_{\varphi,+}$ denote the set of points in D where $H_\varphi(z)$ is positive definite.

Definition 1.1
We say that $z_0 \in D$ has the N-th localization property (for φ) if the following two conditions are satisfied

$$\begin{aligned} &(i) \quad z_0 \in D_{\varphi,+} \\ &(ii) \quad \tilde{\varphi}_N(z_0, z) > 0 \text{ for all } z \in \overline{D} \setminus \{z_0\}. \end{aligned}$$

The set of all points which satisfy this condition is denoted by $D_{\varphi,N}$.

Example 1.2
Let $D \subset C^n$ be a domain and $\varphi \in C^\infty(D, \mathbb{R}, \varphi(z) = \sum_{j=1}^n \lambda_j |z_j|^2$, where $\lambda_1, \ldots, \lambda_n$ are positive real numbers. We have $H_\varphi(z) = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ and hence $D_{\varphi,+} = D$. Furthermore, $D_{\varphi,N} = D$ holds for all $N \in \mathbb{N}_0$.

We have $z_0 \in D_{\varphi,+}$ if and only if the transformed weight $\tilde{\varphi}_N(z_0, \cdot)$ is positive on $\overline{D} \setminus \{0\}$ (see Figure 2.1 in Section 2.5). Note that it does not need to be plurisubharmonic away from z_0. The N-th localization property is carefully studied in Section 2.5. For example, if D is bounded, we have that $D_{\varphi,N}$ is open.

We will show that under certain regularity conditions on φ and ρ we have that $B_{k\varphi,\rho}$ and $P_{k\varphi,\rho}$ have asymptotic expansions up to order N and $N/2$ on $D_{\varphi,N}$ and $D_{\varphi,N} \times D$. Our first main result is the construction of some reproducing kernel function which recovers the value of any holomorphic function $f \in H^0_{k\varphi,\rho}(D)$ at a point $z \in D_{\varphi,N}$ up to some error of order $N+1-n+\varepsilon$ where $\varepsilon > 0$ can be chosen arbitrarily small.

Theorem 1.3
Let $D \subset \mathbb{C}^n$ be a bounded domain, $l \in \mathbb{N}_0$ a non-negative integer and

$$\varphi \in C^{6N+3n+4+l}(D,\mathbb{R}) \cap C^0(\overline{D}) \text{ and } \rho \in C^{4N+2n+2+l}(D,\mathbb{R}) \cap C^0(\overline{D})$$

be two functions such that $\rho > 0$ on $\overline{D}$ and $dV_D(z) = \rho(z)dV_{\mathbb{C}^n}$. For any $\alpha \in \mathbb{N}_0^n$ and $j \in \mathbb{N}$, $|\alpha|, j \leq N$, there exist functions $\lambda^{(j)}_{N,\alpha} \in C^l(D_{\varphi,+})$ where $\lambda^{(j)}_{N,\alpha}(z_0)$ only depend on φ, ρ and their derivatives at $z_0 \in D_{\varphi,+}$, such that $K_{k\varphi,N} : D_{\varphi,+} \times D \to \mathbb{C}$ defined by

$$K_{k\varphi,N}(z,w) = k^n e^{k(\gamma_N(z,z)+\gamma_N(z,w))} \sum_{j=0}^{N} k^{-j} \sum_{|\alpha|\leq N} \lambda^{(j)}_{N,\alpha}(z)(w-z)^\alpha$$

satisfies the following: For any compact set $K \subset D_{\varphi,N}$ and any $\varepsilon > 0$ there exists a constant $C > 0$ such that

$$\left|f(z) - (f, K_{k\varphi,N}(z,\cdot))_{k\varphi,\rho}\right|^2 e^{-k\varphi(z)} \leq Ck^{-(N+1)+n+\varepsilon}\|f\|^2_{k\varphi,\rho}$$

holds for all $k \in [1,\infty)$, $z \in K$ and $f \in H^0_{k\varphi,\rho}(D)$.
Here C is bounded when φ stays in a bounded set in $C^{6N+3n+4}(D,\mathbb{R}) \cap C^0(\overline{D})$ such that $\inf_{(z,w)\in K\times\overline{D}} \tilde{\varphi}_N(z,w)/|w-z|^2$ has a positive lower bound and ρ stays in a bounded set in $C^{4N+2n+2}(D,\mathbb{R}) \cap C^0(\overline{D})$ such that $\inf_{w\in\overline{D}} \rho(w)$ has a positive lower bound.

We have an explicit formula for $\lambda^{(j)}_{N,\alpha}(z_0)$, $j, |\alpha| \leq N$, $j \leq N - \frac{|\alpha|}{2}$ (see Theorem 3.17 and Remark 3.18) in terms of ρ, φ and their derivatives. In the smooth setting we can show that this formula also holds for $j > N - \frac{|\alpha|}{2}$ with some improvements (see Lemma 3.23). Note that for the expansion of the Bergman kernel function $B_{k\varphi,\rho}$ we just need to know $\lambda^{(j)}_{N,0}$.

Definition 1.4
For $\varphi \in C^{6j+3n+4}(D,\mathbb{R})$ and $\rho \in C^{4j+2n+2}(D,\mathbb{R})$, $\rho > 0$, define $b_j = b_j^{\varphi,\rho} \colon D_{\varphi,+} \to \mathbb{R}$ by $b_0 = 1$ and for $j \geq 1$, $z \in D_{\varphi,+}$ set

$$b_j(z) \;=\; \sum_{d=1}^{2j} \sum_{\substack{\alpha\in\mathbb{N}^d \\ |\alpha|=2j}} \sum_{\substack{(\beta^{(1)},\ldots,\beta^{(d)})\in(\mathbb{N}_0^n)^{d-1} \\ |\beta^{(1)}|,\ldots,|\beta^{(d-1)}|\leq j}} (-1)^d \nu^{(\alpha_1)}_{\tau,\beta^{(1)}} \nu^{(\alpha_2)}_{\beta^{(1)},\beta^{(2)}} \cdot \ldots \cdot \nu^{(\alpha_d)}_{\beta^{(d-1)},0}$$

where

$$\nu^{(r)}_{\alpha,\beta} = \frac{\lambda^\beta}{\beta!}\frac{\chi^{(r)}_{|\alpha|,|\beta|}}{\rho(z)} \sum_{l=0}^{r+|\alpha|+|\beta|} \sum_{\substack{|\eta|=l+\frac{r+|\alpha|+|\beta|}{2} \\ \eta\geq \max\{\alpha,\beta\}}} (-1)^l \frac{\eta!}{l!\lambda^\eta}\mu^{(l)}_{\eta-\alpha,\eta-\beta}$$

$$\chi^{(r)}_{p,q} = \begin{cases} 1 & \text{, if } 2 \mid (r+p+q) \text{ and } r \geq |p-q| \ , \\ 0 & \text{, else,} \end{cases}$$

and

$$\mu^{(l)}_{\alpha,\beta} = \sum_{\substack{(\alpha^{(0)},\ldots,\alpha^{(l)})\in(\mathbb{N}_0^n)^{l+1} \\ |\alpha^{(0)}_m|+\ldots+|\alpha^{(l)}_m|=\alpha_m}} \sum_{\substack{(\beta^{(0)},\ldots,\beta^{(l)})\in(\mathbb{N}_0^n)^{l+1} \\ |\beta^{(0)}_m|+\ldots+|\beta^{(l)}_m|=\beta_m}} \frac{\rho_{\alpha^{(0)},\beta^{(0)}}(z)}{\alpha^{(0)}!\beta^{(0)}!}\cdot\prod_{j=1}^{l}\frac{\varphi_{\alpha^{(j)},\beta^{(j)}}(z)}{\alpha^{(j)}!\beta^{(j)}!}$$

$\rho_{\alpha,\beta}(z) = \overline{(X_w)^\alpha}(X_w)^\beta\rho(z)$ and

$$\varphi_{\alpha,\beta}(z) = \begin{cases} \left(\overline{X^\alpha_w}X^\beta_w\varphi\right)(z) & \text{, if } \max\{|\alpha|,|\beta|\}\geq 2, \min\{|\alpha|,|\beta|\}\geq 1, \\ 0 & \text{, else,} \end{cases}$$

with

$$(X_w)^\alpha = \prod_{m=1}^{n} X^{\alpha_m}_{w,m}, \quad X_{w,m} = F_{m1}\frac{\partial}{\partial w_1} + \ldots + F_{mn}\frac{\partial}{\partial w_n}$$

where $F = (F_{lm})_{1\leq l,m\leq n}$ is an invertible complex matrix such that $F^*H_\varphi(z)F = \mathrm{diag}(\lambda_1,\ldots,\lambda_n)$ for some $\lambda_1,\ldots,\lambda_n\in\mathbb{R}_+$ and we set $\lambda^\eta = \lambda_1^{\eta_1}\cdots\lambda_n^{\eta_n}$.

Note that in Definition 1.4 we have $X^\alpha_w = \partial^\alpha_w$ when $F = \mathrm{Id}$ and that $\lambda_1,\ldots,\lambda_n$ strongly depend on the choice of F. From the definition it is not clear that $b_j(z)$ is real valued, independent of F and continuous. This has to be proven.

Lemma 1.5
We have that $b_j(z)$ is well defined and independent of the choice of F. Furthermore, given $\varphi\in C^{6j+3n+4+l}(D)$ and $\rho\in C^{4j+2n+2+l}(D,\mathbb{R})$ for some $l\in\mathbb{N}_0$ we have $b_j\in C^l(D_{\varphi,+})$.

We will show now that the b_js , $b_j = b_j^{\varphi,\rho}$, defined above are precisely the coefficients in the asymptotic expansion of the Bergman kernel function. See Section 2.1 for the notations we use in Theorem 1.6.

Theorem 1.6
Let $D\subset\mathbb{C}^n$ be a bounded domain and

$$\varphi\in C^{6N+3n+4+l}(D,\mathbb{R})\cap C^0(\overline{D}), \quad \rho\in C^{4N+2n+2+l}(D,\mathbb{R})\cap C^0(\overline{D})$$

be two functions such that $\rho>0$ on $\overline{D}$ and $dV_D(z) = \rho(z)dV_{\mathbb{C}^n}$. For any $\varepsilon>0$ and any $0\leq r\leq l$ we have

$$B_{k\varphi,\rho} - \frac{\det(H_\varphi)}{\pi^n\rho}k^n\sum_{j=0}^{N}k^{-j}b_j^{\varphi,\rho} = O(k^{-c_r(N+1)+n+r+\varepsilon}) \text{ in } C^r(D_{\varphi,N})$$

with $c_r = 1 - \frac{r}{l+1}$. *More precisely, for any compact set* $K \subset D_{\varphi,N}$, *any* $0 \le r \le l$ *and any* $\varepsilon > 0$ *there exists a constant* $C = C_{K,\varepsilon,l} > 0$ *such that*

$$\left\| B_{k\varphi,\rho} - \frac{\det(H_\varphi)}{\pi^n \rho} k^n \sum_{j=0}^{N} k^{-j} b_j^{\varphi,\rho} \right\|_{C^r(K)} \le C_{K,\varepsilon,l} k^{-c_r(N+1)+n+r+\varepsilon}$$

holds for all $k \in [1,\infty)$. *Here* C *is bounded when* φ *stays in a bounded set in* $C^{6N+3n+4+l}(D,\mathbb{R}) \cap C^0(\overline{D})$ *such that* $\inf_{(z,w)\in K\times\overline{D}} \tilde{\varphi}_N(z,w)/|w-z|^2$ *has a uniform positive lower bound and* ρ *stays in a bounded set in* $C^{4N+2n+2+l}(D,\mathbb{R}) \cap C^0(\overline{D})$ *such that* $\inf_{w\in\overline{D}} \rho(w)$ *has a uniform positive lower bound.*

We have the following theorem for the off-diagonal expansion.

Theorem 1.7
Let $D \subset \mathbb{C}^n$ *be a bounded domain and*

$$\varphi \in C^{6N+3n+4+l}(D,\mathbb{R}) \cap C^0(\overline{D}), \ \ \rho \in C^{4N+2n+2+l}(D,\mathbb{R}) \cap C^0(\overline{D})$$

be two functions such that $\rho > 0$ *on* $\overline{D}$ *and* $dV_D(z) = \rho(z) dV_{\mathbb{C}^n}$. *Given* $N \in \mathbb{N}_0$ *define* $P_{k\varphi,N} \colon D_{\varphi,N} \times D \to \mathbb{C}$ *by*

$$P_{k\varphi,\rho,N}(z,w) = k^n e^{-k\left(\frac{\varphi(w)}{2} - \overline{\gamma_N(z,w)}\right)} \sum_{j=0}^{N} k^{-j} \overline{\sum_{|\alpha|\le N} \lambda_{N,\alpha}^{(j)}(z)(w-z)^\alpha}$$

satisfies with $\lambda_{N,\alpha}^{(j)}$ *as in Theorem 1.3. For any* $\varepsilon > 0$ *and* $r \le l$ *we have*

$$P_{k\varphi,\rho} - P_{k\varphi,\rho,N} = O(k^{-\frac{c_r}{2}(N+1)+n+r+\varepsilon}) \text{ in } C^r(D_{\varphi,N} \times D)$$

with $c_r = 1 - \frac{r}{l+1}$. *More precisely, for any compact set* $K \subset D_{\varphi,N} \times D$, *any* $r \le l$, *and any* $\varepsilon > 0$ *there exists a constant* $C = C_{K,l,\varepsilon} > 0$ *such that*

$$\|P_{k\varphi,\rho} - P_{k\varphi,\rho,N}\|_{C^r(K)} \le C_{K,l,\varepsilon} k^{-\frac{c_r}{2}(N+1)+n+r+\varepsilon}$$

holds for all $k \in [1,\infty)$. *Here* C *is bounded when* φ *stays in a bounded set in* $C^{6N+3n+4+l}(D,\mathbb{R}) \cap C^0(\overline{D})$ *such that* $\inf_{(z,w)\in K\times\overline{D}} |w-z|^2/\tilde{\varphi}_N(z,w)$ *has a uniform positive lower bound and* ρ *in a bounded set in* $C^{4N+2n+2+l}(D,\mathbb{R}) \cap C^0(\overline{D})$ *such that* $\inf_{w\in\overline{D}} \rho(w)$ *has a uniform positive lower bound.*

Corollary 1.8
Let $K \subset D_{\varphi,N} \times D$ *be compact. Given* $\varepsilon > 0$ *there exist constants* $C, \delta > 0$ *such that* $|P_{k\varphi,\rho}(z,w)| \le C\left(k^n e^{-\delta k|w-z|^2} + k^{-\frac{N+1}{2}+n+\varepsilon}\right)$ *holds for all* $(z,w) \in K$, $k \in [1,\infty)$. *Here* C *and* δ^{-1} *stay bounded under the same conditions on* φ *and* ρ *as in Theorem 1.7.*

Definition 1.9
Given $\varphi, \rho \in C^\infty D \cap C^0(\overline{D})$, $\alpha \in \mathbb{N}_0^n$, $j \in \mathbb{N}_0$ define $b_{j,\alpha} \in C^\infty(D_{\varphi,+})$

$$b_{j,\alpha} = \frac{\alpha! \pi^n \rho}{\det(H_\varphi)} \overline{\lambda^{(j)}_{2j+2|\alpha|,\alpha}}$$

with $\lambda^{(j)}_{N,\alpha}$ as in Theorem 1.3, and set

$$\hat{P}_{k\varphi,\rho,N}(z,w) = \frac{k^n \det(H_\varphi(z))}{\pi^n \rho(z)} e^{-k(\frac{\varphi(w)}{2} - \overline{\gamma_N(z,w)})} \sum_{j=0}^{\lfloor \frac{N}{2} \rfloor} k^{-j} \sum_{|\alpha| \leq \lfloor \frac{N}{2} \rfloor - j} \frac{b_{j,\alpha}(z)}{\alpha!} \overline{(w-z)}^\alpha$$

for $(z,w) \in D_{\varphi,+}$.

Corollary 1.10
With the notations in Theorem 1.7 and Definition 1.4 assuming that φ and ρ are smooth and continuous up to the boundary, i.e. $\varphi, \rho \in C^\infty(D,\mathbb{R}) \cap C^0(\overline{D})$, we have

$$\begin{aligned} P_{k\varphi,\rho} - \hat{P}_{k\varphi,\rho,4N} &= O(k^{-N+n+r-\varepsilon}) \text{ in } C^r(D_{\varphi,4N} \times D), \\ B_{k\varphi,\rho} - \frac{\det(H_\varphi)}{\pi^n \rho} k^n \sum_{j=0}^N k^{-j} b_j &= O(k^{-(N+1)+n+r+\varepsilon}) \text{ in } C^r(D_{\varphi,N}) \end{aligned}$$

for any $N, r \in \mathbb{N}_0$ and any $0 < \varepsilon < \frac{1}{2}$.

Note that we do not assume that $D_{\varphi,+} = D$, that is φ does not need to be positive definite (or positive semi-definite) everywhere on D. In a geometric sense this means that the fiber metric induced by φ is only partially positive. In general the set $D_{\varphi,N}$ is small or even empty depending strongly on N. On the other hand we have that given any $N \in \mathbb{N}_0$ and any point in $z_0 \in D_{\varphi,+}$ there exists an open neighborhood U of z_0 such that $z_0 \in U_{\varphi,N}$ holds. Therefore, our results should be considered as local expansion results. However, using some deep result of Hörmander [22] and Demailly [14] we can apply our local computations to a more general setting. We are going to describe the results obtained for such a setting in the next section. Let us finish this section with the following example.

Example 1.11
Let $D \subset \mathbb{C}^n$ be a bounded domain. Let $U \subset D$ be some open set $\psi \in C_0^\infty(U,\mathbb{R})$ a smooth non-negative function supported in U. Consider the weight $\varphi \in C^\infty(D,\mathbb{R}) \cap C^0(\overline{D})$, $\varphi(z) = \sum_{j=1}^n \lambda_j |z_j|^2 + \psi(z)$, $\lambda_1, \ldots, \lambda_n > 0$. As in Example 1.2 we find $V := D \setminus U \subset D_{\varphi,+}$. Given $\rho \in C^\infty(D) \cap C^0(\overline{D})$, $\rho > 0$ on $\overline{D}$, any $N, r \in \mathbb{N}_0$ and any $0 < \varepsilon < \frac{1}{2}$ we have

$$\begin{aligned} P_{k\varphi,\rho} - \hat{P}_{k\varphi,\rho,4N} &= O(k^{-N+n+r-\varepsilon}) \text{ in } C^r(V \times D), \\ B_{k\varphi,\rho} - \frac{\det(H_\varphi)}{\pi^n \rho} k^n \sum_{j=0}^N k^{-j} b_{j,0} &= O(k^{-(N+1)+n+r+\varepsilon}) \text{ in } C^r(V) \end{aligned} \tag{1.3}$$

with

$$P_{k\varphi,\rho,2N}(z,w) = \frac{k^n}{\pi^n \rho(z)} e^{-\frac{k}{2}\left(\sum_{j=1}^n \lambda_j (|w_j - z_j|^2 - z_j\overline{w_j} + \overline{z_j} w_j)\right)} \sum_{j=0}^N k^{-j} \sum_{|\alpha|\le N-j} \frac{b_{j,\alpha}(z)}{\alpha!} \overline{(w-z)}^\alpha.$$

An explicit formula for $b_{j,\alpha}$ is obtained in Theorem 3.17 and Lemma 3.23. We will state this formula here for the case $n = 1$ and $\lambda_1 = 1$. We find

$$b_{j,\alpha} = \sum_{d=1}^{2j+|\alpha|} \sum_{\substack{\tau\in\mathbb{N}^d \\ |\tau|=2j+|\alpha|}} \sum_{\substack{\beta\in\mathbb{N}_0^{d-1} \\ \beta_1,\ldots,\beta_{d-1}\le 2j+2|\alpha|}} \frac{(-1)^d}{\beta!\rho^d} \nu^{(\tau_1)}_{\alpha,\beta_1} \nu^{(\tau_2)}_{\beta_1,\beta_2} \cdot \ldots \cdot \nu^{(\tau_d)}_{\beta_{d-1},0}$$

where

$$\nu^{(r)}_{p,q} = \chi^{(r)}_{p,q} \frac{\partial_z^{\frac{r+q-p}{2}} \overline{\partial_z^{\frac{r+p-q}{2}}} \rho}{(\frac{r+q-p}{2})!(\frac{r+p-q}{2})!}$$

with $\chi^{(r)}_{p,q}$ as in Definition 1.4. Note that $b_{j,0} = b_j$ holds which implies that the sum for $b_{j,0}$ just need to be considered for $\beta_1, \ldots, \beta_{d-1} \le j$.

1.3 Global Expansion

Let $(L,h) \to (M, dV_M)$ be a holomorphic Hermitian line bundle over a complex manifold M with volume form dV_M. Given complex coordinates (U,z) around a point $p \in M$ and a local holomorphic frame s we can identify $h \sim e^{-\varphi}$ with $\varphi = -\log(h(s,s))$, $dV_M = \tilde{\rho} dV_{\mathbb{C}^n}$ for some positive function $\tilde{\rho}$, and holomorphic sections on U with holomorphic functions in the complex variable z. We say that h is upper semi-continuous or has upper semi-continuous weight if $-\log(h(s,s))$ is upper semi-continuous for any local holomorphic frame s. Let $M_{h,+}$ denote the set of points in M which have a neighborhood where h is of class C^2 and has positive curvature. Let (E, h_E) be another holomorphic line bundle with smooth Hermitian metric. We are interested in studying the Bergman kernel $P_k = P_{h^k\otimes h_E, dV_M}$ and its Bergman kernel function $B_k = B_{h^k\otimes h_E, dV_M}$ for the space $H^0_2(M, L^k \otimes E)$ consisting of all holomorphic sections with finite L^2-norm as mentioned in Section 1.1. We define the following invariants for our setting.

Definition 1.12
Assume that h is of class $C^{6j+3n+4}$ and dV_M is of class $C^{4j+2n+2}$. Define

$$b_j = b_j^{h,h_E,dV_M} \colon M_{h,+} \to \mathbb{R},\ \ b_j^{h,dV_M,h_E}(p) = b_j^{\varphi,\rho}(z(p))$$

where $b_j^{\varphi,\rho}$ is given by the formula in Definition 1.4 with respect to a choice of local trivializations s of L and e of E and local coordinates (U,z) with $\varphi = -\log(h(s,s))$, $\rho = h_E(e,e)\tilde{\rho}$, $dV_M = \tilde{\rho} dV_{\mathbb{C}}$.

Lemma 1.13
The function $b_j = b_j^{h,h_E,dV_M}$ is well defined, that is b_j is independent of the choice of coordinates and trivializations. Furthermore, we have that $b_j^{h,h_E,dV_M} \in C^l(M_{h,+},\mathbb{R})$ if h is of class $C^{6j+3n+4+l}$ and dV_M is of class $C^{4j+2n+2+l}$.

Definition 1.12 together with Lemma 1.13 give us globally defined quantities which decode information of h, h_E and dV_M and are explicitly given in local coordinates. Note that it follows from the definition that b_j does not depend on any choice of a Hermitian metric on M. Since the b_js are our candidates for the coefficients in the expansion of the Bergman kernel function that observation becomes obvious because the definition of B_k only depends on the choice of volume form and line bundle metric.

One of the most important cases where the Bergman kernel and its Bergman kernel function has an asymptotic expansion is when $L \to M$ is a positive holomorphic line bundle over a compact Hermitian manifold M. The Hermitian metric on M induces a volume form dV_M. As mentioned in Section 1.1 a theorem of Catlin [10] and Zelditch [38] (see also [30]) implies that B_k has an asymptotic expansion, that is

$$B_k \sim a_0 k^n + a_1 k^{n-1} + a_2 k^{n-2} + \dots . \tag{1.4}$$

A very difficult task is to calculate the coefficients a_j in terms of local geometric data. It is well-known that the Bergman kernel localizes in that setting (see [30, Section 4.1.2]). Hence we have $a_j = b_j$. In other words we established an explicit formula (in local coordinates, see Definition 1.4 and 1.12) for the coefficients a_j. But this formula does not give a direct link between the coefficients and geometric objects. As mentioned before the coefficients a_j should not depend on the choice of the Hermitian metric on M but on its volume form. However, under the assumption that L is positive the curvature induces a unique Kähler metric ω on M. So it is natural to express the $a_j = b_j^{h,h_E,dV_M}$ in terms of the geometry with respect to that specific Kähler metric. Since the construction of b_j^{h,h_E,dV_M} given in Definition 1.12 holds for any choice of local coordinates, we can choose Kähler normal coordinates (see [6]) and some "good" local frame of E and find that the derivatives of φ and ρ in Definition 1.4 are directly linked to the curvature tensor and the Ricci tensor of ω, the curvature of h_E and their covariant derivatives (see also [37] and Section 1.4).

We will now introduce the setting where we want to prove an asymptotic expansion. In order to get a full asymptotic expansion one has to assume the metric h and the volume form to be smooth, i.e. to be of class C^∞, at least at points one wants to prove the expansion. Let L_0 and E_0 be two holomorphic line bundles over a complete Kähler manifold X with metric ω and let $M \subset X$ be a domain. Fix a smooth Hermitian metric h_E on E_0 and consider the holomorphic line bundles $L = L_0|_M$ and $E = E_0|_M$ over M. Let h be a Hermitian metric on L with upper semi-continuous weight. We will use the notation $c_1(L, h)$ to denote the curvature of the Hermitian line bundle (L, h) at points where the metric h is of class C^2. Locally we can write $c_1(L, h) = -\frac{i}{2}\partial\overline{\partial}\log(h(s, s))$ for any local holomorphic frame s of L.

Definition 1.14
We define the set $M_{h,\infty} \subset M$ by saying $p \in M_{h,\infty}$ if and only if p has an open neighborhood U where h is smooth with positive curvature and there exists a smooth semi-positive metric h_0 on $L_0 \to X$ and k_0 with $h \leq h_0$ on M and $h = h_0$ on U and $kc_1(L_0, h_0) + c_1(E, h_E) \geq 0$ for all $k \geq k_0$.

We will study the Bergman kernel and the Bergman kernel function for the line bundle $L_k = L^k \otimes E \otimes \Lambda^n T^{*(1,0)}M$ where the metric h_ω on $\Lambda^n T^{*(1,0)}M$ is induced by the metric ω and the volume form is given by $dV_M = \rho\frac{\omega^n}{n!}$, where $\rho \in C^0(M, \mathbb{R})$ is positive and bounded. In local coordinates (U, z) we will use the holomorphic frame $dz := dz_1 \wedge \ldots \wedge dz_n$ for $\Lambda^n T^{*(1,0)}M$ and find $dV_M = \rho\tilde{\rho} dV_{\mathbb{C}^n}$, $h_\omega(dz, dz) = 1/\tilde{\rho}$. Hence we find that b_j is independent of ω, so we set $b_j^{h,h_E,\rho} := b_j^{h,h_E\otimes h_\omega,dV_M}$ in this setting. Putting $h_k := h^k \otimes h_E \otimes h_\omega$ to denote the metric of L_k we write P_{h_k,dV_M} for the Bergman kernel and B_{h_k,dV_M} for the Bergman kernel function of the space $H^0_2(M, L_k)$. We have the following results.

Theorem 1.15 (On-Diagonal Expansion)
Let L_0 and E_0 be two holomorphic line bundles over a complete Kähler manifold (X, ω) and let h_E be a smooth Hermitian metric on E_0. Let (M, dV_M) be a domain inside X with volume form $dV_M := \rho\omega^n$ for some bounded, positive function $\rho \in C^0(M, \mathbb{R})$ and consider the holomorphic line bundles $L = L_0|_M$ and $E = E_0|_M$ over M. For any upper semi-continuous metric h on L and $k \in \mathbb{N}$ consider the holomorphic Hermitian line bundle

$$(L_k, h_k) = (L^k \otimes E \otimes \Lambda^n T^{*(1,0)}M, h^k \otimes h_E \otimes h_\omega)$$

and let B_{h_k,dV_M} denote the Bergman kernel function for the space $H^0_2(M, L_k)$.

Given any $N, r \in \mathbb{N}_0$ assuming $\rho \in C^{4m+2n+2}(M_{h,\infty})$ with $m = Nr+N+r^2+r+2$, we have

$$B_{h_k,dV_M} - \frac{k^n}{\pi^n}\frac{c_1(L,h)^n}{n!dV_M}\sum_{j=0}^N b_j^{h,h_E,\rho}k^{-j} = O(k^{-N-1+n}) \text{ in } C^r(M_{h,\infty}).$$

More precisely, given any compact set $K \subset M_{h,\infty}$ and any partial differential operator F of order $\leq r$ there exists a constant $C = C_{K,F}$ such that

$$\left| F\left(B_{h_k,dV_M} - \frac{k^n}{\pi^n}\frac{c_1(L,h)^n}{n!dV_M}\sum_{j=0}^N b_j^{h,h_E,\rho}k^{-j}\right)(p)\right| \leq Ck^{-N-1+n}$$

holds for all $p \in K$ *and all* $k \in \mathbb{N}$. *Here* C *is bounded when* ρ *stays in a bounded in* $C^{4m+2n+2}(M_{h,\infty}, \mathbb{R}) \cap C^0(M)$ *such that* $\inf_{p\in M} \rho(p)$ *has a uniform positive lower bound and* $\sup_{p\in M} \rho(p)$ *has a uniform upper bound.*

Given local coordinates (U, z) around a point $p \in M$ and local holomorphic frames s and e of L and E, we denote by $(U \times U, (z, w))$ the induced coordinates around $(p, p) \in M \times M$ and choose $\hat{s_k}(z, w) := e^{\frac{k}{2}(\varphi(z)+\varphi(w))} s^k(z)e(z)dz(s^k(w)e(w)dw)^*$ as a trivialization of $L_k \boxtimes L_k^*|_{U\times U}$ with $\varphi = -\log(h(s, s))$.

Theorem 1.16 (Near-Diagonal Expansion)
Under the assumptions of Theorem 1.15 given $N, r \in \mathbb{N}_0$, $0 < \varepsilon < \frac{1}{2}$ *and any point* $p \in M_{h,\infty}$ *such that* ρ *is smooth in an open neighborhood around* p, *there exist coordinates* (U, z) *around* p *and local holomorphic frames* s *and* e *of* L *and* E *such that*

$$P_{h_k,dV_M} - \hat{P}_{k\varphi,\tilde{\rho},4N}(z, w)\hat{s}^k(z, w) = O(k^{-N+n+r-\varepsilon}) \text{ in } C^r(U \times U)$$

where $\hat{P}_{k\varphi,\tilde{\rho},4N} : U \times U \to \mathbb{C}$ *is defined in Definition 1.9 and* $\tilde{\rho} = \rho h_E(e, e)$.

Corollary 1.17
For any point $p \in M_{h,\infty}$ *there exist coordinates* (U, z) *around* p *and constants* $C, \delta > 0$ *such that* $|P_{h_k,dV_M}| \le k^n Ce^{-\delta k|z-w|^2} + R(k)$ *with* $R(k) = O(k^{-\infty})$ *in* $C^r(U \times U)$.

Theorem 1.18 (Off-Diagonal Asymptotics)
Let $D \subset M_{h,\infty}$ *be an open subset such that* ρ *is smooth on* D. *One has*

$$P_{h_k,dV_M} = O(k^{-\infty}) \text{ in } C^r(D \times (M \setminus \overline{D})).$$

We will now use these results to generalize the setting in Example 1.11.

Example 1.19
Let $D \subset \mathbb{C}^n$ be any domain. Let $U \subset D$ be some open set, ψ an upper semi-continuous non-negative function supported in U. Consider the weight φ defined by $\varphi(z) = \sum_{j=1}^n \lambda_j |z_j|^2 + \psi(z)$, $\lambda_1, \ldots, \lambda_n > 0$. As in Example 1.11 we find $V := D \setminus U \subset M_{e^{-\varphi},\infty}$. Given a bounded function $\rho \in C^\infty(V) \cap C^0(D)$, $\rho > 0$ on D, we have that (1.3) is still valid.

Example 1.20
Let (L, h) be a holomorphic line bundle with smooth positive Hermitian metric over a compact complex manifold X and let $M \subset X$ be a domain with volume form $dV_M := c_1(L, h)^n|_M$. We denote the restriction of the Hermitian line bundle to M again by (L, h) and consider the Bergman kernel P_k and the Bergman kernel function B_k for the space $H^0_2(M, L^k)$. Note that in general the space $H^0_2(M, L^k)$ contains more sections than $H^0_2(X, L^k)$ and can be infinite dimensional while $H^0_2(X, L^k)$ is always

finite dimensional since X is compact. We find $M_{h,\infty} = M$ and hence deduce from Theorem 1.15 and Theorem 1.16 that P_k and B_k admit asymptotic expansions on M. Furthermore, given an open set $U \subset M$ and a perturbation of h in $M \setminus U$ of the form $he^{-\psi}$ for some non-negative upper semi-continuous function ψ supported in $M \setminus U$, we find that P_k and B_k for the perturbed metric still have the same expansion on U.

1.4 Relation to Previous Results

The subject of Bergman kernel expansion has a long history started by Tian [36] in 1990. We start by giving an overview about the most famous results in this context and the results which are directly linked to this thesis. For a complete reference on Bergman kernel expansion we refer to the book of Ma–Marinescu [30]. In the following let (M, ω) denote a complex Hermitian manifold, L a Hermitian line bundle and E a Hermitian vector bundle over X. Let $H^0_2(M, L^k \otimes E)$ be the space of holomorphic sections in $L^k \otimes E$ with finite L^2-norm. We denote by P_k the Bergman kernel and by B_k, $B_k(x) = P_k(x, x)$, the Bergman kernel function for $H^0_2(M, L^k \otimes E)$. Note that if E has rank > 1 we have that B_k becomes a section in the endomorphism bundle of E. Bergman kernel expansion contains the following subdisciplines:

- On-diagonal expansion, that is the expansion of the Bergman kernel function.
- Near-diagonal expansion, that is the expansion of the Bergman kernel in a small neighborhood around the diagonal of $M \times M$.
- Off-diagonal expansion, that is the behaviour of the Bergman kernel away from the diagonal.
- Computations of the coefficients in the expansions.

In [36] Tian uses a so called peak section method to establish an on-diagonal expansion up to order 2 under the assumption that L is positive, ω is the curvature of L, $E = \mathbb{C}$ is trivial and M is compact. He also shows that - modulo some technical assumptions - this method works when (M, ω) is a complete Kähler manifold.
Catlin [9] and Zelditch [38] study the case when L is positive and M is compact, and prove a full asymptotic expansion for the Bergman kernel function using a deep result on Szegö kernel expansion on pseudoconvex boundaries due to Sjöstrand and Boutet de Monvel [7]. Furthermore, they calculate the coefficient a_0. Note that the work of Catlin [9] also contains a result on near-diagonal expansion.

In [27] Lu generalizes the methods of Tian in order to prove a full asymptotic on-diagonal expansion, calculated the first three coefficients and described an algorithm for computing all coefficients. Lu assumes that M is compact and L is positive.
Under the assumption that M is a bounded pseudoconvex domain in $\mathbb{C}^n$, $L, E \simeq \mathbb{C}$ are trivial, L is semi-positive and ω is the curvature of L, Englis proves in [17] a full on-diagonal and near-diagonal expansion on sets where L is positive using results of Fefferman [18] and asymptotic expansion for Laplace integrals. Since the coefficients in the expansion of Laplace integrals are explicitly given he computed the first four coefficients for the on-diagonal expansion in [16] .
In [13] Dai–Liu–Ma (cf. Ma–Marinescu [30]) prove an asymptotic expansion for generalized Bergman kernels for spin^c Dirac operators on symplectic orbifolds using a heat kernel approach. As a consequence they obtain a strong result on Bergman kernel expansion on complex manifolds under the assumption that M is compact and L is positive. Furthermore, Ma–Marinescu [30] give an algorithm for computing the coefficients for the Bergman kernel expansion (see also their work [29] for computations of the coefficients in the near-diagonal expansion). Their result also works - modulo some technical details - when (M, ω) is complete and L is positive (see [30]).
In [3] Berman-Brendtsson-Sjöstrand give a proof for Bergman kernel expansion using L^2 estimates and the construction of local reproducing kernels up to an asymptotically small error. The local kernel is obtained using techniques from microlocal analysis. They assume that L is positive and M is compact. In [37] Xu give an explicit formula for the coefficients in the expansion of the Bergman kernel function in terms of graphs and combinatorial functions.

All the results mentioned above use the assumption that the metric is smooth and has positive (or at least semi-positive) curvature. Furthermore, (M, ω) is assumed to be compact or complete. General results for smooth metrics with arbitrary curvature are due to Berman [5] and Hsiao–Marinescu [20]. Note that the results of Hsiao–Marinescu also work for $(0, q)$-forms. In [5] (cf. [4]) Berman considers M to be compact and L to be a big line bundle with a smooth metric of arbitrary curvature. As a consequence of his results it turns out that Bergman kernel expansion holds (in the sense of Ma–Marinescu [30]) exactly on the set of points where the metric coincides with its equilibrium metric (modulo some base locus). Hsiao–Marinescu considered in [20] the Bergman kernel for lower energy forms and prove that this object always admits an asymptotic expansion. It follows that for arbitrary M and arbitrary smooth Hermitian metric on L the Bergman kernel P_k and Bergman kernel function B_k has an asymptotic expansion on subsets $D \subset M$ where the Kodaira laplacian has a small (local) spectral gap.

Our approach is related to methods due to Tian, Berman-Berendtsson–Sjöstrand, Berman and Englis and uses an expansion result on oscillatory integrals as proven by Hörmander in [24] (see also Theorem 2.51). Note that in this thesis we restrict ourselves to the case where E has rank one. We will now explain the relation of our results from Section 1.2 and their proofs to other results and methods in detail.

Recall that Theorem 1.3 provides a reproducing kernel for a bounded domain D which reproduces the value of any holomorphic function at any point of $D_{\varphi,N}$ (see Definition 1.1) up to some error which is an $O(k^{-N-1+n+\varepsilon})$. Note that we do not assume the weight φ and the volume form $\rho dV_{\mathbb{C}^n}$ to be smooth, that φ does not need to be plurisubharmonic everywhere on D and that the estimates are uniform in φ, ρ up to some technical conditions. Furthermore, we give an explicit formula for the coefficients for the on-diagonal expansion (see Definition 1.4).
In [3] Berman-Brendtsson–Sjöstrand construct a reproducing kernel with error which is an $O(k^{-N-1+n})$ (see [3, Proposition 2.7]) using techniques from microlocal analysis - which is different from our approach - in the following setting. They consider the domain D to be a ball in $\mathbb{C}^n$ around the origin, fixed a smooth weight φ and a smooth volume form $\rho dV_{\mathbb{C}^n}$ and assume the weight φ to be strictly plurisubharmonic on D. From their method they obtained an algorithm for computing the coefficients for the on-diagonal expansion but do not give an explicit formula in general.
Englis proved a result on Bergman kernel expansion in the following setting (see [17, Corollary 1]). He considered a bounded pseudoconvex domain $D \subset \mathbb{C}^n$ and assumed the fixed data φ and $\rho := \det(H_\varphi)$ to be smooth and bounded and φ to be plurisubharmonic. Under some further technical assumption he proved a full expansion of the Bergman kernel on the set of points where φ is strictly plurisubharmonic using an asymptotic expansion for Laplace integrals. Since the coefficients in the expansion of Laplace integrals are well-known (see [16]) he could calculate the first four coefficients. Although his method is different from ours we have that the expansion for Laplace integrals (see [16, Corollary 2]) is similar to Theorem 2.50 under the assumptions that φ is smooth and $\tilde{\varphi}_N$ can be replaced by the almost analytic extension of φ which actually means to put $N = \infty$. Note that in Theorem 2.50 we also have uniformity in φ and its derivatives thanks to the Hörmander's method of stationary phase.
In [26] Liu-Lu develop an abstract version of the peak section method introduced by Tian [36] and generalized by Lu [27]. As a consequence of their result they obtain an on-diagonal expansion for the Bergman kernel for compact manifolds with positive line bundles under the assumption that the metric is fixed, smooth and has positive curvature. This setting seems to be quite different to our local setting considered in Section 1.2. However, we establish a local version of their abstract peak section

method to prove Theorem 1.3. Note that they need Hörmander's L^2 estimates which we can avoid here. Since on a compact manifold the space of holomorphic sections is finite dimensional it is much easier to prove that the Bergman kernel function can be approximated by their peak section method. In our case the space of holomorphic functions is always infinite dimensional. That leads to the problem that we do not see immediately that our computations approximate the Bergman kernel function. We overcome this difficulty by introducing the local reproducing kernel with small error (see Theorem 1.3). It turns out that this is actually an improvement of the method of Liu–Lu since we also obtain results for the off-diagonal expansion (see Theorem 1.7). We refer to Section 1.5.1 and Remark 1.21 for more information about the relation between our method and the method of Liu–Lu. Note that the results of Dai–Liu–Ma and Ma–Marinescu also provides uniformity in the geometric data. On a compact manifold this implies that the expansion holds under weaker regularity assumptions on the metric, that is the metric does not need to be of class C^∞.

In [37] Xu gives a closed formula for the coefficients in the expansion of the Bergman kernel functions. More precisely, he proves that any coefficient is the weighted sum over a set of special graphs which correspond to geometric objects of the manifold, that is the curvature and its covariant derivatives, with weights given by combinatorial functions which are recursively defined. Although our formula for the coefficients (see Definition 1.4) is less aesthetic those combinatorial functions are contained in an explicit way. However, our formula is in local coordinates and hence the connection to geometric quantities is not obvious. Choosing Kähler coordinates (see [6]) will immediately lead to an expression in terms of geometric quantities.

Let us explain now how the results from Section 1.3 and their proofs are related to previous results. In [3] Berman-Brendtsson-Sjöstrand used L^2 estimates of Hörmander [23] (cf. [14]) to extend their local reproducing kernel to the entire manifold. We follow their procedure with some slight modifications to extend our local reproducing kernel (see Theorem 1.3) to the global setting (see Section 1.5.2 for an outline of the idea). As a consequence we obtain the results from Sections 1.3. Note that we assume very weak regularity conditions on the metric and the volume form outside the set we want to prove asymptotic expansion (see Example 1.19). Furthermore, our results for the on-diagonal expansions provides uniformity in the volume form which does not need to be smooth (see Theorem 1.15). In Section 1.3 assuming that the manifold M is compact and the metric and volume form are smooth and fixed, our result follows from the results of Berman in [5]. If the surrounding manifold X is complete and the metric is the restriction of a smooth positive metric on L_0 to L our results follow from Hsiao–Marinescu [20].

1.5 Sketch of the Proofs

In this section we give a sketch of the proof of the results announced in Section 1.2 and Section 1.3. In Section 1.5.1 we give a description of the proof of Theorem 1.3 in a simple case and explain how Theorem 1.7 and Theorem 1.6 follows from Theorem 1.3 as a simple consequence. Furthermore, we explain how to get the formula (see Definition 1.4) for the coefficients in the expansion of the Bergman kernel function. In Section 1.5.2 we show how the results from Section 1.3 can be deduced from Theorem 1.3 and the L^2 estimates due to Hörmander.

1.5.1 Reproducing Kernels with Asymptotically Small Errors

We will start with the proof of Theorem 1.3 and then explain how Theorem 1.7 and Theorem 1.6 follow. We start by sketching the proof while pointing out the main ideas. Note that the fundamental idea of this proof is due to Tian [36] and was generalized by Lu [27] and Liu–Lu [26] (see Remark 1.21).

For making the idea clear we consider the case $n = 1$, that is D is a bounded domain in $\mathbb{C}$ and we assume $0 \in D$. Assume that $\varphi, \rho \in C^\infty(D, \mathbb{R}) \cap C^0(\overline{D})$ be two smooth real valued functions which are continuous up to the boundary of D with $\rho > 0$. Let $H^0_{k\varphi,\rho}(D)$ be the space of holomorphic functions with finite L^2-norm $\|\cdot\|_{k\varphi,\rho}$ and let $K_{k\varphi,\rho}$, $P_{k\varphi,\rho}$ and $B_{k\varphi,\rho}$ be the reproducing function, the Bergman kernel and Bergman kernel function for that space (see Section 2.4 for the definition and construction of these objects). We then want to show how to get a pointwise asymptotic expansion of $B_{k\varphi,\rho}(0)$ up to order $N \in \mathbb{N}$ assuming that φ has the N-th localization property in $z_0 = 0$ (see Section 2.5). For any $0 \leq \alpha \leq N$ consider the function $v_{\alpha,k}(z) = (z - z_0)^\alpha e^{-k\gamma_N(z_0,z)} = z^\alpha e^{-k\gamma_N(0,z)}$ and set $V_{N,k} = \operatorname{span}_{\mathbb{C}}\{v_{\alpha,k}\}$ (see Section 2.5 for the definition of γ_N and $\tilde{\varphi}_N$). Our assumptions ensures that $V_{N,k}$ is an $(N+1)$-dimensional subspace of $H^0_{k\varphi,\rho}(D)$. We can split the proof into two steps:

(i) Calculate the Bergman kernel function $B_{N,k}(0)$ at $z_0 = 0$ for the space $V_{N,k}$ and show that $B_{N,k}(0)$ admits an asymptotic expansion up to order N, that is

$$|B_{N,k}(0) - a_0 k + a_1 k^0 + a_2 k^{-1} + \ldots + a_N k^{-N+1}| \leq C_0 k^{-N}$$

for real numbers $a_0, \ldots, a_N \in \mathbb{R}$ and a constant $C_0 > 0$ independent of $k \in [1, \infty)$.

(ii) Show that for any $0 < \varepsilon < 1$ there exists a constant $C > 0$ such that for all $k \in [1, \infty)$ we have $|B_{N,k}(0) - B_{k\varphi,\rho}(0)| \leq C k^{-N-1+n+\varepsilon}$.

Step (i): We have $B_{N,k}(0) = e^{-k\varphi(0)} \sum_{j=0}^N |s_j(0)|^2$ where $\{s_j\}_{j=0}^N$ is any orthonormal basis for $V_{N,k}$ with respect to the inner product on $V_{N,k}$ given by the restriction of the inner product on $H^0_{k\varphi,\rho}(D)$. We observe $v_{\alpha,k}(0) = 0$ for all $1 \leq \alpha \leq N$. If we choose an orthonormal basis $\{s_j\}_{j=0}^N$ with $\{s_j\}_{j=1}^N \subset \operatorname{span}_{\mathbb{C}}\{v_{\alpha,k}\}_{\alpha=1}^N$ we have $B_{N,k}(0) = e^{-k\varphi(0)}|s_0(0)|^2$. Then the ansatz is to find $u_k = \sum_{\alpha=0}^N \lambda_{\alpha,k} v_{\alpha,k} \in H^0_{k\varphi,\rho}(D)$ such that $(u_k, v_{\alpha,k})_{k\varphi} = 0$ for all $1 \leq \alpha \leq N$ and $(u_k, v_{0,k}) = 1$. So we are seeking for the solution of the linear equation

$$A_k \lambda_k = (1, 0, 0, \dots, 0)^T$$

where $A_k = ((v_{\beta,k}, v_{\alpha,k})_{k\varphi})_{0 \leq \alpha,\beta \leq N}$ and $\lambda_k = (\lambda_{0,k}, \dots, \lambda_{N,k})$. By Cramer's rule we get $\lambda_{\alpha,k} = (-1)^\alpha \frac{\det A_{\alpha,k}}{\det A_k}$ where $A_{\alpha,k}$ is the $N \times N$ submatrix of A_k obtained by eliminating the first row and the α^{th} column from A_k. We write

$$(v_{\alpha,k}, v_{\beta,k})_{k\varphi} = \int_D \chi(z) z^\alpha \overline{z}^\beta e^{-k\tilde{\varphi}_N(0,z)} dV_D(z) + \int_D (1 - \chi(z)) z^\alpha \overline{z}^\beta e^{-k\tilde{\varphi}_N(0,z)} dV_D(z)$$

for some cutoff function χ supported in D such that $\chi \equiv 1$ in a neighborhood of 0. The N-th localization property ensures that the second term on the right-hand side is an $O(k^{-\infty})$. For the first term on the right-hand side we use the method of stationary phase due to Hörmander (see Section 2.6) to show that this term has an asymptotic expansion with an explicit formula for the coefficients. Expanding the determinants we get as a conclusion an asymptotic expansion for $B_{N,k}(0) = \lambda_{0,k} = \frac{\det A_{0,k}}{\det A_k}$. In order to get an explicit formula for the coefficients in that expansion we do some rescaling of the entries of A_k. More precisely, we set $\tilde{A}_k = S_k^{-1} A_k S_k$ for some diagonal matrix S_k to get $\tilde{A}_k = \operatorname{Id} + C_1 k^{-1} + \ldots + C_N k^{-N} + O(k^{-N+1})$ for some matrices $C_1, \dots, C_N$ independent of k. Using the ansatz $\tilde{A}_k^{-1} = \operatorname{Id} + B_1 k^{-1} + \ldots + B_N k^{-N} + O(k^{-N-1})$ we could uniquely determine the k-independent matrices $B_1, \dots, B_N$ from $C_1, \dots, C_N$ using $\tilde{A}_k \tilde{A}_k^{-1} = \operatorname{Id}$. Since we know $C_1, \dots, C_N$,S_k explicitly and $B_{N,k}(0) = \lambda_{0,k}$ is the first entry of the vector $S_k \tilde{A}_k^{-1} S_k^{-1}(1, 0, 0, \dots, 0)^T$ we end up with an explicit formula for the coefficients in the expansion of $B_{N,k}(0)$. A detailed proof of that formula is given in Section 3.2.

Step (ii): Here we show that u_k satisfies some reproducing property up to order N, that is

$$|f(0) - (f, u_k)_{k\varphi}|^2 e^{-k\varphi(0)} \leq C k^{-N-1+n+\varepsilon} \|f\|^2_{k\varphi}, \quad f \in H^0_{k\varphi,\rho}(D) \tag{1.5}$$

for some $0 < \varepsilon < 1$ and some constant $C > 0$ independent of k. Let $W_{N,k}$ be the space of all elements in $H^0_{k\varphi,\rho}(D)$ which vanish up to order N in 0. We have the decomposition $H^0_{k\varphi,\rho}(D) = V_{N,k} \oplus W_{N,k}$. By construction (1.5) is true for all $f \in V_{N,k}$. Proving an L^2-norm estimate for the stationary phase formula in Section 2.7 and using the information about $\lambda_{\alpha,k}$, $0 \leq \alpha \leq N$ we find that (1.5) holds for the elements in $W_{N,k}$. To proof that (1.5) is true for all $f \in H^0_{k\varphi,\rho}(D)$ we need to establish a relation between the orthogonal complement $V_{N,k}^\perp$ of $V_{N,k}$ in $H^0_{k\varphi,\rho}(D)$ and $W_{N,k}$. More precisely, we prove that the restriction of some projection $\operatorname{Id} - T_{N,k} \colon H^0_{k\varphi,\rho}(D) \to W_{N,k}$

coming from some modified Taylor expansion in 0 is uniformly bounded in k (see Section 2.8, Theorem 2.66). From (1.5) and the reproducing property of the kernel $K_{\kappa\varphi,\rho}(0,z)$ (see Definition 2.34) a direct calculation shows

$$|B_{k\varphi,\rho}(0) - B_{N,k}(0)| \leq \|K_{k\varphi,\rho}(0,\cdot) - u_k\|^2_{k\varphi} \leq Ck^{-N-1+n+\varepsilon}.$$

We just showed a pointwise expansion in $z_0 = 0$ here. However, in this thesis we will show that the estimate is uniform in z_0, ρ and φ staying in some bounded sets which satisfy certain conditions (see Theorem 1.7 and 1.6).

Using similar arguments as above and basic L^2-norm estimates from Section 2.3 we get the estimates for $|P_{k\varphi,\rho,N} - P_{k\varphi,\rho}|$ in Theorem 1.7. To get an estimate in the C^l-norm we use Hörmander's trick (see Lemma 3.11) and an apriori estimate for the Bergman kernel (see Corollary 2.39).

Remark 1.21
Our approach is related to the generalization due to Liu–Lu [26] of the methods used in [36] and [27]. In [26] Liu–Lu proved a result on the asymptotic expansion in k of the Bergman kernel function for sections in the k-th tensor power of a positive holomorphic line bundle twisted with some holomorphic vector bundle over a compact manifold. The main difference compared to our setting is that the space of holomorphic sections on a compact manifold is finite dimensional. Hence they worked with matrices A_k as above of increasing size but which contains all the information of the space of holomorphic sections. This makes it easier to prove that an inverse of a submatrix of sufficiently large but fixed size approximates already the Bergman kernel function. In our setting the space of holomorphic functions has always infinite dimension. In order to prove that our computations approximate the Bergman kernel function we construct a reproducing kernel modulo sum error in (1.5). Using this kernel we are able to deduce a result on the off-diagonal expansion from our method.

1.5.2 Approximation of Bergman Kernels on Manifolds

In this section we will outline the idea for the proofs of Theorem 1.15 and Theorem 1.16. For the proof of Lemma 1.13 see Lemma 3.29. The idea is basically to extend the reproducing kernel from Theorem 1.3 as a holomorphic section in the second argument to the entire manifold. Note that the extension of locally reproducing kernels to global settings via L^2 estimates by Hörmander was also used by Berman–Brendtsson–Sjöstrand in [3] and Berman in [5]. In principle we follow the prove of [3, Theorem 3.1]. For making the idea clear we will just proof the expansion at one point and consider the following simplified setting. Let (X, ω) be a complete

Kähler manifold of complex dimension $n = 1$ and $M \subset X$ a domain. Let (L_0, h_0) be a positive holomorphic line bundle over X and let h be a Hermitian metric on $L = L_0|_M$ such that $h \leq h_0$ on M and $h = h_0$ on some open set $U \subset M$. Choose $dV_M = \rho\omega$ to be the volume form for some positive function $\rho \in C^\infty(M, \mathbb{R})$ such that ρ is bounded by a constant $C_0 > 0$. Set $L_k = L^k \otimes T^{*(1,0)}X$. We are interested in the Bergman kernel P_k and its Bergman kernel function B_k for the space $H^0_2(M, L_k)$, $k \in \mathbb{N}$ at a point $p \in U$. Choose coordinates (D, z) around p with $z(p) = 0$ and hence identify D with a subset of $\mathbb{C}$ around 0. Assume that s is a local holomorphic frame of L defined on D, define $\varphi = -\log(|s|^2_h)$ and set $s_k = s^k \otimes dz$. We have that s_k is a local holomorphic frame of L_k. By shrinking D we can ensure that 0 has the N-th localization property for φ. Given a holomorphic section $\tilde{f} \in H^0_2(M, L_k)$ we have $\tilde{f} = f_k s_k$ on D for some holomorphic function $f_k \in H^0_{k\varphi,\rho}(D)$. Setting $\tilde{u}_k = u_k s_k$ with u_k as in Section 1.5.1 and using (1.5) we find since $h_k(\tilde{f}, \tilde{u}_k)dV_M = f_k\overline{u_k}e^{-k\varphi}\rho dV_{\mathbb{C}}$ that

$$|\tilde{f} - s_k(\tilde{f}, \tilde{u}_k)_{k,D}|_{h_k} \leq Ck^{-N-1+n+\varepsilon}\|\tilde{f}\|^2_{h_k,dV_M}, \quad \tilde{f} \in H^0_2(M, L_k)$$

where $(\cdot,\cdot)_{k,D}$ indicates the inner product obtained by integration over D. We showed that $\tilde{u}_k$ has locally also a reproducing property up to some error. But we cannot proceed as in Section 1.5.1 to show that $u_k(0)$ approximates $B_k(p)$ since $\tilde{u}_k$ fails to be holomorphic on M. To overcome this difficulty we multiply $\tilde{u}_k$ with a cutoff function supported and D and equal to one in an open neighborhood V of 0 and consider $\overline{\partial}(\chi\tilde{u}_k) = (\overline{\partial}\chi)\tilde{u}_k \in \Omega^{(0,1)}(X, L^k_0 \otimes T^{*(1,0)}X)$. Thanks to the L^2 estimates of Hörmander [23] in Demailly [14] we find by our assumptions a smooth section v on X with values in $L^k_0 \otimes T^{*(1,0)}X$ such that $\overline{\partial}v_k = (\overline{\partial}\chi)\tilde{u}_k$ and

$$\int_X |v_k|^2_{h_0,\omega}dV_X \leq \int_M |(\overline{\partial}\chi)|^2|\tilde{u}_k|^2_{h_k}dV_X.$$

Since $|\overline{\partial}\chi|^2$ is supported in an annulus around 0, properties of u_k and $h \leq h_0$ it follows $\|v_k\|^2_{h_k,dV_M} \leq C_0\int_X |v_k|^2_{h_0,\omega}dV_X = O(k^{-\infty})$. Furthermore, we have that $\chi\tilde{u}_k - v_k$ is holomorphic on M and v is holomorphic on V. We conclude that

$$|\tilde{f} - s_k(\tilde{f}, \tilde{u}_k)_{h_k,dV_M}|_{h_k} \leq Ck^{-N-1+n+\varepsilon}\|\tilde{f}\|^2_{h_k,dV_M}, \quad \tilde{f} \in H^0_2(M, L_k) \tag{1.6}$$

and can proceed as in Section 1.5.1 to prove that $|B_k(p) - u_k(0)| \leq Ck^{-N-1+n+\varepsilon}$. An approximation for the Bergman kernel P_k can also be obtained from (1.6) using similar arguments as in Section 1.5.1.

Chapter 2

Bergman Kernels in $\mathbb{C}^n$

This chapter contains the fundamental framework for the proofs of the results announced in Section 1.2 and Section 1.3. More precisely, we present a careful study of the space of holomorphic sections with weighted finite L^2-norm. The chapter is organized as follows. In Section 2.1 we state basic definitions from analysis and introduce the notations which are used during this thesis. Section 2.2 and Section 2.3 contain the notion of holomorphic functions, basic properties and basic asymptotic L^2-norm estimates for them. In Section 2.4 we introduce the Bergman kernel and proof some apriori estimates. Sections 2.5 - 2.8 contain the partial steps for the proof of Theorem 1.3 (see Section 1.5 for more explanations).

2.1 Analysis and Basic Notations

2.1.1 Analysis in $\mathbb{R}^n$

We use the following notations: $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}$ (resp. $\mathbb{C}$) is the set of real (resp. complex) numbers. Given a complex number $z = a + ib \in \mathbb{C}$, where $a, b \in \mathbb{R}$ and i denotes the complex unit ($i^2 = -1$), we write $\mathrm{Re}(z) = a$, $\mathrm{Im}(z) = b$, $\overline{z} = a - ib$ and set

$$|z|^2 = z\overline{z} = a^2 + b^2, \qquad |z| = \sqrt{|z|^2}.$$

A subset $X \subset \mathbb{R}^n$ is called domain if it is open and connected. For $U \subset X$ we write $U \subset\subset X$ if U is relatively compact in X, that is the closure of U is a compact subset of X. Given an open set $X \subset \mathbb{R}^n$ and a non-negative integer $l \in \mathbb{N}_0$ the set of l-times continuously differentiable complex valued functions is denoted by $C^l(X)$ and we set $C^\infty(X) = \bigcap_{l \in \mathbb{N}} C^l(X)$. Furthermore, for $l = 0$ we denote by $C^0(\overline{X})$ the continuous functions from $\overline{X}$ to $\mathbb{C}$, where $\overline{X}$ is the closure of X. The set $C^l_0(X)$, $l \in \mathbb{N}_0 \cup \{\infty\}$, indicates the subspace of $C^l(X)$ which consists of all functions which vanish outside a compact subset of X. Given a function $f : X \to \mathbb{C}$, the support of f, denoted by $\mathrm{supp}(f)$, is the closure of the set of points where f is non-vanishing. The space $C^l(X, \mathbb{R})$ consists of all real valued function in $C^l(X)$ and hence we define $C^\infty(X, \mathbb{R})$, $C^l_0(X, \mathbb{R})$, $C^0(\overline{X}, \mathbb{R})$ as above. Furthermore, we set $C^l(X, \mathbb{C}^n) = C^l(X)^n$, $C^l(X, \mathbb{R}^n) = C^l(X, \mathbb{R})^n$.

An element $\alpha \in \mathbb{N}_0^n$ is called a multi-index. We set

$$|\alpha| = \alpha_1 + \ldots + \alpha_n, \qquad \alpha! = \alpha_1! \ldots \alpha_n!.$$

Given another multi-index $\beta \in \mathbb{N}_0^n$ we further set

$$\begin{aligned}
\alpha + \beta &= (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n),\\
\alpha \geq \beta &\Leftrightarrow \alpha_j \geq \beta_j \text{ for all } 1 \leq j \leq n,\\
\alpha - \beta &= (\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n), \text{ if } \alpha \geq \beta,\\
\max\{\alpha, \beta\} &= (\max\{\alpha_1, \beta_1\}, \ldots, \max\{\alpha_n, \beta_n\}),\\
\binom{\alpha}{\beta} &= \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n}.
\end{aligned}$$

Let $x = (x_1, \ldots, x_n)$ the canonical coordinate map of $\mathbb{R}^n$. We will use the notations

$$x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}, \qquad d_x^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

If there is no reason for confusion $x = (x_1, \ldots, x_n)$ will sometimes denote a point or vector in $\mathbb{R}^n$. The Euclidean norm of x is given by $|x| = \sqrt{|x_1|^2 + \ldots + |x_n|^2}$ and given $r > 0$ and $x_0 \in \mathbb{R}^n$ the open ball of radius r around x_0 is $\mathbb{B}_r(x_0) = \{x \in \mathbb{R}^n \mid |x - x_0| < r\}$. The differential of a function $f \in C^1(X)$ is denoted by

$$df = \frac{\partial f}{\partial x_1} dx_1 + \ldots + \frac{\partial f}{\partial x_n} dx_n$$

and we set $|d_x f| = \sqrt{|\frac{\partial f}{\partial x_1}|^2 + \ldots + |\frac{\partial f}{\partial x_n}|^2}$. Given a measurable function $f : X \to \mathbb{C}$ (measurable in the sense of the Borel σ-algebra on $\mathbb{R}^n$) its integral with respect to the Lebesgue measure on $\mathbb{R}^n$ is denoted by

$$\int_X f dV_{\mathbb{R}^n} = \int_X f(x) dV_{\mathbb{R}^n}(x)$$

whenever it exists and we identify $dV_{\mathbb{R}^n}$ with the standard volume form on $\mathbb{R}^n$, that is $dV_{\mathbb{R}^n} = dx_1 \wedge \ldots \wedge dx_n$. Given a map $F \colon X \to Y \subset \mathbb{R}^m$, $F \in C^1(X, \mathbb{R}^m)$, we denote its Jacobi matrix at a point $x \in X$ by $D_x F = (\frac{\partial F_j(x)}{\partial x_l})_{1 \leq j,l \leq n}$ and write DF to describe the map $DF \in C^0(X, \mathbb{R}^{n \cdot m})$, $x \mapsto D_x F$.

For any subset $U \subset X$ and $f \in C^l(X)$ we define $\|f\|_{C^l(U)} \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ by

$$\|f\|_{C^l(U)} = \sum_{|\alpha| \leq l} \sup_{x \in U} |d_x^\alpha(f)(x)|.$$

Note that if $U \subset X$ is compact or relatively compact we have that $\|f\|_{C^l(U)}$ is finite and $f \mapsto \|f\|_{C^l(U)}$ defines a seminorm on $C^l(X)$. We define a topology on $C^l(X)$ in the following way:

Definition 2.1
A subset $A \subset C^l(X)$ is closed if and only if for every sequence $\{f_k\}_{k\in\mathbb{N}} \subset A$ and $f \in C^l(X)$ which satisfy

$$\lim_{k\to\infty} \|f - f_k\|_{C^l(K)} = 0$$

for any compact subset $K \subset X$ one has $f \in A$.

This gives rise for a topology on $C^\infty(X) = \bigcap_{l\in\mathbb{N}} C^l(X)$ by saying a subset $A \subset C^\infty(X)$ is closed if A is a closed set in C^l-topology for any $l \in \mathbb{N}$. A subset $S \subset C^l(X)$ is said to be bounded in $C^l(X)$ if for any compact set $K \subset X$ there exists a constant $C > 0$ such that $\|f\|_{C^l(K)} \leq C$ holds for all $f \in S$. Moreover, assuming that $D \subset\subset X$ is an open relatively compact subset of X we say that $S \subset C^l(D) \cap C^0(\overline{D})$ is bounded if S is bounded as a subset of $C^l(D)$ and there exists a constant $C > 0$ such that $\|f\|_{C^0(\overline{D})} \leq C$ holds for all $f \in S$. The topology on $C^l(X)$ induces a topology on $C^l(X, \mathbb{R})$ and a subset of $C^l(X, \mathbb{R})$ or $C^l(D, \mathbb{R}) \cap C^0(\overline{D})$ is bounded if it is bounded considered as a subset of $C^l(X)$ or $C^l(D) \cap C^0(\overline{D})$, respectively. A subset of $C^l(X, \mathbb{R}^n)$ or $C^l(X, \mathbb{C}^n)$ is bounded the image of its projection to any component is a bounded set in $C^l(X)$.

Now we are going to introduce the Landau symbols.

Definition 2.2
Given a set A, a function $g\colon A \to \mathbb{R}$, a family of functions $\{f_a\}_{a\in A} \subset C^l(X)$ and an open set $D \subset X$ we say $f_a = O(g(a))$ in $C^l(D)$ if for any compact subset $K \subset D$ there exists a constant $C_K > 0$ such that $\|f_a\|_{C^l(K)} \leq C_K g(a)$ holds for any $a \in A$. We say $f_a = O(g(a))$ in $C^\infty(D)$ if $f_a = O(g(a))$ in $C^l(D)$ holds for all $l \in \mathbb{N}$.
Given a function $h\colon X \to \mathbb{R}$ we write $f_a(x) = O(h(x))$ on D, uniformly in $a \in A$ if for any compact subset $K \subset D$ there exists a constant $C_K > 0$ such that $|f_a(x)| \leq C_K h(x)$ holds for all $x \in K$ and $a \in A$.

Usually we use the Landau symbols for the setting $A = [1, \infty) \times B$ and $g(k, b) = k^s$ where B is a parameter set and $s \in \mathbb{R}$ is a real number. We say $f_k = O(k^s)$ in $C^l(D)$ uniformly in $b \in B$ instead of $f_k = f_{k,b} = O(k^s)$ in $C^l(D)$ if the parameter dependence is not explicitly written in the index. Given $B = B_1 \times \cdots \times B_d$ we also write "uniformly in $b_1 \in B_1, \ldots, b_d \in B_d$" instead of "uniformly in $b \in B$" where $b = (b_1, \ldots, b_d)$. We further write $f_k = O(k^{-\infty})$ if $f_k = O(k^s)$ holds for all $s \in \mathbb{R}$.

Definition 2.3
Let A be a set, $\{f_{k,a}\}_{(k,a)\in[1,\infty)\times A} \subset C^l(X)$ a family of functions and $D \subset X$. We say that $f_{k,a}$ has an asymptotic expansion of order $N \in \mathbb{N}_0$ in $C^l(D)$ topology uniformly in $a \in A$ if there exist $s \in \mathbb{R}$ and functions (called coefficients) $f_a^{(0)}, \ldots, f_a^{(N)} \in C^l(D)$

such that $f_a^{(0)} \neq 0$ for all $a \in A$ and $\{f_a^{(j)}\}_{a\in A}$ is a bounded subset of $C^l(D)$ for $1 \leq j \leq N$ with

$$k^{-s} f_{k,a} - \sum_{j=0}^{N} f_a^{(j)} k^{-j} = O(g(k)) \text{ in } C^l(D)$$

(uniformly in $a \in A$) for some function $g\colon [1,\infty) \to \mathbb{R}$ with $\limsup_{k\to\infty} k^{-N} g(k) \leq 0$. The function $f_a^{(0)}$ is called leading coefficient.

Remark 2.4
A simple induction shows that the coefficients in an asymptotic expansion are unique.

Lemma 2.5
Assume that $f_{a,k} \in C^l(D)$, $(k,a) \in [1,\infty) \times A$ has an asymptotic expansion of order N in $C^l(D)$-topology uniformly in $a \in A$ such that $|f_{a,k}|$ and the modulus of the leading coefficient is locally bounded and locally positively bounded from below, that is for any compact set there exists a constant $C > 0$ with $1/C \leq |f_{a,k}(x)|, |f_a^{(0)}(x)| \leq C$ for all $k \in [1,\infty)$, $a \in A$ and $x \in K$. Then $1/f_{a,k}$ has an asymptotic expansion of order N in $C^l(D)$-topology uniformly in $a \in A$. Furthermore, the coefficients in the expansion of $1/f_{a,k}$ at a point $x \in D$ only depend on the coefficients in the expansion of $f_{a,k}$ in that point x.

Proof. The assumptions on the leading coefficient $f_a^{(0)}$ in the expansion of $f_{a,k}$ imply that $f_{a,k}/f_a^{(0)} \in C^l(D)$ has an asymptotic expansion of order N in $C^l(D)$-topology uniformly in $a \in A$. Hence it is enough to prove the statement for $f_a^{(0)} = 1$. We define $\{g_a^{(j)}\}_{j=0}^N \subset C^l(D)$ as follows: $g_a^{(0)} = 1$ and recursively $-g_a^{(j)} = \sum_{m=1}^{j} g_a^{(j-m)} f_a^{(m)}$, where $f_a^{(j)}$, $0 \leq j \leq N$ are the coefficients in the expansion of $f_{k,a}$. It follows from induction that $\{g_a^{(j)}\}_{a\in A}$ is a bounded subset of $C^l(D)$ for $1 \leq j \leq N$ and the construction ensures $R_{1,a,k} := 1 - (\sum_{j=0}^N k^{-j} g_a^{(j)})(\sum_{j=0}^N k^{-j} f_a^{(j)}) = O(k^{-N-1})$. Since $f_{k,a}$ has an asymptotic expansion we have

$$R_{2,a,k} := f_{k,a} - \sum_{j=0}^{N} k^{-j} f_a^{(j)} = O(h_1(k)) \text{ in } C^l(D)$$

for some function $h_1\colon [1,\infty) \to \mathbb{R}$ with $\limsup_{k\to\infty} k^{-N} h_1(k) = 0$, which immediately implies $f_{k,a} = O(1)$ in $C^l(D)$. We calculate

$$f_{a,k} \sum_{j=0}^{N} k^{-j} g_a^{(j)} - 1 = R_{1,a,k} + R_{2,a,k} \sum_{j=0}^{N} k^{-j} g_a^{(j)} = O(h(k)) \text{ in } C^l(D)$$

with $h_2(k) := \max\{h_1(k), k^{-N-1}\}$. Since $|f_{a,k}|$ is locally bounded and locally positively bounded from below, we find $\frac{1}{f_{k,a}} = O(1)$ in $C^l(D)$. Hence we find

$$\sum_{j=0}^{N} k^{-j} g_a^{(j)} - \frac{1}{f_{k,a}} = O(h_2(k)) \text{ in } C^l(D)$$

with $\limsup_{k\to\infty} k^{-N}h(k) \leq 0$. From the construction of $g_a^{(j)}$ the second part of the statement follows immediately. $\square$

We end this section by stating a modified version of the well-known theorem about local diffeomorphisms.

Theorem 2.6
Let $X \subset \mathbb{R}^n$ be open, $x_0 \in X$ a point, $l \in \mathbb{N}_0$ a non-negative integer and $S \subset C^{l+2}(X, \mathbb{R}^n)$ a bounded set. Assume that there exists a constant $C_0 > 0$ such that $|D_{x_0}(F)v| \geq C_0|v|$ holds for all $F \in S$ and $v \in \mathbb{R}^n$. Then there exists an open neighborhood $U \subset X$ around x_0 such that for any $F \in S$ we have that $F|_U \colon U \to F(U)$ is a C^{l+2}-diffeomorphism. Furthermore, there exist constants $C, \varepsilon > 0$ with $\|F^{-1}|_U\|_{C^{l+2}(F(U))} \leq C$ and $B_\varepsilon(F(x_0)) \subset F(U)$ for all $F \in S$.

Proof. If S has only one element the statement is in fact the well-known theorem about the existence of local diffeomorphisms around the point x_0. To prove the modified version we will repeat the first part of the prove of the standard version given in [8, Theorem 2.1] to show that we can choose a fixed U where the restrictions of all F are invertible at the same time. The remaining claims follow then from the standard version, the chain rule and Taylor expansion.
Put $X_{x_0} = \{x - x_0 \mid x \in X\}$. Given $F \in S$ we find that $\tilde{F} \colon X_{x_0} \to \mathbb{R}^n$, $\tilde{F}(x) = D_{x_0}(F)^{-1}F(x) - F(x_0)$ satisfies $\tilde{F}(0) = 0$ and $D_0(\tilde{F}) = \mathrm{Id}$. Setting $\tilde{S} = \{\tilde{F} \mid F \in S\}$ we find that $\tilde{S}$ is bounded in the C^{l+2}-norm. The assumption $|D_{x_0}(F)v| \geq C_0|v|$ and the construction of $\tilde{F}$ ensures that proof of the original statement follows from proving the statement for $\tilde{S}$. Hence without loss of generality we can assume $x_0 = 0$ and $F(0) = 0$, $D_0(F) = \mathrm{Id}$ for all $F \in S$. Since S is bounded in C^2-norm and $D_0(F) = \mathrm{Id}$ there exists for any open set $X' \subset\subset X$ a constant $C_1 > 0$ such that $\|\mathrm{Id} - D_xF\| \leq C_1|x|$ holds for all $F \in S$ and $x \in X'$, so we can choose $r > 0$ such that $\|\mathrm{Id} - D_xF\| \leq \frac{1}{2}$ holds for all $F \in S$ and $x \in \overline{\mathbb{B}_{2r}}(0) \subset X$. For any $y \in \mathbb{R}^n$ and any $F \in S$ consider the map $g_{y,F} \colon \overline{\mathbb{B}_{2r}(0)} \to \mathbb{R}^n$, $g_{y,F}(x) = y + x - F(x)$. Given $x, x_1, x_2 \in \overline{\mathbb{B}_{2r}}(0)$ and $y \in \mathbb{B}_r(0)$ we find $|g_{y,F}(x_1) - g_{y,F}(x_1)| \leq \frac{1}{2}|x_1 - x_2|$ and $|g_{y,F}(x)| \leq |g_{y,F}(x) - g_{y,F}(0)| + |y| \leq 2r$. We conclude that $g_{y,F}$ is a contraction which maps $\overline{\mathbb{B}_{2r}(0)}$ into itself. Applying the Banach fixed-point theorem shows that the equation $g_{y,F}(x) = x$ has a unique solution $x_{y,F}$ for any $y \in B_r(0)$ and $F \in S$. For $F \in S$ set $V_F = F^{-1}(\mathbb{B}_r(0)) \cap \mathbb{B}_{2r}(0)$. We find that the restriction of F, $F|_{V_F} \colon \to \mathbb{B}_r(0)$ has an inverse given by $G_F := F|_{V_F}^{-1}(y) = x_{y,F}$. Since F is a C^1 map and by assumption we have that D_xF is invertible for all $x \in \overline{\mathbb{B}_{2r}(0)}$ we deduce that $\tilde{F} := F|_V \colon \to B_r(0)$ is a C^1-diffeomorphism with

$$D(\tilde{F}^{-1}) = (D\tilde{F})^{-1} \circ \tilde{F}^{-1}. \tag{2.1}$$

Using $|D_x(F)v| \geq \frac{1}{2}|v|$ for all $x \in \overline{\mathbb{B}_{2r}(0)}$, $F \in S$ and $v \in \mathbb{R}^n$ we find that $|\det(DF)|$ has a uniform positive lower bound. We conclude that $\{(DF)^{-1} \mid F \in S\} \subset C^{l+1}(\mathbb{B}_{2r}(0))$ is bounded and deduce from (2.1) using induction that $\tilde{S} = \{G_F \mid F \in S\} \subset C^{l+2}(\mathbb{B}_r(0))$ is bounded. It remains to show that 0 is an interior point of $\bigcap_{F\in S} F^{-1}(B_r(0))$. By the assumptions on r we find that $|F(x) - F(0)| \leq \frac{3}{2}|x|$ holds for all $x \in \overline{\mathbb{B}_{2r}(0)}$ and $F \in S$. Hence, we have that $F(U) \subset B_r(0)$ for all $F \in S$ with $U = B_{r/2}(0)$. Since $\tilde{S}$ satisfies the same properties as S we have $|G_F(y) - 0| \leq C_2|y|$ for some constant $C_2 > 0$ independent of $F \in S$ and $y \in \mathbb{B}_r(0)$. Choosing $\varepsilon > 0$ sufficiently small leads to $G_F(y) \in U$ for all $F \in S$ and $y \in \mathbb{B}_\varepsilon(0)$. The complete claim follows from $F|_U^{-1} = G_F|_{F(U)}$. $\square$

2.1.2 Analysis in $\mathbb{C}^n$

Identify $(\mathbb{C}^n, z = (z_1, \ldots, z_n))$ with $(\mathbb{R}^{2n}, x = (x_1, \ldots, x_{2n}))$ via $z_j = x_j + ix_{j+n}$ for $j \in \{1, \ldots, n\}$ and set $\overline{z}_j = \overline{z_j} = x_j - ix_{j+n}$, $j = 1, \ldots, n$. From this identification we get that an open set $X \subset \mathbb{C}^n$ can be treated as an open set in $\mathbb{R}^{2n}$ and therefore integration of measurable functions over X, the space $C^l(X)$ with its topology, the notion of Landau symbols for families in this space and its subspaces (like $C^l_0(X)$, $C^l(X, \mathbb{R})$, etc.) are defined as in Section 2.1.1. We set

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial x_{j+n}}\right), \qquad dz_j = dx_j + idx_{j+n}$$
$$\frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial x_{j+n}}\right), \qquad d\overline{z}_j = dx_j - idx_{j+n}.$$

Given a function $f \in C^1(X)$ we have

$$df = \sum_{j=1}^{2n} \frac{\partial f}{\partial x_j} dx_j = \partial f + \overline{\partial} f$$

where ∂ and $\overline{\partial}$ are defined by

$$\partial f = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j \qquad \text{and} \qquad \overline{\partial} f = \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j.$$

Furthermore, we will use the notations

$$z^\alpha = z_1^{\alpha_1} \cdot \ldots \cdot z_n^{\alpha_n}, \qquad \partial_z^\alpha = \left(\frac{\partial}{\partial z_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial z_n}\right)^{\alpha_n},$$
$$\overline{z}^\alpha = \overline{z}_1^{\alpha_1} \cdot \ldots \cdot \overline{z}_n^{\alpha_n}, \qquad \overline{\partial}_z^\alpha = \left(\frac{\partial}{\partial \overline{z}_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial \overline{z}_n}\right)^{\alpha_n}$$

where $\alpha \in \mathbb{N}^n$ is a multi-index and set $|z| = \sqrt{|z_1|^2 + \ldots + |z_n|^2}$. Usually, the variables z or w will denote complex coordinates, points or vectors in $\mathbb{C}^n$ where their real counterparts with respect to the identification of $\mathbb{C}^n$ and $\mathbb{R}^{2n}$ are denoted by x or y, respectively. Therefore we also write $d_z^\alpha := d_x^\alpha$ or $d_w^\alpha := d_y^\alpha$ for $\alpha \in \mathbb{N}_0^{2n}$ and $|d_z f| := |d_x f|$ or $|d_w f| := |d_y f|$ for $f \in C^1(X)$. Furthermore, the volume form on $\mathbb{C}^n$, which is identified with the volume form on $\mathbb{R}^{2n}$, is denoted by $dV_{\mathbb{C}^n}$ and we have $dV_{\mathbb{C}^n} = \left(\frac{i}{2}\right)^n dz_1 \wedge \ldots \wedge dz_n \wedge d\overline{z}_1 \wedge \ldots \wedge d\overline{z}_n = dx_1 \wedge \ldots \wedge dx_{2n} = dV_{\mathbb{R}^{2n}}$.

Example 2.7
Let $\gamma = (\gamma_1, \ldots, \gamma_n) : (0,1) \to \mathbb{C}^n$ be a C^1 map and $f \in C^1(\mathbb{C}^n)$. We have

$$\begin{aligned}\frac{\partial}{\partial t} f \circ \gamma(t) &= \sum_{j=1}^n \frac{\partial f}{\partial x_j}(\gamma(t)) \frac{\partial \operatorname{Re} \gamma_j}{\partial t}(t) + \sum_{j=1}^n \frac{\partial f}{\partial x_{j+n}}(\gamma(t)) \frac{\partial \operatorname{Im} \gamma_j}{\partial t}(t) \\ &= \sum_{j=1}^n \frac{\partial f}{\partial z_j}(\gamma(t)) \frac{\partial \gamma_j}{\partial t}(t) + \sum_{j=1}^n \frac{\partial f}{\partial \overline{z}_j}(\gamma(t)) \overline{\frac{\partial \gamma_j}{\partial t}}(t).\end{aligned}$$

Given a point $a \in \mathbb{C}$ and a real number $r > 0$ the open disc of radius r around a is denoted by $\mathbb{D}_r(a) = \{z \in \mathbb{C} \mid |z-a| < r\}$ and we write $\mathbb{D}_r^n(z) = \mathbb{D}_r(z_1) \times \ldots \times \mathbb{D}_r(z_n)$ to denote the polydisc of radius r around $z \in \mathbb{C}^n$. An open ball in $\mathbb{C}^n$ of radius $r > 0$ around $z \in \mathbb{C}^n$ is denoted by $\mathbb{B}_r(z)$ with $\mathbb{B}_r(z) = \{w \in \mathbb{C}^n \mid |w - z| < r\}$. We further set $\mathbb{D}_r^n = \mathbb{D}_r^n(0)$ and $\mathbb{B}_r = \mathbb{B}_r(0)$.

Let us start with the following theorem about Taylor expansion which is just a reformulation of Taylor's formula well-known in the real case.

Theorem 2.8 (Taylor's Formula)
Given a function $f \in C^{l+1}(X)$ we have

$$f(w) = \sum_{|\alpha+\beta| \le l} \frac{1}{\alpha!\beta!} \left(\partial_z^\alpha \overline{\partial}_z^\beta f(z)\right) (w-z)^\alpha \overline{(w-z)}^\beta + O(|w-z|^{l+1}).$$

More precisely, for any compact set $K \subset X \times X$ there exists a constant $C > 0$ such that

$$\left| f(w) - \sum_{|\alpha+\beta| \le l} \frac{1}{\alpha!\beta!} \left(\partial_z^\alpha \overline{\partial}_z^\beta f(z)\right) (w-z)^\alpha \overline{(w-z)}^\beta \right| \le C |w-z|^{l+1}$$

where C is bounded when f stays in a bounded set in $C^{l+1}(X)$.

Proof. First, we prove the statement on an open ball $B \subset\subset X$ which is relatively compact in X. Given $z, w \in B$ define $\tilde{f} : [0,1] \to \mathbb{C}$, $\tilde{f}(t) = f(t(w-z)+z)$. By using induction and Example 2.7 we find

$$\left(\frac{\partial}{\partial t}\right)^m \tilde{f}(t) = \sum_{|\alpha|+|\beta|=m} \frac{m!}{\alpha!\beta!} (\partial_z^\alpha \overline{\partial}_z^\beta f)(t(w-z)+z)(w-z)^\alpha \overline{(w-z)}^\beta$$

for any $0 \le m \le l+1$. Taylor's Formula in one real variable together with an estimate for its remainder (see [8, Theorem 3.4]) implies

$$\left| \tilde{f}(1) - \sum_{m=0}^l \frac{1}{m!} \left(\frac{\partial}{\partial t}\right)^m \tilde{f}(0) \right| \le \frac{1}{(l+1)!} \sup_{t \in [0,1]} \left| \left(\frac{\partial}{\partial t}\right)^{l+1} \tilde{f}(t) \right|.$$

As a conclusion we find a constant $C > 0$ such that

$$A := \left| f(w) - \sum_{|\alpha+\beta|\leq l} \frac{1}{\alpha!\beta!} \left(\partial_z^\alpha \overline{\partial}_z^\beta f(z)\right) (w-z)^\alpha \overline{(w-z)}^\beta \right| \leq C\|f\|_{C^{l+1}(B)}|w-z|^{l+1}$$

holds for all $z, w \in B$ and $f \in C^{l+1}(X)$.

Now let $K \subset X$ be a compact set and consider a smaller open ball $B' \subset\subset B$ contained in B. There exists an $\varepsilon > 0$ such that for all $z \in B'$ and $w \in K \setminus B$ we have $|w - z| \geq \varepsilon$. Furthermore, we find a constant $C' > 0$ such that $A \leq C'\|f\|_{C^{l+1}(B)}$ holds for all $z \in B'$, $w \in K$ and $f \in C^{l+1}(X)$ and hence we find $A \leq \max\{\varepsilon^{-n}C', C\}\|f\|_{C^{l+1}(B)}|w-z|^{l+1}$. As a conclusion, we get that the claim of Theorem 2.8 holds on $B' \times K \subset X \times X$. We finish the proof, using the fact that any compact subset of $X \times X$ is contained in a compact set $K_1 \times K_2$, where $K_1, K_2 \subset X$ are compact. $\square$

Lemma 2.9
Given $f, g \in C^l(X)$ and $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq l$, we have

$$d_z^\alpha(f\cdot g) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} d_z^{\alpha-\beta} f d_z^\beta g,\ \partial_z^\alpha \overline{\partial}_z^\beta (f\cdot g) = \sum_{\alpha'\leq\alpha}\sum_{\beta'\leq\beta} \binom{\alpha}{\alpha'}\binom{\beta}{\beta'} \partial_z^{\alpha-\alpha'}\overline{\partial}_z^{\beta-\beta'} f \partial_z^{\alpha'}\overline{\partial}_z^{\beta'} g.$$

Lemma 2.10
Let $f \in C^l(X \times X)$ be a function and define $\tilde{f} \in C^l(X)$ by $\tilde{f}(z) = f(z,z)$. For any $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq l$, we have

$$d_z^\alpha \tilde{f}(z) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} d_z^{\alpha-\beta} d_w^\beta f(z,w)\ |_{z=w}\ .$$

Lemma 2.9 and Lemma 2.10 follow from a straightforward induction.

2.1.3 Real and Complex Hessians

Given a function $f \in C^2(X)$ we consider its (real) Hessian $\mathrm{Hess}_f(z)$ at a point $z \in X$, which is a $2n \times 2n$-matrix defined by $\mathrm{Hess}_f(z) = \left(\frac{\partial^2 f}{\partial x_j \partial x_l}(z)\right)_{1\leq j,l\leq 2n}$. Its complex analogue is defined as follows.

Definition 2.11
The complex Hessian of a function $f \in C^2(X)$ at the point $z \in X$ is defined by

$$H_f(z) = \left(\frac{\partial^2 f}{\partial z_l \partial \overline{z}_j}(z)\right)_{1\leq l,j\leq n}.$$

The following lemma gives a relation between the real and the complex Hessian.

Lemma 2.12
Given $f \in C^2(X)$ and $z \in X$ one has

$$\begin{pmatrix} \mathrm{Id}_n & -i\,\mathrm{Id}_n \\ \mathrm{Id}_n & i\,\mathrm{Id}_n \end{pmatrix} \mathrm{Hess}_f(z) \begin{pmatrix} \mathrm{Id}_n & \mathrm{Id}_n \\ i\,\mathrm{Id}_n & -i\,\mathrm{Id}_n \end{pmatrix} = 4 \begin{pmatrix} H_f(z) & G_f(z) \\ \overline{G_{\overline{f}}(z)} & \overline{H_{\overline{f}}(z)} \end{pmatrix}$$

where we set $G_f(z) = \left(\frac{\partial^2 f}{\partial z_j \partial z_l}(z)\right)_{1\le j,l\le n}$.

Proof. A direct calculation shows that the entry a_{lj}, $1 \le l, j \le 2n$ of the matrix on the left-hand side is given by

$$a_{lj} = \begin{cases} \left(\frac{\partial^2}{\partial x_l \partial x_j} + i\frac{\partial^2}{\partial x_l \partial x_{j+n}} - i\frac{\partial^2}{\partial x_{l+n}\partial x_j} + \frac{\partial^2}{\partial x_{l+n}\partial x_{j+n}}\right) f(z) & \text{, if } l, j \le n, \\ \left(\frac{\partial^2}{\partial x_{l-n}\partial x_{j-n}} - i\frac{\partial^2}{\partial x_{l-n}\partial x_j} + i\frac{\partial^2}{\partial x_l \partial x_{j-n}} + \frac{\partial^2}{\partial x_l \partial x_j}\right) f(z) & \text{, if } l, j \ge n, \\ \left(\frac{\partial^2}{\partial x_l \partial x_{j-n}} - i\frac{\partial^2}{\partial x_l \partial x_j} - i\frac{\partial^2}{\partial x_{l+n}\partial x_{j-n}} - \frac{\partial^2}{\partial x_{l+n}\partial x_j}\right) f(z) & \text{, if } l \le n, j \ge n, \\ \left(\frac{\partial^2}{\partial x_{l-n}\partial x_j} + i\frac{\partial^2}{\partial x_{l-n}\partial x_{j+n}} + i\frac{\partial^2}{\partial x_l \partial x_j} - \frac{\partial^2}{\partial x_l \partial x_{j+n}}\right) f(z) & \text{, if } l \ge n, j \le n. \end{cases}$$

The claim follows from $\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial x_{j+n}} = 2\frac{\partial}{\partial z_j}$ and $\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial x_{j+n}} = 2\frac{\partial}{\partial \overline{z}_j}$ for $1 \le j \le n$. ☐

Corollary 2.13
Let $z_0 \in X$ be a point. Assume that $f \in C^2(X, \mathbb{R})$ satisfies $\partial_z^\alpha f(z_0) = 0$ for all $\alpha \in \mathbb{N}_0^n$, $|\alpha| = 2$. One has $\det\left(\frac{1}{2}\,\mathrm{Hess}_f(z_0)\right) = |\det H_f(z)|^2$.

Proof. Using the same notations as in Lemma 2.12, the assumptions on f imply $G_{\overline{f}}(z) = G_f(z) = 0$. Furthermore, one has

$$\begin{pmatrix} \mathrm{Id}_n & -i\,\mathrm{Id}_n \\ \mathrm{Id}_n & i\,\mathrm{Id}_n \end{pmatrix} \begin{pmatrix} \mathrm{Id}_n & \mathrm{Id}_n \\ i\,\mathrm{Id}_n & -i\,\mathrm{Id}_n \end{pmatrix} = 2\,\mathrm{Id}_{2n}\,.$$

Then the claim follows from Lemma 2.12 and basic properties of the determinant.
☐

In real analysis the real Hessian matrix plays in important role with respect to the notion of convexity or convex functions. The analogue in complex analysis is called pseudoconvexity and is related to so called plurisubharmonic functions.

Definition 2.14
A function $f \in C^2(X, \mathbb{R})$ is called strictly plurisubharmonic (spsh) in $z_0 \in X$ if $H_\varphi(z_0)$ is positive definite. We say f is spsh on $D \subset X$ if it is spsh in any point $z_0 \in D$. We denote by $D_{f,+}$ (or sometimes D_+) the largest subset of X where f is spsh.

Example 2.15
Let $z_0 \in X$ be a point and $f \in C^2(X, \mathbb{R})$ a real valued function. If $\mathrm{Hess}_f(z_0)$ is positive definite (i.e. f is convex in a neighborhood of z_0) then f is spsh in z_0. Assuming that $\partial_z^\alpha f(z_0) = 0$ holds for all $\alpha \in \mathbb{N}_0^n$, $|\alpha| = 2$ we find that $\mathrm{Hess}_f(z_0)$ is positive definite if and only if f is spsh in z_0.

In order to describe coefficients which appear in the expansion of some integrals (see Section 2.6) we are interested in second order differential operators of the form

$$\langle A d_x, d_x \rangle := \sum_{1 \le l,j \le 2n} a_{lj} \frac{\partial^2}{\partial x_l \partial x_l} \quad \text{and} \quad \langle B\partial_z, \overline{\partial}_z \rangle := \sum_{1 \le l,j \le n} b_{jl} \frac{\partial^2}{\partial z_l \partial \overline{z}_j}$$

where $A = (a_{lj})_{1 \le l,j \le 2n}$ and $B = (b_{lj})_{1 \le l,j \le n}$ are matrices with complex entries of size $2n \times 2n$ and $n \times n$, respectively. We need the following relation between these objects.

Lemma 2.16
Let $z_0 \in X$ be a point. Assume that $f \in C^2(X, \mathbb{R})$ is strictly plurisubharmonic in z_0 and satisfies $\partial_z^\alpha f(z_0) = 0$ for all $\alpha \in \mathbb{N}_0^n$, $|\alpha| = 2$. One has

$$\langle (\mathrm{Hess}_f(z_0))^{-1} d_x, d_x \rangle = 2 \langle H_f^{-1}(z_0) \partial_z, \overline{\partial}_z \rangle.$$

Proof. Using Example 2.15 we find that the inverse of $\mathrm{Hess}_f(z_0)$ exists. From Lemma 2.12, the proof of Corollary 2.13 and the assumptions on f we have

$$\frac{1}{4} \begin{pmatrix} \mathrm{Id}_n & \mathrm{Id}_n \\ i\,\mathrm{Id}_n & -i\,\mathrm{Id}_n \end{pmatrix} \begin{pmatrix} H_f^{-1}(z) & 0 \\ 0 & \overline{H_f^{-1}(z)} \end{pmatrix} \begin{pmatrix} \mathrm{Id}_n & -i\,\mathrm{Id}_n \\ \mathrm{Id}_n & i\,\mathrm{Id}_n \end{pmatrix} = \mathrm{Hess}_f^{-1}(z). \tag{2.2}$$

We write $\mathrm{Hess}_f^{-1}(z) = (a_{lj})_{1 \le l,j \le 2n}$ and $H_f^{-1}(z) = (b_{lj})_{1 \le l,j \le n}$. Given $x = (x_1, \dots, x_{2n}) \in \mathbb{R}^{2n}$ we deduce the following identity from (2.2):

$$4 \sum_{1 \le l,j \le 2n} a_{lj} x_l x_j = \sum_{1 \le l,j \le n} b_{lj} (x_l + i x_{n+l})(x_j - i x_{n+j}) + \sum_{1 \le l,j \le n} \overline{b_{lj}} (x_l - i x_{n+l})(x_j + i x_{n+j}).$$

Since $\overline{b_{lj}} = b_{jl}$, $\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial x_{j+n}}\right)$ for $1 \le l,j \le n$ and by replacing x_j with $\frac{\partial}{\partial x_j}$ we find

$$\sum_{1 \le l,j \le 2n} a_{lj} \frac{\partial}{\partial x_l} \frac{\partial}{\partial x_j} = 2 \sum_{1 \le l,j \le n} b_{jl} \frac{\partial}{\partial z_l} \frac{\partial}{\partial \overline{z}_j}$$

and the claim follows. □

2.2 Basic Properties of Holomorphic Functions

Definition 2.17
A function $f \in C^1(X)$ is called holomorphic if $\overline{\partial} f = 0$, that is

$$\frac{\partial}{\partial \overline{z}_j} f = 0, \ \text{for } j = 1, 2, \dots, n.$$

The set of all holomorphic functions on X is denoted by $\mathcal{O}(X)$.

A map $F = (F_1, \dots, F_m) \colon X \to \mathbb{C}^m$ is holomorphic if each of its components is a holomorphic function. The property of being holomorphic is preserved under holomorphic maps.

Lemma 2.18
Let $Y \subset \mathbb{C}^m$ be an open set, $F = (F_1, \dots, F_m) \colon X \to Y$ a holomorphic map and $f \in \mathcal{O}(Y)$ a holomorphic function. We have $f \circ F \in \mathcal{O}(X)$.

Proof. Since f is holomorphic we find by Example 2.7 that $\frac{\partial}{\partial x_j} f \circ F = \sum_{l=1}^n \frac{\partial f}{\partial w_l} \frac{\partial F_l}{\partial x_j}$ holds for all $j = 1, \dots, 2n$. Thus, we conclude $\frac{\partial}{\partial \bar{z}_j} f \circ F = \sum_{l=1}^n \frac{\partial f}{\partial w_l} \frac{\partial F_l}{\partial \bar{z}_j} = 0$ for all $j = 1, \dots, n$. $\square$

A map $F =\colon X \to Y$ between two open sets $X, Y \subset \mathbb{C}^n$ is called biholomorphic if it is bijective with holomorphic inverse.

Lemma 2.19
Let $X, Y \subset \mathbb{C}^n$ be two open sets and $F = (F_1, \dots, F_n) \colon X \to Y$ a C^1-diffeomorphism which is holomorphic. We have that $F^{-1} \colon Y \to X$ is holomorphic, that is F is biholomorphic.

Proof. Let $D(F)$ be the real Jacobi matrix of F seen as a map between two open sets in $\mathbb{R}^{2n}$ using the identification $\mathbb{C}^n \simeq \mathbb{R}^n$ as before, that is $D(F)$ is the Jacobi matrix of the map $(x_1, \dots, x_n, y_1, \dots, y_n) \mapsto (\operatorname{Re} F_1(z), \dots, \operatorname{Re} F_n(z), \operatorname{Im} F_1(z), \dots, \operatorname{Im} F_n(z))$ with $z = (z_1, \dots, z_n)$, $z_j = x_j + i y_j$, $1 \leq j \leq n$. We find

$$\begin{pmatrix} \mathrm{Id}_n & i\,\mathrm{Id}_n \\ \mathrm{Id}_n & -i\,\mathrm{Id}_n \end{pmatrix} D(F) \begin{pmatrix} \mathrm{Id}_n & \mathrm{Id}_n \\ -i\,\mathrm{Id}_n & i\,\mathrm{Id}_n \end{pmatrix} = 2 \begin{pmatrix} G(F) & \overline{G(\overline{F})} \\ G(\overline{F}) & \overline{G(F)} \end{pmatrix}, \; G(F) = \left(\frac{\partial F_l}{\partial z_j} \right)_{1 \leq l,j \leq n}.$$

Since F is holomorphic we find $G(\overline{F}) = \overline{G(\overline{F})} = 0$. Furthermore, $D(F)$ is invertible at any point $z \in X$. This implies that $G(F)$ is an invertible complex $n \times n$-matrix at any point $z \in X$. Doing the same procedure for the inverse map F^{-1} and using $D(F)D(F^{-1}) = \mathrm{Id}_{2n}$ we end up with

$$\mathrm{Id}_{2n} = \begin{pmatrix} G(F) & 0 \\ 0 & \overline{G(F)} \end{pmatrix} \begin{pmatrix} G(F^{-1}) & \overline{G(\overline{F^{-1}})} \\ G(\overline{F^{-1}}) & \overline{G(F^{-1})} \end{pmatrix} = \begin{pmatrix} G(F)G(F^{-1}) & G(F)\overline{G(\overline{F^{-1}})} \\ \overline{G(F)}G(\overline{F^{-1}}) & \overline{G(F)G(F^{-1})} \end{pmatrix}.$$

Since $G(F)$ is invertible at any point $z \in X$ we conclude $\overline{G(\overline{F^{-1}})} = G(\overline{F^{-1}}) = 0$ which shows that F^{-1} is holomorphic. $\square$

The boundary of a disc $\mathbb{D}_r(a) \subset \mathbb{C}$ is given by $\partial \mathbb{D}_r(a) = \{z \in \mathbb{C} \mid |z - a| = r\}$. Given a polydisc $\mathbb{D}_r^n(z) = \mathbb{D}_r(z_1) \times \ldots \times \mathbb{D}_r(z_n)$ of radius $r > 0$ around $z \in \mathbb{C}^n$ its distinct boundary $\partial_0 \mathbb{D}_r^n(z)$ is defined by $\partial_0 \mathbb{D}_r^n(z) = \partial \mathbb{D}_r(z_1) \times \ldots \times \partial \mathbb{D}_r(z_n)$.

We start with a fundamental theorem of Cauchy for holomorphic functions in one complex variable.

Theorem 2.20 (Cauchy's Integral Theorem)
Let X be an open subset of $\mathbb{C}$ and $f \in \mathcal{O}(X)$ a holomorphic function. Given an open set $D \subset\subset X$ with piecewise C^1-boundary ∂D one has

$$2\pi i f(z) = \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta$$

for any $z \in D$.

Proof. Choose $\varepsilon_0 > 0$ such that $\mathbb{D}_\epsilon(z) \subset\subset D$ holds for all $\varepsilon < \varepsilon_0$. We have that $D \setminus \mathbb{D}_\epsilon(z)$ has a piecewise C^1-boundary. We then apply Stokes' formula (see [14, (1.18)]) and find

$$\begin{aligned} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta - \int_{\partial \mathbb{D}_\epsilon(z)} \frac{f(\zeta)}{\zeta - z} d\zeta &= \int_{D \setminus \mathbb{D}_\epsilon(z)} d\left(\frac{f(\zeta)}{\zeta - z}\right) d\zeta \\ &= \int_{D \setminus \mathbb{D}_\epsilon(z)} \frac{\partial}{\partial \overline{\zeta}} \left(\frac{f(\zeta)}{\zeta - z}\right) d\overline{\zeta} \wedge d\zeta = 0 \end{aligned}$$

since $\zeta \mapsto (\zeta - z)^{-1} f(\zeta)$ is holomorphic on $D \setminus \mathbb{D}_\epsilon(z)$ and $d\zeta \wedge d\zeta = 0$. Thus, we can write

$$\left| 2\pi i f(z) - \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta \right| = \left| \int_{\partial \mathbb{D}_\varepsilon(z)} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta \right| \leq 2\pi \sup_{\zeta \in \mathbb{D}_\varepsilon(z)} |f(\zeta) - f(z)|$$

for all $\varepsilon < \varepsilon_0$. The statement then follows from the continuity of f.

□

From Theorem 2.20 all the basic properties for holomorphic functions in several complex variables follow.

Corollary 2.21 (Cauchy's Integral Theorem in Several Complex Variables)
Let $\mathbb{D}_r^n(z) \subset\subset X$ be a polydisc. Given a holomorphic function $f \in \mathcal{O}(X)$ one has

$$(2\pi i)^n f(z) = \int_{\partial_0 \mathbb{D}_r^n(w)} \frac{f(\zeta_1, \ldots, \zeta_n)}{(\zeta_1 - z_1) \cdot \ldots \cdot (\zeta_n - z_n)} d\zeta_1 \ldots d\zeta_n$$

for all $z \in \mathbb{D}_r^n(z)$.

Proof. See [33, Theorem 1.3]. □

Corollary 2.22
We have $\mathcal{O}(X) \subset C^\infty(X)$ and therefore $\partial_z^\alpha(\mathcal{O}(X)) \subset \mathcal{O}(X)$ with

$$\partial_z^\alpha f(z) = \frac{\alpha!}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}_r^n(w)} \frac{f(\zeta_1, \ldots, \zeta_n)}{(\zeta_1 - z_1)^{\alpha_1 + 1} \cdot \ldots \cdot (\zeta_n - z_n)^{\alpha_n + 1}} d\zeta_1 \ldots d\zeta_n$$

for any polydisc, $\mathbb{D}_r^n(w) \subset\subset X$, and any $f \in \mathcal{O}(X)$.

Proof. Using the fact that f is continuous we can differentiate under the integral sign in Corollary 2.21 to prove the statement. □

Corollary 2.23 (Taylor Expansion)
Fix $w \in X$. Given $\mathbb{D}^n_r(w) \subset\subset X$ and $f \in \mathcal{O}(X)$ we have

$$f(z) = \sum_{\alpha \in \mathbb{N}^n} \frac{(\partial_z^\alpha f)(w)}{\alpha!}(z-w)^\alpha \text{ for all } z \in \mathbb{D}^n_r(w)$$

where the power series on the right-hand side converges absolutely for all $z \in \mathbb{D}^n_r(w)$.

Proof. See [33, Theorem 1.18]. □

Corollary 2.24
We have that $\mathcal{O}(X)$ is closed in the C^0-topology. More precisely, given a sequence $\{f_m\}_{m\in\mathbb{N}} \subset \mathcal{O}(X)$ of holomorphic functions which converges locally uniformly to some function $g : X \to \mathbb{C}$, we have $g \in \mathcal{O}(X)$ and the sequence converges in C^∞-topology.

Proof. See [33, Theorem 1.9]. □

Corollary 2.25 (Identity Theorem)
Let $f \in \mathcal{O}(X)$ be a holomorphic function and let $z \in X$ be a point such that $(\partial_z^\alpha f)(z) = 0$ for all $\alpha \in \mathbb{N}^n$. If X is connected one has $f = 0$.

Proof. See [33, Theorem 1.19]. □

Corollary 2.26 (Maximum Principle)
Let $f \in \mathcal{O}(X)$ be a holomorphic function such that $|f|^2$ has a local maximum in X. If X is connected one has that f is constant.

Proof. See [33, Corollary 1.22]. □

From the definition of holomorphic function it follows immediately that f is holomorphic in each variable z_j when the other variables are kept fix. The converse is also true.

Theorem 2.27 (Hartogs 1906)
Let $X_1 \subset \mathbb{C}^n$ and $X_2 \subset \mathbb{C}^m$ be two open sets and $f : X_1 \times X_2 \to \mathbb{C}$ a function satisfying $f(\cdot, w) \in \mathcal{O}(X_1)$ for all $w \in X_2$ and $f(z, \cdot) \in \mathcal{O}(X_2)$ for all $z \in X_1$. Then one has $f \in \mathcal{O}(X_1 \times X_2)$.

Proof. See [23, Theorem 2.2.8]. □

2.3 L^2-Norm Estimates for Holomorphic Functions

Lemma 2.28
Let $K \subset \mathbb{C}^n$ be a compact set, $X \subset \mathbb{C}^n$ an open neighborhood of K, $\varphi : X \to \mathbb{R}$ an upper semi-continuous function and $\alpha \in \mathbb{N}_0^n$ a multi-index. There exist a constant $C > 0$ such that

$$|\partial_z^\alpha f(z)|^2 \leq C \int_X |f(w)|^2 e^{-\varphi(w)} dV_{\mathbb{C}^n}(w)$$

holds for all $z \in K$ and all $f \in \mathcal{O}(X)$.

Proof. Let $z_0 \in K$ be a point, $\delta > 0$ such that $\mathbb{D}_{3\delta}(z_0) \subset X$. Given a holomorphic function $f \in \mathcal{O}(X)$ and $r = (r_1, \ldots, r_n)$, $0 < r_j < \delta$, $j = 1, \ldots, n$, we find

$$(2\pi i)^n \partial_z^\alpha f(z) = \alpha! \int_0^{2\pi} \ldots \int_0^{2\pi} \frac{f(z + re^{i\theta})}{(re^{i\theta})^\alpha} d\theta$$

for any $z \in \mathbb{D}_\delta(z_0)$ where $re^{i\theta} = (r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_1})$ and $d\theta = d\theta_1 \ldots d\theta_n$. Applying the standard estimate for integrals and the Cauchy–Schwarz inequality one gets

$$\begin{aligned} \frac{(2\pi)^n r^\alpha}{\alpha!} |\partial_z^\alpha f(z)| &\leq \int_0^{2\pi} \ldots \int_0^{2\pi} |f(z + re^{i\theta})| d\theta \\ &\leq (2\pi)^{\frac{n}{2}} \sqrt{\int_0^{2\pi} \ldots \int_0^{2\pi} |f(z + re^{i\theta})|^2 d\theta}. \end{aligned}$$

Squaring both sides, integrating with respect to $r_1 dr_1 \ldots r_n dr_n$ and using polar coordinates leads to

$$\frac{\pi^n}{\alpha!} e^{-M} |\partial_z^\alpha f(z)|^2 \int_0^{\delta^2} r_1^{\alpha_1} dr_1 \cdot \ldots \cdot \int_0^{\delta^2} r_n^{\alpha_n} dr_n \leq \int_{\mathbb{D}_\delta(z)} |f(w)|^2 e^{-\varphi(w)} dV_{\mathbb{C}^n}(w)$$

where $M = \sup_{z \in \mathbb{D}_{2\delta}(z_0)} \varphi(z)$. Let $C > 0$ be the constant defined by

$$C = \alpha! e^M \left(\pi^n \int_0^{\delta^2} r_1^{\alpha_1} dr_1 \cdot \ldots \cdot \int_0^{\delta^2} r_n^{\alpha_n} dr_n \right)^{-1}.$$

Then $|\partial_z^\alpha f(z)|^2 \leq C \int_X |f(w)|^2 e^{-\varphi(w)} dV_{\mathbb{C}^n}(w)$ holds for all $z \in \mathbb{D}_\delta(z_0)$ and all $f \in \mathcal{O}(X)$. By compactness, K can be covered by a finite number of polydiscs contained in X and hence the lemma follows. □

Lemma 2.29
Let $K \subset \mathbb{C}^n$ be a compact set, $X \subset \mathbb{C}^n$ an open neighborhood of K, $\varphi \in C^3(X, \mathbb{R})$, $\rho \in C^0(X, \mathbb{R})$, $\rho > 0$ two real valued functions and $\alpha \in \mathbb{N}_0^n$ a multi-index. There exists a constant $C > 0$ such that

$$|\partial_z^\alpha f(z)|^2 e^{-k\varphi(z)} \leq C k^{n+2|\alpha|} \int_X |f(w)|^2 e^{-k\varphi(w)} \rho(w) dV_{\mathbb{C}^n}(w)$$

holds for all $z \in K$, $k \in [1, \infty)$ and $f \in \mathcal{O}(X)$. Here C is bounded when φ stays in a bounded set in $C^3(X, \mathbb{R})$ and ρ stays in a subset of $C^0(X, \mathbb{R})$ such that $\inf_{z \in X} \rho(z)$ has a positive lower bound.

Proof. Let $z_0 \in K$ be a point, $\delta > 0$ such that $\mathbb{D}^n_{3\delta}(z_0) \subset X$ and $S_1 \subset C^3(X,\mathbb{R})$, $S_2 \subset C^0(X,\mathbb{R})$ some sets such that S_1 is bounded and $\inf_{z\in X}\rho(z) \geq C_1$ holds for all $\rho \in S_2$ where $C_1 > 0$ is some constant. Given a holomorphic function $f \in \mathcal{O}(X)$ and a point $z \in \mathbb{D}^n_\delta(z_0)$ we define $g_{k,z} \in \mathcal{O}(X)$ by $g_{k,z}(w) = f(w)e^{-k\gamma(z,w)}$ where

$$\gamma(z,w) = \frac{\varphi(z)}{2} + \sum_{1\leq|\alpha|\leq 2} \frac{1}{\alpha!}\left(\partial_z^\alpha \varphi\right)(z)(w-z)^\alpha.$$

Using Taylor expansion (see Theorem 2.8) we find

$$|\varphi(w) - \gamma(z,w) - \overline{\gamma(z,w)}| \leq M|w-z|^2$$

and hence $\varphi(w) \leq \gamma(z,w) + \overline{\gamma(z,w)} + M|w-z|^2$ for all $(z,w) \in \mathbb{D}^n_\delta(z_0) \times \mathbb{D}^n_{2\delta}(z_0)$ and all $\varphi \in S_1$ where $M > 0$ is some constant. Similar to the proof of Lemma 2.28 we find

$$\frac{\pi^n}{\alpha!}|(\partial_w^\alpha g_{k,z})(z)|^2 \int_0^{\delta^2} r^{\alpha_1} e^{-kMr_1} dr_1 \cdot \ldots \cdot \int_0^{\delta^2} r^{\alpha_n} e^{-kMr_n} dr_n \leq \int_{\mathbb{D}^n_\delta(z)} |f(w)|^2 e^{-k\varphi(w)} dV_{\mathbb{C}^n}(w).$$

One has $\int_0^\infty r^l e^{-kMr}dr = l!(Mk)^{-l-1}$ and

$$\int_{\delta^2}^\infty r^l e^{-kMr} dr \leq (l+1)!k^{-1}(1+M^{-1-l})(1+\delta^{2l})e^{-\delta^2 Mk}$$

so that we find a constant $C_2 > 0$ such that

$$\frac{\pi^n}{\alpha!}\int_0^{\delta^2} r^{\alpha_1} e^{-kMr_1} dr_1 \cdot \ldots \cdot \int_0^{\delta^2} r^{\alpha_n} e^{-kMr_n} dr_n \geq C_2^{-1} k^{-|\alpha|-n}$$

holds for all $k \in (1,\infty)$. Furthermore, we have that $\rho \geq C_1$ on $\mathbb{D}^n_{2\delta}(z_0)$ holds for all $\rho \in S_2$ and hence

$$|(\partial_w^\alpha g_{k,z})(z)|^2 \leq C_2 C_1^{-1} k^{n+|\alpha|} \int_{\mathbb{D}^n_\delta(z)} |f(w)|^2 e^{-k\varphi(w)} \rho(w) dV_{\mathbb{C}^n}(w). \tag{2.3}$$

We prove the original statement by induction with respect to $|\alpha|$. Given $\alpha = 0$ we have $|(\partial_w^\alpha g_{k,z})(z)|^2 = |f(z)|^2 e^{-k\gamma(z,z)} = |f(z)|^2 e^{-k\varphi(z)}$. Now assume that the statement holds for all $\beta \in \mathbb{N}_0^n$, $|\beta| < N \in \mathbb{N}$. Given $\alpha \in \mathbb{N}_0^n$, $|\alpha| = N$ we find

$$(\partial_w^\alpha g_{k,z})(z) = e^{-k\gamma_z(z)}(\partial_w^\alpha f)(z) + \alpha! \sum_{\beta<\alpha} \frac{1}{\beta!(\alpha-\beta)!}(\partial_w^\beta f)(z)(\partial_w^{\alpha-\beta} e^{-k\gamma_z})(z).$$

Using the induction hypothesis, (2.3) and $|(\partial_w^\beta e^{-k\gamma_z})(z)| \leq C_3 k^{|\beta|} e^{-k\varphi(z)/2}$ for some constant $C_3 > 0$ independent of $k \in [1,\infty)$, $z \in \mathbb{D}^n_\delta(z_0)$, $\varphi \in S_1$ and $\rho \in S_2$ we conclude

$$|(\partial_w^\alpha f)(z)|^2 e^{-k\varphi(z)} \leq C k^{2|\alpha|+n} \int_{\mathbb{D}_\delta(z)} |f(w)|^2 e^{-k\varphi(w)} \rho(w) dV_{\mathbb{C}^n}(w)$$

for all $k \in [1,\infty)$ and $z \in \mathbb{D}_\delta(z_0)$ where $C > 0$ is a constant independent of $k \in [1,\infty)$, $z \in \mathbb{D}^n_\delta(z_0)$, $\varphi \in S_1$ and $\rho \in S_2$. We have that K can be covered by a finite number of polydiscs contained in X and hence the conclusion of the lemma follows. $\square$

Lemma 2.30

Let $\mathbb{D}^n_{3\delta} \subset \mathbb{C}^n$ *be a polydisc of radius* 3δ *around* 0 *for some* $\delta > 0$, $\varphi \in C^3(\mathbb{D}^n_{3\delta}, \mathbb{R})$, $\rho \in C^0(\mathbb{D}^n_{3\delta}, \mathbb{R})$, $\rho > 0$ *two real valued functions. For any* $\varepsilon > 0$ *and* $\alpha \in \mathbb{N}^n_0$ *there exists a constant* $C > 0$ *such that*

$$|f(z)|^2 e^{-k\varphi(z)} \leq C k^{|\alpha|+n+\varepsilon} \int_{\mathbb{D}^n_{2\delta}} |w^\alpha f(w)|^2 e^{-k\varphi(w)} \rho(w) dV_{\mathbb{C}^n}(w)$$

holds for all $z \in \mathbb{D}^n_\delta$, $k \in [1, \infty)$ *and* $f \in \mathcal{O}(\mathbb{D}^n_{3\delta})$. *Here* C *is bounded when* φ *stays in a bounded set in* $C^3(\mathbb{D}^n_{3\delta}, \mathbb{R})$ *and* ρ *stays in a bounded set in* $C^0(\mathbb{D}^n_{3\delta}, \mathbb{R})$ *such that* $\inf_{z\in\mathbb{D}^n_{3\delta}} \rho(z)$ *has a positive lower bound.*

Proof. Let $S_1 \subset C^3(\mathbb{D}^n_{3\delta}, \mathbb{R})$ and $S_2 \subset C^0(\mathbb{D}^n_{3\delta}, \mathbb{R})$ be bounded subsets such that $\inf_{z\in\mathbb{D}^n_{3\delta}} \rho(z) \geq C_0$ holds for all $\rho \in S_2$ where $C_0 > 0$ is some constant. We will prove the claim via induction with respect to n. Consider the case $n = 1$ and set

$$A_k(f) := \int_{\mathbb{D}_{2\delta}} |w^\alpha f(w)|^2 e^{-k\varphi(w)} \rho(w) dV_{\mathbb{C}}(w),$$

where $\alpha \in \mathbb{N}_0$. Then for any $\tau \leq \delta$ one finds

$$\begin{aligned}
A_k(f) &\geq \int_{\mathbb{D}_{2\delta}\setminus\mathbb{D}_\tau} |w^\alpha f(w)|^2 e^{-k\varphi(w)} \rho(w) dV_{\mathbb{C}}(w) \\
&\geq \tau^{2|\alpha|} \int_{\mathbb{D}_{2\delta}\setminus\mathbb{D}_\tau} |f(w)|^2 e^{-k\varphi(w)} \rho(w) dV_{\mathbb{C}}(w) \\
&= \tau^{2|\alpha|} \left(\int_{\mathbb{D}_{2\delta}} |f(w)|^2 e^{-k\varphi(w)} \rho(w) dV_{\mathbb{C}}(w) - \int_{\mathbb{D}_\tau} |f(w)|^2 e^{-k\varphi(w)} \rho(w) dV_{\mathbb{C}}(w) \right).
\end{aligned}$$

Applying Lemma 2.29, there exists a constant $C_1 > 0$ independent of $k \in [1, \infty)$, $f \in \mathcal{O}(\mathbb{D}^n_{3\delta})$, $\varphi \in S_1$ and $\rho \in S_2$ such that

$$\sup_{a\in\mathbb{D}_\delta} |f(a)|^2 e^{-k\varphi(a)} \leq C_1 k \int_{\mathbb{D}_{2\delta}} |f(w)|^2 e^{-k\varphi(w)} \rho(w) dV_{\mathbb{C}}(w)$$

holds. Using this, the standard estimate for integrals, $\rho \leq C_2 > 0$ on $\mathbb{D}_{2\delta}$ for all $\rho \in S_2$ and $\mathbb{D}_\tau \subset \mathbb{D}_\delta$ we get

$$\begin{aligned}
A_k(f) &\geq \tau^{2|\alpha|} \left(\sup_{a\in\mathbb{D}_\delta} |f(a)|^2 e^{-k\varphi(a)} C_1^{-1} k^{-1} - \sup_{a\in\mathbb{D}_\tau} |f(a)|^2 e^{-k\varphi(a)} \int_{\mathbb{D}_\tau} 1\rho(w) dV_{\mathbb{C}}(w) \right) \\
&\geq k^{-1} \tau^{2|\alpha|} \sup_{a\in\mathbb{D}_\delta} |f(a)|^2 e^{-k\varphi(a)} \left(C_1^{-1} - C_2 k \int_{\mathbb{D}_\tau} 1 dV_{\mathbb{C}} \right) \\
&= k^{-1} \tau^{2|\alpha|} \sup_{a\in\mathbb{D}_\delta} |f(a)|^2 e^{-k\varphi(a)} \left(C_1^{-1} - C_2 k \pi \tau^2 \right) \qquad (2.4)
\end{aligned}$$

and the calculation above is true for any $\tau \leq \delta$. Then set $\tau = \delta k^{-\frac{1}{2}-\frac{\varepsilon}{2|\alpha|}}$ for some $\varepsilon > 0$. One has

$$A_k(f) \geq \delta^{2|\alpha|} k^{-(|\alpha|+1)-\varepsilon} \sup_{a\in\mathbb{D}_\delta} |f(a)|^2 e^{-k\varphi(a)} \left(C_1^{-1} - C_2 \delta^2 k^{-\frac{\varepsilon}{|\alpha|}} \pi \right).$$

For k large enough one can achieve that $\delta^{2|\alpha|}(C_1^{-1} - C_2\delta^2 k^{-\frac{\varepsilon}{|\alpha|}}\pi) \geq C_3$ for some constant $C_3 > 0$. As a conclusion one finds $C > 0$ and $k_0 \in (1, \infty)$ such that

$$\sup_{a\in\mathbb{D}_\delta} |f(a)|^2 e^{-k\varphi(a)} \leq C k^{|\alpha|+1+\varepsilon} \int_{\mathbb{D}_{2\delta}} |w^\alpha f(w)|^2 e^{-k\varphi(w)} \rho(w) dV_{\mathbb{C}}(w) \tag{2.5}$$

holds for all $k \in (k_0, \infty)$, $f \in \mathcal{O}(\mathbb{D}_{3\delta})$, $\varphi \in S_1$ and $\rho \in S_2$. For $k \leq k_0$ we can choose τ in (2.4) sufficiently small and conclude that $C > 0$ can be chosen large enough such that (2.5) holds for all $k \in [1, \infty)$.

Now assume the statement holds for $n - 1$. Let $\alpha \in \mathbb{N}_0^n$ be a multi-index and write $\alpha = (\alpha', \alpha_n)$ where $\alpha' \in \mathbb{N}_0^{n-1}$. Given $f \in \mathcal{O}(\mathbb{D}_{3\delta}^n)$ one has $(w' \mapsto f(w', a)) \in \mathcal{O}(\mathbb{D}_{3\delta}^{n-1})$ for any $a \in \mathbb{D}_{2\delta}$. Furthermore, the sets $\{\varphi(\cdot, a) \mid a \in \mathbb{D}_{2\delta},\ \varphi \in S_1\}$ and $\{\rho(\cdot, a) \mid a \in \mathbb{D}_{2\delta}, \rho \in S_2\}$ are bounded subsets of $C^3(\mathbb{D}_{3\delta}^{n-1}, \mathbb{R})$ and $C^0(\mathbb{D}_{3\delta}^{n-1}, \mathbb{R})$, respectively. Using the induction hypothesis one finds a constant $C' > 0$ such that

$$|f(z', a)|^2 e^{-k\varphi(z',a)} \leq C' k^{|\alpha'|+n-1+\varepsilon} \int_{\mathbb{D}_{2\delta}^{n-1}} |w'^{\alpha'} f(w', a)|^2 e^{-k\varphi(w',a)} \rho(w', a) dV_{\mathbb{C}^{n-1}}(w')$$

for all $z' \in \mathbb{D}_\delta^{n-1}$, $k \in [1, \infty)$, $f \in \mathcal{O}(\mathbb{D}_{3\delta}^n)$, $\varphi \in S_1$, $\rho \in S_2$ and $a \in \mathbb{D}_{2\delta}$. Because both sides are continuous and bounded in a, we can integrate them with respect to $|a|^{2\alpha_n} dV_{\mathbb{C}}$ on $\mathbb{D}_{2\delta}$ and get

$$\int_{\mathbb{D}_{2\delta}} |a|^{2\alpha_n} |f(z', a)|^2 e^{-k\varphi(z',a)} dV_{\mathbb{C}} \leq C' k^{|\alpha'|+n-1+\varepsilon} \int_{\mathbb{D}_{2\delta}^n} |w^\alpha f(w)|^2 e^{-k\varphi(w)} \rho(w) dV_{\mathbb{C}^n}(w).$$

Since $\{\varphi(z', \cdot) \mid z' \in \mathbb{D}_\delta^{n-1},\ \varphi \in S_1\}$ is a bounded set in $C^3(\mathbb{D}_{3\delta}, \mathbb{R})$ and $(a \mapsto f(z', a)) \in \mathcal{O}(\mathbb{D}_{3\delta})$ for any $z' \in \mathbb{D}_\delta^{n-1}$, we can apply the case $n = 1$, that is (2.5), on the left-hand side. Thus, for any $\varepsilon' > 0$ there exists a constant $C'' > 0$ such that

$$\sup_{a\in\mathbb{D}_\delta} |f(z', a)|^2 e^{-k\varphi(z',a)} \leq C'' k^{|\alpha|+n+\varepsilon+\varepsilon'} \int_{\mathbb{D}_{2\delta}^n} |w^\alpha f(w)|^2 e^{-k\varphi(w)} \rho(w) dV_{\mathbb{C}^n}(w)$$

holds for all $z' \in \mathbb{D}_\delta^{n-1}$, $k \in [1, \infty)$, $f \in \mathcal{O}(\mathbb{D}_{3\delta}^n)$, $\varphi \in S_1$, $\rho \in S_2$. □

Remark 2.31
In Lemma 2.30, choosing δ sufficiently small depending on φ and ρ, we observe by some slight modification in the proof that the constant C can be replaced by $\delta^{2|\alpha|}C'$ where $C' > 0$ is a constant which is independent of $\alpha \in \mathbb{N}_0^n$.

2.4 Reproducing Kernels and Bergman Kernel Functions

Let $D \subset \mathbb{C}^n$ be a domain and $\rho \in C^0(D, \mathbb{R})$ a continuous function such that $\rho > 0$ on D holds. We choose the volume form on D to be $dV_D = \rho dV_{\mathbb{C}^n}$. Given a weight, i.e. an upper semi-continuous function $\varphi : D \to \mathbb{R}$ which is bounded from below, we consider the space

$$H^0_\varphi(D) = H^0_{\varphi,\rho}(D) = \{f \in \mathcal{O}(D) \mid \int_D |f(w)|^2 e^{-\varphi(w)} dV_D(w) < \infty\}$$

together with an L^2- inner product defined by

$$(f,g)_{\varphi,\rho} = \int_D f(w)\overline{g(w)} e^{-\varphi(w)} dV_D(w)$$

and set $\|f\|_{\varphi,\rho} = \sqrt{(f,f)_{\varphi,\rho}}$.

Lemma 2.32
The pair $(H^0_{\varphi,\rho}(D), (\cdot,\cdot)_{\varphi,\rho})$ is a separable complex Hilbert space.

Proof. We have that $L^2(D,\varphi) = \overline{C_0^\infty(D)}^{\|\cdot\|_{\varphi,\rho}}$ together with $(\cdot,\cdot)_{\varphi,\rho}$ is a separable complex Hilbert space. Thus, it lasts out to show that $H^0_{\varphi,\rho}(D) = \mathcal{O}(D) \cap L^2(D,\varphi)$ is closed. Let $\{f_l\}_{l\in\mathbb{N}} \subset H^0_{\varphi,\rho}(D)$ be a Cauchy sequence with respect to the L^2-norm $\|\cdot\|_{\varphi,\rho}$. We find that $\varphi - \ln(\rho)$ defines an upper semi-continuous function on D which is bounded from below. Then, using Lemma 2.28 with respect to the weight $\varphi - \ln(\rho)$, we find that for any compact subset $K \subset D$ there exists a constant $C > 0$ such that $\sup_{z\in K} |f_l(z) - f_m(z)| \leq C\|f_l - f_m\|_{\varphi,\rho}$ holds for all $l, m \in \mathbb{N}$. We conclude that $\{f_l\}_{l\in\mathbb{N}}$ converges locally uniformly to some function $g \in C^0(D) \cap L^2(D,\varphi)$ which implies $g \in \mathcal{O}(D)$ by Corollary 2.24. □

Lemma 2.33
Given $z \in D$ there exists an unique function $g_z \in H^0_{\varphi,\rho}(D)$ such that $(f, g_z)_{\varphi,\rho} = f(z)$ holds for all $f \in H^0_{\varphi,\rho}(D)$.

Proof. Let $z \in D$ be a point. We apply Lemma 2.28 with respect to the weight $\varphi - \ln(\rho)$ to find that the map $f \mapsto f(z)$ defines a $\mathbb{C}$-linear continuous map on $H^0_{\varphi,\rho}(D)$. Since $H^0_{\varphi,\rho}(D)$ is a complex Hilbert space we can apply the representation theorem of Riezs and find a unique function $g_z \in H^0_{\varphi,\rho}(D)$ such that $(f, g_z)_{\varphi,\rho} = f(z)$ holds for all $f \in H^0_{\varphi,\rho}(D)$. □

Lemma 2.33 gives rise for the following definition.

Definition 2.34
We denote by $K_\varphi = K_{\varphi,\rho} : D \times D \to \mathbb{C}$ the function which satisfies $K_\varphi(z,\cdot) \in H^0_{\varphi,\rho}(D)$ and $(f, K_{\varphi,\rho}(z,\cdot)) = f(z)$ for any $f \in H^0_{\varphi,\rho}(D)$ and $z \in D$. Furthermore, set $P_{\varphi,\rho}(z,w) = e^{-\frac{1}{2}(\varphi(z)+\varphi(w))}\overline{K_{\varphi,\rho}(z,w)}$ and $B_{\varphi,\rho}(z) = P_{\varphi,\rho}(z,z)$. The function $P_\varphi = P_{\varphi,\rho} : D \times D \to \mathbb{C}$ is called reproducing kernel for the space $H^0_{\varphi,\rho}(D)$ or just Bergman kernel. The function $B_\varphi = B_{\varphi,\rho} : D \to \mathbb{R}$ is called Bergman kernel function.

The following two lemmata describe basic properties for the Bergman kernel or the Bergman kernel function.

Lemma 2.35
One has $K_{\varphi,\rho} \in C^\infty(D \times D)$. Furthermore, given any orthonormal basis $\{s_j\}_{j\geq 1}$ of $H^0_{\varphi,\rho}(D)$ one has $K_{\varphi,\rho}(z,w) = \sum_{j\geq 1} \overline{s_j(z)} s_j(w)$ where the sum on the right hand side converges with respect to the topology on $C^\infty(D \times D)$.

Proof (cf. [11, Theorem 6.3.2] for the case $\varphi \equiv 0$ and $\rho \equiv 1$). Let $\{s_j\}_{j\geq 1}$ be an orthonormal basis of $H^0_{\varphi,\rho}(D)$ which always exists since $H^0_{\varphi,\rho}(D)$ is separable. By definition one has $K_{\varphi,\rho}(z,\cdot) = \sum_{j\geq 1} \overline{s_j(z)} s_j(\cdot)$ where the sum on the right-hand side converges with respect to $\|\cdot\|_{\varphi,\rho}$. Using the same argument as in the proof of Lemma 2.32 we find that $K_{\varphi,\rho}(z,w) = \sum_{j\geq 1} \overline{s_j(z)} s_j(w)$ where the sum on the right-hand side converges pointwise. From this fact we observe that $K_{\varphi,\rho}(z,w) = \overline{K_{\varphi,\rho}(w,z)}$ holds. Set $\tilde{D} = \{\overline{z} \mid z \in D\}$. Since $K_{\varphi,\rho}(z,\cdot)$ is a holomorphic function on D we conclude that $(z,w) \mapsto K_{\varphi,\rho}(\overline{z},w)$ defines a function on $\tilde{D} \times D$ which is holomorphic in the variable z and w separately and hence, using Theorem 2.27, it defines a holomorphic function on $\tilde{D} \times D$. As a conclusion we get $K_{\varphi,\rho} \in C^\infty(D \times D)$.

We still have to prove the second part of the statement. Considering $K_{\varphi,\rho}(z,z)$ for $z \in D$ we find that $z \mapsto \sum_{j=1}^\infty |s_j(z)|^2$ defines a smooth function on D. Applying a theorem of Dini we get that the sum converges locally uniformly. Since $|\overline{s_j(\overline{z})} s_j(w)| \leq |s_j(\overline{z})|^2 + |s_j(w)|^2$ for any $j \in \mathbb{N}$ we have that on any compact subset $K \subset \tilde{D} \times D$,

$$
\begin{aligned}
0 &= \lim_{N\to\infty} \sup_{(z,w)\in K} \sum_{j=N}^{\infty} |s_j(\overline{z})|^2 + |s_j(w)|^2 \\
&\geq \lim_{N\to\infty} \sup_{(z,w)\in K} |\sum_{j=N}^{\infty} \overline{s_j(\overline{z})} s_j(w)|
\end{aligned}
$$

holds. Thus, $\left((z,w) \mapsto \sum_{j=1}^N \overline{s_j(\overline{z})} s_j(z)\right)_{N\in\mathbb{N}}$ is a sequence of holomorphic functions which converges locally uniformly. By Corollary 2.24 all its derivatives converges locally uniformly too which proves the second part of the statement. □

Lemma 2.36
For any $z \in D$ one has

$$
B_{\varphi,\rho}(z) = \sup_{f \in H^0_{\varphi,\rho}(D)\setminus\{0\}} \frac{|f(z)|^2 e^{-\varphi(z)}}{\|f\|^2_{\varphi,\rho}}.
$$

Proof. Given $z \in D$ one has $e^{-\varphi(z)} \|K_{\varphi,\rho}(z,\cdot)\|^2_{\varphi,\rho} = B_{\varphi,\rho}(z)$. Let $f \in H^0_{\varphi,\rho}(D)$ be arbitrary. Applying the Cauchy-Schwarz inequality to $(f, K_{\varphi,\rho}(z,\cdot))_{\varphi,\rho} = f(z)$ we deduce $e^{-\varphi(z)}|f(z)|^2 \le B_{\varphi,\rho}(z)\|f\|^2_{\varphi,\rho}$ and get

$$B_{\varphi,\rho}(z) \ge \sup_{f\in H^0_{\varphi,\rho}(D)\setminus 0} \frac{|f(z)|^2 e^{-\varphi(z)}}{\|f\|^2_{\varphi,\rho}}.$$

On the other hand one has $K_{\varphi,\rho}(z,\cdot) \in H^0_{\varphi,\rho}(D)$ and $K_{\varphi,\rho}(z,z) = \|K_{\varphi,\rho}(z,\cdot)\|^2_{\varphi,\rho}$ which leads to

$$B_{\varphi,\rho}(z) = \frac{|K_{\varphi,\rho}(z,z)|^2 e^{-\varphi(z)}}{\|K_{\varphi,\rho}(z,\cdot)\|^2_{\varphi,\rho}} \le \sup_{f\in H^0_{\varphi,\rho}(D)\setminus 0} \frac{|f(z)|^2 e^{-\varphi(z)}}{\|f\|^2_{\varphi,\rho}}.$$

□

Replacing the weight φ by the weight $k\varphi$, $k \in [1,\infty)$, we would like to study the asymptotic behavior of $P_{k\varphi,\rho}$ and $B_{k\varphi,\rho}$ when k goes to infinity. We should consider some examples for such quantities to get an idea how this behavior could look like.

Example 2.37

Let $D = \mathbb{C}^n$ be the complex Euclidean space. Consider the weight $\varphi \in C^\infty(D)$, $\varphi(z) = \sum_{j=1}^n \lambda_j |z_j|^2$ where $\lambda_j \in \mathbb{R}$, $j = 1,\dots,n$ and set $\rho = 1$. For any $k \in [1,\infty)$ one has

$$\dim H^0_{k\varphi,\rho}(\mathbb{C}^n) = \begin{cases} \infty & \text{, if } \lambda_1,\dots,\lambda_n > 0\ , \\ 0 & \text{, else,} \end{cases}$$

and the Bergman kernel and the Bergman kernel function for the case $\lambda_1,\dots,\lambda_n > 0$ are given by

$$P_{k\varphi,\rho}(z,w) = \frac{k^n}{\pi^n}\lambda_1\cdot\ldots\cdot\lambda_n e^{-\frac{k}{2}\sum_{j=0}^n \lambda_j(|w_j-z_j|^2 + w_j\overline{z}_j - \overline{w}_j z_j)},\ B_{k\varphi,\rho}(z) = \frac{k^n}{\pi^n}\lambda_1\cdot\ldots\cdot\lambda_n.$$

Consider the weight $\varphi \in \mathbb{C}^\infty(\mathbb{C})$, $\varphi(z) = \ln(1+|z|^2)$ and assume that the volume form is given by $\rho dV_{\mathbb{C}^n}$, where $\rho \in C^\infty(\mathbb{C}^n)$, $\rho(z) = (1+|z|^2)^{-2}$. For any $k \in [1,\infty)$ one has $\dim H^0_{k\varphi,\rho}(\mathbb{C}^n) = k$ and one has

$$P_{k\varphi,\rho}(z,w) = \frac{k}{\pi}\left(\frac{1+z\overline{w}}{\sqrt{(1+|z|^2)(1+|w|^2)}}\right)^k, \qquad B_{k\varphi,\rho}(z) = \frac{k}{\pi}.$$

Let $D = \mathbb{D}^n_r(0)$ be the polydisc in $\mathbb{C}^n$ with radius $r > 0$ around 0. Set $\rho = \varphi = 1$. For any $k \in [1,\infty)$ one has $\dim H^0_{k\varphi,\rho}(\mathbb{D}^n_r(0)) = \infty$ with

$$P_{k\varphi,\rho}(z,w) = \frac{1}{\pi^n}\prod_{j=1}^n \frac{r^2}{(r^2 - z_j\overline{w}_j)^2}\ , \qquad B_{k\varphi,\rho}(z) = \frac{1}{\pi^n}\prod_{j=1}^n \frac{r^2}{(r^2 - |z_j|^2)^2}.$$

The precise formulas in Example 2.37 can be verified by a direct computation since the monomials $\{z^\alpha | \alpha \in \mathbb{N}_0^n\}$ are already orthogonal with respect to the inner products.

The next lemma together with its consequences give upper bounds for the k-dependent Bergman kernel $P_{k\varphi,\rho}$ and its kernel function $B_{k\varphi,\rho}$ and their derivatives when k goes to infinity.

Lemma 2.38
Assume $\varphi \in C^3(D,\mathbb{R})$. Given $\alpha, \beta \in \mathbb{N}_0^{2n}$ one has $e^{-\frac{k}{2}h} d_z^\alpha d_w^\beta K_{k\varphi,\rho} = O(k^{n+|\alpha|+|\beta|})$ on $D\times D$, where $h(z,w) = \varphi(z)+\varphi(w)$. More precisely, for any compact set $K \subset D\times D$ there exists a constant $C > 0$ such that

$$e^{-\frac{k}{2}(\varphi(z)+\varphi(w))}|d_z^\alpha d_w^\beta K_{k\varphi,\rho}(z,w)| \leq Ck^{n+|\alpha|+|\beta|}$$

holds for all $(z,w) \in K$ and C is bounded when φ stays in a bounded set in $C^3(D,\mathbb{R})$ and ρ stays in a subset of $C^0(D,\mathbb{R})$ such that $\rho > 0$ holds and $\inf_{z\in V}\rho(z)$ has a positive lower bound where $V \subset D$ is some open set satisfying $K \subset V\times V$.

Proof. Let $\alpha', \beta' \in \mathbb{N}_0^{2n}$ be two multi-indices. Lemma 2.35 implies that $(z,w) \mapsto K_{k\varphi,\rho}(z,w)$ is antiholomorphic in the first argument and holomorphic in the second argument and hence $d_z^{\alpha'} d_w^{\beta'} K_{k\varphi,\rho} = c\overline{\partial_z^\alpha}\partial_w^\beta K_{k\varphi,\rho}$ for some complex number $c \in \mathbb{C}\setminus 0$ and multi-indices $\alpha, \beta \in \mathbb{N}^n$ defined by $\alpha_j = \alpha'_j + \alpha'_{n+j}$ and $\beta_j = \beta'_j + \beta'_{n+j}$, $1 \leq j \leq n$. Because of $|\alpha| = |\alpha'|$ and $|\beta| = |\beta'|$ it just lasts out to estimate $\overline{\partial_z^\alpha}\partial_w^\beta K_{k\varphi,\rho}$.

From Lemma 2.28 we get that $f \mapsto (\partial_w^\alpha f)(z)$ defines a $\mathbb{C}$-linear continuous map on $H^0_{k\varphi,\rho}(D)$. As in the proofs of Lemma 2.33 and Lemma 2.35 we find that there exists a unique element $g_{k,z} \in H^0_{k\varphi,\rho}(D)$ such that $(\partial_w^\alpha f)(z) = (f, g_{k,z})_{k\varphi,\rho}$ holds for all $f \in H^0_{k\varphi,\rho}(D)$ and that one has the identity $g_{k,z}(w) = \sum_{j\geq 1}\overline{(\partial_w^\alpha s_j^{(k)})(z)}s_j^{(k)}(w)$ where $\{s_j^{(k)}\}_{j\geq 1}$ is an orthonormal basis of $H^0_{k\varphi,\rho}(D)$. Let $K_1, K_2 \subset D$ be two compact sets and let $S_1 \subset C^3(D,\mathbb{R})$ and $S_2 \subset C^0(D,\mathbb{R})$ be two sets such that S_1 is bounded and $\rho > 0$, $\inf_{z\in V}\rho(z) \geq C_0$ holds for all $\rho \in S_2$ where $C_0 > 0$ is some constant and V is an open neighborhood of $K_1 \cup K_2$. We apply Lemma 2.29 and find a constant $C_1 > 0$ such that

$$e^{-k\varphi(z)}|(\partial_w^\alpha g_{k,z})(z)|^2 \leq C_1 k^{n+2|\alpha|}\|g_{k,z}\|^2_{k\varphi,\rho}$$

holds for all $z \in K_1$, $k \in [1,\infty)$, $\varphi \in S_1$ and $\rho \in S_2$ which leads to $\|g_{k,z}\|^2_{k\varphi,\rho}e^{-k\varphi(z)} \leq C_1 k^{n+2|\alpha|}$ where we use the identity $(\partial_w^\alpha g_{k,z})(z) = \|g_{k,z}\|^2_{k\varphi,\rho}$. Applying Lemma 2.29 again we find a constant $C_2 > 0$ such that

$$e^{-k\varphi(w)-k\varphi(z)}|(\partial_w^\beta g_{k,z})(w)|^2 \leq C_2 k^{n+2|\beta|}\|g_{k,z}\|^2_{k\varphi,\rho}e^{-k\varphi(z)} \leq C_1C_2k^{2n+2|\alpha|+2|\beta|}$$

holds for all $(z,w) \in K_1\times K_2$, $k \in [1,\infty)$, $\varphi \in S_1$ and $\rho \in S_2$. Since $\sum_{j\geq 1}\overline{s_j^{(k)}(w)}s_j^{(k)}(z)$ converges in C^∞ topology, we have

$$(\partial_w^\beta g_{k,z})(w) = \sum_{j\geq 1}\overline{(\partial_z^\alpha s_j^{(k)})(z)}(\partial_w^\alpha s_j^{(k)})(w) = \overline{\partial_z^\alpha}\partial_w^\beta K_{k\varphi,\rho}(z,w).$$

Thus, we conclude that $|e^{-\frac{k}{2}(\varphi(w)-k\varphi(z))}(\overline{\partial_z^\alpha}\partial_w^\beta K_{k\varphi,\rho})(z,w)| \leq Ck^{n+|\alpha|+|\beta|}$ holds for all $(z,w) \in K_1 \times K_2$, $k \in [1,\infty)$, $\varphi \in S_1$ and $\rho \in S_2$ with $C = \sqrt{C_1C_2}$. Since any compact set $K \subset D \times D$ is contained in a compact set of the form $K_1 \times K_2$, the claim follows. □

Corollary 2.39
Let $\varphi \in C^N(D,\mathbb{R})$ be arbitrary. Given $\alpha, \beta \in \mathbb{N}_0^{2n}$, $|\alpha|, |\beta| \leq N$, one has $d_z^\alpha d_w^\beta P_{k\varphi,\rho} = O(k^{n+|\alpha|+|\beta|})$ on $D\times D$ and $d_z^\alpha B_{k\varphi,\rho} = O(k^{n+|\alpha|})$ on D. More precisely, given compact sets $K_1, K_2 \subset D$ there exist constants $C_1, C_2 > 0$ such that

$$|d_z^\alpha d_w^\beta P_{k\varphi,\rho}(z,w)| \leq C_1k^{n+|\alpha|+|\beta|} , \qquad |d_z^\alpha B_{k\varphi,\rho}(z)| \leq C_2k^{n+|\alpha|}$$

hold for all $z \in K_1$, $w \in K_2$, $k \in [1,\infty)$ and C_1, C_2 are bounded when φ stays in a bounded set $C^N(D,\mathbb{R})$ and ρ stays in a subset of $C^0(D,\mathbb{R})$ with $\rho > 0$ and $\inf_{z\in V}\rho(z)$ has a positive lower bound where $V \subset D$ is an open neighborhood of $K_1 \cup K_2$.

Proof. Let $S_1 \subset C^N(D,\mathbb{R})$ be a bounded set and $S_2 \subset C^0(D,\mathbb{R})$ a subset such that $\rho > 0$ and $\inf_{z\in V}\rho(z) \geq C'$ hold for all $\rho \in S_2$ where $C' > 0$ is a constant. Define a function $g_{k\varphi} : D\times D \to \mathbb{R}$ by $g_{k\varphi}(z,w) = e^{-\frac{k}{2}(\varphi(z)+\varphi(w))}$. Let $\alpha'\beta' \in \mathbb{N}_0^{2n}$ be two multi-indices satisfying $\alpha' \leq \alpha$ and $\beta' \leq \beta$. One has $(g_{k\varphi})^{-1}(d_z^{\alpha'}d_w^{\beta'}g_{k\varphi}) = O(k^{|\alpha'|+|\beta'|})$ on $D \times D$ where the underlying estimates are uniform in $\varphi \in S_1$. Set

$$A_{\alpha',\beta'}(z,w) = (d_z^{\alpha-\alpha'}d_w^{\beta-\beta'}K_{k\varphi,\rho})(z,w)(d_z^{\alpha'}d_w^{\beta'}g_k)(z,w)$$

and observe, using Lemma 2.38, that $A_{\alpha',\beta'} = O(k^{n+|\alpha|+|\beta|})$ on $D \times D$ holds. More precisely, for any compact sets $K_1, K_2 \subset D$ there exists a constant $C > 0$ such that

$$|A_{\alpha',\beta'}(z,w)| \leq Ck^{n+|\alpha|+|\beta|}$$

holds for all $z \in K_1$, $w \in K_2$, $k \in [1,\infty)$, $\varphi \in S_1$ and $\rho \in S_2$. Then the estimates for the derivatives of $P_{k\varphi,\rho}$ follow from the identity

$$d_z^\alpha d_w^\beta P_{k\varphi,\rho} = \beta!\alpha! \sum_{\beta'\leq\beta}\sum_{\alpha'\leq\alpha}(\alpha'!(\alpha-\alpha')!\beta'!(\beta-\beta')!)^{-1}A_{\alpha',\beta'}.$$

The estimates for the derivatives of $B_{k\varphi,\rho}$ follow from the fact that we can write

$$(d_z^\alpha B_{k\varphi,\rho})(z) = \alpha! \sum_{\beta\leq\alpha}((\alpha-\beta)!\beta!)^{-1}(d_z^{\alpha-\beta}d_w^\beta P_{k\varphi,\rho})(z,z).$$

□

For the rest of this section we are going to study some transformation behavior of the Bergman kernel. Therefore, let $D \subset \mathbb{C}^n$ be a domain, $\rho \in C^0(D,\mathbb{R})$ a positive function and $\varphi\colon D \to \mathbb{R}$ upper semi-continuous and bounded from below.

Lemma 2.40
Let $f \in \mathcal{O}(D)$ be a holomorphic function and set $\psi = \varphi + 2f + 2\overline{f}$. We have $P_{\psi,\rho}(z,w) = e^{f(z)-f(w)+\overline{f(w)}-\overline{f(z)}}P_{\varphi,\rho}(z,w)$, $z, w \in D$, and hence $B_{\varphi,\rho} = B_{\psi,\rho}$.

Proof. Given an orthonormal basis $\{s_j\}_{j=1}^d$ of $H^0_{k\varphi,\rho}(D)$, $d = \dim H^0_{k\varphi,\rho}(D)$, we find that $\{s_j e^{2f}\}_{j=1}^d$ is an orthonormal basis of $H^0_{k\psi,\rho}(D)$. We calculate

$$P_{\psi,\rho}(z,w) = e^{-\frac{\psi(z)+\psi(w)}{2}}\sum_{j=1}^{d} e^{2f(z)+2\overline{f(w)}}s_j(z)\overline{s_j(w)} = e^{f(z)-f(w)+\overline{f(w)}-\overline{f(z)}}P_{\varphi,\rho}(z,w).$$

□

Lemma 2.41
Let $U \subset \mathbb{C}^n$ be a domain and $G = (G_1, \ldots, G_n)\colon U \to D \subset \mathbb{C}^n$ be a biholomorphic map. Set $\varphi' = \varphi \circ G$ and $\rho' = |\det(F)|^2\rho \circ G$ with $F = (\frac{G_j}{\partial z_l})_{1\le j,l\le n}$. We have $P_{\varphi',\rho'}(z,w) = P_{\varphi,\rho}(G(z), G(w))$, $z, w \in D$, and hence $B_{\varphi',\rho'} = B_{\varphi,\rho} \circ G$.

Proof. Given an orthonormal basis $\{s_j\}_{j=1}^d$ of $H^0_{\varphi,\rho}(D)$, $d = \dim H^0_{\varphi,\rho}(D)$, we find that $\{s_j \circ G\}_{j=1}^d$ is an orthonormal basis of $H^0_{\varphi',\rho'}(U)$ since

$$(f,g)_{\varphi,\rho} = \int_U f \circ G\overline{g \circ G}e^{-\varphi'}|\det(F)|^2\rho' dV_{\mathbb{C}^n}$$

by the standard transformations for integrals. □

2.5 The Localization Property

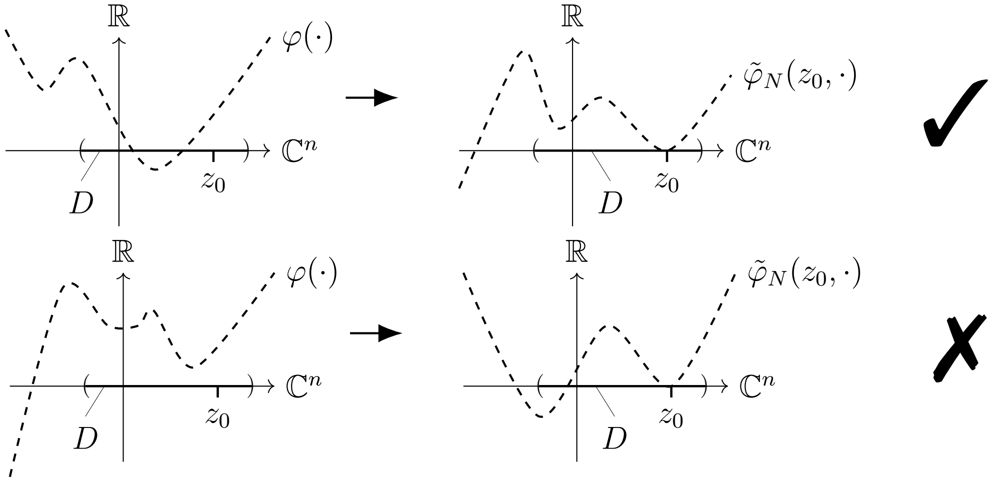
Throughout this section we consider the following setting: Let $D \subset \mathbb{C}^n$ be a bounded domain and $\varphi \in C^{M+3}(D,\mathbb{R}) \cap C^0(\overline{D})$ be a real valued function which is continuous up to the boundary, where $M \in \mathbb{N} \cup \{\infty\}$. The volume form of D is denoted by $dV_D = \rho dV_{\mathbb{C}^n}$ where $\rho \in C^0(\overline{D})$, $\rho > 0$, is continuous and positive on $\overline{D}$.

For $N \in \mathbb{N}_0$, $N \le M$, let $\gamma_N, \tilde{\varphi}_N : D \times \overline{D} \to \mathbb{C}$ be defined by

$$\gamma_N(z,w) = \frac{\varphi(z)}{2} + \sum_{1\le|\alpha|\le N+2}\frac{1}{\alpha!}\frac{\partial^{|\alpha|}\varphi(z)}{\partial^\alpha z}(w-z)^\alpha$$

and $\tilde{\varphi}_N(z,w) = \varphi(w) - \gamma_N(z,w) - \overline{\gamma_N(z,w)}$.

Recall that $D_{\varphi,+}$ denotes the set of all points z in D where the complex Hessian $H_\varphi(z)$ of φ is positive definite (see Definition 2.14). The localization property is defined as follows.



As illustrated in the two upper pictures z_0 has the N-th localization property for φ because the transformation $\tilde{\varphi}_N(z_0,\cdot)$ is positive on $\overline{D}\setminus\{z_0\}$. In the two pictures below, z_0 does not have the N-th localization property since there exists at least one point $z\in\overline{D}$ with $\tilde{\varphi}_N(z_0,z)<0$.

Figure 2.1: Illustration of the N-th localization property

Definition 2.42
Let $N\in\mathbb{N}_0$ be a non-negative integer. A point $z\in D$ has the N-th localization property (for φ) if the following two conditions are satisfied,

$$(i)\ z\in D_{\varphi,+},$$
$$(ii)\ \tilde{\varphi}_N(z,w)>0 \text{ for all } w\in\overline{D}\setminus\{z\}.$$

The set of all points which have the N-th localization property is denoted by $D_{\varphi,N}$ (or sometimes D_N) and given $N>M$ we set $D_{\varphi,N}=\emptyset$.

Example 2.43
Let $\varphi\in C^\infty(D,\mathbb{R})\cap C^0(\overline{D})$ be defined by $\varphi(z)=\sum_{j=1}^n\lambda_j|z_j|^2$ for $\lambda_j\in\mathbb{R}_+$, $j=1,\ldots,n$. Then any point $z\in D$ has the N-th localization property for arbitrary $N\in\mathbb{N}_0$. In other words, $D_{\varphi,N}=D$ for all $N\in\mathbb{N}_0$.

Proof. Let $z\in D$ be a point. Since the complex Hessian $H_\varphi(z)$ in z is a diagonal matrix $\mathrm{diag}(\lambda_1,\ldots,\lambda_n)$ with $\lambda_j>0$, $1\le j\le n$ we find $z\in D_{\varphi,+}$. Furthermore, one has

$$|w_j-z_j|^2=|w_j|^2+|z_j|^2-w_j\overline{z_j}-z_j\overline{w_j}=|w_j|^2-|z_j|^2-\overline{z_j}(w_j-z_j)-z_j\overline{(w_j-z_j)},$$

$\frac{\partial\varphi}{\partial w_j}(z)=\lambda_j\overline{z_j}$ and $\partial_w^\alpha\varphi=0$ for $|\alpha|>1$. Hence, for any $N\in\mathbb{N}_0$ one finds

$$\sum_{j=1}^n\lambda_j|w_j-z_j|^2=\varphi(w)-\gamma_N(z,w)-\overline{\gamma_N(z,w)}=\tilde{\varphi}_N(z,w).$$

$\square$

Lemma 2.44
Given any $N \in \mathbb{N}_0$ the set $D_{\varphi,N}$ is open. Moreover, for any compact set $K \subset D_{\varphi,N}$ there exists a constant $C > 0$ such that $\tilde{\varphi}_N(z,w) \geq C|w-z|^2$ holds for all $z \in K$ and $w \in \overline{D}$.

Proof. Let $N \in \mathbb{N}_0$ be a non-negative integer. If $D_{\varphi,N} = \emptyset$ there is nothing to show. Otherwise take $z_0 \in D_{\varphi,N}$. We will show that there exists an open neighborhood $D' \subset D$ around z_0 such that any $z \in D'$ has the N-th localization property and that $\tilde{\varphi}_N(z,w) \geq C|w-z|^2$ holds for all $z \in D'$ and $w \in \overline{D}$ for some constant $C > 0$. The complete statement follows from the fact that K can be covered by finitely many open sets.

By Taylor expansion in w at some point $z \in D$ we find

$$\tilde{\varphi}_N(z,w) = (w-z)H_\varphi(z)\overline{(w-z)}^T + R(z,w)$$

and for any relative compact sets $V, V' \subset\subset D$ there exists a constant C_1 such that $|R(z,w)| \leq C_1|w-z|^3$ holds for all $(z,w) \in V \times V'$ (see Theorem 2.8). Since the eigenvalues of $H_\varphi(z)$ depend continuously on z one has that $D_{\varphi,+}$ is open and there exist an open neighborhood $U \subset D_{\varphi,+}$ around z_0 and a constant $C_2 > 0$ such that $(w-z)H_\varphi(z)\overline{(w-z)}^T \geq C_2|w-z|^2$ for all $(z,w) \in U \times \overline{D}$. Thus, choosing a ball $\mathbb{B}_\varepsilon(z_0) \subset U$ of radius $\varepsilon > 0$ around z_0 we find that

$$\tilde{\varphi}_N(z,w) \geq C_2|w-z|^2 - C_1|w-z|^3 \geq |w-z|^2(C_2 - 2\varepsilon C_1)$$

holds for all $w, z \in \mathbb{B}_\varepsilon(z_0)$. For ε sufficiently small there exists a constant $C_3 > 0$ such that $\tilde{\varphi}_N(z,w) \geq C_3|w-z|^2$ holds for all $w, z \in \mathbb{B}_\varepsilon(z_0)$. By the assumption on φ we find $\delta > 0$ such that $\tilde{\varphi}_N(z_0,w) \geq 2\delta$ for all $w \in \overline{D} \setminus \mathbb{B}_\varepsilon(z_0)$. Since $\tilde{\varphi}_N$ is continuous on $D \times \overline{D}$ there exists $0 < \varepsilon' < \varepsilon$ such that $\tilde{\varphi}_N(z,w) \geq \delta$ for all $z \in \mathbb{B}_{\varepsilon'}(z_0)$ and $w \in \overline{D} \setminus \mathbb{B}_\varepsilon(z_0)$ and hence $\tilde{\varphi}_N(z,w) > 0$ for all $z \in \mathbb{B}_{\varepsilon'}(z_0)$ and $w \in \overline{D} \setminus \{z\}$. Set $t = \sup_{(z,w) \in \mathbb{B}_{\varepsilon'}(z_0) \times D} |w-z|$ then for $C = \min\{C_3, \delta/t^2\}$ we have $\tilde{\varphi}_N(z,w) \geq C|w-z|^2$ for all $z \in \mathbb{B}_{\varepsilon'}(z_0)$ and $w \in \overline{D}$. □

Example 2.45
Let $N \in \mathbb{N}_0$ be a non-negative integer $K \subset D_{\varphi,N}$ a compact set and $\alpha \in \mathbb{N}_0^n$ a multi-index. There exists a constant $C > 0$ such that

$$\left| \int_D |(w-z)^\alpha|^2 e^{-k\tilde{\varphi}_N(z,w)} dV_D(w) \right| \leq Ck^{-n-|\alpha|}$$

holds for all $z \in K$ and $k \in [1, \infty)$. Here C is bounded when φ stays in a bounded set $S \subset C^{N+3}(D, \mathbb{R})$ such that $\inf_{(z,w) \in K \times D} \tilde{\varphi}_N(z,w)/|w-z|^2$ has a uniform positive lower bound.

Proof. Let $z_0 \in K$ be a point. Using Lemma 2.44 we find an open neighborhood $D' \subset\subset D_N$ around z_0 and a constant $C_1 > 0$ such that $\tilde{\varphi}_N(z,w) \geq C_1|w-z|^2$ holds for all $z \in D'$ and $w \in \overline{D}$. For example, choose $C_1 > 0$ such that $C_1 \leq \inf_{(z,w)\in K\times D} |w-z|^2/\tilde{\varphi}_N(z,w)$ holds for all $\varphi \in S$ to obtain the second part of the statement. Furthermore, there exists a constant $C_2 > 0$ such that $\rho \leq C_2$. Thus, one has

$$|\int_D |(w-z)^\alpha|^2 e^{-k\tilde{\varphi}_N(z,w)} dV_D(w)| \leq C_2 \int_D |(w-z)^\alpha|^2 e^{-kC_1|w-z|^2} dV_{\mathbb{C}^n}(w).$$

We set

$$A_j(z) = \int_{U_j} |(w-z)^\alpha|^2 e^{-kC_1|w-z|^2} dV_{\mathbb{C}^n}(w)\ ,\ j = 1,2$$

where $U_1 = \mathbb{C}^n$ and $U_2 = \mathbb{C}^n \setminus D$ and write

$$\int_D |(w-z)^\alpha|^2 e^{-kC_1|w-z|^2} dV_{\mathbb{C}^n}(w) = A_1(z) - A_2(z).$$

Let $\delta = \operatorname{dist}(D', \partial D) > 0$ be the distance between D' and the boundary of D. Since $\int_{U_2} |(w-z)^\alpha|^2 e^{-C_1(|w-z|^2-\delta^2)} dV_{\mathbb{C}}^n(w) \leq C_3$ holds for all $z \in D'$ where $C_3 > 0$ is some constant and $|w-z|^2 - \delta^2 \geq 0$ for all $z \in D'$ and $w \in U_2$ one finds $A_2(z) \leq C_3 e^{-\delta^2 C_1 k}$. One has

$$2\int_0^\infty r^{2m+1} e^{-tr^2} dr = \int_0^\infty r^m e^{-tr} dr = \frac{m!}{t^{m+1}}$$

for $m \in \mathbb{N}_0$ and $t > 0$. Then, using polar coordinates leads to

$$A_1(z) = \pi^n \prod_{j=1}^n \int_0^\infty r^{\alpha_j} e^{-C_1 kr} dr = k^{-n-|\alpha|} \frac{\pi^n \alpha!}{C_1^{|\alpha|+n}}$$

and hence there exists a constant $C_4 > 0$ such that

$$A_1(z) - A_2(z) \leq C_4 k^{-n-|\alpha|}$$

for all $z \in D'$ and all sufficiently large $k \in [1,\infty)$. We conclude that there exists a constant $C > 0$ such that

$$|\int_D |(w-z)^\alpha|^2 e^{-k\tilde{\varphi}_N(z,w)} dV_D(w)| \leq Ck^{-n-|\alpha|}$$

holds for all $z \in D'$ and all $k \in [1,\infty)$. $\square$

The localization property ensures that the asymptotic behavior of some integrals considered in the following sections comes from integrals with compactly supported integrands. More precisely, one has has the following.

Lemma 2.46

Let $N \in \mathbb{N}_0$ be a non-negative integer and $K \subset D_N$ a compact set. Given any cutoff *function* $\chi \in C_0^\infty(D)$*,* $0 \leq \chi \leq 1$*,* $\chi \equiv 1$ *in an open neighborhood of* K*, there exist constants* $\delta, C > 0$ *such that*

$$\left|\int_D (1-\chi(w))h(z,w)e^{-k\tilde{\varphi}_N(z,w)}dV_D(w)\right| \leq e^{-k\delta}\operatorname{Vol}(D)\|\rho\|_{C^0(\overline{D})} \sup_{(z,w)\in K\times D}|h(z,w)|$$

holds for all $z \in K$*,* $k \in [1,\infty)$ *and* $h \in C^0(D \times \overline{D})$*. Here* δ *can be chosen as*

$$\delta = C' \inf_{(z,w)\in K\times\overline{D}} \tilde{\varphi}_N(z,w)/|w-z|^2$$

where $C' > 0$ *is a constant which only depends on* K *and* χ*.*

Proof. Choose an open neighborhood $D' \subset D$ around K such that $\chi \equiv 1$ on D' holds. By Lemma 2.44 there is a constant $C_1 > 0$ such that $\tilde{\varphi}_N(z,w) \geq C_1|w-z|^2$ holds for all $(z,w) \in K \times \overline{D}$ and C_1 can be chosen to be

$$C_1 = \inf_{(z,w)\in K\times\overline{D}} \tilde{\varphi}_N(z,w)/|w-z|^2.$$

Hence we find $\tilde{\varphi}_N(z,w) \geq C_1 \operatorname{dist}(K, \partial D')^2$ for all $(z,w) \in K \times \overline{D} \setminus D'$ which leads to

$$\left|\int_D (1-\chi(w))h(z,w)e^{-k\tilde{\varphi}_N(z,w)}dV_D(w)\right| \leq e^{-k\delta}\int_{D\setminus D'}|h(z,w)\rho(w)|dV_{\mathbb{C}^n}(w)$$

for any $z \in K$, $k \in [1,\infty)$ and $h \in C^0(D \times \overline{D})$. Applying the standard estimate for integrals finishes the proof. □

The following lemma is important for applying the results obtained in Section 3.1 to the manifold setting. It shows that under some conditions we can always assume that the localization property holds in a local sense.

Lemma 2.47
Let $N \in \mathbb{N}_0$ *be a non-negative integer,* $S \subset C^{N+3}(D,\mathbb{R})$ *a bounded set,* $D' \subset D$ *open and* $C' > 0$ *a constant such that* $H_\varphi(z) - C'\mathrm{Id}$ *is positive definite for all* $z \in D'$ *and* $\varphi \in S$*. Given* $z_0 \in D'$ *there exists an open neighborhood* $V \subset D'$ *around* z_0 *such that for any* $z \in V$*,* $\varphi \in S$ *we have that* z *satisfies the* N*-th localization property for* $\varphi|_{\overline{V}}$*. More precisely, there exists a constant* $C > 0$ *such that* $\tilde{\varphi}_N(z,w) \geq C|w-z|^2$ *and* $|d_w\tilde{\varphi}_N(z,w)| \geq C|w-z|$ *holds for all* $(z,w) \in V \times \overline{V}$ *and all* $\varphi \in S$*.*

Proof. By Taylor expansion in w at some point $z \in D'$ we find

$$\tilde{\varphi}_N(z,w) = (w-z)H_\varphi(z)\overline{(w-z)}^T + R(z,w).$$

Choose a ball $\mathbb{B}_\varepsilon(z_0) \subset\subset D'$ of radius $\varepsilon > 0$ around z_0. There exists a constant C_1 such that $|R(z,w)| \leq C_1|w-z|^3$ for all $z, w \in \mathbb{B}_\varepsilon(z_0)$ and $\varphi \in S$ by Theorem 2.8. By the assumptions on S there exists a constant $C' > 0$ such that

$(w-z)H_\varphi(z)\overline{(w-z)}^T \geq C'|w-z|^2$ for all $(z,w) \in \mathbb{B}_\varepsilon(z_0) \times D$ and $\varphi \in S$. Thus, we find that

$$\tilde{\varphi}_N(z,w) \geq C'|w-z|^2 - C_1|w-z|^3 \geq |w-z|^2(C' - 2\varepsilon C_1)$$

holds for all $z, w \in \mathbb{B}_\varepsilon(z_0)$ and $\varphi \in S$. Similar, we find a constant $C_2 > 0$ such that $|d_w\tilde{\varphi}_N(z,w)| \geq |w-z|(2C' - 2\varepsilon C_2)$ holds for all $z, w \in \mathbb{B}_\varepsilon(z_0)$ and $\varphi \in S$. Choosing $\varepsilon > 0$ sufficiently small there exists a constant $C > 0$ such that $\tilde{\varphi}_N(z,w) \geq C|w-z|^2$ and $|(\partial_w + \overline{\partial}_w)\tilde{\varphi}_N(z,w)| \geq C|w-z|$ hold for all $w, z \in \mathbb{B}_\varepsilon(z_0)$ and $\varphi \in S$. Putting $V = \mathbb{B}_{\varepsilon/2}(z_0)$, the claim follows. □

Remark 2.48

Choosing $S \subset C^{N+3}(D, \mathbb{R})$ compact in Lemma 2.47 it is sufficient to assume that $H_\varphi(z)$ is positive definite in a neighborhood of $\overline{D'}$. Since the eigenvalues of $H_\varphi(z)$ depend continuously on $(z, \varphi) \in D \times C^{N+3}(D, \mathbb{R})$ there exists a constant C' such that $H_\varphi(z) - C'\mathrm{Id}$ is positive definite for all $z \in D'$ and all $\varphi \in S$.

2.6 The Method of Stationary Phase

As before, we assume that $D \subset \mathbb{C}^n$ is a bounded domain with volume form $dV_D = \rho dV_{\mathbb{C}^n}$ where $\rho \in C^0(\overline{D})$, $\rho > 0$ on $\overline{D}$.

Definition 2.49

Given $z \in D$, $\alpha, \beta \in \mathbb{N}_0^n$, $k \in [1, \infty)$ and $\varphi \in C^{N+3}(D, \mathbb{R}) \cap C^0(\overline{D})$ set

$$a_{\alpha,\beta,k\varphi}(z) = \int_D \overline{(w-a)}^\alpha (w-z)^\beta e^{-k\tilde{\varphi}_N(z,w)} dV_D(w),$$

where $\tilde{\varphi_N}$ is defined as in Section 2.5.

We like to study the asymptotic behavior of $a_{\alpha,\beta,k\varphi}(z)$ when k goes to infinity. Recall that $D_{\varphi,N} \subset D$ is the set of points which satisfy the N-th localization property for $\varphi \in C^{N+3}(D, \mathbb{R}) \cap C^0(\overline{D})$ (see Definition 2.42). We have the following theorem.

Theorem 2.50

Let $M, N \in \mathbb{N}_0$ be two non-negative integers, $N \leq 3M+1$, $\varphi \in C^{3M+4}(D, \mathbb{R}) \cap C^0(\overline{D})$ and $\rho \in C^{2M+2}(D) \cap C^0(\overline{D})$. Given $\alpha, \beta \in \mathbb{N}_0^n$, $|\alpha|, |\beta| \leq M$, one has $a_{\alpha,\beta,k\varphi} = O(k^{-\max\{|\alpha|,|\beta|\}-n})$ in $C^0(D_{\varphi,N})$ and in particular

$$a_{\alpha,\beta,k\varphi} - \frac{\pi^n}{\det(H_\varphi)} \sum_{j=\max\{|\alpha|,|\beta|\}}^{M} k^{-j-n} a_{\alpha,\beta}^{(j)} = O(k^{-M-1}) \text{ in } C^0(D_{\varphi,N})$$

where

$$a_{\alpha,\beta}^{(j)}(z) = \sum_{\mu=0}^{2j} \frac{(-1)^\mu}{\mu!(\mu+j)!} \langle H_\varphi(z)^{-1}\partial_w, \overline{\partial}_w\rangle^{\mu+j} (h_{N,z}(w)^\mu \rho(w) \overline{(w-z)}^\alpha (w-z)^\beta)|_{w=z}$$

and $h_{N,z}(w) = \tilde{\varphi}_N(z,w) - (w-z)^T H_\varphi(z)\overline{(w-z)}$. *More precisely, given a compact set* $K \subset D_{\varphi,N}$ *and an open neighborhood* $D' \subset\subset D_{\varphi,N}$ *of* K *there exists a constant* $C > 0$ *such that*

$$\left| a_{\alpha,\beta,k\varphi}(z) - \frac{\pi^n}{\det(H_\varphi(z))} \sum_{j=\max\{|\alpha|,|\beta|\}}^{M} k^{-j-n} a_{\alpha,\beta}^{(j)}(z) \right| \leq Ck^{-M-1}$$

holds for all $z \in K$ *and* $k \in [1,\infty)$. *Here* C *is bounded when* φ *stays in a bounded set in* $C^{3M+4}(D,\mathbb{R}) \cap C^0(\overline{D})$ *such that* $\inf_{(z,w)\in D'\times\overline{D}} \tilde{\varphi}_N(z,w)/|z-w|^2$ *has a positive lower bound and* ρ *stays in a bounded set in* $C^{2M+2}(D,\mathbb{R}) \cap C^0(\overline{D})$ *such that* $\inf_{z\in\overline{D}} \rho(z)$ *has a positive lower bound.*

Furthermore, if $|\alpha| = |\beta|$ *we have* $a_{\alpha,\beta}^{(|\alpha|)}(z) = \langle H_\varphi(z)^{-1}\partial_w, \overline{\partial}_w\rangle^{|\alpha|}(\overline{w}^\alpha w^\beta)\rho(z)/|\alpha|!$.

We will prove Theorem 2.50 by adapting the stationary phase formula of Hörmander to our setting.

Theorem 2.51 (Method of Stationary Phase [24, Theorem 7.7.5])
Let $K \subset \mathbb{R}^n$ *be a compact set,* X *an open neighborhood of* K *and* M *a positive integer. If* $u \in C_0^{2M}(K)$, $f \in C^{3M+1}(X)$ *and* $\operatorname{Im} f \geq 0$ *in* X, $\operatorname{Im} f(x_0) = 0$, $\det f''(x_0) \neq 0$, $f' \neq 0$ *in* $K \setminus \{x_0\}$ *then*

$$\begin{aligned} &\Big| \int u(x)e^{ikf(x)}dx - e^{ikf(x_0)}(\det(kf''(x_0)/2\pi i))^{-\frac{1}{2}} \sum_{j<M} k^{-j} L_j u\Big| \\ &\qquad \leq Ck^{-M} \sum_{|\alpha|\leq 2M} \sup|d_x^\alpha u|, \;\; k > 0. \end{aligned} \tag{2.6}$$

Here C *is bounded when* f *stays in a bounded set in* $C^{3M+1}(X)$ *and* $|x-x_0|/|f'(x)|$ *has a uniform bound. With*

$$g_{x_0}(x) = f(x) - f(x_0) - \langle f''(x_0)(x-x_0), x-x_0\rangle/2$$

which vanishes of third order at x_0 *we have*

$$L_j u = \sum_{\nu-\mu=j} \sum_{2\nu\geq 3\mu} i^{-j+2\nu} 2^{-\nu} \langle f''(x_0)^{-1} d_x, d_x\rangle^\nu (g_{x_0}^\mu u)(x_0)/\mu!\nu!.$$

This is a differential operator of order $2j$ *acting on* u *at* x_0. *The coefficients are rational homogeneous functions of degree* $-j$ *in* $f''(x_0), \ldots, f^{(2j+2)}(x_0)$ *with denominator* $(\det f''(x_0))^{3j}$. *In every term the total number of derivatives of* u *and of* f'' *is at most* $2j$.

Proof. See [24, Theorem 7.7.5]. □

Lemma 2.52
Let $D \subset \mathbb{C}^n$ *be a domain,* $M, N \in \mathbb{N}_0$ *non-negative integers with* $N \leq 3M-2$,

$S \subset C^{3M+1}(D,\mathbb{R})$ a bounded set, $D' \subset D$ open and $C' > 0$ some constant such that $H_\varphi(z) - C'\mathrm{Id}$ is positive definite for all $z \in D'$ and $\varphi \in S$. Furthermore, assume $\rho \in C^{2M}(D,\mathbb{R})$. Given a function $u \in C^{2M}(D \times D)$ set $u_z(w) = u(z,w)$ and

$$b_j(z) = \sum_{\mu=0}^{2j} (-1)^\mu \langle H_\varphi(z)^{-1}\partial_w, \overline{\partial}_w\rangle^{\mu+j}(h^\mu_{N,z}u_z\rho)(z)/\mu!(\mu+j)!$$

where

$$h_{N,z}(w) = \tilde{\varphi}_N(z,w) - (w-z)^T H_\varphi(z)\overline{(w-z)}.$$

Given $z_0 \in D'$ there exist an open neighborhood $D'' \subset\subset D'$ around z_0, a cutoff function $\chi \in C_0^\infty(D,\mathbb{R})$, $0 \le \chi \le 1$, $\chi \equiv 1$ in a neighborhood of D'' and a constant $C > 0$ such that the expression

$$A_k(z) = \int_D u(z,w)\chi(w)e^{-k\tilde{\varphi}_N(z,w)}dV_D(w)$$

satisfies

$$|A_k(z) - \frac{\pi^n}{\det(H_\varphi(z))}\sum_{j<M} k^{-j-n}b_j(z)| \le Ck^{-M}\sup_{z\in D''}\|\chi u_z\rho\|_{C^{2M}(\overline{D})}$$

for all $k \in [1,\infty)$, $z \in D''$, $\varphi \in S$, $\rho \in C^{2M}(D,\mathbb{R})$ and $u \in C^{2M}(D\times D)$.

Proof. We would like to apply Theorem 2.51. Therefore set $f_z(w) = f_{z,\varphi}(w) = i\tilde{\varphi}_N(z,z+w)$ and $x_0 = 0$. By construction we have $\operatorname{Im} f_z(0) = 0$ and $f'_z(0) = 0$. Since $H_{\tilde{\varphi}_N(z,\cdot)}(z) = H_\varphi(z)$ and by Corollary 2.13 we find $\det(f''_z(0)/2i) = \det(H_\varphi(z))^2 > 0$ for all $z \in D'$. Using Lemma 2.47 we find an open ball $\mathbb{B}_{5\varepsilon}(z_0) \subset\subset D'$ of radius $5\varepsilon > 0$ around z_0 and constants $C_1, C_2 > 0$ such that $\tilde{\varphi}_N(z,w) \ge C_1|w-z|^2$ and $|(\partial_w + \overline{\partial}_w)\tilde{\varphi}_N(z,w)| \ge C_2|w-z|$ for all $z,w \in \mathbb{B}_{5\varepsilon}(z_0)$ and $\varphi \in S$. Hence we have $\operatorname{Im} f_z(w) > 0$ and $f'_z(w) \ne 0$ for all $z \in \mathbb{B}_\varepsilon(z_0)$, $w \in \mathbb{B}_{4\varepsilon}(0) \setminus \{0\}$ and $\varphi \in S$. Furthermore, $S' := \{f_{z,\varphi} \mid z \in \mathbb{B}_\varepsilon(z_0), \varphi \in S\}$ defines a bounded set in $C^{3M+1}(\mathbb{B}_{4\varepsilon}(0))$ satisfying $|w|/|f'(w)| \le C_2$ for all $w \in \mathbb{B}_{4\varepsilon}(0)$ and all $f \in S'$. Choose a cutoff function $\chi \in C_0^\infty(\mathbb{B}_{2\varepsilon}(z_0),\mathbb{R})$, $0 \le \chi \le 1$, $\chi \equiv 1$ in an open neighborhood of $\overline{\mathbb{B}_\varepsilon(z_0)}$. One has

$$A_k(z) = \int_{\mathbb{B}_{4\varepsilon}(0)} u(z,z+w)\chi(z+w)\rho(z+w)e^{ikf_z(w)}dV_{\mathbb{C}^n}(w)$$

for all $z \in \mathbb{B}_\varepsilon(z_0)$. For $z \in \mathbb{B}_\varepsilon(z_0)$ set $\tilde{u}_z(w) = u(z,z+w)\chi(z+w)\rho(z+w)$. Then, for all $z \in \mathbb{B}_\varepsilon(z_0)$ we have $\tilde{u}_z \in C^{2M}(\mathbb{B}_{4\varepsilon}(0))$ and $\operatorname{supp}(\tilde{u}_z) \subset \mathbb{B}_{3\varepsilon}(0)$. Set $X = \mathbb{B}_{4\varepsilon}(0)$, $K = \overline{\mathbb{B}_{3\varepsilon}(0)}$ and apply Theorem 2.51. Thus, we find a constant $C > 0$ such that

$$|A_k(z) - \frac{\pi^n}{\det(H_\varphi(z))}\sum_{j<M} k^{-j-n}b_j(z)| \le Ck^{-M}$$

holds for all $k \in [1,\infty)$, $z \in \mathbb{B}_\varepsilon(z_0)$ where

$$b_j(z) = \sum_{\nu-\mu=j} \sum_{2\nu \geq 3\mu} i^{-j+2\nu} 2^{-\nu} \langle f_z''(0)^{-1} d_x, d_x \rangle^\nu (g_{z,0}^\mu \tilde{u}_z)(0)/\mu!\nu!$$

and

$$g_{z,0}(w) = f_z(w) - f_z(0) - \langle f_z''(0)w, w\rangle/2 = ih_{N,z}(z+w).$$

Using $\langle f_z''(0)^{-1} d_x, d_x\rangle = -2i\langle H_\varphi(z)^{-1}\partial_w, \overline{\partial}_w\rangle$ from Lemma 2.16 we find

$$\begin{aligned} b_j(z) &= \sum_{\nu-\mu=j} \sum_{2\nu \geq 3\mu} (-1)^\mu \langle H_\varphi(z)^{-1}\partial_w, \overline{\partial}_w\rangle^\nu (h_{N,z}^\mu u_z \rho)(z)/\mu!\nu! \\ &= \sum_{\mu=0}^{2j} (-1)^\mu \langle H_\varphi(z)^{-1}\partial_w, \overline{\partial}_w\rangle^{\mu+j} (h_{N,z}^\mu u_z \rho)(z)/\mu!(\mu+j)! \end{aligned}$$

where we use that for $\nu = j+\mu$, $\mu, \nu \geq 0$ one has that $2\nu \geq 3\mu$ holds if and only if $\mu \leq 2j$ holds to obtain the last line. $\square$

Proof of Theorem 2.50. Let $M \in \mathbb{N}$ be a positive integer, $D' \subset\subset D$ open and $S_1 \subset C^{3M+4}(D,\mathbb{R}) \cap C^0(\overline{D})$, $S_2 \subset C^{2M+2}(D,\mathbb{R}) \cap C^0(\overline{D})$ be bounded sets such that $\{\inf_{(z,w)\in D'\times\overline{D}} \tilde{\varphi}_N(z,w)/|z-w|^2 \mid \varphi \in S_1\}$, $\{\inf_{z\in\overline{D}} \rho \mid \rho \in S_2\}$ have positive lower bounds. For any $\varphi \in S_1$ we have $D' \subset D_{\varphi,N}$ then. Let $K \subset D'$ be a compact subset. Given $z_0 \in K$ we can apply Lemma 2.52 and find an open neighborhood $D'' \subset\subset D'$ around z_0 and a cutoff function $\chi \in C_0^\infty(D,\mathbb{R})$, $0 \leq \chi \leq 1$, $\chi \equiv 1$ on D'' and a constant $C_1 > 0$ such that

$$\left| \tilde{a}_{\alpha,\beta,k}(z) - \frac{1}{\pi^n \det(H_\varphi(z))} \sum_{j<M} k^{-j-n} a_{\alpha,\beta}^{(j)}(z) \right| \leq C_1 k^{-M}$$

where

$$\tilde{a}_{\alpha,\beta,k}(z) = \int_D \chi(w) \overline{(w-a)}^\alpha (w-z)^\beta e^{-k\tilde{\varphi}_N(z,w)} dV_D(w)$$

and

$$a_{\alpha,\beta}^{(j)}(z) = \sum_{\mu=0}^{2j} \frac{(-1)^\mu}{\mu!(\mu+j)!} \langle H_\varphi(z)^{-1}\partial_w, \overline{\partial}_w\rangle^{\mu+j} (h_{N,z}(w)^\mu \rho(w) \overline{(w-z)}^\alpha (w-z)^\beta)|_{w=z}.$$

Using Lemma 2.46 we find a constant $C_2 > 0$ and $\delta > 0$ such that $|a_{\alpha,\beta,k}(z) - \tilde{a}_{\alpha,\beta,k}(z)| \leq C_2 e^{-\delta k}$ holds for all $z \in D''$, $k \in [1,\infty)$, $\varphi \in S_1$ and $\rho \in S_2$. Thus, there exists a constant $C_3 > 0$ such that

$$\left| a_{\alpha,\beta,k}(z) - \frac{\pi^n}{\det(H_\varphi(z))} \sum_{j<M} k^{-j-n} a_{\alpha,\beta}^{(j)}(z) \right| \leq C_3 k^{-M}$$

holds for all $z \in D''$, $k \in [1,\infty)$, $\varphi \in S_1$ and $\rho \in S_2$.

Now, consider $a^{(j)}_{\alpha,\beta}$ for $j < \max\{|\alpha|, |\beta|\}$. For fixed μ, $0 \le \mu \le 2j$, we write

$$\langle H_\varphi(z)^{-1}\partial_w, \overline{\partial}_w\rangle^{j+\mu} = \sum_{|\alpha'|=|\beta'|=j+\mu} c_{\alpha',\beta'}\overline{\partial}^{\alpha'}_w \partial^{\beta'}_w$$

where $c_{\alpha,\beta} \in \mathbb{C}$ are complex numbers. Given $\alpha', \beta' \in \mathbb{N}^n_0$, $|\alpha'| = |\beta'| = \mu + j$, such that $\alpha' \ge \alpha$ and $\beta' \ge \beta$ holds (this implies $\mu > 0$) we find, using the general Leibniz rule, that

$$\overline{\partial}^{\alpha'}_w \partial^{\beta'}_w (h_{N,z}(w)^\mu \rho(w)\overline{(w-z)}^\alpha (w-z)^\beta)|_{w=z} = \\ \frac{\alpha'!\beta'!}{(\alpha'-\alpha)!(\beta'-\beta)!}\overline{\partial}^{\alpha'-\alpha}_w \partial^{\beta'-\beta}_w (h_{N,z}(w)^\mu \rho(w))|_{w=z}$$

holds. Since $h_{N,z}(w)^\mu\rho(w) = O(|w|^{3\mu})$ and $|\alpha'-\alpha|, |\beta'-\beta| < \mu$ we have that less then 2μ derivatives acting on the function $O(|w|^{3\mu})$ which implies

$$\overline{\partial}^{\alpha'-\alpha}_w \partial^{\beta'-\beta}_w (h_{N,z}(w)^\mu \rho(w))|_{w=z} = 0.$$

If $\alpha' \ge \alpha$ or $\beta' \ge \beta$ fails, we directly observe that

$$\overline{\partial}^{\alpha'}_w \partial^{\beta'}_w (h_{N,z}(w)^\mu \rho(w)\overline{(w-z)}^\alpha (w-z)^\beta)|_{w=z} = 0$$

holds. Hence,

$$\langle H_\varphi(z)^{-1}\partial_w, \overline{\partial}_w\rangle^{\mu+j}(h_{N,z}(w)^\mu \rho(w)\overline{(w-z)}^\alpha (w-z)^\beta)|_{w=z} = 0$$

and we conclude that $a^{(j)}_{\alpha,\beta} = 0$ if $j < \max\{|\alpha|, |\beta|\}$ which proves that $a_{\alpha,\beta,k} \in O(k^{-\max\{|\alpha|,|\beta|\}-n})$ in $C^0(D'')$, uniformly in $\varphi \in S_1$ and $\rho \in S_2$.

Given $\alpha, \beta \in \mathbb{N}^n_0$, $|\alpha| = |\beta|$, consider $a^{(|\alpha|)}_{\alpha,\beta}$. If $\mu > 0$ we can proceed similar as above to observe that

$$\overline{\partial}^{\alpha'}_w \partial^{\beta'}_w (h_{N,z}(w)^\mu \rho(w)\overline{(w-z)}^\alpha (w-z)^\beta)|_{w=z} = 0$$

holds for $|\alpha'| = |\beta'| = \mu + |\alpha|$. Treating the case $\mu = 0$ leads to

$$\begin{aligned}\langle H_\varphi(z)^{-1}\partial_w, \overline{\partial}_w\rangle^{|\alpha|}(\rho(w)\overline{(w-z)}^\alpha (w-z)^\beta)|_{w=z} & \\ &= \alpha!\beta! c_{\alpha,\beta}\rho(z) \\ &= \rho(z)\langle H_\varphi(z)^{-1}\partial_w, \overline{\partial}_w\rangle^{|\alpha|}\overline{w}^\alpha w^\beta.\end{aligned}$$

Covering K by finitely many of those D'' completes the proof. □

Lemma 2.53
Given $z \in D_{\varphi,N}$ *such that* $H_\varphi(z) = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ *one has*

$$a^{(j)}_{\alpha,\beta}(z) = \sum_{\mu=0}^{2j} \sum_{\substack{|\eta|=\mu+j \\ \eta \ge \max\{\alpha,\beta\}}} (1/\lambda)^\eta \frac{(-1)^\mu \eta!}{\mu!(\eta-\alpha)!(\eta-\beta!)}\overline{\partial}^{\eta-\alpha}_w \partial^{\eta-\beta}_w (h^\mu_{N,z}\rho)(z)$$

where $(1/\lambda)^\eta = \lambda_1^{-\eta_1} \cdot \ldots \cdot \lambda_n^{-\eta_n}$.

Proof. One has $\langle H_\varphi(z)^{-1}\partial_w, \overline{\partial}_w\rangle = \sum_{l=1}^n \lambda_l^{-1}\frac{\partial^2}{\partial w_l \partial \overline{w}_l}$ and hence

$$\langle H_\varphi(z)^{-1}\partial_w, \overline{\partial}_w\rangle^{\mu+j} = \sum_{|\eta|=\mu+j}(1/\lambda)^\eta \frac{(\mu+j)!}{\eta!}\overline{\partial}_w^\eta \partial_w^\eta.$$

Given $\alpha, \beta \in \mathbb{N}_0^n$ such that $\eta \geq \max\{\alpha,\beta\}$ holds one finds

$$\begin{aligned}&\overline{\partial}_w^\eta \partial_w^\eta (h_{N,z}(w)^\mu \rho(w)\overline{(w-z)}^\alpha (w-z)^\beta)|_{w=z}\\ &= \frac{\eta!\eta!}{(\eta-\alpha)!(\eta-\beta)!}\overline{\partial}_w^{\eta-\alpha}\partial_w^{\eta-\beta}(h_{N,z}^\mu \rho)(z).\end{aligned}$$

Given $\alpha, \beta \in \mathbb{N}_0^n$ such that $\eta \geq \max\{\alpha,\beta\}$ fails one has

$$\overline{\partial}_w^\eta \partial_w^\eta (h_{N,z}(w)^\mu \rho(w)\overline{(w-z)}^\alpha (w-z)^\beta)|_{w=z} = 0.$$

Thus, we conclude

$$a_{\alpha,\beta}^{(j)}(z) = \sum_{\mu=0}^{2j}\sum_{\substack{|\eta|=\mu+j\\ \eta\geq\max\{\alpha,\beta\}}}(1/\lambda)^\eta \frac{\eta!}{\mu!(\eta-\alpha)!(\eta-\beta!)}\overline{\partial}_w^{\eta-\alpha}\partial_w^{\eta-\beta}(h_{N,z}^\mu \rho)(z).$$

□

2.7 An L^2-Norm Estimate for the Stationary Phase Formula

Let $N \in \mathbb{N}_0$ be a non-negative integer. In this section we study the asymptotic behavior for $k \to \infty$ of the inner product $(\cdot,\cdot)_{k\varphi,\rho}$ between polynomials of degree at most N and holomorphic functions in $H^0_{k\varphi,\rho}(D)$ which vanish up to order N at some point. For brevity we will use the notations $H^0_k(D) := H^0_{k\varphi,\rho}(D)$, $(\cdot,\cdot)_k := (\cdot,\cdot)_{k\varphi,\rho}$ and $\|\cdot\|_k := \|\cdot\|_{k\varphi,\rho}$ in this section. As before, we assume that $D \subset \mathbb{C}^n$ is a bounded domain.

Definition 2.54
Given a point $z_0 \in \mathbb{C}^n$ and a holomorphic function $f \in \mathcal{O}(U)$ defined on an open neighborhood U around z_0 we say that f vanishes up to order $N \in \mathbb{N}_0$ in z_0 if $\partial_z^\alpha f(z_0) = 0$ holds for all $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq N$.

The following result is fundamental for proving the reprodicng property in Theorem 1.3. The idea for Theorem 2.55 in the version given below was inspired by Chin-Yu Hsiao during a discussion in 2016.

Theorem 2.55
For $N \in \mathbb{N}_0$ let $\varphi \in C^{N+4}(D,\mathbb{R})$ be a function and $dV_D = \rho dV_{\mathbb{C}^n}$ be a volume form, that is $\rho \in C^{N+1}(D,\mathbb{R})$ with $\rho > 0$. For any $z_0 \in D_{\varphi,+}$, $\beta \in \mathbb{N}_0^n$ and $0 < \varepsilon < 1$

there exist an open neighborhood $U \subset D$ *around* z_0*, a cutoff function* $\chi \in C_0^\infty(D, \mathbb{R})$*,* $0 \leq \chi \leq 1$*,* $\chi \equiv 1$ *on* U *and a constant* C *such that for all* $z \in U$*,* $k \in [1, \infty)$*, and all functions* $f \in H_k^0(D)$ *which vanish up to order* N *in* z*, one has*

$$\left| \int_D f(w)\overline{(w-z)}^\beta e^{k\overline{\gamma_N(z,w)}} \chi(w) e^{-k\varphi(w)} dV_D(w) \right| \leq Ck^{-\frac{N+1+n-\varepsilon}{2}} \|f\|_k.$$

Here C *is bounded when* φ *stays in a bounded set in* $C^{N+4}(D, \mathbb{R})$ *with a constant* $C' > 0$ *such that* $H_\varphi(z) - C' \operatorname{Id}$ *is positive definite on* $\operatorname{supp}(\chi)$ *and* ρ *stays in a bounded set in* $C^{N+1}(D, \mathbb{R})$ *such that* $\sup_{w \in \operatorname{supp}(\chi)} \rho(w)^{-1}$ *has a uniform bound.*

Corollary 2.56

Given $N \in \mathbb{N}_0$*,* $\varphi \in C^{N+4}(D, \mathbb{R}) \cap C^0(\overline{D})$*,* $dV_D = \rho dV_{\mathbb{C}^n}$ *with* $\rho \in C^{N+1}(D, \mathbb{R}) \cap C^0(\overline{D})$*,* $\rho > 0$*,* $z_0 \in D_{\varphi,N}$*,* $\beta \in \mathbb{N}_0^n$ *and any neighborhood* $D' \subset\subset D_{\varphi,N}$ *around* z_0 *there exists a constant* $C > 0$ *such that*

$$\left| \int_D f(w)\overline{(w-z)}^\beta e^{k\overline{\gamma_N(z,w)}} e^{-k\varphi(w)} dV_D(w) \right| \leq Ck^{-\frac{N+1+n-\varepsilon}{2}} \|f\|_k \tag{2.7}$$

for all $z \in D'$*,* $k \in [1, \infty)$ *and all* $f \in H_k^0(D)$ *which vanish up to order* N *in* z*. Here* C *is bounded when* φ *stays in a bounded set in* $C^{N+4}(D, \mathbb{R}) \cap C^0(\overline{D})$ *such that* $\sup_{(z,w) \in D'' \times \overline{D}} |w-z|^2/\tilde{\varphi}_N(z,w)$ *has a uniform bound for some open neighborhood* D'' *of* $\overline{D'}$ *and* ρ *stays in a bounded set in* $C^{N+1}(D, \mathbb{R}) \cap C^0(\overline{D})$ *such that* $\sup_{w \in \overline{D}} \rho(w)^{-1}$ *has a uniform bound.*

Proof. Given any point $z_0 \in D'$ we apply Theorem 2.55 and find

$$A_1 := \left| \int_D f(w)\overline{(w-z)}^\beta e^{k\overline{\gamma_N(z,w)}} \chi(w) e^{-k\varphi(w)} dV_D(w) \right| \leq Ck^{-\frac{N+1+n-\varepsilon}{2}} \|f\|_k$$

for some cutoff function χ with $\operatorname{supp} \chi \subset D''$, $\chi \equiv 1$ on some open neighborhood $U \subset D''$ of z_0 for all $z \in U$. Setting

$$A_2 := \left| \int_D f(w)\overline{(w-z)}^\beta e^{k\overline{\gamma_N(z,w)}} (1-\chi(w)) e^{-k\varphi(w)} dV_D(w) \right|$$

we have that the left-hand side of (2.7) can be estimated by $A_1 + A_2$. So we just need to show that A_2 will decrease fast enough when k goes to infinity. Cauchy-Schwarz inequality and the assumptions on φ and ρ lead to

$$\begin{aligned} A_2^2 &\leq \left| \int_D |\overline{(w-z)}^\beta (1-\chi(w))|^2 e^{-k\tilde{\varphi}_N(z,w)} dV_D(w) \right| \|f\|_k^2 \\ &\leq C_1 e^{-\delta k} \|f\|_k^2 \end{aligned}$$

where $\delta, C_1 > 0$ only depend on $\sup_{(z,w) \in D'' \times \overline{D}} |w-z|^2/\tilde{\varphi}_N(z,w)$ and $\sup_{w \in \overline{D}} \rho(w)^{-1}$. Since we can cover D' by a finite number of such sets U the claim follows. $\square$

Remark 2.57
Theorem 2.55 and Corollary 2.56 stay true for a fixed weight $\varphi \in C^{N+3}(D,\mathbb{R})$. We need to increase the regularity by one in order to get the uniformity in φ. The reason for that can be found in Proposition 2.59, which is also true for a fixed weight $\varphi \in C^{N+3}(D,\mathbb{R})$ but needs the higher regularity for choosing the neighborhoods independent of φ.

Sketch of the proof. Before we begin with the proof of Theorem 2.55, we would like to outline the idea first by using a simple example. Consider the case $n = 1$, $D = \mathbb{D}$, $z_0 = 0$, $\rho \equiv 1$ and

$$\varphi(z) := |z|^2 + \mu_a(z),\ \ \mu_a(z) := a(z^2\overline{z} + \overline{z}^2 z) + z^3\overline{z}^2 + \overline{z}^3 z^2 + |z|^{N+3}$$

for $a \in \mathbb{R}$. We immediately find that $\gamma_N(0,w) \equiv 0$ and hence that $\tilde{\varphi}_N(0,z) = \varphi(z)$. First, we choose a disk $\mathbb{D}_{6\tau}$ with radius $0 < 6\tau < 1$ around 0 such that $\varphi(z) \geq C_0|z|^2$ holds on $\mathbb{D}_{6\tau}$ for some constant $C_0 > 0$. Then choose a cutoff function $\chi \in C_0^\infty(\mathbb{D}_{2\tau},\mathbb{R})$ with $\chi \equiv 1$ on some open neighborhood $U \subset\subset \mathbb{D}_{2\tau}$ around 0. Given a holomorphic function $f \in \mathcal{O}(\mathbb{D})$ which vanishes up to order N in $z_0 = 0$ we can write $f(z) = z^{N+1}g(z)$ for some holomorphic function $g \in \mathcal{O}(\mathbb{D})$. We then use integration by parts and get

$$\begin{aligned}\int_{\mathbb{D}} \chi(z)f(z)e^{-k\varphi(z)}dV_{\mathbb{C}} &= (-k)^{-N-1}\int_{\mathbb{D}} \chi(z)g(z)e^{k\mu_a(z)}\left(\overline{\partial}^{N+1}e^{-k|z|^2}\right)dV_{\mathbb{C}} \\ &= k^{-N-1}\int_{\mathbb{D}} e^{-k\mu_a(z)}\left(\overline{\partial}^{N+1}\chi(z)e^{k\mu_a(z)}\right)g(z)e^{-k\varphi(z)}dV_{\mathbb{C}}.\end{aligned}$$

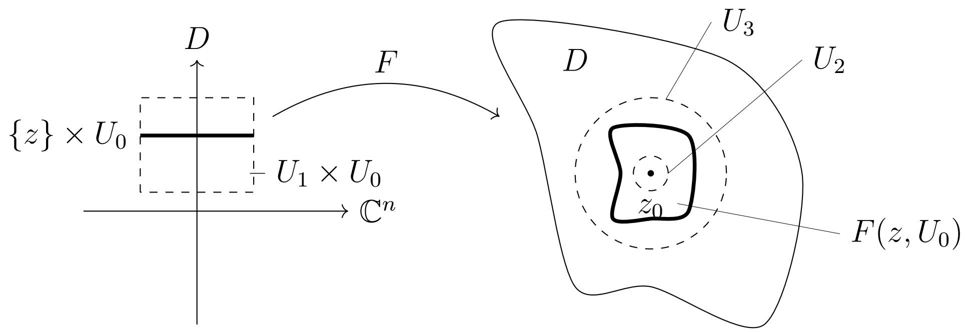
By Lemma 2.30 we have $\sup_{z\in\mathbb{D}_{2\tau}} |g(z)|^2e^{-\frac{3}{2}\varphi(z)} \leq C_1k^{N+1+n+\varepsilon/2}\|f\|_k^2$ for some constant $C_1 > 0$ independent of k and g or f respectively. It follows

$$\left|\int_{\mathbb{D}} \chi(z)f(z)e^{-k\varphi(z)}dV_{\mathbb{C}}\right| \ \leq\ C_1k^{-\frac{N+1-n}{2}+\frac{\varepsilon}{4}}\|f\|_k\left|\int_{\mathbb{D}} G_k(z)e^{-k\frac{1}{4}\varphi(z)}dV_{\mathbb{C}}\right|$$

with $G_k(z) = e^{-k\mu_a(z)}\overline{\partial}^{N+1}\left(\chi(z)e^{k\mu_a(z)}\right)$. We write the integral on the right-hand side as $\int_{\mathbb{D}} G_k(z)e^{-k\frac{1}{4}\varphi(z)}dV_{\mathbb{C}} = A_{1,k} + A_{2,k}$ with

$$A_{1,k} = \int_{\mathbb{D}} G_k(z)\xi(|z|^2k^{1-\varepsilon'})e^{-k\frac{1}{4}\varphi(z)}dV_{\mathbb{C}},\ \ A_{2,k} = \int_{\mathbb{D}} G_k(z)(1-\xi(|z|^2k^{1-\varepsilon'}))e^{-k\frac{1}{4}\varphi(z)}dV_{\mathbb{C}}$$

where $\xi \in C_0^\infty((-2\delta,2\delta),\mathbb{R})$, $0 \leq \xi \leq 1$ and $\xi \equiv 1$ on $(-\delta,\delta)$ is a cutoff function with $\delta = 2\tau^2$ and $\varepsilon' = \varepsilon/8(N+1)$. We find $|A_{2,k}| \leq C_2k^{N+1}e^{-\frac{\delta C_0}{4}k^{\varepsilon'}}$. Using Example 2.45 we have $\int_{\mathbb{D}_{2\tau}} e^{-k\frac{1}{4}\varphi(z)}dV_{\mathbb{C}} \leq C_3k^{-n}$. Then a direct calculation shows that $|A_{1,k}| \leq C_4k^{-n+2\varepsilon'(N+1)}$ assuming $a = 0$. For the general case, that is $a \neq 0$, we have to change the coordinates in order to eliminate all the $z^\alpha\overline{z}^\beta$ terms with $\min\{|\alpha|,|\beta|\} = 1$ in the Taylor expansion of $\tilde{\varphi}_N(z_0,\cdot)$ up to order $N+2$. Those coordinates are provided in Proposition 2.59. A careful analysis of $\xi(|z|^2k^{1-\varepsilon'})G_k(z)$



One has that F maps the subset $U_1 \times U_0 \subset D \times \mathbb{C}^n$ to D such that its restriction to any $\{z\} \times U_0$ is biholomorhpic. Furthermore, the U_2 is always contained in $F(z, U_0)$ and $F(z, U_0)$ is always contained in U_3 for all $z \in U_1$. Note that the left-hand side of the picture represents a subset of $\mathbb{C}^{2n}$ while the right-hand side represents a subset of $\mathbb{C}^n$.

Figure 2.2: Illustration of F in Proposition 2.59

using these coordinates is done in Lemma 2.63. So let us assume $a = 0$. Putting all together we get

$$\begin{aligned}\left|\int_{\mathbb{D}} \chi(z) f(z) e^{-k\varphi(z)} dV_{\mathbb{C}}\right| &\leq C_1 k^{-\frac{N+1-n}{2}+\frac{\varepsilon}{4}} \|f\|_k (C_4 k^{-n+2\varepsilon'(N+1)} + C_2 k^{N+1} e^{-\delta C_0 k^{\varepsilon'}}) \\ &\leq C k^{-\frac{N+1+n-\varepsilon}{2}} \|f\|_k\end{aligned}$$

for some constant $C > 0$ independent of k and f.
Given $n > 1$ we cannot apply Lemma 2.30 directly because in general the holomorphic functions, which vanish up to order N in z_0, do not have the form $z^\alpha g(z)$ in that case. We overcome this difficulty by introducing the meaning of a splitting decomposition (see Definition 2.60) and by modifying Lemma 2.30 (see Lemma 2.62). Moreover, here we just consider the case where z_0, ρ and φ are fixed. To prove the general statement we also need to show that the constant $C > 0$ can be chosen independent of z_0, ρ and φ in some suitable sets.

Remark 2.58
It should be mentioned that in the case $n = 1$ a much simpler proof of Theorem 2.55 could be given. But this simpler method does not generalize to higher dimensions since the zero set of $z \to z^\alpha$, $|\alpha| > 0$, fails to be compact in that case.

For the proof of Theorem 2.55 we need to change coordinates at some point in order to show that some error terms become small. We prove the existence of those so called Kähler coordinates (see [6], [26]) in Proposition 2.59. We have to prove that in some sense those coordinates can be chosen uniformly in φ when φ stays in some bounded set in $C^{N+4}(D)$ (see Figure 2.2 for a visualization).

Proposition 2.59
For $\varphi \in C^{N+4}(D,\mathbb{R})$ and any $z_0 \in D_{\varphi,+}$ and any open neighborhood $U_3 \subset D$ around z_0 there exist an open neighborhood $U_0 \subset \mathbb{C}^n$ around 0, open neighborhoods $U_1, U_2 \subset D_{\varphi,+}$ around z_0 and C^2-map $F = F_\varphi : U_1 \times U_0 \to D$ such that $F(z,0) = z$, $F(z,\cdot) : U_0 \to F(z,U_0)$ is biholomorphic, $U_2 \subset\subset F(z,U_0) \subset\subset U_3$ for all $z \in U_1$ and

$$\tilde{\varphi}(z, F(z,w)) = |w|^2 - \sum_{\substack{|\alpha|,|\beta|\geq 2\\ |\alpha|,|\beta|\leq N+2}} c_{\alpha,\beta}(z) w^\alpha \overline{w}^\beta - \eta(z,w)$$

with $|\eta(z,w)| \leq C|w|^{N+3}$ for some constant $C > 0$ independent of z.
If φ stays in a bounded set $A \subset C^{N+4}(D,\mathbb{R})$ such that $H_\varphi(z) - C' \,\mathrm{Id}$ is positive definite for all $\varphi \in A$ and for all z in an open neighborhood D' of z_0 the sets U_0, U_1, U_2 with $U_1 \subset D'$ and the constant C can be chosen independent of φ. Furthermore, $\{\eta(z,\cdot)\}_{z \in U_1}$ stays in a bounded set in $C^{N+4}(D,\mathbb{C})$ and $\{c_{\alpha,\beta}(z)\}_{z\in U_1}$ stays in a bounded set in $\mathbb{C}$ for all $\alpha, \beta \in \mathbb{N}^n$ with $2 \leq |\alpha|, |\beta| \leq N+2$.

Proof. Let $H_\varphi(z) = (\frac{\partial^2 \varphi(z)}{\partial z_j \partial \overline{z}_l})_{1\leq l,j\leq n}$ be the complex Hessian of φ in $z \in D$. We have that $S_\varphi(z) := (\sqrt{H_\varphi(z)})^{-1}$ is well defined and of class C^{N+2} on D' since we can always assume that locally $\frac{t}{2} \leq v^T H(z) v \leq \frac{3t}{2}$ for all $v \in \mathbb{C}^n$ of unit length and all $\varphi \in A$ for some $t > 0$ to get, by using the Taylor expansion of $x \mapsto \sqrt{t + (x-t)}$ and Cramer's rule, that S_φ is well defined and of class C^{N+2}. Choose open neighborhoods $U' \subset\subset D'$ around z_0 and $U'' \subset \mathbb{C}^n$ around 0 such that the smooth map $F_{\varphi,0} : U' \times U'' \to D$, $F_{\varphi,0}(z,w) = S_\varphi(z) w + z$ is well defined for all $\varphi \in A$. Then, by a Taylor expansion of $\tilde{\varphi}_N(z, F_{\varphi,0}(z,\cdot))$ we can write

$$\tilde{\varphi}_N(z, F_{\varphi,0}(z,w)) = |w|^2 + \eta_1(z,w) + \overline{\eta_1(z,w)} - \eta_0(z,w) - \eta(z,w)$$

$$\eta_1(z,w) = \sum_{j=1}^n \sum_{2\leq|\alpha|\leq N+1} \overline{w}_j b_{j,\alpha}(z) w^\alpha \ , \ \eta_0(z,w) = \sum_{\substack{|\alpha|+|\beta|\geq 3\\ |\alpha|,|\beta|\leq N+2}} c_{\alpha,\beta}(z) w^\alpha \overline{w}^\beta$$

where $b_{j,\alpha}$ and $c_{\alpha,\beta}$ are C^2-functions on U' such that $c_{\alpha,\beta}(z) = 0$ if $|\alpha| \leq 1$ or $|\beta| \leq 1$ and $\eta(z,w) \in O(|w|^{N+3})$. Note that $b_{j,\alpha}, c_{\alpha,\beta}$ and η also depend on φ. Throughout this proof by saying $\eta(z,w) = O(|w|^{N+3})$ we mean $|\eta(z,w)| \leq C|w|^{N+3}$ for some constant C independent of $\varphi \in A$ and (z,w) in the domain of definition of η.
Define a map $\tilde{G}_\varphi : U' \times U'' \to U' \times \mathbb{C}^n$ by $\tilde{G}_\varphi(z,w) = (z, G_\varphi(z,w))$ where

$$G_\varphi(z,w) = \left(w_j + \sum_{2\leq|\alpha|\leq N+1} b_{j,\alpha}(z) w^\alpha \right)_{1\leq j\leq n} .$$

The real Jacobi matrix of $\tilde{G}_\varphi$ in $(z_0, 0)$ is the identity map for all $\varphi \in A$ and hence invertible. Since $A \subset C^{N+4}(D,\mathbb{R})$ is bounded we have that $\{\tilde{G}_\varphi\}_{\varphi\in A}$ is bounded in the C^2-norm. Thus, we can shrink U' and U'' independent of $\varphi \in A$ to ensure that $\tilde{G}_\varphi$ is a C^2-diffeomorphism on its image for all $\varphi \in A$. Furthermore, we find that for any $z \in U'$ the map $G_\varphi(z, \cdot)$ has an invertible differential, is holomorphic and injective and hence biholomorphic on its image (see Lemma 2.19). Since $\{\tilde{G}_\varphi\}_{\varphi\in A}$ is bounded in the C^2-norm we can find open neighborhoods $V' \subset U'$ around z_0 and $V'' \subset \mathbb{C}^n$ around 0 independent of $\varphi \in A$ such that $V' \times V''$ is contained in the image of $\tilde{G}_\varphi$ for all $\varphi \in A$. Denote the restriction of the inverse map of $\tilde{G}_\varphi : U' \times U'' \to \tilde{G}_\varphi(U' \times U'')$ to $V' \times V''$ by $\tilde{G}_\varphi^{-1}$. We have that $\tilde{G}_\varphi^{-1}$ can be written as $\tilde{G}^{-1}(z, w) = (z, F_{\varphi,1}(z, w))$ where $F_{\varphi,1} : V' \times V'' \to U''$ is a C^2-map such that $F_{\varphi,1}(z, 0) = 0$ and $F_{\varphi,1}(z, \cdot)$ is biholomorphic on its image for any $z \in V'$. We find

$$\tilde{\varphi}_N(z, F_{\varphi,0}(z, w)) = |G_\varphi(z, w)|^2 - \eta_0'(z, w) - \eta'(z, w)$$

for functions η', η_0' depending on φ with $\eta'(z, w) \in O(|w|^{N+3})$ and

$$\eta_0'(z, w) = \sum_{\substack{|\alpha|+|\beta|\geq 3 \\ |\alpha|,|\beta|\leq N+2}} c'_{\alpha,\beta}(z) w^\alpha \overline{w}^\beta \tag{2.8}$$

where $c'_{\alpha,\beta}$ are C^2-functions on U' such that $c'_{\alpha,\beta}(z) = 0$ if $|\alpha| \leq 1$ or $|\beta| \leq 1$. Furthermore, we observe that $\eta_0'(z, F_{\varphi,1}(z, w)) + \eta'(z, F_{\varphi,1}(z, w)) = \tilde{\eta}_0(z, w) + \tilde{\eta}(z, w)$ where $\tilde{\eta}, \tilde{\eta}_0$ are C^2-functions such that $\tilde{\eta}(z, w) = O(|w|^{N+3})$ and $\tilde{\eta}_0$ can be written in the form (2.8). Define $\tilde{F}_\varphi : V' \times V'' \to V' \times D$ by $\tilde{F}_\varphi(z, w) = (z, F_\varphi(z, w))$ where $F_\varphi(z, w) = F_{\varphi,0}(z, F_{\varphi,1}(z, w))$. We have that $\tilde{F}_\varphi$ is a diffeomorphism on its image and that $F_\varphi(z, 0) = z$ and $F_\varphi(z, \cdot)$ is biholomorphic on its image for any $z \in V'$ as the composition of the maps $(z, w) \mapsto (z, F_{\varphi,0}(z, w))$ and $\tilde{G}_\varphi^{-1}$. Let $U_3' \subset\subset U_3$ be an open neighborhood around z_0. Since $\tilde{F}_\varphi(z_0, 0) = (z_0, z_0)$ and the properties of $\{\tilde{G}_\varphi\}_{\varphi\in A}$ we find an open subset of the form $W' \times W''$ where $W' \subset V'$ is an open neighborhood around z_0 and $W'' \subset V''$ is an open neighborhood around 0 such that $W' \times W'' \subset \tilde{F}_\varphi^{-1}(V' \times U_3')$ holds for all $\varphi \in A$. Using similar arguments we find open neighborhoods $U_1 \subset W'$ and $U_2' \subset U_3' \cap D'$ around z_0 independent of $\varphi \in A$ satisfying $\tilde{F}_\varphi^{-1}(U_1 \times U_2') \subset W' \times W''$ for all $\varphi \in A$. Now set $U_0 = W''$ and restrict F_φ to $U_1 \times U_0$. We have that $F_\varphi(z, 0) = z$ and $F_\varphi(z, \cdot)$ is biholomorphic on its image for all $z \in U_1$. Furthermore,

$$\tilde{\varphi}(z, F_\varphi(z, w)) = |w|^2 - \tilde{\eta}_0(z, w) - \tilde{\eta}(z, w)$$

where $\tilde{\eta}(z, w) \in O(|w|^{N+3})$ and

$$\tilde{\eta}_0(z, w) = \sum_{\substack{|\alpha|+|\beta|\geq 3 \\ |\alpha|,|\beta|\leq N+2}} \tilde{c}_{\alpha,\beta}(z) w^\alpha \overline{w}^\beta$$

for C^2-functions $\tilde{c}_{\alpha,\beta}$ defined on U_1 such that $\tilde{c}_{\alpha,\beta}(z) = 0$ if $|\alpha| \leq 1$ or $|\beta| \leq 1$. Using standard relations between the derivatives of a map and its inverse (see

Theorem 2.6) and the the properties of A we find that $\{F_{\varphi,1}(z,\cdot)\}_{z\in U_1,\varphi\in A}$ and hence $\{F_\varphi(z,\cdot)\}_{z\in U_1,\varphi\in A}$ is bounded in any C^l-norm. As a conclusion we get that $\{\tilde\eta(z,\cdot)\}_{z\in U_1,\varphi\in A}$ is bounded in $C^{N+4}(D,\mathbb{C})$ and $\{\tilde c_{\alpha,\beta}(z)\}_{z\in U_1,\varphi\in A}$ is bounded in $\mathbb{C}$. Since $\tilde F_\varphi(U_1\times U_0)\subset V'\times U_3'$ we find that $F_\varphi(z,U_0)\subset U_3'\subset\subset U_3$ for all $z\in U_1$ and $\varphi\in A$. Given $z\in U_1$ and $w\in U_2'$ we can find a point $(z',w')\in U_1\times U_0$ such that $\tilde F_\varphi(z',w')=(z,w)$. By the construction of $\tilde F_\varphi$ we have $z=z'$ and hence $w\in F_\varphi(z,U_0)$. This proves that $U_2'\subset F_\varphi(z,U_0)$ holds for all $z\in U_1$ and $\varphi\in A$. Taking an open neighborhood $U_2\subset\subset U_2'$ around z_0 finishes the proof. $\square$

Let $U\subset\mathbb{C}^n$ be a domain and assume that U contains the closure of a polydisc $\mathbb{D}^n_\delta$ of radius δ around 0.

Definition 2.60

Let f be a holomorphic function on U which vanishes up to order $N\in\mathbb{N}_0$ in 0. A (local) decomposition $f(w)=\sum_{|\alpha|=N+1}w^\alpha f_\alpha(w)$ where $f_\alpha\in\mathcal{O}(\mathbb{D}^n_\delta)$, $f_\alpha(w)=\sum_\beta a^{(\alpha)}_\beta w^\beta$, $|\alpha|=N+1$, converges absolutely on $\overline{\mathbb{D}^n_\delta}$ is called a splitting decomposition (of f) if for all $\alpha,\beta\in\mathbb{N}_0^n$, $|\alpha|=N+1$, the following holds: $a^{(\alpha)}_\beta\neq 0$ implies $a^{(\alpha')}_{\beta'}=0$ for all α',β' satisfying $\alpha'+\beta'=\alpha+\beta$, $\alpha\neq\alpha'$, $\beta\neq\beta'$.

Roughly speaking this means that a term w^β cannot be contained in the power series expansion of $w^\alpha f_\alpha(w)$ and $w^{\alpha'}f_{\alpha'}(w)$, $\alpha\neq\alpha'$ at the same time.

Lemma 2.61

Any $f\in\mathcal{O}(U)$ admits a splitting decomposition. Furthermore, given a splitting decomposition $f(w)=\sum_{|\alpha|=N+1}w^\alpha f_\alpha(w)$ and a positive continuous function $\rho:[0,\delta]^n\to\mathbb{R}_+$ one has

$$\int_{\mathbb{D}^n_\tau} w^\alpha f_\alpha(w)\overline{w^\beta f_\beta(w)}\rho(|w_1|^2,\ldots,|w_2|^2)dV_{\mathbb{C}^n}=0$$

for all $\alpha\neq\beta$, $\tau\leq\delta$.

Proof. Assume that f vanishes up to order $N\in\mathbb{N}_0$ in 0. By assumption U contains the closure of a polydisc $\mathbb{D}^n_\tau$ for some $\tau>\delta$ around 0. Thus, we can write $f(w)=\sum_{|\alpha|\geq N+1}a_\alpha w^\alpha$ where the sum on the right hand side converges absolutely on the closure of $\mathbb{D}^n_\delta$. Let $\alpha(1),\alpha(2),\ldots,\alpha(d_{N+1})$ be an enumeration of the elements in $\{\alpha\in\mathbb{N}^n \mid |\alpha|=N+1\}$. For $w\in\mathbb{D}^n_\delta$ define

$$\tilde f_1(w)=\sum_{\substack{|\alpha|\geq N+1\\ \alpha-\alpha(1)\in\mathbb{N}_0^n}} a_\alpha w^\alpha$$

where $\alpha-\alpha(1)\in\mathbb{N}_0^n$ means that $\alpha_j\geq\alpha(1)_j$ holds for all $1\leq j\leq n$. We have that $\tilde f_1$ is a holomorphic function on $\mathbb{D}^n_\delta$ and that its power series converges absolutely on $\overline{\mathbb{D}^n_\delta}$. Take $\alpha(2)$ and do the same construction for $f - \tilde{f}_1$ to obtain $\tilde{f}_2$ and so on. One has that $f_l(w) := \tilde{f}_l(w)/w^{\alpha(l)}$, $1 \le l \le d_{N+1}$, defines a holomorphic function on $\mathbb{D}^n_\delta$. Since for any $|\alpha| \ge N+1$ there exists at least one $1 \le l \le d_{N+1}$ such that $\alpha - \alpha^{(l)} \in \mathbb{N}^n_0$ and by construction we get that $f(w) = \sum_{l=1}^{d_{N+1}} w^{\alpha(l)} f_l(w)$ is a splitting decomposition. To prove the second part of the statement we observe that for $m \in \mathbb{Z} \setminus \{0\}$ one has $\int_0^{2\pi} e^{im\theta} d\theta = 0$. This implies that for $\alpha, \beta \in \mathbb{N}^n_0$, $\alpha \neq \beta$, one gets

$$\begin{aligned} 0 &= \prod_{j=1}^n \left(\int_0^{2\pi} e^{i(\alpha_j - \beta_j)\theta} d\theta \right) \cdot \int_{[0,\tau]^n} \prod_{j=1}^n r_j^{\alpha_j + \beta_j + 1} \rho(r_1^2, \ldots, r_n^2) dr_1 \ldots dr_n \\ &= \int_{\mathbb{D}^n_\tau} w^\alpha \overline{w^\beta} \rho(|w_1|^2, \ldots, |w_2|^2) dV_{\mathbb{C}^n} \end{aligned}$$

for any $\tau \in \mathbb{R}_+$. Consider a splitting decomposition of f, i.e. $f(w) = \sum_{|\alpha|=N+1} w^\alpha f_\alpha(w)$ and $w^\alpha f_\alpha(w) = \sum_{|\beta| \ge N+1} a^{(\alpha)}_\beta w^\beta$ converges absolutely, $|\alpha| = N+1$. By the properties of a splitting decomposition we have that

$$w^\alpha f_\alpha(w) \overline{w^\beta f_\beta(w)} = \sum_{\alpha', \beta' \ge N+1} a^{(\alpha)}_{\alpha'} \overline{a^{(\beta)}_{\beta'}} w^{\alpha'} \overline{w^{\beta'}} = \sum_{\substack{\alpha', \beta' \ge N+1 \\ \alpha' \neq \beta'}} a^{(\alpha)}_{\alpha'} \overline{a^{(\beta)}_{\beta'}} w^{\alpha'} \overline{w^{\beta'}}$$

is a power series which converges absolutely on $\overline{\mathbb{D}^n_\delta}$. Thus, for $\tau \le \delta$ we find

$$\begin{aligned} &\int_{\mathbb{D}^n_\tau} w^\alpha f_\alpha(w) \overline{w^\beta f_\beta(w)} \rho(|w_1|^2, \ldots, |w_2|^2) dV_{\mathbb{C}^n} \\ &= \sum_{\substack{\alpha', \beta' \ge N+1 \\ \alpha' \neq \beta'}} a^{(\alpha)}_{\alpha'} \overline{a^{(\beta)}_{\beta'}} \int_{\mathbb{D}^n_\tau} w^{\alpha'} \overline{w^{\beta'}} \rho(|w_1|^2, \ldots, |w_2|^2) dV_{\mathbb{C}^n} \\ &= 0. \end{aligned}$$

□

Lemma 2.62
Let $S_1 \subset C^{N+4}(D, \mathbb{R})$ and $S_2 \subset C^0(D, \mathbb{R})$ be bounded sets, $D' \subset\subset D$ open and $C_1', C_2' > 0$ two constants such that $H_\varphi(z) - C_1' \operatorname{Id}$ is positive definite and $\rho(z) > C_2'$ for all $\varphi \in S_1$, all $\rho \in S_2$ and all $z \in D'$. Choose U_j, $0 \le j \le 3$, and F as in Proposition 2.59. Assume that U_0 contains the closure of a polydisc $\mathbb{D}^n_{3\delta_0}$ of radius $3\delta_0$ around 0 for some $\delta_0 > 0$. One can find an open neighborhood $U \subset U_1$ around z_0 such that the following holds: For any $\varepsilon > 0$ there exists a constant $C > 0$ such that for all $k \in [1, \infty)$, all $\varphi \in S_1, \rho \in S_2$, all $z \in U$ and any $f \in H^0_k(D)$, which vanishes up to order N in z, and any splitting decomposition of $f \circ F(z, \cdot) e^{-k\gamma(z, F(z, \cdot))}$,

$$f(F(z,w)) e^{-k\gamma(z, F(z,w))} = \sum_{|\alpha|=N+1} w^\alpha g_{\alpha,z,k}(w) \ , \ g_{\alpha,z,k} \in \mathcal{O}(\mathbb{D}^n_{3\delta_0})$$

one has

$$\sum_{|\alpha|=N+1} \sup_{w \in \mathbb{D}^n_{\delta_0}} |g_{\alpha,z,k}(w)|^2 e^{-\frac{3k}{2}\tilde{\varphi}(z, F(z,w))} \le C k^{N+n+1+\varepsilon} \|f\|^2_k.$$

Proof. We can write $\tilde{\varphi}(z, F(z,w)) = |w|^2 + R(z,w)$ where $|R(z,w)| \leq C_0|w|^3$ for some constant $C_0 > 0$ independent of $z \in U_1$ and $\varphi \in S_1$. Thus, we can find $\delta > 0$ such that $|R(z,w)/|w|^2| \leq 1/5$ on $U_1 \times \mathbb{D}^n_{3\delta}$. Setting $C_1 = 6/5$ we conclude that

$$\tilde{\varphi}(z, F(z,w)) \leq C_1|w|^2 \leq \frac{3}{2}\tilde{\varphi}(z, F(z,w))$$

holds for all $w \in \mathbb{D}^n_{3\delta}$, $z \in U$ and $\varphi \in S_1$. Applying Lemma 2.30 for $\varphi(w) = C_1|w|^2$ and $\rho \equiv 1$ we find a constant $C_2 > 0$ independent of $g_{\alpha,z,k}$ and k such that

$$k^{-(N+1+n+\varepsilon)} \sup_{w \in \mathbb{D}^n_\delta} |g_{\alpha,z,k}(w)|^2 e^{-kC_1|w|^2} \leq \frac{C_2}{n} \int_{\mathbb{D}^n_{2\delta}} |w^\alpha g_{\alpha,z,k}(w)|^2 e^{-kC_1|w|^2} dV_{\mathbb{C}^n}(w)$$

holds. By Lemma 2.61 we have

$$\begin{aligned} &\int_{\mathbb{D}^n_{2\delta}} \left| \sum_{|\alpha|=N+1} w^\alpha g_{\alpha,z,k}(w) \right|^2 e^{-kC_1|w|^2} dV_{\mathbb{C}^n}(w) \\ &\qquad = \sum_{|\alpha|=N+1} \int_{\mathbb{D}^n_{2\delta}} |w^\alpha g_{\alpha,z,k}(w)|^2 e^{-kC_1|w|^2} dV_{\mathbb{C}^n}(w). \end{aligned}$$

Write $dV_{z,U_0}(w) := F^*(z,w)(\rho(w)dV_{\mathbb{C}^n}) = \rho(z,w)dV_{\mathbb{C}^n}$ for $\rho \in S_2$. There is a constant $C_3 > 0$ independent of $\rho \in S_2$ such that $C_3\rho(z,w) > 1$ holds for all $(z,w) \in U \times \mathbb{D}^n_{2\delta}$. Thus, one finds

$$\begin{aligned} &k^{-(N+1+n+\varepsilon)} \sum_{|\alpha|=N+1} \sup_{w \in \mathbb{D}^n_\delta} |g_{\alpha,z,k}(w)|^2 e^{-\frac{3k}{2}\tilde{\varphi}(z,F(z,w))} \\ &\leq C_2 \int_{\mathbb{D}^n_{2\delta}} |f(F(z,w)) e^{-k\gamma(z,F(z,w))}|^2 e^{-kC_1|w|^2} dV_{\mathbb{C}^n}(w) \\ &\leq C_2 \int_{\mathbb{D}^n_{2\delta}} |f(F(z,w))|^2 e^{-k(\tilde{\varphi}(z,F(z,w)) + \gamma(z,F(z,w)) + \overline{\gamma(z,F(z,w))})} dV_{\mathbb{C}^n}(w) \\ &\leq C_2C_3 \int_{\mathbb{D}^n_{2\delta}} |f(F(z,w))|^2 e^{-k\varphi(F(z,w))} dV_{z,U_0}(w) \\ &= C_2C_3 \int_{F(z,\mathbb{D}^n_{2\delta})} |f(w)|^2 e^{-k\varphi(w)} dV_D(w) \leq C_2C_3\|f\|_k^2. \end{aligned}$$

□

The following lemma is crucial for the proof of Theorem 2.55.

Lemma 2.63
Let $U_0 \subset \mathbb{C}^n$ be an open neighborhood around 0 and $N \in \mathbb{N}_0$ be a non-negative integer. Furthermore, let $R, S \subset C^{N+3}(U_0)$ be bounded sets with $\operatorname{supp} s \subset U_0$ *for all $s \in S$ and that there is a constant $C_0 > 0$ such that $|\eta(w)| \leq C_0|w|^{N+3}$ holds for all $\eta \in R$ and $w \in U_0$. Let $\xi \in C^\infty_0(\mathbb{R}, \mathbb{R})$ be a cutoff function $0 \leq \xi \leq 1$,*

$\operatorname{supp}(\xi) \subset (-2\delta, 2\delta) \subset (-1,1)$, $\xi \equiv 1$ on $(-\delta,\delta)$ for some $\delta > 0$ such that U_0 contains the closure of a ball of radius 3δ around 0. Set

$$R_0 = \left\{ w \mapsto \sum_{\substack{|\alpha|,|\beta|\geq 2\\ |\alpha|,|\beta|\leq N+2}} c_{\alpha,\beta} w^\alpha \overline{w}^\beta \,\middle|\, c_{\alpha,\beta} \in A \right\} \subset C^\infty(U_0)$$

where $A \subset \mathbb{C}$ is some bounded set. For $\eta \in R$, $\eta_0 \in R_0$ and $s \in S$ set

$$G_{\alpha,k}(w) = e^{-k(\eta_0(w)+\eta(w))} \overline{\partial}_w^\alpha \left(s(w) e^{k(\eta_0(w)+\eta(w))} \right).$$

There exists a constant $C > 0$ such that for all $\alpha \in \mathbb{N}_0^n$, $|\alpha| = N+1$, all $k \in [1,\infty)$ and all $\eta \in R$, $\eta_0 \in R_0$, $s \in S$ one has

$$\sup_{w \in U_0} |\xi(k^{1-\varepsilon}|w|^2) G_{\alpha,k}(w)| \leq C k^{\varepsilon 2(N+1)}.$$

Proof. First we notice that

$$\sup_{|\alpha|\leq N} \sup_{w \in \mathbb{D}_{3\delta}} |\overline{\partial}_w^\alpha s(w)| \leq C_1$$

for some constant $C_1 > 0$ independent of k and $s \in S$. Next, we find for $|\alpha| \leq N+1$ that $|\overline{\partial}_w^\alpha \eta(w)| \leq C_2 |w|^{N+3-|\alpha|}$ for some constant $C_2 > 0$ independent of $w \in \mathbb{D}_{3\delta}$, $\eta \in R$ and α, $|\alpha| \leq N+1$. Set $\xi_k(w) = \xi(k^{1-\varepsilon}|w|^2)$. Since $\operatorname{supp}(\xi) \subset (-2\delta, 2\delta)$, i.e. $\xi(k^{1-\varepsilon}|w|^2) = 0$ for $|w|^2 \geq 2\delta k^{\varepsilon-1}$, we conclude

$$|\overline{\partial}_w^\alpha \eta(w)| \leq C_2 |w|^{N+3-|\alpha|} \leq C_3 k^{-(1-\varepsilon)(N+3-|\alpha|)}$$

for all $w \in \operatorname{supp} \xi_k$, $\eta \in R$ and α, $|\alpha| \leq N+1$, where we choose $C_3 > 0$ such that $C_2 (2\delta)^{N+3-|\alpha|} \leq C_3$ for all $|\alpha| \leq N+1$. One has

$$\begin{aligned} k^m \Big| \prod_{j=1}^m \overline{\partial}_w^{\alpha(j)} \eta(w) \Big| &\leq C_3^m k^{-\frac{m}{2}(1-\varepsilon)(N+3)} k^{\frac{1}{2}|\alpha|(1-\varepsilon)+m} \\ &\leq (1+C_3)^{N+1} k^\varepsilon k^{-\frac{m-1}{2}(N-1)(1-\varepsilon)} \leq C_4 k^\varepsilon \end{aligned}$$

for all $\alpha(1), \ldots, \alpha(m) \in \mathbb{N}_0^n$, $1 \leq m \leq N+1$, $|\sum_{j=1}^m \alpha(j)| \leq N+1$. Thus, there exists C_5 such that

$$\left| e^{-k\eta(w)} \overline{\partial}_w^\alpha e^{k\eta(w)} \right| \leq C_5 k^{(N+1)\varepsilon}$$

for all $\eta \in R$, $w \in \operatorname{supp} \xi_k$, $|\alpha| \leq N+1$ and $k \in [1,\infty)$. For $\alpha', \beta' \in \mathbb{N}_0^n$ such that $|\alpha'|, |\beta'| \geq 2$ one has

$$|\overline{\partial}_w^\alpha w^{\beta'} \overline{w}^{\alpha'}| \leq |w|^{|\beta'|} \leq |w|^2.$$

Thus, we find a constant $C_6 > 0$ such that for all $\eta_0 \in R_0$, $|\alpha| \le N+1$, $k \in [1,\infty)$ and $w \in \operatorname{supp}(\xi_k)$ one has $|\overline{\partial}_w^\alpha \eta_0(w)| \le C_6 k^{-1+\varepsilon}$ and hence there exists a constant $C_7 > 0$ independent of $\eta_0 \in R_0$, $k \in [1,\infty)$ and $w \in \operatorname{supp}(\xi_k)$ such that

$$k^m |\prod_{j=1}^m \overline{\partial}_w^{\alpha(j)} \eta_0(w)| \le C_7 k^{(N+1)\varepsilon}$$

holds for all $\alpha(1), \ldots, \alpha(m) \in \mathbb{N}_0^n$, $1 \le m \le N+1$, $|\sum_{j=1}^m \alpha(j)| \le N+1$. We conclude that there is a constant $C_8 > 0$ such that for any $\eta_0 \in R_0$, $k \in [1,\infty)$, $w \in \operatorname{supp}(\xi_k)$ and $|\alpha| \le N+1$ one has

$$\left| e^{-k\eta_0(w)} \overline{\partial}_w^\alpha e^{k\eta_0(w)} \right| \le C_8 k^{(N+1)\varepsilon}.$$

As a conclusion we find a constant $C > 0$ such that $|G_{\alpha,k}(w)| \le C k^{2\varepsilon(N+1)}$ for all $\eta \in R$, $\eta_0 \in R_0$, $s \in S$, $k \in [1,\infty)$, $w \in \operatorname{supp}(\xi_k)$ and $|\alpha| = N+1$ or in other words

$$\sup_{w \in U_0} |\xi(k^{1-\varepsilon}|w|^2) G_{\alpha,k}(w)| \le C k^{2\varepsilon(N+1)}$$

for all $k \in [1,\infty)$.

□

Proof of Theorem 2.55. Let $S_1 \subset C^{N+4}(D,\mathbb{R})$ and $S_2 \subset C^{N+1}(D,\mathbb{R})$ be bounded sets, $D' \subset\subset D$ open and $C_1', C_2' > 0$ two constants such that $H_\varphi(z) - C_1' \operatorname{Id}$ is positive definite and $\rho(z) > C_2'$ for all $\varphi \in S_1$, all $\rho \in S_2$ and all $z \in D'$. Fix $0 < \varepsilon' < 1$ and $z_0 \in D'$. Choose an open neighborhood $V \subset D'$ around z_0 such that for $\varphi|_V$ the point z_0 has the N-th localization property. Now, there exists an open neighborhood $V' \subset V$ such that $\varphi|_V$ has the N-th localization property for any point $z \in V'$ and any $\varphi \in S_1$. More precisely there exists a constant $C_3' > 0$ with $\tilde{\varphi}_N(z,w) \ge C|w-z|^2$ for all $(z,w) \in V' \times V$ and $\varphi \in S_1$ (see Lemma 2.47). Apply Proposition 2.59 where we assume $U_1 \subset V'$ and $U_3 \subset\subset V$. Choose the cutoff function $\chi \in C_0^\infty(D,\mathbb{R})$ to be supported in U_2 and $\chi \equiv 1$ on some open neighborhood $U' \subset U_2$ around z_0. Set

$$\begin{aligned} A_k(z) &:= \int_D f(w) \overline{(w-z)}^\beta e^{k\overline{\gamma_N(z,w)}} \chi(w) e^{-k\varphi(w)} dV_D(w) \\ &= \int_{U_0} g(z,w) h(z,w) e^{k\overline{\gamma_{N,F}(z,w)}} e^{-k\varphi(F(z,w))} dV_{z,U_0}(w) \end{aligned}$$

where $g(z,w) = f(F(z,w))$ is holomorphic in w and $h(z,w) = \overline{(F(z,w)-z)}^\beta \chi(F(z,w))$, $\gamma_{N,F}(z,w) = \gamma(z,F(z,w))$, $dV_{z,U_0}(\cdot) = F(z,\cdot)^* dV_D$. Note that $\operatorname{supp}(h(z,\cdot)) \subset U_1$ holds for all $z \in U_1$. We further set $\tilde{\varphi}_{N,F}(z,w) = \tilde{\varphi}_N(z,F(z,w))$ and can assume that U_0 contains the closure of a polydisc $\mathbb{D}_{3\tau}^n$ of radius 3τ around 0 for some $\tau > 0$.

Choose a cutoff function $\xi \in C_0^\infty(\mathbb{R},\mathbb{R})$, $0 \le \xi \le 1$, $\operatorname{supp}(\xi) \subset (-2\delta, 2\delta) \subset (-1,1)$, $\xi \equiv 1$ on $(-\delta,\delta)$ where $\sqrt{\delta} < \tau$. Consider the term

$$\begin{aligned} A_{1,k}(z) &:= \left| \int_{U_0} g(z,w)h(z,w)e^{k\overline{\gamma_{N,F}(z,w)}}(1-\xi(|w|^2))e^{-k\varphi(F(z,w))}dV_{z,U_0}(w) \right|^2 \\ &\le \|f\|_k^2 \int_{U_0} |h(z,w)|^2 |(1-\xi(|w|^2))|^2 e^{-k\tilde{\varphi}_{N,F}(z,w)} dV_{z,U_0}(w). \end{aligned}$$

By the assumptions on $\tilde{\varphi}_N$, U_1 and the properties of F from Proposition 2.59 we find a constant C_1 independent of k, $z \in U_1$ and $\varphi \in S_1$ such that $\tilde{\varphi}_{N,F}(z,w) \ge C_1|w|^2$. Then, we conclude

$$\begin{aligned} &\int_{U_0} |h(z,w)|^2(1-\xi(|w|^2))|^2 e^{-k\tilde{\varphi}_N(z,w)} dV_{z,U_0}(w) \\ &\le \int_{U_0} |h(z,w)|^2 |(1-\xi(|w|^2))|^2 e^{-kC_1|w|^2} dV_{z,U_0}(w) \\ &\le e^{-C_1\delta k} \int_{U_0} |h(z,w)|^2 dV_{z,U_0}(w) \end{aligned}$$

where $\int_{U_0} |h(z,w)|^2 dV_{z,U_0}(w)$ is uniformly bounded in $z \in U_1$, $\varphi \in S_1$ and $\rho \in S_2$. Now, consider

$$\begin{aligned} A_{2,k}(z) &:= \int_{U_0} g(z,w)s(z,w)e^{k\overline{\gamma_{N,F}(z,w)}} e^{-k\varphi(F(z,w))} dV_{z,U_0}(w) \\ &= \int_{U_0} g(z,w)e^{-k\gamma_{N,F}(z,w)} s(z,w) e^{-k\tilde{\varphi}_{N,F}(z,w)} dV_{z,U_0}(w) \end{aligned}$$

where $s(z,w) := h(z,w)\xi(|w|^2)$. Since U_0 contains the closure of a polydisc $\mathbb{D}^n_{3\tau}$ of radius 3τ around 0 we find by the assumption on f a splitting decomposition of $g(z,\cdot)e^{-k\gamma_{N,F}(z,\cdot)}$ as

$$g(z,w)e^{-k\gamma_{N,F}(z,w)} = \sum_{|\alpha|=N+1} w^\alpha g_{z,\alpha,k}(w)$$

where $g_{z,\alpha,k}$ are holomorphic functions on $\mathbb{D}^n_{3\tau}$. By the properties of s we can write $A_{2,k}(z) = \sum_{|\alpha|=N+1} B_{\alpha,k}(z)$ where

$$B_{\alpha,k}(z) := \int_{U_0} w^\alpha g_{z,\alpha,k}(w) s(z,w) e^{-k\tilde{\varphi}_{N,F}(z,w)} dV_{z,U_0}(w).$$

We write

$$\tilde{\varphi}_{N,F}(z,w) = |w|^2 - \eta_0(z,w) - \eta(z,w)$$

as in Proposition 2.59 where

$$\eta_0(z,w) = \sum_{\substack{|\alpha|+|\beta|\ge 3 \\ |\alpha|,|\beta| \le N+2}} c_{\alpha,\beta}(z) w^\alpha \overline{w}^\beta$$

such that $c_{\alpha,\beta}(z) = 0$ if $|\alpha| \leq 1$ or $|\beta| \leq 1$ and $\eta(z, w) = O(|w|^{N+3})$. Furthermore, set

$$G_{\alpha,k}(z, w) = \rho(z, w)^{-1} e^{-k(\eta_0(z,w)+\eta(z,w))} \cdot \overline{\partial}_w^\alpha \left(\rho(z, w)s(z, w)e^{k(\eta_0(z,w)+\eta(z,w))}\right)$$

where $\rho(z, w)dV_{\mathbb{C}^n} = dV_{z,U_0}(w)$. One calculates

$$\begin{aligned} B_{\alpha,k}(z) &= (-k)^{-N-1} \int_{U_0} g_{z,\alpha,k}(w)s(z, w)e^{k(\eta_0(z,w)+\eta(z,w))}(\overline{\partial}_w^\alpha e^{-k|w|^2})dV_{z,U_0}(w) \\ &= k^{-N-1} \int_{U_0} g_{z,\alpha,k}(w)G_{\alpha,k}(z, w)e^{-k\tilde{\varphi}_{N,F}(z,w)}dV_{z,U_0}(w) \end{aligned}$$

For $\varepsilon = \varepsilon'/8(N + 1)$, $0 < \varepsilon' < 1$, consider the term

$$\begin{aligned} B_{1,\alpha,k}(z) :=& \left|\int_{U_0} g_{z,\alpha,k}(w)(1 - \xi(k^{1-\varepsilon}|w|^2))G_{\alpha,k}(z, w)e^{-k\tilde{\varphi}_{N,F}(z,w)}dV_{z,U_0}(w)\right| \\ \leq& \sup_{w\in\mathbb{D}_\tau^n} |g_{z,\alpha,k}(w)|e^{-\frac{3k}{4}\tilde{\varphi}_{N,F}(z,w)} \sup_{w\in\mathbb{D}_\tau^n} |G_{\alpha,k}(z, w)| \\ &\cdot \int_{\mathbb{D}_\tau^n} (1 - \xi(k^{1-\varepsilon}|w|^2))^2 e^{-\frac{k}{4}\tilde{\varphi}_{N,F}(z,w)}dV_{z,U_0}(w). \end{aligned}$$

We observe that $|G_{\alpha,k}(z, w)| \leq C_2k^{N+1}$ for some constant $C_2 > 0$ independent of k, $z \in U_1$, $w \in U_0$, $\varphi \in S_1$ and $\rho \in S_2$. Furthermore, we have $\tilde{\varphi}_{N,F}(z, w) \geq C_1|w|^2$. Thus,

$$\int_{\mathbb{D}_\tau^n} (1 - \xi(k^{1-\varepsilon}|w|^2))^2 e^{-\frac{k}{4}\tilde{\varphi}_{N,F}(z,w)}dV_{z,U_0}(w) \leq C_3e^{-C_1\frac{\delta}{4}k^\varepsilon}$$

for some constant $C_3 > 0$ independent of k, z, φ and ρ which implies

$$B_{1,\alpha,k}(z) \leq C_2C_3e^{-C_1\delta k^\varepsilon}k^{N+1} \sup_{w\in\mathbb{D}_\tau^n} |g_{z,\alpha,k}(w)|e^{-\frac{3k}{4}\tilde{\varphi}_{N,F}(z,w)}.$$

Now, consider

$$\begin{aligned} B_{2,\alpha,k}(z) :=& \left|\int_{U_0} g_{z,\alpha,k}(w)G_{\alpha,k}(z, w)\xi(k^{1-\varepsilon}|w|^2)e^{-k\tilde{\varphi}_{N,F}(z,w)}dV_{z,U_0}(w)\right| \\ \leq& \sup_{w\in\mathbb{D}_\tau^n} |g_{z,\alpha,k}(w)|e^{-\frac{3k}{4}\tilde{\varphi}_{N,F}(z,w)} \sup_{w\in\mathbb{D}_\tau^n} |\xi(k^{1-\varepsilon}|w|^2)G_{\alpha,k}(z, w)| \\ &\cdot \int_{\mathbb{D}_\tau^n} e^{-\frac{k}{4}\tilde{\varphi}_{N,F}(z,w)}dV_{z,U_0}(w). \end{aligned}$$

A similar argument as above and Example 2.45 leads to

$$\int_{\mathbb{D}_\tau^n} e^{-\frac{k}{4}\tilde{\varphi}_{N,F}(z,w)}dV_{z,U_0}(w) \leq C_4k^{-n}$$

for some constant $C_4 > 0$ independent of k, z, φ and ρ which implies

$$B_{2,\alpha,k}(z) \leq C_4k^{-n} \sup_{w\in\mathbb{D}_\tau^n} |g_{z,\alpha,k}(w)|e^{-\frac{3k}{4}\tilde{\varphi}_F(z,w)} \sup_{w\in\mathbb{D}_\tau^n} |\xi(k^{1-\varepsilon}|w|^2)G_{\alpha,k}(z, w)|.$$

Using Lemma 2.63 and Lemma 2.62 we find an open neighborhood $U \subset U_1$ around z_0, $k_0 \in \mathbb{N}$ and constants $C_5, C_6 > 0$ such that

$$|\xi(k^{1-\varepsilon}|w|^2)G_{\alpha,k}(z,w)| \leq C_5 k^{2(N+1)\varepsilon}$$

and

$$\sup_{w\in\mathbb{D}_\tau^n} |g_{z,\alpha,k}(w)|^2 e^{-\frac{3k}{2}\tilde{\varphi}_F(z,w)} \leq C_6 k^{N+1+n+\frac{\varepsilon'}{2}} \|f\|_k^2$$

holds for all $z \in U$, $k \geq k_0$, $\alpha \in \mathbb{N}_0^n$, $|\alpha| = N+1$, $w \in U_0$, $\varphi \in S_1$, $\rho \in S_2$ and f with the properties mentioned above and all splitting decompositions for $f(F(z,\cdot))e^{-k\gamma_F(z,\cdot)}$. Then we have

$$\begin{aligned}
|B_{\alpha,k}(z)| \leq & k^{-(N+1+n)+2(N+1)\varepsilon} \sup_{w\in\mathbb{D}_\tau^n} \left(|g_{z,\alpha,k}(w)|e^{-\frac{3k}{4}\tilde{\varphi}_F(z,w)}\right) \\
& \cdot \left(C_2C_3e^{-C_1\delta k^\varepsilon} k^{N+1+n-2(N+1)\varepsilon} + C_4C_5\right) \\
\leq & k^{-\frac{N+1+n}{2}+\frac{8(N+1)\varepsilon+\varepsilon'}{4}} \|f\|_k \left(C_2C_3e^{-C_1\delta k^\varepsilon} k^{N+1+n-2(N+1)\varepsilon} + C_4C_5\right) \\
\leq & k^{-\frac{N+1+n-\varepsilon'}{2}} C_7 \|f\|_k
\end{aligned}$$

some constant $C_7 > 0$ independent of $z \in U$, $k \geq k_0$, $\varphi \in S_1$, $\rho \in S_2$ and f or its decomposition respectively such that $C_2C_3e^{-C_1\delta k^\varepsilon}k^{N+1+n-(N+1)\varepsilon} + C_4C_5 \leq C_7$ for all $k \geq k_0$. Putting all together we find

$$\begin{aligned}
\left|\int_D f(w)\overline{(w-z)}^\beta e^{k\overline{\gamma(z,w)}}\chi(w)e^{-k\varphi(w)}dV_D(w)\right| & \leq A_{1,k}(z) + \sum_{|\alpha|=N+1} |B_{\alpha,k}(z)| \\
& \leq Ck^{-\frac{N+1+n-\varepsilon'}{2}} \|f\|_k
\end{aligned}$$

for some constant C independent of $z \in U$, $k \geq k_0$, $\varphi \in S_1$, $\rho \in S_2$ and f. From Lemma 2.29 Cauchy–Schwarz inequality and the compactness of $[1, k_0]$ we can choose C such that the statement holds for all $k \in [1, \infty)$. □

2.8 Decomposition of $H_k^0(D)$

Let $D \subset \mathbb{C}^n$ be a bounded domain with volume form $dV_D = \rho dV_{\mathbb{C}^n}$ where $\rho \in C^{2(N+n+1)}(D) \cap C^0(\overline{D})$ is positive on $\overline{D}$ and consider a weight $\varphi \in C^{3(N+n)+4}(D) \cap C^0(\overline{D})$. Given $k \in [1,\infty)$ and $z \in D$ let $V_{z,N,k} \subset H_k^0(D) = H_{k\varphi,\rho}^0(D)$ be the linear subspace spanned by the linear independent set

$$\{w \mapsto (w-z)^\alpha e^{k\gamma_N(z,w)}\}_{|\alpha|\leq N}.$$

Let $W_{z,N,k} \subset H_{k\varphi,\rho}^0(D)$ the linear subspace which consists of all elements in $H_{k\varphi,\rho}^0(D)$ vanishing up to order N in z, i.e

$$W_{z,N,k} := \left\{ f \in H_{k\varphi,\rho}^0(D) \mid \frac{\partial^{|\alpha|}f}{\partial^\alpha w}(z) = 0 \text{ for all } \alpha \in \mathbb{N}_0^n\ ,\ |\alpha| \leq N \right\}.$$

For any holomorphic function f on D we can write the Taylor expansion in z up to order N as

$$T_{z,N}(f)(w) = \sum_{|\alpha|\le N} \frac{1}{\alpha!}\frac{\partial^{|\alpha|} f}{\partial^\alpha w}(z)(w-z)^\alpha.$$

Then, define the linear map $T_{z,N,k} : H^0_{k\varphi,\rho}(D) \to H^0_{k\varphi,\rho}(D)$ by

$$T_{z,N,k}(f)(w) = e^{k\gamma_N(z,w)} T_{z,N}(e^{-k\gamma_N(z,\cdot)} f(\cdot))(w).$$

We will show that $H^0_{k\varphi,\rho}(D)$ can be written as the direct sum of $V_{z,N,k}$ and $W_{z,N,k}$ with respective projection $T_{z,N,k}$.

Lemma 2.64

One has $H^0_{k\varphi,\rho}(D) = V_{z,N,k} \oplus W_{z,N,k}$ where the respective projections are given by $T_{z,N,k}$ or $I - T_{z,N,k}$ respectively. More precisely, $T_{z,N,k}$ is a projection, i.e $T_{z,N,k} \circ T_{z,N,k} = T_{z,N,k}$, such that $\operatorname{ran}(T_{z,N,k}) = V_{z,N,k}$ and $\ker(T_{z,N,k}) = W_{z,N,k}$.

Proof. By construction one has $\operatorname{ran}(T_{z,N,k}) \subset V_{z,N,k}$. Given a polynomial p, $p(w) = \sum_{|\alpha|\le N} c_\alpha (w-z)^\alpha$, of degree lower than or equal to N one has $T_{z,N}p = p$. For $f \in V_{z,N,k}$ write

$$f(w) = e^{k\gamma_N(z,w)} \sum_{|\alpha|\le N} c_\alpha (w-z)^\alpha$$

and consider the polynomial p defined by $p(w) = e^{-k\gamma_N(z,w)} f(w)$. One gets

$$T_{z,N,k} f(w) = e^{k\gamma_N(z,w)}(T_{z,N}p)(w) = e^{k\gamma_N(z,w)} p(w) = f(w)$$

and hence $V_{z,N,k} \subset \operatorname{ran}(T_{z,N,k})$ as well as $T_{z,N,k} \circ T_{z,N,k} = T_{z,N,k}$. Since $w \mapsto e^{-k\gamma_N(z,w)}$ does not vanish we have $f \in W_{z,N,k}$ if and only if $e^{-k\gamma_N(z,\cdot)} f(\cdot) \in W_{z,N,k}$. Furthermore, one observes that $f \in W_{z,N,k}$ if and only if $T_{z,N} f = 0$. Thus, we conclude that $\ker(T_{z,N,k}) = W_{z,N,k}$. □

We would like to study the restriction of $\|\cdot\|_{k\varphi,\rho}$ to $V_{z,N,k}$. Therefore, we set $v_{\alpha,k,z}(w) = k^{\frac{|\alpha|+n}{2}} (w-z)^\alpha e^{k\gamma_N(z,w)}$ for $\alpha \in \mathbb{N}_0^n$, $|\alpha| \le N$, $k \in [1,\infty)$ and $z, w \in D$. Then, $\{v_{\alpha,k,z}\}_{|\alpha|\le N}$ is a basis for $V_{z,N,k\varphi}$ and we define a norm $\|\cdot\|_{k;2}$ by $\|f\|^2_{k;2} = \sum_{|\alpha|\le N} |c_\alpha|^2$ for $f = \sum_{|\alpha|\le N} c_\alpha v_{\alpha,k,z} \in V_{z,N,k\varphi}$.

Lemma 2.65

Given $N \in \mathbb{N}_0$ and $D' \subset\subset D_N$ there exists a constant $C > 0$ such that

$$\|f\|_{k\varphi,\rho} \ge C \|f\|_{k;2}$$

holds for all $k \in [1,\infty)$ and $f \in V_{z,N,k\varphi}$. Moreover, $C > 0$ can be chosen independent of φ and ρ if φ stays in a bounded set $S_1 \subset C^{3(N+n)+4}(D) \cap C^0(\overline{D})$ such that $\inf_{(z,w)\in D'\times D} \tilde{\varphi}_N(z,w)/|w-z|^2$ has a positive lower bound and ρ stays in a bounded set $S_2 \subset C^{2(N+n+1)}(D) \cap C^0(\overline{D})$ such that $\inf_{z\in D} \rho(z)$ has a positive lower bound.

Proof. Let $R_M(\mathbb{C}^n)$ be the space of homogeneous polynomials of degree M in n complex variables, i.e.

$$R_M(\mathbb{C}^n) = \{f \in \mathbb{C}[z_1, \ldots, z_n] \mid f(\lambda z) = \lambda^M f(z) \text{ for all } \lambda \in \mathbb{C}\}.$$

First, we observe that $(f,g)^{\mathrm{Id}} = \Delta^M f(z)\overline{g(z)} \in \mathbb{C}$ defines a Hermitian inner product on $R_M(\mathbb{C}^n)$, because $\Delta^M z^\alpha \overline{z}^\beta = \alpha!\beta!\delta_{\alpha,\beta}$, $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| = |\beta| = M$, and $\{z^\alpha \mid |\alpha| = M\}$ is a basis for $R_M(\mathbb{C}^n)$, where $\Delta = \langle \partial_z, \overline{\partial}_z\rangle = \sum_{j=1}^n \frac{\partial^2}{\partial z_j \partial \overline{z}_j}$. Given $A \in \mathrm{Gl}_n(\mathbb{C})$ we have that $f \mapsto f \circ A$ defines an automorphism on $R_M(\mathbb{C}^n)$. Since $\langle A^{-1}\partial_z, \overline{A^{-1}\partial}_z\rangle^M f(z)\overline{g(z)} = \Delta^M f(Az)\overline{g(Az)}$ we find out that

$$(f,g)^A = \langle A^{-1}\partial_z, \overline{A^{-1}\partial}_z\rangle^M f(z)\overline{g(z)}$$

defines a Hermitian inner product on $R_M(\mathbb{C}^n)$ as well. For $0 \leq M \leq N$ let $R_{z,M,k}$ be the linear span of the linear independent set $\{v_{\alpha,k,z}\}_{|\alpha|=M}$. By Theorem 2.50 we find

$$|(v_{\alpha,k,z}, v_{\beta,k,z})_{k\varphi,\rho} - \rho(z)\det(H_\varphi(z))^{-1}\langle H_\varphi(z)^{-1}\partial_w, \overline{\partial}_w\rangle^M w^\alpha \overline{w}^\beta| \leq C_1 k^{-1}$$

for some constant $C_1 > 0$ independent of $k \in [1,\infty)$, $z \in D'$, $\varphi \in S_1$, $\rho \in S_2$ and $\alpha, \beta \in \mathbb{N}_0^n$, $|\alpha|, |\beta| = M$. Given $f \in R_{z,M,k}$ write $f = \sum_{|\alpha|=M} c_\alpha v_{\alpha,k,z}$ and we observe that the function g defined by $g(w) = e^{-k\tilde{\varphi}_N(z,z+w)} f(z+w)$ satisfies $g \in R_M(\mathbb{C}^n)$. We conclude that

$$|\|f\|_k^2 - \rho(z)\det(H_\varphi(z))^{-1}\langle H_\varphi(z)^{-1}\partial_w, \overline{\partial}_w\rangle^M g(w)\overline{g(w)}| \leq C_2 k^{-1} \sum_{|\alpha|=M} |c_\alpha|^2$$

for some constant $C_2 > 0$ independent of $k \in [1,\infty)$, $z \in D'$, $\varphi \in S_1$, $\rho \in S_2$ and $f \in R_{z,M,k}$. Since $\langle H_\varphi(z)^{-1}\partial_w, \overline{\partial}_w\rangle = \langle H_\varphi(z)^{-\frac{1}{2}}\partial_w, \overline{H_\varphi(z)^{-\frac{1}{2}}\partial_w}\rangle$ we find, using the considerations above, that

$$\langle H_\varphi(z)^{-1}\partial_w, \overline{\partial}_w\rangle^M g(w)\overline{g(w)} \geq C_3 \sum_{|\alpha|=M} |c_\alpha|^2$$

holds for some constant $C_3 > 0$ independent of $z \in D'$ and $g \in R_M(\mathbb{C}^n)$. Thus, one has

$$\|f\|_{k\varphi,\rho}^2 \geq (C_3 - k^{-1}C_2)\|f\|_{k;2}^2.$$

By using Theorem 2.50 we find for $\alpha, \beta \in \mathbb{N}_0^n$, $|\alpha|, |\beta| \leq N$, $|\alpha| \neq |\beta|$ that

$$|(v_{\alpha,k,z}, v_{\beta,k,z})_{k\varphi,\rho}| \leq C_4 k^{-\frac{1}{2}}$$

holds where $C_4 > 0$ is a constant independent of $k \in [1,\infty)$, $z \in D'$, $\varphi \in S_1$ and $\rho \in S_2$. Thus, we find a constant $C_5 > 0$ such that $|(f,g)_{k\varphi,\rho}| \leq C_5 k^{-\frac{1}{2}}(\|f\|_{k;2}^2 + \|g\|_{k;2}^2)$

for all $k \in [1,\infty)$, $s \in D'$, $\varphi \in S_1$, $\rho \in S_2$ and $f \in R_{z,M,k}$, $g \in R_{z,M',k}$ such that $M \neq M'$. Using the decomposition $V_{z,N,k} = \bigoplus_{M=0}^N R_{z,M,k}$ we find constants $C_6, C_7 > 0$ with $\|f\|_{k\varphi,\rho} \geq (C_6 - C_7 k^{-\frac{1}{2}})\|f\|_{k;2}$. Thus, there exist $k_0 \in \mathbb{N}$ and $C_8 > 0$ such that $\|f\|_{k\varphi,\rho} \geq C\|f\|_{k,2}$ holds for all $k \geq k_0$, $z \in D'$, $\varphi \in S_1$, $\rho \in S_2$ and $f \in V_{z,N,k}$. We have $\|f\|_{k\varphi,\rho}/\|f\|_{k;2} > 0$ for all $k \in [1,\infty)$ and $f \in V_{z,N,k} \setminus \{0\}$ and there exists a constant $C_9 > 0$ with $-C_9 \leq \inf_{(z,w)\in D'\times\overline{D}} \tilde{\varphi}_N \leq \sup_{(z,w)\in D'\times\overline{D}} \tilde{\varphi}_N \leq C_9$, $\inf_{z\in\overline{D}} \rho > 1/C_9$ for all $\varphi \in S_1$ and $\rho \in S_2$. Together with the compactness of $\{f \in V_{z,N,k} \mid \|f\|_{k;2} = 1\}$ and $[1,k_0]$ we find a constant $C_{10} > 0$ with $\|f\|_{k\varphi,\rho}/\|f\|_{k;2} \geq C_{10}$ for all $f \in V_{z,N,k}$, $k \in [1,k_0]$, $\varphi \in S_1$ and $\rho \in S_2$. Putting $C = \min\{C_8, C_{10}\}$ finishes the proof. □

We have the decomposition $H^0_{k\varphi,\rho}(D) = V_{z,N,k} \oplus V_{z,N,k}^{\perp}$ where $V_{z,N,k}^{\perp}$ denotes the orthogonal complement of $V_{z,N,k}$ in $H^0_{k\varphi,\rho}(D)$. The main result we want to prove in this section shows that the restriction of $\operatorname{Id} - T_{z,N,k}$ to $V_{z,N,k}^{\perp}$ is bounded by a constant independent of k and locally uniformly in $z \subset D'$. In particular, we will show the following.

Theorem 2.66
Given $D' \subset\subset D_N$ there exists a constant $C > 0$ such that

$$\|(\operatorname{Id} - T_{z,N,k})f\|_{k\varphi,\rho} \leq C\|f\|_{k\varphi,\rho}$$

holds for all $k \in [1,\infty)$, $z \in D'$ and $f \in V_{z,N,k}^{\perp}$. Moreover, $C > 0$ can be chosen independent of φ and ρ if φ stays in a bounded set $S_1 \subset C^{3N+3n+4}(D) \cap C^0(\overline{D})$ such that $\inf_{(z,w)\in D'\times D} \tilde{\varphi}_N(z,w)/|w-z|^2$ has a positive lower bound and ρ stays in a bounded set $S_2 \subset C^{2N+2n+2}(D) \cap C^0(\overline{D})$ such that $\inf_{z\in D} \rho(z)$ has a positive lower bound.

In order to prove Theorem 2.66 we need the following lemma which shows that $V_{z,N,k}$ and $W_{z,N,k}$ become asymptotically orthogonal, i.e. $W_{z,N,k} \to V_{z,N,k}^{\perp}$ when k goes to infinity.

Lemma 2.67
In the situation of Theorem 2.66 we have that for any $0 < \varepsilon < 1$ there exists a constant $C > 0$ such that

$$|(f,g)_{k\varphi,\rho}| \leq C k^{-\frac{1-\varepsilon}{2}} \|f\|_{k\varphi,\rho}\|g\|_{k\varphi,\rho}$$

holds for all $k \in [1,\infty)$, $\varphi \in S_1$, $\rho \in S_2$, $z \in D'$, $f \in V_{z,N,k}$ and $g \in W_{z,N,k}$.

Proof. We define another norm on $V_{z,N,k}$ which is the maximum norm with respect to the basis

$$\{w \mapsto (w-z)^\alpha e^{k\gamma(z,w)}\}_{|\alpha|\leq N},$$

i.e. $\|f\|_{k;\infty} := \max_{|\alpha|\leq N} |c_\alpha|$ where $f(w) = \sum_{|\alpha|\leq N} c_\alpha (w-z)^\alpha e^{k\gamma_N(z,w)}$. One has $\|f\|_{k;\infty} \leq k^{\frac{N+n}{2}} \|f\|_{k;2}$ and hence, using Lemma 2.65, we find a constant $C_2 > 0$ such that $\|f\|_{k;\infty} \leq C_2 k^{\frac{N+n}{2}} \|f\|_{k\varphi,\rho}$ holds for all $k \in [1,\infty)$, $z \in D'$, $\varphi \in S_1$, $\rho \in S_2$, and $f \in V_{z,N,k}$. Then, we can apply Corollary 2.56 and get for $f \in V_{z,N,k}$ and $g \in W_{z,N,k}$ that

$$|(f,g)_{k\varphi,\rho}| \leq C_1 k^{-\frac{N+n+1-\varepsilon}{2}} \|f\|_{k;\infty} \|g\|_{k\varphi,\rho} \leq C k^{-\frac{1-\varepsilon}{2}} \|f\|_{k\varphi,\rho} \|g\|_{k\varphi,\rho}$$

holds. In other words, for any $0 < \varepsilon < 1$ there exists a constant $C > 0$ such that $|(f,g)_{k\varphi,\rho}| \leq C k^{-\frac{1-\varepsilon}{2}} \|f\|_{k\varphi,\rho} \|g\|_{k\varphi,\rho}$ holds for all $k \in [1,\infty)$, $z \in D'$, $\varphi \in S_1$, $\rho \in S_2$, $f \in V_{z,N,k}$ and $g \in W_{z,N,k}$. $\square$

Proof of Theorem 2.66. Given $f \in V_{z,N,k}^\perp$ write $f = f_1 + f_2$ where $f_1 = T_{z,N,k} f \in V_{z,N,k}$ and $f_2 = (I - T_{z,N,k}) f \in W_{z,N,k}$ respectively. We find $0 = (f_1, f)_{k\varphi,\rho} = \|f_1\|^2_{k\varphi,\rho} + (f_1, f_2)_{k\varphi,\rho}$ and hence, using Lemma 2.67 for some $0 < \varepsilon < 1$, $\|f_1\|^2_{k\varphi,\rho} = |(f_1, f_2)_{k\varphi,\rho}| \leq C_1 k^{-\frac{1-\varepsilon}{2}} \|f_1\|_{k\varphi,\rho} \|f_2\|_{k\varphi,\rho}$ which implies $\|f_1\|_{k\varphi,\rho} \leq C_1 k^{-\frac{1-\varepsilon}{2}} \|f_2\|_{k\varphi,\rho}$. Thus, we have $\|f\|_{k\varphi,\rho} \geq \|f_2\|_{k\varphi,\rho} - \|f_1\|_{k\varphi,\rho} \geq \|f_2\|_{k\varphi,\rho}(1 - C_1 k^{-\frac{1-\varepsilon}{2}})$ and we can find $k_0 \in \mathbb{N}$ and $C > 0$ such that $\|(I - T_{z,N,k}) f\|_{k\varphi,\rho} \leq C \|f\|_{k\varphi,\rho}$ for all $k \geq k_0$, $z \in D'$, $\varphi \in S_1$, $\rho \in S_2$ and $f \in V_{z,N,k}^\perp$. From the compactness of $[1, k_0]$ and the assumption that S_1 and S_2 are bounded sets we find by using Lemma 2.28 that $C > 0$ can be chosen such that the statement holds for all $k \in [1,\infty)$. $\square$

Chapter 3

Bergman Kernel Expansion

In this chapter we prove the main results announced in Section 1.2 and Section 1.3. We construct the local asymptotically reproducing kernel in Section 3.1. Theorem 1.3 then follows from Lemma 3.4 and Lemma 3.8. In Section 3.2 we establish a formula for the coefficients in the Bergman kernel expansion (see Definition 1.4 and Example 1.11). The main calculations for the explicit formulas are performed in the proof of Theorem 3.17. With that formulas Theorem 1.6 and Theorem 1.7 follow from Lemma 3.9 and Lemma 3.12 (see Theorem 3.20, Theorem 3.21 and Corollary 3.26).

In Section 3.3 we introduce some basic notations from complex geometry in order to apply Theorem 1.3 in the manifold case (see Lemma 3.30). Using Hörmander's L^2 estimates in a version due to Demailly (see Theorem 3.31) we prove Theorem 1.15 and Theorem 1.16 in Section 3.4 (see Theorem 3.36 and Theorem 3.38).

3.1 Local Expansion of the Bergman Kernel

Let $D \subset \mathbb{C}^n$ be a bounded domain and let $\varphi \in C^{6N+3n+4}(D,\mathbb{R}) \cap C^0(\overline{D})$, $\rho \in C^{4N+2n+2}(D,\mathbb{R}) \cap C^0(\overline{D})$ be two real valued functions such that $\rho > 0$ on $\overline{D}$ holds. Define a volume form on D by $dV_D(z) = \rho(z) dV_{\mathbb{C}^n}$. For any $k \in [1,\infty)$ set

$$H^0_{k\varphi,\rho}(D) = \{f \in \mathcal{O}(D) \mid \|f\|_{k\varphi,\rho} < \infty\}$$

where the norm $\|\cdot\|_{k\varphi,\rho}$ is induced by the weighted inner product $(\cdot,\cdot)_{k\varphi,\rho}$ given by

$$(f,g)_{k\varphi,\rho} = \int_D f(z)\overline{g(z)} e^{-k\varphi(z)} dV_D(z) \text{ , for all } f,g \in L^2(D).$$

Thus, $H^0_{k\varphi,\rho}(D)$ is the space of holomorphic functions on D with finite L^2-norm. Let $K_k := K_{k\varphi,\rho}$ the reproducing kernel, $P_k := P_{k\varphi,\rho}$ the Bergman kernel and $B_k := B_{k\varphi,\rho}$ the Bergman kernel function for the space $H^0_{k\varphi,\rho}(D)$ (see Definition 2.34). For brevity, we will also use the notations $H^0_k(D) := H^0_{k\varphi,\rho}(D)$, $(\cdot,\cdot)_k := (\cdot,\cdot)_{k\varphi,\rho}$ and $\|\cdot\|_k := \|\cdot\|_{k\varphi,\rho}$ in this section. Let $\gamma_N, \tilde{\varphi}_N : D \times \overline{D} \to \mathbb{C}$ be defined by

$$\gamma_N(z,w) = \frac{\varphi(z)}{2} + \sum_{1 \le |\alpha| \le N+2} \frac{1}{\alpha!} \frac{\partial^{|\alpha|}\varphi(z)}{\partial^\alpha z} (w-z)^\alpha$$

and $\tilde{\varphi}_N(z,w) = \varphi(w) - \gamma_N(z,w) - \overline{\gamma_N(z,w)}$. Given $\alpha, \alpha', \beta \in \mathbb{N}_0^n$ and $z \in D$ we set

$$\begin{aligned} a_{\alpha,\beta,k}(z) &= \int_D \overline{(w-z)^\alpha}(w-z)^\beta e^{-k\tilde{\varphi}_N(z,w)} dV_D(w) \\ A_{N,k}(z) &= (a_{\alpha,\beta,k}(z))_{|\alpha|,|\beta|\le N} \text{ and } A_{N,\alpha',k}(z) = (a_{\alpha,\beta,k}(z))_{|\alpha|,|\beta|\le N, |\alpha|\ne 0, \beta\ne\alpha'} \end{aligned}$$

where we define an order on $\mathbb{N}_0^n$ by saying $\alpha < \beta$ if $|\alpha| < |\beta|$ and using lexicographic order for $|\alpha| = |\beta|$. We have that $A_{N,k}(z)$ and $A_{N,\alpha',k}(z)$ are square matrices and that $A_{N,k}(z)$ is invertible as the restriction of $(\cdot,\cdot)_{k\varphi,\rho}$ to the finite dimensional subspace

$$V_{N,z,k} = \operatorname{span}_{\mathbb{C}}\{w \mapsto (w-z)^\alpha e^{k\gamma(z,w)}\}_{|\alpha|\le N} \subset H^0_{k\varphi,\rho}(D).$$

Hence, for any $\alpha \in \mathbb{N}_0^n$, $|\alpha| \le N$, we can define

$$\lambda_{N,\alpha,k}(z) = (-1)^{\mathrm{NUM}(\alpha)} \frac{\det A_{N,\alpha,k}(z)}{\det A_{N,k}(z)}$$

where $\mathrm{NUM}\colon \mathbb{N}_0^n \to \mathbb{N}_0$ is the inverse of the enumeration of the elements in $\mathbb{N}_0^n$ with respect to the order on $\mathbb{N}_0^n$. Now, define $K_{N,k} = K_{k\varphi,\rho,N} : D \times \overline{D} \to \mathbb{C}$

$$K_{N,k}(z,w) = e^{k(\overline{\gamma_N(z,z)} + \gamma_N(z,w))} \sum_{|\alpha|\le N} \lambda_{N,\alpha,k}(z)(w-z)^\alpha$$

and set similar to Section 2.4

$$P_{N,k}(z,w) = e^{-k\frac{\varphi(z)+\varphi(w)}{2}} \overline{K_{N,k}(z,w)} \text{ and } B_{N,k}(z) = P_{N,k}(z,z)$$

for $z \in D$, $w \in \overline{D}$.

Lemma 3.1
One has $(f, K_{N,k}(z,\cdot))_k = f(z)$ for all $z \in D$ and $f \in V_{N,z,k}$.

Proof. Let $z \in D$ be a point. Consider the vector $\lambda_{N,k}(z) = (\lambda_{N,\alpha,k}(z))^T_{|\alpha|\le N}$. By Cramer's rule we find that $\lambda_{N,k}(z)$ solves $A_{N,k}(z)\lambda_{N,k}(z) = (1,0,0,\dots,0)^T$ which implies

$$\int_D K_{N,k}(z,w)\overline{(w-z)}^\alpha e^{k\overline{\gamma_N(z,w)}} e^{-k\varphi(w)} dV_D(w) = \begin{cases} 0 & \text{, if } 0 < |\alpha| \le N, \\ e^{k\gamma_N(z,z)} & \text{, if } \alpha = 0. \end{cases}$$

Given $f \in V_{N,z,k}$ write $f(w) = \sum_{|\alpha|\le N} c_\alpha (w-z)^\alpha e^{\gamma_N(z,w)}$ and hence, using that $\gamma_N(z,z) \in \mathbb{R}$ holds, one finds $(f, K_{N,k}(z,\cdot)_k) = c_0 e^{k\overline{\gamma(z,z)}} = f(z)$. □

Lemma 3.2
For any $z \in D$ one has $\|K_{N,k}(z,\cdot)\|_k^2 = K_{N,k}(z,z)$ and $B_{N,k}(z) = \lambda_{N,0,k}(z)$.

Proof. Since $K_{N,k}(z,\cdot)\in V_{N,z,k}$ we can apply Lemma 3.1 and find

$$\|K_{N,k}(z,\cdot)\|_k^2=(K_{N,k}(z,\cdot),K_{N,k}(z,\cdot))_k=K_{N,k}(z,z)=\lambda_{N,0,k}(z)e^{k(\overline{\gamma(z,z)}+\gamma(z,z))}.$$

Furthermore, one has $B_{N,k}(z)=e^{-k\varphi(z)}K_{N,k}(z,z)=\lambda_{N,0,k}(z)$. $\square$

Lemma 3.3
Given $\varphi\in C^{6N+3n+4+l}(D)\cap C^0(\overline{D})$ one has $\lambda_{N,\alpha,k}\in C^{5N+3n+1+l}(D)$ for all $\alpha\in\mathbb{N}_0^n$, $|\alpha|\le N$.

Recall that $D_{\varphi,N}$ is the set which consists of all points $z\in D$ satisfying the N-th localization property, that is

$$\begin{aligned}&(i)\quad z\in D_{\varphi,+},\\&(ii)\quad \tilde{\varphi}_N(z,w)>0\text{ for all }w\in\overline{D}\setminus\{z\}.\end{aligned}$$

For the rest of the section we define the following. Let $D'\subset\subset D$ be an open set and $S_1\subset C^{6N+3n+4}(D,\mathbb{R})\cap C^0(\overline{D})$, $S_2\subset C^{4N+2n+2}(D,\mathbb{R})\cap C^0(\overline{D})$ be bounded sets such that $\{\inf_{(z,w)\in D'\times\overline{D}}\tilde{\varphi}_N(z,w)/|z-w|^2\mid\varphi\in S_1\}$, $\{\inf_{z\in\overline{D}}\rho\mid\rho\in S_2\}$ have positive lower bounds. It immediately follows that $D'\subset D_{\varphi,N}$ for $\varphi\in S_1$.

Lemma 3.4
Let $l\in\mathbb{N}_0$ be a non-negative integer. For any $\varphi\in S_1\cap C^{6N+3n+4+l}(D)$, $\rho\in S_2\cap C^{4N+2n+2+l}(D)$ and $\alpha\in\mathbb{N}_0^n$, $|\alpha|\le N$, there exist functions $\lambda_{N,\alpha}^{(0)},\dots,\lambda_{N,\alpha}^{(N)}\in C^l(D_{\varphi,+})$ where $\lambda_{N,\alpha}^{(j)}(z)$ depends only on the derivatives of φ and ρ in $z\in D_{\varphi,+}$ such that

$$\lambda_{N,\alpha,k}-k^n\sum_{j=0}^{N}k^{-j}\lambda_{N,\alpha}^{(j)}=O(k^{-N-1+n})\ \text{in } C^0(D')\ \text{uniformly in } \varphi\in S_1\ \text{and } \rho\in S_2.$$

Furthermore, we have $\lambda_{N,0}^{(0)}(z)=\pi^{-n}\rho(z)^{-1}\det\left(H_\varphi(z)\right)$.

Proof. Let F_N be the set of all multi-indices $\alpha\in\mathbb{N}_0^n$, $|\alpha|\le N$ and denote by $\mathrm{Perm}(F_N)$ the group of permutations of the elements in F_N where we define an order on F_N as before by saying $\alpha<\beta$ if $|\alpha|<|\beta|$ and using lexicographic order for $|\alpha|=|\beta|$. Using the Laplace rule for the determinant we find

$$\det(A_{N,k})=\sum_{\tau\in\mathrm{Perm}(F_N)}(-1)^{\mathrm{sign}(\tau)}\prod_{\alpha\in F_N}a_{\alpha,\tau(\alpha),k}.$$

We can apply Theorem 2.50 and find $a_{\alpha,\tau(\alpha),k}=O(k^{-n-\max\{|\alpha|,\tau(\alpha)\}})$ which implies $a_{\alpha,\tau(\alpha),k}=O(k^{-n-|\alpha|})$ with

$$k^{n+|\alpha|}a_{\alpha,\tau(\alpha),k}-\frac{\pi^n}{\det(H_\varphi)}\sum_{j=0}^{N}a_{\alpha,\tau(\alpha)}^{(j+|\alpha|)}k^{-j}=O(k^{-N-1})\text{ in }C^0(D')\tag{3.1}$$

uniformly in $\varphi \in S_1$ and $\rho \in S_2$. Thus, $k^{d_N} \prod_{\alpha \in F_N} a_{\alpha,\tau(\alpha),k} = O(1)$ and hence $k^{d_N} \det(A_{N,k}) = O(1)$ where $d_N = \sum_{|\alpha| \le N}(|\alpha| + n)$. From Lemma 2.65 we get that for any compact subset $K \subset D'$ there exists a constant $C > 0$ such that $\det(A_{N,k}(z)) \ge Ck^{-d_N}$ holds for all $z \in K$, $k \in [1,\infty)$, $\varphi \in S_1$ and $\rho \in S_2$. We conclude that

$$k^{d_N} \det(A_{N,k}) - \sum_{j=0}^{N} c_j k^{-j} = O(k^{-N-1}) \text{ in } C^0(D')$$

uniformly in $\varphi \in S_1$ and $\rho \in S_2$ holds where $c_0, \ldots, c_N$ are continuous functions defined on D' and c_0 positive lower bound uniform in $\varphi \in S_1$ and $\rho \in S_2$ on any compact subset of D'. Applying the Leibniz criterion for determinants we find

$$\det(A_{N,\beta,k}) = \sum_{\substack{\tau \in \mathrm{Perm}(F_N) \\ \tau(0)=\beta}} (-1)^{\mathrm{sig}(\tau)+NUM(\beta)} \prod_{\substack{\alpha \in F_N \\ \alpha \neq 0}} a_{\alpha,\tau(\alpha),k}.$$

As before NUM: $\mathbb{N}_0^n \to \mathbb{N}_0$ is the inverse of the enumeration of the elements in $\mathbb{N}_0^n$ with respect to the order on $\mathbb{N}_0^n$. Using the same arguments as above, we find

$$k^{d_N - n} \det(A_{N,\beta,k}) - \sum_{j=0}^{N} c_{\beta,j} k^{-j} = O(k^{-N-1}) \text{ in } C^0(D')$$

uniformly in $\varphi \in S_1$ and $\rho \in S_2$ where $c_{\beta,0}, \ldots, c_{\beta,N}$ are continuous functions defined on D'. We apply Lemma 2.5 for $\frac{k^{-d_N}}{\det(A_{N,k})}$ to get after multiplication with $k^{d_N - n} \det(A_{N,\beta,k})$

$$\lambda_{N,\beta,k} - k^n \sum_{j=0}^{N} \tilde{\lambda}^{(j)}_{N,\beta} k^{-j} = O(k^{-N-1+n}) \text{ in } C^0(D')$$

uniformly in $\varphi \in S_1$ and $\rho \in S_2$. Since

$$\det(A_{N,k}(z)) - a_{0,0,k} \det(A_{N,0,k}) = O(k^{-d_N - 1})$$

we find by (3.1) that $\tilde{\lambda}^{(0)}_{N,0}(z) = \pi^{-n} \det(H_\varphi(z))(a^{(0)}_{0,0}(z))^{-1}$ holds which implies by Theorem 2.50 $\tilde{\lambda}^{(0)}_{N,0}(z) = \pi^{-n} \det(H_\varphi(z))\rho(z)^{-1}$. It remains to show the first part of the statement, that is for $l \ge 0$, $|\alpha| \le N$, $0 \le j \le N$, $\varphi \in S_1 \cap C^{6N+4+l}(D)$ and $\rho \in S_2 \cap C^{4N+l+l}(D)$ we have $\tilde{\lambda}^{(j)}_{N,\alpha} = \lambda^{(j)}_{N,\alpha}|_{D'}$ where $\lambda^{(j)}_{N,\alpha} \in C^l(D_{\varphi,+})$ such that $\lambda^{(j)}_{N,\alpha}(z)$ only depends on the derivatives of φ and ρ at $z \in D_{\varphi,+}$. Given $z_0 \in D_{\varphi,+}$ we find by Lemma 2.47 an open neighborhood $U \subset D$ around z_0 such that $z_0 \in U_{\varphi|_{\overline{U}},N}$ holds. We apply the already proven part of Lemma 3.4 to the setting $D = U$, $S_1 = \{\varphi|_{\overline{U}}\}$ and $S_2 = \{\rho|_{\overline{U}}\}$ and find $\lambda^{(j),U}_{N,\alpha} \in C^l(U_{\varphi|_{\overline{U}},N})$, $0 \le j \le N$, because of $a^{(m)}_{\alpha,\beta} \in C^l(U_{\varphi|_{\overline{U}},N})$ by Theorem 2.50. Given another open neighborhood $V \subset U$ around z_0 we have $U_{\varphi|_{\overline{U}},N} \cap V \subset V_{\varphi|_{\overline{V}},N}$ and since $a^{(j)}_{\alpha,\beta,N}(z)$ can be expressed in terms of the derivatives of φ and ρ at $z \in U_{\varphi|_{\overline{U}},N}$ we verify that $\lambda^{(j),U}_{N,\alpha}(z_0) = \lambda^{(j),V}_{N,\alpha}(z_0)$ holds. In this way we can define $\lambda^{(j)}_{N,\alpha}\colon D_{\varphi,+} \to \mathbb{C}$ and find that $\lambda^{(j)}_{N,\alpha}(z_0)$ only depends on the derivatives of φ and ρ at z_0. The construction also implies that $\lambda^{(j)}_{N,\alpha}$ coincides with $\tilde{\lambda}^{(j)}_{N,\alpha}$ on D'. Furthermore, since $a^{(j)}_{\alpha,\beta,N} \in C^l(U_{\varphi|_{\overline{U}},N})$ for $\varphi \in C^{6N+4+l}(U)$ and $\rho \in C^{4N+2+l}(U)$ it follows that any point z_0 has an open neighborhood where $\lambda^{(j)}_{N,\alpha}$ is a C^l functions. Hence, we have $\lambda^{(j)}_{N,\alpha} \in C^l(D_{\varphi,+})$. $\square$

Corollary 3.5

Given $\varphi \in C^{6N+3n+4+l}(D) \cap S_1$ and $\rho \in C^{4N+2n2+l}(D) \cap S_2$ we have the following. For all $\alpha \in \mathbb{N}_0^n$, $|\alpha| \le N$, there exist functions $\lambda^{(0)}_{N,\alpha}, \dots, \lambda^{(N)}_{N,\alpha} \in C^l(D_{\varphi,+})$ such that

$$\lambda_{N,\alpha,k} - k^n \sum_{j=0}^{N} \lambda^{(j)}_{N,\alpha} k^{-j} = O(k^{-N-1}) \text{ in } C^0(D_{\varphi,N}).$$

Furthermore, we have $\lambda^{(0)}_{N,0}(z) = \pi^{-n}\rho(z)^{-1}\det(H_\varphi(z))$.

Remark 3.6

Given any open set $U \subset \mathbb{C}^n$, $\varphi \in C^{6N+3n+4}(U)$, $\rho \in C^{4N+2n2}(U)$ and $a \in U$ such that $H_\varphi(a)$ is positive definite Lemma 3.4 allows us to define $\lambda^{(j)}_{N,\alpha}(a)$ for $0 \le j \le N$, $\alpha \in \mathbb{N}_0^n$, $|\alpha| \le N$, by taking a bounded open neighborhood $V \subset\subset U$ around a with $a \in V_{\varphi|_{\overline{V}},N}$ (which always exists by Lemma 2.47). Then $\lambda^{(j)}_{N,\alpha}(a)$ is independent of the choice of V.

Lemma 3.7

We have $e^{k\tilde{\varphi}_N}|P_{N,k}|^2 = O(k^{2n})$ in $C^0(D' \times D)$ uniformly in $\varphi \in S_1$ and $\rho \in S_2$. In addition we find for any compact set $K \subset D'$ constants $\delta, C > 0$ such that

$$|P_{N,k}(z,w)| \le Ck^n e^{-\delta k|z-w|^2}$$

holds for all $(z,w) \in K \times \overline{D}$, $k \in [1,\infty)$, $\varphi \in S_1$ and $\rho \in S_2$.

Proof. Write

$$-\varphi(w) + \overline{\gamma_N(z,z)} + \gamma_N(z,w) + \gamma_N(z,z) + \overline{\gamma_N(z,w)} = -\tilde{\varphi}_N(z,w) + \varphi(z).$$

Thus, we find

$$e^{-k(\varphi(z)+\varphi(w))}|K_{N,k}(z,w)|^2 = e^{-k\tilde{\varphi}_N(z,w)} \left| \sum_{|\alpha|\le N} \lambda_{N,\alpha,k}(z)(w-z)^\alpha \right|^2.$$

The first part of the claim follows from $P_{N,k}(z,w) = e^{-k\frac{\varphi(z)+\varphi(w)}{2}}K_{N,k}(z,w)$ and Lemma 3.4. Using the assumptions on S_1 we find $\delta > 0$ such that $\tilde{\varphi}_N(z,w) \ge \delta|w-z|^2$ for all $(z,w) \in D' \times \overline{D}$ and all $\varphi \in S_1$. Since D is bounded the second part of the claim follows. $\square$

Lemma 3.8
For any compact subset $K \subset D'$ and $\varepsilon > 0$ there exists a constant $C > 0$ such that

$$|f(z) - (f, K_{N,k}(z,\cdot))_k|^2 e^{-k\varphi(z)} \leq Ck^{-(N+1)+n+\varepsilon}\|f\|_k^2$$

holds for all $z \in K$, $k \in [1,\infty)$, $\varphi \in S_1$, $\rho \in S_2$ and $f \in H_k^0(D)$.

Proof. Given $k \in [1,\infty)$ and $z \in D'$ we can write $H_k^0(D) = V_{N,z,k} \oplus V_{N,z,k}^{\perp}$ where $V_{N,z,k} \subset H_k^0(D)$ is defined as above and $V_{N,z,k}^{\perp}$ denotes its orthogonal complement in $H_k^0(D)$. Recall that we denote the Taylor expansion up to order N in z of a holomorphic function f on D by $T_{z,N}(f)$, i.e.

$$T_{z,N}(f)(w) = \sum_{|\alpha|\leq N} \frac{1}{\alpha!}\frac{\partial^{|\alpha|} f}{\partial^\alpha w}(z)(w-z)^\alpha.$$

We have another decomposition $H_k^0(D) = V_{N,z,k} \oplus W_{z,k}$ where

$$W_{N,z,k} = \left\{ f \in H_k^0(D) \mid \frac{\partial^{|\alpha|} f}{\partial^\alpha w}(z) = 0 \text{ for all } |\alpha| \leq N \right\}$$

and the respective projection on $V_{N,z,k}$ is given by $T_{z,N,k} : H_k^0(D) \to H_k^0(D)$ by $T_{z,N,k}(f)(w) = e^{k\gamma(z,w)} T_{z,N}(e^{-k\gamma(z,\cdot)} f(\cdot))(w)$ (see Lemma 2.64). By Theorem 2.66 there exists a constant $C_1 > 0$ such that $\|(\mathrm{Id} - T_{z,N,k})f\|_k \leq C_1\|f\|_k$ holds for all $k \in [1,\infty)$, $z \in D'$, $\varphi \in S_1$, $\rho \in S_2$ and $f \in V_{N,z,k}^{\perp}$.

Now, given $g \in V_{N,z,k}$, we have by Lemma 3.1 that $(K_{N,k}((z,\cdot)), g)_k = g(z)$ holds. For $f \in V_{N,z,k}^{\perp}$ we write $f = f_1 + f_2$ with respect to the decomposition $H_k^0(D) = V_{N,z,k} \oplus W_{N,z,k}$ and get for f_2 by Corollary 2.56 that

$$\sum_{|\alpha|\leq N} \left| \int_D f_2(w)\overline{(w-z)^\alpha e^{k\gamma(z,w)}} e^{-\varphi(w)} dV_D(w) \right|^2 \leq C_2 k^{-(N+1)-n+\varepsilon}\|f_2\|_k^2$$

holds for some constant $C_2 > 0$ independent of $k \in [1,\infty)$, $z \in D'$, $\varphi \in S_1$, $\rho \in S_2$ and f_2. Thus, using Lemma 3.4 one has

$$|(f_2, K_{N,k}(z,\cdot))_k|^2 e^{-k\varphi(z)} \leq C_3 k^{-(N+1)+n+\varepsilon}\|f_2\|_k^2 \leq C_1C_3 k^{-(N+1)+n+\varepsilon}\|f\|_k^2$$

for some constant $C_3 > 0$ independent of $k \in [1,\infty)$, $z \in D'$, $\varphi \in S_1$, $\rho \in S_2$ and f_2. Since

$$|(f+g)(z) - (f+g, K_{N,k}(z,\cdot))_k|^2 e^{-k\varphi(z)} = |(f_2, K_{N,k}(z,\cdot))_k|^2 e^{-k\varphi(z)}$$

and $\|f+g\|_k^2 = \|f\|_k^2 + \|g\|_k^2$ the claim follows. □

From this lemma we obtain

Lemma 3.9
For any $0 < \varepsilon < 1$ *one has*

$$\begin{aligned} B_k - B_{N,k} &= O(k^{-N-1+n+\varepsilon}) \text{ in } C^0(D'), \\ P_k - P_{N,k} &= O(k^{-\frac{N+1}{2}+n+\varepsilon}) \text{ in } C^0(D' \times D) \end{aligned}$$

uniformly in $\varphi \in S_1$ *and* $\rho \in S_2$.

Proof. Writing $f(z) = (f, K_k(z,\cdot))_k$ and putting $f(\cdot) = K_k(z,\cdot) - K_{N,k}(z,\cdot)$ we find by using Lemma 3.8 that

$$\begin{aligned} Ck^{-(N+1)+n+\varepsilon} &\geq \|K_k(z,\cdot) - K_{N,k}(z,\cdot)\|_k^2 e^{-k\varphi(z)} \\ &= e^{-k\varphi(z)}(K_k(z,z) + K_{N,k}(z,z) - 2\,\mathrm{Re}(K_{N,k}(z,\cdot), K_k(z,\cdot))_k) \\ &= e^{-k\varphi(z)}K_k(z,z) - e^{-k\varphi(z)}K_{N,k}(z,z) \\ &= B_k(z) - B_{N,k}(z) \geq 0 \end{aligned}$$

where we use the reproducing property of $K_k(z,\cdot)$, that $K_{N,k}(z,z)$ is real and $B_k(z) \geq B_{N,k}(z)$ since B_k is the Bergman kernel function for a larger space (see Lemma 2.36). Since K_k and $K_{N,k}$ are holomorphic in the second argument we can apply Lemma 2.29 and get

$$\begin{aligned} Ck^n\|K_k(z,\cdot) - K_{N,k}(z,\cdot)\|_k^2 &\geq e^{-k\varphi(w)}|K_k(z,w) - K_{N,k}(z,w)|^2 \\ &= e^{k\varphi(z)}|P_k(z,w) - P_{N,k}(z,w)|^2. \end{aligned}$$

□

Corollary 3.10
Let $K \subset D' \times D$ *be compact. Given* $\varepsilon > 0$ *there exist constants* $C, \delta > 0$ *such that* $|P_{k\varphi,\rho}(z,w)| \leq C\left(k^n e^{-\delta k|w-z|^2} + k^{-\frac{N+1}{2}+n+\varepsilon}\right)$ *holds for all* $k \in [1,\infty)$, $(z,w) \in K$, $\varphi \in S_1$ *and* $\rho \in S_2$.

Proof. The statement is a direct consequence of Lemma 3.7 and Lemma 3.9. □

Now, define $\tilde{K}_{N,k} = \tilde{K}_{k\varphi,\rho,N} : D' \times \overline{D} \to \mathbb{C}$

$$\tilde{K}_{N,k}(z,w) = e^{k(\gamma_N(z,z)+\gamma_N(z,w))}k^n \sum_{j=0}^{N} k^{-j} \sum_{|\alpha|\leq N} \lambda_{N,\alpha}^{(j)}(z)(w-z)^\alpha$$

with $\lambda_{N,\alpha}^{(j)}$ as in Lemma 3.4 and set

$$\tilde{P}_{N,k}(z,w) = e^{-k\frac{\varphi(z)+\varphi(w)}{2}}\overline{\tilde{K}_{N,k}(z,w)} \quad \text{and} \quad \tilde{B}_{N,k}(z) = \tilde{P}_{N,k}(z,z)$$

for $z \in D'$, $w \in \overline{D}$. Lemma 3.9 and Lemma 3.4 imply $B_k - \tilde{B}_{N,k} = O(k^{-N-1+n+\varepsilon})$ in $C^0(D')$ which proves that the Bergman kernel function B_k for $H^0_k(D)$ has an asymptotic expansion of order N on D' in C^0-norm. To get an expansion in C^l-norm for $l \geq 0$ we need the following.

Lemma 3.11 (Hörmander's Trick)
Let $U \subset \mathbb{R}^n$ be a domain. For any $V \subset\subset U$ there exists a constant $C > 0$ such that

$$|d_x f(x)|^2 \leq C\|f\|_{C^0(U)}(\|f\|_{C^0(U)} + \|f\|_{C^2(U)})$$

holds for all $x \in V$ and all $f \in C^2(U)$.

Proof. Given a non-negative function $g \in C^2((-\delta,\delta),\mathbb{R})$ one has

$$\delta^2|g'(0)|^2 \leq g(0)(g(0) + 2\sup_{|x|<\delta}\delta^2|g''(x)|).$$

A proof of this statement can be found for example in [24, Lemma 7.7.2]. Replacing g by $g + \sup_{|x|<\delta}|g(x)|$ and considering real and imaginary part separately shows that

$$\delta^2|g'(0)|^2 \leq 8\sup_{|x|<\delta}|g(x)|(\sup_{|x|<\delta}|g(x)| + \sup_{|x|<\delta}\delta^2|g''(x)|)$$

holds for all $g \in C^2(-\delta,\delta)$. Given any $x \in V$ choose $\delta > 0$ such that $\mathbb{B}_\delta(x) \subset U$ holds. For $1 \leq j \leq n$ put $g_j(t) = f(x + te_j)$, where e_j is the vector which has a one at the j-th position and all other entries are zero. We have $\frac{\partial f}{\partial x_j}(x) = g_j'(0)$ and hence

$$\begin{aligned} |d_x f(x)|^2 &= \sum_{j=1}^n |g_j'(0)|^2 \leq 8\sum_{j=1}^n \sup_{|x|<\delta}|g_j(x)|(\delta^{-2}\sup_{|x|<\delta}|g_j(x)| + \sup_{|x|<\delta}|g_j''(x)|) \\ &\leq 8n\|f\|_{C^0(U)}(\delta^{-2}\|f\|_{C^0(U)} + \|f\|_{C^2(U)}). \end{aligned}$$

Choosing $\delta > 0$ such that $\mathbb{B}_\delta(x) \subset U$ holds for all $x \in V$ finishes the proof. □

Lemma 3.12
Given $l \in \mathbb{N}_0$ put $\tilde{S}_1 := S_1 \cap C^{6N+3n+5+l}(D')$ and $\tilde{S}_2 := S_2 \cap C^{2N+2n+3+l}(D')$. Assuming that $\tilde{S}_1$ and $\tilde{S}_2$ are bounded subsets of $C^{6N+3n+5+l}(D')$ and $C^{2N+2n+3+l}(D')$ respectively one has for any $0 < \varepsilon < 1$

$$\begin{aligned} B_k - \tilde{B}_{N,k} &= O(k^{-c_r(N+1)+n+r}) \text{ in } C^r(D'), \\ P_k - \tilde{P}_{N,k} &= O(k^{-c_r\frac{N+1}{2}+n+r}) \text{ in } C^r(D' \times D) \end{aligned}$$

uniformly in $\varphi \in \tilde{S}_1$ and $\rho \in \tilde{S}_2$ for $1 \leq r \leq l$ with $c_r = 1 - \frac{r}{l+1}$.

Proof. We are going to prove a slightly more stronger result, that is for any $\varepsilon > 0$ we have

$$\begin{aligned} B_k - \tilde{B}_{N,k} &= O(k^{-c_r(N+1-\varepsilon)+n+r}) \text{ in } C^r(D'), \\ P_k - \tilde{P}_{N,k} &= O(k^{-c_r\frac{N+1-\varepsilon}{2}+n+r}) \text{ in } C^r(D' \times D) \end{aligned} \tag{3.2}$$

uniformly in $\varphi \in \tilde{S}_1$ and $\rho \in \tilde{S}_2$ for $1 \leq r \leq l$ with $c_r = 1 - r\frac{1-2^{-l}}{l+1}$. By Lemma 2.39 and Lemma 3.4 we have that $d_x^\alpha \tilde{B}_{N,k}, d_x^\alpha B_k = O(k^{n+|\alpha|})$ and $d_x^\alpha d_y^\beta \tilde{P}_{N,k}, d_x^\alpha d_y^\beta P_k = O(k^{n+|\alpha|+|\beta|})$ in $C^0(D')$ or $C^0(D' \times D)$ respectively, uniformly in $\varphi \in \tilde{S}_1$ and $\rho \in \tilde{S}_2$ for any $\alpha, \beta \in \mathbb{N}_0^n$, $|\beta| + |\alpha| \leq l$. Hence we find that (3.2) is true for $c_r = c_r^{(0)} = 0$, $1 \leq r \leq l$. For $m \in \mathbb{N}_0$ we write $c^{(m)} = (c_1^{(m)}, \ldots, c_l^{(m)})^T \in \mathbb{R}^l$. Applying Lemma 3.11 and using the estimate in C^0-norm from Lemma 3.9 we find that if (3.2) was true for $c_r = c_r^{(m)}$, $1 \leq r \leq l$, we have that the statement will be true for $c_r = c_r^{(m+1)}$, $1 \leq r \leq l$, where $c_r^{(m+1)}$ is defined by

$$c_r^{(m+1)} = \begin{cases} \frac{c_2^{(m)}}{2} + \frac{1}{2} & \text{, for } r = 1, \\ \frac{c_{l-1}^{(m)}}{2} + \frac{1}{2^{l+1}} & \text{, for } r = l, \\ \frac{c_{r+1}^{(m)}}{2} + \frac{c_{r-1}^{(m)}}{2} & \text{, else.} \end{cases}$$

We can rewrite this as $c^{(m+1)} = F(c^{(m)})$ where $F \colon \mathbb{R}^l \to \mathbb{R}^l$ is the affine map defined by $F(v) = Av + b$ with

$$A = \frac{1}{2}\begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & & \ddots & \\ & & & 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 2^{-1} \\ 0 \\ \vdots \\ 0 \\ 2^{-l-1} \end{pmatrix}.$$

Setting $c^{(0)} = 0$ and $c^{(m+1)} = F(c^{(m)})$ for $m \in \mathbb{N}_0$, it follows from induction that the statement holds for all $c_r^{(m)}$, $1 \leq r \leq l$, and $m \in \mathbb{N}_0$. We are now going to show that $c^{(m)} \to c = (c_1, \ldots, c_l)$ for $m \to \infty$ with $c_r = 1 - r\frac{1-2^{-l}}{l+1}$, $1 \leq r \leq l$. The constant $\varepsilon > 0$ ensures that the statement (3.2) follows after finitely many iterations.

For $l = 1$ we immediately find $c_1^{(m)} = \frac{1}{2}$ for all $m \geq 1$. Given $l \geq 2$ we start by observing that the restriction $F \colon [0,1]^l \to [0,1]^l$ is well defined. Let $\|A\|$ denote the operator norm of A with respect to the standard Euclidean norm on $\mathbb{R}^l$. Applying the definition of the operator norm to A it follows $\|A\| \leq 1$. Since A is a real symmetric matrix we have that it is diagonalizable with eigenvalues $\lambda_1, \ldots, \lambda_l \in \mathbb{R}$. Furthermore, we have $\|A\| = \max\{|\lambda_1|, \ldots, |\lambda_l|\}$, which implies $\lambda_1, \ldots, \lambda_l \in [-1, 1]$. Solving the the equation $(A \pm \mathrm{Id})v = 0$ recursively shows $\ker(A \pm \mathrm{Id}) = \{0\}$ and hence $\lambda_1, \ldots, \lambda_l \in (-1, 1)$ which implies $\|A\| < 1$. We conclude that F is a contraction.

Since $[0,1]^l \subset \mathbb{R}^l$ is closed we deduce from the Banach fixed-point theorem that $\lim_{m\to\infty} c^{(m)} = c$ where c is given by the solution of $F(c) = c$. Finding c then leads to solving the equation $(2\,\mathrm{Id} - 2A)c = 2b$. This can be written as $c_2 = 2c_1 - 1$, $2c_l - c_{l-1} = 2^{-l}$ and $c_r = 2c_{r-1} - c_{r-2}$, $3 \le r \le l$ which in fact leads to the problem of finding $c_1 \in [0,1]$ such that

$$(-1,2)\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}^{l-2} \begin{pmatrix} c_1 \\ 2c_1 - 1 \end{pmatrix} = 2^{-l} \tag{3.3}$$

holds. Putting

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

we observe $Bv_1 = v_1$, $Bv_2 = v_1 + v_2$, $(c_1, 2c_1 - 1)^T = (3c_1 - 2)v_1 + (c_1 - 1)v_2$. Because of $(-1,2)v_1 = 1$ and $(-1,2)v_2 = 0$ we have that (3.3) is equivalent to $(3c_1 + 2) + (l-2)(c_1 - 1) = 2^{-l}$ which has $c_1 = \frac{l+2^{-l}}{l+1} = 1 - \frac{1-2^{-l}}{l+1}$ as its unique solution. It follows $c_2 = 2c_1 - 1 = 1 - 2\frac{1-2^{-l}}{l+1}$ and since $c_r = 2c_{r-1} - c_{r-2} = (-1,2)B^{r-3}((3c_1 - 2)v_1 + (c_1 - 1)v_2)$, $3 \le r \le l$, we conclude that $c_r = 1 - r\frac{1-2^{-l}}{l+1}$ holds for all $1 \le r \le l$. Choosing $\varepsilon > 0$ sufficiently small proves the original statement. $\square$

3.2 Coefficients

Given $N \in \mathbb{N}_0$ let $\varphi \in C^{6N+3n+4}(U,\mathbb{R})$, $\rho \in C^{4N+2n+2}(U,\mathbb{R})$ two functions defined in a neighborhood U around a point $z_0 \in \mathbb{C}^n$. Before we start to compute the coefficients in the expansion of the Bergman kernel, that is $\lambda_{N,0}^{j,\varphi,\rho}(z_0) := \lambda_{N,0}^{j}(z_0)$ defined in Lemma 3.4 we need to develop some basic tools in order to reduce the computations to simpler cases.

Lemma 3.13
Let $f, g \in \mathcal{O}(U)$ be holomorphic functions and let $\psi \in C^{6N+3n+4}(U,\mathbb{R})$, $\tilde\rho \in C^{6N+3n+4}(U,\mathbb{R})$ be defined by $\psi(z) = \varphi(z) + f(z) + \overline{f(z)}$ and $\tilde\rho(z) = e^{g(z)+\overline{g(z)}}\rho(z)$. We have $H_\psi(z_0)$ is positive definite and $\lambda_{N,0}^{(j),\varphi,\rho}(z_0) = e^{g(z)+\overline{g(z)}}\lambda_{N,0}^{(j),\psi,\tilde\rho}(z_0)$ for all $0 \le j \le N$.

Proof. We have $B_{k\varphi,\rho} = e^{g(z)+\overline{g(z)}}B_{k\psi,\tilde\rho}$ by Lemma 2.40. Since f is holomorphic we find $\overline{\partial} f = \partial \overline{f} = 0$ and hence we conclude $H_\varphi(z_0) = H_\psi(z_0)$ by the definition of the complex Hessian. Using Lemma 2.47 we find an open neighborhood $D \subset\subset U$ around z_0 such that $z_0 \in D_{\varphi,N} \cap D_{\psi,N}$ holds. From Lemma 3.9 it follows that

$$\sum_{j=0}^{N}(\lambda_{N,0}^{(j),\varphi,\rho}(z_0) - \lambda_{N,0}^{(j),\psi,\rho}(z_0))k^{-j} \le Ck^{-N-1-\varepsilon}$$

for some constants $C > 0$ and $0 < \varepsilon < 1$ which proves the statement. $\square$

Lemma 3.14
Given an invertible $n\times n$-matrix F consider the map $G\colon \mathbb{C}^n \to \mathbb{C}^n$ defined by $G(w) = F(w-z_0)+z_0$. We have $\varphi\circ G \in C^{6N+3n+4}(G^{-1}(U),\mathbb{R})$, $\rho\circ G \in C^{4N+2n+2}(G^{-1}U,\mathbb{R})$, $H_{\varphi\circ G}(z_0) = F^* H_\psi(z_0)F$ is positive definite and $\lambda_{N,0}^{(j),\varphi\circ G,\rho\circ G}(z_0) = |\det(F)|^2 \lambda_{N,0}^{(j),\varphi,\rho}(z_0)$ for all $0 \leq j \leq N$. In addition, for $\alpha, \beta \in \mathbb{N}_0^n$ and $\psi \in C^{|\alpha|+|\beta|}(U \cap G^{-1}(U))$ we have $\overline{\partial_w^\alpha} \partial_w^\beta \psi \circ G(z_0) = \overline{X_w^\alpha} X_w^\beta \psi(z_0)$ with

$$(X_w)^\alpha = \prod_{m=1}^{n} (X_{w,m})^{\alpha_m}, \ \ X_{w,m} = F_{m1}\frac{\partial}{\partial w_1} + \ldots + F_{mn}\frac{\partial}{\partial w_n}$$

where $F = (F_{lm})_{1\leq l,m\leq n}$.

Proof. Since G is holomorphic (even affine linear) we find $H_{\varphi\circ G}(z_0) = F^* H_\varphi(z_0)F$ which immediately implies that $H_{\varphi\circ G}(z_0)$ is positive definite. Since $G(z_0) = z_0$ and using Lemma 2.47 we find an open neighborhood $D \subset\subset U \cap G^{-1}(U)$ around z_0 such that $z_0 \in D_{\varphi,N} \cap D_{\varphi\circ G,N}$ holds. We have $|\det(F)|^2 B_{k\varphi|_D} = B_{k\varphi\circ G|_D}$ by Lemma 2.41. As in the proof of Lemma 3.13 the claim follows from Lemma 3.9. The last part of the statement follows from the chain rule and the fact that the differential of G is constant by induction. □

Lemma 3.15
We have $\lambda_{M,0}^{(j)}(z_0) = \lambda_{N,0}^{(j)}(z_0)$ for any $M \leq N$ and $j \leq M$.

Proof. Using Lemma 2.47 we find an open neighborhood $D \subset\subset U$ around z_0 such that $z_0 \in D_{\varphi,N} \cap D_{\varphi,M}$ holds. From Lemma 3.9 it follows that

$$\sum_{j=0}^{M} (\lambda_{M,0}^{(j)}(z_0) - \lambda_{N,0}^{(j)}(z_0))k^{-j} \leq Ck^{-M-1-\varepsilon}$$

for some constants $C > 0$ and $0 < \varepsilon < 1$ which proves the statement. □

Lemma 3.16
Let $d, n \in \mathbb{N}$ be some positive integers and $A_1, \ldots, A_d \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ matrices, $A_j = (a_{l,m}^{(j)})_{1\leq l,m,\leq n}$ for $j = 1, \ldots, d$. Denote by $B = (b_{l,m})_{0\leq l,m,\leq n} \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ the product of the A_j's, i.e $B = A_1 \cdot \ldots \cdot A_d$. One has that

$$b_{l,m} = \sum_{\substack{\alpha\in\mathbb{N}^{d-1} \\ \alpha_1,\ldots,\alpha_{d-1}\leq n}} a_{l,\alpha_1}^{(1)} a_{\alpha_1,\alpha_2}^{(2)} \ldots a_{\alpha_{d-1},m}^{(d)}$$

holds for all $1 \leq l, m \leq n$.

Proof. We prove the statement via induction with respect to d. Given the case $d = 1$ there is nothing to show. Let $d \in \mathbb{N}$ be arbitrary and assume that the statement holds for d. We set $B' = A_1 \cdot \ldots \cdot A_d$, $B' = (b'_{l,m})_{0\le l,m,\le n}$. One has $B = B'A_{d+1}$, i.e. $b_{l,m} = \sum_{j=1}^n b'_{l,j} a^{(d+1)}_{j,m}$. Using the induction hypothesis one finds

$$b'_{l,j} = \sum_{\substack{\alpha\in\mathbb{N}^{d-1}\\ \alpha_1,\ldots,\alpha_{d-1}\le n}} a^{(1)}_{l,\alpha_1} a^{(2)}_{\alpha_1,\alpha_2} \ldots a^{(d)}_{\alpha_{d-1},j}$$

and hence

$$b_{l,m} = \sum_{\substack{\alpha\in\mathbb{N}^{d-1}\\ \alpha_1,\ldots,\alpha_{d-1}\le n}} \sum_{j=1}^{n} a^{(1)}_{l,\alpha_1} a^{(2)}_{\alpha_1,\alpha_2} \ldots a^{(d)}_{\alpha_{d-1},j} a^{(d+1)}_{j,m}$$

which proves that the statement holds for $d+1$. □

Theorem 3.17

Given a point $z_0 \in \mathbb{C}^n$ and $\varphi \in C^{6N+3n+4}(U,\mathbb{R})$, $\rho \in C^{4N+2n+2}(U,\mathbb{R})$ two functions defined in a neighborhood U around z_0 such that $H_\varphi(z_0)$ is positive definite and $\rho(z_0) > 0$. We have the following expression for $\lambda^{(j)}_{N,\alpha}(z_0)$ constructed in Lemma 3.4. We have $\lambda^{(0)}_{N,0}(z_0) = \frac{\det(H_\varphi(z_0))}{\pi^n \rho(z)}$ and assuming $H_\varphi(z_0) = \mathrm{diag}(\lambda_1,\ldots,\lambda_n)$ we find

$$\begin{aligned} \lambda^{(j)}_{N,\tau}(z_0) &= \frac{\det(H_\varphi(z_0))}{\pi^n \rho(z)} \frac{\lambda^\tau}{\tau!} b_{N,\tau,j}(z_0), \qquad (3.4)\\ b_{N,\tau,j}(z_0) &= \sum_{d=1}^{2j+|\tau|} \sum_{\substack{\alpha\in\mathbb{N}^d\\ |\alpha|=2j+|\tau|}} \sum_{\substack{(\beta^{(1)},\ldots,\beta^{(d)})\in(\mathbb{N}_0^n)^{d-1}\\ |\beta^{(1)}|,\ldots,|\beta^{(d-1)}|\le N}} (-1)^d \nu^{(\alpha_1)}_{\tau,\beta^{(1)}} \nu^{(\alpha_2)}_{\beta^{(1)},\beta^{(2)}} \cdot \ldots \cdot \nu^{(\alpha_d)}_{\beta^{(d-1)},0} \end{aligned}$$

for $1 \le j \le N$ if $\tau = 0$ and for $0 \le j \le N - \frac{|\tau|}{2}$ if $1 \le |\tau| \le N$ where

$$\nu^{(r)}_{\alpha,\beta} = \frac{\lambda^\beta}{\beta!} \frac{\chi^{(r)}_{|\alpha|,|\beta|}}{\rho(z)} \sum_{l=0}^{r+|\alpha|+|\beta|} \sum_{\substack{|\eta|=l+\frac{r+|\alpha|+|\beta|}{2}\\ \eta\ge\max\{\alpha,\beta\}}} (-1)^l \frac{\eta!}{l!\lambda^\eta} \mu^{(l)}_{\eta-\alpha,\eta-\beta}$$

$$\chi^{(r)}_{p,q} = \begin{cases} 1 & \text{, if } 2 \mid (r+p+q) \text{ and } r \ge |p-q| \text{ ,}\\ 0 & \text{, else.}\end{cases}$$

and

$$\mu^{(l)}_{\alpha,\beta} = \sum_{\substack{(\alpha^{(0)},\ldots,\alpha^{(l)})\in(\mathbb{N}_0^n)^{l+1}\\ |\alpha^{(0)}_m|+\ldots+|\alpha^{(l)}_m|=\alpha_m}} \sum_{\substack{(\beta^{(0)},\ldots,\beta^{(l)})\in(\mathbb{N}_0^n)^{l+1}\\ |\beta^{(0)}_m|+\ldots+|\beta^{(l)}_m|=\beta_m}} \frac{\overline{\partial_w^{\alpha^{(0)}}} \partial_w^{\beta^{(0)}} \rho(z_0)}{\alpha^{(0)}!\beta^{(0)}!} \cdot \prod_{j=1}^{l} \frac{\overline{\partial_w^{\alpha^{(j)}}} \partial_w^{\beta^{(j)}} h_{N,z_0}(z_0)}{\alpha^{(j)}!\beta^{(j)}!}.$$

Here h_{N,z_0} is given by

$$h_{N,z}(w) = \tilde{\varphi}_N(z,w) - (w-z)^T H_\varphi(z) \overline{(w-z)}.$$

Furthermore, for $\tau = 0$ and $H_\varphi(z_0)$ not necessarily diagonal let F be an invertible matrix such that $F^* H_\varphi(z_0) F = \mathrm{diag}(\lambda_1,\ldots,\lambda_n)$ for some $\lambda_1,\ldots,\lambda_n \in \mathbb{R}_+$. In *that case* (3.4) *stays true when we can replace* $\overline{\partial_w^\alpha}\partial_w^\beta\rho(z_0)$ *by* $\overline{(X_w)^\alpha}(X_w)^\beta\rho(z_0)$, *and* $\overline{\partial_w^\alpha}\partial_w^\beta h_{N,z_0}(z_0)$ *by*

$$\varphi_{\alpha,\beta}(z_0) = \begin{cases} \left(\overline{X_w^\alpha}X_w^\beta\varphi\right)(z) & , \textit{if } \max\{|\alpha|,|\beta|\} \geq 2, \min\{|\alpha|,|\beta|\} \geq 1, \\ 0 & , \textit{else,} \end{cases}$$

with

$$(X_w)^\alpha = \prod_{m=1}^{n} X_{w,m}^{\alpha_m}, \quad X_{w,m} = F_{m1}\frac{\partial}{\partial w_1} + \ldots + F_{mn}\frac{\partial}{\partial w_n}$$

where $F = (F_{lm})_{1\leq l,m\leq n}$ *and noticing that* $X_w^\alpha = \partial_w^\alpha$ *when* $F = \mathrm{Id}$.

Proof. Recall that

$$a_{\alpha,\beta,k}(z_0) = \int_D \overline{(w-z_0)^\alpha}(w-z_0)^\beta e^{-k\tilde{\varphi}_N(z_0,w)}dV_D(w).$$

Define

$$c_{\alpha,\beta,k} = \frac{\det(H_\varphi(z))}{\pi^n\rho(z)}\sqrt{\frac{\lambda^\alpha\lambda^\beta}{\alpha!\beta!}}k^{n+\frac{1}{2}(|\alpha|+|\beta|)}a_{\alpha,\beta,k}(z_0).$$

From Theorem 2.50 we get

$$a_{\alpha,\beta,k}(z_0) - \frac{\pi^n}{\det(H_\varphi(z))}k^{-n}\sum_{j=\max\{|\alpha|,|\beta|\}}^{2N} k^{-j}a_{\alpha,\beta}^{(j)}(z_0) = O(k^{-2N-1-n}).$$

Replacing j by $(j+|\alpha|+|\beta|)/2$ we find since $j+|\alpha|+|\beta| \geq 2\max\{|\alpha|,|\beta|\}$ if and only if $j \geq ||\alpha|-|\beta||$ that

$$c_{\alpha,\beta,k} - \sum_{j=0}^{2N} k^{-\frac{j}{2}}c_{\alpha,\beta}^{(j)} = O(k^{-N-1}), \;\; c_{\alpha,\beta}^{(j)} = \frac{\chi_{|\alpha|,|\beta|}^{(j)}}{\rho(z)}\sqrt{\frac{\lambda^\alpha\lambda^\beta}{\alpha!\beta!}}a_{\alpha,\beta}^{(\frac{j+|\alpha|+|\beta|}{2})}(z_0) \tag{3.5}$$

with

$$\chi_{p,q}^{(j)} = \begin{cases} 1 & , \text{if } 2 \mid (j+p+q), \;\; j \geq |p-q| \;, \\ 0 & , \text{else.} \end{cases}$$

As in Section 3.1 set $C_{N,k}(z) = (c_{\alpha,\beta,k}(z))_{0\leq|\alpha|,|\beta|\leq N}$. Let $S_{N,k}$ be a diagonal matrix with entries $\left(\sqrt{\frac{\det(H_\varphi(z))\lambda^\alpha}{\pi^n\rho(z)\alpha!}}k^{\frac{1}{2}(n+|\alpha|)}\right)_{|\alpha|\leq N}$. One has $S_{N,k}A_{N,k}S_{N,k} = C_{N,k}$ and hence $A_{N,k}^{-1} = S_{N,k}C_{N,k}^{-1}S_{N,k}$. Thus, it lasts out to calculate $C_{N,k}^{-1}$. Therefore, we write $C_{N,k} - \sum_{j=0}^{2N} k^{-\frac{j}{2}}C_j = (O(k^{-N-1}))$ with $C_j = (c_{\alpha,\beta}^{(j)})_{0\leq|\alpha|,|\beta|\leq N}$. Here $(O(k^{-N-1}))$ denotes a matrix of suitable size such that any entry is an $O(k^{-N-1})$. Since $c_{\alpha,\beta}^{(0)} = 1$ if $\alpha = \beta$ and $c_{\alpha,\beta}^{(0)} = 0$ otherwise we find $C_0 = \mathrm{Id}$. Put

$$\tilde{C}_{N,k} = -\sum_{j=1}^{2N} k^{-\frac{j}{2}}C_j \quad \text{and} \quad C'_{N,k} = \sum_{d=0}^{2N}(\tilde{C}_{N,k})^d.$$

One has $\tilde{C}_{N,k} = (O(k^{-\frac{1}{2}}))$ and hence $\left(\tilde{C}_{N,k}\right)^{2N+1} = (O(k^{-N-\frac{1}{2}}))$. Thus we find that $C_{N,k}C'_{N,k} - \mathrm{Id} = C'_{N,k}C_{N,k} - \mathrm{Id} = -\tilde{C}_{N,k}^{2N+1} = \mathrm{Id} + (O(k^{-N-\frac{1}{2}}))$ holds. We need to prove that this implies $C'_{N,k} - C_{N,k}^{-1} = (O(k^{-N-\frac{1}{2}}))$. As in the proof of Lemma 3.4 we find $\delta > 0$ such that $\det(C_{N,k}) \geq \delta$ holds for all $k \in [1, \infty)$ and that $\det(C_{N,k})$ as well as all the subdeterminants of $C_{N,k}$ is an $O(1)$. Applying Cramer's rule we find $C_{N,k}^{-1} = (O(1))$ which implies $C'_{N,k} - C_{N,k}^{-1} = C_{N,k}^{-1}(C_{N,k}C'_{N,k} - \mathrm{Id}) = (O(k^{-N-\frac{1}{2}}))$. Thus, we can write

$$C_{N,k}^{-1} - \sum_{j=0}^{2N} k^{-\frac{j}{2}} C_N'^{(j)} = (O(k^{-N-\frac{1}{2}})), \quad C_N'^{(j)} = \sum_{d=1}^{j} \sum_{\substack{\eta \in \mathbb{N}^d \\ |\eta| = j}} (-1)^d \prod_{l=1}^{d} C_{\eta_l}.$$

Let c_k^α (resp. $c^{\alpha,(j)}$) denote the entry of $C_{N,k}^{-1}$ (resp. $C_N'^{(j)}$) at position $(\alpha, 0)$. One has

$$c_k^\alpha = \sum_{j=0}^{2N} k^{-\frac{j}{2}} c^{\alpha,(j)} + O(k^{-N-\frac{1}{2}}). \tag{3.6}$$

Since $\lambda_{N,\alpha,k}$ is the entry of $A_{N,k}^{-1}$ at position $(\alpha, 0)$ we find

$$k^{-n}\lambda_{N,\alpha,k}(z_0) = \frac{\det(H_\varphi(z_0))}{\pi^n \rho(z_0)} \sqrt{\frac{\lambda^\alpha}{\alpha!}} k^{\frac{|\alpha|}{2}} c_k^\alpha. \tag{3.7}$$

From Lemma 3.4 we have $k^{-n}\lambda_{N,\alpha,k} - \sum_{j=0}^{N} k^{-j} \lambda_{N,\alpha}^{(j)} = O(k^{-N-1})$. Plugging (3.6) into (3.7) we find after comparing the coefficients that

$$\lambda_{N,\alpha}^{(j)}(z_0) = \frac{\det(H_\varphi(z_0))}{\pi^n \rho(z_0)} \sqrt{\frac{\lambda^\alpha}{\alpha!}} c^{\alpha,(2j+|\alpha|)}$$

holds for $j \in \mathbb{N}_0$, $0 \leq j \leq N - \frac{|\alpha|}{2}$. Let us compute now $c^{\alpha,(2j+|\alpha|)}$. It follows from Lemma 3.16 that

$$c^{\alpha,(2j+|\alpha|)} = \sum_{d=1}^{2j+|\alpha|} \sum_{\substack{\tau \in \mathbb{N}^d \\ |\tau| = 2j+|\alpha|}} (-1)^d \sum_{\substack{\eta \in (\mathbb{N}_0^n)^{d-1} \\ |\eta_1|, \ldots, |\eta_{d-1}| \leq N}} c_{\alpha,\eta_1}^{(\tau_1)} c_{\eta_1,\eta_2}^{(\tau_2)} \ldots c_{\eta_{d-1},0}^{(\tau_d)}.$$

From Lemma 2.53 we find

$$a_{\alpha,\beta}^{(j)}(z) = \sum_{l=0}^{2j} \sum_{\substack{|\eta| = l+j \\ \eta \geq \max\{\alpha,\beta\}}} (1/\lambda)^\eta \frac{(-1)^l \eta!}{l!(\eta-\alpha)!(\eta-\beta!)} \overline{\partial}_w^{\eta-\alpha} \partial_w^{\eta-\beta} (h_{N,z}^l \rho)(z)$$

which implies

$$c_{\alpha,\beta}^{(j)} = \frac{\chi_{|\alpha|,|\beta|}^{(j)}}{\rho(z)} \sqrt{\frac{\lambda^\alpha \lambda^\beta}{\alpha!\beta!}} \sum_{l=0}^{j+|\alpha|+|\beta|} \sum_{\substack{|\eta| = l + \frac{j+|\alpha|+|\beta|}{2} \\ \eta \geq \max\{\alpha,\beta\}}} (-1)^l \frac{\eta!}{l!\lambda^\eta} \mu_{\eta-\alpha,\eta-\beta}^{(l)}$$

where $\mu_{\alpha,\beta}^{(l)} = \frac{1}{\alpha!\beta!}\overline{\partial}_w^\alpha \partial_w^\beta (h_{N,z_0}^l \rho)(z_0)$. Using Lemma 2.9 we get

$$\mu_{\alpha,\beta}^{(l)} = \sum_{\alpha'\leq\alpha}\sum_{\beta'\leq\beta}\left(\frac{1}{(\alpha-\alpha')!}\frac{1}{(\beta-\beta')!}\overline{\partial}_w^{\alpha-\alpha'}\partial_w^{\beta-\beta'}h_{N,z_0}^l(z_0)\right)\left(\frac{1}{\alpha'!}\frac{1}{\beta'!}\overline{\partial}_w^{\alpha'}\partial_w^{\beta'}\rho(z_0)\right).$$

Writing $h_{N,z_0}^l = h_{N,z_0}\cdots h_{N,z_0}$ and proceeding inductively we find

$$\mu_{\alpha,\beta}^{(l)} = \sum_{\substack{(\alpha^{(0)},\ldots,\alpha^{(l)})\in(\mathbb{N}_0^n)^{l+1}\\ |\alpha_m^{(0)}|+\ldots+|\alpha_m^{(l)}|=\alpha_m}}\sum_{\substack{(\beta^{(0)},\ldots,\beta^{(l)})\in(\mathbb{N}_0^n)^{l+1}\\ |\beta_m^{(0)}|+\ldots+|\beta_m^{(l)}|=\beta_m}}\frac{\overline{\partial_w^{\alpha^{(0)}}}\partial_w^{\beta^{(0)}}\rho(z_0)}{\alpha^{(0)}!\beta^{(0)}!}\cdot\prod_{j=1}^l\frac{\overline{\partial_w^{\alpha^{(j)}}}\partial_w^{\beta^{(j)}}h_{N,z_0}(z_0)}{\alpha^{(j)}!\beta^{(j)}!}.$$

Now consider the case $\tau = 0$ and $F = \mathrm{Id}$. We want to show that we can replace $(\overline{\partial}_w^\alpha\partial_w^\beta h_{N,z_0})(z_0)$ by $\varphi_{\alpha,\beta}(z_0)$ where $\varphi_{\alpha,\beta}(z_0)$ is defined as above. Define a holomorphic function $f(w) = \sum_{|\alpha|\geq N+2}^{4N}\partial^\alpha\varphi(z_0)(w-z_0)^\alpha$ and set $\psi = \varphi - f - \overline{f} \in C^{6N+3n+4}(U)$,

$$h_{N,z_0}^\psi(w) = \tilde{\psi}_N(z_0,w) - \overline{(w-z_0)}^T H_\psi(z_0)(w-z_0).$$

We notice by the definition of $h_{N,z}^\psi$ that for $|\alpha|,|\beta|\leq 4N$ we have $(\overline{\partial}_w^\alpha\partial_w^\beta h_{N,z_0}^\psi)(z_0) = 0$ if $\min\{|\alpha|,|\beta|\} = 0$ or $\max\{|\alpha|,|\beta|\}\leq 1$. Using Lemma 3.13 we can replace $h_{N,z}$ by $h_{N,z}^\psi$ with $\overline{\partial_w^\alpha}\partial_w^\beta h_{N,z_0}^\psi(z_0) = \varphi_{\alpha,\beta}(z_0)$ where $\varphi_{\alpha,\beta}(z_0)$ is defined as above. Now consider the case where $\tau = 0$ and $H_\varphi(z_0)$ is not diagonal. Given an invertible matrix F with $F^*H_\varphi(z_0)F = \mathrm{diag}(\lambda_1,\ldots,\lambda_n)$ define the map $G\colon\mathbb{C}^n\to\mathbb{C}^n$, $G(w) = F(w-z_0)+z_0$. Since $H_{\varphi\circ G}(z_0)$ is diagonal and by Lemma 3.14 we find that $\lambda_{N,\tau}^j$ can be computed as in the diagonal case with φ and ρ replaced by $\varphi\circ G$ and $\rho\circ G$. The statement follows from $\overline{X_w^\alpha}X_w^\beta\psi(z_0) = \overline{\partial_w^\alpha}\partial_w^\beta\psi\circ G(z_0)$ for any sufficiently often differentiable function ψ. $\square$

Remark 3.18

In the case where H_φ is not diagonal we can still get a formula for $\lambda_{N,\alpha}^{(j)}$, $\alpha\neq 0$, by replacing $a_{\alpha,\beta}^{(\frac{j+|\alpha|+|\beta|}{2})}(z_0)$ in (3.5) with its representation given in Theorem 2.50.

Assuming higher regularity on φ and ρ in Theorem 3.17 would lead to a similar expression also for the $\lambda_{N,\alpha}^{(j)}$ with $j > N - \frac{|\alpha|}{2}$. We will show this in Lemma 3.23 below where we assume φ and ρ to be smooth. Before we turn to the smooth case let us focus on the coefficients for the diagonal expansion, that is the coefficients in the expansion of $B_{k\varphi,\rho}$ in the non-smooth case.

Definition 3.19

Let $D\subset\mathbb{C}^n$ be a domain and $j\in\mathbb{N}_0$ a non-negative integer. Given two functions $\varphi\in C^{6j+3n+4}(D,\mathbb{R})$, $\rho\in C^{4j+2n+j}(D,\mathbb{R})$ we define $b_j\colon D_{\varphi,+}\to\mathbb{R}$ by $b_j(z) = \frac{\pi^n\rho}{\det(H_\varphi)}\lambda_{j,0}^{(j)}(z)$ with $\lambda_{j,0}^{(j)}(z)$ given as in Theorem 3.17.

From Lemma 3.4 it is clear that $b_j \in C^l(D_{\varphi,+},\mathbb{R})$ if $\varphi \in C^{6j+3n+4+l}(D,\mathbb{R})$ and $\rho \in C^{2j+2+l}(D,\mathbb{R})$ holds for some $l \in \mathbb{N}_0$. We have the following theorem for diagonal Bergman kernel expansion.

Theorem 3.20 (On-Diagonal Expansion)
Let $D \subset \mathbb{C}^n$ be a bounded domain, $D' \subset D$ open and $S_1 \subset C^{6N+2n+4+l}(D) \cap C^0(\overline{D})$ and $S_2 \subset C^{2N+2n+2+l}(D) \cap C^0(\overline{D})$ two bounded sets such that

$$\{\inf_{(z,w)\in D'\times\overline{D}} \tilde{\varphi}_N(z,w)/|z-w|^2 \mid \varphi \in S_1\} \quad \textit{and} \quad \{\inf_{z\in\overline{D}} \rho \mid \rho \in S_2\}$$

have positive lower bounds. Then for any $0 \le r \le l$ we have that $B_{k\varphi,\rho}$ has an asymptotic expansion of order $c_r(N+1)-n-r-1$ in $C^r(D')$ uniformly in $\varphi \in S_1$ and $\rho \in S_2$ with $c_r = 1-\frac{r}{l+1}$, coefficients $b_0,\dots,b_N$ of class C^l defined in Definition 3.19 and explicitly computed in Theorem 3.17. More precisely, for any $\varepsilon > 0$ we have

$$B_{k\varphi,\rho} - k^n \frac{\det(H_\varphi)}{\pi^n \rho} \sum_{j=0}^{N} k^{-j} b_j = O(k^{-(N+1)+n+\varepsilon}) \textit{ in } C^0(D')$$

uniformly in $\varphi \in S_1$ and $\rho \in S_2$, and for $1 \le r \le l$ we have

$$B_{k\varphi,\rho} - k^n \frac{\det(H_\varphi)}{\pi^n \rho} \sum_{j=0}^{N} k^{-j} b_j = O(k^{-c_r(N+1)+n+r}) \textit{ in } C^r(D')$$

uniformly in $\varphi \in S_1$ and $\rho \in S_2$.

Furthermore, the $b_j = b_j^{\rho,\varphi}$, $0 \le j \le N$, are polynomials in the derivatives of ρ and the entries of H_φ and the reciprocals of ρ and the eigenvalues of H_φ (see Theorem 3.17).

Proof. The claim follows immediately from Lemma 3.9, Lemma 3.12, Lemma 3.15 and Theorem 3.17. □

Theorem 3.21 (Near-Diagonal Expansion)
Let $D \subset \mathbb{C}^n$ be a bounded domain, $D' \subset D$ open and $S_1 \subset C^{6N+2n+4+l}(D) \cap C^0(\overline{D})$ and $S_2 \subset C^{4N+2n+2+l}(D) \cap C^0(\overline{D})$ two bounded sets such that

$$\{\inf_{(z,w)\in D'\times\overline{D}} \tilde{\varphi}_N(z,w)/|z-w|^2 \mid \varphi \in S_1\} \quad \textit{and} \quad \{\inf_{z\in\overline{D}} \rho \mid \rho \in S_2\}$$

have positive lower bounds. Then for any $0 \le r \le l$ we have that $P_{k\varphi,\rho}$ has an expansion of order $\frac{c_r}{2}(N+1)-n-r-1$ in $C^r(D')$ uniformly in $\varphi \in S_1 \cap C^\infty(D')$ and $\rho \in S_2 \cap C^\infty(D')$ with $c_r = 1-\frac{r}{l+1}$ and C^∞-coefficients $\lambda^{(j)}_{N,\alpha}$, $j \in \mathbb{N}_0, \alpha \in \mathbb{N}_0^n$, such that $\lambda^{(j)}_{N,\alpha} = b_{j,\alpha}$ for $2|\alpha|+2j \le N$ with $\lambda^{(j)}_{N,\alpha}$ constructed in Lemma 3.4 and $b_{j,\alpha}$ defined in Definition 3.24. More precisely, for any $\varepsilon > 0$ we have

$$P_{k\varphi,\rho} - P_{k\varphi,\rho,N} = O(k^{-\frac{N+1}{2}+n+\varepsilon}) \textit{ in } C^0(D' \times D)$$

uniformly in $\varphi \in S_1 \cap C^\infty(D')$ *and* $\rho \in S_2 \cap C^\infty(D')$, *and for* $1 \le r \le l$ *we have*

$$P_{k\varphi,\rho} - P_{k\varphi,\rho,N} = O(k^{-c_r \frac{N+1}{2}+n+r}) \text{ in } C^r(D' \times D)$$

uniformly in $\varphi \in S_1 \cap C^\infty(D')$ *and* $\rho \in S_2 \cap C^\infty(D')$ *with*

$$P_{k\varphi,\rho,N}(z,w) = k^n e^{-\frac{k}{2}(\varphi(w)-2\overline{\gamma_N(z,w)})} \sum_{j=0}^{N} k^{-j} \sum_{|\alpha|\le N} \overline{\lambda^{(j)}_{N,\alpha}(z)}\,\overline{(w-z)}^{\alpha}.$$

Furthermore, for any point $z \in D'$ *where* $H_\varphi(z)$ *is diagonal the* $\lambda^{(j)}_{N,\alpha} = \lambda^{(j),\rho,\varphi}_{N,\alpha}$ *can be computed explicitly for any* j *and any* α *with* $j, |\alpha| \le N$ *(see Theorem 3.17 and Lemma 3.23).*

Proof. The claim follows immediately from Lemma 3.9, Lemma 3.12, Lemma 3.15 and Theorem 3.17. □

Now we will consider the smooth case, that is $\varphi, \rho \in C^\infty(D, \mathbb{R})$ where $D \subset \mathbb{C}^n$ is a domain. From Definition 3.19 we obtain functions $b_0, b_1, \ldots \in C^\infty(D_{\varphi,+}, \mathbb{R})$ which are explicitly computed in terms of the entries of H_φ, ρ and their derivatives.

Lemma 3.22
Let $z_0 \in D$ *be a point such that* $H_\varphi(z_0)$ *is positive definite and* $\alpha \in \mathbb{N}_0^n$ *be a multi-index. We have* $\lambda^{(j)}_{N,\alpha}(z_0) = \lambda^{(j)}_{M,\alpha}(z_0)$ *for all* $M, N, j \in \mathbb{N}_0$ *satisfying* $2|\alpha|+2j \le M \le N$. *Here* $\lambda^{(j)}_{N,\alpha}(z_0)$ *and* $\lambda^{(j)}_{M,\alpha}(z_0)$ *are defined by Lemma 3.4 and Remark 3.6.*

Proof. Fix $M, N \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le M \le N$. First we observe that $\overline{\partial}^\eta_w(\overline{\gamma_M(z,w)} - \overline{\gamma_N(z,w)})|_{w=z} = 0$ for all $|\eta| \le M$ which implies

$$\overline{\partial}^\eta_w e^{-k\frac{1}{2}\varphi(w)+k\overline{\gamma_M(z,w)}}|_{w=z} = \overline{\partial}^\eta_w e^{-k\frac{1}{2}\varphi(w)+k\overline{\gamma_N(z,w)}}|_{w=z}$$

for all $|\eta| \le M$. From Lemma 2.47 we get an open neighborhood $D' \subset\subset D$ around z_0 such that $z_0 \in D'_{\varphi,N} \cap D'_{\varphi,M}$ holds. Using Lemma 3.12 we find a constant $C > 0$ such that

$$|k^{-n}\overline{\partial}^\tau_w(\tilde{P}_{N,k} - \tilde{P}_{M,k})|_{w=z_0}| \le Ck^{-\frac{M}{2}-\varepsilon+|\tau|}$$

holds for all $k \in [1,\infty)$ and some fixed $\varepsilon < \frac{1}{2}$. By Lemma 2.9 and our previous considerations we conclude

$$\begin{aligned}\left|\sum_{\tau\le\eta}\binom{\eta}{\tau}\partial^{\eta-\tau}_w(e^{-k\frac{1}{2}\varphi(w)+k\gamma_N(z,w)})|_{w=z_0}\sum_{j=0}^{M}\tau!k^{-j}(\lambda^{(j)}_{M,\tau}(z_0)-\lambda^{(j)}_{N,\tau}(z_0))\right| \\ \le Ck^{-\frac{M}{2}-\varepsilon+|\eta|}+R(k)\end{aligned} \tag{3.8}$$

with

$$R(k) = \left|\sum_{\tau\le\eta}\binom{\eta}{\tau}\partial^{\eta-\tau}_w(e^{-k\frac{1}{2}\varphi(w)+k\gamma_N(z,w)})|_{w=z_0}\sum_{j=M+1}^{N}\tau!k^{-j}\lambda^{(j)}_{N,\tau}(z_0)\right| \le C'k^{-M-1+|\eta|}$$

for some constant $C' > 0$ independent of k where the sums run over all $\tau \in \mathbb{N}_0^n$, $\tau \leq \eta$. For $|\eta| = 0$ the claim follows from Lemma 3.15. Assume that the claim is true for all $\eta \in \mathbb{N}_0^n$, $|\eta| < |\alpha|$. By (3.8) it follows that

$$\left|\sum_{j=0}^{M} \alpha! k^{-j} (\lambda_{M,\alpha}^{(j)}(z_0) - \lambda_{N,\alpha}^{(j)}(z_0))\right| \leq C k^{-\frac{M}{2}-\varepsilon+|\alpha|} + C' k^{-M-1+|\alpha|}$$

holds for all $k \in [1,\infty)$ and hence $\lambda_{M,\alpha}^{(j)}(z_0) = \lambda_{N,\alpha}^{(j)}(z_0)$ for all $j \leq \frac{M}{2} - |\alpha|$. □

Lemma 3.23
Let $z_0 \in D$ be a point such that $H_\varphi(z_0)$ is positive definite and diagonal. We have that the formula (3.4) for $\lambda_{N,\alpha}^{(j)}(z_0)$ holds also for $N - \frac{|\alpha|}{2} < j \leq N$. Furthermore, for $j \leq \frac{N}{2} - |\alpha|$ we have that (3.4) stays true when we replace $\overline{\partial_w^\alpha} \partial_w^\beta h_{N,z_0}(z_0)$ by

$$\varphi_{\alpha,\beta}(z_0) = \begin{cases} \left(\overline{\partial_w^\alpha} \partial_w^\beta \varphi\right)(z) & , \textit{ if } \max\{|\alpha|,|\beta|\} \geq 2, \min\{|\alpha|,|\beta|\} \geq 1, \\ 0 & , \textit{ else.} \end{cases}$$

Proof. Fix $N \in \mathbb{N}_0$. For the first part of the statement we go through the proof of Theorem 3.17 and see that (since φ and ρ are smooth) we can expand $a_{k,\alpha,\beta}$ up to higher order namely $4N + n + 1$. This implies that (3.6) becomes $c_k^\alpha = \sum_{j=0}^{4N} k^{-\frac{j}{2}} c^{\alpha,(j)} + O(k^{-2N-\frac{1}{2}})$. We conclude

$$\lambda_{N,\alpha}^{(j)}(z_0) = \frac{\det(H_\varphi(z_0))}{\pi^n \rho(z_0)} \sqrt{\frac{\lambda^\alpha}{\alpha!}} c^{\alpha,(2j+|\alpha|)}$$

for all $0 \leq j \leq N$.

For the second part, we define a holomorphic function

$$f(w) = -\frac{1}{2} \sum_{|\alpha| \geq N+2}^{4N} \partial_z^\alpha \varphi(z_0)(w - z_0)^\alpha$$

and set $\psi = \varphi + 2f + 2\overline{f} \in C^\infty(D)$,

$$h_{N,z_0}^\psi(w) = \tilde{\psi}_N(z_0, w) - \overline{(w - z_0)}^T H_\psi(z_0)(w - z_0).$$

As in the proof of Theorem 3.17 we notice that $H_\psi(z_0)$ is diagonal and that by the definition of $h_{N,z}^\psi$ we have $(\overline{\partial}_w^\alpha \partial_w^\beta h_{N,z_0}^\psi)(z_0) = 0$ if $\min\{|\alpha|,|\beta|\} = 0$ or $\max\{|\alpha|,|\beta|\} \leq 1$, $|\alpha|,|\beta| \leq 4N$. Using Lemma 2.40 we get $P_{k\psi,\rho}(z,w) = e^{k(f(z)-f(w)+\overline{f(w)}-\overline{f(z)})} P_{k\varphi,\rho}(z,w)$, $z,w \in D$. Since all the derivatives of f vanish up to order $N+2$ we can proceed as in the proof of Lemma 3.22 and find $\lambda_{N,\alpha}^{(j),\varphi,\rho}(z_0) = \lambda_{N,\alpha}^{(j),\psi,\rho}(z_o)$ for all $j \leq \frac{N}{2} - |\alpha|$. □

Lemma 3.22 and Lemma 3.23 give rise for the following definition.

Definition 3.24
Given $\varphi, \rho \in C^\infty(D, \mathbb{R})$, $\rho > 0$ and $\alpha \in \mathbb{N}_0^n$ define

$$b_{j,\alpha}\colon D_{\varphi,+} \to \mathbb{R},\ \ b_{j,\alpha}(z) = \frac{\rho}{\det(H_\varphi(z))}\overline{\lambda^{(j)}_{2(|\alpha|+j),\alpha}(z)}$$

where in the case when $H_\varphi(z)$ is diagonal $\lambda^{(j)}_{2(|\alpha|+j),\alpha}(z)$ is explicitly given by Theorem 3.17 with the modification in Lemma 3.23. Furthermore, set

$$\hat{P}_{k\varphi,\rho,N}(z,w) = \frac{k^n}{\pi^n}\frac{\det(H_\varphi(z))}{\rho}e^{-\frac{k}{2}(\varphi(w)-2\overline{\gamma_N(z,w)})}\sum_{j=0}^{\lfloor\frac{N}{2}\rfloor}k^{-j}\sum_{|\alpha|\leq\lfloor\frac{N}{2}\rfloor-j}b_{j,\alpha}(z)\overline{(w-z)}^\alpha.$$

Lemma 3.25
One has

$$\hat{P}_{k\varphi,\rho,2N} - P_{k\varphi,\rho,2N} = O(k^{-\frac{N+1}{2}+n+r+\varepsilon}) \text{ in } C^r(D_{\varphi,2N}\times D)$$

with $P_{k\varphi,\rho,2N}$ as in Theorem 3.21.

Proof. Let $K \subset D_{\varphi,2N}$ be compact and $\eta, \tau \in \mathbb{N}_0^{2n}$, $|\eta|+|\tau| \leq r$, two multi-indicies. Set $\psi_k(z,w) = e^{-\frac{k}{2}(\varphi(w)-2\gamma_N(z,w))}$. The $2N$-th localization property ensures that there exist constants $C, \delta > 0$ such that $|\psi(z,w)|^2 \leq Ck^{-\delta k|z-w|^2}$ holds for $(z,w) \in K \times D$ and $k \in [1,\infty)$. This observation leads to

$$\left|d_z^\eta d_w^\tau \psi_k(z,w)\sum_{N+1}^{2N}\overline{\lambda^{(j)}_{N,\alpha}(z)}\overline{(w-z)}^\alpha\right| \leq C_1 k^{-N-1+r}$$

for all $k \in [1,\infty)$ and all $(z,w) \in K\times D$ where $C_1 > 0$ is a constant independent of k, z and w. Given $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq N-j$, we get from Lemma 3.22 that $\frac{\det(H_\varphi)}{\rho}b_{j,\alpha} = \lambda^{(j)}_{2N,\alpha}$. Then consider the term

$$R_{j,\alpha,k}(z,w) := \psi_k(z,w)k^j\overline{\lambda^{(j)}_{N,\alpha}(z)}\overline{(w-z)}^\alpha$$

for $0 \leq j \leq N$ and $N-j < |\alpha| \leq 2N$. Given $0 < \varepsilon' < 1$ we find $|d_z^\eta d_w^\tau R_{j,\alpha,k}|^2 \leq C_2 e^{-\delta k^{\varepsilon'}}$ for $|w-z|^2 > k^{\varepsilon'-1}$. For $|w-z|^2 \leq k^{\varepsilon'-1}$ we have $|(w-z)^\beta|^2 \leq |w-z|^{2|\beta|} \leq k^{|\beta|(\varepsilon'-1)}$. Since $|\alpha| \geq N-j+1$ we find $|d_z^\eta d_w^\tau R_{j,\alpha,k}|^2 \leq C_3 k^{-\frac{N+1}{2}+\varepsilon}$. Here $C_2, C_3, \varepsilon > 0$ are constants independent of k, z and w. The claim follows from

$$\tilde{P}_{k\varphi,\rho,2N}(z,w) - \hat{P}_{k\varphi,\rho,2N}(z,w) = \psi_k(z,w)\left(\sum_{\substack{|\alpha|\leq 2N\\ N-j<|\alpha|}}R_{j,\alpha,k}(z,w) + \sum_{N+1}^{2N}\overline{\lambda^{(j)}_{N,\alpha}(z)}\overline{(w-z)}^\alpha\right).$$

□

Corollary 3.26
Let $D \subset \mathbb{C}^n$ be a bounded domain. Given $\varphi, \rho \in C^\infty(D)\cap C^0(\overline{D})$, $\varepsilon > 0$ and $r \in \mathbb{N}_0$ we have

$$P_{k\varphi,\rho} - \hat{P}_{k\varphi,\rho,2N} = O(k^{-\frac{N+1}{2}+n+r+\varepsilon}) \text{ in } C^r(D_{\varphi,2N}\times D).$$

3.3 Bergman Kernels on Manifolds

Let M be a complex manifold of complex dimension n. Given a smooth vector bundle F over M we denote by $\Gamma(M,F)$ the space of smooth sections $M \to F$. Let $T^{1,0}M$ (resp. $T^{0,1}M$) denote the bundle of holomorphic (resp. antiholomorphic) vectors. Let E be a holomorphic line bundle over M. The space of smooth (p,q)-forms (or forms of type (p,q)) with values in E is defined by $\Omega^{p,q}(M,E) = \Gamma(M, E \otimes \Lambda^p T^{*(1,0)}M \otimes \Lambda^q T^{*(0,1)}M)$ where $T^{*(1,0)}M := (T^{1,0}M)^*$ and $T^{*(0,1)}M := (T^{0,1}M)^*$. We denote by $\partial\colon \Omega^{p,q}(M,E) \to \Omega^{p+1,q}(M,E)$ and $\overline{\partial}\colon \Omega^{p,q}(M,E) \to \Omega^{p,q+1}(M,E)$ the holomorphic and antiholomorphic differential. A Hermitian metric h_E on E is said do be upper semi-continuous (resp. of class C^l) if $-\log(|s|^2_{h_E})$ is an upper semi-continuous (resp. a C^l) function for any local holomorphic frame s of E where $|s|_{h_E} := \sqrt{h_E(s,s)}$ denotes the pointwise norm of s. Given a continuous volume form dV_M on M and an upper semi-continuous locally bounded Hermitian metric h_E on E we denote by $H^0_2(M,E)$ the space of holomorphic sections with finite L^2-norm $\|\cdot\|_{h_E,dV_M}$ induced by the inner product

$$(f,g)_{h_E,dV_M} = \int_M h_E(f,g)dV_M \quad f,g \in L^2_{h_E,dV_M}(M,E).$$

We have that $H^0_2(M,E)$ is a separable Hilbert space. Given an orthonormal basis $\{s_j\}_{j=1}^d$, $d \in \mathbb{N}_0 \cup \{\infty\}$, of $H^0_2(M,E)$ we define the Bergman kernel

$$P_{h_E,dV_M}\colon M \to E \boxtimes E^*,\ \ P_{h_E,dV_M}(x,y) = \sum_{j=1}^d s_j(x) \otimes (s_j(y))^*$$

and the Bergman kernel function

$$B_{h_E,dV_M} : M \to \mathbb{R},\ \ B_{h_E,dV_M} = \sum_{j=1} h_E(s_j,s_j),$$

where we write $v^* := h_E(\cdot,v) \in E^*_x$ for $v \in E_x$, $x \in M$ and choose the metric h^*_E on E^* such that $v \mapsto v^*$ becomes an isometry. Note that P_{h_E,dV_M} and B_{h_E,dV_M} are well defined, independent of the choice of the orthonormal basis $\{s_j\}_{j=1}^d$ and in the case when h_E is smooth we have $P_{h_E,dV_M} \in \Gamma(M, E \boxtimes E^*)$ and $B_{h_E,dV_M} \in C^\infty(M,\mathbb{R})$ (see Section 2.4).

From now on we assume that h_E is smooth. Given another holomorphic Hermitian line bundle L over M and an upper semi-continuous Hermitian metric h on L we are interested in studying the Bergman kernel and the Bergman kernel function for the space $H^0_2(M,L_k)$, $k \in \mathbb{N}$ with $L_k = L^k \otimes E$ and $h_k = h^k \otimes h_E$.

Definition 3.27
Let $M_{h,+}$ denote the subset of M consisting of points which have a neighborhood where h is of class C^2 with positive curvature $c_1(L,h)$.

The curvature $c_1(L,h)$ is a form of type $(1,1)$ and can be locally written as $c_1(L,h) = -\frac{i}{2}\partial\overline{\partial}\log(h(s,s))$ for any local holomorphic frame s of L around points where h is at least of class C^2. We define the following invariants.

Definition 3.28
Assume h is of class $C^{6j+3n+4}$ and dV_M is of class $C^{4j+2n+2}$, Define

$$b_j = b_j^{h,h_E,dV_M} \colon M_{h,+} \to \mathbb{R},\ \ b_j^{h,h_E,dV_M}(p) = b_j^{\varphi,\rho}(z(p))$$

where $b_j^{\varphi,\rho}$ is given by the formula in Definition 1.4 with respect to a choice of local trivializations s of L and e of E and local coordinates (U,z) with $\varphi = -\log(h(s,s))$, $\rho = h_E(e,e)\tilde{\rho}$, $dV_M = \tilde{\rho} dV_{\mathbb{C}^n}$.

Lemma 3.29
The function b_j is well defined, that is b_j is independent of the choice of coordinates and trivializations. Furthermore, we have that $b_j \in C^l(M_{h,+},\mathbb{R})$ if h is of class $C^{6j+3n+4+l}$ and dV_M is of class $C^{4j+2n+2+l}$.

Proof. We need to show that $b_j^{\varphi,\rho}$ is invariant under biholomorhic mappings. Let U and V be open neighborhoods around points p_1 and p_2 in $\mathbb{C}^n$ and let $G\colon U \to V$ be a biholomorphic map. Set $F = (\frac{\partial G_l}{\partial z_m})_{1\le m,l\le n}$ and $c = \det(F(p))$. By shrinking U and V we can achieve that $|\det(F(z)) - c| \le |c|/2$ holds for all $p \in U$, $p_1 \in U_{\varphi,j}$ and $p_2 \in V_{\varphi\circ G,j}$. By Lemma 3.14 we have $B_{\varphi\circ G,|\det(F)|\rho\circ G} = B_{\varphi,\rho}$. Using our assumptions on $\det(F)$ we find $|\det(F)|^2 = e^{\log(\det(F))+\overline{\log(\det(F))}}$. Then the claim follows from Lemma 3.13, Lemma 3.9 and the uniqueness of the coefficients in an asymptotic expansion. The last part of the statement follows from Definition 3.19 and Lemma 3.4. $\square$

Lemma 3.30
Given $N \in \mathbb{N}_0$ assume that $p \in M_{h,+}$ has an open neighborhood $U \subset M$ where h is of class $C^{6N+3n+4}$ and dV_M is of class $C^{4N+2n+2}$. Choose coordinates (D,z), $D \subset\subset U$ around p such that D is identified with a bounded domain in $\mathbb{C}^n$ and local holomorphic frames s and e for L and E such that $z(p) \in D_{\varphi,N}$ holds (see Lemma 2.47) with $\varphi := -\log(|s|_h^2)$. Furthermore, set $\tilde{\rho} = \rho|e|_{h_E}$ where ρ is defined by $dV_M = \rho\cdot\left(\frac{i}{2}\right)^n dz_1\wedge\ldots\wedge dz_n\wedge d\overline{z_1}\wedge\ldots\wedge d\overline{z_n}$. For any compact set $K \subset D_{\varphi,N}$ there exists a constant $C > 0$ such that

$$|\tilde{f}(z) - (\tilde{f}, K_{k\varphi,\tilde{\rho},N}(z,\cdot))_{k\varphi,\rho}|^2 e^{-k\varphi(z)} \le Ck^{-N-1+n+\varepsilon}\|f\|^2_{h_k,dV_M}$$

for all $f \in H^0_2(M,L_k)$, $k \in \mathbb{N}$ and $z \in D_{\varphi,N}$ where $\tilde{f} \in H^0_{k\varphi,\rho}(D)$ is defined by $\tilde{f}(z)s^k \otimes e = f$ and $(\cdot,\cdot)_{k\varphi,\rho}$, $K_{k\varphi,\tilde{\rho},N}$ are as in Section 3.1.
Here C is bounded when ρ stays in a bounded set in $C^{4N+2n+2}(D,\mathbb{R})\cap C^0(\overline{D})$ such that $\inf_{w\in\overline{D}}\rho(w)$ has a positive lower bound.

Proof. The statement follows immediately from Lemm 3.8. $\square$

3.4 Global Bergman Kernel Expansion

We start this section by stating the following theorem on L^2 estimates due to Hörmander [23] in a generalized version given by Demailly [14].

Theorem 3.31 ([14, Theorem VIII–6.5])
Let (X,ω) be a weakly pseudoconvex Kähler manifold, E a hermitian line bundle on X, $\varphi \in C^\infty(X,\mathbb{R})$ a weight function such that the eigenvalues $\lambda_1 \leq \ldots \leq \lambda_n$ of $i\Theta(E) + id'd''\varphi$ are ≥ 0. Then for every form g of type (n,q), $q \geq 1$, with L^2_{loc} (resp. C^∞) coefficients such that $D''g = 0$ and

$$\int_X \frac{1}{\lambda_1 + \ldots + \lambda_n} |g|^2 e^{-\varphi} dV < +\infty.$$

We can find a L^2_{loc} (resp. C^∞) form f of type $(n, q-1)$ such that $D''f = g$ and

$$\int_X |f|^2 e^{-\varphi} dV \leq \int_X \frac{1}{\lambda_1 + \ldots + \lambda_n} |g|^2 e^{-\varphi} dV.$$

Proof. [14, Theorem VIII–6.5] □

Lemma 3.32
The conclusion of Theorem 3.31 is valid if (X,ω) is a complete Kähler manifold provided that g has compact support.

Proof. The claim follows from [14, Theorem VIII–4.5] and [14, VIII–(6.4)]. □

We will reformulate Theorem 3.31 in our notation. Therefore, let (X,ω) be a complete Kähler manifold and let (E, h_E) be a holomorphic Hermitian line bundle with smooth metric h_E.

Corollary 3.33
Assume $c_1(E,h_E) \geq 0$ on X. Let $f \in \Gamma(X, E \otimes \Lambda^n T^{*(1,0)}X)$ be a section compactly supported in $X_{h_E,+}$ and $C > 0$ a constant with $Cc_1(E,h_E) \geq \omega$ on the support of f. Then there exists $u \in \Gamma(X, E \otimes \Lambda^n T^{*(1,0)}X)$ with $f - u \in H^0_2(X, E \otimes \Lambda^n T^{*(1,0)}X)$, such that $\int_X |u|^2_{h_E} dV_X \leq C \int_X |\overline{\partial} f|^2_\omega dV_X$ holds.

We focus on the following setting. Let L_0 be a holomorphic line bundle over a complete Kähler manifold (X,ω). Given a domain $M \subset X$ we consider the line bundle $L = L_0|_M$. Let $(E, h_E) \to X$ be another holomorphic line bundle with smooth Hermitian metric h_E. Choose an upper semi-continuous metric h on L, a function $\rho \in C^0(M)$ which is positive and bounded and define a volume form on M by $dV_M = \rho \frac{\omega^n}{n!}$. Then we consider the Bergman kernel P_{h_k,dV_M} and the Bergman kernel function B_{h_k,dV_M} for the space $H^0_2(M, L_k)$ with $L_k = L^k \otimes E \otimes \Lambda^n T^{*(1,0)}M$, $k \in \mathbb{N}$. In local coordinates (U,z) using the holomorphic frame $dz := dz_1 \wedge \ldots \wedge dz_n$

for $\Lambda^n T^{*(1,0)}M$ we observe $dV_M = \rho\tilde{\rho} dV_{\mathbb{C}^n}$ and $h_\omega(dz, dz) = 1/\tilde{\rho}$. Hence we find that b_j is independent of ω, so we set $b_j^{h,\rho} := b_j^{h,dV_M,h_E\otimes h^\omega}$ in this setting. Note that $b_j^{h,\rho}$ depends on the fixed metric h_E but we do not indicate this here to maintain a brief notation style.

Definition 3.34
We define the set $M_{h,\infty} \subset M$ by saying $p \in M_{h,\infty}$ if and only if p has an open neighborhood U where h is smooth with positive curvature and there exists a smooth Hermitian metric h_0 on $L_0 \to X$ with $h \leq h_0$ on M and $h = h_0$ on U and $k_0 \in \mathbb{N}$ such that

$$kc_1(L_0, h_0) + c_1(E, h_E) \geq 0, \quad k \geq k_0. \tag{3.9}$$

Remark 3.35
In Definition 3.34 assume that E is trivial with flat metric then (3.9) is equivalent to assume that h_0 is semi-positive.

Theorem 3.36 (On-Diagonal Expansion)
For any $\varepsilon > 0$, $N \in \mathbb{N}_0$ and $\rho \in C^{4m+2n+2+r}(M_{h,\infty}, \mathbb{R}) \cap C^0(M)$, $m = Nr + N + r^2 + 2r + 1$, which is positive and bounded one has

$$B_{h_k,dV_M} - \frac{k^n}{\pi^n}\frac{c_1(L,h)^n}{n!dV_M}\sum_{j=0}^{N} b_j^{h,\rho} k^{-j} = O(k^{-N-1+n}) \text{ in } C^r(M_{h,\infty}).$$

More precisely, given any compact set $K \subset M_{h,\infty}$ and any partial differential operator F of order $\leq r$ there exists a constant $C = C_{K,F}$ such that

$$\left| F\left(B_{h_k,dV_M} - \frac{k^n}{\pi^n}\frac{c_1(L,h)^n}{n!dV_M}\sum_{j=0}^{N} b_j^{h,\rho} k^{-j}\right)(p)\right| \leq Ck^{-N-1+n}$$

holds for all $p \in K$ and all $k \in \mathbb{N}$. Here C is bounded when ρ stays in a bounded set in $C^{4m+2n+2+r}(M_{h,\infty}, \mathbb{R}) \cap C^0(M)$ such that $\inf_{p\in M}\rho(p)$ has a uniform positive lower bound and $\sup_{p\in M}\rho(p)$ has a uniform upper bound.

The theorem follows from the following lemma.

Lemma 3.37
Let $U \subset M$ be an open set where h is smooth and has positive curvature. Assume there exists a semi-positive smooth Hermitian metric h_0 on L_0 with $h \leq h_0$ on M, $h = h_0$ on U, and there exists $k_0 \in \mathbb{N}$ such that $kc_1(L_0, h_0) + c_1(E, h_E) \geq 0$ holds on X for all $k \geq k_0$. For any $\varepsilon > 0$, $N \in \mathbb{N}_0$ and $\rho \in C^{4N+2n+2}(U, \mathbb{R}) \cap C^0(M)$ which is positive and bounded one has

$$B_{h_k,dV_M} - \frac{k^n}{\pi^n}\frac{c_1(L,h)^n}{n!dV_M}\sum_{j=0}^{N} k^{-j} b_j^{h,\rho} \quad = \quad O(k^{-N-1+n+\varepsilon}) \text{ in } C^0(U).$$

More precisely, given a compact set $K \subset U$, $N \in \mathbb{N}_0$, $\varepsilon > 0$, a bounded subset $S \subset C^{4N+2n+2}(U) \cap C^0(M)$ and a constant $C_0 > 0$ with $1/C_0 \leq \rho(p) \leq C_0$ for all $p \in M$ and $\rho \in S$, there exists a constant $C = C_{K,S,N,\varepsilon} > 0$ independent of k such that

$$\left| B_{h_k,dV_M}(p) - \frac{k^n}{\pi^n} \frac{c_1(L,h)^n}{n! dV_M} \sum_{j=0}^{N} k^{-j} b_j^{h,\rho}(p) \right| \leq C k^{-N-1+n+\varepsilon}$$

holds for all $k \in \mathbb{N}$, $p \in K$ and $\rho \in S$.

Proof. Fix $N \in \mathbb{N}_0$, $\varepsilon > 0$ and a compact set $K \subset U$. Given any point $p \in K$ choose local coordinates (D, z), $D \subset\subset U$ around p and local frames s and e for L and E such that $z(p) \in D_{\varphi,N}$ (see Lemma 2.47) where $\varphi = -\log(|s|_h^2)$ and set $\tilde{\rho} = \rho |e|_{h_E}$. Furthermore, let $s_k = s^k \otimes e \otimes dz$ be the induced local holomorphic frame for L_k and identify D with an open set in $\mathbb{C}^n$ via the local coordinate z. Since $D_{\varphi,N}$ is open and non-empty we find an open neighborhood $D' \subset\subset D_{\varphi,N}$ around p. With $K'_{k\varphi,\rho,N}(z,w) = s_k(w) K_{k\varphi,\tilde{\rho},N}(z,w)$ we find from Lemma 3.30 that

$$|f - s_k(z)(f, K'_{k\varphi,\tilde{\rho},N}(z,w))_{h_k,dV_M,U}|_{h_k}^2 \leq C_1 k^{-N-1+n+\varepsilon} \|f\|_{h_k,dV_M}^2$$

for all $k \in \mathbb{N}$, all $z \in D'$ and all $f \in H_2^0(M, L_k)$ where C_1 is a constant independent of $k \in \mathbb{N}$, $z \in D'$ and $\rho \in S$. Now take a cutoff function $\chi \in C_0^\infty(M, \mathbb{R})$, $0 \leq \chi \leq 1$ supported in D such that $\chi \equiv 1$ in a neighborhood of the closure of D'. Using $h \leq h_0$ on M and $h = h_0$ on U we get from Corollary 3.33 that for any $k \in \mathbb{N}$, $k \geq k_0$, and any $z \in D'$ we can choose $u_z^{(k)} \in \Gamma(M, L_k)$ such that $\chi(w) K'_{k\varphi,\rho,N}(z,w) - u_z^{(k)}$ is holomorphic on M and

$$\|u_z^{(k)}\|_{h_k,dV_M}^2 \leq C_0^2 C_2 \int_M |\overline{\partial}\chi(w)|_\omega^2 |K'_{k\varphi,\tilde{\rho},N}(z,w)|_{h_k}^2 dV_M$$

where $C_2 > 0$ is a constant independent of $k \geq k_0$, $z \in D'$ and $\rho \in S$. Note that it is actually enough to assume here that $1/C_0 \leq \rho$ holds on $\mathrm{supp}(\chi)$. Since $\overline{\partial}\chi(w)$ and $(1 - \chi(w))$ is zero in a neighborhood of $\overline{D'}$ we find from Lemma 3.7 that

$$e^{-k\varphi(z)} \|u_z^{(k)}\|_{h_k}^2, \;\; e^{-k\varphi(z)} \int_D (1 - \chi(w))^2 |K'_{k\varphi,\tilde{\rho},N}(z,w)|_{h_k}^2 dV_M = O(k^{-\infty}) \tag{3.10}$$

in $C^0(D')$ uniformly in $\rho \in S$. Write

$$K'_{k\varphi,\tilde{\rho},N}(z,w) = \chi(w) K'_{k\varphi,\tilde{\rho},N}(z,w) + (1 - \chi(w)) K'_{k\varphi,\tilde{\rho},N}(z,w) - u_z^{(k)} + u_z^{(k)}.$$

From (3.10) we conclude

$$|f - s_k(z)(f, \chi(w) K'_{k\varphi,\rho,N}(z,w) - u_z^{(k)})_{h_k,dV_M}|_{h_k}^2 \leq C_3 k^{-N-1+n+\varepsilon} \|f\|_{h_k,dV_M} \tag{3.11}$$

for all $k \geq k_0$, $z \in D'$ where C_3 is a constant independent of k, z and ρ. For $z \in D'$ set $v_z^{(k)} = \sum_{j=1}^{d_k} \overline{\tilde{s}_j^{(k)}}(z) s_j^{(k)}$ where $\{s_j^{(k)}\}_{j=1}^{d_k}$ is an orthonormal basis of $H_2^0(M, L_k)$

with $d_k = \dim H^0_2(M, L_k) \in \mathbb{N} \cup \{\infty\}$. Here we use the notation $\tilde f(z)s_k = f$ to identify sections $f \in H^0_2(M, L)$ with holomorphic functions $\tilde f$ on D. We have $\tilde f(z) = (f, v^{(k)}_z)_{h_k,dV_M}$ for all $f \in H^0(M, L_k)$. Putting $f^{(k)}_z = v^{(k)}_z - \chi(w)K'_{k\varphi,\tilde\rho,N}(z, w) + u^{(k)}_z$ we have $f^{(k)}_z \in H^0_2(M, L_k)$ as in the proof of Lemma 3.9 we obtain from (3.11)

$$\|v^{(k)}_z - \chi(w)K'_{k\varphi,\tilde\rho,N}(z, w) + u^{(k)}_z\|^2_{h_k} e^{-k\varphi(z)} \le C_3k^{-N-1+n+\varepsilon} \tag{3.12}$$

for all $k \ge k_0$, $z \in D'$ and $\rho \in S$. Furthermore, we have

$$\|f_z\|^2_{h_k} = (v^{(k)}_z, v^{(k)}_z)_{h_k,dV_M} - \chi(z)\lambda_{N,0,k}(z)e^{k\varphi(z)} + 2\operatorname{Re} u^{(k)}_z(z) + R_{k,z}$$

with

$$\begin{aligned} |R_{k,z}| &= \left|\|\chi(w)K'_{k\varphi,\tilde\rho,N}(z, w) + u^{(k)}_z\|^2_{h_k} - \chi(z)\lambda_{N,0,k}(z)e^{k\varphi(z)}\right| \\ &\le \|(1 - \chi(w))K'_{k\varphi,\tilde\rho,N}(z, w)\|^2_{h_k,D} + \|u^{(k)}_z\|^2_{h_k} \end{aligned}$$

where we used $(K_{k\varphi,\tilde\rho,N}(z, \cdot), K_{k\varphi,\tilde\rho,N}(z, \cdot))_{k\varphi,\tilde\rho} = \lambda_{N,0,k}(z)e^{k\varphi(z)}$ (see Lemma 3.2) and $(f, v^{(k)}_z) = \tilde f(z)$ for any $f \in H^0_2(M, L_k)$ by the reproducing property of the Bergman kernel. Since $\overline\partial u^{(k)}_z = (\overline\partial\chi(w))K'_{k\varphi,\tilde\rho,N}(z, w)$ we have that u_z is holomorphic on D'' with $D' \subset\subset D''$. Hence we can use Lemma 2.29 and find a constant $C_4 > 0$ independent of k with $|u^{(k)}_z(z)|^2_{h_k} e^{-k\varphi(z)} \le C_4k^n e^{-k\varphi(z)}\|u_z\|^2_{h_k} = O(k^{-\infty})$ on D'. Using $B_{h_k,dV_M} = (v_z, v_z)_{h_k} e^{-k\varphi(z)}$, (3.10) and the expansion of $\lambda_{N,0,k}$ in Lemma 3.4 (see also Lemma 3.15 and Definition 3.19) we conclude

$$\left|B_{h_k,dV_M} - \frac{k^n}{\pi^n}\frac{c_1(L, h)^n}{n!dV_M}\sum_{j=0}^{N} k^{-j}b^{h,\rho}_j(q)\right| \le C_5k^{-N-1+n+\varepsilon}$$

on D' for all $k \in \mathbb{N}$, $\rho \in S$. Since K can be covered by finitely many of those sets D' the claim follows. □

Proof of Theorem 3.36. We just need to show that any point in $M_{h,\infty}$ has an open neighborhood where the claim in Theorem 3.36 holds. Given an arbitrary point in $M_{h,\infty}$ we know from the definition that is has an open neighborhood $U \subset M_{h,\infty}$ where the assumptions of Lemma 3.37 are satisfied. In Lemma 3.37 we proved the statement already for $r = 0$ and arbitrary N. Let $C_0 > 0$ be a constant and $S \subset C^{4m+2n+2+r}(U) \cap C^0(M)$, $m = Nr + N + r^2 + 2r + 2$ a bounded set with $1/C_0 \le \rho(p) \le C_0$ for all $p \in M$ and $\rho \in S$. Using Hörmander's trick (see Lemma 3.11) and Lemma 3.12 which is true for that case since all the arguments are local and Corollary 2.39 is valid we find

$$B_{h_k,dV_M} - \frac{k^n}{\pi^n}\frac{c_1(L, h)^n}{n!dV_M}\sum_{j=0}^{m} k^{-j}b^{h,\rho}_j = O(k^{-c_r(m)+n+r+\varepsilon}) \text{ in } C^r(U)$$

uniformly in $\rho \in S$ with $c_r = \frac{1}{r+1}$. Taking $\varepsilon > 0$ small enough the statement follows from $c_r(Nr + N + r^2 + 2r + 2) - r = N + 1 + c_r$ and $\sum_{j=N+1}^{m} k^{-j} b_j = O(k^{-N-1})$ in $C^r(U)$ uniformly in $\rho \in S$ (see Definition 3.19 and Remark 3.18). $\square$

In order to state the next theorem we fix the following notation. Given local coordinates (D, z) around a point $p \in M$ and local holomorphic frames s and e of L and E, we denote by $(D \times D, (z, w))$ the induced coordinates around $(p, p) \in M \times M$ and choose $\hat{s}_k(z, w) := e^{\frac{k}{2}(\varphi(z)+\varphi(w))} s^k(z) e(z) dz (s^k(w) e(w) dw)^*$ as a trivialization of $L_k \boxtimes L_k^*|_{U \times U}$ with $\varphi = -\log(h(s, s))$. Furthermore, set $\tilde{\rho} = \rho |e|^2_{h_E}$ and recall the definition of $\hat{P}_{k\varphi,\tilde{\rho},N}$ (see Definition 1.9). We have the following result on off-diagonal expansion.

Theorem 3.38
Assume that $\rho \in C^\infty(M_{h,\infty}, \mathbb{R}) \cap C^0(M)$ is bounded and positive. Let $p \in M_{h,\infty}$ be a point and (D, z) local coordinates around p with $D \subset M_{h,\infty}$. For any $\varepsilon > 0$, $N \in \mathbb{N}_0$ one has

$$P_{h_k, dV_M} - \hat{s}_k(z, w) \hat{P}_{k\varphi,\tilde{\rho},2N}(z, w) = O(k^{-\frac{N+1}{2}+n+r+\varepsilon}) \text{ in } C^r(D_{\varphi,2N} \times D).$$

Furthermore, for any open set $D \subset M_{h,\infty}$ we have

$$P_{h_k, dV_M} = O(k^{-\infty}) \text{ in } C^r(D \times M \setminus \overline{D}).$$

For the proof we need the following lemma.

Lemma 3.39
Let $U \subset M$ be an open set where h is smooth and has positive curvature. Assume there exists a semi-positive smooth Hermitian metric h_0 on L_0 with $h \leq h_0$ on M, $h = h_0$ on U, and there exists $k_0 \in \mathbb{N}$ such that $kc_1(L_0, h_0) + c_1(E, h_E) \geq 0$ holds on X for all $k \geq k_0$. Let $p \in U$ be a point and (D, z) local coordinates around p with $D \subset U$. For any $\varepsilon > 0$, $N \in \mathbb{N}_0$ and $\rho \in C^\infty(U, \mathbb{R}) \cap C^0(M)$, $C_0^{-1} \leq \rho \leq C_0$ on M for some constant $C_0 > 0$ one has

$$P_{h_k, dV_M} - \hat{s}_k P_{k\varphi,\tilde{\rho},N}(z, w) = O(k^{-\frac{N+1}{2}+n+\varepsilon}) \text{ in } C^0(D_{\varphi,2N} \times D)$$

with $P_{k\varphi,\tilde{\rho},N}$ as in Theorem 3.21. Furthermore, for any open set $D \subset M_{h,\infty}$ we have

$$P_{h_k, dV_M} = O(k^{-\infty}) \text{ in } C^0(D \times M \setminus \overline{D}).$$

Proof. We use the same notation as in the proof of Lemma 3.37. Given any compact subset K of D we take a cutoff function χ with support in D and $\chi \equiv 1$ in a neighborhood of K. Choose $D' \subset\subset D_{\varphi,N} \cap K$. As in the proof of Lemma 3.37 we find

$$\|v_z^{(k)} - \chi(w) K'_{k\varphi,\rho,N}(z, w) + u_z^{(k)}\|^2_{h_k} e^{-k\varphi(z)} \leq C_3 k^{-N-1+n+\varepsilon} \tag{3.13}$$

for all $z \in D'$ and all $k \geq k_0$ where $v_z^{(k)}$ and $u_z^{(k)}$ as in the proof of Lemma 3.37. We apply Lemma 2.29 and find

$$|\tilde{v}_z^{(k)}(w) - K_{k\varphi,\rho,N}(z,w) + \tilde{u}_z^{(k)}(w)|^2 e^{-k(\varphi(z)+\varphi(w))} \leq C_3 k^{-N-1+2n+\varepsilon}$$

for all $(z,w) \in D' \times K$. We have that $u_z^{(k)}$ is holomorphic in a neighborhood of K. We apply Lemma 2.29 again and find by (3.10) that $|\tilde{u}_z^{(k)}(w)|^2 e^{-k(\varphi(z)+\varphi(w))} = O(k^{-\infty})$ in $C^0(D' \times K)$. Since locally we have

$$P_{h_k,dV_M} = s_k(z) \otimes (v_z^{(k)}(w))^* e^{-\frac{k}{2}(\varphi(z)+\varphi(w))}$$

the claim follows from Lemma 3.4. Furthermore, we have that $u_z^{(k)}$ is holomorphic outside the support of χ. By Lemma 3.7 we conclude $|P_{h_k,dV_M}|^2_{h_k \otimes h_k^*} \leq C_4 k^{-N-1+2n+\varepsilon}$ on $K \times M \setminus \operatorname{supp}(\chi)$. Since the statement is true for all N we find by choosing K and the support of χ sufficiently small that

$$P_{h_k,dV_M} = O(k^{-\infty}) \text{ in } C^0(D \times M \setminus \overline{D})$$

is valid for any set $D \subset U$. □

Proof of Theorem 3.38. As explained in the proof of Theorem 3.36 the C^r expansion can be obtained from the C^0 expansion using Hörmander's trick and the apriori estimate for P_{h_k,dV_M} in Corollary 2.39 as in the proof of Lemma 3.12. Then the conclusion of Theorem 3.38 follows from Lemma 3.25 by replacing N with $2N$. □

As a consequence we obtain the following corollary which is actually the result of Catlin [10] and Zelditch [38] for vector bundles of rank one.

Corollary 3.40
Let M be a compact complex manifold, dV_M a smooth volume form on M and L, E two holomorphic line bundles over M with smooth Hermitian metrics such that the metric of L has positive curvature. The Bergman kernel function for the space $H^0_k(M, L^k \otimes E)$ has an asymptotic expansion, that is

$$B_{h^k \otimes h_E, dV_M} - \frac{k^n}{\pi^n} \frac{c_1(L,h)^n}{n! dV_M} \sum_{j=0}^{N} k^{-j} b_j^{h,h_E,dV_M} \quad = \quad O(k^{-N-1+n}) \text{ in } C^r(M)$$

for any $N, r \in \mathbb{N}_0$.

Proof. Since M is compact and L is positive we have that the curvature of L induces a complete Hermitian metric on M. Thus, the curvature of L^k becomes arbitrary large for $k \to \infty$ and M is compact the claim follows from Theorem 3.36 replacing E by $E \otimes (\Lambda^n T^{*(1,0)})^{-1}$. □

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Köln, August 2019 Hendrik Herrmann