Bulk-Boundary Correspondence of Disordered Topological Insulators and Superconductors

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Kurzfassung

Das Hauptergebnis dieser Arbeit ist eine systematische Formulierung und Beweis der Bulk-Boundary Korrespondenz in ungeordneten topologischen Isolatoren und Supraleitern für alle Symmetrieklassen der Altland–Zirnbauer-Klassifizierung.

Für die Beschreibung der symmetrischen Grundzustände in den verschiedenen Altland–Zirnbauer Klassen übernehmen wir das von R. Kennedy und M.R. Zirnbauer eingeführte Bild von Quasiteilchenvakua mit Pseudosymmetrien. Wir konstruieren physikalisch fundierte C*-Algebren, die jeweils die relevanten physikalischen Quasiteilchenvakua von ungeordneten Festkörpern ohne Rand bzw. mit Rand enthalten. Diese Konstruktion basiert auf etablierten Näherungen für die Beschreibung von Festkörpern bei niedrigen Temperaturen. Die Unordnung in diesen Systemen wird mittels Bellissards Ansatz sogenannter homogener Unordnung beschrieben.

Mittels der Beschreibung von Grundzuständen anhand von Quasiteilchenvakua mit Pseudosymmetrien führen wir eine systematische Beschreibung von topologischen Phasen im Inneren sowie am Rand von ungeordneten topologischen Isolatoren und Supraleitern ein. Diese topologischen Phasen identifizieren wir anschliessend mit (reellen) K-Klassen der zuvor konstruierten C*-Algebren zur Beschreibung der Quasiteilchenvakua.

Ein wesentlicher Aspekt dieser Arbeit ist unsere Anpassung der Beschreibung der K-Theorie, sodass die Klassen in natürlicher Weise durch die Quasiteilchenvakua mit Pseudosymmetrien im Inneren bzw. am Rand von ungeordneten topologischen Isolatoren und Supraleitern beschrieben werden. Neben einer physikalisch natürlichen Beschreibung der topologischen Phasen ermöglicht diese Anpassung der K-Theorie auch eine systematische und physikalisch natürliche Formulierung der Bulk-Boundary Korrespondenz. Infolgedessen erhalten wir neue Einblicke in die physikalische Natur der topologischen Phasen im Inneren und am Rand von ungeordneten topologischen Isolatoren und Supraleitern und deren Beziehung zueinander.

Für eine detaillierte Beschreibung dieser topologischen Phasen leiten wir topologische Invarianten her, welche die Eigenschaften dieser topologischen Phasen quantifizieren.

Abstract

In this work we establish a systematic formulation and proof of the bulk-boundary correspondence of disordered topological insulators and superconductors for all symmetry classes of the Tenfold Way.

We establish a physically natural description of the ground states in the bulk and at the boundary of these solids for all symmetry classes of the Tenfold Way. This is based on the description of symmetric ground states in terms of so-called quasi-particle vacua with pseudo-symmetries, as introduced by Kennedy–Zirnbauer in the case of no disorder. In doing so, we construct, on the basis of physically natural assumptions, C^*-algebras containing the relevant physical quasi-particle vacua of disordered solids in the bulk and in the vicinity of a boundary.

We define a physically natural notion of topological phases for the ground states of disordered topological insulators and superconductors in the bulk and at the boundary. The description of ground states in terms of quasi-particle vacua allows us to identify the topological phases with K-classes of the corresponding C^*-algebras of observables. An essential aspect of this work is that we have adjusted the description of the K-theory in such a way that the classes are naturally described by the quasi-particle vacua with pseudo-symmetries in the bulk and at the boundary. This leads to a physically natural description of the topological phases and allows us to formulate the bulk-boundary correspondence systematically. The physically natural and explicit form of the bulk-boundary correspondence allows for a deeper insight into its physical properties.

To describe the topological phases in the bulk and at the boundary of topological insulators in more detail, we then derive topological invariants quantifying the properties of these topological phases.

Contents

Chapter One

Introduction

During the last almost 40 years, since the discovery of the quantum Hall effect (QHE) in 1980 [89], the investigation of topology in solid state systems has become one of the most active research topics in solid state physics. Experimental physicists are driven by the idea to use these topological properties in order to develop new technologies, such as quantum computers. This idea has its origins in the robust nature that the topological phases exhibited in experiments. For example, in the QHE this stability is reflected by the experimentally observed effect that the quantised Hall current at the boundary cannot be destroyed by disorder at the boundary, e.g. by small cuts.

A rigorous mathematical argument for this stability of the QHE was given by Bellissard in 1986. Based on the observation that the topological phases of the QHE can be described by K-classes [7, 85] in the case without disorder, he established a numerical index for this phase and proved that it is invariant under disorder [10]. His approach relies on the assumption that the translational invariance of the system holds on macroscopic scales, whereas it is broken on microscopic scales due to disorder. Because of these properties, this disorder model is also referred to as *homogeneous disorder*.

It took more than 30 years to realise that topological phases are also possible for solid state systems with local symmetries. The first system of such kind was discovered in 2005 [36] with the quantum spin Hall effect in solid state systems with particle number conservation and time-reversal symmetry. During the following years, several other realisations of topological phases in solid states system with other symmetries as well as for systems without particle number conservation were observed theoretically and experimentally. These systems are nowadays summarised under the names *topological insulators* (TI) and *topological superconductors* (TSC), where the later ones denote the classes without particle number conservation.

This zoo of TI and TSC was put in order by Kitaev [50]. In the free-fermion limit with translational invariance and without boundaries, he used the Fourier–Bloch technique to establish a systematic classification of all symmetry classes of the Tenfold Way [2] in terms of topological K-classes over the torus. However, since disorder breaks translational invariance, Kitaev's classification is not valid for systems with disorder.

Based on Bellissard's work on the disordered QHE, a systematic classification approach for disordered TI and TSC was established by [68, 66, 45]. There, the Hamiltonians are considered as elements of a suitable C^*-algebra containing all observables over a lattice in the context of homogeneous disorder. The classes in the K-theory of this C^*-algebra are then interpreted as the topological phases of the system.

Although the topological classification of topological insulators and superconductors with homogeneous disorder is in principle established for all symmetry classes, the picture is still far from complete. In the following, we will give an overview over the open questions that will be addressed in this work.

First, the choice of the C*-algebra describing the observables is the most fundamental step in the classification. It is essential that this choice governs the physical properties of the system at hand in the most precise way. However, there are different opinions about the right choice of this C*-algebra. See [68, 66] and [53, 25] for recent overviews over the two most important approaches.

Second, the relation between the K-classes of the C*-algebra of observables and the physical properties of the corresponding solid state systems are not understood on a systematic level for all symmetry classes. The major reason for this is that the description of real K-theory is more abstract than in the complex case. Kellendonk has established a K-theoretic description of topological phases in terms of Van Daele groups [86, 87], where the relation to the gapped Hamiltonians in the bulk of disordered TI and TSC can be seen explicitly [45].

Third, the bulk-boundary correspondence is well known for TI in the complex symmetry classes A and $A\mathrm{III}$ [48, 78, 66], whereas it is less understood in the remaining eight real symmetry classes of the Tenfold Way. Although there are already some systematic descriptions of bulk-boundary correspondence which are valid for disordered systems in any symmetry class and dimension [59, 53, 17], a satisfactory understanding of the physical properties of the bulk-boundary correspondence is still missing.

Fourth, the computation of numerical topological invariants that quantify the topological phases of disordered TI and TSC is not yet understood to full extent for the real symmetry classes.

Our contributions to these questions can be roughly summarised as follows.

We develop a physically rigorous construction for the C*-algebra of physical observables in the bulk and in the vicinity of a boundary of disordered TI and TSC.

The starting point of our constructions is the single-particle Hilbert space $\mathcal{V}$. In the tight-binding approximation, the states in the bulk of solid state systems are elements of the complex Hilbert space

$$\mathcal{V} = \ell^2(\Lambda) \otimes V.$$

Here, $\Lambda \cong \mathbb{Z}^d$ denotes the Bravais lattice which is equipped with a natural $\mathbb{Z}^d$-action by translations and V is a finite-dimensional Hilbert space denoting the local degrees of freedom. In this work, we will in particular consider TSC, where it is fundamental that the particle number is not conserved. In order to describe such systems without particle number conservation, we will develop our observables in the more general Nambu space picture. The *Nambu space* $\mathcal{W}$ is defined as the real Hilbert space

$$\mathcal{W} = \mathcal{V} \oplus \mathcal{V}^* \cong \ell^2(\Lambda) \otimes (V \oplus V^*),$$

with the canonical real structure $\gamma := \begin{pmatrix} 0 & \mathfrak{h}^{-1} \\ \mathfrak{h} & 0 \end{pmatrix}$, where $\mathfrak{h}$ denotes the Fréchet-Riesz isomorphism from $\mathcal{V}$ to its dual space $\mathcal{V}^*$. All physically reasonable non-interacting observables can be realised as operators in $\mathcal{L}(\mathcal{W})$.

As observed by Kubota [53], the observables in the tight-binding approximation can be identified with the controlled operators in $\mathcal{L}(\mathcal{W})$. The norm closure of the set of these controlled operators is the real uniform Roe C*-algebra $C^*_u(\Lambda, W) \subseteq \mathcal{L}(\mathcal{W})$, where $W := V \oplus V^*$.

We will extend Kubota's approach by combining this description of observables in terms of $C^*_u(\Lambda, W)$ with the homogeneous disorder approach of Bellissard.

In this approach, the disorder of the solid state system is described by a space of disorder configurations Ω, carrying a natural $\mathbb{Z}^d$-action, and the observables in $C^*_u(\Lambda, W)$ are replaced by norm continuous maps

$$O : \Omega \to C^*_u(\Lambda, W),$$

satisfying the covariance relation

$$O_{\omega \cdot x} = u_x^* O_\omega u_x \quad \forall x \in \mathbb{Z}^d, \omega \in \Omega,$$

where the operators u_x for $x \in \mathbb{Z}^d$ are defined by the action of $\mathbb{Z}^d$ on Λ by translations. These covariant maps generate a real C*-algebra, which we denote by $\mathbb{A}$. This real C*-algebra consists of all physical non-interacting observables in the bulk of disordered TI and TSC.

The constructions can be done in the same way on the single-particle level for charge conserving systems. In the charge conserving case, there is no real structure and therefore the resulting C*-algebra of observables A is a complex C*-algebra. In the construction of A, we will also allows for homogeneous magnetic fields, leading to a twisting in the $\mathbb{Z}^d$-action over Λ.

We perform the analogous constructions in the vicinity of a boundary. In this work, we will only consider the boundary generated by cutting the Bravais lattice perpendicular to one translational direction. The resulting physical systems can be described in the same manner as the bulk system by replacing the Bravais lattice Λ by the half-space lattice

$$\widehat{\Lambda} \cong \mathbb{Z}^{d-1} \times \mathbb{N}.$$

This leads to a real C^*-algebra of half-space observables $\widehat{\mathbb{A}}$. In the charge conserving case, we obtain a complex C*-algebra $\widehat{\mathrm{A}}$ of charge conserving half-space observables.

Summarising, we obtain C*-algebras for the bulk and half-space observables of disordered TI and TSC, which are based on natural physical assumptions. To put these C*-algebras of observables in contact with the literature, we will then identify them with crossed product C*-algebras:

$$\mathbb{A} \cong (C(\Omega) \otimes \mathrm{End}(W)) \rtimes \mathbb{Z}^d, \quad \mathrm{A} \cong (C(\Omega) \otimes \mathrm{End}(V)) \rtimes \mathbb{Z}^d.$$

For the C*-algebras of observables over the half-space, we will construct $*$-isomorphisms

$$\widehat{\mathbb{A}} \cong (C(\Omega) \otimes \mathrm{End}(W)) \rtimes (\mathbb{Z}^{d-1} \times \mathbb{N}), \quad \widehat{\mathrm{A}} \cong (C(\Omega) \otimes \mathrm{End}(V)) \rtimes (\mathbb{Z}^{d-1} \times \mathbb{N}).$$

For the complex C*-algebras A and $\widehat{\mathrm{A}}$, the presence of a homogeneous magnetic field leads to a twisting in the corresponding crossed product C*-algebras.

This identification shows that we obtain the same C*-algebras as in [9, 10, 13, 78]. This yields a new, mathematically and physically rigorous justification for the use of their C*-algebras. Moreover, our construction establishes a canonical realisation of the crossed product C*-algebras over a physically reasonable (real) Hilbert space. This aspect will be important for the physical interpretation of our results.

The identification with crossed product C*-algebras also allows us to establish the bulk-boundary short exact sequence

$$0 \to \mathbb{A}_\partial \otimes \mathbb{K}(\ell^2(\mathbb{N})) \longrightarrow \widehat{\mathbb{A}} \longrightarrow \mathbb{A} \to 0 \tag{1.1}$$

of real C*-algebras. The real C*-algebra $\mathbb{A}_\partial$ is defined as the real C*-algebras of observables over the boundary lattice $\Lambda_\partial := \mathbb{Z}^{d-1}$. Analogously, by the identification of the C*-algebras of charge conserving observables with twisted crossed product C*-algebras, we can construct a short exact sequence

$$0 \to \mathrm{A}_\partial \otimes \mathbb{K}(\ell^2(\mathbb{N})) \longrightarrow \widehat{\mathrm{A}} \longrightarrow \mathrm{A} \to 0 \tag{1.2}$$

of complex C*-algebras. As before, the complex C*-algebra A_∂ is defined as the C*-algebra of charge conserving observables over Λ_∂. The C*-algebras $\mathbb{A}_\partial \otimes \mathbb{K}(\ell^2(\mathbb{N}))$ and $\mathrm{A}_\partial \otimes \mathbb{K}(\ell^2(\mathbb{N}))$ can be interpreted in the physical context as (the norm closure of) the algebra of half-space observables that are localised in the vicinity of the boundary.

On the basis of these results, we will then establish a classification scheme for ground states in the bulk and in the vicinity of a boundary of disordered TI and TSC in all symmetry classes of the Tenfold Way [2]. The Tenfold Way consists of eight so-called *real symmetry classes* and two so-called *complex symmetry classes*.

The ground states in the bulk of disordered TI and TSC in the real symmetry classes can be described by *disordered invariant quasi-particle vacua* (IQPV) with pseudo-symmetries. These are given by pairs

$$(J; \phi)$$

of a real skew-Hermitian unitaries $J \in \mathbb{A}$ and unital real $*$-morphisms $\phi : C\ell_{r,s} \to \mathrm{End}(W) \subseteq \mathbb{A}$, such that

$$\{J, \phi(k_a)\} = \{J, \phi(j_\alpha)\} = 0 \quad \forall a = 1, \dots, r, \alpha = 1, \dots, s,$$

where $\{x, y\} = xy + yx$ for all $x, y \in \mathbb{A}$ denotes the anti-commutator, and $k_a(j_\alpha)$ denote the positive (negative) generators of the real Clifford algebra $C\ell_{r,s}$. The unitaries $K_a := \phi(k_a)$ for $a = 1 \dots, r$ and $J_\alpha := \phi(j_\alpha)$ for $\alpha = 1, \dots, s$ are called positive and negative pseudo-symmetries, respectively. The pair (r, s) is called the *symmetry index*.

The ground states in the bulk of disordered TI in the two complex symmetry classes can be described by *complex* or *charge conserving disordered invariant quasi-particle vacua*. These are given in the same way as disordered IQPV, where the $\mathbb{A}$ is replaced by A, $\phi : \mathbb{C}\ell_{r,s} \to \mathrm{End}(V) \subseteq \mathrm{A}$, where $\mathbb{C}\ell_{r,s}$ denotes the complex Clifford algebra with r positive and s negative generators, and there are no reality conditions on J and ϕ.

We establish a canonical map which assigns to a given disordered IQPV of symmetry index (r, s) a class in $KR^{-(s-r+2)}(\mathbb{A})$ for the real symmetry classes, and in $K^{-(s-r+2)}(\mathrm{A})$ for the complex symmetry classes. To that end, we construct Abelian groups $\mathrm{Top}^{r,s}(\mathbb{A})$ and $\mathrm{Top}^{r,s}(\mathrm{A})$, whose elements are formal differences of equivalence classes of disordered IQPV. Thus, this group measures the topological phases of the disordered IQPV *relative to each other*. In practice, we will fix a reference disordered IQPV $(J_{\mathrm{ref}}; \phi)$ and measure the topological phases relative to it.

We will then construct explicit isomorphisms

$$\begin{aligned} &\mathrm{Top}^{r,s}(\mathbb{A}) \cong DKR^{-(s-r+2)}(\mathbb{A}) = KR^{-(s-r+2)}(\mathbb{A}) \\ \text{and} \quad &\mathrm{Top}^{r,s}(\mathrm{A}) \cong DK^{-(s-r+2)}(\mathrm{A}) = K^{-(s-r+2)}(\mathrm{A}), \end{aligned}$$

where $DKR^{-\bullet}(-)$ and $DK^{-\bullet}(-)$ denote the KR- and K-groups in Van Daele's picture of (real) K-theory [86].

As we will explain in Section 3.3, the group $\mathrm{Top}^{r,s}(A)$ can be defined for every real or complex C*-algebra. In this way, we obtain a new picture of real and complex K-theory, which is an important mathematical result on its own. In the present physical context, this new picture of K-theory leads to an explicit, physically natural and systematic definition of bulk topological phases of disordered TI and TSC in any symmetry class as classes in the corresponding (real) K-groups.

The topological properties of ground states of disordered TI and TSC in the vicinity of a boundary are determined by the gapless boundary states. We will show that the ground states in the real symmetry classes can be described by *disordered IQPV with boundary*, which are pairs $(\hat{J};\phi)$, where $\hat{J} \in \widehat{\mathbb{A}}$ is real and skew-Hermitian, and $\phi : C\ell_{r,s} \to \mathrm{End}(W) \subset \widehat{\mathbb{A}}$ is a real unital $*$-morphism such that

$$\{\phi(k_a), \hat{J}\},\ \{\phi(j_\alpha), \hat{J}\}, 1 + \hat{J}^2 \in \mathbb{A}_\partial \otimes \mathbb{K}(\ell^2(\mathbb{N})) \quad \forall a = 1, \ldots, r, \alpha = 1, \ldots, s. \tag{1.3}$$

In the complex symmetry classes, the ground states at the boundary can be described by *complex disordered IQPV with boundary*, which are defined in the same way as disordered IQPV with boundary, where $\mathbb{A}_\partial$ is replaced by A_∂, $C\ell_{r,s}$ is replaced by $\mathbb{C}\ell_{r,s}$ and there are no reality constraints.

We will show that the algebra of half-space observables $\widehat{\mathbb{A}}$ can be identified with a real C*-subalgebra of the real C*-algebra $\mathbb{B}(\mathcal{H}_{\mathbb{A}_\partial})$ of adjointable operators over the real Hilbert space over $\mathbb{A}_\partial$. In the same way, we will show that the algebra of complex half-space observables $\widehat{\mathrm{A}}$ defines a C*-subalgebra of the C*-algebra $\mathbb{B}(\mathcal{H}_{\mathrm{A}_\partial})$ of adjointable operators over the Hilbert space over A_∂.

We refer to the resulting generalisation of disordered IQPV with boundary to pairs $(\hat{J};\phi)$, where $\hat{J} \in \mathbb{B}(\mathcal{H}_{\mathbb{A}_\partial})$ is real and skew-Hermitian, and ϕ is given as before such that Equation (1.3) holds, as *generalised quasi-particle vacua* (GQPV). The corresponding generalisation of complex disordered IQPV with boundary is called *complex GQPV*.

On the basis of these GQPV, we then establish a topological classification scheme by constructing groups $\mathrm{TopG}(\mathbb{A}_\partial)$ and $\mathrm{TopG}(\mathrm{A}_\partial)$, whose elements consist of equivalence classes of GQPV. In doing so, we make sure that the resulting groups indeed measure the topological properties at the boundary of disordered TI and TSC.

The construction of the groups $\mathrm{TopG}(\mathbb{A}_\partial)$ and $\mathrm{TopG}(\mathrm{A}_\partial)$ is closely related to the construction of Kasparov's Fredholm picture of real and complex K-theory [41, 43]. In fact, we will identify these groups with Kasparov's Fredholm picture of real and complex K-groups by constructing explicit isomorphisms

$$\begin{aligned} &\mathrm{TopG}(\mathbb{A}_\partial) \cong KKR(C\ell_{s,r}, \mathbb{A}_\partial \otimes C\ell_{0,1}) \cong KR^{-(s-r+1)}(\mathbb{A}_\partial), \\ \text{and}\quad &\mathrm{TopG}(\mathrm{A}_\partial) \cong KK(\mathbb{C}\ell_{s,r}, \mathrm{A}_\partial \otimes \mathbb{C}\ell_{0,1}) \cong K^{-(s-r+1)}(\mathrm{A}_\partial). \end{aligned}$$

Summarising, we obtain two independent classification schemes in terms of (real) K-groups, one for the bulk and one for the boundary topological phases of disordered TI and TSC.

Following the literature, the bulk-boundary correspondence will be formulated on the level of the K-groups describing the topological phases. By the bulk-boundary short exact sequences (1.1) and (1.2), we obtain connecting morphisms

$$\partial : KR^{-(s-r+2)}(\mathbb{A}) \to KR^{-(s-r+1)}(\mathbb{A}_\partial) \quad \text{and} \quad \partial : K^{-(s-r+2)}(\mathrm{A}) \to K^{-(s-r+1)}(\mathrm{A}_\partial) \tag{1.4}$$

on the level of K-groups. In the literature, one can find explicit forms of the connecting morphism for Van Daele's picture of K-theory [87], as well for Kasparov's Fredholm picture of K-theory [43]. However, a satisfactory formulation of the bulk-boundary correspondence has to identify the bulk topological phases with boundary topological phases. The former can be identified with classes in Van Daele's picture of K-theory, whereas the latter can be explicitly identified with classes in Kasparov's Fredholm picture of K-theory. Therefore, we have to concatenate the connecting morphisms with the isomorphism between those two pictures of K-theory, which we call *Roe's isomorphism* [72]. This strategy can be summarised in the following commutative diagram, where we denote Roe's isomorphism by α:

$$\begin{tikzcd}
\mathrm{Top}^{r,s}(\mathbb{A}) \cong DKR^{-(s-r+2)}(\mathbb{A}) \arrow[r, dashed, "\alpha"] \arrow[d, "\partial"] \arrow[dr] & KKR^{-(s-r+2)}(\mathbb{A}) \arrow[d, dashed, "\partial"'] \\
DKR^{-(s-r+1)}(\mathbb{A}_\partial) \arrow[r, "\alpha"] & KKR^{-(s-r+1)}(\mathbb{A}_\partial) \cong \mathrm{TopG}^{r,s}(\mathbb{A}_\partial)
\end{tikzcd}$$

One of the main results of this work is Theorem 5.3.1. It states that for any disordered IQPV $(J;\phi)$, the bulk-boundary map $\alpha \circ \partial$ sends the bulk class associated with $(J;\phi)$ relative to a reference disordered IQPV $(J_{\mathrm{ref}};\phi)$ to the boundary class associated with the same disordered IQPV. Explicitly, this means that

$$KR^{-(s-r+2)}(\mathbb{A}) = \mathrm{Top}^{r,s}(\mathbb{A}) \ni [J] - [J_{\mathrm{ref}}] \xrightarrow{\alpha\circ\partial} [(\hat{J};\phi)] - [(\hat{J}_{\mathrm{ref}};\phi)] \in \mathrm{TopG}(\mathbb{A}_\partial) = KR^{-(s-r+1)}(\mathbb{A}_\partial),$$

$$K^{-(s-r+2)}(\mathrm{A}) = \mathrm{Top}^{r,s}(\mathrm{A}) \ni [J] - [J_{\mathrm{ref}}] \xrightarrow{\alpha\circ\partial} [(\hat{J};\phi)] - [(\hat{J}_{\mathrm{ref}};\phi)] \in \mathrm{TopG}(\mathrm{A}_\partial) = K^{-(s-r+1)}(\mathrm{A}_\partial).$$

The disordered IQPV with boundary $\hat{J}$ and $\hat{J}_{\mathrm{ref}}$ denote lifts of J and J_{ref}, respectively, along the morphism $\widehat{\mathbb{A}} \to \mathbb{A}$ of the SES (1.1) in the real case, and along the morphism $\widehat{\mathrm{A}} \to \mathrm{A}$ of the SES (1.2) in the complex case. This theorem is our mathematical formulation of 'bulk-boundary correspondence'.

This explicit and simple form of the bulk-boundary correspondence enables us to derive concrete physical properties of the bulk-boundary correspondence, which cannot be seen in the more abstract formulations in the literature. One of the physically most important observations is that the bulk-boundary correspondence determines the existence of stable gapless boundary states.

In order to quantify the bulk and boundary K-classes, we will also construct numerical invariants, which we call *topological invariants* because of the present physical context.

There are recently established formulas by Kellendonk for the computation of those topological invariants *via* Van Daele groups and cyclic cohomology [45]. We will use these results to establish numerical values for the $\mathbb{Z}$-valued invariants of the bulk classes in $KR^{-(s+r-2)}(\mathbb{A})$.

For the topological phases at the boundary, we will derive formulas for the so-called strong topological invariants by the application of index maps [16, 17] onto the K-classes in Kasparov's Fredholm picture, representing the topological phases at the boundary.

The structure of this work is as follows.

In Chapter 2, we will review some fundamental mathematical objects which will be important for the following chapters and cannot be associated with just one of these chapters.

In Section 3.1, we will introduce Kasparov's Fredholm picture of K-theory, as well as KK-theory, also called Kasparov's bivariant K-theory, which is a generalisation of K-theory. This more general theory will be needed because it describes K-theory as well as morphisms between K-groups, which are important for the bulk-boundary correspondence and the derivation of numerical invariants for topological phases.

In Section 3.2, we will introduce Van Daele's picture of K-theory, which is based on Van Daele's generalisation of K-theory to graded C*-algebras [86, 87]. As before, we will first give an introduction to the more general theory, and then explain the resulting picture of K-theory.

In Section 3.3, we will then derive our new picture of real and complex K-theory of trivially graded C*-algebras.

In Chapter 4, we will introduce disordered TI and TSC from the perspective of free-fermion many-particle physics. In Section 4.1, we will put our upcoming constructions on firm physical grounds by giving a detailed review of the many-particle picture for free fermions. In particular, we will explain thoroughly the description of free-fermion ground states in terms of quasi-particle vacua with pseudo-symmetries, as introduced by [49]. The correct usage of this many-body picture is essential for this work, and therefore, we consider such a detailed explanation as necessary.

In Section 4.2, we will construct the C*-algebras of bulk observables $\mathbb{A}$ and A, and derive the isomorphisms to the crossed product C*-algebras. We will also establish the description of symmetric ground states in the bulk of disordered TI or TSC.

In Section 4.3, we will perform the same constructions for disordered TI or TSC with boundary, leading to the C*-algebra of half-space observables $\widehat{\mathbb{A}}$ and $\widehat{\mathrm{A}}$. As in the previous section, we will establish the isomorphisms to the crossed product C*-algebras, as well as the description of symmetric ground states of disordered TI and TSC in the vicinity of a boundary. Moreover, we will construct the bulk-boundary short exact sequence, which establishes a relation between the bulk and half-space observables.

In Chapter 5, we use the established descriptions of ground states in the bulk and at the boundary of disordered TI and TSC, to define topological phases. The bulk topological phases will be defined in Section 5.1 *via* our new picture of K-theory of Section 3.3. At the boundary, we will define in Section 5.2 a group of boundary topological phases. We will establish an explicit isomorphism to Kasparov's Fredholm picture of K-theory. In Section 5.3, we will then formulate and prove the bulk-boundary correspondence. In Sections 5.4 and 5.5, we will derive physical properties of this bulk-boundary correspondence.

In Chapter 6, we will use the explicit form of the topological phases in terms of K-classes to derive the aforementioned numerical topological invariants.

Chapter 7 contains a short outlook on open questions and topics that are closely related to the topics of this work, but which have not been addressed here.

Most of the material presented in this work was published in the preprint *Bulk-boundary correspondence for disordered free-fermion topological phases* by A. Alldridge, C. Max and M. R. Zirnbauer, 2019, [1]. We extend the results of [1] at the following places of this work.

In Section 3.3, we extend the results of [1] by a more rigorous derivation of our new picture of K-theory.

In Sections 4.2 and 4.3, we extend the results of [1] by including non-trivial homogeneous magnetic fields in the construction of the C*-algebras A and $\widehat{\mathrm{A}}$. In the sequel, we will generalise the results of [1] regarding these two C*-algebras to the presence of homogeneous magnetic fields.

In Section 5.1, we extend the construction of the topological phases at the boundary of disordered TI and TSC by a rigorous discussion of the physical properties of this construction.

Chapter 6 and Chapter 7 have not been published in [1].

Chapter Two

Fundamentals

This chapter summarises the mathematical tools that are needed throughout the following chapters at different places. We begin with a review of real and graded C*-algebras, which is addressed to the reader familiar with the general properties of complex C*-algebras.

Thereafter, we introduce real and complex Clifford algebras. Although detailed introductions to this topic can be found in the literature [55, Ch. I], there are various notations and conventions that are not unique and therefore have to be fixed. The properties of Clifford algebras are essential for the systematic description of real K-theory and a good understanding of those is necessary to follow the constructions in the upcoming chapters.

Most of the material presented here was published as part of the preprint *Bulk-boundary correspondence for disordered free-fermion topological phases* by A. Alldridge, C. Max and M. R. Zirnbauer, 2019, [1]. The introduction to Clifford algebras is extended by a discussion of Clifford algebras for vector spaces.

2.1 Real and graded C*-algebras

In this section, we review the fundamental definitions and properties of real and graded C*-algebras that will be needed throughout this work. We assume that the properties of complex C*-algebras are known. This introduction is primarily based on [15, 77]. More details about the topics of this chapter can be found there.

Definition 2.1.1 (Real Hilbert space)**.** *Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$. A* real structure *on $\mathcal{H}$ is an anti-unitary involution γ. That is, $\gamma^2 = 1$ and*

$$\langle\gamma\psi,\gamma\psi'\rangle = \langle\psi',\psi\rangle \quad \forall\psi,\psi' \in \mathcal{H}.$$

A real Hilbert space *is by definition a pair $(\mathcal{H},\gamma)$ of a Hilbert space $\mathcal{H}$ and a real structure γ. A complex linear (or anti-linear) map $\phi : (\mathcal{H}_1,\gamma_1) \longrightarrow (\mathcal{H}_2,\gamma_2)$ is called* real *if it intertwines the real structures,* i.e.

$$\phi\circ\gamma_1 = \gamma_2\circ\phi.$$

Definition 2.1.2 (Real C*-algebra)**.** *A real C*-algebra is a complex C*-algebra equipped with an anti-linear involutive $*$-isomorphism $\overline{(-)} : A \to A$; $x \mapsto \overline{x}$, called* conjugation *or* real structure*, such that $\overline{(-)}$ commutes with $*$.*

Equivalently, we may define real C-algebras as complex C*-algebras equipped with a linear involutive anti-isomorphism* $\mathsf{T} : A \to A;\ a \mapsto a^{\mathsf{T}}$ *such that* T *commutes with* $*$. *The map* T *is called transposition.*

These two equivalent definitions are related by the equation

$$\overline{a} = (a^*)^{\mathsf{T}} \quad \textit{for } a \in A.$$

We will call an element in a real C*-algebra *real* if it is invariant under conjugation. A $*$-morphism $\phi : A_1 \to A_2$ between real C*-algebras is called *real* if it intertwines the real structures, i.e.

$$\phi(\overline{a}) = \overline{\phi(a)} \quad \forall a \in A_1.$$

The spatial tensor product $A_1 \otimes A_2$ of real C*-algebras becomes a real C*-algebra when equipped with the conjugation

$$\overline{a_1 \otimes a_2} \coloneqq \overline{a_1} \otimes \overline{a_2} \quad \forall a_1 \in A_1, a_2 \in A_2.$$

An important point in order to understand the nature of real C*-algebras is that they can be realised over a suitable real Hilbert space, as explained in the following.

Remark 2.1.3. *For any real C*-algebra A, the real subalgebra* $B = A_{\mathbb{R}}$ *consisting of all real elements is a Banach* $*$*-algebra such that for every* $a \in B$, $\|a^*a\| = \|a\|^2$ *and* $1 + a^*a$ *is invertible in the unitisation of B. Building on the work of Arens [4] and Arens–Kaplansky [5] in the commutative case, Ingelstam [34] proved that conversely, if B is a real Banach* $*$*-algebra satisfying the above assumptions, then its complexification* $A = B_{\mathbb{C}}$ *is a real C*-algebra for the conjugation given by the complex conjugation*

$$\overline{a + ib} \coloneqq a - ib \quad \textit{for } a, b \in B.$$

Moreover, he proved that every real C-algebra admits, for some real Hilbert space* $\mathcal{H}$, *a real isometric* $*$*-isomorphism onto a closed subalgebra of* $\mathcal{L}(\mathcal{H})$ *which is invariant under the adjoint and the transpose. Here,* $\mathcal{L}(-)$ *denotes the set of bounded linear operators.*

Although real C*-algebras can thus always be realised on a real Hilbert space, in practice, they also arise through quaternionic rather than real structures.

Definition 2.1.4 (Quaternionic structures of Hilbert spaces)**.** *Let* $\mathcal{H}$ *be a complex Hilbert space. An anti-linear operator* $T : \mathcal{H} \to \mathcal{H}$ *is called quaternionic structure if* $T^2 = -1$. *That is, T is anti-linear and we have*

$$\langle Tx, Ty \rangle = \langle y, x \rangle, \quad \langle Tx, y \rangle = -\langle Ty, x \rangle \quad \forall x, y \in \mathcal{H}.$$

A complex Hilbert space equipped with a quaternionic structure is called a quaternionic Hilbert space. A linear map between quaternionic Hilbert spaces is called quaternionic if it intertwines the quaternionic structures.

Example 2.1.5. *Let* $\mathbb{C}^2$ *be equipped with the quaternionic structure*

$$\mathfrak{t} \coloneqq \begin{pmatrix} 0 & \mathfrak{c} \\ -\mathfrak{c} & 0 \end{pmatrix}.$$

Then Ad_{t} *defines a real structure on* $M_2(\mathbb{C})$ *which is given by*

$$\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} \overline{d} & -\overline{c} \\ -\overline{b} & \overline{a} \end{pmatrix} \quad \forall a, b, c, d \in \mathbb{C}.$$

For the real C-algebra* $(M_2(\mathbb{C}), \mathrm{Ad}_{\mathrm{t}})$*, the subalgebra of real elements is spanned (over the real numbers) by the identity matrix and the matrices* $i\sigma_x, i\sigma_y, i\sigma_z$*, where*

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.1}$$

This subalgebra is thus isomorphic (as an algebra over $\mathbb{R}$*) to the quaternions* $\mathbb{H}$*. For this reason, we will denote the real C*-algebra* $M_2(\mathbb{C})$ *equipped with the conjugation* Ad_{t} *by* $\mathbb{H}_{\mathbb{C}}$*.*

With the notion of real C*-algebras in place, we now introduce graded C*-algebras.

Definition 2.1.6 (Graded real C*-algebra)**.** *A* $\mathbb{Z}/2\mathbb{Z}$*-grading on a* $*$*-algebra* A *is a decomposition*

$$A = A^{(\bar{0})} \oplus A^{(\bar{1})},$$

where $\mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$*, into two closed subspaces such that*

$$A^{(i)}A^{(j)} \subseteq A^{(i+j)}, \quad *(A^{(i)}) \subseteq A^{(i)} \quad \forall i, j \in \mathbb{Z}/2\mathbb{Z}.$$

When equipped with a grading, A *is called graded* $*$*-algebra. Throughout the present work we use the terms '*$\mathbb{Z}/2\mathbb{Z}$*-grading' and 'grading' synonymously. The elements of* $A^{(\bar{0})}$ *are called even, while those of* $A^{(\bar{1})}$ *are called odd. An element is called homogeneous, if it is either even or odd. For homogeneous elements we call*

$$|x| := \begin{cases} \bar{0} & \text{if } x \text{ even}, \\ \bar{1} & \text{if } x \text{ odd}, \end{cases} \quad \text{the degree of } x.$$

If A *is a C*-algebra, then we assume in addition that* $A^{(\bar{0})}$ *and* $A^{(\bar{1})}$ *are closed; in this case,* A *is called a* graded C*-algebra*.*

If A *is a real* $*$*-algebra and* A *is equipped with a grading, then these data are called* graded real $*$-algebra *if* $A^{(\bar{0})}$ *and* $A^{(\bar{1})}$ *are invariant under conjugation. A graded real* $*$*-algebra that is also a graded C*-algebra is called a* graded real C*-algebra*.*

A map between graded $*$*-algebras is called even if it preserves the grading.*

Any (real) C*-algebra A can be considered as a graded (real) C*-algebra equipped with the trivial grading $A \equiv A^{(\bar{0})}$ and $A^{(\bar{1})} = 0$.

For graded $*$-algebras A and B, the algebraic tensor product $A \odot B$ is graded by

$$(A \odot B)^{(k)} := \bigoplus_{i+j=k} A^{(i)} \odot B^{(j)}, \quad \forall k \in \mathbb{Z}/2\mathbb{Z}.$$

If A and B are real, then so is $A \odot B$. If A and B are graded (real) C*-algebras, then the grading on $A \odot B$ extends to the spatial tensor product $A \otimes B$, turning it into a graded (real) C*-algebra.

Apart from the spatial tensor product, there is the *graded tensor product*.

Definition 2.1.7 (Graded tensor product)**.** *Let A and B be graded $*$-algebras and consider the following $*$-algebra structure of $A \odot B$:*

$$(a \otimes b)(a' \otimes b') := (-1)^{|a'||b|} aa' \otimes bb', \quad (a \otimes b)^* := (-1)^{|a||b|} a^* \otimes b^*,$$

for all homogeneous elements $a, a' \in A$ and $b, b' \in B$. This defines a graded $*$-algebra which we denote by $A \widehat{\odot} B$. If A and B are real then so is $A \widehat{\odot} B$.

If A and B are graded C*-algebras, there is the following C*-norm on $A \widehat{\odot} B$ [83]:

$$\left\| \sum_{i=1}^m a_i \otimes b_i \right\|^2 := \sup \frac{(\rho \otimes \lambda)\Big[\big(\sum_{j=1}^n x_j \otimes y_j\big)^* \big(\sum_{i=1}^m a_i \otimes b_i\big)^* \big(\sum_{i=1}^m a_i \otimes b_i\big)\big(\sum_{j=1}^n x_j \otimes y_j\big)\Big]}{(\rho \otimes \lambda)\Big[\big(\sum_{j=1}^n x_j \otimes y_j\big)^* \big(\sum_{j=1}^n x_j \otimes y_j\big)\Big]},$$

where the supremum is taken over all non-zero finite sums $\sum_{j=1}^n x_j \otimes y_j \in A \widehat{\odot} B$ and all states ρ of A and λ of B such that $\rho|_{A^{(\bar{1})}} \equiv 0$ and $\lambda|_{B^{(\bar{1})}} \equiv 0$.

The completion of $A \widehat{\odot} B$ w.r.t. this norm is denoted by $A \widehat{\otimes} B$.

If A or B is a nuclear C*-algebra, $A \widehat{\otimes} B$ is the universal enveloping C*-algebra of $A \widehat{\odot} B$ [15, 14.4.1]. We will consider the graded tensor product almost exclusively in the context, where one of the C*-algebras is a finite-dimensional Clifford algebra which are in particular nuclear.

2.2 Clifford algebras

In this section, we will summarise the most important properties of Clifford algebras. Clifford algebras will play an essential role in the construction of (real) K-theory and the systematic description of symmetric free-fermion many-particle ground states. This introduction is based on [77, Sec. 1.2] and [55, Ch. I].

Definition 2.2.1 (Clifford algebra)**.** *Let V be a complex vector space equipped with a bilinear form q. Let $\mathcal{T}(V) := \bigoplus_{r=0}^\infty V^{\otimes r}$ be the tensor algebra of V. Then the Clifford algebra is defined as the quotient*

$$C\ell(V) := \mathcal{T}(V)/\mathcal{I},$$

where $\mathcal{I} \subseteq \mathcal{T}(V)$ is the ideal generated by all elements of the form $v \otimes v - q(v)$ for $v \in V$.

If V is a real vector space, i.e. if there is an anti-linear involution $\bar{\cdot} : V \to V$, we demand that the quadratic form is real. Then $\mathcal{T}(V)$ and $C\ell(V, q)$ become real algebras equipped with the natural conjugation defined by

$$\overline{v_1 \otimes \cdots \otimes v_k} = \overline{v_1} \otimes \cdots \otimes \overline{v_k} \quad \forall k \geq 1, v_1, \ldots, v_k \in V.$$

There is a natural (real) embedding $j : V \to C\ell(V, q)$ which identifies V with $V^{\otimes 1} \subseteq C\ell(V, q)$. By [55, Ch. I, Prop. 1.1], the Clifford algebra is universal in the sense that for every linear map $\tilde{j} : V \to A$ into a unital complex algebra A such that $\tilde{j}(v)^2 = q(v)1_A$, there is a unique algebra morphism $\varphi : C\ell(V, q) \to A$, such that $\varphi \circ j = \tilde{j}$.

Clifford algebras can be canonically considered as graded algebras by introducing the following natural grading.

Definition 2.2.2 (Natural grading)**.** *The automorphism $\alpha : V \to V;\ v \mapsto -v$ extends to a involutive automorphism $\alpha : C\ell(V,q) \to C\ell(V,q)$. This defines a grading on $C\ell(V,q)$ by*

$$C\ell(V,q)^{(i)} := \tfrac{1}{2}(\,\mathrm{id} + (-1)^i\alpha)C\ell(V,q) \quad \textit{for } i = \bar{0}, \bar{1} \in \mathbb{Z}/2\mathbb{Z}.$$

This grading is called the natural grading *of $C\ell(V,q)$.*

Clifford algebras for finite-dimensional vector spaces can also be defined in terms of generators, as explained in the following proposition.

Proposition 2.2.3. *[77, Cor. 1.2.4] Let $\{e_1,\ldots,e_n\}$ be a basis of V which is orthogonal w.r.t. the symmetric bilinear form $q(v_1,v_2) := \frac{1}{2}(q(v_1+v_2) - q(v_1) - q(v_2))$ associated with q. Then a linear basis of $C\ell(V,q)$ is given by*

$$\{e_{i_1}\cdots e_{i_k} \mid i_1 < \ldots < i_k, 0 \le k \le n\},$$

where $e_{i_1}\cdots e_{i_k} = 1$ if $k = 0$. $C\ell(V,q)$ is uniquely determined by the generators $e_1,\ldots,e_n$ and the relations

$$e_ie_j + e_je_i = 2\delta_{ij}q(e_i) \quad \forall i,j = 1,\ldots,n.$$

By [77, Thm. 1.2.3], for a q-orthogonal decomposition $V = V_1 \oplus V_2$ (i.e. $q(v_1+v_2) = q(v_1) + q(v_2)\,\forall v_1 \in V_1, v_2 \in V_2$), there is a natural isomorphism of oriented graded (real) Clifford algebras

$$C\ell(V_1,q_1)\widehat{\otimes}C\ell(V_2,q_2) \longrightarrow C\ell(V,q); \quad v_1 \otimes v_2 \mapsto v_1v_2,$$

where q_i denotes the restriction of q onto V_i for $i = 1,2$.

From hereon, we assume that the vector space V is a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and that the quadratic form q is of the form $q(v) = \langle\gamma(v), v\rangle$ for some Hermitian unitary anti-linear $\gamma : V \to V$. Then we can define a $*$-involution of $C\ell(V,q)$ by

$$(v_1\cdots v_k)^* = v_k^*\cdots v_1^*, \quad \text{where } v_i^* := \gamma(v)\,\forall i = 1,\ldots,k,$$

for all $k \ge 1$. Then $C\ell(V,q)$ becomes a $*$-algebra. By the following proposition, there is a suitable norm on $C\ell(V,q)$ such that it becomes a C*-algebra w.r.t. this $*$-structure.

Proposition 2.2.4. [77, p. 11] *Let V be a finite n-dimensional vector space and consider the exterior algebra*

$$\Lambda(V) = \bigoplus_{i=0}^{n} \Lambda^i(V).$$

This space can be equipped with the $\mathbb{Z}/2\mathbb{Z}$-grading

$$\Lambda(V)^{(\bar{0})} := \bigoplus_{i \text{ even}} \Lambda^i(V), \quad \Lambda(V)^{(\bar{1})} := \bigoplus_{i \text{ odd}} \Lambda^i(V).$$

Then we obtain an even $*$-representation $\pi : C\ell(V \oplus V, q \oplus -q) \to \mathrm{End}(\Lambda(V))$, given by

$$\pi(v \oplus w) = \varepsilon_{v+w} - \iota_{\gamma(v)-\gamma(w)},$$

where we define for all $v \in V$, $x_1 \wedge \ldots \wedge x_k \in \Lambda^k(V)$ and $i, k = 1, \ldots, n$

$$\varepsilon_v(x_1 \wedge \ldots \wedge x_k) := v \wedge x_1 \wedge \ldots \wedge x_k, \tag{2.2}$$

$$\iota_v(x_1 \wedge \ldots \wedge x_k) := \sum_{i=1}^{k} (-1)^{i+1} \langle v, x_i \rangle x_1 \wedge \ldots \wedge x_{i-1} \wedge x_{i+1} \wedge \ldots \wedge x_k. \tag{2.3}$$

This induces an isomorphism $C\ell(V \oplus V, q \oplus -q) \cong \mathrm{End}(\Lambda(V))$ of graded $*$-algebras.

The choice of an orthonormal basis $\{e_1, \ldots, e_n\}$ in V defines the basis

$$\{e_{i_1} \wedge \ldots \wedge e_{i_k} \mid i_1 < \ldots < i_k,\ 1 \leq k \leq n\}$$

of $\Lambda(V)$ and thus an inner product by demanding this basis to be orthonormal. The corresponding norm on $\Lambda(V)$ defines a C^*-norm on $C\ell(V \oplus V, q \oplus -q)$. A C^*-norm on $C\ell(V, q)$ is induced by the embedding

$$C\ell(V, q) \ni x \mapsto x \otimes 1 \in C\ell(V, q) \widehat{\otimes} C\ell(V, -q) \cong C\ell(V \oplus V, q \oplus -q).$$

We are primarily interested in Clifford algebras over finite-dimensional real or complex Hilbert spaces. By Proposition 2.2.3, these can be described in the following systematic fashion.

Definition 2.2.5 (Clifford algebras $C\ell_{r,s}$, $\mathbb{C}\ell_{r,s}$)**.** *Let $\dim(V) = r + s$ for some $r, s \in \mathbb{N}$ and fix an orthonormal basis $e_1, \ldots, e_{r+s}$. Let the quadratic form be given by $q(e_a) = \|e_a\|^2 = 1$ for $a = 1, \ldots, r$ and $q(e_{r+\alpha}) = -\|e_{r+\alpha}\|^2 = -1$ for $\alpha = 1, \ldots, s$. The assertions above imply that the Clifford algebra $C\ell(V, q)$ is the universal C^*-algebra with unitary generators $k_1, \ldots, k_r, j_1, \ldots, j_s$ and the relations*

$$\left.\begin{aligned} k_a k_b + k_b k_a &= 2\delta_{a,b} \\ j_\alpha j_\beta + j_\beta j_\alpha &= -2\delta_{\alpha,\beta} \\ k_a j_\alpha + j_\alpha k_a &= 0 \end{aligned}\right\} \quad \forall a, b = 1, \ldots, r; \alpha, \beta = 1, \ldots, s.$$

The natural grading defines the generators $k_1, \ldots, k_r j_1, \ldots, j_s$ to be odd. The k_a (j_α) are called positive (negative) generators, respectively. We denote this Clifford algebra by $\mathbb{C}\ell_{r,s}$.

The corresponding real Clifford algebra, equipped with the real structure which is defined by declaring the generators to be real, is denoted by $C\ell_{r,s}$.

These Clifford algebras over finite-dimensional Hilbert spaces can be equipped with so-called *orientations*. Orientation-preserving isomorphisms between Clifford algebras will play an important role in the description of K-theory in terms of Kasparov groups.

Definition 2.2.6 (Standard orientation of $C\ell_{r,s}$ and $\mathbb{C}\ell_{r,s}$)**.** *An* orientation *of $C\ell_{r,s}$ is defined as a homogeneous element $\omega \in C\ell_{r,s}$ such that*

$$\omega^* = \pm\omega,\ \omega^*\omega = 1,\ \overline{\omega} = \omega$$

and $x\omega = (-1)^{|x|(1+|\omega|)}\omega x$ for all homogeneous $x \in C\ell_{r,s}$. The standard orientation of *$C\ell_{r,s}$ is defined by*

$$\omega_{r,s} := k_1 \cdots k_r j_1 \cdots j_s.$$

An orientation of $\mathbb{C}\ell_{r,s}$ is defined analogously without the reality constraint. The standard orientation of *$\mathbb{C}\ell_{r,s}$ is defined by*

$$\omega^{\mathbb{C}}_{r,s} := (-i)^s k_1 \cdots k_r j_1 \cdots j_s.$$

We now collect some periodicity results on these Clifford algebras, *cf.* [6, §4]. Fundamental for all of the following assertions is the real even orientation-preserving isomorphism

$$C\ell_{p,q}\,\widehat{\otimes}\,C\ell_{r,s} \cong C\ell_{p+r,q+s}. \tag{2.4}$$

On the level of generators, it is induced by

$$\begin{aligned}
k_b \otimes 1 &\mapsto k_b && \text{for } b = 1, \dots, p,\\
j_\beta \otimes 1 &\mapsto j_\beta && \text{for } \beta = 1, \dots, q,\\
1 \otimes k_a &\mapsto (-1)^q k_{p+a} && \text{for } a = 1, \dots, r,\\
1 \otimes j_\alpha &\mapsto j_{q+\alpha} && \text{for } \alpha = 1, \dots, s.
\end{aligned}$$

This also defines an orientation-preserving isomorphism for the complex Clifford algebras

$$\mathbb{C}\ell_{p,q}\,\widehat{\otimes}\,\mathbb{C}\ell_{r,s} \cong \mathbb{C}\ell_{p+r,q+s}. \tag{2.5}$$

Remark 2.2.7. *The $*$-morphism $\mathbb{C}\ell_{0,1} \to \mathbb{C}\ell_{1,0}$ defined by $j_1 \mapsto ik_1$ defines an even isomorphism $\mathbb{C}\ell_{0,1} \cong \mathbb{C}\ell_{1,0}$ between graded complex C^*-algebras, which preserves the standard orientations. In combination with Equation* (2.5)*, this induces an even orientation-preserving $*$-isomorphism $\mathbb{C}\ell_{r,s+1} \cong \mathbb{C}\ell_{r+1,s}$. Thus $\mathbb{C}\ell_{r,s+1}$ only depends on the sum $r + s$.*

The complex Clifford algebra $\mathbb{C}\ell_{r,s}$ is therefore often denoted by $\mathbb{C}\ell_{r+s}$. However, we refrain from using this notation in the present work.

The key to many periodicity arguments of Clifford algebras is the so-called $(1,1)$-periodicity which we explain next.

Proposition 2.2.8 (Clifford $(1,1)$-periodicity)**.** *There is an isomorphism of real C^*-algebras $C\ell_{1,1} \longrightarrow M_2(\mathbb{C})$, defined on generators by*

$$k_1 \longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad j_1 \longmapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

If we grade $M_2(\mathbb{C})$ by

$$M_2(\mathbb{C})^{(\bar{0})} := \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\}, \quad M_2(\mathbb{C})^{(\bar{1})} := \left\{ \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \right\},$$

this defines an even isomorphism between graded C^*-algebras. In particular, we have

$$C\ell_{r+1,s+1} \cong C\ell_{r,s}\,\widehat{\otimes}\,C\ell_{1,1} \cong C\ell_{r,s}\,\widehat{\otimes}\,M_2(\mathbb{C}) \quad \forall r, s \in \mathbb{N}.$$

The same assertions hold for complex Clifford algebras.

Due to the $(1,1)$-periodicity, all Clifford algebras $C\ell_{r,s}$ are stably isomorphic to Clifford algebras with $r = 0$ or $s = 0$, depending on the sign of $r - s$. We can further improve this statement by the use of the following proposition.

Proposition 2.2.9. [6, Prop. 4.2] *There are isomorphisms of graded real C^*-algebras*

$$C\ell_{0,r+2} \cong C\ell_{0,2} \otimes C\ell_{r,0}, \quad C\ell_{r+2,0} \cong C\ell_{2,0} \otimes C\ell_{0,r}$$

induced respectively by

$$\begin{cases} j_\alpha \longmapsto j_\alpha \otimes 1, & \text{if } \alpha = 1, 2, \\ j_\alpha \longmapsto j_1 j_2 \otimes k_{\alpha-2}, & \text{if } \alpha = 3, \ldots, r+2, \end{cases}$$

and

$$\begin{cases} k_a \longmapsto k_a \otimes 1, & \text{if } a = 1, 2, \\ k_a \longmapsto k_1 k_2 \otimes j_{a-2}, & \text{if } a = 3, \ldots, r+2. \end{cases}$$

Note that above, we consider the ungraded *tensor product.*

The same assertions hold for complex Clifford algebras.

Recall the definition of the real C*-algebra $\mathbb{H}_\mathbb{C}$ from Example 2.1.5. It is easy to see that the identification $j_1 \mapsto i\sigma_x,\ j_2 \mapsto i\sigma_y$ induces an isomorphism

$$C\ell_{0,2} \cong \mathbb{H}_\mathbb{C}$$

of ungraded real C*-algebras. Similarly, there is an isomorphism

$$C\ell_{2,0} \cong M_2(\mathbb{C})$$

of ungraded real C*-algebras, induced by $k_1 \mapsto \sigma_z$, $k_2 \mapsto \sigma_x$.

From this, we deduce, *cf.* [6, §4], isomorphisms of ungraded real C*-algebras

$$C\ell_{0,8} \cong C\ell_{0,2} \otimes C\ell_{0,2} \otimes C\ell_{2,0} \otimes C\ell_{2,0} \cong \mathbb{H}_\mathbb{C} \otimes \mathbb{H}_\mathbb{C} \otimes M_4(\mathbb{C}).$$

Since $\mathbb{H}_\mathbb{C} \otimes \mathbb{H}_\mathbb{C} \cong M_4(\mathbb{C})$ as real C*-algebras, it follows that

$$C\ell_{0,8} \cong M_{16}(\mathbb{C})$$

as ungraded real C*-algebras. A similar argument also shows

$$C\ell_{8,0} \cong M_{16}(\mathbb{C})$$

as ungraded real C*-algebras. This is the mod 8 *periodicity of real Clifford algebras* [6, Table 1]. Disregarding the real structure, we have $M_2(\mathbb{C}) = \mathbb{H}_\mathbb{C}$; hence, we obtain the isomorphism of (ungraded) C*-algebras

$$\mathbb{C}\ell_{0,2} \cong \mathbb{C}\ell_{2,0} \cong M_2(\mathbb{C}).$$

This is the mod 2 *periodicity of complex Clifford algebras*.

Summarising, up to stable isomorphisms, there are only eight real Clifford algebras $C\ell_{r,s}$ and only two complex Clifford algebras $\mathbb{C}\ell_{r,s}$.

From the above, we also deduce the following isomorphism:

$$C\ell_{0,s+4} \cong C\ell_{0,2} \otimes C\ell_{s+2,0} \cong C\ell_{0,2} \otimes C\ell_{2,0} \otimes C\ell_{0,s} \cong M_2(\mathbb{H}_\mathbb{C}) \otimes C\ell_{0,s},$$

which establishes a symmetry between the Clifford algebras $C\ell_{0,s}$ with $0 \leqslant s \leqslant 3$ and the ones with $4 \leqslant s \leqslant 7$. With these results it is possible to compute $C\ell_{0,s}$ and $\mathbb{C}\ell_{0,s}$ in all cases, see Table 2.1. This reproduces the result from [6, Table 2].

In particular, we obtain the following corollary.

s	$C\ell_{0,s}$	$\mathbb{C}\ell_{0,s}$
0	$\mathbb{C} = \mathbb{R}_{\mathbb{C}}$	$\mathbb{C}$
1	$\mathbb{C}_{\mathbb{C}}$	$\mathbb{C} \oplus \mathbb{C}$
2	$\mathbb{H}_{\mathbb{C}}$	$M_2(\mathbb{C})$
3	$\mathbb{H}_{\mathbb{C}} \oplus \mathbb{H}_{\mathbb{C}}$	$M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$
4	$M_2(\mathbb{H}_{\mathbb{C}})$	$M_4(\mathbb{C})$
5	$M_4(\mathbb{C}_{\mathbb{C}})$	$M_4(\mathbb{C}) \oplus M_4(\mathbb{C})$
6	$M_8(\mathbb{C}) = M_8(\mathbb{R}_{\mathbb{C}})$	$M_8(\mathbb{C})$
7	$M_8(\mathbb{C}) \oplus M_8(\mathbb{C})$	$M_8(\mathbb{C}) \oplus M_8(\mathbb{C})$

Table 2.1: Clifford algebras

Proposition 2.2.10. *If $s \not\equiv 3 \bmod 4$, then $C\ell_{0,s}$ is simple as a real C^*-algebra. Moreover, any two finite-dimensional real $*$-representations are unitarily equivalent, provided only that they have the same dimension.*

If $s \not\equiv 1 \bmod 2$, then $\mathbb{C}\ell_{0,s}$ is simple as a complex C^*-algebra and any two finite-dimensional $*$-representations are unitarily equivalent, provided they have the same dimension.

Proof. Simplicity is clear. Any finite-dimensional (real) $*$-representation is the direct sum of simples, and hence, a multiple of the unique isomorphism class of simple (real) $*$-representations. □

It is also not hard to deduce the following statement, which may be found in [55, Ch. 1, Prop. 5.9 & Prop. 5.10].

Proposition 2.2.11. *Let $s \equiv 3 \bmod 4$ and set $\omega := j_1 \cdots j_s$. There are up to unitary equivalence exactly two simple real $*$-representations ϕ of $C\ell_{0,s}$, and they are distinguished by $\phi(\omega) = 1$ and $\phi(\omega) = -1$, respectively.*

A finite-dimensional real $*$-representation (W, ϕ) of $C\ell_{0,s}$ extends to a real $*$-representation of $C\ell_{0,s+1}$ if and only if the eigenvalues ± 1 of $\phi(\omega)$ have equal multiplicity. In this case, the unitary equivalence class of (W, ϕ) and indeed of its extension to $C\ell_{0,s+1}$ is uniquely determined by the dimension of W.

Similarly, for $s \equiv 1 \bmod 2$ there are two simple $*$-representations of $\mathbb{C}\ell_{0,s}$, which are distinguished by $\phi(\omega^{\mathbb{C}}_{0,s}) = 1$ and $\phi(\omega^{\mathbb{C}}_{0,s}) = -1$, if $(\omega^{\mathbb{C}}_{0,s})^2 = 1$, and by $\phi(\omega^{\mathbb{C}}_{0,s}) = i$ and $\phi(\omega^{\mathbb{C}}_{0,s}) = -i$, if $(\omega^{\mathbb{C}}_{0,s})^2 = -1$. A finite-dimensional $*$-representation (W, ϕ) of $\mathbb{C}\ell_{0,s}$ extends to a $*$-representation of $\mathbb{C}\ell_{0,s+1}$ if and only if the eigenvalues ± 1 resp. $\pm i$ of $\phi(\omega^{\mathbb{C}})$ have equal multiplicity. In this case, the equivalence class of (W, ϕ) and its extension to $\mathbb{C}\ell_{0,s+1}$ is uniquely determined by the dimension of W.

Proof. The first part is immediate from Schur's lemma. In particular, the unitary equivalence class of any finite-dimensional real $*$-representation (W, ϕ) of $C\ell_{0,n}$ is determined by the multiplicity of the eigenvalues ± 1 of $\phi(\omega)$.

Assume that in the representation (W, ϕ), $W_\pm := \ker(\phi(\omega) \mp 1)$ have equal dimension. The automorphism α of Definition 2.2.2 satisfies $\alpha(j_\alpha) = -j_\alpha$ for all $\alpha = 1, \ldots, s$ and therefore

$$\alpha(\omega) = -\omega.$$

It follows that $\phi_+ \circ \alpha$ and ϕ_- are unitarily equivalent, where $\phi_\pm$ is the restriction of ϕ to $W_\pm$. Hence, there is a real unitary isomorphism $u : W_+ \longrightarrow W_-$ such that

$$-u\phi_+(j_\alpha)u^* = u\phi_+(\alpha(j_\alpha))u^* = \phi_-(j_\alpha), \quad \forall \alpha = 1, \ldots, s.$$

We may define

$$\phi(j_{s+1}) \coloneqq \begin{pmatrix} 0 & -u^* \\ u & 0 \end{pmatrix}.$$

This defines the required extension.

Conversely, let (W, ϕ) be a real $*$-representation of $C\ell_{0,s+1}$. Because $\phi(\omega_s)$ anti-commutes with $\phi(j_{s+1})$, the eigenvalues of $\phi(\omega_s)$ have equal multiplicity. The statement about uniqueness is obvious from Proposition 2.2.10, as $s + 1 \equiv 0 \bmod 4$.

The complex case is proved analogously. □

Chapter Three

Real and complex K-theory and KK-theory

There are many equivalent pictures of K-theory that have been developed over the past decades. One of the first definitions of K-theory for real and complex C*-algebras can be found in works by Karoubi from 1969 [39, 40]. Based on this work, Kasparov developed in 1975 a picture of (real) K-theory which we will call Kasparov's Fredholm picture of (real) K-theory [41].

In 1981, Kasparov's generalisation of this picture of (real) K-theory to (real) KK-theory, also called Kasparov's bivariant K-theory, has been published. We will give an introduction to Kasparov's theory in Section 3.1. Some aspects of this more general theory will be needed for the construction of morphisms between different (real) K-groups as well as index maps for topological phases. Thereafter, we will introduce Kasparov's Fredholm picture of K-theory explain its relation to KK-theory. We will also compare this picture for the real and complex K-theory of $\mathbb{C}$ to the Clifford module picture of these groups [55].

Prompted by Kasparov's use of graded C*-algebras, in 1986, Van Daele developed a simple uniform picture of the K-theory of graded real and complex Banach algebras [86, 87]. We will review this theory in Section 3.2. The first one to use Van Daele's theory in the context of the classification of topological phases was Kellendonk [45]. He observed that this picture allows for an explicit construction of topological phases in the bulk of topological insulators and superconductors.

In Section 3.3, we will establish a new formulation of real and complex K-theory, which is closely related to Van Daele's picture of K-theory. In fact, we will construct an explicit isomorphism between our new picture and Van Daele's picture of K-theory. This new picture of K-theory is adjusted to our description of physical ground states in the bulk of topological insulators and superconductors in Chapter 4. It will allow for a physically natural and mathematically systematic definition of topological phases in the bulk for all symmetry classes of the Tenfold Way in terms of K-theory.

Sections 3.1, 3.2 are expository in nature, and introduce basics of KK-theory and Van Daele's K-theory by drawing upon the literature. The material of Subsections 3.1.2, 3.3.2 was published in the preprint *Bulk-boundary correspondence for disordered free-fermion topological phases* by A. Alldridge, C. Max and M. R. Zirnbauer, 2019, [1]. Here, we give a more extensive account of the various different pictures of K-theory used in the sequel, along with the picture-changing isomorphisms that are essential for the derivation of our main results. Note in passing that in the preprint, we use a different definition of Van Daele's K-theory than the one discussed here.

3.1 KK-theory

In this section, we introduce real and complex KK-theory, also called Kasparov's bivariant K-theory, which generalises (real) K-theory. Here, we will focus on the aspects of KK-theory that are actually relevant for the present work. We are especially interested in the aspects of real KK-theory, which are discussed in very detail in [77]. General introductions to complex KK-theory can be found in [15, §17], [35]. This section is based on the references [15, 43, 77].

The fundamental objects of this theory are Hilbert modules, which are defined as follows.

Definition 3.1.1 (Hilbert module)**.** *Let B be a complex C^*-algebra. A pre-Hilbert B-module E is a right B-module with a B-valued sesquilinear positive-definite inner product*

$$\langle\cdot,\cdot\rangle : E \times E \to B$$

such that

$$\langle x, yb\rangle = \langle x, y\rangle b \quad \langle y, x\rangle = \langle x, y\rangle^* \quad \forall x, y \in E,\ b \in B.$$

The completion of E w.r.t. the norm $\|x\|$ is called a Hilbert B-module.

If B is a real C^*-algebra, then a real Hilbert module is defined as in the complex case with the additional condition of existence of an anti-linear involution $\overline{(-)} : E \to E$ such that

$$\overline{xb} = \bar{x}\bar{b}, \quad \overline{\langle x, y\rangle} = \langle \bar{x}, \bar{y}\rangle \quad \forall x \in E, b \in B.$$

The involution $\overline{(-)}$ is called a real structure *on E.*

Example 3.1.2. [15, Ex. 13.1.2] *Let B be a real C^*-algebra.*

- *The B-module $E := B$ equipped with the inner product $\langle b_1, b_2\rangle = b_1^* b_2$ for $b_1, b_2 \in E$ and the real structure of B defines a real Hilbert B-module.*
- *The set*
$$\left\{ (b_n)_{n\geq 0} \in \prod_{n=0}^{\infty} B \,\middle|\, \sum_{n=0}^{\infty} b_n^* b_n \text{ converges} \right\}$$
defines a real Hilbert B-module w.r.t. the inner product
$$\langle b, b'\rangle := \sum_{n=0}^{\infty} b_n^* b_n' \quad \text{for all } b = (b_0, b_1, \dots), b' = (b_0', b_1', \dots) \in \mathcal{H}_B$$
and the real structure $\overline{(b_0, b_1, \dots)} := (\overline{b_0}, \overline{b_1}, \dots)$ for all $(b_0, b_1, \dots) \in \mathcal{H}_B$.

 This Hilbert module is denoted by $\mathcal{H}_B$ and is called the Hilbert space over *B.*

Equally important for us will be the so-called *graded Hilbert modules*, which generalise Hilbert modules. These are defined as follows.

Definition 3.1.3 (Graded Hilbert module)**.** *Let $B = B^{(\bar{0})} \oplus B^{(\bar{1})}$ be a graded complex C^*-algebra. A Hilbert B-module E is* graded *if there is a $\mathbb{Z}/2\mathbb{Z}$-grading $E = E^{(\bar{0})} \oplus E^{(\bar{1})}$ with even part $E^{(\bar{0})}$ and odd part $E^{(\bar{1})}$ such that*

$$E^{(n)}B^{(m)} \subseteq E^{(n+m)} \text{ and } \langle E^{(n)}, E^{(m)}\rangle \subseteq B^{(n+m)}$$

for all $n, m \in \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$.

For a graded real C^*-algebra, a graded real Hilbert B-module *is defined as a real Hilbert B-module with the same additional obstruction on the grading as for the complex case. Furthermore, the real structure of the Hilbert B-module has to preserve the grading.*

An element $x \in E$ is called homogeneous if $x \in E^{(\bar{0})}$ or $x \in E^{(\bar{1})}$. In this case, we call

$$|x| := \begin{cases} \bar{0} & \text{if } x \in E^{(0)}, \\ \bar{1} & \text{if } x \in E^{(1)}, \end{cases} \quad \text{the degree of } x.$$

Before we continue, we present the most important examples of graded Hilbert modules for this work.

Example 3.1.4. *Let B be a graded real C^*-algebra.*

- *The real Hilbert space over B equipped with the induced grading of the C^*-algebra B, i.e.*
$$\mathcal{H}_B^{(i)} = \left\{(b_0, b_1, \dots) \in \mathcal{H}_B \,\middle|\, b_n \in B^{(i)}\right\} \quad \text{for } i = \bar{0}, \bar{1} \in \mathbb{Z}/2\mathbb{Z},$$
defines a graded real Hilbert B-module. This grading is called the natural grading.

- *Let $E = E^{(\bar{0})} \oplus E^{(\bar{1})}$ be a graded real Hilbert B-module. Interchanging $E^{(\bar{0})}$ and $E^{(\bar{1})}$ defines the so-called* opposite grading. *The resulting graded real Hilbert module is denoted by E^{op}.*

- *Let $\mathcal{H}_B$ be naturally graded. Then $\hat{\mathcal{H}}_B := \mathcal{H}_B \oplus \mathcal{H}_B^{op}$ is a graded real Hilbert B-module, too. If B is trivially graded, then $\hat{\mathcal{H}}_B$ is called the* graded Hilbert space over B.

In this work, we will solely consider countably generated Hilbert modules.

Definition 3.1.5 (Countably generated Hilbert module)**.** *Let E be a Hilbert B-module. E is called* countably generated *if there is a countable family of generators $\{x_i\}_{i\in I}$ in E such that $\{\sum_k x_{i_k} b_k \mid b_k \in B\}$ is dense in E.*

The following famous theorem by Kasparov is crucial for the explicit description of Kasparov groups.

Theorem 3.1.6 (Kasparov's Stabilisation Theorem)**.** [42, Thm. 2],[15, Thm. 14.6.1] *Let E be a countably generated graded Hilbert B-module over a real graded C^*-algebra B. Then $E \oplus \hat{\mathcal{H}}_B \cong \hat{\mathcal{H}}_B$. In particular, if E and B are trivially graded, then $E \oplus \mathcal{H}_B \cong \mathcal{H}_B$.*

In view of the intersection product of Kasparov groups, which will be introduced in Subsection 3.1.1, we also introduce the tensor product of Hilbert modules.

Definition 3.1.7 (Tensor product of Hilbert modules)**.** *Let B_1, B_2 be (real) graded C^*-algebras and E_1 and E_2 be graded (real) Hilbert B_1- and B_2-modules, respectively.*

Given a (real) even $*$-morphism $\psi : B_1 \to \mathbb{B}(E_2)$, we can consider E_2 as a left-B_1-module and define $E_1 \odot_\psi E_2$ as the usual algebraic tensor product equipped with the $\mathbb{Z}/2\mathbb{Z}$-grading

$$|x \otimes y| := |x| + |y| \quad \forall\, x \otimes y \in E_1 \odot_\psi E_2.$$

Moreover, we define the following inner product on the B_2-module $E_1 \odot_\psi E_2$:

$$\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle := \langle x_2, \psi(\langle x_1, y_1 \rangle) y_2 \rangle.$$

By dividing out elements of zero length and taking the norm closure, we obtain a Hilbert B_2-module which we denote by $E_1 \otimes_\psi E_2$.

The outer tensor product, denoted by $E_1 \widehat{\otimes} E_2$, is defined as the closure of the algebraic tensor product $E_1 \widehat{\odot} E_2$, equipped with the same grading as before, the right $B_1 \widehat{\odot} B_2$-module structure

$$(x_1 \otimes x_2)(b_1 \otimes b_2) := (-1)^{|x_2||b_1|} x_1 b_1 \otimes x_2 b_2$$

and the inner product

$$\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle := (-1)^{|x_2|(|x_1|+|y_1|)} \langle x_1, y_1 \rangle \otimes \langle x_2, y_2 \rangle \in B_1 \widehat{\otimes} B_2.$$

Whenever one of two factors is trivially graded, we simply write $E_1 \otimes E_2 := E_1 \widehat{\otimes} E_2$.

With the most important properties of Hilbert modules in place, we now consider the adjointable operators over Hilbert modules in more detail. These are defined as follows.

Definition 3.1.8 (Adjointable operators)**.** *A map $T : E \to E$ over a Hilbert module E is called* adjointable *if there is a second map $T^* : E \to E$ satisfying*

$$\langle Tx, y \rangle = \langle x, T^* y \rangle \text{ for all } x, y \in E.$$

An adjointable map is automatically linear and continuous, and its adjoint T^* is unique. The set $\mathbb{B}(E)$ of all adjointable morphisms defines a C^*-algebra w.r.t. the composition and linear combination of linear maps, the adjoint operation and the operator norm. If E is a real Hilbert module, then

$$\overline{T}(x) := \overline{T(\overline{x})} \quad \forall x \in E$$

defines a real structure on $\mathbb{B}(E)$. If E is graded, then a canonical grading of $\mathbb{B}(E)$ is defined by

$$T \in \mathbb{B}(E)^{(n)} \quad \Leftrightarrow \quad TE^{(m)} \subseteq E^{(n+m)} \quad \forall m, n \in \mathbb{Z}/2\mathbb{Z}.$$

The (real) graded C*-algebra $\mathbb{B}(E_1, E_2)$ of all adjointable morphisms between two (real) graded Hilbert modules E_1, E_2 can be defined in the same manner.

An important example for adjointable operators are the ‘rank one’ operators

$$|x\rangle\langle y|(z) := x\langle y, z\rangle \quad \forall\, x, y, z \in E.$$

It is easy to check that $(|x\rangle\langle y|)^* = |y\rangle\langle x|$. For any $T \in \mathbb{B}(E)$, we have

$$T|x\rangle\langle y| = |Tx\rangle\langle y|, \quad |x\rangle\langle y|T = |x\rangle\langle T^* y|,$$

showing that the product is again a rank one operator. Therefore, the rank one operators generate the following non-trivial ideal in $\mathbb{B}(E)$.

Definition 3.1.9 (Compact operators)**.** *Let $\mathcal{F}(E) \subset \mathbb{B}(E)$ be the ideal of linear combinations of rank one operators over a Hilbert B-module E. We call this the ideal of B-finite rank operators.*

*The norm closure of $\mathcal{F}(E)$ is denoted by $\mathbb{K}(E)$. The elements in $\mathbb{K}(E)$ are called B-*compact operators.

The compact operators over a (real) graded Hilbert module E form a (real) graded C*-algebra $\mathbb{K}(E)$, which is an ideal in $\mathbb{B}(E)$. The compact operators $\mathbb{K}(E_1, E_2) \subseteq \mathbb{B}(E_1, E_2)$ between two Hilbert modules E_1, E_2 can be defined in a similar manner.

Compact operators over Hilbert modules can be related to compact operators over Hilbert spaces, as explained in the following lemma.

Lemma 3.1.10. [15, Ex. 13.2.4] *Let B be real C*-algebra. Then $\mathbb{K} \otimes B \cong \mathbb{K}(\mathcal{H}_B)$, where $\mathbb{K}$ denotes the compact operators over an infinite-dimensional separable real Hilbert space.*

Note that the statement is in general not true on the level of adjointable operators, i.e. $\mathbb{B}(\mathcal{H}_B)$ is in general *not* isomorphic to $\mathcal{L}(\mathcal{H}) \otimes B$. Instead, one can identify the adjointable operators with multiplier algebras, which are defined as follows.

Definition 3.1.11 (Stable multiplier algebra)**.** *Let A be a C*-algebra. Then the* multiplier algebra *of A is defined as the maximal C*-algebra containing A as an essential ideal. The* outer multiplier algebra *is defined as $Q(A) := \mathrm{M}(A)/A$.*

The stable multiplier algebra *is defined as $\mathrm{M}^s(A) := \mathrm{M}(A \otimes \mathbb{K})$. Similarly, the* stable outer multiplier algebra *is defined as $Q^s(A) := Q(A \otimes \mathbb{K})$.*

The multiplier algebra can be realised more concretely as the C*-algebra of double centralisers, i.e. pairs (T_1, T_2) of mappings $T_1, T_2 : A \to A$ with

$$xT_1(y) = T_2(x)y \quad \forall x, y \in A.$$

The $*$-involution is given by $(T_1, T_2)^* = (T_2^*, T_1^*)$, where $T_i^*(x) := T_i(x^*)^*$ for all $x \in A$.

If A is real, then $\mathrm{M}(A)$ becomes a real C*-algebra with the real structure $\overline{(T_1, T_2)} = (\overline{T_1}, \overline{T_2})$.

If A is graded, then $\mathrm{M}(A)$ becomes a graded C*-algebra with the grading

$$\mathrm{M}(A)^{(n)} := \{(T_1, T_2) \mid T_i^n(A^{(m)}) \subseteq A^{(n+m)}\ \forall i = 1, 2; m = \bar{0}, \bar{1}\} \quad \textit{for } n = \bar{0}, \bar{1}.$$

For more details on multiplier algebras, see [65]. Multiplier algebras are closely related to the bounded operators over Hilbert modules, as explained in the following.

Theorem 3.1.12. [42, Thm. 1] *Let E be a (real) graded Hilbert module. Then there is a (real) $*$-isomorphism $\mathbb{B}(E) \to \mathrm{M}(\mathbb{K}(E))$ between (real) graded C*-algebras, which is defined by*

$$\mathbb{B}(E) \ni T \mapsto (T_1, T_2) \in \mathrm{M}(\mathbb{K}(E)),$$

where $T_1(|x\rangle\langle y|) = |Tx\rangle\langle y|$, $T_2(|x\rangle\langle y|) = |x\rangle\langle T^* y|$ for all $x, y \in E$. In particular,

$$\mathrm{M}^s(B) := \mathrm{M}(\mathbb{K}(\mathcal{H}) \otimes B) \cong \mathrm{M}(\mathbb{K}(\mathcal{H}_B)) \cong \mathbb{B}(\mathcal{H}_B)$$

for any (real) infinite-dimensional separable Hilbert space $\mathcal{H}$.

Now we have established the basic tools that are needed for the definition of Kasparov groups. The elements of these groups are equivalence classes of Kasparov modules, which are defined as follows.

Definition 3.1.13 (Kasparov modules)**.** *[43, §4: Def. 1] Let A, B be (real) graded C^*-algebras. Let $\mathcal{E}(A, B)$ be the set of triples (E, ψ, F), where E is a graded (real) countably generated Hilbert B-module, $\psi : A \to \mathbb{B}(E)$ is a (real) graded $*$-morphism and $F \in \mathbb{B}(E)$ is an odd (real) operator, such that*

$$[\psi(a), F], (F^2 - 1)\psi(a), (F - F^*)\psi(a) \in \mathbb{K}(E) \quad \forall\, a \in A$$

for the graded commutator $[\cdot, \cdot]$, defined by $[x, y] = xy - (-1)^{|x||y|}yx$ for homogeneous $x, y \in E$. The elements of $\mathcal{E}(A, B)$ are called (real) Kasparov modules.

Let $\mathcal{D}(A, B) \subseteq \mathcal{E}(A, B)$ be the set of degenerate Kasparov modules, which are defined by the property

$$[\psi(a), F] = (F^2 - 1)\psi(a) = (F - F^*)\psi(a) = 0 \quad \forall\, a \in A.$$

Equivalence relations for these Kasparov modules are defined as follows.

Definition 3.1.14 (Equivalence classes of Kasparov modules)**.** *Let A, B be (real) graded C^*-algebras. We define the following equivalence relations on $\mathcal{E}(A, B)$:*

- *Two modules $(E_1, \psi_1, F_1), (E_2, \psi_2, F_2) \in \mathcal{E}(A, B)$ are* unitarily equivalent *if there is an even unitary (real) $u \in \mathbb{B}(E_1, E_2)$ such that $\psi_1(a) = u^*\psi_2(a)u\ \forall\, a \in A$ and $F_1 = u^*F_2u$.*

- *Two modules $(E_1, \psi_1, F_1), (E_2, \psi_2, F_2) \in \mathcal{E}(A, B)$ are* homotopy equivalent *if there is an element $(E, \psi, F) \in \mathcal{E}(A, C([0, 1], B))$ such that $(E \otimes_{\mathrm{ev}_i} B, \mathrm{ev}_i \circ \psi, (\mathrm{ev}_i)_*(F))$ is unitarily equivalent to (E_i, ψ_i, F_i) for $i = 0, 1$, where ev_i denotes the evaluation at $i \in [0, 1]$.*

  In the case $E_1 = E_2$, we call them standard homotopy equivalent *if there is a cycle $(C([0, 1]) \otimes E_1, 1 \otimes \psi, F)$ with $\psi(0) = \psi_1, F(0) = F_1$ and $\psi(1) = \psi_2, F(1) = F_2$, such that $t \mapsto \psi(t)(a)$ and $t \mapsto F(t)$ are strong-$*$-continuous paths for all $a \in A$. They are called* operator homotopy equivalent *if the stronger conditions that $\psi(t)$ is constant and the path $t \mapsto F(t)$ is norm continuous are satisfied.*

We denote the corresponding equivalence classes of Kasparov modules by $\overline{\mathcal{E}}(A, B)$ and the equivalence classes of degenerate Kasparov modules by $\overline{\mathcal{D}}(A, B)$.

The direct sum

$$[(E_1, \psi_1, F_1)] \oplus [(E_2, \psi_2, F_2)] := [(E_1 \oplus E_2, \psi_1 \oplus \psi_2, F_1 \oplus F_2)] \in \overline{\mathcal{E}}(A, B)$$

defines an Abelian semi-group structure on $\overline{\mathcal{E}}(A, B)$. The following lemma implies that the degenerate Kasparov modules represent the neutral element in this Abelian semi-group.

Lemma 3.1.15. [15, Prop. 17.2.3] *All $(E, \psi, F) \in \mathcal{D}(A, B)$ are homotopically equivalent to the Kasparov module* $(0, 0, 0)$.

By the following theorem, the direct sum defines an Abelian group structure on $\overline{\mathcal{E}}(A, B)$.

Theorem 3.1.16. [43, Thm. 1] *$\overline{\mathcal{E}}(A, B)$ equipped with the direct sum defines a group. The inverse of an element $[(E, \psi, F)] \in \overline{\mathcal{E}}(A, B)$ is given by*

$$-[(E, \psi, F)] = [(E^{op}, \psi^{op}, -F)],$$

where ψ^{op} is defined by $\psi^{op}(a) := (-1)^{|a|}a$ for homogeneous $a \in A$.

Definition 3.1.17 (Kasparov groups)**.** *For complex graded C^*-algebras A, B the Kasparov group $KK(A, B)$ is defined as the Abelian group $(\overline{\mathcal{E}}(A, B), \oplus)$.*

If A and B are real, we define the real Kasparov group $KKR(A, B)$ as the corresponding Abelian group of equivalence classes of real Kasparov modules $\overline{\mathcal{E}}(A, B)$.

The elements of Kasparov groups are called Kasparov classes.

Theorem 3.1.18. [15, Thm. 18.5.3] *Let A be separable and B be σ-unital. Then one obtains the same Kasparov groups $KKR(A, B)$ and $KK(A, B)$ if one restricts the equivalence relations in Definition 3.1.14 to unitary equivalence and operator homotopy equivalence. In other words, two (real) Kasparov modules (E_1, ψ_1, F_1) and (E_2, ψ_2, F_2) are equivalent in the corresponding (real) Kasparov group if and only if there are degenerate (real) Kasparov modules (E_1', ψ_1', F_1') and (E_2', ψ_2', F_2') such that*

$$(E_1, \psi_1, F_1) \oplus (E_1', \psi_1', F_1') \quad \text{and} \quad (E_2, \psi_2, F_2) \oplus (E_2', \psi_2', F_2')$$

are connected by an operator homotopy, up to unitary equivalence. This was originally proved by Skandalis [81].

The following proposition is crucial for the explicit description of Kasparov groups and will also be important in our physical applications in Chapter 5.

Proposition 3.1.19. [77, Prop. 2.3.5] *If A is separable and B is σ-unital, then in the definition of Kasparov groups it is sufficient to consider Kasparov modules of the form*

$$\left(\hat{\mathcal{H}}_B, \psi : A \to \mathbb{B}(\hat{\mathcal{H}}_B), F\right) \in \mathcal{E}(A, B)$$

with $F = F^*$ and $\|F\| \leq 1$.

Finally, we mention two important types of natural morphisms between Kasparov groups.

Definition 3.1.20 (Induced morphisms of Kasparov groups)**.** [43, §4, Def. 4] *A real even $*$-morphism $f : A_2 \to A_1$ induces a morphism $f^* : KKR(A_1, B_1) \to KKR(A_2, B_1)$ defined by*

$$f^*[(E, \psi, F)] = [(E, \psi \circ f, F)].$$

A real even $*$-morphism $g : B_1 \to B_2$ defines a morphism $g_* : KKR(A_1, B_1) \to KKR(A_1, B_2)$ defined by

$$g_*[(E, \psi, F)] = \left[(E \otimes_g B_2, g_*(\psi), g_*(F))\right].$$

The analogous statements are true for KK-groups.

Proposition 3.1.21. [77, p. 76], [15, 17.8.6] *The map*

$$\mathcal{E}(A, B) \to \mathcal{E}(A \widehat{\otimes} D, B \widehat{\otimes} D); \ (E, \psi, F) \mapsto (E \widehat{\otimes} D, \psi \otimes 1, F \otimes 1)$$

commutes with direct sums and thus defines a natural morphism

$$\tau_D : KKR(A, B) \to KKR(A \widehat{\otimes} D, B \widehat{\otimes} D).$$

If $h : D_1 \to D_2$ is a real even $*$-morphism of real graded C^*-algebras, then $(1 \otimes h)_* \circ \tau_{D_1} = (1 \otimes h)^* \circ \tau_{D_2}$ as a map from $KKR(A, B) \to KKR(A \widehat{\otimes} D_1, B \widehat{\otimes} D_2)$.

The analogous results hold for KK-groups.

3.1.1 The intersection product

Now we come to the intersection product, also called Kasparov product. It is one of tools that makes Kasparov's theory so powerful for applications in geometry and physics. We will only describe it superficially since a rigorous derivation of the intersection product is beyond the scope of this work. The basic idea of the intersection product is to define for two Kasparov modules $(E_1, \psi_1, F_1) \in \mathcal{E}(A, D)$ and $(E_2, \psi_2, F_2) \in \mathcal{E}(D, B)$ a suitable product module in $\mathcal{E}(A, B)$ of the form

$$(E_1 \otimes_{\psi_2} E_2, \psi_1 \otimes 1, `M(F_1 \otimes 1) + N(1 \otimes F_2)')$$

for some suitable operators $M, N \in \mathbb{B}(E_1 \otimes_{\psi_2} E_2)$. However, this naive idea leads in general to an ill-defined Kasparov module because of the following.

On the one hand, we can form the tensor product $E := E_1 \otimes_{\psi_2} E_2$ and define the induced operator $F_1 \otimes 1$ on E as well as the induced morphism $\psi_1 \otimes 1$ without ambiguity. On the other hand, obtaining a well-defined induced operator for F_2 is more difficult. The naive choice $1 \otimes F_2$ is in general not well defined over $E_1 \otimes_{\psi_2} E_2$, because in general F_2 doesn't commute with the left action of D on E_2. Following the arguments of [15, Sec. 18.3 & 18.4], this problem is solved by introducing so-called *connections*, which can be thought of as operators that are equal to $1 \otimes F_2$ 'up to compacts'.

Definition 3.1.22 (Connections)**.** *Let E_i be a countably generated (real) Hilbert B_i-module for $i = 0, 1$ and let $\psi : B_1 \to \mathbb{B}(E_2)$ be a (real) graded $*$-morphism. We define $E := E_1 \otimes_\psi E_2$.*

*For $x \in E_1$, let $T_x \in \mathbb{B}(E_2, E)$ be defined by $T_x(y) = x \otimes y$. Its adjoint is given by $T_x^*(z \otimes y) = \psi(\langle x, z\rangle)y$. Given an $F_2 \in \mathbb{B}(E_2)$, an operator $F \in \mathbb{B}(E)$ is an F_2-*connection *if for all $x \in E_1$*

$$T_x \circ F_2 - (-1)^{|x||F_2|} F \circ T_x \in \mathbb{K}(E_2, E),$$
$$F_2 \circ T_x^* - (-1)^{|x||F_2|} T_x^* \circ F \in \mathbb{K}(E, E_2).$$

Given such a connection, it remains to find a 'suitable' combination of the operator $F_1 \otimes 1$ and the F_2-connection. By 'suitable', we mean that the resulting Kasparov module is well-defined and, moreover, it defines an intersection product, which is defined as follows.

Definition 3.1.23 (Intersection product)**.** *Let $[x] = [(E_1, \psi_1, F_1)] \in KKR(A, D)$ and $[y] = [(E_2, \psi_2, F_2)] \in KKR(A, D)$. Let $E := E_1 \otimes_{\psi_2} E_2$ and $\psi := \psi_1 \otimes 1 : A \to \mathbb{B}(E)$.*

The Kasparov class $[(E, \psi, F)] \in KKR(A, B)$ is an intersection product of $[x]$ and $[y]$, *denoted by $[x] \otimes_D [y]$, if F is an F_2-connection for E_1 and*

$$\psi(a)[F_1 \otimes 1, F]\psi(a)^* \geq 0 \mod \mathbb{K}(E) \quad \forall a \in A.$$

The intersection product for KK-groups is defined analogously.

To find an explicit form of the intersection product is in general difficult, but its existence is guaranteed by the following famous theorem of Kasparov [43, §4: Thm. 4].

Theorem 3.1.24. *Let A, D, B be real graded C^*-algebras. Then for any $[x] \in KKR(A, D)$ and $[y] \in KKR(D, B)$ there is a unique intersection product $[x] \otimes_D [y] \in KKR(A, B)$. If A is separable and B, D are σ-unital, this defines an associative bilinear operation*

$$\otimes_D : KKR(A, D) \times KKR(D, B) \to KKR(A, B).$$

This product is contravariantly functorial in A and covariantly functorial in B. The intersection product commutes with the natural morphism τ, i.e.

$$\tau_{D'}([x] \otimes_D [y]) = \tau_{D'}([x]) \otimes_D \tau_{D'}([y])$$

for all graded real C^*-algebras D'.

The intersection product can be generalised for separable real graded C^*-algebras A, A' and σ-unital real graded C^*-algebras B, B', D to an associative bilinear intersection product

$$\otimes_D : KKR(A, B \widehat{\otimes} D) \times KKR(D \widehat{\otimes} A', B') \to KKR(A \widehat{\otimes} A', B \widehat{\otimes} B'),$$

with

$$[x] \otimes_D [y] := \tau_{A'}(x) \otimes_{B \widehat{\otimes} D \widehat{\otimes} A'} \tau_B(y),$$

satisfying the same properties as the previous one.

The analogous assertions are true for KK-groups.

For $[\alpha] = [(E_1, \psi_1, F_1)] \in KKR(A, D)$ and $[\beta] = [(E_2, \psi_2, 0)] \in KKR(D, B)$, the intersection product can be written down explicitly:

$$[\alpha] \otimes_D [\beta] = [(E_1 \otimes_{\psi_2} E_2, \psi_1 \otimes 1, F_1 \otimes 1)] \in KKR(A, B).$$

A more general case, where the explicit calculation of the intersection product is possible, is presented in the following proposition, which we cite from [15, Prop. 18.10.1].

Proposition 3.1.25. *Let A, D, B be (real) graded C^*-algebras with A separable and D, B σ-unital. Let $\alpha = (E_1, \psi_1, F_1) \in \mathcal{E}(A, D)$ and $(E_2, \psi_2, F_2) \in \mathcal{E}(D, B)$ such that $F_1 = F_1^*$ and $\|F_1\| \leq 1$. Let G be an F_2-connection on $E = E_1 \otimes_{\psi_2} E_2$. Set $\psi := \psi_1 \otimes 1$ and*

$$F := F_1 \otimes 1 + ((1 - F_1^2)^{1/2} \otimes 1)G.$$

If $[F, \psi(A)] \subseteq \mathbb{K}(E)$, then $\gamma = (E, \psi, F) \in \mathcal{E}(A, B)$ is operator homotopic to the intersection product for α and β, i.e. $[\gamma] = [\alpha] \otimes_D [\beta]$.

Theorem 3.1.26 (Multiplicative unit of intersection product)**.** [43, §4: Thm. 5] *Let $\mathcal{K}$ be a separable real graded Hilbert space and $T_1 : \mathcal{K}^{(\bar{0})} \to \mathcal{K}^{(\bar{1})}$ be a real Fredholm operator such that*

$$1_{\mathcal{K}^{(\bar{1})}} = T_1 T_1^* \text{ and } T_1^* T_1 = 1_{\mathcal{K}^{(\bar{0})}} - p,$$

where p is a projection onto a one-dimensional subspace in $\mathcal{K}^{(\bar{0})}$. We define

$$T := \begin{pmatrix} 0 & T_1^* \\ T_1 & 0 \end{pmatrix}.$$

Then $c_1 := (\mathcal{K}, \mathrm{id}_{\mathbb{C}}, T) \in \mathcal{E}(\mathbb{C}, \mathbb{C})$ defines a class $[c_1] \in KKR(\mathbb{C}, \mathbb{C})$. This class is the multiplicative unit w.r.t. the intersection product in the sense

$$[x] \otimes_{\mathbb{C}} [c_1] = [c_1] \otimes_{\mathbb{C}} [x] = [x] \quad \forall\, [x] \in KKR(A, B).$$

By dropping the real structure, we obtain the multiplicative unit in $KK(A, B)$.

A major benefit of the intersection product is that we can interpret Kasparov classes as morphisms between Kasparov groups. In particular, this gives us a tool to describe isomorphisms between Kasparov groups in a systematic fashion as explained in the following theorem, cited from [43, §4: Thm. 6].

Theorem 3.1.27. *Let A be a real graded separable C^*-algebra and B, D, E be real graded σ-unital C^*-algebras.*

1. *Let $[\alpha] \in KKR(D,E)$ and $[\beta] \in KKR(E,D)$ such that $[\alpha] \otimes_E [\beta] = \pm\tau_D(c_1) \in KKR(D,D)$ and $[\beta] \otimes_D [\alpha] = \pm\tau_E(c_1) \in KKR(E,E)$. Then the morphisms*

$$\begin{aligned} \otimes_D [\alpha] &: KKR(A, B\widehat{\otimes} D) \longrightarrow KKR(A, B\widehat{\otimes} E)\\ \text{and } \otimes_E [\beta] &: KKR(A, B\widehat{\otimes} E) \longrightarrow KKR(A, B\widehat{\otimes} D)\end{aligned}$$

are isomorphisms. If D and E are separable, then

$$\begin{aligned} [\beta]\otimes_D &: KKR(A\widehat{\otimes} D, B) \longrightarrow KKR(A\widehat{\otimes} E, B)\\ \text{and } [\alpha]\otimes_E &: KKR(A\widehat{\otimes} E, B) \longrightarrow KKR(A\widehat{\otimes} D, B)\end{aligned}$$

are also isomorphisms.

2. *Assume that D and E are separable and let $[\alpha] \in KKR(D\widehat{\otimes} E, \mathbb{C})$, $[\beta] \in KKR(\mathbb{C}, D\widehat{\otimes} E)$ such that $[\beta] \otimes_D [\alpha] = \pm\tau_E(c_1) \in KKR(E,E)$ and $[\beta] \otimes_E [\alpha] = \pm\tau_D(c_1) \in KKR(D,D)$. Then the morphisms*

$$\begin{aligned} [\beta]\otimes_D &: KKR(A\widehat{\otimes} D, B) \longrightarrow KKR(A, B\widehat{\otimes} E),\\ [\beta]\otimes_E &: KKR(A\widehat{\otimes} E, B) \longrightarrow KKR(A, B\widehat{\otimes} D),\\ \otimes_D [\alpha] &: KKR(A, B\widehat{\otimes} D) \longrightarrow KKR(A\widehat{\otimes} E, B),\\ \otimes_E [\alpha] &: KKR(A, B\widehat{\otimes} E) \longrightarrow KKR(A\widehat{\otimes} D, B)\end{aligned}$$

are isomorphisms.

The analogous statements for KK-groups are true, too.

An important application of Theorem 3.1.27 is Kasparov's construction of the following isomorphisms proving the stability of Kasparov groups.

Theorem 3.1.28. [43, §4: Thm. 5 & §5: Thm. 1] *For any separable real graded Hilbert space $\mathcal{H}$ and σ-unital real graded C^*-algebras A, B there are isomorphisms*

$$KKR(A\widehat{\otimes}\mathbb{K}(\mathcal{H}), B) \cong KKR(A,B) \cong KKR(A, B\widehat{\otimes}\mathbb{K}(\mathcal{H})).$$

These isomorphisms are defined by the intersection product with the Kasparov classes

$$\begin{aligned} [\alpha] &= [(\mathcal{K}\widehat{\otimes}\mathcal{H}, 1\otimes \mathrm{id} : \mathbb{K}(\mathcal{H}) \to \mathbb{B}(\mathcal{K}\widehat{\otimes}\mathcal{H}), T\otimes 1)] \in KKR(\mathbb{K}(\mathcal{H}), \mathbb{C}) && (3.1)\\ \text{and} \quad [\beta] &= [(\mathcal{K}\widehat{\otimes}\mathbb{K}(\mathcal{H}), \mathrm{id}_{\mathbb{C}}, T_2)] \in KKR(\mathbb{C}, \mathbb{K}(\mathcal{H})), && (3.2)\end{aligned}$$

where T and $\mathcal{K}$ are defined as in Theorem 3.1.26 and $T_2 \in \mathbb{B}(\mathcal{K}\widehat{\otimes}\mathcal{H})$ is isomorphic to T via a real even isomorphism $\mathcal{K}\widehat{\otimes}\mathcal{H} \cong \mathcal{K}$.

The same assertions without the real structure are true for KK-groups.

Before we come to the description of K-groups in terms of Kasparov groups, we make a short comment on the long exact sequences of Kasparov groups.

For that, we consider a short exact sequence of graded (real) C*-algebras

$$0 \longrightarrow I \xrightarrow{\iota} A \xrightarrow{\pi} A/I \longrightarrow 0. \tag{3.3}$$

Then we have the following theorem.

Theorem 3.1.29. [77, Thm. 2.5.6] *If the SES* (3.3) *is semisplit, i.e. there is an even contractive completely positive map $s : A/I \to A$ with $\pi \circ s = \mathrm{id}_{A/I}$, then there is a class $[\partial] \in KKR(A/I, I \widehat{\otimes} C\ell_{1,0})$ such that*

$$\begin{tikzcd}
{} \arrow[r, dotted] & KKR(D,I) \arrow[r, "\iota_*"] & KKR(D,A) \arrow[r, "\pi_*"] & KKR(D,A/I) \arrow[dll, "\partial"] \\
& KKR(D, I\widehat{\otimes} C\ell_{1,0}) \arrow[r, "\iota_*"] & KKR(D, A\widehat{\otimes} C\ell_{1,0}) \arrow[r, "\pi_*"] & KKR(D, A/I\widehat{\otimes} C\ell_{1,0}) \arrow[r, dotted] & {}
\end{tikzcd} \tag{3.4}$$

is a long exact sequence for D separable and

$$\begin{tikzcd}
{} \arrow[r, dotted] & KKR(A/I\widehat{\otimes} C\ell_{1,0}, D) \arrow[r, "\pi^*"] & KKR(A\widehat{\otimes} C\ell_{1,0}, D) \arrow[r, "\iota^*"] & KKR(I\widehat{\otimes} C\ell_{1,0}, D) \arrow[dll, "\partial"] \\
& KKR(A/I, D) \arrow[r, "\pi^*"] & KKR(A, D) \arrow[r, "\iota^*"] & KKR(I, D) \arrow[r, dotted] & {}
\end{tikzcd} \tag{3.5}$$

is a long exact sequence for A and I separable, where the connecting maps are given by $\otimes_{A/I}[\partial]$ and $[\partial]\otimes_{I\widehat{\otimes} C\ell_{1,0}}$.

The analogous results are true for the complex case.

We will not need the explicit form of the connecting class $[\partial]$ in the present work. A detailed explicit derivation of this class can be found in [43, §7].

3.1.2 Kasparov's Fredholm picture

In this section, we explain the description of (real) K-groups of trivially graded (real) C*-algebras in terms of Kasparov groups. Since the topics of this section will be crucial for the following chapters, we will explain things in more detail than in the previous subsections.

We begin with the definition of Kasparov's Fredholm picture of K-groups, as introduced in [41] at the beginning of page 790 on the basis of the work by Karoubi [38, 40]. The following construction of this picture follows essentially the same strategy as in the definition of Kasparov groups. Therefore, we keep the following construction short, referring for more details to [41].

Definition 3.1.30 (Kasparov's Fredholm picture of KR- and K-theory)**.** *Let A be a trivially graded σ-unital real C*-algebra. Suppose we are given a real $*$-morphism $\phi : C\ell_{r,s} \to \mathrm{M}^s(A)$ and a real operator $F \in \mathrm{M}^s(A)$ such that*

$$F^* + F,\ 1 + F^2,\ \{\phi(k_a), F\},\ \{\phi(j_\alpha), F\} \in A \otimes \mathbb{K} \quad \forall\, a = 1, \dots, r, \alpha = 1, \dots, s,$$

where $\{\cdot,\cdot\}$ *denotes the anti-commutator. Denote the set of all such pairs* (F,ϕ) *by* $\mathbb{E}^{r,s}(A)$. *A pair is called* degenerate *if* $F^2 = -1$, $F^* = -F$ *and* $\{\phi(k_a), F\} = \{\phi(j_\alpha), F\} = 0$ *for all* $a = 1,\ldots,r, \alpha = 1,\ldots,s$. *The set of degenerate pairs is denoted by* $\mathbb{D}^{r,s}(A)$.

We define the following equivalence relations:

- *Two pairs* $(F_1,\phi_1),(F_2,\phi_2)$ *are* homotopy equivalent *if there is a pair* $[(F,\phi)] \in \mathbb{E}(C\ell_{r,s}, C([0,1],A))$ *such that* $(F(0),\phi(0)) = (F_1,\phi_1)$, $(F(1),\phi(1)) = (F_2,\phi_2)$.

  They are called operator homotopically equivalent *if the path* $t \mapsto \hat{J}(t) \in \mathbb{B}(\mathcal{H}_A)$ *is norm continuous and the path* $t \mapsto \phi(t)$ *is constant.*

- *Two pairs* $(F_1,\phi_1),(F_2,\phi_2)$ *are* unitarily equivalent *if there is a real unitary isomorphism* $u \in \mathrm{M}^s(A)$ *such that* $\phi_1 = u^*\phi_2 u, F_1 = u^* F_2 u$.

The set of equivalence classes induced by these two equivalence relations forms an Abelian semi-group w.r.t. the direct sum

$$[(F_1,\phi_1)] \oplus [(F_2,\phi_2)] := [(F_1 \oplus F_2, \phi_1 \oplus \phi_2)].$$

This group operation is well-defined since $\mathrm{M}^s(A) \oplus \mathrm{M}^s(A) \cong \mathrm{M}^s(A)$. *Let* $\overline{\mathbb{E}}^{r,s}(A)$ *be the so constructed semi-group and let* $\overline{\mathbb{D}}^{r,s}(A)$ *denote the corresponding semi-group of degenerate Kasparov cycles. Then we finally define the Fredholm picture of KR-theory as the following quotient which forms an Abelian group:*

$$KR^{r,s}(A) := \overline{\mathbb{E}}^{r,s}(A)/\overline{\mathbb{D}}^{r,s}(A).$$

The construction for a complex C^**-algebra* A *is possible in completely the same way as for the real case. The only difference is that the* $*$*-morphism* ϕ *and the operator* F *have no reality condition and the Clifford algebras are complex. The resulting group is denoted by* $K^{r,s}(A)$.

In the following, we will show that these Abelian groups are isomorphic to Kasparov groups. We first discuss the corresponding Kasparov groups and then show the equality.

Definition 3.1.31 **(**$K_{p,q}K^{r,s}R(A)$, $K_{p,q}K^{r,s}(A)$**).** *Let* A *be a real* σ*-unital trivially graded* C^**-algebra. Then we define*

$$K_{p,q}K^{r,s}R(A) := KKR(C\ell_{p,q}, A \otimes C\ell_{r,s}),$$

where the Clifford algebras are considered as real graded C^**-algebras. For a complex trivially graded* C^**-algebra* A*, we define*

$$K_{p,q}K^{r,s}(A) := KK(\mathbb{C}\ell_{p,q}, A \otimes \mathbb{C}\ell_{r,s}).$$

Theorem 3.1.32. [77, Cor. 2.4.10] *Let* A, B *be* C^**-algebras with* A *separable and* B σ*-unital. Then*

$$KK(A \otimes \mathbb{C}\ell_{p,q}, B \otimes \mathbb{C}\ell_{r,s}) \cong KK(A \otimes \mathbb{C}\ell_{p-r,q-s}, B).$$

If A, B *are real, then*

$$KKR(A \otimes C\ell_{p,q}, B \otimes C\ell_{r,s}) \cong KKR(A \otimes C\ell_{p-r,q-s}, B).$$

Proof. We only discuss the real case as the complex case is almost the same. Since $C\ell_{r+s,r+s} \cong \mathrm{End}(\Lambda(\mathbb{C}^{r+s}))$, we know by Theorem 3.1.28 that there is an isomorphism

$$\psi : KKR(A \otimes C\ell_{r+s,r+s}, B \otimes C\ell_{r+s,r+s}) \to KKR(A, B).$$

It follows

$$\psi \circ \tau_{C\ell_{r,s}} \circ \tau_{C\ell_{s,r}} = \mathrm{id}_{KKR(A,B)}, \quad \tau_{C\ell_{r,s}} \circ \psi \circ \tau_{C\ell_{s,r}} = \mathrm{id}_{KKR(A\otimes C\ell_{r,s}, B\otimes C\ell_{r,s})}.$$

Thus by Theorem 3.1.27, we obtain isomorphisms

$$KKR(A \otimes C\ell_{r,s} \otimes C\ell_{s,r}, B) \cong KKR(A, B) \cong KKR(A \otimes C\ell_{r,s}, B \otimes C\ell_{r,s}).$$

□

Theorem 3.1.32 in particular implies that the groups $K_{p,q}K^{r,s}R(A)$ and $K_{p,q}K^{r,s}(A)$ only depend on the difference $p - q - (r - s)$. The isomorphism of Theorem 3.1.32 will also be important for Chapter 5. Therefore, we give a short exposition of the explicit form of this isomorphism in the following.

Consider the Kasparov class $[(\hat{\mathcal{H}}_A \widehat{\otimes} C\ell_{r,s}, 1_{\mathbb{C}}, F)] \in KKR(\mathbb{C}, A \otimes C\ell_{r,s})$ with $F^* = F$ and $\|F\| \leq 1$. These conditions are no restrictions on the generality of the class according to Theorem 3.1.19 and the fact that the morphism $\mathbb{C} \to \mathbb{B}(\hat{\mathcal{H}}_A \widehat{\otimes} C\ell_{r,s})$ can be assumed to be unital, *cf.* [43, §4, Remark 2]. The natural morphism $\tau_{C\ell_{s,r}}$ maps this class onto

$$[(\hat{\mathcal{H}}_A \widehat{\otimes} C\ell_{r,s} \widehat{\otimes} C\ell_{s,r}, 1 \otimes 1 \otimes \mathrm{id}_{C\ell_{s,r}}, F)] \in KKR(C\ell_{s,r}, A \otimes C\ell_{r,s} \widehat{\otimes} C\ell_{s,r}).$$

The isomorphism of Equation (2.4) induces an isomorphism $\hat{\mathcal{H}}_A \widehat{\otimes} C\ell_{r,s} \widehat{\otimes} C\ell_{s,r} \cong \hat{\mathcal{H}}_A \widehat{\otimes} C\ell_{r+s,r+s}$. This leads to the Kasparov class

$$[(\hat{\mathcal{H}}_A \widehat{\otimes} C\ell_{r+s,r+s}, 1 \otimes \mathrm{id}_{C\ell_{s,r}}, F)] \in KKR(C\ell_{s,r}, A \otimes C\ell_{r+s,r+s}).$$

We identify $C\ell_{r+s,r+s} \cong \mathrm{End}(\Lambda(\mathbb{C}^{r+s}))$, *cf.* Proposition 2.2.4, and apply the stability isomorphism of Theorem 3.1.28, which is given by the intersection product from the right with the class

$$[\alpha] = [(\mathcal{K} \widehat{\otimes} \Lambda(\mathbb{C}^{r+s}), 1 \otimes \mathrm{id}_\Lambda, T \otimes 1)] \in KKR(\mathrm{End}(\Lambda(\mathbb{C}^{r+s})), \mathbb{C}),$$

where $\mathrm{id}_\Lambda : \mathrm{End}(\Lambda(\mathbb{C}^{r+s})) \to \mathrm{End}(\Lambda(\mathbb{C}^{r+s}))$ denotes the identity. The intersection product can be calculated explicitly using Proposition 3.1.25, see also [77, Thm. 2.4.7] for more details. It is given by

$$[(\hat{\mathcal{H}}_A \widehat{\otimes} \Lambda(\mathbb{C}^{r+s}) \widehat{\otimes} \mathcal{K}, 1 \otimes \mathrm{id}_{C\ell_{s,r}} \otimes 1, F \otimes 1 + (1 - F^2)^{1/2} \otimes T)]$$

in $KKR(C\ell_{s,r}, A)$. The operator F is now considered as an operator in $\mathbb{B}(\hat{\mathcal{H}}_A \widehat{\otimes} \Lambda(\mathbb{C}^{r+s}))$.

Let p denote the one-dimensional projection in $\mathcal{K}^{(\bar{0})}$ such that $T^2 = 1 - p$. Then we can decompose the Hilbert module as

$$\hat{\mathcal{H}}_A \widehat{\otimes} \Lambda(\mathbb{C}^{r+s}) \widehat{\otimes} \mathcal{K} \cong \left(\hat{\mathcal{H}}_A \widehat{\otimes} \Lambda(\mathbb{C}^{r+s}) \widehat{\otimes} p(\mathcal{K})\right) \oplus \left(\hat{\mathcal{H}}_A \widehat{\otimes} \Lambda(\mathbb{C}^{r+s}) \widehat{\otimes} (p(\mathcal{K})^\perp \oplus \mathcal{K}^{(\bar{1})})\right).$$

On the first summand, the operator is given by F and we obtain a Hermitian unitary operator on the second summand. Therefore, the second summand is degenerate and the Kasparov class is equal to

$$[(\hat{\mathcal{H}}_A \widehat{\otimes} \Lambda(\mathbb{C}^{r+s}), 1 \otimes \mathrm{id} : C\ell_{s,r} \to \mathbb{B}(\hat{\mathcal{H}}_A) \widehat{\otimes} \mathrm{End}(\Lambda(\mathbb{C}^{r+s})), F)] \in KKR(C\ell_{s,r}, A),$$

where we identified $p\mathcal{K} \cong \mathbb{C}$ which thus vanishes in the tensor product.

By Theorem 3.1.6, there is a real even isomorphism $\hat{\mathcal{H}}_A \mathbin{\widehat{\otimes}} \Lambda(\mathbb{C}^{r+s}) \cong \hat{\mathcal{H}}_A$ which maps the Kasparov class above onto a Kasparov class of the form

$$[(\hat{\mathcal{H}}_A, \psi : C\ell_{s,r} \to \mathbb{B}(\hat{\mathcal{H}}_A), F')] \in KKR(C\ell_{s,r}, A),$$

where $\psi : C\ell_{s,r} \to \mathbb{B}(\hat{\mathcal{H}}_A)$ denotes the resulting $*$-morphism.

Thus, we have established the explicit form of the isomorphism from $KKR(\mathbb{C}, A \otimes C\ell_{r,s})$ to $KKR(C\ell_{s,r}, A)$. All other isomorphisms in Theorem 3.1.32 can be written down explicitly by using this result.

In particular, we conclude from the explicit form of the isomorphism above that all classes in $KKR(C\ell_{s,r}, A)$ and $KK(\mathbb{C}\ell_{s,r}, A)$ can be represented in the form

$$[(\hat{\mathcal{H}}_A, \psi, F)],$$

such that ψ graded-commutes with the operator F. This result can also be obtained more directly as explained in the following proposition.

Proposition 3.1.33. *Any element of $KKR(C\ell_{s,r}, A)$ and $KK(\mathbb{C}\ell_{s,r}, A)$ for σ-unital A can be represented by a Kasparov module $(\hat{\mathcal{H}}_A, \psi, F)$ such that all of the following expressions vanish:*

$$\psi(k_a)F + F\psi(k_a), \quad \psi(j_\alpha)F + F\psi(j_\alpha), \quad F^* - F \quad \forall a = 1, \dots, p, \alpha = 1, \dots, q.$$

Proof. By Proposition 3.1.19, we already know that all classes can be represented by Kasparov modules of the form

$$(\hat{\mathcal{H}}_A, \psi, F),$$

such that F is Hermitian and $\psi(k_a)F + F\psi(k_a), \psi(j_\alpha)F + F\psi(j_\alpha) \in \mathbb{K}(\hat{\mathcal{H}}_A)$ for all $a = 1, \dots, s, \alpha = 1, \dots, r$.

If there is some $1 \leqslant a \leqslant s$ such that $\psi(k_a)F + F\psi(k_a) \neq 0$, we may assume that a is minimal for this property. Form

$$\tilde{F} := \frac{1}{2}(1 - \mathrm{Ad}(\psi(k_a)))(F).$$

One computes easily that $\tilde{F}$ is Hermitian and $\tilde{F}^2 - 1 \in \mathbb{K}(\hat{\mathcal{H}}_A)$. As for $b < a$, $\mathrm{Ad}(\psi(k_b))$ commutes with $\mathrm{Ad}(\psi(k_a))$, we see that $\mathrm{Ad}(k_b)(\tilde{F}) = -\tilde{F}$ for all $b \leqslant a$.

Similarly, $\psi(j_\alpha)F + F\psi(j_\alpha) \in \mathbb{K}(\hat{\mathcal{H}}_A)$. As $\tilde{F}$ is a "compact perturbation" of F, $(\hat{\mathcal{H}}_A, \psi, \tilde{F})$ represents the same class as $(\hat{\mathcal{H}}_A, \psi, F)$ [15, Prop. 17.2.5]. Thus, we may assume that F anti-commutes with all the $\psi(k_a)$. Arguing similarly, we may assume also that it anti-commutes with all the $\psi(j_\alpha)$. □

By the results hitherto, we can now introduce the following notation without ambiguity.

Definition 3.1.34 ((Real) K-theory as Kasparov groups)**.** *Since the groups $K_{p,q}K^{r,s}R(A)$ and $K_{p,q}K^{r,s}(A)$ only depend on the difference $p - q - (r - s)$, we introduce the notation*

$$KKR^{-i}(A) := K_{p,q}K^{r,s}R(A) \quad \textit{and} \quad KK^{-i}(A) := K_{p,q}K^{r,s}(A)$$

for $i = p - q - (r - s)$. Note that we write the groups cohomologically.

The groups $KKR^{-i}(A)$ and $KK^{-i}(A)$ are one possible definition for the KR-groups of real C*-algebras and K-groups of complex C*-algebras, respectively.

The 8-fold periodicity of real Clifford algebras, the 2-fold periodicity of complex Clifford algebras and the stability of Kasparov groups imply the following result.

Theorem 3.1.35. [43, §5, Thm. 5] *For all $i \in \mathbb{Z}$ we have*

$$KKR^{-i}(A) \cong KKR^{-(i+8)}(A) \text{ and } KK^{-i}(A) \cong KK^{-(i+2)}(A).$$

Now we have a sufficient understanding of the groups $KKR^{-i}(-)$ and $KR^{-i}(-)$ in order to prove the following theorem, which establishes an explicit relation between Kasparov's Fredholm picture of K-theory and Kasparov groups.

Theorem 3.1.36. [43, §6: Cor. 1] *For a real σ-unital C*-algebra A, we have*

$$KR^{r,s}(A) \cong KKR^{-(s-r+1)}(A).$$

If A is a complex σ-unital C-algebra, then $K^{r,s}(A) \cong KK^{-(s-r+1)}(A)$.*

Proof. We prove this theorem by showing that the group $KR^{r,s}(A)$ is isomorphic to the Kasparov group $KKR(C\ell_{s+1,r}, A)$. We focus our proof on the real case, as the complex case can be proved in completely the same way.

We present the proof in a form which will be useful for later considerations. In a first step, we show how to construct a class in $KKR(C\ell_{s,r}, A \otimes C\ell_{0,1})$ out of a given class in $KR^{r,s}(A)$. In a second step, we then show that this defines an isomorphism using $KKR(C\ell_{s,r}, A \otimes C\ell_{0,1}) \cong KKR(C\ell_{s+1,r}, A)$.

Let $[(F, \phi)] \in KR^{s,r}(A)$, i.e. $F \in \mathrm{M}^s(A) \cong \mathbb{B}(\mathcal{H}_A)$ and $\phi : C\ell_{s,r} \to \mathrm{M}^s(A) \cong \mathbb{B}(\mathcal{H}_A)$ is a real $*$-morphism between real ungraded C*-algebras. We define the following even real $*$-morphism onto $\mathbb{B}(\mathcal{H}_A \otimes C\ell_{0,1})$:

$$\psi : C\ell_{s,r} \to \mathbb{B}(\mathcal{H}_A \otimes C\ell_{0,1}); \quad k_\alpha \mapsto \phi(j_\alpha) \otimes j_1, \quad j_a \mapsto \phi(k_a) \otimes j_1,$$

for all $\alpha \in \{1, \dots, s\}$ and $a \in \{1, \dots, r\}$. Here, $C\ell_{s,r}$ and $C\ell_{0,1}$ are equipped with the natural grading. We define the real odd operator

$$G := F \otimes j_1 \in \mathbb{B}(\mathcal{H}_A \otimes C\ell_{0,1}).$$

It follows

$$G^* - G,\ G^2 - 1,\ \{\psi(k_\alpha), G\},\ \{\psi(j_a), G\} \in \mathbb{K}(\mathcal{H}_A \otimes C\ell_{0,1}) \quad \forall\, \alpha = 1, \dots, q, a = 1 \dots, p.$$

Thus, we obtain the Kasparov class

$$[(\mathcal{H}_A \otimes C\ell_{0,1}, \psi, G)] \in KKR(C\ell_{s,r}, A \otimes C\ell_{0,1}).$$

To get in contact with our discussion above this theorem, we now apply the natural morphism $\tau_{C\ell_{1,0}}$ on this class leading to

$$\Big[\big(\mathcal{H}_A \otimes C\ell_{1,1}, \psi \otimes (-\mathrm{id}_{C\ell_{1,0}}), G\big)\Big] \in KKR(C\ell_{s,r} \widehat{\otimes} C\ell_{0,1}, A \otimes C\ell_{1,1}),$$

where we also applied the orientation-preserving isomorphism $C\ell_{0,1}\widehat{\otimes}C\ell_{1,0}\cong C\ell_{1,1}$; $j_1\otimes 1\mapsto j_1$, $1\otimes k_1\mapsto -k_1$. Moreover, we apply the orientation-preserving isomorphism $C\ell_{s,r}\widehat{\otimes}C\ell_{1,0}\cong C\ell_{s+1,r}$ defined by $k_\alpha\otimes 1\mapsto k_\alpha$ $j_a\otimes 1\mapsto j_a$ $1\otimes k_1\mapsto(-1)^{r+1}k_{s+1}$.

Let $\psi' : C\ell_{s+1,r}\to\mathbb{B}(\mathcal{H}_A\otimes C\ell_{1,1})$ be the corresponding real even $*$-morphism. Then we end up with the Kasparov class

$$[(\mathcal{H}_A\otimes C\ell_{1,1},\psi',G)]\in KKR(C\ell_{s+1,r},A\otimes C\ell_{1,1}).$$

Finally, we apply the stability isomorphism given by the intersection product with the class $[\alpha]\in KKR(C\ell_{1,1},\mathbb{C})$ from the right. For the explicit form of this class and the calculation of the intersection product, we refer to the discussion above this theorem. The result is the Kasparov class

$$[(\mathcal{H}_A\otimes\Lambda(\mathbb{C}),\psi',F\otimes j_1)]=\left[\left(\hat{\mathcal{H}}_A,\psi'',\left(\begin{smallmatrix}0&-F\\F&0\end{smallmatrix}\right)\right)\right]\in KKR(C\ell_{s+1,r},A),\tag{3.6}$$

where the isomorphism above is induced by $\mathcal{H}_A\otimes\Lambda(\mathbb{C})\cong\mathcal{H}_A\oplus\mathcal{H}_A^{op}=\hat{\mathcal{H}}_A$. The $*$-morphism $\psi'' : C\ell_{s+1,r}\to\mathbb{B}(\hat{\mathcal{H}}_A)$ is given by

$$\psi''(k_\alpha)=\begin{pmatrix}0&-J_\alpha\\J_\alpha&0\end{pmatrix},\quad\psi''(k_{s+1})=(-1)^{r+1}\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad\psi''(j_a)=\begin{pmatrix}0&-K_a\\K_a&0\end{pmatrix}$$

for $\alpha=1,\dots,s$ and $a=1,\dots,r$.

According to Theorem 3.1.19, we can represent all classes in $KKR(C\ell_{s+1,r},A)$ in the form $[(\hat{\mathcal{H}}_A,\psi : C\ell_{s+1,r}\to\mathbb{B}(\hat{\mathcal{H}}_A),T)]$, such that $T^*=T$, $\|T\|\le 1$. Moreover, we can demand $[\psi(x),T]=0$ for all $x\in C\ell_{s+1,r}$, where $[\cdot,\cdot]$ denotes the graded commutator. In particular, this implies the weaker statement that all Kasparov classes can be represented in the form of Equation (3.6).

After all, we conclude that the map from $KR^{s,r}(A)$ to $KKR(C\ell_{s+1,r},A)=KKR^{-(s-r+1)}(A)$ is a well-defined isomorphism. □

Corollary 3.1.37. *In Definition 3.1.30, the restriction of the equivalence relations to unitary equivalence and operator homotopy equivalence does not change the resulting groups* $KR^{r,s}(A)$ *and* $K^{r,s}(A)$.

Moreover, any given class in $KR^{r,s}(A)$ *and* $K^{r,s}(A)$ *can be represented by a pair* $(F,\phi)\in\mathbb{E}^{r,s}(A)$ *satisfying* $F^*=-F$ *and*

$$\phi(k_a)F+F\phi(k_a),\phi(j_\alpha)F+F\phi(j_\alpha)=0\quad\forall\, a=1,\dots,r,\alpha=1,\dots,s.$$

All equivalence relations can be realised over such pairs, i.e. restriction of $\mathbb{E}^{r,s}(A)$ *to pairs satisfying the above conditions does not change the resulting groups* $KR^{r,s}(A)$ *and* $K^{r,s}(A)$.

Proof. This is an immediate consequence of Theorem 3.1.36 and the corresponding results for Kasparov groups in Proposition 3.1.19, Theorem 3.1.18 and Proposition 3.1.33. □

3.1.3 The *KR*- and *K*-theory of $\mathbb{C}$

The *KR*- and *K*-groups of $\mathbb{C}$ can be calculated explicitly due to its simple structure, where in the real case, we consider $\mathbb{C}$ as a real C*-algebra equipped with its natural complex conjugation. These groups constitute an important tool for the calculation of (real) *K*-groups of more involved C*-algebras, such as the group C*-algebra of $\mathbb{Z}^d$ in Section 5.5.

In this subsection, we will compare the groups $KR^{r,s}(\mathbb{C})$ and $K^{r,s}(\mathbb{C})$ with the so-called Clifford module picture of the (real) K-groups of $\mathbb{C}$, as introduced in [6]. For that, we first give a short survey of this Clifford module picture. Then we establish in Theorem 3.1.40 an isomorphism between those two pictures of $KR^{-(s-r+1)}(\mathbb{C})$, which is based on [55, Ch. 3, Sec. 10] and [16, App. B].

Definition 3.1.38 (Clifford module)**.** *A $C\ell_{r,s}$-module is a real finite-dimensional vector space W with a real $*$-representation*

$$\phi : C\ell_{r,s} \to \mathrm{End}(W).$$

A $\mathbb{C}\ell_{r,s}$-module is a complex finite-dimensional vector space W with a $*$-representation

$$\phi : \mathbb{C}\ell_{r,s} \to \mathrm{End}(W).$$

Two $\mathbb{C}\ell_{r,s}$-modules ($C\ell_{r,s}$-modules) $(W_1, \phi_1), (W_2, \phi_2)$ are equivalent *if their representations are equivalent, i.e. if there is a (real) unitary $u : W_1 \to W_2$ such that $u\phi_1 u^* = \phi_2$. Let $\mathfrak{M}_{r,s}$ ($\mathfrak{M}\mathfrak{R}_{r,s}$) denote the Grothendieck group of equivalence classes of (real) $C\ell_{r,s}$-modules.*

Example 3.1.39. *We have seen in Proposition 2.2.4 that there is a real $*$-representation of $C\ell_{n,n}$ on $\Lambda(\mathbb{C}^n)$. Thus $\Lambda(\mathbb{C}^n)$ is a real $C\ell_{n,n}$-module.*

There is a natural morphism

$$i : \mathfrak{M}\mathfrak{R}_{r,s+1} \to \mathfrak{M}\mathfrak{R}_{r,s},$$

given by restricting the $C\ell_{r,s+1}$-representation to a $C\ell_{r,s}$-representation. Therefore, the quotient

$$\mathfrak{M}\mathfrak{R}_{r,s}/i(\mathfrak{M}\mathfrak{R}_{r,s+1}),$$

is always well-defined. The analogous statement holds for the complex case. By the Atiyah–Bott–Shapiro isomorphism [55, Ch. 1,Thm. 10.11 & Prop. 5.20]

$$\mathfrak{M}\mathfrak{R}_{r,s}/i(\mathfrak{M}\mathfrak{R}_{r,s+1}) \cong KR^{-(s-r+1)}(\{\mathrm{pt}\}) \equiv KR^{-(s-r+1)}(\mathbb{C}),$$
$$\mathfrak{M}_{r,s}/i(\mathfrak{M}_{r,s+1}) \cong K^{-(s-r+1)}(\{\mathrm{pt}\}) \equiv K^{-(s-r+1)}(\mathbb{C}).$$

we know that these quotients of Clifford modules indeed describe the real and complex K-groups of $\mathbb{C}$.

The details of the assertions above can be found in [55, Ch. I: §5, §9, §10].

By the detailed results of Section 2.2, we can describe the groups $KR^{-s}(\mathbb{C})$ for $0 \le s \le 7$ and $K^{-s}(\mathbb{C})$ for $0 \le s \le 1$ even more explicitly by following the arguments of [55, Ch. I, §9]. In the following, we will discuss the non-trivial groups case-by-case. The resulting real and complex K-groups of $\mathbb{C}$ are listed in Table 3.1.

- The group $K^0(\mathbb{C}) \cong \mathfrak{M}^{\mathbb{C}}_{1,0}/i(\mathfrak{M}^{\mathbb{C}}_{1,1})$ is generated by equivalence classes of complex $\mathbb{C}\ell_{1,0}$-modules. By Proposition 2.2.11, there are two inequivalent irreducible representations $\rho_\pm : \mathbb{C}\ell_{1,0} \to W_\pm$ of $\mathbb{C}\ell_{1,0} \cong \mathbb{C} \oplus \mathbb{C}$. In particular, we have $\dim_{\mathbb{C}}(W_\pm) = 1$. Any representation $\phi : \mathbb{C}\ell_{1,0} \to \mathrm{End}(W)$ can be decomposed as $\phi = (\rho_+)^{\oplus n_+} \oplus (\rho_-)^{\oplus n_-}$. Therefore, the equivalence class is uniquely determined by $n_+, n_- \in \mathbb{N}$.

  As explained in Lemma 2.2.11, this representation is extendable to a $\mathbb{C}\ell_{1,1}$-representation if and only if $n_+ = n_-$. The class $[W] \in K^0(\mathbb{C})$ is thus determined by

  $$n_+ - n_- \in \mathbb{Z}.$$

s	$KR^{-s}(\mathbb{C})$	$K^{-s}(\mathbb{C})$
0	$\mathbb{Z}$	$\mathbb{Z}$
1	$\mathbb{Z}/2\mathbb{Z}$	0
2	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}$
3	0	0
4	$\mathbb{Z}$	$\mathbb{Z}$
5	0	0
6	0	$\mathbb{Z}$
7	0	0

Table 3.1: KR- and K-groups of $\mathbb{C}$

- The group $KR^0(\mathbb{C}) \cong \mathfrak{MR}_{1,0}/i(\mathfrak{MR}_{1,1})$ is generated by equivalence classes of real $C\ell_{1,0}$-modules. By the same arguments as for the class $K^0(\mathbb{C})$, we deduce that the class of a given real Hilbert module W is determined by the dimensions of the irreducible parts as
$$n_+ - n_- \in \mathbb{Z}.$$
- The group $KR^{-1}(\mathbb{C}) \cong \mathfrak{MR}_{0,0}/i(\mathfrak{MR}_{0,1})$ is simply generated by finite-dimensional real vector spaces. A class $[W]$ is trivial if there is a representation $\phi : C\ell_{0,1} \to \mathrm{End}(W)$. Since $C\ell_{0,1} \cong \mathbb{C}$, such a representation is the same as a complex structure on $W_{\mathbb{R}} := \mathrm{Re}(W) := \{w \in W \mid \overline{w} = w\}$. It follows that the class $[W]$ is trivial if and only if the real dimension of $W_{\mathbb{R}}$ is even. Since $\dim_{\mathbb{R}}(W_{\mathbb{R}}) = \dim_{\mathbb{C}}(W)$, the class $[W] \in KR^{-2}(\mathbb{C})$ is determined by
$$\dim_{\mathbb{C}}(W) \bmod 2 \in \mathbb{Z}/2\mathbb{Z}.$$
- The group $KR^{-2}(\mathbb{C}) \cong \mathfrak{MR}_{0,1}/i(\mathfrak{MR}_{0,2})$ is generated by equivalence classes of $C\ell_{0,1}$-modules. As we have seen above, the datum of a $C\ell_{0,1}$-module W is equivalent to a complex vector space $W_{\mathbb{R}} = \mathrm{Re}(W)$. Since $C\ell_{0,2} \cong \mathbb{H}_{\mathbb{C}}$, the space W admits a $C\ell_{0,2}$-representation if and only if $W_{\mathbb{R}}$ can be equipped with a quaternionic structure. This is possible if and only if its real dimension is divisible by four. Thus the class in $KR^{-2}(\mathbb{C})$ is determined by
$$\tfrac{1}{2}\dim_{\mathbb{R}}(W_{\mathbb{R}}) \bmod 2 = \tfrac{1}{2}\dim_{\mathbb{C}}(W) \bmod 2 \in \mathbb{Z}/2\mathbb{Z}.$$
- The group $KR^{-4}(\mathbb{C}) \cong \mathfrak{MR}_{0,3}/i(\mathfrak{MR}_{0,4})$ consists of equivalence classes of real $C\ell_{0,3}$-modules. There are two inequivalent irreducible representations $\rho_\pm : C\ell_{0,3} \to \mathrm{End}(W_\pm)$. Any $C\ell_{0,3}$-representation ϕ can be decomposed as $\phi = \rho_+^{\oplus n_+} \oplus \rho_-^{\oplus n_-}$. As explained in Lemma 2.2.11, ϕ is extendable to a $C\ell_{0,4}$-action if and only if $n_+ = n_-$. The class of a given $C\ell_{0,3}$-module $W = (W_+)^{n_+} \oplus (W_-)^{n_-}$ is thus determined by
$$n_+ - n_- \in \mathbb{Z}.$$

All other real and complex K-groups of $\mathbb{C}$ turn out to be trivial and are therefore not discussed here.

The following theorem establishes the explicit connection between the Clifford module picture and Kasparov's Fredholm picture of the real and complex K-groups of $\mathbb{C}$. The proof of this theorem follows the arguments of [16, App. B].

Theorem 3.1.40. *Let $[(F,\phi)] \in KR^{r,s}(\mathbb{C})$. This class corresponds to the class in $KR^{-(s-r+1)}(\mathbb{C})$ that is determined by the Clifford module $\ker(F)$ with representation $\phi : C\ell_{r,s} \to \ker(F)$. This defines an isomorphism $KR^{r,s}(\mathbb{C}) \cong \mathfrak{MR}_{r,s}/i(\mathfrak{MR}_{r,s+1})$.*

The analogous result is true for complex K-theory.

Proof. Here, we will focus on the construction of the map from $KR^{r,s}(\mathbb{C})$ to $\mathfrak{MR}_{r,s}/i(\mathfrak{MR}_{r,s+1})$. The proof of the fact that this map indeed defines a well-defined isomorphism is quite technical and we will therefore refer to [55] for these results.

Let $[(F,\phi)] \in KR^{r,s}(\mathbb{C})$, such that $\phi : C\ell_{r,s} \to \mathcal{L}(\mathcal{H})$ is a real unital $*$-morphism over some real Hilbert space $\mathcal{H}$ and $F \in \mathcal{L}(\mathcal{H})$ is a real skew-Hermitian operator such that

$$F^2 + 1 \in \mathbb{K}(\mathcal{H}) \quad \text{and} \quad \phi(k_\alpha)F + F\phi(k_\alpha) = \phi(j_a)F + F\phi(j_a) = 0$$

for all $\alpha = 1,\dots,s, a = 1,\dots,r$. By Corollary 3.1.37, all classes in $KR^{r,s}(\mathbb{C})$ can be represented by such a pair. By [56, Thm. 1.2.5] and [82, Thm. 3.1.5], there is a unique partial isometry V such that $V : (\ker F)^\perp \to (\ker F)^\perp$ is an isometry and $F = V|F|$ where $|F| := (F^*F)^{1/2}$. The map $[0,1] \ni t \mapsto F_t := V|F|^t$ defines an operator homotopy between (F,ϕ) and (V,ϕ). Thus, the class $[(F,\phi)]$ is equivalent to the class

$$[(V,\phi)] \in KR^{r,s}(\mathbb{C}).$$

We split the Hilbert space as $\mathcal{H} = (1 - V^*V)\mathcal{H} \oplus (V^*V)\mathcal{H}$. The splitting of $\mathcal{H}$ leads to the decomposition of the class as

$$[(V,\phi)] = [(V|_{(1-V^*V)\mathcal{H}}, \phi|_{(1-V^*V)\mathcal{H}})] \oplus [(V|_{V^*V\mathcal{H}}, \phi_{V^*V\mathcal{H}})]\,.$$

The operator V vanishes in the first class and is an isometry in the second class. Thus the second class is degenerate and the remaining non-trivial term is equal to

$$[(V,\phi)] = [(0, \phi|_{\ker F})] \in KR^{-(s-r)}(\mathbb{C}).$$

Thus, we end up with the finite-dimensional vector space $\ker(F)$, equipped with the $*$-represen-tation $\phi : C\ell_{r,s} \to \mathcal{L}(\ker(F))$. This defines an element in $\mathfrak{MR}_{r,s}$.

It follows immediately that unitarily equivalent pairs in $KR^{r,s}(\mathbb{C})$ are mapped onto unitarily equivalent Clifford modules. By the results of [55, Ch. 3, Prop. 10.6], this indeed defines a well-defined morphism from $KR^{r,s}(\mathbb{C})$ to $KR^{-(s-r+1)}(\mathbb{C}) = \mathfrak{MR}_{r,s}/i(\mathfrak{MR}_{r,s+1})$ and [55, Ch. 3, Thm. 10.8] shows that this morphism defines an isomorphism.

The result for the complex case can be proved in the same way. □

3.2 Van Daele K-theory for graded C^*-algebras

Here we recapitulate the construction of Van Daele groups as worked out in [86, 87]. Although the definition of Van Daele groups holds for any Banach algebra, we will restrict ourselves to the construction for C^*-algebras.

We begin with the definition of ORHU resp. OHU, which are the fundamental objects of Van Daele groups.

Definition 3.2.1 (ORHU, OHU)**.** *Let A be a graded unital real C^*-algebra. Then we define $\mathcal{F}(A)$ to be the set of all* odd real Hermitian unitaries (ORHU) *in A. Since $\mathcal{F}(A)$ is a subset of A, we can use the norm topology of A to define the set of path-connected components of ORHU*

$$F(A) := \pi_0(\mathcal{F}(A)).$$

If A is just a complex graded C^*-algebra, we define $\mathcal{F}(A)$ to be the set of all odd Hermitian unitaries (OHU) *in A. The corresponding set of path-connected components of OHU is also denoted by $F(A)$.*

Remark 3.2.2. *Note that the set $\mathcal{F}(A)$ might be empty in general, e.g. when A is trivially graded. In the original definition [86, Def. 2.1] of $\mathcal{F}(A)$, the elements don't have to be Hermitian. However, [86, Prop. 2.5] shows that our definition and the original one lead to the same set $F(A)$ if A is a C^*-algebra.*

We say that two elements in $\mathcal{F}(A)$ are *homotopic* if they define the same element in $F(A)$. For the matrix algebra $M_n(A)$ over A with $n \geq 1$ we define

$$\mathcal{F}_n(A) := \mathcal{F}(M_n(A)), \quad F_n(A) := F(M_n(A)).$$

Then we can define the *direct sum* by

$$\oplus : \mathcal{F}_n(A) \times \mathcal{F}_m(A) \to \mathcal{F}_{n+m}(A); \quad (x, y) \mapsto x \oplus y := \left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right).$$

This direct sum is associative and induces a map $F_n(A) \times F_m(A) \to F_{n+m}(A)$ such that $[x \oplus y] = [y \oplus x]$ in $F_{n+m}(A)$ for all $x \in \mathcal{F}_n$ and $y \in \mathcal{F}_m(A)$ [86, Prop. 2.7].

Assume that $\mathcal{F}(A)$ is not empty, i.e. there is an element $e \in \mathcal{F}(A)$. Then $F_n(A)$ is an inductive system of sets with connecting maps $F_n(A) \to F_{n+1}(A)$; $[x] \mapsto [x] \oplus [e] = [x \oplus e]$. The inductive limit of this system defines a semi-group, which we denote as follows.

Definition 3.2.3 (Van Daele (semi-)group)**.** *Let A be a graded C^*-algebra. Given an element $e \in \mathcal{F}(A)$, we define*

$$DK_e(A) := \varinjlim_n F_n(A).$$

If A is real, we define

$$DKR_e(A) := \varinjlim_n F_n(A).$$

By definition, two elements $[x], [y] \in DKR_e(A)$ with $x \in \mathcal{F}_n(A)$ and $y \in \mathcal{F}_m(A)$ are equal if there are $k, l \in \mathbb{N}$ such that $[x \oplus e_k] = [y \oplus e_l]$ in $F_N(A)$, where $N := n + k = m + l$.

Van Daele shows in [86, Prop. 2.9] that $DKR_e(A)$ is an Abelian semi-group with $[x]+[y] = [x \oplus y]$ and the neutral element $0 = [e]$. If the element $e \in \mathcal{F}(A)$ is homotopic to $-e$ in $\mathcal{F}(A)$, then we have the following important result.

Proposition 3.2.4. [86, Prop. 2.11 & Prop. 2.12] *If e and $-e$ are homotopic in $\mathcal{F}(A)$ then $DKR_e(A)$ and $DK_e(A)$ are groups. For $x \in \mathcal{F}_n(A)$, the inverse element of $[x]$ is given by $-[x] = [-e_n x e_n]$, where $e_n = e \oplus \cdots \oplus e$ denotes the n-fold direct sum of e.*

Moreover, the groups $DKR_e(A)$ and $DK_e(A)$ are, up to isomorphism, independent of e and therefore we also write $DKR(A)$ and $DK(A)$, respectively.

Remark 3.2.5. *Proposition 3.2.4 has been improved by Van Daele in [88] by showing that the condition of e and $-e$ to be homotopic can be replaced by the weaker condition that there exists an even (real) invertible element $u \in A$ such that $u^{-1}eu = -e$.*

The above definitions only make sense if $\mathcal{F}(A) \neq \emptyset$. However, in general we may consider $M_2(A) = A \hat{\otimes} M_2(\mathbb{C})$, where $M_2(\mathbb{C})$ is equipped with the real structure of component-wise complex conjugation and the grading where the diagonal elements are even and the off-diagonal elements are odd. The grading on $M_2(A)$ is then given by the grading of the graded tensor product. Then $M_2(A)$ admits the ORHU $e = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The matrix $u = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ satisfies $u^{-1}eu = -e$. Therefore, $DKR_e(M_2(A))$ and $DK_e(M_2(A))$ are groups by Remark 3.2.5.

Thus, without any assumptions on the ORHU for a graded real C*-algebra, we can define the Van Daele KR-group for $M_2(A) = A \otimes M_2(\mathbb{C})$ which is constructed as above:

$$DKR_e(M_2(A)), \quad \text{where } e = 1 \otimes \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right).$$

The analogous construction works for complex C*-algebras by dropping the real structures.

In this work, we will mostly encounter the case that we are given a reference element $e_0 \in \mathcal{F}(A)$ that may not be not homotopic to $-e_0$. Then one can again consider the matrix algebra $M_2(A) = A \otimes M_2(\mathbb{C})$ over A, but the real C*-algebra $M_2(\mathbb{C})$ with the component-wise complex conjugation shall now be equipped with the trivial grading. Then we define $e := e_0 \oplus (-e_0) \in \mathcal{F}_2(A)$. This element is homotopic to $-e$ *via* the path

$$[0, \pi] \ni t \mapsto \begin{pmatrix} e_0 \cos(t) & e_0 \sin(t) \\ e_0 \sin(t) & -e_0 \cos(t) \end{pmatrix}.$$

This generalises Definition 3.2.3 because of Proposition 3.2.4 and the following stability of Van Daele groups.

Proposition 3.2.6 (Stability of Van Daele groups)**.** [86, Prop. 3.2], [88] *Let A be a graded real or complex C*-algebra and let $e \in \mathcal{F}(A)$ be homotopic to $-e$ within $\mathcal{F}(A)$. Then*

$$\mathcal{F}_n(A) \longrightarrow \mathcal{F}_n(M_m(A)); \quad x \mapsto x \oplus e_{m-1}$$

induces an isomorphism $DKR_e(A) \cong DKR_{e_m}(M_m(A))$ resp. $DK_e(A) \cong DK_{e_m}(M_2(A))$, where $e_k = e \oplus \cdots \oplus e$ denotes the k-fold direct sum for $k \in \mathbb{N}$ and $M_m(A)$ is equipped with the component-wise real structure and grading.

This proposition still holds under the weaker conditions of Remark 3.2.5.

Thus, we have defined the Van Daele groups for any graded (real) unital C*-algebra, and because of the above observations we can denote this group by $DKR(A)$ or $DK(A)$ without ambiguity.

In later considerations, we will also need the following relation between the semi-group $DKR_e(A)$ or $DK_e(A)$ and the corresponding Van Daele group $DKR(A)$ or $DK(A)$, respectively.

Proposition 3.2.7. [86, Prop. 3.3] *If there is an element $e_0 \in \mathcal{F}(A)$, then $DKR_e(M_2(A)) = DKR(A)$ with $e = e_0 \oplus (-e_0)$ is the Grothendieck group of the semi-group $DKR_{e_0}(A)$. The isomorphism is defined by*

$$[x] - [y] \mapsto \left[\left(\begin{smallmatrix} x & 0 \\ 0 & -(e_0)_n y (e_0)_n \end{smallmatrix}\right)\right],$$

for $[x], [y] \in DKR_e(A)$ with $y \in \mathcal{F}_n(A)$. In particular, the elements in $[x] \in DKR_{e_0}(A)$ are mapped onto $\left[\left(\begin{smallmatrix} x & 0 \\ 0 & -e_0 \end{smallmatrix}\right)\right] \in DKR(A)$.

The analogous statement holds for the complex case.

By Proposition 3.2.7, we could also define the Van Daele groups as the Grothendieck groups of the semi-groups in Definition 3.2.3. This point of view will be important in Section 3.3.

Now we extend the definition of Van Daele groups to non-unital C*-algebras.

Definition 3.2.8 (Unitisation)**.** *The unitisation of a C*-algebra is defined as* $A^+ := \{(a, \lambda) \mid a \in A, \lambda \in \mathbb{C}\}$ *equipped with the following multiplication and* $*$*-involution:*

$$(a, \lambda)(b, \mu) := (ab + \lambda b + a\mu, \lambda\mu), \quad (a, \lambda)^* := (a^*, \lambda^*).$$

If A *is real, we define the unitisation as the real C*-algebra* A^+ *with the conjugation* $\overline{(a, \lambda)} = (\overline{a}, \overline{\lambda})$*. We denote by* $\varphi : A^+ \to \mathbb{C}$ *the canonical* $*$*-morphism given by* $\varphi(a, \lambda) = \lambda$*.*

By the use of the unitisation, we generalise the definition of Van Daele groups as follows.

Definition 3.2.9 (Van Daele groups for non-unital C*-algebra)**.** *If* A *is unital, then* $DKR(A)$ *is isomorphic to the kernel of* $DKR(A^+) \xrightarrow{\varphi_*} DKR(\mathbb{C})$ *[86, Prop. 3.6]. Thus, for any not necessarily unital C*-algebra* A*, we can define*

$$DKR(A) := \ker(\varphi_*) \subseteq DKR(A^+),$$

generalising the definition for unital C-algebras. The analogous statement is true for the complex case.*

Subsection 3.2.1 will be essentially based on the existence of long exact sequences of Van Daele groups for any short exact sequence of graded C*-algebras. Therefore, we give a short exposition of this fact in the following.

Let

$$0 \to I \xrightarrow{\iota} A \xrightarrow{\pi} A/I \to 0$$

be a SES of (real) graded C*-algebras. Since the Van Daele groups are functorial, the $*$-morphisms ι and π induce morphisms $\iota_* : DKR(I) \to DKR(A)$ and $\pi_* : DKR(A) \to DKR(A/I)$ that form an exact sequence [86, Prop. 4.1]

$$DKR(I) \xrightarrow{\iota_*} DKR(A) \xrightarrow{\pi_*} DKR(A/I).$$

The analogous assertions hold for DK-groups.

In order to obtain a long exact sequence, Van Daele constructs in [87] a connecting morphism

$$\partial : DKR(A/I) \to DKR(A/I \otimes C\ell_{1,0})$$

for the real case, such that

$$\begin{tikzcd}
{} \arrow[r, dotted] & DKR(I) \arrow[r, "\iota_*"] & DKR(A) \arrow[r, "\pi_*"] & DKR(A/I) \arrow[dll, rounded corners, "\partial"] & \\
 & DKR(I\widehat{\otimes} C\ell_{1,0}) \arrow[r, "\iota_*"] & DKR(A\widehat{\otimes} C\ell_{1,0}) \arrow[r, "\pi_*"] & DKR(A/I\widehat{\otimes} C\ell_{1,0}) \arrow[r, dotted] & {}
\end{tikzcd} \tag{3.7}$$

is a 24-fold periodic long exact sequence. Here, the real Clifford algebra $C\ell_{1,0}$ is equipped with its natural grading.

For complex C*-algebras, this leads to a 6-fold periodic long exact sequence of DK-groups.

Van Daele also proved that the connecting morphism is of the following form.

Theorem 3.2.10. [87, Prop. 3.4] *Let $[x] \in DKR(A/I)$ and assume that $x \in \mathcal{F}_n(M_4((A/I)^+))$ such that $\varphi(x) = e_n$. Let k be the generator of $C\ell_{1,0}$ and $k_n = k \oplus \ldots \oplus k$ be the n-fold direct sum of k and define $\nu := \frac{1}{\sqrt{2}}(1 + k_n e_n)$. Let $a \in M_n(M_4(A^+))$ be any odd real element such that $\pi(a) = x$. Then*

$$\partial([x]) = [-\nu^* \exp(\pi a k_n) k_n \nu] \in DKR(I \widehat{\otimes} C\ell_{1,0}).$$

The connecting morphism for DK-groups has the same form without reality condition on $a \in M_n(M_4(A^+))$.

Finally, we come to the formulation of K-groups for trivially graded real or complex C*-algebra in terms of Van Daele groups.

Definition 3.2.11 (Van Daele's picture of (real) K-theory)**.** *Let $C\ell_{r+1,s}$ and $\mathbb{C}\ell_{r+1,s}$ be graded and let A be a trivially graded real C*-algebra. Then we define*

$$DKR^{-(s-r)}(A) := DKR(A \otimes C\ell_{r+1,s}). \tag{3.8}$$

For a complex trivially graded C-algebra A, we define*

$$DK^{-(s-r)}(A) := DK(A \otimes \mathbb{C}\ell_{r+1,s}). \tag{3.9}$$

The groups $DK^{-(s-r)}(A)$ and $DKR^{-(s-r)}(A)$ indeed depend only on the difference $s - r$ because of the stability of Van Daele groups, cf. Proposition 3.2.6, and the $(1,1)$-periodicity of Clifford algebras, cf. Proposition 2.2.8.

3.2.1 Roe's isomorphism

Here we will prove that Van Daele's picture and Kasparov's Fredholm picture of real resp. complex K-theory indeed give rise to isomorphic groups. The proof is based on the work [73] by Roe. Therefore, we call the isomorphism between the two pictures *Roe's isomorphism*.

Since we have already established all necessary results and definitions, we can start right away with the formulation of the theorem.

Theorem 3.2.12. [73] *Let A be a trivially graded C*-algebra. If A is real, we have*

$$DKR^{-i}(A) \cong KKR^{-i}(A) \quad \forall i \in \mathbb{Z}.$$

Otherwise, if A is complex, there is an isomorphism

$$DK^{-i}(A) \cong KK^{-i}(A) \quad \forall i \in \mathbb{Z}.$$

By the results of Theorem 3.2.12, we can introduce the following notation.

Definition 3.2.13 ($KR^{-i}(A)$, $K^{-i}(A)$)**.** *Let A be a C*-algebra. Then we denote its K-groups by*

$$K^{-i}(A) := DK^{-i}(A) = KK^{-i}(A).$$

If A is real, we denote its real K-groups by

$$KR^{-i}(A) := DKR^{-i}(A) = KKR^{-i}(A).$$

We will use this notation whenever we do not wish to specify a picture for these groups.

Proof of Theorem 3.2.12. Since the real and the complex case can be proved in almost the same way, we only prove the real case. To be precise, we will prove that $DKR(A \otimes C\ell_{r+1,s}) \cong KKR(\mathbb{C}, A \otimes C\ell_{r,s})$.

By Proposition 3.1.19, any class in $KKR(\mathbb{C}, A \otimes C\ell_{r,s})$ can be represented by a Kasparov module of the form $(\hat{\mathcal{H}}_A \widehat{\otimes} C\ell_{r,s}, 1_{\mathbb{C}}, F)$ with $F^* = F$ and $\|F\| \leq 1$. Moreover, $F \in \mathbb{B}(\hat{\mathcal{H}}_A \widehat{\otimes} C\ell_{r,s}) = \mathrm{M}^s(A \otimes C\ell_{r+1,s+1})$ is odd, real and $F^2 - 1 \in \mathbb{K}(\hat{\mathcal{H}}_A \widehat{\otimes} C\ell_{r,s}) = \mathbb{K}(\mathcal{H}_A \otimes C\ell_{r+1,s+1})$. Thus F defines, by definition, an ORHU in $\mathcal{F}(Q^s(A) \otimes C\ell_{r+1,s+1})$. This defines a map

$$KKR(\mathbb{C}, A \otimes C\ell_{r,s}) \to DKR(Q^s(A) \otimes C\ell_{r+1,s+1}), \tag{3.10}$$

which is a well-defined isomorphism because of the following:

On the left hand side, it is sufficient to only consider operator homotopy equivalence of Kasparov modules, because two unitary equivalent Kasparov modules $(\hat{\mathcal{H}}_A \widehat{\otimes} C\ell_{r,s}, 1_{\mathbb{C}}, F_1)$ and $(\hat{\mathcal{H}}_A \otimes C\ell_{r,s}, 1_{\mathbb{C}}, F_2)$ are already operator homotopy equivalent by [15, Prop. 12.2.2] and homotopy equivalence and operator homotopy equivalence are equivalent by Theorem 3.1.18.

Any operator homotopy is determined by a norm continuous path $[0,1] \ni t \mapsto F(t) \in \mathbb{B}(\hat{\mathcal{H}}_A \otimes C\ell_{r,s})$ between F_1 and F_2 in the set of all real Hermitian odd operators such that $F(t)^2 - 1 \in \mathbb{K}(\hat{\mathcal{H}}_A \otimes C\ell_{r,s})$ for all $t \in [0,1]$. Under the above identification with ORHU in $\mathcal{F}(Q^s(A) \otimes C\ell_{r+1,s+1})$, this is the same as a homotopy of ORHU between F_1 and F_2 in $\mathcal{F}(Q^s(A) \otimes C\ell_{r+1,s+1})$ in the sense of Definition 3.2.1. Moreover, the identification commutes with direct sums and thus the morphism (3.10) is indeed well-defined.

It can be checked in the same way that the inverse of the morphism (3.10) is also well-defined and hence this morphism indeed defines an isomorphism.

It remains to find an isomorphism between $DKR(Q^s(A) \otimes C\ell_{r+1,s+1})$ and $DKR(A \otimes C\ell_{r+1,s})$. For that, consider the short exact sequence

$$0 \to \mathbb{K} \otimes A \to \mathrm{M}^s(A) \to Q^s(A) \to 0.$$

The KR-theory of $\mathrm{M}^s(A)$ is trivial, because the multiplier algebra is contractible [24], [15, §12.2]. In the corresponding long exact sequence, we thus obtain the exact sequence

$$0 \to DKR(Q^s(A) \otimes C\ell_{r+1,s+1}) \xrightarrow{\partial_Q} DKR(\mathbb{K} \otimes A \otimes C\ell_{r+2,s+1}) \to 0.$$

This means that the connecting morphism is an isomorphism. Using $(1,1)$-periodicity of the Clifford algebra (Proposition 2.2.8) and stability of DKR-theory (Theorem 3.2.6), we end up with an isomorphism

$$\partial_Q : DKR(Q^s(A) \otimes C\ell_{r+1,s+1}) \to DKR(A \otimes C\ell_{r+1,s}). \tag{3.11}$$

The composition of the isomorphisms (3.10) and (3.11) exhibits the desired isomorphism $KKR(\mathbb{C}, A \otimes C\ell_{r,s}) \cong DKR(A \otimes C\ell_{r+1,s})$. □

3.3 A new picture of real K-theory

A physically natural and well-motivated picture for the formulation of topological phases is essential in order to be able to connect the different K-classes to physical properties of the systems. To that end, we will now construct a picture of KR- and K-theory for trivially graded C*-algebras which is formulated in terms of so-called quasi-particle vacua with pseudo-symmetries, representing the ground states in the bulk of topological insulators and superconductors, *cf.* Subsection 4.1.2.

This section is split into three subsections. In Subsection 3.3.1, we will construct for a given unital real or complex C*-algebra A and a unital $*$-morphism $\phi : C\ell_{r,s} \to A$ or $\phi : \mathbb{C}\ell_{r,s} \to A$, respectively, the groups $\mathrm{Top}^{\phi}_{J_{\mathrm{ref}}}(A)$ that are the core of our new picture of K-theory. This construction is based on the assumption that the set $\mathcal{FF}^{\phi}(A)$, *cf.* Definition 3.3.1, is not empty. This is the case in our physical applications in Chapter 5, but from a purely mathematical point of view, this condition cannot be true in general.

In Subsection 3.3.2, we will identify the groups of Subsection 3.3.1 with Van Daele groups.

By combination of the properties of Van Daele groups of Section 3.2 with the isomorphism of Subsection 3.3.2, we will then show in Subsection 3.3.3 that these groups only depend on the C*-algebra A and the index $r - s$. We will also generalise the constructions of Subsection 3.3.1 to the case $\mathcal{FF}^{\phi}(A) = \emptyset$ and to non-unital C*-algebras. In the end, this will lead to a systematic picture of these groups. Due to the identification with Van Daele groups, we thus obtain a new picture of K-theory.

3.3.1 Construction of the groups $\mathrm{Top}^{\phi}_{J_{\mathrm{ref}}}(A)$

The following constructions are closely related to the construction of Van Daele groups in [86], *cf.* Section 3.2. Throughout this section, we consider *all C*-algebras as trivially graded.*

The starting point for our picture of (real) K-theory is the following set.

Definition 3.3.1 ($\mathcal{FF}^{\phi}(A)$)**.** *Let A be a unital C*-algebra and suppose we are given a unital $*$-morphism $\phi : \mathbb{C}\ell_{r,s} \to A$. Let $K_a := \phi(k_a)$ for $a = 1, \dots, r$ and $J_\alpha := \phi(j_\alpha)$ for $\alpha = 1, \dots, s$.*

Then $\mathcal{FF}^{\phi}(A)$ is defined as the set of all skew-Hermitian unitaries $J \in A$ such that

$$\{J, K_a\} = \{J, J_\alpha\} = 0 \quad \forall a = 1, \dots, r, \alpha = 1, \dots, s,$$

where $\{\cdot, \cdot\}$ denotes the anti-commutator. If A is real, then we demand that J and $\phi : C\ell_{r,s} \to A$ are real.

Throughout this section, we will assume that the set $\mathcal{FF}^{\phi}(A)$ is not empty. This is in accordance with our physical application in Chapter 5. We will generalise the results of the present subsection to the general case in Subsection 3.3.3.

Remark 3.3.2. *The notation '$\mathcal{FF}$' in the definition above is motivated by the fact that the $J \in \mathcal{FF}^{\phi}(A)$ correspond in a physical context to quasi-particle vacua of free fermions in gapped systems. The $*$-morphism ϕ encodes the 'local' symmetries of the ground state. More details can be found in Chapter 4 and Section 5.1.*

We define a homotopy equivalence on this set as follows.

Definition 3.3.3 ($\mathrm{FF}^{\phi}(A)$)**.** *Two elements $J, J' \in \mathcal{FF}^{\phi}(A)$ are called homotopically equivalent if there is a norm continuous path in $\mathcal{FF}^{\phi}(A)$ connecting J and J'.*

The set of path-connected components of $\mathcal{FF}^{\phi}(A)$ is denoted by $FF^{\phi}(A) := \pi_0(\mathcal{FF}^{\phi}(A))$.

These definitions can canonically be generalised to matrix algebras over A, as explained in the following definition.

Definition 3.3.4 ($\mathcal{FF}^{\phi}_n(A)$, $FF^{\phi}_n(A)$)**.** *Let $\phi_n := \phi \oplus \ldots \oplus \phi \in M_n(A)$ denote the n-fold direct sum and define*

$$\mathcal{FF}^{\phi}_n(A) := \mathcal{FF}^{\phi_n}(M_n(A)), \quad FF^{\phi}_n(A) := \pi_0(\mathcal{FF}^{\phi}_n(A)).$$

The direct sum of two elements $J, J' \in \mathcal{FF}^{\phi}(A)$ is defined as

$$J \oplus J' := \begin{pmatrix} J & 0 \\ 0 & J' \end{pmatrix} \in \mathcal{FF}^{\phi}_2(A).$$

In order to compare elements in $FF^{\phi}_n(A)$ with elements in $FF^{\phi}(A)$, or, more generally, in $FF^{\phi}_m(A)$ for any $m \neq n$, we have to establish a canonical embedding of $FF^{\phi}_n(A)$ into $FF^{\phi}_{n+1}(A)$ for all $n \in \mathbb{N}$. Therefore, we use the direct sum in combination with a reference element $J_{\mathrm{ref}} \in \mathcal{FF}^{\phi}(A)$.

Lemma 3.3.5. *The direct sum in $FF^{\phi}(A)$ is Abelian, i.e. $[J \oplus J'] = [J' \oplus J] \in FF^{\phi}_2(A)$.*

Proof. Observe that conjugation with $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ exchanges $\left(\begin{smallmatrix} J' & 0 \\ 0 & J \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} J & 0 \\ 0 & J' \end{smallmatrix}\right)$, whereas conjugation with $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ leaves them invariant. Thus, the continuous path $[0, \pi/2] \ni t \mapsto \left(\begin{smallmatrix} \sin(t) & \cos(t) \\ \cos(t) & -\sin(t) \end{smallmatrix}\right)$ in $\mathrm{U}(M_2(A))$ between $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ defines a homotopy in $\mathcal{FF}^{\phi}_2(A)$ between $\left(\begin{smallmatrix} J' & 0 \\ 0 & J \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} J & 0 \\ 0 & J' \end{smallmatrix}\right)$. □

Definition 3.3.6 ($FF^{\phi}_{\infty}(A)$)**.** *Let $J_{\mathrm{ref}} \in \mathcal{FF}^{\phi}(A)$ be given. Then we define an inductive system* via *the morphisms*

$$\mathcal{FF}^{\phi}_n(A) \longrightarrow \mathcal{FF}^{\phi}_{n+1}(A);\ J \mapsto J \oplus J_{\mathrm{ref}} = \begin{pmatrix} J & 0 \\ 0 & J_{\mathrm{ref}} \end{pmatrix}. \tag{3.12}$$

The inductive limit of the sets $FF^{\phi}_n(A)$ with the inductive maps (3.12) *is denoted as*

$$FF^{\phi}_{\infty}(A) := \varinjlim_n FF^{\phi}_n(A)$$

The inductive limit $\mathcal{FF}^{\phi}_{\infty}(A)$ naturally forms a semi-group when equipped with the group structure of direct sums. By Lemma 3.3.5, this semi-group structure is Abelian. What is missing for this semi-group in order to become a group is that in general not all elements are invertible. We overcome this obstacle by considering the associated Grothendieck group.

Definition 3.3.7 ($\mathrm{Top}^{\phi}_{J_{\mathrm{ref}}}(A)$)**.** *The Grothendieck group of the Abelian semi-group $FF^{\phi}_{\infty}(A)$ is denoted by*

$$\mathrm{Top}^{\phi}_{J_{\mathrm{ref}}}(A) := \mathrm{Groth}(FF^{\phi}_{\infty}(A)).$$

We have thus established the Abelian group $\mathrm{Top}^{\phi}_{J_{\mathrm{ref}}}(A)$ for any given triple $(A, \phi, J_{\mathrm{ref}})$. This Abelian group $\mathrm{Top}^{\phi}_{J_{\mathrm{ref}}}(A)$ is the desired group which will be proved to exhibit a new picture of K-theory of (real) C*-algebras.

We refrain from deriving any further properties of these groups at this point. Instead, in the next section we will first identify the groups of this subsection with Van Daele groups and then derive all relevant properties for our groups by using this isomorphism.

$$\begin{tikzcd}
\mathcal{F}\mathcal{F}^{\phi}(A) \arrow[r, "\cong"] \arrow[d, "\text{Hom. equiv.}"'] & \mathcal{F}(P^{\phi}(A \otimes C\ell_{r,s+1})P^{\phi}) \arrow[d, "\text{Hom. equiv.}"] \\
FF^{\phi}(A) \arrow[r, "\cong"] \arrow[d, "\text{Ind. limit}"'] & F(P^{\phi}(A \otimes C\ell_{r,s+1})P^{\phi}) \arrow[d, "\text{Ind. limit}"] \\
FF^{\phi}_{\infty}(A) \arrow[r, "\cong"] \arrow[d, "\text{Groth. group}"'] & DKR_e(P^{\phi}(A \otimes C\ell_{r,s+1})P^{\phi}) \arrow[d, "\text{Groth. group}"] \\
\mathrm{Top}^{\phi}_{J_{\mathrm{ref}}}(A) \arrow[r, "\cong"] & DKR(P^{\phi}(A \otimes C\ell_{r,s+1})P^{\phi})
\end{tikzcd}$$

Figure 3.1: We will construct this commutative diagram. Once we have established the first row, this diagram is essentially a consequence of the constructions being the same on the two sides. Note that in the last row, we consider the Van Daele group as the Grothendieck group of the (semi-)group $DKR_e(P^{\phi}(A \otimes C\ell_{r,s+1})P^{\phi})$, as explained in Proposition 3.2.7.

3.3.2 Isomorphism to Van Daele groups

Now we bring our groups in contact with the existing definitions of K-groups. We will show that the group $\mathrm{Top}^{\phi}_{J_{\mathrm{ref}}}(A)$ is isomorphic to a Van Daele group, as shown in Figure 3.1.

Definition 3.3.8 ($P^{\phi}, Q_a, P_{\alpha}$)**.** *For given unital $*$-morphism $\phi : C\ell_{r,s} \to A$, we define commuting projections Q_a, P_{α} in $A \otimes C\ell_{r,s+1}$ (in $A \otimes \mathbb{C}\ell_{r,s+1}$ if A is complex) by*

$$2Q_a - 1 = (-1)^s K_a \otimes k_a j_1, \quad 2P_{\alpha} - 1 = J_{\alpha} \otimes j_1 j_{\alpha+1} \tag{3.13}$$

for all $a = 1, \ldots, r, \alpha = 1, \ldots, s$, where, as before, $K_a := \phi(k_a)$ and $J_{\alpha} := \phi(j_{\alpha})$. Furthermore, we define

$$P^{\phi} := \prod_{a=1}^{r} Q_a \prod_{\alpha=1}^{s} P_{\alpha}. \tag{3.14}$$

Remark 3.3.9. *The signs of the unitaries in Equation* (3.13) *are a matter of convention. Our convention allows us to formulate the bulk-boundary correspondence in a canonical way. In Subsection 3.3.3, we will see that the resulting Van Daele groups are independent of these signs.*

Proposition 3.3.10. *If A is a real C^*-algebra, the assignment*

$$\mathcal{F}\mathcal{F}^{\phi}(A) \longrightarrow \mathcal{F}(P^{\phi}(A \otimes C\ell_{r,s+1})P^{\phi}); \quad J \longmapsto (J \otimes j_1)P^{\phi}$$

defines a bijective map. If A is complex, one has a corresponding bijection

$$\mathcal{F}\mathcal{F}^{\phi}(A) \longrightarrow \mathcal{F}(P^{\phi}(A \otimes \mathbb{C}\ell_{r,s+1})P^{\phi}); \quad J \longmapsto (J \otimes j_1)P^{\phi}.$$

The proof of Proposition 3.3.10 uses the following lemma.

Lemma 3.3.11. *Let C be a unital graded real C*-algebra and $x \in C$ be an ORHU. Consider the projection $q \in C \widehat{\otimes} C\ell_{0,1}$ defined by $2q - 1 = x \otimes j_1$. (Note that $x \otimes j_1$ is Hermitian due to the appearance of the graded tensor product.) Then*

$$\mathcal{F}(q(C \widehat{\otimes} C\ell_{0,1})q) = \{(y \otimes 1)q \mid y \in \mathcal{F}(C), xy + yx = 0\}.$$

Moreover, the ORHU $y \in C$ representing an ORHU $(y \otimes 1)q \in q(C \widehat{\otimes} C\ell_{0,1})q$ is unique.

Similarly, let $x \in C$ be an odd real skew-Hermitian unitary and $p \in C \widehat{\otimes} C\ell_{1,0}$ be the projection defined by $2p - 1 = x \otimes k_1$. Then

$$\mathcal{F}(p(C \widehat{\otimes} C\ell_{1,0})p) = \{(y \otimes 1)p \mid y \in \mathcal{F}(C), xy + yx = 0\},$$

where once again, the representatives y are uniquely determined.

The analogue statements are true if C is complex.

Proof. We only prove the first statement, the proof of the second being nearly identical. Moreover, the proof of the complex statement works in the same way as for the real case and therefore we focus on the real case.

The odd elements of $C \widehat{\otimes} C\ell_{0,1}$ are exactly those of the form

$$z = \tfrac{1}{2}(y \otimes 1) + \tfrac{1}{2}(y' \otimes j_1)$$

where y is odd and y' is even. Then z lies in the corner $q(C \widehat{\otimes} C\ell_{0,1})q$ if and only if

$$(x \otimes j_1)z = z = z(x \otimes j_1),$$

that is, if and only if $y' = -xy = yx$. If this is the case, then $z = (y \otimes 1)q$ and z is Hermitian if and only if so is y. Finally, in that case, z is unitary in the corner if and only if

$$\tfrac{1}{2}(1 \otimes 1) + \tfrac{1}{2}(x \otimes j_1) = q = z^2 = (y \otimes 1)^2 q = \tfrac{1}{2}(y^2 \otimes 1) + \tfrac{1}{2}(y^2 x \otimes j_1),$$

which in turn is equivalent to $y^2 = 1$. By construction, y is uniquely determined by z. □

With Lemma 3.3.11 at hand, we now prove Proposition 3.3.10.

Proof of Proposition 3.3.10. It is straightforward to check that the Q_a and P_α are indeed commuting projections. We will prove the statement by two separate inductions with respect to r and s. In case $r = s = 0$, we have $P^\phi = 1, \phi = 1$, and hence there is nothing to prove.

Assume that $r = 0$, $s \geqslant 1$, and that the statement has been proved for $(0, s-1)$ and any unital real C*-algebra A. Let $P^{0,s-1} := \prod_{\alpha=1}^{s-1} P_\alpha$. Since $1 \otimes j_{s+1}$ commutes with $P^{0,s-1}$, Equation (2.4) implies that

$$P^\phi(A \otimes C\ell_{0,s+1})P^\phi \cong P_s(A' \widehat{\otimes} C\ell_{0,1})P_s \tag{3.15}$$

where

$$A' := P^{0,s-1}(A \otimes C\ell_{0,s})P^{0,s-1}.$$

Here, the $C\ell_{0,1}$ factor on the right-hand side of Equation (3.15) is generated by j_{s+1}, and the $C\ell_{0,s}$ factor on the right-hand side of the latter is generated by $j_1, \ldots, j_s$.

Under the isomorphism in the former equation, $J_s \otimes j_1 j_{s+1}$ is mapped to $x \otimes j_{s+1}$, where $x := J_s \otimes j_1$. We may apply the first part of Lemma 3.3.11 with $C = A'$ to the right-hand side of the former equation. We conclude that there is a bijection, defined by the equation $a = bP_s$, between the

$$a \in \mathcal{F}(P^\phi(A \otimes C\ell_{0,s+1})P^\phi),$$

and the $b \in \mathcal{F}(P^{0,s-1}(A \otimes C\ell_{0,s})P^{0,s-1})$ anti-commuting with $J_s \otimes j_1$. By the inductive assumption, the $b \in \mathcal{F}(P^{0,s-1}(A \otimes C\ell_{0,s})P^{0,s-1})$ are in bijection *via* $b = (J \otimes j_1)P^{0,s-1}$ with the real Hermitian unitaries $J \in A$ anti-commuting with $J_1, \dots, J_{s-1}$. The b anti-commuting with $J_s \otimes j_1$ are in bijection with the J anti-commuting with $J_1, \dots, J_s$.

This proves the statement for $(0, s)$. If now $r \geqslant 1$, then similarly

$$P^\phi(A \otimes C\ell_{r,s+1})P^\phi \cong Q_r(A'' \widehat{\otimes} C\ell_{1,0})Q_r$$

where

$$P^{r-1,s} := \prod_{a=1}^{r-1} Q_a \prod_{\alpha=1}^{s} P_\alpha \quad \text{and} \quad A'' := P^{r-1,s}(A \otimes C\ell_{r-1,s+1})P^{r-1,s}.$$

Here, the $C\ell_{1,0}$ factor on the right-hand side of the former equation is generated by k_r, whereas the $C\ell_{r-1,s+1}$ factor in the latter is generated by $k_1, \dots, k_{r-1}, j_1, \dots, j_{s+1}$. The assertion now follows by induction on r, applying the second part of Lemma 3.3.11 to $C = A''$ and $x = (-1)^s K_r \otimes j_1$. □

We have established the bijective map in the first row in Figure 3.1. In the next lemma, we will show that this map is a homeomorphism and therefore induces a well-defined bijection between the path-connected components in the second row of Figure 3.1.

Lemma 3.3.12. *Let $\mathcal{F}\mathcal{F}^\phi(A)$ and $\mathcal{F}(P^\phi(A \otimes C\ell_{r,s+1})P^\phi)$ be equipped with the topologies induced by the norm on A and $A \otimes C\ell_{r,s+1}$. Then the bijective map*

$$\mathcal{F}\mathcal{F}^\phi(A) \longrightarrow \mathcal{F}(P^\phi(A \otimes C\ell_{r,s+1})P^\phi);\ J \mapsto (J \otimes j_1)P^\phi$$

is a homeomorphism.

Proof. Inspecting the proof of Proposition 3.3.10, it is sufficient to see that the projection onto the first factor of $C\widehat{\otimes}C\ell_{0,1} = C \oplus Cj_1$ is continuous. The same holds for $C\widehat{\otimes}C\ell_{1,0} = C \oplus Ck_1$. □

Thus the isomorphism of Proposition 3.3.10 induces a bijection between the corresponding sets of homotopy equivalence classes. Even more, it defines an isomorphism of resulting Abelian inductive limits and subsequently of Abelian groups as explained in the following theorem.

Theorem 3.3.13. *If A is real, the homeomorphism $\mathcal{F}\mathcal{F}^\phi(A) \longrightarrow \mathcal{F}(P^\phi(A \otimes C\ell_{r,s+1})P^\phi)$; $J \mapsto (J \otimes j_1)P^\phi$ induces an isomorphism of Abelian monoids*

$$FF^\phi_\infty(A) \longrightarrow DKR_{e_0}(P^\phi(A \otimes C\ell_{r,s+1})P^\phi),$$

with reference ORHU $e_0 := (J_{\mathrm{ref}} \otimes j_1)P^\phi$. If A is complex, this induces an isomorphism

$$FF^\phi_\infty(A) \longrightarrow DK_{e_0}(P^\phi(A \otimes \mathbb{C}\ell_{r,s+1})P^\phi).$$

These isomorphisms define isomorphisms between the associated Grothendieck groups.

Proof. Note that $M_n(P^\phi(A \otimes \mathbb{C}\ell_{r,s+1})P^\phi) = (1_n \otimes P^\phi)(M_n(A) \otimes C\ell_{r,s+1})(1_n \otimes P^\phi)$. For all $J \in \mathcal{FF}_n^\phi(A)$ and $J' \in \mathcal{FF}_m^\phi(A)$ we have

$$((J \oplus J') \otimes j_1)(1_{n+m} \otimes P^\phi) = (J \otimes j_1)(1_n \otimes P^\phi) \oplus (J' \otimes j_1)(1_m \otimes P^\phi),$$

i.e. the isomorphism commutes with the direct sum. In particular,

$$((J \oplus J_{\text{ref}}) \otimes j_1)(1_{n+1} \otimes P^\phi) = (J \otimes j_1)(1_n \otimes P^\phi) \oplus e_0.$$

Therefore, the homeomorphism of Lemma 3.3.12 defines an isomorphism between the inductive systems $(FF_n^\phi(A))_{n\geqslant 1}$ and $(F_n(P^\phi(A \otimes \mathbb{C}\ell_{r,s+1})P^\phi))_{n\geqslant 1}$. Thus we obtain an isomorphism of the inductive limits

$$FF_\infty^\phi(A) \xrightarrow{\cong} DKR_{e_0}(P^\phi(A \otimes \mathbb{C}\ell_{r,s+1})P^\phi).$$

The claim about the Grothendieck group now follows directly by the naturality of the Grothendieck functor. □

We have thus established the isomorphism

$$\mathrm{Top}^\phi_{J_{\text{ref}}}(A) \longrightarrow DKR(P^\phi(A \otimes C\ell_{r,s+1})P^\phi),$$

which completes the diagram in Figure 3.1. We summarise this result and the explicit form of the isomorphism for later reference in the following corollary.

Corollary 3.3.14. *The group* $\mathrm{Top}^\phi_{J_{\text{ref}}}(A)$ *is isomorphic to the group* $DKR_e(P^\phi(A \otimes C\ell_{r,s+1})P^\phi)$*, as defined in Definition 3.2.3, with reference ORHU* $e = e_0 \oplus -e_0 = (J_{\text{ref}} \otimes j_1)P^\phi \oplus -(J_{\text{ref}} \otimes j_1)P^\phi$.

This isomorphism maps a class $[J] - [J_{\text{ref}}] \in \mathrm{Top}^\phi_{J_{\text{ref}}}(A)$ *onto* $[(J \otimes j_1)P^\phi] - [(J_{\text{ref}} \otimes j_1)P^\phi] \in DKR_e(P^\phi(A \otimes C\ell_{r,s+1})P^\phi)$.

Proof. This is an immediate consequence of Proposition 3.2.7. □

Our next goal is to identify the Van Daele group $DKR(P^\phi(A \otimes C\ell_{r,s+1})P^\phi)$ with the Van Daele group $DKR(A \otimes C\ell_{r,s+1})$. This is primarily a consequence of the following observation.

Lemma 3.3.15. *If* $\mathcal{FF}^\phi(A) \neq \emptyset$*, then the graded real* C^**-algebra* $A \otimes C\ell_{r,s+1}$ *is isomorphic to the graded real* C^**-algebra of* $2^{r+s} \times 2^{r+s}$ *matrices over* $P^\phi(A \otimes C\ell_{r,s+1})P^\phi$ *with entry-wise grading.*

Proof. There are many ways to define an isomorphism. Here we make a specific choice that will allow us to compute its effect on K-theory. We shall index the rows and columns of $2^{r+s} \times 2^{r+s}$ matrices by indices ε running over $\{\pm\}^{r+s}$. Set $Q_\alpha^+ := Q_\alpha$ and $Q_\alpha^- := 1 - Q_\alpha$, and similarly for P^a. This enables us to define

$$P_\varepsilon^\phi := Q_1^{\varepsilon_1} \cdots Q_r^{\varepsilon_r} P_1^{\varepsilon_{r+1}} \cdots P_s^{\varepsilon_{r+s}}.$$

The projections P_ε^ϕ satisfy

$$P_\varepsilon^\phi P_{\varepsilon'}^\phi = \delta_{\varepsilon\varepsilon'} P_\varepsilon^\phi, \qquad \sum_{\varepsilon \in \{\pm\}^{r+s}} P_\varepsilon^\phi = 1. \tag{3.16}$$

Now we make explicit use of the condition $\mathcal{FF}^\phi(A) \neq \emptyset$. Namely, this implies that the projections P^ϕ_ε are all unitarily equivalent, as can be seen from the following careful choice of unitaries implementing the equivalence. Observe first that

$$\left.\begin{aligned} \mathrm{Ad}(K_a J_{\mathrm{ref}} \otimes 1)(P^\phi) &= P^\phi_{\varepsilon-2e_a} \\ \mathrm{Ad}(J_\alpha J_{\mathrm{ref}} \otimes 1)(P^\phi) &= P^\phi_{\varepsilon-2e_{r+\alpha}} \end{aligned}\right\} \quad \forall a = 1,\dots,r, \alpha = 1,\dots,s.$$

Hence, setting

$$u_\varepsilon := \prod_{1\leqslant a\leqslant r, \varepsilon_a=-} (K_a J_{\mathrm{ref}} \otimes 1) \prod_{1\leqslant \alpha\leqslant s, \varepsilon_{r+\alpha}=-} (J_\alpha J_{\mathrm{ref}} \otimes 1),$$

we obtain

$$u_\varepsilon P^\phi = P^\phi_\varepsilon u_\varepsilon, \quad u_\varepsilon (J_{\mathrm{ref}} \otimes j_1) = \varepsilon (J_{\mathrm{ref}} \otimes j_1) u_\varepsilon.$$

Define a map

$$\Psi : A \otimes C\ell_{r,s+1} \longrightarrow M_{2^{r+s}}(\mathbb{C}) \otimes P^\phi (A \otimes C\ell_{r,s+1}) P^\phi$$

by

$$\Psi(x) := \sum_{\varepsilon,\varepsilon' \in \{\pm\}^{r+s}} E_{\varepsilon\varepsilon'} \otimes P^\phi u^*_\varepsilon x u_{\varepsilon'} P^\phi = \sum_{\varepsilon,\varepsilon' \in \{\pm\}^{r+s}} E_{\varepsilon\varepsilon'} \otimes u^*_\varepsilon P^\phi_\varepsilon x P^\phi_{\varepsilon'} u_{\varepsilon'}.$$

Here, $E_{\varepsilon\varepsilon'}$ are the standard matrix units. It is clear that Ψ commutes with $*$ and the conjugation. A computation using Equation (3.16) implies that Ψ is in fact a real $*$-morphism. Since the matrix units form a basis of $M_{2^{r+s}}(\mathbb{C})$, it is easy to see that Ψ is bijective. □

Remark 3.3.16. *The result of Lemma 3.3.15 is not surprising, as by [69, Ex. 3.6 & Prop. 3.28], the corner $P^\phi(A \otimes C\ell_{r,s+1})P^\phi$ is Morita equivalent to the closed two-sided ideal generated by P^ϕ. If $\mathcal{FF}^\phi(A) \neq \emptyset$, one can show that this ideal is already all of $A \otimes C\ell_{r,s+1}$. Since A is unital by assumption, the Brown–Green–Rieffel Theorem [69, Thm. 5.55] applies, and the Morita equivalence implies stable isomorphism.*

It follows by Lemma 3.3.15 and Proposition 3.2.6 that

$$\mathrm{Top}^\phi_{J_{\mathrm{ref}}}(A) = DKR(P^\phi(A \otimes C\ell_{r,s+1})P^\phi) = DKR(A \otimes C\ell_{r,s+1}) \tag{3.17}$$

in the real case, and

$$\mathrm{Top}^\phi_{J_{\mathrm{ref}}}(A) = DK(P^\phi(A \otimes \mathbb{C}\ell_{r,s+1})P^\phi) = DK(A \otimes \mathbb{C}\ell_{r,s+1}) \tag{3.18}$$

in the complex case. The next proposition about the explicit form of these isomorphisms will be important for the proof of the bulk-boundary correspondence in Section 5.3 and the computation of topological invariants in Chapter 6.

Proposition 3.3.17. *The class $[J] - [J_{\mathrm{ref}}] \in \mathrm{Top}^\phi_{J_{\mathrm{ref}}}(A)$ is under the isomorphisms of Proposition 3.2.6 and Lemma 3.3.15 mapped onto*

$$[(J\otimes j_1)P^\phi + (J_{\mathrm{ref}}\otimes j_1)(1-P^\phi)] - [J_{\mathrm{ref}}\otimes j_1] = \Big[((J \otimes j_1)P^\phi + (J_{\mathrm{ref}} \otimes j_1)(1 - P^\phi)) \oplus -(J_{\mathrm{ref}} \otimes j_1)\Big] \tag{3.19}$$

in $DKR(A \otimes C\ell_{r,s+1})$ resp. $DK(A \otimes \mathbb{C}\ell_{r,s+1})$ with reference ORHU $J_{\mathrm{ref}} \otimes j_1 \oplus -(J_{\mathrm{ref}} \otimes j_1)$.

Proof. We only prove the real case since the complex case is the same.

By Corollary 3.3.14, the isomorphism $\mathrm{Top}^{\phi}_{J_{\mathrm{ref}}}(A) \cong DKR(P^{\phi}(A \otimes C\ell_{r,s+1})P^{\phi})$ maps the class $[J] - [J_{\mathrm{ref}}] \in \mathrm{Top}^{\phi}_{J_{\mathrm{ref}}}(A)$ onto the class

$$[x] := \left[(J \otimes j_1)P^{\phi} \oplus -e_0\right] \in DKR(P^{\phi}(A \otimes C\ell_{r,s+1})P^{\phi}),$$

where the reference ORHU is $e = e_0 \oplus -e_0$ with $e_0 = (J_{\mathrm{ref}} \otimes j_1)P^{\phi}$.

Unless $r + s > 0$, there is nothing more to prove.

Set $n := 2^{r+s-1}$. We abbreviate $p := P^{\phi}$, and $C := A \otimes C\ell_{r,s+1}$. The map from Proposition 3.2.6 sends x to $x \oplus e_{n-1} \in \mathcal{F}_{2n}(pAp)$. This is the $2n \times 2n$ matrix

$$E_{++} \otimes (J \otimes j_1)p + \sum_{\varepsilon \neq (+,\dots,+)} \varepsilon E_{\varepsilon\varepsilon} \otimes e_0,$$

where E_{++} is the matrix unit corresponding to $\varepsilon = (+, \dots, +)$.

Let Ψ be the isomorphism from Lemma 3.3.15. As $J_{\mathrm{ref}} \otimes j_1$ commutes with $p = P^{\phi}$ and $\mathrm{Ad}(u_\varepsilon)(J_{\mathrm{ref}} \otimes j_1) = \varepsilon(J_{\mathrm{ref}} \otimes j_1)$, we find

$$\Psi(J_{\mathrm{ref}} \otimes j_1) = \sum_{\varepsilon,\varepsilon' \in \{\pm\}^{r+s}} \varepsilon' E_{\varepsilon\varepsilon'} \otimes (J_{\mathrm{ref}} \otimes j_1)u_\varepsilon^* P^{\phi}_{\varepsilon} P^{\phi}_{\varepsilon'} u_{\varepsilon'} = \sum_{\varepsilon \in \{\pm\}^{r+s}} \varepsilon E_{\varepsilon\varepsilon} \otimes (J_{\mathrm{ref}} \otimes j_1)P^{\phi}.$$

Thus we see that $\Psi(J_{\mathrm{ref}} \otimes j_1) = e_n$. A similar computation shows that

$$\Psi((J \otimes j_1)p + (J_{\mathrm{ref}} \otimes j_1)(1 - p)) = x \oplus e_{n-1}.$$

Therefore, $x \otimes e_{n-1}$ is identified with the desired element in $M_{2n}(C)$. □

3.3.3 The general picture

In this section, we will use the properties of Van Daele groups of Section 3.2 in order to derive a more systematic and generalised picture for the group $\mathrm{Top}^{\phi}_{J_{\mathrm{ref}}}(A)$.

Proposition 3.3.18. *Up to isomorphisms, the group* $\mathrm{Top}^{\phi}_{J_{\mathrm{ref}}}(A)$ *is independent of the choice of the reference element* J_{ref} *and the unital* $*$*-morphism* ϕ*, i.e. given another unital (real)* $*$*-morphism* $\phi' : C\ell_{r,s} \to A$ *with reference element* $J'_{\mathrm{ref}} \in \mathcal{FF}^{\phi'}(A)$*, there is an isomorphism* $\mathrm{Top}^{\phi}_{J_{\mathrm{ref}}}(A) \cong \mathrm{Top}^{\phi'}_{J'_{\mathrm{ref}}}(A)$.

Proof. If A is real, different choices ϕ and ϕ' with reference elements J_{ref} and J'_{ref}, respectively, lead to the groups $\mathrm{Top}^{\phi}_{J_{\mathrm{ref}}}(A) = DKR(P^{\phi}(A \otimes C\ell_{r,s+1})P^{\phi})$ and $\mathrm{Top}^{\phi'}_{J'_{\mathrm{ref}}}(A) = DKR(P^{\phi'}(A \otimes C\ell_{r,s+1})P^{\phi'})$. By Equation (3.17), it follows

$$\mathrm{Top}^{\phi'}_{J'_{\mathrm{ref}}}(A) = DKR(A \otimes C\ell_{r,s+1}) = \mathrm{Top}^{\phi}_{J_{\mathrm{ref}}}(A).$$

The complex case can be proved in the same way. □

Proposition 3.3.18 in particular implies that the definition of $\mathrm{Top}^{\phi}_{J_{\mathrm{ref}}}(A)$ is independent of the signs in Equations (3.13), as promised in Remark 3.3.9. Moreover, this proposition implies that the following notation is well-defined.

Definition 3.3.19 ($\mathrm{Top}^{r,s}(A)$)**.** *In view of Proposition 3.3.18, we introduce the notation*

$$\mathrm{Top}^{r,s}(A) := \mathrm{Top}^{\phi}_{J_{\mathrm{ref}}}(A).$$

We will use this notation whenever we do not wish to specify the morphism ϕ and the reference element J_{ref}.

In analogy to the stability of Van Daele groups, we obtain the following stability of the group $\mathrm{Top}^{r,s}(-)$.

Corollary 3.3.20. *The map $\mathcal{FF}^{\phi}(A) \longrightarrow \mathcal{FF}^{\phi_2}(M_2(A))$; $J \mapsto J \oplus J_{\mathrm{ref}}$ induces an isomorphism $\mathrm{Top}^{r,s}(A) \cong \mathrm{Top}^{r,s}(M_2(A))$.*

Proof. The assertion is an immediate consequence of the stability of Van Daele groups, *cf. Proposition 3.2.6 and the explicit form of the identification in Proposition 3.3.17.* □

Hitherto, the definition of $\mathrm{Top}^{\phi}_{J_{\mathrm{ref}}}(A)$, and hence the definition of $\mathrm{Top}^{r,s}(A)$, is based on the assumption $\mathcal{FF}^{\phi}(A) \neq \emptyset$. As promised at the beginning of this section, we will now generalise the definition $\mathrm{Top}^{r,s}(A)$ to the case $\mathcal{FF}^{\phi}(A) = \emptyset$. For that, we first construct the following group.

Definition 3.3.21. *Let A be a real C^*-algebra and let $\phi : C\ell_{r,s} \to A$ be a unital $*$-morphism. We consider the real unital $*$-morphism $\Phi : C\ell_{r,s} \to M_2(A)$ defined by*

$$\Phi(k_a) := \begin{pmatrix} K_a & 0 \\ 0 & -K_a \end{pmatrix}, \quad \Phi(j_\alpha) := \begin{pmatrix} J_\alpha & 0 \\ 0 & -J_\alpha \end{pmatrix} \quad \forall a = 1, \ldots, r, \alpha = 1, \ldots, s.$$

Then we have $\mathcal{FF}^{\Phi}(M_2(A)) \neq \emptyset$ since $\tilde{J}_{\mathrm{ref}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathcal{FF}^{\Phi}(M_2(A))$. Thus, the group $\mathrm{Top}^{\Phi}_{\tilde{J}_{\mathrm{ref}}}(M_2(A))$ is well-defined.

The construction for the complex case is the same.

All results for the group $\mathrm{Top}^{\phi}_{J_{\mathrm{ref}}}(A)$ hold analogously for $\mathrm{Top}^{\Phi}_{\tilde{J}_{\mathrm{ref}}}(M_2(A))$. In the following, we will show that this group indeed exhibits a generalisation for the group $\mathrm{Top}^{r,s}(A)$, i.e.

$$\mathrm{Top}^{r,s}(A) := \mathrm{Top}^{\Phi}_{\tilde{J}_{\mathrm{ref}}}(M_2(A))$$

is a well-defined generalisation. For that, we have to show that if there is an element $J_{\mathrm{ref}} \in \mathcal{FF}^{\phi}(A)$, then the group $\mathrm{Top}^{\Phi}_{\tilde{J}_{\mathrm{ref}}}(M_2(A))$ is isomorphic to $\mathrm{Top}^{\phi}_{J_{\mathrm{ref}}}(A)$. But this follows almost immediately by the above results.

Lemma 3.3.22. *Let A be a real C^*-algebra with reference element $J_{\mathrm{ref}} \in \mathcal{FF}^{\phi}(M_2(A))$. Then $\mathrm{Top}^{\Phi}_{\tilde{J}_{\mathrm{ref}}}(M_2(A)) \cong \mathrm{Top}^{\phi}_{J_{\mathrm{ref}}}(A)$.*

Proof. By Proposition 3.3.18, there is an isomorphism $\mathrm{Top}^{\Phi}_{\tilde{J}_{\mathrm{ref}}}(M_2(A)) \cong \mathrm{Top}^{\phi_2}_{(J_{\mathrm{ref}})_2}(M_2(A))$. Then the assertion follows by Corollary 3.3.20. □

Thus, both constructions lead, up to isomorphisms, to the same group $\mathrm{Top}^{r,s}(A)$.

Altogether, we have constructed a group $\mathrm{Top}^{r,s}(A)$ which solely depends, up to isomorphisms, on the unital C*-algebra A and the natural numbers r, s. By Theorem 3.3.13 and Lemma 3.3.15, we obtain the equations

$$\mathrm{Top}^{r,s}(A) \cong DKR^{-(s-r+2)}(A) \quad \text{or} \quad \mathrm{Top}^{r,s}(A) \cong DK^{-(s-r+2)}(A)$$

for a real or complex C*-algebra A, respectively. Thus, we have indeed constructed a new picture of real and complex K-theory for unital C*-algebras. In particular, we see here that the group $\mathrm{Top}^{r,s}(A)$ actually only depends on the difference $s - r$.

We close this chapter with the generalisation to non-unital C*-algebras.

Lemma 3.3.23. *Let A be a C^*-algebra. Let A^+ be its unitisation, cf. Definition 3.2.8, and let $\varphi : A^+ \to \mathbb{C}$ be the canonical $*$-morphism. Then we define*

$$\mathrm{Top}^{r,s}(A) := \ker(\varphi_*) \subseteq \mathrm{Top}^{r,s}(A^+),$$

where $\varphi_* : \mathrm{Top}^{r,s}(A^+) \to \mathrm{Top}^{r,s}(\mathbb{C})$ denotes the induced morphism given by

$$\mathrm{Top}^{\phi}_{J_{\mathrm{ref}}}(A^+) \ni [(J, \lambda)] - [(J_{\mathrm{ref}}, \lambda_{\mathrm{ref}})] \longmapsto [\lambda] - [\lambda_{\mathrm{ref}}] \in \mathrm{Top}^{\varphi(\phi)}_{\varphi(J_{\mathrm{ref}})}(\mathbb{C}).$$

This definition generalises the definition of $\mathrm{Top}^{r,s}(A)$ *for unital C^*-algebras.*

Proof. We only have to show that $\ker \varphi_* = \mathrm{Top}^{r,s}(A)$ for a unital C*-algebra A. The map

$$A \oplus \mathbb{C} \longrightarrow A^+; \quad a \oplus \lambda \mapsto (a - \lambda, \lambda)$$

defines a $*$-isomorphism for a unital C*-algebra A. Thus, in this case we have $\mathrm{Top}^{r,s}(A^+) = \mathrm{Top}^{r,s}(A \oplus \mathbb{C})$. It is straightforward to check that $\mathrm{Top}^{r,s}(A \oplus \mathbb{C}) = \mathrm{Top}^{r,s}(A) \times \mathrm{Top}^{r,s}(\mathbb{C})$. On the right hand side, the morphism φ_* is simply given by restriction onto the second factor. This implies $\ker \varphi_* = \mathrm{Top}^{r,s}(A)$. □

Chapter Four

Disordered Topological Insulators and Superconductors

The notions of topological insulators and topological superconductors were established over the last two decades in order to describe a new class of solid state systems with certain topological features that have been discovered experimentally and theoretically during that time. These systems can be roughly described as solid state systems with an energy gap around the chemical potential in the bulk and a gapless energy spectrum around the chemical potential in the vicinity of the boundary.

Retrospectively, the first topological insulator was discovered with the quantum Hall effect (QHE) in 1980 [89]. The QHE was experimentally realised in a two-dimensional electron-gas at the surface of semiconductors, e.g. silicon or gallium. A strong magnetic field localises the electrons in the bulk of this two-dimensional sample, whereas the electrons at the one-dimensional boundary are delocalised. This leads to an insulating bulk and a conducting boundary. At that time, the extraordinary observation was that the conducting property at the boundary is stable w.r.t. disorder and its quantised phases behave fundamentally different to thermodynamic phases. This was the first example of a 'topological phase'. Nowadays, there are several experimental realisations of the quantum Hall effect with other samples. In the modern formulation, these are called topological insulators in symmetry class A.

For along time, the quantum Hall effect was the only known realisation of a topological insulator, theoretically and experimentally. However, in 2005, Kane and Mele, [36, 37] predicted a new type of topological insulators, the so-called *quantum spin Hall effect* (QSHE). The most important difference to the QHE is that the states in the QSHE are time-reversal invariant. Thus, the QSHE gives rise to topological insulators in symmetry class $A\mathrm{II}$. Moreover, in contrast to the QHE, the boundary of the QSHE exhibits a quantised spin current and meanwhile there is no charge current. The first experimental observations of the QSHE were made in mercury telluride quantum wells [14, 52] in 2007 and in bismuth antimony alloys [27, 33] in 2008.

It should be noted that hitherto, the notion of a topological insulator did not exist. The QSHE made apparent that the QHE is just a certain type of a larger class of materials showing topological features. This larger class was then called topological insulators.

Throughout the following years, the idea to consider systems with symmetries led to the discovery of several other realisations of topological insulators. At that time, it was also realised that these topological phenomena are not restricted to systems with conserved particle number, but could also occur in superconductors. This lead to the notion of *topological superconductors* [76]. In fact, much of the excitement about topological quantum matter stems from the technological promise of robust boundary modes (so-called Majorana modes) at the ends of superconducting wires. In modern terms, this corresponds to a topological superconductor in symmetry class D. Another example of topological superconductors in class D are two-dimensional substructures in strontium ruthenate (SR_2RuO_4) [58].

The variety of different models for topological insulators and superconductors was then put in a systematic order by Kitaev's famous work [50]. Kitaev was the first who used the Fourier–Bloch theory to identify the topological phases of topological insulators and superconductors with classes in the topological K-theory of the torus. There, the ground states of gapped systems are interpreted as vector bundles over the d-dimensional torus. He presented a systematic approach to the classification of the bulk phase of topological insulators and superconductors without disorder or particle-particle interactions in arbitrary dimensions and all symmetry classes of the Tenfold Way. The relatively abstract approach by Kitaev was in the following years formulated in physically more concrete ways. Kennedy–Zirnbauer [49] gave a rigorous translation of translationally invariant free-fermion ground states into vector bundles over the torus for all symmetry classes of the Tenfold Way. Their results are fundamental for the concepts of this chapter.

In contrast to the works mentioned above, we are interested in *disordered* topological insulators and superconductors where the translational invariance is inherently broken. Therefore, Kitaev's approach is not applicable for our considerations. A rigorous mathematical approach for such disordered systems was already established in 1986 by Bellissard [10] in the context of the disordered quantum Hall effect. We will refer to Bellissard's approach as *homogeneous disorder* for the following reason. His approach is based upon the idea that, although disorder breaks the translational invariance on microscopic scales, the average over all disorder configurations should result in translationally invariant observables.

Based on the idea of homogeneous disorder, we will establish a framework for disordered topological insulators and superconductors in the following way.

We begin with an introduction to the description of ground states of free fermions without particle number conservation. These can be described via operators, so called *quasi-particle vacua*, over the so-called *Nambu space*. The precise formulation of this picture is fundamental in order to get the correct classification scheme for topological insulators and superconductors. The implementation of symmetries in this Nambu space picture is based on the work by Kennedy–Zirnbauer [49]. There the symmetries are encoded in so-called *pseudo-symmetries* leading to a systematic picture for the different symmetry classes in the Tenfold Way.

Once we have established this Nambu space picture, we will use it to construct a framework for physical observables of disordered topological insulators and superconductors. We will begin with the construction for observables in the bulk, i.e. in the absence of boundaries. Observables of solids at zero temperature are in our context described by the so-called *tight-binding approximation*. As observed by Kubota [53], operators in the tight-binding approximation can be identified with elements in the uniform Roe C*-algebra [72, Ch. 4].

We then add homogeneous disorder as well as homogeneous magnetic fields to the picture. We will show that the resulting set of physically reasonable bulk observables generates a C*-algebra, which we call the *algebra of bulk observables*. In the following, we will then prove that this C*-algebra is $*$-isomorphic to the crossed product C*-algebra that is usually used in the literature to describe observables with homogeneous disorder [9, 10, 13, 45, 66, 78].

Thereafter, we perform the analogous constructions in the presence of a boundary. We will consider an infinitely long straight boundary perpendicular to one translational direction of the underlying Bravais lattice. The observables over this system with boundary form a C*-algebra which we call the *algebra of half-space observables*. The final result of this section will be that the algebras of bulk and half-space observables can be arranged into a short exact sequence (SES), which we call the *bulk-boundary short exact sequence*. This SES will be essential for the bulk-boundary correspondence in Chapter 5.

Most of the material in this chapter was published in the preprint *Bulk-boundary correspondence for disordered free-fermion topological phases* by A. Alldridge, C. Max and M. R. Zirnbauer, 2019, [1]. We extend the results of the preprint by including non-trivial homogeneous magnetic fields in the complex symmetry classes.

4.1 Symmetric fermionic quasi-particle vacua

Here, we establish the description of free-fermion many-particle systems with symmetries. This section is based on [1, 49].

The assumption that our systems are non-interacting means that there is an *effective description* in terms of non-interacting fermions. In a more technical language, this means that we can describe the systems in a so-called Hartree–Fock–Bogoliubov mean-field approximation. The principle idea behind this approximation is that the system is described in terms of effective fermions that govern the physical properties of the system and exhibit no particle-particle interaction. Since these fermions don't have to resemble the real physical fermions (e.g. electrons) in the system, they are called *quasi-particles*.

Topological insulators exhibit particle number conservation and can therefore be described in a single-particle picture in the non-interacting case. Since we consider charged particles, particle number conservation is equivalent to charge conservation. In contrast to that, superconductivity is a true many-body phenomenon where the broken charge conservation is fundamental. Therefore, we have to describe superconductors in a many-particle picture. Although these two systems are physically fundamentally different, they can both be described in the Nambu space picture which we will describe in the following.

4.1.1 The Nambu space of fields

The starting point is a complex Hilbert space $\mathcal{V}$ representing the single-particle states of the quasi-particles in a given solid state system. A fundamental feature of these systems is that there is a *chemical potential* μ, which is defined as the energy level at which the insertion or removal of quasi-particles costs no work. This defines a splitting of the single-particle Hilbert space as $\mathcal{V} = \mathcal{V}_+ \oplus \mathcal{V}_-$, where $\mathcal{V}_+$ ($\mathcal{V}_-$) consists of single-particle states with energies above (below) μ. In particular, this means that $\mathcal{V}_\pm$ are complex Hilbert spaces such that the inner products satisfy

$$\langle v_+ + v_-, v'_+ + v'_- \rangle_{\mathcal{V}} = \langle v_+, v'_+ \rangle_{\mathcal{V}_+} + \langle v_-, v'_- \rangle_{\mathcal{V}_-} \quad \forall v_+, v'_+ \in \mathcal{V}_+, v_-, v'_- \in \mathcal{V}_-.$$

The states in $\mathcal{V}_+$ are called *conduction states*, whereas those in $\mathcal{V}_-$ are called *valence states*. Without loss of generality, we can assume that $\mu = 0$.

The ground state of the system is given by the state where all states in $\mathcal{V}_-$ are occupied and all states in $\mathcal{V}_+$ are unoccupied. Deviations from this ground state are constructed by removing states from $\mathcal{V}_-$ or adding states in $\mathcal{V}_+$. We use this idea of describing the states relative to the ground states for the following descriptions of many-particle states. Namely, we now consider the corresponding *Fock space* for $\mathcal{V}$, which is defined as the Hilbert space completion of

$$\Lambda(\mathcal{V}_+ \oplus \mathcal{V}_-^*) = \bigoplus_{n\geqslant 0} \Lambda^n(\mathcal{V}_+ \oplus \mathcal{V}_-^*) \cong \bigoplus_{p,q\geqslant 0} \Lambda^p(\mathcal{V}_+) \otimes \Lambda^q(\mathcal{V}_-^*),$$

where $\Lambda(-)$ denotes the exterior algebra as explained in Proposition 2.2.4.

Remark 4.1.1. *Note that the bare vacuum, i.e. the state without any quasi-particles, would be a poor starting point for the description of states in solid state physics.*

The space $\Lambda^0(\mathcal{V}_+ \oplus \mathcal{V}_-^*) \cong \mathbb{C}$ represents the ground state. The space $\Lambda^p(\mathcal{V}_+) \oplus \Lambda^q(\mathcal{V}_-^*)$ contains the states with p particles in $\mathcal{V}_+$ and q holes in $\mathcal{V}_-$ relative to the ground state. Accordingly, the space $\Lambda^n(\mathcal{V}_+ \oplus \mathcal{V}_-^*) = \bigoplus_{k=0}^n \Lambda^{n-k}(\mathcal{V}_+) \otimes \Lambda^k(\mathcal{V}_-^*)$ for $n \geqslant 1$ denotes the subspace of all many-particle states that can be reached from the ground state by adding $n-k$ states in $\mathcal{V}_+$ and removing k states in $\mathcal{V}_-$ for any $0 \leqslant k \leqslant n$. Removing valence states corresponds on the Hilbert space level to dual elements of $\mathcal{V}_-$. Therefore, the direct sum $\mathcal{V}_+ \oplus \mathcal{V}_-^*$, where $\mathcal{V}_-^*$ denotes the dual space of $\mathcal{V}_-$, is the correct space to consider. If the quasi-particles are charged, as it is the case in our considerations, then the holes have the opposite charge and therefore the number $p-q$ measures the *total charge* relative to the ground state.

The insertion of a quasi-particle in state $v_+ \in \mathcal{V}_+$ or a hole in state $\varphi_- \in \mathcal{V}_-^*$ corresponds to the following operators:

$$\begin{aligned} \varepsilon_{v_+} &: \Lambda^p(\mathcal{V}_+) \to \Lambda^{p+1}(\mathcal{V}_+); \quad v_1 \wedge \ldots \wedge v_p \mapsto v_+ \wedge v_1 \wedge \ldots \wedge v_p, \\ \varepsilon_{\varphi_-} &: \Lambda^q(\mathcal{V}_-^*) \to \Lambda^{q+1}(\mathcal{V}_-^*); \quad \varphi_1 \wedge \ldots \wedge \varphi_q \mapsto \varphi_- \wedge \varphi_1 \wedge \ldots \wedge \varphi_q. \end{aligned}$$

The removal of quasi-particles and holes corresponds to the operators

$$\tilde{\iota}_{\varphi_+} : \Lambda^p(\mathcal{V}_+) \to \Lambda^{p-1}(\mathcal{V}_+);\ v_1 \wedge \ldots \wedge v_p \mapsto \sum_{i=1}^{p} (-1)^{i-1} \varphi_+(v_i) v_1 \wedge \ldots \wedge v_{i-1} \wedge v_{i+1} \wedge \ldots \wedge v_p,$$

$$\tilde{\iota}_{v_-} : \Lambda^q(\mathcal{V}_-^*) \to \Lambda^{q-1}(\mathcal{V}_-^*),\ \varphi_1 \wedge \ldots \wedge \varphi_q \mapsto \sum_{i=1}^{q} (-1)^{i-1} \varphi_i(v_-) \varphi_1 \wedge \ldots \wedge \varphi_{i-1} \wedge \varphi_{i+1} \wedge \ldots \wedge \varphi_q,$$

for given $\varphi_+ \in \mathcal{V}_+^*$ and $v_- \in \mathcal{V}_- \subseteq (\mathcal{V}_-^*)^*$. Note that these operators are linear in $\varphi_\pm$ and $v_\pm$.

Remark 4.1.2. *The operators $\varepsilon_{v_+}, \tilde{\iota}_{v_-}$ raise the charge by one unit, whereas the operators $\varepsilon_{\varphi_-}, \tilde{\iota}_{\varphi_+}$ decrease the charge by one unit. Subsequently, the concatenations $\varepsilon_{v_+}\varepsilon_{\varphi_-}$, $\varepsilon_{v_+}\tilde{\iota}_{\varphi_+}$, $\tilde{\iota}_{v_-}\varepsilon_{\varphi_-}$ and $\tilde{\iota}_{v_-}\tilde{\iota}_{\varphi_+}$ are charge preserving.*

Definition 4.1.3 (Field operators)**.** *A* field operator ψ *is defined as an element of* $\mathcal{L}(\Lambda(\mathcal{V}_+ \oplus \mathcal{V}_-^*))$ *which is induced by a linear combination of these linear operators, i.e.*

$$\psi = \varepsilon_{v_+} \otimes 1 + 1 \otimes \varepsilon_{\varphi_-} + \tilde{\iota}_{\varphi_+} \otimes 1 + 1 \otimes \tilde{\iota}_{v_-} \in \mathcal{L}\left(\bigoplus_{p,q \geqslant 0} \Lambda^p(\mathcal{V}_+) \otimes \Lambda^q(\mathcal{V}_-^*) \right)$$

for some $v_\pm \in \mathcal{V}_\pm$ *and* $\varphi_\pm \in \mathcal{V}_\pm^*$*. Let* $\mathcal{W}$ *be the complex vector space of all such field operators.*

Any two field operators $\psi = \varepsilon_{v_+} \otimes 1 + 1 \otimes \varepsilon_{\varphi_-} + \tilde{\iota}_{\varphi_+} \otimes 1 + 1 \otimes \tilde{\iota}_{v_-}, \psi' = \varepsilon_{v'_+} \otimes 1 + 1 \otimes \varepsilon_{\varphi'_-} + \tilde{\iota}_{\varphi'_+} \otimes 1 + 1 \otimes \tilde{\iota}_{v'_-} \in \mathcal{W}$ satisfy the *canonical anti-commutation relations* (CAR)

$$\{\psi, \psi'\} := \psi\psi' + \psi'\psi = \varphi_+(v'_+) + \varphi_-(v'_-) + \varphi'_+(v_+) + \varphi'_-(v_-). \tag{4.1}$$

This induces a canonical bilinear form on $\mathcal{W}$, the so-called *CAR form*

$$\{\cdot, \cdot\} : \mathcal{W} \otimes \mathcal{W} \to \mathbb{C}; \quad \{\psi, \psi'\} = \varphi_+(v'_+) + \varphi_-(v'_-) + \varphi'_+(v_+) + \varphi'_-(v_-). \tag{4.2}$$

We can use the Hilbert space structures of $\mathcal{V}_\pm$ to introduce the anti-linear Fréchet-Riesz isomorphisms $\mathfrak{h} : \mathcal{V}_\pm \to \mathcal{V}_\pm^*$ defined by

$$\mathfrak{h}(x) := \langle x, \cdot \rangle_{\mathcal{V}_\pm} \quad \forall x \in \mathcal{V}_\pm.$$

These Fréchet-Riesz isomorphisms define an anti-linear involution $\gamma : \mathcal{W} \to \mathcal{W}$ by

$$\gamma(\varepsilon_{v_+} \otimes 1 + 1 \otimes \varepsilon_{\varphi_-} + \tilde{\iota}_{\varphi_+} \otimes 1 + 1 \otimes \tilde{\iota}_{v_-}) := \tilde{\iota}_{\mathfrak{h}v_+} \otimes 1 + 1 \otimes \tilde{\iota}_{\mathfrak{h}^{-1}\varphi_-} + \varepsilon_{\mathfrak{h}^{-1}\varphi_+} \otimes 1 + 1 \otimes \varepsilon_{\mathfrak{h}v_-} \tag{4.3}$$

for all $v_\pm \in \mathcal{V}_\pm, \varphi_\pm \in \mathcal{V}_\pm^*$. This defines a real structure on $\mathcal{W}$. It can be easily checked that γ is symmetric with respect to the CAR form. The complex vector space $\mathcal{W}$, equipped with the real structure γ and the CAR form defines the so-called *Nambu space of fields*.

For later reference, we summarise the above observations in the following definition.

Definition 4.1.4 (Nambu space of fields)**.** *The* Nambu space of fields *is defined as the real vector space* $(\mathcal{W}, \gamma)$*, equipped with the bilinear CAR form of Equation* (4.2)*, where* $\mathcal{W}$ *denotes the complex vector space of field operators and* γ *is the real structure from Equation* (4.3)*.*

The Nambu space of fields is isomorphic to the real vector space $\mathcal{V}_+ \oplus \mathcal{V}_-^* \oplus \mathcal{V}_+^* \oplus \mathcal{V}_-$ via the isomorphism defined by

$$\varepsilon_{v_+} \otimes 1 + 1 \otimes \varepsilon_{\varphi_-} + \tilde{\iota}_{\varphi_+} \otimes 1 + 1 \otimes \tilde{\iota}_{v_-} \mapsto v_+ + \varphi_- + \varphi_+ + v_-. \tag{4.4}$$

This is indeed a complex linear isomorphism because the operators ε and $\tilde{\iota}$ are both linear in the index. By reordering of the summands, we thus get

$$\mathcal{W} \cong \mathcal{V} \oplus \mathcal{V}^*, \qquad \text{where } \mathcal{V} = \mathcal{V}_+ \oplus \mathcal{V}_-.$$

Lemma 4.1.5. *The isomorphism* $\mathcal{W} \cong \mathcal{V} \oplus \mathcal{V}^*$ *of Equation* (4.4) *maps the CAR form onto the bilinear form*

$$\{v + \varphi, v' + \varphi'\} = \varphi(v') + \varphi'(v) \quad \forall v, v' \in \mathcal{V}, \varphi, \varphi' \in \mathcal{V}^*. \tag{4.5}$$

The real structure γ *is mapped onto*

$$\gamma(v + \varphi) = \mathfrak{h}(v) + \mathfrak{h}^{-1}(\varphi) \quad \forall v \in \mathcal{V}, \varphi \in \mathcal{V}^*. \tag{4.6}$$

Proof. Let $v_\pm, v'_\pm \in \mathcal{V}_\pm, \varphi_\pm, \varphi'_\pm \in \mathcal{V}^*_\pm$ be arbitrary. For the sake of readability, let $F : \mathcal{W} \to \mathcal{V} \oplus \mathcal{V}^*$ denote the isomorphism. Then

$$
\begin{aligned}
&\left\{v_+ + v_- + \varphi_+ + \varphi_-, v'_+ + v'_- + \varphi'_+ + \varphi'_-\right\} \\
=&\left\{F^{-1}(v_+ + v_- + \varphi_+ + \varphi_-), F^{-1}(v'_+ + v'_- + \varphi'_+ + \varphi'_-)\right\} \\
=&\left\{\varepsilon_{v_+} \otimes 1 + 1 \otimes \varepsilon_{\varphi_-} + \tilde{\iota}_{\varphi_+} \otimes 1 + 1 \otimes \tilde{\iota}_{v_-}, \varepsilon_{v'_+} \otimes 1 + 1 \otimes \varepsilon_{\varphi'_-} + \tilde{\iota}_{\varphi'_+} \otimes 1 + 1 \otimes \tilde{\iota}_{v'_-}\right\} \\
=&\varphi_+(v'_+) + \varphi_-(v'_-) + \varphi'_+(v_+) + \varphi'_-(v_-)
\end{aligned}
$$

The form of γ follows immediately. □

We have constructed the Nambu space $\mathcal{W}$ equipped with the bilinear symmetric CAR form $\{\cdot,\cdot\}$ and the real structure γ. With these information, we can now define the following canonical inner product.

Definition 4.1.6 (Inner product on Nambu space)**.** *The inner product on $\mathcal{W}$ is defined by*

$$\langle w, w' \rangle := \{\gamma w, w'\}.$$

By construction, the adjoint of an operator $T \in \mathcal{L}(\mathcal{W})$ is given by $T^* = \overline{T}^{\mathsf{T}}$.

Lemma 4.1.7. *The inner product of Definition 4.1.6 defines the structure of a real Hilbert space on $\mathcal{W}$. Under the isomorphism $\mathcal{W} = \mathcal{V} \oplus \mathcal{V}^*$, this inner product becomes of the form*

$$\langle v + \varphi, v' + \varphi' \rangle = \langle v, v' \rangle_{\mathcal{V}} + \langle \varphi, \varphi' \rangle_{\mathcal{V}^*}.$$

Proof. The form of the inner product over $\mathcal{V} \oplus \mathcal{V}^*$ follows immediately by the explicit form of the CAR form of Equation (4.5) and the real structure γ of Equation (4.6).

This defines a Hilbert space structure on $\mathcal{W}$, because the inner products on $\mathcal{V}$ and $\mathcal{V}^*$ are positive definite and hence the inner product on $\mathcal{W}$ is positive definite, too. □

Remark 4.1.8. *The positive definiteness of the inner product relies crucially on the fact that we consider fermions. In fact, to perform the same constructions for bosons, we would have to replace the symmetric bilinear CAR form $\{\cdot,\cdot\}$ by the anti-symmetric bilinear form*

$$[v + \varphi, v' + \varphi'] = \varphi(v') - \varphi'(v) \quad \forall v + \varphi, v' + \varphi' \in \mathcal{V} \oplus \mathcal{V}^*.$$

The resulting inner product would then no longer be positive definite.

By the observations above, we can conclude that in general all physical many-body operators, including particle-particle interactions, are elements of the Clifford algebra $C\ell(\mathcal{W}, q)$, where q is the quadratic form induced by the CAR form. However, we assume that there is a quasi-particle picture of non-interacting fermions for our systems, which is equivalent to the existence of a description of the system via a *free-fermion Hamiltonian*.

Remark 4.1.9. *In the literature, one often finds the claim that the generated Clifford algebra is $C\ell(\mathcal{V} \oplus \mathcal{V}, q)$. This point of view requires to fix a real or quaternionic structure on $\mathcal{V}$ which induces a linear isomorphism $\mathcal{V} \cong \mathcal{V}^*$. But such a real or quaternionic structure is unphysical if chosen arbitrarily. This isomorphism becomes physically reasonable if we consider systems where such a structure is given, for example in systems with time-reversal symmetry,* cf. *Subsection 4.1.2.*

All free-fermion Hamiltonians are by definition elements of the complex vector space of non-interacting operators, which is defined as follows.

Definition 4.1.10 (Non-interacting operators)**.** *An element in $C\ell(\mathcal{W}, q)$ is called* non-interacting *if it is an element of the complex vector space generated by the elements $\psi\psi' - \psi'\psi \in C\ell(\mathcal{W}, q)$ for $\psi, \psi' \in \mathcal{W}$.*

One of the most important points of the Nambu space picture is that Equation (4.1) implies

$$\tfrac{1}{2}[\psi\psi' - \psi'\psi, w] = \{\psi', w\}\psi - \{\psi, w\}\psi' \quad \forall \psi, \psi', w \in \mathcal{W},$$

where $[\cdot,\cdot]$ denotes the commutator and $\{\cdot,\cdot\}$ the bilinear CAR form. This allows us to identify the non-interacting operators with elements of $\mathcal{L}(\mathcal{W})$ by

$$[\psi, \psi'] \mapsto T_{[\psi,\psi']} \quad \text{where} \quad T_{[\psi,\psi']}(w) := [[\psi, \psi'], w]. \tag{4.7}$$

This morphism gives rise to an isomorphism to the Lie algebra $\mathfrak{so}(\mathcal{W})$.

Proposition 4.1.11. *The non-interacting operators generate a real Lie algebra in $\mathcal{L}(\mathcal{W})$ w.r.t. the commutator which is isomorphic to $\mathfrak{so}(\mathcal{W})$.*

Proof. Let $\psi, \psi' \in \mathcal{W}$ be arbitrary. It is easy to see that the operators $T_{[\psi,\psi']}$ generate a complex vector space which is closed under the commutator $[\cdot,\cdot]$ and the real structure Ad_γ, thus defining a real Lie algebra.

It remains to show that this Lie algebra is isomorphic to $\mathfrak{so}(\mathcal{W})$. As a set, the Lie algebra $\mathfrak{so}(\mathcal{W})$ consists of all skew-symmetric operators in $\mathcal{L}(\mathcal{W})$.

Let $\phi := \frac{1}{2}[\psi, \psi']$. Then

$$\{T_\phi w, w'\} = \{\psi', w\}\{\psi, w'\} - \{\psi, w\}\{\psi', w'\} = -\{w, T_\phi w'\}.$$

Thus T_ϕ is skew-symmetric w.r.t. the CAR form. Since the Lie algebra structure is already given by the commutator, it follows that the Lie algebra is a subalgebra of $\mathfrak{so}(\mathcal{W})$.

In order to show equality, let $\{e_i\}$ be an orthonormal Hilbert basis of $\mathcal{W}$ and let

$$O = \sum_{i,j} O_{i,j} e_i \mathfrak{h}(e_j) \in \mathfrak{so}(\mathcal{W})$$

be arbitrary. Since O is skew-symmetric, we have $O_{i,j} = -O_{j,i}$ for all i, j. Define $T := \frac{1}{2}\sum_{i,j} T_{O_{i,j}\gamma(e_i)\gamma(e_j)}$. By

$$T_{O_{i,j}\gamma(e_i)\gamma(e_j)} e_k = O_{i,j}\langle e_j, e_k\rangle e_i - O_{i,j}\langle e_i, e_k\rangle e_j = O_{i,j}\delta_{j,k} e_i - O_{i,j}\delta_{i,k} e_j \quad \forall i, j, k,$$

it follows

$$T e_k = \frac{1}{2}\sum_i O_{i,j}\delta_{j,k} e_i - O_{j,i}\delta_{j,k} e_i = \sum_i O_{i,j} e_i = O e_k$$

and thus $T = O$.

Finally, note that the elements in $\mathcal{W}_{\mathbb{R}} = \{w \in \mathcal{W} \mid \gamma(w) = w\}$ induce real operators in $\mathfrak{so}(\mathcal{W})$ since $\gamma T_{[\psi,\psi']}\gamma = T_{[\gamma\psi,\gamma\psi']}$ for all $\psi, \psi' \in \mathcal{W}$. Thus, the real operators in $\mathfrak{so}(\mathcal{W})$ can be identified with $\mathfrak{so}(\mathcal{W}_{\mathbb{R}})$. □

Remark 4.1.12. *In physics, the field operators are often expressed in terms of creation and annihilation operators. For $v_\pm \in \mathcal{V}_\pm$, these are given by the creation operator*

$$c^\dagger_{v_+ + v_-} = \varepsilon_{v_+} \otimes 1 + 1 \otimes \tilde{\iota}_{v_-}$$

and the annihilation operator

$$c_{v_+ + v_-} = \tilde{\iota}_{\mathfrak{h}(v_+)} \otimes 1 + 1 \otimes \varepsilon_{\mathfrak{h}(v_-)}.$$

The real structure γ satisfies $\gamma(c^\dagger_{v_+ + v_-}) = c_{v_+ + v_-}$ and the CAR for these operators are given by

$$\{c^\dagger_x, c^\dagger_y\} = 0, \quad \{c_x, c_y\} = 0, \quad \{c^\dagger_x, c_y\} = c^\dagger_x c_y + c_y c^\dagger_x = \langle y, x\rangle_{\mathcal{V}} \quad \forall x, y \in \mathcal{V}. \tag{4.8}$$

For $\psi = \varepsilon_{v_+} \otimes 1 + 1 \otimes \varepsilon_{\varphi_-} + \tilde{\iota}_{\varphi_+} \otimes 1 + 1 \otimes \tilde{\iota}_{v_-}, \psi' = \varepsilon_{v'_+} \otimes 1 + 1 \otimes \varepsilon_{\varphi'_-} + \tilde{\iota}_{\varphi'_+} \otimes 1 + 1 \otimes \tilde{\iota}_{v'_-} \in \mathcal{W}$, the creation and annihilation operators satisfy the relations

$$\psi = c^\dagger_x + c_y, \quad \psi' = c^\dagger_{x'} + c_{y'},$$

where $x = v_+ + v_-, x' = v'_+ + v'_-$ and $y = \mathfrak{h}^{-1}(\varphi_+) + \mathfrak{h}^{-1}(\varphi_-), y' = \mathfrak{h}^{-1}(\varphi'_+) + \mathfrak{h}^{-1}(\varphi'_-)$. It follows

$$\psi\psi' - \psi'\psi = 2c^\dagger_x c^\dagger_{x'} + 2c_y c_{y'} + c^\dagger_x c_{y'} - c_{y'} c^\dagger_x + c_y c^\dagger_{x'} - c^\dagger_{x'} c_y.$$

Thus the non-interacting operators are given by linear combinations of operators of the form

$$c^\dagger_x c_y - c_y c^\dagger_x, \quad c^\dagger_x c^\dagger_y, \quad c_x c_y \quad \forall x, y \in \mathcal{V}.$$

The Lie group corresponding to $\mathfrak{so}(\mathcal{W})$ inside $C\ell(\mathcal{W}, q)$, *cf.* [55, Def. 2.3., Prop. 6.2], is given by

$$\mathrm{Spin}(\mathcal{W}) = \exp(\mathfrak{so}(\mathcal{W})) = \{g \in C\ell(\mathcal{W}, q)^{(\bar{0})} \mid q(g) = \pm 1\}.$$

The Lie group $\mathrm{Spin}(\mathcal{W})$ contains all physically relevant unitary operators in the free-fermion limit. Its canonical action on the Nambu space of fields is given by

$$\rho : \mathrm{Spin}(\mathcal{W}) \to \mathrm{SO}(\mathcal{W}); \quad \rho(g)w := gwg^{-1} \quad \forall w \in \mathcal{W}. \tag{4.9}$$

A *free-fermion Hamiltonian* $\mathcal{H} \in C\ell(\mathcal{W})$ generates by definition the time evolution operator
$\exp(-it\mathcal{H}/\hbar) \in \mathrm{Spin}(\mathcal{W})$ which has to induce a real orthogonal action $\rho(\exp(-it\mathcal{H}/\hbar)) \in \mathrm{SO}(\mathcal{W})$. This reality condition is motivated physically by the obstruction that the time evolution operator has to preserve the CAR. Therefore, the induced non-interacting Hamiltonian $H = d\rho(\mathcal{H}) \in \mathcal{L}(\mathcal{W})$ has to be imaginary w.r.t. the real structure γ on $\mathcal{W}$. We summarise this important observation in the following definition.

Definition 4.1.13 (Free-fermion Hamiltonian)**.** *A free-fermion Hamiltonian is a skew-symmetric element $H \in \mathcal{L}(\mathcal{W})$ which is imaginary with respect to the canonical real structure* Ad_γ *on $\mathcal{L}(\mathcal{W})$, i.e.*

$$\overline{H} = \gamma H \gamma = -H.$$

(Recall that $\gamma^2 = 1$.) By definition of the Hilbert space structure of $\mathcal{W}$, all free-fermion Hamiltonians are Hermitian. A free-fermion Hamiltonian is gapped *if it is invertible. A gapped free-fermion Hamiltonian is* flattened *if $H^2 = 1$.*

If we consider the Nambu space in the form $\mathcal{W} = \mathcal{V} \oplus \mathcal{V}^*$, then we deduce by the form of the real structure $\gamma = \left(\begin{smallmatrix} 0 & \mathfrak{h}^{-1} \\ \mathfrak{h} & 0 \end{smallmatrix}\right) : \mathcal{V} \oplus \mathcal{V}^* \to \mathcal{V} \oplus \mathcal{V}^*$ that any free-fermion Hamiltonian can be written in the form

$$H = \begin{pmatrix} A & B \\ -\mathfrak{h}B\mathfrak{h} & -\mathfrak{h}A\mathfrak{h}^{-1} \end{pmatrix} \in \mathcal{L}(\mathcal{V} \oplus \mathcal{V}^*),$$

where $A \in \mathcal{L}(\mathcal{V})$ is Hermitian and $B : \mathcal{V}^* \to \mathcal{V}$ is a *linear* morphism with $B^* = -\mathfrak{h}B\mathfrak{h}$.

Remark 4.1.14. *An invertible Hamiltonian H is called gapped because there is a $\Delta > 0$ such that there are no eigenstates of the Hamiltonian with eigenvalues (=eigenenergies) in $(-\Delta, \Delta)$. Thus, there is a gap in the spectrum of H around the chemical potential $\mu = 0$.*

To any gapped free-fermion Hamiltonian H, we can canonically associate a flattened free-fermion Hamiltonian $H|H|^{-1}$.

The ground state of a gapped free-fermion Hamiltonian is uniquely determined by the (-1)-eigenspace of the corresponding flattened Hamiltonian $H|H|^{-1}$. In principle, one could thus proceed with the description of ground states *via* flattened Hamiltonians. However, since all Hamiltonians are imaginary, it is from a mathematical point of view much more reasonable to describe ground states via the real operators

$$J := -iH|H|^{-1}.$$

This leads to the notion of quasi-particle vacua.

Definition 4.1.15 (Quasi-particle vacuum)**.** *A quasi-particle vacuum (QPV) is a skew-Hermitian real unitary $J \in \mathcal{L}(\mathcal{W})$, i.e.*

$$J^* = -J = -\overline{J} = J^{-1}.$$

Remark 4.1.16. *The original definition of QPV in [49] was formulated in terms of stable ground states of non-interacting quasi-particles. We briefly explain the bijection between stable ground states and QPV.*

A given QPV $J \in \mathcal{L}(\mathcal{W})$ has two eigenvalues $\pm i$. Let $\mathcal{A} := \ker(J - i)$. Thus, J defines a polarisation $\mathcal{W} = \mathcal{A}^c \oplus \mathcal{A}$ with $\mathcal{A}^c := \ker(J + i)$. The corresponding flattened Hamiltonian H satisfies

$$Ha = -a \quad \forall a \in \mathcal{A}, \qquad Hc = c \quad \forall c \in \mathcal{A}^c.$$

Thus, the elements in $\mathcal{A}$ correspond to eigenstates below the chemical potential $\mu = 0$. The ground state is thereby given by the state where all states in $\mathcal{A}$ are occupied. Because J is orthogonal, the eigenspaces are isotropic, i.e.

$$\{\mathcal{A}, \mathcal{A}\} = 0 = \{\mathcal{A}^c, \mathcal{A}^c\}.$$

This defines a ground state in the sense of [49].

For the converse construction, define

$$J := iP_{\mathcal{A}} - iP_{\mathcal{A}^c},$$

where $P_{\mathcal{A}}$ and $P_{\mathcal{A}^c}$ are the orthogonal projections onto $\mathcal{A}$ and $\mathcal{A}^c$. It follows immediately that $J^* = J^{-1} = -J$, and $\overline{P_{\mathcal{A}}} = P_{\mathcal{A}^c}$ implies $\overline{J} = J$.

In the literature, ground states of translationally invariant gapped systems are often described in terms of vector bundles over the Brillouin torus. In Appendix 4.B, we will give the reader who is familiar with this description a comparison to our description in terms of QPV.

The description of ground states in terms of QPV makes sense as long as we consider stable ground states corresponding to Hamiltonians with an energy gap above the chemical potential μ. In contrast to that, the boundary of a topological insulator or superconductor will in general violate this condition. Therefore, we will have to introduce a more general definition which governs the possibility of gapless localised states. Since we do not yet have the necessary tools to give this definition, we postpone this issue to Section 4.3.

4.1.2 Pseudo-symmetries of quasi-particle vacua

As explained at the beginning of this chapter, the incorporation of symmetries led to the discovery of various candidates of topological insulators and superconductors. In this section, we will recapitulate the fundamental properties of symmetries and explain how the symmetry classes in the Tenfold Way [2] are described in terms of QPV. This will lead to the notion of *pseudo-symmetries*. This section is based on [49].

We begin with the definition of symmetries in the sense of [49].

Definition 4.1.17 (Symmetry)**.** *Suppose we are given a free-fermion Hamiltonian $H \in \mathcal{L}(\mathcal{W})$ for a given physical system with Nambu space $\mathcal{W}$. A linear or anti-linear real unitary isomorphism $S : \mathcal{W} \to \mathcal{W}$ is called symmetry and the Hamiltonian H is called S-symmetric if*

$$[H, S] = 0. \tag{4.10}$$

By Definition 4.1.17, we consider only *true* symmetries which preserve the structure of the underlying space and the Hamiltonian. This is in keep with the notion of symmetries used by Kennedy–Zirnbauer [49]. However, one also finds a different definition for symmetries in the context of topological insulators in the literature, see e.g. [76]. In the appendix to this chapter, we give a translation between these two pictures.

In the following example, we present the most important physical examples of symmetries for this work.

Example 4.1.18 (Local symmetries)**.** *We list some examples of symmetries which are so-called local symmetries. Local symmetries can be roughly defined as symmetries that act only locally in space. Since we haven't introduced any notion of space yet, we postpone a more quantitative definition of locality to the next sections.*

The symmetries presented here will later be important, because by suitable combinations of these symmetries one can represent all of the ten symmetry classes in the Tenfold Way [2].

1. *Time-reversal is algebraically defined by an anti-linear unitary automorphism $T : \mathcal{V} \to \mathcal{V}$ [71, §2.2]. Since we consider free-fermions of spin $\frac{1}{2}$, inversion of spin implies $T^2 = -1$ [71, §4.3]. In other words, T defines a quaternionic structure on $\mathcal{V}$. The canonical lift of this automorphism to a symmetry over $\mathcal{W}$, which we still denote by T, is given by*

$$T := \begin{pmatrix} T & 0 \\ 0 & \mathfrak{h} T \mathfrak{h}^{-1} \end{pmatrix}.$$

2. *Another set of symmetries may be induced by spin rotations. The operators* $j_1 = iS_1, j_2 = iS_2, j_3 = iS_3$ *are the generators of the group* SU(2) *of spin-rotations. The operators* S_1, S_2, S_3 *are initially defined over the single-particle space* $\mathcal{V}$*, satisfying the relations*

$$S_1 S_2 = -S_2 S_1 = -iS_3,\ S_\mu^2 = 1 \text{ and } S_\mu^* = S_\mu \quad \forall \mu \in \{1,2,3\}.$$

Since time-reversal T *inverts the spin, it anti-commutes with the* S_1, S_2, S_3 *and subsequently it commutes with* j_1, j_2, j_3*. The canonical lift to symmetries on* $\mathcal{W}$ *is given by*

$$j_\mu = \begin{pmatrix} j_\mu & 0 \\ 0 & \mathfrak{h} j_\mu \mathfrak{h}^{-1} \end{pmatrix} \text{ for } \mu = 1,2,3.$$

3. *The charge operator*

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathcal{L}(\mathcal{W})$$

defines the symmetry iQ *that corresponds to the conservation of charge. Note that this operator corresponds to the operator* $\mathcal{Q}$ *over the Fock space that is defined by* $\mathcal{Q}\big|_{\Lambda^p(\mathcal{V})\oplus\Lambda^q(\mathcal{V}^*)} = p - q$ *for all* $p, q \geqslant 0$*. Thus this operator indeed measures the charge.*

4. *Our last symmetry we mention here is generated by a particle-hole transformation. A particle-hole transformation is defined as an anti-unitary operation* C *arising as follows. Consider a unitary* $S \in \mathrm{U}(\mathcal{V})$*,* $S^2 = 1$*,* $[S, T] = 0$ *and* $[S, S_\mu] = 0$ *for* $\mu = 1,2,3$*. A physical example of such an operator* S *are sublattice symmetries. Then the particle-hole transformation* C *is defined as*

$$C := \begin{pmatrix} 0 & S\mathfrak{h}^{-1} \\ \mathfrak{h}S & 0 \end{pmatrix}.$$

Additionally, we demand that C *commutes with the spin symmetries* $j_1, j_2, j_3 : \mathcal{W} \to \mathcal{W}$*.*

If this particle-hole transformation is a symmetry, then we also call it a particle-hole symmetry.

For the classification of symmetric ground states in terms of QPV, we have to translate symmetric Hamiltonians into symmetric QPV. Anti-linear symmetries of Hamiltonians no longer commute with the corresponding QPV, instead they anti-commute. Thus, to obtain a systematic description of the symmetry classes, we have to establish a different treatment for symmetries of QPV. Such a systematic translation of symmetric Hamiltonians into symmetric QPV for all symmetry classes of the Tenfold Way is obtained by considering so-called *pseudo-symmetries*, as introduced in [49].

Definition 4.1.19 (Symmetric QPV)**.** *A* QPV of symmetry index (r, s) *(or simply* of index (r, s)*) is a tuple* $(J; K_1, \ldots, K_r, J_1, \ldots, J_s)$*, where* $J \in \mathcal{L}(\mathcal{W})$ *is a QPV and* $K_1, \ldots, K_r, J_1, \ldots, J_s \in \mathcal{L}(\mathcal{W})$ *are real unitaries defining a unital* $*$*-morphism* $\phi : C\ell_{r,s} \to \mathcal{L}(\mathcal{W})$ *with* $\phi(k_a) = K_a$ *for all* $a = 1, \ldots, r$ *and* $\phi(j_\alpha) = J_\alpha$ *for all* $\alpha = 1, \ldots, s$ *such that*

$$JK_a + K_a J = JJ_\alpha + J_\alpha J = 0 \quad \forall a = 1, \ldots, r, \alpha = 1, \ldots, s. \tag{4.11}$$

If $r = 0$ and $0 \leqslant s \leqslant 7$, then we also call this a QPV of symmetry class s. *We shall also refer to it by the Cartan–Killing label in Table 4.1 below. (For instance, "J is a QPV of class D.") We call the J_α* negative *and the K_a* positive pseudo-symmetries *of J.*

The definition of pseudo-symmetries differs slightly from the one in [49]. Though, by the following remark, it can be easily seen that our definition is equivalent to the one in [49].

Remark 4.1.20. *The condition in Definition 4.1.19 that the pseudo-symmetries $K_1, \dots, K_r$ and $J_1, \dots, J_s$ generate the unital real $*$-morphism $\phi : C\ell_{r,s} \to \mathcal{L}(\mathcal{W})$ is equivalent to demanding that they satisfy the relations*

$$\{K_a, K_b\} = 2\delta_{a,b}, \quad \{J_\alpha, J_\beta\} = -2\delta_{\alpha,\beta}, \quad \{K_a, J_\alpha\} = 0$$

for all $a, b = 1, \dots, r$ and $\alpha, \beta = 1, \dots, s$, where $\{\cdot, \cdot\}$ denotes the anti-commutator.

In the following, we will use the properties of Clifford algebras to establish a systematic structure for these symmetric QPV, which leads to the Tenfold Way.

Remark 4.1.21 (Periodicity of QPV)**.** *Let $J \in \mathcal{L}(\mathcal{W})$ be a symmetric QPV of symmetry index $(r+1, s+1)$. By the $(1,1)$-periodicity of Clifford algebras*

$$C\ell_{r+1,s+1} \cong C\ell_{r,s} \otimes M_2(\mathbb{C}),$$

we can write the QPV in the form $\begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix}$ with pseudo-symmetries

$$\begin{pmatrix} K_a & 0 \\ 0 & -K_a \end{pmatrix}, \begin{pmatrix} J_\alpha & 0 \\ 0 & -J_\alpha \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

for $a = 1, \dots, r$ and $\alpha = 1, \dots, s$. Therefore, a QPV of index $(r+1, s+1)$ can always be reduced to a QPV of index (r, s). In combination with the 8-fold periodicity of Clifford algebras

$$C\ell_{r,s+8} \cong C\ell_{r+8,s} \cong C\ell_{r,s} \otimes M_{2^4}(\mathbb{C}),$$

we can reduce all symmetric QPV to symmetric QPV of symmetry class s with $0 \leq s \leq 7$.

A special role will be played by those classes where the charge (or, equivalently, the number of particles) is preserved. This holds when the QPV J and all its pseudo-symmetries commute with the charge operator Q of Example 4.1.18.(3), i.e. they are block diagonal w.r.t. the splitting $\mathcal{W} \cong \mathcal{V} \oplus \mathcal{V}^*$:

$$J = \begin{pmatrix} \tilde{J} & 0 \\ 0 & \mathfrak{h}\tilde{J}\mathfrak{h}^{-1} \end{pmatrix}; \; K_a = \begin{pmatrix} \tilde{K}_a & 0 \\ 0 & \mathfrak{h}\tilde{K}_a\mathfrak{h}^{-1} \end{pmatrix}, \; J_\alpha = \begin{pmatrix} \tilde{J}_\alpha & 0 \\ 0 & \mathfrak{h}\tilde{J}_\alpha\mathfrak{h}^{-1} \end{pmatrix}$$

for $a \in \{1, \dots, r\}$ and $\alpha = \{1, \dots, s\}$. We call such QPV *charge-conserving*. Charge conserving symmetric QPV can be reduced to the following so-called complex symmetric QPV.

Definition 4.1.22 (Complex symmetric QPV)**.** *A* QPV of complex symmetry index (r, s)*, or simply a* complex QPV of index (r, s)*, consists of a complex QPV $J \in \mathcal{L}(\mathcal{V})$ and unitaries $K_1, \dots, K_r, J_1, \dots, J_s \in \mathcal{L}(\mathcal{V})$, defining a unital $*$-morphism $\phi : \mathbb{C}\ell_{r,s} \to \mathcal{L}(\mathcal{V})$ with $\phi(k_a) = K_a$ and $\phi(j_\alpha) = J_\alpha$ for all $a = 1, \dots, r$ and $\alpha = 1, \dots, s$ such that Equation* (4.11) *holds; we call these K_a, J_α* complex pseudo-symmetries.

For $r = 0$ and $s = 0, 1$, we say that $(J; J_\alpha)$ are in complex symmetry class s.

Remark 4.1.23 (Periodicity of complex QPV)**.** *The pseudo-symmetries of a complex QPV define a representation of the complex C^*-algebra $\mathbb{C}\ell_{r,s}$. Because of the 2-fold periodicity of complex Clifford algebras*

$$\mathrm{C}\ell_{r,s+2} \cong \mathrm{C}\ell_{r+2,s} \cong \mathrm{C}\ell_{r,s} \otimes M_2(\mathbb{C}),$$

all complex symmetric QPV can be reduced to QPV of complex symmetry class s with $s \in \{0,1\}$.

Remark 4.1.24. *In view of the correspondence between operators K_a, J_α and Clifford representations ϕ, we will interchangeably write a QPV of a given (complex) symmetry index as $(J;\phi)$, where ϕ is the $*$-morphism representing the (complex) pseudo-symmetries.*

Tenfold Way in terms of pseudo-symmetries

We review the translation of time-reversal symmetry, particle-hole symmetry, spin-rotation symmetries and the symmetry leading to particle number conservation into pseudo-symmetries, as presented in [49]. This translation will in particular show that the definition of symmetric QPV will be exhaustive to describe all symmetry classes of the Tenfold Way in terms of symmetric QPV.

s	class	symmetry group	generators	comments
0	D	trivial	none	-
1	DIII	$\mathbb{Z}_4$	T	time-reversal
2	AII	$\mathbb{Z}_4 \ltimes \mathrm{U}(1)_c$	T, iQ	charge
3	CII	$\mathbb{Z}_4 \ltimes \mathrm{U}(1)_c \times \mathbb{Z}_2$	T, iQ, C	particle-hole symmetry
4	C	$\mathrm{SU}(2)_s$	j_1, j_2, j_3	spin rotations
5	CI	$\mathrm{SU}(2)_s \times \mathbb{Z}_4$	j_1, j_2, j_3, T	
6	AI	$\mathrm{SU}(2)_s \times (\mathbb{Z}_4 \ltimes \mathrm{U}(1)_c)$	j_1, j_2, j_3, T, iQ	
7	BDI	$\mathrm{SU}(2)_s \times (\mathbb{Z}_4 \ltimes \mathrm{U}(1)_c) \times \mathbb{Z}_2$	j_1, j_2, j_3, T, iQ, C	
0	A	$\mathrm{U}(1)_c$	iQ	charge
1	AIII	$\mathrm{U}(1)_c \times \mathbb{Z}_2$	iQ, C	particle-hole symmetry

Table 4.1: Symmetry classes of the Tenfold Way. The index s denotes the number of negative pseudo-symmetries. The first eight rows describe the real symmetry classes and the last two rows describe the complex symmetry classes. The symbol $\mathrm{U}(1)_c$ denotes the group U(1) generated by the operator iQ, and $\mathrm{SU}(2)_s$ denotes the group SU(2) of spin-rotations generated by j_1, j_2, j_3.

As observed by Kitaev for translationally invariant topological insulators, these ten classes of the Tenfold Way are organised into two cycles, one of length two (complex classes) and one of length eight (real classes), see Table 4.1.

Remark 4.1.25. *The terminology used to label the symmetry classes is based on the work by Altland–Zirnbauer [2] who used it in their classification of disordered Hamiltonians in the context of mesoscopic metals and superconductors, according to the type of the corresponding σ-model target spaces. The labels (D, D*III*, etc.) used in this context refer to the Cartan–Killing classification of Riemannian symmetric spaces.*

Class D, $s = 0$. In symmetry class D we have no symmetries. Thus this class is described by all QPV $J \in \mathcal{L}(\mathcal{W})$. These give rise to the symmetric space $O/(U \times U)$ which is a symmetric space of Cartan–Killing type $D\mathrm{III}$.

Class $D\mathrm{III}$, $s = 1$. In class $D\mathrm{III}$, we have time-reversal symmetry $T : \mathcal{W} \to \mathcal{W}$, which is defined as explained in Example 4.1.18.(1) and commutes with the flattened Hamiltonian H, i.e. $[H, T] = 0$. Thus, time-reversal has to anti-commute with the corresponding QPV $J = -iH$. Since J is real, it commutes with γ and therefore anti-commutes with the product

$$J_1 := J_T := \gamma T.$$

The time-reversal T is also real, i.e. $\gamma T = T\gamma$ and this implies $\overline{J}_1 = J_1 = -J_1^*$ and $J_1^2 = -1$.

We define the operator $J_{\mathrm{ref}} := i\gamma T Q$, where Q is the charge operator as defined in Example 4.1.18.(3). T and Q commute with γ and they anti-commute with each other. This implies $\overline{J}_{\mathrm{ref}} = J_{\mathrm{ref}}$, $J_{\mathrm{ref}}^* = QT\gamma i = iQ\gamma T = -J_{\mathrm{ref}}$ and $J_{\mathrm{ref}}^2 = i\gamma T Q i\gamma T Q = -1$. Analogously, one can check $\{J_1, J_{\mathrm{ref}}\} = 0$. Thus J_{ref} is a QPV in class $D\mathrm{III}$. We refer to J_{ref} as a *reference QPV*.

Now we can apply Lemma 4.1.26 below for $x_1 = J_{\mathrm{ref}}$, $x_2 = J_1$ in order to obtain a real $*$-isomorphism

$$\mathcal{L}(\mathcal{W}) \cong \mathcal{L}(\mathcal{W}^{\sim}) \otimes \mathbb{H}_{\mathbb{C}}.$$

Since $x_2x_1 = -iQ = \left(\begin{smallmatrix} -i & 0 \\ 0 & i \end{smallmatrix}\right)$ is already diagonal and $\gamma x_2 = T$, it is easy to see that $\mathcal{L}(\mathcal{W}^{\sim})$ is equipped with the quaternionic structure T. Furthermore, it maps $J_1 \mapsto i\sigma_y$ and thus a QPV J that anti-commutes with J_1 becomes of the form

$$J = y_1 \otimes i\sigma_x + y_2 \otimes i\sigma_z,$$

where $y_1, y_2 \in \mathcal{L}(\mathcal{W}^{\sim})$ are Hermitian real (i.e. time-reversal symmetric) elements such that $y_1^2 + y_2^2 = 1$, $y_1y_2 = y_2y_1$. Alternatively, one can describe this QPV J via the symmetric unitary

$$U := y_1 + iy_2 \in \mathcal{L}(\mathcal{W}^{\sim}),$$

where the transposition over $\mathcal{L}(\mathcal{W}^{\sim})$ is given by $U^{\mathsf{T}} = (\overline{U})^* = T^*U^*T$.

Thus, the target space of the corresponding σ-model is the symmetric space symmetric space U/Sp, which is of Cartan–Killing type $A\mathrm{II}$.

Lemma 4.1.26. *Let $x_1, x_2 \in \mathcal{L}(\mathcal{W})$ be two anti-commuting real skew-Hermitian unitaries. Then $(\mathcal{W}^{\sim}, T)$, where $\mathcal{W}^{\sim} := \ker(x_2x_1 - i)$ and $T := -\gamma x_2$, is a quaternionic Hilbert space. There is an isomorphism $(\mathcal{W}, \gamma) \cong (\mathcal{W}^{\sim} \otimes \mathbb{C}^2, T \otimes \mathfrak{t})$ of real vector spaces such that under the induced isomorphism of real C^*-algebras*

$$\mathcal{L}(\mathcal{W}) \cong \mathcal{L}(\mathcal{W}^{\sim}) \otimes \mathbb{H}_{\mathbb{C}},$$

we have $x_1 \longmapsto 1 \otimes i\sigma_x$ and $x_2 \longmapsto 1 \otimes i\sigma_y$. The isomorphism maps real unitaries $J \in \mathcal{L}(\mathcal{W})$ anti-commuting with x_1 onto operators $y_1 \otimes i\sigma_y + y_2 \otimes i\sigma_z \in \mathcal{L}(\mathcal{W}^{\sim}) \otimes \mathbb{H}_{\mathbb{C}}$, where $y_1, y_2 \in \mathcal{L}(\mathcal{W}^{\sim})$ are real Hermitian elements such that

$$y_1^2 + y_2^2 = 1, \quad y_1y_2 = y_2y_1.$$

Proof. Let $x_3 := x_2 x_1 \in \mathcal{L}(\mathcal{W})$. Then x_3 is a real skew-Hermitian unitary that anti-commutes with x_1 and x_2. In particular, x_3 has the eigenvalues $\pm i$. Setting $\mathcal{W}^\pm := \ker(x_3 \mp i)$, we obtain a splitting $\mathcal{W} = \mathcal{W}^+ \oplus \mathcal{W}^-$ with

$$x_1 = \begin{pmatrix} 0 & iu^* \\ iu & 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 0 & u^* \\ -u & 0 \end{pmatrix}, \quad x_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

The map $u : \mathcal{W}^+ \longrightarrow \mathcal{W}^-$ is a unitary isomorphism. Therefore, we can use it to define the isomorphism

$$\phi := \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} : \mathcal{W}^+ \oplus \mathcal{W}^+ = \mathcal{W}^+ \otimes \mathbb{C} \longrightarrow \mathcal{W}^+ \oplus \mathcal{W}^-.$$

We compute

$$\phi^* x_1 \phi = 1 \otimes i\sigma_x, \quad \phi^* x_2 \phi = 1 \otimes i\sigma_y.$$

Since the anti-unitary map γx_2 anti-commutes with x_3, it leaves $\mathcal{W}^+$ invariant. We let $T := -\gamma x_2|_{\mathcal{W}^+}$. Then T is a quaternionic structure on $\mathcal{W}^\sim := \mathcal{W}^+$ and

$$-uT = -x_2 \gamma x_2|_{\mathcal{W}^+} = -\gamma x_2^2|_{\mathcal{W}^+} = \gamma|_{\mathcal{W}^+},$$

so that

$$\phi(T \otimes \mathrm{c})\phi^* = \phi \begin{pmatrix} 0 & T \\ -T & 0 \end{pmatrix} \phi^* = \begin{pmatrix} 0 & Tu^* \\ -uT & 0 \end{pmatrix} = \gamma.$$

This proves the claim. □

Class $A\mathrm{II}$, $s = 2$. In class $A\mathrm{II}$, we have a time-reversal symmetric Hamiltonian H with preserved charge, i.e $[H, T] = 0 = [H, Q]$, where Q is defined as in Example 4.1.18.(3). The first pseudo-symmetry is $J_1 = \gamma T$. A second pseudo-symmetry can be defined by

$$J_2 := J_Q := i\gamma T Q.$$

As in class $D\mathrm{III}$ we have $\overline{J_2} = J_2$, $J_2^* = -J_2$, $J_2^2 = -1$ and $\{J_1, J_2\} = 0$. Applying Lemma 4.1.26 for $x_1 = J_2$ and $x_2 = J_1$, we obtain by the same arguments as for class $D\mathrm{III}$ that a QPV in class $A\mathrm{II}$, which anti-commutes with J_1 and J_2 is given by

$$J = x \otimes i\sigma_z \in \mathcal{L}(\mathcal{W}^\sim) \otimes \mathbb{H}_\mathbb{C},$$

where $x \in \mathcal{L}(\mathcal{W}^\sim)$ is a real Hermitian unitary.

The target space of the corresponding σ-model is the symmetric space $\mathrm{Sp}/(\mathrm{Sp} \times \mathrm{Sp})$, which is of Cartan–Killing type $C\mathrm{II}$.

Class $C\mathrm{II}$, $s = 3$. In class $C\mathrm{II}$, we have a time-reversal symmetric, charge preserving Hamiltonian which is particle-hole symmetric. This means $[H, C] = 0$, where $C : \mathcal{W} \to \mathcal{W}$ is defined as in Example 4.1.18.(4) and satisfies $C^2 = 1$, $CT = TC$, $CiQ = iQC$ and $C\gamma = \gamma C$. Here, we are given the two pseudo-symmetries $J_1 = \gamma T$ and $J_2 = i\gamma T Q$ and we can define a third pseudo-symmetry by

$$J_3 := J_C := i\gamma C Q.$$

It follows that J_3 is a real skew-Hermitian unitary and $\{J_1, J_3\} = \{J_2, J_3\} = 0$.

Additionally, we define $J_{\mathrm{ref}} \in \mathcal{L}(\mathcal{W})$ to be a reference QPV that anti-commutes with J_1, J_2, J_3. Then we can use these four operators for the application of Lemma 4.1.27 below, where $x_1 = J_2, x_2 = J_1, x_3 = J_3, x_4 = J_{\mathrm{ref}}$. Then we obtain an isomorphism

$$\mathcal{L}(\mathcal{W}) \cong \mathcal{L}(\mathcal{W}^{\approx}) \otimes M_2(\mathbb{C}) \otimes \mathbb{H}_{\mathbb{C}},$$

under which the pseudo-symmetries are mapped onto

$$J_1 \mapsto 1 \otimes 1 \otimes i\sigma_y, \quad J_2 \mapsto 1 \otimes 1 \otimes i\sigma_x, \quad J_3 \mapsto 1 \otimes k_1 j_1 \otimes i\sigma_z.$$

The reference QPV J_{ref} is mapped onto $1 \otimes k_1 \otimes i\sigma_z$. Any QPV $J \in \mathcal{L}(\mathcal{W})$ of class $C\mathbb{I}$ anti-commutes with J_1, J_2, J_3 and thus has to be of the form

$$J = y_+ \otimes k_1 \otimes i\sigma_z + y_- \otimes j_1 \otimes i\sigma_z \in \mathcal{L}(\mathcal{W}^{\approx}) \otimes M_2(\mathbb{C}) \otimes \mathbb{H}_{\mathbb{C}},$$

where $y_\pm \in \mathcal{L}(\mathcal{W}^{\approx})$ are commuting real, $y_\pm^* = \pm y_\pm$ and $y_+^2 - y_-^2 = 1$.

Alternatively, one can describe J via the Hermitian unitary $U := y_+ + iy_- \in \mathcal{L}(\mathcal{W}^{\approx})$.

Here we obtain the symmetric space USp, which is of Cartan–Killing type C.

Lemma 4.1.27. *Let $x_1, x_2, x_3, x_4 \in \mathcal{L}(\mathcal{W}) \subseteq \mathbb{A}$ be anti-commuting real skew-Hermitian unitaries. Consider $\mathcal{W}^{\sim}$ and $\mathcal{L}(\mathcal{W}^{\sim})$, defined as in Lemma 4.1.26 with respect to x_1, x_2, and let $y_3, y_4 \in \mathcal{L}(\mathcal{W}^{\sim})$ such that $y_j \otimes i\sigma_z$ corresponds to x_j for $j = 3, 4$.*

Then $\mathcal{W}^{\approx} := \ker(y_3 - 1)$ is a quaternionic subspace of $\mathcal{W}^{\sim}$, and there is an isomorphism $\mathcal{W}^{\sim} \cong \mathcal{W}^{\approx} \otimes \mathbb{C}^2$ of quaternionic Hilbert spaces which maps the quaternionic structure T on $\mathcal{W}^{\sim}$ onto $T \otimes \mathfrak{c}_2$ on $\mathcal{W}^{\approx} \otimes \mathbb{C}^2$, where $\mathfrak{c}_2$ is component-wise conjugation on $\mathbb{C}^2$.

Moreover, the isomorphism is such that, combining the two isomorphisms, we obtain an induced isomorphism of real C^*-algebras

$$\mathcal{L}(\mathcal{W}) \cong \mathcal{L}(\mathcal{W}^{\approx}) \otimes M_2(\mathbb{C}) \otimes \mathbb{H}_{\mathbb{C}},$$

under which for $k_1 := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $j_1 := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$

$$x_1 \longmapsto 1 \otimes 1 \otimes i\sigma_x, \quad x_2 \longmapsto 1 \otimes 1 \otimes i\sigma_y, \quad x_3 \longmapsto 1 \otimes k_1 j_1 \otimes i\sigma_z, \quad x_4 \longmapsto 1 \otimes k_1 \otimes i\sigma_z.$$

This isomorphism maps real skew-Hermitian unitaries $J \in \mathcal{L}(\mathcal{W})$ anti-commuting with x_1, x_2, x_3 onto operators $y_+ \otimes k_1 \otimes i\sigma_z + y_- \otimes j_1 \otimes i\sigma_z$, where $y_\pm \in \mathbb{A}^{\approx}$ are commuting real elements such that

$$y_\pm^* = \pm y_\pm, \quad y_+^2 - y_-^2 = 1.$$

Proof. By Lemma 4.1.26 and the discussion of class $A\mathbb{I}$, the operators y_3 and y_4 are anti-commuting real Hermitian unitaries. The reality constraint means that they commute with T. Hence, the eigenspaces $\mathcal{W}^{\sim}_\pm := \ker(y_3 \mp 1)$ are quaternionic subspaces, and $y_4 : \mathcal{W}^{\sim}_\pm \longrightarrow \mathcal{W}^{\sim}_\mp$.

Let $\mathcal{W}^{\approx} := \mathcal{W}^{\sim}_+$ and $v := y_4|_{\mathcal{W}^{\approx}}$. We obtain a unitary isomorphism

$$\varphi := \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix} : \mathcal{W}^{\approx} \otimes \mathbb{C}^2 \longrightarrow \mathcal{W}^{\sim}.$$

Clearly, φ is real. Moreover,

$$\varphi^* y_3 \varphi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \mathrm{id}_{\mathcal{W}^{\approx}} \otimes \sigma_z, \quad \varphi^* y_4 \varphi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \mathrm{id}_{\mathcal{W}^{\approx}} \otimes \sigma_x.$$

$\square$

Classes $s \geq 4$. Here we are given a Hamiltonian that commutes with the spin rotation generators $j_1, j_2, j_3 : \mathcal{W} \to \mathcal{W}$ as defined in Example 4.1.18.(2). The spin rotation generators are linear and thus also commute with the QPV J. To define pseudo-symmetries, we have to amplify $\mathcal{W}$ to $\mathcal{W} \otimes \mathbb{C}^2$, where $\mathbb{C}^2$ is equipped with the natural real structure $\mathfrak{c}_2$. On this space, we consider the QPV

$$J \otimes \sigma_x$$

and define the pseudo-symmetries

$$J_\mu := j_\mu \otimes \sigma_z \text{ for } \mu = 1,2,3 \text{ and } J_4 := 1 \otimes i\sigma_y.$$

The j_μ are real and anti-commute pairwise. Thus $\{J_i, J_j\} = -2\delta_{i,j}$, $\overline{J}_i = J_i$, $J_i^* = -J_i$ for all $i, j \in \{1, \ldots, 4\}$. The other pseudo-symmetries can defined by

$$J_5 := J_T \otimes \sigma_x,\ J_6 := J_Q \otimes \sigma_x,\ J_7 := J_C \otimes \sigma_x,$$

if the corresponding symmetries are present. The j_μ commute with γ, T, Q, C and therefore also with J_T, J_Q and J_C. Thus J_5, J_6, J_7 anti-commute with the pseudo-symmetries $J_1, \ldots, J_4$.

Class A, $s = 0$. Here we consider the fundamental example of a complex QPV which are by definition skew-Hermitian unitaries in $\mathcal{L}(\mathcal{V})$. These give rise to the symmetric space $U/(U \times U)$ which is a symmetric space of Cartan–Killing type $A\mathrm{III}$.

Class $A\mathrm{III}$, $s = 1$. In class $A\mathrm{III}$, we have a charge-conserving QPV anti-commuting with a particle-hole symmetry C of the form

$$C = \begin{pmatrix} 0 & S\mathfrak{h}^{-1} \\ \mathfrak{h}S & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{W}).$$

This defines the complex pseudo-symmetry $J_1 := J_C := iS$ for the corresponding complex QPV $J \in \mathcal{L}(\mathcal{V})$, i.e. the pair $(J; J_1)$ defines a QPV of complex symmetry class $s = 1$. Let $J_{\mathrm{ref}} \in \mathcal{L}(\mathcal{V})$ be any reference QPV anti-commuting with J_1. Then, as in symmetry class $D\mathrm{III}$, we can use Lemma 4.1.26 for $x_1 = J_{\mathrm{ref}}$ and $x_2 = J_1$ to deduce that there is an isomorphism

$$\mathcal{L}(\mathcal{V}) \cong \mathcal{L}(\tilde{\mathcal{V}}) \otimes M_2(\mathbb{C}),$$

where $J_1 \mapsto 1 \otimes i\sigma_y$ and $J_{\mathrm{ref}} \mapsto 1 \otimes i\sigma_x$. An arbitrary QPV $J \in \mathcal{L}(\mathcal{V})$ anti-commuting with J_1 thus has to be of the form

$$J = y_1 \otimes i\sigma_x + y_2 \otimes i\sigma_z$$

where $y_1, y_2 \in \mathcal{L}(\tilde{\mathcal{V}})$ are commuting Hermitian elements with $y_1^2 + y_2^2 = 1$. Alternatively, J can be described via the unitary $U := y_1 + iy_2$.

Here, the target space is the symmetric space U, which is of Cartan–Killing type A.

4.2 Bulk systems with disorder

In this section, we will construct the mathematical framework to describe the ground states in the bulk of disordered topological insulators and superconductors. We begin with the construction of a C*-algebra that contains all non-interacting lattice Hamiltonians in the tight-binding approximation with homogeneous disorder.

4.2.1 The tight-binding approximation

In the bulk, a solid state system is described over a d-dimensional crystalline lattice in $\mathbb{R}^d$. As is well known in solid state physics, one can describe any d-dimensional lattice by the following two ingredients [51]:

- A Bravais lattice
$$\Lambda := \left\{ \sum_{i=1}^{d} n_i e_i \,\middle|\, n_1, \ldots, n_d \in \mathbb{Z} \right\} \cong \mathbb{Z}^d,$$
spanned by the linear independent set of so-called primitive vectors $e_1, \ldots, e_d \in \mathbb{R}^d$. If there are no magnetic fields, the Bravais lattice carries a canonical action by the group $\mathbb{Z}^d$, which is generated by the translations of the lattice along the primitive vectors:
$$(m_1, \ldots, m_d) \cdot \left(\sum_{i=1}^{d} n_i e_i \right) := \sum_{i=1}^{d} (n_i + m_i) e_i. \tag{4.12}$$
In the presence of a homogeneous magnetic field, this group action is twisted, as we will explain later.

- A basis U, which is a bounded finite discrete set in $\mathbb{R}^d$. The boundedness of the set U reflects the fact that it describes the local spacial degrees of freedom around each point of the Bravais lattice.

With these two ingredients, the lattice can be represented as $\Lambda \times U \cong \mathbb{Z}^d \times U$.

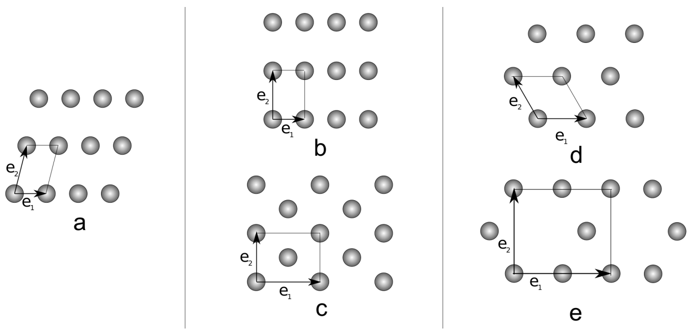

Figure 4.1: Examples of Bravais lattices in two dimensions. The primitive vectors are denoted by e_1 and e_2. In Examples a, b and d, the basis U is given by a single point. In Example c (e), the basis consists of two (three) points.

We consider solid state systems in the tight-binding approximation, which means that the quasi-particles are localised at the lattice sites up to some finite orbital degrees of freedom. The resulting complex Hilbert space of single-particle states is therefore given as follows.

Definition 4.2.1 (Single-particle space $\mathcal{V}$)**.** *The complex Hilbert space of single-particle states in the tight-binding approximation is defined as*
$$\mathcal{V} := \ell^2(\Lambda) \otimes V,$$
where V is a finite-dimensional complex Hilbert space, which is called the local space.

The local space contains the Hilbert space $\ell^2(U)$ as well as the relevant internal degrees of freedom such as spin or orbital degrees of freedom. The property of V to be of finite dimension is one of the defining properties of the tight-binding approximation. This approximation is physically justified as long as we are at very low temperatures, where only the states close to the chemical potential are relevant.

Throughout the rest of this work, the space $\mathcal{V}$ will always denote the complex Hilbert space of Definition 4.2.1 and $\mathcal{W}$ will always denote the Nambu space associated to $\mathcal{V}$, equipped with the canonical real structure γ.

There is an isometric isomorphism $\varphi : \mathcal{V}^* \to \ell^2(\Lambda) \otimes V^*$, given by

$$\langle \varphi(\psi_1), \psi_2 \rangle := \sum_{x \in \Lambda} \langle \psi_1(x), \psi_2(x) \rangle , \quad \forall\, \psi_1 \in \ell^2(\Lambda) \otimes V^*,\, \psi_2 \in \mathcal{V}^*.$$

Setting $W := V \oplus V^*$, this induces an isometric isomorphism $\ell^2(\Lambda) \otimes W \cong \mathcal{W}$.

Proposition 4.2.2. *Consider the canonical real structure on $W = V \oplus V^*$. Then the real structure on $\mathcal{W}$ satisfies the equation*

$$\psi^\gamma(x) = \psi(x)^\gamma, \quad \forall \psi \in \mathcal{W}, x \in \Lambda.$$

Here, on the left-hand side, $\gamma = \gamma_{\mathcal{W}}$, whereas on the right-hand side, $\gamma = \gamma_W$.

Corollary 4.2.3. *The real C^*-algebra $\mathcal{L}(\mathcal{W})$ is $*$-isomorphic to $\mathcal{L}(\ell^2(\Lambda)) \otimes \mathrm{End}(W)$ with real structure $\overline{T_1 \otimes T_2} = \mathfrak{c}_\Lambda T_1 \mathfrak{c}_\Lambda \otimes \gamma T_2 \gamma$, where $\mathfrak{c}_\Lambda$ denotes the point-wise complex conjugation on $\ell^2(\Lambda)$.*

Proof. By Proposition 4.2.2, we know there is a real isometric isomorphism

$$\mathcal{W} = \ell^2(\Lambda) \otimes (V \oplus V^*),$$

where the right-hand side is equipped with the real structure $\mathfrak{c}_\Lambda \otimes \gamma$. It can easily be checked that this induces the real $*$-isomorphism $\mathcal{L}(\mathcal{W}) \cong \mathcal{L}(\ell^2(\Lambda)) \otimes \mathrm{End}(W)$. □

4.2.2 The C*-algebra of disordered bulk observables

A key feature of Hamiltonians and their corresponding QPV in the tight-binding approximation is that they have a finite hopping range. Mathematically, this property is captured by the notion of *controlled operators* as defined below.

Definition 4.2.4 (Controlled Operators)**.** *Let U be a finite-dimensional Hilbert space, $D \subseteq \Lambda$ a subset. Let $O(x, y) \in \mathrm{End}(W)$, for $x, y \in D$, denote the kernel function of $O \in \mathcal{L}(\ell^2(D) \otimes U)$, defined by*

$$O(x, y) := \langle x, Oy \rangle_{\ell^2(D)}.$$

Then we call an operator $O \in \mathcal{L}(\ell^2(D) \otimes U)$ controlled *if there is an $R > 0$ such that for all $x, y \in D$ with $\|x - y\| > R$, we have*

$$O(x, y) = 0.$$

The norm closure of the set of controlled operators in $\mathcal{L}(\ell^2(D) \otimes U)$ is denoted by $C^*_u(D, U)$.

Remark 4.2.5. *The previous definition is due to Roe [72, Def. 4.22] in a much more general form. The subscript u in $C^*_u(D,U)$ refers to the word "uniform". Indeed, $C^*_u(D,U) \cong C^*_u(D,\mathbb{C}) \otimes \mathrm{End}(U)$ where $C^*_u(D,\mathbb{C})$ is usually denoted by $C^*_u(D)$ and called the* uniform Roe algebra *(of the metric space D) in the literature. In the context of topological classification of solid state systems in the tight-binding approximation, the uniform Roe algebra has first been used by Kubota [53].*

The following statement is immediate, see [72, Cor. 4.24] for the first part.

Proposition 4.2.6. *Let U be a finite-dimensional Hilbert space and $D \subseteq \Lambda$ a subset. Then $C^*_u(D,U)$ is a closed $*$-subalgebra of $\mathcal{L}(\ell^2(D) \otimes U)$ and hence a C^*-algebra.*

If U has a real or quaternionic structure, then $C^*_u(D,U)$ is invariant under the induced real structure on $\mathcal{L}(\ell^2(D) \otimes U)$, and thus, a real C^*-algebra.

Proof. Only the statement about the real structure requires proof. By definition, the real structure acts by point-wise complex conjugation over $\ell^2(D)$, hence

$$\overline{O}(x,y) = \overline{O(x,y)} \quad \forall O \in \mathcal{L}(\ell^2(D) \otimes U), x,y \in D.$$

This implies that the set of controlled operators is invariant under conjugation. Since the conjugation is isometric, the assertion follows. □

Homogeneous disorder

As long as our system is microscopically translation invariant in the bulk, the smallest C*-algebra containing all free-fermion Hamiltonians of interest on $\mathcal{W}$ is the algebra of $\mathbb{Z}^d$-invariants in $C^*_u(\Lambda, W)$. In this case, it would be fair to declare this to be the algebra of bulk observables.

However, a solid state system being *microscopically* translation invariant is not a reasonable assumption, as all such systems display some disorder which breaks the translational invariance. One may however assume that the translational invariance still holds at a macroscopic level. A general way to model this assumption was suggested by Bellissard [9]. Starting from some basic Hamiltonian, he considers all translates thereof and the closure Ω of this translational orbit in the space of operators (in a suitable topology). If the Hamiltonian is macroscopically homogeneous, then, together with its translates, it should [12] be a function on Ω which is translationally *covariant* (or, in a more modern language, equivariant) in the sense that it intertwines the action of the translation group on Ω and the space of operators.

In this work, we make a general ansatz and do not a priori specify the details of Ω. Following [66], we define Ω as follows.

Definition 4.2.7 (Space of disorder configurations)**.** *A space of disorder configurations is a compact Hausdorff space Ω with a continuous action of the Abelian group $\mathbb{Z}^d$. We denote the action by the right action $\omega \mapsto \omega \cdot x$ for $\omega \in \Omega$ and $x \in \mathbb{Z}^d$. Furthermore, we assume that there is a fully supported $\mathbb{Z}^d$-invariant Borel probability measure $\mathbb{P}$ on Ω.*

Example 4.2.8. *Bellissard's approach works quite generally for any kind of disorder, as long as the observables are in average translationally invariant. Here we point out the two most important applications of this approach, giving rise to concrete realisations of Ω.*

- *Disorder induced by doping (synthetic insertion of atoms in a solid), is one of the most important types of disorder in solid state physics. This kind of disorder is commonly described by a random potential around each lattice site. The resulting space of disorder configurations for the lattice $\Lambda \times U$ is given by*

$$\Omega = \prod_{x \in \Lambda} \Omega_x,$$

where Ω_x is a contractible compact Hausdorff space representing all disorder contributions (random potential, spacial fluctuations) at lattice site x. It is assumed that the homogeneity of solid state systems is preserved on macroscopic scales. Therefore, it is physically reasonable to assume $\Omega_0 := \Omega_x = \Omega_y$ for all $x, y \in \Lambda$. More details can be found in [66].

We obtain a canonical $\mathbb{Z}^d$-action on Ω by

$$((\omega_x)_{x\in\Lambda}) \cdot y := (\omega_{x+y})_{x\in\Lambda} \quad \forall y \in \mathbb{Z}^d, (\omega_x)_{x\in\Lambda} \in \prod_{x\in\Lambda} \Omega_0.$$

This dynamical system, known in mathematics as a Bernoulli shift*, was first suggested by Bellissard as a model for the disorder space (see [66] for an exposition).*

In [54], numerical calculations lead to explicit results for the space Ω which support the above assumptions. In this reference, this type of disorder is called thermal disorder, referring to the idea that thermal fluctuations correspond to spacial fluctuations of the localised quasi-particles around each lattice site.

- *The topological classification of quasi-crystals relies on Bellissard's approach, too. There, the quasi-crystals are described* via *an effective model as higher-dimensional lattices, where the space Ω is given by a Cantor set, equipped with a $\mathbb{Z}^d$-action [11, 44, 47]. Note that in this case, the space Ω has its origin in the nature of the effective description of quasi-crystals and does not come from disorder in a physical sense.*

Homogeneous magnetic fields

Before we state the covariance condition in its full generality, we digress briefly on magnetic fields. We incorporate them in our exposition because they are relevant for the complex symmetry classes. For example, the first experiments on the quantum Hall effect and its interpretation relied crucially on the existence of a homogeneous magnetic field.[1]

The homogeneity of the magnetic field implies that the quantised nature of the magnetic field has to be compatible with the underlying lattice. This is satisfied for the QHE, because the underlying lattice is actually determined by the magnetic field. However, if the lattice is determined by the atoms in the solid, this condition can only be achieved by fine-tuning.

We adapt the treatment of magnetic fields in the tight-binding approximation from [66]. The presence of a spatially homogeneous magnetic field is modelled by a real skew-symmetric $d \times d$-matrix $\mathbf{B} = (B_{\mu\nu})$. Throughout this work, we consider the so-called symmetric gauge, *cf.* [66, Ch. 2 & Ch. 3],

$$\sigma(x, y) := e^{\frac{i}{2}(x, \mathbf{B} y)} \quad \forall x, y \in \Lambda, \tag{4.13}$$

[1] Actually, the Hall conductance in the QHE may be quantised even if the total magnetic flux per lattice plaquette vanishes. This phenomenon was discovered by Haldane and is nowadays known as the anomalous quantum Hall effect [31].

where $(x, By) := \sum_{\mu,\nu=1}^{d} x_\mu (B)_{\mu\nu} y_\nu$. In particular, we have $\sigma(x, x) = \sigma(x, x^{-1}) = 1$ for all $x \in \mathbb{Z}^d$. The corresponding *magnetic translations* u_x^σ on $\mathcal{V}$ are defined as

$$(u_x^\sigma \psi)(y) := \sigma(y, x)\psi(y - x), \quad \forall x \in \mathbb{Z}^d, y \in \Lambda, \psi \in \mathcal{V}.$$

This defines a twisted projective unitary representation of $\mathbb{Z}^d$. The magnetic translation u_x^σ extends to the real operator on $\mathcal{W}$, given by

$$(u_x^\sigma \psi)(y) := \sigma(y, x)\psi(y - x), \quad \forall x \in \mathbb{Z}^d, y \in \Lambda, \psi \in \mathcal{W},$$

where

$$\sigma(x, y) := \begin{pmatrix} e^{\frac{i}{2}(x,\mathbf{B}y)} & 0 \\ 0 & e^{-\frac{i}{2}(x,\mathbf{B}y)} \end{pmatrix}.$$

Although the operators u_x^σ, $x \in \mathbb{Z}^d$, define a twisted unitary representation of $\mathbb{Z}^d$, this representation is in general not projective, as the cocycle σ does not take values in U(1). This is only true when σ solely takes its values in $\{\pm 1\}$. But this corresponds physically to a 2π-flux per unit cell which is equivalent to no flux. For this reason, we will allow non-trivial homogeneous magnetic fields only in the complex symmetry classes; in the real symmetry classes, we assume that the magnetic field vanishes.

This restriction is in keep with the physical applications. Indeed, the only real symmetry classes where magnetic fields can in principle occur are *D* (corresponding to gapped superconductors or superfluids with no symmetries) and *C* (corresponding to gapped superconductors or superfluids with fermions of spin $\frac{1}{2}$ and SU(2) spin-rotation symmetry). But the physical realisation of magnetic fields in these symmetry classes is given by a so-called vortex phase, which is far from homogeneous.

Definition of C*-algebra of bulk observables

Definition 4.2.9 (Covariance algebra)**.** *Let Ω be a space of disorder configurations. Let U be a finite-dimensional Hilbert space and $\sigma : \mathbb{Z}^d \times \mathbb{Z}^d \longrightarrow$* U(1) *a magnetic cocycle as defined in Equation* (4.13) *(which might be trivial). Let u_x^σ, for $x \in \mathbb{Z}^d$, be defined by*

$$(u_x^\sigma \psi)(y) := \sigma(y, x)\psi(y - x), \quad \forall x \in \mathbb{Z}^d, y \in \Lambda, \psi \in \ell^2(\Lambda) \otimes U.$$

We define $A^{U,\sigma}$ to be the set of all maps $O : \Omega \longrightarrow C_u^*(\Lambda, U)$ that are continuous in the norm topology and covariant *(or equivariant) in the sense that*

$$O_{\omega \cdot x} = (u_x^\sigma)^* O_\omega u_x^\sigma, \quad \forall x \in \mathbb{Z}^d, \omega \in \Omega. \tag{4.14}$$

The following lemma is straightforward.

Lemma 4.2.10. *Retain the assumptions of Definition 4.2.9. With point-wise operations*

$$(O_1 + \lambda O_2)_\omega := (O_1)_\omega + \lambda (O_2)_\omega,\ (O^*)_\omega := (O_\omega)^*,\ \forall O, O_1, O_2 \in A^{U,\sigma}, \lambda \in \mathbb{C}, \omega \in \Omega,$$

and the norm defined by

$$\|O\| := \sup_{\omega\in\Omega} \|O_\omega\|, \quad \forall O \in A^{U,\sigma},$$

the set $A^{U,\sigma}$ is a C^*-algebra. If U is real or quaternionic and $\sigma \equiv 1$, then A^U is a real C^*-algebra with real structure defined by

$$(\overline{O})_\omega := \overline{(O_\omega)}, \quad \forall O \in A^{U,\sigma}, \omega \in \Omega.$$

Definition 4.2.11 (C*-algebras of bulk observables)**.** *The covariance condition in Equation* (4.14) *encodes the notion that the bulk observables of interest are macroscopically homogeneous. We therefore call the real C*-algebra*

$$\mathbb{A} := A^{W}$$

the algebra of bulk observables *(or* bulk algebra *for short), where W denotes the Nambu space of the local space V.*

As we have seen in the context of Definition 4.1.22, charge-conserving observables can be reduced to operators within the complex C*-algebra $\mathcal{L}(\mathcal{V})$. Subsequently, the algebra of complex (or charge-conserving) bulk observables *is defined as the complex C*-algebra*

$$\mathrm{A} := A^{V,\sigma}.$$

4.2.3 The bulk algebra as a crossed product C*-algebra

In the previous subsection, we have constructed the bulk algebra under the premise that all definitions shall be based on well-motivated physical assumptions. The aim of this section is to identify this algebra of (complex) bulk observables with a more abstract C*-algebra, a so-called (twisted) crossed product C*-algebra. This will show that our C*-algebra is indeed $*$-isomorphic to the algebra of observables with homogeneous disorder that is used in the literature, as explained at the beginning of this chapter. Moreover, this subsection serves as a preparation for our derivation of the bulk-boundary correspondence.

Twisted crossed product C*-algebras

We begin with an overview of the basic properties of twisted crossed product C*-algebras, following [64, Sec. 2]. A general overview of complex crossed product C*-algebras can be found in [65, Sec. 7.6]. Our references for twisted crossed product C*-algebras are [22, 64].

Throughout this section, we assume that G is a discrete locally compact group (having in mind the group $G = \mathbb{Z}^d$). In particular, there is the unique normalised unimodular Haar measure on G, defined by

$$\mu(g) = 1 \quad \forall g \in G.$$

The fundamental objects for the twisted crossed product are the following twisted dynamical systems.

Definition 4.2.12 (Twisted dynamical system)**.** *Let G be a group and A be a C*-algebra. A* twisted action *of G on A is a pair (α, σ) of maps*

$$\alpha : G \longrightarrow \mathrm{Aut}(A), \quad \sigma : G \times G \longrightarrow \mathrm{U}(\mathrm{M}(A))$$

such that $\sigma(g_1, 1) = \sigma(1, g_2) = 1$, $\alpha_1 = \mathrm{id}$ and

$$\alpha_{g_1} \circ \alpha_{g_2} = \mathrm{Ad}(\sigma(g_1, g_2)) \circ \alpha_{g_1 g_2}, \quad \alpha_{g_1}(\sigma(g_2, g_3))\sigma(g_1, g_2 g_3) = \sigma(g_1, g_2)\sigma(g_1 g_2, g_3).$$

for all $g_1, g_2, g_3 \in G$. Here, $\mathrm{U}(\mathrm{M}(A))$ *denotes the set of unitary elements in* $\mathrm{M}(A)$*. The quadruple (A, G, α, σ) is called* twisted dynamical system*. If $\sigma \equiv 1$, then α is called an* action *and* $(A, G, \alpha, 1)$ *is called* dynamical system*.*

The (twisted) dynamical system is called real *if A is a real C*-algebra and α_g and $\sigma(g, g')$ are real for all $g, g' \in G$.*

In this work, we are only interested in the following two (twisted) dynamical systems.

Example 4.2.13. *Let Ω be a space of disorder configurations and let σ be a magnetic cocycle. The set $(C(\Omega) \otimes \mathrm{End}(V), \mathbb{Z}^d, \alpha, \sigma)$ defines a twisted dynamical system, where (α, σ) is defined on $C(\Omega) \otimes \mathrm{End}(V)$ by*

$$\alpha_x(f)(\omega) := f(\omega \cdot x) \quad \forall x \in \mathbb{Z}^d, f \in C(\Omega) \otimes \mathrm{End}(V), \omega \in \Omega. \tag{4.15}$$

Similarly, $(C(\Omega) \otimes \mathrm{End}(W), \mathbb{Z}^d, \alpha, 1)$ with $\alpha_x(f)(\omega) := f(\omega \cdot x)\ \forall x \in \mathbb{Z}^d, f \in C(\Omega) \otimes \mathrm{End}(W), \omega \in \Omega$ defines a real dynamical system.

An important tool for the explicit description of twisted dynamical systems and their crossed products are the following covariant pairs and representations.

Definition 4.2.14 (Covariant pair; covariant representation)**.** *A* covariant pair *(φ, U) for (A, G, α, σ) is given by a non-degenerate $*$-morphism $\varphi : A \to B$ for a C^*-algebra B and a map $U : G \to \mathrm{U}(\mathrm{M}(B))$ such that*

$$U_g U_{g'} = \varphi(\sigma(g, g'))U_{gg'}, \quad \varphi(\alpha_g(a))U_g = U_g\varphi(a) \quad \forall g, g' \in G, a \in A.$$

A real covariant pair *(φ, U) of a real twisted dynamical system is defined in the same way with the additional constraint that U_g is real for all $g \in G$ and φ, B are real.*

A (real) covariant representation *of a twisted dynamical system (A, G, α, σ) is a (real) covariant pair (π, U) with $B = \mathcal{L}(\mathcal{H})$ for some (real) Hilbert space $\mathcal{H}$.*

The following theorem gives us a simple method to construct such covariant representations.

Theorem 4.2.15 (Induced representation)**.** [22, Thm. 4.1] *Let $\tilde{\pi} : A \to \mathcal{L}(\mathcal{H})$ be a $*$-representation. Define $\pi : A \to \mathcal{L}(\ell^2(G, \mathcal{H}))$ and $U : G \to \mathrm{U}(\ell^2(G, \mathcal{H}))$ on the Hilbert space $\ell^2(G, \mathcal{H})$ by*

$$(\pi(a)f)(g) := \tilde{\pi}(\alpha_g(a))f(g), \quad (U_g f)(g') := \tilde{\pi}(\sigma(g', g))f(g'g) \quad \forall a \in A, g, g' \in G \tag{4.16}$$

for all $f \in \ell^2(G, \mathcal{H})$. Then (π, U) is a covariant representation for (A, G, α, σ).

If (A, G, α, σ) is a real twisted dynamical system and $\tilde{\pi}$ is a real $*$-representation, then (4.16) *defines a real covariant representation.*

A crucial aspect for our identification of the algebra of observables with a (twisted) crossed product is the following result about faithful representations.

Theorem 4.2.16. [64, Thm. 3.11] *Let G be amenable and let $\tilde{\pi} : A \to \mathcal{L}(\mathcal{H})$ be a faithful representation. Then the induced representation (π, R) is faithful.*

Twisted representations for the twisted dynamical systems of Example 4.2.13 are given as follows.

Example 4.2.17. *By Theorem 4.2.15, the pair (π, R^σ) defined by*

$$(\pi(f)\psi)(\omega, x) := f(\omega \cdot (-x))\psi(\omega, x), \quad (R^\sigma_y \psi)(\omega, x) := \sigma(x, y)\psi(\omega, x + y)$$

for all $y \in \mathbb{Z}^d, f \in C(\Omega) \otimes \mathrm{End}(V), \psi \in L^2(\Omega, \mathbb{P}) \otimes \mathcal{V}, \omega \in \Omega$ and $x \in \Lambda$, defines a covariant representation of the twisted dynamical system $(C(\Omega) \otimes \mathrm{End}(V), \mathbb{Z}^d, \alpha, \sigma)$.

Analogously, the pair (π, R), defined as

$$(\pi(f)\psi)(\omega, x) := f(\omega \cdot (-x))\psi(\omega, x), \quad (R_y\psi)(\omega, x) := \psi(\omega, x + y)$$

for all $y \in \mathbb{Z}^d, f \in C(\Omega) \otimes \mathrm{End}(W), \psi \in L^2(\Omega, \mathbb{P}) \otimes \mathcal{W}, \omega \in \Omega$ and $x \in \Lambda$, defines a real covariant representation for the real dynamical system $(C(\Omega) \otimes \mathrm{End}(W), \mathbb{Z}^d, \alpha, 1)$.

Now we introduce the notion of twisted crossed products.

Definition 4.2.18 (Twisted crossed product)**.** *A* twisted crossed product *C^*-algebra for a twisted dynamical system (A, G, α, σ) is a C^*-algebra B together with a covariant pair $(\varphi : A \to \mathrm{M}(B), V : G \to \mathrm{U}(\mathrm{M}(B)))$ satisfying:*

- *For any covariant representation (π, U) over $\mathcal{H}$ of (A, G, α, σ), there is a non-degenerate representation of B on $\mathcal{H}$, denoted by $\pi \rtimes_{\alpha,\sigma} U$ such that*

$$\pi = (\pi \rtimes_{\alpha,\sigma} U) \circ \varphi, \quad U = (\pi \rtimes_{\alpha,\sigma} U) \circ V.$$

- *The set $\left\{\sum_{g \in G} \varphi(z(g))V_g \,\middle|\, z \in \ell^1(G, A)\right\}$ is dense in B.*

Proposition 4.2.19. [64, Prop. 2.7] *Let (A, G, α, σ) be a twisted dynamical system. Then there is a crossed product (B, φ, V). It is universal in the sense that if (B', φ', V') is another crossed product, then there is an isomorphism $\psi : B \to B'$ such that $\psi \circ \varphi = \varphi'$ and $\psi \circ V = V'$.*

Thus, we see that the crossed product C*-algebra is the universal C*-algebra defined by generators τ_g for all $g \in G$ and $a \in A$, and the following relations: the C*-algebraic relations of A, and the relations

$$\tau_g \tau_{g'} = \sigma(g, g')\tau_{gg'}, \quad \tau_g^* \tau_g = \tau_g \tau_g^* = 1, \quad \tau_g a = \alpha_g(a)\tau_g, \quad \forall g, g' \in G, a \in A.$$

We denote the twisted crossed product C*-algebra by $A \rtimes_{\alpha,\sigma} G$. Whenever $\sigma \equiv 1$, we drop σ from the notation and call $A \rtimes_\alpha G$ the *crossed product* C*-algebra.

In the real case, we will prove that the algebra of observables is isomorphic to a real crossed product. A real structure on a (twisted) crossed product is in general defined as follows.

Definition 4.2.20 (Real structures on twisted crossed products)**.** *If (A, G, α, σ) is a real twisted dynamical system, we extend the real structure of A to $A \rtimes_{\alpha,\sigma} G$ by declaring the τ_g to be real for all $g \in G$.*

The crossed product C^*-algebra of bulk observables

We now assume that we are given a space Ω of disorder configurations as in Definition 4.2.7 and a magnetic cocycle σ as in Equation (4.13). As explained before, we only allow for non-trivial twisting in the charge-conserving cases. Then we obtain the following result.

Theorem 4.2.21. *Let α, σ be given as in Example 4.2.13. Then the twisted crossed product C^*-algebra $(C(\Omega) \otimes \mathrm{End}(V)) \rtimes_{\alpha,\sigma} \mathbb{Z}^d$ is $*$-isomorphic to* $\mathbb{A}$.

Proof. We can consider A as a C*-subalgebra of $\mathcal{L}(L^2(\Omega, \mathbb{P}) \otimes \mathcal{V})$. For brevity, let $G := \mathbb{Z}^d$.

Let (π, R^σ) denote the twisted covariant pair of Example 4.2.17. This pair induces the $*$-representation $\Phi := \pi \rtimes_{\alpha,\sigma} R^\sigma$ of $(C(\Omega) \otimes \mathrm{End}(V)) \rtimes_{\alpha,\sigma} G$. As G is amenable, it follows by Theorem 4.2.16 that Φ is faithful.

We claim that the image of Φ lies in A. By definition, $\Phi(\tau_g)_\omega$ and $\Phi(f)_\omega$ are controlled for any $g \in G$, $f \in C(\Omega) \otimes \mathrm{End}(V)$, and $\omega \in \Omega$. Hence, $\Phi(a)_\omega \in C^*_u(\Lambda, V)$ for any $a \in (C(\Omega) \otimes \mathrm{End}(V)) \rtimes_{\alpha,\sigma} G$.

It remains to check the covariance condition. We compute for $g, g' \in G$ and $\omega \in \Omega$:

$$\begin{aligned}(u^\sigma_{g'})^* \Phi(\tau_g)_\omega u^\sigma_{g'} &= \sigma(\cdot, -g')u_{-g'}\sigma(\cdot, g)u_{-g}\sigma(\cdot, g')u_{g'} \\ &= \sigma(\cdot, -g')\sigma(\cdot + g', g)\sigma(\cdot + g' + g, g')u_{-g} = R^\sigma_g = \Phi(\tau_g)_{\omega\cdot g'}.\end{aligned}$$

Similarly, for $f \in C(\Omega) \otimes \mathrm{End}(V)$, $g \in G$, and $\omega \in \Omega$, we have

$$(u^\sigma_g)^* \Phi(f)_\omega u^\sigma_g = \sum_{x\in\Lambda} \sigma(x, -g)\sigma(x, g) f(\omega \cdot x)|x - g\rangle\langle x - g| = \Phi(f)_{\omega\cdot g}.$$

This shows that $\mathrm{im}\,\Phi \subseteq$ A. We have to prove the equality.

As G is an amenable group, there is an approximate unit (ϕ_n) for $c_0(G)$ consisting of finitely supported functions of positive type [72, Lem. 11.19]. We may assume that $\phi_n(0) = 1$. For every n, there is a finite-dimensional Hilbert space $\mathcal{H}_n$, a unitary representation π_n of G on $\mathcal{H}_n$, and a unit vector $\xi_n \in \mathcal{H}_n$ such that

$$\phi_n(g) = \langle \xi_n | \pi_n(g) \xi_n \rangle \quad \forall g \in G.$$

Let (ξ^i_n) be an orthonormal basis of $\mathcal{H}_n$. Set

$$f^i_n(g) := \langle \xi^i_n | \pi_n(g) \xi_n \rangle \quad \forall g \in G,$$

so that f^i_n is bounded on G. For $T \in \mathcal{L}(\mathcal{V})$, we define

$$S_n(T) := \sum_i M^*_{f^i_n} T M_{f^i_n} \in \mathcal{L}(\mathcal{V}).$$

Let $T(x, y) := \langle x, Ty\rangle_\Lambda$ denote the kernel of T. Then

$$S_n(T) = \sum_{x,y\in\Lambda} \sum_i \langle \pi_n(x)\xi_n | \xi^i_n \rangle\langle \xi^i_n | \pi_n(y)\xi_n\rangle T(x, y)|x\rangle\langle y| = \sum_{x,y\in\Lambda} \phi_n(y - x)T(x, y)|x\rangle\langle y|.$$

In particular, $S_n(1) = 1$, since $\phi_n(0) = 1$. From the definition, it is clear that S_n is completely positive, see also [72, Lem. 11.17]. In particular, S_n is positive, and therefore

$$\|S_n\| \leqslant \|S_n(1)\| = 1.$$

That is, S_n is a contraction. As S_n leaves controlled operators invariant, this implies that S_n leaves $C^*_u(\Lambda, V)$ invariant.

Let $T \in$ A. We set $S_n(T)_\omega := S_n(T_\omega)$. Since S_n commutes with u^σ_g for every $g \in G$, this defines an element of A. Let $\varepsilon > 0$ be arbitrary. As Ω is compact, there are finitely many $\omega_i \in \Omega$ and open neighbourhoods $U_i \subseteq \Omega$ of ω_i covering Ω such that

$$\|T_\omega - T_{\omega_i}\| \leqslant \frac{\varepsilon}{6} \quad \forall \omega \in U_i$$

and all i. There exist a finite $R > 0$ and some R-controlled operators $T_i \in C^*_u(\Lambda, V)$ such that

$$||T_{\omega_i} - T_i|| \leqslant \frac{\varepsilon}{6}$$

for all i. By [72, Lem. 4.27], there is a constant $C_R > 0$ only depending on R such that

$$||T_i - S_n(T_i)|| \leqslant C_R \cdot \sup_{||x|| \leqslant R} \left|1 - \phi_n(x)\right| \cdot \sup_{||x-y|| \leqslant R} \left\|T_i(x,y)\right\|$$

for all n and i. Since ϕ_n converges to 1 uniformly on $\{||\cdot \leqslant R||\}$, there is an $N \in \mathbb{N}$ such that for all $n \geqslant N$ and all i, the right-hand side is $\leqslant \frac{\varepsilon}{3}$.

If now $n \geqslant N$ and $\omega \in \Omega$, then there is some i such that $\omega \in \Omega$. It follows that

$$\begin{aligned}
&||T_\omega - S_n(T_\omega)|| \\
\leqslant &||T_\omega - T_{\omega_i}|| + ||T_{\omega_i} - T_i|| + ||T_i - S_n(T_i)|| + ||S_n(T_i - T_{\omega_i})|| + ||S_n(T_{\omega_i} - T_\omega)|| \\
\leqslant &2||T_\omega - T_{\omega_i}|| + 2||T_{\omega_i} - T_i|| + ||T_i - S_n(T_i)|| \leqslant \frac{4\varepsilon}{6} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Therefore, the sequence $S_n(T)$ converges to T in A.

For any $n \in \mathbb{N}$ and $g \in \operatorname{supp}(\phi_n)$, define $f_{n,g} \in C(\Omega) \otimes \operatorname{End}(V)$ by $f_{n,g}(\omega) := T_\omega(0, g)\ \forall \omega \in \Omega$. Then

$$\begin{aligned}
\operatorname{im}\Phi \ni \sum_{\phi_n(g) \neq 0} \phi_n(g)\pi(f_{n,g})R^\sigma_{-g} &= \sum_{\phi_n(g)\neq 0, x \in \Lambda} \phi_n(g) T_{\omega \cdot x}(0, g)|x\rangle\langle x|R^\sigma_{-g} \\
&= \sum_{\phi_n(g-x)\neq 0} \phi_n(g - x) T_\omega(x, g)|x\rangle\langle g| = S_n(T).
\end{aligned}$$

As the image of Φ is closed, T is contained in it. Thus, Φ is surjective. Since we already know that Φ is injective, this proves the theorem. □

Theorem 4.2.22. *Let α be given as in Example 4.2.13 with $\sigma \equiv 1$. Then the crossed product C^*-algebra $(C(\Omega) \otimes \operatorname{End}(W)) \rtimes_\alpha \mathbb{Z}^d$ is isomorphic to $\mathbb{A}$ as a real C^*-algebra.*

Proof. Here we consider $\mathbb{A}$ as a real C*-subalgebra of the real C*-algebra $\mathcal{L}(L^2(\Omega, \mathbb{P}) \otimes \mathcal{W})$ equipped with usual conjugation over $\mathcal{W}$ and the point-wise complex conjugation over $L^2(\Omega, \mathbb{P})$.

We consider the covariant pair (π, R) of Example 4.2.17. Since $\mathbb{Z}^d$ is amenable, it follows by Theorem 4.2.16 that the induced $*$-representation $\Phi := \pi \rtimes_\alpha R$ of $(C(\Omega) \otimes \operatorname{End}(W)) \rtimes_\alpha \mathbb{Z}^d$ is faithful. Since R_x is real for every $x \in \mathbb{Z}^d$ and π is real, it follows that Φ is real, too.

The proof of $\operatorname{im}\Phi = \mathbb{A}$ is the same as for the bulk algebra of complex observables. □

4.2.4 Disordered invariant quasi-particle vacua

Here we finally establish a description of the ground states of disordered topological insulators and superconductors in the bulk. We have already understood in Section 4.1 how to describe stable free-fermion ground states in terms of QPV. Due to our detailed derivation of the algebra of observables for disordered topological insulators and superconductors, we can now make the following definition.

Definition 4.2.23 (Symmetric disordered IQPV)**.** *A* disordered invariant quasi-particle vacuum (IQPV) of symmetry index (r, s) *is given by a tuple*

$$(J; K_1, \dots, K_r, J_1 \dots, J_s),$$

where $J \in \mathbb{A}$ *and* $K_1, \dots, K_r, J_1, \dots, J_s \in \mathrm{End}(W) \subseteq \mathbb{A}$ *such that* $(J_\omega; K_1, \dots, K_r, J_1 \dots, J_s)$ *is a QPV of symmetry index* (r, s) *for all* $\omega \in \Omega$*, cf. Definition 4.1.19.*

A complex disordered IQPV of symmetry index (r, s) *is given by a tuple*

$$(J; K_1 \dots, K_r, J_1, \dots, J_s),$$

where $J \in \mathrm{A}$ *and* $K_1, \dots, K_r, J_1, \dots, J_s \in \mathrm{End}(V) \subseteq \mathrm{A}$ *such that* $(J_\omega; K_1, \dots, K_r, J_1 \dots, J_s)$ *is a complex QPV of index* (r, s) *for all* $\omega \in \Omega$*, cf. Definition 4.1.22.*

As before, we will interchangeably use the notation $(J; K_1, \dots, K_r, J_1, \dots, J_s)$ *and* $(J; \phi)$*, where* ϕ *is the induced unital* $*$*-morphism from* $C\ell_{r,s}$ *or* $\mathbb{C}\ell_{r,s}$ *to* $\mathbb{A}$ *or* A*, respectively.*

The name IQPV was introduced in [49]. There it denotes the translationally invariant QPV. To emphasize that we consider systems with disorder where the translational invariance is broken on microscopic scale, we equip the IQPV with the prefix 'disordered'.

We emphasise once more that, under the premise $\sigma \equiv 1$, a charge-conserving disordered IQPV of symmetry index (r, s) is the same as a disordered IQPV of symmetry index (r, s) such that $[J, Q] = [K_a, Q] = [J_\alpha, Q] = 0$ for all $\alpha = 1, \dots, s$ and $a = 1, \dots, r$.

In Definition 4.2.23, we demanded in particular that the pseudo-symmetries act locally in the sense that they are elements of $\mathrm{End}(W) \subseteq \mathbb{A}$ in the real case and $\mathrm{End}(V) \subseteq \mathrm{A}$ in the complex case. In particular, the pseudo-symmetries thereby commute with all lattice translations.

We have seen in Section 4.1.2 that all symmetry classes in the Tenfold Way can be described by pseudo-symmetries that are generated by local symmetries. It can easily be checked that all physical examples of pseudo-symmetries that we have introduced in Subsection 4.1.2 define elements in $\mathrm{End}(W)$ or $\mathrm{End}(V)$. Thus, this locality restriction on the pseudo-symmetries still allows us to give an exhaustive classification of these symmetry classes.

However, it is in general not possible to translate symmetries acting non-trivially on the Bravais lattice Λ, so called *crystalline symmetries* such as translation symmetries, into pseudo-symmetries. There exists a plethora of classification schemes in the context of such crystalline symmetries for clean translationally invariant systems [3, 79, 80, 21]. See [63] for a pedagogical introduction to this topic.

4.3 Boundary systems with disorder

In this section, we will construct a mathematical approach for disordered topological insulators and superconductors with boundary. The construction is almost the same as for the bulk algebra. The major difference is that we include a boundary into our picture which manifestly breaks the translational invariance of the system. Once we have established the algebra of observables, we will construct an explicit relation between this algebra and the bulk algebra of the previous section. This will result in the bulk-boundary short exact sequence. At the end of this section, we will introduce the notion of disordered IQPV with boundary, describing the ground states of disordered topological insulators and superconductors in the vicinity of a boundary.

4.3.1 The C*-algebra of half-space observables

Here, we construct the algebra of disordered observables over a half-space lattice in the tight-binding approximation in a similar manner as in Subsection 4.2.2.

We begin our construction by introducing a boundary. In this work, we will consider the most fundamental prototype of a boundary, which has also been considered in [78, 48]. It is obtained by cutting the lattice $\Lambda \times U$ perpendicular to one translational direction in Λ, as depicted in Figure 4.2.

Hitherto, all directions in the lattice ware equivalent and therefore we can now assume without loss of generality that we cut the lattice perpendicular to the dth direction. The remaining lattice is called the *half-space* and is given by

$$\widehat{\Lambda} \times U \quad \text{with} \quad \widehat{\Lambda} := (\mathbb{Z}^{d-1} \times \mathbb{N}),$$

where $\mathbb{N} = \{0, 1, 2, \ldots\}$. In particular, we assume that the cut preserves the basis U. The sub-lattice

$$\Lambda_\partial \times U \quad \text{with} \quad \Lambda_\partial := \mathbb{Z}^{d-1} \equiv \mathbb{Z}^{d-1} \times \{0\} \subseteq \widehat{\Lambda}$$

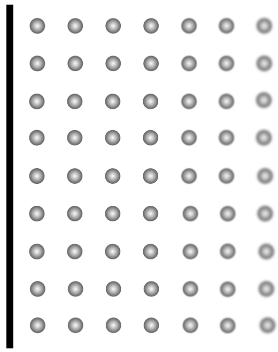

Figure 4.2: Half-space lattice of the cubic Bravais lattice.

is called the *boundary*. The lattice translations parallel to the boundary induce a (twisted) action of $\mathbb{Z}^{d-1}$ on $\Lambda_\partial \times U$.

Definition 4.3.1 (Half-space projection q)**.** *The orthogonal projection*

$$q : \ell^2(\Lambda) \longrightarrow \ell^2(\widehat{\Lambda}) \tag{4.17}$$

restricts the bulk states onto the half-space states and thus represents the cutting process described above. We refer to this projection as the half-space projection.

The Hilbert space of single-particle states over the half-space in the tight-binding approximation is given by

$$\widehat{\mathcal{V}} := \ell^2(\widehat{\Lambda}) \otimes V,$$

and the corresponding Nambu space is

$$\widehat{\mathcal{W}} := \widehat{\mathcal{V}} \oplus \widehat{\mathcal{V}}^* \cong \ell^2(\widehat{\Lambda}) \otimes W.$$

Our strategy for the construction of the algebra of half-space observables is the same as for the algebra of bulk observables. The major difference to the bulk is that over the half-space the lattice translations along the dth direction cannot be unitary operators anymore. We briefly explain this in more detail.

By tensoring the half-space projection q with id_V or id_W, we obtain orthogonal projections

$$q_V : \mathcal{V} \longrightarrow \widehat{\mathcal{V}} \quad \text{and} \quad q_W : \mathcal{W} \longrightarrow \widehat{\mathcal{W}}.$$

In the following, we use the notation $q \equiv q_V \equiv q_W$ as this should not lead to confusion. Let us now consider the effect of this projection on the twisted lattice translations

$$(u_x^\sigma \psi)(y) = \sigma(y, x)\psi(y - x) \quad \forall x \in \mathbb{Z}^d, y \in \Lambda, \psi \in \mathcal{V}.$$

We define

$$\hat{u}_x^\sigma := q u_x^\sigma q \quad \forall x \in \mathbb{Z}^d.$$

The lattice translations $\hat{u}_x^\sigma$ commute with q if $x \in \mathbb{Z}^{d-1} \times \{0\} \subset \mathbb{Z}^{d-1} \times \mathbb{Z} = \mathbb{Z}^d$ and are therefore still unitary. However, the lattice translation $u_d^\sigma := u_{e_d}^\sigma$ along the dth direction, where $e_d = (0, \ldots, 0, 1) \in \mathbb{Z}^{d-1} \times \mathbb{N}$, is mapped onto the operator

$$\hat{u}_d^\sigma = q u_d^\sigma|_{\widehat{\mathcal{V}}} \in \mathcal{L}(\ell^2(\widehat{\Lambda}))$$

over the half-space, which is no longer unitary. Instead, it is a partial isometry as $(\hat{u}_d^\sigma)^* \hat{u}_d^\sigma = 1$ and $1 - \hat{u}_d^\sigma (\hat{u}_d^\sigma)^*$ is the orthogonal projection onto $\mathbb{Z}^{d-1} \times \{0\}$.

For $\sigma \equiv 1$, the lattice translations $\hat{u}_x^\sigma = \hat{u}_x \in \mathcal{L}(\widehat{\mathcal{W}})$ can be defined in the same manner.

With these half-space translations, we can define the algebra of half-space observables as follows.

Definition 4.3.2 (Algebra of half-space observables)**.** *Let* Ω *be a space of disorder configurations. Let* $U \in \{V, W\}$ *and* $\sigma : \mathbb{Z}^d \times \mathbb{Z}^d \longrightarrow \mathrm{U}(1)$ *be a magnetic cocycle as defined in Equation* (4.13) *(possibly trivial). Define* $\widehat{A}^{U,\sigma}$ *as the closed subalgebra of the set* $C(\Omega, C_u^*(\widehat{\Lambda}, U))$ *of norm continuous maps* $O : \Omega \longrightarrow C_u^*(\widehat{\Lambda}, U)$ *generated by those maps* O *that are covariant in the sense that*

$$O_{\omega \cdot x} = (\hat{u}_x^\sigma)^* O_\omega \hat{u}_x^\sigma, \quad \forall x \in \mathbb{Z}^{d-1} \times \mathbb{N}, \omega \in \Omega.$$

Equipped with the point-wise operations and the sup-norm, $\widehat{A}^{U,\sigma}$ defines a C*-algebra. For $U = V$, we define

$$\widehat{\mathrm{A}} := \widehat{A}^{V,\sigma}.$$

This complex C*-algebra is called the *algebra of charge-conserving half-space observables*, or simply the *complex half-space algebra*.

For $\sigma \equiv 1$ and $U = W$, we define

$$\widehat{\mathbb{A}} := \widehat{A}^W,$$

equipped with the point-wise conjugation. This real C*-algebra is called the *algebra of half-space observables*, or simply the *half-space algebra*.

4.3.2 The bulk-boundary short exact sequence

In this section, we will construct a short-exact sequence that exhibits the relations between the half-space, boundary and bulk in the following way.

We proved that the bulk algebra is the universal C*-algebra generated by $C(\Omega) \otimes \mathrm{End}(W)$ (resp. $C(\Omega) \otimes \mathrm{End}(V)$) and the (twisted) lattice translations over Λ. For the half-space algebra, we will now establish a similar result. This will enable us to define a canonical surjective $*$-morphism ρ from the half-space algebra onto the bulk algebra giving rise to the bulk-boundary SES.

In more detail, we will show that the half-space algebra is isomorphic to a so-called monoidal twisted crossed product C*-algebra. We take the pragmatic approach of explaining only the most important definitions in order to derive the desired results. This is because the theory of crossed product C*-algebras for (twisted) monoidal semi-groups is more involved than for groups. The advantage of a more detailed explanation for the reader to understand this work is not of such extent that it would be justified. Furthermore, this theory is not expected to be applicable to more general geometries of the boundary. There is already the work [18] that has developed an approach using groupoid C*-algebras in order to describe lattices with more general boundaries.

We begin our short exposition by introducing twisted monoidal dynamical systems.

Definition 4.3.3 (Twisted monoidal dynamical system)**.** *A twisted monoidal dynamical system $(A, \widehat{G}, \alpha, \sigma)$ is given by a C*-algebra A, a left-cancellative unital monoid $\widehat{G}$ and a pair of maps*

$$\alpha : \widehat{G} \to \mathrm{Aut}(A), \quad \sigma : \widehat{G} \times \widehat{G} \to \mathrm{U}(\mathrm{M}(A)),$$

such that for all $g_1, g_2, g_3 \in \widehat{G}$ we have $\sigma(g_1, 1) = \sigma(1, g_2) = 1$, $\alpha_1 = \mathrm{id}$ and

$$\alpha_{g_1} \circ \alpha_{g_2} = \mathrm{Ad}(\sigma(g_1, g_2)) \circ \alpha_{g_1 g_2}, \quad \alpha_{g_1}(\sigma(g_2, g_3))\sigma(g_1, g_2 g_3) = \sigma(g_1, g_2)\sigma(g_1 g_2, g_3).$$

If A is a real C-algebra, we demand that $\alpha_g, \sigma(g, g')$ are real for all $g, g' \in \widehat{G}$ and we call $(A, \widehat{G}, \alpha, \sigma)$ a real twisted monoidal dynamical system.*

The following example contains the relevant twisted monoidal dynamical systems for the present work.

Example 4.3.4. *Let Ω be a space of disorder configurations, $C^U := C(\Omega) \otimes \mathrm{End}(U)$ for $U \in \{V, W\}$, and let α, σ be given as in Example 4.2.13. We already know that $(C^V, \mathbb{Z}^d, \alpha, \sigma)$ for defines a twisted dynamical system. Therefore, it is easy to see that $(C^V, \mathbb{Z}^{d-1} \times \mathbb{N}, \alpha, \sigma)$, where α, σ are restricted to $\mathbb{Z}^{d-1} \times \mathbb{N}$, defines a twisted monoidal dynamical system.*

In the case $U = W$ and $\sigma \equiv 1$, $(C^W, \mathbb{Z}^d, \alpha, 1)$ defines a real monoidal dynamical system.

In the same manner as for the twisted dynamical systems, we now introduce covariant pairs and representations.

Definition 4.3.5 (Twisted covariant pair)**.** *A covariant pair $(\varphi, \widehat{U})$ for $(A, \widehat{G}, \alpha, \sigma)$ is given by a non-degenerate $*$-morphism $\varphi : A \to B$ into a C*-algebra B and an isometric map $\widehat{U} : \widehat{G} \to \mathrm{M}(B)$ such that*

$$\widehat{U}_g \widehat{U}_{g'} = \varphi(\sigma(g, g'))\widehat{U}_{gg'}, \quad \varphi(\alpha_g(a))\widehat{U}_g = \widehat{U}_g \varphi(a) \quad \forall g, g' \in \widehat{G}, a \in A.$$

A real covariant pair $(\varphi, \widehat{U})$ *of a real twisted monoidal dynamical system is defined in the same way with the additional obstruction that $\widehat{U}_g$ is real for all $g \in G$ and φ, B are real.*

A (real) covariant representation of $(A, \widehat{G}, \alpha, \sigma)$ is a (real) covariant pair for $(A, \widehat{G}, \alpha, \sigma)$ with $B = \mathcal{L}(\mathcal{H})$ for some (real) Hilbert space $\mathcal{H}$.

Note that for twisted monoidal dynamical systems there is in general no result as Theorem 4.2.16 about faithful covariant representations.

Covariant representations for the twisted monoidal dynamical systems of Example 4.3.4 are given as follows.

Example 4.3.6. *A covariant representation $(\hat{\pi}, \widehat{R^\sigma})$ for $(C^V, \mathbb{Z}^{d-1} \times \mathbb{N}, \alpha, \sigma)$ is given by restriction of the covariant pair (π, R^σ). In more detail, $\hat{\pi} : C^V \to \mathcal{L}(L^2(\Omega, \mathbb{P}) \otimes \widehat{\mathcal{V}})$ is the $*$-representation defined by*

$$(\hat{\pi}(f)_\omega \psi)(x) = f(\omega \cdot (-x))\psi(x) \quad \forall f \in C(\Omega) \otimes \mathrm{End}(V), x \in \widehat{\Lambda}, \psi \in \widehat{\mathcal{V}}$$

and $\widehat{R}_x^\sigma := q R_x^\sigma q \in \mathcal{L}(L^2(\Omega,\mathbb{P}) \otimes \widehat{\mathcal{V}})$ *satisfies*

$$\widehat{R}_x^\sigma(\widehat{R}_x^\sigma)^* = 1, \quad \widehat{R}_x^\sigma \widehat{R}_y^\sigma = \sigma(x,y)\widehat{R}_{x+y}^\sigma, \quad \hat{\pi}(\alpha_x(f))\widehat{R}_x^\sigma = \widehat{R}_x^\sigma \hat{\pi}(f) \quad \forall x,y \in \mathbb{Z}^{d-1} \times \mathbb{N}.$$

In the case $U = W$, $\sigma \equiv 1$, *a real covariant pair* $(\hat{\pi}, \widehat{R})$ *for* $(C^W, \mathbb{Z}^{d-1} \times \mathbb{N}, \alpha)$ *can be defined analogously.*

Remark 4.3.7. *Observe that the equation* $\widehat{R}_x^\sigma(\widehat{R}_x^\sigma)^* = 1$ *for all* $x \in \mathbb{Z}^{d-1} \times \mathbb{N}$ *in the example above differs from the condition* $(\hat{u}_x^\sigma)^* \hat{u}_x^\sigma = 1$ *for all* $x \in \mathbb{Z}^{d-1} \times \mathbb{N}$ *for the lattice shifts in Definition 4.3.2. The rationale here is that operators that* commute *with (truncated) left translations are* generated *by (truncated) right translations.*

Finally, crossed product C*-algebras for twisted monoidal dynamical systems are defined as follows.

Definition 4.3.8 (Crossed product of twisted monoidal dynamical system)**.** *Let* $C^U \rtimes_{\alpha,\sigma} (\mathbb{Z}^{d-1} \times \mathbb{N})$ *be the universal C*-algebra that is generated by the generators* $f \in C^U$ *and* $\hat{\tau}_x$, *for* $x \in \mathbb{Z}^{d-1} \times \mathbb{N}$, *the* $*$*-algebraic relations for* $C(\Omega) \otimes \mathrm{End}(U)$ *and the relations*

$$\hat{\tau}_x \hat{\tau}_x^* = 1, \quad \hat{\tau}_x \hat{\tau}_y = \sigma(x,y)\hat{\tau}_{x+y}, \quad \alpha_x(f)\hat{\tau}_x = \hat{\tau}_x f \quad \forall x,y \in \mathbb{Z}^{d-1} \times \mathbb{N}, f \in C(\Omega) \otimes \mathrm{End}(U).$$

If $U = W$, $\sigma \equiv 1$, *the C*-algebra* $C^W \rtimes_\alpha (\mathbb{Z}^{d-1} \times \mathbb{N})$ *becomes a real C*-algebra when equipped with the induced real structure from* C^W *and by declaring the* $\hat{\tau}_x$ *to be real.*

The algebra of half-space observables can now be identified with such a crossed product, as explained in the following proposition.

Proposition 4.3.9. *The C*-algebra* $\widehat{\mathrm{A}}$ *is isomorphic to the crossed product C*-algebra* $(C(\Omega) \otimes \mathrm{End}(V)) \rtimes_{\alpha,\sigma} (\mathbb{Z}^{d-1} \times \mathbb{N})$ *and the real C*-algebra* $\widehat{\mathbb{A}}$ *is isomorphic to* $(C(\Omega) \otimes \mathrm{End}(W)) \rtimes_\alpha (\mathbb{Z}^{d-1} \times \mathbb{N})$.

Proof. For the sake of brevity we use the notation $\widehat{G} := \mathbb{Z}^{d-1} \times \mathbb{N}$. We will only prove the case $U = V$, where we have a possibly non-trivial twisting. The case $U = W$, $\sigma \equiv 1$ can be proved similarly. It is easy to check that the constructed $*$-isomorphism then becomes a real $*$-isomorphism.

Let $(\hat{\pi}, \widehat{R}^\sigma)$ be the covariant representation of Example 4.3.6. By definition, $C^V \rtimes_{\alpha,\sigma} \widehat{G}$ is universal for covariant pairs of the dynamical system $(C^V, \mathbb{Z}^{d-1} \times \mathbb{N}, \alpha, \sigma)$ and therefore $(\hat{\pi}, \widehat{R}^\sigma)$ defines a unique $*$-morphism

$$\widehat{\Phi} := \hat{\pi} \rtimes_{\alpha,\sigma} \widehat{R}^\sigma : (C(\Omega) \otimes \mathrm{End}(V)) \rtimes_{\alpha,\sigma} \widehat{G} \longrightarrow \mathcal{L}(L^2(\Omega,\mathbb{P}) \otimes \widehat{\mathcal{V}}).$$

For the sake of brevity, we denote the crossed product by $\widehat{B} := (C(\Omega) \otimes \mathrm{End}(V)) \rtimes_{\alpha,\sigma} \widehat{G}$.

In the following, we will show that $\widehat{\Phi}$ defines a $*$-isomorphism from $\widehat{B}$ to $\widehat{\mathrm{A}}$. We begin by proving the injectivity of $\widehat{\Phi}$. Thereby, we follow an idea by Murphy [62, Thm. 4.4], who proved the case $\sigma \equiv 1$ and $d = 1$.

Let $T \cong \mathrm{U}(1)^d$ be the dual group of $\mathbb{Z}^d$. For $\gamma \in T$, define $V_\gamma \in \mathrm{U}(\widehat{\mathcal{V}})$ by

$$V_\gamma(\psi)(x) := \gamma(x)\psi(x).$$

Then V is a unitary representation of T, and clearly V_γ commutes with $\operatorname{im}\hat{\pi}$. It is similarly straightforward that

$$V_\gamma \widehat{R^\sigma_x} = \gamma(x)\widehat{R^\sigma_x}\, V_\gamma, \quad \forall \gamma \in T, x \in \widehat{G}.$$

We may also define an action δ of T on B by letting $C(\Omega) \otimes \operatorname{End}(V)$ be point-wise fixed and setting

$$\delta_\gamma(x) := \gamma(x)x \quad \forall \gamma \in T, x \in \widehat{G}.$$

Then $\widehat{\Phi}$ is T-equivariant.

Now, it is sufficient to prove that $\widehat{\Phi}$ is injective when restricted to the fixed-point subalgebra of B^δ of the action δ. Indeed, consider normalised Haar measure on T. If $a \in \ker\widehat{\Phi}$, then

$$0 = \int_T \operatorname{Ad}(V_\gamma)(\widehat{\Phi}(a^*a))\, dt = \widehat{\Phi}\Big(\int_T \delta_\gamma(a^*a)\, d\gamma\Big),$$

so if $\widehat{\Phi}$ is injective on the δ-invariants, then

$$\int_T \delta_\gamma(a^*a)\, d\gamma = 0$$

which implies $a^*a = 0$ and hence $a = 0$, in view of the positivity of the integral.

We now show the injectivity of $\widehat{\Phi}$ on invariants. A simple calculation [62, p. 342] shows that

$$\int_T \delta_\gamma(\hat{\tau}_{x_1} \cdots \hat{\tau}_{x_n})\, d\gamma = \begin{cases} \hat{\tau}_{x_1} \cdots \hat{\tau}_{x_n}, & \text{if } x_1 + \cdots + x_n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

But if $x_1 + \cdots + x_n = 0$, then $x_1, \ldots, x_n \in G_\partial := \mathbb{Z}^{d-1} \times \{0\} \subseteq \mathbb{Z}^{d-1} \times \mathbb{N} = \widehat{G}$.

Let B_∂ denote the universal C*-algebra generated by $f \in C(\Omega) \otimes \operatorname{End}(V)$ and $\hat{\tau}_x$ for $x \in G_\partial$, subject to the C*-algebraic relations of $C(\Omega) \otimes \operatorname{End}(V)$, the algebraic relations of G_∂, and the relations

$$\tau_x^*\tau_x = \tau_x\tau_x^* = 1, \quad \tau_x\tau_y = \sigma(x, y)\tau_{x+y}, \quad \alpha_x(f)\tau_x = \tau_x f$$

for all $f \in C(\Omega) \otimes \operatorname{End}(V)$, $x, y \in G_\partial$. There is a canonical $*$-morphism $\Phi_\partial : B_\partial \longrightarrow \widehat{B}$ defined by $\Phi_\partial(\tau_x) = \hat{\tau}_x$ for all $x \in G_\partial$ and $\Phi_\partial(f) = f$ for all $f \in C(\Omega) \otimes \operatorname{End}(V)$. By Theorem 4.2.21, $\widehat{\Phi} \circ \Phi_\partial$ is injective. Hence, B_∂ is a closed $*$-subalgebra of $\widehat{B}$ and the restriction of $\widehat{\Phi}$ to B_∂ is injective. As $\int_T \delta_\gamma\, d\gamma$ is the identity on $\widehat{B}^\delta$, the above computation shows that $\widehat{B}^\delta \subseteq B_\partial$, which proves that the restriction of $\widehat{\Phi}$ to $\widehat{B}^\delta$ is injective. Hence, indeed, $\widehat{\Phi}$ itself is injective.

It remains to prove that $\operatorname{im}\widehat{\Phi} = \widehat{\mathrm{A}}$. To that end, we refer back to the proof of Theorem 4.2.21. The statement $\operatorname{im}\widehat{\Phi} \subseteq \widehat{\mathrm{A}}$ follows in much the same way as the corresponding statement of A. For the converse inclusion, we begin with some preparations which are largely similar to the corresponding steps in the bulk:

Recall the definition of S_n and the functions f_n^i. We may consider $\mathcal{L}(\widehat{\mathcal{V}})$ as a subset of $\mathcal{L}(\mathcal{V})$ by the use of the projection q, so that we may restrict S_n to $\mathcal{L}(\widehat{\mathcal{V}})$. For $T \in \mathcal{L}(\widehat{\mathcal{V}})$, we find that

$$\widehat{S}_n(T) := S_n(T) = \sum_i M^*_{\hat{f}_n^i} T M_{\hat{f}_n^i}, \quad \hat{f}_n^i := f_n^i|_{\widehat{G}} \quad \forall T \in \mathcal{L}(\widehat{\mathcal{V}}).$$

In particular, $\widehat{S}_n$ is a completely positive endomorphism of $\mathcal{L}(\widehat{\mathcal{V}})$. Arguing as in the proof of Theorem 4.2.21, we see that $\widehat{S}_n$ is a contraction.

Moreover, we see from the proof of Theorem 4.2.21 that

$$\widehat{S}_n(T) = \sum_{x,y\in\widehat{\Lambda}} \phi_n(x-y)T(x,y)|x\rangle\langle y| \quad \forall T \in \mathcal{L}(\widehat{\mathcal{V}}).$$

In particular, $\widehat{S}_n$ leaves controlled operators on $\ell^2(\widehat{\Lambda})$ invariant, and therefore also $C^*_u(\widehat{\Lambda}, V)$. Defining, for $T \in \widehat{\mathrm{A}}$, $\widehat{S}_n(T)$ by $\widehat{S}_n(T)_\omega := \widehat{S}_n(T_\omega)$ for all $\omega \in \Omega$, we obtain a contractive, completely positive endomorphism of $\widehat{\mathrm{A}}$. As in the proof of Theorem 4.2.21, it follows that $S_n(T)$ converges to T for any $T \in \widehat{\mathrm{A}}$.

The remainder of the proof differs slightly from that of Theorem 4.2.21. Let $T : \Omega \longrightarrow C^*_u(\widehat{\Lambda}, V)$ be norm continuous and satisfy the covariance in Definition 4.3.2. Setting

$$x \leqslant y \quad :\Longleftrightarrow \quad y - x \in \widehat{\Lambda}$$

defines a total preorder on $\widehat{\Lambda}$. Therefore, we may write $\widehat{S}_n(T) = \mathrm{I} + \mathrm{II}$ where

$$\mathrm{I}_\omega = \sum_{x,y\in\widehat{\Lambda},x\leqslant y} \phi_n(x-y)T_\omega(x,y)|x\rangle\langle y|$$

and the sum defining II extends over all $x, y \in \widehat{\Lambda}$, $x > y$. For $x \leqslant y$, we have

$$\langle x|\widehat{R}^\sigma_{y-x} = \sigma(x,y)\langle y| = \sigma(y,x)^*\langle y|,$$

so that

$$\mathrm{I}_\omega = \sum_{x,y\in\widehat{\Lambda},x\leqslant y} \phi_n(x-y)T_{\omega\cdot x}(0, y-x)\sigma(y,x)^*|x\rangle\langle y| = \sum_{g\in\widehat{G}\cap\operatorname{supp}\phi_n} \pi(f^{\mathrm{I}}_{n,g})_\omega \widehat{R}^\sigma_g,$$

where $f^{\mathrm{I}}_{n,g}(\omega) := \phi_n(g)T_\omega(0,g)$, $\forall n \geqslant 0, g \in \widehat{G}, \omega \in \Omega$. Similarly, for $x > y$, we have

$$(\widehat{R}^\sigma_{x-y})^* v|y\rangle = \sigma(y,x)^* v|x\rangle = \sigma(x,y)v|x\rangle \quad \forall v \in V,$$

so that

$$\mathrm{II}_\omega = \sum_{x,y\in\widehat{\Lambda},x>y} \phi_n(x-y)\sigma(x,y)T_{\omega\cdot y}(x-y,0)|x\rangle\langle y| = \sum_{g\in(\widehat{G}\setminus G_\partial)\cap(-\operatorname{supp}\phi_n)} (\widehat{R}^\sigma_g)^*\pi(f^{\mathrm{II}}_{n,g})_\omega,$$

where $f^{\mathrm{II}}_{n,g}(\omega) := \phi_n(-g)T_\omega(g,0)$, $\forall n \geqslant 0, g \in \widehat{G}, \omega \in \Omega$. This proves that $\widehat{S}_n(T) \in \operatorname{im}\widehat{\Phi}$; since $\operatorname{im}\widehat{\Phi}$ is closed, it also contains T, proving that $\widehat{\Phi}$ is surjective, and hence, the assertion. □

Due to Proposition 4.3.9, we can now construct the bulk-boundary SES. Besides the algebras of bulk and half-space observables, the bulk-boundary SES will contain a C*-algebra corresponding to the observables localised around the boundary. In analogy to the algebra of bulk observables, we define the algebra of observables that are exactly localised at the boundary as follows.

Definition 4.3.10 (Algebra of boundary observables)**.** *Let Ω, U and u^σ_x be given as in Definition 4.2.9. In analogy to the bulk algebra, we define $A^{U,\sigma}_\partial$ to be the C*-algebra generated by all norm continuous maps $O : \Omega \longrightarrow C^*_u(\Lambda_\partial, U)$ that are covariant in the sense that*

$$O_{\omega\cdot x} = (u^\sigma_x)^* O_\omega u^\sigma_x \quad \forall x \in \mathbb{Z}^{d-1}, \omega \in \Omega.$$

The real C^*-algebra $\mathbb{A}_\partial := A_\partial^W$ is called the algebra of boundary observables and the complex C^*-algebra $\mathrm{A}_\partial := A_\partial^{V,\sigma}$ is called the algebra of complex (or charge-conserving) observables.

Note that the boundary algebras consist of observables that are strictly localised at the boundary.

In the remainder of this section and in Chapter 5, we will make frequent use of the operators $R^\sigma_{e_d}$ and R_{e_d}, as well as their induced automorphism of A_∂ and $\mathbb{A}_\partial$, respectively. Therefore, we introduce for brevity the following notation.

Definition 4.3.11 ($R^\sigma_d, R_d, \alpha_d$)**.** *Consider the covariant pairs (π, R^σ) and $(\hat{\pi}, \widehat{R^\sigma})$ of the bulk and half-space algebra of complex observables. We introduce the notation*

$$R^\sigma_d := R^\sigma_{e_d}, \quad \widehat{R^\sigma_d} := \widehat{R^\sigma_{e_d}} \quad \text{where } e_d = (0, \ldots, 0, 1) \in \mathbb{Z}^{d-1} \times \mathbb{N}.$$

The unitary operator R^σ_d defines the $*$-automorphism

$$\alpha_d(a) := (R^\sigma_d)^* a R^\sigma_d \quad \forall a \in \mathrm{A}_\partial.$$

Analogously, we introduce for the real covariant pairs (π, R) and $(\hat{\pi}, \widehat{R})$ of the bulk and half-space algebra of observables the notation $R_d := R_{e_d}$ and $\widehat{R_d} := \widehat{R_{e_d}}$. The unitary operator R_d defines the real $*$-automorphism $\alpha_d(a) := R_d^* a R_d \; \forall a \in \mathbb{A}_\partial$.

An important role will in the following also be played by the following boundary projection.

Definition 4.3.12 (Boundary projection e)**.** *The* boundary projection *is defined by $e := 1 - (\widehat{R^\sigma_d})^* \widehat{R^\sigma_d} \in \mathcal{L}(L^2(\Omega, \mathbb{P}) \otimes \widehat{\mathcal{V}})$. The name of this projection is motivated by the fact that for all $\psi \in L^2(\Omega, \mathbb{P}) \otimes \widehat{\mathcal{V}}$*

$$e\psi(\omega, x) = \begin{cases} \psi(\omega, x), & \text{if } x \in \Lambda_\partial, \\ 0, & \text{otherwise.} \end{cases}$$

In the real case, we define $e := 1 - \widehat{R}_d^* \widehat{R}_d \in \mathcal{L}(L^2(\Omega, \mathbb{P}) \otimes \widehat{\mathcal{W}})$.

We are now ready to formulate and proof the following theorem, which is the core of the bulk-boundary SES that will be explained thereafter.

Theorem 4.3.13. *There is a unique surjective $*$-morphism $\rho : \widehat{\mathrm{A}} \longrightarrow \mathrm{A}$ such that*

$$\rho(\hat{\pi}(f)) = \pi(f), \quad \rho(\widehat{R^\sigma_x}) = R^\sigma_x, \quad \forall f \in C(\Omega) \otimes \mathrm{End}(V), x \in \mathbb{Z}^{d-1} \times \mathbb{N}.$$

The kernel of ρ is the closed ideal (e) generated by the boundary projection e. It is isomorphic to $\mathrm{A}_\partial \otimes \mathbb{K}(\ell^2(\mathbb{N}))$.

In the real case, there is a unique real surjective $*$-morphism $\rho : \widehat{\mathbb{A}} \longrightarrow \mathbb{A}$ such that

$$\rho(\hat{\pi}(f)) = \pi(f), \quad \rho(\widehat{R_x}) = R_x, \quad \forall f \in C(\Omega) \otimes \mathrm{End}(W), x \in \mathbb{Z}^{d-1} \times \mathbb{N}.$$

The kernel of ρ is here as a real C^*-algebra isomorphic to $\mathbb{A}_\partial \otimes \mathbb{K}(\ell^2(\mathbb{N}))$.

Proof. We prove the complex case where the twisting is non-trivial. The real case can be proved in the same way.

The unique existence and the surjectivity of a $*$-morphism as claimed follow from Proposition 4.3.9 and Theorem 4.2.21. It is also clear from the definitions that the projection e is contained in the kernel of ρ.

We abbreviate $\widehat{S} := \hat{u}_d$ and $\hat{e} := 1 - \widehat{S}\widehat{S}^* = |0\rangle\langle 0|$. By Theorem 4.2.21, there is a unique $*$-morphism $\nu : \mathrm{A} \longrightarrow \mathrm{A} \otimes M_2(\mathcal{L}(\ell^2(\mathbb{N})))$, such that

$$\nu(\pi(f)) := \pi(f) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \nu(R^\sigma_{(x,n)}) := \begin{cases} R^\sigma_{(x,n)} \otimes \begin{pmatrix} \widehat{S}^* & 0 \\ \hat{e} & \widehat{S} \end{pmatrix}^n, & \text{if } n \geqslant 0, \\ R^\sigma_{(x,n)} \otimes \begin{pmatrix} \widehat{S} & \hat{e} \\ 0 & \widehat{S}^* \end{pmatrix}^{-n}, & \text{if } n < 0, \end{cases}$$

for all $f \in C(\Omega) \otimes \mathrm{End}(W)$, $x \in \mathbb{Z}^{d-1}$, $n \in \mathbb{Z}$. In fact, the covariance condition is easy to compute once we note that

$$\begin{pmatrix} \widehat{S}^* & 0 \\ \hat{e} & \widehat{S} \end{pmatrix} \begin{pmatrix} \widehat{S} & \hat{e} \\ 0 & \widehat{S}^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \widehat{S} & \hat{e} \\ 0 & \widehat{S}^* \end{pmatrix} \begin{pmatrix} \widehat{S}^* & 0 \\ \hat{e} & \widehat{S} \end{pmatrix}.$$

Similarly, by Proposition 4.3.9, there is a unique $*$-morphism $\hat{\nu} : \widehat{\mathrm{A}} \longrightarrow \mathrm{A} \otimes \mathcal{L}(\ell^2(\mathbb{N}))$ such that

$$\hat{\nu}(\hat{\pi}(f)) := \pi(f) \otimes 1, \quad \hat{\nu}(\widehat{R^\sigma_{(x,n)}}) := R^\sigma_{(x,n)} \otimes (\widehat{S}^*)^n, \tag{4.18}$$

for all $f \in C(\Omega) \otimes \mathrm{End}(W)$, $x \in \mathbb{Z}^{d-1}$, $n \in \mathbb{N}$. Arguing as in the proof of Proposition 4.3.9, using the obvious action of the dual group of $\mathbb{Z}$ on $\mathcal{L}(\ell^2(\mathbb{N}))$, we see that $\hat{\nu}$ is injective, and hence defines a $*$-isomorphism onto its image. By the definition of ν, the image of the upper left component ν_{11} is contained in the image of $\hat{\nu}$. We may therefore define

$$s : \mathrm{A} \longrightarrow \widehat{\mathrm{A}}, \quad s := (\hat{\nu})^{-1} \circ \nu_{11}.$$

This map is not a $*$-morphism, but it is a completely positive contraction intertwining the $*$-operations. Moreover, we have

$$\nu_{11}(xy) = \nu_{11}(x)\nu_{11}(y) + \nu_{12}(x)\nu_{21}(y), \quad \forall x, y \in \mathrm{A},$$

where $\nu_{12}(x)\nu_{21}(y)$ is contained in the ideal generated by $1 \otimes \hat{e}$. By $\hat{\nu}^{-1}$, this ideal is mapped onto the ideal generated by e. It follows that s is a section of ρ, as it is sufficient to check this statement on generators.

We claim that $x - s(\rho(x))$ is contained in the closed ideal generated by e. By the same argument as in the previous paragraph, it is sufficient to check on generators x that $\hat{\nu}(x) - \nu_{11}(\rho(x))$ is contained in the closed ideal generated by $1 \otimes \hat{e}$. But this expression actually vanishes on generators.

Thus, let $x \in \widehat{\mathrm{A}}$, $\rho(x) = 0$. Then we have

$$x = x - s(\rho(x)) \in (e),$$

proving that $\ker \rho = (e)$.

It remains to prove that this ideal is indeed isomorphic to $\mathrm{A}_\partial \otimes \mathbb{K}(\ell^2(\mathbb{N}))$. Define a $*$-morphism

$$\nu_\partial : \mathrm{A}_\partial \otimes \mathbb{K}(\ell^2(\mathbb{N})) \longrightarrow \mathrm{A} \otimes \mathcal{L}(\ell^2(\mathbb{N})), \quad \nu_\partial(a \otimes |m\rangle\langle n|) := (R_d^{\sigma *})^m a (R_d^\sigma)^n \otimes |m\rangle\langle n|,$$

Then ν_∂ is a $*$-morphism on $\mathrm{A}_\partial \otimes \mathcal{F}(\ell^2(\mathbb{N}))$ where $\mathcal{F}$ denotes the finite rank operators.

Both $\mathrm{A}_\partial \otimes \mathbb{K}(\ell^2(\mathbb{N}))$ and $\mathrm{A} \otimes \mathcal{L}(\ell^2(\mathbb{N}))$ are represented faithfully on $\mathcal{W} \otimes \ell^2(\mathbb{N})$, and the operator norm on this space is invariant under left and right multiplications by operators of the form $u \otimes 1$ where u is unitary on $\mathcal{W}$. This shows that ν_∂ is an isometry on $\mathrm{A}_\partial \otimes \mathcal{F}(\ell^2(\mathbb{N}))$, and hence it extends to $\mathrm{A}_\partial \otimes \mathbb{K}(\ell^2(\mathbb{N}))$ as an isometric $*$-morphism.

Observe that

$$(R_d^{\sigma *})^m a (R_d^\sigma)^n \otimes |m\rangle\langle n| = \hat{\nu}(\widehat{R_d^\sigma})^{*m}(a \otimes 1)(1 \otimes \hat{e})\hat{\nu}(\widehat{R_d^\sigma})^n.$$

Thus, it is clear that the image of ν_∂ is equal to the closed ideal of $\mathrm{im}(\hat{\nu})$ generated by $1 \otimes \hat{e}$, proving the claim. □

With the results of Theorem 4.3.13 we have finally established the relation between the bulk, half-space and boundary algebra. This can be summarised in the short exact sequence

$$0 \to \mathbb{A}_\partial \otimes \mathbb{K}(\ell^2(\mathbb{N})) \xrightarrow{\iota} \widehat{\mathbb{A}} \xrightarrow{\rho} \mathbb{A} \to 0 \tag{4.19}$$

of real C^*-algebras, where $\iota := \hat{\nu}^{-1} \circ \nu_\partial : \mathbb{A}_\partial \otimes \mathbb{K}(\ell^2(\mathbb{N})) \to \widehat{\mathbb{A}}$. The corresponding SES for the complex observables is given by

$$0 \to \mathrm{A}_\partial \otimes \mathbb{K}(\ell^2(\mathbb{N})) \xrightarrow{\iota} \widehat{\mathrm{A}} \xrightarrow{\rho} \mathrm{A} \to 0. \tag{4.20}$$

The algebras $\mathbb{A}_\partial \otimes \mathbb{K}(\ell^2(\mathbb{N}))$ and $\mathrm{A}_\partial \otimes \mathbb{K}(\ell^2(\mathbb{N}))$ can be interpreted in the physical context as (the norm closure of) the algebra of half-space observables that are localised around the boundary.

We also constructed a section of the SES $s : \mathbb{A} \to \widehat{\mathbb{A}}$ resp. $s : \mathrm{A} \to \widehat{\mathrm{A}}$ given by

$$s(x) = \hat{\nu}^{-1}(\nu_{11}(x)). \tag{4.21}$$

More explicitly, this section is defined by

$$s(\pi(f)) = \hat{\pi}(f), \quad s(R_{x,n}) = \begin{cases} \widehat{R}_{x,n} & \text{for } n \geq 0, \\ \widehat{R}^*_{x,-n} & \text{for } n < 0, \end{cases} \text{ for all } (x,n) \in \mathbb{Z}^{d-1} \times \mathbb{Z}$$

in the real case, and

$$s(\pi(f)) = \hat{\pi}(f), \quad s(R^\sigma_{x,n}) = \begin{cases} \widehat{R}^\sigma_{x,n} & \text{for } n \geq 0, \\ \widehat{R}^{\sigma *}_{x,-n} & \text{for } n < 0, \end{cases} \text{ for all } (x,n) \in \mathbb{Z}^{d-1} \times \mathbb{Z}$$

in the complex case.

Now we have all tools at hand that are needed in Chapter 5 for the definition of topological phases at the boundary of topological insulators and superconductors, and, subsequently, the formulation and proof of the bulk-boundary correspondence. In particular, we can now describe the ground states of these solids with boundary in an explicit way, as explained in the next subsection.

4.3.3 Disordered invariant quasi-particle vacua with boundary

Now we will finally define a mathematical framework for ground states of disordered topological insulators and superconductors with boundary. We will adapt the description of ground states in terms of disordered IQPV to these physical systems with boundary.

In doing so, we have to take care of the following two important physical observations:

Suppose we are given a topological insulator or superconductor which is described in the bulk by a disordered IQPV with a finite hopping range $R > 0$. Then in the description of the ground state including a boundary, we have to recover the properties of this disordered IQPV of the bulk if we go far away from the boundary with respect to this length scale R.

In contrast to that, in the vicinity of the boundary, we expect the spectrum to become gapless at the chemical potential μ due to localised boundary states. In other words, we expect that the gap condition (*cf.* Definition 4.1.13) is violated at the boundary and we call the corresponding states the *gapless states*. These states are the most important difference between the bulk and the boundary theory. Thus, in the vicinity of the boundary, we have to generalise the notion of disordered IQPV in order to allow for such localised gapless states.

These two physical aspects are taken care of by the following definition.

Definition 4.3.14 (Symmetric disordered IQPV with boundary)**.** *A disordered IQPV with boundary of symmetry index* (r, s) *is a real skew-Hermitian* $\hat{J} \in \widehat{\mathbb{A}}$ *such that*

$$J := \rho(\hat{J}) \in \mathbb{A}$$

defines a disordered IQPV of symmetry index (r, s)*.*

A complex disordered IQPV with boundary of symmetry index (r, s) *is defined as a skew-Hermitian* $\hat{J} \in \widehat{\mathrm{A}}$ *such that* $J := \rho(\hat{J}) \in \mathrm{A}$ *defines a complex disordered IQPV of symmetry index* (r, s)*.*

The properties of ρ enable us to characterise disordered IQPV with boundary more explicitly.

Proposition 4.3.15. *All disordered IQPV with boundary of symmetry index* (r, s) *are determined by tuples*

$$(\hat{J}; \phi),$$

where $\hat{J} \in \widehat{\mathbb{A}}$ *is a real skew-Hermitian element and* $\phi : C\ell_{r,s} \to \widehat{\mathbb{A}}$ *is a unital real* $*$*-morphism such that*

$$\{\hat{J}, \phi(k_a)\}, \{\hat{J}, \phi(j_\alpha)\}, \hat{J}^2 + 1 \in \ker(\rho) \cong \mathbb{A}_\partial \otimes \mathbb{K}(\ell^2(\mathbb{N})), \tag{4.22}$$

for all $a = 1, \dots, r$ *and* $\alpha = 1, \dots, s$*, where* $\{\cdot, \cdot\}$ *denotes the anti-commutator.*

Complex disordered IQPV with boundary of symmetry index (r, s) *are determined by tuples* $(\hat{J}; \phi)$ *of skew-Hermitian* $\hat{J} \in \widehat{\mathrm{A}}$ *and unital* $*$*-morphism* $\phi : \mathbb{C}\ell_{r,s} \to \widehat{\mathrm{A}}$ *satisfying the conditions in Equation* (4.22) *for* $\mathbb{A}_\partial$ *replaced by* A_∂*.*

Proof. We only prove the real case because the proof of the complex case is almost identical.

Let $\hat{J} \in \widehat{\mathbb{A}}$ be a disordered IQPV with boundary of symmetry index (r, s). For the disordered IQPV $J := \rho(\hat{J}) \in \mathbb{A}$ there are pseudo-symmetries $K_1, \dots, K_r, J_1, \dots, J_s \in \mathrm{End}(W) \subset \mathbb{A}$, such that

$$(J; K_1, \dots, K_r, J_1, \dots, J_s)$$

is a disordered IQPV of symmetry index (r, s). By definition, there is a unital real $*$-morphism $\phi : C\ell_{r,s} \to \mathbb{A}$ such that $\phi(k_a) = K_a$ and $\phi(j_\alpha) = J_\alpha$ for all $a = 1, \ldots, r$ and $\alpha = 1, \ldots, s$.

The restriction of ρ to $C(\Omega) \otimes \mathrm{End}(W)$ is equal to the identity. Therefore, ϕ defines by the same arguments a unital $*$-morphism $\phi : C\ell_{r,s} \to \widehat{\mathbb{A}}$. Since ρ is a $*$-morphism, the equations $\{\rho(\hat{J}), \phi(k_a)\} = \{\rho(\hat{J}), \phi(j_\alpha)\} = 0 \, \forall \, a = 1, \ldots r, \alpha = 1, \ldots, s$ imply

$$\{\hat{J}, \phi(k_a)\}, \{\hat{J}, \phi(j_\alpha)\} \in \ker(\rho) = \mathbb{A}_\partial \otimes \mathbb{K}(\ell^2(\mathbb{N})) \quad \forall \, a = 1, \ldots r, \alpha = 1, \ldots, s.$$

By the same argument, we conclude $\hat{J}^2 + 1 \in \ker(\rho) \cong \mathbb{A}_\partial \otimes \mathbb{K}(\ell^2(\mathbb{N}))$. □

By Proposition 4.3.15, we see that a disordered IQPV with boundary can be considered as a disordered IQPV up to contributions of localised boundary states. These boundary contribution will be fundamental for the topological properties at the boundary. In fact, our definition of topological phases at the boundary in Chapter 5 will be essentially based on the fact that topological insulators and superconductors with boundary are considered as topologically trivial if there are no such localised boundary states closing the gap.

4.A Appendix: The SRFL-scheme

In the literature, symmetries are often organised according to a scheme introduced by Schnyder–Ryu–Furusaki–Ludwig [76], which is different from the approach of Kennedy and Zirnbauer that we follow. We speak of the SRFL scheme. It is based on two anti-unitary endomorphisms $\Theta, \Xi : V \longrightarrow V$ and a unitary operator Π proportional to $\Xi\Theta$ satisfying the relations

$$\Theta^2 = \pm 1, \quad \Xi^2 = \pm 1, \quad \Pi^2 = 1.$$

In the literature, Θ is called *time-reversal*, Ξ is called *particle-hole conjugation*, and Π the *chiral* or *sub-lattice symmetry*. As we have already reserved these expressions for certain physical symmetries with potentially different commutation relations, we will not use this terminology here.

class	Θ	Ξ	Π
A	0	0	0
$A\mathrm{III}$	0	0	1
D	0	1	0
$D\mathrm{III}$	−1	1	1
$A\mathrm{II}$	−1	0	0
$C\mathrm{II}$	−1	−1	1
C	0	−1	0
$C\mathrm{I}$	1	−1	1
$A\mathrm{I}$	1	0	0
$BD\mathrm{I}$	1	1	1

Table 4.2: Symmetry classes according to Schnyder–Ryu–Furusaki–Ludwig. An entry 0 means that the corresponding symmetry is absent, whereas a non-zero entry ±1 indicates that the symmetry is present; in this case, the entry equals the square of the corresponding symmetry operator.

According to the SRFL scheme, a Hamiltonian is called Θ-*symmetric*, Ξ-*symmetric*, or Π-*symmetric*, if respectively,

$$\Theta H \Theta^* = H, \quad \Xi H \Xi^* = -H, \quad \Pi H \Pi^* = -H.$$

All possible combinations of the presence or absence of these symmetries can be summarised in the following table.

Class D

The only restriction here is that a flat-band Hamiltonian H anti-commutes with the real structure γ on W. Hence, if we set $\Xi := \gamma$, then $\Xi^2 = 1$ and any Ξ-symmetric Hamiltonian gives rise to a QPV of class D.

Class $D\mathrm{III}$

Here, a time-reversal symmetry T on $\mathcal{V}$, $T^2 = -1$, is present, and extended to $\mathcal{W}$ as explained in Example 4.1.18. We may set $\Xi := \gamma$, $\Theta := T$, and $\Pi := i\Theta\Xi$. Then $\Theta^2 = -1$, $\Xi^2 = 1$, and $\Pi^2 = 1$, and any H that is symmetric for Θ, Ξ, and Π gives rise to a QPV of class $D\mathrm{III}$.

Class $A\mathrm{II}$

In this class we have the symmetries T and iQ. As explained in the discussion of class $A\mathrm{II}$ in Subsection 4.1.2, a QPV of class $A\mathrm{II}$ is the same as an operator $x \in \mathcal{L}(\mathcal{V})$, $x^2 = 1$, which commutes with T. Hence, if we set $\Theta := T$, then $\Theta^2 = -1$ and any Θ-symmetric flat-band Hamiltonian H determines a QPV of class $A\mathrm{II}$ *via* $x := H$.

Class $C\mathrm{II}$

In addition to the previous symmetries, the particle-hole symmetry C is present and given by $C = \gamma S$ where S is linear, $S^2 = 1$, and $[S, T] = 0$. As explained in the discussion of class $C\mathrm{II}$ in Subsection 4.1.2, a QPV of class $C\mathrm{II}$ is the same as an operator $x \in \mathcal{L}(\mathcal{V})$, $x^2 = 1$, such that $Tx = xT$ and $Sx = -xS$. We may set $\Theta := T$, $\Pi := S$, $\Xi := \Theta\Pi$. Then $\Theta^2 = -1$, $\Pi^2 = 1$, $\Xi^2 = -1$, and any flat-band Hamiltonian H symmetric for Θ, Ξ, and Π determines a QPV of class $C\mathrm{II}$ *via* $x := H$.

Classes $s \geqslant 4$

For $s \geqslant 4$, we use the two spin rotation generators j_1, j_2 and Lemma 4.1.26 to induce an isomorphism $\mathcal{W} \cong \mathcal{W}^{\sim} \otimes \mathbb{C}^2$ and a splitting $\mathcal{L}(\mathcal{W}) \cong \mathcal{L}(\mathcal{V}) \otimes \mathbb{H}_{\mathbb{C}}$. In this splitting, γ corresponds to $T \otimes \mathfrak{t}$ where T is a quaternionic structure on $W^{\sim}$ and $\mathfrak{t}$ the standard quaternionic structure on $\mathbb{C}^2$.

QPV of class $s \geqslant 4$ correspond to operators $j \in \mathcal{L}(\mathcal{W}^{\sim})$ commuting with T such that $j^2 = -1$ and j anti-commutes with the first $s-4$ of the operators j_T, j_Q, and j_C, corresponding to J_T, J_Q, and J_C, respectively. We may proceed as above, with T playing the role of γ, to establish the correspondence in all real symmetry classes.

Complex class A

The complex classes are those where both J and the group of symmetries commute with Q. As we have noted, this implies that J and any other symmetries present leave $\mathcal{V}$ invariant and are determined by their restriction to this space. In class A, there are no further symmetries present, so the Hamiltonians H on $\mathcal{V}$ determine QPV in the complex symmetry class A.

Complex class $A\mathrm{III}$

In complex class $A\mathrm{III}$, we have the additional symmetry $C = \gamma S$. QPV in this class correspond to operators on $\mathcal{V}$ anti-commuting with S. If we set $\Pi := S$, then $\Pi^2 = 1$, and Π-symmetric Hamiltonians determine QPV in the complex symmetry class $A\mathrm{III}$.

4.B Appendix: Translationally invariant systems

Here, we give the reader who is familiar with the classification of translationally invariant systems without magnetic fields a comparison to these systems.

In the absence of disorder, a QPV corresponds [49, Def. 2.4] to a vector bundle $\mathcal{A}$, *cf.* Remark 4.1.15, on the *Brillouin zone* $\widehat{\mathbb{Z}}^d$ of rank $\operatorname{rk}\mathcal{A} = \frac{1}{2}\dim W$ contained in the trivial vector bundle with fibre W. The Brillouin zone is the d-dimensional torus $\mathbb{T}^d = \widehat{\mathbb{Z}}^d$ dual to $\mathbb{Z}^d$. It is defined by $\mathbb{T}^d := \mathbb{R}^d/(\mathbb{Z}^d)^\perp$, where

$$(\mathbb{Z}^d)^\perp := \left\{x \in \mathbb{R}^d \,\middle|\, (x, \mathbb{Z}^d) \subseteq \mathbb{Z}\right\}.$$

The fibres of $\mathcal{A}$ are subject to the *Fermi constraint*

$$\{\mathcal{A}_{-k}, \mathcal{A}_k\} = 0 \quad \forall k \in \mathbb{T}^d. \tag{4.23}$$

The QPV J corresponding to $\mathcal{A}$ is associated with the *flattened* (or *flat-band*) *Hamiltonian* H of $\mathcal{A}$.

The Abelian group $\mathbb{Z}^d$ acts on $\mathcal{V} = \ell^2(\Lambda) \otimes V$ by translations, induced by the following real operators over the Nambu space

$$(u_x\psi)(y) := \psi(y - x) \quad \forall x \in \mathbb{Z}^d, y \in \Lambda, \psi \in \mathcal{W}.$$

Any operator $H \in \mathcal{L}(\mathcal{W})$ that commutes with the translations u_x for all $x \in \mathbb{Z}^d$ corresponds to an operator $\hat{H}$ over the Brillouin zone $\hat{\mathcal{W}} := L^2(\mathbb{T}^d) \otimes W$ of the form

$$(\hat{H}\hat{\psi})(k) = H(k)\hat{\psi}(k) \quad \forall \hat{\psi} \in \hat{\mathcal{W}}, k \in \mathbb{T}^d,$$

for some function $H(\cdot) \in L^\infty(\mathbb{T}^d)$, and *vice versa*.

So for a vector bundle $\mathcal{A}$ representing a QPV, the corresponding flat-band Hamiltonian H is defined as the unique operator on $\hat{\mathcal{W}}$ which for every $k \in \mathbb{T}^d$ satisfies

$$H(k) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

in terms of the orthogonal decomposition $\hat{\mathcal{W}}_k = \mathcal{A}_k \oplus \mathcal{A}_k^c$.

The Fermi constraint in Equation (4.23) translates to the condition

$$H(-k) = -H(k)^{\mathsf{T}} \quad \forall k \in \mathbb{T}^d, \tag{4.24}$$

see [49, Eq. 2.16]. To translate this into a condition on H, we write out the CAR form in momentum space. Recall that for any $\psi \in (\ell^1 \cap \ell^2)(\Lambda) \otimes W$, the Fourier transform is defined as

$$\hat{\psi}(k) := \sum_{x \in \Lambda} e^{-2\pi i (k,x)} \psi(x) \quad \forall k \in \mathbb{T}^d.$$

Then, for sufficiently regular ψ_1, ψ_2, we have

$$\{\hat{\psi}_1, \hat{\psi}_2\} = \int dk_1 \, dk_2 \sum_{x \in \Lambda} e^{2\pi i (k_1 + k_2, x)} \{\hat{\psi}_1(k_1), \hat{\psi}_2(k_2)\} = \int dk \, \{\hat{\psi}_1(-k), \hat{\psi}_2(k)\},$$

the domain of integration being $\mathbb{T}^d$. Hence, Equation (4.24) is equivalent to

$$H = -H^{\mathsf{T}}.$$

The same condition holds for J because the transposition is linear. This established the relations between the conditions on ground states in the vector bundle picture over the Brillouin torus and the description in terms of quasi-particle vacua in the general setting.

Chapter Five

Topological Phases and the Bulk-Boundary Correspondence

This is the most important chapter of the present work, containing the definitions of topological phases in the bulk and at the boundary, as well as the formulation and proof of the bulk-boundary correspondence.

The bulk-boundary correspondence can be roughly understood as a relation between the topological properties of the stable ground state in the bulk of a topological insulator or superconductor and the topological properties and the existence of gapless states at its boundary.

The first principle of bulk-boundary correspondence was established in the context of anomalies in quantum field theory. It was observed that the effective bulk and boundary field theories are not well defined in isolation. This means that these theories would violate conservation laws which have to be satisfied. This problem can only be solved by considering bulk and boundary together such that the anomalies cancel out each other.

The first example of this bulk-boundary correspondence was observed in the quantum Hall effect (QHE). As depicted in Figure 5.1, the QHE exhibits states that are localised at the boundary and move into a fixed direction due to the strong magnetic field. These boundary states give rise to the quantised Hall conductance. In the bulk, the electrons move in circular orbits around the flux quanta, forming so-called Landau states. In terms of anomalies of effective quantum field theories, the bulk-boundary correspondence of the QHE can be understood as the fact that the edge theory exhibits a so-called chiral anomaly. This anomaly manifests itself in a non-conserved boundary charge. The charge is only conserved if the boundary theory is accompanied with a suitable effective field theory of the bulk. A recent survey of the assertions hitherto can be found in [26].

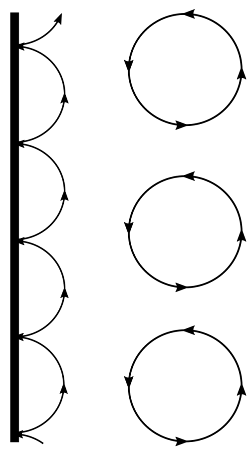

Figure 5.1: Semi-classical picture of bulk-boundary correspondence in the QHE.

On the basis of the non-commutative geometry approach for the bulk theory of the QHE by Bellissard, see [13] for a review, this physical principle of bulk-boundary correspondence in the QHE was put on firm mathematical grounds by Kellendonk, Richter and Schulz-Baldes [48, 78]. From a physical perspective, they established a rigorous proof of the equality of the bulk Hall conductance and the edge Hall conductance, which is also valid in the context of disorder. Mathematically, their results were fundamental for the descriptions of bulk-boundary correspondence during the following years because they first observed that the bulk-boundary correspondence can be interpreted as a relation of topological invariants, *cf.* Chapter 6, of the K-theories of the bulk and the boundary.

In 2014, Schulz-Baldes and Prodan proved the bulk-boundary correspondence for all topological insulators of symmetry class $A\mathrm{III}$ [68, 66]. Thus, the bulk-boundary correspondence for the complex symmetry classes $A, A\mathrm{III}$ is well known. However, its formulation and proof is based on complex K-theory and a direct translation of these results to the real symmetry classes is not possible.

During the following years, there have been various physical results addressing the bulk-boundary correspondence in the real symmetry classes for some special choices of the dimension, the symmetry class and the Hamiltonian. An outstanding mathematical work thereof is by Graf–Prota [29], addressing the bulk-boundary correspondence in two-dimensional systems in symmetry class $A\mathrm{II}$.

In contrast to that, there are nowadays only few approaches addressing the bulk-boundary correspondence in the context of disorder for all real symmetry classes and all dimensions in a systematic manner.

One of the first more systematic approaches was established by Loring in 2015 [57]. He defined topological invariants for topological insulators and superconductors in terms of almost commuting Hermitian matrices for all ten symmetry classes of the Tenfold Way in two dimensions, as well as for one symmetry class in three dimensions. The bulk-boundary correspondence was then established by comparing d and $d+1$ dimensional systems. However, the drawback of this approach is that, besides the restriction to low dimensional systems, the relation of the topological invariants to the physical systems remained unclear.

During the years 2016 through 2018, Hannabuss, Mathai and Thiang established in a series of papers a bulk-boundary correspondence [60, 59, 84, 32], which works for all dimensions and all symmetry classes in the Tenfold Way. Their approach relies on the concept of T-duality which describes the topological phases as classes in the K-theory of the torus. In contrast to the classification by the use of Fourier-Bloch theory, this T-duality has also been established in the context of disorder and the presence of homogeneous magnetic fields, where the translational invariance is broken. The bulk-boundary correspondence is then induced by a restriction map over the torus.

In 2017, Kubota [53] worked out a formulation of bulk-boundary correspondence which is more directly formulated for tight-binding Hamiltonians over a lattice and which also holds in any dimension and symmetry class. His construction is based on the description of tight-binding observables in terms of uniform Roe C*-algebras, an idea that we have adapted in our constructions. The bulk-boundary correspondence is then established by considering a so-called coarse Mayer–Vietoris exact sequence. On the one hand, the use of the uniform Roe C*-algebra is appealing since it reflects the properties of tight-binding observables. On the other hand, the K-theory of the uniform Roe C*-algebra is involved and cannot be computed in full detail in all dimensions. Kubota overcomes this problem in his formulation of the bulk-boundary correspondence by going over to the larger non-uniform Roe C*-algebra, whose

K-theory is well known, *cf.* [25]. But the use of this larger C*-algebra is problematic from a physical point of view as it contradicts the assumptions of the tight-binding approximation.

Last but not least, we mention the approach by Bourne–Carey–Rennie [16] and Bourne–Kellendonk–Rennie [17]. The bulk-boundary correspondence is there constructed in terms of unbounded Kasparov modules and the intersection product. The results of [17] will play an important role in Chapter 6.

The main result of this chapter is that we prove the bulk-boundary correspondence for all symmetry classes in the Tenfold Way, including disorder and magnetic fields. One of the essential aspects of this chapter is that the topological phases in the bulk and at the boundary, as well as the bulk-boundary correspondence, are formulated in an explicit and physically natural way. This enables us to draw conclusions for the physical properties of the bulk-boundary correspondence which cannot be seen in the more abstract formulations mentioned above. Moreover, we can treat all symmetry classes of the Tenfold Way in a systematic picture. From our point of view, it leads also to a simplified formulation of the bulk-boundary correspondence of the complex symmetry classes [66].

The outline of this chapter is as follows.

We begin in Section 5.1 with the definition of topological phases for the gapped bulk phases of topological insulators and superconductors. Since we have already established a picture of real K-theory in terms of disordered IQPV with pseudo-symmetries in Section 3.3, we can almost immediately define the notion of topological phases in the bulk.

In Section 5.2, we will independently define topological phases for disordered IQPV with boundary. The topological properties at the boundary are determined by the gapless localised boundary states. In order to define a notion of topological phase which indeed solely depends on the topological properties of these boundary states, we establish topological phases for disordered IQPV with boundary on a more general level of so-called *generalised quasi-particle vacua* (GQPV). The resulting group of topological phases of GQPV of a given symmetry index will then be proved to be isomorphic to a (real) K-group of the algebra of boundary observables by constructing an explicit isomorphism to Kasparov's Fredholm picture of K-theory.

A priori, the resulting topological phases in the bulk and at the boundary are independent of each other. In Section 5.3, we will show that there is a canonical correspondence – the bulk-boundary correspondence – between the topological phases in the bulk and the ones at the boundary. The bulk-boundary correspondence is essentially based on the bulk-boundary SES and the corresponding long exact sequences of (real) K-groups.

In Sections 5.4 and 5.5, we will derive some properties of the bulk-boundary correspondence and discuss their physical meaning. We will consider topological phases and the bulk-boundary correspondence in the case of disorder by doping of Example 4.2.8 in more detail. The topological phases will be proved to be invariant under this kind of disorder, i.e. the disordered physical systems exhibit the same topological phases as in the clean case. This result is not true for other kinds of disorder. Furthermore, we will discuss the role of the reference disordered IQPV and the local space for the topological classification of topological insulators and superconductors.

Most of the material of this chapter was published in the preprint *Bulk-boundary correspondence for disordered free-fermion topological phases* by A. Alldridge, C. Max and M. R. Zirnbauer, 2019 [1]. In Section 5.2, we give a more detailed construction of the topological phases at the boundary. In particular, we further generalise the notion of topological phases of IQPV with boundary in [1] to topological phases of so-called generalised quasi-particle vacua.

5.1 Topological phases in the bulk

In this section, we define topological phases in the bulk of disordered topological insulators and superconductors for all symmetry classes of the Tenfold Way.

By construction, the set of all disordered IQPV, symmetric w.r.t. to a set of positive pseudo-symmetries $K_1, \dots, K_r \in \mathrm{End}(W)$ and negative pseudo-symmetries $J_1, \dots, J_s \in \mathrm{End}(W)$, is equal to the set

$$\mathcal{FF}^{\phi}(\mathbb{A}),$$

where $\phi : C\ell_{r,s} \to \mathbb{A}$ denotes the corresponding unital real $*$-morphism defined by $\phi(k_a) = K_a$ and $\phi(j_\alpha) = J_\alpha$ for all $a = 1, \dots, r$ and $\alpha = 1, \dots, s$. In the charge-conserving case, the set of complex disordered IQPV, symmetric w.r.t. $\phi : \mathbb{C}\ell_{r,s} \to \mathrm{A}$, is equal to the set

$$\mathcal{FF}^{\phi}(\mathrm{A}).$$

The sets $\mathrm{FF}^{\phi}(\mathbb{A})$ and $\mathrm{FF}^{\phi}(\mathrm{A})$ establish a natural notion of homotopy equivalence between the disordered IQPV which are symmetric w.r.t. to a given $\phi : C\ell_{r,s} \to \mathbb{A}$, resp. $\phi : C\ell_{r,s} \to \mathbb{A}$.

By the choice of a reference disordered IQPV $(J_{\mathrm{ref}}; \phi)$, we can now go along the lines of Subsection 3.3.1 in order to construct the groups of topological phases $\mathrm{Top}^{\phi}_{J_{\mathrm{ref}}}(\mathbb{A})$ and $\mathrm{Top}^{\phi}_{J_{\mathrm{ref}}}(\mathrm{A})$. The reference disordered IQPV defines the trivial topological phase. The topological phase of an arbitrary disordered IQPV (J, ϕ) will thus always be measured relative to the topological phase of the reference disordered IQPV.

Thus, our new picture of K- and KR-theory from Section 3.3 defines the following canonical notion of topological phases for symmetric disordered IQPV.

Definition 5.1.1 (Topological phase of symmetric disordered IQPV)**.** *Let $(J; \phi)$ be a disordered IQPV of symmetry index (r, s) and fix a reference disordered IQPV $(J_{\mathrm{ref}}; \phi)$. Then J and J_{ref} are elements in $\mathcal{FF}^{\phi}(\mathbb{A})$ and we call*

$$[(J; \phi)] \coloneqq [J] - [J_{\mathrm{ref}}] \in \mathrm{Top}^{\phi}_{J_{\mathrm{ref}}}(\mathbb{A}) = \mathrm{Top}^{r,s}(A) \tag{5.1}$$

the topological phase of the disordered IQPV $(J; \phi)$ in the bulk.

Similarly, if $(J; \phi)$ and $(J_{\mathrm{ref}}; \phi)$ are complex disordered IQPV of symmetry index (r, s), the topological phase is defined by

$$[(J; \phi)] \coloneqq [J] - [J_{\mathrm{ref}}] \in \mathrm{Top}^{\phi}_{J_{\mathrm{ref}}}(\mathrm{A}) = \mathrm{Top}^{r,s}(A). \tag{5.2}$$

With this definition, we have thus established a systematic and physically natural notion of topological phases for disordered topological insulators and superconductors.

For technical reasons, it will in the following often be necessary to consider the topological phases in the bulk as Van Daele classes. In particular, this will be needed in the proof of the bulk-boundary correspondence and the application of the results by Kellendonk on the topological invariants [45]. By Proposition 5.3, the classes (5.1) in $\mathrm{Top}^{\phi}_{J_{\mathrm{ref}}}(\mathbb{A})$ and (5.2) in $\mathrm{Top}^{\phi}_{J_{\mathrm{ref}}}(\mathrm{A})$ correspond to the classes

$$\left[(J \otimes j_1)P^{\phi} + (J_{\mathrm{ref}} \otimes j_1)(1 - P^{\phi})\right] - \left[J_{\mathrm{ref}} \otimes j_1\right] \tag{5.3}$$

in $DKR(\mathbb{A} \otimes C\ell_{r,s+1}) = KR^{-(s-r+2)}(\mathbb{A})$ and $DK(\mathrm{A} \otimes \mathbb{C}\ell_{r,s+1}) = K^{-(s-r+2)}(\mathrm{A})$, respectively.

We emphasize once more the important observation that a disordered IQPV of index (r, s) defines a class in $KR^{-(s-r+2)}(\mathbb{A})$. If the disordered IQPV is complex, it defines a class in $K^{-(s-r+2)}(\mathrm{A})$

5.2 Topological phases at the boundary

Here we will define topological phases for topological insulators and superconductors with boundary. The main result of this section will be that the topological phases at the boundary can be identified with (real) K-classes in terms of Kasparov's Fredholm picture of (real) K-theory. In order to derive this result, we start from the physical description of disordered IQPV with boundary and establish notions of topological equivalence which are physically natural. On the basis of the resulting equivalence classes, we then construct a group of topological phases. This construction is closely related to the construction of the KR-groups in Kasparov's Fredholm picture and therefore it is not surprising that these groups are indeed isomorphic to (real) K-groups. The principal idea behind the structure of this section is to convince the reader of the physical naturality of our description of topological phases at the boundary. This could not be achieved by an ad hoc definition of topological phases in terms of (real) K-classes.

We already know that the algebra of localised boundary observables $\mathbb{A}_\partial \otimes \mathbb{K}(\ell^2(\mathbb{N}))$ is isomorphic to the ideal generated by e inside $\widehat{\mathbb{A}}$. Now we use this fact for realising $\widehat{\mathbb{A}}$ inside the operators over the Hilbert space over $\mathbb{A}_\partial$.

Proposition 5.2.1. *There is a real injective $*$-morphism $\widehat{\mathbb{A}} \hookrightarrow \mathrm{M}^s(\mathbb{A}_\partial)$. The concatenation of this morphism with the $*$-isomorphism $\mathrm{M}^s(\mathbb{A}_\partial) \cong \mathbb{B}(\mathcal{H}_{\mathbb{A}_\partial})$ of Theorem 3.1.12 is given by*

$$\widehat{\mathbb{A}} \ni \widehat{R}_d \mapsto \widehat{S}^* \in \mathbb{B}(\mathcal{H}_{\mathbb{A}_\partial}); \quad \widehat{\mathbb{A}} \ni a \mapsto \sum_{n \in \mathbb{N}} \alpha_d^n(a)|n\rangle\langle n| \in \mathbb{B}(\mathcal{H}_{\mathbb{A}_\partial}) \quad \forall a \in \mathbb{A}_\partial \subset \widehat{\mathbb{A}}.$$

The analogous statement is true for $\widehat{\mathrm{A}}$, where

$$\widehat{\mathrm{A}} \ni \widehat{R}_d^\sigma \mapsto \widehat{S}^* \in \mathbb{B}(\mathcal{H}_{\mathrm{A}_\partial}); \quad \widehat{\mathrm{A}} \ni a \mapsto \sum_{n \in \mathbb{N}} \alpha_d^n(a)|n\rangle\langle n| \in \mathrm{B}(\mathcal{H}_{\mathrm{A}_\partial}) \quad \forall a \in \mathrm{A}_\partial \subset \widehat{\mathrm{A}}.$$

Proof. We will only prove the real case, since the complex case can be proved in exactly the same way. The twisting does not play a role in the proof.

We have already established the injective $*$-morphism $\hat{\nu} : \widehat{\mathbb{A}} \to \mathbb{A} \otimes \mathcal{L}(\ell^2(\mathbb{N}))$ in Equation (4.18). Thus, $\widehat{\mathbb{A}}$ is isomorphic to the real C*-subalgebra $\operatorname{im}\hat{\nu} \subseteq \mathbb{A} \otimes \mathcal{L}(\ell^2(\mathbb{N}))$ which is generated by

$$R_d^* \otimes \widehat{S} \quad \text{and} \quad a \otimes 1 \quad \forall a \in \mathbb{A}_\partial. \tag{5.4}$$

We have also already identified the ideal $(e) \cong \mathbb{A}_\partial \otimes \mathbb{K}(\ell^2(\mathbb{N}))$ with the ideal $(1 \otimes \hat{e}) \subseteq \operatorname{im}\hat{\nu}$ via the $*$-morphism ν_∂. It maps an element $b \otimes |m\rangle\langle n| \in \mathbb{A}_\partial \otimes \mathbb{K}(\ell^2(\mathbb{N}))$ onto $(R_d^*)^m b R_d^n \otimes |m\rangle\langle n| \in \mathbb{A} \otimes \mathcal{L}(\ell^2(\mathbb{N}))$. The product of this element with the generators in Equation (5.4) is given by

$$\begin{aligned} (R_d^* \otimes \widehat{S})((R_d^*)^m b R_d^n \otimes |m\rangle\langle n|) &= (R_d^*)^{m+1} b R_d^n \otimes |m+1\rangle\langle n| \\ \text{and} \quad (a \otimes 1)((R_d^*)^m b R_d^n \otimes |m\rangle\langle n|) &= (R_d^*)^m \left[(R_d^*)^{-m} a (R_d^*)^m\right] b R_d^n \otimes |m\rangle\langle n|. \end{aligned} \tag{5.5}$$

Since $(R_d^*)^{-1} = R_d$, we have $\left[(R_d^*)^{-m} a (R_d^*)^m\right] = \alpha_d^m(a)$.

Suppose now that we are given an element $x \in \widehat{\mathbb{A}}$ such that $\hat{\nu}(x)b = 0$ for all $b \in (1 \otimes \hat{e})$. In particular, this implies that

$$\hat{\nu}(x)(1 \otimes |m\rangle\langle n|) = 0 \quad \forall m, n \in \mathbb{N}.$$

By Equations (5.5), it becomes apparent that this implies $\hat{\nu}(x) = 0$. Since $\hat{\nu}$ is injective, this implies $x = 0$. Thus, the ideal is essential and it follows by [65, Prop. 3.12.8] that there is a real injective $*$-morphism $\widehat{\mathbb{A}} \to \mathrm{M}(\mathbb{A}_\partial \otimes \mathbb{K}(\ell^2(\mathbb{N}))) = \mathrm{M}^s(\mathbb{A}_\partial)$, which is given by

$$\widehat{\mathbb{A}} \ni a \mapsto (T_1^a, T_2^a) \in \mathrm{M}^s(\mathbb{A}_\partial),$$

where $T_1^a(x) = ax, T_2^a(x) = xa$ for all $x \in \mathbb{A}_\partial \otimes \mathbb{K}(\ell^2(\mathbb{N}))$. By the above arguments, this morphism is for the generators of $\widehat{\mathbb{A}}$ explicitly given by

$$T_1^{\widehat{R_d^*}}(b \otimes |m\rangle\langle n|) = \nu_\partial^{-1}\Big((R_d^*)^{m+1} b R_d^n \otimes |m+1\rangle\langle n|\Big) = b \otimes |m+1\rangle\langle n| = \Big(\sum_{k \in \mathbb{N}} 1 \otimes |k+1\rangle\langle k|\Big)(b \otimes |m\rangle\langle n|),$$
$$T_1^a(b \otimes |m\rangle\langle n|) = \nu_\partial^{-1}\Big((R_d^*)^m \alpha_d^m(a) b R_d^n \otimes |m\rangle\langle n|\Big) = \alpha_d^m(a) b \otimes |m\rangle\langle n| = \Big(\sum_{k \in \mathbb{N}} \alpha_d^k(a) \otimes |k\rangle\langle k|\Big)(b \otimes |m\rangle\langle n|)$$

for all $a \in \mathbb{A}_\partial \subseteq \widehat{\mathbb{A}}, b \otimes |m\rangle\langle n| \in \mathbb{A}_\partial \otimes \mathbb{K}(\ell^2(\mathbb{N}))$ and T_2 can be derived similarly.

Finally, the isomorphism $\mathrm{M}^s(\mathbb{A}_\partial) \cong \mathbb{B}(\mathcal{H}_{\mathbb{A}_\partial})$ of Theorem 3.1.12 defines the embedding of $\widehat{\mathbb{A}} \hookrightarrow \mathbb{B}(\mathcal{H}_{\mathbb{A}_\partial})$ given by

$$\widehat{\mathbb{A}} \ni \widehat{R_d^*} \mapsto \widehat{S} = \sum_{n \in \mathbb{N}} |n+1\rangle\langle n| \in \mathbb{B}(\mathcal{H}_{\mathbb{A}_\partial}) \quad \text{and} \quad \widehat{\mathbb{A}} \supseteq \mathbb{A}_\partial \ni a \mapsto \sum_{n \in \mathbb{N}} \alpha_d^n(a)|n\rangle\langle n| \in \mathbb{B}(\mathcal{H}_{\mathbb{A}_\partial}).$$

□

The fact that $\widehat{\mathbb{A}}$ can be considered as a real C*-subalgebra of $\mathrm{M}^s(\mathbb{A}_\partial) \cong \mathbb{B}(\mathcal{H}_{\mathbb{A}_\partial})$ will be crucial for the definition of topological phases at the boundary. It canonically generalises[1] the notion of disordered IQPV with boundary to operators in $\mathbb{B}(\mathcal{H}_{\mathbb{A}_\partial})$.

Definition 5.2.2 (Generalised quasi-particle vacuum)**.** *A generalised quasi-particle vacuum (GQPV) of symmetry index (r,s), or simply of index (r,s), is a tuple $(\hat{J};\phi)$ given by a skew-Hermitian real $\hat{J} \in \mathbb{B}(\mathcal{H}_{\mathbb{A}_\partial})$ and a unital real $*$-morphism $\phi : C\ell_{r,s} \to \mathbb{B}(\mathcal{H}_{\mathbb{A}_\partial})$ such that*

$$\{\hat{J}, \phi(k_a)\},\ \{\hat{J}, \phi(j_\alpha)\},\ \hat{J}^2 + 1 \in \mathbb{K}(\mathcal{H}_{\mathbb{A}_\partial}) \quad \forall a = 1, \ldots, r, \alpha = 1, \ldots, s. \tag{5.6}$$

Recall that the k_a, j_α denote the generators of $C\ell_{r,s}$ and $\{\cdot,\cdot\}$ denotes the anti-commutator.

A complex generalised quasi-particle vacuum *of symmetry index (r,s) is defined in the same way by dropping the reality conditions, replacing $\mathbb{A}_\partial$ by A_∂ and $C\ell_{r,s}$ by $\mathbb{C}\ell_{r,s}$.*

Remark 5.2.3. *In Definition 5.2.2 one may replace the Hilbert space over $\mathbb{A}_\partial$ by any infinite-dimensional countably generated Hilbert module over $\mathbb{A}_\partial$. A future topic of research might be to use this flexibility in order to develop classification schemes for other physical systems.*

[1] This is indeed a generalisation because $\widehat{\mathbb{A}} \subseteq \mathrm{M}^s(\mathbb{A}_\partial)$ is in general a proper subalgebra. This is true because the K-theory of $M^s(\mathbb{A}_\partial)$ is trivial, whereas K-theory of $\widehat{\mathbb{A}}$ is in general non-trivial.

The generalisation of disordered IQPV with boundary to GQPV has the following physical interpretation. By construction, all IQPV with boundary give rise to IQPV via the surjective $*$-morphism $\rho : \widehat{\mathbb{A}} \to \mathbb{A}$. This feature distinguishes them from arbitrary GQPV $\hat{J} \in \mathbb{B}(\mathcal{H}_{\mathbb{A}_\partial})$, which in general don't have such a correspondence to IQPV in the bulk, but still have to satisfy $\hat{J}^2 = -1 \bmod \mathbb{K}(\mathcal{H}_{\mathbb{A}_\partial})$.

Physically, this means that we drop the condition that the operators have to be translationally covariant in the direction perpendicular to the boundary after going into the bulk, i.e. very far away from the boundary. This is reasonable because we are interested in the topological properties of the gapless states at the boundary where the translational covariance in this direction is inherently broken. By this classification approach, we ensure that the resulting topological phases are determined solely by the properties of these boundary states, which are a priori independent of the details in the bulk, as long as it is gapped. With this fact, the bulk-boundary correspondence of Section 5.3 becomes even more fascinating from a physical point of view.

We now introduce the following two equivalence relations on the set of GQPV.

Definition 5.2.4 (Equivalence relations of GQPV)**.** *Let $(\hat{J};\phi)$ be a GQPV of index (r,s).*

- *It is* operator homotopically equivalent *to a GQPV $(\hat{J}';\phi)$ if there is a norm continuous path of real skew-Hermitian operators $\hat{J}(t) \in \mathbb{B}(\mathcal{H}_{\mathbb{A}_\partial})$ for $t \in [0,1]$ connecting $\hat{J}$ and $\hat{J}'$ such that the conditions in Equation* (5.6) *are satisfied for all $t \in [0,1]$.*

- *It is* unitarily equivalent *to the GQPV $(u\hat{J}u^*; u\phi u^*)$ for any real unitary $u \in \mathbb{B}(\mathcal{H}_{\mathbb{A}_\partial})$.*

Let $E^{r,s}(\mathbb{A}_\partial)$ denote the corresponding set of equivalence classes which are obtained by these two equivalence relations.

The equivalence relations for complex GQPV can be defined in the same manner by dropping the reality constraints. The set of equivalence classes of complex GQPV of index (r,s) is denoted by $E^{r,s}(\mathrm{A}_\partial)$.

Definition 5.2.5 (Topologically trivial GQPV)**.** *A (complex) GQPV is called topologically trivial if it is equivalent, in the sense of Definition 5.2.4, to a (complex) GQPV $(\hat{J};\phi)$ satisfying*

$$\{\hat{J}, \phi(k_a)\} = \{\hat{J}, \phi(j_\alpha)\} = 1 + \hat{J}^2 = 0 \quad \forall a = 1,\dots,r, \alpha = 1,\dots,s.$$

Let $D^{r,s}(\mathbb{A}_\partial) \subseteq E^{r,s}(\mathbb{A}_\partial)$ and $D^{r,s}(\mathrm{A}_\partial) \subseteq E^{r,s}(\mathrm{A}_\partial)$ denote the corresponding equivalence classes of topologically trivial (complex) GQPV.

The topologically trivial classes correspond physically to GQPV which have no stable gapless states. For disordered IQPV with boundary, this means that there are no stable gapless boundary states. Thus, this notion of topological triviality is in accordance with the physical notion of topological triviality.

Proposition 5.2.6. *Consider $E^{r,s}(\mathbb{A}_\partial)$ and $D^{r,s}(\mathbb{A}_\partial)$ equipped with the Abelian semi-group structure induced by the direct sum*

$$[(\hat{J};\phi)] \oplus [(\hat{J}';\phi')] := \left[\left(\hat{J} \oplus \hat{J}'; \phi \oplus \phi'\right)\right].$$

Then the quotient

$$\mathrm{TopG}^{r,s}(\mathbb{A}_\partial) := E^{r,s}(\mathbb{A}_\partial)/D^{r,s}(\mathbb{A}_\partial)$$

is an Abelian group. The same assertions are true for the quotient

$$\mathrm{TopG}^{r,s}(\mathrm{A}_\partial) := E^{r,s}(\mathrm{A}_\partial)/D^{r,s}(\mathrm{A}_\partial).$$

We call the elements of these groups the topological phases of GQPV *of symmetry index* (r, s).

Proof. Note that the direct sum is well defined because $\mathbb{B}(\mathcal{H}_{\mathbb{A}_\partial}) \oplus \mathbb{B}(\mathcal{H}_{\mathbb{A}_\partial}) \subseteq \mathbb{B}(\mathcal{H}_{\mathbb{A}_\partial} \oplus \mathcal{H}_{\mathbb{A}_\partial}) \cong \mathbb{B}(\mathcal{H}_{\mathbb{A}_\partial})$ and it is Abelian because conjugation with the real unitary $u := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathbb{B}(\mathcal{H}_{\mathbb{A}_\partial} \oplus \mathcal{H}_{\mathbb{A}_\partial})$ exchanges the two summands of a direct sum.

There is a neutral element if the set $D^{r,s}(\mathbb{A}_\partial)$ is not empty. In order to show this, consider

$$(j_{s+r+1}; \mathrm{id}_{C\ell_{r,s}}) \in C\ell_{r+s+1,r+s+1} \cong M_{2^{r+s+1}}(\mathbb{C}).$$

By the isomorphisms $M_{2^{r+s+1}}(\mathbb{C}) \otimes \mathbb{B}(\mathcal{H}_{\mathbb{A}_\partial}) = \mathbb{B}(\mathbb{C}^{2^{r+s+1}} \otimes \mathcal{H}_{\mathbb{A}_\partial}) \cong \mathbb{B}(\mathcal{H}_{\mathbb{A}_\partial})$, this pair defines a topologically trivial GQPV in $E^{r,s}(\mathbb{A}_\partial)$.

It remains to prove the existence of the inverse. To that end, we consider a class $[(\hat{J}; \phi)] \in \mathrm{TopG}^{r,s}(\mathbb{A}_\partial)$ and define

$$-[(\hat{J}; \phi)] := [(-\hat{J}; \phi^{op})] \in \mathrm{TopG}^{r,s}(\mathbb{A}_\partial),$$

where $\phi^{op} : C\ell_{r,s} \to \mathbb{B}(\mathcal{H}_{\mathbb{A}_\partial})$ is defined by $\phi^{op}(k_a) := -\phi(k_a)$ for all $a = 1, \ldots, r$ and $\phi^{op}(j_\alpha) := -\phi(j_\alpha)$ for all $\alpha = 1, \ldots, s$. The direct sum of this class with the original one is given by

$$[(\hat{J}; \phi)] \oplus -[(\hat{J}; \phi)] = \left[\left(\left(\begin{smallmatrix} \hat{J} & 0 \\ 0 & -\hat{J} \end{smallmatrix}\right); \left(\begin{smallmatrix} \phi & 0 \\ 0 & \phi^{op} \end{smallmatrix}\right)\right)\right].$$

and the operator homotopy

$$[0, \pi/2] \ni t \longmapsto \begin{pmatrix} \hat{J}\cos(t) & -\sin(t) \\ \sin(t) & -\hat{J}\cos(t) \end{pmatrix}$$

connects this class with the topologically trivial class $\left[\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right); \left(\begin{smallmatrix} \phi & 0 \\ 0 & \phi^{op} \end{smallmatrix}\right)\right)\right]$.

Thus, the sum is equal to 0 in $\mathrm{TopG}^{r,s}(\mathbb{A}_\partial)$. □

In the following theorem, we will show that the group of topological phases of GQPV is isomorphic to a real or complex K-group in Kasparov's Fredholm picture.

Theorem 5.2.7. *For all* $r, s \in \mathbb{N}$, *we have*

$$\mathrm{TopG}^{r,s}(\mathbb{A}_\partial) = KR^{r,s}(\mathbb{A}_\partial) = KR^{-(s-r+1)}(\mathbb{A}_\partial) \quad \textit{and} \quad \mathrm{TopG}^{r,s}(\mathrm{A}_\partial) = K^{r,s}(\mathrm{A}_\partial) = K^{-(s-r+1)}(\mathrm{A}_\partial).$$

Proof. This theorem follows almost immediately by construction of our groups $\mathrm{TopG}^{r,s}(\mathbb{A}_\partial)$ and $\mathrm{TopG}^{r,s}(\mathrm{A}_\partial)$. We only prove the real case since the complex case is completely the same.

By definition, the set of GQPV is equal to the set of pairs $(F, \phi) \in \mathbb{E}^{r,s}(\mathbb{A}_\partial)$ with $F^* = -F$, and the set of topologically trivial GQPV is equal to $\mathbb{D}^{r,s}(\mathbb{A}_\partial)$.

By the second assertion of Corollary 3.1.37, all classes in $KR^{r,s}(\mathbb{A}_\partial)$ can be represented by pairs $(F; \phi)$ with $F^* = -F$. Therefore, the first assertion of Corollary 3.1.37 implies that the equivalence relations of Definition 5.2.4 are equivalent to the equivalence relations in Definition 3.1.30. Thus, the map

$$\mathrm{TopG}^{r,s}(\mathbb{A}_\partial) \ni [(\hat{J}; \phi)] \longmapsto [(\hat{J}; \phi)] \in KR^{r,s}(\mathbb{A}_\partial)$$

defines a well-defined isomorphism. □

Corollary 5.2.8. *For all $r, s \in \mathbb{N}$, we have*

$$\mathrm{TopG}^{r,s}(\mathbb{A}_\partial) = KKR^{-(s-r+1)}(\mathbb{A}_\partial) \quad \textit{and} \quad \mathrm{TopG}^{r,s}(\mathrm{A}_\partial) = KK^{-(s-r+1)}(\mathrm{A}_\partial).$$

This isomorphism maps a class $[(\hat{J};\phi)] \in \mathrm{TopG}^{r,s}(\mathbb{A}_\partial)$ onto the Kasparov class

$$\left[\left(\mathcal{H}_{\mathbb{A}_\partial} \otimes C\ell_{0,1}, \varphi, \hat{J} \otimes j_1\right)\right] \in KKR(C\ell_{s,r}, \mathbb{A}_\partial \otimes C\ell_{0,1}) = KKR^{-(s-r+1)}(\mathbb{A}_\partial), \tag{5.7}$$

where $\varphi : C\ell_{s,r} \to \mathbb{B}(\mathcal{H}_{\mathbb{A}_\partial}) \otimes C\ell_{0,1}$ with

$$\varphi(k_\alpha) = \phi(j_\alpha) \otimes j_1 \quad \forall a \in \{1, \ldots, r\} \quad \textit{and} \quad \varphi(j_a) = \phi(k_a) \otimes j_1 \quad \forall \alpha \in \{1, \ldots, s\}. \tag{5.8}$$

Similarly, a class $[(\hat{J};\phi)] \in \mathrm{TopG}^{r,s}(\mathrm{A}_\partial)$ is mapped onto the Kasparov class

$$\left[\left(\mathcal{H}_{\mathrm{A}_\partial} \otimes \mathbb{C}\ell_{0,1}, \varphi, \hat{J} \otimes j_1\right)\right] \in KK(\mathbb{C}\ell_{s,r}, \mathrm{A}_\partial \otimes \mathbb{C}\ell_{0,1}) = KK^{-(s-r+1)}(\mathrm{A}_\partial), \tag{5.9}$$

where $\varphi : \mathbb{C}\ell_{s,r} \to \mathbb{B}(\mathcal{H}_{\mathrm{A}_\partial}) \otimes \mathbb{C}\ell_{0,1}$ satisfies Equation (5.8).

Proof. This corollary is an immediate consequence of Theorem 5.2.7 and the explicit isomorphisms $KR^{r,s}(\mathbb{A}_\partial) \cong KKR(C\ell_{s,r}, \mathbb{A}_\partial \otimes C\ell_{0,1})$ and $K^{r,s}(\mathrm{A}_\partial) \cong KK(\mathbb{C}\ell_{s,r}, \mathrm{A}_\partial \otimes \mathbb{C}\ell_{0,1})$ of the proof of Theorem 3.1.36. □

With the results above, we have now the full power of Kasparov's theory at hand in order to classify topological phases of GQPV. In particular, we obtain the following corollary.

Corollary 5.2.9. *Every topological phase in $\mathrm{TopG}^{r,s}(\mathbb{A}_\partial)$ can be represented by a GQPV $(\hat{J};\phi)$ with*

$$\{\hat{J}, \phi(k_a)\} = \{\hat{J}; \phi(j_\alpha)\} = 0 \quad \forall a = 1, \ldots, r, \alpha = 1, \ldots, s. \tag{5.10}$$

Proof. This corollary immediately follows by Theorem 5.2.7 and Corollary 3.1.37. □

Corollary 5.2.9 is important for the physical interpretation of topological phases of GQPV. It implies that it is sufficient to consider only those GQPV $(\hat{J};\phi)$ that satisfy Equation (5.10) in order to get an exhaustive topological classification.

In the context of disordered IQPV with boundary, we can make the statement of Corollary 5.2.9 even more constructive. To that end, suppose we are given a disordered IQPV $(J;\phi)$ of symmetry index (r,s) in $\mathbb{A}$. Then we construct a corresponding disordered IQPV with boundary via the section $s : \mathbb{A} \to \widehat{\mathbb{A}}$ of Equation (4.21):

$$\hat{J} := s(J).$$

This section is equal to the identity, when restricted to $C(\Omega) \otimes \mathrm{End}(W)$. Therefore, we have

$$s(J)\phi(k_a) + \phi(k_a)s(J) = s(J\phi(k_a) + \phi(k_a)J) = 0 \quad \forall a = 1, \ldots, r.$$

Analogously, we derive $\{\hat{J}; \phi(j_\alpha)\} = 0$ for all $\alpha = 1, \ldots, s$, thus obtaining Equation (5.10).

In general, the choice of a disordered IQPV with boundary to a given disordered IQPV in the bulk is far from unique. However, due to the following lemma, there is a canonical topological phase at the boundary for a given disordered IQPV in the bulk.

Lemma 5.2.10. *Let (J,ϕ) be a disordered IQPV of index (r,s) and let $(\hat{J},\phi)$ be a disordered IQPV with boundary corresponding to (J,ϕ), i.e. $\rho(\hat{J}) = J$. Then $[(\hat{J},\phi)] = [(s(J),\phi)] \in$* $\mathrm{TopG}^{r,s}(\mathbb{A}_\partial)$.

The same holds for the complex case.

Proof. By assumption, we have $\rho(\hat{J}) = \rho((s(J))$, which implies $s(J) - \hat{J} \in \mathbb{K}(\mathcal{H}_{\mathbb{A}_\partial})$. Thus

$$[0,1] \ni t \longmapsto \hat{J} + t(s(J) - \hat{J})$$

defines an operator homotopy in the set of GQPV. □

By Lemma 5.2.10, we can make the following definition without ambiguity.

Definition 5.2.11 (Boundary class attached to disordered IQPV)**.** *Let $[(J;\phi)]$ be the topological phase of a bulk disordered IQPV of symmetry index (r,s) with respect to a disordered reference disordered IQPV $(J_{\mathrm{ref}};\phi)$. Define $\hat{J} := s(J)$ and $\hat{J}_{\mathrm{ref}} := s(J_{\mathrm{ref}})$. Then we call*

$$[(J;\phi)]_\partial := [(\hat{J};\phi)] - [(\hat{J}_{\mathrm{ref}};\phi)] \in \mathrm{TopG}^{r,s}(\mathbb{A}_\partial)$$

the boundary class for $(J;\phi)$. *By Theorem 5.2.7, the boundary phase defines a class in $KR^{r,s}(\mathbb{A}_\partial) = KR^{-(s-r+1)}(\mathbb{A}_\partial)$. In terms of Kasparov classes, it is given by the difference of*

$$[(\mathcal{H}_{\mathbb{A}_\partial} \otimes C\ell_{0,1}, \varphi : C\ell_{s,r} \to \mathbb{B}(\mathcal{H}_{\mathbb{A}_\partial} \otimes C\ell_{0,1}), \hat{J} \otimes j_1)] \in KKR(C\ell_{s,r}, \mathbb{A}_\partial \otimes C\ell_{0,1})$$

and

$$[(\mathcal{H}_{\mathbb{A}_\partial} \otimes C\ell_{0,1}, \varphi : C\ell_{s,r} \to \mathbb{B}(\mathcal{H}_{\mathbb{A}_\partial} \otimes C\ell_{0,1}), \hat{J}_{\mathrm{ref}} \otimes j_1)] \in KKR(C\ell_{s,r}, \mathbb{A}_\partial \otimes C\ell_{0,1}),$$

where $\varphi : C\ell_{s,r} \to \mathbb{B}(\mathcal{H}_{\mathbb{A}_\partial} \otimes C\ell_{0,1})$ is defined by $\varphi(k_\alpha) = J_\alpha \otimes j_1$ and $\varphi(j_a) = K_a \otimes j_1$.

For complex disordered IQPV $(J;\phi)$, we define

$$[(J;\phi)]_\partial := [(\hat{J};\phi)] - [(\hat{J}_{\mathrm{ref}};\phi)] \in \mathrm{TopG}^{r,s}(\mathrm{A}_\partial)$$

This boundary class defines a class in $K^{r,s}(\mathrm{A}_\partial) = K^{-(s+r+1)}(\mathrm{A}_\partial)$.

We close this section with two examples where the boundary class can be represented in an even simpler form.

Example 5.2.12. *For the symmetry class BD*I*, we have $r - s = 1$, where r and s label the number of positive and negative pseudo-symmetries. Up to stable equivalence, we can consider the case $r = 1$, $s = 0$. Then the boundary class $[(J;\phi)]_\partial$ is given by the difference of*

$$[(\mathcal{H}_{\mathbb{A}_\partial} \otimes C\ell_{0,1}, \varphi : C\ell_{0,1} \to \mathbb{B}(\mathcal{H}_{\mathbb{A}_\partial} \otimes C\ell_{0,1}), \hat{J} \otimes j_1)] \in KKR(C\ell_{0,1}, \mathbb{A}_\partial \otimes C\ell_{0,1}),$$

and

$$[(\mathcal{H}_{\mathbb{A}_\partial} \otimes C\ell_{0,1}, \varphi : C\ell_{0,1} \to \mathbb{B}(\mathcal{H}_{\mathbb{A}_\partial} \otimes C\ell_{0,1}), \hat{J}_{\mathrm{ref}} \otimes j_1)] \in KKR(C\ell_{0,1}, \mathbb{A}_\partial \otimes C\ell_{0,1}),$$

where $\varphi(j_1) = K_1 \otimes j_1$. The first class is equivalent to

$$[(\mathcal{H}_{\mathbb{A}_\partial}, 1_{\mathbb{C}}, \hat{J}K_1)] \in KKR(\mathbb{C}, \mathbb{A}_\partial),$$

where $\mathcal{H}_{\mathbb{A}_\partial}$ is graded by the grading operator K_1. Using an index morphism [77, Thm. 2.2.8], we can identify this Kasparov class with the class

$$[P_+] - [P_-] \in KR^0(\mathbb{A}_\partial),$$

where $P_\pm$ denote the orthogonal projections onto the sub-modules

$$\ker\left[\tfrac{1}{2}(1 \pm K_1)\hat{J}K_1\tfrac{1}{2}(1 \mp K_1)\right] \subseteq \mathcal{H}_{\mathbb{A}_\partial}.$$

If $K_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, then $\hat{J}K_1 = \left(\begin{smallmatrix} 0 & \hat{u} \\ \hat{u}^* & 0 \end{smallmatrix}\right)$ and the class in $K^0(\mathbb{A}_\partial)$ is of the form

$$[P_{\ker(\hat{u})}] - [P_{\ker(\hat{u}^*)}] \in KR^0(\mathbb{A}_\partial).$$

The reference class can be simplified in the same way.

In symmetry class AIII, we also have $r - s = 1$ and therefore we can proceed in the same way as for symmetry class BDI. For a given complex disordered IQPV $J \in \mathrm{A}$ with one positive pseudo-symmetry K_1, the Kasparov class at the boundary is given by

$$[(\mathcal{H}_{\mathrm{A}_\partial}, 1_{\mathbb{C}}, \hat{J}K_1)] - [(\mathcal{H}_{\mathrm{A}_\partial}, 1_{\mathbb{C}}, \hat{J}_{\mathrm{ref}}K_1)] \in KK(\mathbb{C}, \mathrm{A}_\partial).$$

This class can be simplified in completely the same way as in symmetry class BDI.

5.3 The bulk-boundary correspondence

In this section, we will establish a canonical correspondence between bulk and boundary topological phases of topological insulators and superconductors.

As shown in Section 5.1, the bulk topological phases define elements in $KR^{-(s-r+2)}(\mathbb{A})$. By Section 5.2, the boundary topological phases give rise to elements in $KR^{-(s-r+1)}(\mathbb{A}_\partial)$. These two groups are connected by a connecting map which is obtained as follows.

The bulk-boundary SES in Equation (4.19) induces the long exact sequence

$$\begin{tikzcd} \cdots \arrow[r, dotted] & KR^{-(s-r+2)}(\mathbb{A}_\partial) \arrow[r, "\iota_*"] & KR^{-(s-r+2)}(\widehat{\mathbb{A}}) \arrow[r, "\rho_*"] & KR^{-(s-r+2)}(\mathbb{A}) \arrow[dll, "\partial", rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- (X.center) \tikztonodes -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\ & KR^{-(s-r+1)}(\mathbb{A}_\partial) \arrow[r, "\iota_*"] & KR^{-(s-r+1)}(\widehat{\mathbb{A}}) \arrow[r, "\rho_*"] & KR^{-(s-r+1)}(\mathbb{A}) \arrow[r, dotted] & \cdots \end{tikzcd} \tag{5.11}$$

of KR-groups. Accordingly, the complex bulk-boundary SES in Equation (4.20) induces the following long exact sequence of K-groups:

$$\begin{tikzcd} \cdots \arrow[r, dotted] & K^{-(s-r+2)}(\mathrm{A}_\partial) \arrow[r, "\iota_*"] & K^{-(s-r+2)}(\widehat{\mathrm{A}}) \arrow[r, "\rho_*"] & K^{-(s-r+2)}(\mathrm{A}) \arrow[dll, "\partial", rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- (X.center) \tikztonodes -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\ & K^{-(s-r+1)}(\mathrm{A}_\partial) \arrow[r, "\iota_*"] & K^{-(s-r+1)}(\widehat{\mathrm{A}}) \arrow[r, "\rho_*"] & K^{-(s-r+1)}(\mathrm{A}) \arrow[r, dotted] & \cdots \end{tikzcd} \tag{5.12}$$

In particular, we obtain connecting maps

$$\partial : KR^{-(s-r+2)}(\mathbb{A}) \to KR^{-(s-r+1)}(\mathbb{A}_\partial) \quad \text{and} \quad \partial : K^{-(s-r+2)}(\mathrm{A}) \to K^{-(s-r+1)}(\mathrm{A}_\partial), \tag{5.13}$$

which we call the *bulk-boundary maps*. These bulk-boundary maps are well known and can even be written down explicitly in terms of Van Daele groups, *cf.* Theorem 3.2.10, and Kasparov groups, *cf.* Theorem 3.1.29.

On the one hand, we have seen in Section 5.1 that the bulk topological phases are canonically defined as elements of $\mathrm{Top}^{r,s}(\mathbb{A})$ and $\mathrm{Top}^{r,s}(\mathrm{A})$, and these can be explicitly identified with the classes of Equation (5.3) in Van Daele's picture of real and complex K-theory, respectively.

On the other hand, the topological phases at the boundary are defined as classes in the group $\mathrm{TopG}^{r,s}(\mathbb{A}_\partial)$ or $\mathrm{TopG}^{r,s}(\mathrm{A}_\partial)$, which is explicitly isomorphic to Kasparov's Fredholm picture of KR- or K-theory, respectively.

Therefore, a satisfactory explicit formulation of the bulk-boundary correspondence has to identify classes in these two different pictures. This is obtained by the composition of the bulk-boundary maps with Roe's isomorphism α from Van Daele groups to Kasparov groups, *cf.* Subsection 3.2.1. This strategy can be summarised in the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Top}^{r,s}(\mathbb{A}) \cong DKR^{-(s-r+2)}(\mathbb{A}) \arrow[r, dashed, "\alpha"] \arrow[d, "\partial"] \arrow[dr] & KKR^{-(s-r+2)}(\mathbb{A}) \arrow[d, dashed, "\partial"'] \\
DKR^{-(s-r+1)}(\mathbb{A}_\partial) \arrow[r, "\alpha"] & KKR^{-(s-r+1)}(\mathbb{A}_\partial) \cong \mathrm{TopG}^{r,s}(\mathbb{A}_\partial)
\end{tikzcd}$$

The main result of this section is that ∂ maps the bulk class attached to a disordered IQPV with symmetries onto the corresponding boundary class, as explained in the following theorem. (The same holds for the complex symmetry classes, too.)

Theorem 5.3.1 (Bulk-boundary correspondence)**.** *Fix a reference disordered IQPV* $(J_{\mathrm{ref}};\phi)$ *of (complex) symmetry index* (r,s)*. For any disordered IQPV* $(J;\phi)$ *of (complex) symmetry index* (r,s)*, we have*

$$\partial[(J;\phi)] = [(J;\phi)]_\partial. \tag{5.14}$$

That is, the bulk-boundary map ∂ *maps the bulk class* $[(J;\phi)]$ *attached to* $(J;\phi)$ *onto the boundary class* $[(J;\phi)]_\partial$ *attached to* $(J;\phi)$*, cf. Definition 5.2.11.*

5.3.1 Proof of the bulk-boundary correspondence

This subsection is devoted to the proof of Theorem 5.3.1. Let us begin by fixing some conventions. Throughout the proof, we will consider the Clifford algebras with their natural grading. All other C*-algebras are ungraded. We will only prove the real case here since the complex case can be proved in completely the same way.

Recall Roe's isomorphism α from Subsection 3.2.1. Since $\alpha\circ\partial = \partial\circ\alpha$, we can compute the effect of ∂ on the bulk class in DKR-theory and subsequently apply α^{-1} to obtain a class in KKR-theory.

The non-trivial part of the definition of α is given by the connecting map ∂_Q in DKR-theory. We relate it to the bulk-boundary map. To that end, recall that $\mathbb{A}_\partial\otimes\mathbb{K}$ (where $\mathbb{K} = \mathbb{K}(\ell^2(\mathbb{N}))$) is an essential ideal in $\widehat{\mathbb{A}}$, as was shown in the proof of Proposition 5.2.1. Therefore, the canonical map $\mathbb{A}_\partial\otimes\mathbb{K}\longrightarrow \mathrm{M}^s(\mathbb{A}_\partial)$ factors as $\Psi\circ\iota$ where $\Psi:\widehat{\mathbb{A}}\longrightarrow\mathrm{M}^s(\mathbb{A}_\partial)$ is a unique injective real $*$-morphism.

We obtain a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{A}_\partial \otimes \mathbb{K} \arrow[r, "\iota"] \arrow[d, equal] & \widehat{\mathbb{A}} \arrow[r, "\rho"] \arrow[d, "\Psi"] & \mathbb{A} \arrow[r] \arrow[d, "\widetilde{\Psi}"] & 0 \\
0 \arrow[r] & \mathbb{A}_\partial \otimes \mathbb{K} \arrow[r] & \mathrm{M}^s(\mathbb{A}_\partial) \arrow[r, "\pi"] & Q^s(\mathbb{A}_\partial) \arrow[r] & 0
\end{tikzcd}$$

From the naturality of connecting maps (which is a general fact for δ-functors, but can also be derived directly from the explicit form of the connecting map given in Theorem 3.2.10), the following diagram commutes:

$$\begin{tikzcd}
DKR(\mathbb{A} \otimes C\ell_{r,s+1}) \arrow[r, "\partial"] \arrow[d, "\widetilde{\Psi}_*"] & DKR(\mathbb{A}_\partial \otimes C\ell_{r+1,s+1}) \\
DKR(Q^s(\mathbb{A}_\partial) \otimes C\ell_{r,s+1}), \arrow[ur, "\partial_Q"'] &
\end{tikzcd}$$

Therefore,

$$(\alpha^{-1} \circ \partial)([x] - [e]) = [(\mathcal{H}_{\mathbb{A}_\partial} \otimes C\ell_{r,s+1}, 1_{\mathbb{C}}, y)] - [(\mathcal{H}_{\mathbb{A}_\partial} \otimes C\ell_{r,s+1}, 1_{\mathbb{C}}, f)]$$

for $y = \Psi(\hat{x})$, $f = \Psi(\hat{e})$, $\rho(\hat{x}) = x$ and $\rho(\hat{e}) = e$. Applying these considerations to the bulk class with reference ORHU $e = J_{\mathrm{ref}} \otimes j_1 \in \mathcal{F}(\mathbb{A} \otimes C\ell_{r,s+1})$, we find by Equation (5.3) that as an element of $KKR(\mathbb{C}, \mathbb{A}_\partial \otimes C\ell_{r,s+1})$, the image $\partial[(J;\phi)]$ of the bulk class under ∂ equals

$$[(\mathcal{H}_{\mathbb{A}_\partial} \otimes C\ell_{r,s+1}, 1_{\mathbb{C}}, (\hat{J} \otimes j_1)P + (\hat{J}_{\mathrm{ref}} \otimes j_1)(1-P))] - [(\mathcal{H}_{\mathbb{A}_\partial} \otimes C\ell_{r,s+1}, 1_{\mathbb{C}}, \hat{J}_{\mathrm{ref}} \otimes j_1)]$$

where we write $P := P^\phi$. This in turn equals

$$[(P(\mathcal{H}_{\mathbb{A}_\partial} \otimes C\ell_{r,s+1}), 1_{\mathbb{C}}, (\hat{J} \otimes j_1)P)] - [(P(\mathcal{H}_{\mathbb{A}_\partial} \otimes C\ell_{r,s+1}), 1_{\mathbb{C}}, (\hat{J}_{\mathrm{ref}} \otimes j_1)P)].$$

We denote the two parts of this difference by ∂_{I} and ∂_{II}, respectively. To compare the expression $\partial_{\mathrm{I}} - \partial_{\mathrm{II}}$ to the boundary class, we may treat the parts independently. In what follows, we shall focus on ∂_{I}, the case of ∂_{II} being almost identical.

We set $B := \mathbb{A}_\partial \otimes C\ell_{0,1}$ for brevity. Then the boundary class is given in $KKR(C\ell_{s,r}, B)$, by Corollary 5.2.8. We will use some standard isomorphisms of Kasparov's theory to transfer the class computed above to this Kasparov group. As a first step, we apply the Clifford algebra isomorphism $C\ell_{r,s+1} \cong C\ell_{0,1} \widehat{\otimes} C\ell_{r,s}$, given by

$$j_1 \longmapsto j_1 \otimes 1, \quad j_{\alpha+1} \otimes 1 \longmapsto 1 \otimes j_\alpha, \quad k_a \longmapsto -1 \otimes k_a.$$

This isomorphism preserves the orientation of the Clifford algebra. The image of the projection P now takes the form

$$P' := \prod_{a=1}^{r} \tfrac{1}{2}(1 + (-1)^s K_a \otimes j_1 \otimes k_a) \prod_{\alpha=1}^{s} \tfrac{1}{2}(1 + J_\alpha \otimes j_1 \otimes j_\alpha).$$

The natural isomorphism $\tau_{C\ell_{s,r}}$, *cf.* Proposition 3.1.21, maps ∂_{I} onto

$$\tau_{C\ell_{s,r}}(\partial_{\mathrm{I}}) = [((P' \otimes 1)(\mathcal{H}_B \widehat{\otimes} C\ell_{r,s} \widehat{\otimes} C\ell_{s,r}), \mathrm{id}_{C\ell_{s,r}}, (J \otimes j_1 \otimes 1 \otimes 1)(P' \otimes 1))]$$

in $KKR(C\ell_{s,r}, B\widehat{\otimes}C\ell_{r,s}\widehat{\otimes}C\ell_{s,r})$. We now apply the orientation-preserving isomorphism $C\ell_{r,s}\widehat{\otimes}C\ell_{s,r} \cong C\ell_{s+r,s+r}$ given by

$$k_a \otimes 1 \longmapsto (-1)^s k_{s+a}, \quad j_\alpha \otimes 1 \longmapsto j_\alpha, \quad 1 \otimes k_\alpha \longmapsto (-1)^s k_\alpha, \quad 1 \otimes j_a \longmapsto j_{s+a},$$

for $a = 1, \ldots r$ and $\alpha = 1, \ldots, s$. This isomorphism converts $P' \otimes 1$ into the projection

$$P'' = \prod_{a=1}^{r} \frac{1}{2}(1 + K_a \otimes j_1 \otimes k_{s+a}) \prod_{\alpha=1}^{s} \tfrac{1}{2}(1 + J_\alpha \otimes j_1 \otimes j_\alpha),$$

so it identifies $\partial_{\mathrm{I}}\widehat{\otimes}\tau$ with the class

$$[(P''(\mathcal{H}_B\widehat{\otimes}C\ell_{s+r,s+r}), \varphi, (\hat{J} \otimes j_1 \otimes 1)P'')] \in KKR(C\ell_{s,r}, B\widehat{\otimes}C\ell_{s+r,s+r}),$$

where φ is defined by

$$\varphi(k_\alpha) := (-1)^s \otimes 1 \otimes k_\alpha, \quad \varphi(j_a) = 1 \otimes 1 \otimes j_{s+a} \quad \forall a = 1, \ldots, r, \alpha = 1, \ldots, s.$$

The next step is to apply a unitary equivalence. We define commuting unitaries

$$U_\alpha := \tfrac{1}{\sqrt{2}}(1 - J_\alpha \otimes j_1 \otimes k_\alpha), \quad V_a := \tfrac{1}{\sqrt{2}}(1 - K_a \otimes j_1 \otimes j_{s+a})$$

for $a = 1, \ldots, r$ and $\alpha = 1, \ldots, s$. Then we have

$$\mathrm{Ad}(U_\alpha)(1 \otimes 1 \otimes k_\alpha) = -J_\alpha \otimes j_1 \otimes 1, \quad \mathrm{Ad}(V_a)(1 \otimes 1 \otimes j_{s+a}) = K_a \otimes j_1 \otimes 1,$$

whereas $1 \otimes 1 \otimes c$ is fixed for any other generator c of $C\ell_{r+s,r+s}$. Similarly, $\mathrm{Ad}(U_\alpha)$ fixes any $K_a \otimes j_1 \otimes 1$ and $J_\beta \otimes j_1 \otimes 1$ for $\beta \neq \alpha$, and $\mathrm{Ad}(V_a)$ fixes any $J_\alpha \otimes j_1 \otimes 1$ and $K_b \otimes j_1 \otimes 1$ for $b \neq a$. Both actions fix $\hat{J} \otimes j_1 \otimes 1$.

Hence, applying the unitary equivalence given by $V_1 \cdots V_r U_1 \cdots U_s$ identifies $\tau_{C\ell_{s,r}}(\partial_{\mathrm{I}})$ with the class

$$[(\mathcal{H}_B\widehat{\otimes}Q(C\ell_{r+s,r+s}), \psi \otimes 1, \hat{J} \otimes j_1 \otimes Q)] \in KKR(C\ell_{s,r}, B\widehat{\otimes}C\ell_{r+s,r+s})$$

where $Q = \prod_{i=1}^{r+s} \frac{1}{2}(1 + k_i j_i)$ and ψ is given by

$$\psi(k_\alpha) := (-1)^{s+1} J_\alpha \otimes j_1, \quad \psi(j_a) := K_a \otimes j_1 \quad \forall a = 1, \ldots, r, \alpha = 1, \ldots, s.$$

The automorphism of $C\ell_{s,r}$ given by $k_\alpha \longmapsto (-1)^{s+1} k_\alpha$ and $j_a \longmapsto j_a$ preserves orientation, as $(-1)^{s(s+1)} = 1$, and thus acts trivially on Kasparov groups. We may therefore remove the signs in the definition of ψ without changing the class.

The final step of the proof will be to apply stability. By Theorem 3.1.28, the stability isomorphism is given by the intersection product with

$$\alpha := [((\mathcal{K} \oplus \mathcal{K}^{op})\widehat{\otimes}\Lambda(\mathbb{C}^{r+s}), 1 \otimes \rho, T \otimes 1)] \in KKR(C\ell_{r+s,r+s}, \mathbb{C}),$$

where $\mathcal{K}$ and T are defined as in Theorem 3.1.26 and $\rho : C\ell_{r+s,r+s} \to \mathrm{End}(\Lambda(\mathbb{C}^{r+s}))$ is the even real $*$-representation of $C\ell_{r+s,r+s}$ of Proposition 2.2.4. As we have already explained in the discussion below Theorem 3.1.32, the intersection product is given by

$$[(\mathcal{H}_B \otimes \rho(Q)(\Lambda(\mathbb{C}^{r+s})), \psi \otimes 1, (\hat{J} \otimes j_1) \otimes 1)] \in KKR(C\ell_{s,r}, B).$$

But $\rho(Q)(\Lambda(\mathbb{C}^{r+s}))$ is even, has dimension one and contributes neither to the action nor to the Fredholm operator, so we conclude

$$\tau_{C\ell_{s,r}}(\partial_{\mathrm{I}})\,\widehat{\otimes}_{C\ell_{r+s,r+s}}\,\alpha = [(\mathcal{H}_B, \psi, \hat{J} \otimes j_1)] \in KKR(C\ell_{s,r}, B).$$

The same holds for ∂_{I} replaced by ∂_{II} and $\hat{J}$ replaced by $\hat{J}_{\mathrm{ref}}$. Because

$$[(J;\phi)]_\partial = [(\mathcal{H}_B, \psi, \hat{J} \otimes j_1)] - [(\mathcal{H}_B, \psi, \hat{J}_{\mathrm{ref}} \otimes j_1)],$$

this completes the proof of the theorem.

5.4 Corollaries to the bulk-boundary correspondence

In this section, we summarise some important properties of the bulk-boundary correspondence which are in particular relevant for its physical interpretation.

In the context of topological insulators and superconductors, a choice of the reference disordered IQPV which is especially relevant are disordered IQPV which are local in the strong sense that $J_{\mathrm{ref}} \in C(\Omega) \otimes \mathrm{End}(W)$. The corresponding ground state of such a local reference disordered IQPV consists of combinations of localised states. In the physical context, these are referred to as *atomic limit topological insulators* or *atomic limit superconductors*. Such a local reference disordered IQPV always exists in the present physical context, where we only consider local (pseudo-)symmetries over the bulk algebra $\mathbb{A}$. In materials with other crystalline symmetries than translational symmetries, such an atomic limit, resp. local reference disordered IQPV, may not exist [21].

The bulk-boundary correspondence becomes of the following more simple form if the reference disordered IQPV is chosen to be local.

Corollary 5.4.1. *Fix $r, s \geqslant 0$ and assume that the reference disordered IQPV in Theorem 5.3.1 is local,* i.e. $J_{\mathrm{ref}} \in C(\Omega) \otimes \mathrm{End}(W)$ *(resp.* $C(\Omega) \otimes \mathrm{End}(V)$ *in the complex case). Then the boundary class simplifies to*

$$[(J;\phi)]_\partial = [(\hat{J};\phi)],$$

so that bulk-boundary correspondence reads

$$\partial[(J;\phi)] = [(\hat{J};\phi)].$$

Proof. By assumption, J_{ref} commutes with q, so that $\hat{J}_{\mathrm{ref}}$ is unitary in $\widehat{\mathbb{A}}$. Hence, the class $[(\hat{J}_{\mathrm{ref}};\phi)] \in \mathrm{TopG}^{r,s}(\mathbb{A}_\partial)$ is trivial. The same holds for complex disordered IQPV. □

Corollary 5.4.2. *Fix $r, s \geqslant 0$ and assume that the reference disordered IQPV in Theorem 5.3.1 is local (see the previous corollary). Let $(J;\phi)$ be a disordered IQPV of (complex) symmetry index (r, s) such that its boundary class is trivial,* i.e. $\partial[(J;\phi)] = 0$. *Then $\hat{J}$ is operator homotopic, in the sense of Definition 5.2.4, to a (real) skew-Hermitian unitary in $\mathbb{B}(\mathcal{H}_{\mathbb{A}_\partial})$ anti-commuting with the K_a, J_α.*

Proof. This is immediate from Theorem 5.3.1, Corollary 5.4.1 and Corollary 5.2.9. □

Remark 5.4.3. *Although the statement of Corollary 5.4.2 is trivial from a mathematical point of view, it nevertheless is an important physical observation. Namely, expressed in physical terms, it states that the boundary topological phase attached to a bulk disordered IQPV is topologically trivial if and only if it exhibits no stable localised boundary states violating the gap condition.*

Remark 5.4.4. *By the exactness of the long exact sequence in K-theory in Equation* (5.11)*, the image of the map ∂ is equal to the kernel of*

$$\iota_* : KR^{-(s-r+1)}(\mathbb{A}_\partial) \longrightarrow KR^{-(s-r+1)}(\widehat{\mathbb{A}}) \quad \textit{resp.} \quad \iota_* : K^{-(s-r+1)}(\mathrm{A}_\partial) \longrightarrow K^{-(s-r+1)}(\widehat{\mathrm{A}}).$$

As explained in [77, Thm. 1.5.5] for the real case and in [66, Ch. 4] for the complex case, this coincides with $\ker(\mathrm{id} - (\alpha_d)_*)$ *under an isomorphism* $KR^{-(s-r+1)}(\widehat{\mathbb{A}}) \cong KR^{-(s-r+1)}(\mathbb{A}_\partial)$ *resp.* $K^{-(s-r+1)}(\widehat{\mathrm{A}}) \cong K^{-(s-r+1)}(\mathrm{A}_\partial)$*. This kernel is trivial if and only if α_d is stably homotopic to the identity. While this holds for the clean case without disorder, it may fail in general. Thus, ∂ may not be surjective.*

We can compare the boundary phases corresponding to different sides of a boundary. To that end, observe that $1 - q$ is the projection corresponding to the left half-space semilattice $-\widehat{\Lambda}$. In the following proposition, let $\hat{J}_+ := \hat{J}$ and $\hat{J}_- := (1-q)J(1-q)$.

Proposition 5.4.5. *Let $(J_{\mathrm{ref}}; \phi)$ be a reference disordered IQPV of (complex) symmetry index (r, s). For any disordered IQPV $(J; \phi)$ of (complex) symmetry index (r, s), we have*

$$[(J;\phi)]_\partial^+ = -[(J;\phi)]_\partial^-.$$

Here, we set $[(J;\phi)]_\partial^\pm := [(\hat{J}_\pm;\phi)] - [(\hat{J}_{\mathrm{ref},\pm};\phi)]$.

Proof. It is sufficient to prove that the sum $[(\hat{J}_+;\phi)] + [(\hat{J}_-;\phi)]$ is trivial for any J. The sum is represented by

$$(\hat{J}_+ \oplus \hat{J}_-, 1_2 \otimes \phi)$$

Here, by definition

$$\hat{J}_+ \oplus \hat{J}_- = qJq \oplus (1-q)J(1-q),$$

where q is the half-space projection of Equation (4.17). The right-hand side is the starting point F_0 of the path

$$[0,1] \ni t \longmapsto F_t := \begin{pmatrix} qJq & tqJ(1-q) \\ t(1-q)Jq & (1-q)J(1-q) \end{pmatrix}.$$

The operators F_t are skew-Hermitian and

$$F_t^2 = \begin{pmatrix} q(-1 + (t^2-1)J(1-q)J)q & 0 \\ 0 & (1-q)(-1 + (t^2-1)JqJ) \end{pmatrix}.$$

The operator $qJ(1-q)Jq$ lies in the kernel of the canonical projection $\rho = \rho_+$ of the right half-space algebra. Indeed, the expression $qO_1(1-q)O_2q$ vanishes if O_1 and O_2 are among the generators $f \in C(\Omega) \otimes \mathrm{End}(W)$ and $u_1, \dots, u_{d-1}$, together with their adjoints, as all of these commute with q. Furthermore, we compute

$$(1-q)u_d q = 0, \quad (1-q)u_d^* q = u_d^* e.$$

Because J is the limit of non-commutative polynomials in the generators f and $u_1, \ldots, u_d$, together with their adjoints, it follows that $qJ(1-q)Jq$ lies in the closed two-sided ideal generated by e. This is the kernel of ρ_+ by Theorem 4.3.13. A similar statement holds for the lower right corner of the matrix representing F_t^2. Hence, (F_t) is an operator homotopy, so since $F_1^2 = -1$, it follows that the class represented by F_0 is trivial. □

5.5 Discussion

We have derived a framework for the bulk and boundary classification of disordered free-fermion topological phases and proved that these are naturally related. In this final section, we complement our previous results by drawing some physical conclusions.

Invariance under disorder

So far, the precise nature of the space Ω of disorder configurations has not been important. Bulk-boundary correspondence holds for any choice of Ω. However, the K-theory of the bulk and boundary algebras will in general depend on Ω.

Here we assume that the space of disorder configurations is given by a Bernoulli shift

$$\Omega = \Omega_0^\Lambda = \prod_{x\in\Lambda} \Omega_0 \tag{5.15}$$

for some compact convex Hausdorff space Ω_0 with the canonical action of $\mathbb{Z}^d$ by translation in the parameter set Λ. As explained in Example 4.2.8, a physical realisation of such a disorder space is given by disorder by doping. Then we have the following important result.

Proposition 5.5.1. *Assume that Ω is a Bernoulli shift with contractible Ω_0. Let $\mathbb{A}^0$, $\widehat{\mathbb{A}}^0$, and $\mathbb{A}_\partial^0$ denote the algebras $\mathbb{A}$, $\widehat{\mathbb{A}}$, and $\mathbb{A}_\partial$, respectively, for the case of the singleton disorder space. There is a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{A}_\partial \otimes \mathbb{K}(\ell^2(\mathbb{N})) \arrow[r, "\iota"] \arrow[d] & \widehat{\mathbb{A}} \arrow[r, "\rho"] \arrow[d] & \mathbb{A} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{A}_\partial^0 \otimes \mathbb{K}(\ell^2(\mathbb{N})) \arrow[r, "\iota^0"] & \widehat{\mathbb{A}}^0 \arrow[r, "\rho^0"] & \mathbb{A}^0 \arrow[r] & 0
\end{tikzcd} \tag{5.16}$$

where the rows are the short exact sequence (4.19) *for $\mathbb{A}$ and $\mathbb{A}^0$, respectively, and the vertical maps are homotopy equivalences of real C^*-algebras.*

In particular, the bulk-boundary map ∂ is surjective and

$$KR^{-(s-r+2)}(\mathbb{A}) = KR^{-(s-r+2)}(\mathbb{A}^0) = \bigoplus_{i=0}^{d} \binom{d}{i} KR^{-(s-r+2-i)}(\mathbb{C}), \tag{5.17}$$

$$K^{-(s-r+2)}(\mathrm{A}) = K^{-(s-r+2)}(\mathrm{A}^0) = \mathbb{Z}^{2^{d-1}}, \tag{5.18}$$

Remark 5.5.2. *Equation* (5.17) *identifies $KR^{-(s-r+2)}(\mathbb{A})$ with the real K-theory of the d-dimensional (Brillouin) torus. The KR-groups of $\mathbb{C}$ have been discussed in Subsection 3.1.3. The component of the bulk class of a disordered IQPV $(J;\phi)$ of symmetry index (r,s) corresponding to the summand for $i = d$ is what is often referred to in the literature as the* strong topological invariant. *The other components are summarily called* weak topological invariants. *As Theorem 5.3.1 and Proposition 5.5.1 show, at least as far as the type of disorder considered here goes, there is no indication that this distinction is mathematically justified.*

Proof of Proposition 5.5.1. We consider the real case first. By assumption, there is a point $\omega_0 \in \Omega$ and a homotopy $h = (h_t)$ of maps $\Omega_0 \longrightarrow \Omega_0$ where $h_0 = \omega_0$ is the constant map and $h_1 = \mathrm{id}_{\Omega_0}$.

Consider the inclusion $\eta : \mathrm{End}(W) \longrightarrow C(\Omega) \otimes \mathrm{End}(W)$ and the evaluation

$$\mathrm{ev}_{\omega_0} : C(\Omega) \otimes \mathrm{End}(W) \longrightarrow \mathrm{End}(W) : f \longmapsto f(x \longmapsto \omega_0)$$

at the constant sequence $x \longmapsto \omega_0$. We have $\mathrm{ev}_{\omega_0} \circ \eta = 1$. Moreover, for any $t \in [0, 1]$,

$$\phi_t(f)(\omega) := f(h_t(\omega)) \quad \forall f \in C(\Omega) \otimes \mathrm{End}(W), \omega \in \Omega$$

defines a homotopy (ϕ_t) of real $*$-endomorphisms of $C(\Omega) \otimes \mathrm{End}(W)$ such that

$$\phi_0 = \eta \circ \mathrm{ev}_{\omega_0}, \quad \phi_1 = \mathrm{id}_{C(\Omega)\otimes\mathrm{End}(W)}.$$

By definition, the morphisms ev_{ω_0}, η and ϕ_t are $\mathbb{Z}^d$-equivariant, and by restriction, also equivariant with respect to $\mathbb{Z}^{d-1}$ and $\mathbb{Z}^{d-1} \times \mathbb{N}$. By Theorem 4.2.22, Proposition 4.3.9, and Theorem 4.3.13, the existence of the Diagram (5.16) follows.

The homotopy invariance of K-theory gives $KR^{-\bullet}(\mathbb{A}) = KR^{-\bullet}(\mathbb{A}^0)$. In view of [77, Thm. 1.5.5], ∂ is surjective. Since by Theorem 4.2.22,

$$\mathbb{A}^0 = (\mathbb{C} \rtimes \mathbb{Z}^d) \otimes \mathrm{End}(W) = C^*(\mathbb{Z}^d) \otimes \mathrm{End}(W),$$

where $C^*(\mathbb{Z}^d)$ is the group C*-algebra of $\mathbb{Z}^d$, we find that

$$KR^{-\bullet}(\mathbb{A}) = KR^{-\bullet}(C^*(\mathbb{Z}^d)) = KR^{-\bullet}\Big(\bigotimes\nolimits^d C(\mathbb{S}^1)\Big).$$

From this, Equation (5.17) follows by [77, Thm. 1.5.4].

The complex case can be proved analogously and we end up with $K^{-\bullet}(\mathrm{A}) = K^{-\bullet}(\mathrm{A}^0)$, where $\mathrm{A}^0 = \mathrm{End}(V) \rtimes_\sigma \mathbb{Z}^d$. By the untwisting trick by Packer and Raeburn [64, Thm. 3.4], A^0 is stably isomorphic to $C^*(\mathbb{Z}^d) \otimes \mathrm{End}(V)$. It follows $K^{-\bullet}(\mathrm{A}^0) = K^{-\bullet}(C^*(\mathbb{Z}^d))$. □

The role of the trivial phase

Our construction of bulk classes depends on the choice of a trivial phase. Indeed, we have constructed bulk classes as deviations from an arbitrarily chosen reference disordered IQPV. This circumstance is imposed by the group structure of K-theory, which measures differences of certain homotopy classes, rather than the classes themselves.

From a physical point of view, free-fermion topological phases do not directly have a group structure, whereas differences of such phases do. Therefore, it is advantageous for the physical interpretation of the bulk classes as free-fermion topological phases that the choice of reference disordered IQPV is not fixed by the mathematical model, and instead allows for adjustments according to the physical system of interest.

The situation is different for the boundary classes. Any reference disordered IQPV J_{ref} that has a *unitary* lift $\hat{J}_{\mathrm{ref}}$ (in particular, any local reference disordered IQPV in the sense of Corollaries 5.4.1 and 5.4.2) will itself already represent a trivial Kasparov class, trivial in the associated K-group. This corresponds to the understanding in physics that a boundary free-fermion topological phase is trivial if there are no stable localised boundary states that close the gap.

To state bulk-boundary correspondence cleanly, our definition of boundary classes avoids the requirement that the reference disordered IQPV be trivial in this sense. It instead measures the deviation from an arbitrary reference boundary class. The bulk-boundary map is sensitive to the choice of reference phase and preserves this dependence.

The role of the local space

Throughout this work, we have relied on a tight-binding approximation in which the relevant degrees of freedom close to the chemical potential split into a spatial contribution, corresponding to the translational lattice of atomic sites $\mathbb{Z}^d$, and a finite-dimensional local space V resp. $W = V \oplus V^*$ independent of the spatial part. It is important that the choice of this splitting is performed in such a way that the only relevant spatial symmetries (in mean) of the Hamiltonians of interest are the lattice translations.

This is shown by the following *Gedankenexperiment* due to Fu–Kane–Mele [28]. Assume that W is doubled to $W_2 := W \oplus W$ while $\mathbb{Z}^d$ is reduced to $(2\mathbb{Z})^d$ by merging sites of the Bravais lattice adjacent in one selected spatial direction. Let $\mathbb{A}_2$ be the bulk algebra defined in terms of W_2 and $(2\mathbb{Z})^d$, and suppose we are in the setting of Proposition 5.5.1. Although the K-groups of $\mathbb{A}$ and $\mathbb{A}_2$ are isomorphic by the proposition, the natural "doubling" map $\phi_2 : \mathbb{A} \longrightarrow \mathbb{A}_2$ is not an isomorphism, not even on K-theory. Indeed, Fu–Kane–Mele show for clean systems that, although the strong invariant is preserved, some weak invariants are annihilated by doubling; by the proposition, this also holds in the presence of disorder. Fu–Kane–Mele view the annihilation of weak invariants as an indication that they are not stable under disorder. However, as we have seen, all topological phases are preserved by the kind of disorder we consider. This is corroborated by the work of Ringel–Kraus–Stern [70].

The problem lies in the fact that ϕ_2 maps a disordered IQPV J in $\mathbb{A}$ to one in the algebra $\mathbb{A}_2$ which forgets the invariance (in mean) of J under the translations removed by the doubling process. By incorporating these lost translations into the definition of the algebra as symmetries of $\mathcal{W}_2 = \ell^2((2\mathbb{Z})^d) \otimes W_2 \cong \mathcal{W}$, we obtain a new algebra, isomorphic to $\mathbb{A}$ under the doubling map, thus removing the perceived inconsistencies.

Chapter Six
Topological Invariants

In order to assign physical properties to topological phases of topological insulators and superconductors it is essential to define *topological invariants*. As topological invariants we refer to numbers that quantify topological phases of topological insulators and superconductors.

This chapter is split into two sections. In the first section, we will apply the tools of cyclic cohomology in the context of Van Daele's picture of KR- and K-theory to define topological invariants in the bulk. In principle, this section is an application of the results of Kellendonk [46].

In the second section, we will compute the so-called strong topological invariants at the boundary on the level of Kasparov's Fredholm picture of KR- and K-theory and their corresponding Kasparov groups. This section combines the index maps established in [16, 17] in terms of Kasparov classes with our new explicit form of the topological phases at the boundary.

6.1 Topological invariants without torsion in the bulk

Here, we apply Kellendonk's generalisation of cyclic cohomology to Van Daele groups [46] on the bulk topological phases of disordered IQPV.

The idea of using cyclic cohomology in order to derive numerical values for K-classes goes back to Connes [23, Ch. 3.3]. He constructed the cyclic cohomology groups $HC^n(A)$ and showed that these are dual to complex K-theory in the sense that there is a non-degenerate linear pairing

$$K^{-n}(A) \otimes HC^n(A) \longrightarrow \mathbb{C} \quad \text{for } n \in \mathbb{N}.$$

This pairing is determined by a so-called character, which will also be explained below. In the context of topological insulators, this pairing has first been used by Bellissard in the context of the QHE. Bellissard showed that the quantised Hall conductance (complex symmetry class A) is equal to such a pairing, up to a prefactor of $\frac{e^2}{h}$, where e is the electron charge and h denotes Planck's constant, see [13] for an overview. By the work of Prodan and Schulz-Baldes [68, 66], this pairing is also well understood for the complex symmetry class $A\mathrm{III}$. We have seen that these symmetry classes A and $A\mathrm{III}$ are described by complex K-groups and therefore the pairing can be applied without ambiguity.

The pairing with real K-groups is more involved because these groups might have torsion. The linearity of the pairing implies that it vanishes for torsion elements. Therefore, for the real symmetry classes one has to perform two calculations in order to get a complete set of topological invariants. Connes pairing allows for the calculation of numerical values for the invariants without torsion in the real case, too.

The invariants with torsion can be computed *via* Kellendonk's results [46]. He showed how to modify this pairing for $\mathbb{Z}/2\mathbb{Z}$-valued invariants. However, to the present date, we are missing a satisfactory explicit result for these invariants of the bulk topological phases in our context. Therefore, we refrain from considering them in this work.

In the following, we will first present the extension of Connes' pairing to real K-classes in Van Daele's picture, as it has been derived in [46], and then apply it for our bulk classes. Since the complex topological invariants have already been derived in very detail in [66], we will not consider them here.

6.1.1 Cyclic cohomology for Van Daele's picture of K-theory

Here, we summarise the results of [46] that enable us compute the pairing for Van Daele's picture of K- and KR-theory.

We begin with the definition of cycles and characters as defined in [23, Ch. 3].

Definition 6.1.1 (Cycle, character)**.** *An n-dimensional cycle over a trivially graded algebra $\mathcal{A}$ is a triple $\left(\Omega, d, \int\right)$, with*

- *a $\mathbb{Z}$-graded algebra $\Omega = \bigoplus_{k\in\mathbb{Z}} \Omega_k$ together with a morphism $\varphi : \mathcal{A} \to \Omega_0$,*
- *a derivation of degree 1, i.e. $d : \Omega_k \to \Omega_{k+1}$ for all $k \in \mathbb{Z}$ with $d^2 = 0$,*
- *a closed graded trace $\int_{\mathcal{A}} : \Omega_n \to \mathbb{C}$.*

The character $\xi : \Omega_{n+1} \to \mathbb{C}$ of the cycle $\left(\Omega, d, \int\right)$ is defined as

$$\xi(a_0, \dots, a_n) := \int_{\mathcal{A}} \varphi(a_0) d\varphi(a_1) \cdots d\varphi(a_n).$$

Proposition 6.1.2. [23, Ch. 3.1, Prop. 4] *The character of an n-dimensional cycle is a closed graded trace of dimension n.*

Example 6.1.3. *The most simple example is the algebra $M_m(\mathbb{C})$. For $\Omega_0 = M_m(\mathbb{C})$ and $\Omega_k = 0$ for all $0 \neq k \in \mathbb{Z}$, the triple $(\Omega, 0, \mathrm{Tr}_m)$ defines a cycle over $M_m(\mathbb{C})$.*

Example 6.1.4. [46, 7.1] *By Theorem 4.2.22, all elements in $\mathbb{A}$ are (the limit of) non-commutative polynomials*

$$\sum_{x\in\mathbb{Z}^d} f_x R_x \in \mathbb{A}, \tag{6.1}$$

where $f_x \in C(\Omega) \otimes \mathrm{End}(W)$ for all $x \in \mathbb{Z}^d$. We define the derivations

$$\partial_j \sum_{x\in\mathbb{Z}^d} f_x R_x := i \sum_{x\in\mathbb{Z}^d} x_j f_x R_x \tag{6.2}$$

for all $j = 1, \dots, d$ and $x = (x_1, \dots, x_d)$. (The prefactor i is introduced in order to let ∂_j be a $*$-derivation.) This is of course not a well-defined map over $\mathbb{A}$ since not all polynomials of the form (6.2) *have to converge. Thus let $C^m(\mathbb{A})$ be the linear subspace of $\mathbb{A}$ that is spanned by m-times differentiable elements in $\mathbb{A}$ and define*

$$\mathcal{A} := C^\infty(\mathbb{A}) := \bigcap_{m \in \mathbb{N}} C^m(\mathbb{A}).$$

For given $n = (n_1, \dots, n_d) \in \mathbb{N}^d$, we define the semi-norm

$$\|a\|_n := \|\partial^n a\| \quad \textit{where} \quad \partial^n = \partial_1^{n_1} \cdots \partial_d^{n_d}. \tag{6.3}$$

Then by [66, Prop. 3.3.3], $\mathcal{A}$ is a $*$-algebra which, equipped with the topology induced by the semi-norms (6.3)*, is a dense Fréchet sub-algebra of $\mathbb{A}$ which is stable under holomorphic calculus.*

A trace on $\mathbb{A}$ is defined by

$$\mathcal{T}\left(\sum_{x \in \mathbb{Z}^d} f_x R_{x_1} \cdots R_{x_d}\right) := \int_\Omega f_0 \, d\mathbb{P}. \tag{6.4}$$

This trace obviously satisfies $\mathcal{T} \circ \partial_j = 0$ for all $j = 1, \dots, d$.

Fix an orthonormal basis $\lambda_1, \dots, \lambda_n$ of $\mathbb{C}^n$ and a subset $I = \{i_1, \dots, i_n\} \subseteq \{1, \dots, d\}$ with $|I| = n$. Define

$$\Omega := \mathcal{A} \otimes \Lambda(\mathbb{C}^m),$$

equipped with the derivation

$$d := \sum_{j=1}^m \partial_{i_j} \otimes \lambda_j,$$

and the trace $\int$ on Ω defined by

$$\int (a \otimes (\lambda_1 \wedge \ldots \wedge \lambda_n)) = \mathcal{T}(a), \qquad \int ((a \otimes (\lambda_1 \wedge \ldots \wedge \lambda_k)) = 0 \quad \forall k < n,$$

for all $a \otimes \lambda \in \mathcal{A} \otimes \Lambda(\mathbb{C}^m)$. Then $(\Omega, d, \int)$ defines an n-dimensional cycle over $\mathcal{A}$.

Definition 6.1.5 (Graded trace over $C\ell_{r,s}$)**.** *Let $\kappa : C\ell_{r,s} \to \mathbb{C}$ be the graded trace defined by*

$$\kappa(k_1 \cdots k_r j_1 \cdots j_s) := 2^{\frac{r+s}{2}} i^{\frac{r-s}{2}} \quad \textit{and} \quad \kappa(k_{a_1} \cdots k_{a_{r'}} j_{\alpha_1} \cdots j_{\alpha_{s'}}) = 0 \textit{ if } r' < r \textit{ or } s' < s.$$

Here the square root of i is defined as $i^{\frac{1}{2}} := e^{i\frac{\pi}{4}}$.

Now we introduce the extension of cycles as in [46, Def. 4.4].

Definition 6.1.6 (Extension of cycles)**.** *Let $\left(\Omega, d, \int_{\mathcal{A}}\right)$ be a cycle over an algebra $\mathcal{A}$. The extension of this cycle to $M_m(\mathcal{A}) \otimes C\ell_{r,s}$ is defined by the cycle*

$$\left(M_m(\Omega) \otimes C\ell_{r,s}, d \otimes \mathrm{id}, \int\right),$$

where $\int := \int_{\mathcal{A}} \circ \mathrm{Tr}_m \circ \kappa$ with $\kappa(a \otimes c) := \kappa(c) a$ for all $a \in M_m(\mathcal{A}), c \in C\ell_{r,s}$.

By this extension of cycles, we can canonically define the extended character over $A \otimes C\ell_{r,s}$. The following definition is a specialised version of [46, Def. 4.7].

Definition 6.1.7 (Extended character)**.** *Let $\left(\Omega, d, \int_{\mathcal{A}}\right)$ be an n-dimensional cycle of a trivially graded C^*-algebra A, with domain algebra $\mathcal{A}$ and character ξ.*

Let $e \in \mathcal{A} \otimes C\ell_{r,s}$ be a reference ORHU satisfying $de = 0$. Then the pairing with an element $[x] \in DK(A \otimes C\ell_{r,s})$, where $x \in \mathcal{F}_m(A \otimes C\ell_{r,s})$ is defined as

$$\xi\#\mathrm{Tr}_m\#\kappa(x - e_m, \ldots x - e_m) = \int_{\mathcal{A}} \mathrm{Tr}_m \kappa((x - e_m)(dx)^n).$$

By [46, Cor. 4.2 & Lem. 4.5], this pairing is constant on path connected components of $\mathcal{F}_m(\mathcal{A} \otimes C\ell_{r,s})$ and thus well-defined.

6.1.2 Bulk topological invariants without torsion

Here, we will finally apply the formulas that we have explained in Subsection 6.1.1 for the calculation of topological invariants for the bulk topological phase of a disordered IQPV $(J; \phi)$ of symmetry index (r, s).

Throughout this subsection we assume that the reference disordered IQPV (J_{ref}, ϕ) is local, i.e. $J_{\mathrm{ref}} \in C(\Omega) \otimes \mathrm{End}(W)$, in particular $dJ_{\mathrm{ref}} = 0$. As we have seen in Section 5.1, the topological phase

$$[J] - [J_{\mathrm{ref}}] \in \mathrm{Top}^{\phi}_{J_{\mathrm{ref}}}(\mathbb{A})$$

of $(J; \phi)$ relative to $(J_{\mathrm{ref}}; \phi)$ corresponds to the Van Daele class

$$[x] := \left[\begin{pmatrix} (J \otimes j_1)P^{\phi} + (J_{\mathrm{ref}} \otimes j_1)(1 - P^{\phi}) & 0 \\ 0 & -J_{\mathrm{ref}} \otimes j_1 \end{pmatrix}\right] \in DKR(\mathbb{A} \otimes C\ell_{r,s+1}). \tag{6.5}$$

We can associate topological invariants without torsion to this topological phase as explained in the following theorem.

Theorem 6.1.8 (Topological bulk invariants without torsion)**.** *Assume that $J \in \mathcal{A} \subseteq \mathbb{A}$ and J_{ref} is local.*

Let $\left(\Omega, d, \int_{\mathcal{A}}\right)$ be the n-dimensional cycle of Example 6.1.4 with character ξ for an index set $I = \{i_1, \ldots, i_n\} \subseteq \{1, \ldots, d\}$. If $r + s + n$ is even, the pairing with the class $[x] \in DKR(\mathbb{A} \otimes C\ell_{r,s+1})$ of Equation (6.5) is given by

$$\xi\#\mathrm{Tr}_W\#\kappa([x]) = \Lambda_{r,s,n} \sum_{\sigma \in S_I} \mathrm{sign}(\sigma)(\mathrm{tr}_W \circ \mathcal{T})\left(\Big(\prod_{a=1}^{r} K_a\Big)\Big(\prod_{\alpha=1}^{s} J_\alpha\Big)(J - J_{\mathrm{ref}}) \prod_{j=1}^{n} \partial_{i_{\sigma(j)}} J\right),$$

where $\Lambda_{r,s,n} := 2^{\frac{r+s+1}{2}} i^{\frac{r-s-1}{2}} (-1)^{\frac{1}{2}(r^2+s^2+n)}$ and S_I denotes the permutations of the index set I.

The pairing vanishes if $r + s + n$ is odd.

Proof. Let $e := J_{\mathrm{ref}} \otimes j_1 \oplus (-J_{\mathrm{ref}} \otimes j_1)$. The pairing is by definition given by

$$\xi\#\mathrm{Tr}_W\#\kappa([x]) = \int \kappa((x - e)(dx)^n).$$

Since $de = 0$, it follows $dx = (dJ \otimes j_1)P^\phi \oplus 0$, and thus the pairing is equal to

$$\int \kappa\Big(((J - J_{\mathrm{ref}})(dJ)^n \otimes j_1^{n+1})P^\phi\Big).$$

Note that we used that $J \otimes j_1$ commutes with P^ϕ, and therefore also $dJ \otimes j_1$ commutes with P^ϕ, because P^ϕ acts non-trivially only on $\mathrm{End}(W)$. The graded trace on $C\ell_{r,s+1}$ vanishes for terms which are not proportional to the product of all generators $k_1 \cdots k_r j_1 \cdots j_{s+1}$. Thus the only relevant term of the projection is given by

$$\Big(\prod_{a=1}^{r}(-1)^s K_a \otimes k_a j_1\Big)\Big(\prod_{\alpha=1}^{s} J_\alpha \otimes j_1 j_{\alpha+1}\Big).$$

It follows

$$\xi\#\mathrm{Tr}_W\#\kappa([x]) = \int \kappa\Big(\Big(\prod_{a=1}^{r}(-1)^s K_a \otimes k_a j_1\Big)\Big(\prod_{\alpha=1}^{s} J_\alpha \otimes j_1 j_{\alpha+1}\Big)(J - J_{\mathrm{ref}})(dJ)^n \otimes j_1^{n+1}\Big).$$

The graded trace κ of $C\ell_{r,s+1}$ vanishes, if $r + s + n$ is odd, because then the j_1 occurs even times. Otherwise, if $r + s + n$ is even, the pairing is given by

$$\begin{aligned}
&\int \kappa\Big(\Big(\prod_{a=1}^{r}(-1)^s K_a \otimes k_a j_1\Big)\Big(\prod_{\alpha=1}^{s} J_\alpha \otimes j_1 j_{\alpha+1}\Big)(J - J_{\mathrm{ref}})(dJ)^n \otimes j_1^{n+1}\Big)\\
=&(-1)^{\frac{1}{2}(r^2+s^2+n)} \int \kappa\Big(\Big(\prod_{a=1}^{r} K_a\Big)\Big(\prod_{\alpha=1}^{s} J_\alpha\Big)(J - J_{\mathrm{ref}})(dJ)^n \otimes k_1 \cdots k_r j_1 \cdots j_{s+1}\Big)\\
=&2^{\frac{r+s+1}{2}} i^{\frac{r-s-1}{2}} (-1)^{\frac{1}{2}(r^2+s^2+n)} \int \Big(\prod_{a=1}^{r} K_a\Big)\Big(\prod_{\alpha=1}^{s} J_\alpha\Big)(J - J_{\mathrm{ref}})(dJ)^n,
\end{aligned}$$

where we used that $\kappa(k_1 \cdots k_r j_1 \cdots j_{s+1}) = 2^{\frac{r+s+1}{2}} i^{\frac{r-s-1}{2}}$. The prefactor is equal to $\Lambda_{r,s,n}$. Finally, the operator $(dJ)^n \in \mathcal{A} \otimes \Lambda(\mathbb{C}^n)$ is given by

$$(dJ)^n = \sum_{\sigma \in \mathcal{S}_I} \mathrm{sign}(\sigma) \prod_{j=1}^{n} \partial_{\sigma(j)} J \otimes (\lambda_1 \wedge \ldots \wedge \lambda_n).$$

Thus, we obtain

$$\xi\#\mathrm{Tr}_W\#\kappa([x]) = \Lambda_{r,s,n} \sum_{\sigma \in \mathcal{S}_I} \mathrm{sign}(\sigma)(\mathrm{tr}_W \circ \mathcal{T})\left(\Big(\prod_{a=1}^{r} K_a\Big)\Big(\prod_{\alpha=1}^{s} J_\alpha\Big)(J - J_{\mathrm{ref}}) \prod_{j=1}^{n} \partial_{i_{\sigma(j)}} J\right).$$

□

6.2 Strong topological invariants at the boundary

In this section, we derive explicit formulas for the strong topological invariants of the topological phases at the boundary. This section is based on the results of [16, 17]. Our contribution to this topic is that we compute the explicit form of the strong topological invariants for our novel boundary topological phases.

We first discuss the one-dimensional case which is simpler than the higher-dimensional case. The resulting formula for the topological invariant can be computed more explicitly than in the higher-dimensional case. Moreover, the one-dimensional case is important for physical considerations (e.g. superconducting chains in class D). The higher-dimensional strong topological invariants will be computed by reduction to systems that can be treated with the same techniques as the one-dimensional case.

Throughout this section, we assume for the sake of simplicity that the reference disordered IQPV is local. The assertions below can easily be generalised to general reference disordered IQPV by performing the calculations below in the same way for the topological phase at the boundary induced by the reference disordered IQPV J_{ref}. The resulting topological invariant at the boundary is then given by the difference of the topological invariants of J and J_{ref}.

6.2.1 One-dimensional systems

For a one-dimensional Bravais lattice $\Lambda = \mathbb{Z}$, we have $\mathbb{A}_\partial = C(\Omega) \otimes \mathrm{End}(W)$ and $\mathrm{A}_\partial = C(\Omega) \otimes \mathrm{End}(V)$. Since the reference disordered IQPV J_{ref} is assumed to be local, we know by the bulk-boundary correspondence that the boundary class for a disordered IQPV $(J; \phi)$ is given by

$$[(\mathcal{H}_{C(\Omega)\otimes\mathrm{End}(W)} \otimes C\ell_{0,1}, \varphi, \hat{J} \otimes j_1)] \in KKR(C\ell_{s,r}, C(\Omega) \otimes \mathrm{End}(W) \otimes C\ell_{0,1}),$$

where $\varphi : C\ell_{s,r} \to \mathbb{B}(\mathcal{H}_{C(\Omega)\otimes\mathrm{End}(W)} \otimes C\ell_{0,1})$ is defined as in Definition 5.2.11.

We now fix a disorder configuration $\omega \in \Omega$ and consider the evaluation map $\mathrm{ev}_\omega : C(\Omega) \to \mathbb{C}; f \mapsto f(\omega)$. The induced morphism of Kasparov groups $(\mathrm{ev}_\omega)_*$, *cf.* Definition 3.1.20, maps the class above onto

$$[(\mathcal{H}_{\mathrm{End}(W)} \otimes C\ell_{0,1}, \varphi, \hat{J}_\omega \otimes j_1)] \in KKR(C\ell_{s,r}, \mathrm{End}(W) \otimes C\ell_{0,1}),$$

where $\hat{J}_\omega := (\mathrm{ev}_\omega)_*(\hat{J})$. If Ω is as in Proposition 5.5.1, then this class is independent of the choice of ω. However, for more general Ω, the class depends on the choice of ω and the resulting strong topological invariants could vary for different disorder configurations. The local Nambu space W is a finite-dimensional real vector space, and therefore, we can apply the stability isomorphism of Kasparov groups in order to reduce the group to $KKR(C\ell_{s,r}, C\ell_{0,1})$. This isomorphism is given by the intersection product from the right with the element

$$\alpha := [((\mathcal{K} \oplus \mathcal{K}^{op}) \otimes W, 1 \otimes \mathrm{id}, T \otimes 1)] \in KKR(\mathrm{End}(W), \mathbb{C}),$$

where $\mathcal{K}$ and T are given as in Theorem 3.1.28. The result is the class

$$[(\ell^2(\mathbb{N}, W) \otimes C\ell_{0,1}, \varphi, \hat{J}_\omega \otimes j_1)] \in KKR(C\ell_{s,r}, C\ell_{0,1}),$$

where we used that $\mathcal{H}_{\mathbb{C}} = \ell^2(\mathbb{N})$. By Theorem 3.1.36, this class is equivalent to the tuple

$$[(\hat{J}_\omega, \phi)] \in KR^{r,s}(\mathbb{C}) = KR^{-(s-r+1)}(\mathbb{C}).$$

By Theorem 3.1.40, this class is equivalent to the class in the Clifford module picture that is generated by the $C\ell_{r,s}$-module

$$(\ker \hat{J}_\omega, \phi),$$

where $\phi : C\ell_{r,s} \to \mathcal{L}(\ker \hat{J}_\omega)$ denotes the usual real $*$-morphism induced by pseudo-symmetries.

Depending on the value of $s-r$, one can now proceed to determine the class in the group $KR^{-(s+1-r)}(\mathbb{C})$. In the following, we will consider the fundamental irreducible examples for each case with a non-trivial KR-group using the results of Section 3.1.3.

1. Class BDI, $s-r = -1 \bmod 8$. The fundamental example is a disordered IQPV of symmetry index $(1,0)$, i.e. there is one positive pseudo-symmetry K_1. The equivalence class of the Clifford module is then determined by the number

$$\dim_{\mathbb{C}}(\ker \hat{J}_\omega)_+ - \dim_{\mathbb{C}}(\ker \hat{J}_\omega)_- \in \mathbb{Z},$$

   where $(\ker \hat{J}_\omega)_\pm := P_\pm(\ker \hat{J}_\omega)$ with $P_\pm := \frac{1}{2}(1 \pm K_1)$. In particular, the trivial phase is determined by $\dim(\ker \hat{J}_\omega)_+ = \dim(\ker \hat{J}_\omega)_-$.

2. Class D, $s-r=0$. Here we consider the case where we have no pseudo-symmetries. The kernel $\ker(\hat{J}_\omega)$ is now just a real vector space, and the topological invariant is therefore determined by

$$\dim_{\mathbb{C}}(\ker \hat{J}_\omega) \bmod 2 \in \mathbb{Z}/2\mathbb{Z}.$$

   This symmetry class corresponds to superconducting chains without any symmetries. The $\mathbb{Z}/2\mathbb{Z}$-invariant determines whether there is a localised Majorana mode at the end of the chain, and the $\mathbb{Z}/2\mathbb{Z}$-value reflects the topological instability of pairs of Majorana modes.

3. Class DIII, $s-r = 1 \bmod 8$. The fundamental example is a disordered IQPV with one negative pseudo-symmetry J_1 and no positive pseudo-symmetries. Then J_1 defines a complex structure on $U := \mathrm{Fix}_\gamma(\ker(\hat{J}_\omega))$. The topological invariant is here determined by

$$\dim_{\mathbb{C}}(U) \bmod 2 = \tfrac{1}{2}\dim_{\mathbb{C}}(\ker \hat{J}_\omega) \bmod 2 \in \mathbb{Z}/2\mathbb{Z}.$$

4. Class CII, $s-r = 3 \bmod 8$. We consider the fundamental example of a disordered IQPV $J \in \mathbb{A}$ with three negative pseudo-symmetries J_1, J_2, J_3. We have seen in Lemma 4.1.27 and the discussion of class CII in Subsection 4.1.2 that the algebra of observables can be split as $\mathbb{A} \cong \mathbb{A}^\sim \otimes \mathbb{H}_{\mathbb{C}}$, such that $J = x \otimes i\sigma_3$ and $J_1 = 1 \otimes i\sigma_1$, $J_2 = 1 \otimes i\sigma_2$ and $J_3 = x_3 \otimes i\sigma_3$. Thus $\ker(\hat{J}_\omega) = \ker(\hat{x}_\omega) \otimes \mathbb{C}^2$. The x_3 plays algebraically the same role as the positive pseudo-symmetry in the class BDI. Thus, the topological invariant is determined by

$$\dim_{\mathbb{C}}(\ker \hat{x}_\omega)_+ - \dim_{\mathbb{C}}(\ker \hat{x}_\omega)_- \in \mathbb{Z},$$

   where $(\ker \hat{x}_\omega)_\pm := P_\pm(\ker \hat{x}_\omega)$ with $P_\pm := \frac{1}{2}(1 \pm x_3)$.

5. Complex class AIII, $s-r = 1 \bmod 2$. We consider the case of a complex disordered IQPV $J \in \mathrm{A}$ with one complex pseudo-symmetry K_1. We can argue in the same way as for the class BDI that the topological invariant is given by

$$\dim_{\mathbb{C}}(\ker \hat{J}_\omega)_+ - \dim_{\mathbb{C}}(\ker \hat{J}_\omega)_- \in \mathbb{Z},$$

   where $(\ker \hat{J}_\omega)_\pm := P_\pm(\ker \hat{J}_\omega)$ with $P_\pm := \frac{1}{2}(1 \pm K_1)$.

6.2.2 Higher-dimensional systems

In this section, we will derive a formula for the strong topological invariants for all symmetry classes and dimensions $d \geqslant 2$. The resulting formula is based on our explicit form of the boundary classes and the spectral triples which have been derived in [17].

Spectral triples are formulated in [17] in terms of *unbounded* Kasparov classes. Unbounded Kasparov theory is a generalisation of the usual Kasparov theory in the sense that there is a canonical surjection from unbounded onto bounded Kasparov groups, where bounded Kasparov theory coincides by definition with Kasparov theory as introduced in Chapter 3. Although we will in the end not work with the unbounded theory, we have to give a short review over the fundamental definitions of unbounded Kasparov modules in order to explain the results of [17].

Definition 6.2.1 (Unbounded Kasparov module)**.** *Let A, B be graded real C^*-algebras and let E be a Hilbert B-module. A regular operator over E is a densely defined closed operator $D : \mathrm{Dom}(D) \to E$, such that D^* is densely defined in E and $1 + D^*D$ is densely defined. The operator is self-adjoint if* $\mathrm{Dom}(D) = \mathrm{Dom}(D^*)$.

If there is an even real $*$-morphism $\pi : A \to \mathbb{B}(E)$, then (E, D) defines a real unbounded A-B-Kasparov module if $D : \mathrm{Dom}(D) \to E$ is an unbounded regular real odd Hermitian operator such that for all $a \in \mathcal{A}$

$$\pi(a)(1 + D^2)^{-1/2} \in \mathbb{K}(E)$$

and the set of elements $a \in A$ such that

$$[D, \pi(a)] \in \mathbb{B}(E)$$

is dense in A, where $[\cdot, \cdot]$ denotes the graded commutator. Unbounded Kasparov modules for complex C^*-algebra A, B can be defined in the same manner by dropping the reality constraints.

More details to this topic can be found in [61, Sec. 5]. Crucial for us is the following result about the aforementioned surjection from unbounded onto bounded Kasparov groups [8].

Theorem 6.2.2. *Let (E, D) be a (real) unbounded A-B-Kasparov module and define $F := D(1 + D^2)^{-1/2}$. Then*

$$(E, \pi, F) \in \mathcal{E}(A, B)$$

is a (real) Kasparov module. This defines a surjection from (real) unbounded Kasparov modules onto (real) Kasparov modules, which is natural w.r.t. the intersection product in the sense that it maps the intersection product on the unbounded level, cf. *[61, Sec. 6], onto the one on the bounded level.*

These basic definitions are sufficient in order to obtain the desired Kasparov modules *via* the following unbounded Kasparov modules.

Definition 6.2.3 (Representations $\pi^\omega, \pi^\omega_\partial, \hat{\pi}^\omega$)**.** *Let*

$$\pi : \mathbb{A} \to \mathcal{L}(L^2(\Omega, \mathbb{P}) \otimes \mathcal{W})$$

denote the natural real $*$-representation of $\mathbb{A}$. Fix a disorder configuration $\omega \in \Omega$ and let $\mathrm{ev}_\omega : L^2(\Omega, \mathbb{P}) \to \mathbb{C}$ denote the restriction onto the disorder configuration ω. Then $\pi^\omega := \mathrm{ev}_\omega \circ \pi : \mathbb{A} \to \mathcal{L}(\mathcal{W})$ defines a representation of $\mathbb{A}$ over $\mathcal{W}$.

Analogously, we obtain real $*$-representations

$$\pi_\partial^\omega := \mathrm{ev}_\omega \circ \pi_\partial : \mathbb{A}_\partial \to \mathcal{L}(\mathcal{W}_\partial) \quad \textit{and} \quad \hat{\pi}^\omega := \mathrm{ev}_\omega \circ \hat{\pi} : \widehat{\mathbb{A}} \to \mathcal{L}(\widehat{\mathcal{W}})$$

via the natural $*$-representations $\pi_\partial : \mathbb{A}_\partial \to \mathcal{L}(L^2(\Omega, \mathbb{P}) \otimes \mathcal{W}_\partial)$ and $\hat{\pi} : \widehat{\mathbb{A}} \to \mathcal{L}(L^2(\Omega, \mathbb{P}) \otimes \widehat{\mathcal{W}})$.

Representations for the algebras of complex observables can be defined analogously.

Proposition 6.2.4. [17, Prop. 3.2] *Consider the graded real Hilbert space $\mathcal{W} \otimes \Lambda(\mathbb{C}^d)$. Identify $C\ell_{d,d} \cong \mathrm{End}(\Lambda(\mathbb{C}^d))$ and define the unbounded odd real operator*

$$D^{(d)} := \sum_{j=1}^{d} X_j \otimes k_j,$$

where X_j is the position operator in the jth direction of $\mathbb{Z}^d$. Then $(\mathcal{W} \otimes \Lambda(\mathbb{C}^d), D^{(d)})$, equipped with the $*$-morphism $\pi^\omega \otimes \mathrm{id}_{C\ell_{0,d}}$ defines a real unbounded $(\mathbb{A} \otimes C\ell_{0,d})$-$\mathbb{C}$-Kasparov module. The corresponding real Kasparov module is given by

$$\left(\mathcal{W} \otimes \Lambda(\mathbb{C}^d), \pi^\omega \otimes \mathrm{id}_{C\ell_{0,d}}, F^{(d)}\right) \in \mathcal{E}(\mathbb{A} \otimes C\ell_{0,d}, \mathbb{C}).$$

where

$$F^{(d)} := D^{(d)}(1 + (D^{(d)})^2)^{-1/2} = \sum_{j=1}^{d} \frac{X_j}{\sqrt{1 + |X|^2}} \otimes k_j$$

for $|X|^2 := \sum_{j=1}^d X_j^2$.

Analogously, we define the unbounded odd real operator $D^{(d-1)} := \sum_{j=1}^{d-1} X_j \otimes k_j \in \mathcal{W}_\partial \otimes \Lambda(\mathbb{C}^{d-1})$. Then $(\mathcal{W}_\partial \otimes \Lambda(\mathbb{C}^{d-1}), D^{(d-1)})$ equipped with the $*$-morphism $\pi_\partial^\omega \otimes \mathrm{id}_{C\ell_{0,d-1}}$ defines a real unbounded $(\mathbb{A}_\partial \otimes C\ell_{0,d-1})$-$\mathbb{C}$-Kasparov module. The corresponding real Kasparov module is given by

$$\left(\mathcal{W}_\partial \otimes \Lambda(\mathbb{C}^{d-1}), \pi_\partial^\omega \otimes \mathrm{id}_{C\ell_{0,d-1}}, F^{(d-1)}\right) \in \mathcal{E}(\mathbb{A}_\partial \otimes C\ell_{0,d-1}, \mathbb{C}).$$

where $F^{(d-1)} := D^{(d-1)}(1 + (D^{(d-1)})^2)^{-1/2}$.

We summarise the resulting Kasparov classes in the following definition.

Definition 6.2.5 ($\lambda_d^\omega, \lambda_{d-1}^\omega$)**.** *We denote the Kasparov classes associated to the real unbounded Kasparov modules from Proposition 6.2.4 by*

$$[\lambda_d^\omega] := \left[\left(\mathcal{W} \otimes \Lambda(\mathbb{C}^d), \pi^\omega \otimes \mathrm{id}_{C\ell_{0,d}}, F^{(d)}\right)\right] \in KKR(\mathbb{A} \otimes C\ell_{0,d}, \mathbb{C}),$$

and

$$[\lambda_{d-1}^\omega] := \left[\left(\mathcal{W}_\partial \otimes \Lambda(\mathbb{C}^{d-1}), \pi_\partial \otimes \mathrm{id}_{C\ell_{0,d-1}}, F^{(d-1)}\right)\right] \in KKR(\mathbb{A}_\partial \otimes C\ell_{0,d-1}, \mathbb{C}).$$

In the complex case, we can analogously define classes in $KK(\mathrm{A} \otimes C\ell_{0,d}, \mathbb{C})$ resp. $KK(\mathrm{A}_\partial \otimes C\ell_{0,d-1}, \mathbb{C})$.

Via the intersection product, these Kasparov classes can be interpreted as morphisms between (real) K-groups, i.e.

$$\lambda^{\omega}_{d-1} : KR^{-(s-r+1)}(\mathbb{A}_\partial) \xrightarrow{\otimes[\lambda^{\omega}_{d-1}]} KR^{-(s-r+2-d)}(\mathbb{C}).$$

In the following, we will calculate the image of this map for the classes induced by disordered IQPV with boundary. The result is the following theorem.

Theorem 6.2.6. *Let* $[(J;\phi)]_\partial \in KKR^{-(s-r+1)}(\mathbb{A}_\partial)$ *be the boundary class attached to a given disordered IQPV* (J,ϕ) *of index* (r,s)*. Assume that the reference disordered IQPV is local. Then the strong topological invariant at the boundary is given by*

$$\lambda^{\omega}_{d-1}([(J;\phi)]_\partial) = [\ker(T)] \in KR^{-(s-r+2-d)}(\mathbb{C}),$$

where

$$T := \hat{\pi}^{\omega}(\hat{J}) \otimes \omega_{d-1} + (1+\hat{\pi}^{\omega}(\hat{J})^2)^{1/2} \sum_{j=1}^{d-1} \frac{X_j}{\sqrt{1+|X|^2}} \otimes \omega_{d-1} k_j \in \mathcal{L}(\widehat{\mathcal{W}} \otimes \Lambda(\mathbb{C}^{d-1})).$$

Here, $\Lambda(\mathbb{C}^{d-1})$ *is considered as trivially graded,* $\omega_{d-1} := k_1 \cdots k_{d-1} j_1 \cdots j_{d-1}$ *and* $\ker(T)$ *is equipped with the* $C\ell_{r+d-1,s}$*-representation* ϕ *defined by* $\phi(k_a) := K_a \otimes \omega_{d-1}$ *for* $a = 1,\ldots,r$*,* $\phi(k_{r+n}) := 1 \otimes j_n\omega_{d-1}$ *for* $n = 1,\ldots,d-1$ *and* $\phi(j_\alpha) := J_\alpha \otimes \omega_{d-1}$ *for* $\alpha = 1,\ldots,s$*.*

Analogously, for a complex disordered IQPV $(J;\phi)$ *we have*

$$\lambda^{\omega}_{d-1}([(J;\phi)]_\partial) = [\ker(T)] \in K^{-(s-r+2-d)}(\mathbb{C}),$$

where $T \in \mathcal{L}(\widehat{\mathcal{V}} \otimes \Lambda(\mathbb{C}^d))$ *and* $\phi : \mathbb{C}\ell_{r+d-1,s} \to \mathcal{L}(\widehat{\mathcal{V}} \otimes \Lambda(\mathbb{C}^d))$ *have the same form as in the real case.*

Proof. The complex case can be proved in exactly the same way as the real case and will therefore not be discussed in the following.

Recall from Definition 5.2.11 that the boundary topological phase attached to a disordered IQPV $(J;\phi)$ and local reference disordered IQPV corresponds to the Kasparov class

$$[(J;\phi)]_\partial = [(\mathcal{H}_{\mathbb{A}_\partial} \otimes C\ell_{0,1}, \varphi, \hat{J} \otimes j_1)] \in KKR(C\ell_{s,r}, \mathbb{A}_\partial \otimes C\ell_{0,1}),$$

where $\hat{J} = s(J)$. The intersection product of this class and the class $[\lambda^{\omega}_{d-1}]$ is of the form

$$\tau_{C\ell_{0,d-1}}([(J;\phi)]_\partial) \otimes_{\mathbb{A}_\partial \otimes C\ell_{0,1} \widehat{\otimes} C\ell_{0,d-1}} \tau_{C\ell_{0,1}}([\lambda^{\omega}_{d-1}]) \in KKR(C\ell_{s,r} \widehat{\otimes} C\ell_{0,d-1}, C\ell_{0,1}).$$

where τ denotes the natural isomorphism of Definition 3.1.21. We have

$$\begin{aligned}\tau_{C\ell_{0,d-1}}([(J;\phi)]_\partial) &= [(\mathcal{H}_{\mathbb{A}_\partial} \otimes C\ell_{0,1} \widehat{\otimes} C\ell_{0,d-1}, \varphi \otimes \mathrm{id}_{C\ell_{0,d-1}}, \hat{J} \otimes j_1 \otimes 1)] \\ \tau_{C\ell_{0,1}}([\lambda^{\omega}_{d-1}]) &= \Big[\Big(\mathcal{W}_\partial \otimes \Lambda(\mathbb{C}^{d-1}) \widehat{\otimes} C\ell_{0,1}, \pi_\partial \otimes \mathrm{id}_{C\ell_{0,d-1}} \otimes \mathrm{id}_{C\ell_{0,1}}, F^{(d-1)} \otimes 1\Big)\Big]\end{aligned}$$

We apply on $\tau_{C\ell_{0,d-1}}([(J;\phi)]_\partial)$ the orientation-preserving $*$-isomorphism $C\ell_{0,1} \widehat{\otimes} C\ell_{0,d-1} \cong C\ell_{0,d-1} \widehat{\otimes} C\ell_{0,1}$ defined by $j_1 \otimes 1 \mapsto 1 \otimes j_1$ and $1 \otimes j_n \mapsto -j_n \otimes 1$ for $n = 1,\ldots,d-1$, leading to

$$\tau_{C\ell_{0,d-1}}([(J;\phi)]_\partial) = [(\mathcal{H}_{\mathbb{A}_\partial} \otimes C\ell_{0,d-1} \widehat{\otimes} C\ell_{0,1}, \varphi', \hat{J} \otimes 1 \otimes j_1)],$$

where $\varphi' : C\ell_{s,r} \otimes C\ell_{0,d-1} \to \mathbb{B}(\mathcal{H}_{\mathbb{A}_\partial} \otimes C\ell_{0,d-1} \widehat{\otimes} C\ell_{0,1})$ is given by

$$\varphi'(k_\alpha \otimes 1) = J_\alpha \otimes 1 \otimes j_1, \quad \varphi'(j_a \otimes 1) = K_a \otimes 1 \otimes j_1, \quad \varphi'(1 \otimes j_n) = 1 \otimes -j_n \otimes 1$$

for $\alpha = 1, \ldots, s$, $a = 1, \ldots, r$ and $n = 1, \ldots, d-1$.

Now the tensor product of the Hilbert modules is given by

$$\begin{aligned}&\left(\mathcal{H}_{\mathbb{A}_\partial} \otimes C\ell_{0,d-1} \widehat{\otimes} C\ell_{0,1}\right) \otimes_{\pi^\omega_\partial \otimes \mathrm{id}_{C\ell_{0,d-1}} \widehat{\otimes} \mathrm{id}_{C\ell_{0,1}}} \left(\mathcal{W}_\partial \otimes \Lambda(\mathbb{C}^{d-1}) \widehat{\otimes} C\ell_{0,1}\right)\\ \cong&\mathcal{H}_{\mathbb{A}_\partial} \otimes_{\pi^\omega_\partial} \mathcal{W}_\partial \otimes \Lambda(\mathbb{C}^{d-1}) \widehat{\otimes} C\ell_{0,1},\end{aligned}$$

where we used that $C\ell_{0,d-1} \cdot \Lambda(\mathbb{C}^{d-1}) = \Lambda(\mathbb{C}^{d-1})$. Since the representation π^ω_∂ is unital, we have

$$\mathcal{H}_{\mathbb{A}_\partial} \otimes_{\pi^\omega_\partial} \mathcal{W}_\partial \cong \ell^2(\mathbb{N}) \otimes \mathcal{W}_\partial = \widehat{\mathcal{W}}.$$

Thus, we obtain the graded real Hilbert space $\widehat{\mathcal{W}} \otimes \Lambda(\mathbb{C}^{d-1}) \widehat{\otimes} C\ell_{0,1}$.

The orientation-preserving $*$-isomorphism $C\ell_{s,r} \otimes C\ell_{0,d-1} \cong C\ell_{s,r+d-1}$ of Equation (2.4) maps the $*$-morphism φ' onto $\psi : C\ell_{s,r+d-1} \to \mathbb{B}(\widehat{\mathcal{W}} \otimes \Lambda(\mathbb{C}^{d-1}) \widehat{\otimes} C\ell_{0,1})$ with $\psi(k_\alpha) = \varphi'(k_\alpha \otimes 1)$, $\psi(j_a) = \varphi'(j_a \otimes 1)$ and $\psi(j_{r+n}) = \varphi'(1 \otimes j_n)$ for all $\alpha = 1, \ldots, s$, $a = 1, \ldots, r$ and $n = 1, \ldots, d-1$.

It remains to determine the product operator. We follow [17, Sec. 3.3].

First we need a X_j-connection over $\mathcal{H}_{\mathbb{A}_\partial} \otimes_{\pi^\omega_\partial} \mathcal{W}_\partial$. Such a connection, denoted by $1 \otimes_\nabla X_j$, has already been derived in [17, Sec. 3.3]:

$$(1 \otimes_\nabla X_j)(|m\rangle \otimes a) \otimes x := (|m\rangle \otimes a) \otimes X_j(x) + (|m\rangle \otimes 1) \otimes [X_j, \pi^\omega_\partial(a)]x = (|m\rangle \otimes 1) \otimes X_j \pi^\omega_\partial(a)x$$

for all $(|m\rangle \otimes a) \otimes x \in \mathcal{H}_{\mathbb{A}_\partial} \otimes_{\pi^\omega_\partial} \mathcal{W}_\partial$. The isomorphism $\mathcal{H}_{\mathbb{A}_\partial} \otimes_{\pi^\omega_\partial} \mathcal{W}_\partial \cong \ell^2(\mathbb{N}) \otimes \mathcal{W}_\partial$ is on the level of elementary tensors given by

$$(|m\rangle \otimes a) \otimes x \mapsto |m\rangle \otimes \pi^\omega_\partial(a)x.$$

The operator $1 \otimes_\nabla X_j$ is thereby mapped onto $1 \otimes X_j$. The corresponding $F^{(d-1)}$-connection is given by

$$F := \sum_{j=1}^{d-1} \frac{X_j}{\sqrt{1 + |X|^2}} \otimes k_j \in \mathcal{L}(\widehat{\mathcal{W}} \otimes \Lambda(\mathbb{C}^{d-1})).$$

It is easy to see that the operator $\hat{J} \otimes 1$ over $\mathcal{H}_{\mathbb{A}_\partial} \otimes_{\pi^\omega_\partial} \mathcal{W}_\partial$ is mapped onto $\hat{\pi}^\omega(\hat{J})$ via the isomorphism $\mathcal{H}_{\mathbb{A}_\partial} \otimes_{\pi^\omega_\partial} \mathcal{W}_\partial \cong \widehat{\mathcal{W}}$.

Now we have everything in place in order to apply Proposition 3.1.25. The premises of this proposition are satisfied, and therefore the intersection product is given by the class

$$\left[\left(\widehat{\mathcal{W}} \otimes \Lambda(\mathbb{C}^{d-1}) \widehat{\otimes} C\ell_{0,1}, \psi, \hat{\pi}^\omega(\hat{J}) \otimes 1 \otimes j_1 + \left(1 \otimes 1 + \hat{\pi}^\omega(\hat{J})^2 \otimes 1\right)^{1/2} F \otimes 1\right)\right] \tag{6.6}$$

in $KKR(C\ell_{s,r+d-1}, C\ell_{0,1})$.

In order to classify the resulting class in $KR^{-(s-r-d+2)}(\mathbb{C})$ as in Subsection 3.1.3, it remains to calculate the class in $KR^{r+d-1,s}(\mathbb{C}) \cong KR^{-(s-r-d+2)}(\mathbb{C})$ which is determined by the inverse image of the Kasparov class above under the isomorphism in Theorem 3.1.36.

To that end, we first apply the even unitary isomorphism

$$u := 1 \otimes 1 \oplus 1 \otimes j_1 : \Lambda(\mathbb{C}^{d-1})^{(\bar{0})} \widehat{\otimes} C\ell_{0,1} \oplus \Lambda(\mathbb{C}^{d-1})^{(\bar{1})} \widehat{\otimes} C\ell_{0,1} \longrightarrow \Lambda(\mathbb{C}^{d-1})^{(\bar{0})} \otimes C\ell_{0,1} \oplus |\Lambda(\mathbb{C}^{d-1})^{(\bar{1})}| \otimes C\ell_{0,1} \cong |\Lambda(\mathbb{C}^{d-1})| \otimes C\ell_{0,1},$$

where, for the sake of readability, we denote by $|\Lambda(\mathbb{C}^{d-1})^{(\bar{1})}|$ and $|\Lambda(\mathbb{C}^{d-1})|$ the Hilbert spaces $\Lambda(\mathbb{C}^{d-1})^{(\bar{1})}$ and $\Lambda(\mathbb{C}^{d-1})$ equipped with the trivial grading.

Since the grading on $\Lambda(\mathbb{C}^{d-1})$ is defined by the grading operator $\omega_{d-1} = k_1 \cdots k_{d-1} j_1 \cdots j_{d-1}$, we can also write this isomorphism as

$$u = P_+ \otimes 1 + P_- \otimes j_1 : \Lambda(\mathbb{C}^{d-1}) \widehat{\otimes} C\ell_{0,1} \longrightarrow |\Lambda(\mathbb{C}^{d-1})| \otimes C\ell_{0,1},$$

where $P_\pm := \frac{1}{2}(1 \pm \omega_{d-1})$ denote the projections onto the even/odd part of $\Lambda(\mathbb{C}^{d-1})$. We have

$$u(1 \otimes j_1)u^* = \omega_{d-1} \otimes j_1, \quad u(k_n \otimes 1)u^* = \omega_{d-1} k_n \otimes j_1, \quad u(j_n \otimes 1)u^* = \omega_{d-1} j_n \otimes j_1 \quad \forall n = 1, \ldots, d-1.$$

This isomorphism maps the Kasparov class (6.6) onto

$$\left[\left(\widehat{\mathcal{W}} \otimes |\Lambda(\mathbb{C}^{d-1})| \otimes C\ell_{0,1}, \psi', \hat{\pi}^\omega(\hat{J}) \otimes \omega_{d-1} \otimes j_1 + ((1 + \hat{\pi}^\omega(\hat{J})^2)^{1/2} \otimes \omega_{d-1})F \otimes j_1\right)\right] \quad (6.7)$$

in $KKR(C\ell_{s,r+d-1}, C\ell_{0,1})$, where $\psi'(k_\alpha) = J_\alpha \otimes \omega_{d-1} \otimes j_1$ for $\alpha = 1, \ldots, s$, $\psi'(j_a) = K_a \otimes \omega_{d-1} \otimes j_1$ for $a = 1, \ldots, r$ and $\psi'(j_{r+n}) = 1 \otimes -\omega_{d-1} j_n \otimes j_1 = 1 \otimes j_n \omega_{d-1} \otimes j_1$ for $n = 1, \ldots, d-1$.

It can now easily be checked that the inverse of the isomorphism of Theorem 3.1.36 maps the Kasparov class (6.7) onto the KR-class

$$\left[\left(\hat{\pi}^\omega(\hat{J}) \otimes \omega_{d-1} + ((1 + \hat{\pi}^\omega(\hat{J})^2)^{1/2} \otimes \omega_{d-1})F, \phi\right)\right] \in KR^{r+d-1,s}(\mathbb{C}) = KR^{-(s-r-d+2)}(\mathbb{C}),$$

where $\phi : C\ell_{r+d-1,s} \to \mathcal{L}(\widehat{\mathcal{W}} \otimes |\Lambda(\mathbb{C}^{d-1}|)$ with $\phi(k_a) = K_a \otimes \omega_{d-1}$ for $a = 1, \ldots, r$, $\phi(k_{r+n}) = 1 \otimes j_n \omega_{d-1}$ for $n = 1, \ldots, d-1$ and $\phi(j_\alpha) = J_\alpha \otimes \omega_{d-1}$ for $\alpha = 1, \ldots, s$.

Finally, by the results of Subsection 3.1.3, the class is determined by the Clifford module

$$\ker\left(\hat{\pi}^\omega(\hat{J}) \otimes \omega_{d-1} + ((1 + \hat{\pi}^\omega(\hat{J})^2)^{1/2} \otimes \omega_{d-1})F\right) \subseteq \widehat{\mathcal{W}} \otimes |\Lambda(\mathbb{C}^{d-1})|,$$

equipped with Clifford representation ϕ. This proves the theorem.

□

For a given value for the dimension d, a numerical value for the topological invariants of the above theorem can be determined as in the one-dimensional case in Subsection 6.2.1.

We close this section with the following important observation about strong topological invariants at the boundary.

Proposition 6.2.7. *Let*

$$\partial_i : KR^{-(s-r+2)}(\mathbb{A}) \to KR^{-(s-r+1)}(\mathbb{A}_{\partial_i}) \quad \textit{and} \quad \partial_i : K^{-(s-r+2)}(\mathrm{A}) \to K^{-(s-r+1)}(\mathrm{A}_{\partial_i})$$

for $i \in \{1, \ldots, d\}$ *denote the bulk-boundary maps obtained by introducing a boundary perpendicular to the ith direction. Here,* $\mathbb{A}_{\partial_i}$ *and* A_{∂_i} *denote the boundary algebras generated by* $C(\Omega) \otimes \mathrm{End}(W)$ *resp.* $C(\Omega) \otimes \mathrm{End}(V)$ *and the lattice translations parallel to this boundary. Then*

$$\lambda^\omega_{d-1} \circ \partial_i = (-1)^{i-j} \lambda^\omega_{d-1} \circ \partial_j \quad \forall i, j \in \{1, \ldots, d\}, i \neq j, \omega \in \Omega.$$

Proof. We only prove the real case since the proof of the complex case is literally the same. By [17, Thm. 3.4], we have

$$[\partial_d] \otimes_{\mathbb{A}_\partial} [\lambda_{d-1}^\omega] = (-1)^{d-1}[\lambda_d^\omega] \in KKR(\mathbb{A} \otimes C\ell_{0,d}, \mathbb{C}). \tag{6.8}$$

The proof of this theorem in particular shows that

$$[\partial_d] \otimes_{\mathbb{A}_\partial} [\lambda_{d-1}^\omega] = \left[\left(\mathcal{W} \otimes \Lambda(\mathbb{C}^d), \pi^\omega \otimes \mathrm{id}_{C\ell_{0,d}}, F^{(d)}\right)\right].$$

The sign occurring in Equation (6.8) is due to the reordering of the Clifford generators $k_1 \mapsto k_d$ and $k_{j+1} \mapsto k_j$ for $j = 1, \ldots, d-1$, which changes the orientation by the factor $(-1)^{d-1}$.

Therefore, the inspection of the proof of this theorem yields the more general result

$$[\partial_i] \otimes_{\mathbb{A}_{\partial_i}} [\lambda_{d-1}^\omega] = (-1)^{i-1}[\lambda_d^\omega]$$

for all $i = 1, \ldots, d$. It follows

$$[\partial_i] \otimes_{\mathbb{A}_{\partial_i}} [\lambda_{d-1}^\omega] = (-1)^{i-1}[\lambda_d^\omega] = (-1)^{i-1}(-1)^{-j+1}[\partial_j] \otimes_{\mathbb{A}_{\partial_j}} [\lambda_{d-1}^\omega] = (-1)^{i-j}[\partial_j] \otimes_{\mathbb{A}_{\partial_j}} [\lambda_{d-1}^\omega].$$

The formulation in the proposition then follows by the definition of λ_{d-1}^ω and the functoriality of the KKR-groups. □

Proposition 6.2.7 implies that the strong topological invariant is, up to a sign, the same on *all* boundaries. Since topological invariants in the physical context are expected to correspond to boundary currents (e.g. of charge, heat or spin), this sign may be interpreted as an orientation for these currents. For example, in the quantum Hall effect, the strong topological invariant is reflected at the boundary by oriented charge currents along the boundary.

CHAPTER SEVEN

Outlook

In this final chapter we mention some open questions and topics that have not been addressed in this work but are closely related.

Crystalline symmetries

We have established a rigorous approach for the topological classification of disordered topological insulators and superconductors by assuming that the observables are translationally equivariant and symmetric w.r.t. local symmetries. Thus, in our approach, the only kind of symmetries that act non-trivially on the lattice, so-called crystalline symmetries, were translations. However, experiments on topological insulators made apparent that other crystalline symmetries, such as rotation- or mirror-symmetries, can also play an important role for the topological phases.

During the last years, the role of more general crystalline symmetries on the topological phases has been extensively studied for clean systems using the tools of the Fourier–Bloch theory, see [63, 79, 80, 3] for an overview. A rigorous classification of disordered topological insulators and superconductors with crystalline symmetries is missing to the present date. Our construction of the algebra of observables allows for the inclusion of additional symmetries in the same way as for the translational symmetries. Therefore, we consider our approach as suitable for establishing such a classification scheme in the disordered case. Moreover, our construction is formulated explicitly in terms of operators over the physical lattice, thus allowing for a more explicit view on the influence of crystalline symmetries on topological phases.

Higher order topological insulators

We have established the bulk-boundary correspondence for a flat boundary perpendicular to a translational direction of the underlying lattice. It has been proved in [18], on an abstract level, that the bulk-boundary correspondence still holds if the boundary is flat and perpendicular to an arbitrary direction, i.e. not necessarily perpendicular to a translational direction.

An even more general boundary can be considered by allowing boundaries with corners and edges. Such boundaries are physically interesting for the following reason.

In order to consider a bulk-boundary correspondence for systems with crystalline symmetries, e.g. a rotational symmetry, one has to consider boundaries that are invariant under these symmetries. This can only be achieved by boundaries with corners. It has been observed experimentally that in this case the boundary states can exhibit a special type of localisation. Namely, it might happen that the boundary states are localised at the corners or edges corresponding to sub-lattices of co-dimension larger than one. In this case, one calls such systems *higher order topological insulators* or *higher order topological superconductors*. The most prominent experimental realisations of higher order topological insulators are bismuth [75] and SnTe [74].

There are currently no systematic results on the bulk-boundary correspondence of these higher order systems including disorder.

Topological invariants

We have presented a collection of topological invariants that quantify the topological phases in the bulk and at the boundary. As explained there, this picture is not yet complete. In the bulk, we are missing a systematic explicit formula for the torsion valued topological invariants. Kellendonk has already established a general approach for these torsion valued invariants. However, the application of this approach is in our context not yet established on an explicit level.

At the boundary, there is currently no canonical approach for the calculation of the weak topological invariants. This problem is addressed in [20, 67].

Strong disorder

An issue that has not been addressed in this work is the extension of bulk-boundary correspondence and topological phases into the regime of strong disorder. Disorder is called strong if there are states which are localised due to disorder, so-called *Anderson localisation*. These localised states violate the gap condition in the bulk, which is fundamental for the classification of topological phases in terms of K-theory. Therefore, it is currently not possible to include strong disorder into the K-theoretic picture of topological phases.

In the literature, the strong disorder regime is usually approached by first deriving formulas for the numerical topological invariants in the bulk and at the boundary and then showing that these are invariant under strong disorder in a certain sense. This strategy has already been worked out for the complex symmetry classes in [66]. For the real symmetry classes, there are only partial results [19, §5].

Systems with particle-particle interactions

Many interesting phenomena in solid state physics stem from the particle-particle interactions of the electrons. In our consideration, we relied crucially on the assumptions that there is an effective model of free-fermion quasi-particles that governs the physical features of the underlying physical system. However, such an effective model does not exist in general. The most prominent physical example where there is in general no effective free-fermion quasi-particle picture is the *fractional quantum Hall effect* (FQHE) [90, §7.2,7.3]. Nevertheless, there are topological phenomena in the FQHE which are similar to the ones of the QHE [30]. Therefore, a systematic topological classification of interacting systems would be of great importance.

To the present date, there are no systematic K-theoretic approaches for interacting systems.

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Erklärung

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